UNIVERSITY OF TECHNOLOGY SYDNEY

FOR MEMBERS OF SI-TAFE NSW, SAH, AFTRS

APPLICATION TO REGISTER FOR BORROWING

ASSOCIATE MEMBERSHIP

UTS: LIBRARY

Application to register for borrowing at UTS Library by current members of the following institutions:

- Sydney Institute –TAFE NSW
- Sydney Adventist Hospital
- Australian Film, Television and Radio School

Your registration will be valid until 28 February of the following year.

☑ New application or ☐ Renewal

Family Name: ______ Given Name: ______

Title: ☑ Mr ☐ Ms ☐ Mrs ☐ Miss ☐ Dr ☐ Prof ☐ Other

Address: 1/277 LIVING STONE ROAD, MARRICKVILLE

Postcode: NSW2204

Tel:(hm) 02-95730285 (work) 02-9598 6247 (mobile) 0404289750

Email address: (Library notices will be sent by email only)

u.kyawnaing@tafe.nsw.edu.au

Student

Student number: ______

Faculty: ______

Course: ______

My home institution is: ______

Staff

Staff number: 200435

Faculty: Electrotechnology, StGeorge TAFE Sydney Institute

My home institution is: SYDNEY INSTITUTE TAFE-NSW

Important: Please read and sign

1. I understand that my use of the services and facilities of the University Library of the University of Technology, Sydney, is subject to Section 18-Use of the University Library, of the UTS Student and Related Rules

2. I acknowledge that information may be sent to my home Library if I do not return items to UTS Library, and that further sanctions may then apply at my home Library.

3. It is the Library's policy to send all notices via email. I understand it is my responsibility to check my email regularly fro notification of recalls and that failure to provide an email address for the receipt of Library notices will not be accepted as a reason for my being unaware of matters so notified. I may ring (02) 9514 3666 to obtain information on my record or access my borrowing record online.

Signature: [signature] Date: ______

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Signature: KyawNaing Date: ____________

Library staff use only

☐ Sydney Institute - TAFE NSW

☐ Sydney Adventist Hospital

☐ Australian Film, Television and Radio School

Please tick each step when completed:

☐ 1. Photo ID sighted

Current membership of home institution verified.

☐ 2. Payment collected: ☐ $50 for students ☐ $50 for staff (SI-TAFE NSW only)

☐ 3. UTS Photo ID card created for card-type Library associate EMERALD

ID number created AB

Barcode on Photo ID

Name of Library staff member:______________________________ Date:__________

- ATTACH PRINTOUT FROM HOME LIBRARY
- REGISTRATION FORM TO BE SENT TO BACKROOM STAFF TO VERIFY PTYPE, CONTACT DETAILS AND ID NUMBER EG ABMXXXXXX
- FORMS TO BE FILED BY INSTITUTION.

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CONTENTS

TOPIC 1

FURTHER ALGEBRA AND ARITHMETIC

(A) Harder Maths (2 Unit)

A sound knowledge of HSC Maths (2 Unit) algebra and arithmetic is essential.

Example 1: Rationalise the denominators and simplify:

(i) $\dfrac{1+\sqrt{5}}{2-\sqrt{5}}$ (ii) $\dfrac{1}{5+\sqrt{2}}+\dfrac{1}{5-\sqrt{2}}$

Solution 1:

(i) $$\frac{1+\sqrt{5}}{2-\sqrt{5}}=\frac{1+\sqrt{5}}{2-\sqrt{5}}\times\frac{2+\sqrt{5}}{2+\sqrt{5}}$$
$$=\frac{2+\sqrt{5}+2\sqrt{5}+5}{4-5}$$
$$=\frac{7+3\sqrt{5}}{-1}=-\left(7+3\sqrt{5}\right)\ \#$$

(ii) $$\frac{1}{5+\sqrt{2}}+\frac{1}{5-\sqrt{2}}=\frac{1}{5+\sqrt{2}}\times\frac{5-\sqrt{2}}{5-\sqrt{2}}+\frac{1}{5-\sqrt{2}}\times\frac{5+\sqrt{2}}{5+\sqrt{2}}$$
$$=\frac{5-\sqrt{2}}{25-2}+\frac{5+\sqrt{2}}{25-2}$$
$$=\frac{5-\sqrt{2}+5+\sqrt{2}}{23}=\frac{10}{23}\ \#$$

Example 2: Solve:

(i) $x^2-6x-7<0$ (ii) $x^2+2x-2-\dfrac{3}{x^2+2x}=0$

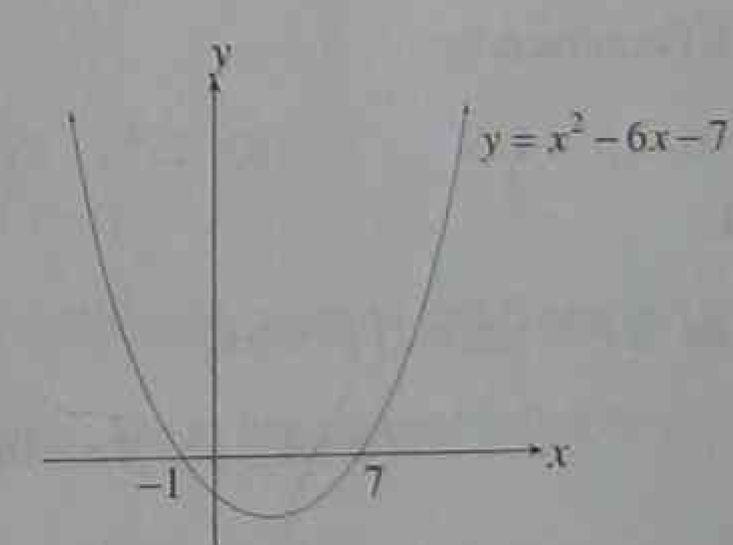
Solution 2:

(i) $x^2-6x-7<0$

$(x-7)(x+1)<0$

From the graph, $y<0$,

when $-1<x<7$ #



(ii) $x^2+2x-2-\dfrac{3}{x^2+2x}=0$

Let $y=x^2+2x$

$$\text{i.e.}\quad y-2-\frac{3}{y}=0$$

$$y^2-2y-3=0$$

$$(y-3)(y+1)=0$$

$$\text{i.e.}\quad \left(x^2+2x-3\right)\left(x^2+2x+1\right)=0$$

$$(x+3)(x-1)(x+1)^2=0$$

$$x=-3,-1,\ 1 \quad \#$$

Example 3: Simplify as far as possible.

(i) $1-\dfrac{1}{\dfrac{2x}{x-1}-1}$ (ii) $\dfrac{9^x-4^x}{9^x+6^x}$

Solution 3:

(i)
$$1-\frac{1}{\dfrac{2x}{x-1}-1}=1-\frac{1}{\dfrac{2x-(x-1)}{x-1}}$$
$$=1-\frac{x-1}{x+1}$$
$$=\frac{(x+1)-(x-1)}{x+1}=\frac{2}{x+1} \quad \#$$

(ii)
$$\frac{9^x-4^x}{9^x+6^x}=\frac{\left(3^x-2^x\right)\left(3^x+2^x\right)}{3^x\left(3^x+2^x\right)}$$
$$=\frac{3^x-2^x}{3^x}=1-\frac{2^x}{3^x}=1-\left(\frac{2}{3}\right)^x \quad \#$$

Example 4: Factorise fully:

(i) $2p^3+128$ (ii) $2x^2y-8y^3$

Solution 4:

(i)
$$2p^3+128=2\left(p^3+64\right)$$
$$=2(p+4)\left(p^2-4p+16\right) \quad \#$$

(ii)
$$2x^2y-8y^3=2y\left(x^2-4y^2\right)$$
$$=2y(x-2y)(x+2y) \quad \#$$

(B) Harder Inequalities

(i) Standard Inequalities

Standard inequalities are those which do not involve absolute values. The essential steps in solving these inequalities can be summarised as follows:

Step 1: Exclude solutions by finding the value(s) of x for which the denominator is zero.

Step 2: Multiply both sides by the square of the denominator (since a square is always positive, the inequality sign is unchanged).

Step 3: Solve the resultant equation for x. This will often require a sketch.

Step 4: Combine the solution from Step 3 with any excluded values from Step 1 to arrive at the final answer.

Example 1: Solve:

(i) $\dfrac{1}{x-1}\leq 3$ (ii) $\dfrac{x}{x^2-1}\geq 0$

Solution 1:

(i) Step 1: $x-1\neq 0,\ \therefore\ x\neq 1.$

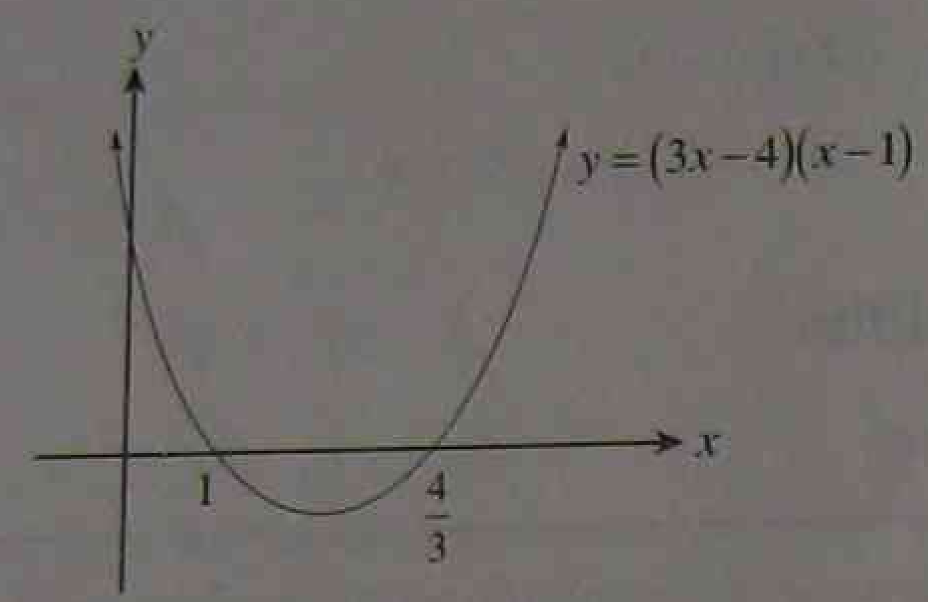
Step 2: Multiply both sides by $(x-1)^2$

$$\text{i.e.}\quad (x-1)\leq 3(x-1)^2$$
$$x-1\leq 3\left(x^2-2x+1\right)$$
$$\text{i.e.}\quad 0\leq 3x^2-7x+4$$
$$0\leq (3x-4)(x-1)$$

Step 3: To solve this it is best to sketch the function first (see over).

Step 4: Noting $x\neq 1$ from Step 1, hence the required solution is:

$$x<1,\ x\geq 1\frac{1}{3} \quad \#$$



(ii) Step 1: $x^2-1=0 \quad \therefore \quad x \neq \pm 1$

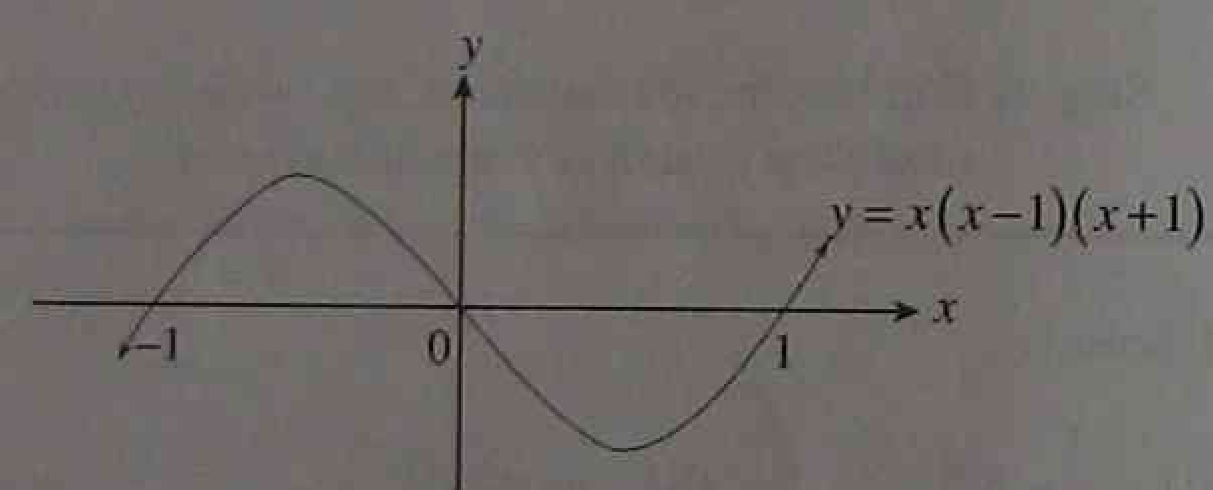
Step 2: Multiple both sides by $\left(x^2-1\right)^2$.

i.e. $x\left(x^2-1\right) \geq 0$

$x(x-1)(x+1) \geq 0$ #

Step 3: To solve this, it is best to sketch the function first (see below).

Step 4: Noting $x \neq \pm 1$, hence the required solution is $-1<x \leq 0$ or $x>1$ #



(ii) Inequalities Involving Absolute Values

Properties of aboslute values and inequalities:

- If $|x+a|<|x+b|$ or $|x+a|>|x+b|$, then $(x+a)^2<(x+b)^2$ or $(x+a)^2>(x+b)^2$
- If $|x+a|<x+b$, then $(x+a)^2<(x+b)^2$ provided $x+b>0$ i.e. $x>-b$
- If $|x+a|>x+b$, then it does **not** follow that $(x+a)^2>(x+b)^2$
- $|a|=\sqrt{a^2}$, since by definition $\sqrt{a^2} \geq 0$ for all values of a.
- $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}$

Example 1: Solve:

(i) $|x-2|<x$ (ii) $\frac{1}{|2x-1|}<3$

(iii) $\left|\frac{x+1}{x-2}\right|<2$ (iv) $\frac{\left|x^2-4\right|}{x+2} \geq 1$

Solution 1:

(i) $|x-2|<x$

Let $|x-2|=\sqrt{(x-2)^2}$

i.e. $\sqrt{(x-2)^2}<x$, provided $x>0$

Squaring both sides, gives:

$(x-2)^2<x^2$

$x^2-4x+4<x^2$

$4x>4$

i.e. $x>1$ which satisfies $x>0 \quad \therefore \quad x>1$ #

(ii) $\frac{1}{|2x-1|}<3$

$2x-1 \neq 0$ i.e. $x \neq \frac{1}{2}$

Multiplying both sides by $|2x-1|$, gives:

$1<3|2x-1|$

Let $|2x-1|=\sqrt{(2x-1)^2}$

i.e. $1<3\sqrt{(2x-1)^2}$

Squaring both sides, gives:

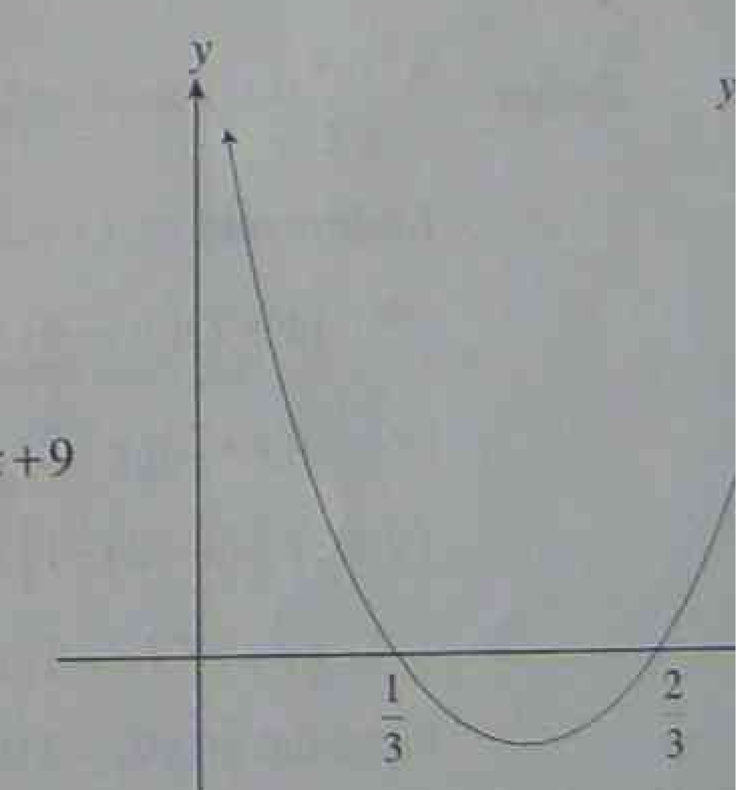
$1<9(2x-1)^2$

$1<9\left(4x^2-4x+1\right)=36x^2-36x+9$

i.e. $36x^2-36x+8>0$

$9x^2-9x+2>0$

$(3x-2)(3x-1)>0$



From the graph, $y>0$

when $x<\frac{1}{3}$ or $x>\frac{2}{3}$ #

(iii) $\left|\frac{x+1}{x-2}\right|<2$ i.e. $\frac{|x+1|}{|x-2|}<2$

$x-2\neq 0$ i.e. $x\neq 2$

Multiplying both sides by $|x-2|>0$, gives:

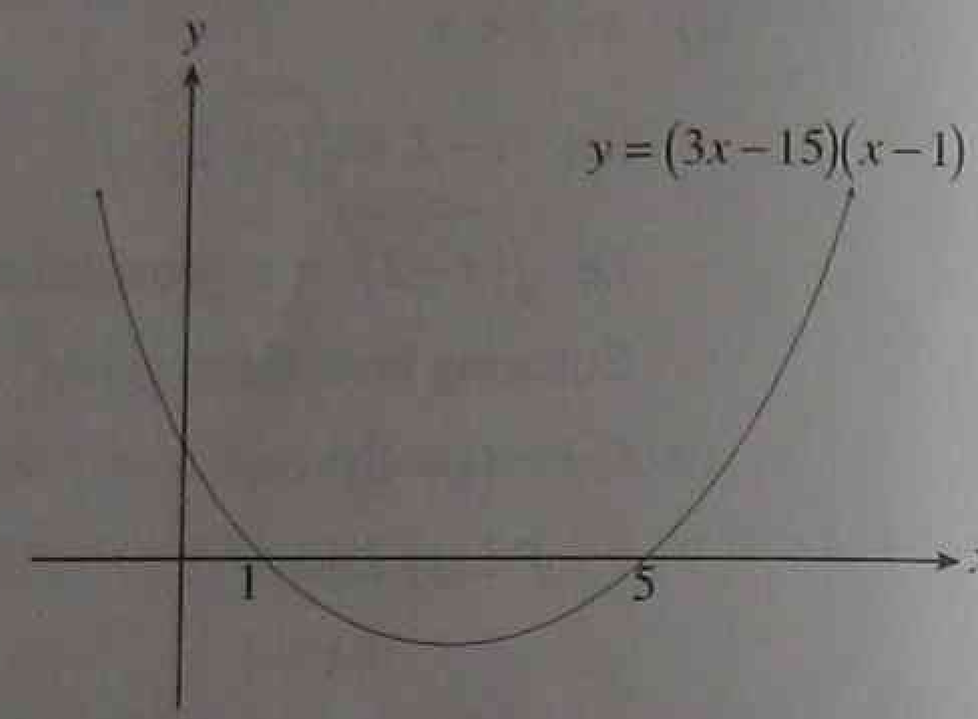
$|x+1|<2|x-2|$

Let $|x+1|=\sqrt{(x+1)^2}$ and $|x-2|=\sqrt{(x-2)^2}$

i.e. $\sqrt{(x+1)^2}<2\sqrt{(x-2)^2}$

Squaring both sides, gives:

$$(x+1)^2<4(x-2)^2$$
$$x^2+2x+1<4(x^2-4x+4)$$
$$x^2+2x+1<4x^2-16x+16$$
$$\text{i.e. } 0<3x^2-18x+15$$
$$0<(3x-15)(x-1)$$



From the graph $0<y$,

when $x<1$ or $x>5$. #

(iv) $\frac{x^2-4}{x+2}\geq 1, x+2\neq 0$ i.e. $x\neq -2$

Since $x^2-4\geq 0$, $\therefore$ $x+2>0$ for $\frac{x^2-4}{x+2}\geq 1$.

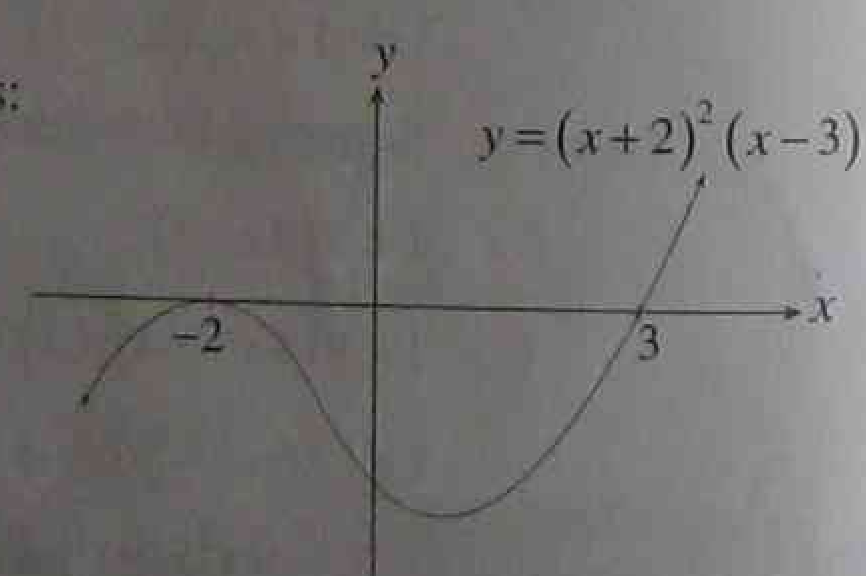
i.e. $x>-2$

Now, consider each case separately:

Either: $\frac{x^2-4}{x+2}\geq 1, x\neq -2$ and $x>-2$

Multiplying by $(x+2)^2>0$, gives:

$$(x+2)(x^2-4)\geq(x+2)^2$$
$$(x+2)(x+2)(x-2)\geq(x+2)^2$$
$$(x+2)^2[(x-2)-1]\geq 0$$
$$(x+2)^2(x-3)\geq 0$$



From the graph, $y\geq 0$, when $x\geq 3$.

Combining the solution

i.e. $x>-2$ and $x\geq 3$

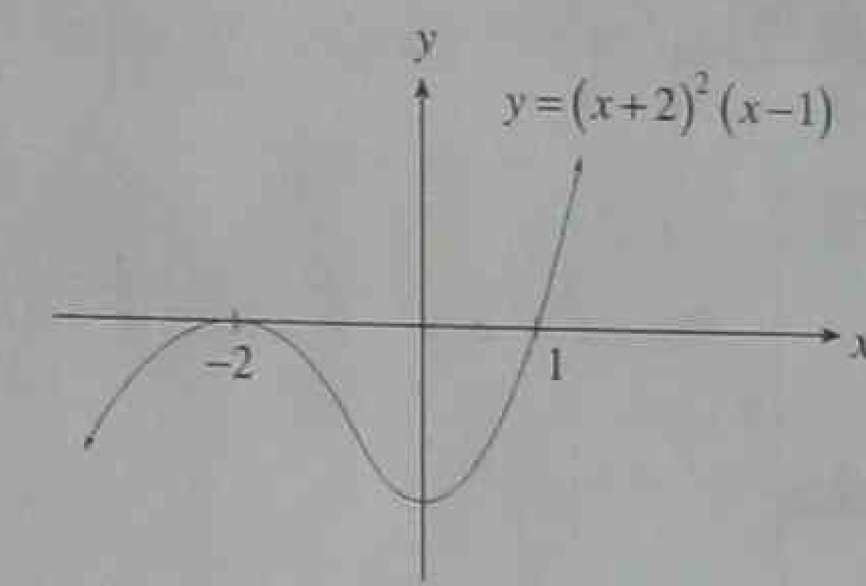
$\therefore$ $x\geq 3$. #

Or: $\frac{-(x^2-4)}{x+2}\geq 1$, $x\neq -2$ and $x>-2$

$\frac{(x^2-4)}{x+2}\leq -1$ (multiplying by -1 reverses the inequality sign)

Multiplying by $(x+2)^2>0$, gives:

$$(x+2)(x^2-4)\leq-(x+2)^2$$
$$(x+2)^2(x-2)\leq-(x+2)^2$$
$$(x+2)^2[(x-2)+1]\leq 0$$
$$(x+2)^2(x-1)\leq 0$$



From the graph, $y\leq 0$ when $x\leq 1$.

Combining the solutions:

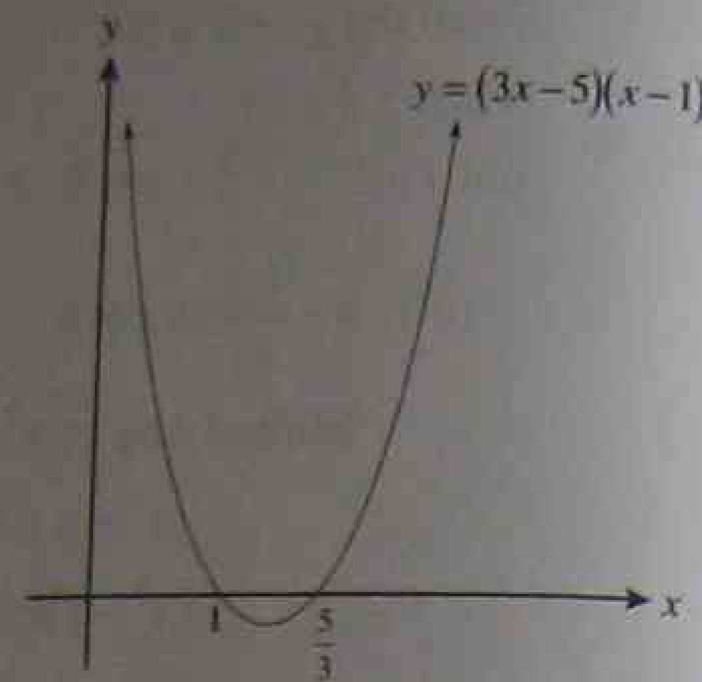
i.e. $x>-2$, $x\neq -2$ and $x\leq 1$

$\therefore$ $-2<x\leq 1$

Hence, complete solution sets for $\left|\frac{x^2-4}{x+2}\right|\geq 1$ are $-2<x\leq 1$ and $x\geq 3$. #

Squaring both sides,

$$(x-2)^2 \le (2x-3)^2$$
$$x^2-4x+4 \le 4x^2-12x+9$$
$$0 \le 3x^2-8x+5$$
$$0 \le (3x-5)(x-1)$$



From the graph $y \ge 0$ when $x \le 1$ or $x \ge \frac{5}{3}$

but $x > \frac{3}{2}$ $\therefore$ $x \ge \frac{5}{3}$. #

(iii) $x+1 = 2x$

$\sqrt{(x+1)^2} = 2x$, provided $x \ge 0$ squaring both sides, gives:

$$(x+1)^2 = 4x^2$$
$$x^2+2x+1 = 4x^2$$
$$3x^2-2x-1=0$$
$$(3x+1)(x-1)=0$$

$\therefore$ $x = -\frac{1}{3}, 1$ but $x \ge 0$ $\therefore$ $x=1$ is the only solution. #

(iv) $x-1 + x-2 = 7$, rearranging

$$x-1 = 7 - x-2$$
$$\sqrt{(x-1)^2} = 7-\sqrt{(x-2)^2}$$

Squaring both sides, gives:

$$(x-1)^2 = 49-14\sqrt{(x-2)^2}+(x-2)^2$$
$$x^2-2x+1 = 49-14\sqrt{(x-2)^2}+x^2-4x+4$$

i.e. $14\sqrt{(x-2)^2} = 52-2x$

$$7\sqrt{(x-2)^2} = 26-x$$

Squaring both sides, gives:

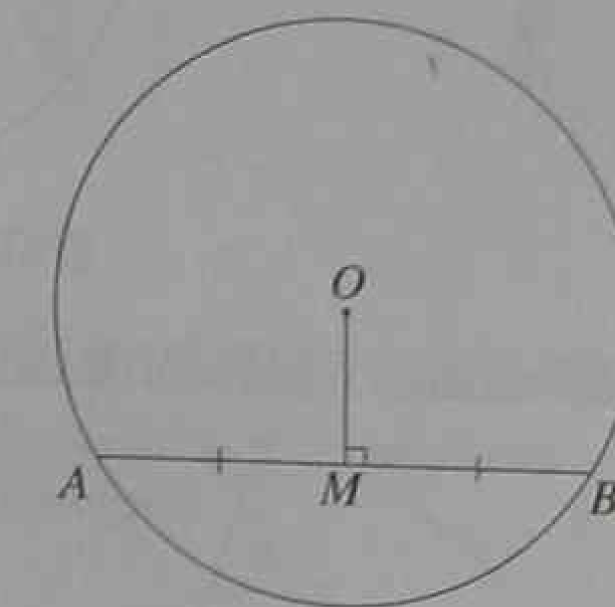
$$49(x-2)^2 = 676-52x+x^2$$
$$49\left(x^2-4x+4\right) = 676-52x+x^2$$
$$48x^2-144x-480=0$$
$$x^2-3x-10=0$$
$$(x-5)(x+2)=0$$

$\therefore$ $x=5$ or $x=-2$ #

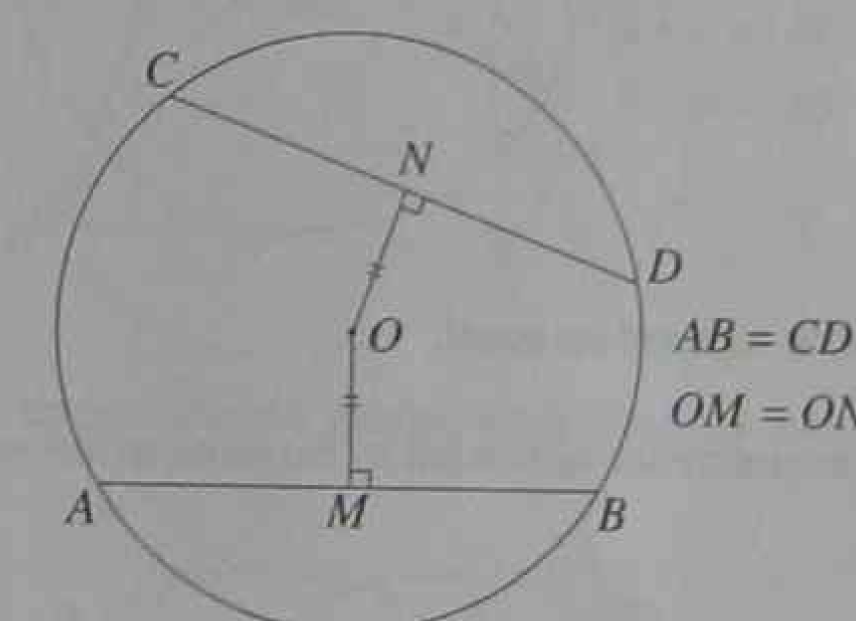
CIRCLE GEOMETRY

(A) Chord Properties

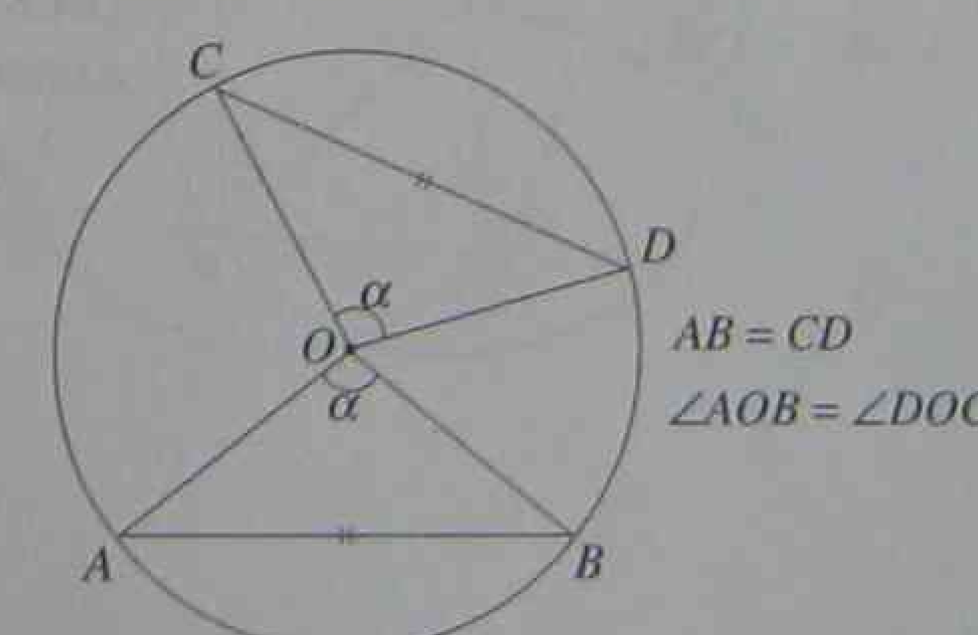
1. The perpendicular from the centre of the circle to the chord bisects the chord, **and conversely** the line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.



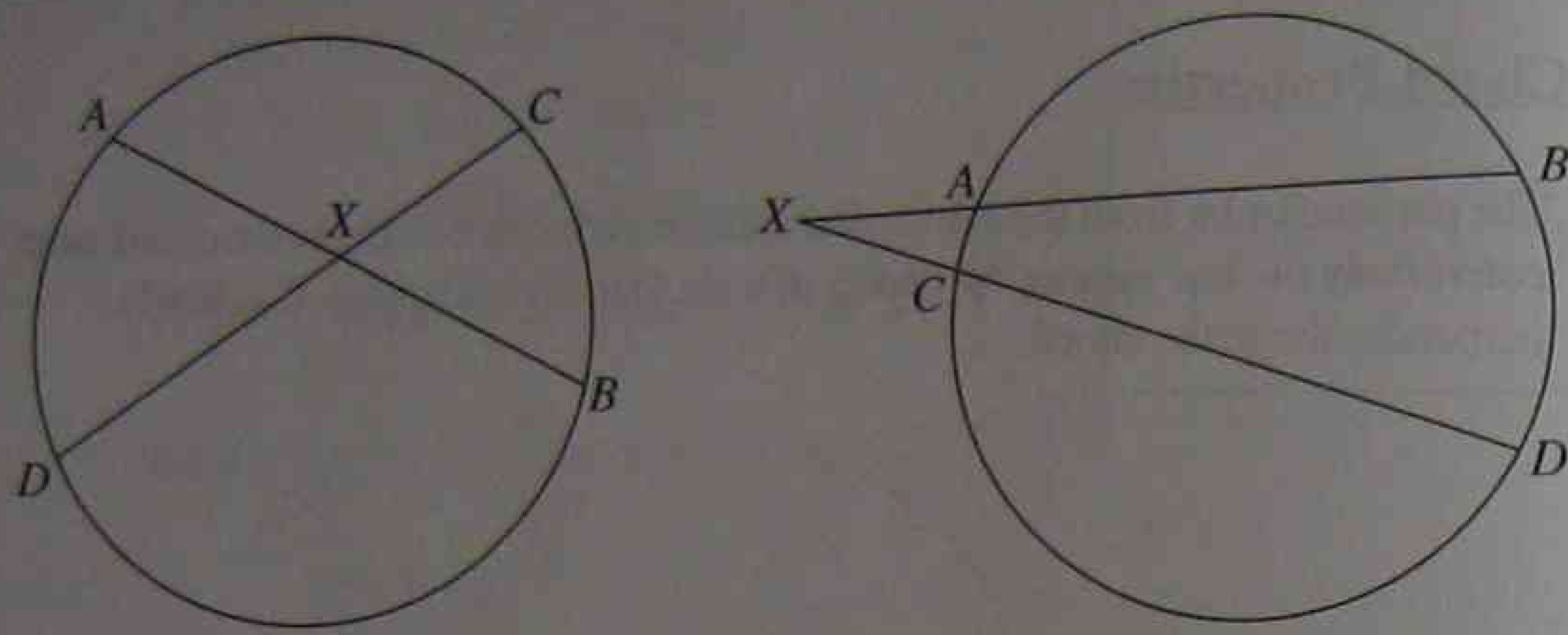
2. Chords in a circle which are equidistant from the centre are equal, **and conversely** equal chords in a circle (or in equal circles) are equidistant from the centre.



3. Equal angles at the centre of a circle stand on equal chords, **and conversely** equal chords subtend equal angles at the centre of a circle.

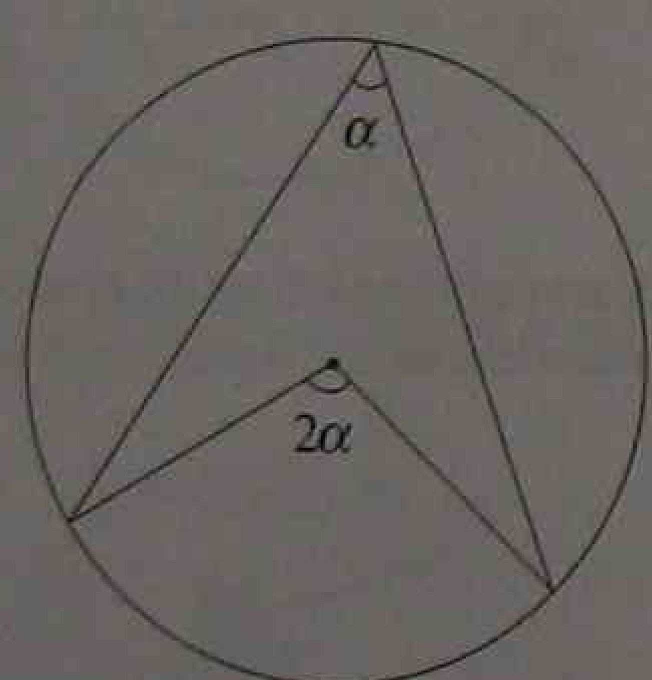


4. The products of the intercepts of two intersecting chords are equal.
   i.e. $AX.XB = CX.XD$



(B) Angle Properties

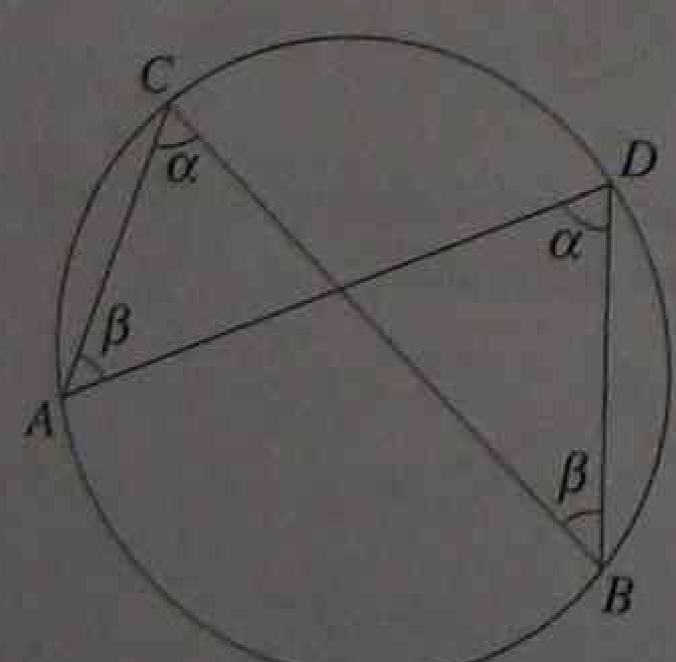
5. The angle at the centre is twice the angle at the circumference subtended by the same arc.



6. Angles in the same segment are equal.

   or

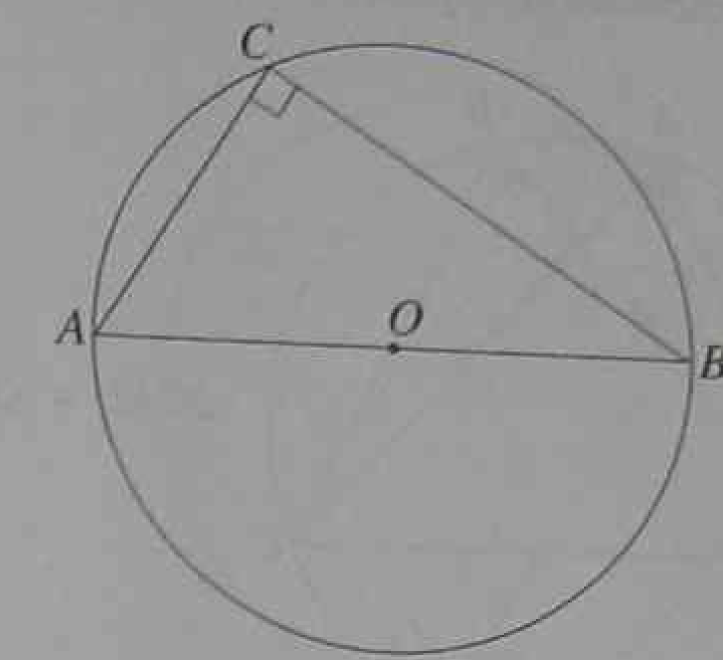
   Angles at the circumference subtended by the same arc are equal.



$\angle ACB = \angle ADB$

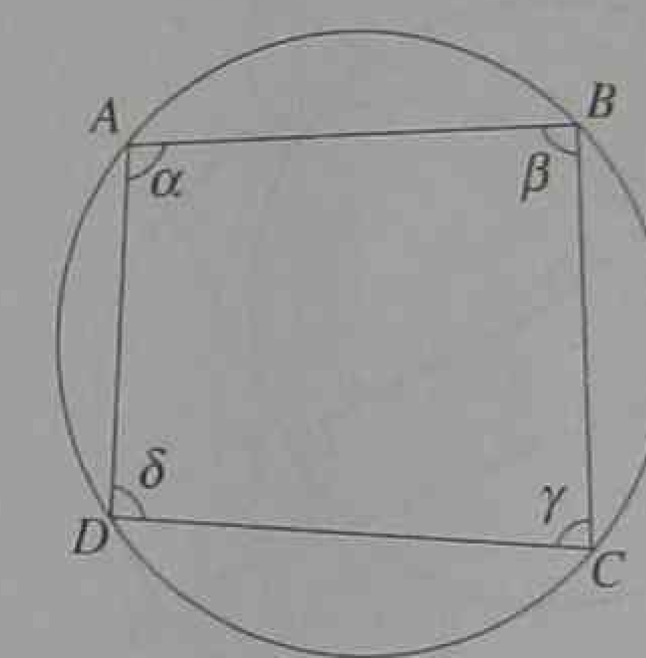
$\angle CBD = \angle CAD$

7. The angle in a semi-circle is a right angle.



(C) Cyclic Quadrilaterals

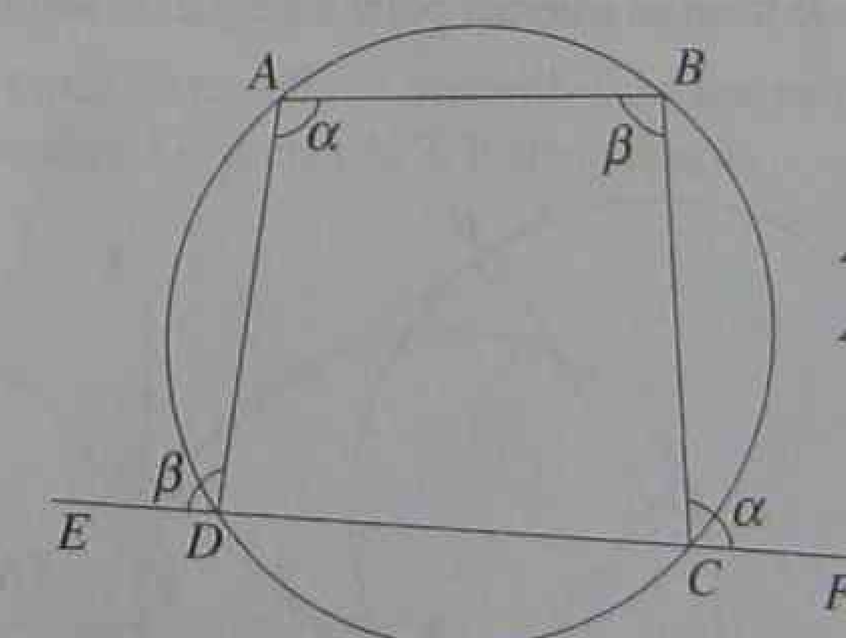
8. Opposite angles of a cyclic quadrilateral are supplementary (i.e. sum to 180°)
9. If the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic (also a test for 4 points to be concyclic).



$\alpha + \gamma = 180°$

$\beta + \delta = 180°$

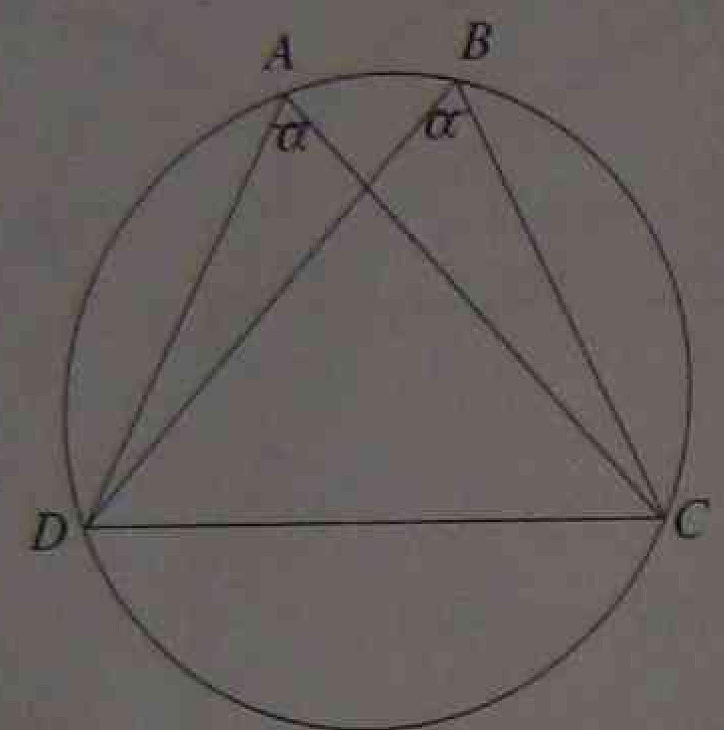
10. The exterior angle to a vertex of a cyclic quadrilateral equals the interior opposite angle.



$\angle ABC = \angle ADE$

$\angle BCF = \angle BAD$

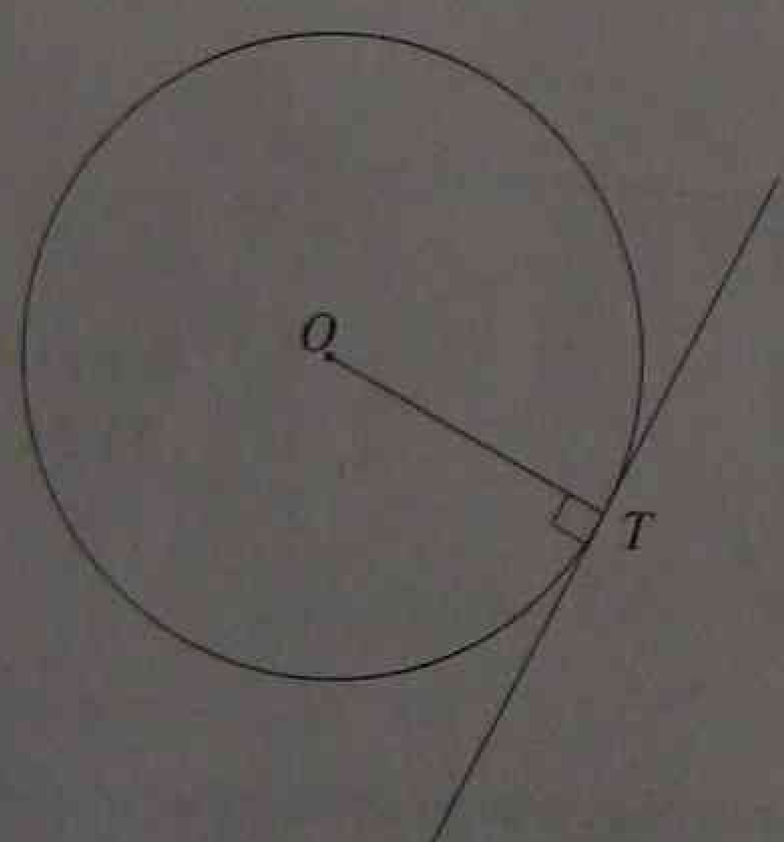
11. If an interval subtends equal angles at two points on the same side of it, then the end points of the interval and the two points are concyclic.



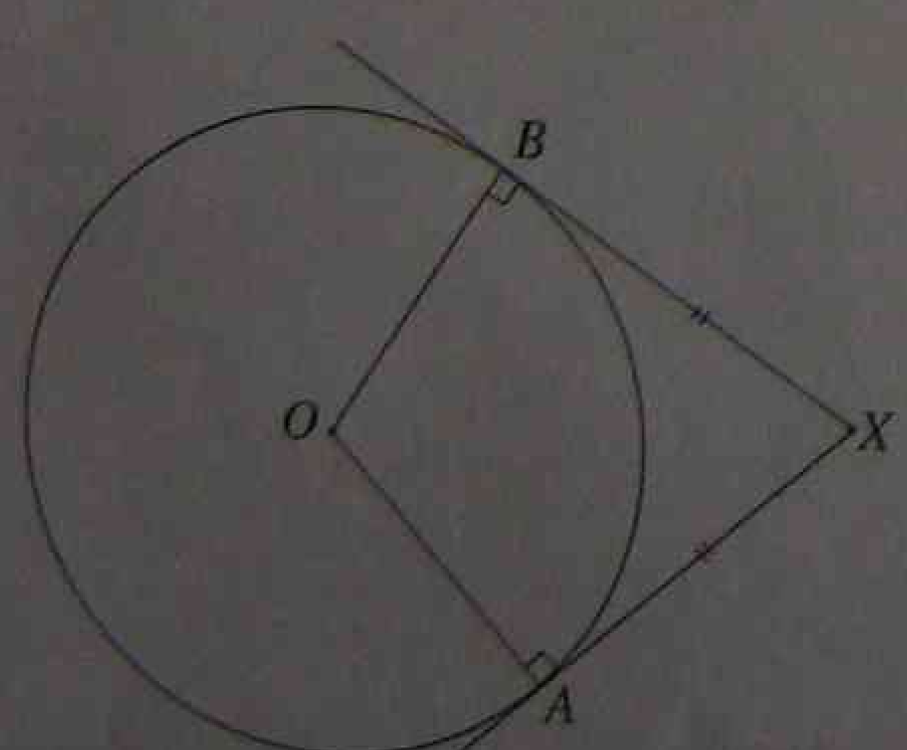
$\angle CAD = \angle CBD$

(D) Tangent Properties

12. The tangent is perpendicular to the radius through the point of contact.

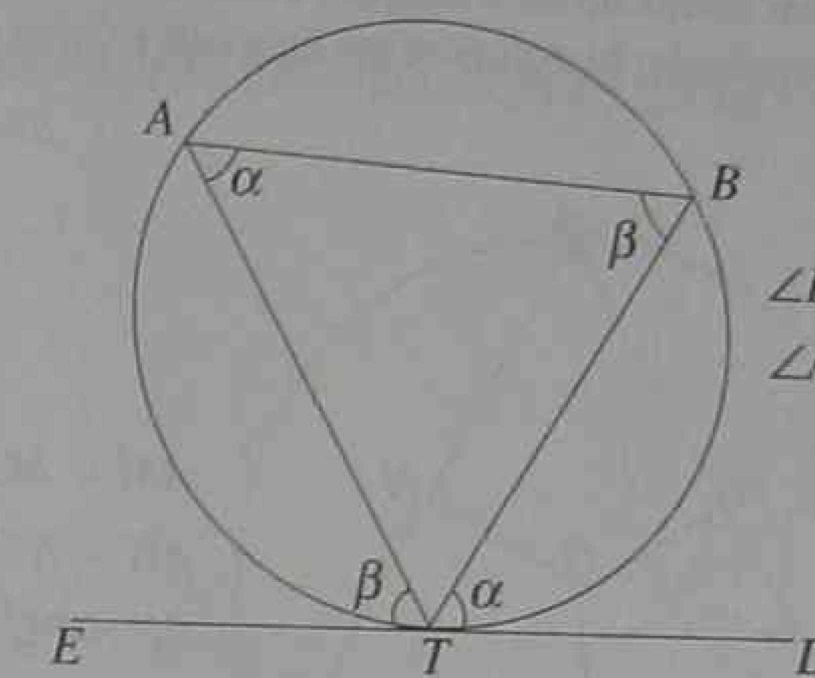


13. Two tangents drawn to a circle from an external point are equal in length.



$XA = XB$

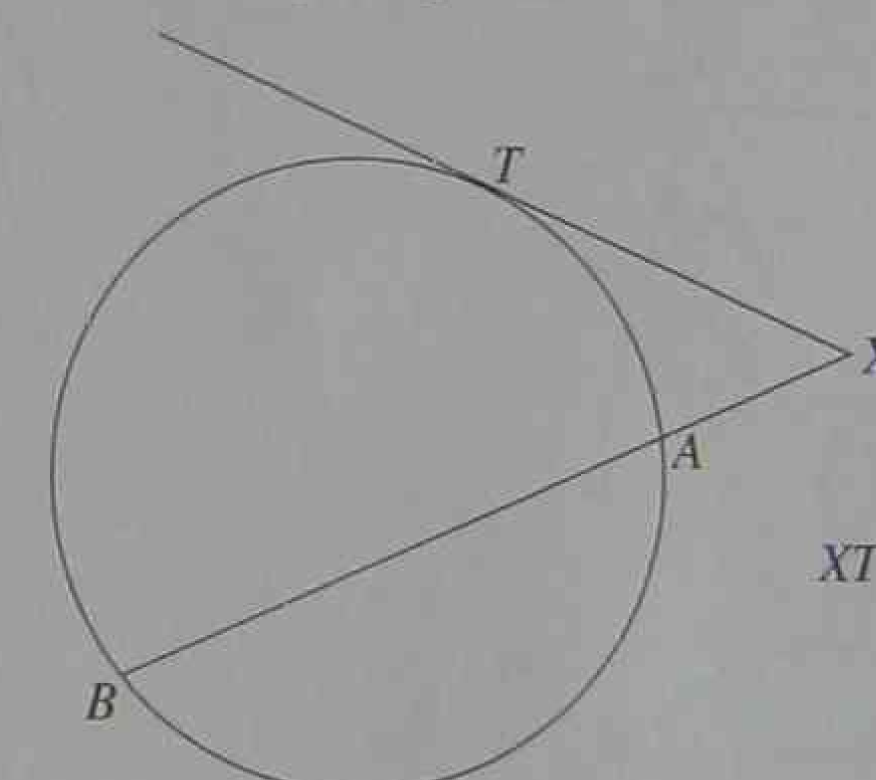
14. The angle between a tangent and a chord is equal to the angle in the alternate segment.



$\angle BAT = \angle BTD$

$\angle ABT = \angle ATE$

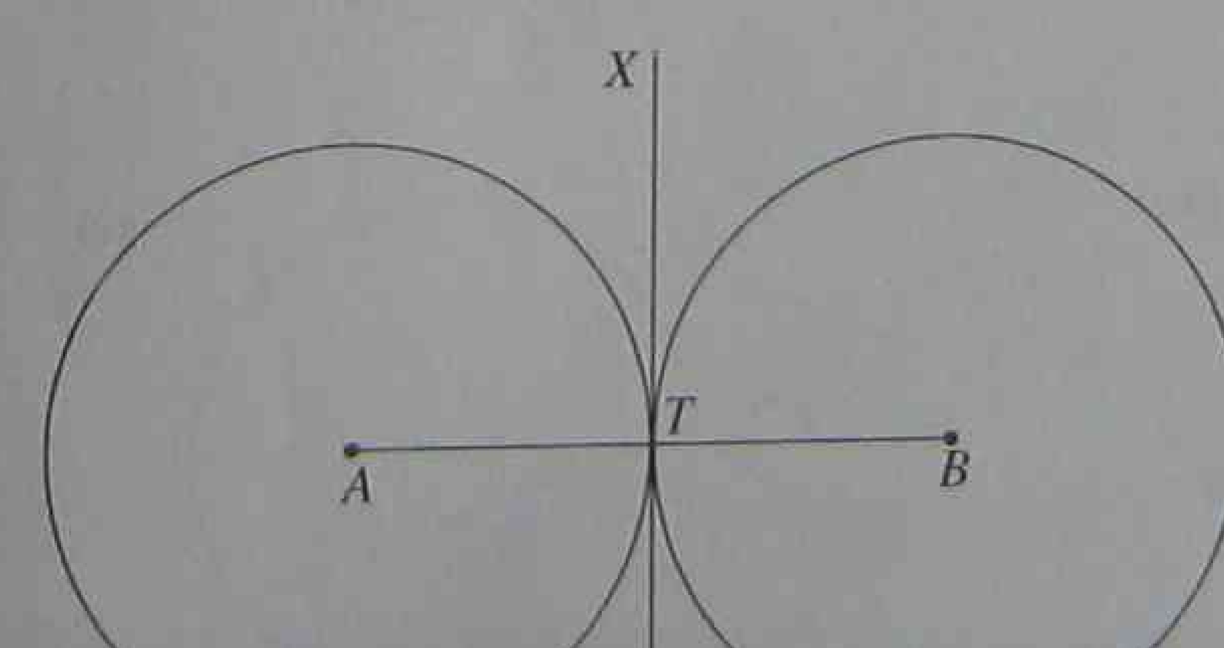
15. The square of the length of the tangent from an external point is equal to the product of the intercept of the secant passing through this point.

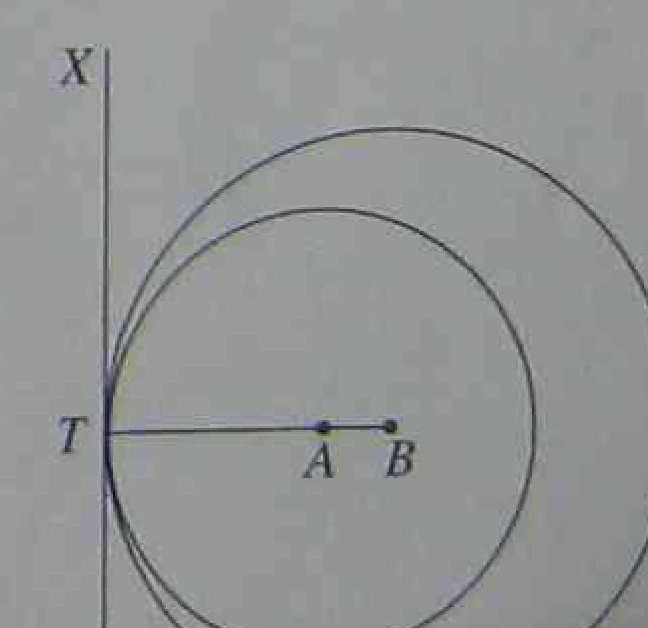


$XT^2 = AX.BX$

(E) Two Circle Properties

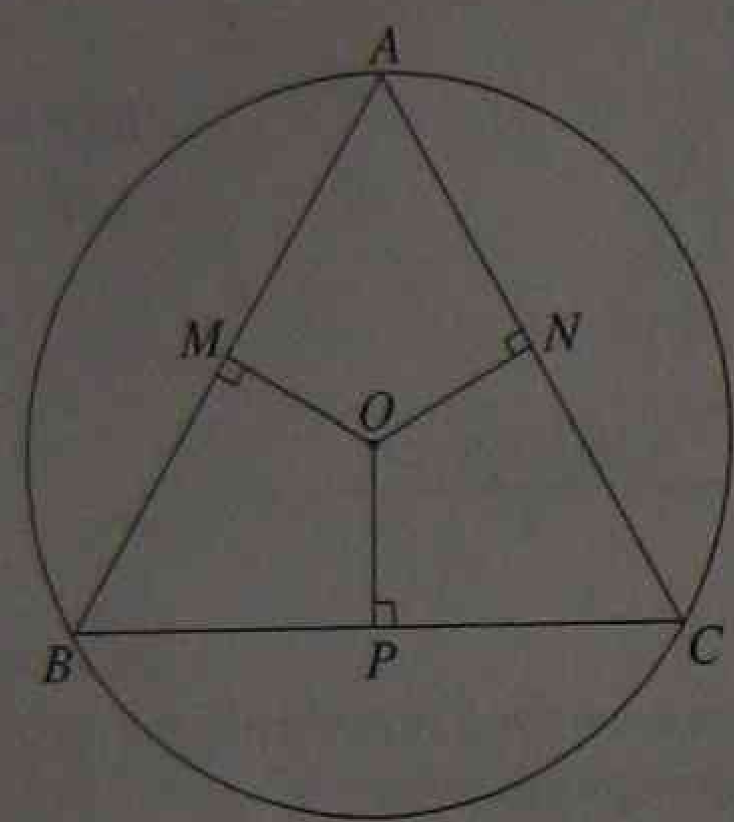
16. If two circles touch (externally or internally), the line joining their centres passes through the point of contact. i.e. points A, T, B are collinear.





(F) Property of Non-Collinear Points

17. Any of three non-collinear points lie on a unique circle, whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.

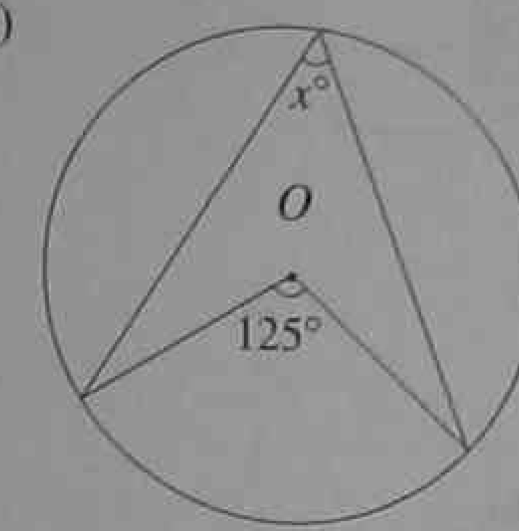


$AM = MB$
$AN = NC$
$BP = PC$

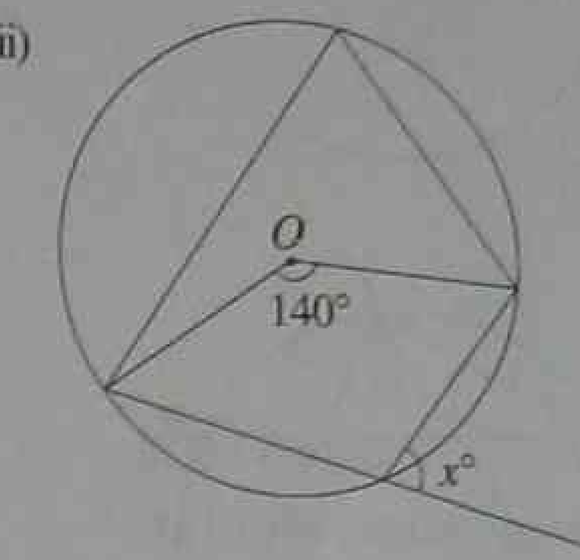
REVIEW EXERCISES

1. Find the value of x in each of the following diagrams, giving reasons for your answer. O is the centre of the circle.

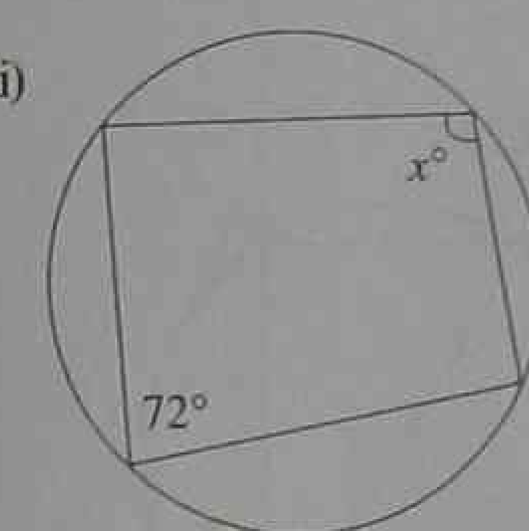
(i)



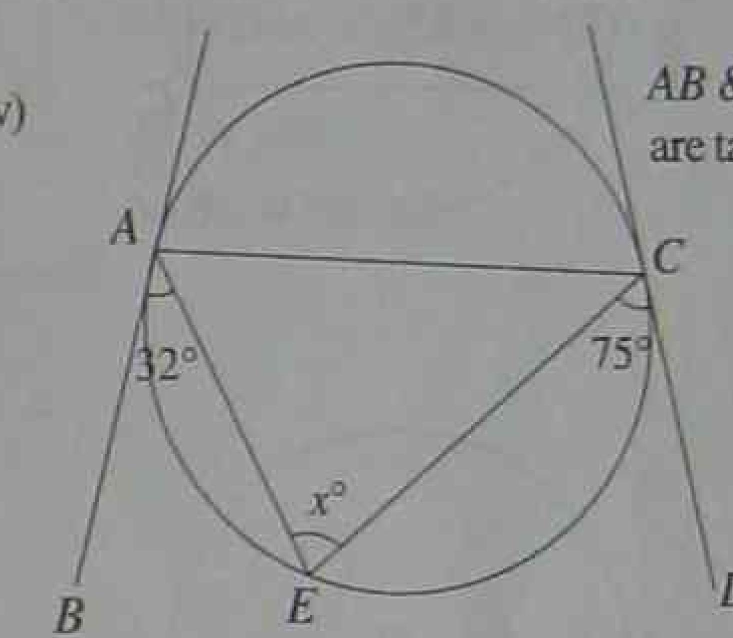
(ii)



(iii)

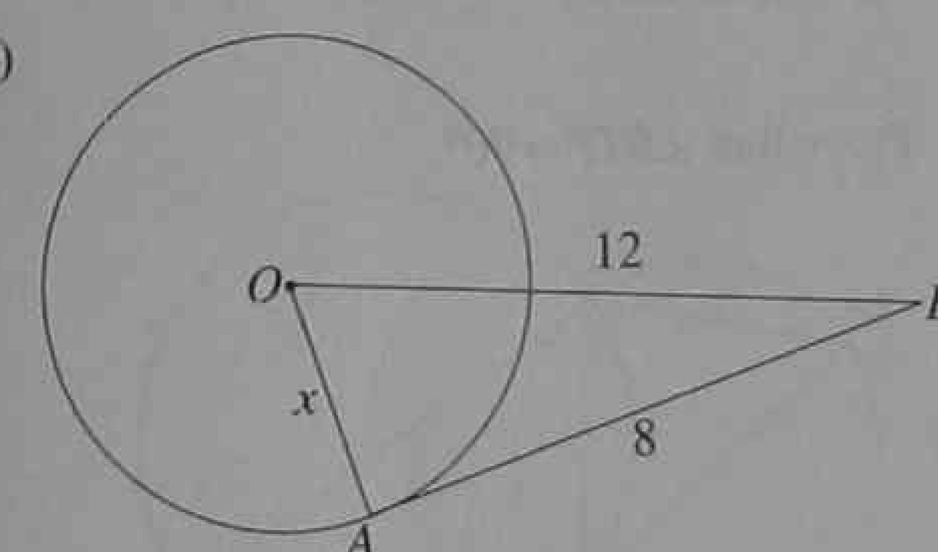


(iv)

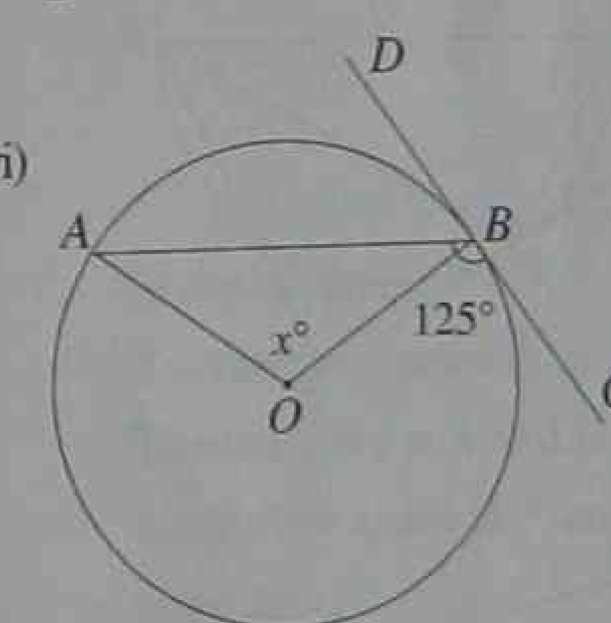


AB & CD are tangents

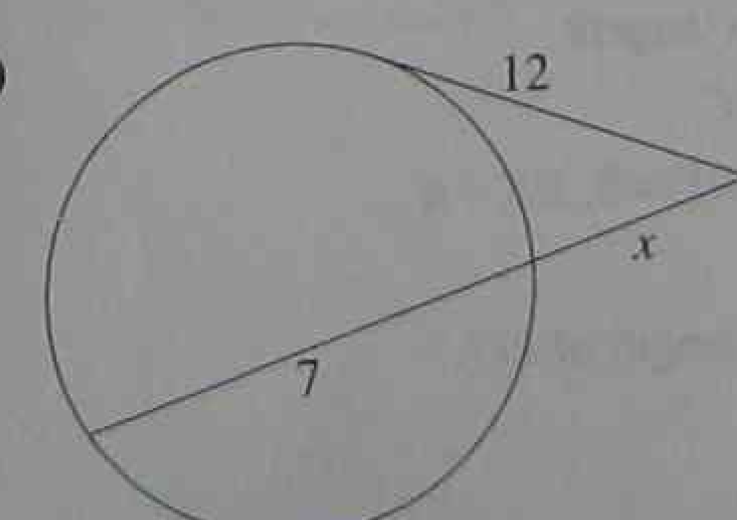
(v)



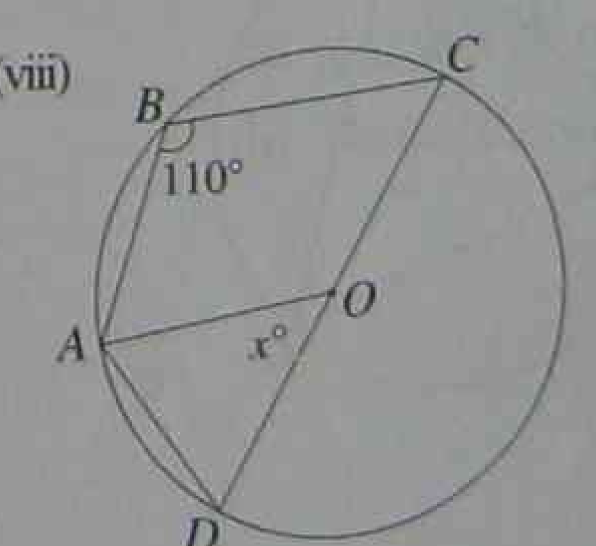
(vi)



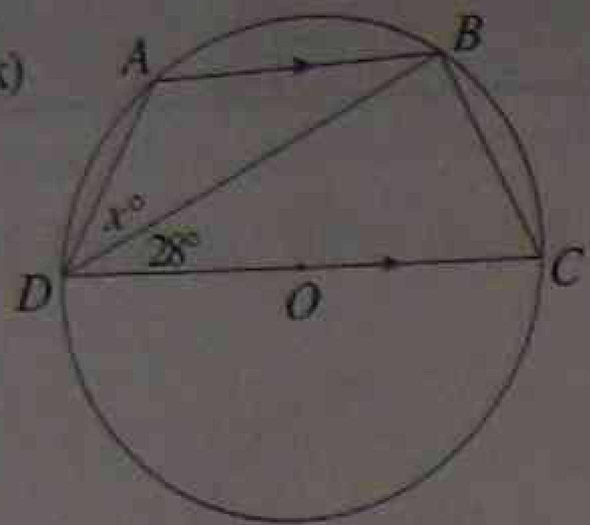
(vii)



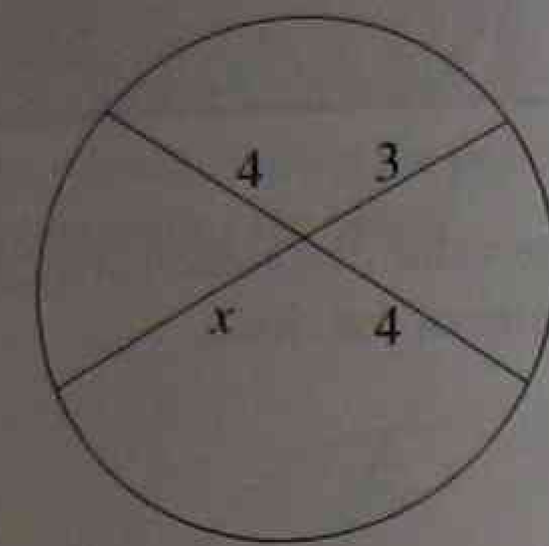
(viii)



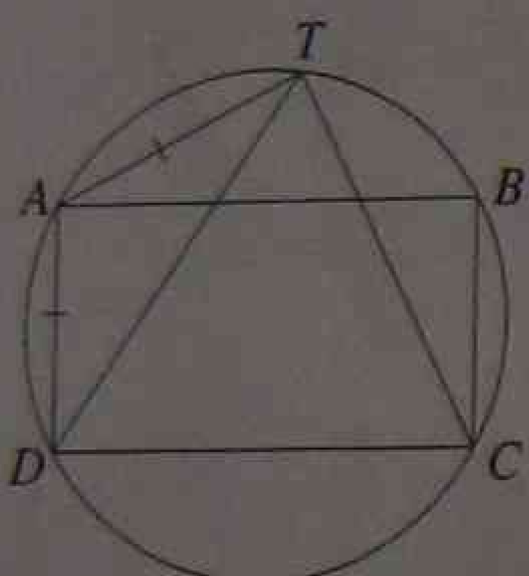
(ix)



(x)



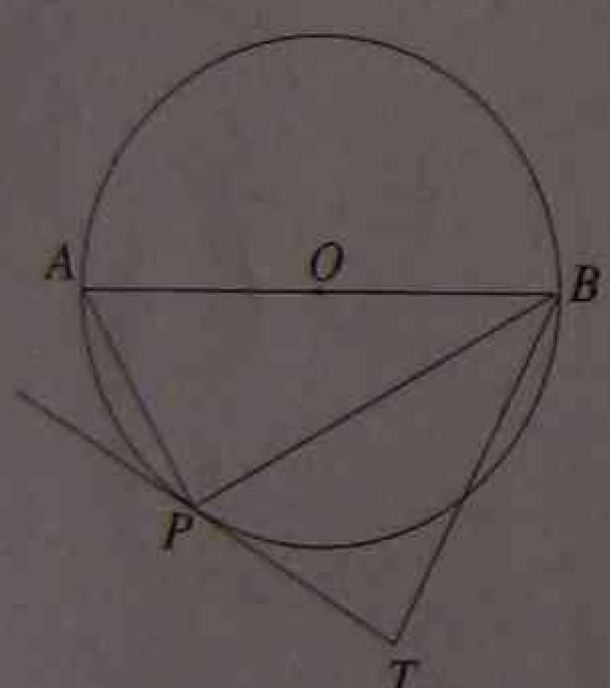
2.



In the diagram:
$ABCD$ is a rectangle
$AT = AD$

Prove that $TC = DC$.

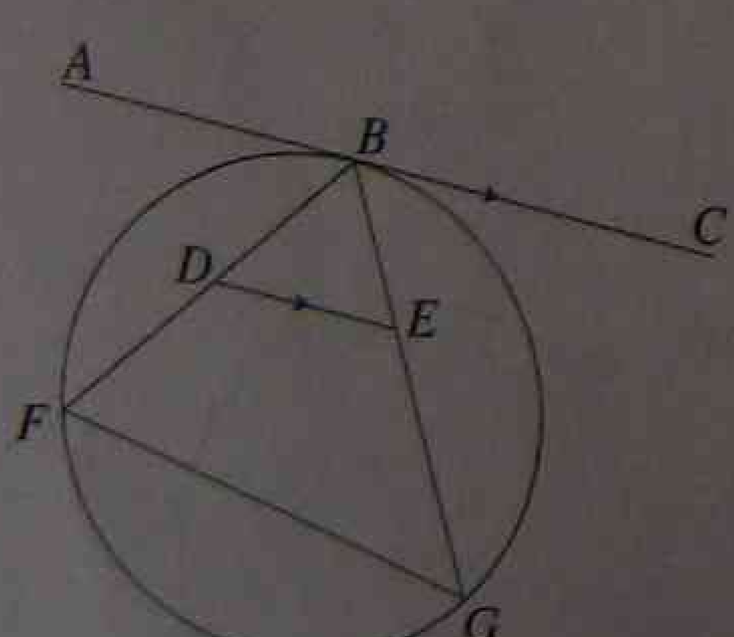
3.



In the diagram:
PT is a tangent
PB bisects $\angle ABT$

Prove that $\angle BTP = 90°$.

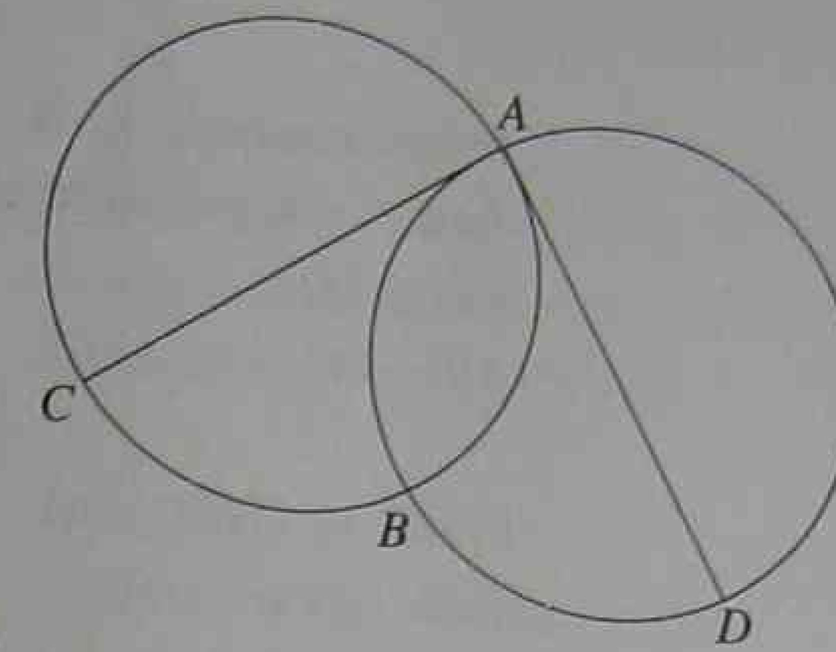
4.



In the diagram:
ABC is a tangent
$DE \parallel ABC$
$BD = 5$, $DF = 3$, $BE = 4$

Find the length of EG.

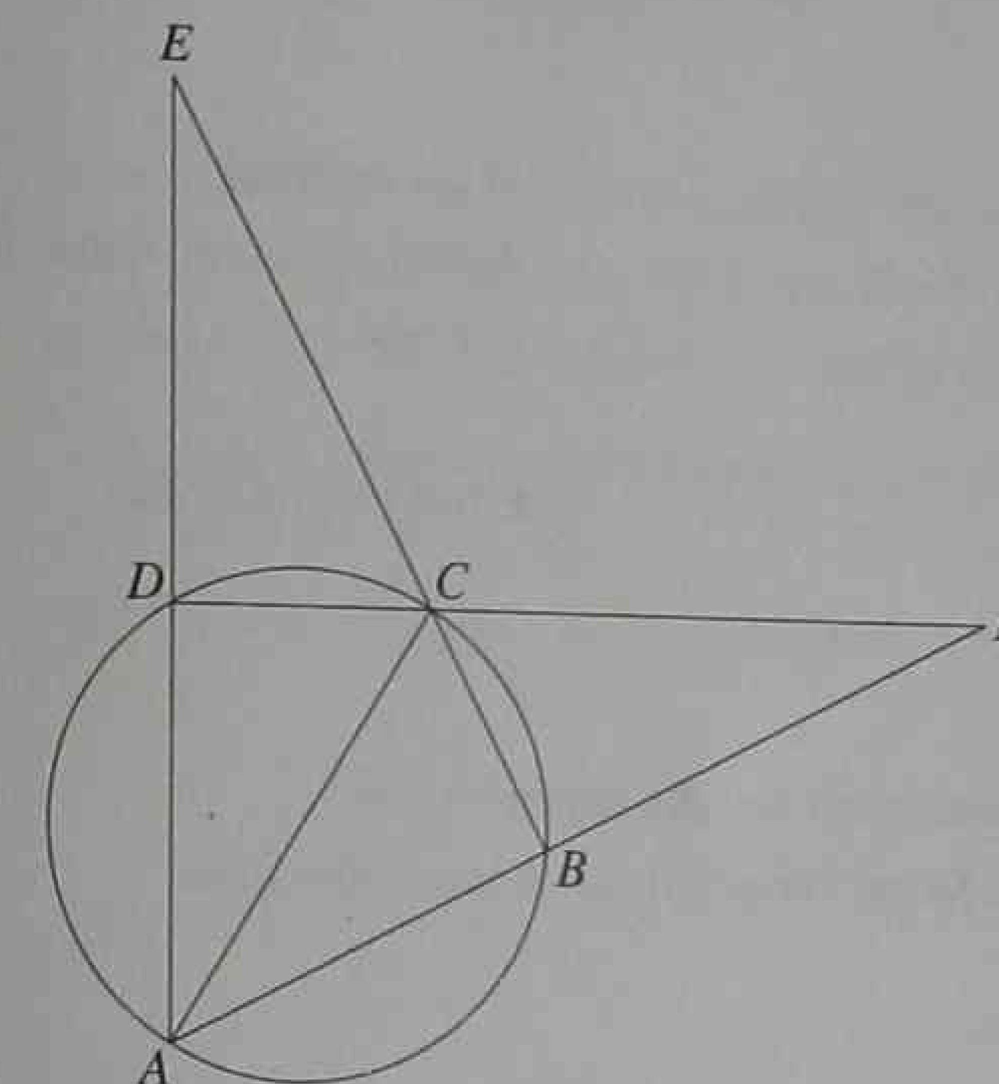
5.



AC and AD are diameters in the respective circles.

Prove that the points C, B, D are collinear.

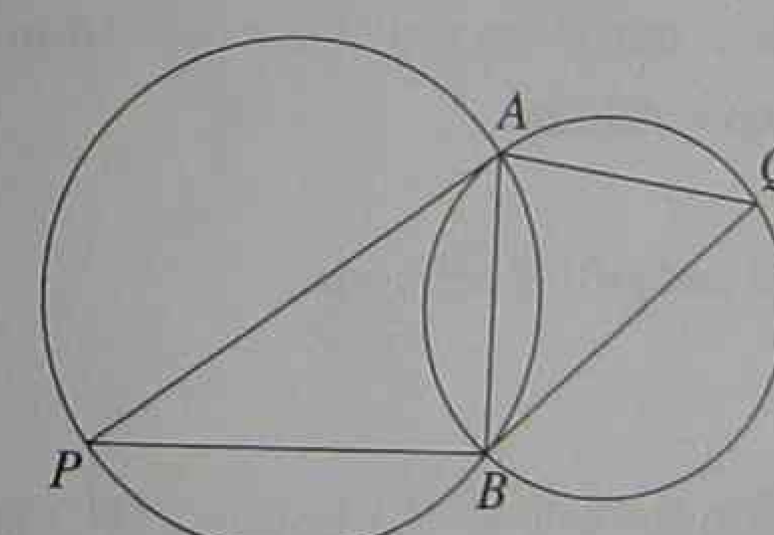
6.



In the diagram:
$ABCD$ is a cyclic quadrilateral.
$AB = AD$, $\angle AEB = \angle AFD$.

Prove that:
$\Delta ABC \equiv \Delta ADC$.

7.



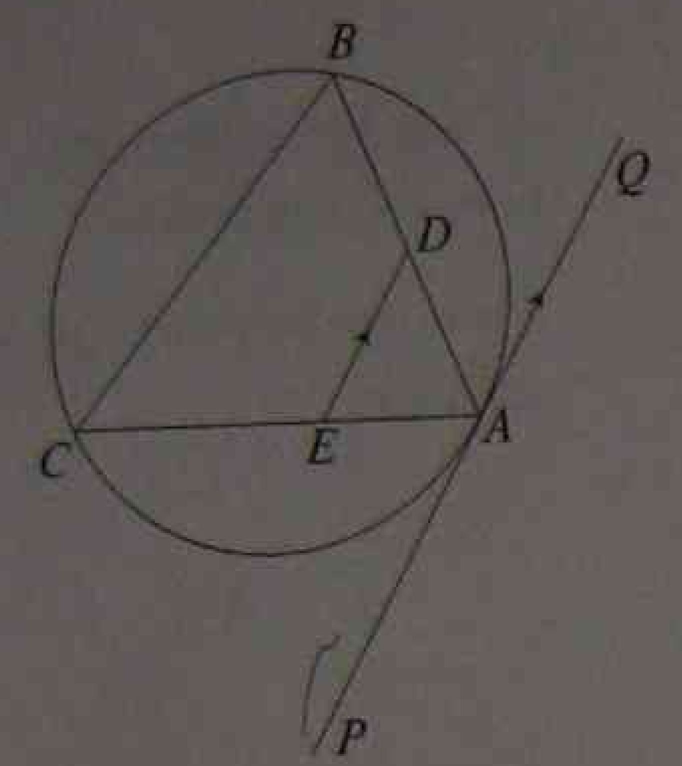
In the diagram:
Two circles intersect in A and B.
The tangents at A and B meet the circles again in P and Q respectively.

Prove that:

(i) $\Delta AQB \ ||| \ \Delta APB$.

(ii) Hence, show that $AB^2 = AQ \cdot B$

8.



In the diagram:

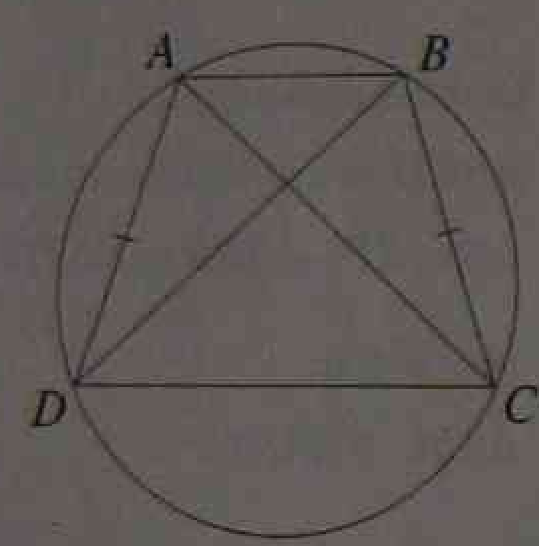
PAQ is a tangent to the circle.

$PAQ \parallel ED$

$AB = AC$.

Prove that $BDEC$ is a cyclic quadrilateral.

9.



In the diagram:

$ABCD$ is a cyclic quadrilateral.

$AD = BC$.

Prove that $AC = BD$.

10. $ABCD$ is a cyclic quadrilateral, such that the tangent to the circle at A is parallel to BD. Draw a diagram, and hence show that:

(i) AC bisects $\angle BCD$.

(ii) ΔABD is isosceles.

11. Two circles, centres O and P, touch externally at point C only. AB is a common tangent to the two circles. The common tangent at C meets AB in D.

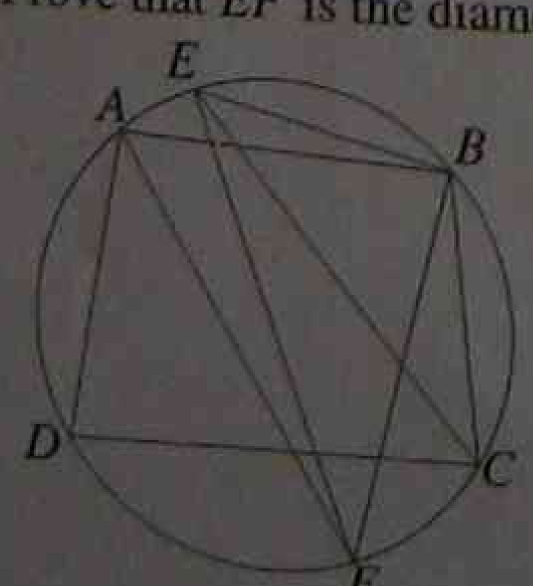
(i) Draw a diagram showing the above details.

(ii) Prove that $DA = DB$.

(iii) Prove that quadrilaterals $ADCO$ and $DBPC$ are cyclic.

(iv) Hence, prove that $\angle ODP = 90°$.

12 In the diagram below, $ABCD$ is a cyclic quadrilateral. FA bisects $\angle DAB$ and EC bisects $\angle BCD$. Prove that EF is the diameter of the circle.



13. In the diagram:

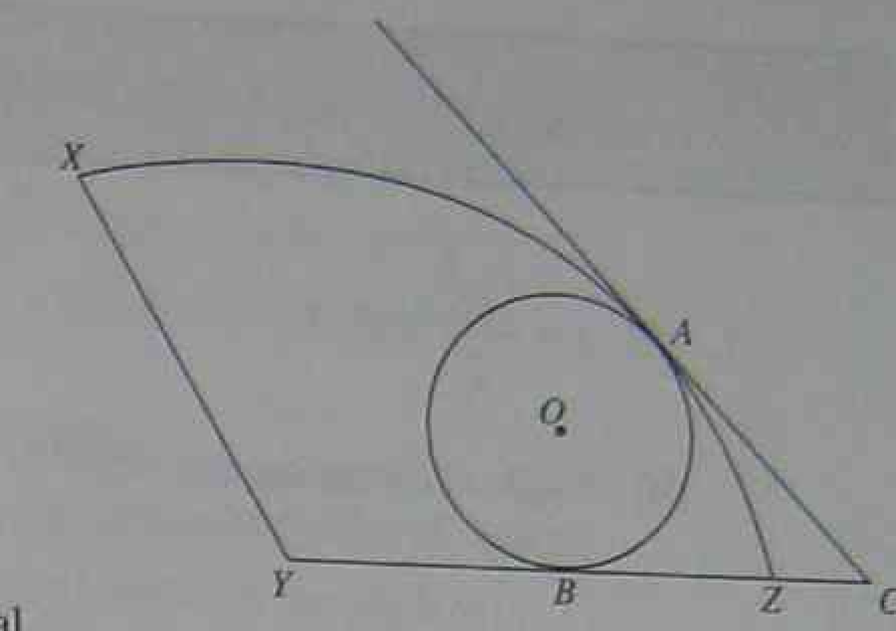
XYZ is a sector of a circle centre Y.

A circle, centre O, touches the arc XZ at A and the radius YZ at B.

Prove that:

(i) Points Y, O, A are collinear.

(ii) $OACB$ is a cyclic quadrilateral.



14. Two circles touch internally at a point P. The smaller circle passes through the centre O of the larger circle. PQ is any chord on the larger circle, intersecting the smaller circle at K. The tangents at P and Q of the larger circle meet at T.

(i) Sketch a diagram indicating the above details.

(ii) Prove that $PK = KQ$.

(iii) Prove that O, K and T are collinear.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $x=\frac{125°}{2}=62.5°$ (angle at centre is twice angle at circumference subtended by the same arc.) #

(ii) Angle at cicumference = 70° (as above)
$x=70°$ (exterior angle of cyclic quad. is = to interior opposite angle.) #

(iii) $x=180°-72°=108°$ (opp. $\angle$'s of a cyclic quad. supplementery) #

(iv) $\angle EAC=75°$, $\angle ACE=32°$ ($\angle$ between tang. and chord = $\angle$ in alt. segment)
$x=180°-75°-32°=73°$ ($\angle$ sum of $\Delta AEC=180°$) #

(v) $\angle OAB=90°$ (tang. at A $\perp$ radius OA)
$\therefore\ x=\sqrt{12^2-8^2}=\sqrt{80}=4\sqrt{5}$ units #

(vi) $\angle OBC=90°$ (tang. at B $\perp$ radius OB)
$\angle OBA=125°-90°=35°$
$\angle OAB=35°$ (base $\angle$'s of isosceles ΔOAB =)
$\therefore\ x=180°-2\times 35°=110°$ ($\angle$ sum of $\Delta=180°$) #

(vii) $12^2=x(x+7)$ (square of tang. = product of secant from an external point)
i.e. $144=x^2+7x$
$x^2+7x-144=0$
$(x+16)(x-9)=0$
$\therefore\ x=9$ (as $x>0$) #

(viii) $\angle ODA=70°$ (opp. $\angle$'s of cyclic quad. $ABCD$ sum to 180°)
$\angle OAD=70°$ (base $\angle$'s of isosceles ΔOAD =)
$x=180°-2\times 70°=40°$ ($\angle$ sum of $\Delta=180°$) #

(ix) $\angle ABD=28°$ (alt. $\angle$s =, $AB \parallel DC$)
$\angle DBC=90°$ (angle in a semi-circle = 90°)
$\angle ABC=\angle ABD+\angle DBC=90°+28°=118°$
$x°=180°-118°-28°=34°$ (opp. $\angle$'s of cyclic quad. $ABCD$ sum to 180°) #

(x) $3x=4\times 4$
i.e. $x=\frac{16}{3}=5\frac{1}{3}$ units (product of two intersecting chords =) #

2. $\angle ABC=90°$ (right angle in rectangle $ABCD$)
Hence, AC is diameter. ($\angle$ in semi-circle = 90°)
$\angle ATC=90°$ ($\angle$ in semi-circle = 90°)
$AD=BC$ (opp. sides of rectangle =)
Hence, $AT=BC$
$\therefore\ \Delta ABC\equiv\Delta ATC$ (RHS)
$\therefore\ AB=TC$ (corresponding sides of $\equiv\Delta$'s)
but $AB=DC$ (opp. sides of rectangle =)
$\therefore\ TC=DC$. #

3. $\angle ABP=\angle PBT$ (given PB bisects $\angle ABT$)
$\angle BPT=\angle BAP$ ($\angle$ between tang. and chord BP = $\angle$ in alt. segment)
$\angle APB=90°$ ($\angle$ in a semi-circle=90°)
$\therefore\ \angle PTB=90°$ (remaining $\angle$ in equiangular Δ's APB and PTB) #

4. $\angle CBG=\angle BED$ (alt. $\angle$'s= $ABC \parallel DE$)
$\angle CBG=\angle BFG$ ($\angle$ between tang. and chord GB = $\angle$ in alt. segment)
$\angle FBG=\angle DBE$ (common $\angle$ to ΔFBG and ΔDBE)
$\therefore\ \Delta FBG\,|||\,\Delta DBE$ (AAA)
Hence, $\frac{BE}{BF}=\frac{BD}{BG}$ (corresponding sides of ||| Δ's in = proportion)
i.e. $\frac{4}{5+3}=\frac{5}{BG}$
i.e. $BG=\frac{8\times 5}{4}=10$
$\therefore\ EG=BG-BE=10-4=6$ units. #

5. $\angle ABD=\angle ABC=90°$ ($\angle$'s in respective semi-circles = 90°)
Since $\angle ABC+\angle ABD=180°$
$\therefore$ points C, B and D are collinear. #

6. $\angle EAF=\angle DAF$ (common $\angle$)
$\angle AEB=\angle AFD$ (given)
$\therefore\ \angle ADF=\angle ABC$ (remaining $\angle$ of Δ's with two $\angle$'s =)
$\angle ADF+\angle ABC=180°$ (opp. $\angle$'s of cyclic quad. sum to 180°)
$\therefore\ \angle ADC=\angle ABC=90°$
Now, AC is common.

$AD = AB$ (given)

$\therefore\ \Delta ADC \equiv ABC$ (RHS) #

7. (i) $\angle PAB = \angle AQB$ ($\angle$ between tang. at A and chord AB = to $\angle$ in alt. segment)

$\angle ABQ = \angle APB$ ($\angle$ between tang. at B and chord AB = to $\angle$ in alt. segment)

$\therefore\ \Delta AQB \,|||\, \Delta APB$ (AAA) #

(ii) $\dfrac{AQ}{AB} = \dfrac{AB}{BP}$ (corresponding sides of ||| Δ's in = proportion)

i.e. $AB^2 = AQ \cdot BP$ #

8. $\angle BAQ = \angle EDA$ (alt. $\angle$'s =, $PAQ \parallel DE$)

$\angle ACB = \angle BAQ$ ($\angle$ between tangent at A and chord AB = to $\angle$ in alt. segment)

$\therefore\ \angle EDA = \angle ACB$, (exterior $\angle$ at vertex = opp. interior $\angle$)

hence, $BDEC$ is a cyclic quadrilateral. #

9. Let AC and BD intersect at X.

$\angle ACD = \angle BDC$ (= chords subtend equal $\angle$'s at circumference)

$\therefore$ in $\Delta\, XDC$, $DX = XC$ (= sides opp. equal $\angle$'s)

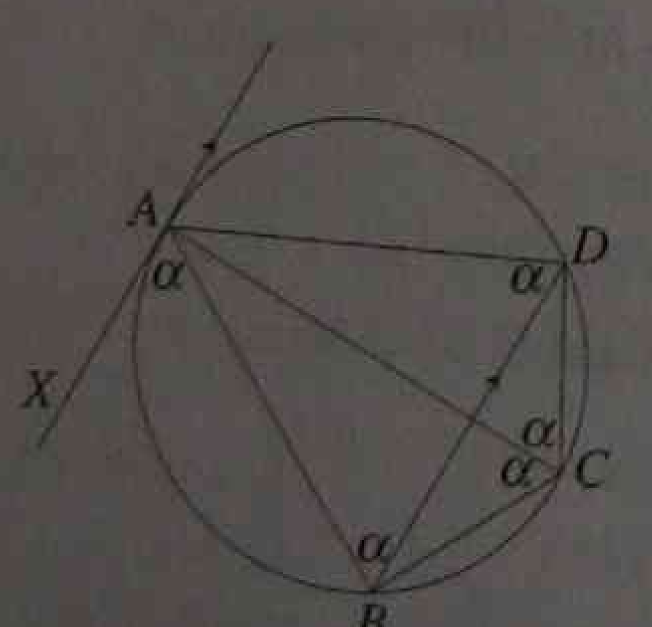
$\angle CAB = \angle DBA$ (= chords subtend equal $\angle$'s at circumference)

$\therefore$ in $\Delta\, AXB$, $AX = XB$ (= sides opp. equal $\angle$'s)

$\therefore\ AX + XC = BX + XD$

i.e. $AC = BD$ #

10. (i)



$\angle XAB = \angle ABD$ (alt. $\angle$'s =, $XA \parallel BD$)

$\angle ABD = \angle ACD$ ($\angle$'s in same segment)

$\therefore\ \angle XAB = \angle ACD$

$\angle XAB = \angle ADB$ ($\angle$ between tangent and chord at A is = to $\angle$ in alt. segment)

$\angle ADB = \angle ACB$ ($\angle$'s in same segment)

$\therefore\ \angle XAB = \angle ACB$

hence, $\angle ACD = \angle ACB$ (since each is = to $\angle XAB$)

$\therefore\ AC$ bisects $\angle BCD$ #

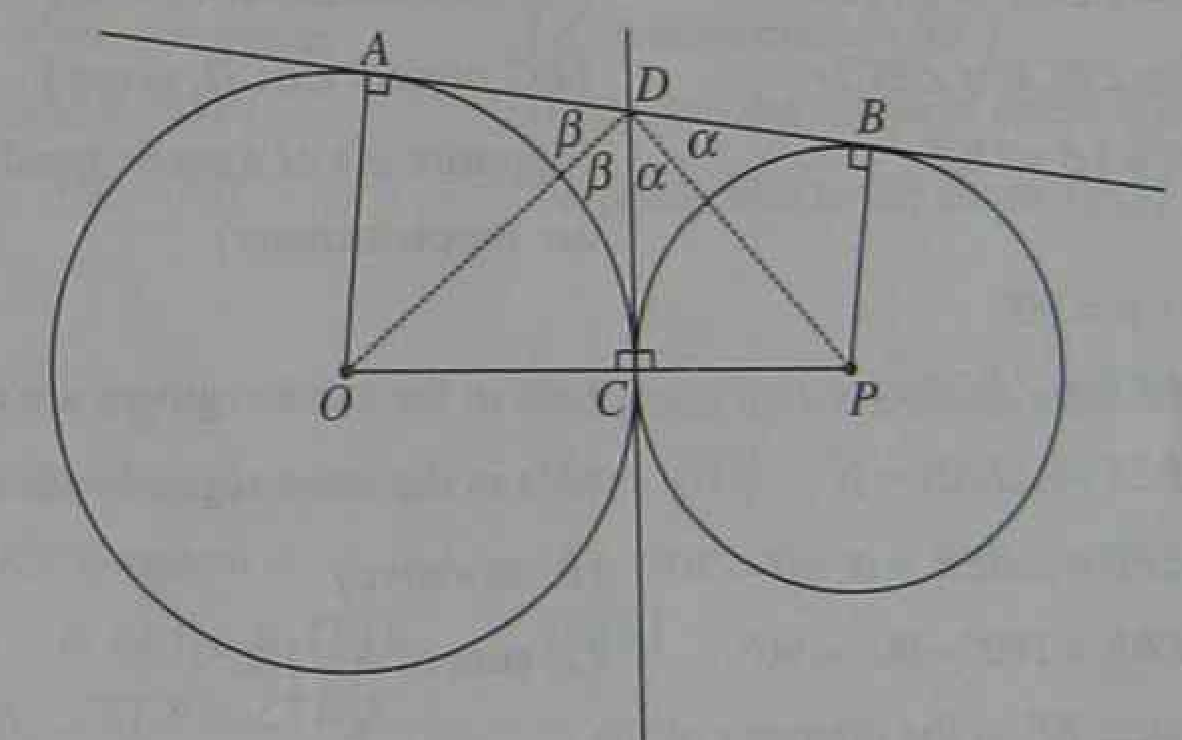
(ii) From above,

$\angle ABD = \angle ADB$ (since each is = to $\angle XAB$)

$\therefore\ AB = AD$ (= sides opp. = $\angle$'s)

$\therefore\ \Delta ABD$ is isosceles. #

11. (i)



(ii) $DC = DB, DC = DA$ (= tangents drawn from external point D)

$\therefore\ DA = DB$ (since each is = to DC) #

(iii) O, C, P are collinear (if two circles touch externally, the line joini[ng] their centres passes through their point of co[ntact])

$\angle OAD = OCD = \angle DCP = \angle DBP = 90^\circ$ (tangent $\perp$ radius at point of contact)

$\therefore$ quadrilaterals $ADCO$ and $DBPC$ are cyclic (opp. $\angle$'s sum to 180°) #

(iv) Join OD and DP.

Let $\angle BDP = \alpha$ and $\angle ADO = \beta$

Now, $DB = DC$ (from (ii))

$PC = PB$ (equal radii)

DP is common side

$\therefore$ $\Delta BDP \equiv \Delta PDC$ (SSS)

$\therefore$ $\angle CDP = \alpha$ (corresponding $\angle$'s of $\equiv$ Δ's $=$)

Similarly, $\Delta OAD \equiv \Delta ODC$ (SSS)

$\therefore$ $\angle ODC = \beta$

Hence, $2\alpha + 2\beta = 180°$ ($\angle$'s in a straight line $= 180°$)

$\therefore$ $\alpha + \beta = 90°$

i.e. $\angle ODP = 90°$ #

12. Let:

$\alpha = \angle DAF = \angle FAB$ (AF bisects $\angle DAB$, given)

$\beta = \angle BCE = \angle ECD$ (EC bisects $\angle BCD$, given)

$2\alpha + 2\beta = 180°$ (opposite $\angle$'s of a cyclic quad. are supplementary)

$\alpha + \beta = 90°$

$\angle EFB = \angle ECB = \alpha$ ($\angle$'s in the same segment are $=$)

$\angle BEF = \angle FAB = \beta$ ($\angle$'s in the same segment are $=$)

$\angle EFB + \angle BEF = \alpha + \beta = 90°$ (from above)

$\angle EBF = 180° - 90° = 90°$ ($\angle$ sum of $\Delta = 180°$)

Hence, EF is the diameter of the circle. ($\angle$ in semi-circle is $=90°$) #

13. (i) Y, O, A are collinear. (If two circles touch internally, the line joining their centres passes through their point of contact.) #

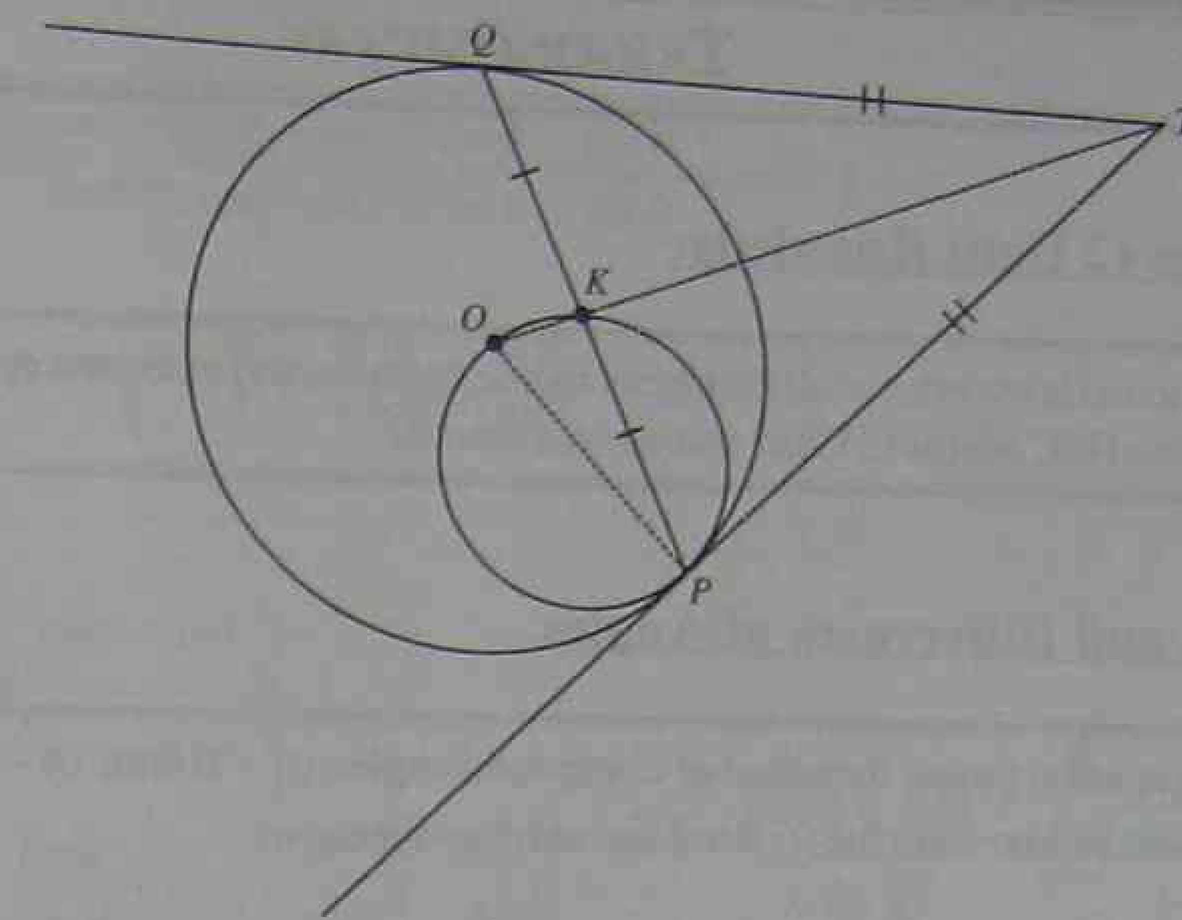
(ii) Join OA and OB.

$\angle OAC = \angle OBC = 90°$ (radius $\perp$ tangent at point of contact)

$\angle OAC + \angle OBC = 180°$

$\therefore$ $OACB$ is a cyclic quad. (opp. $\angle$'s sum to $180°$) #

14. (i)



(ii) Join OP and OK.

OP passes through centre of smaller circle. (when circles touch, line joining centres passes through point of contact)

($\angle$ in semi-circle $= 90°$)

$\angle OKP = 90°$ (line from the centre of circle (OPQ) perpendicular to the chord (PQ) bisects it) #

$\therefore$ $PK = KQ$

(iii) $TP = TQ$ (tangents drawn from an external point $=$)

$PK = KQ$ (from (ii))

KT common

$\therefore$ $\Delta TKP \equiv \Delta TKQ$ (SSS)

$\therefore$ $\angle TKP = \angle TKQ$

$\angle TKP + \angle TKQ = 180°$ ($\angle$'s in a straight line sum to $180°$)

$\therefore$ $\angle TKP = \angle TKQ = 90°$

Now, $\angle OKP = 90°$, (from (ii))

hence, $\angle OKP + \angle TKP = 180°$

$\therefore$ points O, K and T are collinear. #

TRIGONOMETRY

(A) Maths (2 Unit) Knowledge

A sound knowledge of all the trigonometric formulae and other concepts covered in the HSC Maths (2 Unit) course are essential.

(B) Sums and Differences of Angles

There are six main formulae of compound angles $(A+B)$ and $(A-B)$ which should be known in the forward and reverse directions.

1. $\sin(A+B)=\sin A\cos B+\cos A\sin B$
2. $\sin(A-B)=\sin A\cos B-\cos A\sin B$
3. $\cos(A+B)=\cos A\cos B-\sin A\sin B$
4. $\cos(A-B)=\cos A\cos B+\sin A\sin B$
5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$
6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Note: Carefully observe the signs.

Example 1: Show that $\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$.

Solution 1:

$$\begin{aligned}LHS &= \sin(\alpha+\beta)\sin(\alpha-\beta)\\ &=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\ &=\sin^2\alpha(1-\sin^2\beta)-\sin^2\beta(1-\sin^2\alpha)\\ &=\sin^2\alpha-\sin^2\beta \quad \text{as required \#}\end{aligned}$$

Example 2: Show that $\dfrac{\sin(90^\circ-\theta)}{\cos(90^\circ-\theta)}=\cot\theta$.

Solution 2:

$$\frac{\sin(90^\circ-\theta)}{\cos(90^\circ-\theta)}=\frac{\sin90^\circ\cos\theta-\cos90^\circ\sin\theta}{\cos90^\circ\cos\theta+\sin90^\circ\sin\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta \ \#$$

Example 3: Show that $\tan\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$, for $0<\theta<\dfrac{\pi}{4}$.

Solution 3:

$$\begin{aligned}\tan\left(\theta+\frac{\pi}{4}\right)&=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\\ &=\frac{\frac{\sin\theta}{\cos\theta}+1}{1-\frac{\sin\theta}{\cos\theta}}\\ &=\frac{\sin\theta+\cos\theta}{\cos\theta}\times\frac{\cos\theta}{\cos\theta-\sin\theta}\\ &=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\ \#\end{aligned}$$

$\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\cos\theta} = \dfrac{\sin\theta+\cos\theta}{\cos\theta} \div \dfrac{\cos\theta-\sin\theta}{\cos\theta}$

Example 4: If $\sin x=\dfrac{3}{5}, 0<x<\dfrac{\pi}{2}$ and $\sin y=\dfrac{5}{13}, \dfrac{\pi}{2}<y<\pi$.

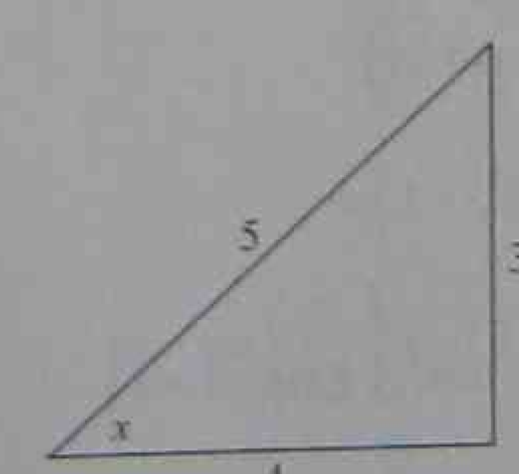
Find the exact values of:

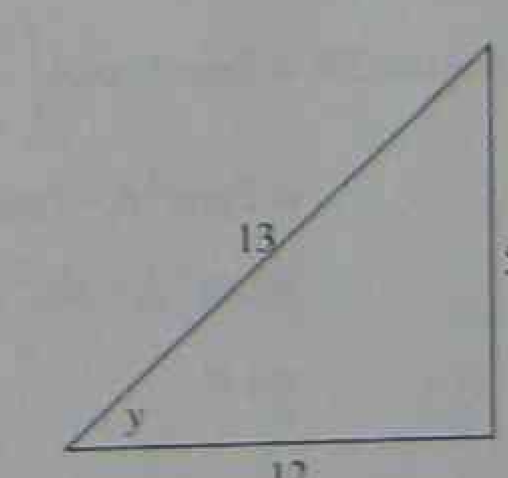
(i) $\sin(x-y)$

(ii) $\cos(x-y)$

(iii) $\tan(x-y)$

Solution 4:





(note y is in second quadrant)

(i) $\sin(x-y)=\sin x\cos y+\cos x\sin y$

$$=\frac{3}{5}\times\left(-\frac{12}{13}\right)+\frac{4}{5}\times\left(\frac{5}{13}\right) \quad \left(\text{note: cos is negative in } 2^{\text{nd}} \text{ quad.}\right)$$

$$=-\frac{16}{65} \ \#$$

(ii) $\cos(x-y)=\cos x\cos y+\sin x\sin y$

$$=\frac{4}{5}\times\left(-\frac{12}{13}\right)+\frac{3}{5}\times\frac{5}{13}=-\frac{33}{65} \ \#$$

(iii) $\tan(x-y)=\dfrac{\sin(x-y)}{\cos(x-y)}=-\dfrac{16}{65}\times-\dfrac{65}{33}=\dfrac{16}{33}$ #

(C) Double Angle Formulae

The formulae for $\cos 2A$, $\sin 2A$ and $\tan 2A$ can be obtained explicitly from the compound angle formulae by letting $B=A$.

$$\sin 2A=2\sin A\cos A$$

$$\cos 2A=\cos^2 A-\sin^2 A \quad \text{or}$$
$$=\cos^2 A-\left(1-\cos^2 A\right)=2\cos^2 A-1 \quad \text{or}$$
$$=\left(1-\sin^2 A\right)-\sin^2 A=1-2\sin^2 A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Example 1: Simplify $\sin 2A\cot A-\cos 2A$.

Solution 1:

$$\sin 2A\cot A-\cos 2A=2\sin A\cos A\left(\frac{\cos A}{\sin A}\right)-\left(\cos^2 A-\sin^2 A\right)$$
$$=2\cos^2 A-\cos^2 A+\sin^2 A$$
$$=\cos^2 A+\sin^2 A$$
$$=1 \ \#$$

$$\left[\text{note } \cos^2 A+\sin^2 A=1\right]$$

Example 2: Prove that $\dfrac{1-\cos 2x}{\sin 2x}=\tan x$.

Solution 2:

$$\text{LHS}=\frac{1-\cos 2x}{\sin 2x}$$
$$=\frac{1-\left(\cos^2 x-\sin^2 x\right)}{2\sin x\cos x}$$
$$=\frac{\left(\sin^2 x+\cos^2 x\right)-\cos^2 x+\sin^2 x}{2\sin x\cos x}=\frac{2\sin^2 x}{2\sin x\cos x}=\frac{\sin x}{\cos x}=\tan x=\text{RHS} \ \#$$

Example 3: (i) Show that $\dfrac{1-\cos 2x}{1+\cos 2x}=\tan^2 x$.

(ii) Hence, show that $\tan\left(\dfrac{\pi}{12}\right)=2-\sqrt{3}$.

Solution 3:

(i) $\text{LHS}=\dfrac{1-\cos 2x}{1+\cos 2x}$

$$=\frac{\left(\sin^2 x+\cos^2 x\right)-\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^2 x+\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\right)}=\frac{2\sin^2 x}{2\cos^2 x}=\tan^2 x=\text{RHS} \ \#$$

(ii) $\tan^2\left(\dfrac{\pi}{12}\right)=\dfrac{1-\cos\frac{\pi}{6}}{1+\cos\frac{\pi}{6}}$ (using the result in (i))

$$=\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}$$
$$=\frac{2-\sqrt{3}}{2}\times\frac{2}{2+\sqrt{3}}$$
$$=\frac{2-\sqrt{3}}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$=\left(2-\sqrt{3}\right)^2$$

$$\therefore \ \tan\left(\frac{\pi}{12}\right)=\sqrt{\left(2-\sqrt{3}\right)^2}=2-\sqrt{3} \ \#$$

✓ **Example 4**: Prove that $\dfrac{\sin 3x}{\sin x}+\dfrac{\cos 3x}{\cos x}=4\cos 2x$.

Solution 4:

$$\begin{aligned}
\text{LHS} &= \frac{\sin 3x}{\sin x} + \frac{\cos 3x}{\cos x} \\
&= \frac{\sin 3x \cos x + \cos 3x \sin x}{\sin x \cos x} \\
&= \frac{\sin(3x+x)}{\sin x \cos x} && (\text{using: } \sin(A+B) = \sin A\cos B + \cos B \sin A) \\
&= \frac{\sin 4x}{\sin x \cos x} \\
&= \frac{\sin(2x+2x)}{\sin x \cos x} \\
&= \frac{2\sin 2x \cos 2x}{\sin x \cos x} && (\text{using: } \sin 2A = 2\sin A\cos A) \\
&= \frac{2(2\sin x\cos x)\cos 2x}{\sin x \cos x} && (\text{using: } \sin 2A = 2\sin A\cos A) \\
&= 4\cos 2x = \text{RHS}\ \#
\end{aligned}$$

(D) Half Angle Formulae and 't' Formulae

By writing $A = \frac{A}{2} + \frac{A}{2}$, we obtain the half angle formulae and hence the t-formulae.

$$\sin A = \sin\left(\frac{A}{2}+\frac{A}{2}\right) = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\begin{aligned}
\cos A = \cos\left(\frac{A}{2}+\frac{A}{2}\right) &= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\
&= 2\cos^2\frac{A}{2} - 1 \\
&= 1 - 2\sin^2\frac{A}{2}
\end{aligned}$$

$$\tan A = \tan\left(\frac{A}{2}+\frac{A}{2}\right) = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

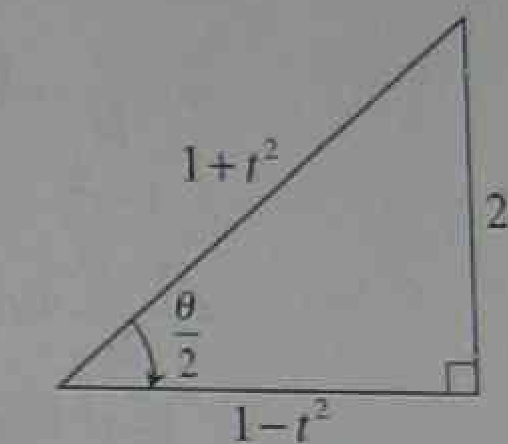
If we substitute $A = \theta$ and $t = \tan\frac{\theta}{2}$ into the above formulae, we get:

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$$

Referring to the diagram:

$$\sin\theta = \frac{opp.}{hyp.} = \frac{2t}{1+t^2} \text{ and}$$

$$\cos\theta = \frac{adj.}{hyp.} = \frac{1-t^2}{1+t^2}$$



Example 1: Simplify $\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$.

Solution 1:

$$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2} = \cos\theta\ \#$$

Example 2: Show that $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\left(\dfrac{\theta}{2}\right)$.

Solution 2:

Using the t-formulae:

$$\begin{aligned}
\text{LHS} = \frac{1-\cos\theta}{1+\cos\theta} &= \frac{1-\left(\frac{1-t^2}{1+t^2}\right)}{1+\left(\frac{1-t^2}{1+t^2}\right)} \\
&= \frac{(1+t^2)-(1-t^2)}{1+t^2} \times \frac{1+t^2}{(1+t^2)+(1-t^2)} \\
&= \frac{2t^2}{2} = t^2 = \tan^2\left(\frac{\theta}{2}\right) = \text{RHS}\#
\end{aligned}$$

Example 3: Show that $\cot\theta + \tan\frac{\theta}{2} = \operatorname{cosec}\theta$.

Solution 3:

Using the t-formula:

$$= \frac{1}{\frac{2t}{1-t^2}} + t$$

$$= \frac{1-t^2}{2t} + t$$

$$= \frac{1-t^2+2t^2}{2t} = \frac{1+t^2}{2t} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta = \text{RHS } \#$$

(E) Triple Angle Formulae

By writing $3A = A + 2A$, the triple angle formulae can be obtained. These results need not be committed to memory as they are simple to derive when required.

$$\begin{aligned}\sin(3A) &= \sin(A+2A)\\ &= \sin A\cos 2A + \cos A \sin 2A\\ &= \sin A\left(1-2\sin^2 A\right) + \cos A\left(2\sin A\cos A\right)\\ &= \sin A - 2\sin^3 A + 2\sin A\cos^2 A\\ &= \sin A - 2\sin^3 A + 2\sin A\left(1-\sin^2 A\right)\\ &= 3\sin A - 4\sin^3 A\end{aligned}$$

Similarly, the following ratios can be found:

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Example 1:

(i) Show that $\cos 3A = 4\cos^3 A - 3\cos A$.

(ii) Hence, find the smallest positive value of x which satisfies the equation:

$8x^3 - 6x + \sqrt{3} = 0$

Express your answer correct to 3 decimal places.

Solution 1:

(i) $\text{LHS} = \cos(3A)$

$$\begin{aligned}&= \cos(2A + A)\\ &= \cos 2A\cos A - \sin 2A \sin A\end{aligned}$$

$$\begin{aligned}&= 2\cos^3 A - \cos A - 2\cos A\left(1-\cos^2 A\right)\\ &= 4\cos^3 A - 3\cos A \ \#\end{aligned}$$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

$2\cos 3A = 8\cos^3 A - 6\cos A$

Let $x = \cos A$, so that $8x^3 - 6x + \sqrt{3} = 0$, becomes

$8\cos^3 A - 6\cos A + \sqrt{3} = 0$

i.e. $2\cos 3A + \sqrt{3} = 0$

$$\cos 3A = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad 3A = \frac{5\pi}{6}$$

i.e. $$A = \frac{5\pi}{18}$$

$$\therefore \quad x = \cos\left(\frac{5\pi}{18}\right) = 0.643 \text{ to 3 d.p. } \#$$

(F) Three Dimensional Trigonometry

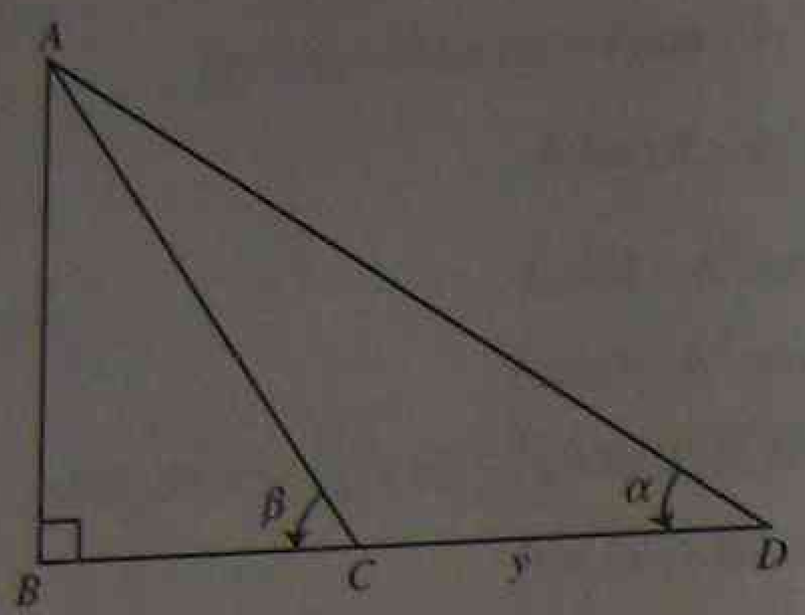
One common application requires students to find the height of a tower whose base is unknown, by means of two (or three) observations made from points in line with the base of the tower.

There are a number of basic guidelines that can be used:

- For clarity, it is best to draw the N-S and E-W axes at **22.5°** and **112.5°** to the vertical. If necessary, for further clarity, redraw the vertical or base triangles.
- It is often easier to work with **cot** as opposed to tan.
- Relate the height (h) and angles of elevation to each other in each vertical triangle.
- Relate the sides in the base triangle to each other to determine the height. If triangle is right-angled, use Pythagoras' Theorem. If not, use the sine or cosine rule.

Example 1: Using the diagram below on the next page,

(i) show that $AD = \dfrac{y\sin\beta}{\sin(\beta-\alpha)}$



Solution 1:

(i) $\angle DAC = (\beta - \alpha)$ (exterior $\angle$ of Δ=sum of opp. interior $\angle$'s)

$\angle ACD = (180 - \beta)$ ($\angle$'s in a straight line sum to $180°$)

Using the sine rule in $\Delta\ ACD$:

$$\frac{AD}{\sin(180° - \beta)} = \frac{y}{\sin(\beta - \alpha)}$$

$$\therefore\ AD = \frac{y\sin\beta}{\sin(\beta - \alpha)}\ \#\qquad \left(\text{note: } \sin(180° - \beta) = \sin\beta\right)$$

(ii) $$\sin\alpha = \frac{AB}{AD} = \frac{AB}{\dfrac{y\sin\beta}{\sin(\beta-\alpha)}}$$

$$\text{i.e. } AB = \frac{y\sin\alpha\sin\beta}{\sin(\beta - \alpha)}\ \#$$

Example 2: A hill of height h m is observed from two points X and Y 400 m apart on level ground. From point X due east of the tower, the angle of elevation to the top of the hill is 24° and from point Y in the direction of 225°T from the base of the hill, the angle of elevation is 32°.

(i) Draw a diagram showing the above data.

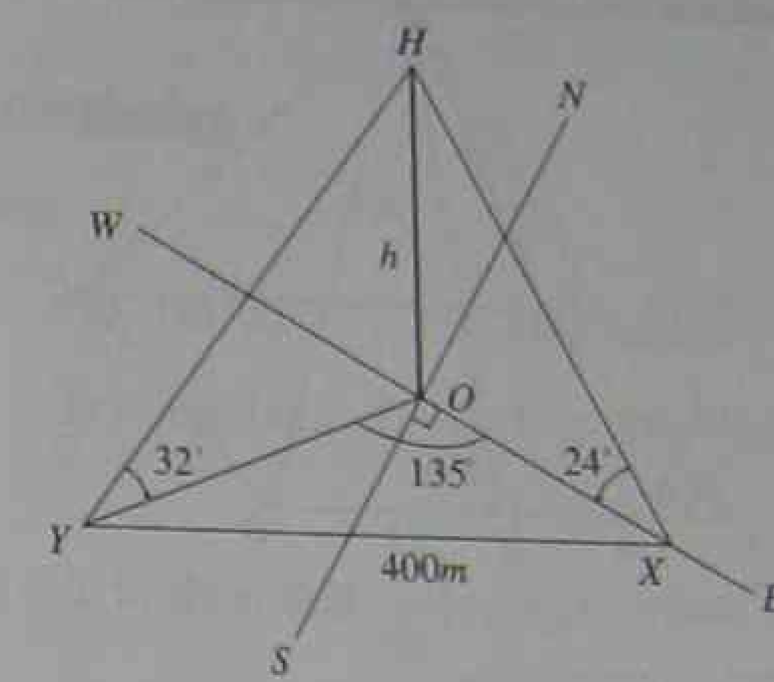
(ii) Show that the height of the hill is given by:

$$h = \frac{400}{\sqrt{\cot^2 32° + \cot^2 24° - 2\cot 32° \cot 24° \cos 135°}}$$

hence, find h correct to 2 decimal places.

Solution 2:

(i)



(ii) $\angle YOX = 90° + (225° - 180°) = 135°$

Looking at $\Delta\ YOH$:

$$\tan 32° = \frac{h}{OY}\ \text{ i.e. } OY = \frac{h}{\tan 32°} = h\cot 32°$$

Looking at $\Delta\ HOX$:

$$\tan 24° = \frac{h}{OX}\ \text{ i.e. } OX = \frac{h}{\tan 24°} = h\cot 24°$$

Now, using the cosine rule in $\Delta\ YOX$:

$$\begin{aligned} 400^2 &= YO^2 + OX^2 - 2\times(YO)\times(OX)\times\cos 135° \\ &= h^2\cot^2 32° + h^2\cot^2 24° - 2h^2\cot 32°\cot 24°\cos 135° \end{aligned}$$

$$\text{i.e. } h^2 = \frac{400^2}{\cot^2 32° + \cot^2 24° - 2\cot 32°\cot 24°\cos 135°}$$

$$\therefore\ h = \frac{400}{\sqrt{\cot^2 32° + \cot^2 24° - 2\cot 32°\cot 24°\cos 135°}}$$

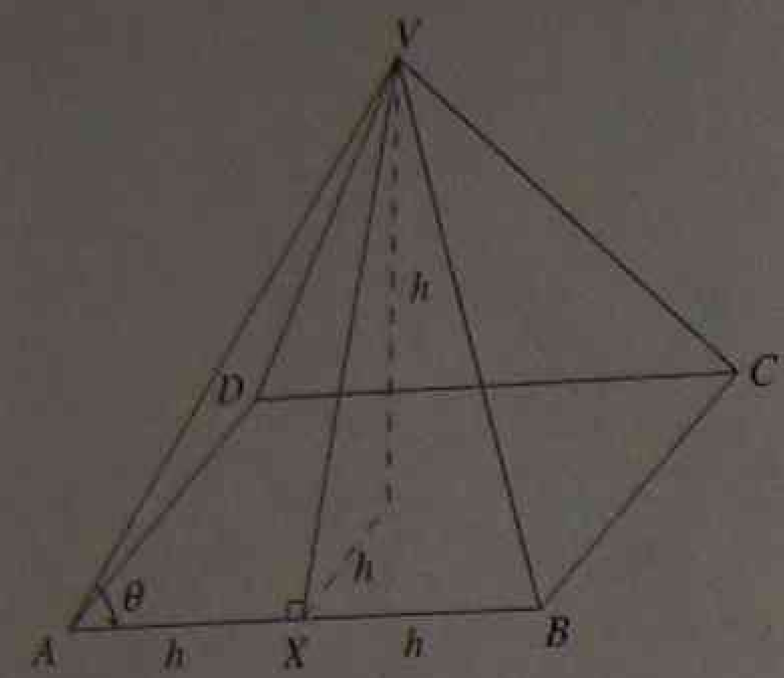
$$= 112.89 \text{ m correct to 2 d.p. } \#$$

Example 3: A right square pyramid has base $ABCD$ and vertex V. Its perpendicular height h is half the length of the base edge AB.

(i) Show that the angle θ between the base edge AB and the sloping edge AV is given by $\theta = \dfrac{1}{\sqrt{3}}$.

(ii) Show that the total surface area of the pyramid is $4h^2\left(1+\sqrt{2}\right)\text{units}^2$.

Solution 3:



(i) $VX=\sqrt{h^2+h^2}=\sqrt{2h^2}=\sqrt{2}h$

Looking at ΔAVX:

$$AV^2=AX^2+VX^2$$
$$=h^2+\left(\sqrt{2}h\right)^2=3h^2$$
$$\therefore\ AV=\sqrt{3}h$$
$$\cos\theta=\frac{h}{\sqrt{3}h}=\frac{1}{\sqrt{3}}\ \#$$

(ii) Area of base $=2h\times 2h=4h^2$

Area of $\Delta AVB=\frac{1}{2}\text{base}\times\text{height}$

$$=\frac{1}{2}\times 2h\times\sqrt{2}h$$
$$=\sqrt{2}h^2$$

$\therefore$ Total surface area $=4h^2+4\sqrt{2}h^2=4h^2\left(1+\sqrt{2}\right)$ units^2 #

Trigonometry

Review Exercises

(A) Maths (2 Unit) Knowledge

1. Differentiate the following:

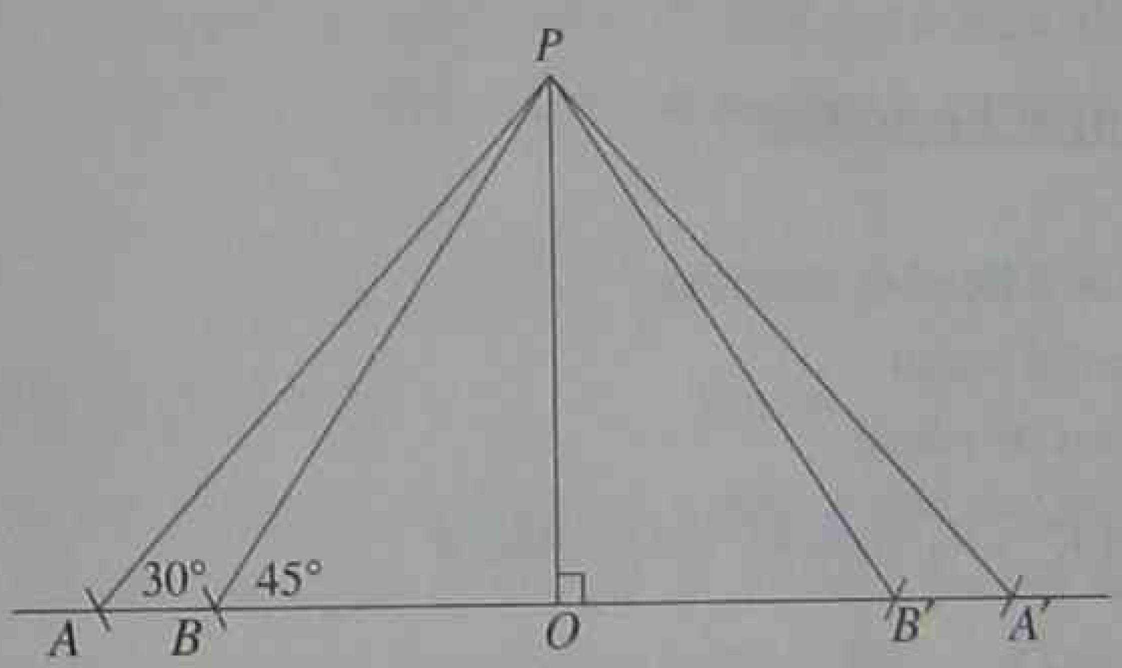
(i) $e^{2x}\cos 3x$ (ii) $\tan^2 2x$ (iii) $\sqrt{\sin x}$

(iv) $e^{\sin x}$ (v) $\log_e\left(\sin^2 x\right)$ (vi) $\cos^3(2x+1)$

2. (i) Use the quotient rule to show that $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$.

(ii) Hence, evaluate, correct to 2 decimal places, $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\text{cosec}^2 x\,dx$.

3.



In the figure shown, APA' and BPB' are wire ropes supporting tower OP. The ropes AP (and $A'P$) and BP (and $B'P$) have angles of elevation 30° and 45° respectively. The distance between the base pegs A and B and also A' and B' is 8 metres. Find:

(i) the height of the tower.

(ii) the distance of peg A from the base of the tower.

(iii) the total length of wire rope required to support the tower.

4. If $ABCD$ is a rhombus, show that: $4AB^2=AC^2+BD^2$.

Trigonometry

(B) Sums and Differences of Angles

5. Prove the following identities:

   (i) $\dfrac{\cos^2\alpha-\cos^2\beta}{\cos^2\alpha\cos^2\beta}=\tan^2\beta-\tan^2\alpha$.

   (ii) $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta$.

   (iii) $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}=\tan\alpha$.

6. Show that $\sin(x+y)+\sin(x-y)=2\sin x\cos y$.

   Hence, use this result to find $\int \sin 4x\cos 2x\,dx$

7. Using the expansion of $\tan(A+B)$, show that $\tan\left(\dfrac{5\pi}{12}\right)=2+\sqrt{3}$.

8. (i) Write down in surd form the values of: $\sin 45°$, $\sin 60°$, $\cos 45°$, $\cos 60°$.

   (ii) Hence, find the exact value of $\sin 105°$.

(C) Double Angle Formulae

9. Prove the following identities:

   (i) $\dfrac{\sin 2\theta-\sin\theta}{1+\cos 2\theta+\cos\theta}=\tan\theta$.

   (ii) $\tan(45°+\theta)-\tan(45°-\theta)=2\tan 2\theta$.

   (iii) $\dfrac{1+\sin 2x}{1-\sin 2x}=\tan^2\left(x+\dfrac{\pi}{4}\right)$.

   (iv) $\dfrac{\cos x-\cos(x+2\theta)}{2\sin\theta}=\sin(x+\theta)$.

10. Prove $\dfrac{\sin 2A}{1-\cos 2A}=\cot A$, hence find the exact value to $\tan 22.5°$.

11. If $\cos A=\dfrac{3}{4}$ and $\sin B=\dfrac{1}{2\sqrt{2}}$ where $0<A<\dfrac{\pi}{2}$ and $0<B<\dfrac{\pi}{2}$

    (i) Show that $A=2B$.

    (ii) Find the exact value of $\sec(A-B)$.

(D) Half Angle Formulae and 't' Formulae

12. If $\sin\dfrac{t}{2}=\dfrac{4}{5}$, find the value of $\sin t$ if $180°<t<270°$.

13. Prove that $\dfrac{2}{1+\cos x}=\sec^2\dfrac{x}{2}$.

14. Prove that $\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x}=\tan\left(\dfrac{x}{2}\right)$.

(E) Triple Angle Formulae

15. Prove that $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$.

(F) Three Dimensional Trigonometry

16. The angle of elevation to the top of a mast from a point A due east of it is 40° and from a point B due south of it is 35°. If A and B are a distance of 500 m apart, show that the height, h metres, of the mast is given by:

$$h=\frac{500}{\sqrt{\cot^2 40+\cot^2 35°}}$$

    Find h to the nearest metre.

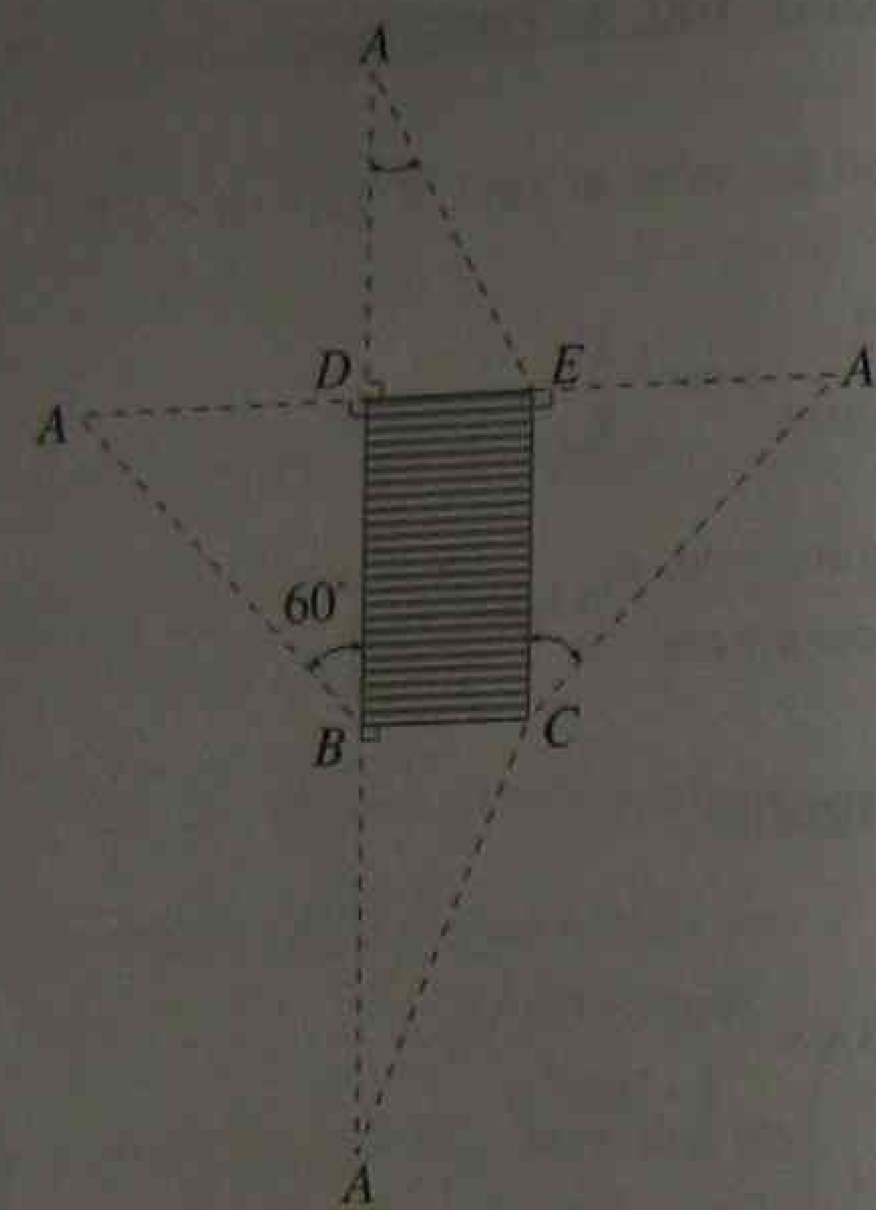
17. The elevation to the top of a hill from a point X due east of it is 28° and from a point Y due south of X is 15°. If X and Y are 800 metres apart, prove that the height of the hill is given by:

$$h=\frac{800}{\sqrt{\cot^2 15°-\cot^2 28°}}$$

    Hence, find h correct to 2 significant figures.

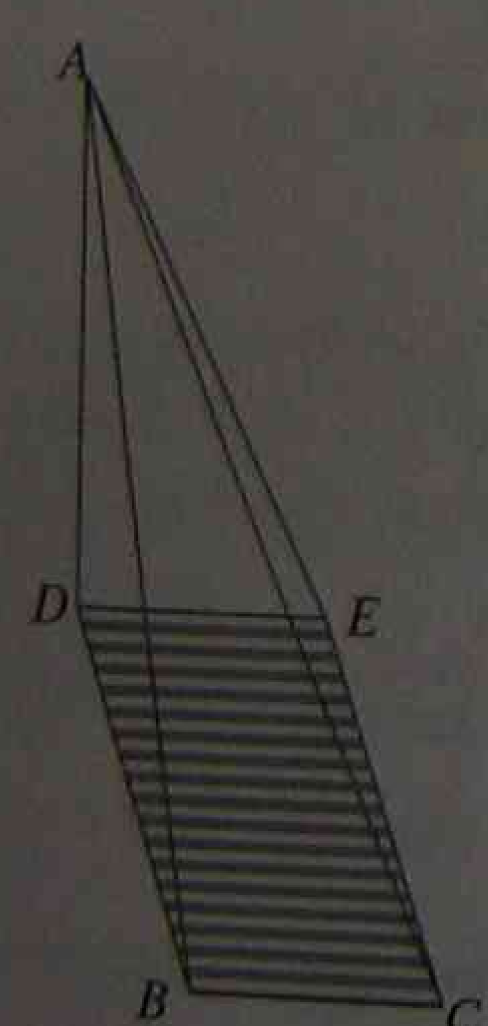
18. The angles of elevation to the top of a hill taken from three points X, Y and Z along a flat path, are respectively 30°, 45° and 30°. If $XY=600$ m and $YZ=300$ m, find the height of the hill.

19. The diagram shows the unfolded plan view of a quadrangular pyramid, which is formed by joining four right-angled triangles and a rectangle togeth



When the net is folded, all the points indicated by A meet together as shown below:

Given, $\angle ABD = 60^\circ$ and $\tan\angle ACE = 2$:



(i) Show that $\angle DAE = 30^\circ$.

(ii) Hence, find the angle that the face ACE makes with the base.

Worked Solutions to Review Exercises

1. (i) $\frac{d}{dx}\left(e^{2x}\cos 3x\right) = e^{2x}\cdot -3\sin 3x + \cos 3x\cdot 2e^{2x} = e^{2x}\left(2\cos 3x - 3\sin 3x\right)$ #

(ii) $\frac{d}{dx}\left(\tan^2 2x\right) = 2\tan 2x\cdot 2\sec^2 2x = 4\tan 2x\sec^2 2x$ #

(iii) $\frac{d}{dx}\left(\sqrt{\sin x}\right) = \frac{d}{dx}(\sin x)^{\frac{1}{2}} = \frac{1}{2}(\sin x)^{-\frac{1}{2}}\cdot\cos x = \frac{\cos x}{2\sqrt{\sin x}}$ #

(iv) $\frac{d}{dx}\left(e^{\sin x}\right) = \cos x e^{(\sin x)}$ #

(v) $\frac{d}{dx}\left(\log_e\left(\sin^2 x\right)\right) = \frac{2\sin x\cos x}{\sin^2 x} = 2\cot x$ #

(vi) $\frac{d}{dx}\left(\cos^3(2x+1)\right) = 3\cos^2(2x+1)\cdot -2\sin(2x+1) = -6\sin(2x+1)\cos^2(2x+1)$ #

2. (i) $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x\cdot -\sin x - \cos x\cdot\cos x}{\sin^2 x}$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-\left(\sin^2 + \cos^2\right)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \#$$

(ii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

integrating both sides w.r.t. x:

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d}{dx}(\cot x)\,dx = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} -\operatorname{cosec}^2 x\,dx$$

$$\therefore \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \operatorname{cosec}^2 x\,dx = -\left[\cot x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = -\left[\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) - 0\right] = \frac{1}{\sqrt{3}} \quad \#$$

3. $\angle BPB' = 90^\circ$, $\angle APB = \angle A'PB' = 15^\circ$

(i) $\dfrac{PB}{\sin 30^\circ} = \dfrac{8}{\sin 15^\circ}$ i.e. $PB = \dfrac{4}{\sin 15^\circ}$

$\sin 45^\circ = \dfrac{OP}{PB}$ i.e. $OP = PB\sin 45^\circ = \dfrac{4\sin 45^\circ}{\sin 15^\circ} = 10.9$ m to 1 d.p. #

(ii) $\cos 45^\circ = \dfrac{OB}{PB}$ i.e. $OB = PB\cos 45^\circ = 10.9$ m

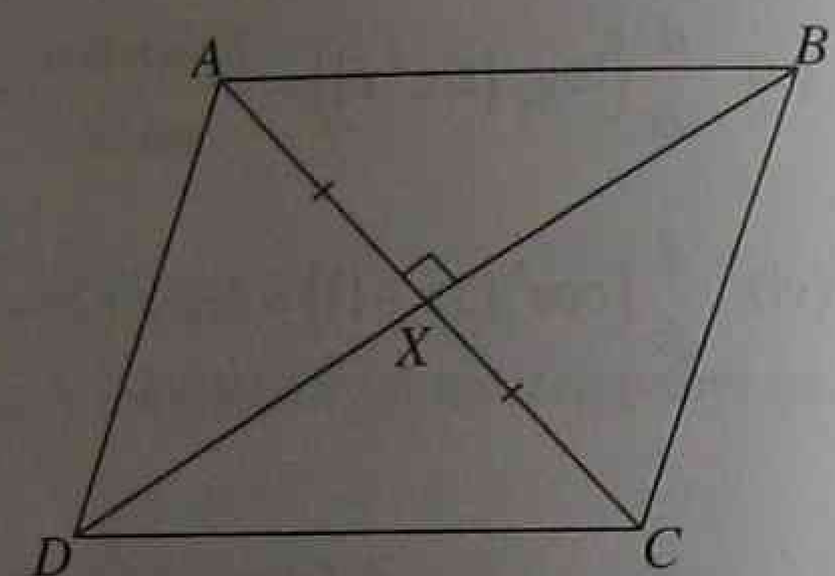
$\therefore\ AO = AB + OB = 8 + 10.9 = 18.9$ m to 1 d.p. #

(iii) $\sin 30^\circ = \dfrac{OP}{PA}$ i.e. $PA = \dfrac{OP}{\sin 30^\circ} = 21.8$ m

$\therefore$ Total length $= 2PA + 2PB = 74.4$ m #

4.
$$AB^2 = AX^2 + BX^2$$
$$= \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2$$
$$= \frac{AC^2}{4} + \frac{BD^2}{4}$$

i.e. $4AB^2 = AC^2 + BD^2$ #



5. (i)
$$\text{LHS} = \frac{\cos^2\alpha - \cos^2\beta}{\cos^2\alpha\cos^2\beta} = \frac{\cos^2\alpha}{\cos^2\alpha\cos^2\beta} - \frac{\cos^2\beta}{\cos^2\alpha\cos^2\beta}$$
$$= \frac{1}{\cos^2\beta} - \frac{1}{\cos^2\alpha}$$
$$= \sec^2\beta - \sec^2\alpha$$
$$= \left(1+\tan^2\beta\right) - \left(1+\tan^2\alpha\right)$$
$$= \tan^2\beta - \tan^2\alpha = \text{RHS} \quad \#$$

(ii)
$$\cos(\alpha+\beta)\cos(\alpha-\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$$
$$= \cos^2\alpha\left(1-\sin^2\beta\right) - \sin^2\beta\left(1-\cos^2\alpha\right)$$
$$= \cos^2\alpha - \sin^2\beta \quad \#$$

(iii)
$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\alpha + \sin\alpha\sin\beta}$$
$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \tan\alpha \quad \#$$

6.
$$\sin(x+y) + \sin(x-y) = \sin x\cos y + \cos x\sin y + \sin x\cos y - \cos x\sin y$$
$$= 2\sin x\cos y$$

thus, $\displaystyle\int \sin 4x\cos 2x\,dx = \frac{1}{2}\int \sin 6x\,dx + \frac{1}{2}\int \sin 2x\,dx$
$$= -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C \quad \#$$

7. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$, thus

$$\tan\left(\frac{5\pi}{12}\right) = \tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}$$
$$= \frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}\cdot 1}$$
$$= \frac{1+\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}-1}$$
$$= \frac{1+\sqrt{3}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$= \frac{\left(\sqrt{3}+1\right)^2}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3} \quad \#$$

8. (i) $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$, $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$, $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$, $\cos 60^\circ = \dfrac{1}{2}$ #

(ii)
$$\sin(105^\circ) = \sin(60^\circ + 45^\circ)$$
$$= \sin 60^\circ\cos 45^\circ + \cos 60^\circ\sin 45^\circ$$
$$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\times\frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \#$$

9. (i) $\text{LHS} = \dfrac{\sin 2\theta - \sin\theta}{1+\cos 2\theta - \cos\theta}$

$$= \frac{2\sin\theta\cos\theta - \sin\theta}{\sin^2\theta + \cos^2\theta + \cos^2\theta - \sin^2\theta - \cos\theta}$$

$$= \frac{\sin\theta\left(2\cos\theta - 1\right)}{\cos\theta\left(2\cos\theta - 1\right)} = \tan\theta = \text{RHS} \ \#$$

(ii) $\text{LHS} = \tan\left(45^\circ + \theta\right) - \tan\left(45^\circ - \theta\right)$

$$= \frac{\tan 45^\circ + \tan\theta}{1 - \tan 45^\circ \tan\theta} - \frac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{\left(1+\tan\theta\right)^2 - \left(1-\tan\theta\right)^2}{\left(1-\tan\theta\right)\left(1+\tan\theta\right)}$$

$$= \frac{\left(1+2\tan\theta+\tan^2\theta\right) - \left(1-2\tan\theta+\tan^2\theta\right)}{1-\tan^2\theta}$$

$$= \frac{4\tan\theta}{1-\tan^2\theta} \qquad \left(\text{note: } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= 2\tan 2\theta = \text{RHS} \ \#$$

(iii) $\text{LHS} = \dfrac{1+\sin 2x}{1-\sin 2x} = \dfrac{\cos^2 x + \sin^2 x + 2\sin x\cos x}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$$= \frac{\left(\cos x + \sin x\right)^2}{\left(\cos x - \sin x\right)^2}$$

$$= \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2$$

$$= \left(\frac{1+\tan x}{1-\tan x}\right)^2 \qquad (\div \text{ by } \cos x)$$

$$= \tan^2\left(x + \frac{\pi}{4}\right) = \text{RHS} \ \#$$

$$\left(\text{note: } \tan\left(x+\frac{\pi}{4}\right) = \frac{\tan x + \tan\frac{\pi}{4}}{\ldots} = \frac{1+\tan x}{\ldots}\right)$$

(iv) $\text{LHS} = \dfrac{\cos x - \cos\left(x+2\theta\right)}{2\sin\theta} = \dfrac{\cos x - \left(\cos x\cos 2\theta - \sin x\sin 2\theta\right)}{2\sin\theta}$

$$= \frac{\cos x - \cos x\cos 2\theta + \sin x\sin 2\theta}{2\sin\theta}$$

$$= \frac{\cos x - \cos x\left(1 - 2\sin^2\theta\right) + \sin x \cdot 2\sin\theta\cos\theta}{2\sin\theta}$$

$$= \frac{2\sin^2\theta\cos x + 2\sin\theta\cos\theta\sin x}{2\sin\theta}$$

$$= \sin\theta\cos x + \cos\theta\sin x$$

$$= \sin\left(x+\theta\right) = \text{RHS} \ \#$$

10. $\text{LHS} = \dfrac{\sin 2A}{1-\cos 2A} = \dfrac{2\sin A\cos A}{\left(\sin^2 A + \cos^2 A\right) - \left(\cos^2 A - \sin^2 A\right)}$

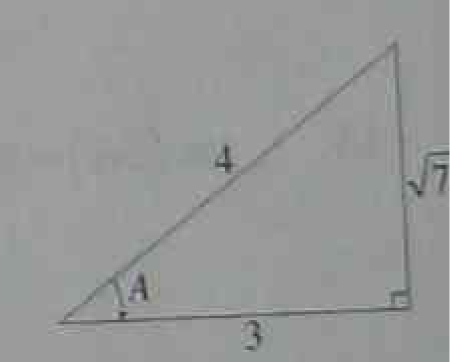
$$= \frac{2\sin A\cos A}{2\sin^2 A} = \frac{\cos A}{\sin A} = \cot A = \text{RHS} \ \#$$

$$\cot 22.5^\circ = \frac{\sin 45^\circ}{1-\cos 45^\circ} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}$$

$\therefore \ \tan 22.5 = \sqrt{2} - 1 \ \#$

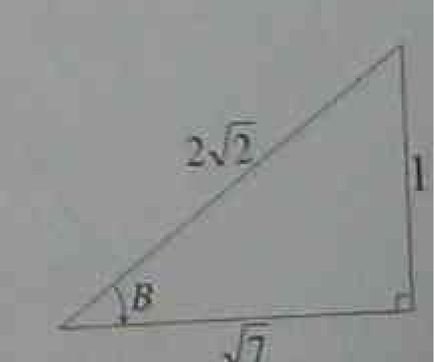
11. (i) $\cos 2B = 1 - 2\sin^2 B = 1 - 2\times\left(\dfrac{1}{2\sqrt{2}}\right)^2 = 1 - \dfrac{1}{4} = \dfrac{3}{4} = \cos A$

$\therefore \ A = 2B \ \#$



(ii) $\sec\left(A-B\right) = \dfrac{1}{\cos\left(A-B\right)}$

$$= \frac{1}{\cos A\cos B + \sin A\sin B}$$



$$= \frac{1}{\frac{3}{4}\times\frac{\sqrt{7}}{2\sqrt{2}} + \frac{\sqrt{7}}{4}\times\frac{1}{2\sqrt{2}}}$$

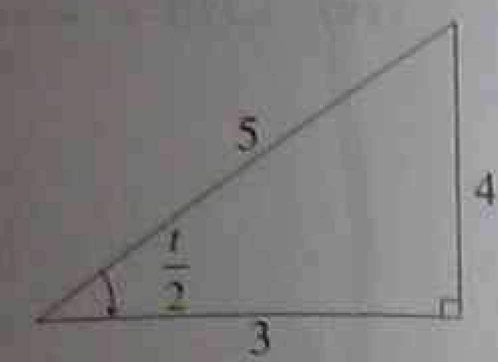
$$= \frac{8\sqrt{2}}{4\sqrt{7}} = \frac{2\sqrt{2}}{\sqrt{7}} \ \#$$

12. $\sin t = 2\sin\frac{t}{2}\cos\frac{t}{2}$

$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$

$= -\frac{24}{25}$ #

[note: cos is negative as $\frac{t}{2}$ is in the 2nd quadrant.]



13. Using the t-formulae, where $t = \tan\frac{x}{2}$:

$$\text{LHS} = \frac{2}{1+\cos x}$$

$$= \frac{2}{1+\frac{1-t^2}{1+t^2}}$$

$$= \frac{2}{\frac{1+t^2+1-t^2}{1+t^2}} = \frac{2\left(1+t^2\right)}{2} = 1+t^2 = 1+\tan^2\frac{x}{2} = \sec^2\frac{x}{2} = \text{RHS} \ \#$$

14. Using the t-formulae, where $t = \tan\frac{x}{2}$:

$$\text{LHS} = \frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2+2t-1+t^2}{1+t^2} \times \frac{1+t^2}{1+t^2+2t+1-t^2}$$

$$= \frac{2t+2t^2}{2+2t} = \frac{2t(1+t)}{2(1+t)} = t = \tan\frac{x}{2} = \text{RHS} \ \#$$

15. $\tan(3A) = \tan(2A+A) = \dfrac{\tan 2A+\tan A}{1-\tan A\tan 2A}$

$$= \frac{\frac{2\tan A}{1-\tan^2 A}+\tan A}{1-\tan A\cdot\frac{2\tan A}{1-\tan^2 A}}$$

$= \ 2\tan A+\tan A-\tan^3 A \qquad 1-\tan^2 A$

16. Looking at ΔBOH:

$\tan 35° = \frac{h}{OB}$ i.e. $OB = h\cot 35°$

Looking at ΔAOH:

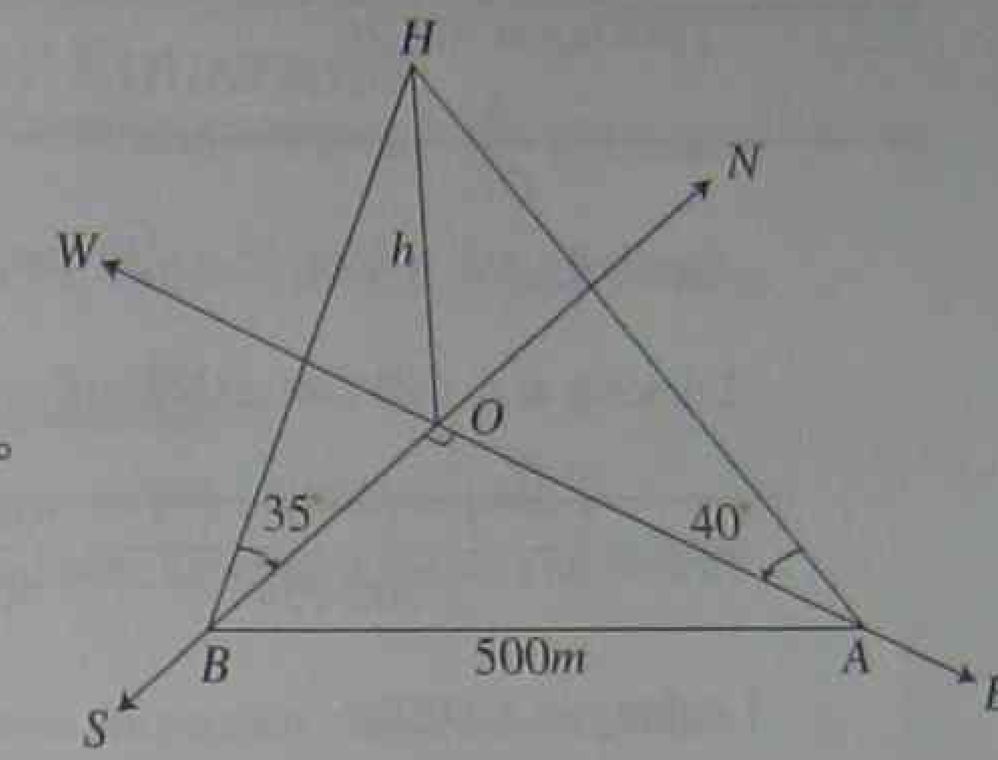
$\tan 40° = \frac{h}{OH}$ i.e. $OH = h\cot 40°$

Looking at ΔOAB:

$OA^2 + OB^2 = 500^2$

$(h\cot 40°)^2 + (h\cot 35°)^2 = 500^2$

$\therefore \ h = \dfrac{500}{\sqrt{\cot^2 40° + \cot^2 35°}} = 268.80... = 269$ to the nearest metre #



17. Looking at ΔOHX:

$\tan 28° = \frac{h}{OX}$ i.e. $OX = h\cot 28°$

Looking at ΔOHY:

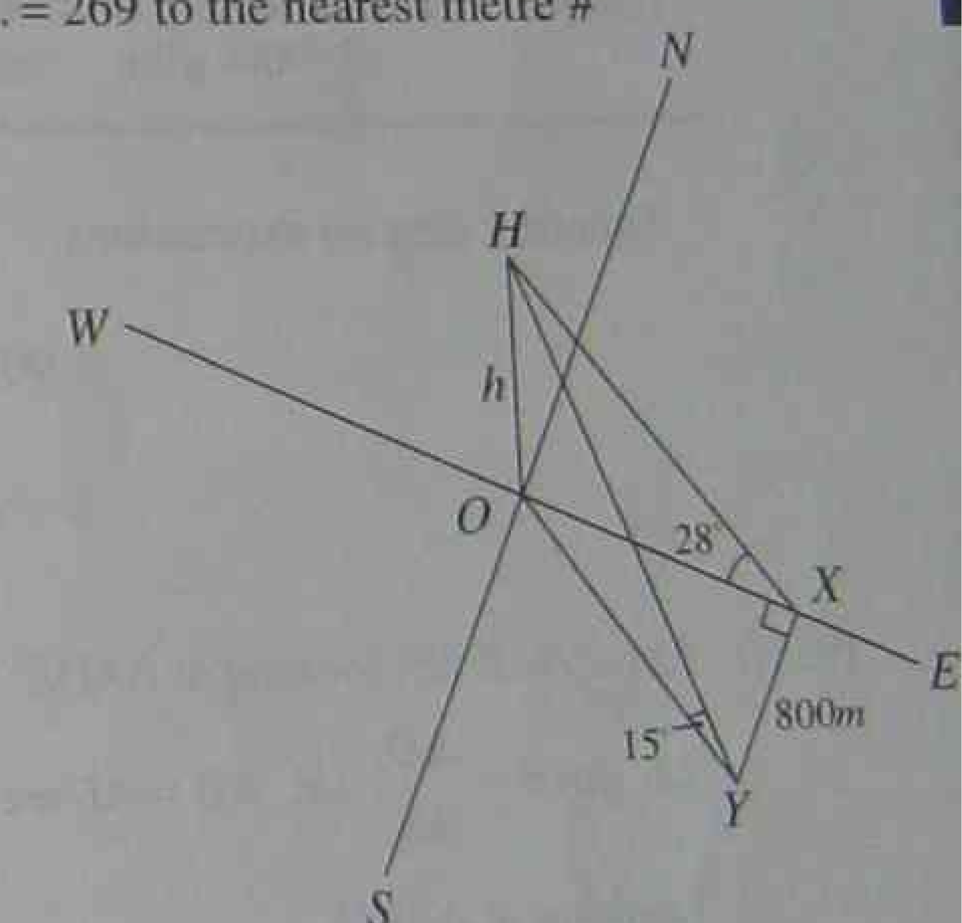
$\tan 15° = \frac{h}{OY}$ i.e. $OY = h\cot 15°$

Looking at ΔOXY:

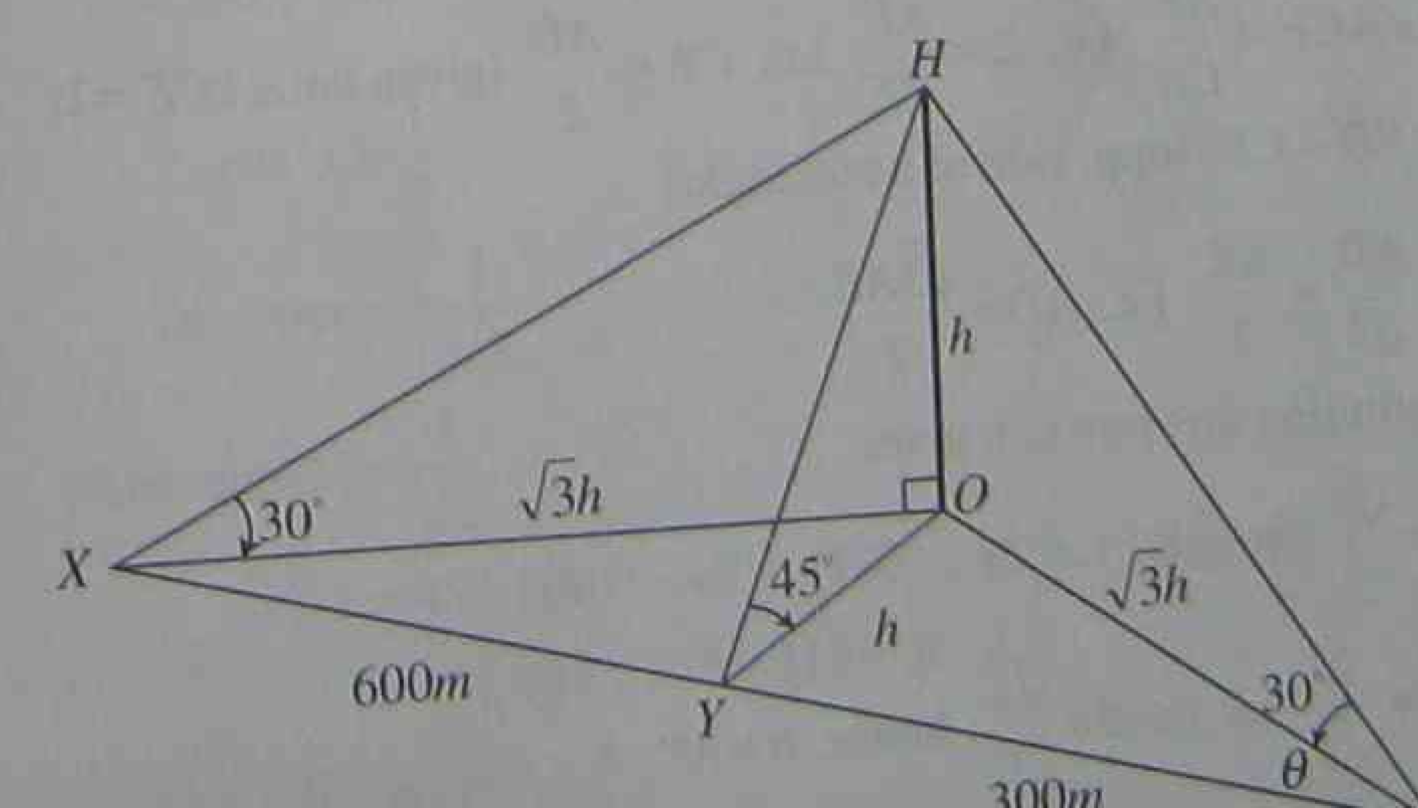
$OY^2 = OX^2 + 800^2$

i.e $h^2\cot^2 15° = h^2\cot^2 28° + 800^2$

$h = \dfrac{800}{\sqrt{\cot^2 15° - \cot^2 28°}} = 248.17... = 250$ m correct to 2 s.f. #



18.



Looking at ΔOZH:

$\tan 30° = \dfrac{h}{OZ}$ i.e. $OZ = h\cot 30° = \sqrt{3}h$

Similarly, $OX = h\cot 30° = \sqrt{3}h$ and $OY = h\cot 45° = h$

Looking at $\Delta\ OZY$, let $\angle OZY = \theta$:

$$\cos\theta = \frac{\left(\sqrt{3}h\right)^2 + (300)^2 - (h)^2}{2\cdot 300.\sqrt{3}h} = \frac{300^2 + 2h^2}{600\sqrt{3}h} \quad \text{(using the cosine rule)}$$

Looking at $\Delta\ OZX$:

$$\cos\theta = \frac{\left(\sqrt{3}h\right)^2 + (900)^2 - \left(\sqrt{3}h\right)^2}{2\cdot 900\cdot\sqrt{3}h} = \frac{900^2}{1800\sqrt{3}h} \quad \text{(using the cosine rule)}$$

Equating the two expressions: $\dfrac{300^2 + 2h^2}{600\sqrt{3}h} = \dfrac{900^2}{1800\sqrt{3}h}$

$$90{,}000 + 2h^2 = 270{,}000$$

$$h^2 = 90{,}000$$

$$h = 300 \text{ m } (\text{as } h > 0)\ \#$$

19. (i) Let $\angle DAE = \theta$, looking at ΔADE:

i.e. $\cos\theta = \dfrac{AD}{AE}$ i.e. $AE = AD\sec\theta$ ....(1)

Looking at ΔABD:

$\tan 60° = \dfrac{AD}{BD}$ i.e. $BD = AD\cot 60°$ i.e. $BD = \dfrac{AD}{\sqrt{3}}$

Looking at ΔACE:

$\tan\angle ACE = \dfrac{AE}{CE}$ i.e. $2 = \dfrac{AE}{CE}$ i.e. $CE = \dfrac{AE}{2}$ (given $\tan\angle ACE = 2$)

but $BD = CE$ (opp. side of a rectangle)

i.e. $\dfrac{AD}{\sqrt{3}} = \dfrac{AE}{2}$ i.e. $AD = \dfrac{\sqrt{3}AE}{2}$

Substituting this into (1), gives:

$AE = \dfrac{\sqrt{3}}{2}AE\sec\theta$

$\sec\theta = \dfrac{2}{\sqrt{3}}$ i.e. $\cos\theta = \dfrac{\sqrt{3}}{2}$ hence, $\theta = 30°$ #

(ii) Angle that ACE makes with the base $= 90° - \angle DAE = 90° - 30° = 60°$ #

TOPIC 4

Trigonometric Equations

There are essentially three main types of trigonometric equations in the Extension 1 course.

(A) Type 1: Simple Trigonometric Equations

There is no direct method that can be used here, however there are a number of general rules to follow:

- If more than two trigonometric ratios are present, then convert all the ratios into one of the basic ratios (i.e. sin, cos, tan) then solve.
- Equations of this type can often be expressed as a quadratic in sin, cos or tan and then solved by algebraic methods.

Example 1: Solve the following for $0° \le x \le 360°$.

(i) $\sqrt{3}\sin x = -\cos x$ (ii) $\cos^2 x = \dfrac{1}{2}$

(iii) $2\sin 3x - \sqrt{3} = 0$ (iv) $\cot x - 3\tan x = 2$

Solution 1:

(i) $\sqrt{3}\sin x = -\cos x$

$\tan x = -\dfrac{1}{\sqrt{3}}$

now, $\tan 30° = \dfrac{1}{\sqrt{3}}$

$\therefore x = (180° - 30°), (360° - 30°)$

$= 150°, 330°$ #

(ii) $\cos^2 x = \dfrac{1}{2}$

i.e. $\cos x = \pm\dfrac{1}{\sqrt{2}}$

Now, $\cos 45° = \dfrac{1}{\sqrt{2}}$

$\therefore x = 45°, (180° - 45°), (180° + 45°), (360° - 45°)$

$= 45°, 135°, 225°, 315°$ #

(iii) $2\sin 3x - \sqrt{3} = 0$

$\sin 3x = \dfrac{\sqrt{3}}{2}$

$$\text{Now, } \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Note also that since $0^\circ \le x \le 360^\circ \ \therefore\ 0^\circ \le 3x \le 1080^\circ$

$\therefore\ 3x = 60^\circ, 120^\circ, 420^\circ, 480^\circ, 780^\circ, 840^\circ$

$\therefore\ x = 20^\circ, 40^\circ, 140^\circ, 160^\circ, 260^\circ, 280^\circ$ #

(iv) $\cot x - 3\tan x = 2$

$$\frac{1}{\tan x} - 3\tan x = 2$$

$$1 - 3\tan^2 x = 2\tan x$$

i.e. $3\tan^2 x + 2\tan x - 1 = 0$

$$(3\tan x - 1)(\tan x + 1) = 0$$

$\therefore\ 3\tan x = 1$ or $\tan x = -1$

$\tan x = \dfrac{1}{3}$ $\qquad x = 135^\circ, 315^\circ$

$x = 18^\circ 26', 198^\circ 26'$

$\therefore\ x = 18^\circ 26', 135^\circ, 198^\circ 26', 315^\circ$ #

(B) Type 2: Trigonometric Equations Involving Double and Half Angle Results

These equations require the knowledge of double angle and half angle results.

- Special care should be taken to ensure that all solutions in the given domain are obtained. e.g. for $0^\circ \le x \le 360^\circ$, then :

$\cos 2x = 1 \qquad 0^\circ \le 2x \le 720^\circ$

$\cos\left(\dfrac{x}{2}\right) = \dfrac{1}{2} \qquad 0^\circ \le \dfrac{x}{2} \le 180^\circ$

- Never cancel trigonometric functions by division as that can remove a solution. The approach is to factorise then solve.

Example 1: Solve the following, for $0 \le x \le 2\pi$

(i) $\sin 2x = \sin x$ (ii) $1 + \cos x = \cos\left(\dfrac{x}{2}\right)$ (iii) $\sin 2x = \tan x$

Solution 1:

$$2\sin x\cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

i.e. $\sin x = 0$ or $2\cos x = 1$

$x = 0, \pi, 2\pi$ $\qquad \cos x = \dfrac{1}{2}$

$$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}$$

$\therefore\ x = 0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}, 2\pi$ #

(ii) $1 + \cos x = \cos\left(\dfrac{x}{2}\right) \qquad \left(\cos x = 2\cos^2\left(\dfrac{x}{2}\right) - 1\right)$

$$1 + 2\cos^2\left(\frac{x}{2}\right) - 1 = \cos\left(\frac{x}{2}\right)$$

$$2\cos^2\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) = 0$$

$$\cos\left(\frac{x}{2}\right)\left[2\cos\left(\frac{x}{2}\right) - 1\right] = 0, \text{ now } 0 \le \frac{x}{2} \le \pi$$

$\cos\left(\dfrac{x}{2}\right) = 0$ or $2\cos\left(\dfrac{x}{2}\right) = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2} \qquad \cos\left(\dfrac{x}{2}\right) = \dfrac{1}{2}$

$\qquad \dfrac{x}{2} = \dfrac{\pi}{3}$

i.e. $x = \pi \qquad$ i.e. $x = \dfrac{2\pi}{3}$

$\therefore\ x = \pi, \dfrac{2\pi}{3}$ #

(iii) $\sin 2x = \tan x$

$$2\sin x\cos x = \frac{\sin x}{\cos x}$$

$$2\sin x\cos^2 x - \sin x = 0$$

$$\sin x\left(2\cos^2 x - 1\right) = 0$$

$\sin x = 0$ or $2\cos^2 x = 1$

$\qquad \cos^2 x = \dfrac{1}{2}$

$x = 0, \pi, 2\pi$

$$\cos x = \pm\frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \left(\pi - \frac{\pi}{4}\right), \left(\pi + \frac{\pi}{4}\right), \left(2\pi - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\therefore\ x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi\ \#$$

(C) Type 3: Equations of the Form $a\sin x \pm b\cos x = c$

There are two methods for solving equations of the type $a\sin x \pm b\cos x = c$.

Method 1: Use 't' Formulae

Replace $\sin x$ and $\cos x$ by their 't' formulae learnt in topic 3 and solve for t.

i.e. $t = \tan\left(\frac{x}{2}\right)$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

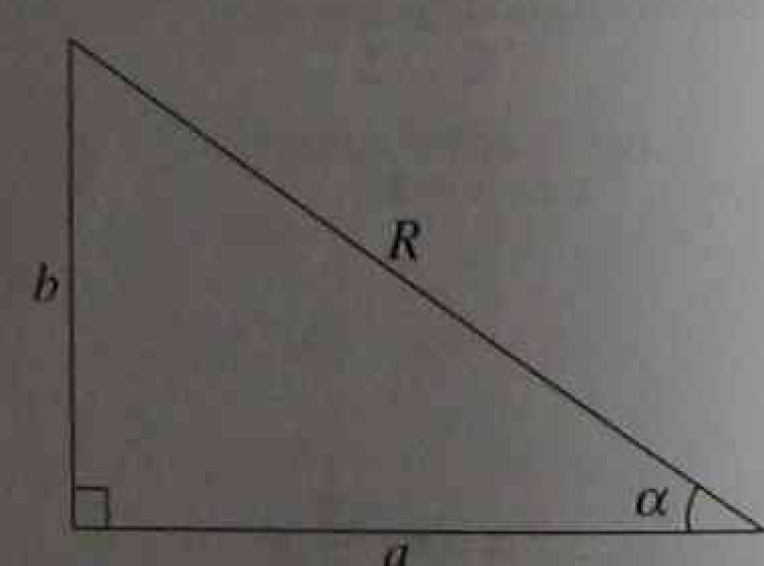
Note when using the 't' formulae must always test for $x = \pi$ (or 180°),

as $\tan\left(\frac{\pi}{2}\right)$ (or 90°) is undefined.

Method 2: The Transformation Method

We can rewrite:

$$a\sin x + b\cos x = R\sin(x+\alpha)$$
$$a\sin x - b\cos x = R\sin(x-\alpha)$$
$$a\cos x + b\sin x = R\cos(x-\alpha)$$
$$a\cos x - b\sin x = R\cos(x+\alpha)$$



where in each case:

$R = \sqrt{a^2+b^2}$ and

$\tan a = \frac{b}{a}$, $0^\circ < \alpha < 90^\circ$

It should be noted, that since the transformation method is often the simpler and more reliable method to use, most type 3 equations in this topic have been solved that way.

Example 1: Solve $\sqrt{3}\sin x - \cos x = 1$, for $0^\circ \le x \le 360^\circ$ using:

(i) the transformation method.

(ii) the t-formulae.

Solution 1:

(i) Transformation method.

Let $\sqrt{3}\sin x - \cos x = R\sin(x-\alpha)$

$a = \sqrt{3}, b = 1$

$\therefore\ R = \sqrt{a^2+b^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\tan\alpha = \frac{b}{a} = \frac{1}{\sqrt{3}}$ $\therefore\ \alpha = 30^\circ$

$\therefore\ \sqrt{3}\sin x - \cos x = 2\sin(x - 30^\circ)$

i.e. $2\sin(x-30^\circ) = 1$ $\therefore\ \sin(x-30^\circ) = \frac{1}{2}$

$x - 30^\circ = 30^\circ, 150^\circ$

$\therefore\ x = 60^\circ, 180^\circ$ #

(ii) The t-formulae.

Using $t = \tan\left(\frac{x}{2}\right)$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

and substituting into the equation:

$\sqrt{3}\sin x - \cos x = 1$, gives

$\frac{2\sqrt{3}t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$ (multiplying by $(1+t^2)$)

$2\sqrt{3}t - (1-t^2) = 1+t^2$

$2\sqrt{3}t - 1 + t^2 = 1 + t^2$

i.e. $2\sqrt{3}t = 2$

$t = \frac{1}{\sqrt{3}}$

i.e. $\tan\left(\frac{x}{2}\right) = \frac{1}{\sqrt{3}}$, gives:

$\frac{x}{2} = 30^\circ$, noting $0^\circ \le \frac{x}{2} \le 180^\circ$ $\therefore\ x = 60^\circ$

with t-method, test $x = 180^\circ$.

i.e. $\sqrt{3}\sin 180^\circ - \cos 180^\circ = 0 + 1 = 1$

$\therefore\ x = 180^\circ$ is also a solution.

$\therefore$ Complete solutions are $x = 60^\circ, 180^\circ$ #

(D) General Solutions

At times students are required to solve an equation for **all** values of x satisfying the trigonometric equation. In these circumstances the general solution for x is required:

- If $\sin x = k$ and $\alpha = \sin^{-1} k$, then $x = n\pi + (-1)^n \alpha$
- If $\cos x = k$ and $\alpha = \cos^{-1} k$, then $x = 2n\pi \pm \alpha$
- If $\tan x = k$ and $\alpha = \tan^{-1} k$, then $x = n\pi + \alpha$

Where $n = 0, \pm 1, \pm 2, \ldots$ and α is the smallest positive or negative angle that satisfies the given equation.

Trigonometric Equations

Example 1: Find the general solution of the following equations:

(i) $\tan x = \sqrt{3}$ (ii) $\sin^2 x = \sin x$

Solution 1:

(i) $\tan x = \sqrt{3}$ $\alpha = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

$\therefore\ x = n\pi + \dfrac{\pi}{3},\quad n = 0, \pm 1, \pm 2$ #

(ii) $\sin^2 x = \sin x$

$\sin^2 x - \sin x = 0$

$\sin x(\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = 1$

$\therefore\ x = n\pi$ or $x = n\pi + (-1)^n \dfrac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, \ldots$ #

Review Exercises

(A) Type 1: Simple Trigonometric Equations

1. Solve the following equation for θ where $0 \le \theta \le 2\pi$:

(i) $2\sin 2\theta = -1$ (ii) $\tan^2\theta = \tan\theta$

(iii) $\sin^2\theta - 5\sin\theta - 2\cos^2\theta = 0$ (iv) $\cos^3\theta = \dfrac{1}{2}\cos\theta$

2. Solve the following equations for x where $0^\circ \le x \le 360^\circ$, expressing your answer where applicable to the nearest minute.

(i) $\sec 2x = 3$ (ii) $\tan x - 3\cot x = 0$

(iii) $2\cos^2 x + \sin x - 1 = 0$ (iv) $\cos^2 x - \sin 2x - 3\sin^2 x = 0$

Trigonometric Equations

(B) Type 2: Trigonometric Equations Involving Double and Half Angle Results

3. Solve the following equations for x where $0 \le x \le 2\pi$.

(i) $2\cos 2\theta + 2\sin\theta = 0$ (ii) $\sin 2\theta = \sqrt{3}\cos\theta$

(iii) $\sin\theta - \sin\left(\dfrac{\theta}{2}\right) = 0$ (iv) $\sin 4\theta = \cos 2\theta$

4. Solve the following equations for x where $0^\circ \le x \le 360^\circ$ expressing your answers, where applicable, to the nearest minute.

(i) $\sin 2x = \dfrac{\cos x}{4}$ (ii) $\cos 2x = \sin x$

(iii) $\tan 2x + \tan x = 0$ (iv) $\sin 3x = \sin x$

(C) Type 3: Equations of the Form $a\sin x + b\cos x = c$

5. Solve θ where $0 \le \theta \le 2\pi$, expressing your answer in exact form where applicable correct to 2 decimal places:

(i) $\sin\theta + \sqrt{3}\cos\theta = -1$ (ii) $2\sin\theta - \cos\theta = 2$

(iii) $\sqrt{3}\cos\theta - \sin\theta = 1$ (iv) $\cos\theta + \sin\theta = -\dfrac{1}{2}$

6. Solve for x where $0^\circ \le x \le 360^\circ$ expressing your answer to the nearest minute:

(i) $2\sin x + 3\cos x = 1$ (ii) $3\cos x + 4\sin x = 3$

TOPIC 5

COORDINATE GEOMETRY

(A) Angle Between Two Lines

The acute angle between two lines with respective gradients m_1 and m_2, is given by:

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Example 1: Show that the acute angle α, between the tangents at the point $(1, 1)$ to the curves $y = x^3$ and $y = \sqrt{x}$, is given by: $\tan\alpha = 1$.

Solution 1:

$y = x^3,\ y' = 3x^2$ at $x = 1,\ y' = 3$

$y = \sqrt{x},\ y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ at $x = 1,\ y' = \frac{1}{2}$

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{3 - \frac{1}{2}}{1 + \frac{3}{2}}\right|$$

$$= \frac{5}{2} \times \frac{2}{5}$$

$$= 1 \ \#$$

Example 2: Find the obtuse angle between the lines $x + y = 1$ and $2x - 3y = 0$, to the nearest minute.

Solution 2:

Line: $x + y = 1$

$y = 1 - x$, which has gradient $= -1$

Line: $2x - 3y = 0$

$y = \frac{2}{3}x$, which has gradient $= \frac{2}{3}$

i.e. $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{2}{3} - (-1)}{1 + -\frac{2}{3}}\right| = \frac{5}{3} \times \frac{3}{1} = 5$

i.e. $\theta = \tan^{-1}(5) = 78°41'$

$\therefore$ Obtuse angle $= 180° - \theta = 101°19'$ #

(B) Division of an Interval AB

The coordinates of a point $P(x, y)$ dividing a given interval AB with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ is given by:

Internally: $x = \frac{mx_2 + nx_1}{m + n}$, $y = \frac{my_2 + ny_1}{m + n}$

Externally: $x = \frac{mx_2 - nx_1}{m - n}$, $y = \frac{my_2 - ny_1}{m - n}$

Note:
- For internal division, the point P lies inside the interval AB.
- For external division, the point P lies outside the interval AB.

Example 1: Find the co-ordinates of the point P which divides the interval AB with end points $A(-1, 2)$ and $B(3, -5)$ internally in the ratio $2:3$.

Solution 1:

$$x = \frac{mx_2 + nx_1}{m + n} = \frac{2\times3 + 3\times-1}{2+3} = \frac{6-3}{5} = \frac{3}{5}$$

$$y = \frac{my_2 + ny_1}{m + n} = \frac{2\times-5 + 3\times2}{2+3} = \frac{-10+6}{5} = -\frac{4}{5}$$

$\therefore$ P has coordinates $\left(\frac{3}{5}, -\frac{4}{5}\right)$ #

Example 2: Given $A(-3, 1)$ and $B(4, 2)$, find the point which divides AB in the ratio $3:-4$.

Solution 2:

Since the ratio is negative $\therefore$ this is external division with $m = 3$ and $n = 4$

i.e. $x = \frac{mx_2 - nx_1}{m - n} = \frac{3\times4 - 4\times-3}{3-4} = \frac{12+12}{-1} = -24$

$y = \frac{my_2 - ny_1}{m - n} = \frac{3\times2 - 4\times1}{3-4} = \frac{6-4}{-1} = -2$

$\therefore$ P has coordinates $(-24, -2)$ #

REVIEW EXERCISES

(A) Angle Between Two Lines

1. If θ is the acute angle made between the lines $4x+y+4=0$ and $4x-y+12=0$, find the exact value of $\tan\theta$.

2. Find the acute angle (to the nearest minute) between the curves $y=e^{2x}$, and $y=e^x+2$ at their point of intersection.

3. The curves $xy=2e$ and $y=\ln x^2$ intersect at $P(e, 2)$. If ϕ is the acute angle between these curves, show that $\tan\phi=\dfrac{4e}{e^2-4}$.

4. Find the acute angle (to the nearest minute) between the tangents to the curves $y=\sin x$ and $y=\cos x$ at the point of intersection.

5. Find the size of the obtuse angle (to the nearest degree) between the two curves $y=\ln(x^2+1)$ and $y=2\ln(x+2)$ at their point of intesection.

6. The angle between the line $y=\dfrac{x}{A}$ and the tangent to the curve $y=Ax^2$ at $x=1$ is 45°. Find the value(s) of A.

(B) Division of an Interval AB

7. Find the coordinates of the point P which divides the interval AB internally in the ratio 1:2, where A and B have coordinates $(3, -2)$ and $(6, 7)$ respectively.

8. Given $A(2, 7)$ and $B(-1, -1)$, find the point P which divides AB externally in the ratio 2:5.

9. Given that the point $P(1, 1)$ divides the interval AB internally in the ratio 3:1. Find the coordinates of B if A has coordinates $(-2, 3)$.

10. A point $P(11, -28)$ divided the interval AB in the ratio m:3. If A and B have coordinates $(-1, 8)$ and $(2, 1)$ respectively, find the value of m.

11. The point $P(-10, 21)$ di

Coordinate Geometry

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $4x+y+4=0$ i.e. $y=-4x-4$, which has gradient $=-4$

$4x-y+12=0$ i.e. $y=4x+12$, which has gradient $=4$

$$\therefore \tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{4-(-4)}{1+4(-4)}\right|=\left|\frac{8}{-15}\right|=\frac{8}{15}\ \#$$

2. $e^{2x}=e^x+2$

$e^{2x}-e^x-2=0$

$(e^x+1)(e^x-2)=0$

$e^x>0\ \therefore\ e^x=2$ gives the only solution

i.e. $x=\ln 2$

Now: $y=e^{2x}$, $y'=2e^{2x}$ at $x=\ln 2$ $y'=8$

$y=e^x+2$, $y'=e^x$ at $x=\ln 2$ $y'=2$

$$\therefore \tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{8-2}{1+16}\right|=\frac{6}{17}$$

$\therefore\ \theta=19°26'$ to the nearest minute #

3. $xy=2e$ i.e. $y=\dfrac{2e}{x}$, $y'=\dfrac{-2e}{x^2}$ at $x=e$ $y'=-\dfrac{2}{e}$

$y=\ln x^2$ i.e. $y=2\ln x$, $y'=\dfrac{2}{x}$ at $x=e$ $y'=\dfrac{2}{e}$

$$\therefore \tan\phi=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{\frac{2}{e}-\left(-\frac{2}{e}\right)}{1+\left(\frac{2}{e}\right)\left(-\frac{2}{e}\right)}\right|=\left|\frac{\frac{4}{e}}{1-\frac{4}{e^2}}\right|=\left|\frac{4}{e}\times\frac{e^2}{e^2-4}\right|=\frac{4e}{e^2-4}\ \#$$

4. $\sin x=\cos x$ i.e. $\tan x=1\ \therefore\ x=\dfrac{\pi}{4}$

Now, $y=\sin x$, $y'=\cos x$ at $x=\dfrac{\pi}{4}$ $y'=\dfrac{1}{\sqrt{2}}$

and $y=\cos x$, $y'=-\sin x$ at $x=\dfrac{\pi}{4}$ $y'=-\dfrac{1}{\sqrt{2}}$

$$\therefore \tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)}{1+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)}\right|=\left|\frac{\frac{2}{\sqrt{2}}}{1-\frac{1}{2}}\right|=\frac{2}{\sqrt{2}}\times\frac{2}{1}=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

5. $\ln(x^2+1)=2\ln(x+2)$

i.e. $\ln(x^2+1)=\ln(x+2)^2$

i.e. $x^2+1=(x+2)^2$

$x^2+1=x^2+4x+4$

$\therefore \quad x=-\frac{3}{4}$

Now, $y=\ln(x^2+1)$, $y'=\frac{2x}{x^2+1}$ at $x=-\frac{3}{4}$ $y'=-\frac{24}{25}$

and $y=2\ln(x+2)$, $y'=\frac{2}{x+2}$ at $x=-\frac{3}{4}$ $y'=\frac{8}{5}$

$$\therefore \quad \tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|=\left|\frac{\frac{8}{5}-\left(-\frac{24}{25}\right)}{1+\left(\frac{8}{5}\right)\left(-\frac{24}{25}\right)}\right|=4.776...$$

$\therefore \quad \theta=78°10'28'$

Hence, the obtuse angle $=180°-\theta=102°$ to the nearest degree. #

6. $y=Ax^2$, $y'=2Ax$ at $x=1$ $y'=2A$

$y=\frac{x}{A}$, $y'=\frac{1}{A}$ at $x=1$ $y'=\frac{1}{A}$

$\therefore \quad \tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$

$$\text{i.e.} \quad \tan(45°)=\left|\frac{2A-\frac{1}{A}}{1+2A\left(\frac{1}{A}\right)}\right|=\left|\frac{\frac{2A^2-1}{A}}{3}\right|=\frac{2A^2-1}{3A}$$

i.e. $1=\frac{2A^2-1}{3A}$

$3A=2A^2-1$

$2A^2-3A-1=0$

i.e. $A=\frac{3\pm\sqrt{9-4\times2\times(-1)}}{4}=\frac{3\pm\sqrt{17}}{4}$

$\therefore \quad A=\frac{3+\sqrt{17}}{4}$ or $A=\frac{3-\sqrt{17}}{4}$ #

7. $x=\frac{mx_2+nx_1}{m+n}=\frac{1\times6+2\times3}{1+2}=\frac{12}{3}=4$

$y=\frac{my_2+ny_1}{m+n}=\frac{1\times7+2\times-2}{1+2}=\frac{3}{3}=1$

8. $x=\frac{mx_2-nx_1}{m-n}=\frac{2\times(-1)-5\times(2)}{2-5}=\frac{-2-10}{-3}=4$

$y=\frac{my_2-ny_1}{m-n}=\frac{2\times(-1)-5\times(7)}{2-5}=\frac{-2-35}{-3}=\frac{37}{3}$

$\therefore \quad P$ is $\left(4, \frac{37}{3}\right)$ #

9. $x=\frac{mx_2+nx_1}{m+2}$

$1=\frac{3x_2+1\times-2}{4}=\frac{3x_2-2}{4} \quad \therefore \quad x_2=2$

$y=\frac{my_2+ny_1}{m+n}$

$1=\frac{3y_2+1\times3}{4}=\frac{3y_2+1\times3}{4} \quad \therefore \quad y_2=\frac{1}{3}$

$\therefore$ coordinates of B are $\left(2, \frac{1}{3}\right)$ #

10. As P is outside the interval $\therefore$ the division is external:

i.e. $x=\frac{mx_2-nx_1}{m-n}$

$11=\frac{m\times2-3\times-1}{m-3}=\frac{2m+3}{m-3}$

i.e. $11m-33=2m+3$

$9m=36$

$m=4$, since external $m=-4$

$\therefore \quad AB$ is divided in the ratio $-4:3$ #

11. As P is outside the interval $\therefore$ the division is external:

i.e. $x=\frac{mx_2-nx_1}{m-n}$

$-10=\frac{m\times2-n\times-2}{m-n}=\frac{2m+2n}{m-n}$

i.e. $-10m+10n=2m+2n$

$8n=12m$

$\frac{m}{n}=\frac{8}{12}=\frac{2}{3}$

$\therefore \quad m=-2, n=3$ as division is external #

PARAMETRIC EQUATIONS

(A) The Parametric Equation

The parabola $x^2 = 4ay$ may be represented parametrically by:

$$x = 2at,\ y = at^2$$

Where the point $\left(2at,\ at^2\right)$ is a variable point which lies on the parabola.

Example 1: Find the cartesian equation of the curve whose parametric equations are:

(i) $x = 6t,\ y = 3t^2$ (ii) $x = t,\ y = \dfrac{t^2}{2}$

Solution 1:

(i) $x = 6t, y = 3t^2$

Squaring x, gives:

$$x^2 = 36t$$
$$= 12\left(3t^2\right) = 12y$$

$\therefore$ Cartesian equation is $x^2 = 12y$ #

(ii) $x = t,\ y = \dfrac{t^2}{2}$

squaring x, gives: $x^2 = t^2$

$$= 2\left(\frac{t^2}{2}\right) = 2y$$

$\therefore$ Cartesian equation is $x^2 = 2y$ #

Example 2: Find the parametric equations of the following curve:

(i) $x^2 = 5y$ (ii) $x^2 = -y$

Solution 2:

(i) $x^2 = 5y$

Since $a = \dfrac{5}{4}$, the parametric equations are:

$$x = 2at = 2\left(\frac{5}{4}\right)t = \frac{5}{2}t \text{ and } y = at^2 = \frac{5}{4}t^2 \text{ \#}$$

(i) $x^2 = -y$

Since $a = -\dfrac{1}{4}$, the parametric equations are:

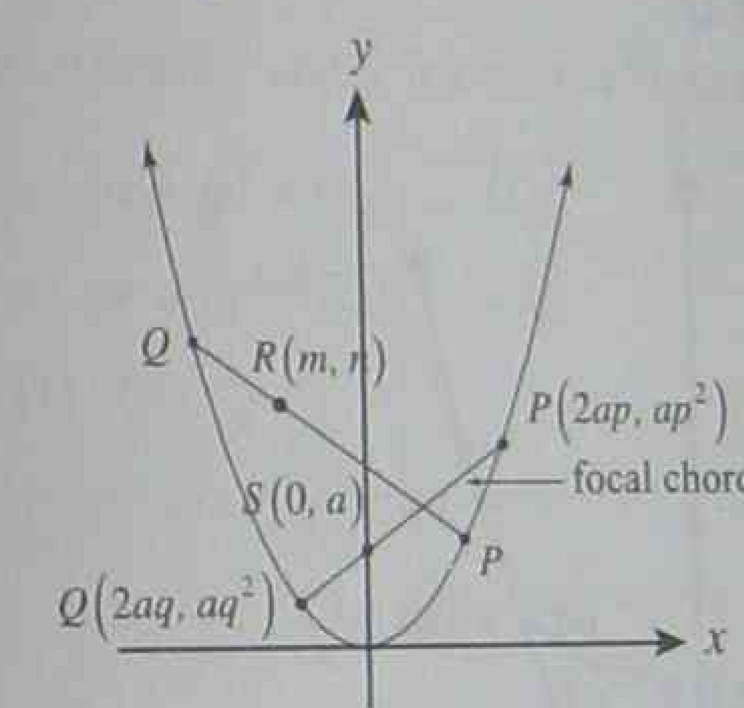
$$x = 2at = 2\left(-\frac{1}{4}\right)t = -\frac{1}{2}t \text{ and } y = at^2 = -\frac{1}{4}t^2 \text{ \#}$$

(B) Useful Results

It is not necessary to commit these results to memory, you should be able to deriv them when required.

The following results are based on the standard parabola $x^2 = 4ay$. Poin $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$ are two variable points which lie on the parabol

(i) Properties of the Chord PQ



1. The gradient of PQ is: $\dfrac{1}{2}(p+q)$

proof : $m = \dfrac{y_2 - y_1}{x_2 - x_1},\ P\left(2ap,\ ap^2\right),\ Q\left(2aq,\ aq^2\right)$

$$\therefore\ m = \frac{aq^2 - ap^2}{2aq - 2ap} = \frac{a(q-p)(q+p)}{2a(q-p)} = \frac{1}{2}(p+q) \text{ \#}$$

2. The equation of the chord PQ is: $y - \dfrac{1}{2}(p+q)x + apq = 0$

proof : $(y - y_1) = m(x - x_1),\ P\left(2ap,\ ap^2\right),\ m = \dfrac{1}{2}(p+q)$

$$\therefore\ \left(y - ap^2\right) = \frac{1}{2}(p+q)[x - 2ap]$$

$$y-ap^2=\frac{1}{2}(p+q)x-ap^2-apq$$

$$\therefore\ y-\frac{1}{2}(p+q)x+apq=0\ \#$$

3. The condition for a focal chord is found by substituting $S(0, a)$ into 2 above, to obtain $pq=-1$.

proof : i.e. If $y-\frac{1}{2}(p+q)x+apq=0$ passes through focus $S(0,a)$,

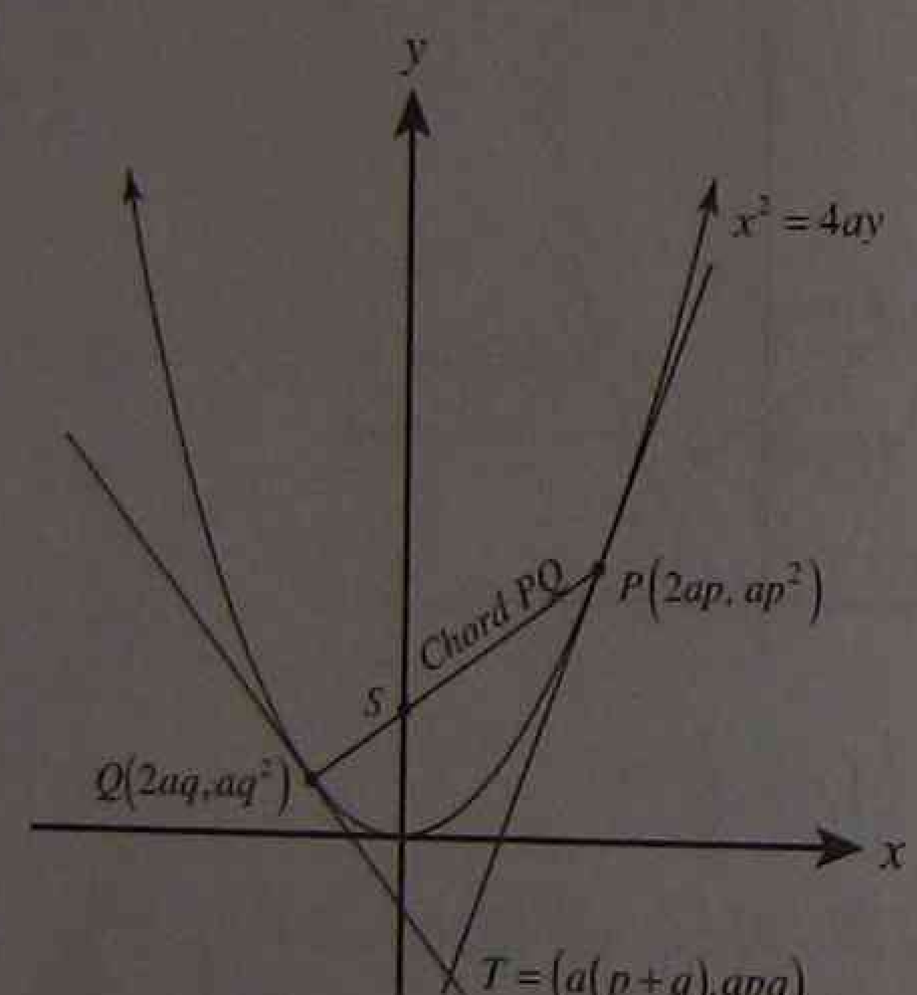
then $a-\frac{1}{2}(p+q)\cdot 0+apq=0$

$$apq=-a$$

$$\therefore\ pq=-1\ \#$$

4. If the chord passes through a specific point $R(m, n)$, then substitute the coordinates of R into 2. to obtain a relation involving p and q.

(ii) Properties of Tangents at P and Q



1. The gradient of the tangents at P and Q are p and q respectively.

proof : $x^2=4ay$, $P(2ap, ap^2)$, $Q(2aq, aq^2)$

$$y=\frac{x^2}{4a}$$

$$\frac{dy}{dx}=\frac{2x}{4a}=\frac{x}{2a}$$

at $P(2ap, ap^2)$, $\frac{dy}{dx}=\frac{2ap}{2a}=p$ (gradient of tangent at P)

2. The equations of the tangents at P and Q are respectively:
$y-px+ap^2=0$ and $y-qx+aq^2=0$

proof : $x^2=4ay$, $P(2ap, ap^2)$, gradient of tangent at $P=p$

$$(y-y_1)=m(x-x_1)$$

$$(y-ap^2)=p(x-2ap)$$

$$y-ap^2=px-2ap^2$$

$$\therefore\ y-px+ap^2=0\ \#$$

$x^2=4ay$, $Q(2aq, aq^2)$, gradient of tangent at $Q=q$

$$(y-y_1)=m(x-x_1)$$

$$(y-aq^2)=q(x-2aq)$$

$$y-aq^2=qx-2aq^2$$

$$\therefore\ y-qx+aq^2=0\ \#$$

3. The tangents at points P and Q intersect at $T(a(p+q), apq)$.

proof : $y-px+ap^2=0$............(1)

$y-qx+aq^2=0$............(2)

(1)−(2)

$$(-p+q)x+ap^2-aq^2=0$$

$$(q-p)x=aq^2-ap^2$$

$$x=\frac{a(q-p)(q+p)}{(q-p)}=a(p+q)$$

Substituting into (1) gives,

$$y-ap(p+q)+ap^2=0$$

$$y=apq$$

$\therefore$ T has coordinates $(a(p+q), apq)\#$

4. If PQ is a focal chord, then the tangents at P and Q intersect at right angles on the directrix.

proof: From before, if PQ is a focal chord then $pq=-1$ ((i)2 & 3)

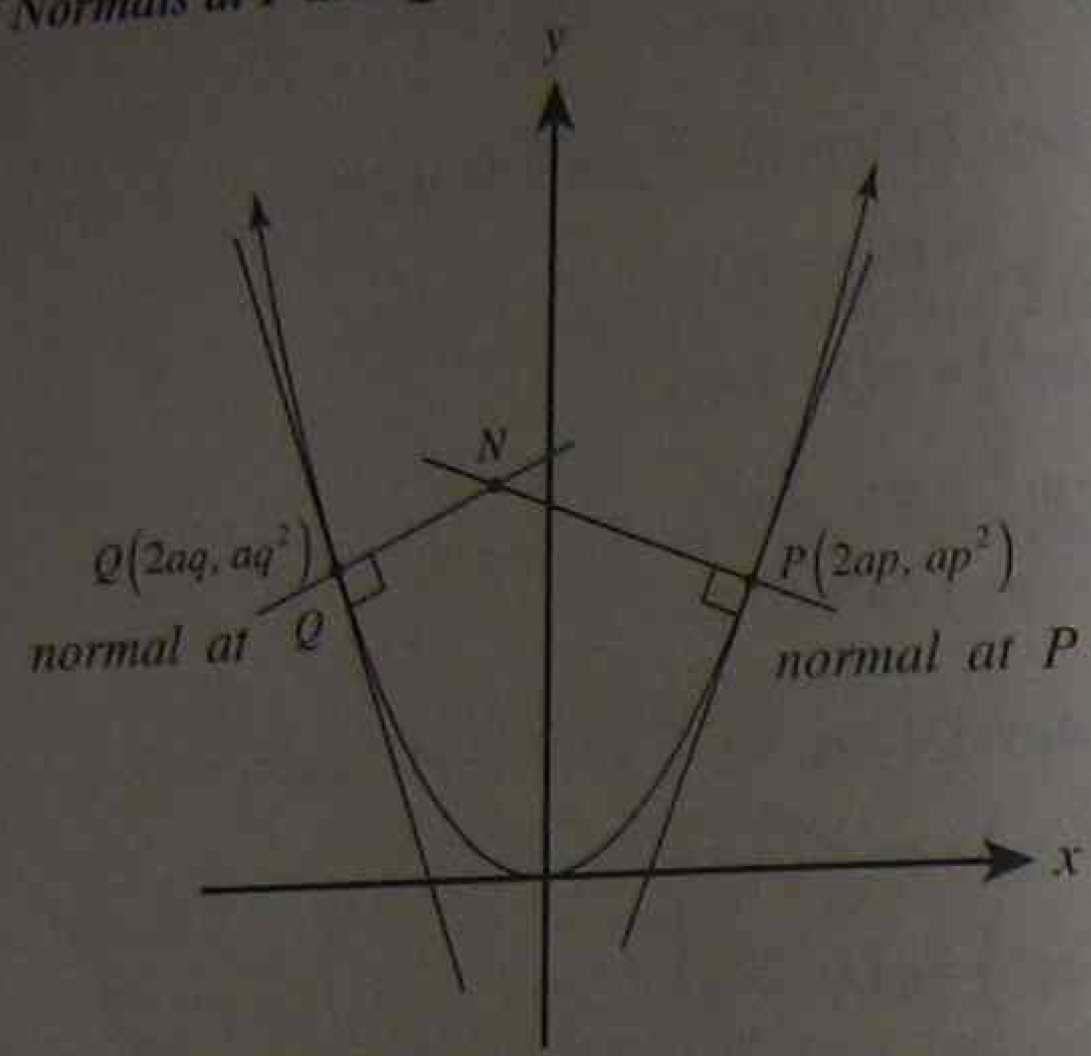
i.e. The gradient of tangent at $P\times$ gradient of tangent $Q=-1$

Thus, tangents at P and Q are perpendicular.

Also, coordinates of $T(a(p+q), apq)$ become $T(a(p+q), -a)$

Thus, T lies on the directrix $y=-a$.

(iii) Properties of Normals at P and Q



1. The gradients of the normals at P and Q are $-\dfrac{1}{p}$ and $-\dfrac{1}{q}$ respectively.

proof : $x^2 = 4ay,\ P(2ap, ap^2),\ Q(2aq, aq^2)$

$$y = \frac{x^2}{4a},\ \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

at P, $\dfrac{dy}{dx} = p =$ gradient of tangent $\therefore$ gradient of normal $= -\dfrac{1}{p}$

at Q, $\dfrac{dy}{dx} = q =$ gradient of tangent $\therefore$ gradient of normal $= -\dfrac{1}{q}$ #

2. The equations of the normals at P and Q are respectively:
$x + py = ap^3 + 2ap$ and $x + qy = aq^3 + 2aq$.

proof : $P(2ap, ap^2)$, gradient of normal at $P = -\dfrac{1}{p}$

$$(y - y_1) = m(x - x_1)$$

$$(y - ap^2) = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap \quad \therefore\ x + py = ap^3 + 2ap \ \#$$

$Q(2aq, aq^2)$, gradient of normal at $Q = -\dfrac{1}{q}$

$$(y - y_1) = m(x - x_1)$$

$$y - aq^2 = -\frac{1}{q}(x - 2aq)$$

$$qy - aq^3 = -x + 2aq$$

$$\therefore\ x + qy = aq^3 + 2aq \ \#$$

3. The normals at P and Q intersect at: $N\left(-apq(p+q),\ a\left(p^2 + pq + q^2 + 2\right)\right)$

proof : $x + py = ap^3 + 2ap$ ....(1)

$x + qy = aq^3 + 2aq$ ....(2)

$(1) - (2)$

$$(p - q)y = a(p^3 - q^3) + 2a(p - q)$$

$$(p - q)y = a(p - q)(p^2 + pq + q^2) + 2a(p - q)$$

$$\therefore\ y = a(p^2 + pq + q^2 + 2)$$

Substituting into (1) gives:

$$x + ap(p^2 + pq + q^2 + 2) = ap^3 + 2ap$$

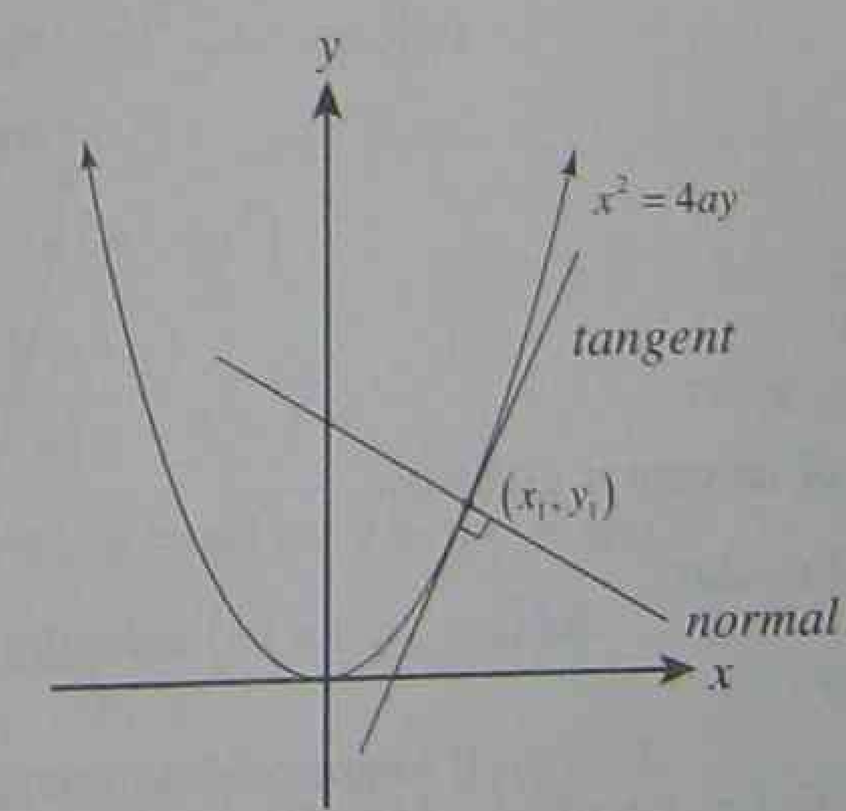
$$x + ap^3 + ap^2q + apq^2 + 2ap = ap^3 + 2ap$$

$$x = -ap^2q - apq^2 = -apq(p + q)$$

$\therefore$ N has coordinates $\left(-apq(p+q),\ a\left(p^2 + pq + q^2 + 2\right)\right)$ #

(iv) Equations of Tangent and Normal at the Point (x_1, y_1)

The Equation of the tangent and normal can also be derived using normal cartesian coordinates (x_1, y_1).



1. The equation of the tangent at (x_1, y_1) is: $2a(y + y_1) = x_1 x$.

proof : $x^2 = 4ay$

$$y = \frac{x^2}{4a},\ \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

At $x = x_1$, gradient of tangent is $\dfrac{x_1}{2a}$.

$\therefore$ The equation of the tangent is given by: $(y-y_1)=m(x-x_1)$.

$$y-y_1=\frac{x_1}{2a}(x-x_1)$$

$$2ay-2ay_1=x_1x-x_1^2 \quad \text{noting } x_1^2=4ay_1$$

$$2ay-2ay_1=x_1x-4ay_1$$

$$\therefore\ 2a(y+y_1)=x_1x\ \#$$

2. The equation of the normal at (x_1, y_1) is: $(y-y_1)=\dfrac{-2a}{x_1}(x-x_1)$.

proof : From (1) the gradient of tangent $=\dfrac{x_1}{2a}$, $\therefore$ gradient of normal $=-\dfrac{2a}{x_1}$

$\therefore$ the equation of the normal is given by:

$$(y-y_1)=m(x-x_1)$$

$$(y-y_1)=\frac{-2a}{x_1}(x-x_1)\ \#$$

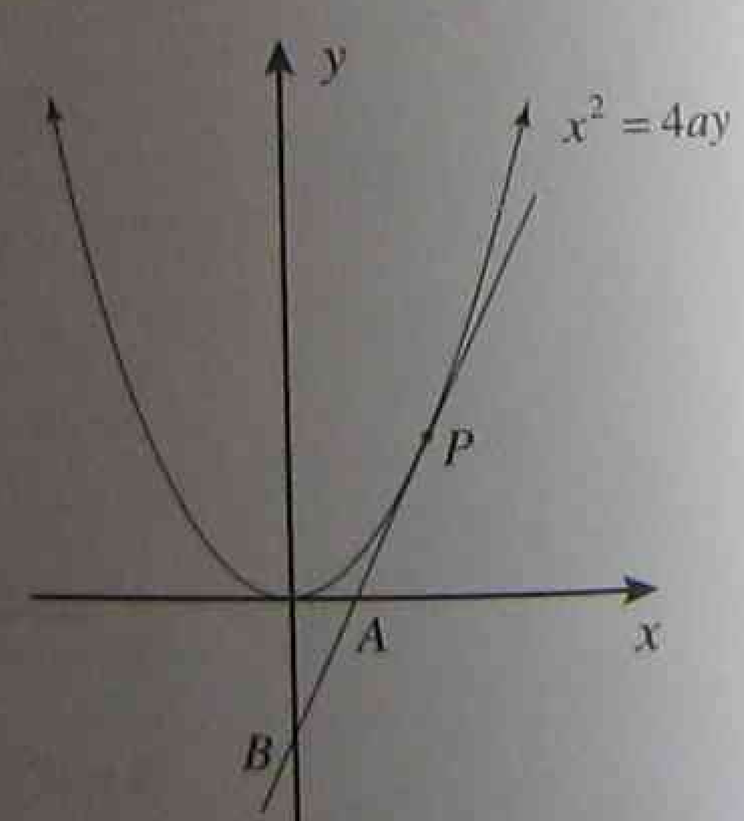
Example 1: $P(2at, at^2)$ is a point on the parabola $x^2=4ay$.

The line l is a tangent at P.

(i) Prove that the equation of l is $y=tx-at^2$.

(ii) If l cuts the x-axis at A and the y-axis at B, find the ratio in which P divides AB.

Solution 1:



(i) $y=\dfrac{x^2}{4a}$

$y'=\dfrac{2x}{4a}=\dfrac{x}{2a}$

when $x=2at$, $y'=t$

$\therefore$ equation of tangent is given by:

$(y-at^2)=t(x-2at)$

$y=tx-at^2\ \#$

(ii) For A, let $y=0$ i.e. $x=at$ $\therefore$ $A(at, 0)$

For B, let $x=0$ i.e. $y=-at^2$ $\therefore$ $B(0, -at^2)$

$PA=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}=\sqrt{(at)^2+(at^2)^2}=at\sqrt{1+t^2}$

$PB=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}=\sqrt{(2at)^2+(2at^2)^2}=2at\sqrt{1+t^2}$

$PA:PB=-1:2$ as P is external to AB #

Parametric Equations

Example 2: The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2=4ay$ with focus $S(0,a)$.

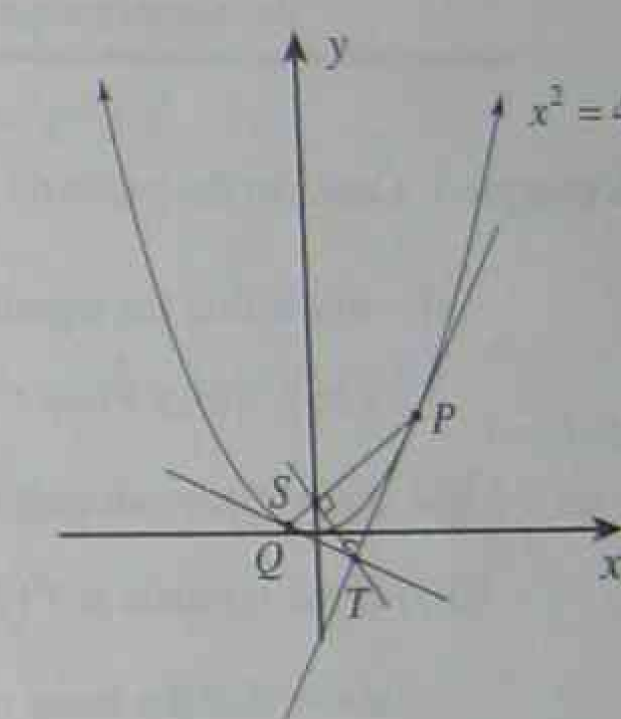
(i) Find the gradient of the chord PQ.

(ii) Find the co-ordintaes of T, the point where the tangents at P and Q intersect.

(iii) If the tangents intersect at right angles, prove that ST is perpendicular to PQ.

Solution 2:

(i) $m_{PQ}=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{aq^2-ap^2}{2aq-2ap}=\dfrac{a(q-p)(q+p)}{2a(q-p)}=\dfrac{p+q}{2}$ #



(ii) $y=\dfrac{x^2}{4a}$

$\dfrac{dy}{dx}=\dfrac{2x}{4a}=\dfrac{x}{2a}$

Thus, gradient of tangent at $P=p$.

Thus, gradient of tangent at $Q=q$.

$\therefore$ equation of the tangent at P is given by:

$(y-y_1)=m(x-x_1)$

$(y-ap^2)=p(x-2ap)$

$y=px-ap^2$ ....(1)

Similarly the equation of the tangent at Q is given by:

$y=qx-aq^2$ ....(2)

(1)−(2) gives:

$0=(p-q)x-a(p^2-q^2)$

$x=\dfrac{a(p-q)(p+q)}{(p-q)}$

$\therefore\ x=a(p+q),\ y=ap(p+q)-ap^2=apq$

$\therefore$ T has coordinates $(a(p+q), apq)$#

(iii) Tangents intersect at right angles if $pq=-1$.

$m_{ST}=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{apq-a}{a(p+q)}=\dfrac{pq-1}{(p+q)}$, $m_{PQ}=\dfrac{p+q}{2}$

$m_{ST}\times m_{PQ}=\dfrac{pq-1}{(p+q)}\times\dfrac{p+q}{2}=\dfrac{pq-1}{2}=\dfrac{-1-1}{2}=-1$

$\therefore$ $ST\perp PQ$, if tangents are perpendicular. #

(C) Finding the Locus of a Point

The locus of a point M may be found using the following method.

Step 1: Let $x = x-$coordinate of M ....(1)
$y = y-$coordinate of M ....(2)

Step 2: If either equation (1) or (2) above is independent of the parameters p and q, then this is the locus, if not go to step 3.

Step 3: To find the locus, eliminate the parameters p and q by algebra. This often involves squaring equation (1) or (2) (usually(1)) and then manipulating the algebra such that x is expressed in terms of y and a only.

Example 1: Consider the parabola $x^2 = 4y$.

(i) Show that the equation of the normal to the parabola $x^2 = 4y$ at the point $(2t, t^2)$ is $x + ty = t^3 + 2t$.

(ii) How many normals can be drawn from the focus $(0, 1)$ to the parabola?

(iii) If the normals at $P(2p, p^2)$ and $Q(2q, q^2)$ intersect at right angles at R, show that the locus of R is the parabola $y = x^2 + 3$.

Solution 1:

(i) $x^2 = 4y$

$y = \frac{x^2}{4}$, when $x = 2t$, $y' = t$ $\therefore$ grad. of normal $= \frac{-1}{t}$

$\therefore$ equation of normal is given by:

$$(y - t^2) = \frac{-1}{t}(x - 2t)$$

$$ty - t^3 = -x + 2t$$

$$x + ty = t^3 + 2t \ \#$$

(ii) Let $(0, 1)$ be a point on the normal,

i.e. $0 + t = t^3 + 2t$

$$t^3 + t = 0$$

$$t(t^2 + 1) = 0$$

i.e. $t = 0$, only 1 value of t satisfies the above equation

(iii) Grad. of normal at $P = \frac{-1}{p}$

Grad. of normal at $Q = \frac{-1}{q}$

$\therefore \frac{-1}{p} \times \frac{-1}{q} = -1$ (as normals at right $\angle$'s)

i.e. $pq = -1$

equation of normal at P: $x + py = p^3 + 2p$....(1)

equation of normal at Q: $x + qy = q^3 + 2q$....(2)

(1)−(2) : $y(p-q) = p^3 - q^3 + 2(p-q)$

$$y = p^2 + pq + q^2 + 2$$

Substitute into (1) : $x = p^3 + 2p - p^3 - p^2q - pq^2 - 2p$

$$= -p^2q - pq^2$$

$$= -pq(p+q)$$

$$= (p+q) \ \therefore R(p+q, p^2 + pq + q^2 + 2)$$

Squaring x, gives : $x^2 = (p+q)^2 = p^2 + 2pq + q^2 = (p^2 + pq + q^2) + pq$

$$= (y-2) - 1$$

$$= y - 3$$

$\therefore$ locus of R is given by $y = x^2 + 3$ #

Example 2: $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

(i) If PQ passes through the point $R(2a, 3a)$, show that $pq = p + q - 3$.

(ii) If M is the mid-point of PQ, show that the coordinates of M are $\left(a(pq+3), \frac{a}{2}(pq+3)^2 - apq\right)$.

(iii) Hence, find the locus of M.

Solution 2:

(i) $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

Equation of PQ is given by:

$$(y - ap^2) = \frac{p+q}{2}(x - 2ap)$$

$$y-ap^2=\left(\frac{p+q}{2}\right)x-\frac{p+q}{2}(2ap)$$

$$y-ap^2=\left(\frac{p+q}{2}\right)x-ap^2-apq$$

$$y=\left(\frac{p+q}{2}\right)x-apq$$

Substituting $R(2a,3a)$ gives:

$$3a=\left(\frac{p+q}{2}\right)2a-apq$$

$$3a=a(p+q)-apq$$

$$3=p+q-pq$$

i.e. $pq=p+q-3$ #

(ii) $M\left(\frac{2ap+2aq}{2},\frac{ap^2+aq^2}{2}\right)=M\left(a(p+q),\frac{a}{2}(p^2+q^2)\right)$

$$=M\left(a(pq+3),\frac{a}{2}\left((p+q)^2-2pq\right)\right)$$

$$=M\left(a(pq+3),\frac{a}{2}\left((pq+3)^2-2pq\right)\right) \#$$

(iii) Let $x=a(pq+3)$ and $y=\frac{a}{2}\left((pq+3)^2-2pq\right)$

$$x^2=a^2(pq+3)^2$$

$$=a^2\left(\frac{2y}{a}+2pq\right) \qquad \left(\text{note: } (pq+3)^2=\frac{2y}{a}+2pq\right)$$

$$=a^2\left(\frac{2y}{a}+2\left(\frac{x}{a}-3\right)\right) \qquad \left(\text{note: } pq=\frac{x}{a}-3\right)$$

$$=2ay+2a^2\left(\frac{x}{a}-3\right)$$

$$=2ay+2ax-6a^2$$

i.e. $x^2-2ax=2a(y-3a)$ #

REVIEW EXERCISES

(A) The Parametric Equation

1. Find the cartesian equation of the curve whose parametric equations are:

(i) $x=\frac{t}{3},\ y=\frac{t^2}{12}$ (ii) $x=\sin 2t,\ y=\cos^2 t$

2. Find the parametric equations of the following:

(i) $x^2=-16y$ (ii) $2x^2=3y$

(B) Useful Results

3. Two points $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$ lie on the parabola $x^2=4ay$. PQ passes through the focus S.
   (i) Show that $pq=-1$
   (ii) Show that the tangents at P and Q intersect on the line $y=-a$.
   (iii) Show that $PQ=a\left(p+\frac{1}{p}\right)^2$.

4. The normal at $P(2ap,\ ap^2)$ to the parabola $x^2=4ay$ meets the curve again at $Q(2aq,\ aq^2)$.
   (i) Show that $q=\frac{-(2+p^2)}{p}$.
   (ii) Find the coordinates of P so that the lines OP and OQ are at right angles, where O is the origin.

5. (i) Show that the equation of the normal to the parabola $x^2=4ay$, at the point $P(2ap,\ ap^2)$ on the parabola, is $x+py=2ap+ap^3$.
   (ii) Show that the condition this normal passes through the point $L(x_1,\ y_1)$ is $f(p)=0$, where $f(p)=ap^3+(2a-y)p-x_1$.
   (iii) Find the three normals passing through $(0,11a)$.
   (iv) Find $f'(p)$, and show that if $a>0$ and $y\le 2a$, then $f'(p)\ge 0$ for all p. Hence, deduce the number of normals to the parabola passing through

6. The tangent at $P\left(2ap,\ ap^2\right)$ to the parabola $x^2=4ay$ intersects the y-axis at T.
   (i) Find the coordinates of point T.
   (ii) Hence, prove that $SP=ST$, where S is the focus of the parabola.
   (iii) Hence, prove that $\angle SPT$ is equal to the angle made between the tangent and the line $x=2ap$.

(C) Finding the Locus of a Point

7. The normals to the parabola $x^2=4ay$ at the points $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$ intersect at R. If the chord PQ varies in such a way that it always passes through the point $(0,2a)$, show that R lies on the parabola.

8. The normal to the parabola $x^2=4ay$ at the point $P\left(2ap,\ ap^2\right)$ intersects the y-axis at T. Find the locus of the midpoint M of the interval PT.

9. $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$ lie on the parabola $x^2=4ay$. If lines passing through P and Q meet at right angles at the vertex (V):
   (i) show that $pq=-4$.
   (ii) hence, find the equation of the locus of the midpoint M of the chord PQ.

10. A variable line through the point $(0,\ a)$ cuts the parabola $x^2=4ay$ at $A\left(2ap,\ ap^2\right)$ and $B\left(-\frac{2a}{p},\ \frac{a}{p^2}\right)$. The line OA, where O is the origin, cuts the line through B and another point $C(0,-a)$ at R. Show that the locus of R has equation $x^2+8y^2+4ay=0$.

11. $P\left(2ap,\ ap^2\right)$ lies on the parabola $x^2=4ay$. From P, perpendiculars are drawn to the x and y axes meeting them at A and B respectively. T is the midpoint of PB and M is the midpoint of TA.
   (i) Find the coordinates of point M.
   (ii) Hence, find the locus of M.

12. The points $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$ lie on the parabola $x^2=4ay$.

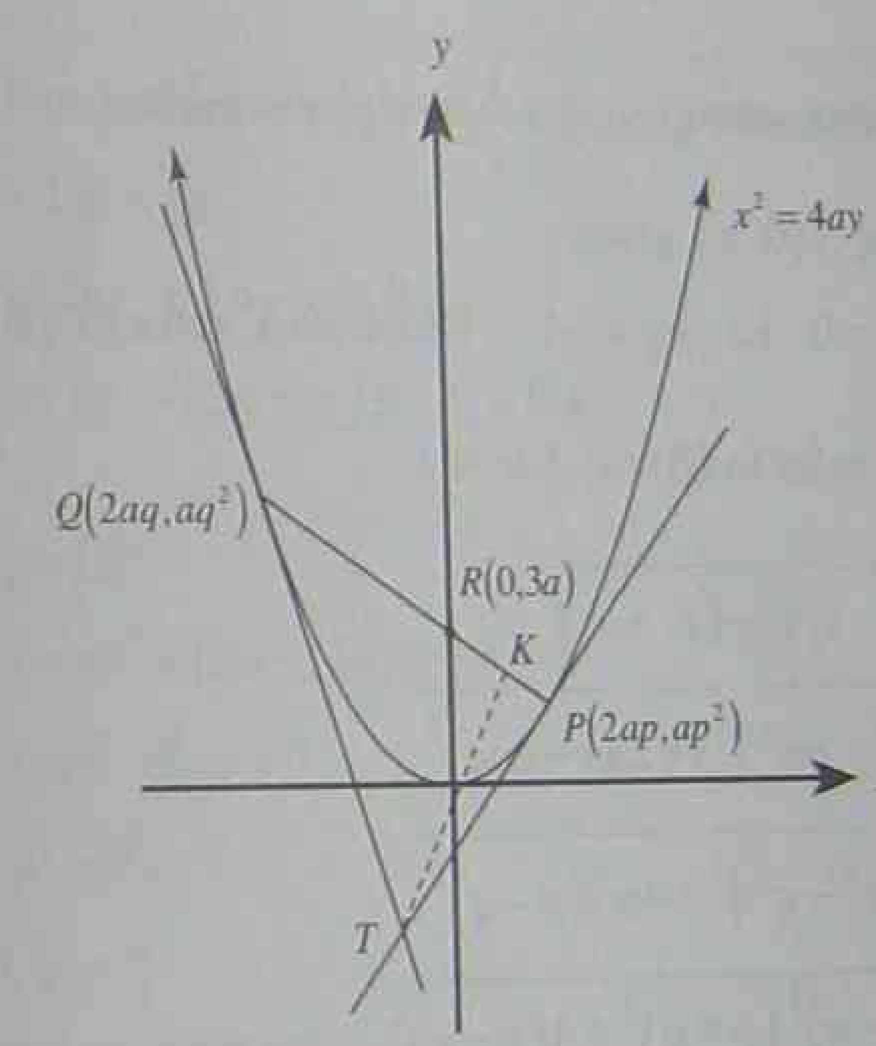
(i) Write down the equations of the tangents to the parabola at the points $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$.

(ii) Show that T has coordinates $\left(a(p+q),\ aqp\right)$.

(iii) Show that the equation of the chord PQ is given by $y=\frac{1}{2}(p+q)x-apq$. Hence, show that $pq=-3$.

(iv) Find the equation of the line TK.

(v) Hence, find the locus of K.



WORKED SOLUTION TO REVIEW EXERCISES

1. (i) $x=\frac{t}{3}$ $\quad\therefore\ x^2=\frac{t^2}{9}$ but $12y=t^2$ $\therefore\ x^2=\frac{12y}{9}=\frac{4}{3}y$ #

(ii) $x=\sin 2t=2\sin t\cos t$

$x^2=4\sin^2 t\cos^2 t=4\left(1-\cos^2 t\right)\cos^2 t=4(1-y)y=4y-4y^2$ $\therefore\ x^2=4y-4y^2$ #

2. (i) $x^2=-16y$ i.e. $a=-4$ $\therefore\ x=-8t,\ y=-4t^2$ #

(ii) $2x^2=3y$ i.e. $x^2=\frac{3}{2}y$ i.e. $a=\frac{3}{8}$ $\therefore\ x=\frac{3}{4}t,\ y=\frac{3}{8}t^2$ #

3. (i) Equation of chord PQ is: $y-\frac{1}{2}(p+q)x+apq=0$.

Substituting $S(0,a)$ gives:

$a-0+apq=0$ i.e. $pq=-1$ (Bookwork, refer to (B)(i) 2 & 3) #

(ii) Bookwork, refer to (B) (ii) 3 & 4 #

(iii) $PQ=\sqrt{(y_2-y_1)^2+(x_2+x_1)^2}$

$=\sqrt{\left(ap^2-aq^2\right)^2+(2ap-2aq)^2}$

$=\sqrt{a^2\left(p^2-q^2\right)^2+4a^2(p-q)^2}$

$=a\sqrt{(p-q)^2(p+q)^2+4(p-q)^2}$

$=a(p-q)\sqrt{p^2+2pq+q^2+4}$ (note: $4=-4pq$)

$=a(p-q)\sqrt{p^2-2pq+q^2}$

$=a(p-q)\sqrt{(p-q)^2}$

$=a(p-q)^2$ $\quad\left(\text{noting: } q=-\frac{1}{p}\right)$

$\therefore\ PQ=a\left(p+\frac{1}{p}\right)^2$ as required. #

4. (i) Gradient of $PQ=\frac{(p+q)}{2}$, gradient of normal at $P=-\frac{1}{p}$

$\therefore\ \frac{p+q}{2}=-\frac{1}{p}$, i.e. $q=\frac{-\left(p^2+2\right)}{p}$ #

(ii) Gradient of $OP=\frac{ap^2}{2ap}=\frac{p}{2}$, gradient of $OQ=\frac{aq^2}{2aq}=\frac{q}{2}$

$\therefore\ \frac{p}{2}\times\frac{q}{2}=-1$ $\quad(OP\perp OQ)$

$p\times\frac{-\left(p^2+2\right)}{p}=-4$ $\quad\left(\text{from (i) } q=-\frac{\left(p^2+2\right)}{p}\right)$

$p^2+2=4$

$p^2=2$ i.e. $p=\pm\sqrt{2}$

$\therefore$ coordinates of P are $\left(2\sqrt{2}a,\ 2a\right)$ or $\left(-2\sqrt{2}a,\ 2a\right)$ #

5. (i) Bookwork, refer to (B) (iii) 1 & 2 #

(ii) $x+py=2ap+ap^3$ passes through $L(x_1,y_1)$ substituting $L(x_1,y_1)$, gives:

$x_1+py_1=2ap+ap^3$

i.e. $ap^3+(2a-y_1)p-x_1=0$

Let $f(p)=ap^3+(2a-y_1)p-x_1=0$ #

(iii) $x_1=0,\ y_1=11a$

$\therefore\ f(p)=ap^3+(2a-11a)p-0=0$

$=ap^3-9ap=0$

i.e. $ap\left(p^2-9\right)=0$

$\therefore\ p=0,3,-3$

$\therefore$ the three normals are:

1. $x=0$
2. $x+3y=2a\times3+a(3)^3$ i.e. $x+3y=33a$
3. $x-3y=2a\times-3+a(-3)^3$ i.e. $x-3y=-33a$ #

(iv) $f(p)=ap^3+(2a-y_1)p-x_1$, $f'(p)=3ap^2+(2a-y_1)$

If $a>0$ and $p^2>0$, then $3ap^2>0$ and since $y_1\le 2a$, then $2a-y_1\ge 0$.

$\therefore\ f'(p)\ge 0$ for all p, provided $a>0$ and $y_1\le 2a$.

Since $f'(p)\ge 0$ $\therefore$ the curve $f(p)$ is increasing for all values of p. Also, $f(0)=-x_1$, and the function $f(p)$ has no asymptotes.

$\therefore\ f(p)=0$ has only one solution. Hence, the number of normals to the parabola passing through $L(x_1,y_1)$, if $y_1\le 2a$, is only one. #

6. (i) Equation of tangent is: $y-px+ap^2=0$ at $x=0,\ y=-ap^2$

$\therefore\ T\left(0,-ap^2\right)$ # (Bookwork, refer to (B) (ii) 1&2)

(ii) $S(0,a),\ P\left(2ap,ap^2\right),\ T\left(0,-ap^2\right)$

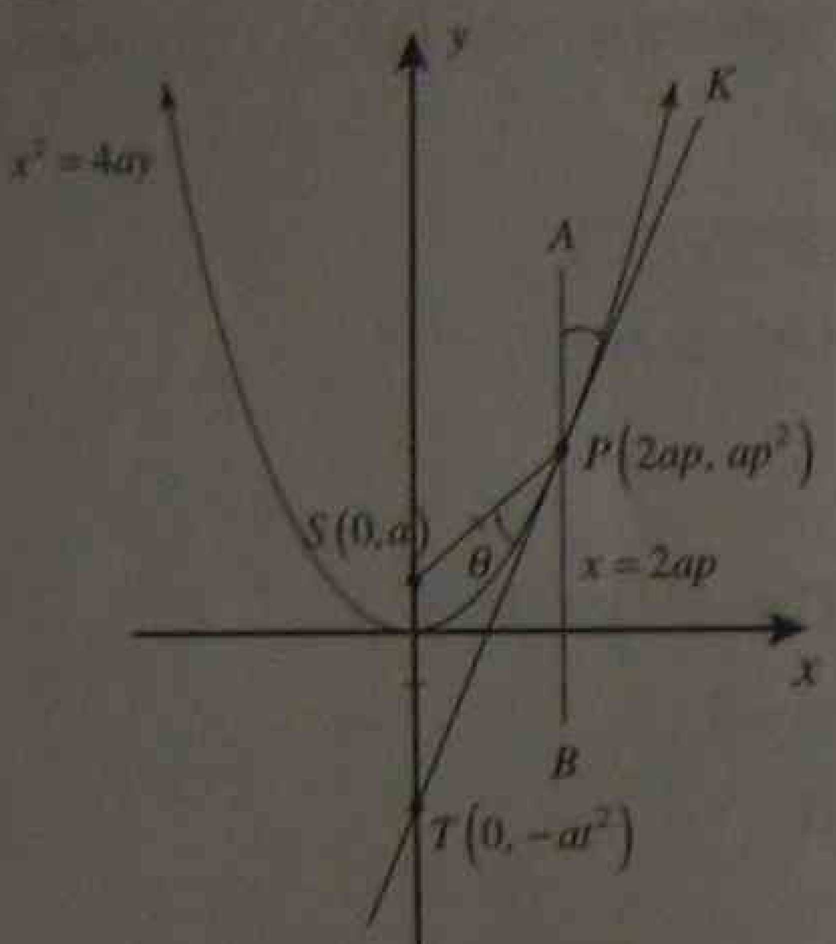
$$ST=a+\left|-ap^2\right|=a\left(p^2+1\right)$$

$$SP=\sqrt{\left(ap^2-a\right)^2+(2ap)^2}$$

$$=a\sqrt{p^4-2p^2+1+4p^2}$$

$$=a\sqrt{p^4+2p^2+1}=a\sqrt{\left(p^2+1\right)^2}=a\left(p^2+1\right)=ST\ \#$$

(iii) This is best answered using geometry:



Let $\angle SPT=\theta$

$SP=ST$	(proven in ii)
$\angle STP=\theta$	(base $\angle$'s of isosceles $\Delta=$)
$\angle TPB=\theta$	(alt. $\angle$'s $=$, $ST \parallel AB$)
$\angle APK=\theta$	(vertically opp. $\angle$'s equal)

$\therefore\ \angle SPT=\angle APK$ #

7. Normals intersect at $R\left(-apq(p+q),\ a\left(p^2+pq+q^2+2\right)\right)$ (refer to (B) (iii) 1-3)

The equation of PQ is: $y-\frac{1}{2}(p+q)x+apq=0$ (refer to (B) (i) 1-2)

and substituting $(0,-2a)$ into equation of PQ gives: $-2a-0+apq=0$, i.e. $pq=2$.

Now, locus of $R\left(-apq(p+q),a\left(p^2+pq+q^2+2\right)\right)$ is given by:

$y=a\left(p^2+pq+q^2+2\right)=a\left(p^2+q^2+4\right)$ (note $pq=2$)

$x=-apq(p+q)=-2a(p+q)$

$$\therefore\ x^2=4a^2\left(p^2+2pq+q^2\right)$$

$$=4a^2\left(\frac{y}{a}-4+4\right) \qquad \left(\text{note } \frac{y}{a}-4=p^2+q^2\right)$$

$$=4ay$$

$\therefore$ R lies on the parabola. #

8. Equation of normal at P is: $x+py=ap^3+2ap$ (refer to (B) (iii) 1-2)

at $x=0,\ y=ap^2+2a$ i.e. $T\left(0,ap^2+2a\right)$

$$M\left(\frac{0+2ap}{2},\frac{ap^2+2a+ap^2}{2}\right)=M\left(ap,\ a+ap^2\right)$$

$x=ap,\ y=a+ap^2$

$$\therefore\ x^2=a^2p^2$$

$$=a\left(ap^2\right)$$

$$\therefore\ x^2=a(y-a)\ \#$$

9. (i) Gradient of $VP=\dfrac{ap^2-0}{2ap-0}=\dfrac{p}{2}$, gradient of $VQ=\dfrac{q}{2}$

for right angle $\dfrac{p}{2}\times\dfrac{q}{2}=-1$ i.e. $pq=-4$ #

(ii) $M\left(\dfrac{2ap+2aq}{2},\dfrac{ap^2+aq^2}{2}\right)=M\left(a(p+q),\dfrac{a\left(p^2+q^2\right)}{2}\right)$

$$x=a(p+q),\ y=\frac{a\left(p^2+q^2\right)}{2}$$

$$x^2=a^2\left(p^2+q^2+2pq\right)$$

$$=a^2\left(\frac{2y}{a}-8\right) \qquad \left(\text{note: } pq=-4,\ p^2+q^2=\frac{2y}{a}\right)$$

$$=2a(y-4a)\ \#$$

10. Gradient of BC: $\dfrac{-a-\dfrac{a}{p^2}}{0+\dfrac{2a}{p}}=\dfrac{-ap^2-a}{p^2}\times\dfrac{p}{2a}=-\dfrac{\left(p^2+1\right)}{2p}$

Equation of BC: $(y+a)=-\left(\dfrac{p^2+1}{2p}\right)x$

Gradient of OA: $\dfrac{ap^2-0}{2ap-0}=\dfrac{p}{2}$

Equation of OA: $(y-0)=\dfrac{p}{2}(x-0)$ i.e. $y=\dfrac{p}{2}x$

Subtituting $p=\dfrac{2y}{x}$ into BC gives:

$$y+a=-\left(\frac{\left(\frac{2y}{x}\right)^2+1}{2\left(\frac{2y}{x}\right)}\right)x$$

$$\frac{4y^2}{x}+\frac{4ay}{x}=-\frac{4y^2}{x}-x$$

$$4y^2+4ay=-4y^2-x^2$$

i.e. $x^2+8y^2+4ay=0$ #

11. (i) $A(2ap,0)$, $B\left(0,ap^2\right)$, $P\left(2ap,ap^2\right)$

$$T=\left(\frac{2ap+0}{2},\frac{ap^2+ap^2}{2}\right)=\left(ap,ap^2\right)$$

$$M=\left(\frac{ap+2ap}{2},\frac{ap^2+0}{2}\right)=\left(\frac{3ap}{2},\frac{ap^2}{2}\right) \#$$

(ii) $x=\dfrac{3ap}{2}$, $y=\dfrac{ap^2}{2}$

$$x^2=\frac{9a^2p^2}{4}=\frac{9}{4}a\left(ap^2\right)$$

$$=\frac{9}{4}a(2y)$$

$$=\frac{9}{2}ay \#$$

12. (i) Tangent at P is: $y-px+ap^2=0$, tangent at Q is: $y-qx+aq^2=0$ #

(ii) Bookwork, refer to (B)(iii)3 #

(iii) Bookwork, refer to (B)(i)1 & 2

Now, the chord PQ passes through $R(0,3a)$.

$\therefore$ $3a=0-apq$

i.e. $pq=-3$ #

(iv) Gradient of $PQ=\dfrac{p+q}{2}$ (refer to (B)(i)1)

$\therefore$ gradient of $TQ=\dfrac{-2}{p+q}$

Thus, equation of TK is given by:

$$(y-apq)=\frac{-2}{p+q}\left(x-a(p+q)\right)$$

$(p+q)y+2x=apq(p+q)+2a(p+q)$ #

(v) From (iii) $pq=-3$, thus equation of line TK and chord PQ become:

$(p+q)y+2x=-a(p+q)$ ....(1)

$2y=(p+q)x+6a$ ....(2)

To find the locus of K, we need to eliminate the parameters p and q in the above equation,

$(p+q)(y+a)=-2x$ ....rearranging (1)

$(p+q)x=2y-6a$ ....rearranging (2)

(1)÷(2) gives:

$$\frac{y+a}{x}=\frac{-2x}{2y-6a}$$

i.e. $2y^2+2ay-6ay-6a^2=-2x^2$

$$2y^2+2x^2-4ay=6a^2$$

$$y^2+x^2-2ay=3a^2$$

$$x^2+(y-a)^2=4a^2$$

which is a circle centre $(0,a)$ and radius $2a$. #

TOPIC 7

PERMUTATIONS AND COMBINATIONS

(A) The Basic Counting Principle

The basic counting principle states that: If there are k different ways of doing one operation, and l different ways of doing another operation, then there are $k \times l$ different ways of performing the two operations. This multiplication effect can be extended to any number of operations.

Example 1: Mary has 6 blouses and 4 skirts in her wardrobe. In how many ways can she select an outfit?

Solution 1:

Total number of different outfits

$= (\text{No. of ways of selecting a blouse}) \times (\text{No. of ways of selecting a skirt})$

$= 6 \times 4 = 24$ ways #

Example 2: Using the digits 1, 3, 5, 7, 9 with no repetitions, how many:

(i) 2-digit numbers can be formed?

(ii) 4-digit numbers can be formed?

Solution 2:

(i) Total number of different 2 digit numbers

$= (\text{No. of ways of selecting } 1^{st} \text{ no.}) \times (\text{No. of ways of selecting } 2^{nd} \text{ no.})$

$= 5 \times 4 = 20$ ways #

(ii) Extending the principle in (i):

Total number of 4-digit numbers $= 5 \times 4 \times 3 \times 2 = 120$ ways #

(B) Ordered Arrangements

The number of different arrangements which can be formed using n different object is $n!$.

Where $n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$

Example 1: How many different arrangements can be formed using all the letters of the word HARMONY?

Example 2: In how many ways can 3 mathematics books, 2 English books and 1 science book be arranged on a bookshelf?

Solution 2:

There are 6 different books in total.

$\therefore$ Total number of arrangements $= 6! = 720$ #

(C) Permutations - 'Ordered Selections'

A permutation of a group of objects is any arrangement of any number of them in a definite order.

If n is the number of objects in the group and r is the number of them arranged in a definite order at one time, the total number of permutations P of n objects, taken r at a time, is given by:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example 1: Using the letters of the word CABINET, how many:

(i) 3-letter arrangements are possible?

(ii) 4-letter arrangements are possible?

(iii) 5-letter arrangements are possible?

Solution 1:

(i) There are 7 letters of which 3 are to be arranged in a definite order:

$\therefore$ total no. of arrangements $= {}^{7}P_{3} = 210$ ways #

(ii) Extending the same principle for 4 letters:

$\therefore$ total no. of arrangements $= {}^{7}P_{4} = 840$ ways #

(iii) Extending the same principle for 5 letters:

$\therefore$ total no. of arrangements $= {}^{7}P_{5} = 2{,}520$ ways #

Example 2: There are eight empty seats on a bus and five people enter. In how many ways can they be seated?

Solution 2:

There are 8 seats and 5 people to be seated in them in a definite order:

(D) Special Cases of Arrangements and Permutations

There are a number of special cases involving arrangements and permutations, in which the standard formulae covered earlier need to be modified.

(i) Permutations with Replacement

The number of permutations of n different objects taken r at a time, with replacement, is given by: n^r.

Example 1: In NSW, number plates, for cars consist of 3 letters followed by 3 numbers. How many such numbers are possible, if repetitions are allowed?

Solution 1:

There are 26 letters and 10 digits (0-9 inclusive), thus the 3 letters can be formed in: $26\times26\times26=26^3$ ways and the 3 digits can be formed in: $10\times10\times10=10^3$ ways. Thus, total number of ways $=26^3\times10^3=17,576,000$ #

Example 2: In NSW, a postal code consists of four digits where the first digit is 2. How many different postal codes are possible?

Solution 2:

The first digit must be 2, thus there are 3 remaining places to fill:

Thus, total number of different codes $=1\times10^3=1000$ #

(ii) Arrangements of n Objects in a Line Where Some Objects are Alike

The number of different arrangements obtained by arranging n objects of which s are the same and another t are the same etc. is given by:

$$\frac{n!}{s!t!\ldots}$$

Example 1: Find the number of different arrangements possible using all the letters of WOOLLOOMOOLOO?

Solution 1:

There are 13 letters of which 8 letters are O, 3 letters are L.

Example 2: How many permutations can be formed using all the letters of the word CALCULATION if:

(i) there are no restrictions?

(ii) the word must begin and end with the same letter?

(iii) the word must end with ION in that order?

Solution 2:

(i) There are 11 letters, of which there are: 2 C's, 2 L's, 2 A's.

Thus, total number of arrangements $=\frac{11!}{2!2!2!}=4,989,600$ #

(ii) If the word begins and ends with C, then that leaves 9 letters of which there are 2 A's and 2 L's.

Thus, total number of words beginning and ending with C is $=\frac{9!}{2!2!}=90{,}720$

Similarly, for A and L.

Thus, total number of words beginning and ending with the same letter $=90,720\times3=272,160$ #

(iii) If the word ends with ION, then that leaves 8 letters of which there are 2 C's, 2 A's and 2 L's.

Thus, the total number of words ending with ION is $=\frac{8!}{2!2!2!}=5,040$ #

(iii) Arrangements in a Circle

The number of ways of arranging different objects around a circle is given by: $(n-1)!$

Example 1: Four married couples sit at a round table. In how many different ways can they be seated if:

(i) there are no restrictions?

(ii) men and women alternate?

(iii) each husband and wife must sit next to each other?

(iv) each husband must sit opposite his wife?

Solution 1:

(i) There are 8 people, with no restrictions, the number of arrangements is $7!=5,040$ #

...remaining men can be seated in 3! ways. The 4 ... the men in 4! ways.

(iii) Treat each couple as one distinct entity hence, the 4 couples can be seated in 3! ways. However, each married couple may be alternated in $2!\times 2!\times 2!\times 2!$ ways.

Thus, total number of ways $= 3!\times(2!)^4 = 96$ #

(iv) Fix one man and one woman. The remaining three couples may then be seated in 3! ways. However each of the remaining 3 couples may be alternated in $2!\times 2!\times 2!$ ways.

Thus, total number of ways $= 3!\times(2!)^3 = 48$ #

(E) Combinations - 'Unordered Selections'

A combination of a group of objects is any number of them selected without regard to their order.

If n is the number of objects in the group and r is the number selected at one time, then the total number of combinations C of n objects taken r at a time is given by:

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Example 1: A committee of 7 people is to be selected from 8 women and 10 men. Find the probability that in a particular committee women are to have majority?

Solution 1:

Women will have majority in the following way:

Committe	Number of Combinations
$4W3M$	$^8C_4\times{}^{10}C_3 = 8,400$
$5W2M$	$^8C_5\times{}^{10}C_2 = 2,520$
$6W1M$	$^8C_6\times{}^{10}C_1 = 280$
$7W0M$	$^8C_7\times{}^{10}C_0 = 8$
	Total = 11,208

Total number of ways 7 members can be selected with no restrictions is $^{18}C_7$.

Thus, required probability $= \frac{11,208}{^{18}C_7} = 0.352$ to 3 d.p. #

Permutations & Combinations

Review Exercises

(A) The Basic Counting Principle

1. If seven dice of six faces each are tossed, how many different outcomes are possible?
2. How many 2-digit numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, 8 if:
   (i) repetitions are not allowed?
   (ii) repetitions are allowed?
3. A sports club of 50 members wished to select a president, vice-president, secretary and treasurer. Assuming that no person can hold two offices, in how many ways can the selections be made?
4. How many different license plates can be made if a plate consists of:
   (i) three different letters followed by two different digits.
   (ii) three letters followed by two numbers, not necessarily different.
   (iii) three consecutive letters and three consecutive numbers. Assuming A follows Z and 0 follows 9.
5. How many 4-digit numbers have digits that are all different?
6. Three married couples go to the movies to watch a film. If four different films are available on this particular night, find how many different arrangements are there if:
   (i) there are no restrictions on which movie any one person can watch.
   (ii) the women decide to watch one movie and the men decide to watch another.
   (iii) one married couple must watch the same movie together.

(B) Ordered Arrangements

7. In how many ways can the letters of the word INSURABLE be arranged?
8. A five-digit number is formed from the digits 1, 2, 3, 4, 5 with no digit being repeated. Find the probability that the number:
   (i) is odd.
   (ii) is divisible by 5.
   (iii) is greater than 40,000.

(C) Permutations - 'Ordered Selections'

9. A bowl contains an apple, a peach, a pear, an apricot, a banana, a plum, and an orange. In how many ways can the fruit be distributed among five children, if each child is to rece one piece of fruit.

10. In the United States, a postal code consists of five digits. In Canada, a postal code consists of a letter, a digit, a letter, a digit, a letter, and a digit. How many different postal codes are possible in each country?

11. Using the letters of the word PRODUCT, how many:

    (i) 3-letter arrangements can be made?

    (ii) 4-letter arrangements can be made?

    (iii) 5-letter arrangements can be made?

12. A flag is to be formed using three different colours in a specific order. How many different ways are possible if there are nine colours available?

(D) Special Cases of Arrangements and Permutations

13. (i) How many different numbers greater than 200,000 can be made from all the digits 1, 3, 3, 3, 4, and 5?

    (ii) What is the probability that if a number is selected from (i), it will be odd?

14. In how many ways can the letters of the word CARRIER be arranged if:

    (i) The first letter is R?

    (ii) The first two letters are RR?

    (iii) The first letter is R, and the next letter is not R?

15. Find the probability that if the letters of the word PARALLEL are randomly arranged that the L's will **not** be together.

16. On six identical cardboards each letter of the word SYDNEY is written. These cardboards are then placed in a hat. The cardboards are chosen at random, one at a time without replacement from the hat and the letter chosen is noted. Find the probability that:

    (i) The chosen letters will be in the order SYDNEY.

    (ii) The first 3 letters are SYD in that order.

    (iii) Repeat the above question if the cardboards are replaced after each selection.

17. In how many ways can 4 keys be arranged on a key ring?

18. Kent is to celebrate his 18th birthday by having a dinner party for himself and nine of his friends (five girls and four boys). In how many ways can the people be seated at a round table if:

    (i) There are no restrictions?

    (ii) The boys and girls are to be seated alternately?

    (iii) Kent is to be seated between two particular girls.?

19. Mum, Dad and their six children (3 boys and 3 girls) are to be seated at a circular table at random. What is the probability that:

(E) Combinations- 'Unordered Selections'

20. In how many ways can a committee of 5 people be selected from 12 people if:

    (i) 2 particular persons **must** be included.?

    (ii) 2 particular persons **must not** be included?

21. An organisation has 25 members, 4 of whom are students. In how many ways can a committee of 3 members be selected so as to include at least one student.

22. In how many ways can 8 different gifts be distributed to 2 children such that each chi receives an odd number of gifts.

23. A box contains 8 different batteries of which 5 are good and 3 are defective. Cindy two batteries for the remote control of the TV and then selects another two for the re control for the video. If both batteries must be good for each remote control to wo the probability that :

    (i) both remote controls work.

    (ii) only the remote control for the TV works.

24. A box contains 20 balls, of which 5 are red, 5 are white, 5 are blue, and 5 are gree Suppose a sample of five balls are chosen without replacement.

    (i) Find the probability that the sample contains balls of the same colour.

    (ii) Find the probability that the sample contains three balls of one colour and two balls of another colour.

Worked Solutions to Review Exercises

1. Total no. of outcomes $= 6\times6\times6\times6\times6\times6\times6 = 6^7 = 279{,}936$ #

2. (i) First number can be selected in 8 ways and the second number in 7 ways.
$\therefore$ total number of 2-digit numbers $= 8\times7 = 56$ #

(ii) First and second numbers can be selected in 8 ways.
$\therefore$ number of 2-digit numbers $= 8\times8 = 64$ #

3. Total number of ways $= 50\times49\times48\times47 = 5{,}527{,}200$ #

4. (i) Total number of ways of selecting 3 different letters and 2 different digits
$= (26)\times(25)\times(24)\times(10)\times(9) = 1{,}404{,}000$ #

(ii) If no restriction, total no. of ways $= (26)^3(10)^2 = 1{,}757{,}600$ #

(iii) The first letter can be selected in 26 ways, then the second and third letters can be chosen in 1 way as they must be consecutive.
Similarily, the first digit can be selected in 10 ways, then the remaining digits must follow.
$\therefore$ total number of arrangements $= 26\times10 = 260$ #

5. The first digit can be selected in 9 ways (as the number can not commence with zero).
The second digit can be selected in 9 ways, the third in 8 and fourth in 7 ways.
$\therefore$ total number of 4-digit numbers $= 9\times9\times8\times7 = 4{,}536$ #

6. (i) Each person has 4 choices.
$\therefore$ total number of arrangement with no restrictions $= (4)^6 = 4{,}096$ #

(ii) The three women choose from 4 movies, then the three men choose from 3 movies.
$\therefore$ total number of arrangements $= 4\times3 = 12$ #

(iii) If the married couple can be considered a single entity, then there are five separate entities and four movies.
$\therefore$ total number of arrangement $= (4)^5 = 1{,}024$ #

7. Total number of ways $= 9! = 362{,}880$ #

8. Total number of 5-digit numbers $= 5! = 120$.
(i) Odd numbers are those that end with 1, 3 or 5. In each case the remaining 4 digits can be arranged in 4! ways.

$\therefore$ required probability $= \frac{72}{120} = 0.6$ #

(ii) A number is divisible by 5 if it ends with 5. In this case the first 4 digits can be arranged in 4! ways.

Thus, the required probability $= \frac{4!}{120} = \frac{24}{120} = 0.2$ #

(iii) All numbers commencing with 4 or 5 are greater than 40,000.
In each case the remaining 4 digits can be arranged in 4! ways.
$\therefore$ Total number of 5-digit numbers greater than $40{,}000 = (4!)\times2 = 48$

$\therefore$ Required probability $= \frac{48}{120} = 0.4$ #

9. The 1st child can choose from 7 pieces of fruit.
The 2nd child can choose from 6 pieces of fruit remaining.
The 3rd child can choose from 5 pieces of fruit, etc.

$\therefore$ Total number of ways $= 7\times6\times5\times4\times3 = 2{,}520$ #

10. **USA:** Five digits for each digit there are 10 choices.
$\therefore$ Total number of possibilities $= (10)^5 = 100{,}000$ #

Canada: Three letters and three digits for each letter there are 26 choices and for each digit there are 10 choices.
$\therefore$ Total number of possibilities $= (26)^3(10)^3 = 17{,}576{,}000$ #

11. (i) ${}^7P_3 = 210$ #
(ii) ${}^7P_4 = 840$ #
(iii) ${}^7P_5 = 2{,}520$ #

12. Since the colours are in a specific order, thus this is a permutation i.e. total number of possibilities ${}^9P_3 = 504$ #

13. (i) All numbers are greater than 200,000 except for those commencing with 1.
Now, total number of different 6-digit numbers $= \frac{6!}{3!} = 120$

Total number of different 6-digit numbers $< 200{,}000 = \frac{5!}{3!} = 20$

Total number of different 6-digit number $> 200{,}000 = 120 - 20 = 100$ #

$\therefore$ Total number of different 6-digit number > 200,000 are odd.

(ii) First, need to determine how many numbers greater than 200,000 are odd.
Since all digits are odd except 4, it is easier to work with the complimentary outcome, i.e. that the number is even. If the number ends with 4, it is even.

$\therefore$ Total number of even numbers $= \dfrac{5!}{3!} = 20$

Of these however, some will have commenced with 1 and need to be deducted as the number will be less than 200,000.

i.e. total number of even numbers commencing with $1 = \dfrac{4!}{3!} = 4$

$\therefore$ Total number of odd numbers $> 200,000 = 100 - (20-4) = 84$

$\therefore$ Required probability $= \dfrac{84}{100} = 0.84$ #

14. (i) If the first letter is R, then the remaining 6 letters can be arranged in

$\dfrac{6!}{2!} = 360$ ways #

(ii) If the first two letters are RR, then the remaining 5 letters can be arranged in $5! = 120$ ways #

(iii) The second letter can be chosen from 4 letters that are not R, and the remaining 5 letters containing 2 R's can be arranged in $\dfrac{5!}{2!}$ ways

$\therefore$ Total number of ways $= 4\times\left(\dfrac{5!}{2!}\right) = 240$ #

15. Again, it is easier to find the complimentary probability i.e. total number of arrangements in which the L's are together $= \dfrac{6!}{2!} = 360$

Total number of arrangements without any restriction $= \dfrac{8!}{3!2!} = 3,360$

$\therefore$ Total number of arrangments in which L's are **not** together $= 3,360 - 360 = 3,000$

Thus, the required probability $= \dfrac{3,000}{3,360} = \dfrac{25}{28}$ #

16. (i) To obtain the word "SYDNEY", draw the letters S, Y, D, N, E, Y exactly in that order. Now, this can be done in 1 way.

Total number of possibilities $= \dfrac{6!}{2!} = 360$

$\therefore$ Required probability $= \dfrac{1}{360}$ #

(ii) Total number of possibilities beginning with SYD $= 3! = 6$ ways.

$\therefore$ Required probability $= \dfrac{6}{360} = \dfrac{1}{60}$ #

(iii) The probability of getting SYDNEY in that order is best done as follows:

$$P(\text{SYDNEY}) = P(1^{\text{st}} \text{ letter S}) \times P(2^{\text{nd}} \text{ letter Y}) \times \ldots \times P(6^{\text{th}} \text{ letter Y})$$

$$= \frac{1}{6}\times\frac{2}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{2}{6} = \frac{4}{6^6} = \frac{1}{11,664} \ \#$$

$$P(\text{SYD}) = P(1^{\text{st}} \text{ letter S}) \times P(2^{\text{nd}} \text{ letter Y}) \times P(3^{\text{rd}} \text{ letter D})$$

$$= \frac{1}{6}\times\frac{2}{6}\times\frac{1}{6} = \frac{1}{108} \ \#$$

17. This is equivalent to arranging 4 persons in a circle.
$\therefore$ Total number of possibilities $= 3! = 6$ #

18. (i) If Kent is fixed, then there are 9 other guests which can be arranged in 9! ways $= 362,880$ #

(ii) If Kent is fixed, then the remaining 4 boys may be seated in 4! ways and the remaining 5 girls in 5! ways.

$\therefore$ Total number of ways $= 5!\times4! = 2,880$ #

(iii) If Kent and the two girls are fixed, then the remaining 7 guests may be seated in 7! ways.
Also, the two girls seated next to Kent may be alternated in 2! ways

$\therefore$ Total number of ways $= 7!\times2! = 10,080$ #

19. (i) There are 8 persons altogether of which 3 are to be seated together. Fixing the 3 girls together, then the remaining 5 people can be arranged in 5! ways.
The 3 girls can also be arranged in 3! ways.

$\therefore$ Total number of ways $= 5!\times3! = 720$ #

(ii) Let the mum be fixed, then there is only one place for the dad to sit. The remaining 6 children can be seated in 6! ways.

$\therefore$ Total number of ways $= 6! = 720$ #

20. (i) If 2 persons must be included, then choose 3 persons from the remaining 10.

$\therefore$ Total number of ways $= {}^{10}C_3 = 120$ #

(ii) If 2 persons must **not** be included, then need to choose 5 persons from the remaining 10.

$\therefore$ Total number of ways $= {}^{10}C_5 = 252$ #

21. Total number of ways of choosing a committee of $3 = {}^{25}C_3 = 2,300$

Committees which do not include students $= {}^{21}C_3 = 1,330$

$\therefore$ The number of committees of 3 which have at least one student

$= 2300 - 1330 = 970$ #

22. Let the 2 children be A and B, then the 8 gifts may be distributed as follows:

A 1 3 5 7
B 7 5 3 1

$\therefore$ Total number of ways $= {}^8C_1\times{}^7C_7+{}^8C_3\times{}^5C_5+{}^8C_5\times{}^3C_3+{}^8C_7\times{}^1C_1$
$= {}^8C_1+{}^8C_3+{}^8C_5+{}^8C_7=8+56+56+8=128$ #

23. (i) The total number of possible selections $= {}^3C_2\times{}^6C_2=420$
If both remote controls work then she must select all the batteries from the 5 good ones $= {}^5C_2\times{}^3C_2=30$

$\therefore$ $P(\text{both remotes work})=\dfrac{30}{420}=\dfrac{1}{14}$ #

(ii) There are 2 possible outcomes where the TV remote works and:

(1) video remote doesn't work with 2 defective batteries.

(2) video remote doesn't work with 1 defective battery and 1 good battery.

$\therefore$ Total number of outcomes $= {}^5C_2\times{}^3C_2+{}^5C_2\times{}^3C_1\times{}^3C_1=120$

$\therefore$ $P(\text{the remote of the TV works only})=\dfrac{120}{420}=\dfrac{2}{7}$ #

24. (i) The ways of choosing 5 balls of the same colour = 4.
The total number of ways of choosing 5 balls from 20 = ${}^{20}C_5$

$\therefore$ Required probability is given by $\dfrac{4}{{}^{20}C_5}=\dfrac{1}{3876}$ #

(ii) Number of ways of choosing 3 balls of the same colour $= {}^5C_3=10$ ways
Number of ways of choosing 2 balls of the same colour $= {}^5C_2=10$ ways
Number of ways of choosing 2 colours $= {}^4C_1\times{}^3C_1=12$ ways
The total number of ways of choosing 5 balls from 20 = ${}^{20}C_5$

$\therefore$ Required probability is given by $\dfrac{{}^5C_3\times{}^5C_2\times{}^4C_1\times{}^3C_1}{{}^{20}C_5}=\dfrac{1,200}{15,504}=\dfrac{25}{323}$ #

TOPIC 8

Mathematical Induction

(A) Step-by-Step Method

Step 1: Test the result is true for $n=1$.

Step 2: Assume the result is true for $n=k$.

Step 3: Prove the result is true for $n=k+1$. Often involves proving that $S_{k+1}=S_k+T_{k+1}$.

Step 4: Thus, if the result is true for $n=k$ it is true for $n=k+1$. It has been shown that the result is true for $n=1$, hence, it is correct for $n=2$. It is therefore correct for $n=3$ and so on for all positive integers n.

There are a number of different types of mathematical induction questions covered in the Extension 1 course.

(B) Type 1: Proving a Summation Pattern

Example 1: Prove that $\sum_{r=1}^{n}\dfrac{1}{(2r-1)(2r+1)}=\dfrac{n}{2n+1}$ using mathematical induction.

Solution 1:

Step 1: Test result for $n=1$.

$$LHS=\frac{1}{(2-1)(2+1)}=\frac{1}{3}$$

$$RHS=\frac{1}{2+1}=\frac{1}{3}=LHS$$

Step 2: Assume the result is true for $n=k$.

i.e. $\sum_{r=1}^{k}\dfrac{1}{(2r-1)(2r+1)}=\dfrac{k}{2k+1}$

Step 3: Prove the result is true for $n=k+1$.

$$S_k+T_{k+1}=\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{k(2k+3)+1}{\ldots(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$= \frac{k+1}{2k+3} = S_{k+1}$$

Step 4: As in part (A).

(C) Type 2: Proving an Expression is Divisible by a Certain Number

Example 1: Show by mathematical induction that 7^n+2 is divisible by 3.

Solution 1:

Step 1: Test result for $n=1$:

$7^1+2=9$ which is divisible by 3.

Step 2: Assume the result is true for $n=k$.

i.e. $7^k+2=3M$ where M is an integer.

Step 3: Prove the result is true for $n=k+1$.

$$\text{i.e. } 7^{k+1}+2 = 7^1\cdot 7^k+2$$
$$=7\cdot(3M-2)+2 \qquad (\text{note: } 7^k=3M-2)$$
$$=21M-12$$
$$=3(7M-4)$$

which is divisible by 3

Step 4: As in part (A).

(D) Type 3: Other Applications

Example 1: Prove by induction that $\cos(x+n\pi)=(-1)^n\cos x,\ n\geq 1$.

Solution 1:

Step 1: Test result for $n=1$

$LHS=\cos(x+\pi)=-\cos x$

$RHS=(-1)^1\cos x=-\cos x=LHS$

Step 2: Assume the result is true for $n=k$

Step 3: Prove the result is true for $n=k+1$

i.e. need to show that:

$$\cos(x+(k+1)\pi)=(-1)^{k+1}\cos x$$
$$\text{LHS}=\cos(x+(k+1)\pi)$$
$$=\cos((x+k\pi)+\pi)$$
$$=-\cos(x+k\pi)$$
$$=-(-1)^k\cos x \qquad (\text{from step 2})$$
$$=(-1)^{k+1}\cos x=\text{RHS}$$

Step 4: As in part (A).

Example 2: Prove by induction that $7^n>2^n+5^n$ for $n\geq 2$

Solution 2:

Step 1: Test result for $n=2$.

$\text{LHS}=7^2=49$

$\text{RHS}=2^2+5^2=29<49$

$\therefore$ the result is true for $n=2$

Step 2: Assume the result is true for $n=k$, i.e. $7^k>2^k+5^k$.

Step 3: Prove the result is true for $n=k+1$

i.e. need to show that

$$7^{k+1}>2^{k+1}+5^{k+1}$$
$$\text{LHS}=7^{k+1}=7^1.7^k$$
$$>7(2^k+5^k) \qquad (\text{from step 2.})$$
$$=(2+5)(2^k+5^k)$$
$$=2^{k+1}+5^{k+1}+5.2^k+2.5^k$$
$$>2^{k+1}+5^{k+1}=\text{RHS}$$

Step 4: Thus, if the result is true for $n=k$, it is true for $n=k+1$. It has been shown that the result is true for $n=1$, hence, it is correct for $n=2$. It is therefore correct for $n=3$ and so on for all positive integers n.

REVIEW EXERCISES

(A) Type 1: Proving a Summation Pattern:

1. Prove the following summation pattern using mathematical induction:

(i) $1^2+3^2+...+(2n-1)^2=\frac{4n^3-n}{3}$

(ii) $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$

(iii) $\sum_{r=1}^{n}rr!=(n+1)!-1$

(iv) $1.2^2+2.3^2+3.4^2+...+n(n+1)^2=\frac{n(n+1)(n+2)(3n+5)}{12}$

(v) $2\times1!+5\times2!+10\times3!+...+(n^2+1)\times n!=n\times(n+1)!$

(B) Type 2: Proving an Expression is Divisible by a Certain Number

2. By writing a simplified expression first, prove using mathematical induction, that the sum of the cubes of three consecutive integers is divisible by 9.

3. Prove by mathematical induction that:

(i) $3^{4n}-1$ is divisible by 80, for all positive integer values of n.

(ii) $3^{3n}+2^{n+2}$ is divisible by 5, for all positive integer values of n.

(iii) $13\times6^n+2$ is divisible by 10, for all positive integer values of n.

4. Use mathematical induction to show that for positive integer values of n, $n\geq2$, the expression, $(x+1)^n-nx-1$ is divisible by x^2.

(C) Type 3: Other Applications

5. Use mathematical induction to show that:

$(4\cdot1-2)(4\cdot2-2)....(4\cdot n-2)=(n+1)(n+2)....(n+n)$ for all positive integers of n.

6. Given $y=a\cos\left(\frac{\pi}{2}x\right)+b\sin\left(\frac{\pi}{2}x\right)+c$, use mathematical induction to show tha for all positive integers of n:

$$\frac{d^ny}{dx^n}=\left(\frac{\pi}{2}\right)^n\left[a\cos\left(\frac{\pi}{2}x+n\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+n\frac{\pi}{2}\right)\right]$$

where $\frac{d^ny}{dx^n}$ is the n^{th} derivative of y.

7. A sequence of terms (U_n) is defined such that:

$U_{n+1}=3\cdot U_n-2$ for $n\geq1$, and $U_1=4$.

Prove by mathematical induction that $U_n=3^n+1$.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) Step 1: Test result for $n=1$.

$\text{LHS}=1^2=1$.

$\text{RHS}=\dfrac{4.1^3-1}{3}=\dfrac{3}{3}=1=\text{LHS}$

Step 2: Assume the result is true for $n=k$.

i.e. $1^2+3^2+...+(2k-1)^2=\dfrac{4k^3-k}{3}$

Step 3: Prove the result is true for $n=k+1$.

$$
\begin{aligned}
S_k+T_{k+1}&=1^2+3^2+...+(2k-1)^2+(2k+1)^2\\
&=\frac{4k^3-k}{3}+(2k+1)^2\\
&=\frac{4k^3-k}{3}+4k^2+4k+1\\
&=\frac{4k^3-k+12k^2+12k+3}{3}\\
&=\frac{4k^3+12k^2+12k+4-(k+1)}{3} \qquad \text{(note this step)}\\
&=\frac{4(k^3+3k^2+3k+1)-(k+1)}{3}\\
&=\frac{4(k+1)^3-(k+1)}{3}=S_{k+1}
\end{aligned}
$$

Step 4: Thus, if the result is true for $n=k$, it is true for $n=k+1$. It has been shown that the result is true for $n=1$, hence it is correct for $n=2$. It is therefore correct for $n=3$ and so on for all positive integers n.

(ii) Step 1: Test result for $n=1$.

$\text{LHS}=\dfrac{1}{1\times2\times3}=\dfrac{1}{6}$

$\text{RHS}=\dfrac{1}{4}-\dfrac{1}{2\times2\times3}=\dfrac{1}{4}-\dfrac{1}{12}=\dfrac{1}{6}=\text{LHS}$

Step 2: Assume the result is true for $n=k$.

Step 3: Prove the result is true for $n=k+1$.

$$
\begin{aligned}
S_k+T_{k+1}&=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}-\frac{(k+3)-2}{2(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}-\frac{k+1}{2(k+1)(k+2)(k+3)}\\
&=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}=S_{k+1}
\end{aligned}
$$

Step 4: As Q1 (i)

(iii) Step 1: Test result for $n=1$.

$\text{LHS}=1\cdot1!=1$

$\text{RHS}=2!-1=1=\text{LHS}$

Step 2: Assume the result is true for $n=k$.

i.e. $\displaystyle\sum_{r=1}^{k} r\,r!=(k+1)!-1$

Step 3: Prove the result is true for $n=k+1$.

$$
\begin{aligned}
S_k+T_{k+1}&=(k+1)!-1+(k+1)(k+1)!\\
&=(k+1)![k+1+1]-1\\
&=(k+1)!(k+2)-1=(k+2)!-1=S_{k+1}
\end{aligned}
$$

Step 4: As Q1 (i)

(iv) Step 1: Test result for $n=1$.

$\text{LHS}=1\cdot2^2=4$

$\text{RHS}=\dfrac{1\times2\times3\times8}{12}=\dfrac{48}{12}=4=\text{LHS}$

Step 2: Assume the result is true for $n=k$.

i.e. $1\cdot2^2+2\cdot3^2+...+k(k+1)^2=\dfrac{k(k+1)(k+2)(3k+5)}{12}$

Step 3: Prove the result is true for $n=k+1$.

$$
\begin{aligned}
S_k+T_{k+1}&=\frac{k(k+1)(k+2)(3k+5)}{12}+(k+1)(k+2)^2\\
&=\frac{k(k+1)(k+2)(3k+5)+12(k+1)(k+2)^2}{12}
\end{aligned}
$$

$$= (k+1)(k+2)\left[\frac{3k^2+17k+24}{12}\right]$$

$$= \frac{(k+1)(k+2)(k+3)(3k+8)}{12} = S_{k+1}$$

Step 4: As Q1 (i).

(v) Step 1: Test result for $n=1$.

$LHS = 2\times1! = 2$

$RHS = 1\times2! = 2 = LHS$

Step 2: Assume the result is true for $n=k$.

$2\times1!+5\times2!+\ldots+\left(k^2+1\right)\times k! = k\times(k+1)!$

Step 3: Prove the result is true for $n=k+1$.

$$S_k + T_{k+1} = k\times(k+1)! + \left((k+1)^2+1\right)\times(k+1)!$$
$$=(k+1)!\left[k+k^2+2k+2\right]$$
$$=(k+1)!\left[k^2+3k+2\right]$$
$$=(k+1)!(k+2)(k+1)$$
$$=(k+2)!(k+1)=S_{k+1}$$

Step 4: As Q1 (i).

2. Let the three consecutive integers be $n-1$, n, $n+1$

$$(n-1)^3+n^3+(n+1)^3 = n^3-3n^2+3n-1+n^3+3n^2+3n+1$$
$$=3n^3+6n$$

Step 1: Test result for $n=1$.

$LHS = 3+6=9$

$\therefore$ $3n^3+6n$ is divisible by 9 for $n=1$.

Step 2: Assume the result is true for $n=k$.

i.e. $3k^3+6k=9M$, where M is an integer.

Step 3: Prove the result is true for $n=k+1$.

i.e. $3(k+1)^3+6(k+1)=3k^3+9k^2+9k+3+6k+6$
$$=3k^3+6k+\left(9k^2+9k+9\right)$$
$$=9M+\left(9k^2+9k+9\right)$$
$$=9\left(M+k^2+k+1\right)$$

which is divisible by 9

Step 4: Hence, if the statement is divisible by 9 for $n=k$, it is also divisible by 9 for $n=k+1$. It is divisible by 9 for $n=1$, so it is divisible by 9 for $n=2$. If it is divisible by 9 for $n=2$, it is also true for $n=3$ and so on, for all positive integers n.

3. (i) Step 1: Test result for $n=1$.

$3^4-1=80$ which is divisible by 80.

Step 2: Assume the result is true for $n=k$.

i.e. $3^{4k}-1=80M$, where M is an integer.

Step 3: Prove the result is true for $n=k+1$.

i.e. $3^{4(k+1)}-1=3^4\cdot3^{4k}-1$
$$=81(80M+1)-1$$
$$=6480M+80$$
$$=80(81M+1)$$ which is divisible by 80. #

Step 4: As Q2, replace 9 by 80.

(ii) Step 1: Test result for $n=1$.

$3^3+2^3=27+8=35$ which is divisible by 5.

Step 2: Assume the result is true for $n=k$.

i.e. $3^{3k}+2^{k+2}=5M$, where M is an integer.

Step 3: Prove the result is true for $n=k+1$.

i.e. $3^{3(k+1)}+2^{k+1+2}=3^{3k+3}+2^{k+3}$
$$=3^3\cdot3^{3k}+2^{k+3}$$
$$=27\left(5M-2^{k+2}\right)+2^{k+3}$$
$$=135M-27\cdot2^{k+2}+2^{k+3}$$
$$=135M+2^{k+2}(2-27)$$
$$=135M-25\cdot2^{k+2}$$
$$=5\left(27M-5\cdot2^{k+2}\right)$$ which is divisible by 5.

Step 4: As Q2, replace 9 by 5.

(iii) Step 1: Test result for $n=1$.

$13\times6+2=80$ which is divisible by 10.

Step 2: Assume the result is true for $n=k$.

i.e. $13\times6^k+2=10M$, where M is an integer.

Step 3: Prove the result is true for $n=k+1$.

i.e. $13\times 6^{k+1}+2$

$$=6\times 13\times 6^k+2 \qquad \left(13\times 6^k=10\text{M}-2\right)$$
$$=6(10\text{M}-2)+2$$
$$=60\text{M}-12+2$$
$$=60\text{M}-10$$
$$=10(6\text{M}-1) \text{ which is divisible by 10.}$$

Step 4: As Q2, replace 9 by 10.

4. Step 1: Test result for $n=2$.

i.e. $(x+1)^2-2x-1=x^2+2x+1-2x-1=x^2$, which is divisible by x^2.

Step 2: Assume the result is true for $n=k$.

i.e. $(x+1)^k-kx-1=x^2P(x)$

i.e. $(x+1)^k=x^2P(x)+kx+1$

Step 3: Prove the result is true for $n=k+1$

i.e. $(x+1)^{k+1}-(k+1)x-1$

$$=(x+1)(x+1)^k-(k+1)x-1$$
$$=(x+1)\left(x^2P(x)+kx+1\right)-(k+1)x-1$$
$$=(x+1)x^2P(x)+kx^2+x+kx+1-kx-x-1$$
$$=(x+1)x^2P(x)+kx^2$$
$$=x^2\left[(x+1)P(x)+k\right] \text{ which is divisible by } x^2.$$

Step 4: Thus it has been proven that the result is true from $n=2$, and hence it is true for $n=3$, and so on for all positive integer values of n.

5. Step 1: Test result for $n=1$.

LHS $=4.1-2=2$

RHS $=1+1=2=$ LHS

Step 2: Assume the result is true for $n=k$.

i.e. $(4\cdot 1-2)(4\cdot 2-2)\ldots(4k-2)=(k+1)(k+2)\ldots(k+k)$

Step 3: Prove the result is true for $n=k+1$.

$S_k\times T_{k+1}$

$$=(k+1)(k+2)(k+3)\ldots(k+k)\left(4(k+1)-2\right)$$
$$=(k+1)(k+1+1)(k+1+2)\ldots(k+1+k-1)(4k+2)$$
$$=(k+1+1)(k+1+2)\ldots(k+1+k-1)2(2k+1)(k+1)$$
$$=(k+1+1)(k+1+2)\ldots(k+1+k-1)(2k+1)(2k+2)$$
$$=(k+1+1)(k+1+2)\ldots(k+1+k-1)(k+1+k)(k+1+k+1)$$
$$=S_{k+1}$$

Step 4: As Q1 (i).

6. Step 1: Test result for $n=1$

$$y=a\cos\left(\frac{\pi}{2}x\right)+b\sin\left(\frac{\pi}{2}x\right)+c$$
$$\frac{dy}{dx}=-a\cdot\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)+b\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$$
$$=\left(\frac{\pi}{2}\right)\left[a\cos\left(\frac{\pi}{2}x+1\cdot\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+1\cdot\frac{\pi}{2}\right)\right]$$

$\therefore$ the formula is true for $n=1$.

Step 2: Assume the result is true for $n=k$.

i.e. $$\frac{d^ky}{dx^k}=\left(\frac{\pi}{2}\right)^k\left[a\cos\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)\right]$$

Step 3: Prove the result is true for $n=k+1$.

$$\frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$$
$$=\frac{d}{dx}\left[\left(\frac{\pi}{2}\right)^k\left[a\cos\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)\right]\right]$$
$$=\left(\frac{\pi}{2}\right)^k\left[-a.\frac{\pi}{2}\sin\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)+b.\frac{\pi}{2}\cos\left(\frac{\pi}{2}x+k\frac{\pi}{2}\right)\right]$$
$$=\left(\frac{\pi}{2}\right)^{k+1}\left[a\cos\left(\frac{\pi}{2}x+k\frac{\pi}{2}+\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+k\frac{\pi}{2}+\frac{\pi}{2}\right)\right]$$
$$=\left(\frac{\pi}{2}\right)^{k+1}\left[a\cos\left(\frac{\pi}{2}x+(k+1)\frac{\pi}{2}\right)+b\sin\left(\frac{\pi}{2}x+(k+1)\frac{\pi}{2}\right)\right]$$
$$=\frac{d^{k+1}y}{dx^{k+1}}$$

Step 4: As Q1 (i).

7. Step 1: Test result for $n=1$.

$U_2 = 3U_1 - 2 = 3\times 4 - 2 = 10$

$U_2 = 3^2 + 1 = 10$

$\therefore$ the formula is true for $n=1$.

Step 2: Assume the result is true for $n=k$.

i.e. assume $U_k = 3^k + 1$.

Step 3: Prove the result is true for $n=k+1$.

$U_{k+1} = 3U_k - 2$

$= 3(3^k + 1) - 2$

$= 3^{k+1} + 1$

Step 4: As Q1 (i).

TOPIC 9

POLYNOMIALS

(A) Features of a Polynomial

The general form of the polynomial $P(x)$ is given by:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

Features of the Polynomial $P(x)$

1. a_0, a_1,.........., a_n are called ***coefficients***.
2. a_n is called for the ***leading coefficient***.
3. $a_n x^n$ is called the leading term.
4. $P(x)$ is said to have ***degree*** n.

   Mathematically, it is expressed as $deg\left[P(x)\right] = n$.
5. If $a_n = 1$, $P(x)$ is said to be a ***monic*** polynomial.
6. A number a is called a *zero* of $P(x)$ if $P(a) = 0$. It is also known as a root or solution of $P(x) = 0$.

   Note that this also means that $P(x)$ cuts the x-axis at $x = a$.
7. A polynomial of degree n cannot have more than n distinct real zeros.

(B) Long Division of Polynomials

Long division of polynomials is similar to the way long division in ordinary arithmetic is performed.

Example 1: Divide $6x^3 - 5x^2 - 2x + 1$ by $(x-1)$. Hence, factorise $6x^3 - 5x^2 - 2x + 1$ completely.

Polynomials

Solution 1:

$$
\begin{array}{rl}
 & 6x^2+x-1 \\
x-1 \!\!\!\! & \overline{)\,6x^3-5x^2-2x+1} \\
 & \underline{6x^3-6x^2} \\
 & x^2-2x \\
 & \underline{x^2-x} \\
 & -x+1 \\
 & \underline{-x+1} \\
 & 0
\end{array}
$$

$\therefore\ 6x^3-5x^2-2x+1=(x-1)(6x^2+x-1)=(x-1)(3x-1)(2x+1)$

Example 2: Divide $P(x)=4x^3-8x^2+5x-1$ by $(2x-1)$. Hence, solve $P(x)=0$.

Solution 2:

$$
\begin{array}{rl}
 & 2x^2-3x+1 \\
(2x-1) \!\!\!\! & \overline{)\,4x^3-8x^2+5x-1} \\
 & \underline{4x^3-2x^2} \\
 & -6x^2+5x \\
 & \underline{-6x^2+3x} \\
 & 2x-1 \\
 & \underline{2x-1} \\
 & 0
\end{array}
$$

$$
\begin{aligned}
\therefore\ P(x) &= 4x^3-8x^2+5x-1 \\
&= (2x-1)(2x^2-3x+1) \\
&= (2x-1)(2x-1)(x-1) \\
&= (2x-1)^2(x-1)
\end{aligned}
$$

$\therefore\ P(x)=0$ when $x=1,\ x=\dfrac{1}{2}$ #

(C) The Factor Theorem

If $P(a)=0$, then $(x-a)$ is a factor of $P(x)$, and vice versa. If $(x-a)$ is a factor of $P(x)$, then $P(a)=0$.

Example 1: If $(x+2)$ is a factor of $P(x)=x(x+6)(x-1)-k$, find the value of k.

Solution 1:

If $(x+2)$ is a factor $\therefore\ P(-2)=0$

i.e. $P(-2)=-2(4)(-3)-k$

$\qquad = 24-k$

Hence, if $P(-2)=0$, then $k=24$ #

Example 2: Consider the polynomial $P(x)=2x^3-6x^2+2x+4$.

(i) Show that $P(2)=0$.

(ii) Hence, solve $P(x)=0$.

Solution 2:

(i) $P(2)=2(2)^3-6(2)^2+2(2)+4$

$\qquad =16-24+4+4=0$ #

(ii) Since $P(2)=0\ \therefore\ (x-2)$ is a factor of $P(x)$,

dividing $P(x)$ by $(x-2)$ gives:

$$
\begin{array}{rl}
 & 2x^2-2x-2 \\
(x-2) \!\!\!\! & \overline{)\,2x^3-6x^2+2x+4} \\
 & \underline{2x^3-4x^2} \\
 & -2x^2+2x \\
 & \underline{-2x^2+4x} \\
 & -2x+4 \\
 & \underline{-2x+4} \\
 & 0
\end{array}
$$

$\therefore\ P(x)=2(x-2)(x^2-x-1)=0$

when $x-2=0$ or $x^2-x-1=0$

$x=2 \qquad x=\dfrac{1\pm\sqrt{1+4\times1\times1}}{2}$

$\qquad =\dfrac{1\pm\sqrt{5}}{2}$

$\therefore\ P(x)=0$, when $x=2,\ \dfrac{1-\sqrt{5}}{2},\ \dfrac{1+\sqrt{5}}{2}$ #

(D) The Remainder Theorem

(i) General Case

A polynomial $P(x)$ of degree n divided by another polynomial $A(x)$ of degree m will give a quotient $Q(x)$ of degree $n-m$ and a remainder $R(x)$ of at most degree $m-1$:

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$$

Multiplying throughout by $A(x)$, gives:

$$P(x) = A(x)Q(x) + R(x).$$

Important Feature

$$Deg[R(x)] < Deg[A(x)] = m$$

This means that the degree of the remainder $R(x)$ must always be less than degree of the divisor $A(x)$.

For example, if $A(x)$ is a quadratic $(i.e. m = 2)$ then $R(x)$ is of the form $ax+b$ (i.e. a polynomial of degree 1). This theorem applies to all values of m.

(ii) Specific Case Where A(x) is a Linear Polynomial

From (i): $Deg[R(x)] < Deg[A(x)] = m$

Thus, if $A(x)$ is a linear polynomial then the remainder $R(x)$ is a constant R.

In particular, if $A(x) = (x-a)$ then :

$$P(x) = (x-a)Q(x) + R \qquad \text{(from (i))}$$

The above identity is true for all values of x. Thus, for $x = a$, $R = P(a)$

In summary, if a polynomial $R(x)$ is divided by $(x-a)$, then the remainder R is found by calculating $P(a)$. This is known as the **Remainder Theorem.**

Example 1: The polynomial $P(x) = (x-a)^3 + b$ has a zero at $x = 1$, and when divided by x, the remainder is -7. Find all possible

Solution 1:

$$P(1) = 0 \text{ i.e. } P(1) = (1-a)^3 + b = 0$$
$$= 1 - 3a + 3a^2 - a^3 + b = 0$$

$$P(0) = -7 \text{ i.e. } P(0) = (-a)^3 + b = -7$$
$$= -a^3 + b = -7$$

$\therefore \; 1 - 3a + 3a^2 - 7 = 0$

$3a^2 - 3a - 6 = 0$

i.e. $a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\therefore \; a = 2, -1$

when: $a = 2$, $b = 2^3 - 7 = 1$

$a = -1$, $b = (-1)^3 - 7 = -8$

$\therefore$ solutions are: $a = 2$, $b = 1$ or $a = -1$, $b = -8$ #

Example 2 : When the polynomial $P(x)$ is divided by $A(x) = x^2 - 1$, it gives quotient $Q(x)$ and a remainder $R(x)$.

(i) Write the general form of $R(x)$. Explain your answer.

(ii) Show that $P(-1) = R(-1)$.

(iii) Given $P(-1) = -12$ and when $P(x)$ is divided by $(x-1)$, the remainder is -1. Find $R(x)$.

Solution 2:

(i) $R(x) = ax + b$. Since $deg(R(x)) < deg[A(x)]$
and $A(x)$ is a quadratic polynomial (i.e. degree 2),
then $R(x)$ is a linear polynomial - i.e. $R(x) = ax + b$ (i.e. degree 1). #

(ii) $$\frac{P(x)}{x^2-1} = Q(x) + \frac{R(x)}{(x^2-1)}$$

i.e. $P(x) = (x^2-1)Q(x) + R(x)$ ....(1)

Now, $P(-1) = 0 \times Q(-1) + R(-1)$

$\therefore \; P(-1) = R(-1)$ #

(iii) $P(-1)=R(-1)=-12$ i.e. $R(-1)=-a+b=-12$

Also, since on division by $(x-1)$, the remainder is -1 $\therefore$ $P(1)=-1$

Substituting $x=1$ into (1) above gives: $P(1)=0\times Q(1)+R(1)=-1$

i.e. $P(1)=R(1)=-1$

$\therefore$ $R(1)=a+b=-1$ and $R(-1)=-a+b=-12$

i.e. $a=-b-1$

Substituting this into $-a+b=-12$, gives:

$$-(-b-1)+b=-12$$
$$2b+1=-12$$
$$\text{i.e. } b=-\frac{13}{2}$$
$$a=-\left(-\frac{13}{2}\right)-1=\frac{13}{2}-1=\frac{11}{2}$$

$$\therefore\ R(x)=\frac{11}{2}x-\frac{13}{2}=\frac{11x-13}{2}\ \#$$

(E) Double Roots and Zeros of a Polynomial

If the polynomial $P(x)=0$ has a double root at $x=a$, then $P(a)=P'(a)=0$.

This means that $P(x)=(x-a)^2Q(x)$, where $Q(x)$ is another polynomial.

If $P(x)$ has degree n and n ***distinct real zeros*** $x_1, x_2, \ldots\ldots\ldots, x_n$, then:

$$P(x)=a_n(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_n)$$

Note: a_n is the coefficient of x^n.

Example 1: The polynomial $P(x)=x^3+bx^2+cx+d$ has a double root at $x=-2$ and a single root at $x=2$. Find the values of b, c and d.

Solution 1:

$$P(x)=(x+2)^2(x-2)$$
$$=(x^2+4x+4)(x-2)$$
$$=x^3-2x^2+4x^2-8x+4x-8$$

(F) Sum and Product of the Roots

If α, β and γ are the roots of: $ax^3+bx^2+cx+d=0$, then:

$$\alpha+\beta+\gamma=-\frac{b}{a}$$
$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$
$$\alpha\beta\gamma=-\frac{d}{a}$$

Example 1: If α, β, γ are the roots of the equation $2x^3-3x+1=0$. Find the value of:

(i) $\alpha+\beta+\gamma$ (ii) $\alpha\beta\gamma$ (iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

Solution 1:

(i) $\alpha+\beta+\gamma=-\frac{b}{a}=0$ #

(ii) $\alpha\beta\gamma=-\frac{d}{a}=-\frac{1}{2}$ #

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{\frac{-3}{2}}{\frac{-1}{2}}=-\frac{3}{2}\times-\frac{2}{1}=3$ #

Example 2: If the roots α, β, γ of the equation $x^3+3x^2-9x-27=0$ are in geometric progression:

(i) Show that $\alpha\gamma=\beta^2$.

(ii) Show that $\beta=3$.

(iii) Hence, find the values of α and γ.

(iv) Hence, factorise $x^3+3x^2-9x-27$ completely.

Solution 2:

(i) If α, β and γ are in geometric progression

then $\frac{\gamma}{\beta}=\frac{\beta}{\alpha}$

i.e. $\alpha\gamma=\beta^2$ #

(ii) $\alpha\beta\gamma=-\dfrac{d}{a}=27$ i.e. $\alpha\gamma=\dfrac{27}{\beta}$

i.e. $\beta^2=\dfrac{27}{\beta}$ i.e. $\beta^3=27$

$\beta=3$ #

(iii) $\alpha+\beta+\gamma=-\dfrac{b}{a}=-3$

i.e. $\alpha+\gamma+3=-3$ i.e. $\alpha+\gamma=-6$............(1)

Also, $\alpha\gamma=\dfrac{27}{3}$ i.e. $\alpha\gamma=9$...........(2)

Solving equations (1) and (2) simultaneously, gives:

$\gamma=\dfrac{9}{\alpha}$ from (2), substituting into (1) gives: $\alpha+\dfrac{9}{\alpha}=-6$

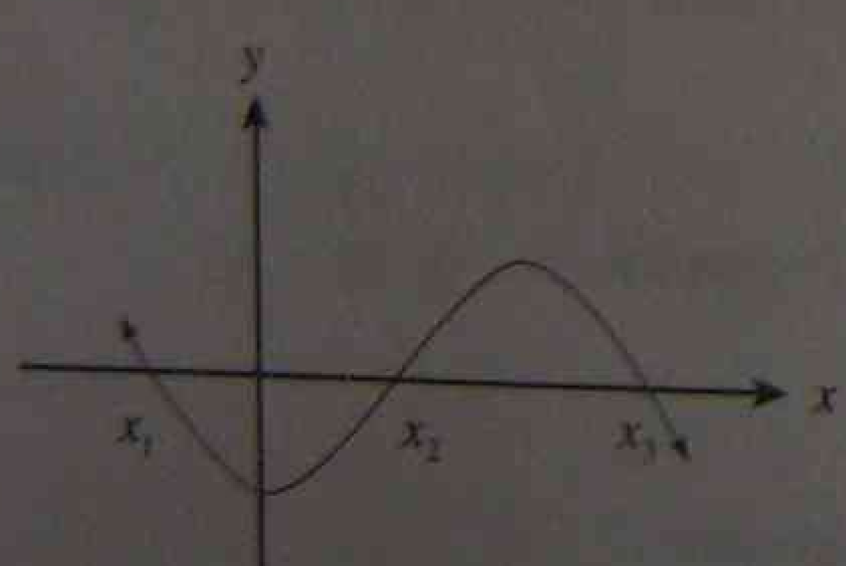
$$\alpha^2+9=-6\alpha$$
$$\alpha^2+6\alpha+9=0$$
$$(\alpha+3)^2=0$$

$\therefore\ \alpha=-3, \gamma=-3$ #

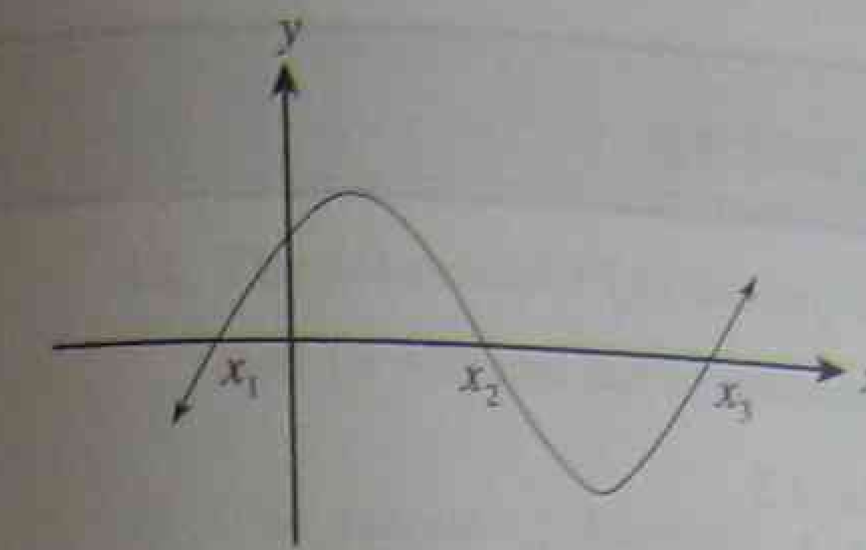
(iv) $x^3+3x^2-9x-27=(x+3)^2(x-3)$ #

(G) Graphing Cubic Polynomials

If $P(x)=ax^3+bx^2+cx+d$, has 3 distinct roots at x_1, x_2 and x_3, respectively, then:

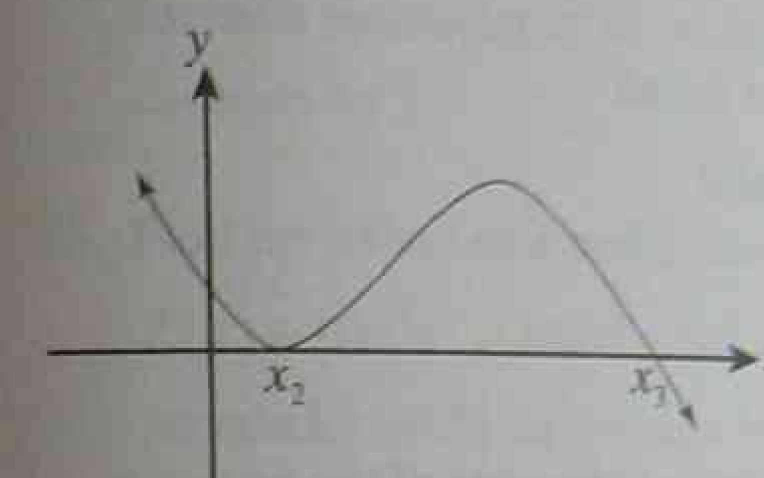


$$P(x)=ax^3+bx^2+cx+d$$
$$=a(x-x_1)(x-x_2)(x-x_3),\ a<0$$

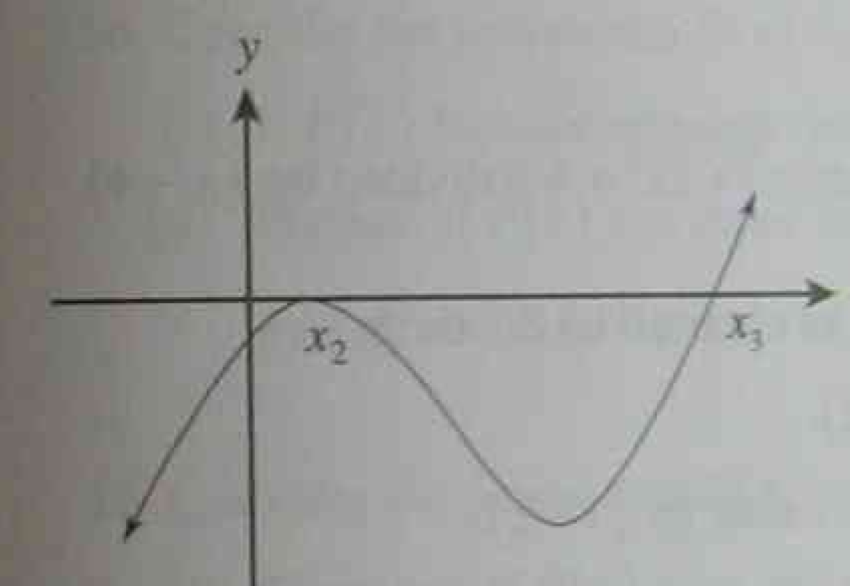


$$P(x)=ax^3+bx^2+cx+d$$
$$=a(x-x_1)(x-x_2)(x-x_3),\ a>0$$

If $P(x)=ax^3+bx^2+cx+d$, had a double root at x_2 and another root at x_3 then:



$$P(x)=ax^3+bx^2+cx+d$$
$$=a(x-x_2)^2(x-x_3),\ a<0$$



$$P(x)=ax^3+bx^2+cx+d$$
$$=a(x-x_2)^2(x-x_3),\ a>0$$

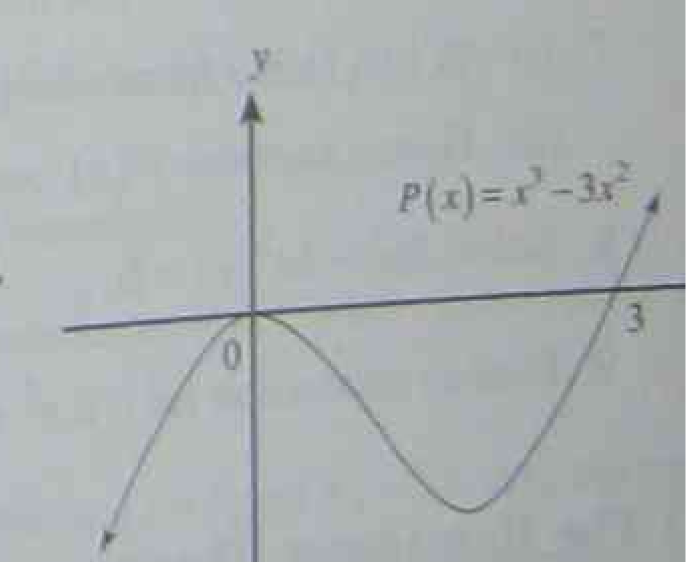
Example 1: $P(x)=x^3-3x^2$.

(i) Sketch $P(x)$, indicating its points of intersection with the axes.

(ii) Hence, solve the inequality $P(x)\geq 0$.

Solution 1:

(i) $P(x)=x^3-3x^2=x^2(x-3)$,
which is a cubic polynomial with $a>0$,
a double root at $x=0$ and another root
at $x=3$.



(ii) From the graph $P(x)\geq 0$ for $x\geq 3$
and $x=0$ where $P(0)=0$. #

Review Exercises

(A) Features of a Polynomial

1. Consider the polynomial $P(x)=2x^4-3x^2+5$
   (i) What is the leading coefficient of $P(x)$? Is it a monic quartic?
   (ii) What is the degree of $P(x)$?
   (iii) What is the maximum number of real roots of $P(x)$?

2. If $x^3+3x=ax(x-1)^2+bx(x-1)+cx+d$ for all values of x. Find the values of a, b, c and d.

(B) Long Division of Polynomials

3. Divide x^3-7x+6 by $(x-1)$, and hence find all the roots of the equation $x^3-7x+6=0$.

4. Find the remainder when the polynomial x^5+2x^2+4 is divided by $(x-4)$.

5. (i) Find the quotient and the remainder obtained by dividing $P(x)=x^3+bx^2-bx+4$ by $(x-2)$.
   (ii) Hence, find b such that $P(x)$ is divisible by $(x-2)$.
   (iii) Hence, solve $P(x)=0$.

(C) The Factor Theorem

6. The coefficients of a and b are such that the polynomial x^4+ax+b is divisible by $(x-1)$ and $(x-2)$. Find the values of a and b.

7. (i) If $(x-1)$ is a factor of $P(x)=x^3+ax^2-2x-4$, find the values of a.
   (ii) Hence, factorise $P(x)$ completely.

8. Solve $2x^3-3x^2+1=0$.

9. Find an expression for c if x^2+px+q is exactly divisible by $(2x+c)$.

(D) The Remainder Theorem

10. Find th...

11. Find the remainder when $(x-a)(x+a+1)$ is divided by $(x+a)$.

12. The polynomial $P(x)$ has degree 3 and has factors $(x-1)^2$ and $(x+2)$ and has a remainder of 40 when divided by $(x-3)$. Find $P(x)$.

13. The quadratic polynomial $ax^2+bx-14$ leaves a remainder of -12 when divided by $(x-1)$, and has $(x+2)$ as a factor. Find the values of a and b.

14. Find a and b, so that the polynomial $P(x)=a^2x^4+3x^3+b^2x^2+4abx+4ab$ leaves a remainder of 10 on division by $x+1$ and a remainder of 24 on division by x.

15. Find the values of a and b, so that the polynomial $P(x)=10a^2x^4-30ax^3-8abx+4ab$ has $(ax-1)$ as a factor and leaves a remainder of 20 on division by x.

16. Consider the polynomial $P(x)=ax^3+bx^2+cx+d$.
   (i) If $P(x)$ leaves a remainder of 12 on division by x, show that $d=12$.
   (ii) Further, if $P(x)$ is a monic cubic and leaves a remainder of $(x-3)$ on div by (x^2+1), find $P(x)$.

17. Consider the polynomial $P(x)=ax^3+bx^2+cx+d$.
   (i) Show that if $P(x)$ is an odd polynomial for all x, then $b=d=0$.
   (ii) Further, if $P(x)$ is monic and has $(x+3)$ as a factor, find $P(x)$.

18. Consider the polynomial $P(x)=ax^4+bx^3+cx^2+dx+e$.
   (i) Show that if $P(x)$ is an even function for all x, then $b=d=0$.
   (ii) Further, if $P(x)$ is a monic quartic with a root at $x=2$ and on division by $(x-1)$ leaves a remainder of 3, find $P(x)$.

19. (i) State the Remainder Theorem for Polynominals.
   (ii) If $P(x)$ is divided by $A(x)$ giving a remainder $R(x)$, what can be said ab degree of $R(x)$?
   (iii) If a certain polynomial $P(x)$ is divided by $(x+1)$and $(x-3)$, then the re remainders are 6 and -2. Find the remainder when $P(x)$ is divided by (x^2-2x-3).

(E) Double Roots and Zeros of a Polynomial

20. A polynomial $P(x)=ax^3+bx^2+cx+d$ has a double root at $x=-2$ and a maximum of 6 when $x=0$. Find $P(x)$.

21. If $P(x)=(x-a)^2 Q(x)$, show that $P(x)$ has a double root at $x=a$.

22. Factorise $P(x)=2x^3-6x^2-18x-10$ completely.

(F) Sum and Product of the Roots

23. If α, β, γ are the roots of the equation $2x^3-4x^2+6x-1=0$, find the values of:
    (i) $\alpha+\beta+\gamma$ (ii) $\alpha\beta\gamma$ (iii) $\alpha\beta+\alpha\gamma+\beta\gamma$
    (iv) $\alpha^2+\beta^2+\gamma^2$ (v) $(\alpha+1)(\beta+1)(\gamma+1)$

24. The equation $x^3+ax^2+b=0$ has 3 distinct roots, such that one is the sum of the other two. Show that $a^3-8b=0$.

25. Two roots of the equation $P(x)=4x^3+bx^2+9x+2=0$ are alike and the other is twice their sum.
    (i) Show that the value of $b=12$.
    (ii) Hence, factorize $P(x)$ completely.

26. Find the roots of the equation $x^3-12x^2+12x+80=0$, given that they are 3 consecutive terms in an arithmetic series.

(G) Graphing Cubic Polynomials

27. Solve $8x-2x^3<0$.

28. Consider the polynomical $P(x)=2x^3-7x^2+2x+3$.
    (i) Show that $x=3$ is a root of $P(x)$.
    (ii) Hence, factorise $P(x)$ completely.
    (iii) Hence, solve $P(x)\geq 0$ for all x.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) Leading coefficient = 2. No it is not a monic, as leading coefficient is not 1. #
   (ii) $Deg\left[P(x)\right]=4$ #
   (iii) Maximum number of roots is 4. #

2. $$x^3+3x=ax(x-1)^2+bx(x-1)+cx+d$$
$$=ax\left(x^2-2x+1\right)+bx^2-bx+cx+d$$
$$=ax^3+(b-2a)x^2+(a-b+c)x+d$$
$a=1$, $b-2a=0$, $a-b+c=3$, $d=0$
$b-2=0$ $\quad 1-2+c=3$
$b=2$ $\quad c=4$
$\therefore\ a=1, b=2, c=4, d=0$ #

3. $$\begin{array}{r} x^2+x-6 \\ (x-1)\overline{\big)\, x^3-7x+6} \\ \underline{x^3-x^2} \\ x^2-7x \\ \underline{x^2-x} \\ -6x+6 \\ \underline{-6x+6} \\ 0 \end{array}$$

$$\therefore\ x^3-7x+6=(x-1)\left(x^2+x-6\right)$$
$$=(x-1)(x+3)(x-2)$$
$\therefore\ (x-1)(x+3)(x-2)=0$, when $x=1, 2, -3$ #

4.
$$
\begin{array}{r}
x^4+4x^3+16x^2+66x+264 \\
(x-4)\overline{)\,x^5+2x^2+4} \\
\underline{x^5-4x^4} \\
4x^4+2x^2 \\
\underline{4x^4-16x^3} \\
16x^3+2x^2 \\
\underline{16x^3-64x^2} \\
66x^2+4 \\
\underline{66x^2-264x} \\
264x+4 \\
\underline{264x-1056} \\
1060
\end{array}
$$

$\therefore$ Remainder $=1060$ #

5. (i)
$$
\begin{array}{r}
x^2+(b+2)x+(b+4) \\
(x-2)\overline{)\,x^3+bx^2-bx+4} \\
\underline{x^3-2x^2} \\
(b+2)x^2-bx \\
\underline{(b+2)x^2-2(b+2)x} \\
(b+4)x+4 \\
\underline{(b+4)x-2(b+4)} \\
2b+12
\end{array}
$$

Quotient $=x^2+(b+2)x+(b+4)$, remainder $=2b+12$. #

(ii) For $P(x)$ to be divisible by $(x-2)$, then remainder $=0$. i.e. $2b+12=0$

i.e. $b=-6$ #

(iii) $P(x)=x^3-6x^2+6x+4=(x-2)(x^2-4x-2)$

Now, solve $P(x)=0$

i.e. $(x-2)(x^2-4x-2)=0$

i.e. $x-2=0$ or $x^2-4x-2=0$

$x=2$ $\qquad x=\frac{4\pm\sqrt{16+8}}{2}=\frac{4\pm\sqrt{24}}{2}=2\pm\sqrt{6}$

6. $P(1)=0$ i.e. $1+a+b=0$ ....(1)

$P(2)=0$ i.e. $16+2a+b=0$ ....(2)

$(2)-(1)$

$15+a=0$

$a=-15,\ b=14$ #

7. (i) $P(1)=0$ i.e. $1+a-2-4=0$ i.e. $a=5$ #

(ii) $P(x)=x^3+5x^2-2x-4$

Dividing $P(x)=x^3+5x^2-2x-4$ by $(x-1)$ gives:

$$
\begin{array}{r}
x^2+6x+4 \\
(x-1)\overline{)\,x^3+5x^2-2x-4} \\
\underline{x^3-x^2} \\
6x^2-2x \\
\underline{6x^2-6x} \\
4x-4 \\
\underline{4x-4} \\
0
\end{array}
$$

$\therefore\ P(x)=(x-1)(x^2+6x+4)$ #

8. Let $P(x)=2x^3-3x^2+1$

By trial and error it can be found that: $P(1)=2-3+1=0$.

$\therefore$ $(x-1)$ is a factor of $P(x)$.

Dividing $P(x)$ by $(x-1)$, gives:

$$
\begin{array}{r}
2x^2-x-1 \\
(x-1)\overline{)\,2x^3-3x^2+1} \\
\underline{2x^3-2x^2} \\
-x^2+1 \\
\underline{-x^2+x} \\
-x+1 \\
\underline{-x+1} \\
0
\end{array}
$$

$\therefore\ P(x)=(x-1)(2x^2-x-1)$

$=(x-1)(2x+1)(x-1)=(x-1)^2(2x+1)$

9. If $P(x)=x^2+px+q$, then $P\left(-\frac{c}{2}\right)=0$ (as $(2x+c)$ is a factor).

i.e. $\frac{c^2}{4}-\frac{pc}{2}+q=0$

$c^2-2pc+4q=0$

$\therefore\ c=\frac{2p\pm\sqrt{4p^2-16q}}{2}$

$=p\pm\sqrt{p^2-4q}$ #

10. Remainder $=P(2)=8-6+7=9$ #

11. Let $P(x)=(x-a)(x+a+1)$

Remainder $=P(-a)=(-a-a)(-a+a+1)=-2a$ #

12. Let $P(x)=A(x-1)^2(x+2)$

$P(3)=40$, i.e. $A(4)(5)=40$ i.e. $A=2$

$\therefore\ P(x)=2(x-1)^2(x+2)$ #

13. Let $P(x)=ax^2+bx-14$

$P(1)=-12$ i.e. $a+b-14=-12$

$a+b=2$ ....(1)

$P(-2)=0$ i.e. $4a-2b-14=0$

$2a-b=7$ ....(2)

(2)+(1)

$3a=9$

$a=3,\ b=-1$ #

14. $P(x)=a^2x^4+3x^3+b^2x^2+4abx+4ab$

$P(-1)=10$, i.e. $a^2-3+b^2-4ab+4ab=10$

$a^2+b^2=13$.............(1)

$P(0)=24$, i.e. $4ab=24$

$ab=6$

$b=\frac{6}{a}$ Substituting this into (1) gives:

$a^2+\left(\frac{6}{a}\right)^2=13$

$a^2+\frac{36}{a^2}=13$

$a^4+36=13a^2$

$a^4-13a^2+36=0$

$(a^2-9)(a^2-4)=0$

$\therefore\ a=\pm3,\ \pm2$

$\therefore$ Solution pairs are: $a=3,\ b=2$

$a=-3,\ b=-2$

$a=2,\ b=3$

$a=-2,\ b=-3$ #

15. $P(0)=20$ i.e. $4ab=20$

$ab=5$

$b=\frac{5}{a}$

$P\left(\frac{1}{a}\right)=0$ i.e. $10a^2\left(\frac{1}{a}\right)^4-30a\left(\frac{1}{a}\right)^3-8ab\left(\frac{1}{a}\right)+4ab=0$

i.e. $\frac{10}{a^2}-\frac{30}{a^2}-8b+4ab=0$

Substituting $b=\frac{5}{a}$ gives: $-\frac{20}{a^2}-\frac{40}{a}+20=0$

i.e. $20a^2-40a-20=0$

$a^2-2a-1=0$

$a=\frac{2\pm\sqrt{4+4}}{2}$

$=\frac{2\pm2\sqrt{2}}{2}$

$=1\pm\sqrt{2}$

$\therefore$ Either $a=1+\sqrt{2}$ and $b=\frac{5}{1+\sqrt{2}}$,

or $a=1-\sqrt{2}$ and $b=\frac{5}{1-\sqrt{2}}$ #

16. (i) $P(0)=12$, i.e. $d=12$ #

(ii) Since $P(x)$ is monic, $a=1$.

... (x^2+1) gives:

$$\begin{array}{r} x+b \\ (x^2+1)\overline{\big)\,x^3+bx^2+cx+12} \\ \underline{x^3+x\qquad\qquad} \\ bx^2-x+cx+12 \\ \underline{bx^2+b\qquad\qquad} \\ (c-1)x+(12-b) \end{array}$$

Now, $c-1=1$ i.e. $c=2$ and $12-b=-3$ i.e. $b=15$

$\therefore\ P(x)=x^3+15x^2+2x+12$ #

17. (i) For odd functions, $P(x)=-P(-x)$

$P(-x)=-ax^3+bx^2-cx+d$

$-P(-x)=ax^3-bx^2+cx-d$

$\therefore$ If $P(x)=-P(-x)$, then $b=d=0$.

i.e. $P(x)=ax^3+cx$ #

(ii) Since $P(x)$ is monic, $a=1$. i.e. $P(x)=x^3+cx$

Also, $P(-3)=0$. i.e. $P(-3)=-27-3c=0$

i.e. $27=-3c$

$c=-9$

$\therefore\ P(x)=x^3-9x$ #

18. (i) For even functions, $P(x)=P(-x)$.

$P(-x)=ax^4-bx^3+cx^2-dx+e$

$\therefore$ If $P(x)=P(-x)$, then $b=d=0$.

i.e. $P(x)=ax^4+cx^2+e$ #

(ii) Since $P(x)$ is a monic quartic, $\therefore$ the coefficient of x^4 is 1 i.e. $a=1$

Also $P(2)=0$, $P(1)=3$, $P(x)=x^4+cx^2+e$

i.e. $P(2)=16+4c+e=0$ ....(1)

and $P(1)=1+c+e=3$ ....(2)

(1)−(2)

$15+3c=-3$

$3c=-18$

$c=-6,\ e=8$

$\therefore\ P(x)=x^4-6x^2+8$ #

(ii) $Deg\left[R(x)\right]<Deg\left[A(x)\right]$ #

(iii) $\dfrac{P(x)}{(x^2-2x-3)}=Q(x)+\dfrac{R(x)}{(x^2-2x-3)}$

Since $Deg(A(x))=2$, $\therefore\ R(x)=ax+b$.

i.e. $P(x)=Q(x)(x^2-2x-3)+ax+b$

$=Q(x)(x+1)(x-3)+ax+b$

Now, $P(-1)=-a+b=6$ ....(1)

$P(3)=3a+b=-2$ ....(2)

(2)−(1)

$4a=-8$

$a=-2,\ b=4$

$\therefore\ R(x)=-2x+4$ #

20. $P(x)=ax^3+bx^2+cx+d$

$P'(x)=3ax^2+2bx+c$

Now, $P(-2)=0$, i.e. $-8a+4b-2c+d=0$

and $P'(-2)=0$, i.e. $12a-4b+c=0$

and $P'(0)=0$, i.e. $c=0$

and $P(0)=6$, i.e. $d=6$

$\therefore\ -8a+4b+6=0$ ....(1)

$12a-4b=0$ ....(2)

(1)+(2)

$4a+6=0$

$a=-\dfrac{3}{2},\ b=3\left(-\dfrac{3}{2}\right)=-\dfrac{9}{2}$

$\therefore\ P(x)=-\dfrac{3}{2}x^3-\dfrac{9}{2}x^2+6$ #

21. $P(x)=(x-a)^2Q(x)$

$P(a)=0\quad\therefore\ x=a$ is a root of $P(x)$

Now, $P'(x)=(x-a)^2Q'(x)+Q(x).2(x-a)$

$=(x-a)\left[(x-a)Q'(x)+2Q(x)\right]$

... is a double root of $P(x)$. #

22. $P(x)=2x^3-6x^2-18x-10$, by trial and error,

$P(-1)=-2-6+18-10=0$

$\therefore$ $(x+1)$ is a factor of $P(x)$.

Dividing $P(x)$ by $(x+1)$, gives:

$$\begin{array}{r}
2x^2-8x-10 \\
(x+1)\overline{\big)\,2x^3-6x^2-18x-10} \\
\underline{2x^3+2x^2} \qquad\qquad \\
-8x^2-18x \qquad \\
\underline{-8x^2-8x} \qquad \\
-10x-10 \\
\underline{-10x-10} \\
0
\end{array}$$

$\therefore$ $P(x)=(x+1)(2x^2-8x-10)$

$=(x+1)(2x+2)(x-5)$

$=2(x+1)^2(x-5)$ #

23. (i) $\alpha+\beta+\gamma=-\frac{b}{a}=2$ #

(ii) $\alpha\beta\gamma=-\frac{d}{a}=\frac{1}{2}$ #

(iii) $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=3$ #

(iv) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=(2)^2-2(3)=-2$ #

(v) $(\alpha+1)(\beta+1)(\gamma+1)=(\alpha\beta+\alpha+\beta+1)(\gamma+1)$

$=\alpha\beta\gamma+\alpha\beta+\alpha\gamma+\alpha+\beta\gamma+\beta+\gamma+1$

$=\alpha\beta\gamma+(\alpha\gamma+\beta\gamma+\alpha\beta)+(\alpha+\beta+\gamma)+1$

$=\frac{1}{2}+(3)+(2)+1=6\frac{1}{2}$ #

24. Let the roots be α, β and γ:

$\therefore$ $\alpha+\beta+\gamma=-\frac{b}{a}=-a$ ....(1)

$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=0$ ....(2)

$\alpha\beta\gamma=-\frac{d}{a}=b$ ....(3)

Now, let $\alpha=\beta+\gamma$ (one root sum of the other two)

$\therefore$ the above equations become:

$2\alpha=-a$ ....(1)

$\alpha\beta+\alpha\gamma+\beta\gamma=0$ ....(2)

$\alpha\beta\gamma=b$ ....(3)

From (1) $\alpha=-\frac{a}{2}$, subtituting this into (3) gives: $-\frac{a}{2}(\beta\gamma)=b$

i.e. $\beta\gamma=-\frac{2b}{a}$

Rearranging (2): $\alpha(\beta+\gamma)+\beta\gamma=0$

$$\alpha(\alpha)+\left(-\frac{2b}{a}\right)=0$$

$$\alpha^2-\frac{2b}{a}=0$$

$$\left(-\frac{a}{2}\right)^2-\frac{2b}{a}=0$$

$$\frac{a^2}{4}=\frac{2b}{a}$$

$$\therefore \quad a^3-8b=0 \text{ \#}$$

25. (i) Let the roots of $P(x)$ be $\alpha, \alpha, 4\alpha$.

i.e. $\alpha\times\alpha\times4\alpha=-\frac{2}{4}$ and $\alpha+\alpha+4\alpha=-\frac{b}{4}$

$4\alpha^3=-\frac{1}{2}$ $\qquad$ $6\alpha=-\frac{b}{4}$

$\alpha^3=-\frac{1}{8}$ $\qquad$ i.e. $6\times-\frac{1}{2}=-\frac{b}{4}$

$\alpha=-\frac{1}{2}$ $\qquad$ $b=12$ #

(ii) Roots of $P(x)$ are $-\frac{1}{2}, -\frac{1}{2}, -2$

$\therefore$ $P(x)=4\left(x+\frac{1}{2}\right)^2(x+2)$ #

26. Let the roots be $a-d, a, a+d$, then $(a-d)+(a)+(a+d)=-\frac{b}{a}=12$

i.e. $3a=12$ i.e. $a=4$

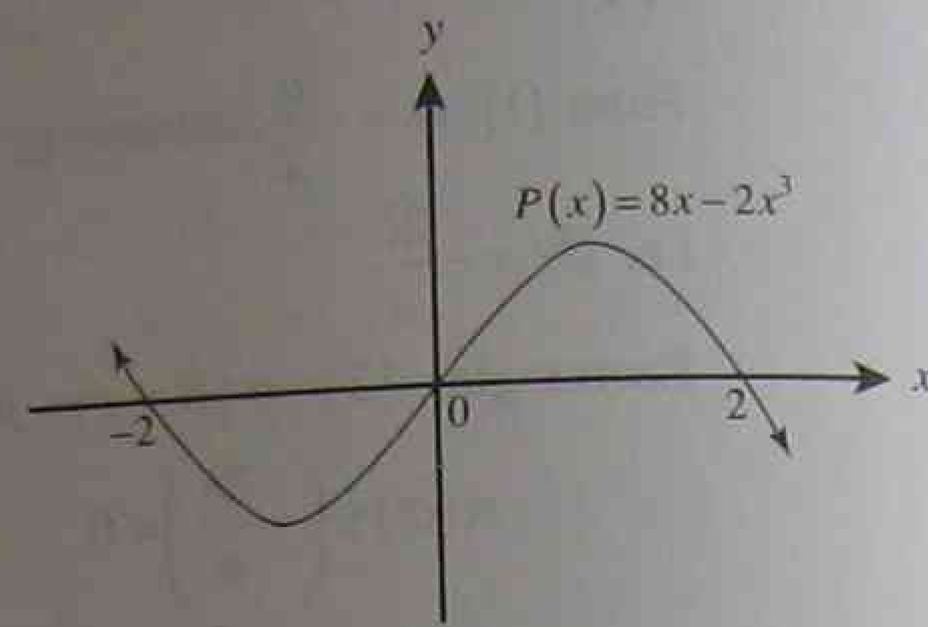
Also, $(a-d)a(a+d)=-\frac{d}{a}=-80$

i.e. $(4-d)4(4+d)=-80$

$$16-d^2=-20$$
$$d^2=36$$
$$d=\pm 6$$

$\therefore$ Roots are $-2, 4, 10.$ #

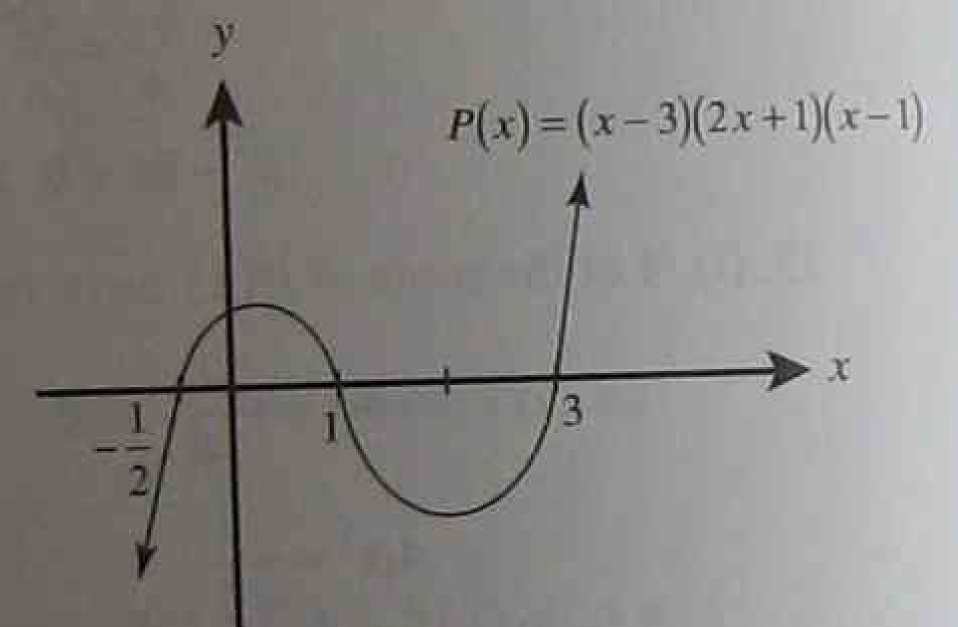
27.
$$8x-2x^3<0$$
$$2x\left(4-x^2\right)<0$$
$$-2x(x-2)(x+2)<0$$



From the graph: $8x-2x^3<0$,
when $-2<x<0$ or $x>2.$ #

28. (i) $P(3)=2(27)-7(9)+2(3)+3=0 \;\therefore\; x=3$ is a root of $P(x)$

(ii)
$$\begin{array}{r}
2x^2-x-1 \\
(x-3)\overline{\big)\,2x^3-7x^2+2x+3} \\
\underline{2x^3-6x^2} \\
-x^2+2x \\
\underline{-x^2+3x} \\
-x+3 \\
\underline{-x+3} \\
0
\end{array}$$



$\therefore\; P(x)=(x-3)\left(2x^2-x-1\right)$
$\qquad\quad =(x-3)(2x+1)(x-1)$ #

(iii) From the graph: $P(x)\geq 0$, when $-\frac{1}{2}\leq x\leq 1$ or $x\geq 3$ #

METHODS OF INTEGRATION

(A) Primitives of $\sin^2 x$ and $\cos^2 x$

Both functions are integrated by using a double-angle formulae,

since $\cos 2x = 2\cos^2 x - 1$

$$\therefore \quad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\therefore \quad \int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]+C$$

Also, $\cos 2x = 1-2\sin^2 x$

$$\therefore \quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\therefore \quad \int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]+C$$

Generalising these results :

$$\int \cos^2 ax\, dx = \frac{1}{2}\int (1+\cos 2ax)\, dx = \frac{1}{2}\left[x+\frac{1}{2a}\sin 2ax\right]+C$$

$$\int \sin^2 ax\, dx = \frac{1}{2}\int (1-\cos 2ax)\, dx = \frac{1}{2}\left[x-\frac{1}{2a}\sin 2ax\right]+C$$

Example 1: Find:

(i) $\displaystyle\int_0^{\pi} 2\sin^2 x\, dx$ (ii) $\displaystyle\int \cos^2 2x\, dx$

(iii) $\displaystyle\int_0^{\frac{\pi}{16}} 4\cos^2 4x\, dx$ (iv) $\displaystyle\int 3\sin^2 3x\, dx$

Solution 1:

(i) $$\int_0^{\pi} 2\sin^2 x\, dx = 2\int_0^{\pi}\frac{1}{2}(1-\cos 2x)\, dx$$
$$=\int_0^{\pi}(1-\cos 2x)\, dx = \left[x-\frac{1}{2}\sin 2x\right]_0^{\pi} = (\pi-0)-(0-0)=\pi \text{ \#}$$

(ii) $\int \cos^2 2x\, dx = \frac{1}{2}\int (1+\cos 4x)\, dx$

$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right] + C$ #

(iii) $\int_0^{\frac{\pi}{16}} 4\cos^2 4x\, dx = 4\int_0^{\frac{\pi}{16}} \frac{1}{2}(1+\cos 8x)\, dx$

$= 2\int_0^{\frac{\pi}{16}} (1+\cos 8x)\, dx$

$= 2\left[x + \frac{1}{8}\sin 8x\right]_0^{\frac{\pi}{16}}$

$= 2\left[\left(\frac{\pi}{16} + \frac{1}{8}\right) - (0+0)\right] = \frac{\pi}{8} + \frac{1}{4}$ #

(iv) $\int 3\sin^2 3x\, dx = 3\int \frac{1}{2}(1-\cos 6x)\, dx = \frac{3}{2}\left[x - \frac{1}{6}\sin 6x\right] + C$ #

(B) Integration by Substitution

Difficult integrals may be integrated using a suitable substitution $u = g(x)$:

$$\int f(x)\, g'(x)\, dx$$

Step 1 : Substitute $u = g(x)$ and $du = g'(x)dx$ to obtain: $\int f(u)\, du$

Step 2 : Integrate with respect to u.

Step 3 : Replace u by $g(x)$ in the final result.

When a substitution is used to evaluate a definite integral, the limits of integration must be changed to refer to the new variable.

Example 1: Use the substitutions given to evaluate the following integrals:

(i) $\int \frac{2x^3}{\sqrt{1+x^4}}\, dx$, using the substitution $u = 1 + x^4$.

(ii) $\int_e^{e^2} \frac{1+\log_e x}{x\log_e x}\, dx$, using the substitution $u = x\log_e x$.

(iii) $\int_{-1}^{1} x^2\sqrt{3x+5}\, dx$, using the substitution $u = 3x+5$.

(iv) $\int_0^2 \frac{dx}{\left(x^2+4\right)^2}$, using the substitution $x = 2\tan\theta$.

Solution 1:

(i) $\int \frac{2x^3}{\sqrt{1+x^4}}\, dx$

$u = 1 + x^4$, $du = 4x^3 dx$ i.e. $\frac{du}{2} = 2x^3 dx$

$\therefore \int \frac{2x^3}{\sqrt{1+x^4}}\, dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int u^{-\frac{1}{2}} du$

$= u^{\frac{1}{2}} + C$

$= \sqrt{1+x^4} + C$ #

(ii) $\int_e^{e^2} \frac{1+\log_e x}{x\log_e x}\, dx$

$u = x\log_e x$, $du = x.\frac{1}{x} + \log_e x.1 = (1+\log_e x)dx$

when $x = e$, $u = e$

when $x = e^2$, $u = 2e^2$

$\therefore \int_e^{2e^2} \frac{1+\log_e x}{x\log_e x}\, dx = \int_e^{2e^2} \frac{du}{u}$

$= \left[\log_e u\right]_e^{2e^2}$

$= \log_e 2e^2 - 1$

$= \log_e 2 + \log_e e^2 - 1$

$= \log_e 2 + 2\log_e e - 1$

$= \log_e 2 + 1$ #

(iii) $\int_{-1}^{1} x^2\sqrt{3x+5}\, dx$

$u = 3x+5$, $du = 3dx$ and $x = \frac{u-5}{3}$

when $x=-1,\ u=2$

when $x=1,\ u=8$

$$\therefore \int_{-1}^{1} x^2\sqrt{3x+5}\ du = \int_{2}^{8}\left(\frac{u-5}{3}\right)^2 \sqrt{u}\ du$$

$$= \frac{1}{9}\int_{2}^{8}\left(u^2-10u+25\right)\sqrt{u}\ du$$

$$= \frac{1}{9}\int_{2}^{8} u^{\frac{5}{2}} - 10u^{\frac{3}{2}} + 25u^{\frac{1}{2}}\ du$$

$$= \frac{1}{9}\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{20}{5}u^{\frac{5}{2}} + \frac{50}{3}u^{\frac{3}{2}}\right]_2^8$$

$$= \frac{1}{9}\left[\begin{array}{l}\left(\frac{2}{7}\left(1024\sqrt{2}\right) - \frac{20}{5}\left(128\sqrt{2}\right) + \frac{50}{3}\left(16\sqrt{2}\right)\right) \\ -\left(\frac{2}{7}\left(8\sqrt{2}\right) - \frac{20}{5}\left(4\sqrt{2}\right) + \frac{50}{3}\left(2\sqrt{2}\right)\right)\end{array}\right]$$

$$= \frac{1}{9}\left[\frac{580}{21}\sqrt{2}\right] = \frac{580}{189}\sqrt{2}\ \#$$

(iv) $\displaystyle\int_0^2 \frac{dx}{\left(x^2+4\right)^2}$

$x = 2\tan\theta \quad dx = 2\sec^2\theta\ d\theta$

when $x=0,\ 2\tan\theta = 0$ and $\theta = 0$

when $x=2,\ 2\tan\theta = 2$ and $\theta = \dfrac{\pi}{4}$

$$\therefore \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{\left(4\tan^2\theta+4\right)^2}\ d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{\left(4\sec^2\theta\right)^2}\ d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{16\sec^4\theta}\ d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\ d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} \cos^2\theta\ d\theta$$

$$= \frac{1}{8}\times\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1+\cos 2\theta\right) d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{16}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{64} + \frac{1}{32}\ \#$$

(C) Other Methods and Integrals

Example 1: Find:

(i) $\int 3\sin^2 x\cos x\ dx$ (ii) $\int \cos^3 x\sin x\ dx$

Solution 1:

(i) $\int 3\sin^2 \cos x\ dx$ (ii) $\int \cos^3 x\sin x\ dx$

$= \sin^3 x + C\ \#$ $\quad = -\dfrac{1}{4}\cos^4 x + C\ \#$

Example 2: Use the table of standard integrals to evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sec 4x\tan 4x\ dx$.

Solution 2:

$$\int_0^{\frac{\pi}{6}} \sec 4x\tan 4x\ dx = \left[\frac{1}{4}\sec 4x\right]_0^{\frac{\pi}{16}}$$

$$= \frac{1}{4}\left[\frac{1}{\cos 4x}\right]_0^{\frac{\pi}{16}}$$

$$= \frac{1}{4}\left[\frac{1}{\frac{1}{\sqrt{2}}} - \frac{1}{1}\right] = \frac{1}{4}\left(\sqrt{2}-1\right)\ \#$$

Example 3: Find $\frac{d}{dx}(x\log_e x - x)$, hence show that $\int_2^4 \log_e x\, dx = \log_e 64 - 2$.

Solution 3:

$$\frac{d}{dx}(x\log_e x - x) = x.\frac{1}{x} + \log_e x \cdot 1 - 1 = \log_e x$$

Integrating both sides from $x = 2$ to $x = 4$, gives:

$$\int_2^4 \frac{d}{dx}(x\log_e x - x)\,dx = \int_2^4 \log_e x\, dx$$

$$\therefore \int_2^4 \log_e x\, dx = \left[x\log_e x - x\right]_2^4$$

$$= (4\log_e 4 - 4) - (2\log_e 2 - 2)$$

$$= \log_e 4^4 - \log_e 2^2 - 2$$

$$= \log_e \frac{256}{4} - 2$$

$$= \log_e 64 - 2 \text{ \#}$$

REVIEW EXERCISES

(A) Primitives of $\sin^2 x$ and $\cos^2 x$

1. Find the following integrals:

(i) $\int 2\cos^2 x\, dx$ (ii) $\int 4\sin^2 3x\, dx$

(iii) $\int_0^{\frac{\pi}{4}} 4\cos^2 6x\, dx$ (iv) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 2x\, dx$

2. Find the volume generated when the curve $y = 1 + \sin 2x$ is rotated about the x-axis between $x = \frac{\pi}{8}$ and $x = \frac{3\pi}{8}$.

3. (i) Given that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, show that $\sin^4 x = \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$

(ii) Hence, find the volume generated when the area bound by the curves $y = \sin x$ and $y = \sin^2 x$ between the lines $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

(B) Integration by Substitution

4. (i) $\int x\sqrt{1-x^2}$, $u = 1 - x^2$ (ii) $\int\left(1 - \cos\frac{x}{2}\right)\sin\frac{x}{2}\, dx$, $u = \cos$

(iii) $\int \frac{e^{2x}}{1+e^x}\, dx$, $u = 1 + e^x$ (iv) $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$, $x = 3\sin\theta$

(v) $\int_0^1 x^3\sqrt{1-x^2}\, dx$, $u = \sqrt{1-x^2}$ (vi) $\int_1^e \frac{dx}{x\sqrt{1+\ln x}}$, $u = 1 + \ln x$

(vii) $\int_4^9 \frac{x}{\sqrt{x}-1}\, dx$, $u = \sqrt{x} - 1$ (viii) $\int_0^2 \sqrt{4-x^2}$, $x = 2\cos\theta$

(C) Other Methods and Integrals

5. Find:

(i) $\int \sin^4 x \cos x\, dx$ (ii) $\int \sin x \cos x\, dx$

6. Use the table of standard integrals to evaluate $\displaystyle\int_6^{15} \frac{dx}{\sqrt{x^2+64}}$

7. (i) Differentiate xe^x and hence evaluate $\displaystyle\int_0^1 xe^x\,dx$.

(ii) Using the result from (i) or otherwise, find the volume of the solid formed when the region bounded by the x-axis and the curve $y = x + e^x$ between $x = 0$ and $x = 1$ is rotated about the x-axis.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $\displaystyle\int 2\cos^2 x\,dx$

$$= 2\int \frac{1}{2}(1+\cos 2x)\,dx = \int (1+\cos 2x)\,dx = x + \frac{1}{2}\sin 2x + C \ \#$$

(ii) $\displaystyle\int 4\sin^2 3x\,dx$

$$= 4\int \frac{1}{2}(1-\cos 6x)\,dx = 2\int (1-\cos 6x)\,dx = 2\left(x - \frac{1}{6}\sin 6x\right) + C$$

$$= 2x - \frac{1}{3}\sin 6x + C \ \#$$

(iii) $\displaystyle\int_0^{\frac{\pi}{4}} 4\cos^2 6x\,dx$

$$= 4\int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 12x)\,dx = 2\int_0^{\frac{\pi}{4}} (1+\cos 12x)\,dx = 2\left[x + \frac{1}{12}\sin 12x\right]_0^{\frac{\pi}{4}} = 2\left[\frac{\pi}{4} - 0\right] =$$

(iv) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 2x\,dx$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\cos 4x)\,dx = \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - 0\right)\right] = \frac{\pi}{8} \ \#$$

2. $$V = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \pi y^2\,dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \pi(1+\sin 2x)^2\,dx$$

$$= \pi\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(1 + 2\sin 2x + \sin^2 2x\right)dx$$

$$= \pi\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(1 + 2\sin 2x + \frac{1}{2}(1-\cos 4x)\right)dx$$

$$= \pi \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\frac{3}{2} + 2\sin 2x - \frac{1}{2}\cos 4x\right) dx$$

$$= \pi \left[\frac{3}{2}x - \cos 2x - \frac{1}{8}\sin 4x\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$= \pi \left[\left(\frac{9\pi}{16} + \frac{1}{\sqrt{2}} + \frac{1}{8}\right) - \left(\frac{3\pi}{16} - \frac{1}{\sqrt{2}} - \frac{1}{8}\right)\right]$$

$$= \pi \left[\frac{3\pi}{8} + \sqrt{2} + \frac{1}{4}\right] = \frac{\pi\left(3\pi + 8\sqrt{2} + 2\right)}{8} \text{ units}^3 \ \#$$

3. (i) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\sin^4 x = \frac{1}{4}(1 - \cos 2x)^2$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right] \qquad \left(\cos^2 2x = \frac{1}{2}(1 + \cos 4x)\right)$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right] = \frac{1}{8}\left[3 - 4\cos 2x + \cos 4x\right] \ \#$$

(ii) $V = \pi \int_0^{\frac{\pi}{2}} \left(y_1^{\,2} - y_2^{\,2}\right) dx$, where $y_1 = \sin x$ and $y_2 = \sin^2 x$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\sin^2 x - \sin^4 x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}(1 - \cos 2x) - \frac{1}{8}(3 - 4\cos 2x + \cos 4x)\right) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{\pi}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 4x)\, dx$$

$$= \frac{\pi}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \frac{\pi^2}{16} \text{ units}^3 \ \#$$

4. (i) $\int x\sqrt{1 - x^2}\, dx$

$u = 1 - x^2$, $du = -2x\, dx$ i.e. $\frac{du}{-2} = x\, dx$

$$\therefore \int x\sqrt{1 - x^2}\, dx = -\frac{1}{2}\int \sqrt{u}\, du = -\frac{1}{2}\int u^{\frac{1}{2}}\, du = -\frac{1}{2} \times \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}\left(1 - x^2\right)^{\frac{3}{2}} + C_1 \ \#$$

(ii) $\int \left(1 - \cos\frac{x}{2}\right)\sin\frac{x}{2}\, dx$

$u = \cos\frac{x}{2}$, $du = -\frac{1}{2}\sin\frac{x}{2}\, dx$ i.e. $-2du = \sin\frac{x}{2}\, dx$

$$\therefore \int \left(1 - \cos\frac{x}{2}\right)\sin\frac{x}{2}\, dx = -2\int (1 - u)\, du = -2\left(u - \frac{u^2}{2}\right) + C$$

$$= -2\cos\frac{x}{2} + \cos^2\frac{x}{2} + C_1 \ \#$$

(iii) $\int \frac{e^{2x}}{1 + e^x}\, dx$

$u = 1 + e^x$, $du = e^x dx$ and $e^x = u - 1$

$$\therefore \int \frac{e^x.e^x}{1 + e^x}\, dx = \int \frac{(u - 1)}{u}\, du = \int 1 - \frac{1}{u}\, du$$

$$= u - \ln(u) + C$$

$$= \left(1 + e^x\right) - \ln\left(1 + e^x\right) + C_1 \ \#$$

(iv) $\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$

$x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$\therefore \int \frac{x^2}{\sqrt{9 - x^2}} = \int \frac{9\sin^2\theta . 3\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\, d\theta$$

$$= \int \frac{27\sin^2\theta\cos\theta}{3\sqrt{1 - \sin^2\theta}}\, d\theta$$

$$= 9\int \frac{\sin^2\theta\cos\theta}{\cos\theta}\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta$$

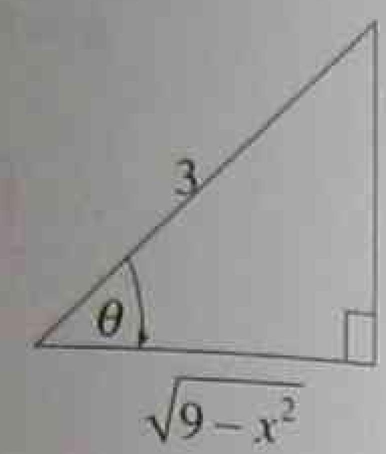
$$= \frac{9}{2}\int (1 - \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \qquad \text{(using } \sin 2\theta = 2\sin\theta\cos\theta\text{)}$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\times\frac{\sqrt{9-x^2}}{3}\right) + C_1$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C_1 \ \#$$



Methods of Integration

(v) $\int_0^1 x^3\sqrt{1-x^2}\ dx$

$u = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$, $du = \frac{-x}{\sqrt{1-x^2}}\ dx$ i.e. $-u\,du = x\,dx$

when $x = 0, u = 1$ and when $x = 1, u = 0$

$$\therefore \int_0^1 x^3\sqrt{1-x^2}\ dx = \int_0^1 x^2 \cdot u \cdot -u\ du = -\int_1^0 (1-u^2)u^2 du \qquad (\text{note: } x^2 = 1-u^2)$$

$$= \int_0^1 u^2 - u^4 du \qquad \left(\text{note:} -\int_b^a = \int_a^b\right)$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \ \#$$

(vi) $\int_1^e \frac{dx}{x\sqrt{1+\ln x}}$

$u = 1 + \ln x$, $du = \frac{dx}{x}$

when $x = 1$, $u = 1$ and when $x = e$, $u = 2$

$$\therefore \int_1^e \frac{dx}{x\sqrt{1+\ln x}} = \int_1^2 \frac{du}{\sqrt{u}} = \int_1^2 u^{-\frac{1}{2}} du = \left[2u^{\frac{1}{2}}\right]_1^2 = 2\left(\sqrt{2}-1\right) \ \#$$

(vii) $\int_4^9 \frac{x}{\sqrt{x}-1} dx$

$u = \sqrt{x} - 1$, $du = \frac{dx}{2\sqrt{x}}$ i.e. $2(u+1)du = dx$ also $x = (u+1)^2$

when $x = 4$, $u = 1$ and when $x = 9$, $u = 2$

$$\therefore \int_4^9 \frac{x}{\sqrt{x}-1}\ dx = \int_1^2 \frac{(u+1)^2 \cdot 2(u+1)}{u} du$$

$$= 2\int_1^2 \frac{(u+1)^3}{u} du$$

$$= 2\int_1^2 \frac{u^3 + 3u^2 + 3u + 1}{u} du$$

$$= 2\int_1^2 \left(u^2 + 3u + 3 + \frac{1}{u}\right) du$$

$$= 2\left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln u\right]_1^2$$

$$= 2\left[\left(\frac{8}{3} + 6 + 6 + \ln 2\right) - \left(\frac{1}{3} + \frac{3}{2} + 3\right)\right]$$

$$= \frac{59}{3} + 2\ln 2 \ \#$$

(viii) $\int_0^2 \sqrt{4-x^2} dx$

$x = 2\cos\theta$, $dx = -2\sin\theta\, d\theta$

when $x = 0$, $2\cos\theta = 0$ i.e. $\theta = \frac{\pi}{2}$

when $x = 2$, $2\cos\theta = 2$ i.e. $\theta = 0$

$$\therefore \int_0^2 \sqrt{4-x^2}\ dx = \int_{\frac{\pi}{2}}^0 \sqrt{4 - 4\cos^2\theta} \cdot -2\sin\theta\ d\theta$$

$$= -4\int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2\theta}\ \sin\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right) d\theta$$

$$= 4\times\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2\theta)\ d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \pi \ \#$$

5. (i) $\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + C$ #

(ii) $\int \sin x \cos x \, dx = \int \frac{\sin 2x}{2} dx$ (using $\sin 2x = 2\sin x \cos x$)

$$= -\frac{1}{4}\cos 2x + C \text{ \#}$$

6. $\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \left[\ln\left(x+\sqrt{x^2+64}\right)\right]_6^{15} = \ln(32) - \ln(16) = \ln\left(\frac{32}{16}\right) = \ln 2$ #

7. (i) $\frac{d}{dx}\left(xe^x\right) = xe^x + e^x$

i.e. $xe^x = \frac{d}{dx}\left(xe^x\right) - e^x$

integrating both sides from $x=0$ to $x=1$, gives:

$$\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 e^x dx$$
$$= (e-0) - \left[e^x\right]_0^1$$
$$= e - (e-1) = 1 \text{ \#}$$

(ii) $V = \int_0^1 \pi y^2 \, dx$

$$= \pi \int_0^1 \left(x+e^x\right)^2 dx$$
$$= \pi \int_0^1 \left(x^2 + 2xe^x + e^{2x}\right) dx$$
$$= \pi\left[\int_0^1 x^2 dx + \int_0^1 2xe^x dx + \int_0^1 e^{2x} dx\right]$$
$$= \pi\left[\left[\frac{x^3}{3}\right]_0^1 + 2\int_0^1 xe^x dx + \left[\frac{e^{2x}}{2}\right]_0^1\right]$$
$$= \pi\left[\frac{1}{3} + 2\times 1 + \left(\frac{e^2}{2} - \frac{1}{2}\right)\right]$$
$$= \pi\left[\frac{1}{3} + 2 + \frac{e^2}{2} - \frac{1}{2}\right] = \pi\left[\frac{1}{3} + \frac{3}{2} + \frac{e^2}{2}\right] = \frac{\pi}{6}\left(2+9+3e^2\right) = \frac{\pi}{6}\left(11+3e^2\right) \text{ units}^3 \text{ \#}$$

Methods of Integration

TOPIC 11

Iterative Methods for Approximating the Roots

There are two methods that can be used to approximate the roots of a polynomial or other equation.

(A) Halving The Interval Method

If $f(x_1)$ and $f(x_2)$ have opposite signs, then at least one root of $f(x)=0$ lies in the interval $x_1 < x < x_2$.

We thus take $x_3 = \frac{x_1+x_2}{2}$ as the first approximation to the root.

Repeated applications are performed to give a closer approximation of the root.

Example 1: Consider the curve $f(x) = x^3 - 5x + 3$.

(i) Show that $f(x) = x^3 - 5x + 3$ has one root between 2 and 1.5.

(ii) Apply the 'halving the interval' method three times to obtain a closer estimate to the root. Express your answer correct to 2 decimal places.

Solution 1:

(i) $f(2) = 2^3 - 5(2) + 3 = 8 - 10 + 3 = 1$

$f(1.5) = 1.5^3 - 5(1.5) + 3 = -1.125$

Since $f(1.5)$ and $f(2)$ have opposite signs, there must be a root between $x = 1.5$ and $x = 2$ #

(ii) Let $x_3 = \frac{2+1.5}{2} = 1.75$

$f(1.75) = -0.390625$ (negative)

$\therefore$ the root of $f(x)$ lies between $x = 1.75$ and $x = 2$.

Let $x_4 = \frac{1.75+2}{2} = 1.875$

$f(1.875) = 0.2168$ (positive)

$\therefore$ the root of $f(x)$ lies between $x = 1.875$ and $x = 1.75$.

Let $x_5 = \dfrac{1.875+1.75}{2} = 1.8125$

$f(1.8125) = -0.1082$ (negative)

$\therefore$ the root of $f(x)$ lies between $x = 1.875$ and $x = 1.8125$

$\therefore$ the approximate value of the root is:

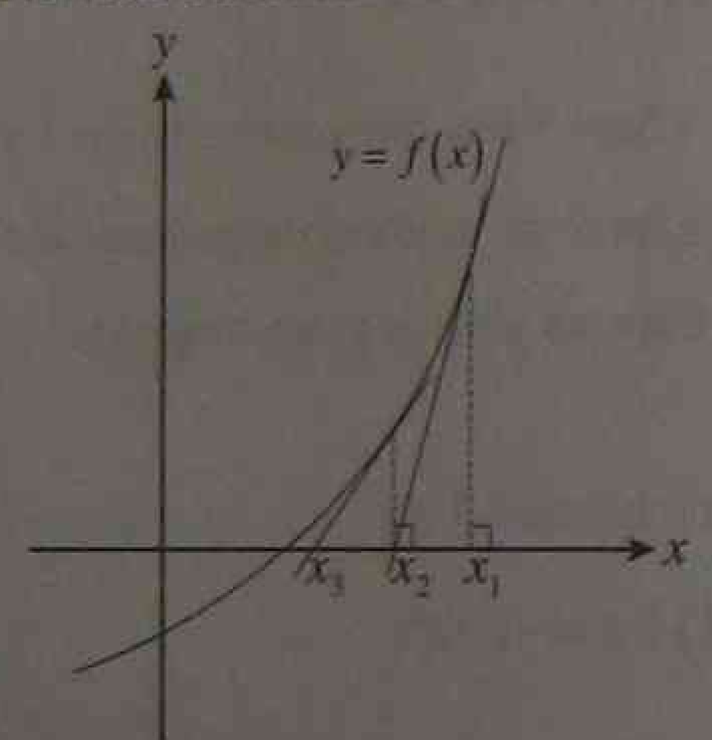
$x = \dfrac{1.875+1.8125}{2} = 1.84$ correct to 2 d.p. #

(B) Newton's Method

If x_1 is an approximation to the root $f(x)=0$ then a better approximation x_2 is obtained as follows:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Graphically Newton's method can be summarised as follows:



A perpendicular is drawn from the inital approximation x_1 to the curve. Where it meets the curve, a tangent is drawn.

The point at which this tangent intersects the x-axis, i.e. x_2 becomes the second approximation. This process is repeated giving better approximations to the root.

Example 1:

Taking $x = 2$ as an approximate root of the equation $x^3 - 7 = 0$. Use one application of Newton's method to find a better approximation. Leave your answer in exact form.

Solution 1:

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x)}$ where $x_1 = 2$

Let $f(x) = x^3 - 7, f(2) = 2^3 - 7 = 1$

$f'(x) = 3x^2,\ f'(2) = 3(2)^2 = 12$

$\therefore\ x_2 = 2 - \dfrac{1}{12} = \dfrac{23}{12}$ #

Example 2:

Take $\theta = 1$ as a first approximation to the root of the equation $\sin\theta + \cos\theta = \theta$. Use Newton's method to find a better approximation to the root correct to two decimal places.

Solution 2:

Let $f(\theta) = \sin\theta + \cos\theta - \theta$

$f'(\theta) = \cos\theta - \sin\theta - 1$

Noting that θ is in radians:

$f(1) = 0.38177329,\ f'(1) = -1.301168679$

$\therefore \theta_2 = \theta_1 - \dfrac{f(\theta)}{f'(\theta)} = 1 - \dfrac{0.38177329}{-1.301168679} = 1.2934...$

$\therefore\ \theta = 1.29$ is a better approximation to the root correct 2 d.p. #

REVIEW EXERCISES

(A) Halving The Interval Method

1. Use three applications of the 'halving the interval method' to find an improved value to the root of the equation $f(x) = 2x - e^{-x}$ which lies between $x = 0.3$ and $x = 0.4$. Express your answer correct to 3 decimal places.

2. Let $f(x) = x^3 - 3x + 1 = 0$.
   (i) Show that $f(x)$ has a root between $x = 1$ and $x = 2$.
   (ii) Use one application of 'halving the interval' method to find a smaller interval containing the root.
   (iii) Which end of the smaller interval in (ii) is a closer approximation to the root. Explain your answer.

3. Let $f(x) = \log_e x + \sin x$.
   (i) Show that $f(x)$ has a root between $x = 0.5$ and $x = 0.6$.
   (ii) Hence, use the 'halving the interval method' to find the value the root correct to one decimal place.

(B) Newton's Method

4. By using one step of Newton's method. Find an approximation (to 2 decimal places) to the root of $f(x) = \dfrac{9}{9x+10} - e^{1.9x}$ which lies close to 0.

5. Use one application of Newton's method to find an improved value to the root of $e^x \ln x - 0.5 = 0$ which lies close to $x = 1.5$ (give your answer to 3 decimal places).

6. Use Newton's Method to show that the root of the equation $x = 2\cos x$ near $\dfrac{\pi}{3}$ is 1.03, correct to two decimal places.

7. Taking $x = 0.2$ as the first approximation, use Newton's method to find a second approximation to the root of $\cos^3(x+2) = -x$, correct to three decimal places.

8. Taking $x = \dfrac{\pi}{6}$ as a first approximation to the root of the equation $2x - \sin 2x = 0$, use Newton's method once, to show that a better approximation to the root is given by $\dfrac{3\sqrt{3} - \pi}{6}$.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $f(0.3) = 2\times 0.3 - e^{-0.3} = -0.14081822$ (negative)

   $f(0.4) = 2\times 0.4 - e^{-0.4} = 0.129679954$ (positive)

   Since $f(0.3)$ and $f(0.4)$ have opposite signs, there must be a root between $x = 0.3$ and $x = 0.4$.

   Let $x_3 = \dfrac{0.3+0.4}{2} = 0.35$

   $f(0.35) = 2\times 0.35 - e^{-0.35} = -0.004688089$ (negative)

   $\therefore$ the root of $f(x)$ lies between $x = 0.35$ and $x = 0.4$.

   Let $x_4 = \dfrac{0.35+0.4}{2} = 0.375$

   $f(0.375) = 2\times 0.375 - e^{-0.375} = 0.06271072$ (positive)

   $\therefore$ the root of $f(x)$ lies between $x = 0.375$ and $x = 0.35$

   Let $x_5 = \dfrac{0.375+0.35}{2} = 0.3625$

   $f(0.3625) = 2\times 0.3625 - e^{-0.3625} = 0.029065686$ (positive)

   $\therefore$ the approximate value of the root is: $x = \dfrac{0.3625+0.35}{2} = 0.356$ correct to 3 d.p. #

2. (i) $f(1) = 1-3+1 = -1$

   $f(2) = 8-6+1 = 3$

   Since $f(1)$ and $f(2)$ have opposite signs, there must be a root between $x = 1$ and $x = 2$. #

   (ii) Let $x_3 = \dfrac{1+2}{2} = 1.5$

   $f(1.5) = (1.5)^3 - 3(1.5) + 1 = -0.125$

   $\therefore$ root lies between $x = 1.5$ and $x = 2$, i.e. $1.5 < x < 2$. #

   (iii) Since $f(1.5)$ is much closer to zero than $f(2)$, then the root must be closer to $x = 1.5$ #

3. (i) $f(0.5) = \log_e(0.5) + \sin(0.5) = -0.213721642$ (negative)

   $f(0.6) = \log_e(0.6) + \sin(0.6) = 0.053816849$ (positive)

   Since $f(0.5)$ and $f(0.6)$ have opposite signs, the root lies between $x = 0.5$ and $x = 0.6$ #

   (ii) Let $x_3 = \dfrac{0.5+0.6}{2} = 0.55$

   $f(0.55) = \log_e(0.55) + \sin(0.55) = -0.075149771$ (negative)

   $\therefore$ root of $f(x)$ lies between $x = 0.6$ and $x = 0.55$.

   $\therefore$ $x = 0.6$ is the value of the root correct to 1 d.p. #

4. $f(x) = \dfrac{9}{9x+10} - e^{1.9x}$

   $= 9(9x+10)^{-1} - e^{1.9x}$

   $f'(x) = \dfrac{-81}{(9x+10)^2} - 1.9e^{1.9x}$

   $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

   $= 0 - \dfrac{f(0)}{f'(0)} = 0 - \dfrac{-0.1}{-2.71} = 0.04$ correct to 2 d.p. #

5. let $f(x) = e^x \ln x - 0.5$

   $f'(x) = e^x \cdot \dfrac{1}{x} + \ln x \cdot e^x = \dfrac{e^x}{x} + \ln x e^x$

   $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

   $= 1.5 - \dfrac{f(1.5)}{f'(1.5)} = 1.5 - \dfrac{1.3171685443}{4.804961257} = 1.226$ correct to 3 d.p. #

6. Let $f(x) = 2\cos x - x$

   $f'(x) = -2\sin x - 1$

   $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \dfrac{\pi}{3} - \dfrac{f\left(\frac{\pi}{3}\right)}{f'\left(\frac{\pi}{3}\right)} = \dfrac{\pi}{3} - \dfrac{-0.047197554}{-2.732050808} = 1.03$ correct to 2 d.p. #

7. Let $f(x)=\cos^3(x+2)+x$

$$f'(x)=-3\cos^2(x+2)\sin(x+2)+1$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$=0.2-\frac{f(0.2)}{f'(0.2)}=0.2-\frac{-0.003817689}{0.159971674}=0.224 \text{ correct to 3 d.p. \#}$$

8. Let $f(x)=2x-\sin 2x$

$$f'(x)=2-2\cos 2x$$

$$f\left(\frac{\pi}{6}\right)=\frac{\pi}{3}-\sin\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-\frac{\sqrt{3}}{2}=\frac{2\pi-3\sqrt{3}}{6}$$

$$f'\left(\frac{\pi}{6}\right)=2-2\cos\left(\frac{\pi}{3}\right)=2-1=1$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$=\frac{\pi}{6}-\frac{f\left(\frac{\pi}{6}\right)}{f'\left(\frac{\pi}{6}\right)}=\frac{\pi}{6}-\frac{2\pi-3\sqrt{3}}{6}=\frac{3\sqrt{3}-\pi}{6} \quad \#$$

INVERSE FUNCTIONS & INVERSE TRIGONOMETRIC FUNCTIONS

Introduction

A function $y=f(x)$ has an inverse function $f^{-1}(x)$, if a horizontal line intersects it in at most only one point (i.e. $f(x)$ is a one-to-one function).

Where the horizontal line intersects the curve in more than one point, then the function will only have an inverse if the domain is suitably restricted.

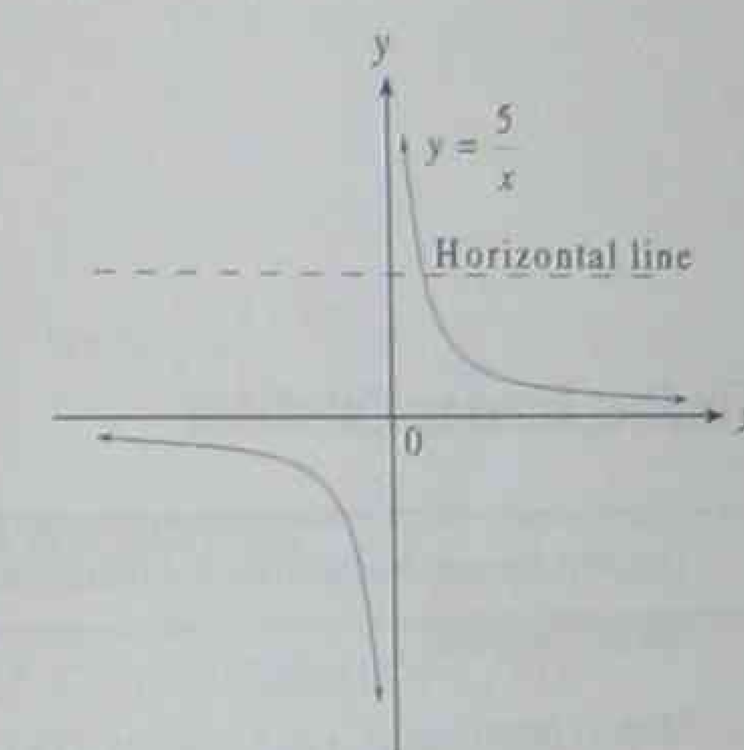
Example 1: For each of the following functions, state whether they have an inverse. If not, find the largest positive domain for which an inverse function exists.

(i) $y=\dfrac{5}{x}$ (ii) $y=\sqrt{4-x^2}$

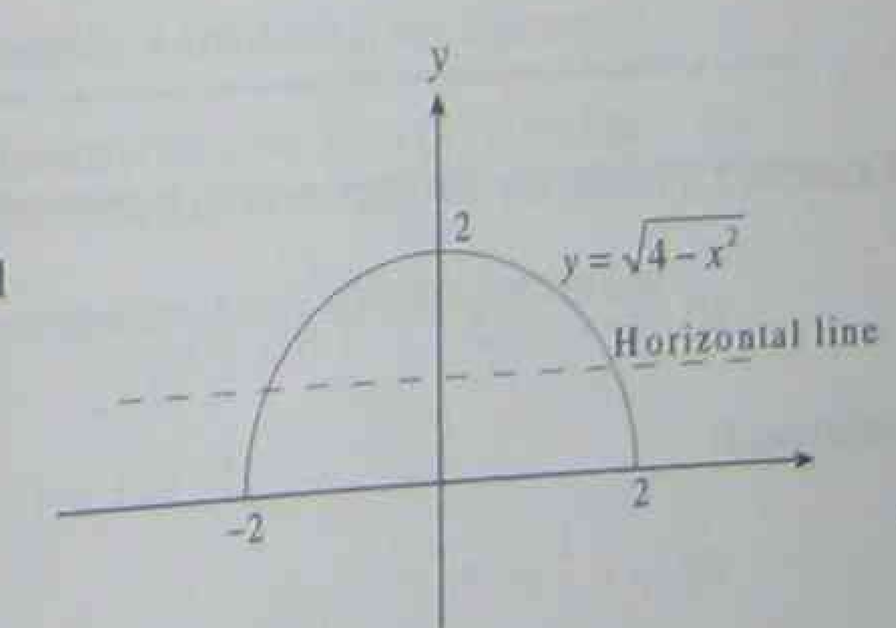
(iii) $y=x^2-6x+9$ (iv) $y=e^x+1$

Solution 1:

(i) From the graph, it can be seen that any horizontal line intersects the curve in no more than one point.



$\therefore\ y=\dfrac{5}{x}$ has an inverse function for all x #



(ii) From the graph, a horizontal line cuts the curve at two points, thus the curve does not have an inverse for $-2\le x\le 2$.

$y=\sqrt{4-x^2}$ is a one-to-one function and thus has an

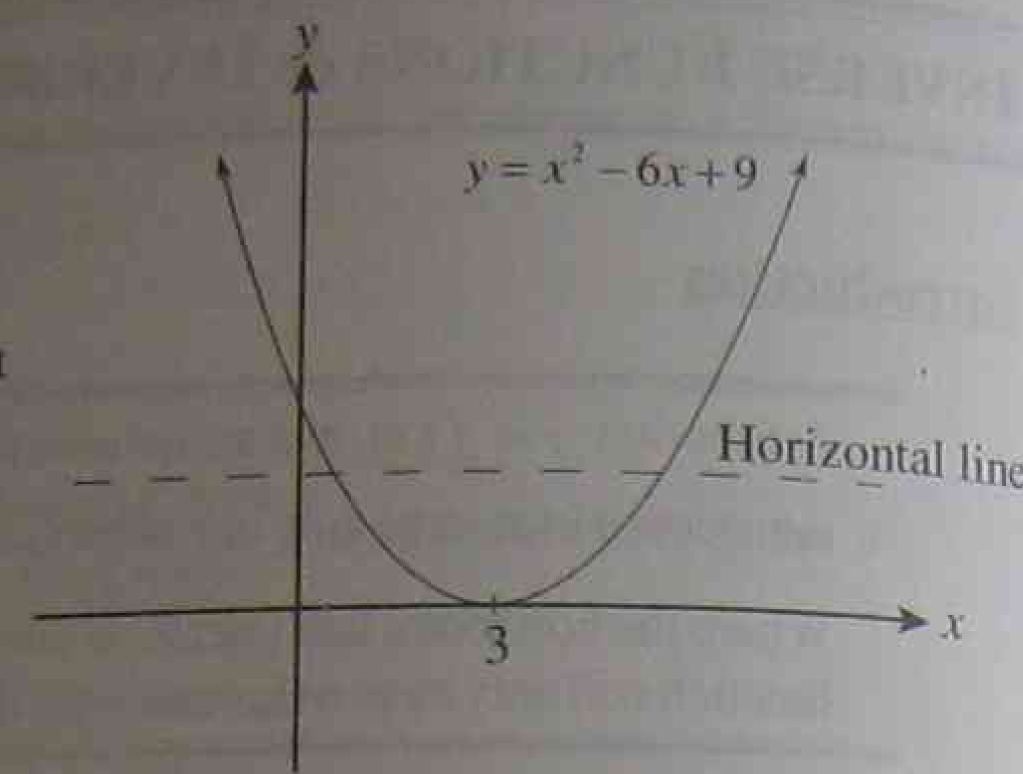
inverse for $x\ge 0$ #

(iii) $y = x^2 - 6x + 9$

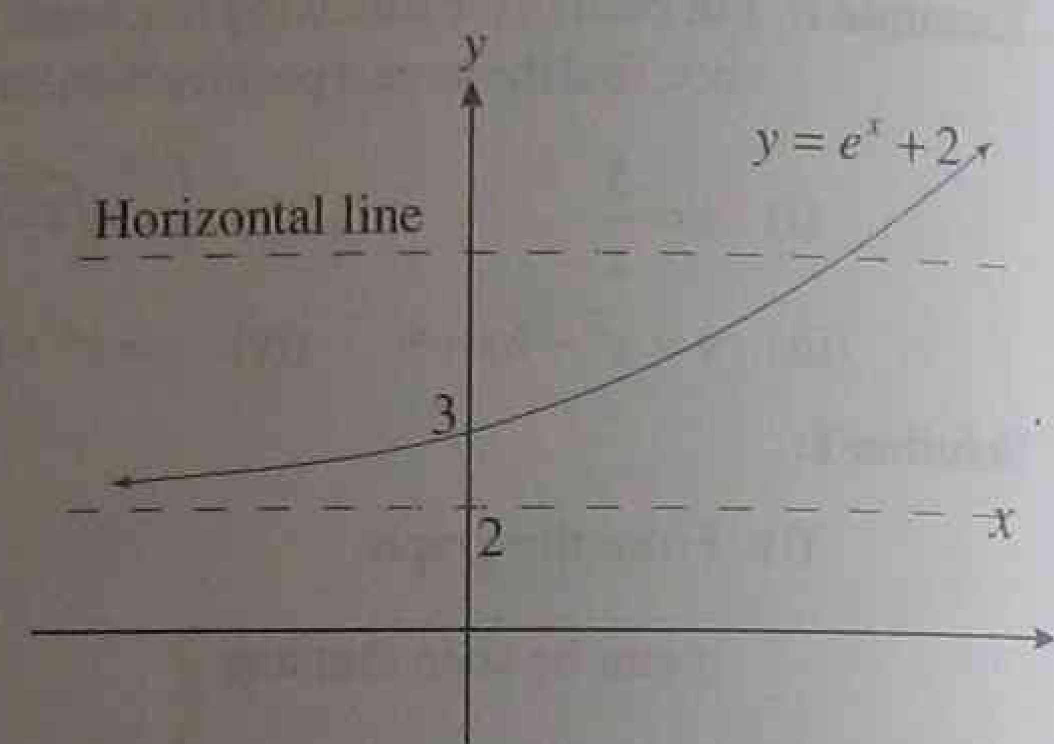
$= (x-3)^2$

From the graph, a horizontal line intersects the curve at two points. Thus the curve does not have an inverse for all x.

$y = x^2 - 6x + 9$ one-to-one function and thus has an inverse for $x \geq 3$ #



(iv) From the graph, any horizontal line cuts the curve in no more than one point. Thus, $y = e^x + 2$ has an inverse function for all x.



(A) Finding Inverse Functions

The step-by-step method below explains how the inverse function of $y = f(x)$ can be obtained:

Step 1: Interchange x and y in $y = f(x)$

Step 2: Rearrange the equation such that y is in terms of x. This is the inverse function and is denoted by $f^{-1}(x)$.

Example 1: Find the inverse function of each of the following:

(i) $y = \frac{1}{4}x - 5$ (ii) $y = \frac{x+1}{1-2x}$ (iii) $y = \frac{e^{2x}-1}{e^{2x}+1}$

Solution 1:

(i) $y = \frac{1}{4}x - 5$

Interchanging x and y to find the inverse of $f(x)$:

$x = \frac{1}{4}y - 5$ i.e. $y = 4(x+5)$ $\therefore f^{-1}(x) = 4(x+5)$ #

(ii) $y = \frac{x+1}{1-2x}$

Interchanging x and y to find the inverse of $f(x)$:

$$x = \frac{y+1}{1-2y}$$

$$x - 2xy = y + 1$$

$$2xy + y = x - 1$$

$$y(2x+1) = x - 1 \qquad \therefore f^{-1}(x) = \frac{x-1}{1+2x} \text{ \#}$$

(iii) $y = \frac{e^{2x}-1}{e^{2x}+1}$

Interchanging x and y to find the inverse of $f(x)$:

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

$$xe^{2y} + x = e^{2y} - 1$$

$$x + 1 = e^{2y}(1-x)$$

$$e^{2y} = \frac{1+x}{1-x}$$

i.e. $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\therefore f^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ #

(B) Inverse Functions Theory and Applications

There are three important features of $f^{-1}(x)$ in relation to $y = f(x)$.

- The **domain** of $f^{-1}(x)$ is the **range** of $y = f(x)$.
- The **range** of $f^{-1}(x)$ is the **domain** of $y = f(x)$.
- The graph of $f^{-1}(x)$ is the reflection of the curve $y = f(x)$ in the line $y = x$.

Example 1: Find the inverse of each of the functions, hence state the domain and range. Sketch both functions.

(i) $y = x^2 - 2,\ x \geq 0$ (ii) $y = \log_e(2x+1)$

Solution 1:

(i) $y = x^2 - 2,\ x \geq 0$ and $y \geq -2$

Interchanging x and y to find $f^{-1}(x)$:

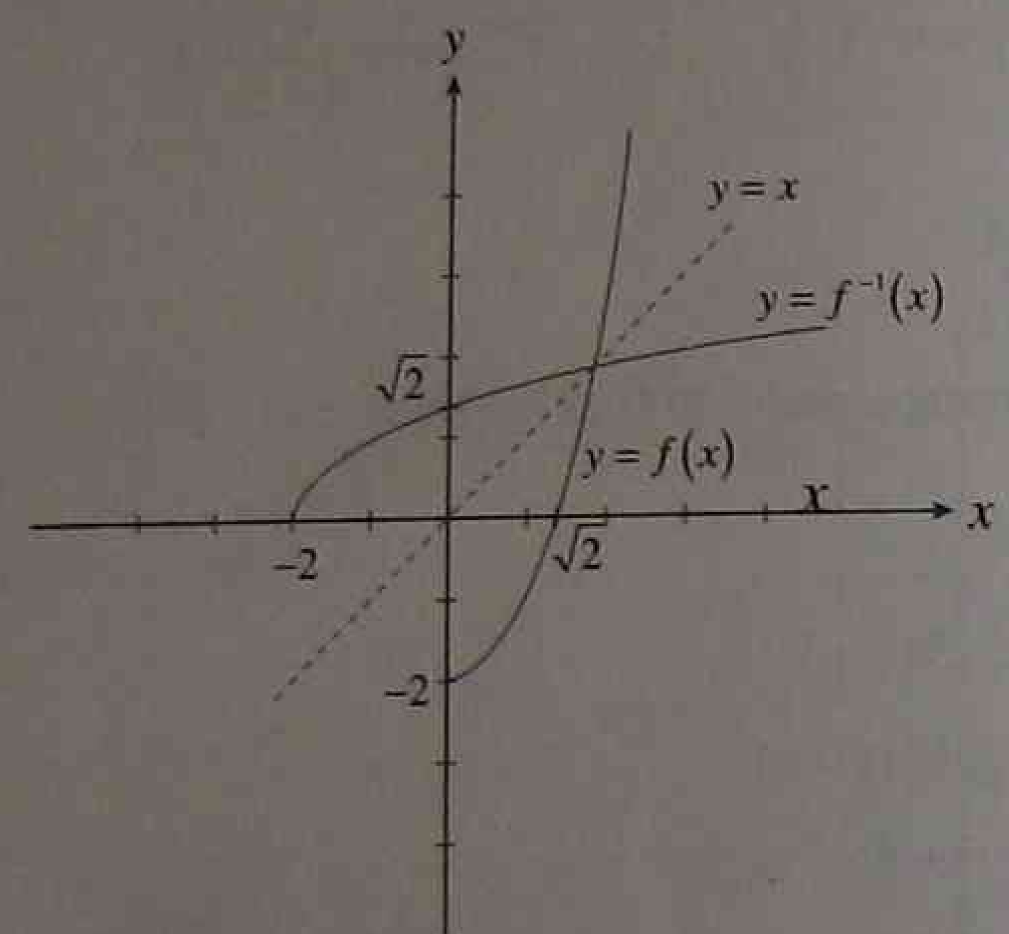
$x = y^2 - 2$

$y^2 = x + 2$

$y = \pm\sqrt{x+2}$, but $y \geq 0$ $\therefore$ $f^{-1}(x) = \sqrt{x+2}$

Domain: $x \geq -2$ $\quad$ (range of $y = x^2 - 2$)

Range: $y \geq 0$ $\quad$ (domain of $y = x^2 - 2$) #



(ii) $y = \log_e(2x+1)$

Domain: $x > -\dfrac{1}{2}$, Range: all real y

Interchanging x and y to find $f^{-1}(x)$:

$x = \log_e(2y+1)$

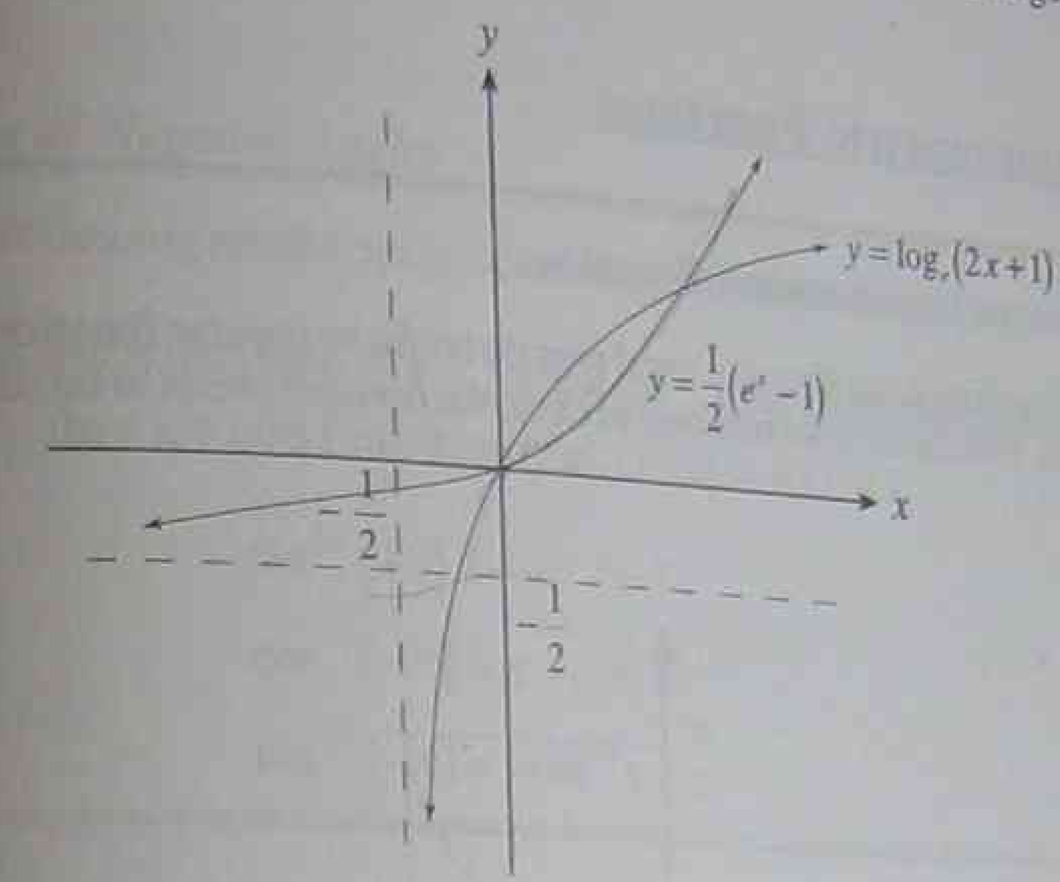
$e^x = 2y+1$

$2y = e^x - 1$

$y = \dfrac{1}{2}(e^x - 1)$

Domain: all real x $\quad$ (range of $y = \log_e(2x+1)$)

Range: $y > -\dfrac{1}{2}$ $\quad$ (domain of $y = \log_e(2x+1)$)#



Example 2: Consider the function $f(x) = x^3 - 2$ for $x \geq 0$.

(i) Find the inverse of $f^{-1}(x)$.

(ii) State the domain of $f^{-1}(x)$.

(iii) Find the slope of the tangent to the inverse function at $x = 6$.

Solution 2:

(i) Let $y = x^3 - 2$.

Interchanging x and y to find $f^{-1}(x)$:

i.e. $x = y^3 - 2$

$y^3 = (x+2)$

$y = (x+2)^{\frac{1}{3}}$ $\quad$ $\therefore$ $f^{-1}(x) = (x+2)^{\frac{1}{3}}$ #

(ii) Domain of $f^{-1}(x)$ is the range of $f(x)$ for $x \geq 0$:

$\therefore$ Domain: $x \geq -2$ $\quad$ (range of $f(x)$)#

(iii) $\dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{1}{3}(x+2)^{-\frac{2}{3}}$

$= \dfrac{1}{3(x+2)^{\frac{2}{3}}}$

at $x = 6$, $\dfrac{d}{dx}\left(f^{-1}(x)\right) = \dfrac{1}{12}$

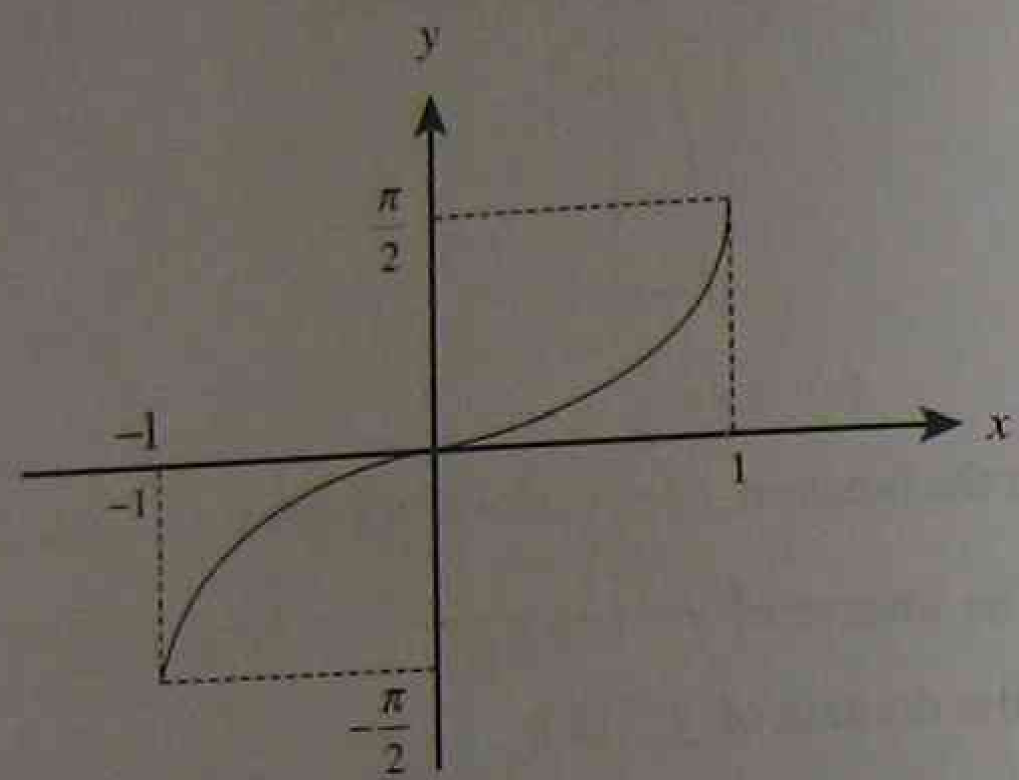
$\therefore$ Slope of tangent to $f^{-1}(x)$ at $x = 6$ is $\dfrac{1}{12}$ #

(C) Inverse Trigonometric Functions

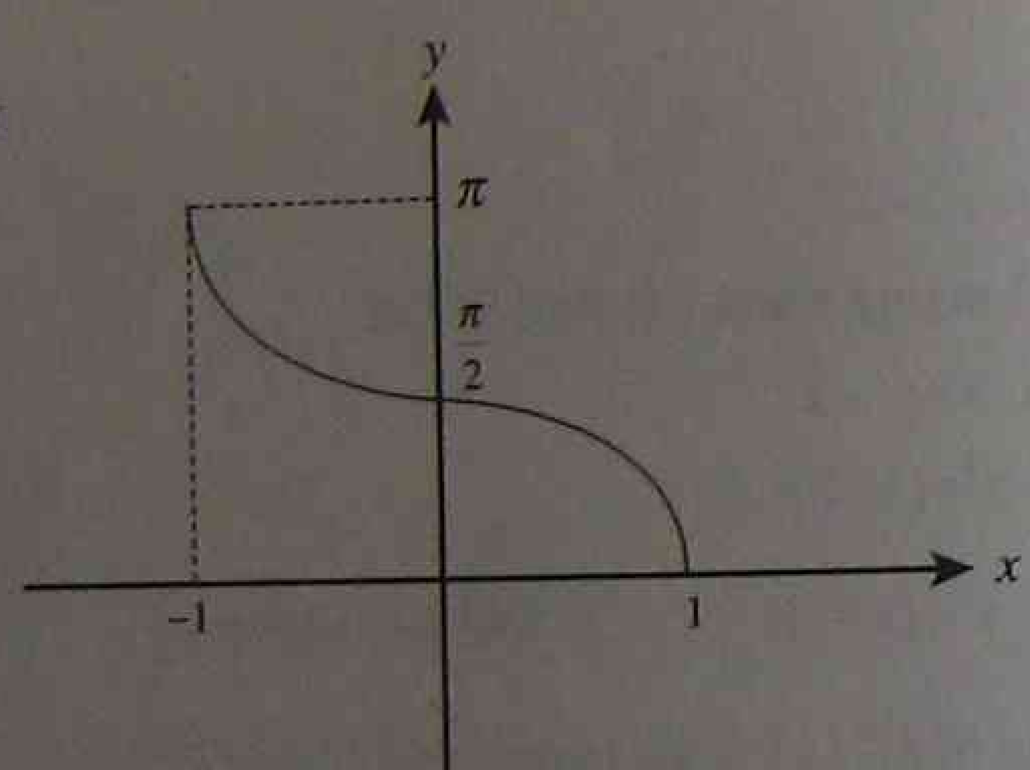
(i) Graphs of Inverse Trigonometric Functions

The trigonometric functions $\sin x$, $\cos x$ and $\tan x$ do not have inverse functions in their natural domains. For the inverse function to exist, the domain needs to be suitably restricted.

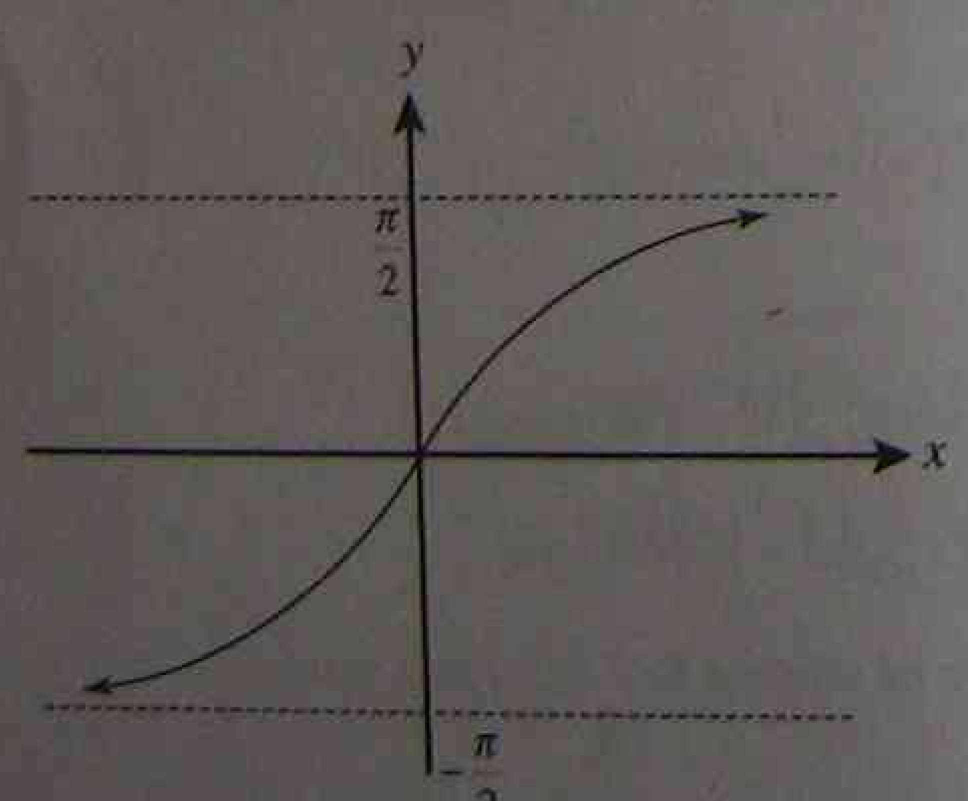
(i) $y=\sin^{-1}x$



(ii) $y=\cos^{-1}x$



(iii) $y=\tan^{-1}x$



(ii) Results of Negative Angles

The following results can be deduced by refering to the graphs.

Note: Since the graph of $\sin^{-1}x$ and $\tan^{-1}x$ are symmetrical about the origin, they are **odd functions** (i.e. $f(-x)=-f(x)$).

$$\sin^{-1}(-x)=-\sin^{-1}x$$
$$\cos^{-1}(-x)=\pi-\cos^{-1}x$$
$$\tan^{-1}(-x)=-\tan^{-1}x$$

Example 1: Find the exact value of the following:

(i) $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$.

(ii) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+2\tan^{-1}\left(-\sqrt{3}\right)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution 1:

(i) $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$

$=-\sin^{-1}\left(\frac{1}{2}\right)+\left(\pi-\cos^{-1}\left(\frac{1}{2}\right)\right)\quad=-\frac{\pi}{6}+\left(\pi-\frac{\pi}{3}\right)=\frac{\pi}{2}$ #

(ii) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+2\tan^{-1}\left(-\sqrt{3}\right)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-2\tan^{-1}\left(\sqrt{3}\right)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$=-\frac{\pi}{3}-2\left(\frac{\pi}{3}\right)+\frac{\pi}{4}\quad=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$ #

Example 2: Find the exact value of the following:

(i) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\right)$

(ii) $\tan\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

(iii) $\cos\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)$

Solution 2:

(i) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)\right)$

$=\sin\left(\frac{\pi}{6}+\left(\pi-\cos^{-1}\left(\frac{1}{2}\right)\right)\right)=\sin\left(\frac{\pi}{6}+\pi-\frac{\pi}{3}\right)=\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$ #

(ii) $\tan\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=\tan\left(2\times\frac{\pi}{6}\right)=\sqrt{3}$ #

(iii) $\cos\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)$

Let $\theta=\cos^{-1}\left(\frac{2}{3}\right)$. i.e. $\cos\theta=\frac{2}{3}$

$\therefore\ \cos\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)=\cos(2\theta)$

$=\cos^2\theta-\sin^2\theta$

$=\cos^2\theta-\left(1-\cos^2\theta\right)$

$=2\cos^2\theta-1$

$=2\left(\frac{2}{3}\right)^2-1=-\frac{1}{9}$ #

Example 3: Evaluate the following expression:

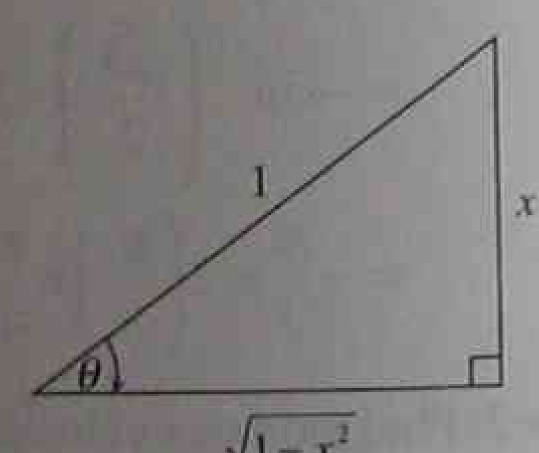
(i) $\cos\left(\sin^{-1}x\right)$ (ii) $\sin\left(\cos^{-1}\frac{t}{4}\right)$

Solution 3:

(i) Let $\theta=\sin^{-1}x$ i.e. $\sin\theta=x$

from the diagram $\cos\theta=\sqrt{1-x^2}$.

$\therefore\ \cos\left(\sin^{-1}x\right)=\cos\theta=\sqrt{1-x^2}$ #

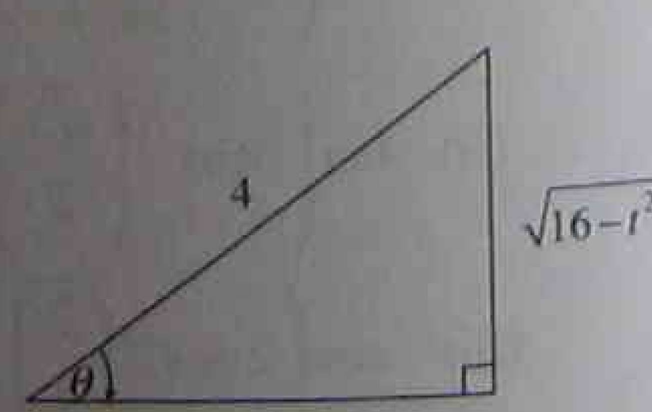


(ii) Let $\theta=\cos^{-1}\left(\frac{t}{4}\right)$ i.e. $\cos\theta=\frac{t}{4}$

from the diagram

$\sin\theta=\sqrt{1-\frac{t^2}{16}}=\frac{\sqrt{16-t^2}}{4}$

$\therefore\ \sin\left(\cos^{-1}\frac{t}{4}\right)=\sin\theta=\frac{\sqrt{16-t^2}}{4}$ #



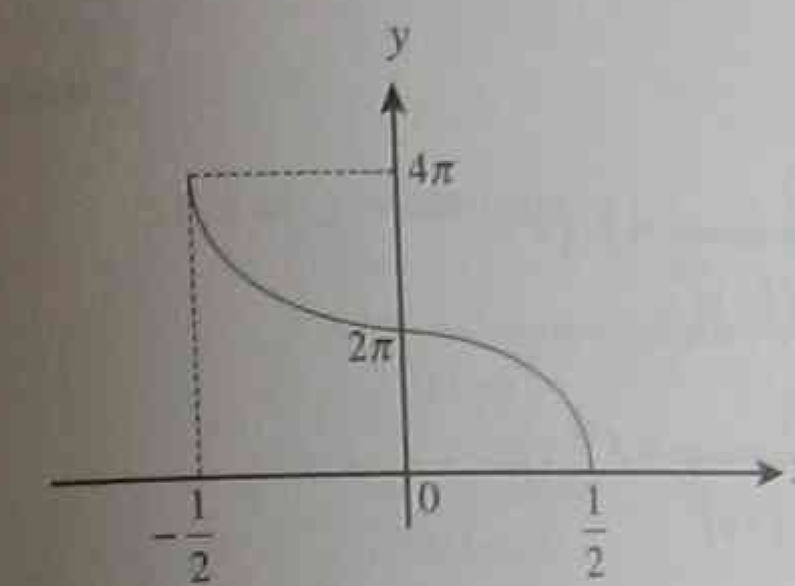
Example 4: Sketch the following inverse trigonometric functions, stating their domain and range:

(i) $y=4\cos^{-1}(2x)$ (ii) $y=3\sin^{-1}\left(\frac{x}{2}\right)$

(iii) $y=4\tan^{-1}(x)$ (iv) $y=2\cos^{-1}\left(\frac{x}{3}\right)$

Solution 4:

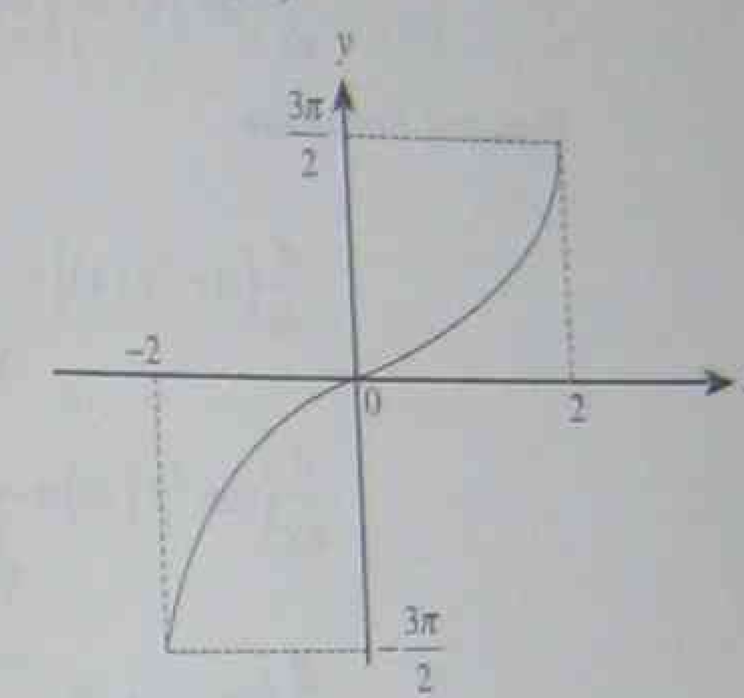
(i) $y=4\cos^{-1}(2x)$



Domain: $-\frac{1}{2}\le x\le\frac{1}{2}$

Range: $0\le y\le 4\pi$

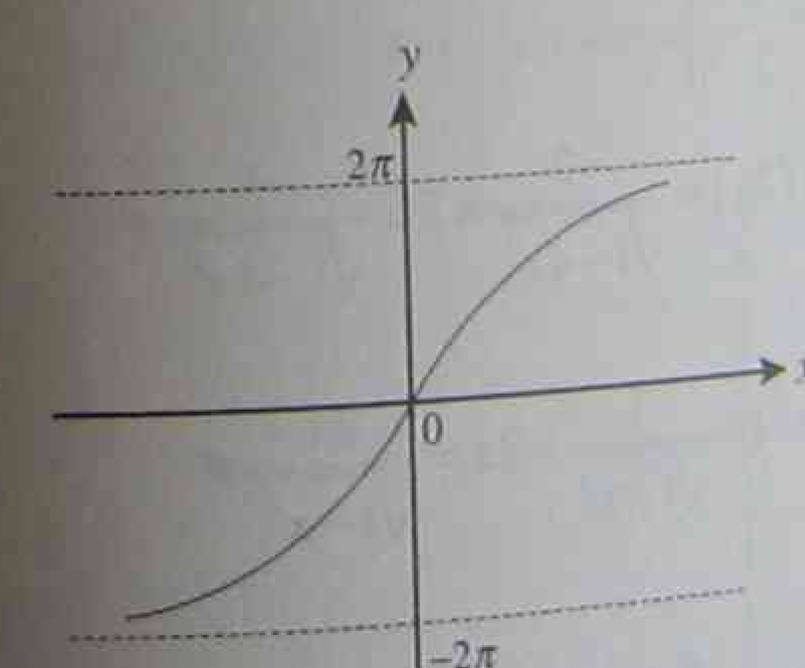
(ii) $y=3\sin^{-1}\left(\frac{x}{2}\right)$



Domain: $-2\le x\le 2$

Range: $-\frac{3\pi}{2}\le y\le\frac{3\pi}{2}$

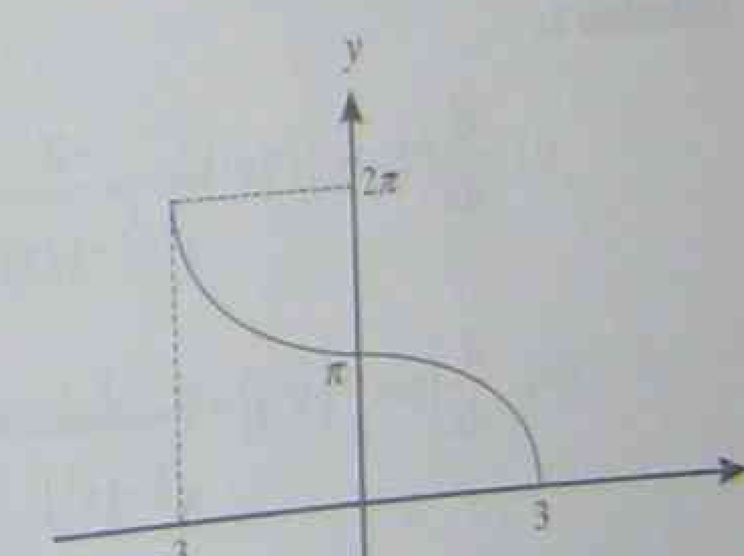
(iii) $y=4\tan^{-1}(x)$



Domain: all real x

Range: $-2\pi<y<2\pi$

(iv) $y=2\cos^{-1}\left(\frac{x}{3}\right)$



Domain: $-3\le x\le 3$

Range: $0\le y\le 2\pi$

(D) Differentiating Inverse Trigonometric Functions

Standard Formulae

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

General Formulae

$$\frac{d}{dx}\left(\sin^{-1}f(x)\right)=\frac{1}{\sqrt{1-[f(x)]^2}}\times f'(x)$$

$$\frac{d}{dx}\left(\cos^{-1}f(x)\right)=\frac{-1}{\sqrt{1-[f(x)]^2}}\times f'(x)$$

$$\frac{d}{dx}\left(\tan^{-1}f(x)\right)=\frac{1}{1+[f(x)]^2}\times f'(x)$$

Example 1: Differentiate the following:

(i) $2\cos^{-1}(2x)$ (ii) $\sin^{-1}(x^2)$

(iii) $\tan^{-1}(e^x)$ (iv) $\tan^{-1}\left(\frac{3}{x}\right)$

Solution 1:

(i) $$\frac{d}{dx}\left(2\cos^{-1}(2x)\right)=\frac{-2}{\sqrt{1-(2x)^2}}\times\frac{d}{dx}(2x)=\frac{-2}{\sqrt{1-4x^2}}\times 2=\frac{-4}{\sqrt{1-4x^2}}\ \#$$

(ii) $$\frac{d}{dx}\left(\sin^{-1}(x^2)\right)=\frac{1}{\sqrt{1-(x^2)^2}}\times\frac{d}{dx}(x^2)=\frac{1}{\sqrt{1-x^4}}\times 2x=\frac{2x}{\sqrt{1-x^4}}\ \#$$

(iii) $$\frac{d}{dx}\left(\tan^{-1}(e^x)\right)=\frac{1}{1+(e^x)^2}\times\frac{d}{dx}(e^x)=\frac{1}{1+e^{2x}}\times e^x=\frac{e^x}{1+e^{2x}}\ \#$$

(iv) $$\frac{d}{dx}\left(\tan^{-1}\left(\frac{3}{x}\right)\right)=\frac{1}{1+\left(\frac{3}{x}\right)^2}\times\frac{d}{dx}\left(\frac{3}{x}\right)$$

$$=\frac{1}{1+\frac{9}{x^2}}\times\frac{d}{dx}\left(3x^{-1}\right)=\frac{x^2}{x^2+9}\times\frac{-3}{x^2}=\frac{-3}{x^2+9}\ \#$$

Example 2: (i) Find $\frac{d}{dx}\left(\cos\left(\sin^{-1}x\right)\right)$, for $-1\le x\le 1$.

(ii) Hence, sketch the curve $y=\cos\left(\sin^{-1}x\right)$, for $-1\le x\le 1$.

Solution 2:

(i) Let $y=\cos\left(\sin^{-1}x\right)$

$$y'=\frac{1}{\sqrt{1-x^2}}\times-\sin\left(\sin^{-1}x\right)$$

$$=\frac{-x}{\sqrt{1-x^2}}\ \#$$

(ii) Let $y'=0$, to find stationary points:

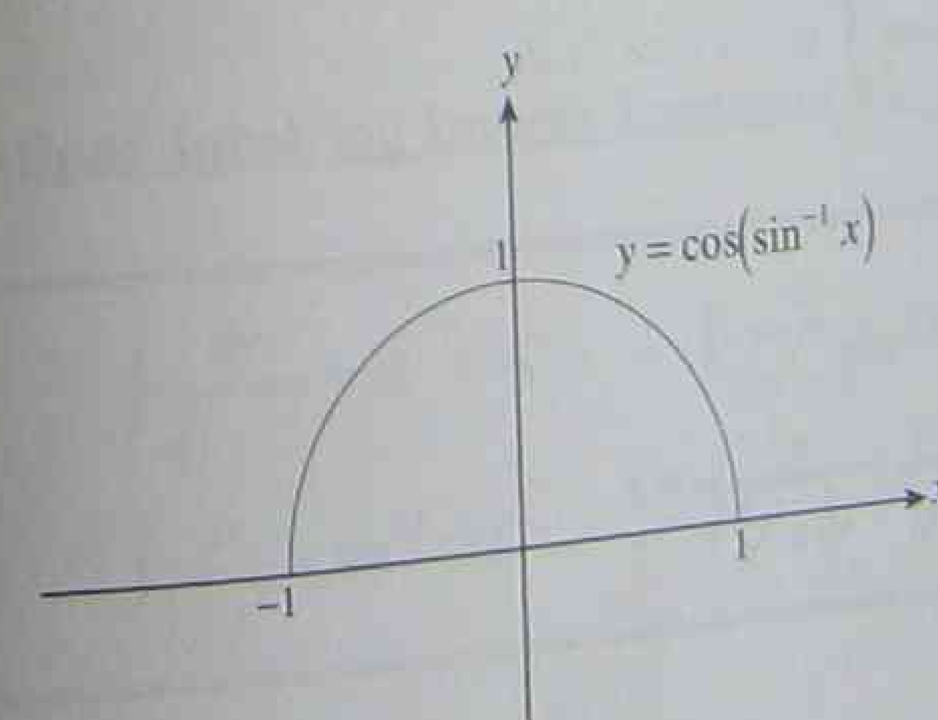
i.e. $\frac{-x}{\sqrt{1-x^2}}=0$ i.e. $x=0$

$\therefore$ $x=0$ is a maximum turning point

when: $x=0,\ y=\cos(0)=1$

$x=1,\ y=\cos\left(\frac{\pi}{2}\right)=0$

$x=-1,\ y=\cos\left(-\frac{\pi}{2}\right)=0$



Example 3: Find the equation of the tangent to the curve $y=\tan^{-1}\sqrt{x}$ at $x=3$.

Solution 3:

$$y=\tan^{-1}\sqrt{x}$$

$$y'=\frac{1}{1+\left(\sqrt{x}\right)^2}\times\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}\,(1+x)}$$

Substituting $x=3$ into y' to find the gradient of the tangent (m):

i.e. $m=\dfrac{1}{2\sqrt{3}\,(1+3)}=\dfrac{1}{8\sqrt{3}}$

at $x=3$, $y=\tan^{-1}\sqrt{3}=\dfrac{\pi}{3}$

$\therefore$ equation of the tangent is given by:

$$(y-y_1)=m(x-x_1)$$

$$\left(y-\frac{\pi}{3}\right)=\frac{1}{8\sqrt{3}}(x-3)$$

$$y-\frac{\pi}{3}=\frac{1}{8\sqrt{3}}x-\frac{\sqrt{3}}{8}$$

$$\therefore \quad y=\frac{1}{8\sqrt{3}}x+\frac{\pi}{3}-\frac{\sqrt{3}}{8}\ \#$$

Example 4: If $f(x)=\sin^{-1}x+\cos^{-1}x,\ -1\le x\le 1$.

(i) Show that $f'(x)=0$.

(ii) Hence, sketch the graph of $y=\sin^{-1}x+\cos^{-1}x$.

(iii) Hence, evaluate $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x+\cos^{-1}x\ dx$.

Solution 4:

(i) $f(x)=\sin^{-1}x+\cos^{-1}x$

$$f'(x)=\frac{1}{\sqrt{1-x^2}}+\frac{-1}{\sqrt{1-x^2}}=0\ \#$$

(ii) Integrating both sides, w.r.t. x gives:

$$\int f'(x)\,dx=\int 0\,dx$$

$f(x)=C$, where C is a constant.

i.e. $\sin^{-1}x+\cos^{-1}x=C$

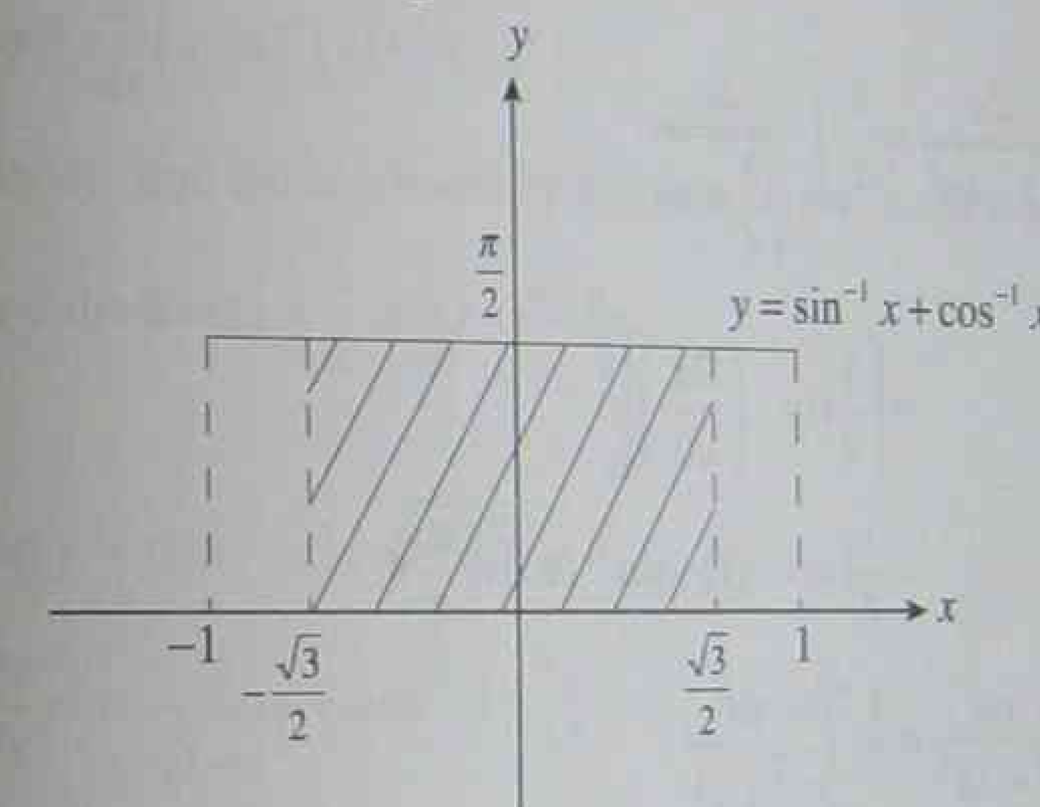
To find C, we substitute different values of x into the above equation:

$x=0$, $\sin^{-1}0+\cos^{-1}0=\dfrac{\pi}{2}=C$

$x=1$, $\sin^{-1}1+\cos^{-1}1=\dfrac{\pi}{2}=C$

$x=-1$, $\sin^{-1}(-1)+\cos^{-1}(-1)=\dfrac{\pi}{2}=C$

$\therefore$ $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$ for $-1\le x\le 1\ \#$



(iii) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x+\cos^{-1}x\ dx$ = Shaded area of rectangle above $=2\times\dfrac{\sqrt{3}}{2}\times\dfrac{\pi}{2}=\dfrac{\sqrt{3}\pi}{2}$ units

(E) Integrations Involving Inverse Trigonometric Functions

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$$

$$\int\frac{dx}{1+x^2}=\tan^{-1}x+C \qquad \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

Example 1: Find the following integrals:

(i) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ (ii) $\displaystyle\int \frac{dx}{1+4x^2}$

(iii) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{3-4x^2}}$ (iv) $\displaystyle\int_{-\sqrt{7}}^{\sqrt{7}} \frac{dx}{7+3x^2}$

Solution:

(i) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + C$ #

(ii) $\displaystyle\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}\,dx = \frac{1}{4}\times\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + C = \frac{1}{2}\tan^{-1}(2x) + C$ #

(iii) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{\frac{3}{4}-x^2}}$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}\left(\sin^{-1}(1) - \sin^{-1}(0)\right) = \frac{\pi}{4} \ \#$$

(iv) $\displaystyle\int_{-\sqrt{7}}^{\sqrt{7}} \frac{dx}{7+3x^2} = \frac{1}{3}\int_{-\sqrt{7}}^{\sqrt{7}} \frac{dx}{\frac{7}{3}+x^2}$

$$= \left[\frac{1}{3}\times\frac{1}{\frac{\sqrt{7}}{\sqrt{3}}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right)\right]_{-\sqrt{7}}^{\sqrt{7}}$$

$$= \left[\frac{1}{3}\times\frac{\sqrt{3}}{\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right)\right]_{-\sqrt{7}}^{\sqrt{7}}$$

$$= \frac{1}{\sqrt{21}}\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(-\sqrt{3})\right) = \frac{1}{\sqrt{21}}\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = \frac{2\pi}{3\sqrt{21}} \ \#$$

Example 2: Find the area bounded by the curve $y = \dfrac{4}{\sqrt{16-x^2}}$, the x-axis and the lines $x = 2$ and $x = 4$.

Solution 2:

$$\text{Area} = \int_2^4 \frac{4}{\sqrt{16-x^2}}\,dx = 4\int_2^4 \frac{1}{\sqrt{16-x^2}}\,dx$$

$$= 4\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$= 4\left(\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= \frac{4\pi}{3} \text{ units}^2 \ \#$$

Example 3:

(i) Find $\dfrac{d}{dx}\left(x\sin^{-1}(x) + \sqrt{1-x^2}\right)$.

(ii) Hence, find the area bound by the curve $y = \sin^{-1}x$, the x-axis and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$.

Solution 3:

(i) Let $y = x\sin^{-1}(x) + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\cdot\frac{1}{\sqrt{1-x^2}} + \sin x^{-1}\cdot 1 + \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\cdot -2x$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x \ \#$$

(ii) $\displaystyle \text{Area} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x\,dx$

Now, from (i):

$$\frac{d}{dx}\left(x\sin^{-1}(x) + \sqrt{1-x^2}\right) = \sin^{-1}x$$

Integrating w.r.t. x, gives:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{d}{dx}\left(x\sin^{-1}(x) + \sqrt{1-x^2}\right)dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x\,dx$$

$$\therefore \left[x\sin^{-1}(x)+\sqrt{1-x^2}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x\,dx$$

$$\text{i.e. } \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x\,dx = \left(\frac{\sqrt{3}}{2}\times\frac{\pi}{3}+\frac{1}{2}\right)-\left(\frac{1}{2}\times\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}\pi}{6}+\frac{1}{2}-\frac{\pi}{12}-\frac{\sqrt{3}}{2}$$

$$=\frac{2\sqrt{3}\pi-\pi+6-6\sqrt{3}}{12}$$

$$=\frac{\pi\left(2\sqrt{3}-1\right)+6\left(1-\sqrt{3}\right)}{12}\text{ units}^2 \ \#$$

Review Exercises

(A) Finding Inverse Functions

1. Find the inverse for each of the following functions. State the domain and range of $f^{-1}(x)$ and hence, sketch both functions on the same set of axes.

   (i) $y = 2x - 2$ (ii) $y = x^2 + 1$ for $x > 0$

   (iii) $y = x^2$ for $x \le 0$ (iv) $y = \frac{1}{x}$ for $x > 0$

(B) Inverse Functions Theory and Applications

2. Consider the function $f(x) = x^2 - 2x + 1$.

   (i) Sketch the parabola $y = f(x)$.

   (ii) What is the largest positive domain for which the function has an inverse function $f^{-1}(x)$?

   (iii) Sketch the inverse function $y = f^{-1}(x)$ on the same set of axes as (i). State its domain and range.

3. Let $y = x^3 - 3x^2 - 1$ for $x \ge 2$. Find the slope of the tangent to the inverse function at $x = -1$.

4. Consider the function $f(x) = \frac{x-1}{x+1}$, for $x \ge 0$

   (i) Find $f'(x)$.

   (ii) Hence or otherwise show that $f(x)$ admits an inverse function.

   (iii) Find $\lim\limits_{x\to\infty} \frac{x-1}{x+1}$.

   (iv) Hence without finding $f^{-1}(x)$, sketch $y = f(x)$ and $y = f^{-1}(x)$ for $x \ge 0$ on th same set of axes.

(C) Inverse Trigonometric Functions

5. Find the exact values of the following:

   (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

   (iii) $\cos^{-1}\left(\sin\frac{2\pi}{3}\right)$ (iv) $\tan^{-1}(\cos 2\pi)$

6. Prove the following expressions:

(i) $\tan^{-1}\left(\frac{2}{3}\right)=\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$ (ii) $2\cos^{-1}\left(\frac{3}{4}\right)=\cos^{-1}\left(\frac{1}{8}\right)$

(iii) $2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{4}{3}\right)$ (iv) $\tan^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}$

7. Find the exact values:

(i) $\sin\left(2\tan^{-1}\frac{1}{5}\right)$ (ii) $\tan\left(2\cos^{-1}\frac{1}{3}\right)$

8. Evaluate the following expressions:

(i) $\tan\left(2\tan^{-1}x\right)$ (ii) $\sin\left(\tan^{-1}\sqrt{x^2-2x}\right)$, $x\geq 2$

9. Consider the function $f(x)=2\sin^{-1}\left(\frac{x+2}{5}\right)$.

(i) Evaluate $f\left(\frac{1}{2}\right)$.

(ii) Find the domain and range of $f(x)$.

(iii) Hence, sketch the graph of $y=f(x)$.

10. Consider the function $f(x)=\cos^{-1}\left(\sqrt{1-x}\right)$.

(i) State the domain and range of $f(x)$.

(ii) Draw the graph of $y=f(x)$, labelling any key features.

11. Evaluate $\tan\left(\tan^{-1}\frac{1}{4}+\tan^{-1}2\right)$.

12. Sketch the graph $y=\sin\left(\sin^{-1}x\right)$.

13. Given $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, solve the following simultaneous equations.

(i) $\sin^{-1}x+\cos^{-1}y=\frac{5\pi}{12}$ and $\sin^{-1}y-\cos^{-1}x=\frac{\pi}{12}$

(ii) $\sin^{-1}x\cdot\sin^{-1}y=\frac{\pi^2}{12}$ and $\sin^{-1}x+\sin^{-1}y=\frac{7\pi}{12}$

(D) Differentiating Inverse Trigonometric Functions

14. Differentiate the following:

(i) $\sin^{-1}(3x)$ (ii) $\cos^{-1}\sqrt{x}$

(iii) $\tan^{-1}\sqrt{x^2-1}$ (iv) $\cos^{-1}\left(\frac{1}{x}\right)$

15. Find the equation of the normal to the curve $y=\sin^{-1}\sqrt{1-x}$ at $x=\frac{1}{2}$.

16. Find $\frac{d}{dx}\left(\cos\left(\tan^{-1}x\right)\right)$. Express your answer in terms of x only.

(E) Integrations Involving Inverse Trigonometric Functions

17. Find the following integrals:

(i) $\int_{-\sqrt{3}}^{1}\frac{dx}{9+3x^2}$ (ii) $\int_{0}^{1}\frac{dx}{\sqrt{4-x^2}}$

(iii) $\int\frac{e^x}{\sqrt{1-e^{2x}}}\,dx$, using the substitution $u=e^x$.

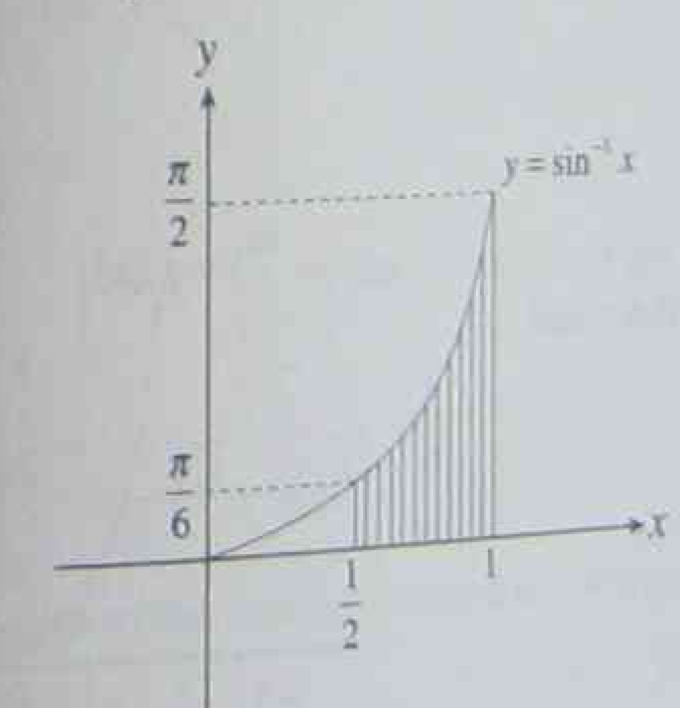
18. (i) Given $4x^2+4x+2=(ax+b)^2+c$, find the values of a, b and c.

(ii) Hence, evaluate $\int_{-1}^{0}\frac{dx}{4x^2+4x+2}$ using the substitution $u=2x+1$.

19. (i) Show that $\frac{d}{dx}\left(\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right)=\frac{16}{\left(4+x^2\right)^2}$.

(ii) Hence, evaluate $\int_{0}^{2}\frac{dx}{\left(4+x^2\right)^2}$.

20. Find the exact area between the curve $y=\sin^{-1}x$, the x-axis and the ordinates $x=\frac{1}{2}$ and $x=1$, as shown by the shaded area in the diagram below.



21. By completing the square, rewrite $4x-x^2$ and hence evaluate $\int_{1}^{4}\frac{dx}{\sqrt{4x-x^2}}$.

WORKED SOLUTIONS TO REVIEW EXERCISES

1\. (i) $y = 2x - 2$

Interchanging the x and y, to find $f^{-1}(x)$:

$x = 2y - 2$

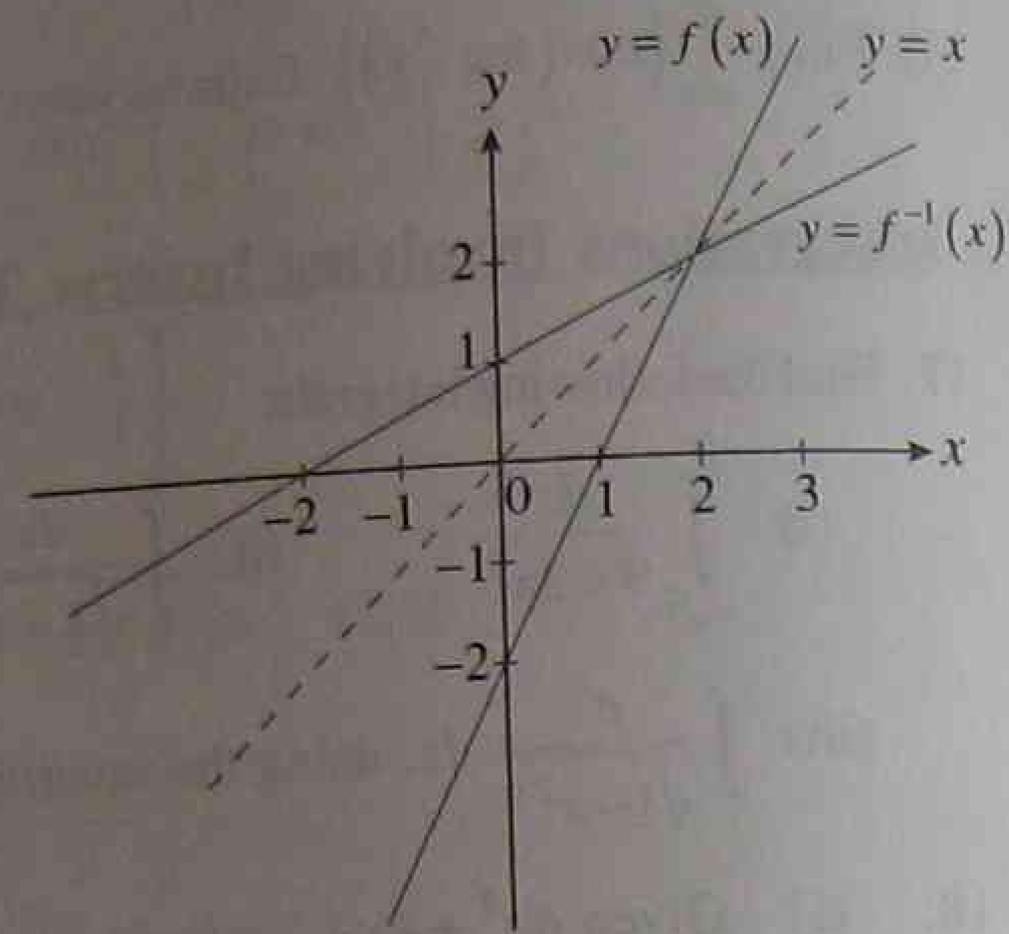
$2y = x + 2$

$y = \frac{1}{2}(x+2)$

i.e. $f^{-1}(x) = \frac{1}{2}(x+2)$

Domain: all real x

Range: all real y #



(ii) $y = x^2 + 1,\ x \geq 0$

Interchanging the x and y, to find $f^{-1}(x)$:

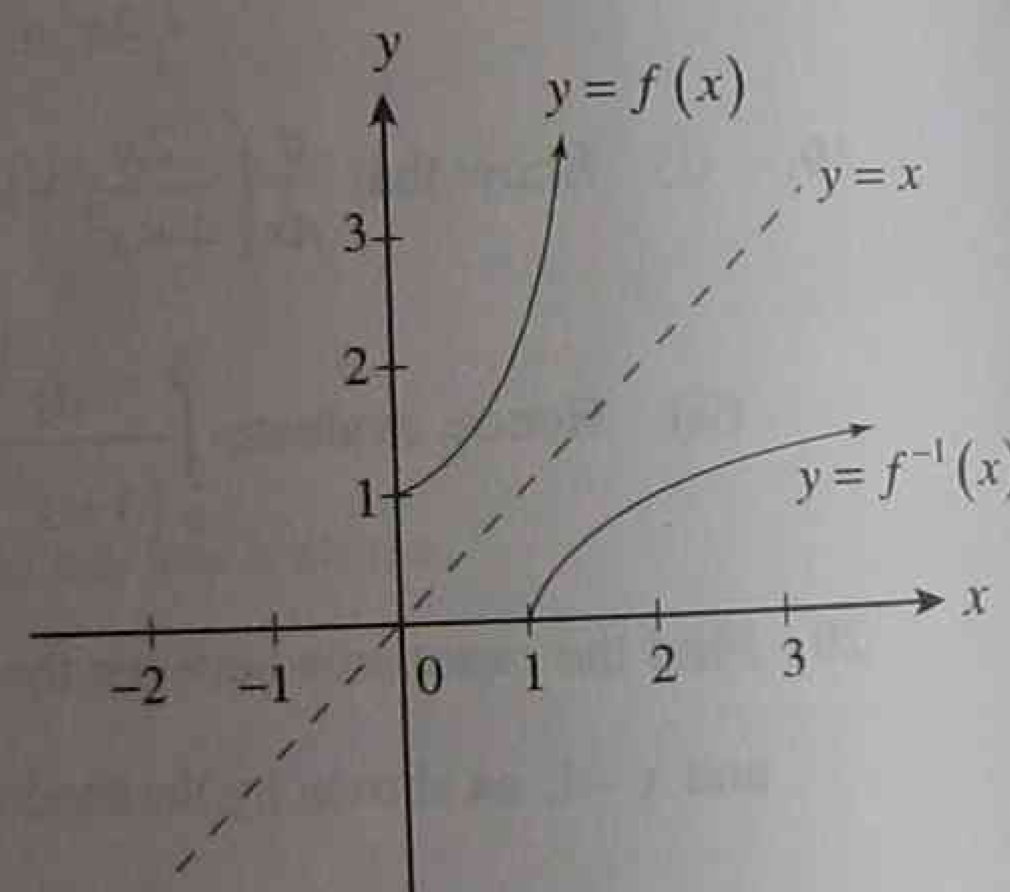
$x = y^2 + 1$

$y^2 = x - 1$

$= \sqrt{x-1}$ as $y \geq 0$

Domain: $x \geq 1$

Range: $y \geq 0$ #



(iii) $y = x^2,\ x \leq 0$

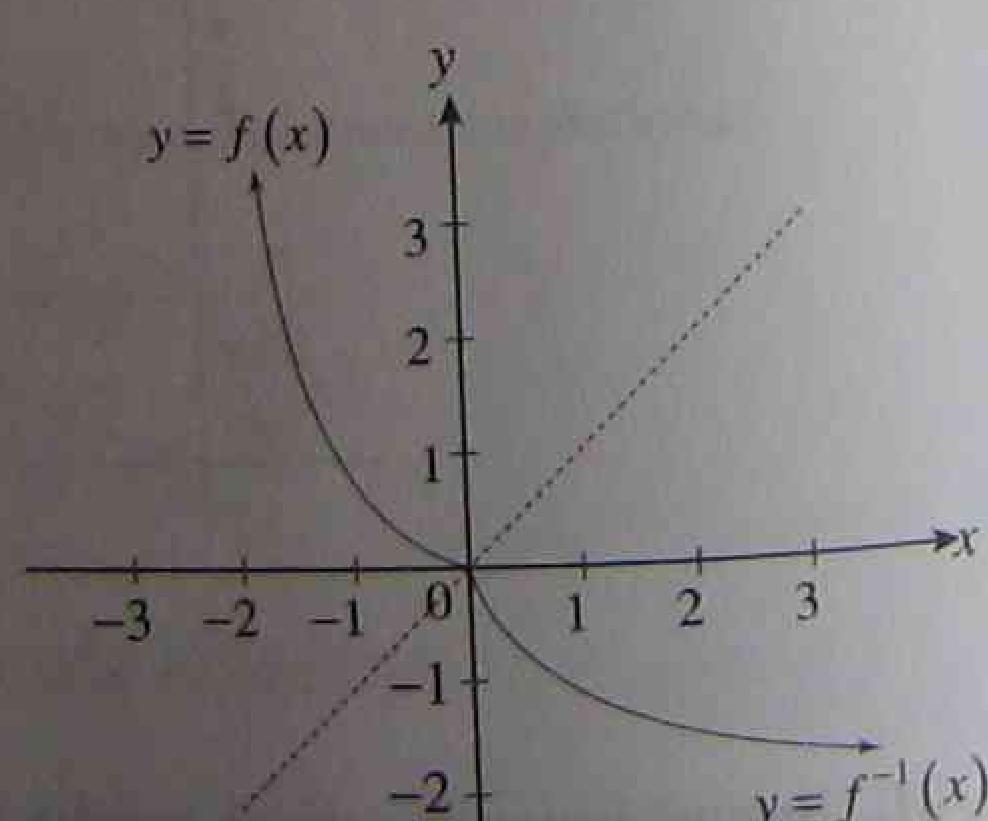
Interchanging the x and y, to find $f^{-1}(x)$:

$x = y^2$

$y = -\sqrt{x}$ as $y \leq 0$

Domain: $x \geq 0$

Range: $y \leq 0$ #



(iv) $y = \frac{1}{x},\ x > 0$

Interchanging x and y, to find $f^{-1}(x)$:

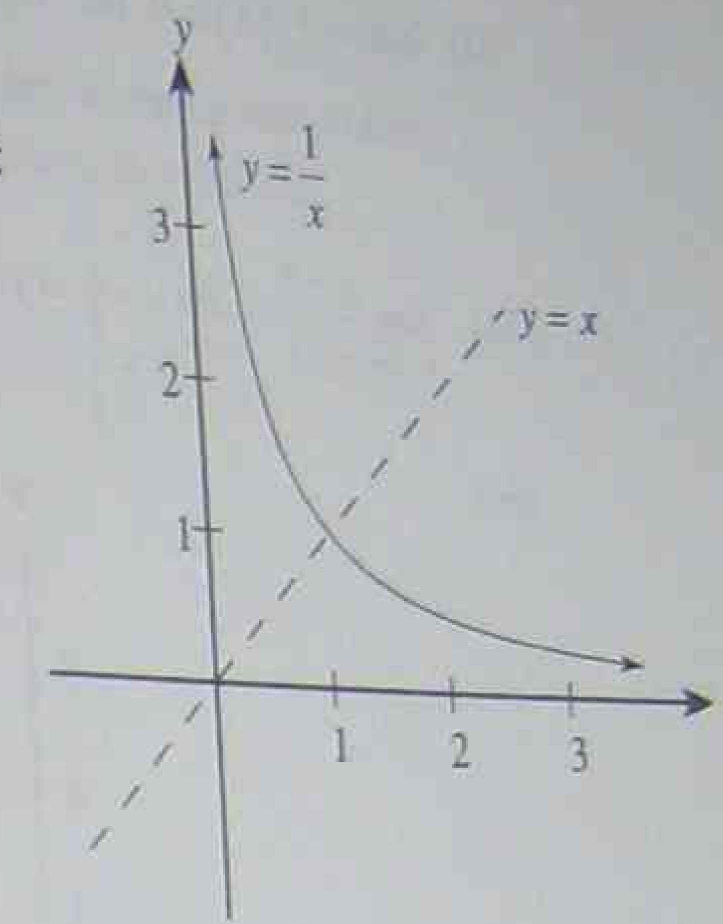
$x = \frac{1}{y}$

i.e. $y = \frac{1}{x}$

Domain: $x > 0$

Range: $y > 0$

(note: $f(x)$ is its own inverse) #

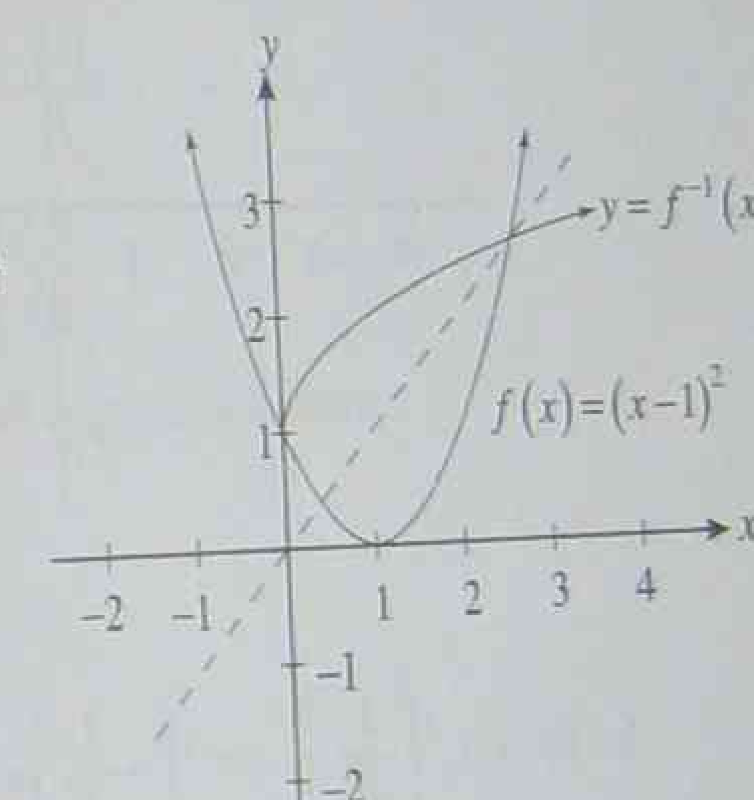


2\. (i) $f(x) = x^2 - 2x + 1 = (x-1)^2$ #

(ii) For $x \geq 1$, $f(x)$ is a one-to-one function and $\therefore$ has an inverse. #

(iii) Domain: $x \geq 0$

Range: $y \geq 1$#



3\. $y = x^3 - 3x^2 - 1$, for $x \geq 2$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{dx}{dy} = \frac{1}{3x^2 - 6x}$ (note: $\frac{dx}{dy}$ is the derivative of the inverse function)

$\therefore$ at $x = -1$, $\frac{dx}{dy} = \frac{1}{3(-1)^2 - 6(-1)} = \frac{1}{9}$

$\therefore$ slope of tangent to the inverse function at $x = -1$ is $\frac{1}{9}$ #

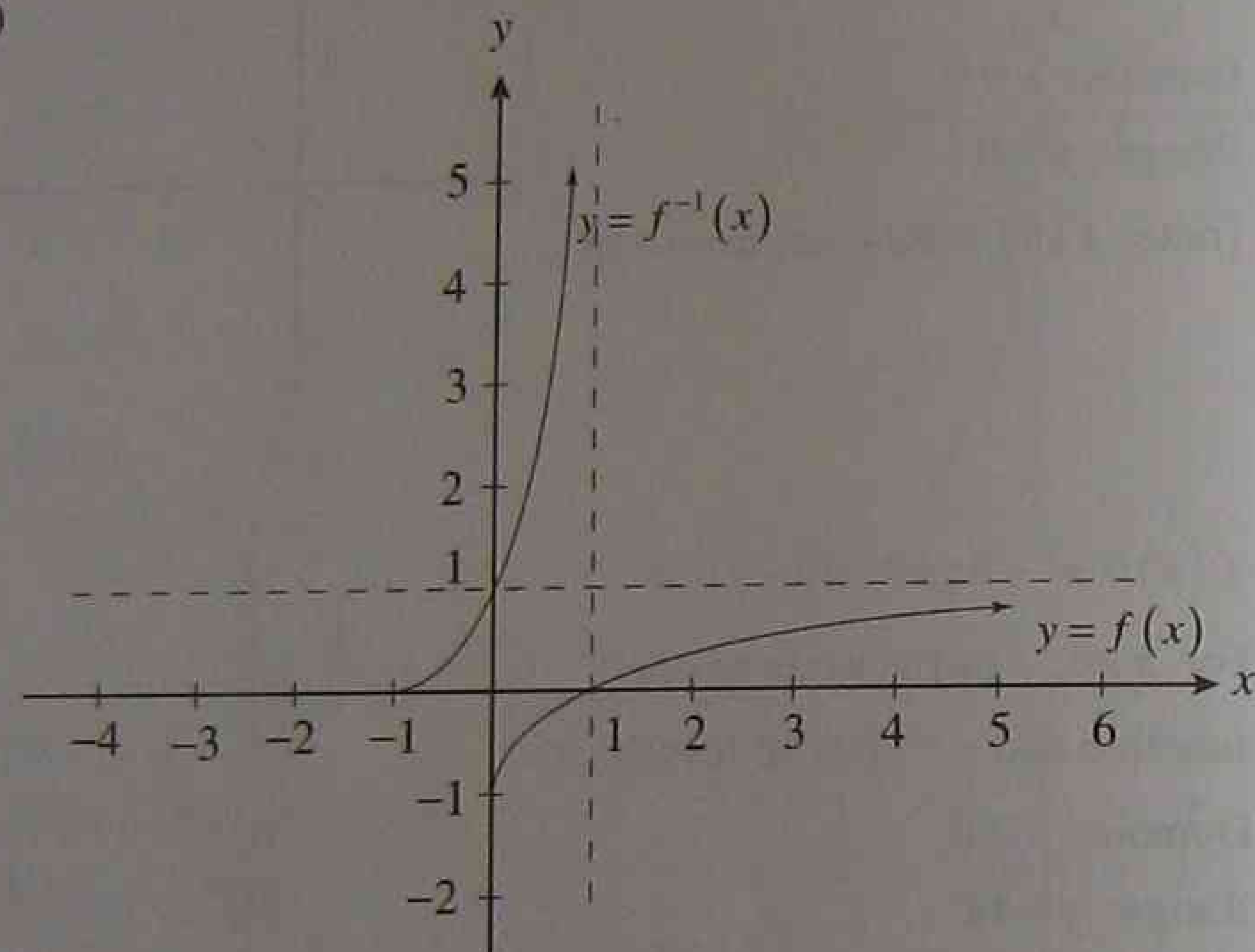
4\. (i) $f(x) = \frac{x-1}{x+1}$

$\frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{2}{(x+1)^2}$ #

(ii) Since $f'(x)>0$ for $x\ge 0$ $\therefore$ $f(x)$ is an increasing function for all $x\ge 0$ and is thus a one-to-one function. Hence, $y=f(x)$ admits an inverse.

(iii) $\displaystyle \lim_{x\to\infty}\frac{x-1}{x+1}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=1$ #

(iv)



5. (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\frac{\pi}{4}+\left(-\frac{\pi}{6}\right)=\frac{\pi}{12}$ #

(ii) $\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\left(\pi-\cos\left(\frac{1}{2}\right)\right)-\sin\left(\frac{1}{\sqrt{2}}\right)$

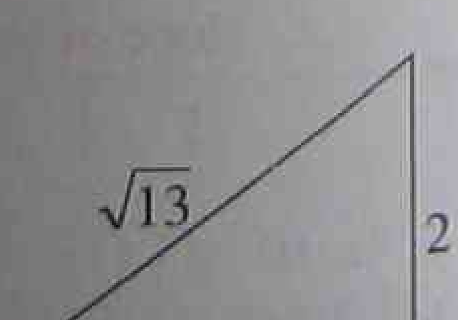
$=\left(\pi-\frac{\pi}{3}\right)-\left(\frac{\pi}{4}\right)=\frac{5\pi}{12}$ #

(iii) $\cos^{-1}\left(\sin\frac{2\pi}{3}\right)=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$ #

(iv) $\tan^{-1}(\cos 2\pi)=\tan^{-1}(1)=\frac{\pi}{4}$ #

6. (i) From the diagram:

$\theta=\tan^{-1}\left(\frac{2}{3}\right)$ or $\theta=\cos^{-1}\frac{3}{\sqrt{13}}$



(ii) Need to prove $2\cos^{-1}\left(\frac{3}{4}\right)=\cos^{-1}\left(\frac{1}{8}\right)$

Taking cos of both sides, required expression becomes:

$\cos\left(2\cos^{-1}\frac{3}{4}\right)=\frac{1}{8}$

let $\theta=\cos^{-1}\frac{3}{4}$ i.e. $\cos\theta=\frac{3}{4}$

$$\begin{aligned}LHS=\cos(2\theta)&=\cos^2\theta-\sin^2\theta\\&=2\cos^2\theta-1\\&=2\times\left(\frac{3}{4}\right)^2-1\\&=2\times\frac{9}{16}-1\\&=\frac{1}{8}\text{ as required \#}\end{aligned}$$

(iii) Need to prove $2\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{4}{3}\right)$

Taking tan of both sides of required expression becomes:

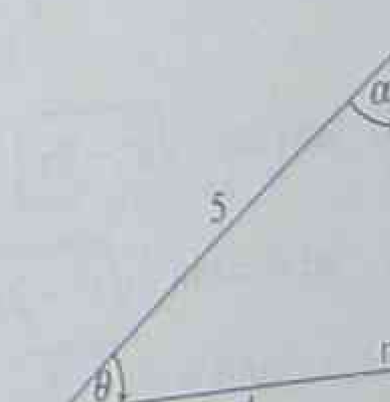
$\tan\left(2\tan^{-1}\frac{1}{2}\right)=\frac{4}{3}$

Let $\theta=\tan^{-1}\frac{1}{2}$ i.e. $\tan\theta=\frac{1}{2}$

$LHS=\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\dfrac{1}{1-\frac{1}{4}}=\frac{4}{3}$ #

(iv) $\tan^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}$

from the diagram:



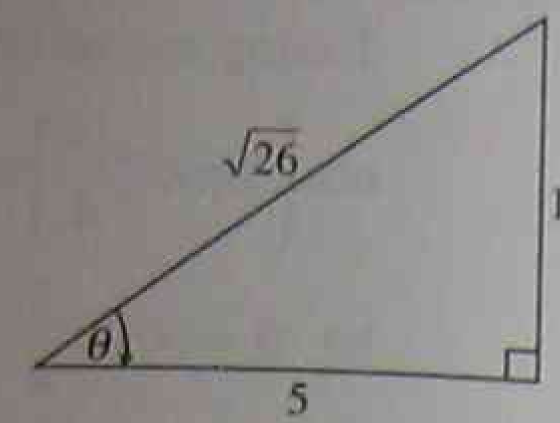
$\tan^{-1}\left(\frac{3}{4}\right)=\theta$ and $\sin^{-1}\left(\frac{4}{5}\right)=\alpha$

Now, $\theta+\alpha=\frac{\pi}{2}$

$\therefore\ \tan^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}$ #

7. (i) $\sin\left(2\tan^{-1}\frac{1}{5}\right)$

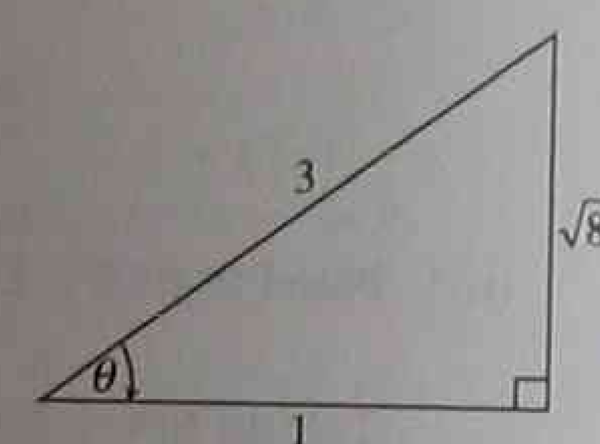
let $\theta=\tan^{-1}\left(\frac{1}{5}\right)$ i.e. $\tan\theta=\frac{1}{5}$



$$\therefore \sin\left(2\tan^{-1}\frac{1}{5}\right)=\sin 2\theta$$
$$=2\sin\theta\cos\theta$$
$$=2\left(\frac{1}{\sqrt{26}}\right)\left(\frac{5}{\sqrt{26}}\right)$$
$$=\frac{10}{26}=\frac{5}{13}\ \#$$

(ii) $\tan\left(2\cos^{-1}\frac{1}{3}\right)$

let $\theta=\cos^{-1}\left(\frac{1}{3}\right)$ i.e. $\cos\theta=\frac{1}{3}$



$$\therefore \tan\left(2\cos^{-1}\frac{1}{3}\right)=\tan 2\theta$$
$$=\frac{2\tan\theta}{1-\tan^2\theta}$$
$$=\frac{2\sqrt{8}}{1-\left(\sqrt{8}\right)^2}=\frac{2\sqrt{8}}{1-8}=-\frac{2\sqrt{8}}{7}\ \#$$

8. (i) $\tan\left(2\tan^{-1}x\right)$

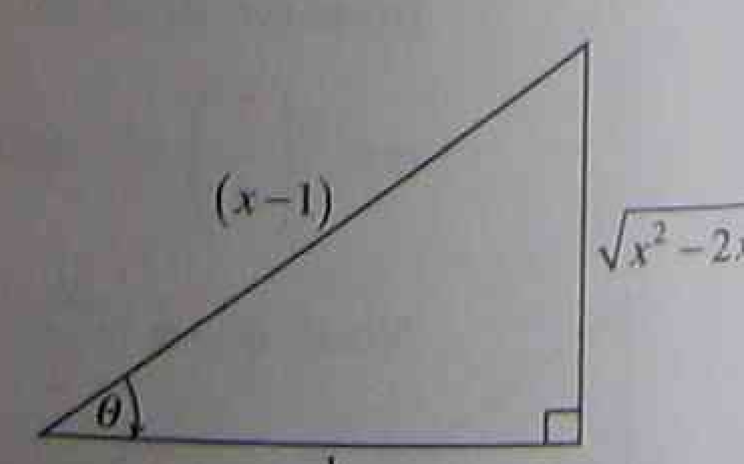
let $\theta=\tan^{-1}x$ $\therefore$ $\tan\theta=x$

$$\therefore \tan\left(2\tan^{-1}x\right)=\tan(2\theta)$$
$$=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2x}{1-x^2}\ \#$$

(ii) $\sin\left(\tan^{-1}\sqrt{x^2-2x}\right)$

let $\theta=\tan^{-1}\sqrt{x^2-2x}$

$\therefore$ $\tan\theta=\sqrt{x^2-2x}$



$$\therefore \sin\left(\tan^{-1}\sqrt{x^2-2x}\right)=\sin\theta$$
$$=\frac{\sqrt{x^2-2x}}{x-1}\ \#$$

9. (i) $f(x)=2\sin^{-1}\left(\frac{0.5+2}{5}\right)=2\sin^{-1}\left(\frac{1}{2}\right)=2\cdot\frac{\pi}{6}=\frac{\pi}{3}\ \#$

(ii) Domain: $-1\le\frac{x+2}{5}\le 1$

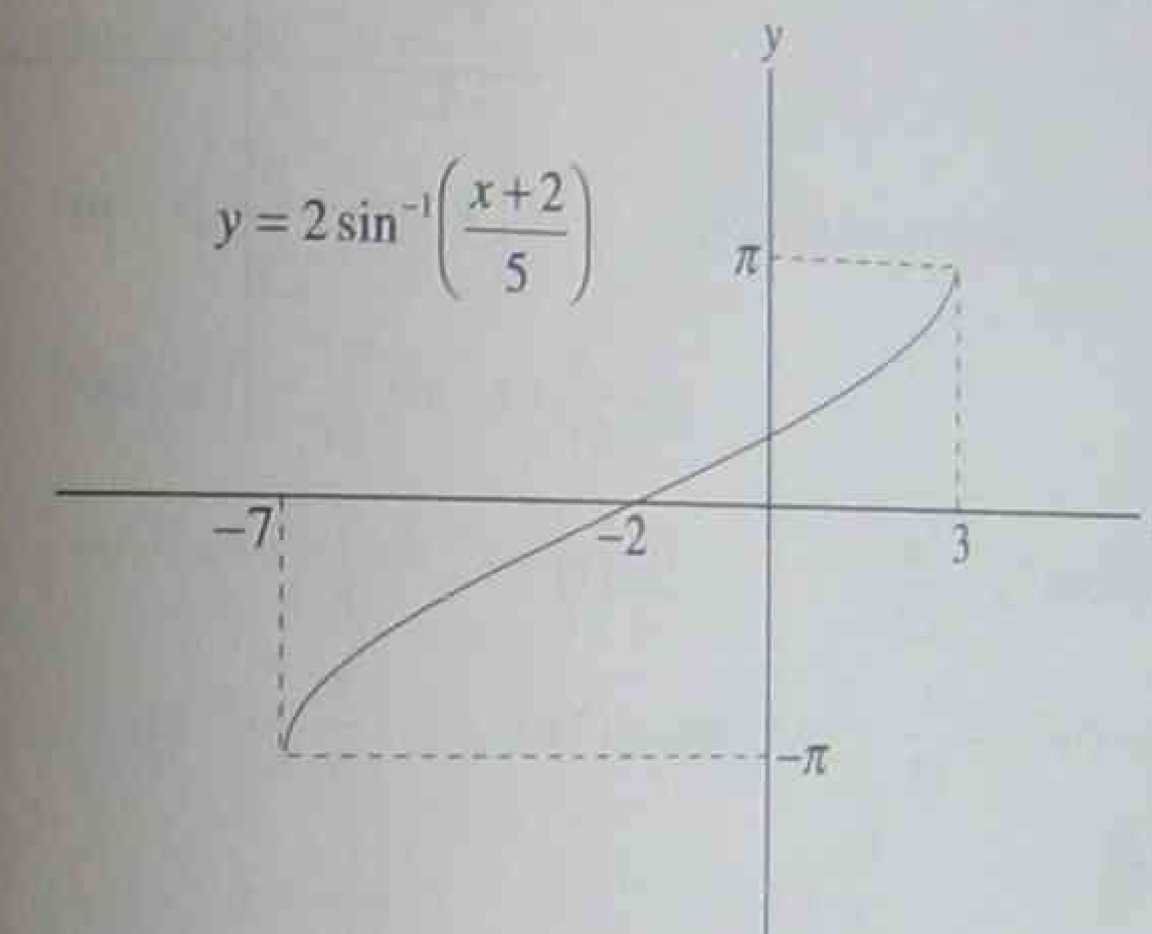
i.e. $-5\le x+2\le 5$

$-7\le x\le 3\ \#$

Range: $-\frac{\pi}{2}\le\sin^{-1}\left(\frac{x+2}{5}\right)\le\frac{\pi}{2}$

i.e. $-\pi\le 2\sin^{-1}\left(\frac{x+2}{5}\right)\le\pi\ \#$

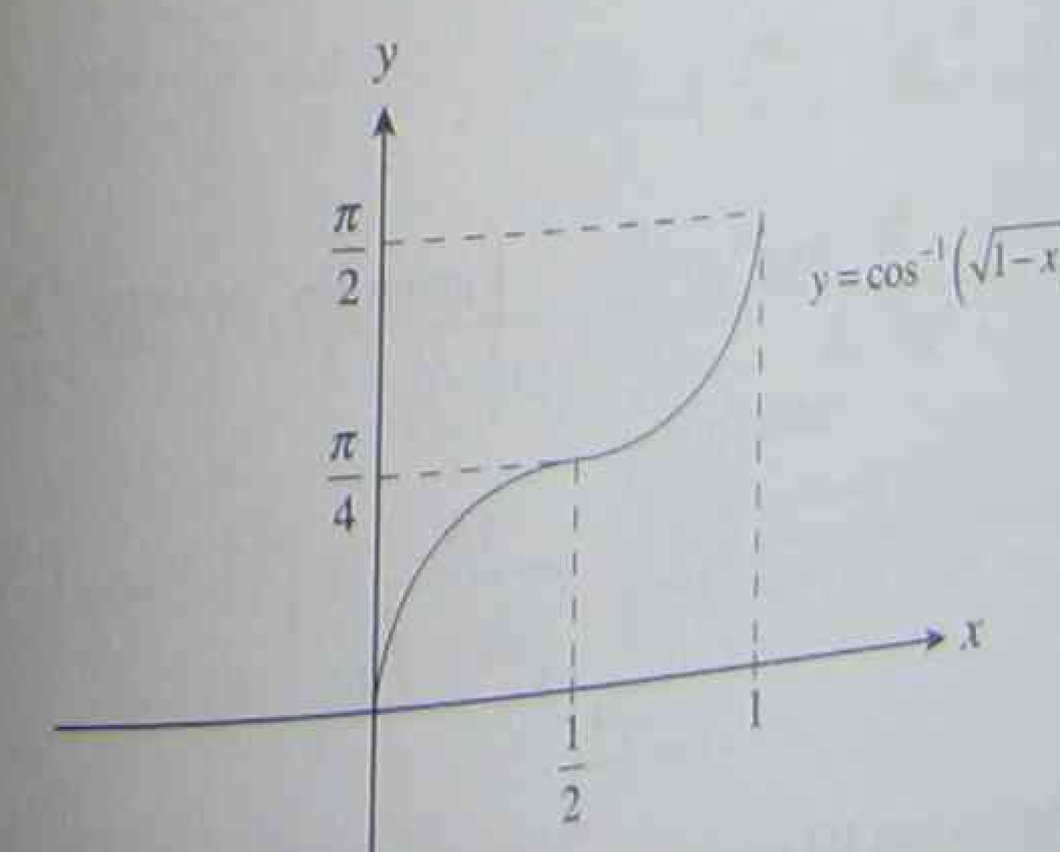
(iii)



10. (i) Domain is given by $0\le\sqrt{1-x}\le 1$ i.e. $0\le x\le 1\ \#$

Range is given by $0\le f(x)\le\frac{\pi}{2}\ \#$

(ii)



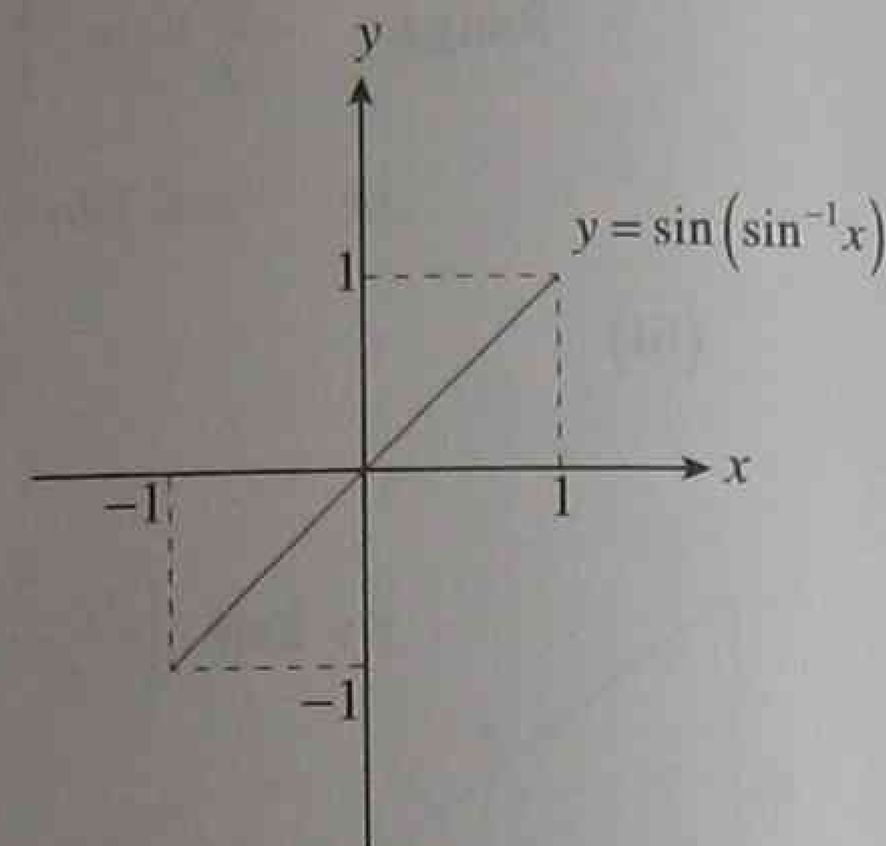
11. $\tan\left(\tan^{-1}\frac{1}{4}+\tan^{-1}2\right)=\dfrac{\tan\left(\tan^{-1}\frac{1}{4}\right)+\tan\left(\tan^{-1}2\right)}{1-\tan\left(\tan^{-1}\frac{1}{4}\right)\tan\left(\tan^{-1}2\right)}$

$$=\frac{\frac{1}{4}+2}{1-\frac{1}{4}\times 2}=\frac{9}{4}\times\frac{2}{1}=\frac{9}{2}\ \#$$

12. $y=\sin\left(\sin^{-1}x\right)$

Domain: $-1\le x\le 1$

$y=x$ for $-1\le x\le 1$



13. (i) $\sin^{-1}x+\cos^{-1}y=\dfrac{5\pi}{12}$ ....(1)

$\sin^{-1}y-\cos^{-1}x=\dfrac{\pi}{12}$ ....(2)

(1)−(2):

$\sin^{-1}x+\cos^{-1}y-\sin^{-1}y+\cos^{-1}x=\dfrac{\pi}{3}$

i.e. $\dfrac{\pi}{2}+\cos^{-1}y-\sin^{-1}y=\dfrac{\pi}{3}$ $\left(\text{using }\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}\right)$

i.e. $\sin^{-1}y-\cos^{-1}y=\dfrac{\pi}{6}$

$\sin^{-1}y+\cos^{-1}y-2\cos^{-1}y=\dfrac{\pi}{6}$

$\dfrac{\pi}{2}-2\cos^{-1}y=\dfrac{\pi}{6}$ $\left(\text{using }\sin^{-1}y+\cos^{-1}y=\dfrac{\pi}{2}\right)$

$2\cos^{-1}y=\dfrac{\pi}{3}$

$\cos^{-1}y=\dfrac{\pi}{6}$

Inverse Functions

$\sin^{-1}x+\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{5\pi}{12}$

i.e. $\sin^{-1}x=\dfrac{5\pi}{12}-\dfrac{\pi}{6}=\dfrac{\pi}{4}$

$\therefore\ x=\dfrac{1}{\sqrt{2}}$

Hence solutions are $x=\dfrac{1}{\sqrt{2}},\ y=\dfrac{\sqrt{3}}{2}$ #

(ii) $\sin^{-1}x\cdot\sin^{-1}y=\dfrac{\pi^2}{12}$ ....(1)

$\sin^{-1}x+\sin^{-1}y=\dfrac{7\pi}{12}$ ....(2)

$\sin^{-1}x\left(\dfrac{7\pi}{12}-\sin^{-1}x\right)=\dfrac{\pi^2}{12}$ (substituting (2) into (1))

$\left(\sin^{-1}x\right)^2-\dfrac{7\pi}{12}\sin^{-1}x+\dfrac{\pi^2}{12}=0$

$\left(\sin^{-1}x-\dfrac{\pi}{4}\right)\left(\sin^{-1}x-\dfrac{\pi}{3}\right)=0$

$\therefore\ \sin^{-1}x=\dfrac{\pi}{4}$ or $\sin^{-1}x=\dfrac{\pi}{3}$

$x=\dfrac{1}{\sqrt{2}}$ $\quad x=\dfrac{\sqrt{3}}{2}$

and $\sin^{-1}y=\dfrac{\pi}{3}$ and $\sin^{-1}y=\dfrac{\pi}{4}$

$y=\dfrac{\sqrt{3}}{2}$ $\quad y=\dfrac{1}{\sqrt{2}}$

$\therefore\ x=\dfrac{1}{\sqrt{2}}$ and $y=\dfrac{\sqrt{3}}{2}$ or $x=\dfrac{\sqrt{3}}{2}$ and $y=\dfrac{1}{\sqrt{2}}$ #

14. (i) $\dfrac{d}{dx}\left[\sin^{-1}(3x)\right]=\dfrac{1}{\sqrt{1-(3x)^2}}\times\dfrac{d}{dx}(3x)=\dfrac{3}{\sqrt{1-9x^2}}$ #

(ii) $\dfrac{d}{dx}\left[\cos^{-1}\sqrt{x}\right]=\dfrac{-1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\times\dfrac{d}{dx}\left(\sqrt{x}\right)=\dfrac{-1}{\sqrt{1-x}}\cdot\dfrac{1}{2\sqrt{x}}=\dfrac{-1}{2\sqrt{x-x^2}}$ #

(iii) $\frac{d}{dx}\left[\tan^{-1}\sqrt{x^2-1}\right]=\frac{1}{1+\left(\sqrt{x^2-1}\right)^2}\times\frac{d}{dx}\left(\sqrt{x^2-1}\right)$

$$=\frac{1}{x^2}\times\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}\times 2x=\frac{1}{x\sqrt{x^2-1}}\ \#$$

(iv) $\frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{x}\right)\right]=\frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\times\frac{d}{dx}\left(\frac{1}{x}\right)$

$$=\frac{-1}{\sqrt{\frac{x^2-1}{x^2}}}\times\frac{-1}{x^2}=\frac{-x}{\sqrt{x^2-1}}\times\frac{-1}{x^2}=\frac{1}{x\sqrt{x^2-1}}\#$$

15. $y=\sin^{-1}\sqrt{1-x}$ at $x=\frac{1}{2}$ $y=\frac{\pi}{4}$

$$y'=\frac{1}{\sqrt{1-\left(\sqrt{1-x}\right)^2}}\times\frac{d}{dx}\left(\sqrt{1-x}\right)$$

$$=\frac{1}{\sqrt{x}}\times\frac{-1}{2\sqrt{1-x}}=\frac{-1}{2\sqrt{x-x^2}}$$

at $x=\frac{1}{2}$, $y'=\frac{-1}{2\sqrt{\frac{1}{4}}}=-1$ $\therefore$ gradient of normal $=1$

$\therefore$ equation of normal is:

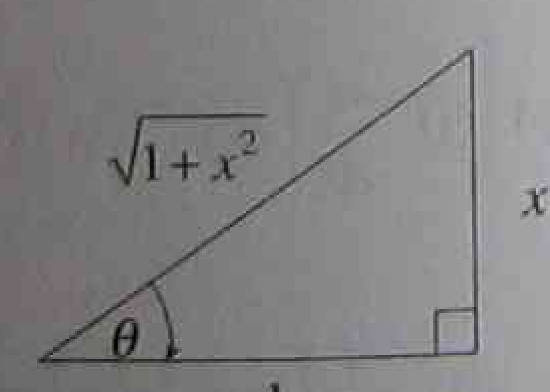
$$(y-y_1)=m(x-x_1)$$

$$\left(y-\frac{\pi}{4}\right)=1\left(x-\frac{1}{2}\right)$$

$$y=x+\frac{\pi}{4}-\frac{1}{2}\#$$

16. Let $\tan^{-1}x=\theta$

i.e. $x=\tan\theta$



$\therefore$ $\cos\left(\tan^{-1}x\right)=\cos\theta=\frac{1}{\sqrt{1+x^2}}$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{\ldots}\right)=\frac{d}{dx}\left((1+\ldots)^{-\frac{1}{2}}\right)=1(\ldots)^{-\frac{3}{2}}\ldots -x$

17. (i) $\int_{-\sqrt{3}}^{1}\frac{dx}{9+3x^2}=\frac{1}{3}\int_{-\sqrt{3}}^{1}\frac{dx}{3+x^2}=\frac{1}{3}\times\left[\frac{1}{\sqrt{3}}tan^{-1}\frac{x}{\sqrt{3}}\right]_{-\sqrt{3}}^{1}$

$$=\frac{1}{3\sqrt{3}}\left[\frac{\pi}{6}-\left(-\frac{\pi}{4}\right)\right]=\frac{5\pi}{36\sqrt{3}}\ \#$$

(ii) $\int_0^1\frac{dx}{\sqrt{4-x^2}}=\left[\sin^{-1}\frac{x}{2}\right]_0^1=\left(\sin^{-1}\frac{1}{2}-\sin^{-1}0\right)=\frac{\pi}{6}\#$

(iii) $\int\frac{e^x}{\sqrt{1-e^{2x}}}dx$ let $u=e^x$

$\frac{du}{dx}=e^x$, i.e. $du=e^x dx$

$$=\int\frac{1}{\sqrt{1-u^2}}du=\sin^{-1}u+C=\sin^{-1}\left(e^x\right)+C\ \#$$

18. (i) $4x^2+4x+2=(ax+b)^2+c$

now, $4x^2+4x+2=(2x+1)^2+1$

i.e. $a=2, b=1, c=1\#$

(ii) $\int_{-1}^{0}\frac{dx}{4x^2+4x+2}=\int_{-1}^{0}\frac{dx}{(2x+1)^2+1}$ let $u=2x+1$, $\frac{du}{dx}=2$, $dx=\frac{du}{2}$

$=\frac{1}{2}\int_{-1}^{1}\frac{du}{u^2+1}$ at $x=-1, u=-1$

$=\frac{1}{2}\left[tan^{-1}u\right]_{-1}^{1}$ $x=0, u=1$

$$=\frac{1}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi}{4}\#$$

19. (i) $\frac{d}{dx}\left(\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right)=\frac{\left(4+x^2\right)\cdot 2-2x\cdot 2x}{\left(4+x^2\right)^2}+\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$

$$=\frac{8+2x^2-4x^2}{\left(4+x^2\right)^2}+\frac{2}{4+x^2}$$

$$=\frac{8-2x^2+2\left(4+x^2\right)}{\ldots}=\frac{16}{\left(4+x^2\right)^2}\#$$

(ii) From (i): $\frac{d}{dx}\left(\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right)=\frac{16}{\left(4+x^2\right)^2}$

Integrating both sides, gives:

$$\int_0^2 \frac{16}{\left(4+x^2\right)^2}\,dx=\left[\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right]_0^2$$

i.e. $\int_0^2 \frac{1}{\left(4+x^2\right)^2}\,dx=\frac{1}{16}\left[\left(\frac{1}{2}+\frac{\pi}{4}\right)-(0+0)\right]=\frac{1}{16}\left(\frac{2+\pi}{4}\right)=\frac{2+\pi}{64}$ #

20. $y=\sin^{-1}x$

$\therefore\ x=\sin y$

Integrating over the y-values:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin y\,dy=\left[-\cos y\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=(-0)-\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$$

Now, shaded area $=\frac{\pi}{2}\times 1-\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{\pi}{6}=\frac{5\pi}{12}-\frac{\sqrt{3}}{2}$ units^2 #

21. $4x-x^2=-\left(x^2-4x\right)$

$=-\left[(x-2)^2-4\right]$

$=4-(x-2)^2$

$$\therefore\ \int_1^4 \frac{dx}{\sqrt{4x-x^2}}=\int_1^4 \frac{dx}{\sqrt{4-(x-2)^2}}=\left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^4=\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)=\frac{2\pi}{3}\ \#$$

TOPIC 13

Applications of Calculus to the Physical World

(A) Related Rates of Change

Related rates of change problems often require students to find the rate of change of a particular quantity (e.g. volume, surface area, etc.) with respect to time using another rate given in the question.

To link the rate given in the question and the rate to be calculated together we use the Chain Rule:

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

In particular:

$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$$

$$\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}$$

Example 1: A spherical balloon is expanding so that the radius, r, is increasing at a constant 2 cm per second. At what rate is the volume V, increasing when the radius is 8 cm?

Solution 1:

Volume of sphere is given by: $V=\frac{4}{3}\pi r^3$

Now: $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

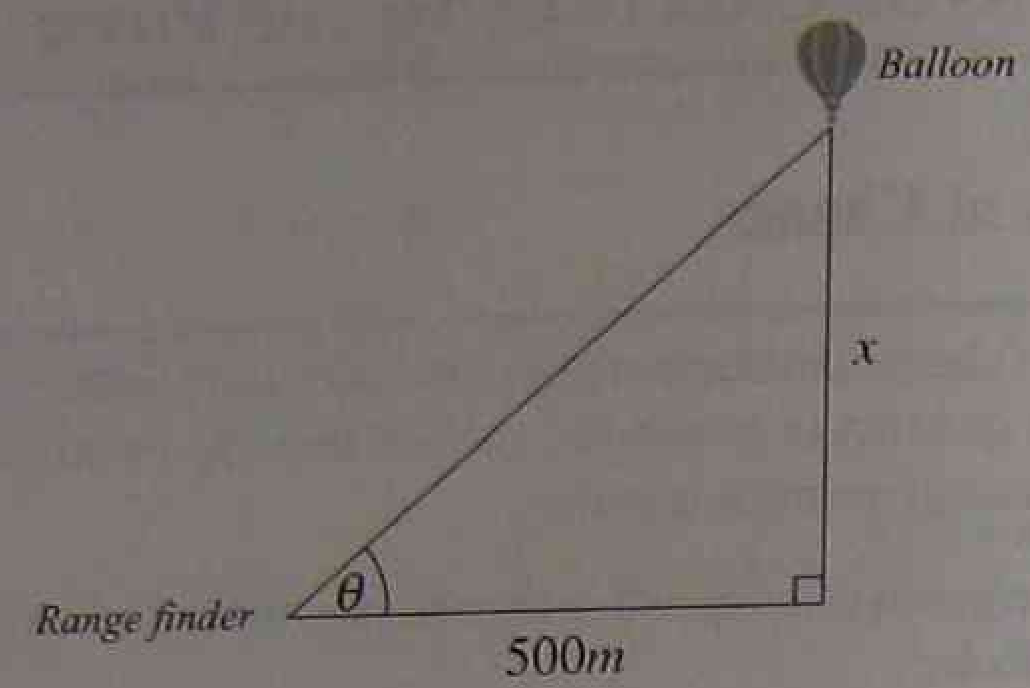
$\frac{dV}{dr}=4\pi r^2$ and $\frac{dr}{dt}=2$ (given)

$\therefore\ \frac{dV}{dt}=4\pi r^2\times 2$, at $r=8$

$=8\pi(8)^2$

$=512\pi$ cm^3/s #

Example 2: A hot-air balloon is rising straight up from a level field and is detected by a range finder 500m from the lift-off point. At that point, the angle of elevation $\frac{\pi}{\ldots}$ and the angle is increasing at the rate of 0.1 rad/min.



Solution 2: From the diagram:

$$\tan\theta = \frac{x}{500} \quad \text{i.e. } x = 500\tan\theta$$

Now, $\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$

$$\frac{dx}{d\theta} = 500\sec^2\theta, \quad \frac{d\theta}{dt} = 0.1 \text{ rad./min} \quad \text{when } \theta = \frac{\pi}{6}$$

$$\text{i.e. } \frac{dx}{dt} = 500\sec^2\theta \times 0.1$$

$$= 50\left(1+\tan^2\theta\right)$$

$$= 50\left(1+\left(\tan\frac{\pi}{6}\right)^2\right)$$

$$= 50\left(1+\frac{1}{3}\right)$$

$$= \frac{200}{3} \text{ metres/min. \#}$$

Example 3: Wheat is stored in a silo shaped as a cone with the vertex at the bottom, the sloping walls being at an angle of 45^0 to the horizontal.

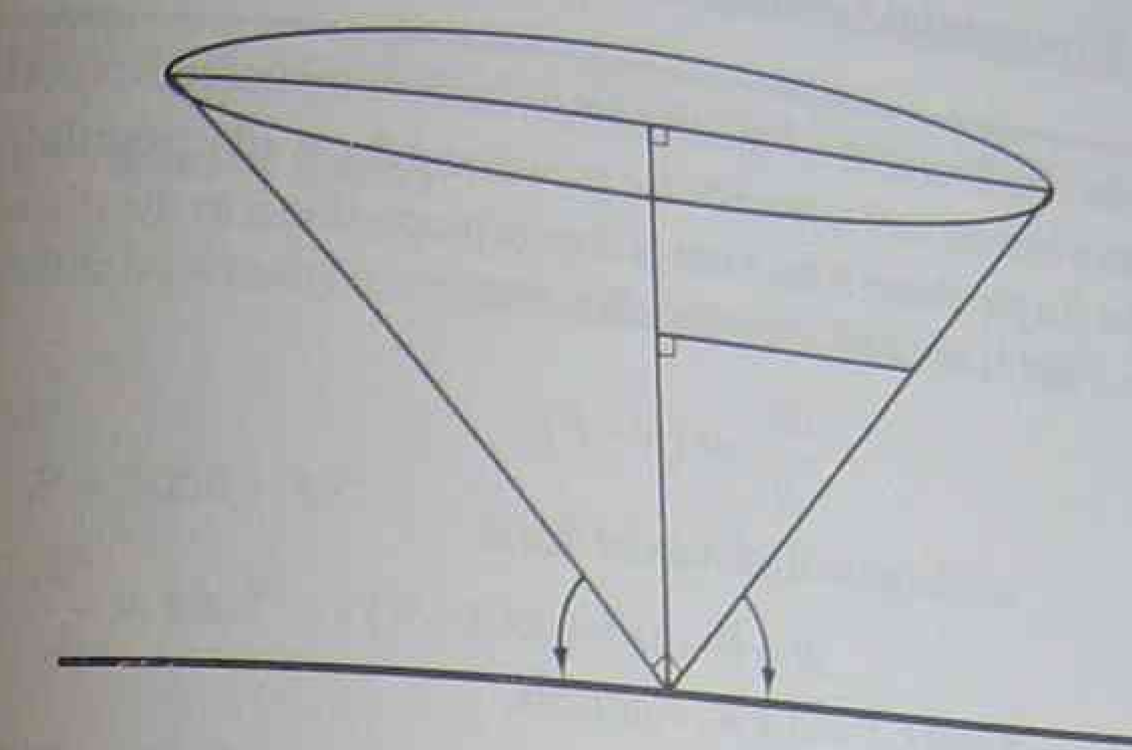
(i) Show that the volume V of wheat is given by:

$$V = \frac{\pi h^3}{3}$$

where h is the height of wheat in the silo.

(ii) If wheat is being drained from the bottom of the silo at 0.1 m^3/s, find the rate at which the height is decreasing when the area of the top surface is 20cm^2. Express your answer in cm/s.

Solution 3:



(i) From the diagram $r = h$ as Δ is isoceles:

Now, $V = \dfrac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi (h)^2 h$$

$$= \frac{\pi h^3}{3} \text{ \#}$$

(ii) $V = \dfrac{\pi h^3}{3}$

$$\frac{dV}{dh} = \frac{3\pi h^2}{3} = \pi h^2$$

Also, $\pi r^2 = 20$ (given)

i.e. $\pi h^2 = 20$ (since $r = h$)

i.e. $h^2 = \dfrac{20}{\pi}$

Now, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

$$-0.1 = \pi h^2 \times \frac{dh}{dt}$$

$$-0.1 = \pi \times \left(\frac{20}{\pi}\right) \times \frac{dh}{dt}$$

i.e. $\dfrac{dh}{dt} = -0.005$ m/s

Hence, the level of the wheat is falling at 0.5 cm/s#

(B) Further Exponential Growth and Decay

Extension 1 students need to consider slightly more complicated exponential functions which are used to describe the growth and decay of a population. In particular, if a population N has a rate of change proportional to the difference between N and a constant P, then this means that:

$$\frac{dN}{dt} = k(N-P)$$

and the general solution for this is

$$N = P + Ae^{kt}$$

where k, P and A are constants.

To verify that $N = P + Ae^{kt}$ is a solution of the differential equation $\frac{dN}{dt}$, we do the following:

$$N = P + Ae^{kt}$$

$$\frac{dN}{dt} = Ake^{kt} = Ak\left(\frac{N-P}{A}\right) = k(N-P) \qquad \left(\text{Note: } e^{kt} = \frac{N-P}{A}\right)$$

Students need to also be aware of the relationship between N and P as t tends to infinity:

For the equation $N = P + Ae^{kt}$:

if $k < 0$ as $t \to \infty$, then $N \to P$

if $k > 0$ and $A > 0$ as $t \to \infty$, then $N \to \infty$

A common application of growth and decay is Newton's law of cooling. Newton's law states that the rate of which a body cools in air is proportional to the difference between its temperature T and the constant temperature S of the surrounding air.

i.e. $$\frac{dT}{dt} = k(T-S)$$

Example 1: The population P of a town increases at a rate proportional to the number by which the town's population exceeds 5000. This can be expressed by the following differential equation:

$$\frac{dP}{dt} = k(P-5000)$$

(i) Show that $P = 5000 + Ae^{kt}$ is a solution of the equation.

(ii) Initially the population has 8,000, but after 3 years it had increased to 8,500. Find the values of A and k.

(iii) After how many more years will the population reach 15,000?

Solution 1:

(i) $P = 5000 + Ae^{kt}$

$$\frac{dP}{dt} = kAe^{kt} = k(P-5000) \ \#$$

(ii) $P = 5000 + Ae^{kt}$

at $t = 0, P = 8000$

i.e. $8000 = 5000 + A$

i.e. $A = 3000$

$\therefore\ P = 5000 + 3000e^{kt}$

at $t = 3, P = 8500$

i.e. $8500 = 5000 + 3000e^{kt}$

$3500 = 3000e^{3k}$

i.e. $k = \frac{1}{3}\ln\left(\frac{35}{30}\right) = 0.0514$ to 4 d.p. #

(iii) $P = 5000 + 3000e^{kt}$, where $k = \frac{1}{3}\ln\left(\frac{35}{30}\right)$

i.e. need to find t when $P = 15{,}000$

i.e. $15{,}000 = 5000 + 3000e^{kt}$

$\frac{10}{3} = e^{kt}$

i.e. $t = \frac{1}{k}\ln\left(\frac{10}{3}\right) = 23.43$ years

$\therefore$ The population reaches 15,000 after 20.43 more years or in the 21st year from now. #

Example 2: At any time t, the rate of cooling of the temperature T of a body, when the surrounding temperature is P, is given by the equation:

(i) Show that $T = P + Ae^{kt}$, for some constant A, satisfies the equation.

(ii) A hard-boiled egg at $98°$ C is left to cool. After 7 minutes, the egg's temperature is $32°$ C. Given the surrounding temperature is $22°$ C, find how much longer it will take for the egg to cool to $25°$ C, giving your answer correct to the nearest minute.

Solution 2:

(i) $T = P + Ae^{kt}$

$$\frac{dT}{dt} = kAe^{kt} = k(T-P) \ \#$$

(ii) $T = P + Ae^{kt}$

at $t=0$, $P=22$, $T=98°$

i.e. $98 = 22 + A$ i.e. $A = 76$

$\therefore \ T = 22 + 76e^{kt}$

at $t=7$, $T=35°$

$\therefore \ 35 = 22 + 76e^{7k}$

$$e^{7k} = \frac{13}{76}$$

i.e. $k = \frac{1}{7}\ln\left(\frac{13}{76}\right) = -0.2523$

Now, when $T = 25°$

$$25 = 22 + 76e^{kt}$$

$$e^{kt} = \frac{3}{76}$$

$\therefore \ t = \frac{1}{k}\ln\left(\frac{3}{76}\right) = 12.81 \approx 13$ min.

$\therefore$ The egg cools to 25°C after a further 6 minutes #

(C) Rectilinear Motion

In the HSC Maths course, velocity (v) and acceleration (a) were always expressed as a function of time.

In the Extension 1 course, it is possible for v and a to be expressed as a function of x.

An important formula frequently required in solving these questions, links a and v together with respect to x.

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) \quad \text{or} \quad a = v\cdot\frac{dv}{dx}$$

Derivation of the result $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Using the chain rule:

$$a = \frac{dv}{dt}$$

$$= \frac{dv}{dx} \times \frac{dx}{dt}$$

$$= v\frac{dv}{dx} \qquad \text{since } v = \frac{dx}{dt}$$

noting $v = \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$$\therefore \quad a = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

This important formula can also be used to find an expression for a with respect to x given an expression for v.

$$\frac{1}{2}v^2 = \int a\,dx$$

Example 1: The velocity of a particle in m/sec. is given by $v = 2x + 3$, where x is its displacement in metres from the origin. Find the acceleration of the particle at $x = 1$.

Solution 1:

Acceleration: $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$= \frac{d}{dx}\left(\frac{1}{2}(2x+3)^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(4x^2+12x+9)\right)$$

$$= \frac{d}{dx}\left(2x^2+6x+\frac{9}{2}\right)$$

$$= 4x + 6$$

at $x = 1$, $a = 10$ m/s^2 #

Example 2: For a particle moving in a straight line, the acceleration at time t in m/s^2 is given by:

$$\frac{d^2x}{dt^2}=\frac{2}{1+4x^2}\text{, where } x \text{ is the displacement in metres.}$$

Find an expression for the velocity v, after the particle has come to rest at $x=\frac{\pi}{2}$.

Solution 2:

Velocity: $\frac{1}{2}v^2=\int a\,dx$

$$v^2=2\int\frac{2}{1+4x^2}\,dx$$

$$=4\int\frac{1}{1+4x^2}\,dx$$

$$=4\times\frac{1}{4}\int\frac{1}{\frac{1}{4}+x^2}\,dx$$

$$=2\tan^{-1}(2x)+C$$

at $x=\frac{\pi}{2}$, $v=0$

i.e. $0=2\tan^{-1}0+C$ i.e. $C=0$

$\therefore\ v^2=2\tan^{-1}(2x)$

$\therefore\ v=\pm\sqrt{2\tan^{-1}(2x)}$

however, since $\frac{d^2x}{dt^2}>0\ \therefore\ v>0$

hence, $v=\sqrt{2\tan^{-1}(2x)}$ #

Example 3: The acceleration of a particle moving in a straight line is given by:

$$\frac{d^2x}{dt^2}=4x-6x^2$$

where x is the displacement in metres and t is the time in seconds. Initially the particle is at rest at the origin.

(i) Show that $v^2=4x^2(1-x)$

Solution 3:

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=4x-6x^2$ $\left[\text{using } \frac{d^2x}{dt^2}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)\right]$

$$\frac{1}{2}v^2=\int 4x-6x^2\,dx$$

$$=2x^2-2x^3+C$$

at $x=0,\ v=0$ i.e. $C=0$

$\therefore\ v^2=4x^2-4x^3$

$=4x^2(1-x)$ #

(ii) $v^2\geq 0$

$\therefore\ 4x^2(1-x)\geq 0$ i.e. $x\leq 1$ #

(iii) At $x=\frac{1}{2}$,

$$v^2=4\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)=\frac{1}{2}$$

$\therefore\ v=\frac{1}{\sqrt{2}}$ as $v>0$ (given)

and $a=4\left(\frac{1}{2}\right)-6\left(\frac{1}{2}\right)^2=\frac{1}{2}$

since $v>0$ and $a>0$ $\therefore$ the particle moves to the right of $x=\frac{1}{2}$ with increasing speed. After $x=\frac{2}{3}$, the acceleration of the particle becomes negative and thus starts to retard the speed of the particle as it approaches $x=1$. At $x=1$, the particle comes to rest $(v=0)$ and begins to move to the left of $x=1$ $(a<0)$ and continues to move to the left, $(v<0)$. #

(D) Simple Harmonic Motion (SHM)

Simple Harmonic Motion describes the motion of a particle which moves in a straight line, such that its acceleration is always directed towards the centre of oscillation and is proportional to its distance from the centre.

Mathematically this is represented as:

This formula is used to prove that the motion of a particle is Simple Harmonic.

(i) Displacement

The displacement function is often expressed in the form:

$$x = a\cos(nt+\alpha)$$

where a, n and a are constants ($a>0$, $n>0$).

(ii) Velocity and Acceleration

Consider $x=a\cos(nt+a)$ differentiating w.r.t. x, gives:

$$\dot{x} = v = \frac{dx}{dt} = -an\sin(nt+\alpha)$$

Squaring the velocity, gives:

$$\begin{aligned} v^2 &= n^2a^2\sin^2(nt+\alpha) \\ &= n^2a^2\left(1-\cos^2(nt+\alpha)\right) \\ &= n^2a^2 - n^2a^2\cos^2(nt+\alpha) \\ &= n^2a^2 - n^2x^2, \text{ i.e.} \end{aligned}$$

$$v^2 = n^2\left(a^2-x^2\right)$$

Now, differentiating v w.r.t. x, gives:

$$\ddot{x} = a = \frac{d^2x}{dt^2} = -an^2\cos(nt+\alpha)$$

$$= -n^2x$$

(iii) Period and Frequency

The period, T, is the time taken for one full oscillation (i.e. time taken for the particle to return to its starting position).

$$T = \frac{2\pi}{n}$$

The frequency, f, is the number of oscillations the particle makes in 1 second.

$$f = \frac{1}{T} = \frac{n}{2\pi}$$

(iv) Extensions on the Standard Case

If the particle is describing simple harmonic motion about a point other than the origin, say $x=b$, then the following equations of motion will apply:

(v) Important Features of SHM

(1) a is called the **amplitude** and is measured by finding the distance from the centre of the motion to one of the extremities of the path.

(2) $(nt+\alpha)$ is called the **phase** of the motion and is measured in radians.

(3) In the standard case where $\alpha=0$, the initial position of the particle is $x=a$. Thus by convention, the particle commences its motion at one of the extremities.

(4) To verify that the motion of a particle is simple harmonic, we need to prove that acceleration of the particle is of the form $\ddot{x}=-n^2x$.

(5) Acceleration and displacement are maximum at the extremities and zero at the centre of the motion.

(6) Velocity is zero at the extremities and maximum at the centre of the motion. This can be seen directly from the formula $v^2=n^2\left(a^2-x^2\right)$.

(7) Since $v^2\ge 0$, for v to exist, then the formula $v^2=n^2\left(a^2-x^2\right)=0$ can be solved to determine the points between which the particle is oscillating.

(8) If acceleration is of the form $\ddot{x}=-n^2(x-b)$, then to show that the motion is simple harmonic follow this approach:

Let $x-b=X$

$$\therefore \frac{d}{dt}(x-b)=\frac{d}{dt}X \text{ i.e. } \dot{x}=\dot{X}$$

$$\text{and } \frac{d}{dt}(\dot{x})=\frac{d}{dt}(\dot{X}), \text{ i.e. } \ddot{x}=\ddot{X}$$

Thus, from $\ddot{x}=-n^2(x-b)$

we obtain $\ddot{X}=-n^2X$

which represents SHM of period $\frac{2\pi}{n}$ seconds

(9) By convention, to the right of the origin means x is positive and to the left of the origin means x is negative.

(10) At times (rarely) the equation $x=a\sin(nt+a)$ may also be used. In this case, the following formulae will apply:

Example 1: A particle moves in such a way that its displacement x metres from the origin O at time t seconds, is given by: $x = 2\cos\left(4t + \frac{\pi}{6}\right)$

(i) Show that the particle moves in SHM.
(ii) Find the period and amplitude of the motion.
(iii) Find when the particle first passes the origin.
(iv) Find the maximum displacement, velocity and acceleration of the particle.

Solution 1:

(i) $x = 2\cos\left(4t + \frac{\pi}{6}\right)$

$$\dot{x} = -2\sin\left(4t + \frac{\pi}{6}\right) \cdot \frac{d}{dt}\left(4t + \frac{\pi}{6}\right)$$

$$= -8\sin\left(4t + \frac{\pi}{6}\right)$$

$$\ddot{x} = -8\cos\left(4t + \frac{\pi}{6}\right) \cdot \frac{d}{dt}\left(4t + \frac{\pi}{6}\right)$$

$$= -32\cos\left(4t + \frac{\pi}{6}\right)$$

$$= -16\left[2\cos\left(4t + \frac{\pi}{6}\right)\right]$$

$$= -16x$$

since $\ddot{x}$ is in the form $-n^2x$ $\therefore$ the particle move in SHM #

(ii) $x = a\cos(nt + \alpha), x = 2\cos\left(4t + \frac{\pi}{6}\right)$

$\therefore$ amplitude $= 2$, period $= \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$ #

(iii) Particle passes the origin when $x = 0$

i.e. $0 = 2\cos\left(4t + \frac{\pi}{6}\right)$

i.e. $4t + \frac{\pi}{6} = \frac{\pi}{2}$

$4t = \frac{\pi}{2} - \frac{\pi}{6}$

$\therefore$ $t = \frac{\pi}{12}$ seconds #

(iv) $x = 2\cos\left(4t + \frac{\pi}{6}\right)$, $\dot{x} = -8\sin\left(4t + \frac{\pi}{6}\right)$, $\ddot{x} = -16x$

Displacement: $-1 \le \cos\left(4t + \frac{\pi}{6}\right) \le 1$ $\therefore$ $-2 \le x \le 2$

hence, maximum displacement is 2m to the right of O

Velocity: $-1 \le \sin\left(4t + \frac{\pi}{6}\right) \le 1$ $\therefore$ $-8 \le \dot{x} \le 8$

hence, maximum velocity is 8 m/s

Acceleration: $-2 \le x \le 2$ $\therefore$ $-32 \le \ddot{x} \le 32$

hence, maximum acceleration is 32 m/s^2 #

Example 2: A particle moves in SHM about the origin and has a speed of 5 cm/sec. when passing through the centre of its path O. The period is π seconds. Find the speed of the particle when it is 2 cm from O.

Solution 2:

period: $T = \frac{2\pi}{n}$ i.e. $\pi = \frac{2\pi}{n}$ $\therefore n = 2$

using the formula: $v^2 = n^2\left(a^2 - x^2\right)$

at $v = 5, x = 0, n = 2$

i.e. $25 = 4a^2$ i.e. $a^2 = \frac{25}{4}$

at $x = 2$, $a^2 = \frac{25}{4}$ (now, we can find v at $x = 2$)

i.e. $v^2 = 4\left(\frac{25}{4} - 4\right)$

$= 9$

i.e. $v = \pm 3$

$\therefore$ the speed of the particle $|v|$ is 3 m/s #

Example 3: The velocity v m/s of a particle moving in a straight line is given by:

$$v^2 = 8 + 4x - 4x^2$$

where x is the displacement in metres.

(i) Show that $\ddot{x} = -4\left(x - \frac{1}{2}\right)$.

(ii) Hence, prove that the motion is simple harmonic.

... the particle oscillating.

(iv) Hence, find the period and amplitude of the motion.

(v) Find the maximum speed of the particle.

Solution 3:

(i) $$v^2 = 8 + 4x - 4x^2$$

$$\frac{1}{2}v^2 = 4 + 2x - 2x^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 4x$$

i.e. $$\ddot{x} = -4\left(x - \frac{1}{2}\right) \#$$

(ii) $$\ddot{x} = -4\left(x - \frac{1}{2}\right)$$

Let $X = x - \frac{1}{2}$

differentiating both sides with respect to t, gives:

$$\therefore \quad \frac{d}{dt}(X) = \frac{d}{dt}\left(x - \frac{1}{2}\right)$$

i.e. $\dot{X} = \dot{x}$

differentiating again with respect to t, gives:

i.e. $$\frac{d}{dt}(\dot{X}) = \frac{d}{dt}(\dot{x})$$

$$\ddot{X} = \ddot{x}$$

$\therefore$ $\ddot{X} = -4X$, which is of the form $\ddot{x} = -n^2 x$

$\therefore$ the motion is simple harmonic. #

(iii) Let $v^2 = 0$, to find points of oscillation

i.e. $8 + 4x - 4x^2 = 0$

$$-4\left(x^2 - 1 - 2\right) = 0$$

$$-4(x-2)(x+1) = 0$$

$\therefore$ $x = -1, 2$

$\therefore$ the particle oscillates between $x = -1$ and $x = 2$ #

Calculus

(iv) From $\ddot{x} = -4\left(x - \frac{1}{2}\right)$

$n = 2$ and centre of motion is $x = \frac{1}{2}$

Period: $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ seconds

Amplitude: by definition, the amplitude is the distance from the centre to one of the extremities.

i.e. $a = 2 - \frac{1}{2} = 1\frac{1}{2}$ metres. #

(v) Maximum speed occurs at the centre (by definition) or when acceleration is zero i.e. $x = \frac{1}{2}$

$$\therefore \quad v^2 = 8 + 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^2 = 9$$

$\therefore$ $v = \pm 3$

hence, maximum speed of particle $|v|$ is 3 m/s #

Example 4: An object moves so that its acceleration in m/s^2 at position x metres from the origin is given by:

$$\frac{d^2x}{dt^2} = -4\pi^2 x$$

Initially the particle is stationary at a position one metre from the origin.

(i) Show that the position of the particle at any time t is given by $x = \cos(2\pi t)$.

(ii) Find the maximum speed of the particle and the first time when this occurs.

Solution 4:

(i) $$\frac{d^2x}{dt^2} = -4\pi^2 x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4\pi^2 x$$

$$\frac{1}{2}v^2 = -2\pi^2 x^2 + C_1$$

when $x = 1$, $v = 0$ $\therefore$ $C_1 = 2\pi^2$

$$\therefore \quad v^2 = 4\pi^2 - 4\pi^2 x^2 = 4\pi^2\left(1 - x^2\right)$$

Integrating both sides with respect to x gives:

$$t = \pm\frac{1}{2\pi}\times -\cos^{-1}x + C_2$$

when $t=0, x=1 \quad \therefore \quad C_2 = 0$

$$\therefore \quad t = \pm\frac{1}{2\pi}\cos^{-1}x$$

$$\pm 2\pi t = \cos^{-1}x$$

$$x = \cos(\pm 2\pi t)$$

as $\cos\theta = \cos(-\theta)$

$\therefore \quad x = \cos(2\pi t)$ #

(ii) $x = \cos(2\pi t)$

$v = \dot{x} = -2\pi\sin(2\pi t)$

since $-1 \le \sin(2\pi t) \le 1 \quad \therefore \quad -2\pi \le v \le 2\pi$

$\therefore$ maximum speed $|v|$ is 2π m/s

which occurs at the equilibrium position i.e. $x = 0$

i.e. $\cos(2\pi t) = 0$

$$2\pi t = \frac{\pi}{2}$$

$\therefore t = \dfrac{\pi}{4}$ seconds #

(E) Projectile Motion

In the 3-Unit course, the study of projectile motion involves analysing the equation of motion of a particle projected upwards at an angle α to the horizontal. Ignoring air resistance, it is assumed that the only force acting on the particle is its weight vertically downwards. The horizontal and vertical components of the motion are individually analysed.

(i) The Basic Formulae

Given there is no force acting horizontally (air resistance is neglected) and a force mg acting vertically on the particle, then the equations of the motions are:

$$m\ddot{x} = 0 \quad \text{and} \quad m\ddot{y} = -mg$$

By integrating these questions successively and using the boundary conditions

Horizontilly

$\ddot{x} = 0$

$\dot{x} = C_1$

When $t = 0,\ x = V\cos\alpha$

$\therefore \quad C_1 = V\cos\alpha$

i.e. $\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + C_2$

at $t = 0,\ x = 0$

$\therefore \quad C_2 = 0$

i.e. $x = Vt\cos\alpha$

Vertically

$\ddot{y} = -g$

$\dot{y} = -gt + C_1$

When $t = 0,\ y = V\sin\alpha$

$\therefore \quad C_1 = V\sin\alpha$

i.e. $\dot{y} = -gt + V\sin\alpha$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha + C_2$

at $t = 0,\ y = 0$

$\therefore \quad C_2 = 0$

i.e. $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

In summary :

$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = V\cos\alpha$	$\dot{y} = -gt + v\sin\alpha$
$x = Vt\cos\alpha$	$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

(ii) Time of Flight - T

This occurs at $y = 0$ (i.e. when the particle returns to the horizontal plane).

i.e. $0 = -\dfrac{1}{2}gt^2 + Vt\sin\alpha$

i.e. $t\left(-\dfrac{1}{2}gt + V\sin\alpha\right) = 0$

i.e. $t = 0,\ \dfrac{2V\sin\alpha}{g}$

$$T = \frac{2V\sin\alpha}{g}$$

(iii) Horizontal Range - R

From $x = Vt\cos\alpha$, the range $R = VT\cos\alpha$

i.e. $R = V\cdot\dfrac{2V\sin\alpha}{g}\cdot\cos\alpha = \dfrac{V^2 2\sin\alpha\cos\alpha}{g}$

$$R = \frac{V^2\sin 2\alpha}{g}$$

(iv) Maximum Height - H

The maximum height reached occurs when

$\dot{y}=0$ i.e. $-gt+V\sin\alpha=0$, when $t=\dfrac{V\sin\alpha}{g}$

Substituting this into $y=-\dfrac{1}{2}gt^2+Vt\sin\alpha$, the greatest height H is given by:

$$H=-\frac{1}{2}\frac{V^2\sin^2\alpha}{g}+\frac{V^2\sin^2\alpha}{g}$$

$$H=\frac{V^2\sin^2\alpha}{2g}$$

(v) Cartesian Equation of the Path

From $x=Vt\cos\alpha$, we get $t=\dfrac{x}{V\cos\alpha}$

Substituting this into $y=-\dfrac{1}{2}gt^2+Vt\sin\alpha$, we obtain:

$$y=-\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2+V\sin\alpha\left(\frac{x}{V\cos\alpha}\right)$$

$$=\frac{-gx^2}{2V^2}\left(\frac{1}{\cos^2\alpha}\right)+x\tan\alpha$$

$$y=x\tan\alpha-\frac{gx^2}{2V^2}\left(\sec^2\alpha\right)$$

$$y=x\tan\alpha-\frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$$

Note: This is known as the Cartesion equation of a parabola.

Applications of Calculus

Example 1: A projectile is fired at a speed of 840 m/sec at an angle of 60° to the horizontal. Find:

(i) the maximum height attained by the projectile.

(ii) the horizontal range to the nearest metre.

(iii) the time it takes to get to a point 21 km downrange.

Solution 1:

$v=840$ m/s

$\dot{x}=840\cos 60°=420$ m/s

The equations of motion are:

Vertically

$\ddot{y}=-10$

$\dot{y}=-10t+C_1$

when $t=0$, $\dot{y}=420\sqrt{3}$, then $C_1=420\sqrt{3}$

i.e. $\dot{y}=-10t+420\sqrt{3}$

thus $y=-5t^2+420\sqrt{3}t+C_2$

when $t=0, y=0$ $\therefore$ $C_2=0$

$\therefore$ $y=-5t^2+420\sqrt{3}t$

Horizontally

$\ddot{x}=0$

$\dot{x}=C_1$

when $t=0$, $\dot{x}=420$ $\therefore$ $C_1=420$

i.e. $\dot{x}=420$

thus $x=420t+C_2$

when $t=0$, $x=0$ $\therefore$ $C_2=0$

$\therefore$ $x=420t$

(i) Maximum height is reached when $\dot{y}=0$

i.e. $-10t+420\sqrt{3}=0$

$t=42\sqrt{3}$

Substituting this into y gives:

$y=-5\left(42\sqrt{3}\right)^2+420\sqrt{3}\left(42\sqrt{3}\right)=26,460$

$\therefore$ maximum height attained = 26,460 m

(ii) The object will hit the ground when $y=0$

i.e. $-5t^2+420\sqrt{3}t=0$

$t\left(420\sqrt{3}-5t\right)=0$

i.e. $t=0$, $84\sqrt{3}$

Substituting this into x gives:

$x=420\times 84\sqrt{3}=61,106.75$

$\therefore$ horizontal range of projectile is 61,107 m to the nearest metre #

... to reach 21,000 m is:

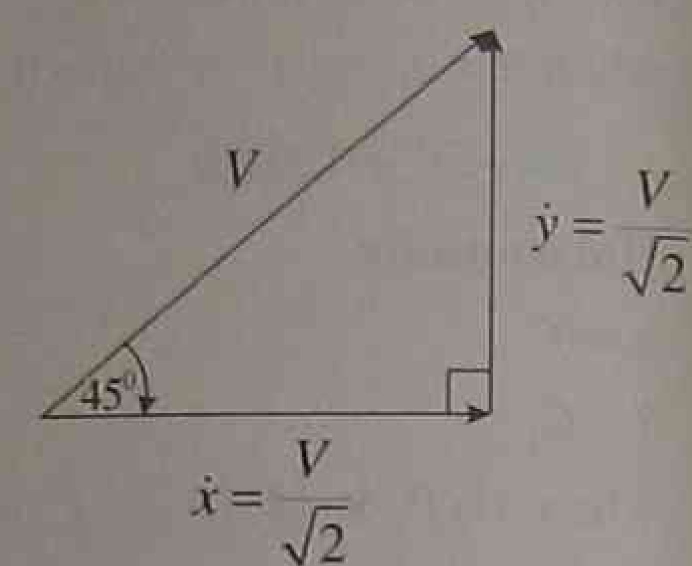
Example 2: An athlete in the game of shot put, fires a shot at an angle of 45° to the horizontal from 2 m above the ground with an initial speed of V m/s. If the shot lands 20 m away and taking $g = 10\,\text{m/s}^2$, find:

(i) The initial velocity of projection.
(ii) The maximum height reached by the shot above ground level.
(iii) The angle and speed at which the shot strikes the ground.

Solution 2:

$$\dot{y} = V\sin 45^\circ = \frac{V}{\sqrt{2}}$$

$$\dot{x} = V\cos 45^\circ = \frac{V}{\sqrt{2}}$$



The equations of motion are:

Vertically

$$\ddot{y} = -10$$

$$\dot{y} = -10t + C_1$$

when $t = 0,\ \dot{y} = \frac{V}{\sqrt{2}} \quad \therefore\ C_1 = \frac{V}{\sqrt{2}}$

i.e. $\dot{y} = -10t + \frac{V}{\sqrt{2}}$

thus $y = -5t^2 + \frac{V}{\sqrt{2}}t + C_2$

when $t = 0,\ y = 2 \quad \therefore\ C_2 = 2$

$\therefore\ y = -5t^2 + \frac{V}{\sqrt{2}}t + 2$

Horizontally

$$\ddot{x} = 0$$

$$\dot{x} = C_1$$

when $t = 0, \dot{x} = \frac{V}{\sqrt{2}} \quad \therefore\ C_1 = \frac{V}{\sqrt{2}}$

i.e. $\dot{x} = \frac{V}{\sqrt{2}}$

thus $x = \frac{V}{\sqrt{2}}t + C_2$

when $t = 0,\ x = 0 \quad \therefore\ C_2 = 0$

$\therefore\ x = \frac{V}{\sqrt{2}}t$

(i) The shot strikes the ground when $x = 20$ and $y = 0$:

i.e. $20 = \frac{V}{\sqrt{2}}t$ i.e. $t = \frac{20\sqrt{2}}{V}$

Substituting this into y gives:

$$y = -5\left(\frac{20\sqrt{2}}{V}\right)^2 + \frac{V}{\sqrt{2}}\left(\frac{20\sqrt{2}}{V}\right) + 2 = 0$$

i.e. $\frac{-4000}{V^2} + 20 + 2 = 0$

i.e. $22V^2 = 4000$

$V = 13.5$ m/s

$\therefore$ The initial velocity of projection is 13.5 m/s to 1 d.p #

(ii) Maximum height is attained when $\dot{y} = 0$

i.e. $-10t + \frac{V}{\sqrt{2}} = 0$ i.e. $t = \frac{V}{10\sqrt{2}}$

Substituting this into y gives:

$$y = -5\left(\frac{V}{10\sqrt{2}}\right)^2 + \frac{V}{\sqrt{2}}\left(\frac{V}{10\sqrt{2}}\right) + 2$$

$$= \frac{-5V^2}{200} + \frac{V^2}{20} + 2$$

$$= \frac{5V^2}{200} + 2 \qquad \left(V^2 = \frac{4000}{22}\right)$$

$$= \frac{5}{200} \times \frac{4000}{22} + 2$$

$$= 6.5$$

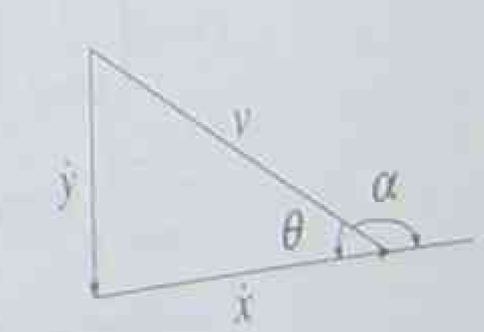
$\therefore$ maximum height attained by shot put is 6.5 m #

(iii) Shot strikes the ground when $t = \frac{20\sqrt{2}}{V}$

at this point $\dot{y} = \frac{-200\sqrt{2}}{V} + \frac{V}{\sqrt{2}} = -11.44$ m/s and $\dot{x} = \frac{V}{\sqrt{2}} = 9.53$ m/s

$$v^2 = \dot{x}^2 + \dot{y}^2$$

$$= (9.53)^2 + (-11.44)^2$$



$\therefore\ v = 14.89$ m/s to 2 d.p.

from the diagram $\tan\theta = \frac{|\dot{y}|}{\dot{x}} \quad \therefore\ \theta = 50^\circ 12'$

$= 180^\circ - \theta = 129^\circ 48'$ #

Example 3: A particle is projected from level ground with velocity 20 m/s at an angle θ to the horizontal. The acceleration due to gravity is asssumed to be 10 m/s^2.

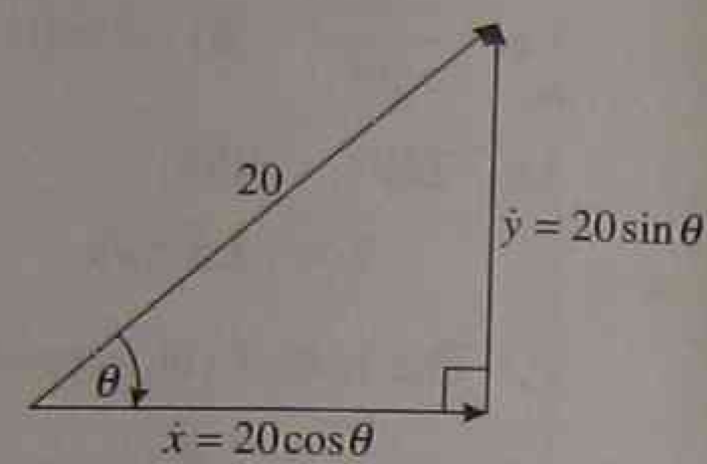
(i) Show that $x = 20t\cos\theta$ and $y = 20t\sin\theta - 5t^2$.

(ii) If the projectile is to clear a board 5 metres high located 20 metres away. Find the range of values that θ must lie in.

Solution 3:

(i) $\dot{y} = 20\sin\theta$

$\dot{x} = 20\cos\theta$



The equations of motion are:

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

when $t = 0,\ \dot{y} = 20\sin\theta \quad \therefore\ C_1 = 20\sin\theta$

i.e. $\dot{y} = -10t + 20\sin\theta$

thus $y = -5t^2 + 20t\sin\theta + C_2$

when $t = 0,\ y = 0 \quad \therefore\ C_2 = 0$

$\therefore\ y = -5t^2 + 20t\sin\theta$

Horizontally

$\ddot{x} = 0$

$\dot{x} = C_1$

when $t = 0,\ \dot{x} = 20\cos\theta \quad \therefore\ C_1 = 20\cos\theta$

i.e. $\dot{x} = 20\cos\theta$

thus $x = 20t\cos\theta + C_2$

when $t = 0,\ x = 0 \quad \therefore\ C_2 = 0$

$\therefore\ x = 20t\cos\theta$ #

(ii) Now the particle **just** clears the board 5m high which is 20m away at $x = 20,\ y = 5$

Substituting this into y gives:

$$y = -5\left(\frac{1}{\cos\theta}\right)^2 + 20\left(\frac{1}{\cos\theta}\right)\sin\theta$$

$$= \frac{-5}{\cos^2\theta} + 20\tan\theta$$

$$= -5\sec^2\theta + 20\tan\theta$$

$$= -5\left(1 + \tan^2\theta\right) + 20\tan\theta$$

$$= -5\tan^2\theta + 20\tan\theta - 5$$

Let $y = 5$ and solve for θ.

i.e. $-5\tan^2\theta + 20\tan\theta - 10 = 0$

$\tan^2\theta - 4\tan\theta + 2 = 0$

$$\therefore\ \tan\theta = \frac{4 \pm \sqrt{16 - 4\times1\times2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

$\therefore$ for $\theta = 30°22'$ or $\theta = 73°41'$, it will just pass over the top of the board.

Thus, for the particle to clear the board, it must be projected between $30°22' \le \theta \le 73°41'$ #

Review Exercises

(A) Related Rates of Change

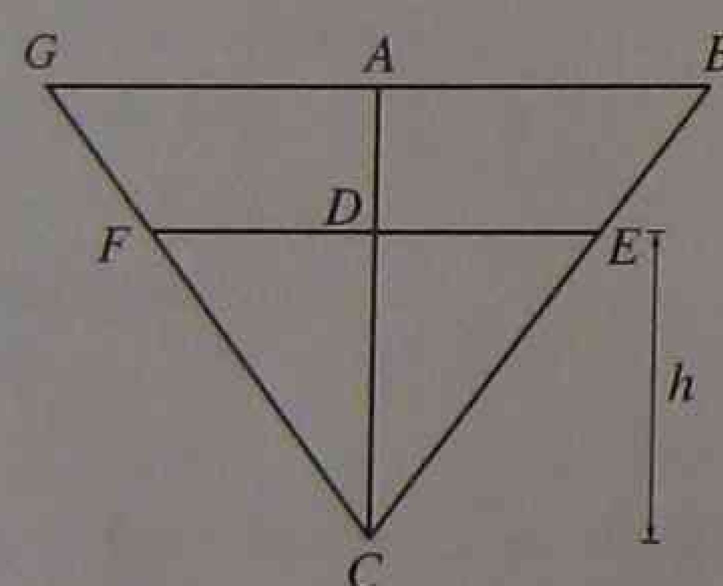
1. The volume of a spherical balloon is expanding at the rate of 8 cubic centimetres per second. Find the rate of increase of the radius when the volume is 200 cubic centimetres.

2. Water runs into a conical tank at the rate of 1 m^3/min. The tank stands point down and has a height of 10 m and a radius at the top of 5 m. How fast is the level rising when the water is 6 m deep?

3. When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 30 cm?

4. A trough is in the shape of a triangular prism. It is 5 metres long, and each cross-section is an equilateral triangle of side length 1 metre, as shown.

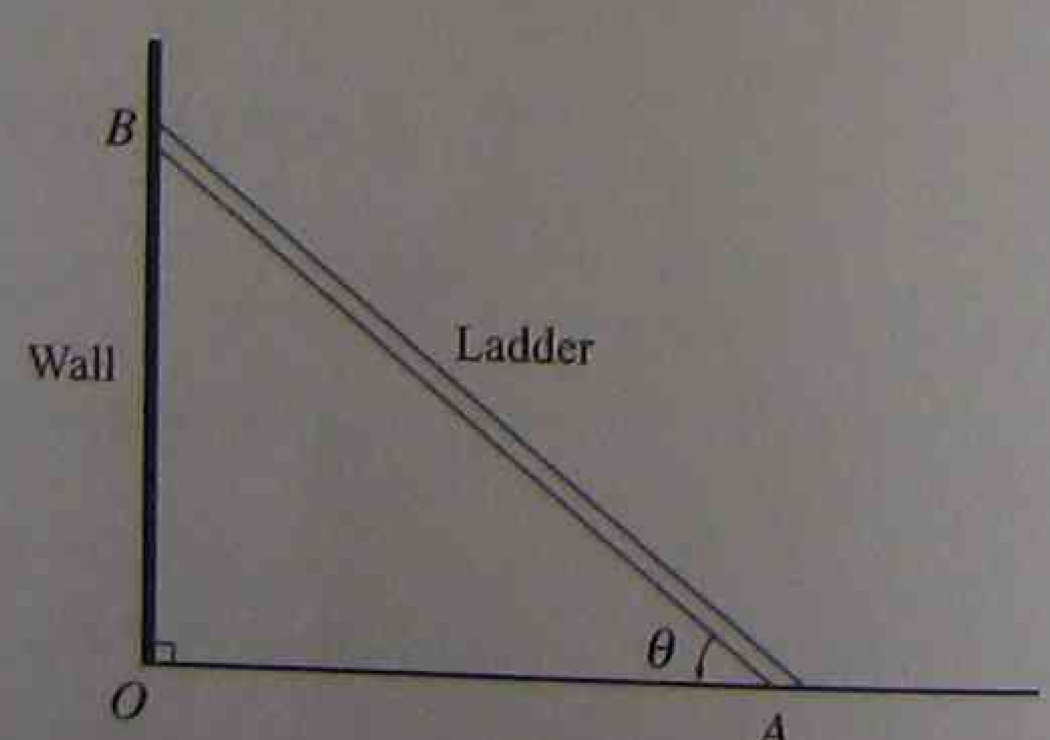
   (i) Show that the volume of the water is given by $V = \dfrac{5h^2}{\sqrt{3}}$.



The trough is filled with water to a depth h metres.

   (ii) Water runs out from a hole in the bottom at the constant rate of 0.1 m^3/min. Find the rate at which the water level is falling when it is 20 cm deep.

5. A 4 metre ladder is leaning against a wall when its base starts to slide away. By the time the base is 2 metres from the wall, the base is moving at the rate of 1 m/sec.



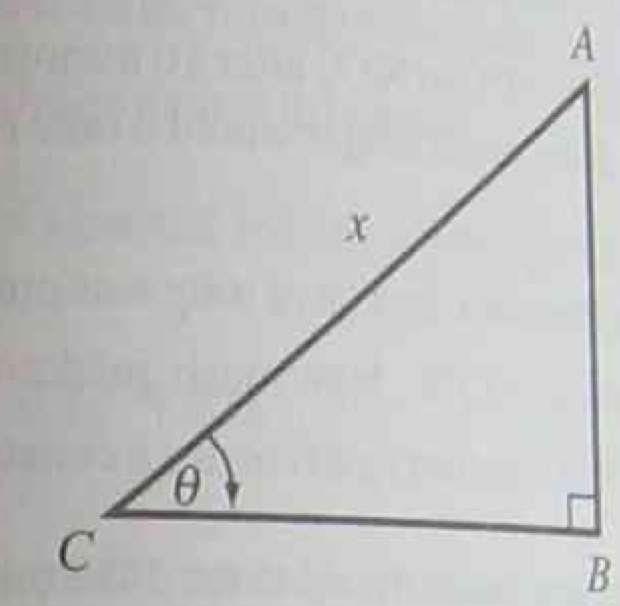
Find:

   (i) at what speed the top of the ladder is sliding down the wall.

   (ii) the rate at which the area of the triangle formed by the ladder, wall and ground is changing.

   (iii) the rate at which the angle θ between the ladder and the ground is changing.

6. Consider ΔABC below:



   (i) Show that the perimeter P of the triangle is $P = x(1+\sin\theta+\cos\theta)$.

   (ii) Show that the area A of the triangle is $A = \dfrac{x^2 \sin 2\theta}{4}$.

   (iii) If $x = 1$ m and θ is increasing at the rate at 0.1 rad/sec find the rate at which the perimeter and the area of the triangle are changing when $x = \dfrac{\pi}{3}$.

(B) Further Exponential Growth and Decay

7. The population of a rare species of birds is changing according to the equation:

$$\frac{dN}{dt} = -k(N-2500), \text{ for some constant } k$$

   (i) Verify that $N = 2{,}500 + Ae^{-kt}$ satisfies the equation for some constant A.

   (ii) If initially there are 5,000 birds but after 5 years there are only 4,250 left, find when the number of birds would have fallen to 3,000.

   (iii) Sketch the graph of the population against time.

8. In a school of 500 fish, the number N infected at time t years is given by:

$$N = \frac{500}{1+Ae^{-500t}}$$

   (i) If initially 5 fish were infected, on which day will 100 be infected?

...ill be infected eventually?

9. The rate at which a body cools in air is proportional to the difference between its temperature T and surrounding temperature S, and is given by:

$$\frac{dT}{dt} = -k(T - S)$$

where t is the time in hours and k is a constant.

(i) Show that $T = S + Ae^{-kt}$, for some constant A satisfies the equation.

(ii) A cup of soup cooled from 90°C to 60°C after 10 minutes in a room whose temperature was 20°C. How much longer would it take for the soup to cool to 35°C?

(iii) If instead of being left in the room, the cup of soup was put immediately in a freezer whose temperature is $-15°\text{C}$. How much quicker will the soup reach the desired temperature of $35°\text{C}$, assuming k remains a constant.

10. The rate at which a body cools in air is proportional to the difference between its temperature T and the surrounding temperature S, and is given by:

$$\frac{dT}{dt} = -k(T - S)$$

where t is the time in hours and k is a constant.

(i) Show that $T = S + Ae^{kt}$, for some constant A satisfies the equation.

(ii) A pan whose temperature was $46°\text{C}$ was put into a refrigerator. Ten minutes later, the pan's temperature was $39°\text{C}$; 10 minutes after that, it was $33°\text{C}$. Find the temperature of the refrigerator to the nearest degree.

(C) Rectilinear Motion

11. The acceleration of a particle moving in a straight line is given by:

$$\frac{d^2x}{dt^2} = -\frac{1}{2}e^{-4x}$$

where x is the displacement from O. Let v represent the velocity in m/s.

(i) Find an expression for v given $v = \frac{1}{2}$, when $x = 0$.

(ii) Find the time T taken for the particle to travel from $x = 0$ to $x = 1$.

(iii) Does the particle ever return to the origin?

12. The velocity of a particle in m/s, moving in a straight line from the origin O is given by:

$$v = \sqrt{\frac{6x^2 + 8x}{x^2 + 1}}$$

where x is the displacement from O.

(i) Find an expression for the acceleration a in terms of x.

(ii) Find the maximum velocity of the particle.

(iii) Describe the motion of the particle.

13. The acceleration of a particle moving in a straight line from O is given by:

$$\frac{d^2x}{dt^2} = 8x^3 + 32x$$

where x is the distance in metres from O at time t seconds.

(i) Find an expression for v, if $v = 8$ at $x = 0$.

(ii) If the particle is initially at the origin, show that its displacement x is given by $x = 2\tan 4t$.

14. A particle moves in a straight line from a fixed point O on the x-axis. Its acceleration is given by:

$$\frac{d^2x}{dt^2} = 8x - 12$$

where x is the displacement in metres and t is the time in seconds.

(i) Show that $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(ii) Hence, show that $v^2 = 8x^3 - 24x - 32$ if $v = 4\sqrt{3}$ at $x = 5$.

(iii) What is the minimum displacement of the particle?

(D) Simple Harmonic Motion

15. The displacement x metres of a particle moving in simple harmonic motion is given by:

$$x = 5\sin\left(\frac{\pi}{\ldots}t\right)$$

(i) What is the period and amplitude of the motion?

(ii) What is the speed v of the particle as it moves through the equilibrium position?

16. A particle is moving with simple harmonic motion in a straight line. It has an amplitude of 4 metres and a period of 3 seconds. How long would it take for the body to travel from one of the extremities of its path of motion to a point 1 metre away?

17. An object moves in a horizontal line so that its acceleration is given by:

$$\frac{d^2x}{dt^2} = -4x$$

where x is the displacement from a fixed point O. Initially the object is 5 metres from O and has a velocity of 10 m/s.

(i) Show that the velocity of the object is given by $v = 2\sqrt{50 - x^2}$.

(ii) What is the greatest velocity of the object?

18. An object moves so that its acceleration metres per second at a position x metres from the origin is given by:

$$\frac{d^2x}{dt^2} = -9\pi^2 x$$

Initially, the particle is at the origin, moving with a velocity of 7 metres per second. Find:

(i) The velocity of the particle at any time t.

(ii) The greatest distance from the origin.

(iii) The time when it first returns to the origin.

19. The velocity v cm/s of a particle moving in SHM along the x-axis is given by:

$$v^2 = 2 + 20x - 5x^2$$

(i) Between which two points is the particle oscillating?

(ii) What is centre and amplitude of the motion?

(iii) What is the period of oscillation?

20. The displacement x metres of a particle from the origin O is given by:

$$x = \frac{1}{2}(\cos 2t + \sin 2t)$$

where t is the time in seconds.

(i) Show that the motion is simple harmonic.

21. The rise and fall of the water level in a harbour changes in simple harmonic motion. Low tide is at 5:30 a.m. and high tide is at 11:30 a.m. and the level of the water at each of these times is 6 metres and 14 metres respectively. If a ship requires 11 metres of water to safely enter the harbour, find:

(i) the period of the motion.

(ii) the amplitude of the motion.

(iii) the earliest time between 5:30 a.m. and 11:30 a.m. that the ship may safely enter the harbour.

(E) Projectile Motion

22. An object is projected from the top of a vertical cliff 25 m above the horizontal ground at an angle α where $\tan\alpha = \frac{3}{4}$, with an initial speed of 40 m/s. Taking $g = 10\,\text{m/s}^2$, find:

(i) the maximum height attained by the object.

(ii) the time taken for the object to next return to a height of 25 m above the ground.

(iii) the speed of the object 2 seconds after its projection.

23. A ball is kicked from a point O on the edge of a roof of a building which is 15 metres above the ground. The ball is kicked at an angle of 30° to the horizontal with a velocity of $20\sqrt{3}$ m/s. Taking $g = 10\,\text{m/s}^2$, find:

(i) the exact distance at which the ball strikes the ground.

(ii) if the ball was instead kicked horizontally from O find the speed of projection if it is to attain the same horizontal distance. (Standard results about projectile motion may be quoted without proof.)

24. A stone is projected at 40 m/s at an angle of elevation θ from the top of a lighthouse 30 m above the ground down into the water below. It strikes the water 120 m away from the base of the lighthouse. Take $g = 10\,\text{m/s}^2$.

Given that $x = 40t\cos\theta$ and $y = -5t^2 + 40t\sin\theta + 30$, find the angle(s) of projection θ to the nearest degree.

25. Two buildings are situated 100 metres apart on level ground. Their heights are respectively 180 and 200 metres. An object is projected from the top of the shorter building at an angle of 45°, striking the top of the taller building at P. Taking $g = 10\,\text{m/s}^2$, find:

(i) the time taken for the object to reach point P.

(ii) the initial velocity of projection.

Applications of

26. An aircraft is flying at a constant height of 1,500 m with a constant velocity of 720 km/h
A shell is fired directly when the plane passes overhead at an angle θ with initial velocity of 1200 m/s. Taking $g = 10\,\text{m/s}^2$, find:

(i) The value of θ to the nearest minute if the shell strikes the aircraft.

(ii) The time(s) when the shell strikes the aircraft. Express your answer in exact form.

27. (i) Prove that the horizontal range of a particle projected upwards at an angle x to the horizontal plane with velocity V m/s is:

$$\frac{V^2 \sin 2\alpha}{g} \text{ metres}$$

where g m/s^2 is the acceleration due to gravity.

(ii) Find the angle α for the range to be maximised.

(iii) A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second. As the jets of water rotate through 360°, at the same time the angle of the jets varies from 15° to 60° to the horizontal.

Show that, from a fixed point O on level ground, the sprinkler will wet the surface of an annular region with centre O and area $\dfrac{3\pi V^4}{4g^2}$ square metres.

Worked Solutions to Review Exercises

1. Volume of the sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Now, $\dfrac{dV}{dt} = \dfrac{dv}{dr} \cdot \dfrac{dr}{dt}$

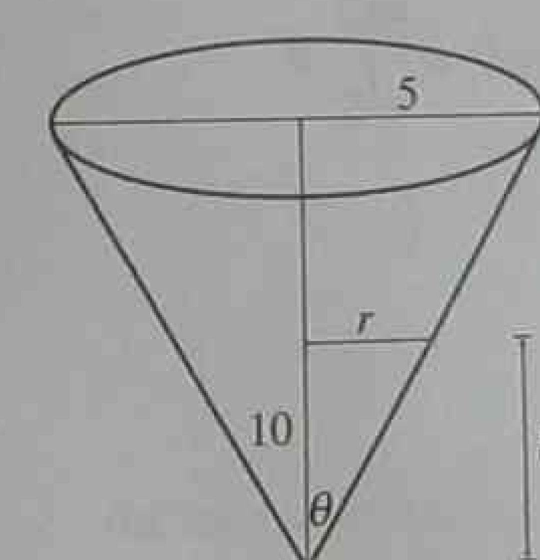
$\dfrac{dV}{dr} = 4\pi^2$ and $200 = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{150}{\pi}$$

$$r = 3.63 \text{ cm}$$

$\therefore\ 8 = 4\pi r^2 \cdot \dfrac{dr}{dt}$

i.e. $\dfrac{dr}{dt} = \dfrac{2}{\pi r^2} = \dfrac{2}{\pi (3.63)^2} = 0.048$ cm/s #

2. 

From the diagram:

$\tan\theta = \dfrac{5}{10} = \dfrac{1}{2}$

also $\tan\theta = \dfrac{r}{h}$

$\therefore\ \dfrac{1}{2} = \dfrac{r}{h}$ i.e. $r = \dfrac{h}{2}$

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

now, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$

i.e. $1 = \dfrac{3\pi h^2}{12} \times \dfrac{dh}{dt}$ (at $h = 6$)

i.e. $\dfrac{dh}{dt} = \dfrac{1}{9\pi} = 0.035$ m^3/min to 3 d.p. #

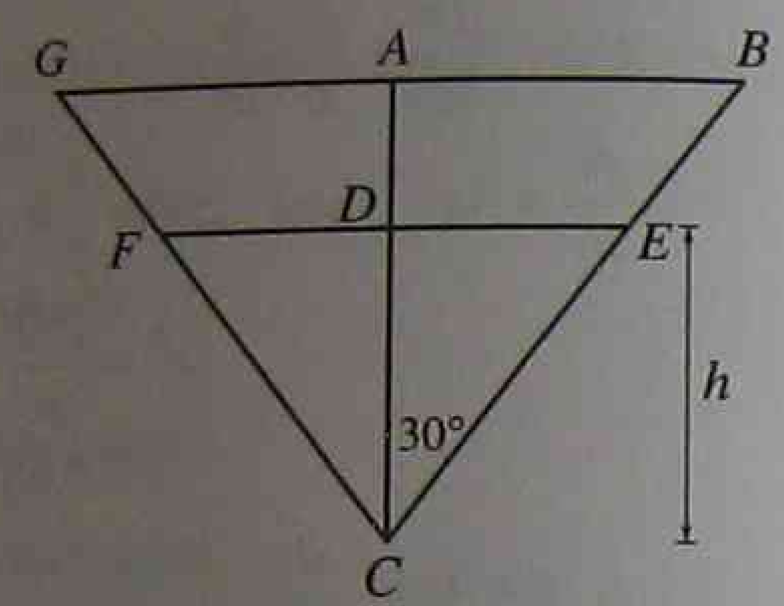
3. Area of plate: $A=\pi r^2$, $\frac{dr}{dt}=0.01$

then $\frac{dA}{dr}=2\pi r$

$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$ when $r=30$

$=2\pi r\times 0.01=0.6\pi=1.88\text{ cm}^2/\text{min. to 2 d.p.}$ #

4. (i) Looking at $\Delta\, DEC$:



$\cos 30°=\frac{h}{CE}$

i.e. $CE=\frac{2h}{\sqrt{3}}$

By symmetry $FC=\frac{2h}{\sqrt{3}}$

Area of $\Delta\, FEC=\frac{1}{2}ab\sin C$

$=\frac{1}{2}\times\frac{2h}{\sqrt{3}}\times\frac{2h}{\sqrt{3}}\times\sin 60°=\frac{h^2}{\sqrt{3}}$

$\therefore$ Volume of the water is given by $V=\frac{5h^2}{\sqrt{3}}$ #

(ii) $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $\frac{dV}{dt}=0.1$

i.e. $0.1=\frac{10h}{\sqrt{3}}\times\frac{dh}{dt}$ at $h=0.2$ m

$\therefore\ \frac{dh}{dt}=\frac{0.1\sqrt{3}}{10h}=\frac{0.1\sqrt{3}}{2}=0.087\text{ m}^3/\text{min to 3 d.p.}$ #

5. (i) Let $OA=x$, $OB=y$

then $x^2+y^2=16$

$\therefore\ y=\sqrt{16-x^2}=(16-x^2)^{\frac{1}{2}}$

Now, $\frac{dy}{dx}=\frac{1}{2}(16-x^2)^{-\frac{1}{2}}\cdot -2x=\frac{-x}{\sqrt{16-x^2}}$

when $x=2$ $\frac{dx}{dt}=1$ $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$

$=\frac{-x}{\sqrt{16-x^2}}\times 1$ at $x=2$

$=\frac{-2}{\sqrt{12}}=\frac{-1}{\sqrt{3}}$

$\therefore$ the top of the ladder is moving down the wall at $\frac{1}{\sqrt{3}}$ m/s #

(ii) Area of triangle: $A=\frac{1}{2}xy=\frac{1}{2}x(16-x^2)^{\frac{1}{2}}$

Now, $\frac{dA}{dx}=\frac{1}{2}x.\frac{-x}{\sqrt{10-x^2}}+\frac{\sqrt{16-x^2}}{2}$

$=\frac{-x^2}{2\sqrt{16-x^2}}+\frac{\sqrt{16-x^2}}{2}$

$\therefore\ \frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$=\left(\frac{-x^2}{2\sqrt{16-x^2}}+\frac{\sqrt{16-x^2}}{2}\right)\times 1$ at $x=2$

$=\frac{-4}{2\sqrt{12}}+\frac{\sqrt{12}}{2}$

$=-\frac{1}{\sqrt{3}}+\sqrt{3}$

$=\frac{2}{\sqrt{3}}$

$\therefore$ the area is increasing at the rate of $\frac{2}{\sqrt{3}}$ m²/s #

(iii) $\tan\theta=\frac{y}{x}$, $y=\sqrt{16-x^2}$

$\therefore\ \theta=\tan^{-1}\left(\frac{\sqrt{16-x^2}}{x}\right)$

Now, $\frac{d\theta}{dx}=\frac{1}{1+\frac{16-x^2}{x^2}}\times\frac{d}{dx}\left(\frac{\sqrt{16-x^2}}{x}\right)=\frac{x^2}{16}\times\left(\frac{x\cdot\frac{-x}{\sqrt{16-x^2}}-\sqrt{16-x^2}}{x^2}\right)$

$$= \frac{1}{16}\left(\frac{-x^2}{\sqrt{16-x^2}} - \sqrt{16-x^2}\right)$$

$$= \frac{1}{16}\left(\frac{-x^2-\left(16-x^2\right)}{\sqrt{16-x^2}}\right)$$

$$= \frac{-16}{16\sqrt{16-x^2}} = \frac{-1}{\sqrt{16-x^2}}$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$$

$$= \frac{-1}{\sqrt{16-x^2}} \times 1 \quad \text{at} \quad x = 2$$

$$= \frac{-1}{2\sqrt{3}}$$

$\therefore$ the angle θ is decreasing at rate of $\frac{1}{2\sqrt{3}}$ rad./sec. #

6. (i) $\sin\theta = \frac{AB}{x} \quad \therefore \quad AB = x\sin\theta$

$\cos\theta = \frac{BC}{x} \quad \therefore \quad BC = x\cos\theta$

$\therefore$ perimeter of $\Delta\, ABC$ is given by:

$P = x + x\sin\theta + x\cos\theta = x\left(1+\sin\theta+\cos\theta\right)$ #

(ii) Area of $\Delta\, ABC = \frac{1}{2}(BC)(AB) = \frac{1}{2}(x\cos\theta)(x\sin\theta)$

$$= \frac{1}{2}x^2\sin\theta\cos\theta = \frac{x^2\sin 2\theta}{4} \ \#$$

(iii) $P = 1+\sin\theta+\cos\theta,\ A = \frac{\sin 2\theta}{4}$, given $x = 1$

$\frac{dP}{d\theta} = \cos\theta - \sin\theta,\ \frac{dA}{d\theta} = \frac{\cos 2\theta}{2}$

$$\therefore \quad \frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt}$$

$$= \left(\cos\theta - \sin\theta\right) \times 0.1 \ \text{ when } \ \theta = \frac{\pi}{3}$$

$$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \times 0.1 = \frac{1-\sqrt{3}}{2} \times 0.1 = -0.037 \text{ m/s}$$

$\therefore$ the perimeter is decreasing at the rate of 0.037 m/s #

Similarly, $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$$= \frac{\cos 2\theta}{2} \times 0.1$$

$$= -0.025$$

$\therefore$ the area is decreasing at the rate of 0.025 m^2/s #

7. (i) $N = 2500 + Ae^{-kt}$

$\frac{dN}{dt} = -kAe^{-kt} = -k\left(N-2500\right)$ #

(ii) $N = 2500 + Ae^{-kt}$

at $t = 0, N = 5000$

i.e. $5000 = 2500 + A \quad \therefore \quad A = 2500$

at $t = 5, N = 4250$

i.e. $4250 = 2500 + 2500e^{-5k}$

$$e^{-5k} = 0.7$$

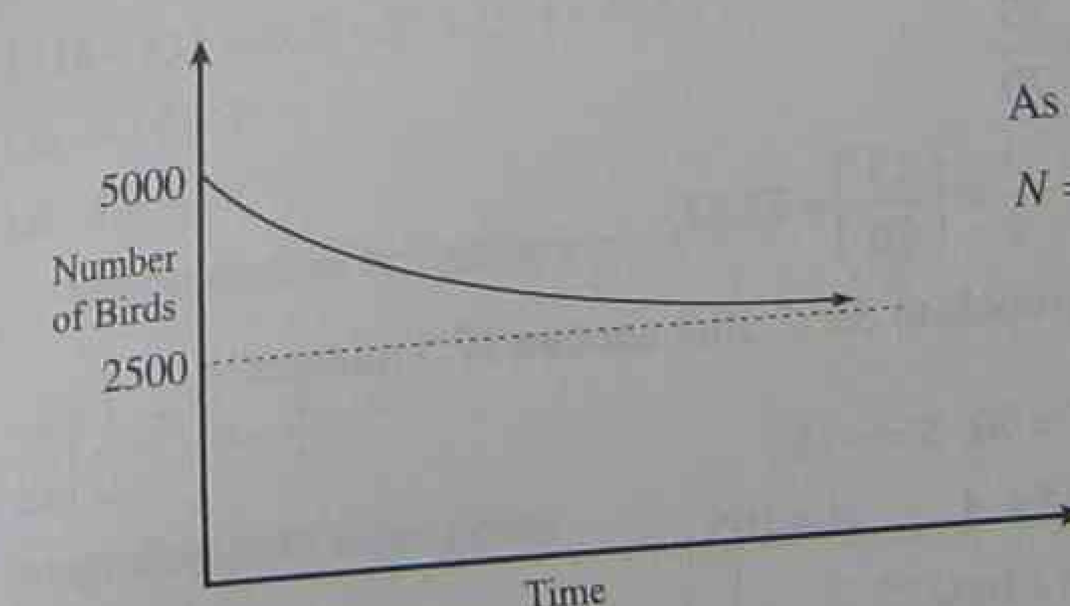
$$k = -\frac{1}{5}\ln\left(0.7\right) = 0.0713..$$

$\therefore \quad 3000 = 2500 + 2500e^{-kt}$

i.e. $e^{-kt} = \frac{1}{5}$

$$t = -\frac{1}{k}\ln\left(\frac{1}{5}\right) = 22.6 \text{ years to 1 d.p. \#}$$

(iii)



As $t \to \infty$, $N \to 2500$

$N = 5000$ at $t = 0$

8. (i) At $t = 0, N = 5$

i.e. $5 = \frac{500}{1+A}$ i.e. $A = 99$

$$\therefore \quad 100 = \frac{500}{1+99e^{-500t}}$$

i.e. $100 + 9900e^{-500t} = 500$

$$e^{-500t} = \frac{4}{99}$$

$$t = -\frac{1}{500}\ln\left(\frac{4}{99}\right) = 0.0064...$$

$$t = 0.0064... \times 365 = 2.3$$

∴ 100 fish will be infected on the 3rd day. #

(ii) $\lim_{t \to \infty} \frac{500}{1+99e^{-500t}} = \frac{500}{1+0} = 500$

∴ eventually all the fish present will be infected #

9. (i) $T = S + Ae^{-kt}$

$$\frac{dT}{dS} = -kAe^{-kt} = -k(T-S) \text{ \#}$$

(ii) At $t=0, T=90, S=20$

i.e. $90 = 20 + A$ ∴ $A = 70$

i.e. $T = 20 + 70e^{-kt}$

at $t = 10, T = 60$

i.e. $60 = 20 + 70e^{-10k}$

$$e^{-10k} = \frac{4}{7}$$

$$k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) = 0.0559...$$

∴ $35 = 20 + 70e^{kt}$

$$e^{kt} = \frac{15}{70}$$

$$t = -\frac{1}{k}\ln\left(\frac{15}{70}\right) = 27.52$$

∴ the cup cools to $35°C$ after another 17.5 minutes. #

(iii) At $t=0, T=90, S=-15$

i.e. $90 = -15 + A$ ∴ $A = 105$

i.e. $T = -15 + 105e^{-kt}$

∴ $35 = -15 + 105e^{-kt}$

Applications of Calculus

∴ the cup will cool to $35°C$ 14.2 minutes quicker if placed in the fridge. #

10. (i) $T = S + Ae^{kt}$

$$\frac{dT}{dt} = kAe^{kt} = k(T-S) \text{ \#}$$

(ii) $T = S + Ae^{kt}$

at $t = 0, T = 46°$

i.e. $46 = S + A$ i.e. $A = 46 - S$

∴ $T = S + (46-S)e^{kt}$

at $t = 10, T = 39°$

i.e. $39 = S + (46-S)e^{10k}$ ______(1)

and at $t = 20, T = 33°$

i.e. $33 = S + (46-S)e^{20k}$ ______(2)

rearranging equation (1) gives:

$$\frac{39-S}{46-S} = e^{10k}$$

$$\therefore \left(\frac{39-S}{46-S}\right)^2 = e^{20k}$$

substituting this into (2) gives:

$$33 = S + (46-S)\left(\frac{39-S}{46-S}\right)^2$$

$$= S + \frac{(39-S)^2}{46-S}$$

$$1518 - 33S = 46S - S^2 + 1521 - 78S + S^2$$

i.e. $-1S = 3°$

i.e. $S = -3°C$

∴ temperature of refrigerator $= -3°C$ #

11. (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-4x}$

integrating both sides gives:

$$\frac{1}{2}v^2 = -\frac{1}{2}\int e^{-4x}\,dx$$

$$= \frac{1}{8}e^{-4x} + C$$

1

i.e. $\frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}\times 1 + C$ i.e. $C = 0$

$\therefore\ v^2 = \frac{1}{4}e^{-4x}$

i.e. $v = \pm\frac{1}{2}e^{-2x}$

but since $v = \frac{1}{2}$ at $x = 0$ $\therefore\ v = \frac{1}{2}e^{-2x}$ #

(ii) $v = \frac{1}{2}e^{-2x}$

$\frac{dx}{dt} = \frac{1}{2}e^{-2x}$ $\left(\text{noting } v = \frac{dx}{dt}\right)$

i.e. $\frac{dt}{dx} = 2e^{2x}$

integrating both sides from $x = 0$ to $x = 1$, gives:

$$T = \int_0^1 2e^{2x}\,dx = \left[e^{2x}\right]_0^1 = \left(e^2 - 1\right) \text{ sec. \#}$$

(iii) $v = \frac{1}{2}e^{-2x}$

now, $e^{-2x} > 0$ for all x $\therefore$ $v > 0$ for all x and

hence, the particle never returns to the origin, but instead always moves further to the right of the origin. #

12. (i) $v = \sqrt{\frac{6x^2 + 8x}{x^2 + 1}}$

$$v^2 = \frac{6x^2 + 8x}{x^2 + 1}$$

$$\frac{1}{2}v^2 = \frac{3x^2 + 4x}{x^2 + 1}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{3x^2 + 4x}{x^2 + 1}\right)$$

$$a = \frac{\left(x^2+1\right).(6x+4) - \left(3x^2+4x\right).2x}{\left(x^2+1\right)^2}$$

$$= \frac{6x^3 + 4x^2 + 6x + 4 - 6x^3 - 8x^2}{\left(x^2+1\right)^2} = \frac{-4x^2 + 6x + 4}{\left(x^2+1\right)^2} \text{ \#}$$

(ii) Maximum velocity occurs when $a = 0$

i.e. $-4x^2 + 6x + 4 = 0$

$$-\left(4x^2 - 6x - 4\right) = 0$$

$$-(2x+1)(2x-4) = 0$$

i.e. $x = -\frac{1}{2}, 2$

Clearly, $x = 2$ will yield a maximum

$\therefore$ maximum velocity is: $v = \sqrt{\frac{24+16}{4+1}} = 2\sqrt{2}$ m/s #

(iii) At $x = 0, v = 0, a = 4$

Thus particle moves to the right and continues to do so ($v > 0$), reaching its maximum velocity at $x = 2$ after which it continues to move to the right with ever decreasing velocity ($a < 0$) and approaches a constant velocity of $\sqrt{6}$ m/s and theoretically never comes to rest. #

13. (i) $\frac{d^2x}{dt^2} = 8x^3 + 32x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x^3 + 32x$$

integrating both sides, gives:

$\frac{1}{2}v^2 = 2x^4 + 16x^2 + C$

at $x = 0$, $v = 8$ i.e. $C = 32$

$\therefore\ v^2 = 4x^4 + 32x^2 + 64$

$$= \left(2x^2 + 8\right)^2$$

$\therefore\ v = \pm\left(2x^2 + 8\right)$

but since $v = 8$ at $x = 0$ $\therefore$ $v = 2x^2 + 8$ #

(ii) $v = 2x^2 + 8$

$$\frac{dx}{dt} = 2x^2 + 8$$

$$\frac{dt}{dx} = \frac{1}{2x^2 + 8}$$

integrating both sides, gives:

$$t = \int \frac{1}{2x^2 + 8}\,dx$$

$$=\frac{1}{2}\int\frac{1}{x^2+4}\,dx$$

$$=\frac{1}{2}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)+C$$

at $x=0,\ t=0\ \therefore\ C=0$

$$\therefore\ t=\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)$$

$$4t=\tan^{-1}\left(\frac{x}{2}\right)$$

$\therefore\ x=2\tan 4t$ #

14. (i) $a=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Using the chain rule:

$$a=\frac{dv}{dt}$$

$$=\frac{dv}{dx}\times\frac{dx}{dt}$$

$$=v\frac{dv}{dx}\qquad\left(\text{since } v=\frac{dx}{dt}\right)$$

noting $v=\dfrac{d}{dv}\left(\dfrac{1}{2}v^2\right)$

$\therefore\ a=\dfrac{d}{dv}\left(\dfrac{1}{2}v^2\right)\times\dfrac{dv}{dx}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ #

(ii) $\dfrac{d^2x}{dt^2}=8x-12$

$$\frac{d}{dv}\left(\frac{1}{2}v^2\right)=8x-12$$

$$\frac{1}{2}v^2=4x^2-12x+C$$

at $x=5,\ v=4\sqrt{3}$

i.e. $\dfrac{1}{2}\left(4\sqrt{3}\right)^2=4\times25-12\times5+C$

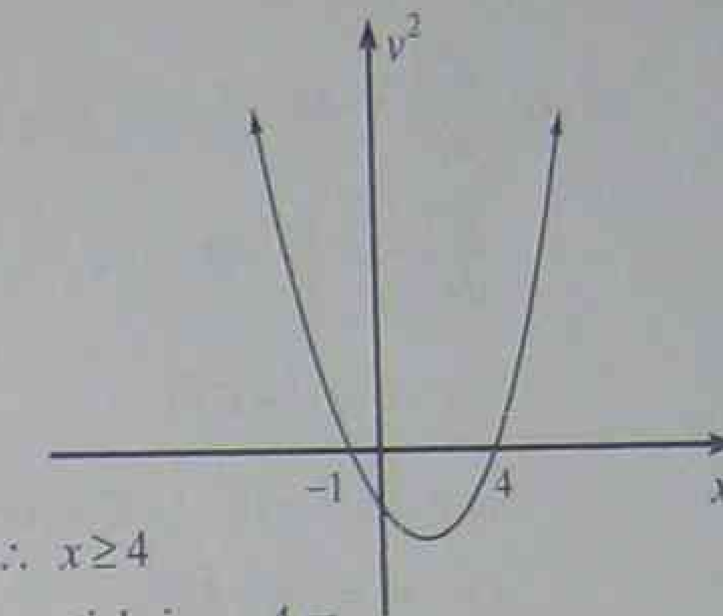
$24=100-60+C\ \therefore\ C=-16$

i.e. $v^2=8x^2-24x-32$ #

(iii) $v^2\geq0$

i.e. $8\left(x^2-3x-4\right)\geq0$

$8(x-4)(x+1)\geq0$



From the graph it can be seen that $v^2\geq0$ when $x\leq-1$ or $x\geq4$

but, it is given that $v=4\sqrt{3}$ at $x=5\ \therefore\ x\geq4$

i.e. The minimum displacement of the particle is $x=4$ m to the right of the origin. #

15. (i) Period: $T=\dfrac{2\pi}{n}=\dfrac{2\pi}{\frac{\pi}{4}}=8$ seconds

Amplitude: $a=5$ metres #

(ii) $v^2=n^2\left(a^2-x^2\right),\ n=\dfrac{\pi}{4},\ a=5$

$$=\left(\frac{\pi}{4}\right)^2\left(25-x^2\right)=\frac{\pi^2}{16}\left(25-x^2\right)$$

at $x=0,\ v^2=\dfrac{\pi^2}{16}(25)=\dfrac{25\pi^2}{16}$

i.e. $v=\pm\dfrac{5\pi}{4}$

$\therefore$ speed $|v|$ of the particle is $\dfrac{5\pi}{4}$ m/s as it passes through the equilibrium position. #

16. $x=a\cos nt$ is the equation of SHM

$a=4,\ T=\dfrac{2\pi}{n}=3\ \therefore\ n=\dfrac{2\pi}{3}$

$\therefore\ x=4\cos\left(\dfrac{2\pi t}{3}\right)$ at $t=0,\ x=4\ \therefore$ particle commences its motion at an extremity

when particle is 1 metre closer to the origin, $x=3$

i.e. $3=4\cos\left(\dfrac{2\pi t}{3}\right)$

$$\cos\left(\frac{2\pi t}{3}\right)=0.75$$

$$t=0.345....$$

$\therefore$ time taken is 0.35 seconds to 2 d.p. #

17. (i) $\frac{d^2x}{dt^2}=-4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-4x$

$\frac{1}{2}v^2=-2x^2+C$

at $x=5,\ v=10\ \therefore\ C=100$

i.e. $v^2=200-4x^2$

$v=\pm\sqrt{200-4x^2}=\pm2\sqrt{50-x^2}$

but at $x=5,\ v=10\ \therefore\ v=2\sqrt{50-x^2}$ #

(ii) Greatest velocity occurs at the equilibrium position or centre. i.e. at $x=0$

$\therefore$ maximum velocity $=2\sqrt{50}=10\sqrt{2}$ m/s #

18. (i) $\frac{d^2x}{dt^2}=-9\pi^2x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-9\pi^2x$

$\frac{1}{2}v^2=\frac{-9\pi^2x^2}{2}+C$

at $x=0,\ v=7\ \therefore\ C=\frac{49}{2}$

i.e. $v^2=49-9\pi^2x^2$

$v=\pm\sqrt{49-9\pi^2x^2}$

but at $x=0,\ v=7\ \therefore\ v=\sqrt{49-9\pi^2x^2}$ #

(ii) The greatest distance occurs when $v=0$

i.e. $49-9\pi^2x^2=0$

$49=9\pi^2x^2$

$x^2=\frac{49}{9\pi^2}$

i.e. $x=\pm\frac{7}{3\pi}$

$\therefore$ the greatest distance $|x|$ is $\frac{7}{3\pi}$ metres #

(iii) Since the particle is initially at the origin, then the next time it returns to the origin it would have completed half its period of oscillation.

$\therefore$ the particle first returns to the origin after $\frac{1}{3}$ sec. #

19. (i) $v^2=2+20x-5x^2$

let $v^2=0$ to find points of oscillation

i.e. $-5x^2+20x+2=0$

i.e. $x=\frac{-20\pm\sqrt{400+4\times5\times2}}{-10}$

$=\frac{-20\pm\sqrt{440}}{-10}=2\pm\frac{\sqrt{110}}{5}$

$\therefore$ the particle oscillates between $x=2-\frac{\sqrt{110}}{5}$ and $x=2+\frac{\sqrt{110}}{5}$ #

(ii) Centre occurs at the mid-point of the extremities

i.e. $x=\frac{1}{2}\left(2-\frac{\sqrt{110}}{5}+2+\frac{\sqrt{110}}{5}\right)=2$ cm

Amplitude: $a=2+\frac{\sqrt{110}}{5}-2=\frac{\sqrt{110}}{5}$ cm #

(iii) $v^2=2+20x-5x^2$

$\frac{1}{2}v^2=1+10x-\frac{5}{2}x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=10-5x=-5(x-2)$

i.e. $n=\sqrt{5}$ and $\therefore$ the period $T=\frac{2\pi}{\sqrt{5}}$ seconds #

20. (i) $x=\frac{1}{2}(\cos 2t+\sin 2t)$

$\dot{x}=\frac{1}{2}(-2\sin 2t+2\cos 2t)=\cos 2t-\sin 2t$

$\ddot{x}=-2\sin 2t-2\cos 2t=-2(\sin 2t+\cos 2t)=-4x$

since $\ddot{x}$ is in the form $-n^2x\ \therefore$ the motion is simple harmonic. #

(ii) $x=\frac{1}{2}\cos 2t+\frac{1}{2}\sin 2t$

$R\cos(2t-\alpha)=R\cos 2t\cos\alpha+R\sin 2t\sin\alpha$

where $R=\sqrt{a^2+b^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\frac{1}{\sqrt{2}}$

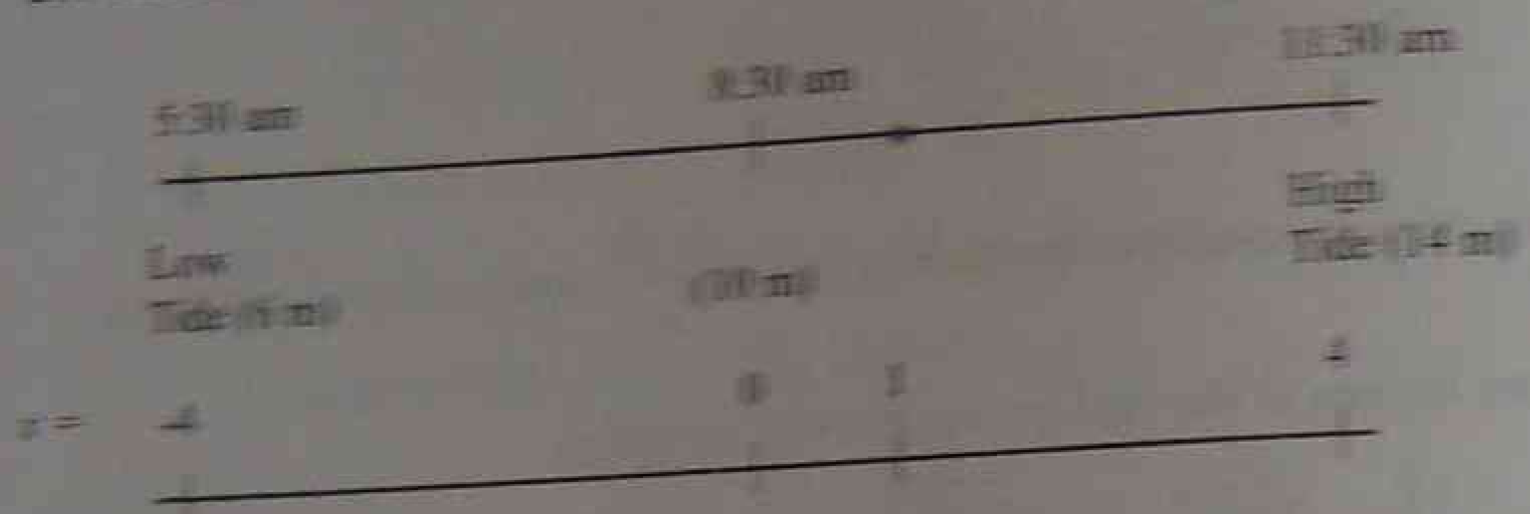
Applications of

and $\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore\ x = \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$ becomes $x = \frac{1}{\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$

$\therefore$ the amplitude of the motion is $\frac{1}{\sqrt{2}}$ m #

21. The motion of the water can be represented as follows:



(i) Period: 2(11:30 a.m. − 5:30 a.m.) = 12 hrs #

(ii) Amplitude: $\frac{14-6}{2} = 4$ m #

(iii) Now, $\frac{2\pi}{n} = 12$ i.e. $n = \frac{\pi}{6}$, $a = 4$

$\therefore\ x = a\cos nt$ becomes:

$x = -4\cos\left(\frac{\pi}{6}t\right)$ (note at $t = 0$ i.e. 5:30 a.m., $x = -4$)

11 m is the equivalent in the above diagram to $x = 1$

i.e. $1 = -4\cos\left(\frac{\pi}{6}t\right)$

i.e. $\cos\left(\frac{\pi}{6}t\right) = -\frac{1}{4}$

$t = 3.4826$ (using radian mode)

$= 3$hrs. 29 min.

$\therefore$ 8:59 a.m. is the earliest time between 5:30 a.m. and 11:30 a.m. that the ship may enter the harbour. #

22. $\dot{y} = 40\sin\alpha = 40\times\frac{3}{5} = 24$

Applications of Calculus

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

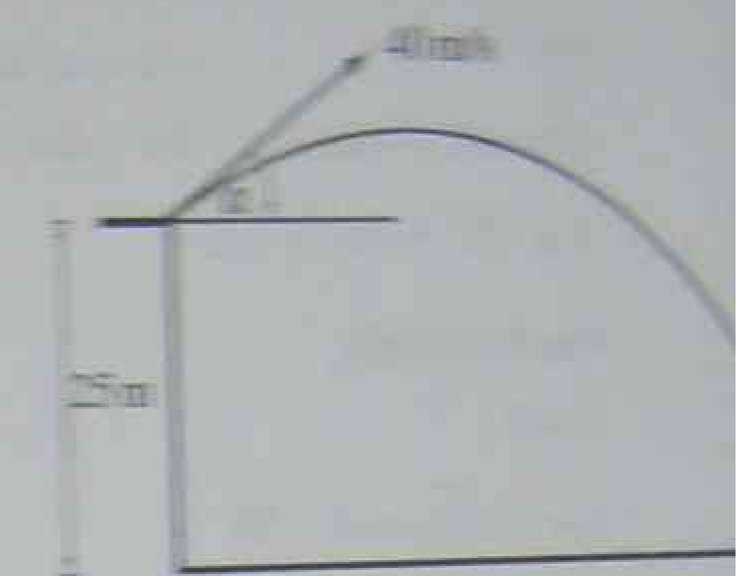
when $t = 0$, $\dot{y} = 24$ $\therefore\ C_1 = 24$

i.e. $\dot{y} = -10t + 24$

thus $y = -5t^2 + 24t + C_2$

when $t = 0$, $y = 25$ $\therefore\ C_2 = 25$

$\therefore\ y = -5t^2 + 24t + 25$



Horizontally

$\ddot{x} = 0$

$\dot{x} = C_1$

when $t = 0$, $\dot{x} = 32$ $\therefore\ C_1 = 32$

i.e. $\dot{x} = 32$

thus, $x = 32t + C_2$

when $t = 0$, $x = 0$ $\therefore\ C_2 = 0$

$\therefore\ x = 32t$

(i) The maximum height attained is when $\dot{y} = 0$

i.e. $-10t + 24 = 0$

$t = 2.4$ seconds

Substituting this into y gives:

$y = -5(2.4)^2 + 24(2.4) + 25 = 53.8$

$\therefore$ maximum height attained = 53.8 m #

(ii) Let $y = 25$ and solve for t:

i.e. $25 = -5t^2 + 24t + 25$

$5t^2 - 24t = 0$

$t(5t - 24) = 0$

i.e. $t = 0, \frac{24}{5}$

$\therefore$ the particle next returns to a height of 25 m at $t = \frac{24}{5} = 4.8$ seco

(iii) Find the vertical and horizontal components of the velocity v at t=

$\dot{y} = -10\times 2 + 24 = 4$

$\dot{x} = 32$

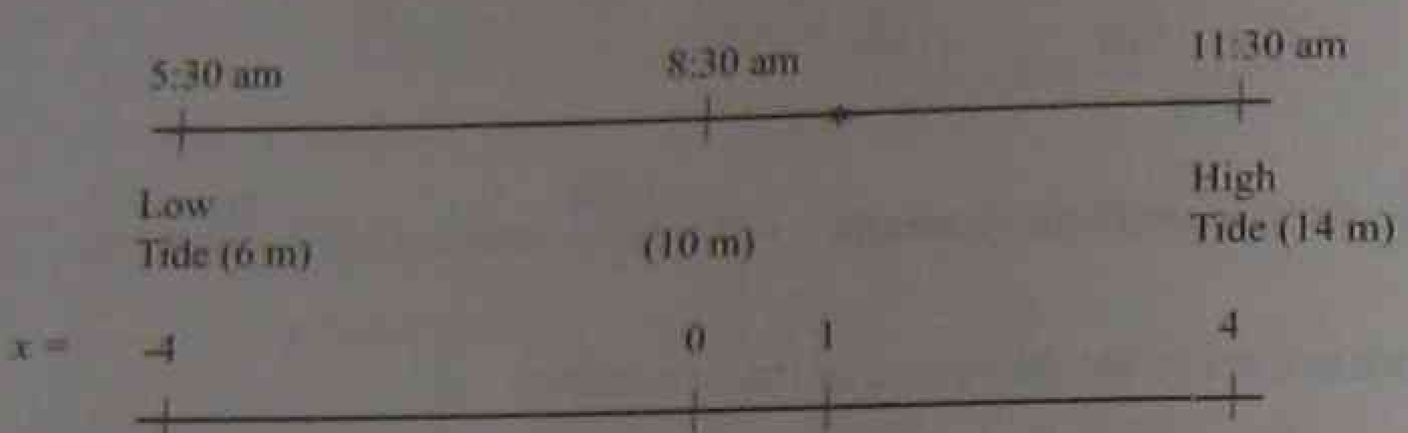
i.e. $v = \sqrt{\dot{y}^2 + \dot{x}^2} = 32.25$

$\therefore$ the speed of the object after 2 seconds is 32.25 m/s correct to and $\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \; x = \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$ becomes $x = \frac{1}{\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$

$\therefore$ the amplitude of the motion is $\frac{1}{\sqrt{2}}$ m #

21. The motion of the water can be represented as follows:



(i) Period: $2(11{:}30 \text{ a.m.} - 5{:}30 \text{ a.m.}) = 12$ hrs #

(ii) Amplitude: $\frac{14-6}{2} = 4$ m #

(iii) Now, $\frac{2\pi}{n} = 12$ i.e. $n = \frac{\pi}{6}$, $a = 4$

$\therefore \; x = a\cos nt$ becomes:

$x = -4\cos\left(\frac{\pi}{6}t\right)$ (note at $t = 0$ i.e. 5:30 a.m. , $x = -4$)

11 m is the equivalent in the above diagram to $x = 1$

i.e. $1 = -4\cos\left(\frac{\pi}{6}t\right)$

i.e. $\cos\left(\frac{\pi}{6}t\right) = -\frac{1}{4}$

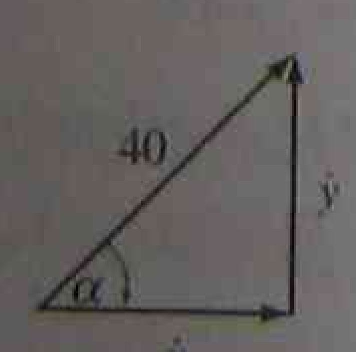
$t = 3.4826$ (using radian mode)

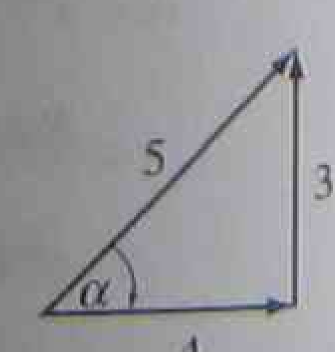
$= 3$ hrs. 29 min.

$\therefore$ 8:59 a.m. is the earliest time between 5:30 a.m. and 11:30 a.m. that the ship may enter the harbour. #

22. $\dot{y} = 40\sin\alpha = 40 \times \frac{3}{5} = 24$

$\dot{x} = 40\cos\alpha = 40 \times \frac{4}{3} = 32$





Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

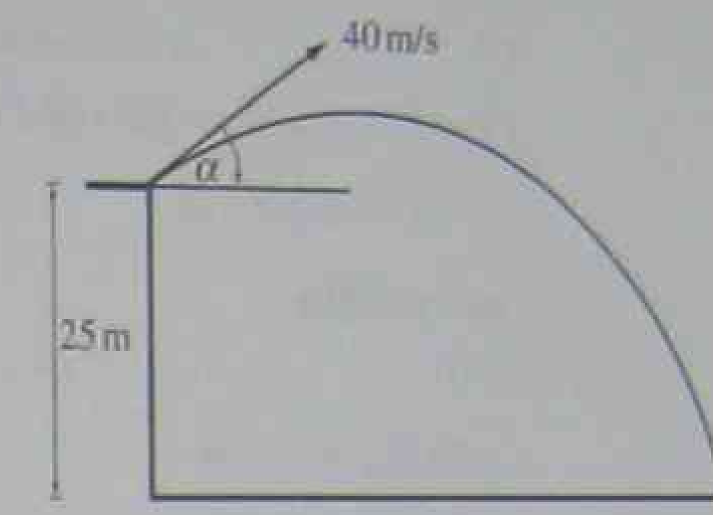
when $t = 0$, $\dot{y} = 24 \;\therefore\; C_1 = 24$

i.e. $\dot{y} = -10t + 24$

thus $y = -5t^2 + 24t + C_2$

when $t = 0$, $y = 25 \;\therefore\; C_2 = 25$

$\therefore \; y = -5t^2 + 24t + 25$



Horizontally

$\ddot{x} = 0$

$\dot{x} = C_1$

when $t = 0$, $\dot{x} = 32 \;\therefore\; C_1 = 32$

i.e. $\dot{x} = 32$

thus, $x = 32t + C_2$

when $t = 0$, $x = 0 \;\therefore\; C_2 = 0$

$\therefore \; x = 32t$

(i) The maximum height attained is when $\dot{y} = 0$

i.e. $-10t + 24 = 0$

$t = 2.4$ seconds

Substituting this into y gives:

$y = -5(2.4)^2 + 24(2.4) + 25 = 53.8$

$\therefore$ maximum height attained $= 53.8$ m #

(ii) Let $y = 25$ and solve for t:

i.e. $25 = -5t^2 + 24t + 25$

$5t^2 - 24t = 0$

$t(5t - 24) = 0$

i.e. $t = 0, \frac{24}{5}$

$\therefore$ the particle next returns to a height of 25 m at $t = \frac{24}{5} = 4.8$ seconds #

(iii) Find the vertical and horizontal compoments of the velocity v at $t = 2$

$\dot{y} = -10 \times 2 + 24 = 4$

$\dot{x} = 32$

i.e. $v = \sqrt{\dot{y}^2 + \dot{x}^2} = 32.25$

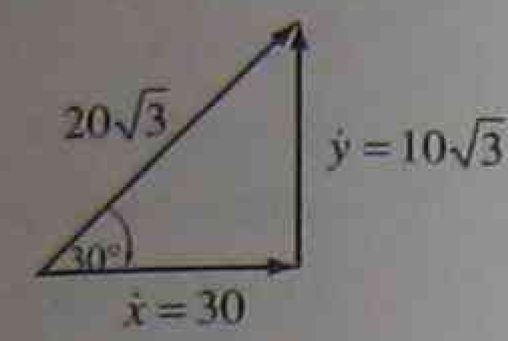
$\therefore$ the speed of the object after 2 seconds is 32.25 m/s correct to 2 d.p. #

23. (i) **Vertically**

$\ddot{y} = -10$

$\dot{y} = -10t + v\sin\theta = -10t + 10\sqrt{3}$

$y = -5t^2 + 10\sqrt{3}t + 15$



Horizonttally

$\ddot{x} = 0$

$\dot{x} = 20\sqrt{3}\cos\theta = 30$

$x = 30t$

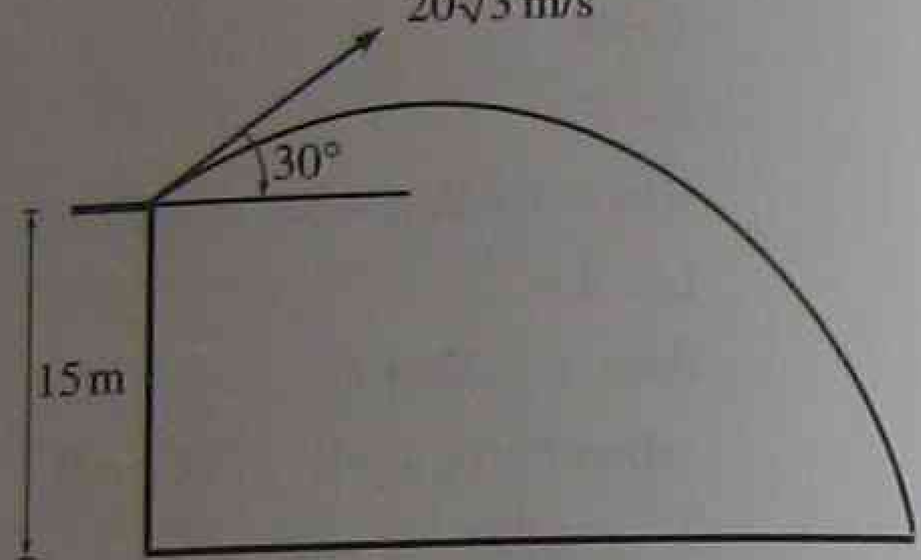
Ball hits the ground when $y = 0$

i.e. $-5t^2 + 10\sqrt{3}t + 15 = 0$

$t = \dfrac{-10\sqrt{3} \pm \sqrt{300+300}}{-10}$

$= \sqrt{3} \pm \sqrt{6}$

but $t > 0$ $\therefore$ $t = \sqrt{3} + \sqrt{6}$



Substituting this into x, gives:

$x = 30\left(\sqrt{3} + \sqrt{6}\right)$

$\therefore$ the ball strikes the ground $30\left(\sqrt{3} + \sqrt{6}\right)$ metres away from the base. #

(ii) **Horizontally**

$x = Vt$

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t$

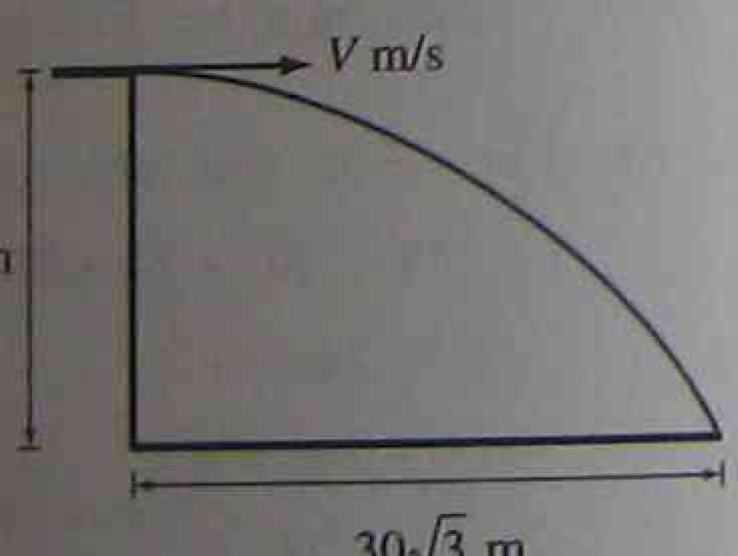
$y = -5t^2 + 15$

ball hits the ground when $y = 0$

i.e. $-5t^2 + 15 = 0$

$t^2 = 3$

$t = \sqrt{3}$ as $t > 0$



$\therefore$ the ball hits the ground after $\sqrt{3}$ seconds

when $t = \sqrt{3}$, $V = \dfrac{x}{t} = \dfrac{30\sqrt{3}\left(1+\sqrt{2}\right)}{\sqrt{3}} = 30\left(1+\sqrt{2}\right)$

24. $x = 40t\cos\theta$ $\qquad y = -5t^2 + 40t\sin\theta + 30$

at $x = 120$, horizontal displacement becomes:

$120 = 40t\cos\theta$ i.e. $t = \dfrac{3}{\cos\theta}$

Substituting this into y gives:

$y = -5\left(\dfrac{3}{\cos\theta}\right)^2 + 40\left(\dfrac{3}{\cos\theta}\right)\sin\alpha + 30$

$= -45\sec^2\theta + 120\tan\theta + 30$

$= -45\left(1 + \tan^2\theta\right) + 120\tan\theta + 30$

$= -45\tan^2\theta + 120\tan\theta - 15$

let $y = 0$ and solve for θ:

i.e. $-45\tan^2\theta + 120\tan\theta - 15 = 0$

$3\tan^2\theta - 8\tan\theta + 1 = 0$

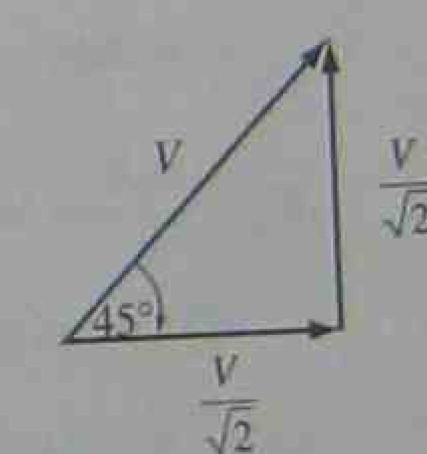
$\therefore$ $\tan\theta = \dfrac{8 \pm \sqrt{64-12}}{6} = \dfrac{8 \pm 2\sqrt{13}}{6} = \dfrac{4 \pm \sqrt{13}}{3}$

i.e. $\theta = 7°29'$ or $\theta = 68°28'$

$\therefore$ if the stone is projected at an angle $\theta = 7°29'$ or $\theta = 68°28'$ to the horizontal, it will strike the water at 120 m away from the base of the lighthouse. #

25. $\dot{y} = V\sin 45° = \dfrac{V}{\sqrt{2}}$

$\dot{x} = V\cos 45° = \dfrac{V}{\sqrt{2}}$



Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

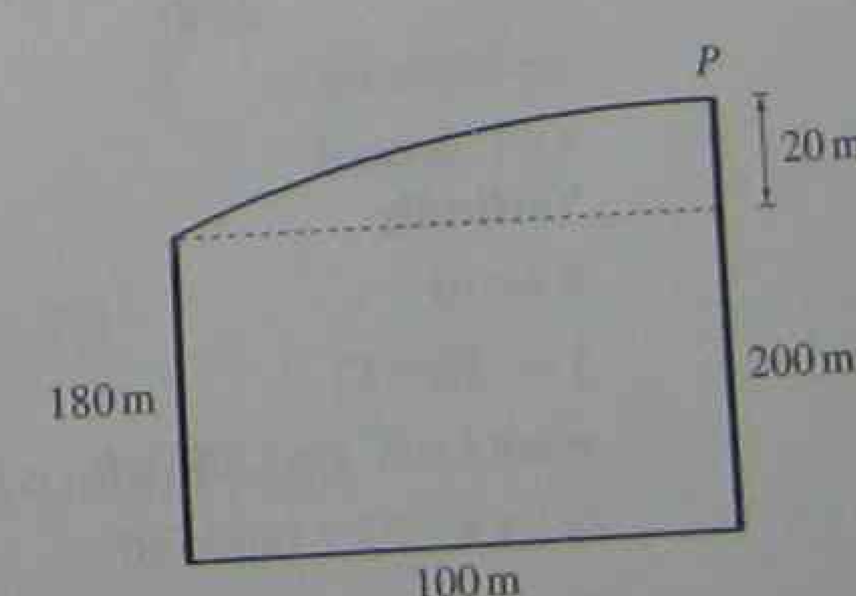
when $t = 0$, $\dot{y} = \dfrac{V}{\sqrt{2}}$ $\therefore$ $C_1 = \dfrac{V}{\sqrt{2}}$

i.e. $\dot{y} = -10t + \dfrac{V}{\sqrt{2}}$

thus, $y = -5t^2 + \dfrac{V}{\sqrt{2}}t + C_2$

when $t = 0$, $y = 180$ $\therefore$ $C_2 = 180$

$\therefore$ $y = -5t^2 + \dfrac{V}{\sqrt{2}}t + 180$



Applications of Calculus

Horizontally

$\ddot{x}=0$

$\dot{x}=C_1$

when $t=0, \dot{x}=\frac{V}{\sqrt{2}}$ $\therefore$ $C_1=\frac{V}{\sqrt{2}}$

i.e. $\dot{x}=\frac{V}{\sqrt{2}}$

thus $x=\frac{V}{\sqrt{2}}t+C_2$

when $t=0, x=0$ $\therefore$ $C_2=0$

$\therefore$ $x=\frac{V}{\sqrt{2}}t$

(i) At P, $y=200$ and $x=100$

$\therefore$ $100=\frac{Vt}{\sqrt{2}}$ and $200=-5t^2+\frac{Vt}{\sqrt{2}}+180$

i.e. $200=-5t^2+100+180$

$5t^2=80$

$t=\pm 4$ but $t>0$ $\therefore$ $t=4$

$\therefore$ time taken to reach point P is 4 sec. #

(ii) when $t=4$, $x=100$

$\therefore$ substituting into $x=\frac{Vt}{\sqrt{2}}$ gives:

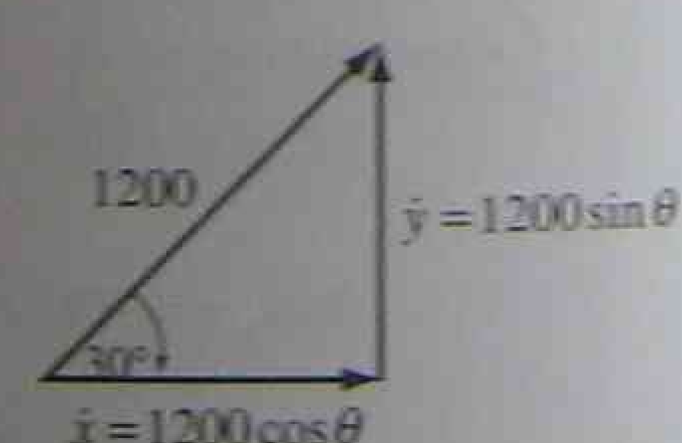
$V=\frac{100\sqrt{2}}{4}=25\sqrt{2}$

$\therefore$ velocity of projection is $25\sqrt{2}$ m/s #

26. $720 \text{ km/hr} = \frac{720\times 1000}{3600} = 200 \text{ m/s}$

$\dot{y}=1200\sin\theta$

$\dot{x}=1200\cos\theta$



Vertically

$\ddot{y}=-10$

$\dot{y}=-10t+C_1$

when $t=0$, $\dot{y}=1200\sin\theta$ $\therefore$ $C_1=1{,}200\sin\theta$

i.e. $\dot{y}=-10t+1200\sin\theta$

Applications of Calculus

Horizontally

$\ddot{x}=0$

$\dot{x}=C_1$

when $t=0$, $\dot{x}=1200\cos\theta$ $\therefore$ $C_1=1200\cos\theta$

i.e. $\dot{x}=1200\cos\theta$

thus $x=1200t\cos\theta+C_2$

when $t=0$, $x=0$ $\therefore$ $C_2=0$

$\therefore$ $x=1200t\cos\theta$

(i) For the shell to strike the aircraft, then the horizontal velocity o shell and the aircraft must be equal.

i.e. $1200\cos\theta=200$

$\cos\theta=\frac{1}{6}$

$\theta=80°24'$

$\therefore$ the required angle of projection is $80°24'$ #

(ii) The shell will strike the aircraft when projected at angle θ at $y=1500$

i.e. $1500=-5t^2+1200t\sin\theta$

$5t^2-1200\sin\theta t+1500=0$

$t^2-240\sin\theta t+300=0$

$t^2-240\left(\frac{\sqrt{35}}{6}\right)t+300=0$

$t^2-40\sqrt{35}t+300=0$

$\therefore$ $t=\frac{40\sqrt{35}\pm\sqrt{56000-4\times 300}}{2}$

$=\frac{40\sqrt{35}\pm\sqrt{54800}}{2}$

$=\frac{40\sqrt{35}\pm 20\sqrt{137}}{2}=20\sqrt{35}\pm 10\sqrt{137}$

$\therefore$ shell strikes aircraft at $t_1=\left(20\sqrt{35}-10\sqrt{137}\right)$ and

$t_2=\left(20\sqrt{35}+10\sqrt{137}\right)$ seconds #

Horizontally

$\ddot{x}=0$

$\dot{x}=C_1$

when $t=0,\ \dot{x}=\dfrac{V}{\sqrt{2}}\ \therefore\ C_1=\dfrac{V}{\sqrt{2}}$

i.e. $\dot{x}=\dfrac{V}{\sqrt{2}}$

thus $x=\dfrac{V}{\sqrt{2}}t+C_2$

when $t=0,\ x=0\ \therefore\ C_2=0$

$\therefore\ x=\dfrac{V}{\sqrt{2}}t$

(i) At P, $y=200$ and $x=100$

$\therefore\ 100=\dfrac{Vt}{\sqrt{2}}$ and $200=-5t^2+\dfrac{Vt}{\sqrt{2}}+180$

i.e. $200=-5t^2+100+180$

$5t^2=80$

$t=\pm 4$ but $t>0\ \therefore\ t=4$

$\therefore$ time taken to reach point P is 4 sec. #

(ii) when $t=4,\ x=100$

$\therefore$ substituting into $x=\dfrac{Vt}{\sqrt{2}}$ gives:

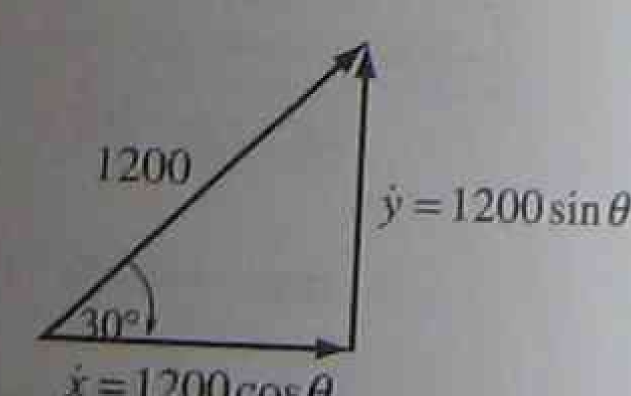
$V=\dfrac{100\sqrt{2}}{4}=25\sqrt{2}$

$\therefore$ velocity of projection is $25\sqrt{2}$ m/s #

26. $720\text{ km/hr}=\dfrac{720\times 1000}{3600}=200\text{ m/s}$

$\dot{y}=1200\sin\theta$

$\dot{x}=1200\cos\theta$



Vertically

$\ddot{y}=-10$

$\dot{y}=-10t+C_1$

when $t=0,\ \dot{y}=1200\sin\theta\ \therefore\ C_1=1{,}200\sin\theta$

i.e. $\dot{y}=-10t+1200\sin\theta$

thus, $y=-5t^2+1200t\sin\theta+C_2$

when $t=0,\ y=0\ \therefore\ C_2=0\ \therefore\ y=-5t^2+1200t\sin\theta$

Horizontally

$\ddot{x}=0$

$\dot{x}=C_1$

when $t=0,\ \dot{x}=1200\cos\theta\ \therefore\ C_1=1200\cos\theta$

i.e. $\dot{x}=1200\cos\theta$

thus $x=1200t\cos\theta+C_2$

when $t=0,\ x=0\ \therefore\ C_2=0$

$\therefore\ x=1200t\cos\theta$

(i) For the shell to strike the aircraft, then the horizontal velocity of the shell and the aircraft must be equal,

i.e. $1200\cos\theta=200$

$\cos\theta=\dfrac{1}{6}$

$\theta=80°24'$

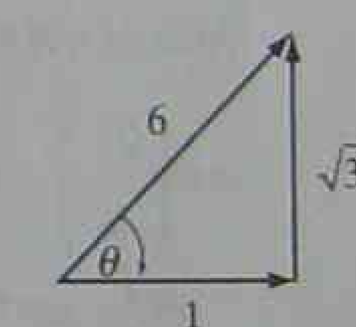
$\therefore$ the required angle of projection is $80°24'$ #

(ii) The shell will strike the aircraft when projected at angle θ at $y=1500$

i.e. $1500=-5t^2+1200t\sin\theta$

$5t^2-1200\sin\theta t+1500=0$

$t^2-240\sin\theta t+300=0$



$t^2-240\left(\dfrac{\sqrt{35}}{6}\right)t+300=0$

$t^2-40\sqrt{35}t+300=0$

$\therefore\ t=\dfrac{40\sqrt{35}\pm\sqrt{56000-4\times 300}}{2}$

$=\dfrac{40\sqrt{35}\pm\sqrt{54800}}{2}$

$=\dfrac{40\sqrt{35}\pm 20\sqrt{137}}{2}=20\sqrt{35}\pm 10\sqrt{137}$

$\therefore$ shell strikes aircraft at $t_1=\left(20\sqrt{35}-10\sqrt{137}\right)$ and

$t_2=\left(20\sqrt{35}+10\sqrt{137}\right)$ seconds#

27. (i) Bookwork refer to section (E) in the notes section of this topic.

(ii) Range $= \dfrac{V^2 \sin 2\theta}{g}$

since V and g are constant $\therefore$ $\sin 2\theta$ must be maximised so that the range is maximised

i.e. $\sin 2\theta = 1$

$\theta = 45°$

$\therefore$ maximum range occurs when $\theta = 45°$ #

(iii) At $\alpha = 15°$, Range $= \dfrac{V^2 \sin 30°}{g} = \dfrac{V^2}{2g}$

at $\alpha = 60°$, Range $= \dfrac{V^2 \sin 120°}{g} = \dfrac{\sqrt{3}V^2}{2g}$

However at $\alpha = 45°$, $R_{\max.} = \dfrac{V^2 \sin 90°}{g} = \dfrac{V^2}{g}$

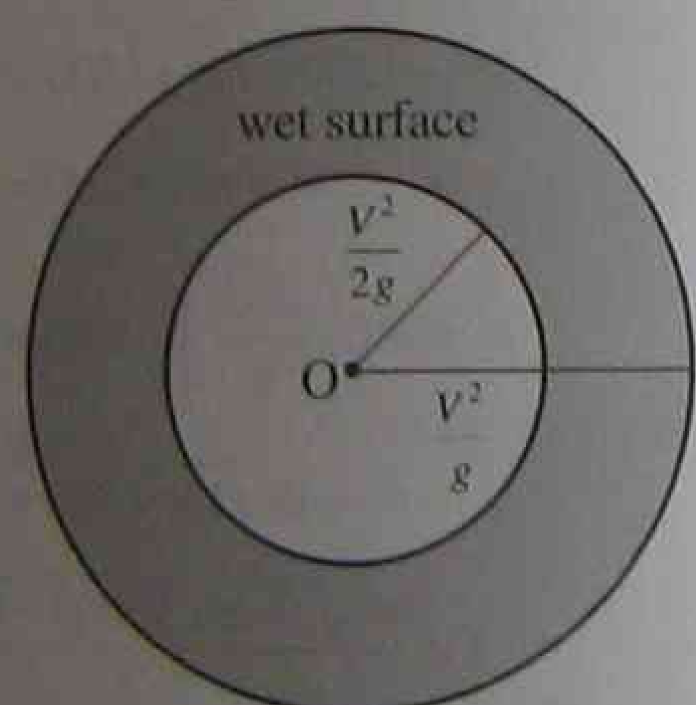
$\therefore$ the sprinkler will wet an annular region of internal and external radii $\dfrac{V^2}{2g}$ and $\dfrac{V^2}{g}$ respectively as shown in the diagram below.

Area of wet surface

$$= \pi\left(\frac{V^2}{g}\right)^2 - \pi\left(\frac{V^2}{2g}\right)^2$$

$$= \frac{\pi V^4}{g^2} - \frac{\pi V^4}{4g^2}$$

$$= \frac{3\pi V^4}{4g^2} \text{ square metres \#}$$



TOPIC 14

THE BINOMIAL THEOREM AND FURTHER PROBABILITY

(A) Binomial Theorem

(i) Expansion of (1+x)ⁿ

For any positive integer n, no matter how large, $(1+x)^n$ is a polynomial of degree n in the variable x. The symbol ${}^nC_k$ is by definition the coefficient of x^k in the expansion $(1+x)^n$.

$$(1+x)^n = 1 + {}^nC_1x + \ldots + {}^nC_2x^2 + \ldots + {}^nC_kx^k + \ldots + {}^nC_nx^n$$

$$\textbf{where: } {}^nC_k = \frac{n!}{k!(n-k)!}$$

(ii) Expansion of (a+b)ⁿ

We can extend the identity of $(1+x)^n$ to an expression of $(a+b)^n$ by putting $x = \dfrac{b}{a}$ and multiplying both sides by a^n:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_nb^n$$
$$= \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

Note: Terms equidistant from the ends are equal.

i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ and in general ${}^nC_r = {}^nC_{n-r}$

(iii) The General Terms of the Expansion of (a+b)ⁿ

Using the preceding result it can be seen that the $(r+1)^{th}$ term is given by:

$$T_{r+1} = C_ra^{n-r}b^r$$

Example 1: Find the term independent of x in the expansion of $\left(x^3 - \dfrac{2}{x}\right)^8$.

Solution 1:

The $(r+1)^{th}$ term in the expansion of $\left(x^3 - \dfrac{2}{x}\right)^8$ is given by:

$$T_{r+1} = {}^8C_r\left(x^3\right)^{8-r}\left(-\frac{2}{x}\right)^r = {}^8C_rx^{24-3r}(-2)^rx^{-r} = {}^8C_rx^{24-4r}(-2)^r$$

for T_{r+1} to be independent of x, then:

$24-4r=0$

i.e. $r=6$

$\therefore\ T_7={}^8C_6(-2)^6=1,792$ #

Example 2: Find the coefficient of x^2 in the expansion $\left(x^2+\dfrac{3}{x}\right)^{10}$.

Solution 2:

The $(r+1)^{th}$ term in the expansion of $\left(x^2+\dfrac{3}{x}\right)^{10}$ is given by:

$$T_{r+1}={}^{10}C_r\left(x^2\right)^{10-r}\left(\frac{3}{x}\right)^r$$

$$={}^{10}C_r x^{20-2r}(3)^r x^{-r}$$

$$={}^{10}C_r x^{20-3r}(3)^r$$

for T_{r+1} to have a term in x^2, then:

$20-2r=2$

$18=2r$

$r=9$

$\therefore\ T_{10}={}^{10}C_9x^2(3)^9$

$=196830x^2$

$\therefore$ co-efficient of x^2 is 196830 #

Example 3: Consider the expansion of $(1+x)^n$: $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+...+{}^nC_nx^n$

Show that:

(i) ${}^nC_0+{}^nC_1+...+{}^nC_n=2^n$

(ii) Show by differentiating both sides that:

$${}^nC_1+2\,{}^nC_2+3\,{}^nC_3...+(n-1)\,{}^nC_{n-1}=n\left(2^{n-1}-1\right)$$

(iii) Show by integrating both sides that:

$$1-\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2-\frac{1}{4}{}^nC_3...+\frac{(-1)^n}{n+1}=\frac{1}{n+1}$$

Solution 3:

(i) Substituting $x=1$ in the expansion of $(1+x)^n$ gives:

$2^n={}^nC_0+{}^nC_1+{}^nC_2+...+{}^nC_n$ #

(ii) $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+...+{}^nC_nx^n$

Differentiating both sides w.r.t. x, gives:

$$\frac{d}{dx}\left[(1+x)^n\right]={}^nC_1+2\,{}^nC_2x+...+n\,{}^nC_nx^{n-1}$$

$$n(1+x)^{n-1}={}^nC_1+2\,{}^nC_2x+...+n\,{}^nC_nx^{n-1}$$

Substituting $x=1$ in the above expression gives:

$$n\cdot2^{n-1}={}^nC_1+2\,{}^nC_2+...+(n-1)\,{}^nC_{n-1}+n\,{}^nC_n$$

$$={}^nC_1+2\,{}^nC_2+...+(n-1)\,{}^nC_{n-1}+n$$

$$n\left(2^{n-1}-1\right)={}^nC_1+2\,{}^nC_2+...+(n-1)\,{}^nC_{n-1}\ \#$$

(iii) $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+...+{}^nC_nx^n$

Integrating both sides from $x=-1$ to $x=0$, gives:

$$\int_{-1}^{0}(1+x)^n=\left[{}^nC_0x+{}^nC_1\frac{x^2}{2}+{}^nC_2\frac{x^3}{3}+...+{}^nC_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0}=(0)-\left(-{}^nC_0+\frac{1}{2}{}^nC_1-\frac{1}{3}{}^nC_2+...+\frac{(-1)^{n+1}}{n+1}\right)$$

$$\frac{1}{n+1}=1-\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2-...+\frac{(-1)^n}{n+1}\ \#$$

Example 4: Using the fact that: $(1+x)(1+x)^n=(1+x)^{n+1}$

(i) Show that ${}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$.

(ii) Hence, prove that ${}^nC_r+2\,{}^nC_{r-1}+{}^nC_{r-2}={}^{n+2}C_r$.

(iii) Hence, find the value of n if ${}^nC_4+2\,{}^nC_3+{}^nC_2=2\left(n^2-1\right)$.

Solution 4:

(i) $(1+x)(1+x)^n=(1+x)^{n+1}$

LHS $=(1+x)\left(1+{}^nC_1x+...+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+...+{}^nC_nx^n\right)$

coefficient of x^r in the above expansion is $={}^nC_r+{}^nC_{r-1}$

$$\text{RHS}=\left(1+{}^{n+1}C_1+\ldots+{}^{n+1}C_r x^r+\ldots+{}^{n+1}C_{n+1}x^{n+1}\right)$$

coefficient of x^r in the above expansion is $={}^{n+1}C_r$

$\therefore\ {}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$ #

(ii) Using the result ${}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$, then:

$$\begin{aligned}{}^{n+2}C_r&={}^{n+1}C_r+{}^{n+1}C_{r-1}\\&=\left({}^nC_r+{}^nC_{r-1}\right)+\left({}^nC_{r-1}+{}^nC_{r-2}\right)\\&={}^nC_r+2\,{}^nC_{r-1}+{}^nC_{r-2}\ \#\end{aligned}$$

(iii) ${}^nC_4+2\,{}^nC_3+{}^nC_2=2\left(n^2-1\right)$

i.e. ${}^{n+2}C_4=2\left(n^2-1\right)$ (using (i), where $r=4$)

$$\frac{(n+2)!}{4!(n-2)!}=2\left(n^2-1\right)$$

$$\frac{(n+2)(n+1)(n)(n-1)}{24}=2\left(n^2-1\right)$$

$$\begin{aligned}(n+2)(n+1)n(n-1)-48(n-1)(n+1)&=0\\(n+1)(n-1)\left[n^2+2n-48\right]&=0\\(n+1)(n-1)(n+8)(n-6)&=0\end{aligned}$$

i.e. $n=-1,1,-8,6$

Since by definition n must be a positive integer and $n\geq 4$,

$\therefore\ n=6$ is the only solution. #

(iv) Finding the Highest Coefficient

In the expansion of $(a+b)^n$, coefficients of the terms commence at 1, increase up to a certain value, and then decrease back to 1. To find the highest coefficient we need to determine the value of r for which the ratio of T_{r+1} to T_r is greater than or equal to 1, this is most efficiently done by using the formula below.

Solve for r in the following inequality:

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\times\frac{b}{a}\geq 1$$

Example 1: In the expansion $(5+7x)^{20}$:

(i) Find the ratio of the coefficients of the $(r+1)^{th}$ term to the r^{th} term.

(ii) Hence, find the greatest coefficient.

Solution 1:

(i) $\dfrac{T_{r+1}}{T_r}=\dfrac{n-r+1}{r}\times\dfrac{b}{a}=\dfrac{21-r}{r}\times\dfrac{7}{5}$ #

(ii) Greatest coefficient occurs when $\dfrac{T_{r+1}}{T_r}\geq 1$

i.e. $7(21-r)\geq 5r$

$12r\leq 147$

$r\leq 12.25$

i.e. when $r=12$

Greatest coefficient $=T_{13}={}^{20}C_{12}(5)^{20-12}\cdot(7)^{12}={}^{20}C_{12}5^8 7^{12}$ #

(B) Binomial Probability

Binomial probability involves repeated trials of an experiment where only two outcomes are possible.

If the two outcomes are denoted by p and q respectively, such that:

$p=$ probability of success in any one trial

$q=$ probability of failure in any one trial

then the probability of exactly r successes in n trials is given by:

$$P(x=r)={}^nC_r p^r q^{n-r} \qquad \text{note: } p+q=1$$

The probability of at least one success in trials is given by:

P(at least one success in n trials) $=1-P$(no sucess in n trials) $=1-q^n$

Example 1: It is known that 5% of the bolts produced by a certain factory are defective. What is the probability that from a random sample of 5 bolts:

(i) all are good?

(ii) Exactly two bolts are defective?

(iii) at least one bolt is defective?

Solution 1:

(i) Let $p=$ probability of success $=0.95$

and $q=$ probability of failure $=0.05$

$\therefore\ P(x=5)={}^5C_5p^5=(0.95)^5=0.774$ to 3 d.p. #

(ii) $P(x=2)={}^5C_3p^3q^2={}^5C_3(0.95)^3(0.05)^2=0.021$ to 3 d.p. #

(iii) $P(\text{at least one bolt is defective})$

$=1-P(\text{all bolts are not defective})$

$=1-(0.95)^5$

$=0.226$ to 3 d.p.#

Example 2: A dice is tossed 4 times. What is the probability that:

(i) 6 appears no more than once?

(ii) An even number appears at least once?

Solution 2:

(i) Let $p=$ probability of $6=\frac{1}{6}$, $\therefore\ q=\frac{5}{6}$

$P(\text{6 appears no more once})=P(x=0)+P(x=1)$

$$=\left(\frac{5}{6}\right)^4+{}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3$$

$$=\frac{125}{144}\ \#$$

(ii) Let $p=$ probability of an even number $=\frac{1}{2}$, $\therefore\ q=\frac{1}{2}$

$P(\text{an even no. appears at least once})=1-q^n$

$$=1-\left(\frac{1}{2}\right)^4=\frac{15}{16}\ \#$$

REVIEW EXERCISES

(A) Binomial Theorem

1. Find the term independent of x in the expansion of $\left(x-\frac{1}{x^2}\right)^{15}$.

2. Find the term independent of x in the expansion of $\left(2x-\frac{1}{4x^3}\right)^{20}$.

3. Find the coefficient of x^4 in the expansion of $\left(x^2+4x+3\right)^8$.

4. In the expansion of $(2x+b)^6$, the coefficients of x^5 and x^6 are equal. Find b.

5. In the expansion $\left(ax-\frac{b}{x^2}\right)^5$ the coefficients of x^2 and x^{-1} are equal. Show that, if a and b are non-zero constants, that $a=-2b$.

6. In the expansion of $(a+b)^{2n+1}$:

   (i) Write down the $(n+1)^{th}$ term, T_{n+1} and $(n+2)^{th}$ term T_{n+2}.

   (ii) Prove that if $a+b=1$, then $T_{n+1}+T_{n+2}={}^{2n+1}C_n a^n b^n$.

7. Find the greatest coefficient in the expansion $(4+3x)^{15}$.

8. Consider the expansion $\left(2+\frac{2x}{3}\right)^{12}$:

   (i) Show that $\frac{T_{r+1}}{T_r}=\frac{13-r}{3r}x$.

   (ii) Hence find, the greatest coefficient of the expansion.

   (iii) Hence find, the greatest term when $x=2$.

9. Consider the expansion: $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\dots+{}^nC_nx^n$

   By using the above expansion, evaluate each of the following expressions:

   (i) $1+3\,{}^nC_1+3\,{}^nC_2+\dots+3^n\,{}^nC_n$

   (ii) $1-{}^nC_1+{}^nC_2+\dots+(-1)^n\,{}^nC_n$

   (iii) $\frac{1}{n!}+\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+\dots+\frac{1}{n!}$

10. Using the fact that: $(1+x)^n(1+x)^n=(1+x)^{2n}$,

show, by equating the coefficients of x^n on both sides that:

$$({}^nC_0)^2+({}^nC_1)^2+\ldots+({}^nC_n)^2={}^{2n}C_n$$

11. (i) Write down the expansion of $(1+x)^{2n}$

(ii) Hence show that: ${}^{2n}C_1+2\,{}^{2n}C_2+\ldots+2n\,{}^{2n}C_n=n4^n$

(B) Binomial Probability

12. A coin is tossed 20 times. Find the probability of obtaining:

(i) exactly 10 heads.

(ii) at most 3 heads.

13. In a factory that makes promotional cubes, it is found that 2% are defective. Find the probability that in a random sample of 20 such cubes:

(i) none are defective.

(ii) at least one is defective

14. A marksman, who keeps a record of his performances, finds that in the long run, he scores a bull's eye on 3 out of 4 occasions. He fires 15 rounds at a target. Assuming that each trial is independent of any other similar trial, find the probability of:

(i) exactly 3 bull's eyes.

(ii) at least 4 bull's eyes.

(iii) a bull's eye with the second round only.

(iv) the most probable number of bull's eyes?

15. Five cards are randomly selected without replacement from a standard deck of 52 cards. Next, five cards are drawn without replacement from a second standard deck of 52 cards. And finally, five cards are drawn without replacement from a third standard deck of 52 cards.

(i) What is the probability that a king of hearts is selected when five cards are drawn from the first pack?

(ii) What is the probability that a king of hearts appears more than once from the cards selected over all three packs?

16. The probability of the birth of a male or female is taken to be 0.5.

(i) Find the probability that a family chosen at random will have two boys and two girls.

(ii) Five families, each with four children, are chosen randomly. Find the probability that exactly two of them will have two boys and two girls.

17. Mark and Linda play a game of darts in which each throws alternately with each hand at the darts board. Mark has a probability of $\frac{2}{3}$ of striking the bull's eye if he throws with his right hand and $\frac{1}{2}$ if he throws with his left hand. Linda has a probability of $\frac{3}{5}$ of striking the bull's eye irrespective of which hand she throws with. One round consists of both Mark and Linda having a throw with each hand.

If ten rounds are played, find the most likely outcome (i.e. number of bull's eyes) for each player and hence, calculate which outcome is more probable.

18. A and B are two kinds of defects of pieces of timber produced in a timber yard. A statistical study on a large number of pieces showed:

- 10% of the pieces had defect A.
- 25% of the pieces having defect A, had defect B also.
- 5% of the pieces not having defect A, had defect B.

A random sample of 4 timber pieces are selected. Find the probability:

(i) all the pieces have defects A and B.

(ii) that exactly one piece had defect B only.

(iii) that at least one has some defect.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $T_{r+1} = C_r a^{n-r} b^r = {}^{15}C_r (x)^{15-r}\left(-x^{-2}\right)^r$

$= {}^{15}C_r (x)^{15-r}\cdot(-1)^r\cdot\left(x^{-2r}\right)$

$= {}^{15}C_r (x)^{15-3r}(-1)^r$

for T_{r+1} to be independant of x, then $15-3r=0$ i.e. $r=5$

$\therefore\ T_6 = {}^{15}C_5(-1)^5 = -3003$ #

2. $T_{r+1} = {}^nC_r a^{n-r}b^r = {}^{20}C_r(2x)^{20-r}\left(-\frac{1}{4}x^{-3}\right)^r$

$= {}^{20}C_r (2)^{20-r}\cdot x^{20-r}\cdot\left(-\frac{1}{4}\right)^r . x^{-3r}$

$= {}^{20}C_r(2)^{20-r}\left(-\frac{1}{4}\right)^r\cdot x^{20-4r}$

for T_{r+1} to have a term in x^4, then $20-4r=4$ i.e. $r=4$

$\therefore\ T_5 = {}^{20}C_4 (2)^{16}\left(-\frac{1}{4}\right)^4 x^4 = 1240320x^4$

$\therefore$ coefficient of x^4 is 1240320 #

3. $\left(x^2+4x+3\right)^8 = \left[(x+1)(x+3)\right]^8 = (x+1)^8(x+3)^8$

Now, let us expand each binomial expression separately:

$(1+x)^8(3+x)^8 = \left(1+8x+28x^2+56x^3+70x^4+\ldots\right)$

$\times\left(3^8+{}^8C_1 3^7 x+{}^8C_2 3^6 x^2+{}^8C_3 3^5 x^3+{}^8C_4 3^4 x^4\ldots\right)$

Terms in x^4 are:

${}^8C_4 3^4 x^4 + 8\times{}^8C_3 3^5 x^4 + 28\times{}^8C_2 3^6 x^4 + 56\times{}^8C_1 3^7 x^4 + 70\times 3^8 x^4$

$= 2125116x^4$

$\therefore$ coefficient of x^4 is 2125116 #

4. $(2x+b)^6 = (2x)^6 + {}^6C_1(2x)^5 b + \ldots..$

equating coefficient of x^6 and x^5 :

$2^6 = {}^6C_1 2^5 b$

$2 = 6b$, i.e. $b = \frac{1}{3}$ #

5. $\left(ax-\frac{b}{x^2}\right)^5$

$T_{r+1} = {}^5C_r(ax)^r\left(-bx^{-2}\right)^{5-r} = {}^5C_r a^r(-b)^{5-r}\left(x^{3r-10}\right)$

for the term in x^2, $3r-10=2$ i.e. $r=4$

$T_5 = {}^5C_4 a^4(-b)x^2$

for the term in x^{-1}, $3r-10=-1$ i.e. $r=3$

$T_4 = {}^5C_3 a^3(-b)^2 x^{-1}$

equating coefficients of x^2 and x^{-1}

${}^5C_4 a^4(-b) = {}^5C_3 a^3(-b)^2$

$-5a^4b = 10a^3b^2$

i.e. $a=-2b$ #

6. (i) $T_{n+1} = {}^{2n+1}C_n a^{n+1}b^n$

$T_{n+2} = {}^{2n+1}C_n a^{2n+1-n-1}\cdot b^{n+1} = {}^{2n+1}C_{n+1}a^n b^{n+1}$ #

(ii) $T_{n+1}+T_{n+2} = {}^{2n+1}C_n a^{n+1}b^n + {}^{2n+1}C_{n+1}a^n b^{n+1}$

$= a^n b^n\left[{}^{2n+1}C_n a + {}^{2n+1}C_{n+1} b\right]$

$= a^n b^n\left[\frac{(2n+1)!}{n!(n+1)!}a + \frac{(2n+1)!}{(n+1)!n!}b\right]$

$= \frac{(2n+1)!}{n!(n+1)!}a^n b^n[a+b]$

$= {}^{2n+1}C_n a^n b^n$ (as $a+b=1$) #

7. $(4+3x)^{15}$

$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\times\frac{b}{a} = \frac{16-r}{r}\times\frac{3}{4} = \frac{48-3r}{4r}$

greatest coefficient occurs when $\frac{T_{r+1}}{T_r}\geq 1$

i.e. $\frac{48-3r}{4r}\geq 1$

$48-3r\geq 4r$

$7r\leq 48$

$r\leq 6.86$

i.e. $r=6$, $\therefore$ greatest coefficient $= T_7 = {}^{15}C_6 4^9 3^6$ #

8. (i) $T_{r+1} = {}^{12}C_r (2)^{12-r}\left(\frac{2x}{3}\right)^r$

$T_r = {}^{12}C_{r-1}(2)^{13-r}\left(\frac{2x}{3}\right)^{r-1}$

$$\therefore \frac{T_{r+1}}{T_r} = \frac{{}^{12}C_r (2)^{12-r}\left(\frac{2x}{3}\right)^r}{{}^{12}C_{r-1}(2)^{13-r}\left(\frac{2x}{3}\right)^{r-1}}$$

$$= \frac{\frac{12!}{(12-r)!r!}\times 2^{12-r}\times\left(\frac{2}{3}\right)^r x^r}{\frac{12!}{(13-r)!(r-1)!}\times 2^{13-r}\times\left(\frac{2}{3}\right)^{r-1} x^{r-1}}$$

$$= \frac{(13-r)}{r}\times\frac{1}{2}\times\frac{2}{3}\times x = \frac{13-r}{3r}x \ \#$$

(ii) The greatest coefficient occurs when:

$\frac{13-r}{3r} \ge 1$

$13 - r \ge 3r$

$4r \le 13$

$r \le 3.25$

i.e. $r = 3$, $\therefore$ greatest coefficient $= T_4 = {}^{12}C_3 (2)^9\left(\frac{2}{3}\right)^3$ #

(iii) From (i) $T_{r+1} = \frac{13-r}{3r}x$

Substituting $x = 2$, then : $\frac{T_{r+1}}{Tr} = \frac{26-2r}{3r} \ge 1$

i.e. $26 - 2r \ge 3r$

$5r \le 26$

$r \le 5.2$

i.e. $r = 5$, $\therefore$ greatest term $= T_6 = {}^{12}C_5 (2)^7\left(\frac{4}{3}\right)^5$ #

9. ${}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n = (1+x)^n$

(i) Substituting $x = 3$, gives:

(ii) Substituting $x = -1$, gives:

$1 - {}^nC_1 + {}^nC_2 + \ldots + (-1)^n\,{}^nC_n = 0$ #

(iii) Substituting $x = 1$, gives:

${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

i.e. $\frac{n!}{0!n!} + \frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!} + \ldots + \frac{n!}{n!0!} = 2^n$

dividing throughout by $n!$ gives:

$\frac{1}{n!} + \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!} + \ldots + \frac{1}{n!} = \frac{2^n}{n!}$ #

10. $(1+x)^n(1+x)^n = (1+x)^{2n}$

$\text{LHS} = \left(1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n\right)\left(1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n\right)$

coefficient of x^n in the above expansion is:

$= 1\times{}^nC_n + {}^nC_1\times{}^nC_{n-1} + {}^nC_2\times{}^nC_{n-2} + \ldots + {}^nC_n\times 1$

Noting ${}^nC_1 = {}^nC_{n-1}$ and in general ${}^nC_{n-r} = {}^nC_r$

$\therefore$ above expression becomes:

$= \left({}^nC_0\right)^2 + \left({}^nC_1\right)^2 + \left({}^nC_2\right)^2 + \ldots + \left({}^nC_n\right)^2$

$\text{RHS} = \left(1 + {}^{2n}C_1 x + \ldots + {}^{2n}C_n x^n + \ldots + {}^{2n}C_{2n}x^{2n}\right)$

coefficient of x^n in the above expansion is $= {}^{2n}C_n$

$\therefore \left({}^nC_0\right)^2 + \left({}^nC_1\right)^2 + \left({}^nC_2\right)^2 + \ldots + \left({}^nC_n\right)^2 = {}^{2n}C_n$ #

11. (i) $(1+x)^{2n} = 1 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + {}^{2n}C_{2n}x^{2n}$

(ii) Differentiating both sides of the above expression gives:

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2\,{}^{2n}C_2 x + \ldots + 2n\,{}^{2n}C_{2n}x^{2n-1}$

Substituting $x = 1$, gives:

$2n(2)^{2n-1} = {}^{2n}C_1 + 2\,{}^{2n}C_2 + \ldots + 2n\,{}^{2n}C_{2n}$

i.e. ${}^{2n}C_1 + 2\,{}^{2n}C_2 + \ldots + 2n\,{}^{2n}C_{2n} = n2^{2n} = n4^n$ #

12. (i) $P(x=10)={}^{20}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10}=0.176$ to 3 d.p. #

(ii) $P(\text{at most 3 heads})=P(x=0)+P(x=1)+P(x=2)+P(x=3)$

$$=\left(\frac{1}{2}\right)^{20}+{}^{20}C_1\left(\frac{1}{2}\right)^{20}+{}^{20}C_2\left(\frac{1}{2}\right)^{20}+{}^{20}C_3\left(\frac{1}{2}\right)^{20}$$

$$=\left(\frac{1}{2}\right)^{20}\left(1+{}^{20}C_1+{}^{20}C_2+{}^{20}C_3\right)=0.001 \text{ to 3 d.p. \#}$$

13. Let $p=$ prob. no defect $=0.98$, $q=$ prob. of defect $=0.02$

(i) $P(\text{no defect})=P(\text{all good})=P(x=10)=(0.98)^{10}=0.817$ to 3 d.p. #

(ii) $P(\text{at least one defective})=1-P(\text{all good})$

$$=1-0.817=0.183 \text{ to 3 d.p. \#}$$

14. Let $p=$ prob. of bull's eye $=\frac{3}{4}$, $q=$ prob. no bull's eye $=\frac{1}{4}$

(i) $P(\text{3 bull's eyes})={}^5C_3p^3q^2=10\times\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2=\frac{135}{512}$ #

(ii) $P(\text{at least 4 bull's eyes})=P(x=4)+P(x=5)$

$$={}^5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)+{}^5C_5\left(\frac{3}{4}\right)^5$$

$$=\frac{405}{1024}+\frac{243}{1024}=\frac{81}{128} \text{ \#}$$

(iii) If we are concerned with the probability of a bull's eye with the second round only, then the pattern of successes and failures is FSFFF.

$P(\text{FSFFF})=\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{1024}$ #

(iv) The greatest term in the binomial expansion $(q+p)^5$ where $q=\frac{1}{4}$ and $p=\frac{3}{4}$ is required.

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\times\frac{p}{q}=\frac{6-r}{r}\times 3\ge 1$$

i.e. $18-3r\ge r$

$4r\le 18$

$r\le 4.5$

i.e. $r=4$ $\therefore$ the most probable number of bull's eyes is 4. #

15. (i) $P(\text{king of hearts is chosen from one pack})=\frac{{}^1C_1\times{}^{51}C_4}{{}^{52}C_5}=0.09615$ #

(ii) Let $p=$ prob. king of hearts is chosen $=0.09615$, and

let $q=$ prob. king of hearts is not chosen $=0.903846$

$\therefore$ $P(\text{king of hearts appears more than once})$

$=P(x=2)+P(x=3)$

$={}^3C_2(0.09615)^2(0.903846)+{}^3C_3(0.09613)^3$

$=0.026$ to 3 d.p. #

16. (i) $P(\text{exactly 2 boys and 2 girls})={}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{6}{16}=\frac{3}{8}$ #

(ii) Let $p=$ prob. of 2 boys and 2 girls $=\frac{3}{8}$, and

let $q=$ prob. of not getting 2 boys and 2 girls $=\frac{5}{8}$

$\therefore$ $P(\text{exactly two of five families with 2 boys and 2 girls})$

$={}^5C_2\left(\frac{3}{8}\right)^2\left(\frac{5}{8}\right)^3$

$=0.342$ to 3 d.p. #

17. Need to consider the performance of Mark and Linda separately:

(a) **Mark:**

(i) For right-handed throws: $p=\frac{2}{3}$, $q=\frac{1}{3}$

Find the greatest term in expansion $(q+p)^{10}$

$$\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\times\frac{p}{q}=\frac{11-r}{r}\times 2\ge 1$$

when $22-2r\ge r$

$3r\le 22$

$r\le 7.32$

i.e. $r=7$,

$\therefore$ $T_8=P(x=7)={}^{10}C_7\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^3=0.2601$

(ii) For left-handed throws: $p=\dfrac{1}{2},\ q=\dfrac{1}{2}$

$$\therefore\ \frac{T_{r+1}}{T_r}=\frac{11-r}{r}\geq 1$$

when $11-r\geq r$

$2r\leq 11$

$r\leq 5.5$

i.e. $r=5$, $\therefore\ T_6=P(x=5)={}^{10}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5=0.2461$

Thus, the most probable outcome for Mark over 10 rounds is 7 bull's eyes with the right and 5 bull's eyes with the left, i.e. 12 in total.

The probability of this occuring $=0.2601\times 0.2461=0.064$

(b) Linda:

(i) For right-handed throws: $p=\dfrac{3}{5},\ q=\dfrac{2}{5}$

$$\therefore\ \frac{T_{r+1}}{T_r}=\frac{11-r}{r}\times\frac{p}{q}=\frac{11-r}{r}\times\frac{3}{2}\geq 1$$

when $33-3r\geq 2r$

$5r\leq 33$

$r\leq 6.6$

i.e. $r=6$, $\therefore\ T_7=P(x=6)={}^{10}C_6\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^4=0.2508$

(ii) For left-handed throws: same as (i) $P(x=6)=0.2508$

Thus, the most probable outcome for Linda over 10 rounds is 6 bull's eyes with the right and 6 bull's eyes with the left, i.e. 12 in total.

The probability of this occuring $=0.2508\times 0.2508=0.063$

$\therefore$ The probability that Mark attains 12 bull's eyes is slightly more probable. #

18. (i) $P(1\text{ piece has defects }A\text{ and }B)$

$=0.10\times 0.25=0.025=\dfrac{1}{40}$

$\therefore\ P(\text{all 4 had defects }A\text{ and }B)=P(x=4)$

$=\left(\dfrac{1}{40}\right)^4$

$=\dfrac{1}{2{,}560{,}000}$ #

(ii) $P(1\text{ piece has defect }B\text{ only})=0.90\times 0.05=0.045=\dfrac{9}{200}$

$\therefore\ P(\text{exactly one of four pieces had defect }B\text{ only})={}^4C_1\left(\dfrac{9}{200}\right)^1\left(\dfrac{191}{200}\right)^3$

$=0.157$ to 3 d.p.

(iii) $P(\text{at least one piece has some defect})=1-P(\text{all pieces have no defects})$

Now,

$P(\text{all pieces have no defects})$

$=1-P(\text{defect }A\text{ only})-P(\text{defect }B\text{ only})-P(\text{defects }A\text{ and }B)$

$=1-(0.10\times 0.75)-0.045-0.025$

$=1-0.075-0.045-0.025=0.855$

$\therefore$ Required probability $=1-(0.855)^4=0.466$ to 3 d.p. #

MISCELLANEOUS

) Curve Sketching

ample 1: Consider the function $f(x)=\dfrac{1}{\sqrt{1-x^2}}$.

(i) Show that $f(x)$ is an even function.

(ii) State the domain and range of $y=f(x)$.

(iii) Find the coordinates of any stationary points and determine their nature.

(iv) Identify any asymptotes of $f(x)$.

(v) Use the above information to sketch the curve.

tion 1:

(i) $f(-x)=\dfrac{1}{\sqrt{1-(-x)^2}}=\dfrac{1}{\sqrt{1-x^2}}=f(x)$

$\therefore\ f(x)$ is an even function. #

(ii) Domain: $1-x^2>0$ i.e. $-1<x<1$

Range: when $x\to\pm1,\ y\to\infty$

when $x=0,\ y=1\ \therefore\ y\geq1$ #

(iii) $f(x)=\left(1-x^2\right)^{-\frac{1}{2}}$

$f'(x)=x\left(1-x^2\right)^{-\frac{3}{2}}$

let $f'(x)=0$ to find stationary points:

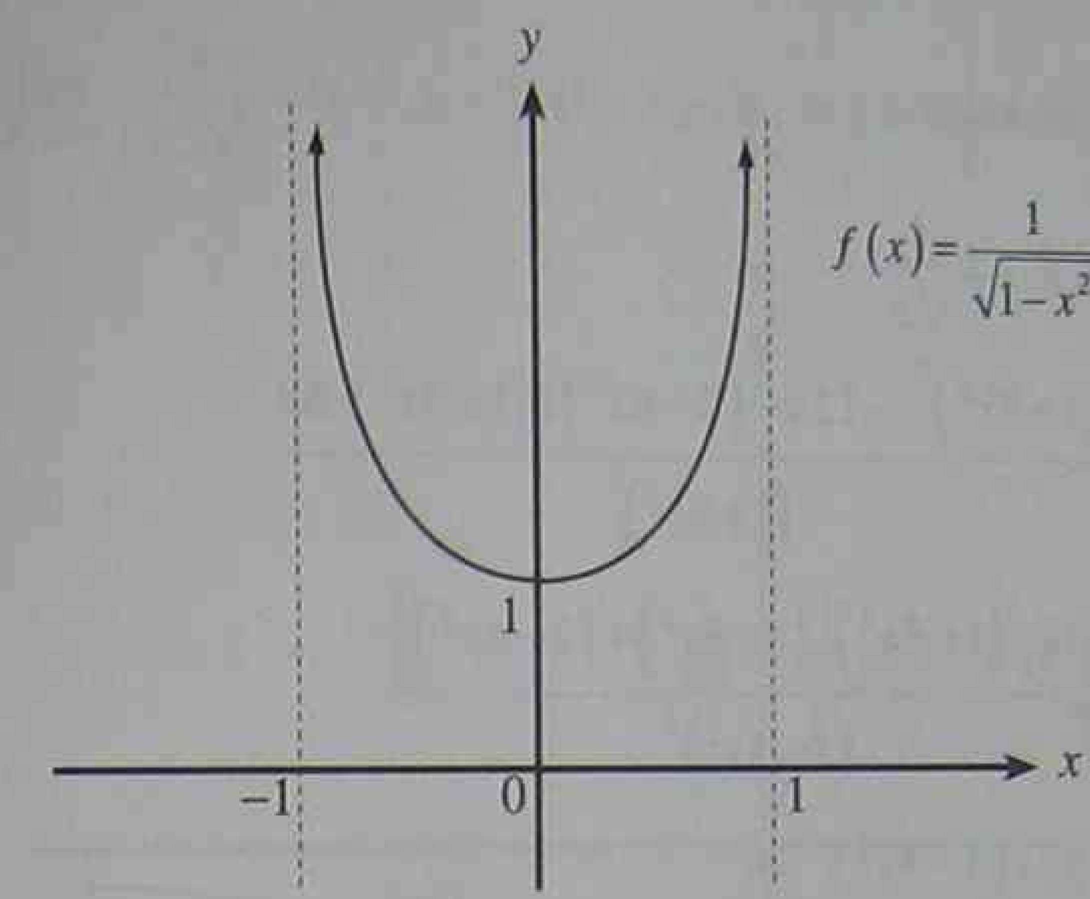
i.e. $\dfrac{x}{\left(1-x^2\right)^{\frac{3}{2}}}=0$ i.e. $x=0$

when $x=0,\ y=1$

x	-0.5	0	+0.5
$f'(x)$	-ve	0	+ve

$\therefore$ (0, 1) is a minimum turning point. #

(v)



Example 2: Consider the function $f(x)=\dfrac{2x}{1+3x^2}$.

(i) Show that $f(x)$ is an odd function.

(ii) Find the coordinates and nature of any stationary points.

(iii) Find the coordinates of any points of inflection.

(iv) Identify any asymptotes of $f(x)$.

(v) Sketch the curve, showing all essential features.

Solution 2:

(i) $f(-x)=\dfrac{-2x}{1+3x^2}\ \therefore\ f(x)=-f(-x)$ and hence $f(x)$ is an odd function. #

(ii) $f(x)=\dfrac{2x}{1+3x^2}$

$$f'(x)=\frac{\left(1+3x^2\right).2-2x.6x}{\left(1+3x^2\right)^2}$$

$$=\frac{2+6x^2-12x^2}{\left(1+3x^2\right)^2}$$

$$=\frac{2-6x^2}{\left(1+3x^2\right)^2}$$

let $f'(x)=0$ to find stationary points

i.e. $2-6x^2=0$

$2=6x^2$

$$x^2=\frac{1}{3}$$

$$x=\pm\frac{1}{\sqrt{3}}$$

$$f''(x)=\frac{\left(1+3x^2\right)^2\cdot -12x-\left(2-6x^2\right)2\left(1+3x^2\right)\cdot 6x}{\left(1+3x^2\right)^4}$$

$$=\frac{-12x\left(1+3x^2\right)\left[\left(1+3x^2\right)+\left(2-6x^2\right)\right]}{\left(1+3x^2\right)^4}$$

$$=\frac{-12x\left(3-3x^2\right)}{\left(1+3x^2\right)^3}$$

$$=\frac{-36x\left(1-x^2\right)}{\left(1+3x^2\right)^3}$$

at $x=\frac{1}{\sqrt{3}}, y=\frac{1}{\sqrt{3}}$ and $f''(x)<0$ $\therefore$ $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is a maximum turning point.

at $x=-\frac{1}{\sqrt{3}}, y=-\frac{1}{\sqrt{3}}$ and $f''(x)>0$ $\therefore$ $\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$ is a minimum turning point. #

(iii) let $f''(x)=0$ to find possible points of inflection:

i.e. $-36x\left(1-x^2\right)=0$

i.e. $x=0,-1,1$

To verify that a point of inflection exists at these points, check the sign of $f''(x)$:

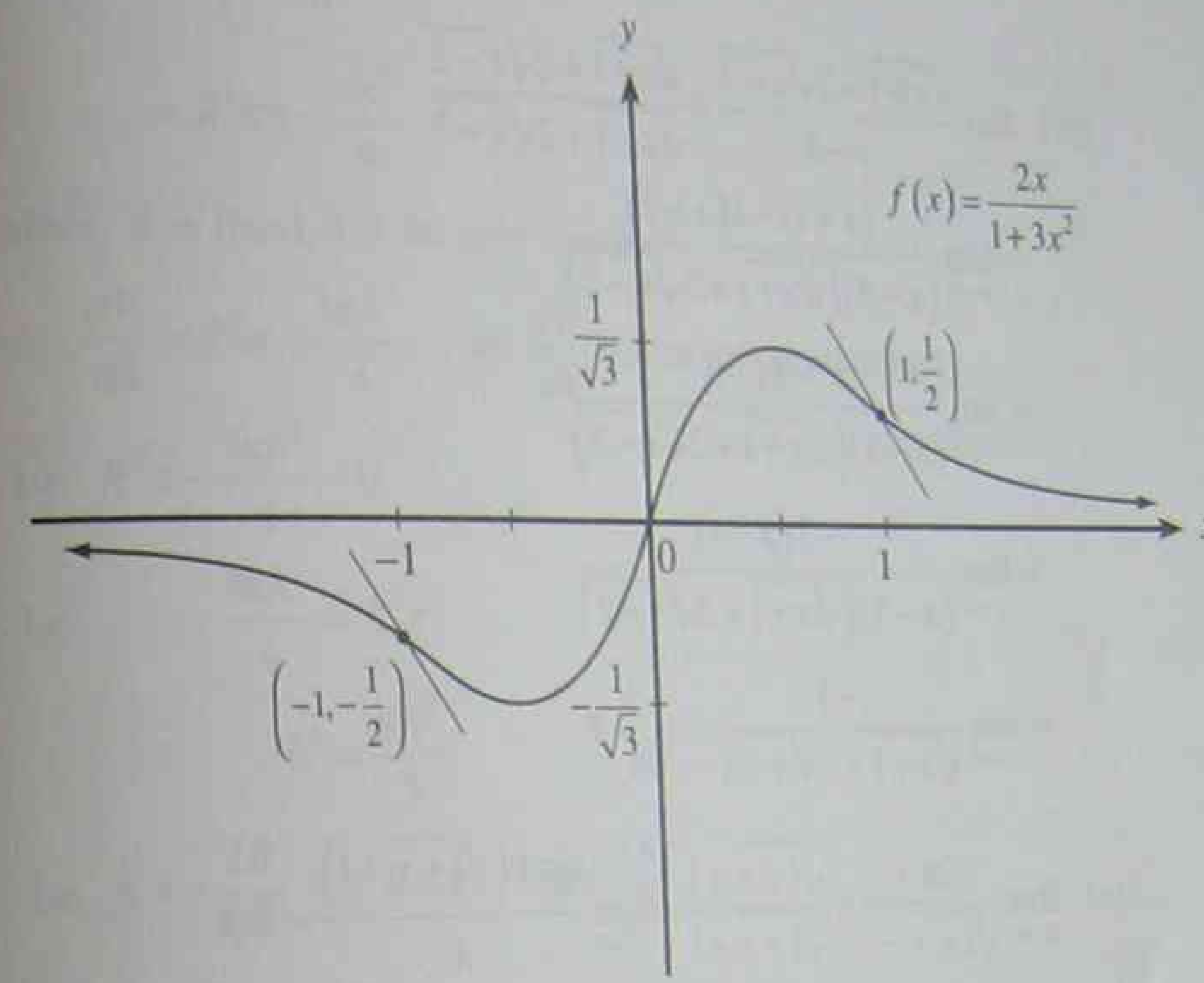
at $(x=0,\ y=0)$, $\left(x=1,\ y=\frac{1}{2}\right)$, $\left(x=-1,\ y=-\frac{1}{2}\right)$

x	-1.5	-1	-0.5	0	0.5	1	1.5
$f''(x)$	-ve	0	+ve	0	-ve	0	+ve

since $f''(x)$ changes sign in passing through all these points $\therefore$ $(0,0)$, $\left(1,\frac{1}{2}\right)$, $\left(-1,-\frac{1}{2}\right)$ are all points of inflection. #

(iv) $\lim_{x\to\pm\infty}\frac{2x}{1+3x^2}=0$ $\therefore$ $y=0$ (i.e. x-axis) is a horizontal asymptote. #

(v)



(B) Limits

Example 1: Find the following limits:

(i) $\lim_{x\to 5}\frac{x^2-25}{x-5}$

(ii) $\lim_{x\to 2}\frac{x^2-7x+10}{x-2}$

(iii) $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

(iv) $\lim_{x\to 3}\frac{\sqrt{x+1}-2\sqrt{x-2}}{x-3}$

(v) $\lim_{x\to 0}\frac{\sin x}{\sqrt{1+x}-1}$

Solution 1:

(i) $\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+5)}{(x-5)}=\lim_{x\to 5}(x+5)=10$ #

(ii) $\lim_{x\to 2}\frac{x^2-7x+10}{x-2}=\lim_{x\to 2}\frac{(x-2)(x-5)}{(x-2)}=\lim_{x\to 2}(x-5)=-3$ #

Miscellaneous

(iii) $\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{x-4}=\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}=\lim\limits_{x\to4}\dfrac{1}{\sqrt{x}+2}=\dfrac{1}{4}$ #

(iv) $\lim\limits_{x\to3}\dfrac{\sqrt{x+1}-2\sqrt{x-2}}{x-3}\times\dfrac{\sqrt{x+1}+2\sqrt{x-2}}{\sqrt{x+1}+2\sqrt{x-2}}$

$$=\lim_{x\to3}\frac{(x+1)-4(x-2)}{(x-3)\left(\sqrt{x+1}+2\sqrt{x-2}\right)}$$

$$=\lim_{x\to3}\frac{-3x+9}{(x-3)\left(\sqrt{x+1}+2\sqrt{x-2}\right)}$$

$$=\lim_{x\to3}\frac{-3(x-3)}{(x-3)\left(\sqrt{x+1}+2\sqrt{x-2}\right)}$$

$$=\lim_{x\to3}\frac{-3}{\sqrt{x+1}+2\sqrt{x-2}}=-\frac{3}{4} \ \#$$

(v) $\lim\limits_{x\to0}\dfrac{\sin x}{\sqrt{1+x}-1}\times\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\lim\limits_{x\to0}\dfrac{\sin x\left(\sqrt{1+x}+1\right)}{x}=2$ #

(C) Maxima/Minima Problems

Example 1: A cylinder of height h and radius r is inscribed in a sphere of radius R.

(i) Show that $r=\dfrac{\sqrt{4R^2-h^2}}{2}$.

(ii) Hence, find the greatest volume of the cylinder.

Solution 1:

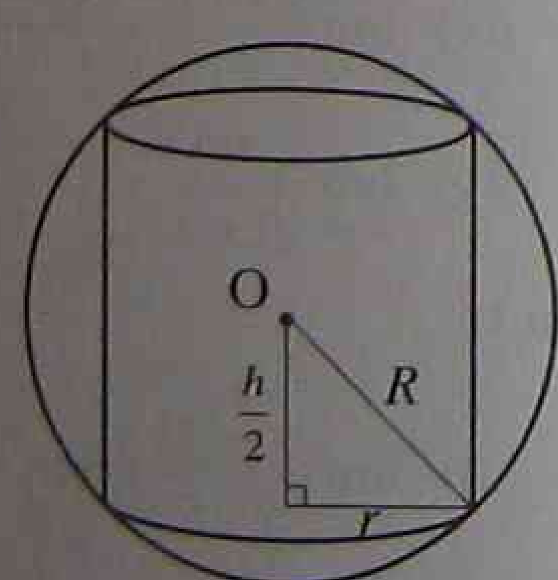
(i) from the diagram, using pythagoras:

$$R^2=\left(\frac{h}{2}\right)^2+r^2$$

$$\therefore\ r=\sqrt{R^2-\frac{h^2}{4}}$$

$$=\sqrt{\frac{4R^2-h^2}{4}}$$

$$=\frac{\sqrt{4R^2-h^2}}{2} \ \#$$



(ii) $V_{\text{cylinder}}=\pi r^2h$

$$=\pi\left(\frac{4R^2-h^2}{4}\right)h$$

$$=R^2\pi h-\frac{\pi h^3}{4}$$

since R is fixed, h is the only variable:

$\therefore\ \dfrac{dV}{dh}=R^2\pi-\dfrac{3\pi h^2}{4}$ let $\dfrac{dV}{dh}=0$ for maxima/minima

i.e. $R^2\pi-\dfrac{3\pi h^2}{4}=0$

i.e. $\dfrac{3\pi h^2}{4}=R^2\pi$

$$h^2=\frac{4R^2}{3}$$

i.e. $h=\pm\dfrac{2R}{\sqrt{3}}$ but $h>0$ $\therefore\ h=\dfrac{2R}{\sqrt{3}}$

Now, $\dfrac{d^2V}{dh^2}=-\dfrac{3\pi h}{2}$

at $h=\dfrac{2R}{\sqrt{3}}$, $\dfrac{d^2V}{dh^2}<0$

$\therefore\ h=\dfrac{2R}{\sqrt{3}}$ yields a maximum volume.

$\therefore$ greatest volume of cylinder $=R^2\pi\left(\dfrac{2R}{\sqrt{3}}\right)-\dfrac{8R^3\pi}{12\sqrt{3}}$

$$=\frac{2R^3\pi}{\sqrt{3}}-\frac{8R^3\pi}{12\sqrt{3}}$$

$$=\frac{24R^3\pi-8R^3\pi}{12\sqrt{3}}=\frac{16R^3\pi}{12\sqrt{3}}=\frac{4\pi R^3}{3\sqrt{3}}\ \text{units}^3 \ \#$$

REVIEW EXERCISES

(A) Curve Sketching

1. Consider the function $f(x)=\frac{-8}{x^2-4}$.

   (i) Show that $f(x)$ is an even function.

   (ii) Find the coordinates of any stationary points and determine their nature.

   (iii) Identify all asymptotes of $f(x)$.

   (iv) Use the above information to sketch the curve.

2. Consider the function $f(x)=\frac{x^3}{x^2-9}$.

   (i) Show that $f(x)$ is an odd function.

   (ii) Find the coordinates and nature of any stationary points.

   (iii) Detemine the values of x for which the function is decreasing.

   (iv) Identify any asymptotes of $f(x)$.

   (v) Sketch the curve, showing all essential features.

3. A function is given by $y=\sin^{-1}\left(\frac{x^2-1}{x^2+1}\right)$.

   (i) Show that for $x\neq 0$, $\frac{dy}{dx}=\frac{2x}{|x|\left(x^2+1\right)}$.

   (ii) Hence, determine for what values of x the function is increasing or decreasing.

   (iii) Find the value of y at $x=0$.

   (iv) Identify any asymptotes of $f(x)$.

   (v) Use the above information to draw a neat sketch of the function.

(B) Limits

4. Evaluate the following limits:

   (i) $\lim_{x\to1}\frac{x^2+3x+2}{x^2-x-2}$ (ii) $\lim_{x\to-2}\frac{-2x-4}{x^3+2x^2}$

   (iii) $\lim_{x\to1}\frac{x^4-1}{x^3-1}$ (iv) $\lim_{x\to0}\frac{\cos x-1}{x^2}$

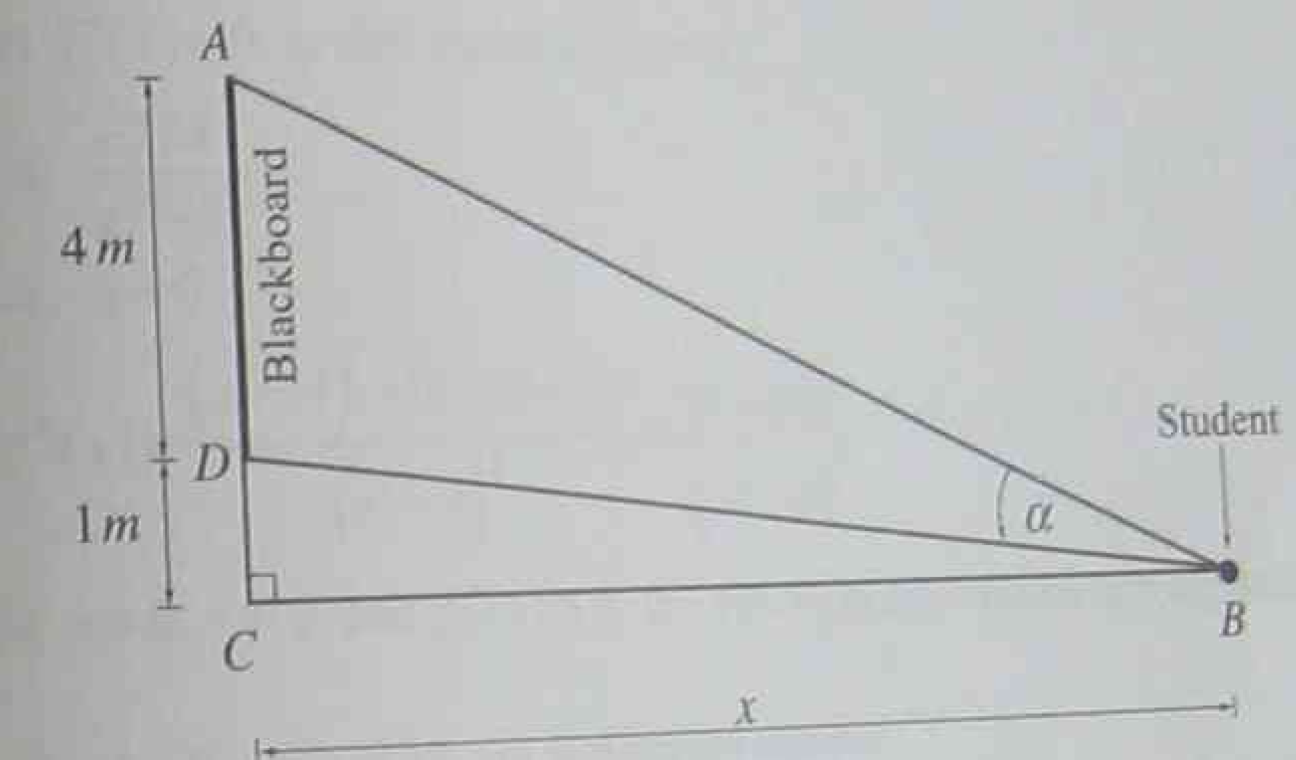
   (v) $\lim_{x\to1}\frac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x-1}}$ (vi) $\lim_{x\to0}\frac{\sin 4x}{x^2}$

(C) Maxima/Minima Problems

5. (i) Show that $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$

   (ii) A student is sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 4 metres high and starts 1 metre from the foot of the wall. If the student is located x metres from the wall, show that the student's viewing angle is given by:

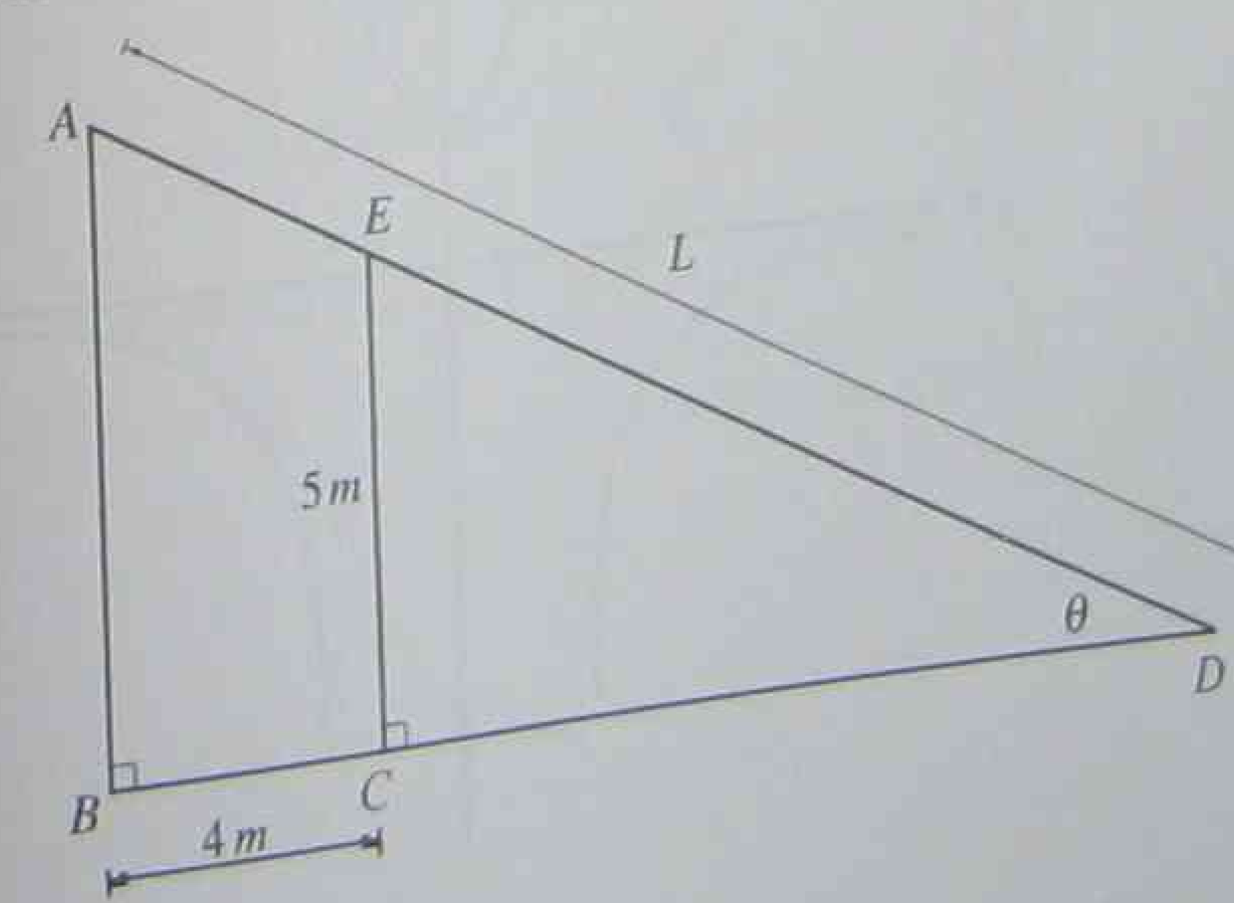
   $$\alpha=\cot^{-1}\left(\frac{x}{5}\right)-\cot^{-1}(x).$$



   (iii) Find the value of x which maximises α.

   (iv) Hence find the maximum angle α subtended to the nearest degree.

6. A 5m fence stands 4m from the wall of a house. A farmer wishes to reach a point A on the wall by the use of a ladder L that can reach from the ground outside the fence to the wall as shown in the diagram below. Let $\angle ADB=\theta$.



(i) Show that $L=\dfrac{5}{\sin\theta}+\dfrac{4}{\cos\theta}$.

(ii) Hence, find the length of the shortest ladder that can reach from the ground outside the fence to the wall. Express your answer correct to 1 decimal place.

Worked Solutions to Review Exercises

1. (i) $f(-x)=\dfrac{-8}{(-x)^2-4}=\dfrac{-8}{x^2-4}=f(x)$

$\therefore\ f(x)$ is an even function. #

(ii) $f(x)=\dfrac{-8}{x^2-4}$

$$f'(x)=\frac{(x^2-4)\cdot 0-8\cdot 2x}{x^2-4}=\frac{-16x}{x^2-4}$$

let $f'(x)=0$ to find stationary points

i.e. $\dfrac{-16x}{x^2-4}=0$

i.e. $x=0$

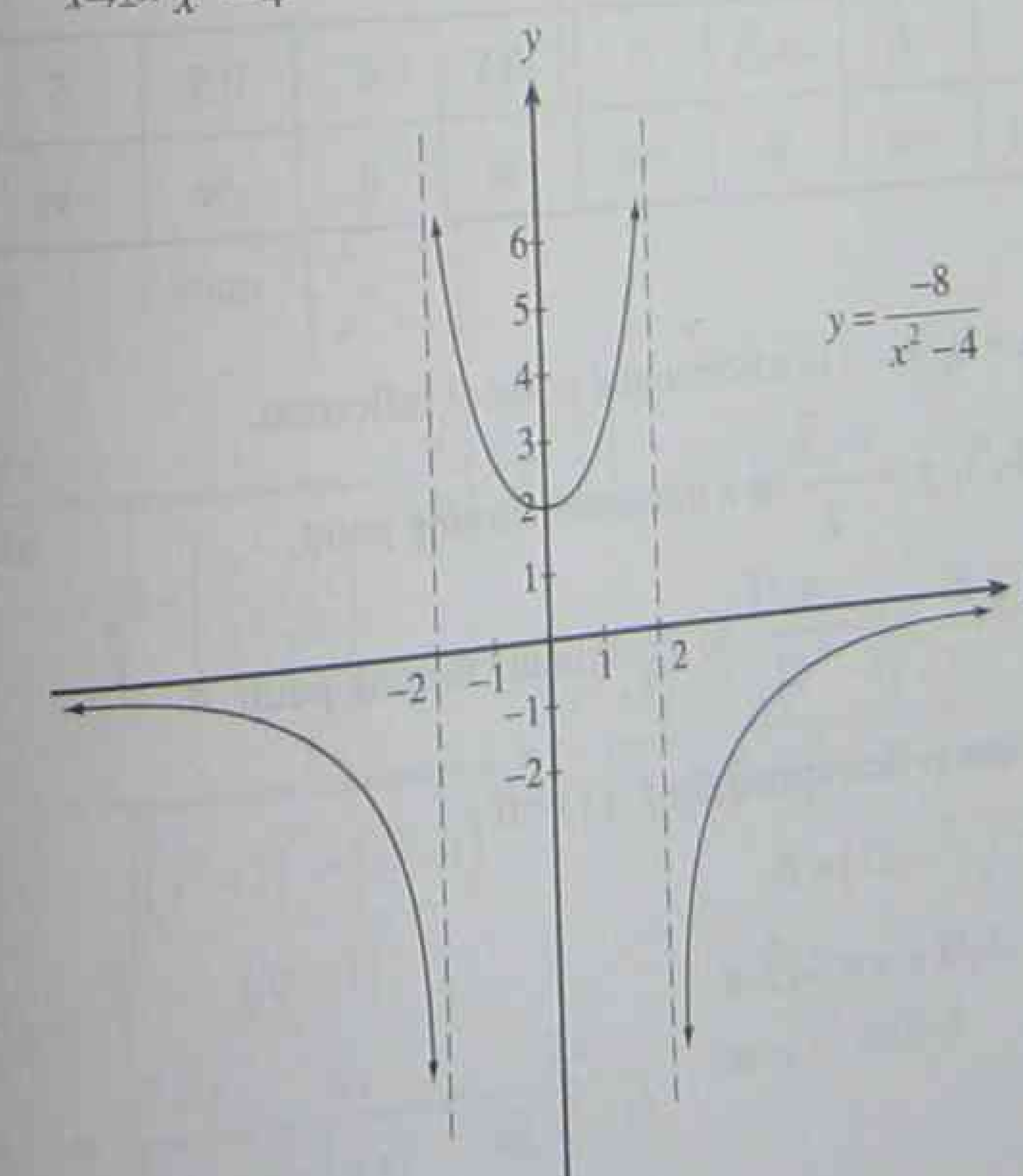
$$f''(x)=\frac{(x^2-4)\cdot -16-16x\cdot 2x}{(x^2-4)^2}=\frac{64-48x^2}{(x^2-4)^2}$$

at $x=0$, $y=2$ and $f''(x)>0$ $\therefore$ $(0,2)$ is a minimum turning point. #

(iii) $x^2-4\neq 0$ $\therefore$ $x=\pm 2$ are vertical asymptotes

also $\lim\limits_{x\to\pm\infty}\dfrac{-8}{x^2-4}=0$ $\therefore$ $y=0$ is a horizontal asymptote. #

(iv)



2. (i) $f(-x)=\dfrac{(-x)^3}{(-x)^2-9}=\dfrac{-x^3}{x^2-9}$

$-f(-x)=\dfrac{x^3}{x^2-9}=f(x)$ $\therefore$ $f(x)$ is an odd function. #

(ii) $f(x)=\dfrac{x^3}{x^2-9}$

$$f'(x)=\frac{(x^2-9).3x^2-x^3.2x}{(x^2-9)^2}$$

$$=\frac{3x^4-27x^2-2x^4}{(x^2-9)^2}$$

$$=\frac{x^2(x^2-27)}{(x^2-9)^2}$$

let $f'(x)=0$ to find stationary points

i.e. $x^2(x^2-27)=0$

i.e. $x=0,-3\sqrt{3},3\sqrt{3}$

To determine the nature of the stationary points we need to examine the sign of $f'(x)$:

x	-6	$-3\sqrt{3}$	-5	-0.5	0	0.5	5	$3\sqrt{3}$	6
$f'(x)$	+ve	0	-ve	-ve	0	-ve	-ve	0	+ve

$\therefore$ $x=0,\ y=0$ is a horizontal point of inflection.

$x=3\sqrt{3},\ y=\dfrac{9\sqrt{3}}{2}$ is a minimum turning point.

$x=-3\sqrt{3},\ y=-\dfrac{9\sqrt{3}}{2}$ is a maximum turning point. #

(iii) Function is decreasing for $f'(x)<0$

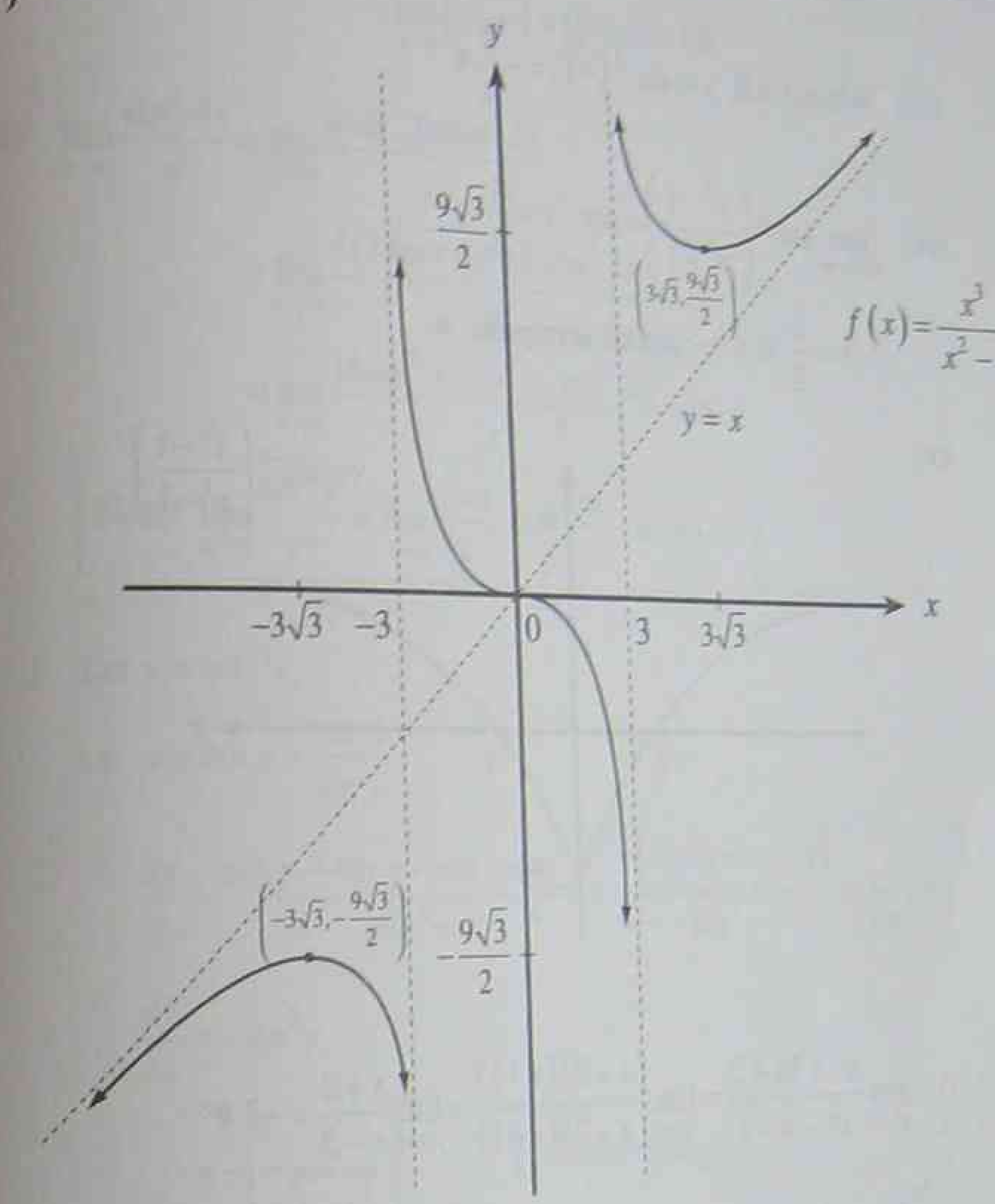
i.e. $x^2(x^2-27)<0$

i.e. $-3\sqrt{3}<x<3\sqrt{3}$ #

(iv) $x^2-9\neq 0$ $\therefore$ $x=\pm 3$ are vertical asymptotes.

Also, $\displaystyle\lim_{x\to\pm\infty}\frac{x^3}{x^2-9}=x$ i.e. $y=x$ is an oblique asymptote of the function. #

(v)



3. (i) $y=\sin^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\frac{1}{\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)^2}}}\times\frac{(x^2+1)\cdot 2x-(x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$=\frac{x^2+1}{\sqrt{4x^2}}\times\frac{4x}{(x^2+1)^2}=\frac{2x}{|x|(x^2+1)}$$ (as $\sqrt{x^2}\geq 0$) #

Miscellaneous

(ii) The function is increasing for $f'(x)>0$ i.e. $x>0$

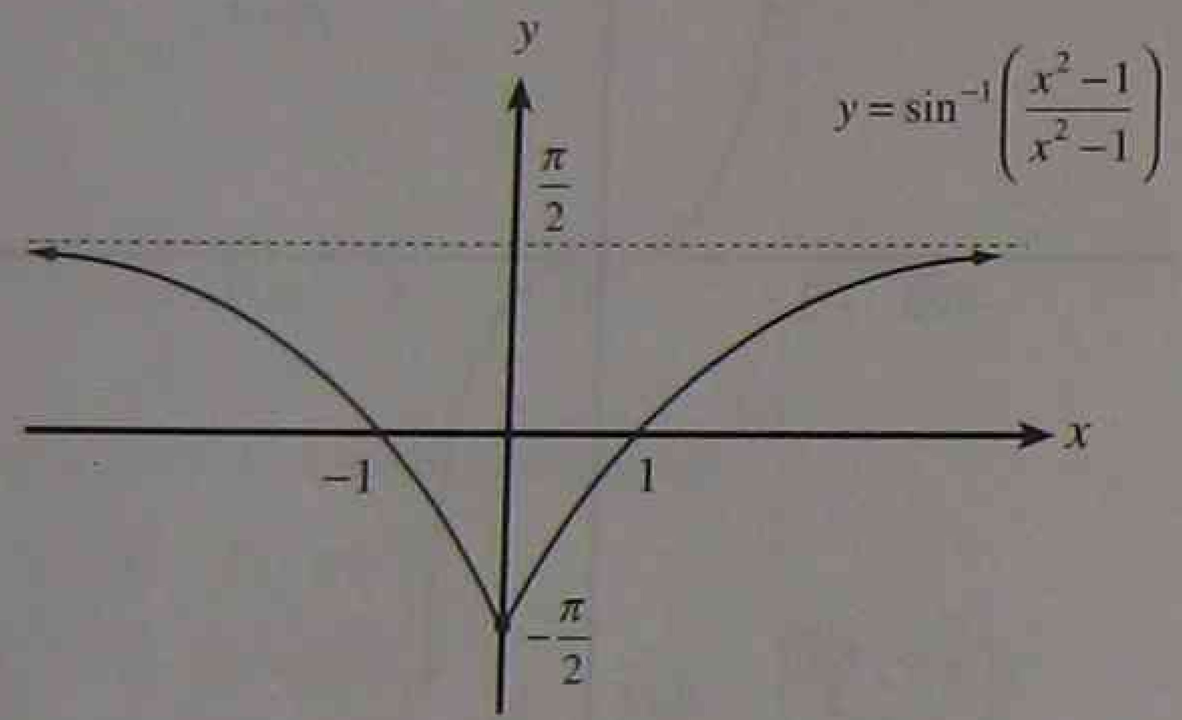
the function is decreasing for $f'(x)<0$ i.e. $x<0$ #

(iii) When $x=0$, $y=\sin^{-1}(-1)=-\dfrac{\pi}{2}$ #

(iv) $\displaystyle\lim_{x\to\pm\infty}\sin^{-1}\left(\frac{x^2-1}{x^2+1}\right)=\sin^{-1}1=\frac{\pi}{2}$

$\therefore\ y=\dfrac{\pi}{2}$ is a horizontal asymptote #

(v)



4. (i) $\displaystyle\lim_{x\to1}\frac{x^2+3x+2}{x^2-x-2}=\lim_{x\to1}\frac{(x+2)(x+1)}{(x-2)(x+1)}=\lim_{x\to1}\frac{x+2}{x-2}=-3$ #

(ii) $\displaystyle\lim_{x\to-2}\frac{-2x-4}{x^3+2x^2}=\lim_{x\to-2}\frac{-2(x+2)}{x^2(x+2)}=\lim_{x\to-2}\frac{-2}{x^2}=-\frac{1}{2}$ #

(iii) $\displaystyle\lim_{x\to1}\frac{x^4-1}{x^3-1}=\lim_{x\to1}\frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)}=\frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)}=\frac{4}{3}$ #

(iv) $\displaystyle\lim_{x\to0}\frac{\cos x-1}{x^2}=\lim_{x\to0}\frac{\cos x-1}{x^2}\times\frac{\cos x+1}{\cos x+1}$

$\displaystyle=\lim_{x\to0}\frac{\cos^2x-1}{x^2(\cos x+1)}$

$\displaystyle=\lim_{x\to0}\frac{-\sin^2x}{x^2(\cos x+1)}=-\frac{1}{2}$ # $\displaystyle\left(\text{as }\lim_{x\to0}\frac{\sin^2x}{x^2}=1\right)$

(v) $\displaystyle\lim_{x\to1}\frac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x-1}}=\lim_{x\to1}\frac{\sqrt{x-1}\left(\sqrt{x+1}+1\right)}{\sqrt{x-1}}$

$\displaystyle=\lim_{x\to1}\left(\sqrt{x+1}+1\right)=\sqrt{2}+1$ #

(vi) $\displaystyle\lim_{x\to0}\frac{\sin^2 4x}{x^2}=\lim_{x\to0}\frac{4\sin^2 2x\cos^2 2x}{x^2}$

$\displaystyle=\lim_{x\to0}\frac{4(2\sin x\cos x)^2\cos^2 2x}{x^2}$

$\displaystyle=\lim_{x\to0}\frac{16\sin^2 x\cdot\cos^2 x\cdot\cos^2 2x}{x^2}=16$ #

$\displaystyle\left[\text{Note: }\lim_{x\to0}\frac{\sin^2x}{x^2}=\lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{\sin x}{x}=1\times1=1\right]$

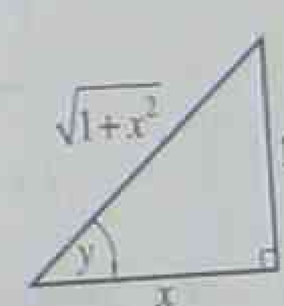
5. (i) Let $y=\cot^{-1}x$

i.e. $x=\cot y=\dfrac{\cos y}{\sin y}$

$\displaystyle\frac{dx}{dy}=\frac{\sin y\cdot-\sin y-\cos y\cdot\cos y}{\sin^2y}=\frac{-\left(\sin^2y+\cos^2y\right)}{\sin^2y}=\frac{-1}{\sin^2y}$

$\displaystyle\therefore\ \frac{dy}{dx}=-\sin^2y$

$\displaystyle=-\left(\frac{1}{\sqrt{1+x^2}}\right)^2$ (from the triangle)



$\displaystyle=\frac{-1}{1+x^2}$ #

(ii) let $\angle DBC=\theta$

$\therefore\ \tan\theta=\dfrac{1}{x}$ i.e. $x=\cot\theta$ i.e. $\theta=\cot^{-1}x$

$\angle ABC=\theta+\alpha$

$\therefore\ \tan(\theta+\alpha)=\dfrac{5}{x}$

i.e. $\cot(\theta+\alpha)=\dfrac{x}{5}$

$\therefore\ \theta+\alpha=\cot^{-1}\left(\dfrac{x}{5}\right)$

i.e. $\alpha=\cot^{-1}\left(\dfrac{x}{5}\right)-\cot^{-1}(x)$ #

(iii) $\alpha = \cot^{-1}\left(\frac{x}{5}\right) - \cot^{-1}(x)$

$$\frac{d\alpha}{dx} = \frac{-1}{1+\left(\frac{x}{5}\right)^2} \times \frac{1}{5} + \frac{1}{1+x^2}$$

$$= \frac{-1}{1+\frac{x^2}{25}} \times \frac{1}{5} + \frac{1}{1+x^2}$$

$$= \frac{-5}{25+x^2} + \frac{1}{1+x^2}$$

let $\frac{d\alpha}{dx} = 0$ to find maxima/minima:

i.e. $\frac{5}{25+x^2} = \frac{1}{1+x^2}$

$$5+5x^2 = 25+x^2$$

$$4x^2 = 20$$

$$x^2 = 5$$

$$x = \sqrt{5} \quad (\text{as } x>0)$$

x	2	$\sqrt{5}$	3
$\frac{d\alpha}{dx}$	+ve	0	-ve

$\therefore$ $x = \sqrt{5}$m yields a maximum α#

(iv) Substituting $x = \sqrt{5}$ into α gives:

$$\alpha = \cot^{-1}\left(\frac{1}{\sqrt{5}}\right) - \cot^{-1}\left(\sqrt{5}\right)$$

let $\gamma = \cot^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and $\beta = \cot^{-1}\left(\sqrt{5}\right)$

i.e. $\cot\gamma = \frac{1}{\sqrt{5}}$ $\qquad$ $\cot\beta = \sqrt{5}$

$\tan\gamma = \sqrt{5}$ $\qquad$ $\tan\beta = \frac{1}{\sqrt{5}}$

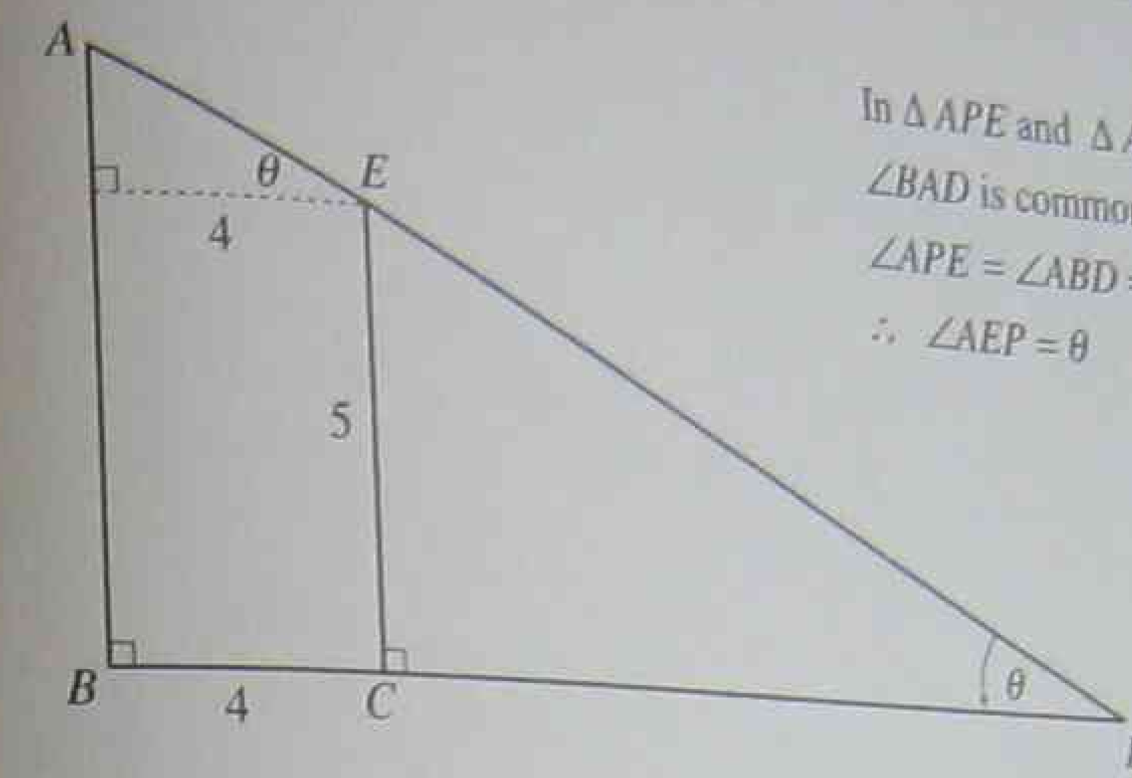
$\gamma = \tan^{-1}\left(\sqrt{5}\right)$ $\qquad$ $\beta = \tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$= 65.905°$ $\qquad$ $= 24.095°$

$\therefore$ $\alpha = 42°$ to the nearest degree#

Miscellaneous

6.



In ΔAPE and ΔABD

$\angle BAD$ is common

$\angle APE = \angle ABD = 90°$

$\therefore$ $\angle AEP = \theta$

(i) Looking at ΔAEP: $\cos\theta = \frac{4}{AE}$ i.e. $AE = \frac{4}{\cos\theta}$

Looking at ΔEDC: $\sin\theta = \frac{5}{DE}$ i.e. $DE = \frac{5}{\sin\theta}$

$\therefore$ $L = AE + DE = \frac{4}{\cos\theta} + \frac{5}{\sin\theta}$ #

(ii) $L = \frac{4}{\cos\theta} + \frac{5}{\sin\theta} = 4(\cos\theta)^{-1} + 5(\sin\theta)^{-1}$

$$\frac{dL}{d\theta} = \frac{-4\times-\sin\theta}{\cos^2\theta} - \frac{5\times\cos\theta}{\sin^2\theta}$$

$$= \frac{4\sin\theta}{\cos^2\theta} - \frac{5\cos\theta}{\sin^2\theta}$$

$$= \frac{4\sin^3\theta - 5\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

Let $\frac{dL}{d\theta} = 0$ to find maxima/minima.

i.e. $4\sin^3\theta = 5\cos^3\theta$

$$\tan^3\theta = \frac{5}{4}$$

i.e. $\theta \approx 47°8'$

θ	45°	47°	50°
$\frac{dL}{d\theta}$	-ve	0	+ve

$\therefore$ $\theta = 47°8'$ yields a minimum.

Hence, the shortest ladder L is given by: $L = \frac{4}{\cos\theta} + \frac{5}{\sin\theta} = 12.7$ m to 1 d.p. #

Miscellaneous

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CONTENTS

TOPIC 1

BASIC ALGEGRA AND ARITHMETIC

(A) Decimals, Significant Figures and Scientific Notation.

Useful steps to follow in solving these problems :

1. Write down initial calculator display in full before rounding off.
2. **Decimal figures** begin with the **first digit after the decimal point**.
3. **Significant figures** begin with the **first non-zero digit**.
4. **Scientific notation** is expressed as a number between 1 and 10 multiplied by a power of ten.
5. **Rounding off :**

   If next digit is ≥ 5, then round-up.
   If next digit is < 5, then round-down.

Example 1: Evaluate the following correct to 4 decimal places:

(i) $\dfrac{(12.5)^3}{12.5+15.3}$ (ii) $\dfrac{\sqrt{3.2+8.77}}{(2.15)^2}$

Solution 1:

(i) $\dfrac{(12.5)^3}{12.5+15.3} = 70.25629496 = 70.2563$ correct to 4 d.p. #

(ii) $\dfrac{\sqrt{3.2+8.77}}{(2.15)^2} = 0.748462688 = 0.7485$ correct to 4 d.p. #

Example 2: Evaluate the following correct to 3 significant figures:

(i) $\dfrac{4.75\times 18.22}{1-\sqrt{11}}$ (ii) $55\times(1000)^2\times 1.25$

Solution 2:

(i) $\dfrac{4.75\times 18.22}{1-\sqrt{11}} = -37.35822925 = -37.4$ correct to 3 s.f. #

(ii) $55\times(1000)^2\times 1.25 = 68,750,000 = 6.88\times 10^7 = 68,800,000$ correct to 3 s.f. #

Example 3: Express the following scientific notation correct to 4 significant figures:

(i) 5 270 100 (ii) 0.0071084 (iii) 387.295

Solution 3:

(i) 5.270×10^{6} # (ii) 7.108×10^{-3} # (iii) 3.873×10^{2} #

(B) Changing Recurring Decimals to Fractions

Steps to follow in solving these questions are:

1. Let $x =$ recurring decimal in expanded form.
2. Let the number of recurring digits $= n$.

   {e.g. $0.\dot{3}\ n=1,\ 0.1\dot{2}\dot{5}\ n=2$ etc.}

3. Multiply recurring decimal by 10^{n}.
4. Subtract (1) from (3) to eliminate recurring part.
5. Solve for x, expressing your answer as a fraction in its simplest form.

Example 1: Change the following recurring decimals into fractions:

(i) $0.\dot{4}$ (ii) $0.2\dot{7}\dot{5}$ (iii) $3.1\dot{1}\dot{2}$

Solution 1:

(i) Let $x = 0.4444$ .... $(n=1)$

$10x = 4.444$ ..... (multiplying by 10^{1})

$\therefore\ 9x = \frac{4}{9}$ (subtracting)

$x = \frac{4}{9}$ #

(ii) Let $x = 0.2757575$ ..... $(n=2)$

$100x = 27.5755757$ ..... (multiply by 10^{2})

$\therefore\ 99x = 27.3$ (subtracting)

i.e. $990x = 273$

$x = \frac{273}{990} = \frac{91}{330}$ #

(iii) Let $x = 3.1121212$ ..... $(n=2)$

$100x = 311.2121212$ ..... (multiplying by 10^{2})

$\therefore\ 99x = 308.1$ (subtracting)

i.e. $990x = 3081$

$x = \frac{3081}{990} = 3\frac{37}{330}$ #

(C) Surds

Important properties of surds:

1. $\sqrt{a}\times\sqrt{a} = \sqrt{a^2} = a$
2. $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$
3. $a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$
4. $a\sqrt{b}\times c\sqrt{b} = ac\sqrt{b^2} = acb$
5. $\sqrt{a^2 b} = \sqrt{a^2}\times\sqrt{b} = a\sqrt{b}$
6. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
7. $(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d}) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

Example 1: Simplify the following as far as possible:

(i) $\sqrt{4}+\sqrt{9}$ (ii) $\sqrt{75}$ (iii) $\sqrt{8}+\sqrt{2}$

(iv) $\sqrt{450}$ (v) $4\sqrt{5}+2\sqrt{7}-\sqrt{5}+2\sqrt{7}$ (vi) $\sqrt{5}\times2\sqrt{5}$

(vii) $2\sqrt{18}-\sqrt{8}+\sqrt{12}$ (viii) $\sqrt{3}+\sqrt{60}$ (ix) $(3\sqrt{10})^2$

(x) $\sqrt{28}+\sqrt{32}-\sqrt{8}$ (xi) $\sqrt{2}(\sqrt{8}-\sqrt{9})$ (xii) $2\sqrt{3}\times\sqrt{27}\times\sqrt{8}$

(xiii) $(\sqrt{3}+2)(\sqrt{3}+1)$ (xiv) $\frac{\sqrt{10}}{\sqrt{5}}$

Solution 1:

(i) $\sqrt{4}+\sqrt{9} = 2+3 = 5$ #

(ii) $\sqrt{75} = \sqrt{25}\times\sqrt{3} = 5\sqrt{3}$ #

(iii) $\sqrt{8}+\sqrt{2} = \sqrt{4\times}\sqrt{2}+\sqrt{2} = 2\sqrt{2}+\sqrt{2} = 3\sqrt{2}$ #

(iv) $\sqrt{450} = \sqrt{9}\times\sqrt{50} = 3\times\ \sqrt{25}\ \times\sqrt{2} = 15\sqrt{2}$ #

(v) $4\sqrt{5}+2\sqrt{7}-\sqrt{5}+2\sqrt{7} = 3\sqrt{5}\ +4\sqrt{7}$ #

(vi) $\sqrt{5}\times2\sqrt{5}\ = 2\times5 = 10$ #

(vii) $2\sqrt{18}-\sqrt{8}+\sqrt{12} = 2\times\sqrt{9}\times\sqrt{2}-\sqrt{2}\times\sqrt{4}+\sqrt{3}\times\sqrt{4}$

$= 6\sqrt{2}-2\sqrt{2}+2\sqrt{3} = 4\sqrt{2}+2\sqrt{3}$ #

(viii) $\sqrt{3}\times\sqrt{60} = \sqrt{180} = \sqrt{36}\times\sqrt{5} = 6\sqrt{5}$ #

(ix) $(3\sqrt{10})^2 = 3\sqrt{10}\times3\sqrt{10} = 9\times10 = 90$ #

(x) $\sqrt{28}+\sqrt{32}-\sqrt{8}=\sqrt{4}\times\sqrt{7}+\sqrt{4}\times\sqrt{8}-\sqrt{4}\times\sqrt{2}$

$$=2\sqrt{7}+2\sqrt{8}-2\sqrt{2}$$
$$=2\sqrt{7}+2\times\sqrt{4}\times\sqrt{2}-2\sqrt{2}$$
$$=2\sqrt{7}+4\sqrt{2}-2\sqrt{2}$$
$$=2\sqrt{7}+2\sqrt{2} \ \#$$

(xi) $\sqrt{2}\left(\sqrt{8}-\sqrt{9}\right)=\sqrt{16}-\sqrt{18}=4-\sqrt{2}\times\sqrt{9}=4-3\sqrt{2}$ #

(xii) $2\sqrt{3}\times\sqrt{27}\times\sqrt{8}=2\sqrt{3}\times\sqrt{9}\times\sqrt{3}\times\sqrt{4}\times\sqrt{2}$

$$=2\sqrt{3}\times3\sqrt{3}\times2\sqrt{2}$$
$$=(2\times3\times2)\left(\sqrt{3}\times\sqrt{3}\right)\left(\sqrt{2}\right)$$
$$=12\times3\times\sqrt{2}$$
$$=36\sqrt{2} \ \#$$

(xiii) $\left(\sqrt{3}+2\right)\left(\sqrt{3}+1\right)=3+\sqrt{3}+2\sqrt{3}+2=5+3\sqrt{3}$ #

(xiv) $\dfrac{\sqrt{10}}{\sqrt{5}}=\sqrt{\dfrac{10}{5}}=\sqrt{2}$ #

(D) Rationalising the Denominator

Multiplying a surd by its conjugate always gives a whole number:

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\sqrt{a^2}-\sqrt{ab}+\sqrt{ab}-\sqrt{b^2}$$
$$=a-b$$

Note $\left(\sqrt{a}+\sqrt{b}\right)$ and $\left(\sqrt{a}-\sqrt{b}\right)$ are conjugates.

Useful hints to follow in rationalising the denominator are:

If the surd is in the form of $\dfrac{1}{\sqrt{a}}$, multiply by $\dfrac{\sqrt{a}}{\sqrt{a}}$

If the surd is in the form of $\dfrac{1}{\sqrt{a}-\sqrt{b}}$ multiply by $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

If surd in the form of $\dfrac{1}{a+\sqrt{b}}$ multiply by $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$

Example 1: Multiply each surd by its conjugate:

(i) $\left(1+\sqrt{2}\right)$ (ii) $\left(2-\sqrt{5}\right)$

(iii) $\left(\sqrt{3}-\sqrt{2}\right)$ (iv) $\left(2\sqrt{3}-\sqrt{2}\right)$

Solution 1:

(i) $\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)=1-\sqrt{2}+\sqrt{2}-2=-1$ #

(ii) $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)=4+2\sqrt{5}-2\sqrt{5}-5=-1$ #

(iii) $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)=3+\sqrt{6}-\sqrt{6}-2=1$ #

(iv) $\left(2\sqrt{3}-\sqrt{2}\right)\left(2\sqrt{3}+\sqrt{2}\right)=12+2\sqrt{6}-2\sqrt{6}-2=10$ #

Example 2: Rationalise the denominator of the following:

(i) $\dfrac{1}{\sqrt{2}}$ (ii) $\dfrac{1}{2\sqrt{5}}$ (iii) $\dfrac{3}{\sqrt{2}}$

Solution 2:

(i) $\dfrac{1}{\sqrt{2}}=\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$ #

(ii) $\dfrac{1}{2\sqrt{5}}=\dfrac{1}{2\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{5}}{10}$ #

(iii) $\dfrac{3}{\sqrt{2}}=\dfrac{3}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{3\sqrt{2}}{2}$ #

Example 3: Rationalise the denominator:

(i) $\dfrac{1}{\sqrt{3}-1}$ (ii) $\dfrac{6}{1-\sqrt{3}}$

(iii) $\dfrac{8}{7-3\sqrt{5}}$ (iv) $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$

Solution 3:

(i) $\dfrac{1}{\sqrt{3}-1}=\dfrac{1}{\sqrt{3}-1}\times\dfrac{\sqrt{3}+1}{\sqrt{3}+1}=\dfrac{\sqrt{3}+1}{3-1}=\dfrac{\sqrt{3}+1}{2}$ #

(ii) $\dfrac{6}{1-\sqrt{3}}=\dfrac{6}{1-\sqrt{3}}\times\dfrac{1+\sqrt{3}}{1+\sqrt{3}}=\dfrac{6\left(1+\sqrt{3}\right)}{1-3}=\dfrac{6\left(1+\sqrt{3}\right)}{-2}=-3\left(1+\sqrt{3}\right)$ #

(iii) $\dfrac{8}{7-3\sqrt{5}}=\dfrac{8}{7-3\sqrt{5}}\times\dfrac{7+3\sqrt{5}}{7+3\sqrt{5}}=\dfrac{56+24\sqrt{5}}{49-45}=\dfrac{56+24\sqrt{5}}{4}=14+6\sqrt{5}$ #

(iv) $\frac{\sqrt{5}+1}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1}=\frac{5+\sqrt{5}+\sqrt{5}+1}{5-1}=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$ #

Example 4: Show that the following fraction is rational: $\frac{4}{2+\sqrt{5}}-\frac{1}{9-4\sqrt{5}}$

Solution 4:

$$\begin{aligned}\frac{4}{2+\sqrt{5}}-\frac{1}{9-4\sqrt{5}}&=\frac{4}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}-\frac{1}{9-4\sqrt{5}}\times\frac{9+4\sqrt{5}}{9+4\sqrt{5}}\\&=\frac{8-4\sqrt{5}}{4-5}-\frac{9+4\sqrt{5}}{81-80}\\&=\frac{8-4\sqrt{5}}{-1}-\frac{9+4\sqrt{5}}{1}\\&=-(8-4\sqrt{5})-(9+4\sqrt{5})\\&=-8+4\sqrt{5}-9-4\sqrt{5}\\&=-17\end{aligned}$$

$\therefore \frac{4}{2+\sqrt{5}}-\frac{1}{9-4\sqrt{5}}$ is rational. #

(E) Quadratic Expansions/Factorisation

1. $(x+a)^2=x^2+2ax+a^2$
2. $(x-a)^2=x^2-2ax+a^2$
3. $(x+a)(x-a)=x^2-a^2$
4. $(x+a)(x+b)=x^2+(a+b)x+ab$
5. $ax+ay=a(x+y)$
6. $x^2-a^2=(x-a)(x+a)$
7. $x^2+(a+b)x+ab=(x+a)(x+b)$
8. $ax+ay+bx+by=a(x+y)+b(x+y)$
   $=(a+b)(x+y)$

Example 1: Expand and simplify

(i) $(x+5)^2$ (ii) $(x+2)(x-3)$

(iii) $(2x-1)^2$ (iv) $(x-2y)(x-y)$

Solution 1:

(i) $(x+5)^2=x^2+10x+25$ #

(ii) $(x+2)(x-3)=x^2-3x+2x-6=x^2-x-6$ #

(iii) $(2x-1)^2=(2x)^2-2(2x)(1)+1^2=4x^2-4x+1$ #

(iv) $(x-2y)(x-y)=x^2-xy-2xy+2y^2=x^2-3xy+2y^2$ #

Example 2: Factorise the following:

(i) $2x^2-8x$ (ii) $5x^2-20$ (iii) $9x^2-1$

(iv) x^2-4x+4 (v) $2x+2y-ax-ay$

Solution 2:

(i) $2x^2-8x=2x(x-4)$ #

(ii) $5x^2-20=5(x^2-4)=5(x-2)(x+2)$ #

(iii) $9x^2-1=(3x-1)(3x+1)$ #

(iv) $x^2-4x+4=(x-2)(x-2)=(x-2)^2$ #

(v) $2x+2y-ax-ay=2(x+y)-a(x+y)=(2-a)(x+y)$ #

(F) Cubic Factorisation

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

Example 1: Factorise the following:

(i) x^3-27 (ii) $3x^3+81$

Solution 1:

(i) $x^3-27=(x-3)(x^2+3x+9)$ #

(ii) $3x^3+81=3(x^3+27)=3(x+3)(x^2-3x+9)$ #

(G) Simplifying Algebraic Expressions

(i) Order of Operations

In simplifying an algebraic expression the following order must be followed:

1) Powers, square roots and brackets.
2) Multiplication and division from left to right.
3) Addition and subtraction from left to right.

(ii) Basic Principles

Follow the 5 basic rules of algebra when simplifying expressions:

1) Terms with like signs multiplied together give a positive,
   e.g. $3\times 2x = 6x$, $\quad -3\times -2x = 6x$.
2) Terms with different signs multiplied together give a negative,
   e.g. $-3\times 2x = -6x$, $\quad 3\times -2x = -6x$
3) When adding or subtracting, collect like terms, i.e. those which have exactly the same pronumeral part,
   e.g. $2xy - 5y + 3xy - 4y = 5xy - 9y$
4) When multiplying like terms, add the powers,
   e.g. $3x\times 2x = 6x^2$, $\quad 4y^2\times y^3 = 4y^5$
5) When dividing like terms, subtract the powers,
   e.g. $\dfrac{4x^2}{x} = 4x$, $\quad \dfrac{5x^3y^2}{xy} = 5x^2y$

(iii) Removing Brackets

To remove brackets we expand and then collect like terms.

- Single Brackets: $a(b\pm c) = ab \pm ac$
  $-a(b\pm c) = -ab \pm ac$
- Double Brackets: $(a+b)(c+d) = a(c+d) + b(c+d)$
  $= ac + ad + bc + bd$

Example 1: Expand and simplify:

(i) $(2x+5)(x-3) - 2x^2$ (ii) $4(k-2)^2 - 2(k+1)^2$

(iii) $5 - 2(3x-5) - 3x(1-x)$ (iv) $4p(p-q) - (p-q)(p+q)$

(v) $\dfrac{8y^{20}}{2y^{10}}$ (vi) $4x(x-1) - \sqrt{4x^2}$

Solution 1:

(i) $(2x+5)(x-3) - 2x^2 = 2x^2 - 6x + 5x - 15 - 2x^2 = -x - 15$ #

(ii) $4(k-2)^2 - 2(k+1)^2 = 4(k^2 - 4k + 4) - 2(k^2 + 2k + 1)$
$= 4k^2 - 16k + 16 - 2k^2 - 4k - 2$
$= 2k^2 - 20k + 14$ #

(iii) $5 - 2(3x-5) - 3x(1-x) = 5 - 6x + 10 - 3x + 3x^2 = 15 - 9x + 3x^2$ #

(iv) $4p(p-q) - (p-q)(p+q) = 4p^2 - 4pq - (p^2 - q^2)$
$= 4p^2 - 4pq - p^2 + q^2$
$= 3p^2 - 4pq + q^2$ #

(v) $\dfrac{8y^{20}}{2y^{10}} = 4y^{10}$ #

(vi) $4x(x-1) - \sqrt{4x^2} = 4x^2 - 4x - 2x = 4x^2 - 6x$ #

(H) Algebraic Fractions:

The principles governing algebraic fractions are the same as those for arithmetic fractions.

1) $\dfrac{ac+bc}{db+da} = \dfrac{c(a+b)}{d(a+b)} = \dfrac{c}{d}$ When **reducing** fractions, factorise numerator and denominator then cancel.

2) $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$ When **adding** or **subtracting** fractions, find a common denominator.

3) $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$ When **multiplying** fractions, multiply numerators and denominators, then simplify.

4) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$ When **dividing** fractions, multiply by the reciprocal of the denominator.

Example 1: Simplify the following fractions:

(i) $\dfrac{2x-4}{2}$ (ii) $\dfrac{5a-b}{2b-10a}$

(iii) $\dfrac{x^2-4x+4}{x-2}$ (iv) $\dfrac{7x-3y}{21x-9y}$

Solution 1:

(i) $\dfrac{2x-4}{2} = \dfrac{2(x-2)}{2} = x - 2$ #

(ii) $\dfrac{5a-b}{2b-10a} = \dfrac{5a-b}{2(b-5a)} = \dfrac{5a-b}{-2(5a-b)} = -\dfrac{1}{2}$ #

(iii) $\frac{x^2-4x+4}{x-2}=\frac{(x-2)^2}{x-2}=x-2$ #

(iv) $\frac{7x-3y}{21x-9y}=\frac{7x-3y}{3(7x-3y)}=\frac{1}{3}$ #

Example 2: Simplify the following:

(i) $\frac{3x}{2}-\frac{2x}{3}$ (ii) $\frac{x}{5}+\frac{1}{x}$

(iii) $\frac{1}{x}-\frac{1}{x+5}$ (iv) $\frac{x}{2y}+\frac{1}{x}-\frac{1}{y}$

Solution 2:

(i) $\frac{3x}{2}-\frac{2x}{3}=\frac{9x-4x}{6}=\frac{5x}{6}$ #

(ii) $\frac{x}{5}+\frac{1}{x}=\frac{x^2+5}{5x}$ #

(iii) $\frac{1}{x}-\frac{1}{x+5}=\frac{x+5-x}{x(x+5)}=\frac{5}{x(x+5)}$ #

(iv) $\frac{x}{2y}+\frac{1}{x}-\frac{1}{y}=\frac{x^2+2y-2x}{2xy}$ #

Example 3: Simplify as far as possible:

(i) $\frac{x}{x-2}\times(5x-10)$ (ii) $\frac{x}{(x^2-4)}\times\frac{x-2}{x+2}$

(iii) $\left(\frac{xy}{2}\right)^4\div\left(\frac{x^2z^3}{y}\right)^2$ (iv) $\frac{1}{2p^2-3pq+q^2}\div\frac{1}{3p^2-3q^2}$

Solution 3:

(i) $\frac{x}{x-2}\times(5x-10)=\frac{x}{x-2}\times5(x-2)=5x$ #

(ii) $\frac{x}{x^2-4}\times\frac{x-2}{x+2}=\frac{x}{(x-2)(x+2)}\times\frac{x-2}{x+2}=\frac{x}{(x+2)^2}$ #

(iii) $\left(\frac{xy}{z}\right)^4\div\left(\frac{x^2z^3}{y}\right)^2=\frac{x^4y^4}{z^4}\div\frac{x^4z^6}{y^2}=\frac{x^4y^4}{z^4}\times\frac{y^2}{x^4z^6}=\frac{y^6}{z^{10}}$ #

(iv) $$\frac{1}{2p^2-3pq+q^2}\div\frac{1}{3p^2-3q^2}=\frac{1}{2p^2-3pq+q^2}\times\frac{3p^2-3q^2}{1}$$
$$=\frac{1}{(2p-q)(p-q)}\times\frac{3(p-q)(p+q)}{1}$$
$$=\frac{3(p+q)}{2p-q}\ \#$$

(I) Substitution

The exact value of an algebraic expression can be evaluated by replacing pronumerals with numbers.

Example 1: Given that $v=\frac{1}{3}\pi r^2h$, find the exact value of h if $v=20,\ r=3$.

Solution 1:

Substituting $v=20,\ r=3$ gives: $20=\frac{1}{3}\pi(3)^2h$

$$20=\frac{1}{3}\times\pi\times9\times h$$
$$h=\frac{20}{3\pi}\ \#$$

Example 2: Given that $2as=v^2-u^2$, find a if $s=16,\ v=31,\ u=17$.

Solution 2:

Substituting $s=16,\ v=31,\ u=17$ gives:

$$2a(16)=(31)^2-(17)^2$$
$$a=\frac{(31)^2-(17)^2}{32}=21\ \#$$

(J) Solving Equations

(i) Linear Equations

In solving linear equations we can:

- add/subtract the same number to both sides.
- multiply/divide each side by the same number.

Example 1: Solve the following:

(i) $5-3(x-1)=2$ (ii) $x-2(x+1)=5$

(iii) $\frac{x}{3}-\frac{2x+1}{5}=2$ (iv) $\frac{3x}{2}=\frac{5x+1}{3}$

Solution 1:

(i) $5-3(x-1)=2$
$5-3x+3=2$
$-3x=-6$
$x=2$ #

(ii) $x-2(x+1)=5$
$x-2x-2=5$
$-x=7$
$x=-7$ #

(iii) $\frac{x}{2}-\frac{2x+1}{5}=2$

$10\left(\frac{x}{2}\right)-10\left(\frac{2x+1}{5}\right)=20$ (multiplying throughout by 10)

$5x-2(2x+1)=20$
$5x-4x-2=20$
$x=22$ #

(iv) $\frac{3x}{2}=\frac{5x+1}{3}$
$3(3x)=2(5x+1)$
$9x=10x+2$
$-x=2$
$x=-2$ #

(ii) Linear Inequalities

In solving linear inequalities, you need to be aware of the following:

- Inequality sign is reversed if inequality is multiplied/divided by a negative number.
- Inequality sign unchanged if enequality multiplied/divided by a positive number.
- Inequality sign unchanged if we add or subtract the same positive or negative number to both sides.

Example 1: Solve the following:

(i) $2-3x>9$ (ii) $\frac{x-2}{} \le 3$

(iii) $1-\frac{2x}{3}>-1$ (iv) $3-2x\geq 5$

Solution 1:

(i) $2-3x>9$
$-3x>7$
$x<-\frac{7}{3}$ #

(ii) $\frac{x-2}{3}\leq 3$
$x-2\leq 9$
$x\leq 11$ #

(iii) $1-\frac{2x}{3}>-1$
$-\frac{2x}{3}>-2$
$-2x>-6$
$x<3$ #

(iv) $3-2x\geq 5$
$-2x\geq 2$
$x\leq -1$ #

(iii) Equations with Absolute Values

- If $|x|=a$, then $x=\pm a$.
- If $|x+a|=b$, then $x+a=\pm b$ i.e. $x=-a\pm b$.

Example 1: Solve:

(i) $|x|=9$ (ii) $|x-2|=5$

(iii) $|6x+4|=16$ (iv) $\left|1-\frac{3x}{5}\right|=8$

Solution 1:

(i) $|x|=9$
$x=9$ or $x=-9$ #

(ii) $|x-2|=5$
$x-2=5$ or $x-2=-5$
$x=7$ $x=-3$ #

(iii) $|6x+4|=16$
$6x+4=16$ or $6x+4=-16$
$6x=12$ $6x=-20$
$x=2$ $x=\frac{-20}{6}=\frac{-10}{3}$ #

(iv) $\left|1-\frac{3x}{5}\right|=8$

$1-\frac{3x}{5}=8$ or $1-\frac{3x}{5}=-8$

$-\frac{3x}{5}=7$ $\quad -\frac{3x}{5}=-9$

$x=-\frac{35}{3}$ $\quad x=15$ #

(iv) *Inequalities with Absolute Values*

There are two types of absolute value inequalities.

Type 1: ***Greater Than*** **Inequalities**

i.e. if $|x+a|\geq b$,

then $x+a\geq b$ or $x+a\leq -b$.

Type 2: ***Less Than*** **Inequalities**

i.e. if $|x+a|\leq b$,

then $-b\leq x+a\leq b$, i.e. $-b-a\leq x\leq b-a$.

Example 1: Solve and sketch the solution to the following inequalities

(i) $|x+1|\leq 4$ $\quad$ (ii) $|2-3x|>1$

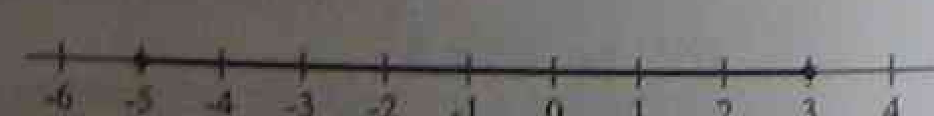
(iii) $|1-2x|\geq 9$ $\quad$ (iv) $\left|\frac{x}{4}-3\right|<2$

Solution 1:

(i) $|x+1|\leq 4$

i.e. $-4\leq x+1\leq 4$

i.e. $-5\leq x\leq 3$

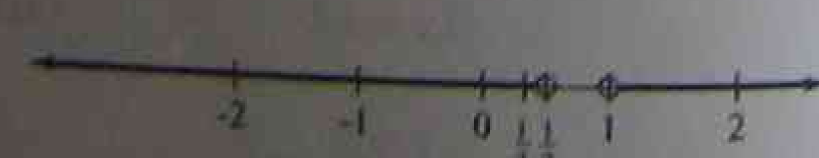


(ii) $|2-3x|>1$

i.e. $2-3x>1$ or $2-3x<-1$

$-3x>-1$ $\quad -3x<-3$

$x<\frac{1}{3}$ $\quad x>1$

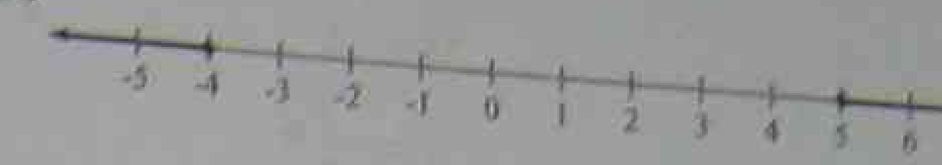


(iii) $|1-2x|\geq 9$

i.e. $1-2x\geq 9$ or $1-2x\leq -9$

$-2x\geq 8$ $\quad -2x\leq -10$

$x\leq -4$ $\quad x\geq 5$



(iv) $\left|\frac{x}{4}-3\right|<2$

i.e. $-2<\frac{x}{4}-3<2$

$-1<\frac{x}{4}<5$

$-4<x<20$

(v) *Quadratic Equations*

Rule: If $a\times b=0$, then either $a=0$ or $b=0$ or both.

In solving quadratic equations:

- Check to see if equation can be factorised; if yes, then it can be solved using the above rule;
- If not, then the solution to the quadratic equation $ax^2+bx+c=0$ is given by:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \text{(Quadratic Formula)}$$

Example 1: Solve:

(i) $x^2-5x-6=0$ $\quad$ (ii) $(x+3)^2=9$

(iii) $5x^2-2x-3=0$ $\quad$ (iv) $5x^2-6x-1=0$

(v) $2x^2-4x+1=0$ $\quad$ (vi) $2x+3-\frac{8}{x}=0$

Solution 1:

(i) $x^2-5x-6=0$

$(x-6)(x+1)=0$

$\therefore\ x-6=0$ or $x+1=0$

$x=6$ $\quad x=-1$ #

(ii) $(x+3)^2=9$

$x+3=\pm 3$

$\therefore\ x+3=3$ or $x+3=-3$

$x=0$ $\quad x=-6$ #

(iii) $5x^2-2x-3=0$

$(5x+3)(x-1)=0$

$\therefore\ 5x+3=0 \quad \text{or} \quad x-1=0$

$5x=-3 \qquad x=1$

$x=-\frac{3}{5}$ #

(iv) $5x^2-6x-1=0$

As quadratic can't be factorised, then we solve using the quadratic formula.

i.e. $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-6)\pm\sqrt{(-6)^2-4\times5\times(-1)}}{2\times5}$$

$$=\frac{6\pm\sqrt{56}}{10}$$

$$=\frac{6\pm2\sqrt{14}}{10}$$

$$=\frac{3\pm\sqrt{14}}{5}$$

$$\therefore\ x=\frac{3+\sqrt{14}}{5} \quad \text{or} \quad x=\frac{3-\sqrt{14}}{5}\ \#$$

(v) $2x^2-4x+1=0$

$(2x-1)(2x-1)=0$

i.e. $(2x-1)^2=0$

$\therefore\ 2x-1=0$

$2x=1$

$x=\frac{1}{2}$ #

(vi) $2x+3-\frac{8}{x}=0$

$2x^2+3x-8=0$ (multiplying through by x)

As quadratic can't be factorised, then we solve using the quadratic formula.

i.e. $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-3\pm\sqrt{(3)^2-4(2)(-8)}}{2\times2}$$

$$=\frac{-3\pm\sqrt{9+64}}{4}$$

$$=\frac{-3\pm\sqrt{73}}{4}$$

$$\therefore\ x=\frac{-3+\sqrt{73}}{4} \quad \text{or} \quad x=\frac{-3-\sqrt{73}}{4}\ \#$$

(K) Simultaneous Equations

There are two methods:

1) **Elimination:** make the coefficients of one of the unknowns equal then add or subtract the two equations to eliminate one of the unknowns.

2) **Substitution:** make one of the unknowns the subject of the equation and substitute into the other equation.

With either method, need to obtain the value for each unknown.
Check your result by direct substitution in the original equation.

Example 1: Solve the following simultaneous equations:

(i) $y=2x-5$ and $y=12x+15$

(ii) $3x-2y=-3$ and $2x+3y=24$

Solution 1:

(i) $y=2x-5$ .... (1)

$y=12x+15$ ....(2)

$0=-10x-20$ ......(1)−(2)

$10x=-20$

$x=-2$

Substituting into (1), gives y:

$y=2\times-2-5=-9$

$\therefore\ x=-2\ ,\ y=-9$ #

(ii)
$$3x-2y=-3 \quad \text{............(1)}$$
$$2x+3y=24 \quad \text{............(2)}$$
$$2\times(1): 6x-4y=-6 \quad \text{............(3)}$$
$$3\times(2): 6x+9y=72 \quad \text{............(4)}$$
$$0-13y=-78 \quad \text{.... (3)}-\text{(4)}$$
$$y=6$$

Substituting into (1), gives x:

$$3x-2\times 6=-3$$
$$3x=9$$
$$x=3$$
$$\therefore\ x=3,\ \ y=6 \ \#$$

(L) Percentage Discount and Increase Problems

This is an application of percentages commonly asked in exams.

Example 1: A tradesman earning \$21.50 an hour receives a 15% pay rise. What is his new gross income for a 40 hour week?

Solution 1:

New hourly rate $= 21.50\times 1.15 = \$24.73$

$\therefore$ gross weekly income $= 40\times 24.73 = \$989.20$ #

Example 2: A painter pays \$9.65 for a litre of paint after receiving a 12.5 % discount. What was the original cost of the paint?

Solution 2:

87.5% is \$9.65

100% is $\$9.65\times\dfrac{100}{87.5} = \$11.03$

$\therefore$ original cost of the paint = \$11.03 #

REVIEW EXERCISES

(A) Decimals, Significant Figures and Scientific Notation

1. Evaluate the following correct to 3 decimal places:

(i) $\dfrac{1+\sqrt{88}}{\sqrt{7}-1}$ (ii) $\dfrac{(2.05)^2}{32^{1/4}}$ (iii) $\dfrac{21.6+\sqrt{20}}{(5.5+4)}$

2. Evaluate the following correct to 3 significant figures:

(i) $\dfrac{6}{1-\sqrt{3}}$ (ii) $\dfrac{\left(10+\sqrt{5}\right)^5}{2{,}000}$ (iii) $\dfrac{11.727\times(0.75)^2}{(10+15)^2}$

3. Express the following in scientific notation to 3 significant figures:

(i) 0.0005257 (ii) 10.705 (iii) $25\times 50\times 75\times 100$

(B) Changing Recurring Decimals to Fractions

4. Change the following recurring decimals into fractions:

(i) $0.3\dot{5}\dot{9}$ (ii) $2.6\dot{7}$ (iii) $0.\dot{1}\dot{5}\dot{5}$

(C) Surds

5. Simplify the following surds as far as possible:

(i) $\sqrt{126}$ (ii) $\sqrt{3}\times 2\sqrt{6}$ (iii) $\sqrt{48}+\sqrt{108}$

(iv) $\left(2\sqrt{7}\right)^2$ (v) $\sqrt{125}+\sqrt{75}-\sqrt{20}$ (vi) $\left(10+\sqrt{5}\right)^2$

(vii) $\left(5-2\sqrt{3}\right)\left(2-\sqrt{3}\right)$ (viii) $\dfrac{\sqrt{32}-\sqrt{72}}{\sqrt{6}}$ (ix) $\dfrac{2\sqrt{2}\times 5\sqrt{6}}{8\sqrt{3}}$

(x) $\dfrac{\sqrt{27}-3\sqrt{3}+2\sqrt{18}}{\sqrt{72}}$

(D) Rationalising the Denominator

6. Rationalise the denominators of the following:

(i) $\dfrac{2\sqrt{3}}{\sqrt{5}}$ (ii) $\dfrac{14}{\sqrt{7}}$ (iii) $\dfrac{16\sqrt{2}}{\sqrt{8}}$

(iv) $\frac{1}{\sqrt{5}-2}$ (v) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ (vi) $\frac{\sqrt{32}+\sqrt{2}}{1+\sqrt{2}}$

7. Show that $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-1}-\frac{1+\sqrt{5}}{\sqrt{5}-\sqrt{3}}$ is rational.

(E) Quadratic Expansions/Factorisation

8. Expand and simplify:

(i) $(m+3)^2$ (ii) $(3x+2)^2$

(iii) $2x(x-5)-x(2-x)$ (iv) $(4m-3)(2m+3n)$

9. Factorise the following:

(i) $3x^2-27$ (ii) $4x^2-1$ (iii) $6x^2+8x-14$

(iv) $2a^2-3a+1$ (v) $4a^2+3a-4b^2+3b$

(F) Cubic Factorisation

10. Factorise the following:

(i) x^3+125 (ii) $2a^4-2a$ (iii) $8p^3+64$

(G) Simplifying Algebraic Expressions

11. Simplify as far as possible:

(i) $(x+5)^2-(x-3)^2$ (ii) $4p^2q^3\times 2pq$

(iii) $(xy)^2\times\sqrt{x^2y^2}$ (iv) $(2x-1)(2x+1)-x(3x-3)$

(v) $\frac{(3xy)^2(2y)}{18x^2}$ (vi) $x^2(x-4y^2)+(2xy)^2$

(H) Algebraic Fractions

12. Simplify as far as possible:

(i) $\frac{x^2-11x+10}{x-1}$ (ii) $\frac{5x^2-xy}{25x^2-y^2}$

(iii) $\frac{4x^2-16}{2-x}$ (iv) $\frac{x-3y}{x^2-9y^2}$

13. Simplify:

(i) $\frac{x}{3}-\frac{3-5x}{2}$ (ii) $\frac{x-2}{5}-\frac{3x+2}{10}$

(iii) $\frac{x-5}{y}-\frac{2-x^2}{xy}$ (iv) $\frac{x}{x-2}+\frac{2}{2-x}+x$

14. Simplify:

(i) $(2xy)^2\times(4yz)^2$ (ii) $\frac{(x+3)(x^2-4)}{(x-3)}\times\frac{(x^2-9)}{(x-2)}$

(iii) $\frac{1}{4x^2-1}\div\frac{1}{6x+3}$ (iv) $\left(\frac{xy}{4z}\right)^3\div\left(\frac{x^2y}{z^2}\right)^2$

(I) Substitution

15. Let $f(x)=x^5-2x^3$, find the exact value of $f\left(2\sqrt{5}\right)$.

16. The kinetic energy, T, of a particular system is given by: $T=\frac{1}{2}(m_1+m_2)\dot{s}^2$.

Find m_1, if $T=30$, $m_2=2$ and $\dot{s}=5$

17. Find the exact value of $\frac{A^4C^2}{B^2}$ given: $A=\left(\frac{1}{2}\right)^2$, $B=\left(\frac{2}{3}\right)^2$ and $C=\left(\frac{4}{3}\right)^4$

(J) Solving Equations

18. Solve:

(i) $2-5(1-x)=0$ (ii) $\frac{x}{7}-\frac{2x}{21}=2$

(iii) $\frac{5x-1}{10x+6}=\frac{2x-4}{4x+12}$ (iv) $\frac{2x-3}{5}-\frac{3x+4}{10}=1$

19. Solve:

(i) $4-7x<12$ (ii) $\frac{2x}{5}>-1$

(iii) $-1-2x\leq\frac{3x}{4}+\frac{2}{3}$ (iv) $\frac{x-2}{4}<5-\frac{x}{3}$

20. Solve:

(i) $|x| = 3$ (ii) $|x-1| = 5$

(iii) $|5-2x| = 15$ (iv) $\left|1-\frac{3x}{2}\right| = 35$

21. Solve and sketch the solution to the following inequalities:

(i) $|x+3| \le 3$ (ii) $|3-5x| > 12$

(iii) $\left|1-\frac{x}{2}\right| \le 4$ (iv) $\left|\frac{x-6}{2}\right| \ge 8$

22. Solve:

(i) $(x-4)^2 = 25$ (ii) $2x^2 - 7x + 3 = 0$

(iii) $5x^2 - 2x - 1 = 0$ (iv) $x + \frac{2}{x} + 3 = 0$

(K) Simultaneous Equations

23. Solve the following simultaneous equations:

(i) $y = 4x - 5$ and $y = 7x - 41$

(ii) $4y - 31 + 3x = 0$ and $2y - x - 3 = 0$

(L) Percentage Discount and Increase Problems

24. After a 20% discount is allowed the cost of a lounge suite is $1250. Find the cost of the lounge when no discount is allowed.

25. The price of a new car has just been increased by 10%. If the current price is $35,000 what was the price of the car prior to the increase to the nearest one hundred dollars?

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $\frac{1+\sqrt{88}}{\sqrt{7}-1} = 6.30765502 = 6.308$ correct to 3 d.p. #

(ii) $\frac{(2.05)^2}{32^{1/4}} = 1.766933593 = 1.767$ correct to 3 d.p. #

(iii) $\frac{21.6+\sqrt{20}}{(5.5+4)} = 2.744435363 = 2.744$ correct to 3 d.p. #

2. (i) $\frac{6}{1-\sqrt{3}} = -8.196152423 = -8.20$ correct to 3 s.f. #

(ii) $\frac{\left(10+\sqrt{5}\right)^5}{2{,}000} = 137.1448202 = 137$ correct to 3 s.f. #

(iii) $\frac{11.727\times(0.75)^2}{(10+15)^2} = 0.0105543 = 0.0106$ correct to 3 s.f. #

3. (i) $0.0005257 = 5.26\times10^{-4}$ correct to 3 s.f. #

(ii) $10.705 = 1.07\times10^1$ correct to 3 s.f. #

(iii) $25\times50\times75\times100 = 9{,}375{,}000 = 9.38\times10^6$ correct to 3 s.f. #

4. (i) Let $x = 0.3595959$ ......$(n=2)$

$100x = 35.9595959$ ......$(\times10^2)$

$\therefore\ 99x = 35.6$

i.e. $x = \frac{356}{990} = \frac{178}{495}$ #

(ii) Let $x = 2.6777$ ......$(n=1)$

$10x = 26.7777$ ......$(\times10^1)$

$\therefore\ 9x = 24.1$

i.e. $x = \frac{241}{90} = 2\frac{61}{90}$ #

(iii) Let $x = 0.155155155$ ...... $(n=3)$

$1000x = 155.155155155$ ...... $(\times10^3)$

$\therefore 999x = 155$

i.e. $x = \frac{155}{999}$ #

5. (i) $\sqrt{126} = \sqrt{9}\times\sqrt{14} = 3\sqrt{14}$ #

(ii) $\sqrt{3}\times2\sqrt{6} = 2\sqrt{18} = 2\sqrt{2}\times\sqrt{9} = 6\sqrt{2}$ #

(iii) $\sqrt{48}+\sqrt{108}=4\sqrt{3}+6\sqrt{3}=10\sqrt{3}$ # (iv) $\left(2\sqrt{7}\right)^2=4\times 7=28$ #

(v) $\sqrt{125}+\sqrt{75}-\sqrt{20}=5\sqrt{5}+5\sqrt{3}-2\sqrt{5}=3\sqrt{5}+5\sqrt{3}$ #

(vi) $\left(10+\sqrt{5}\right)^2=100+20\sqrt{5}+5=105+20\sqrt{5}$ #

(vii) $\left(5-2\sqrt{3}\right)\left(2-\sqrt{3}\right)=10-5\sqrt{3}-4\sqrt{3}+6=16-9\sqrt{3}$ #

(viii) $\dfrac{\sqrt{32}-\sqrt{72}}{\sqrt{6}}=\dfrac{\sqrt{16}\times\sqrt{2}-\sqrt{36}\times\sqrt{2}}{\sqrt{6}}=\dfrac{4\sqrt{2}-6\sqrt{2}}{\sqrt{2}\times\sqrt{3}}=\dfrac{-2}{\sqrt{3}}$ #

(ix) $\dfrac{2\sqrt{2}\times 5\sqrt{6}}{8\sqrt{3}}=\dfrac{10\sqrt{12}}{8\sqrt{3}}=\dfrac{10}{8}\times\sqrt{\dfrac{12}{3}}=\dfrac{5}{4}\times\sqrt{4}=\dfrac{5}{2}$ #

(x) $\dfrac{\sqrt{27}-3\sqrt{3}+2\sqrt{18}}{\sqrt{72}}=\dfrac{\sqrt{9}\times\sqrt{3}-3\sqrt{3}+2\sqrt{9}\times\sqrt{2}}{\sqrt{36}\times\sqrt{2}}=\dfrac{3\sqrt{3}-3\sqrt{3}+6\sqrt{2}}{6\sqrt{2}}=1$ #

6. (i) $\dfrac{2\sqrt{3}}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{2\sqrt{15}}{5}$ # (ii) $\dfrac{14}{\sqrt{7}}\times\dfrac{\sqrt{7}}{\sqrt{7}}=\dfrac{14\sqrt{7}}{7}=2\sqrt{7}$ #

(iii) $\dfrac{16\sqrt{2}}{\sqrt{8}}=\dfrac{16\sqrt{2}}{2\sqrt{2}}=8$ # (iv) $\dfrac{1}{\sqrt{5}-2}\times\dfrac{\sqrt{5}+2}{\sqrt{5}+2}=\dfrac{\sqrt{5}+2}{5-4}=\sqrt{5}+2$ #

(v) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\times\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\dfrac{3-2\sqrt{6}+2}{3-2}=5-2\sqrt{6}$ #

(vi)
$$
\begin{aligned}
\frac{\sqrt{32}+\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{32}+\sqrt{2}}{1+\sqrt{2}}\times\frac{1-\sqrt{2}}{1-\sqrt{2}}&=\frac{\sqrt{32}-\sqrt{64}+\sqrt{2}-2}{1-2}\\
&=\frac{4\sqrt{2}-8+\sqrt{2}-2}{-1}\\
&=-(5\sqrt{2}-10)\\
&=10-5\sqrt{2}\ \#
\end{aligned}
$$

7.
$$
\begin{aligned}
\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-1}-\frac{1+\sqrt{5}}{\sqrt{5}-\sqrt{3}}&=\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}-\frac{1+\sqrt{5}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}\\
&=\frac{3+\sqrt{3}+\sqrt{15}+\sqrt{5}}{2}-\frac{\sqrt{5}+\sqrt{3}+5+\sqrt{15}}{2}\\
&=\frac{3}{2}-\frac{5}{2}=-1\ \#
\end{aligned}
$$

8. (i)

(iii) $2x(x-5)-x(2-x)=2x^2-10x-2x+x^2=3x^2-12x$ #

(iv) $(4m-3)(2m+3n)=8m^2+12mn-6m-9n$ #

9. (i) $3x^2-27=3(x^2-9)=3(x-3)(x+3)$ #

(ii) $4x^2-1=(2x-1)(2x+1)$ #

(iii) $6x^2+8x-14=2(3x^2+4x-7)=2(3x+7)(x-1)$ #

(iv) $2a^2-3a+1=(2a-1)(a-1)$ #

(v)
$$
\begin{aligned}
4a^2+3a-4b^2+3b&=4(a^2-b^2)+3(a+b)\\
&=4(a-b)(a+b)+3(a+b)\\
&=(a+b)(4a-4b+3)\ \#
\end{aligned}
$$

10. (i) $x^3+125=(x+5)(x^2-5x+25)$ #

(ii)
$$
\begin{aligned}
2a^4-2a&=2a(a^3-1)\\
&=2a(a-1)(a^2+a+1)\ \#
\end{aligned}
$$

(iii) $8p^3+64=8(p^3+8)=8(p+2)(p^2-2p+4)$ #

11. (i) $(x+5)^2-(x-3)^2=(x^2+10x+25)-(x^2-6x+9)=16x+16$ #

(ii) $4p^2q^3\times 2pq=8p^3q^4$ #

(iii) $(xy)^2\times\sqrt{x^2y^2}=(x^2y^2)(xy)=x^3y^3$ #

(iv) $(2x-1)(2x+1)-x(3x-3)=4x^2-1-3x^2+3x=x^2+3x-1$ #

(v) $\dfrac{(3xy)^2(2y)}{18x^2}=\dfrac{(9x^2y^2)(2y)}{18x^2}=\dfrac{18x^2y^3}{18x^2}=y^3$ #

(vi) $x^2\left(x-4y^2\right)+\left(2xy\right)^2=x^3-4x^2y^2+4x^2y^2=x^3$ #

12. (i) $\dfrac{x^2-11x+10}{x-1}=\dfrac{(x-10)(x-1)}{(x-1)}=x-10$ #

(ii) $\dfrac{5x^2-xy}{25x^2-y^2}=\dfrac{x(5x-y)}{(5x+y)(5x-y)}=\dfrac{x}{5x+y}$ #

(iii) $\dfrac{4x^2-16}{2-x}=\dfrac{4(x^2-4)}{2-x}=\dfrac{4(x-2)(x+2)}{-(x-2)}=-4(x+2)$ #

(iv) $\dfrac{x-3y}{x^2-9y^2}=\dfrac{x-3y}{(x-3y)(x+3y)}=\dfrac{1}{x+3y}$ #

13. (i) $\dfrac{x}{3}-\dfrac{3-5x}{2}=\dfrac{2x-3(3-5x)}{6}=\dfrac{2x-9+15x}{6}=\dfrac{17x-9}{6}$ #

(ii) $\dfrac{x-2}{5}-\dfrac{3x+2}{10}=\dfrac{2(x-2)-(3x+2)}{10}=\dfrac{(2x-4)-3x-2}{10}=\dfrac{-x-6}{10}$ #

(iii) $\dfrac{x-5}{y}-\dfrac{2-x^2}{xy}=\dfrac{x(x-5)-(2-x^2)}{xy}=\dfrac{x^2-5x-2+x^2}{xy}=\dfrac{2x^2-5x-2}{xy}$ #

(iv) $\dfrac{x}{x-2}+\dfrac{2}{2-x}+x=\dfrac{x(2-x)+2(x-2)}{(x-2)(2-x)}+x\ =\dfrac{2x-x^2+2x-4}{(x-2)(2-x)}+x$

$$=\frac{4x-x^2-4}{(x-2)(2-x)}+x$$

$$=\frac{(x-2)(2-x)}{(x-2)(2-x)}+x=1+x \text{ \#}$$

14. (i) $(2xy)^2\times(4yz)^2=4x^2y^2\times16y^2z^2=64x^2y^4z^2$ #

(ii) $\dfrac{(x+3)(x^2-4)}{(x-3)}\times\dfrac{(x^2-9)}{(x-2)}=\dfrac{(x+3)(x-2)(x+2)}{(x-3)}\times\dfrac{(x-3)(x+3)}{(x-2)}$

$$=(x+3)^2(x+2) \text{ \#}$$

(iii) $\dfrac{1}{4x^2-1}\div\dfrac{1}{6x+3}=\dfrac{1}{4x^2-1}\times\dfrac{6x+3}{1}\ =\dfrac{1}{(2x-1)(2x+1)}\times\dfrac{3(2x+1)}{1}$

$$=\frac{3}{2x-1} \text{ \#}$$

(iv) $\left(\dfrac{xy}{4z}\right)^3\div\left(\dfrac{x^2y}{z^2}\right)^2=\dfrac{x^3y^3}{64z^3}\times\dfrac{z^4}{x^4y^2}=\dfrac{yz}{64x}$ #

15. $f\left(2\sqrt{5}\right)=\left(2\sqrt{5}\right)^5-2\left(2\sqrt{5}\right)^3=\left(2\sqrt{5}\right)^3\left[\left(2\sqrt{5}\right)^2-2\right]$

$$=40\sqrt{5}[20-2]=720\sqrt{5} \text{ \#}$$

16. $T=\dfrac{1}{2}(m_1+m_2)s^2,\ 30=\dfrac{1}{2}(m_1+2)5^2$

$$\frac{60}{25}=m_1+2 \quad \therefore\ m_1=\frac{2}{5} \text{ \#}$$

17. $\dfrac{A^4C^2}{B^2}=\dfrac{\left[\left(\frac{1}{2}\right)^2\right]^4\left[\left(\frac{4}{3}\right)^4\right]^2}{\left[\left(\frac{2}{3}\right)^2\right]^2}=\dfrac{\left(\frac{1}{2}\right)^8\left(\frac{4}{3}\right)^8}{\left(\frac{2}{3}\right)^4}$

$$=\frac{1}{2^8}\times\frac{4^8}{3^8}\times\frac{3^4}{2^4}$$

$$=\frac{1}{2^{12}}\times\frac{(2^2)^8}{3^4}$$

$$=\frac{1}{2^{12}}\times\frac{2^{16}}{3^4}=\frac{2^4}{3^4}=\left(\frac{2}{3}\right)^4=\frac{16}{81} \text{ \#}$$

18. (i)

$$2-5(1-x)=0$$
$$2-5+5x=0$$
$$5x=3$$
$$x=\frac{3}{5} \text{ \#}$$

(ii)

$$\frac{x}{7}-\frac{2x}{21}=2$$
$$21\left(\frac{x}{7}\right)-21\left(\frac{2x}{21}\right)=2\times21 \quad \text{(multiply by 21)}$$
$$3x-2x=42$$
$$x=42 \text{ \#}$$

(iii)

$$\frac{5x-1}{10x+6}=\frac{2x-4}{4x+12}$$
$$(5x-1)(4x+12)=(2x-4)(10x+6)$$
$$20x^2+60x-4x-12=20x^2+12x-40x-24$$
$$84x=-12$$
$$x=\frac{-12}{84}=-\frac{1}{7} \text{ \#}$$

(iv)

$$\frac{2x-3}{5}-\frac{3x+4}{10}=1$$
$$10\frac{(2x-3)}{5}-10\frac{(3x+4)}{10}=10 \quad \text{(multiplying throughout by 10)}$$
$$2(2x-3)-(3x+4)=10$$
$$4x-6-3x-4=10$$
$$x=20 \text{ \#}$$

19. (i) $4-7x<12$

$-7x<8$

$x>\frac{-8}{7}$ #

(ii) $\frac{2x}{5}>-1$

$2x>-5$

$x>-\frac{5}{2}$ #

(iii) $-1-2x\le\frac{3x}{4}+\frac{2}{3}$

$-2x-\frac{3x}{4}\le 1+\frac{2}{3}$

$\frac{-8x-3x}{4}\le\frac{5}{3}$

$-11x\le\frac{20}{3}$

$x\ge-\frac{20}{33}$ #

(iv) $\frac{x-2}{4}<5-\frac{x}{3}$

$\frac{x}{4}-\frac{2}{4}<5-\frac{x}{3}$

$\frac{x}{4}+\frac{x}{3}<5+\frac{1}{2}$

$\frac{3x+4x}{12}<\frac{11}{2}$

$7x<66$

$x<\frac{66}{7}$ #

20. (i) $|x|=3,\ x=\pm 3$ #

(ii) $|x-1|=5$

$x-1=5$ or $x-1=-5$

$x=6$ $\quad x=-4$ #

(iii) $|5-2x|=15$

$5-2x=15$ or $5-2x=-15$

$-2x=10$ $\quad -2x=-20$

$x=-5$ $\quad x=10$ #

(iv) $\left|1-\frac{3x}{2}\right|=35$

$1-\frac{3x}{2}=35$ or $1-\frac{3x}{2}=-35$

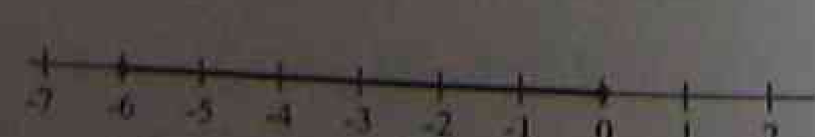
$-\frac{3x}{2}=34$ $\quad -\frac{3x}{2}=-36$

$x=-\frac{68}{3}$ $\quad x=24$ #

21. (i) $|x+3|\le 3$

$-3\le x+3\le 3$

$-6\le x\le 0$ #

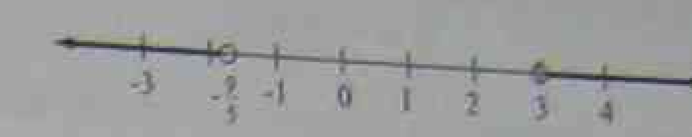


(ii) $|3-5x|>12$

$3-5x>12$ or $3-5x<-12$

$-5x>9$ $\quad -5x<-15$

$x<-\frac{9}{5}$ $\quad x>3$ #



(iii) $\left|1-\frac{x}{2}\right|\le 4$

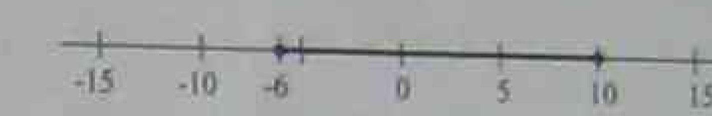
$-4\le 1-\frac{x}{2}\le 4$

$-5\le-\frac{x}{2}\le 3$

$-10\le -x\le 6$

$10\ge x\ge -6$

i.e $-6\le x\le 10$ #



(iv) $\left|\frac{x-6}{2}\right|\ge 8$

$\frac{x-6}{2}\ge 8$ or $\frac{x-6}{2}\le -8$

$x-6\ge 16$ $\quad x-6\le -16$

$x\ge 22$ $\quad x\le -10$

22. (i) $(x-4)^2=25$

$(x-4)=5$ or $(x-4)=-5$

$x=9$ $\quad x=-1$ #

(ii) $2x^2-7x+3=0$

$(2x-1)(x-3)=0$

$x=\frac{1}{2}$ or $x=3$ #

(iii) $5x^2-2x-1=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{(-2)^2-4\times5\times-1}}{10}=\frac{2\pm\sqrt{24}}{10}$$

$$=\frac{2\pm2\sqrt{6}}{10}=\frac{1\pm\sqrt{6}}{5}\ \#$$

(iv) $x+\frac{2}{x}+3=0$

$x^2+3x+2=0$

$(x+2)(x+1)=0$

$x+2=0$ or $x+1=0$

$x=-2$ $\quad x=-1$

23. (i) $y = 4x - 5$ ......(1)

$y = 7x - 41$ ......(2)

(1)−(2), gives:

$0 = -3x + 36$

$3x = 36$

i.e. $x = 12$

Substituting into (1), gives:

$y = 48 - 5 = 43$

$\therefore\ x = 12,\ y = 43$ #

(ii) $4y - 31 + 3x = 0$ ......(1)

$2y - 3 - x = 0$ ......(2)

$3\times(2)$: $6y - 9 - 3x = 0$ ......(3)

(1)+(3), gives:

$10y - 40 = 0$

$10y = 40$

i.e. $y = 4$

Substituting into (1), gives x:

$16 - 31 + 3x = 0$

i.e $3x = 15$

i.e $x = 5$

$\therefore\ x = 5,\ y = 4$ #

24. 80% is \$1,250

100% is $\$1{,}250 \times \dfrac{100}{80} = \$1{,}562.50$

∴ the original cost of the lounge suite is \$1,562.50 #

25. 110% is \$35,000

100% is $\$35{,}000 \times \dfrac{100}{110} = \$31{,}818.18$

∴ the original price of the car is \$31,800 to the nearest hundred dollars. #

TOPIC 2

PLANE GEOMETRY

This topic should be covered together with topic 10 as most schools tend to cover both topics at the same time.

Listed below is a comprehensive coverage of all the geometric rules and properties required in the HSC Maths course.

(A) Angle Properties

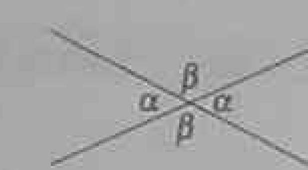
1. Angles in a straight line = 180°

$\alpha + \beta = 180°$

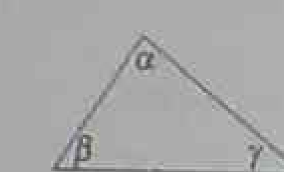
2. Angles which meet at a point = 360°

$\alpha + \beta + \gamma + \theta = 360°$

3. Vertically opposite angles are equal.

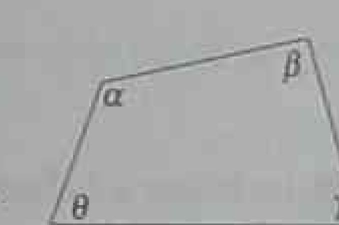


4. Angle sum of a triangle = 180°



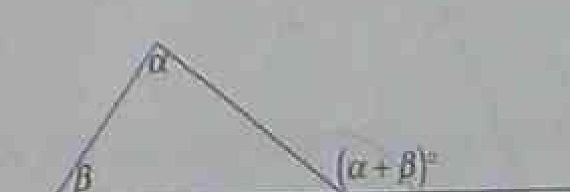
$\alpha + \beta + \gamma = 180°$

5. Angle sum of a quadrilateral = 360°



$\alpha + \beta + \gamma + \theta = 360°$

6. The exterior angle of a triangle is equal to the sum of the two opposite interior angles.



(B) Properties of Polygons

7. Angle sum of any polygon $= (2n - 4) \times 90°$ where n = number of sides.

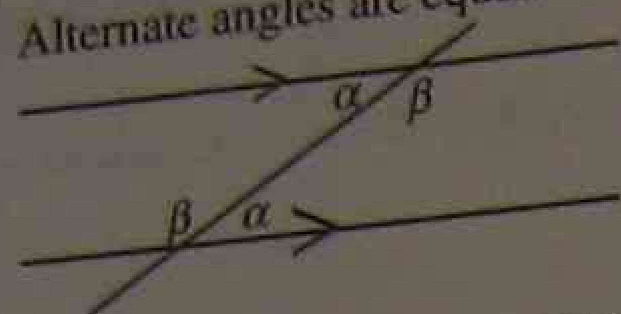
8. Sum of exterior angles of any polygon = 360° (2×6 −4) × 90°

$(a + b + c + d + e + f) = 360°$

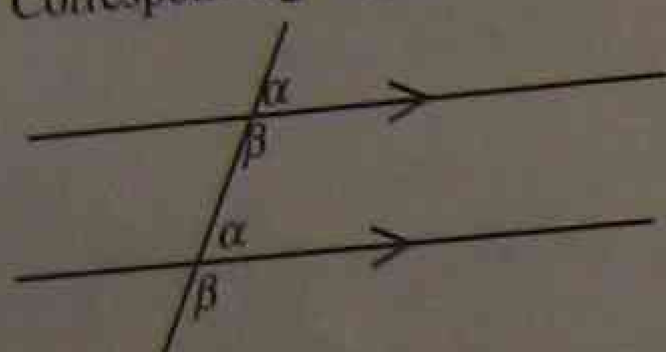
(C) Properties of Parallel Lines

9. Three properties of parallel lines:

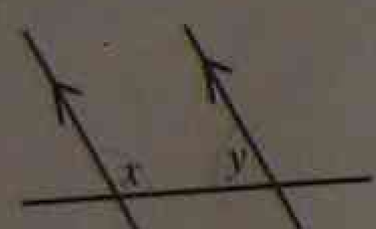
(i) Alternate angles are equal.



(ii) Corresponding angles are equal.



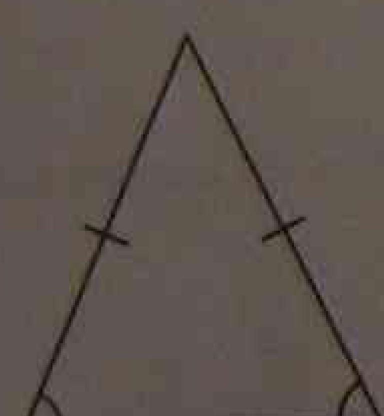
(iii) Corresponding angles are supplementary (i.e. sum to 180°)



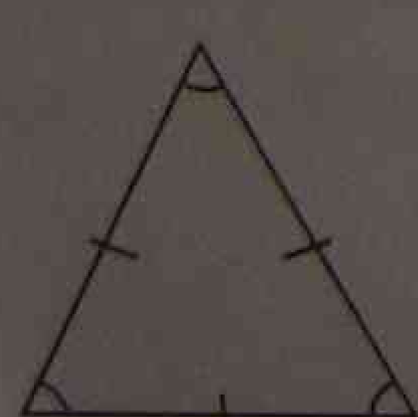
$x + y = 180°$

(D) Special Triangles

10. If two sides are equal then the triangle is ***Isosceles***. Base angles of isosceles triangles are equal.



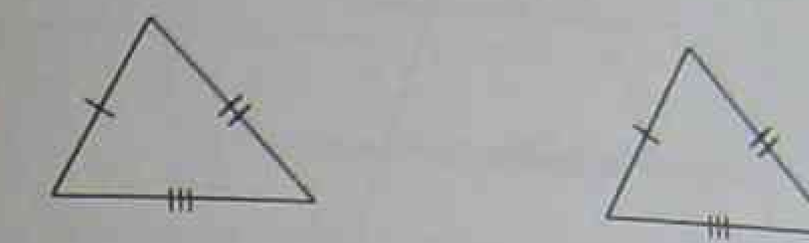
11. If three sides are equal then the triangle is ***Equilateral***. All angles of equilateral triangles are equal to $60°$.



(E) Congruent Triangles

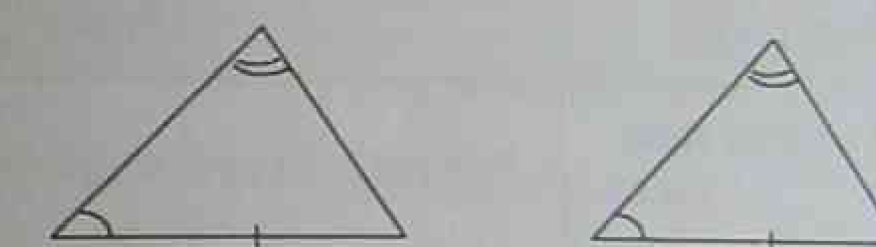
12. Two triangles are congruent (i.e. exactly the same shape and size) if they satisfy any of the following four tests:

(i) 3 corresponding sides are equal **SSS**.



(ii) 2 corresponding sides and the included angle is equal **SAS**.

(iii) 2 corresponding angles and 1 corresponding side is equal **AAS**.



(iv) Hypotenuse and 1 corresponding side equal of a right-angled triangle **RHS**.

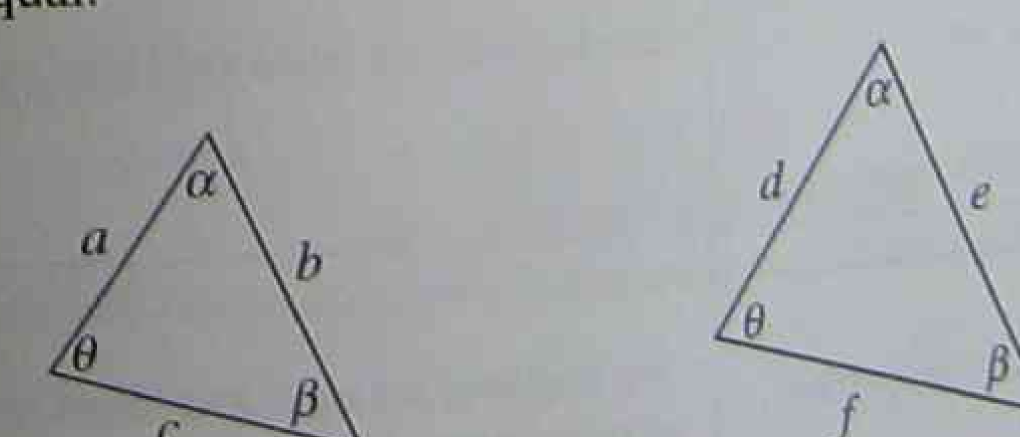
(F) Similar Triangles

13. Two triangles are similar (i.e. same shape but not size) if it satisfies any of the following three tests:

(i) All corresponding angles are equal.

(ii) All corresponding sides are in equal proportion.

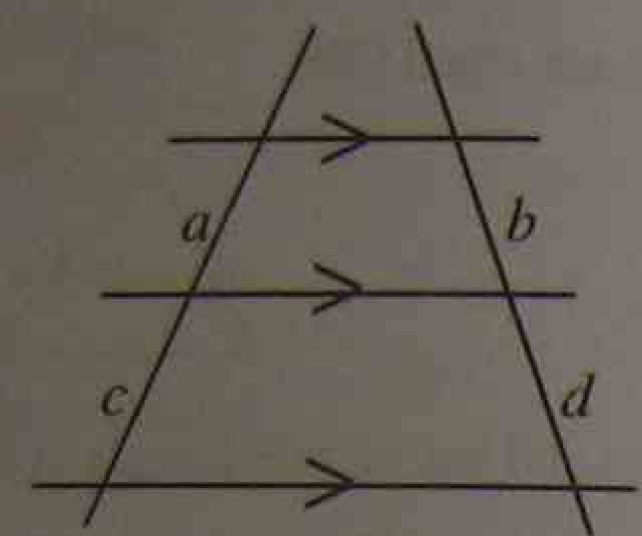
(iii) Two pairs of corresponding sides are in equal proportion and the included angle is equal.



$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

(G) Ratio of Intercepts Theorem

14. A family of parallel lines cutting a number of transversals will preserve the ratio of the sides on each transversal.



$$\frac{a}{c} = \frac{b}{d}$$

(H) Properties and Tests of Quadrilaterals

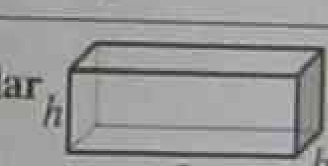
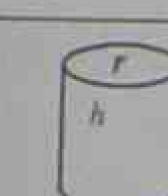
15.

Quadrilateral	Properties	Tests
Parallelogram	* opposite sides are equal * opposite angles are equal * diagonals bisect each other * a diagonal divides a parallelogram into 2 congruent triangles	* both pairs of opposite sides equal **or** * both pairs of opposite angles are equal **or** * both pairs of opposite sides are parallel **or** * one pair of opposite sides equal and parallel **or** * the diagonals bisect each other
Rhombus	* all properties of a parallelogram **plus** * diagonals bisect at right angles * diagonals bisect the angles through which they pass	* diagonals bisect each other at right angles **or** * all sides are equal
Rectangle	* all properties of a parallelogram **plus** * angles are 90° * diagonals are equal	* the diagonals are equal and bisect each other
Square	* all the above properties	* the diagonals ...

(I) Areas

Shape	Area	Formula
Rectangle	length × width	lw
Triangle	$\frac{1}{2}$ base × perpendicular height	$\frac{1}{2}bh$
Parallelogram	base × perpendicular height	bh
Rhombus	$\frac{1}{2}$ × product of diagonals	$\frac{1}{2}xy$
Circle	*pi* × radius squared	πr^2
Trapezium	$\frac{1}{2}$ × sum of parallel sides × perpendicular height	$\frac{1}{2}(a+b)h$

(J) Volumes and Surface Areas

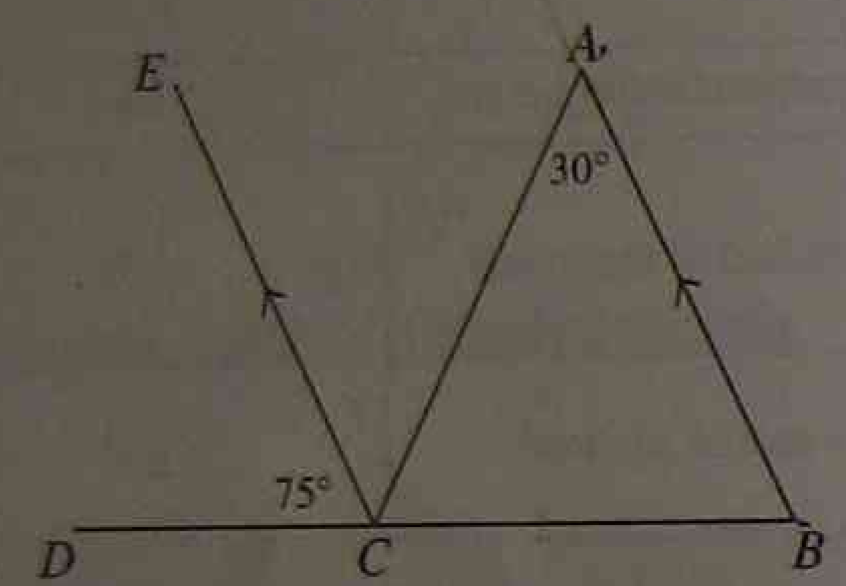
Shape	Surface Area	Volume
Rectangular Prism 	$2(ab+bh+ah)$	abh
Cylinder 	$2\pi rh$ (open) $2\pi rh + 2\pi rh$ (closed)	$\pi r^2 h$
Cone 	πrl (open) $\pi rl + \pi r^2$ (closed)	$\frac{1}{3}\pi r^2 h$
Any Pyramid where	$A =$ area of the base $h =$ the perpendicular height	$\frac{1}{3}Ah$
Any Prism where	$A =$ area of the base $h =$ the perpendicular height	Ah

(K) Worked Examples

Geometric exercises and proofs typically require the use of several rules. Students are expected to know which theorem(s) to apply in solving a particular problem. Problems may be numerical or general in nature.

(i) Simple Numerical Exercises

Example 1:



$AB \parallel EC$

$\angle CAB = 30°$, $\angle DCE = 75°$

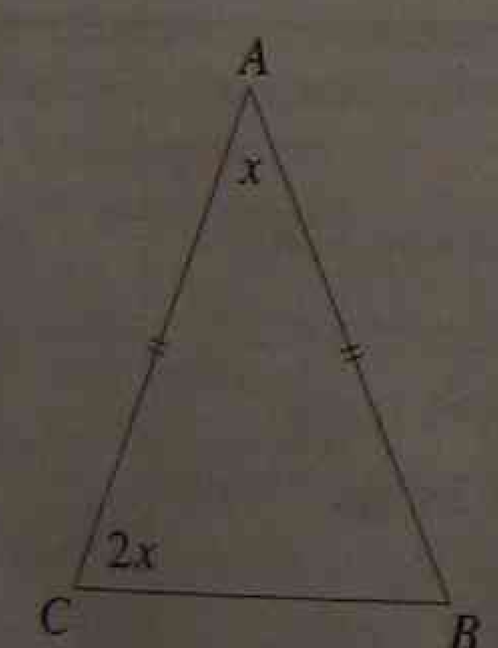
Find $\angle ACB$

Solution 1:

$\angle ECA = \angle CAB = 30°$ (alternate angles equal, $AB \parallel EC$)

$\angle ACB = 180° - 75° - 30° = 75°$ (angle in a straight line $= 180°$) #

Example 2:



$AB = AC$

$\angle CAB = x$ and $\angle ACB = 2x$

Find $\angle ABC$.

Solution 2:

$\angle ABC = 2x$ (base angles of isosceles Δ are equal)

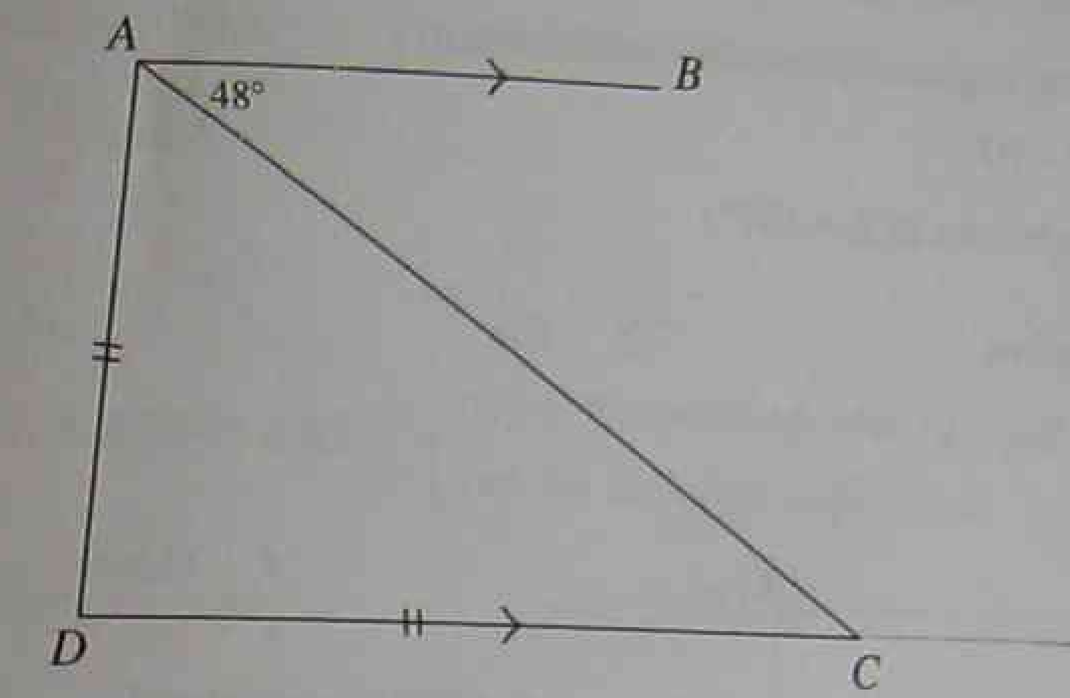
$x + 2x + 2x = 180°$ (angle sum of $\Delta = 180°$)

i.e. $5x = 180°$

$x = 36°$

$\therefore \angle ABC = 2 \times 36° = 72°$ #

Example 3:



$AB \parallel DC$

$AD = DC$

$\angle BAC = 48°$

Find $\angle ADC$

Solution 3:

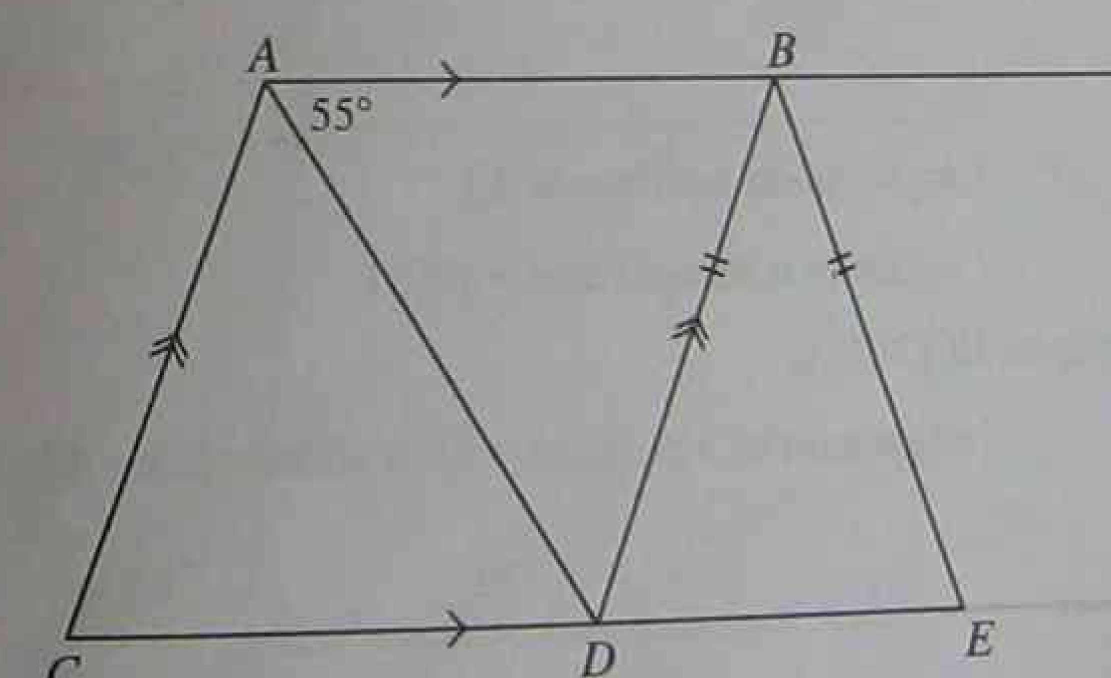
$\angle ACD = 48°$ (alternate angles equal, $AB \parallel DC$)

$\angle DAC = 48°$ (base angles equal of isosceles $\Delta\ ADC$)

$\angle ADC = 180° - 48° - 48°$ (co-interior angles supplementary, $AB \parallel DC$)

$= 84°$ #

Example 4:



$ABCD$ is a rhombus.

$BD = BE$ and $\angle BAD = 55°$

Find $\angle ACD$ and $\angle DBE$.

Solution 4:

$\angle CAD = 55°$ (diagonals of a rhombus bisect the angles through which they pass)

$\angle ACD = 180° - 55° - 55°$

$= 70°$ (co-interior angles supplementary, $AB \parallel CD$)

$\angle BDE = 70^\circ$ (corresponding angles equal, $AC \parallel BD$)

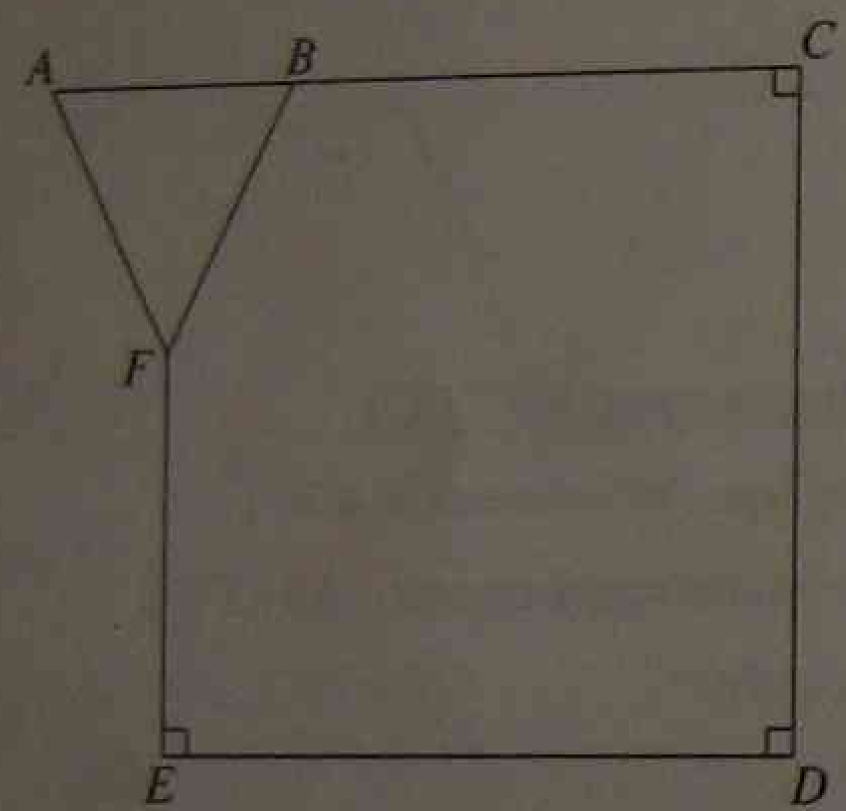
$\angle BED = 70^\circ$ (base angles of isosceles $\Delta\ BDE$ equal)

$\angle DBE = 180^\circ - 70^\circ - 70^\circ$

$= 40^\circ$ (angle sum of $\Delta = 180^\circ$)

(ii) Harder Numerical Exercises

Example 1:



$\Delta\ ABF$ is equilateral

$\angle BCD = \angle CDE = \angle DEF = 90^\circ$

Find $\angle AFE$.

Solution 1:

$\angle ABF = \angle AFB = \angle BAF = 60^\circ$ (angles in an equilateral Δ)

$\angle FBC = 120^\circ$ (angles in a straight line $= 180^\circ$)

Sum of interior angles of pentagon $BCDEF$ is

$= (2n-4)\times 90^\circ$ (angle sum of a polygon with n sides $= (2n-4)\times 90^\circ$)

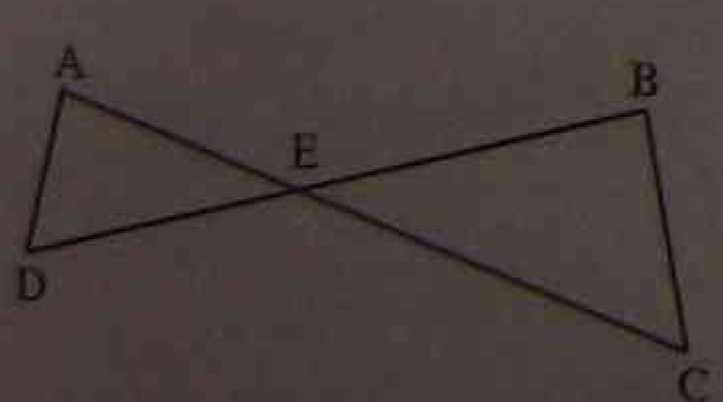
$= (2\times 5 - 4)\times 90^\circ$

$= 540^\circ$

Hence, $\angle BFE = 540^\circ - 3\times 90^\circ - 120^\circ = 150^\circ$ (angle sum of pentagon $= 540^\circ$)

$\therefore\ \angle AFE = 360^\circ - 60^\circ - 150^\circ = 150^\circ$ (angles which meet at a point $= 360^\circ$) #

Example 2:



In the figure: $BE = 4\,cm$, $AE = 3\,cm$

$AD = 5\,cm$, $DE = 6\,cm$

$EC = 8\,cm$.

Find the length of BC, giving reasons.

Solution 2:

$\angle AED = \angle BEC$ (vertically opposite angles equal)

$\dfrac{AE}{BE} = \dfrac{3}{4}$

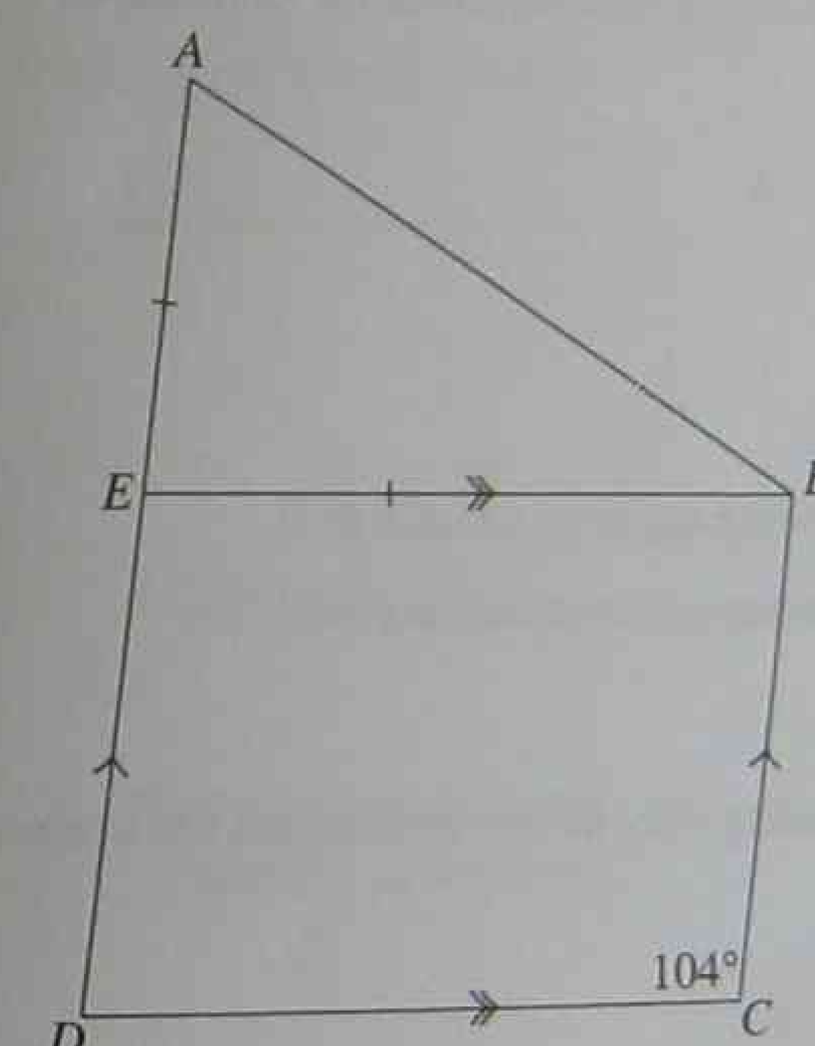
$\dfrac{DE}{EC} = \dfrac{6}{8} = \dfrac{3}{4}$ i.e. $\dfrac{AE}{BE} = \dfrac{DE}{EC}$

thus, $\Delta\ AED \,|||\, \Delta\ BEC$ (two corresponding sides in equal proportion and the included angle equal)

Hence, $\dfrac{AD}{BC} = \dfrac{3}{4}$

i.e. $\dfrac{5}{BC} = \dfrac{3}{4}$ $\therefore\ BC = \dfrac{20}{3} = 6\dfrac{2}{3}$ cm #

Example 3:



In the figure: $BC \parallel AD$, $EB \parallel DC$

$AE = EB$

$\angle BCD = 104^\circ$

Find the size of $\angle ABE$.

Not To Scale

Solution 3:

Quadrilateral $EBCD$ is a parallelogram (both pairs of opposite sides parallel)

thus, $\angle BED = 104^\circ$ (opposite angles of a parallelogram equal)

$\angle AEB = 76^\circ$ (angles in a straight line $= 180^\circ$)

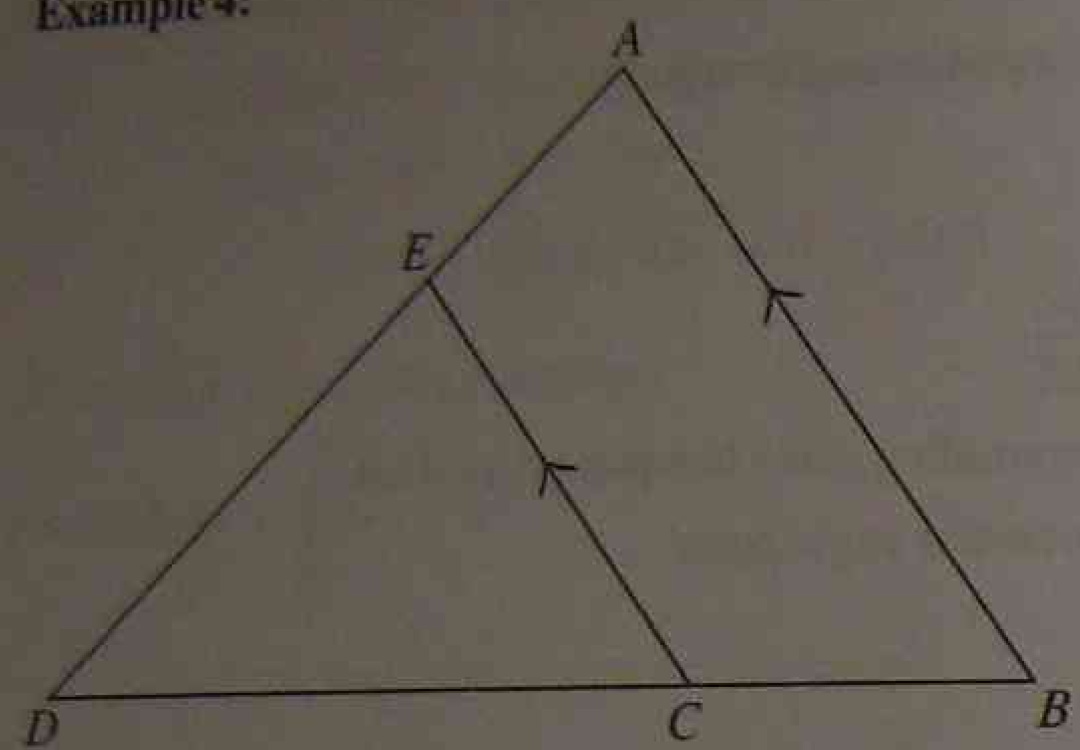
$\angle ABE = \angle EAB$ (base angles of isosceles Δ AEB equal)

$\angle ABE + \angle EAB = 180^\circ - 76^\circ = 104^\circ$ (angle sum of $\Delta = 180^\circ$)

i.e. $2\angle ABE = 104^\circ$

$\therefore\ \angle ABE = 52^\circ$ #

Example 4:



In the figure: $AB \parallel CE$

$AE = 4\text{cm},\ DE = 10\text{cm}$

$BC = 5\text{cm},\ EC = 8$

Find the lengths of DC and AB.

Solution 4:

$\frac{AE}{DE} = \frac{BC}{DC}$ (parallel lines cut transversals in the same ratio)

thus, $\frac{4}{10} = \frac{5}{DC}$

i.e. $DC = \frac{50}{4} = 12.5\text{cm}$

$\angle ADB = \angle EDC$ (common)

$\angle DEC = \angle DAB$ and (corresponding angles equal $AB \parallel EC$)

$\angle ECD = \angle ABD$ (corresponding angles equal $AB \parallel EC$)

thus, $\Delta ADB \,|||\, \Delta EDC$ (AAA)

$\therefore \frac{EC}{AB} = \frac{DE}{DA}$ (corresponding sides of similar triangles are in equal proportion)

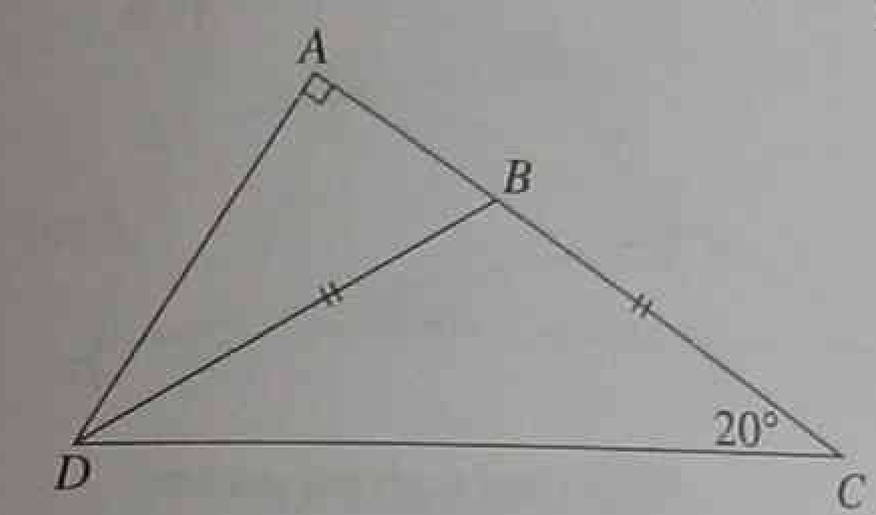
i.e. $\frac{8}{AB} = \frac{10}{14}$

$\therefore AB = \frac{14 \times 8}{10} = 11.2\text{cm}$ #

REVIEW EXERCISES

(A) Simple Numerical Exercises

1.

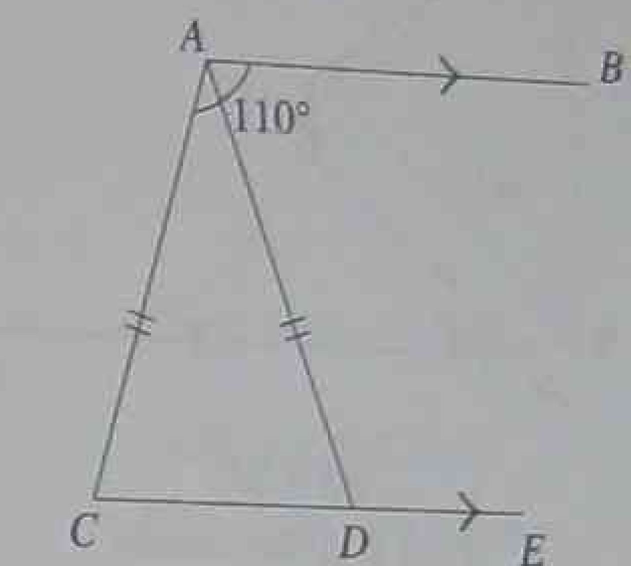


$\angle DAB = 90°,\ \angle BCD = 20°$

$BC = BD$.

Find $\angle ADB$, giving reasons.

2.

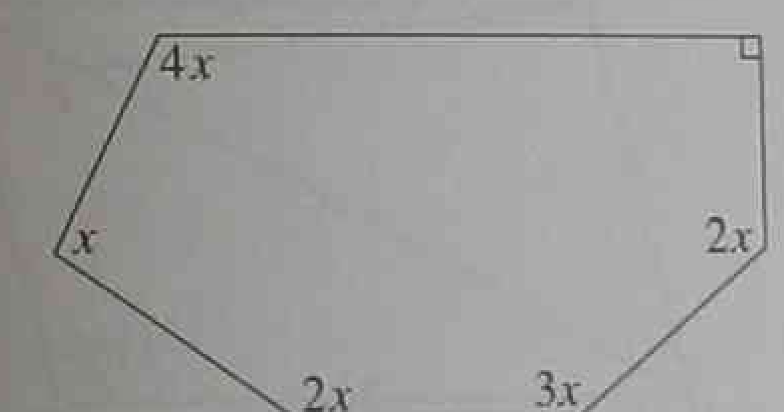


$\angle BAC = 110°$

$AC = AD,\ AB \parallel CE$.

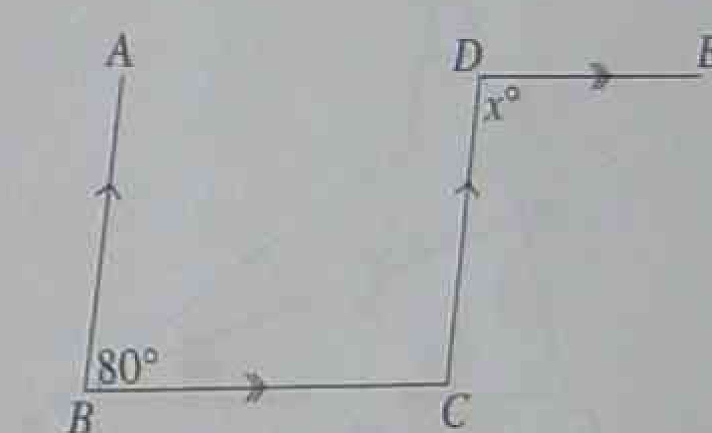
Find $\angle CAD$ giving reasons.

3.



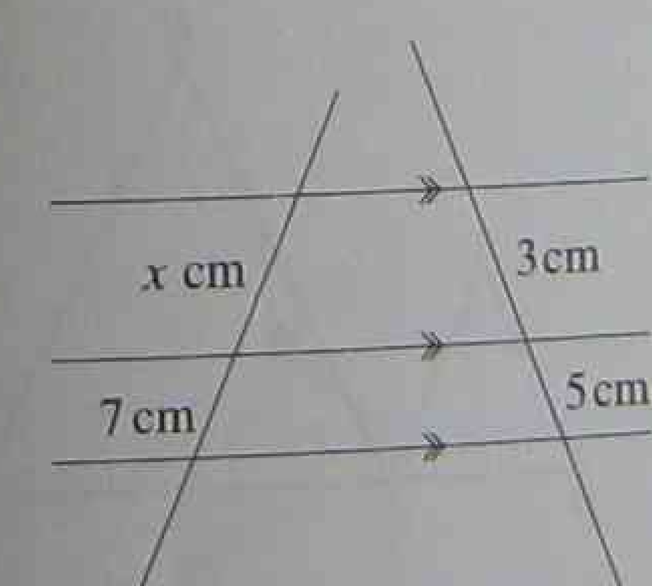
Find the value of x, giving reasons for your answer

4.



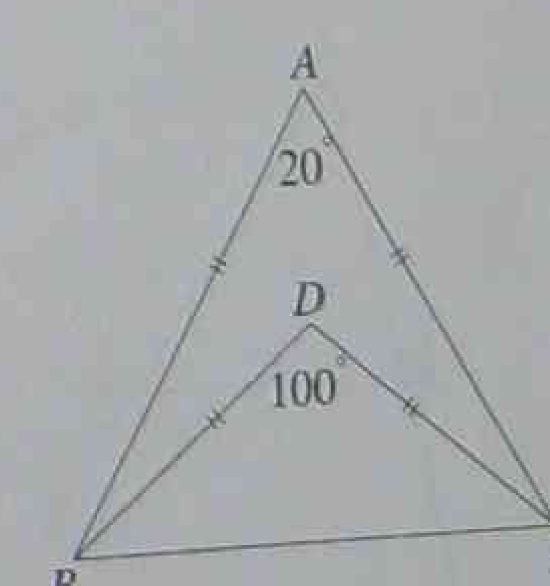
Find the value of x, giving reasons for your answer

5.



Find the value of x, giving reasons for your answer

6.

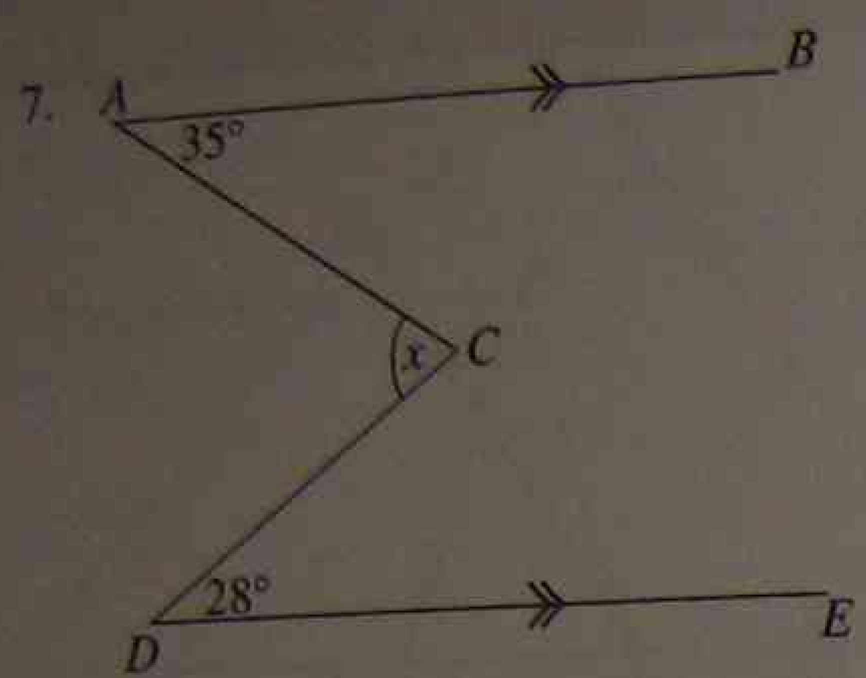


$\angle BAC = 20°,\ \angle BDC = 100°$

Δ's ABC and BDC are isosceles.

Find $\angle ABD$, giving reasons.

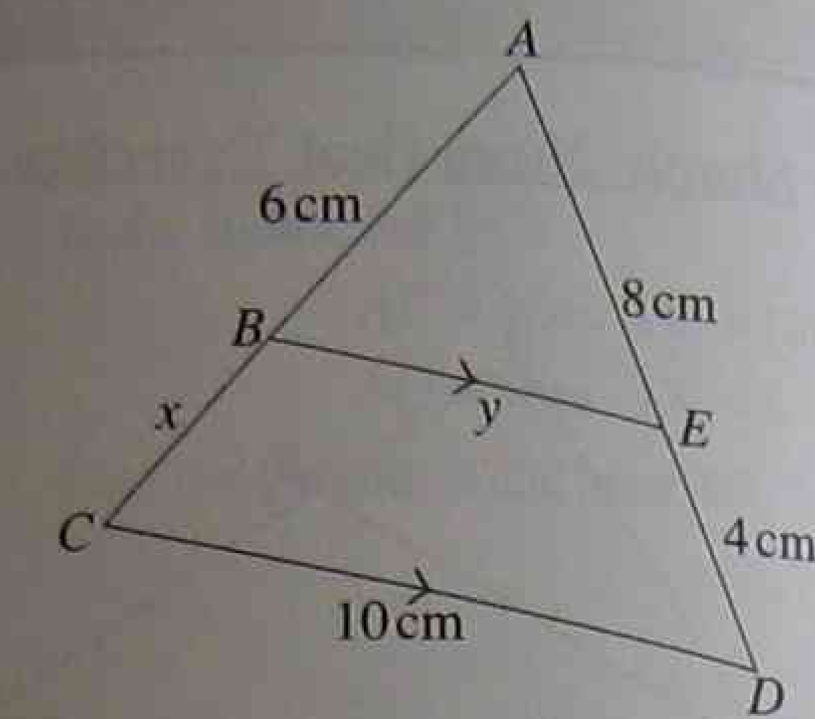
7.



$AB \parallel DE$

Find x, giving reasons

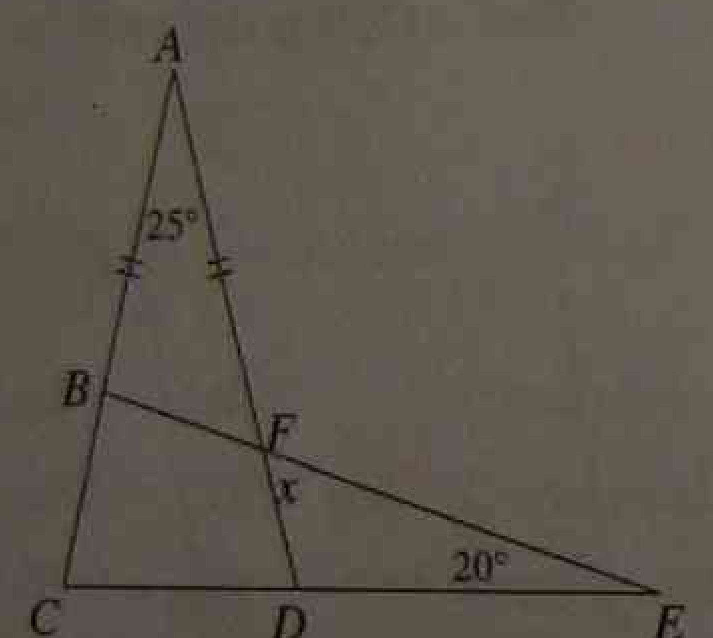
8.



Find x and y, giving reasons

(B) Harder Numerical Exercises

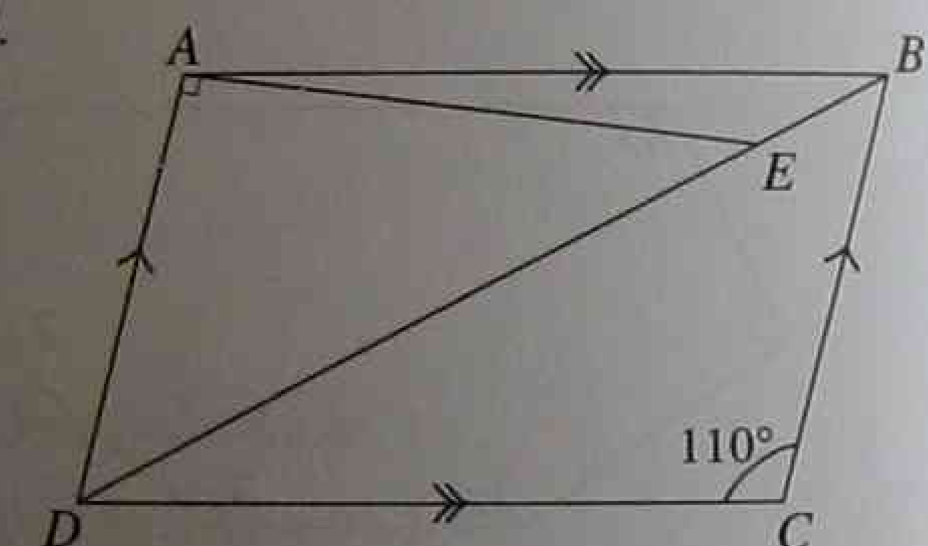
9.



$AC = AD$

Find x, giving reasons

10.

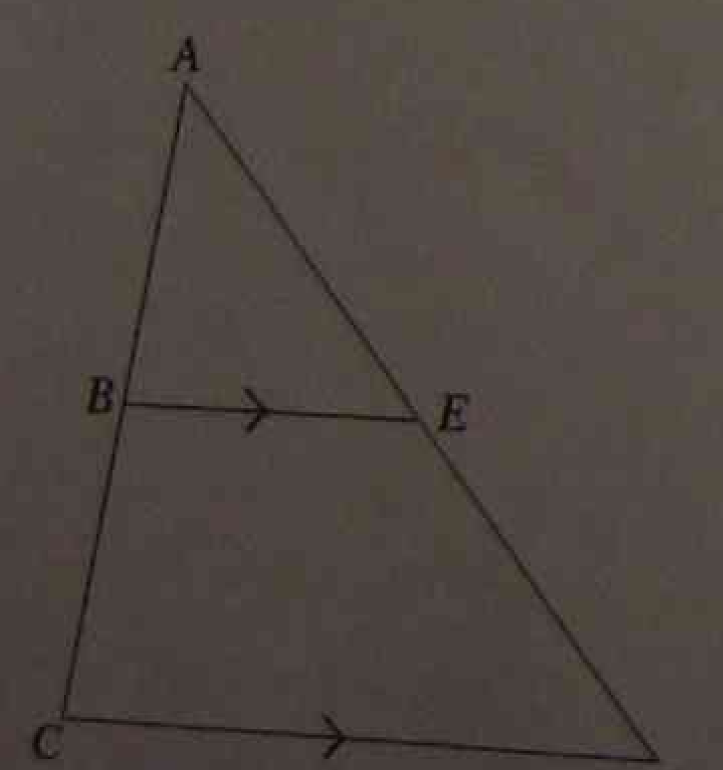


$ABCD$ is a rhombus.

$\angle DAE = 90°$, $\angle DCB = 110°$.

Find $\angle AEB$, giving reasons.

11.

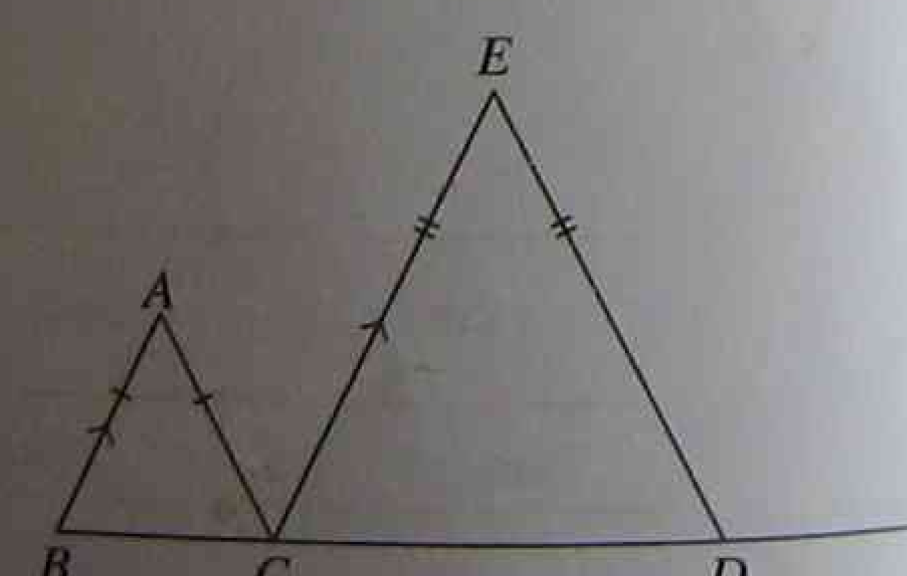


$BE = 7\text{cm}$, $AE = 10\text{cm}$, $DE = 6\text{cm}$

$BE \parallel CD$

Find the length of CD, giving reasons.

12.



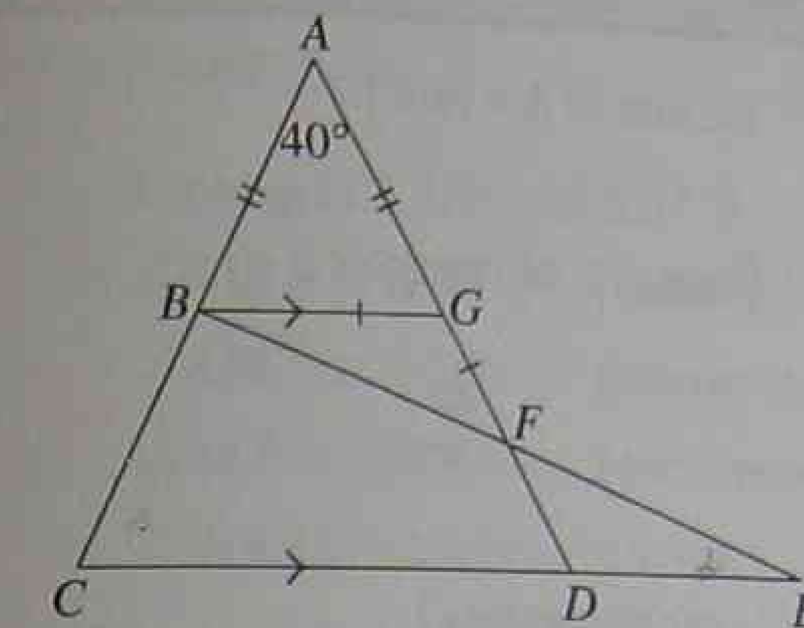
$AB \parallel CE$.

$AB = AC$, $EC = ED$.

$\angle CED = 20°$.

Find $\angle BAC$, giving reasons.

13.

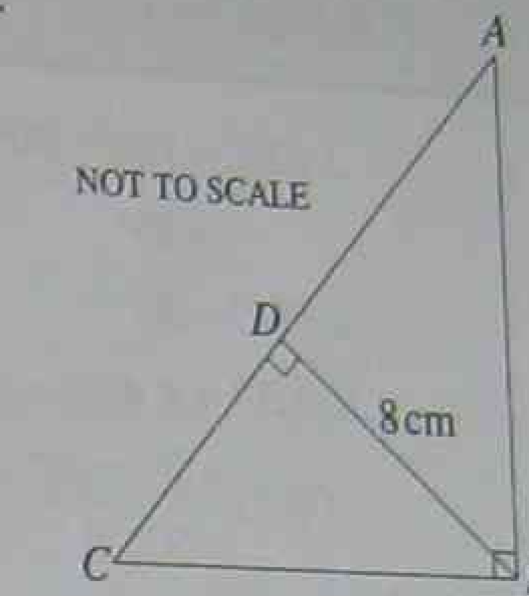


$AB = AG$, $BG = GF$

$\angle CAD = 40°$

Find $\angle FED$, giving reasons.

14.



$\angle ABC = \angle BDC = 90°$

$AB = 10\,\text{cm}$, $BD = 8\,\text{cm}$.

Find the lengths of AD and DC, giving reasons.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $\angle ADC = 180° - 90° - 20°$ ($\angle$sum of $\Delta = 180°$)
$= 70°$
$\angle BDC = 20°$ (base $\angle$'s of isosceles $\Delta =$)
$\angle ADB = \angle ADC - \angle BDC$
$= 70° - 20°$
$= 50°$ #

2. $\angle ACD = 70°$ (co-interior $\angle$'s supplementary)
$\angle ADC = 70°$ (base $\angle$'s of isosceles $\Delta =$)
$\angle CAD = 180° - 70° - 70°$
$= 40°$ ($\angle$sum of $\Delta = 180°$) #

3. Angle sum of any polygon of n sides $= (2n-4)\times 90°$
For $n = 6$, $(2\times 6 - 4)\times 90° = 720°$
i.e. $4x + 3x + 2\times 2x + x + 90° = 720°$
$12x = 630°$
$x = 52.5°$ #

4. $\angle BCD = 100°$ (co-interior $\angle$'s supplementary)
$\angle EDC = 100°$ (alternate $\angle$'s equal) #

5. A family of parallel lines cutting transversals, preserves the ratio of sides on each transversal, i.e.
$\frac{x}{7} = \frac{3}{5}$
$x = \frac{21}{5} = 4.2\,\text{cm}$ #

6. $\angle ABC = \angle ACB$ (base$\angle$'s of isoceles $\Delta =$)
$\angle ABC + \angle ACB = 160°$ ($\angle$sum of $\Delta = 180°$)
$\therefore\ 2\angle ABC = 160°$ i.e. $\angle ABC = 80°$
Also, $\angle DBC = \angle DCB$ (base $\angle$'s of isosceles $\Delta =$)
$\angle DBC + \angle DCB = 80°$ ($\angle$sum of $\Delta = 180°$)
$\therefore\ 2\angle DBC = 80°$ i.e. $\angle DBC = 40°$
thus, $\angle ABD = 80° - 40° = 40°$ #

7. $x = \angle BAC + \angle CDE$ (alternate $\angle$'s equal)
$= 35° + 28°$
$= 63°$

8. Looking at ΔABE and ΔACD
$\angle BAE = \angle CAE$ (common)
$\angle ABE = \angle ACD$ (corresponding $\angle$s equal, $BE \parallel CD$)
$\angle AEB = \angle ADC$ (corresponding $\angle$s equal, $BE \parallel CD$)
$\therefore\ \Delta ABE\ |||\ \Delta ACD$ (AAA)
thus, $\frac{6}{6+x} = \frac{8}{12}$ (corresponding sides in = proportion)
i.e. $72 = 48 + 8x$
$24 = 8x$
$x = 3\,\text{cm}.$
Also, $\frac{y}{10} = \frac{8}{12}$
i.e. $12y = 80$
$y = 6\frac{2}{3}\,\text{cm}$ #

9. $\angle ACD + \angle ADC$ (base $\angle$s of isosceles $\Delta =$)
$\angle ACD + \angle ADC = 180° - 25°$ ($\angle$sum of $\Delta = 180°$)
$= 155°$
$\therefore\ 2\angle ADC = 155°$
i.e. $\angle ADC = 77.5°$
$77.5° = x + 20°$ (exterior $\angle$ of Δ is = to the sum of the two opposite interior $\angle$'s)
$x = 55.5°$

10. $\angle DAB = 110°$ (opposite angles of a rhombus are equal)
$\angle BAE = 110° - 90° = 20°$
$BC = CD$ (opposite sides of rhombus=)
$\angle CDB = \angle CBD$ (base$\angle$'s of isosceles ΔBCD=, $BC = CD$)
i.e. $\angle CBD + \angle CDB = 70°$

i.e. $2\angle CBD = 70^\circ$

$\angle CBD = 35^\circ$

$\angle ABE = 35^\circ$ $\left(\begin{array}{l}\text{diagonals of a rhombus bisects the}\\ \text{angles at the vertex}\end{array}\right)$

$\therefore \quad \angle AEB = 180^\circ - 35^\circ - 20^\circ$ ($\angle$sum of $\Delta ABE = 180^\circ$)

$= 125^\circ$ #

11. $\angle BAE = \angle CAE$ (common)

$\angle ABE = \angle ACD$ (corresponding $\angle$'s equal $BE \parallel CD$)

$\angle AEB = \angle ADC$ (corresponding $\angle$'s equal $BE \parallel CD$)

$\therefore \Delta ABE \parallel\!\mid \Delta ACD$ (AAA)

$\therefore \quad \dfrac{BE}{CD} = \dfrac{AE}{AD}$ $\left(\begin{array}{l}\text{corresponding sides of similar } \Delta\text{'s are}\\ \text{in equal proportion to each other}\end{array}\right)$

i.e. $\dfrac{7}{CD} = \dfrac{10}{16}$

$\therefore \quad CD = 11.2$ cm #

12. $\angle ECD = \angle EDC$ (base $\angle$'s of isosceles $\Delta ECD =$)

$\angle ECD + \angle EDC = 160^\circ$ ($\angle$sum of $\Delta = 180^\circ$)

$\therefore \quad 2\angle ECD = 160^\circ$

i.e. $\angle ECD = 80^\circ$

$\angle BAC = \angle ACE$ (alternate $\angle$'s of equal, $AB \parallel CE$)

$\angle ACB = \angle ABC$ (base $\angle$'s of isosceles $\Delta ABC =$)

$\angle ACB + \angle ABC = 180^\circ - \angle BAC$

$\therefore \quad 2\angle ACB = 180^\circ - \angle BAC$

i.e. $\angle ACB = 90^\circ - \dfrac{\angle BAC}{2}$

Now, $\angle ACB + \angle ACE + \angle ECD = 180^\circ$ (angles in a straight line $= 180^\circ$)

i.e. $\left(90^\circ - \dfrac{\angle BAC}{2}\right) + \angle BAC + 80^\circ = 180^\circ$

$\dfrac{\angle BAC}{2} = 10^\circ$

$\therefore \quad \angle BAC = 20^\circ$

13.

$\angle ACD = \angle ADC$ (base $\angle$'s of isosceles ΔACD are $=$)

$\angle ACD + \angle ADC = 140^\circ$ (angles sum of $\Delta ADC = 180^\circ$)

i.e. $\angle ADC = \dfrac{140^\circ}{2} = 70^\circ$

$\angle FDE = 180^\circ - 70^\circ = 110^\circ$ (angles in a straight line $= 180^\circ$)

$\angle GBF = \angle FED$ (alternate $\angle$'s equal, $BG \parallel CE$)

$\angle GBF = \angle GFB$ (base $\angle$'s of isosceles ΔBGF are $=$)

$\angle GFB = \angle DFE$ (vertically opposite $\angle$'s $=$)

thus, $\angle DFE = \angle FED$

Now, $\angle DFE + \angle FDE + \angle FED = 180^\circ$ (angle sum of $\Delta FED = 180^\circ$)

$\therefore \quad 2\angle FED + 110^\circ = 180^\circ$

$2\angle FED = 70^\circ$

$\angle FED = 35^\circ$ #

14. $AD^2 = 10^2 - 8^2 = 100 - 64 = 36$

$\therefore AD = \sqrt{36} = 6$ cm

$\angle DAB = \angle CAB$ (common angle to ΔABC and ΔABD)

$\angle ABC = \angle BDA = 90^\circ$ (given)

$\therefore \Delta ABC \parallel\!\mid ABD$ (AAA)

thus, $\dfrac{AD}{AB} = \dfrac{AB}{AC}$ $\left(\begin{array}{l}\text{corresponding sides of } \parallel\!\mid \Delta\text{'s are in equal}\\ \text{proportion to each other}\end{array}\right)$

i.e. $\dfrac{6}{10} = \dfrac{10}{6 + DC}$

$36 + 6DC = 100$

$6DC = 64$

$DC = 10\frac{2}{3}$ cm #

TOPIC 3

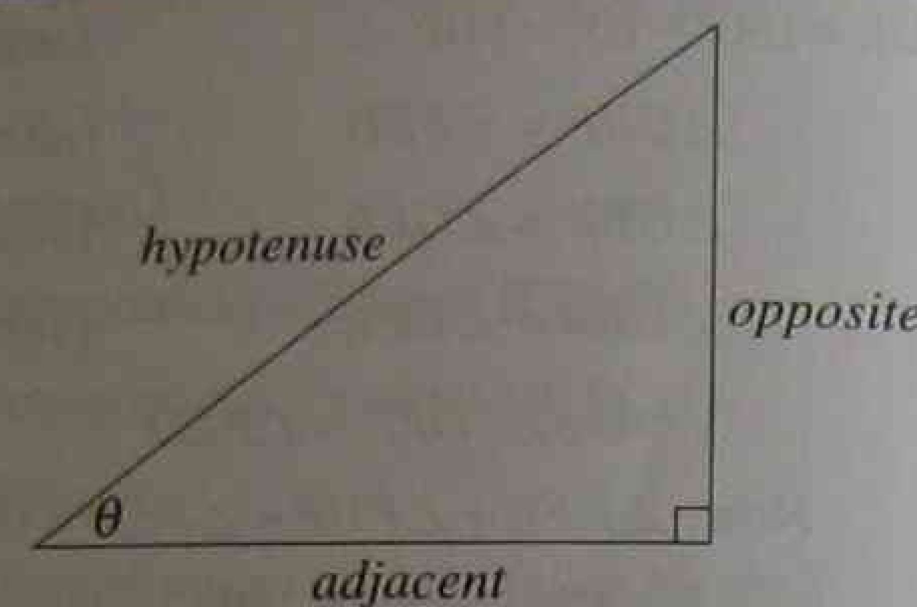
TRIGONOMETRIC RATIOS

(A) Basic Ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

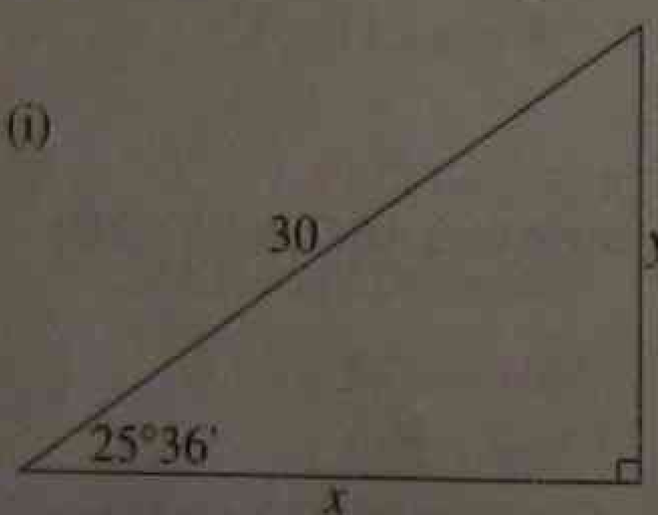
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta}$$

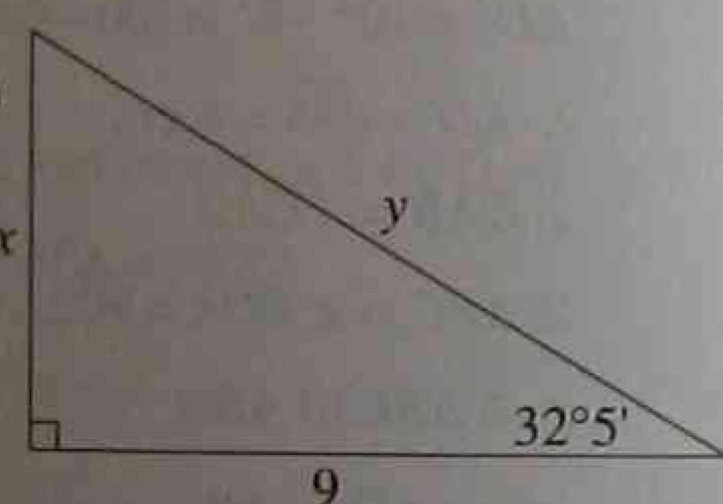


Example 1: Find the values of x and y. Give your answer correct to 2 decimal places.

(i)



(ii)



Solution 1:

(i) $\cos 25°36' = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{30}$

$\therefore \quad x = 30\cos 25°36'$

$= 27.05$ units correct to 2 d.p. #

$\sin 25°36' = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{30}$

$\therefore \quad y = 30\sin 25°36'$

$= 12.96$ units correct to 2 d.p. #

(ii) $\tan 32°5' = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{9}$

$\therefore \quad x = 9\tan 32°5'$

$= 5.64$ units correct to 2 d.p. #

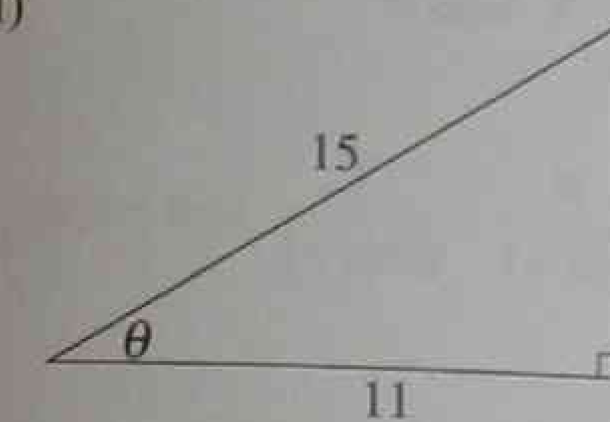
$\cos 32°5' = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{9}{y}$

$\therefore \quad y = \frac{9}{\cos 32°5'}$

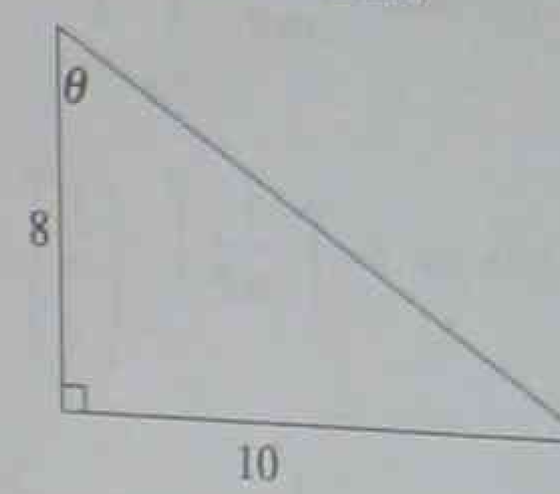
$= 10.62$ units correct to 2 d.p. #

Example 2: Find the value of θ. Give your answer correct to the nearest minute.

(i)



(ii)



Solution 2:

(i) $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{11}{15}$

$\therefore \quad \theta = 42°50'$ to the nearest minute #

(ii) $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{10}{8}$

$\therefore \quad \theta = 51°20'$ to the nearest minute #

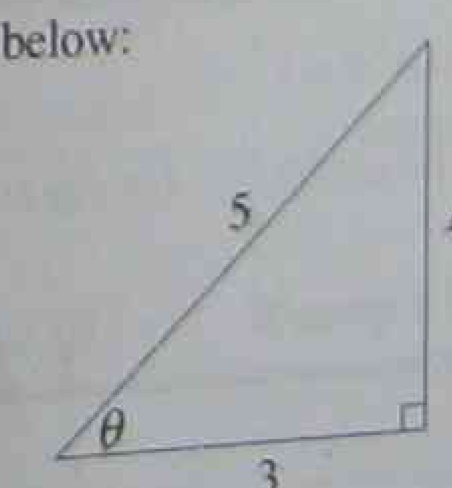
(B) Reciprocal and Complementary Ratios

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Co-ratios : $\sin(90-\theta) = \cos\theta, \quad \cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cot\theta, \quad \cot(90-\theta) = \tan\theta$

Example 1: Consider the diagram below:



Find the exact values of:

Solution 1:

(i) $\cos\theta = \frac{3}{5}$ #

(ii) $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$ #

(iii) $\cot^2\theta = \left(\frac{1}{\tan\theta}\right)^2 = \left(\frac{1}{\frac{4}{3}}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$ #

Example 2: Let $\sin\theta = \frac{1}{3}$ and $\cos\theta = \frac{2\sqrt{2}}{3}$ where θ is acute. Find the exact values of:

(i) $\mathrm{cosec}\,\theta$ (ii) $\sin(90-\theta)$ (iii) $\cot(90-\theta)$

Solution 2:

(i) $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta} = \left(\frac{1}{\frac{1}{3}}\right) = 3$ #

(ii) $\sin(90-\theta) = \cos\theta$

$= \frac{2\sqrt{2}}{3}$ #

(iii) $\cot(90-\theta) = \tan\theta$

$= \frac{\sin\theta}{\cos\theta} = \left(\frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) = \frac{1}{3} \times \frac{3}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$ #

(C) Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1\ ^*$

$\tan^2\theta + 1 = \sec^2\theta$ (÷* by $\cos^2\theta$)

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$ (÷* by $\sin^2\theta$)

Example 1: Simplify $\sqrt{(1-\cos^2)(\tan^2\theta+1)}$

Solution 1:

$$\sqrt{(1-\cos^2)(\tan^2\theta+1)} = \sqrt{\sin^2\theta\sec^2\theta}$$
$$= \sin\theta\sec\theta$$
$$= \sin\theta \times \frac{1}{\cos\theta}$$
$$= \tan\theta \quad \#$$

Example 2: Prove that $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$

Solution 2:

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$$
$$= \frac{2}{1-\sin^2\theta}$$
$$= \frac{2}{\cos^2\theta}$$
$$= 2\sec^2\theta \ \#$$

Example 3: Simplify $\frac{\cos\theta}{1+\cos\theta} + \frac{\sin^2\theta}{(1+\cos\theta)^2}$

Solution 3:

$$\frac{\cos\theta}{1+\cos\theta} + \frac{\sin^2\theta}{(1+\cos\theta)^2} = \frac{\cos\theta(1+\cos\theta)+\sin^2\theta}{(1+\cos\theta)^2}$$
$$= \frac{\cos\theta+\cos^2\theta+\sin^2\theta}{(1+\cos\theta)^2}$$
$$= \frac{\cos\theta+1}{(1+\cos\theta)^2}$$
$$= \frac{1+\cos\theta}{(1+\cos\theta)^2} \qquad \text{(rearranging)}$$
$$= \frac{1}{1+\cos\theta} \quad \#$$

(D) ASTC- All Stations To Central

Diagram 1 indicates which trigonometric ratios are positve in which quadrant i.e. ASTC (All stations To Central).

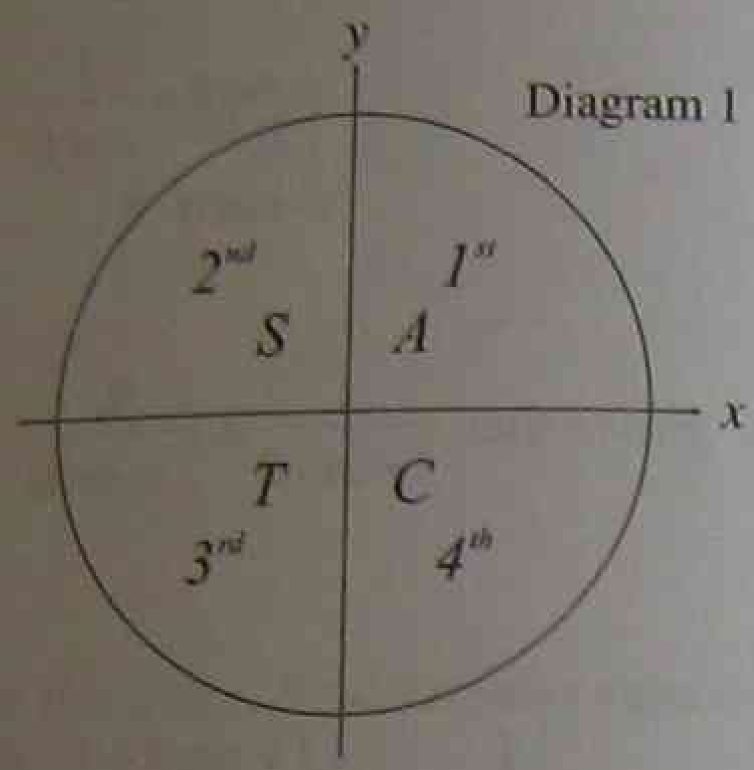


Diagram 1 "ASTC"

All ratios are positive in the 1^{st} quadrant.

Sin only positive in the 2^{nd} quadrant.

Tan only positive in the 3^{rd} quadrant.

Cos only positive in the 4^{th} quadrant.

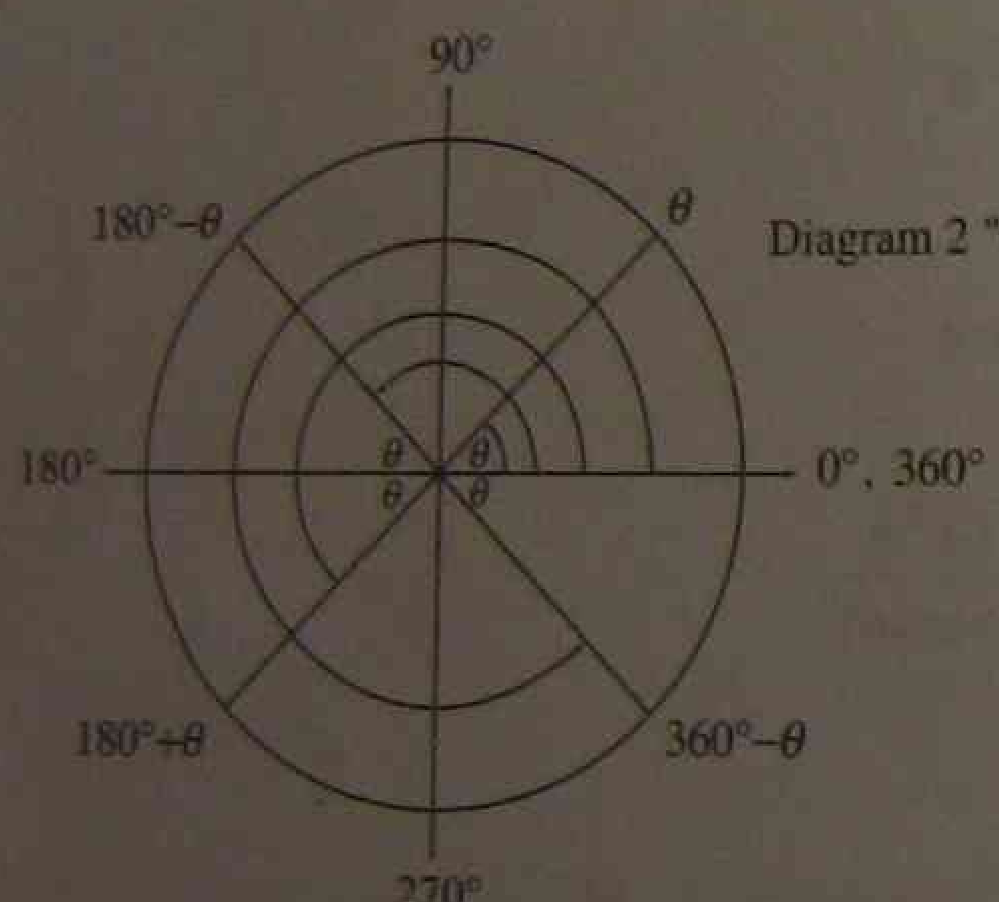


Diagram 2 "Criss-Cross Angles"

In solving trigonometric equations, it's the "criss-cross angles" that are most relevant (indicated as θ in diagram 2).

Angles are generally measured anti-clockwise from the positive x-axis, thus the positive solutions for the various ratios are:

- for sin, the solutions are θ and $180°-\theta$
- for cos, the solutions are θ and $360°-\theta$
- for tan, the solutions are θ and $180°+\theta$

Note that $(360°-\theta)$ may also be written as $(-\theta)$:

- $\cos(-\theta)$

Example 1: If $\cos\theta = 0.766$, find the values of:

(i) $\cos(360°-\theta)$ (ii) $\cos(180°+\theta)$

(iii) $\cos(-\theta)$ (iv) $\cos(180°-\theta)$

Solution 1:

(i) $\cos(360°-\theta) = \cos\theta = 0.766$ #

(ii) $\cos(180°+\theta) = -\cos\theta = -0.766$ #

(iii) $\cos(-\theta) = \cos\theta = 0.766$ #

(iv) $\cos(180°-\theta) = -\cos\theta = -0.766$ #

Example 2: If $\tan A = -2.75$, find the values of:

(i) $\tan(180°+A)$ (ii) $\tan(-A)$

(iii) $\tan(360°+A)$ (iv) $\tan(90°-A)$

Solution 2:

(i) $\tan(180°+A) = \tan A = -2.75$ #

(ii) $\tan(-A) = \tan(360°-A) = -\tan A = 2.75$ #

(iii) $\tan(360°+A) = \tan A = -2.75$ #

(iv) $\tan(90°-A) = \cot A = \dfrac{1}{\tan A} = -0.364$ #

Example 3: If $\sin A = 0.0872$ and $0° \le A \le 360°$, find all values of A to the nearest minute.

Solution 3:

$\sin A = 0.0872$

$\therefore\ A = 5°$ to the nearest degree

Since **sin** is positive in the 1^{st} and 2^{nd} quadrants

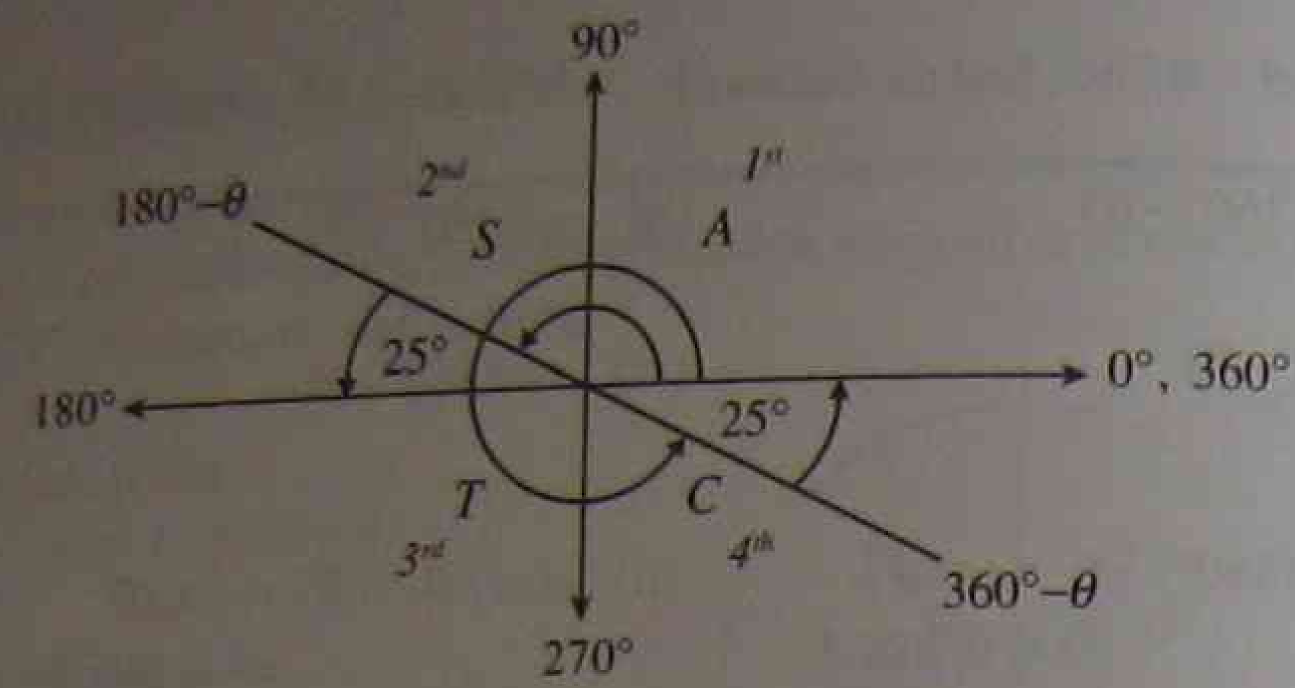
$\therefore\ A = 5°$ or $A = 180° - 5° = 175°$ #

Example 4: If $\tan B = -0.4663$ and $0° \le B \le 360°$, find all values of θ to the nearest minute.

Solution 4:

$\tan B = -0.4663$

Since tan is negative $\therefore$ B is in the 2^{nd} and 4^{th} quadrants.



From the above diagram the "*criss - cross*" angle is 25°.

$\therefore$ the two positive solutions are $180°-\theta$ and $360°-\theta$

i.e. $B=180°-25°$ or $B=360°-25°$

$=155°$ $=335°$ #

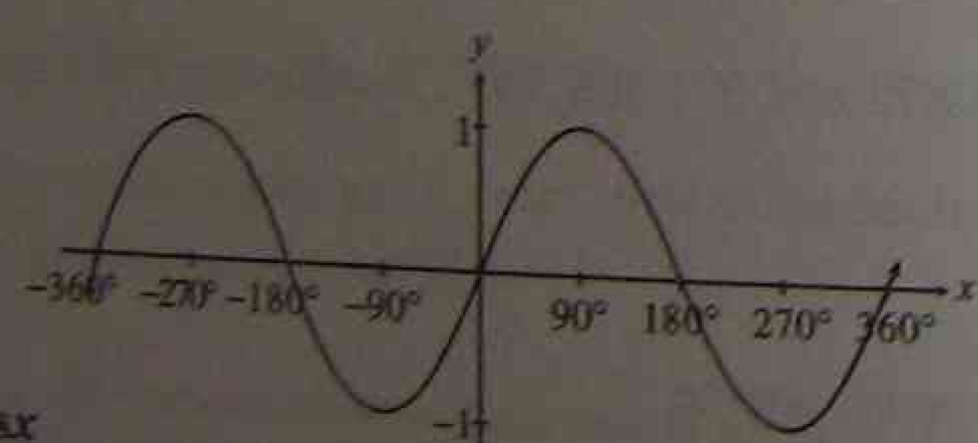
(E) Graphs of sin$x$, cos$x$, tan$x$

The graphs of sin$x$, cos$x$ and tan$x$ are periodic. This means that they repeat themselves over and over again.

Sketching these graphs is best done by drawing up a table of values at the critical points and then fitting the curve through those points. When one period is drawn (i.e. curve returns to its starting point), the curve can then be repeated.

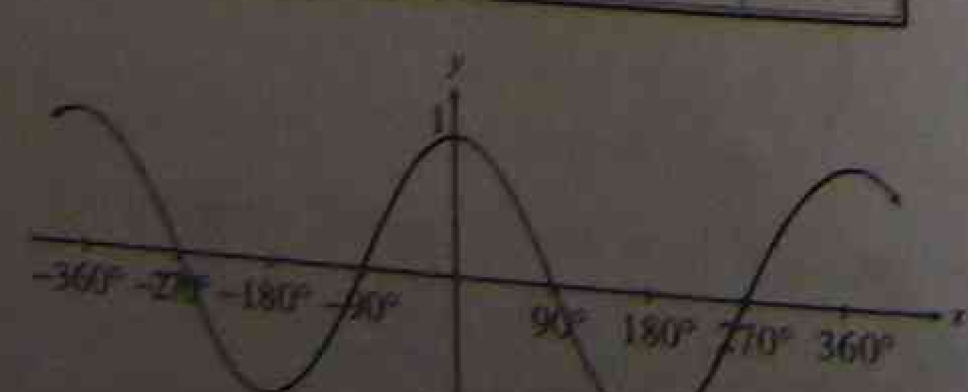
(i) $y=\sin x$

x	0°	90°	180°	270°	360°
$\sin x$	0	1	0	-1	0



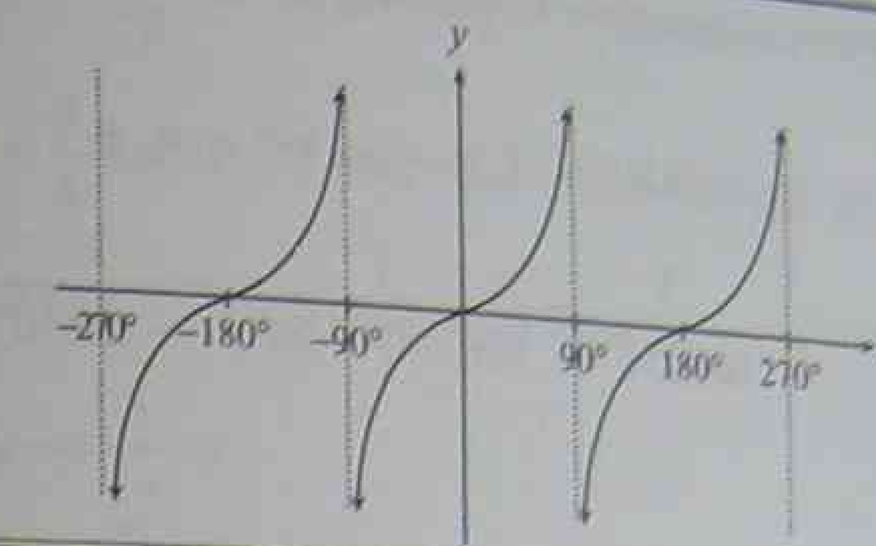
(ii) $y=\cos x$

x	0°	90°	180°	270°	360°
$\cos x$	1	0	-1	0	1



(iii) $y=\tan x$

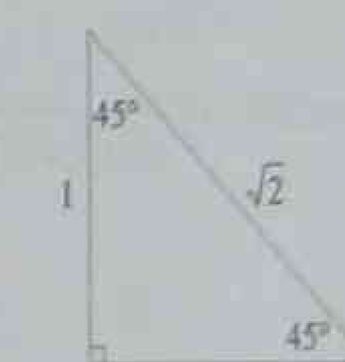
x	-90°	0°	90°	180°	270°
$\tan x$	undefined	0	undefined	0	undefined

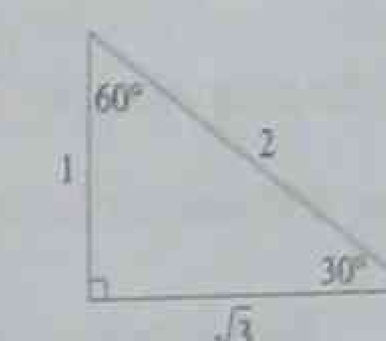


(F) Exact Ratios

Exact ratios are derived from the 45°,45° and 30°,60° triangles.

x	0°	30°	45°	60°	90°
sin$x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos$x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan$x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	error





Students could also learn the calculator displays:

- $\frac{1}{\sqrt{2}}=0.707...$
- $\frac{\sqrt{3}}{2}=0.866...$
- $\frac{1}{\sqrt{3}}=0.577...$
- $\sqrt{3}=1.732...$

Example 1: Find the exact value of each ratio:

(i) $\cos 315°$ (ii) $\sin 330°$ (iii) $\tan 135°$

(iv) $\cot 150°$ (v) $\sec 120°$ (vi) $\text{cosec} 240°$

Solution 1:

(i) $\cos(315^\circ) = \cos(360^\circ - 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$ #

(ii) $\sin(330^\circ) = \sin(360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$ #

(iii) $\tan(135^\circ) = \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1$ #

(iv) $\cot(150^\circ) = \frac{1}{\tan(150^\circ)} = \frac{1}{\tan(180^\circ - 30^\circ)} = \frac{1}{-\tan 30^\circ}$

$= \frac{1}{\frac{-1}{\sqrt{3}}} = -\sqrt{3}$ #

(v) $\sec(120^\circ) = \frac{1}{\cos(120^\circ)} = \frac{1}{\cos(180^\circ - 60^\circ)} = \frac{1}{-\cos 60^\circ}$

$= \frac{1}{\frac{-1}{2}} = -2$ #

(vi) $\text{cosec}(240^\circ) = \frac{1}{\sin(240^\circ)} = \frac{1}{\sin(180^\circ + 60^\circ)} = \frac{1}{-\sin 60^\circ}$

$= \frac{1}{\frac{-\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$ #

Example 2: Find the exact values of:

(i) $\sec(-60^\circ)\sin(120^\circ)\cos(240^\circ)$ (ii) $\sin^2(225^\circ)\sec^2(315^\circ)$

Solution 2:

(i) $\sec(-60^\circ)\sin(120^\circ)\cos(240^\circ) = \frac{1}{\cos(-60^\circ)} \times \sin(180^\circ - 60^\circ) \times \cos(180^\circ + 60^\circ)$

$= \frac{1}{\cos 60^\circ} \times \sin 60^\circ \times -\cos 60^\circ$

$= -\frac{\sqrt{3}}{2}$ #

(ii) $\sin^2(225^\circ)\sec^2(315^\circ) = \sin^2(180^\circ + 45^\circ)\sec^2(360^\circ - 45^\circ)$

$= (-\sin(45^\circ))^2 \times \frac{1}{\cos^2(45^\circ)}$

$= \left(-\frac{1}{\sqrt{2}}\right)^2 \times \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{2} \times \frac{1}{\frac{1}{2}} = \frac{1}{2} \times 2 = 1$ #

(G) Relations between Trigonometric Ratios

Given the exact ratio for one trigonometric ratio of an angle, we can use Pythagoras' theorem to find the other ratios.

Example 1: Given that $\sin\theta = \frac{3}{4}$ and $0^\circ < \theta < 90^\circ$, find the correct value of:

(i) $\cos\theta$ (ii) $\sec\theta + \tan\theta$

Solution 1:

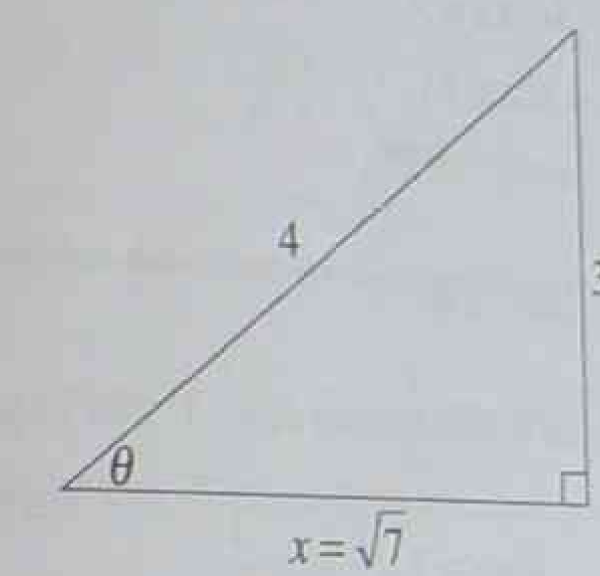
From the triangle:

$x^2 + 3^2 = 4^2$

$x^2 + 9 = 16$

$x^2 = 7$

$x = \sqrt{7}$



(i) $\cos\theta = \frac{\sqrt{7}}{4}$ #

(ii) $\sec\theta + \tan\theta = \frac{1}{\frac{\sqrt{7}}{4}} + \frac{3}{\sqrt{7}}$

$= \frac{4}{\sqrt{7}} + \frac{3}{\sqrt{7}} = \frac{7}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{7\sqrt{7}}{7} = \sqrt{7}$ #

Example 2: Given $\cos\theta = -\frac{2}{5}$ and $\sin\theta < 0$, find the exact value of $\cot\theta$.

Solution 2:

Since cos is negative and sin is negative $\therefore$ θ in the 3rd quadrant.

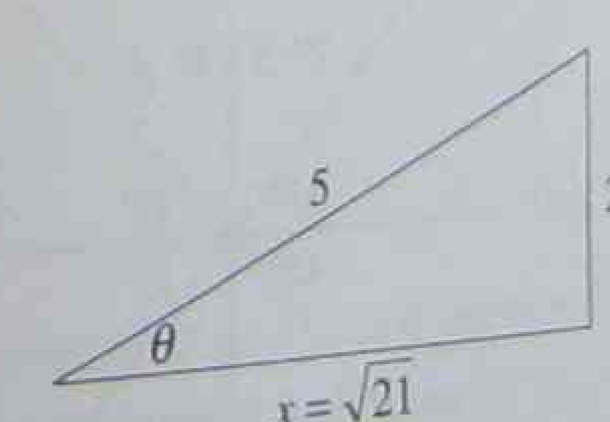
From the triangle:

$x^2 + 2^2 = 5^2$

$x^2 + 4 = 25$

$x^2 = 21$

$x = \sqrt{21}$



Thus, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{2}{\sqrt{21}}} = \frac{\sqrt{21}}{2}$ #

(H) Solving Simple Trigonometric Equations

Make the trigonometric ratio the subject of the equation and solve for all angles in the given domain.

Example 1: Find all values of θ, $0° \le \theta \le 360°$, for which:

(i) $5\cos x = 3 + 2\cos x$ (ii) $2\sin^2 x = 1$

Solution 1:

(i) $5\cos x = 3 + 2\cos x$

$3\cos x = 3$

$\cos x = 1$

$x = 0°, 360°$ #

(ii) $2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2}$

$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

$\sin x = \frac{1}{\sqrt{2}}$ or $\sin x = -\frac{1}{\sqrt{2}}$ #

Noting sin is positive in the 1^{st} and 2^{nd} quadrants:

now, $\sin 45° = \frac{1}{\sqrt{2}}$

$\therefore x = 45°, (180° - 45°)$

$= 45°, 135°$ #

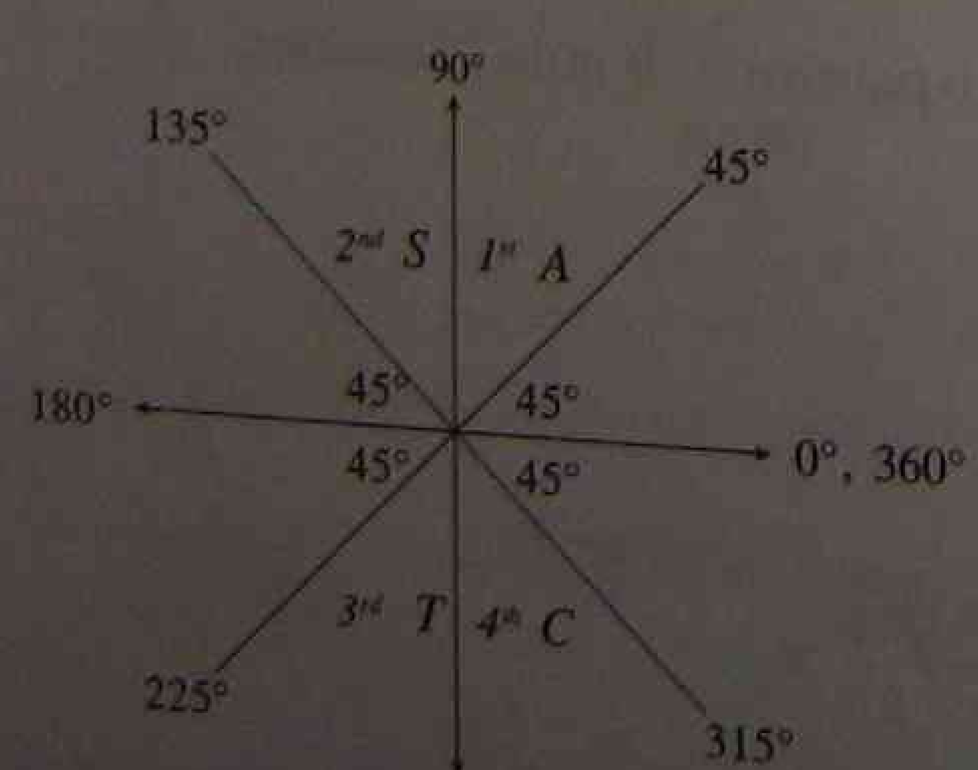
Noting sin is negative in the 3^{rd} and 4^{th} quadrants:

now, $\sin 45° = \frac{1}{\sqrt{2}}$

$\therefore x = (180° + 45°), (360° - 45°)$

$= 225°, 315°$ #

Note: diagramatically, this can be shown as follows:

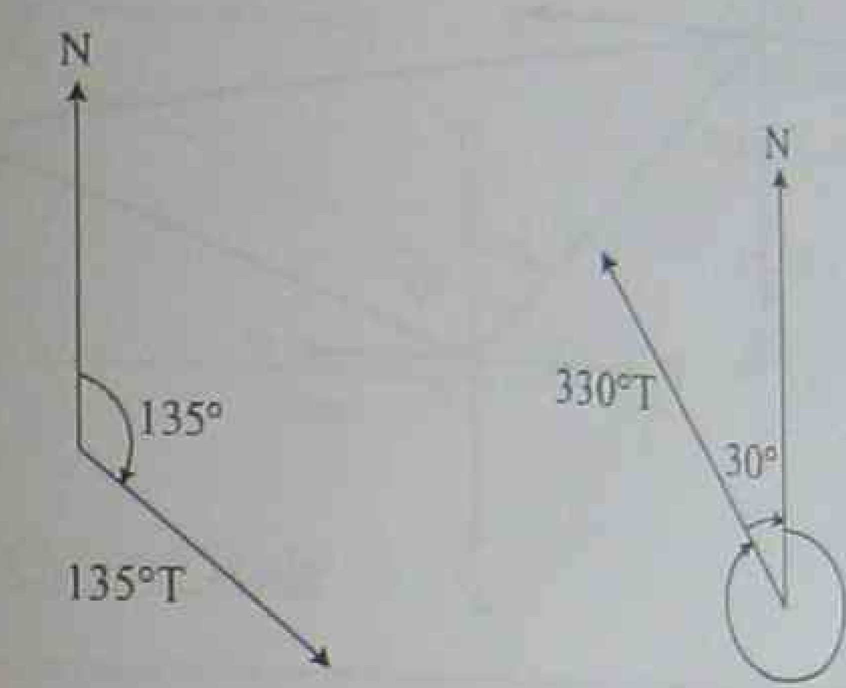


- "criss-cross angle" = 45°
- sin positive in 1^{st} & 2^{nd} i.e. 45°, 135°
- sin negative in 3^{rd} & 4^{th} i.e. 225°, 315°

Trigonometric Ratios

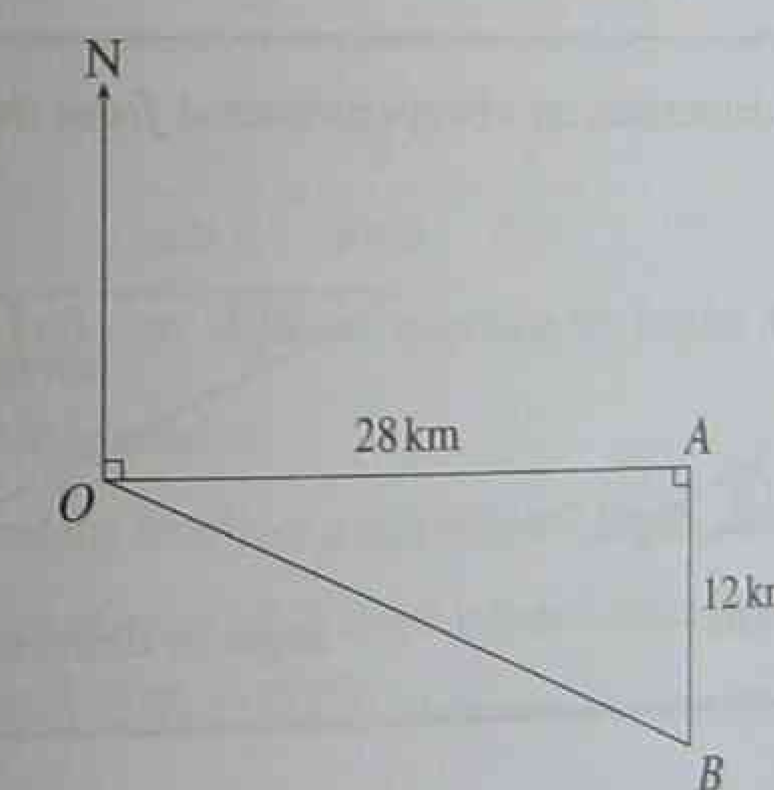
(I) Bearings

Bearings are always measured ***clockwise*** from ***north***. They are expressed in the form: XXX °T , where $000 \le XXX \le 360$.



Example 1: A boat travels 28 km on a bearing of 090°T, then turns and travels 12 km on a bearing of 180°T. Find the straight line distance between the start and finish correct to 2 decimal places.

Solution 1:



Using Pythagoras in ΔOAB:

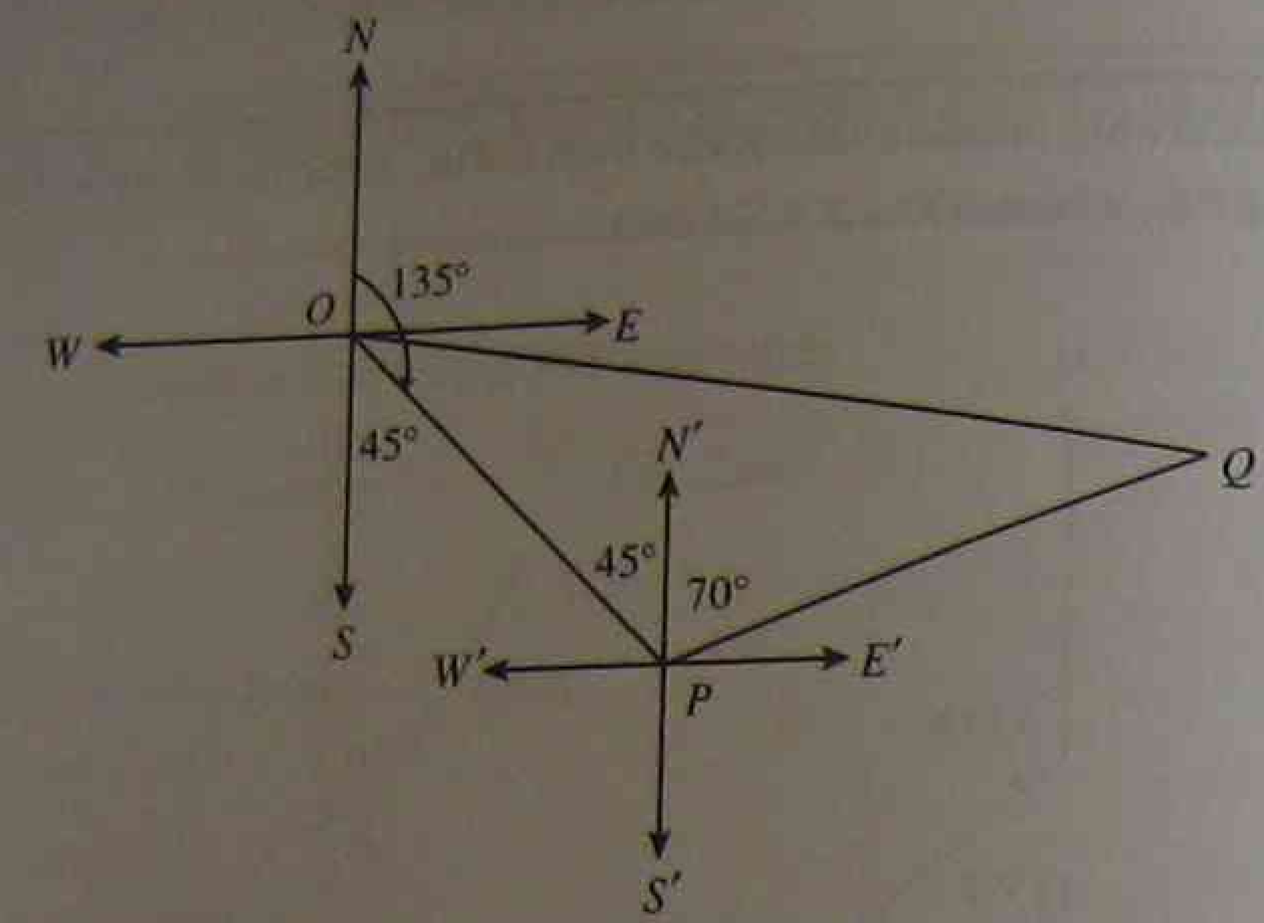
$OB^2 = 28^2 + 12^2$

$OB = \sqrt{28^2 + 12^2}$

$= 30.46$ km correct to 2 d.p. #

Example 2: A boat travels from O, on a bearing of 135°T, to a point P and then changes direction and travels on a bearing of 70°T to a point Q. Find the size of $\angle OPQ$.

Solution 2:



$\angle SOP = 180° - 135° = 45°$

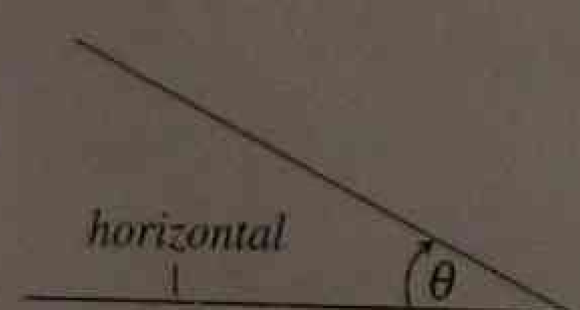
$\angle SOP = \angle OPN' = 45°$ (alternate $\angle$'s equal between parallel lines, $NS \parallel N'S'$)

$\angle N'PQ = 70°$ (given)

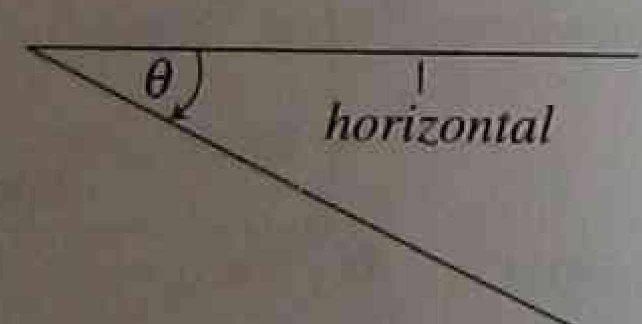
$\therefore \angle OPQ = 45° + 70° = 115°$ #

(J) Angles of Elevation and Depression

Angles of elevation and depression are always measured ***from the horizontal***.



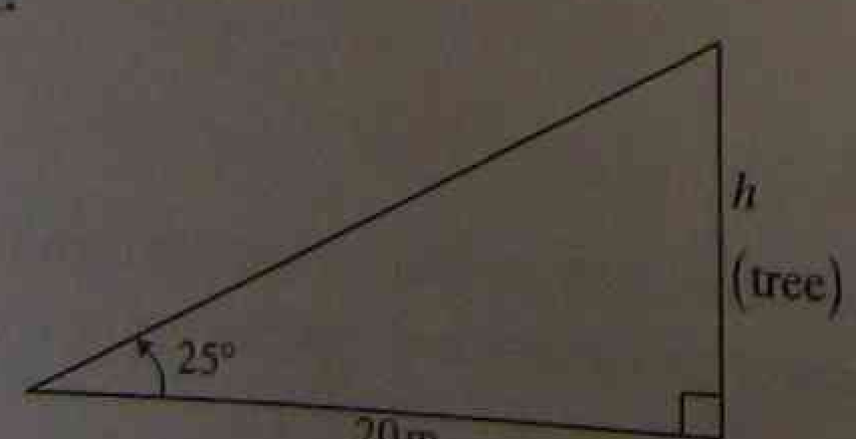
angle of elevation



angle of depression

Example 1: The angle of elevation of a tree from a point A, 20 metres away, is 25°. Calculate the height of the tree, correct to 3 significant figures.

Solution 1:



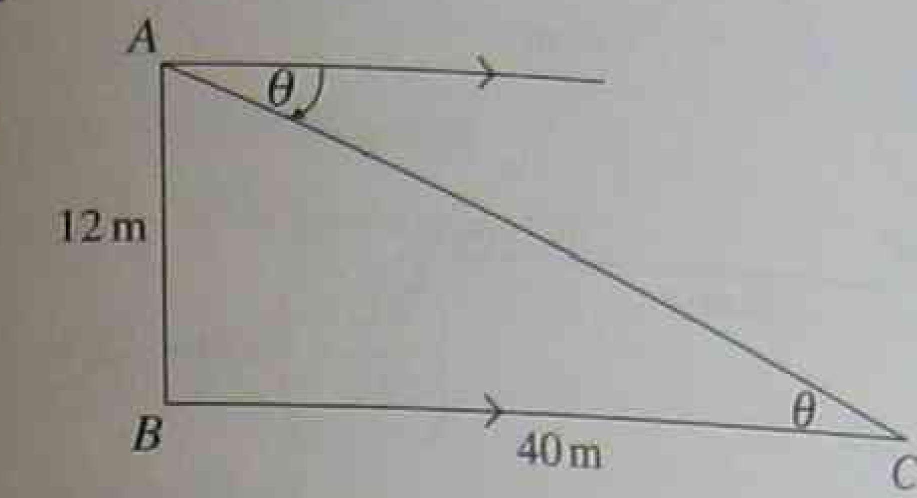
$$\tan 25° = \frac{h}{20}$$

$$h = 20 \tan 25°$$

$$= 9.33 \text{ m} \quad \#$$

Example 2: From the top of a tower 12 m high, a man observes an object at ground level located a distance of 40 m from the base of the tower. Find the angle of depression to the nearest minute.

Solution 2:



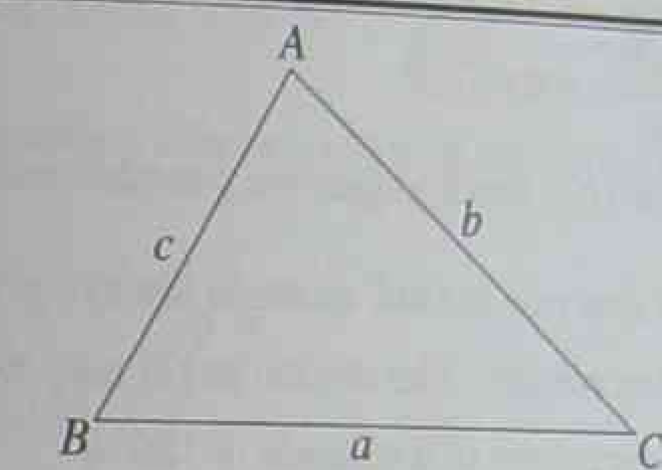
Let angle of depression $= \theta$

$\angle ACB = \theta$ (alternate $\angle$'s equal)

$\therefore \tan\theta = \frac{12}{40} = 0.3$

$\therefore \theta = 16°42'$ to the nearest minute #

(K) The Sine and Cosine Rule



(i) Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b and c are the sides opposite the angles A, B, and C.

The Sine rule is used to:

(1) find the length of a side where 2 angles and 1 side are given.

(2) find the size of an angle where 2 sides and 1 angle opposite one of the 2 sides is given.

(ii) Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

where b, c are two given sides, A is the included angle and a is the remaining side.

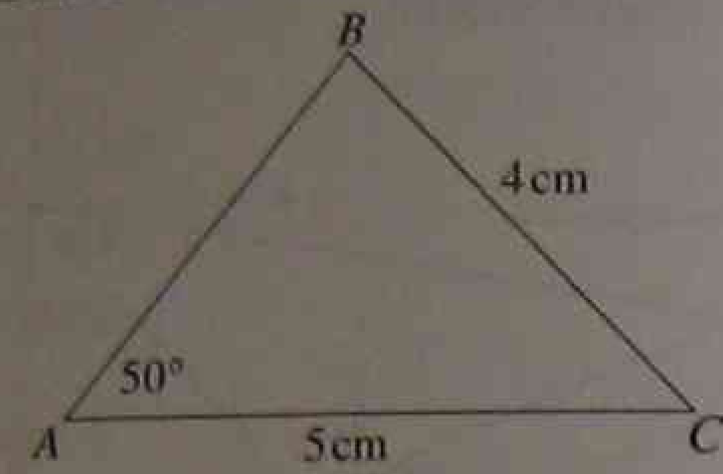
The Cosine rule is used to:

(1) find the length of a side where 2 sides and the included angle is given.

(2) find the size of an angle where all 3 sides and are given.

(i) Sine Rule

Example 1: Find the size of $\angle ABC$ to the nearest degree.



Solution 1:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

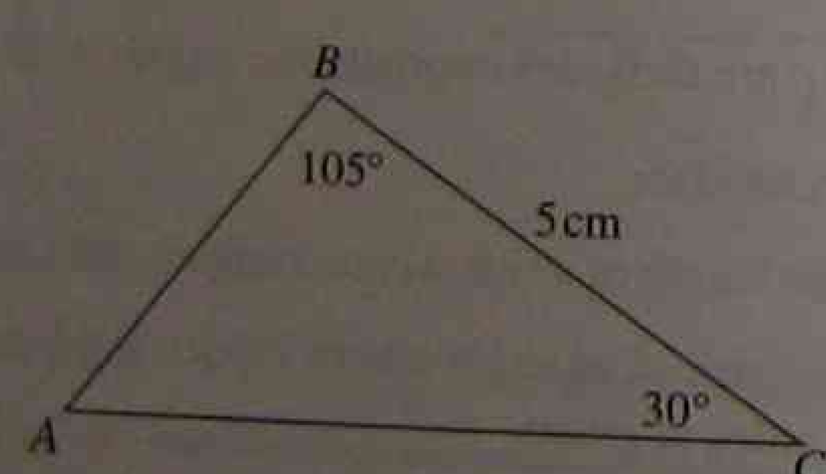
i.e. $$\frac{4}{\sin 50^\circ} = \frac{5}{\sin B}$$

$\therefore$ $$\sin B = \frac{5\sin 50^\circ}{4} = 0.957...$$

$B = 73^\circ$ or 107° (both angles are possible solutions) #

[**Note**: Using the sine rule to find an angle always gives two answers, an acute and an obtuse. The obtuse angle may not always be a solution since the angle sum of a triangle is 180°.]

Example 2: In the triangle below, use the sine rule to find the length of side AB. Express your answer in exact form.



Solution 2:

$$\angle BAC = 180^\circ - 105^\circ - 30^\circ = 45^\circ$$

$\therefore$ $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AB}{\sin 30^\circ} = \frac{5}{\sin 45^\circ}$$

$$AB = \frac{5\sin 30^\circ}{\sin 45^\circ} = \frac{5\times\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{5}{2}\times\frac{\sqrt{2}}{1} = \frac{5\sqrt{2}}{2} \text{ cm } \#$$

(ii) Cosine Rule

Example 1: In ΔABC, $BC = 4$cm, $AC = 6$cm and $\angle ACB = 150^\circ$. Find the exact length of AB.

Solution 1:

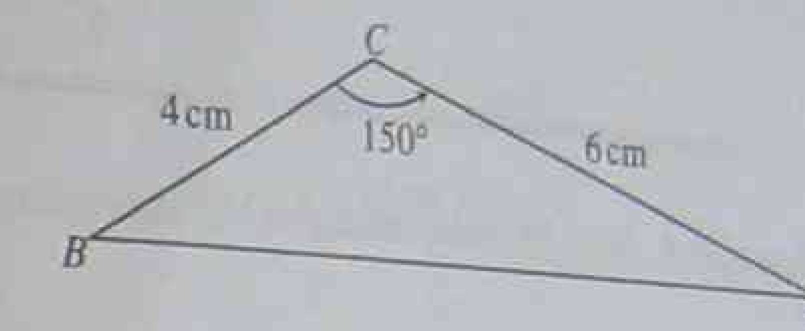
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$AB^2 = 4^2 + 6^2 - 2\times4\times6\cos150^\circ$$

$$= 52 - 48\times\left(-\frac{\sqrt{3}}{2}\right)$$

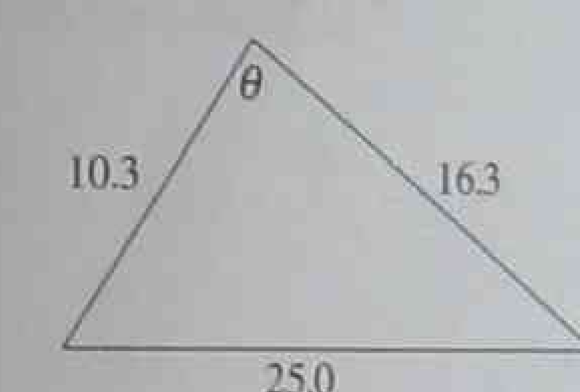
$$= 52 + 24\sqrt{3}$$

$$= 4\left(13 + 6\sqrt{3}\right)$$



$$\therefore AB = \sqrt{4(13+6\sqrt{3})} = 2\sqrt{13+6\sqrt{3}} \text{ cm } \#$$

Example 2: The following figure shows a section of a roof. Find θ to the nearest degree.



Solution 2:

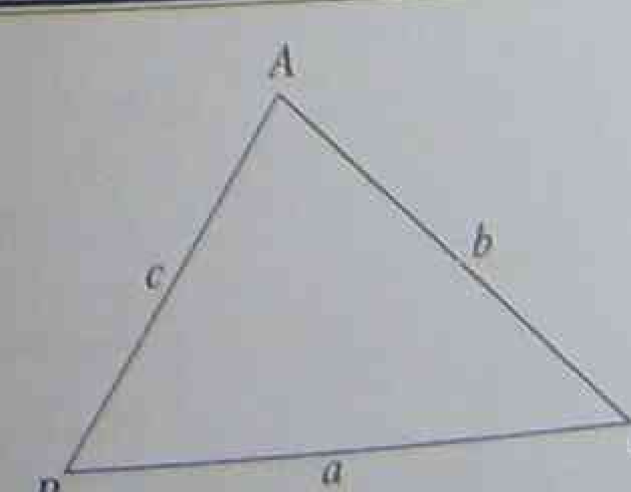
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

i.e. $$\cos\theta = \frac{(10.3)^2 + (16.3)^2 - (25)^2}{2\times10.3\times16.3}$$

$$= -0.754124724$$

$\therefore$ $\theta = 139^\circ$ to the nearest degree #

(L) The Area of a Triangle



The area of any triangle can be found using the formula:

$$A = \frac{1}{2}ab\sin C$$

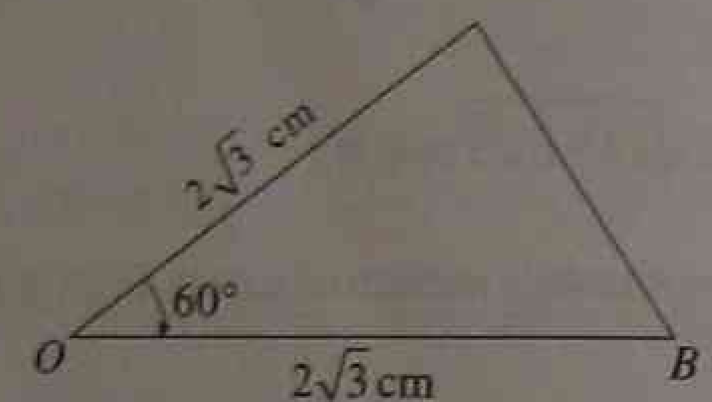
where a, b are two sides and C is the included angle.

Similarly, the area of the above triangle may be expressed as:

$$A = \frac{1}{2}ac\sin B$$

$$A = \frac{1}{2}bc\sin A$$

Example 1: Find the area of triangle *OAB* expressing your answer in exact form.



Solution 1:

$$A = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}\times 2\sqrt{3}\times 2\sqrt{3}\times\sin 60^\circ$$

$$= 6\times\left(\frac{\sqrt{3}}{2}\right)$$

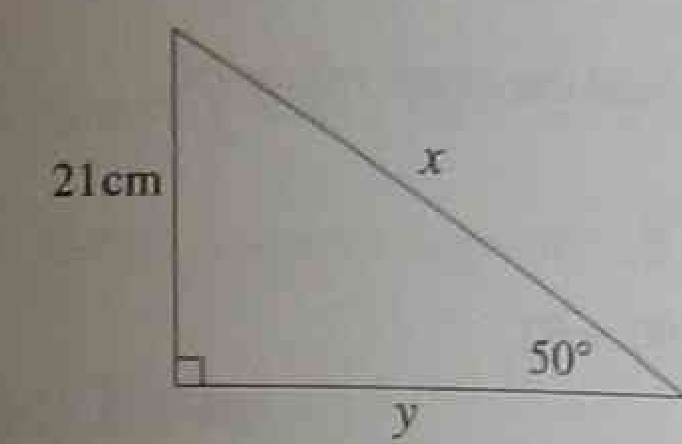
$$= 3\sqrt{3}\text{ cm}^2 \ \#$$

REVIEW EXERCISES

(A) Basic Ratios

1. Find the values of x and y. Give your answer correct to one decimal place.

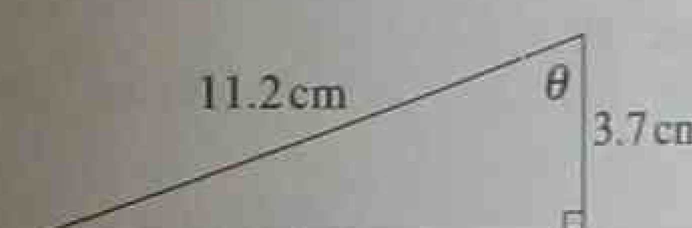
(i)



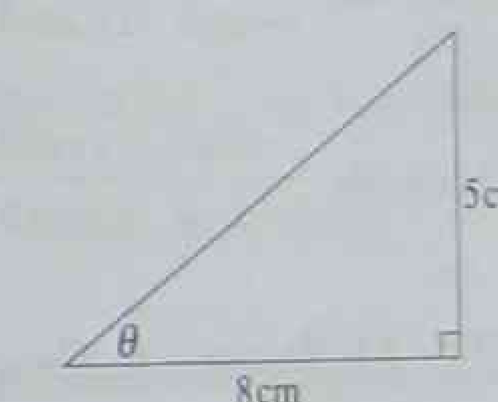
(ii)

2. Find the value of θ to the nearest degree.

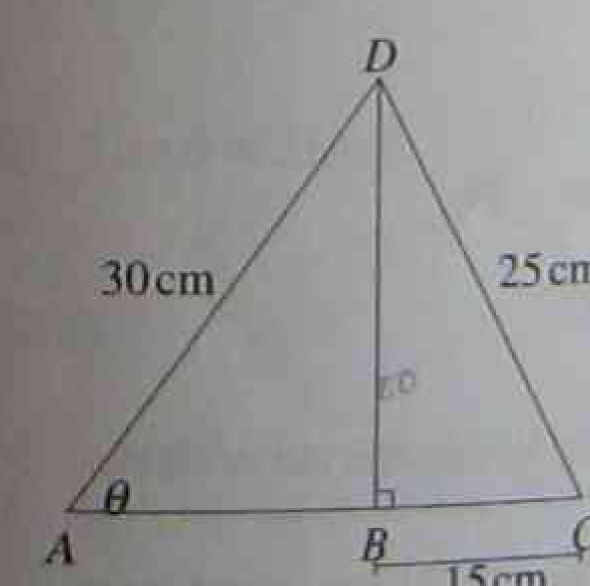
(i)



(ii)

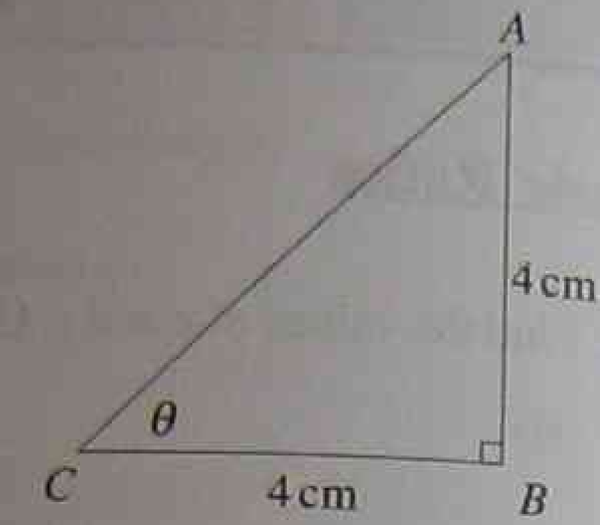


(iii)



(B) Reciprocal and Complimentary Ratios

3. Consider the triangle ABC.



(i) Show that $AC = 4\sqrt{2}$ cm.

(ii) Hence, find the exact values of:

(a) $\sec\theta$

(b) $\operatorname{cosec}(90^\circ - \theta)$

4. $\cos\theta = \dfrac{2}{3}$ and $\sin\theta = \dfrac{\sqrt{5}}{3}$ where θ is acute. Find the exact values of:

(i) $\sin(90^\circ - \theta)$ (ii) $\sec^2\theta$

(iii) $\cot\theta$ (iv) $\cot(90^\circ - \theta)$

(C) Pythagorean Identities

5. Show that $\sqrt{\dfrac{1}{\tan^2\theta} - \dfrac{1}{\sec^2\theta}} = \cos\theta\cot\theta$.

6. Show that $\dfrac{2\cos\theta}{1-\cos\theta} - \dfrac{2\cos\theta}{1+\cos\theta} = 4\cot^2\theta$.

7. Show that $\dfrac{1-\cos\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{1-\sin\theta} = 2(\tan\theta + \sec^2\theta)$.

(D) ASTC- All Stations to Central

8. If $\sin\theta = -0.725$, find the values of:

(i) $\sin(180^\circ - \theta)$ (ii) $\sin(360^\circ - \theta)$

(iii) $\sin(-\theta)$ (iv) $\sin(180^\circ + \theta)$

9. If $\tan\theta = \dfrac{2}{7}$, find the values:

(i) $\tan(360^\circ - \theta)$ (ii) $\tan(180^\circ - \theta)$

(iii) $\tan(90^\circ - \theta)$ (iv) $\tan(-\theta)$

10. If $\tan A = 2.245$ and $0^\circ \le A \le 360^\circ$, find A correct to the nearest minute.

11. If $\cos\theta = -0.159$ and $0^\circ \le \theta \le 360^\circ$, find θ correct to the nearest degree.

12. Let $\sin\theta = 0.375$:

(i) Find the values of θ to the nearest degree.

(ii) Hence, find the values of:

(a) $\sin(202^\circ)$ (b) $\cos(68^\circ)$

(c) $\sec(248^\circ)$ (d) $\operatorname{cosec}(338^\circ)$

(E) Graphs of sin$x$, cos$x$ and tan$x$

13. Graph $y = \cos x$ for $0^\circ \le x \le 360^\circ$.

14. Graph $y = \tan x$ for $-90^\circ \le x \le 270^\circ$.

(F) Exact Ratios

15. If $\cos x = \dfrac{3}{5}$ for $0^\circ \le x \le 90^\circ$, find the exact values of:

(i) $\sin x$ (ii) $\tan x$ (iii) $\cot x$

(G) Relations between Trigonometric Ratios

16. For acute angles A, B it is given that $\sin A = \dfrac{7}{9}$ and $\cos B = \dfrac{1}{5}$, find the exact values of:

(i) $\cos A$ (ii) $\sin B$ (iii) $\sec A + \tan B$

17. If $\tan\theta = \dfrac{15}{8}$, find the exact values of $\sec\theta$ and $\cot\theta$ if $\sin\theta < 0$.

(H) Solving Simple Trigonometric Equations

18. Solve the following equations for $0^\circ \le \theta \le 360^\circ$:

(i) $2\cos\theta = \sqrt{3}$ (ii) $\sqrt{3}\sin\theta = -\cos\theta$

(iii) $3\sin\theta = 2\tan\theta$ (iv) $\operatorname{cosec}^2\theta = \dfrac{4}{3}$

(I) Bearings

19. A ship sails due south from A to B for a distance of 15 nautical miles then due west to C for 10 nautical miles. What is the bearing of A from C? Answer to the nearest degree.

20. The bearing of B from A is 262°T and B is 41 m from A. Find the distance in metres correct to 2 decimal places that B is south of A.

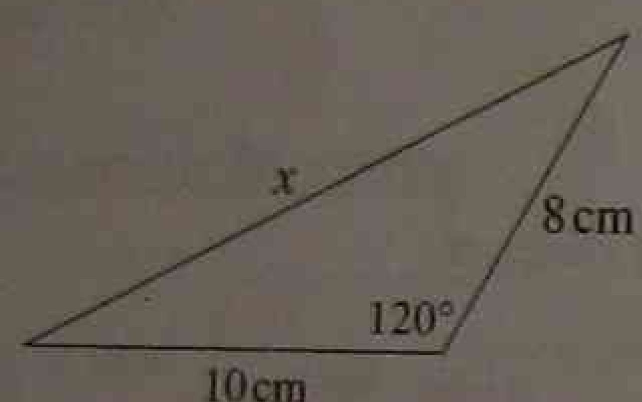
(J) Angles of Elevation and Depression

21. A tree 12.5 metres high casts a shadow 15.8 metres long. Find the angle of elevation of the sun correct to the nearest minute.

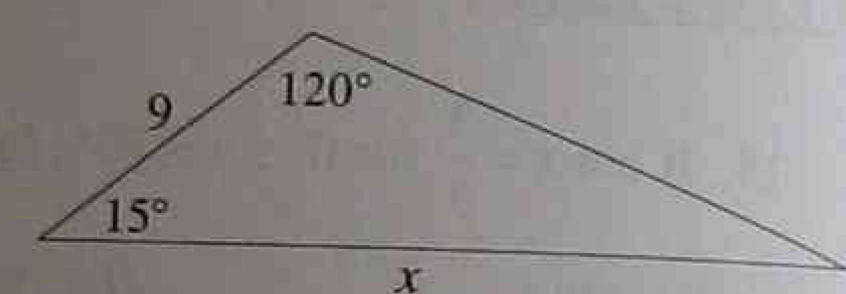
22. A person on top of a building 45 m in height wishes to find the height of an adjacent building. If the angle of depression is 48°22' and the buildings are a distance of 10 m apart, calculate the height of the adjacent building.

(K) The Sine and Cosine Rule

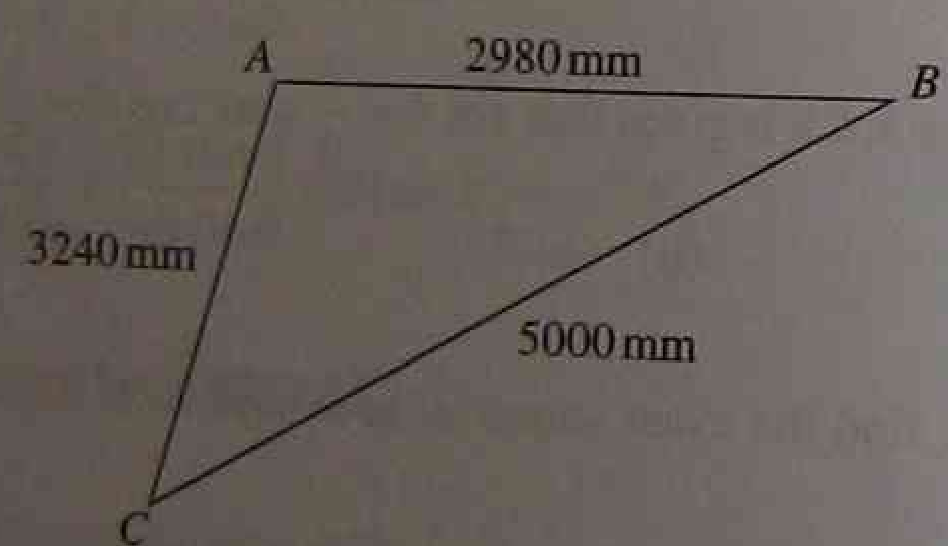
23. Find the value of x. Express your answer in exact form.



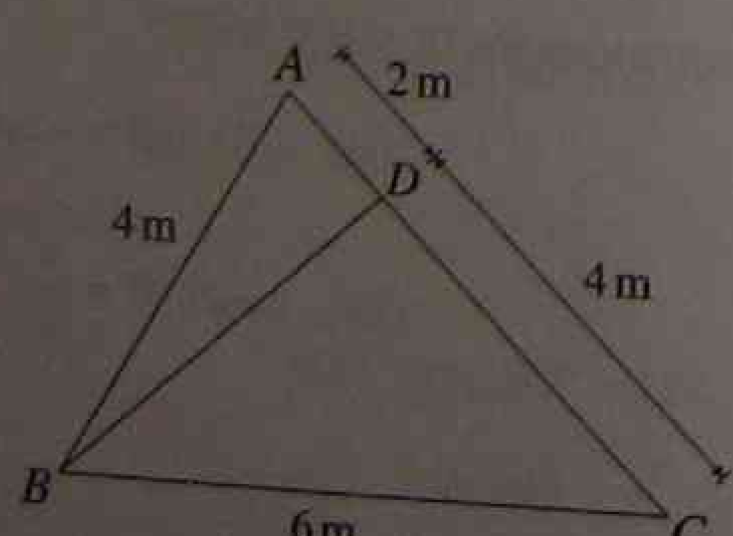
(i)



(ii)



24. Find the smallest angle of triangle ABC shown below:



25.

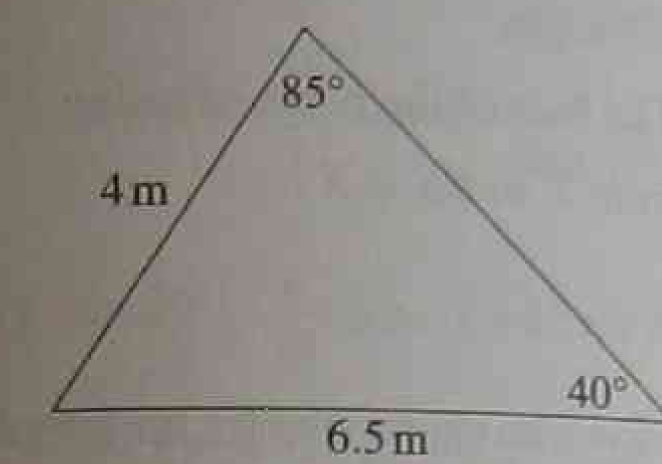
(i) Find the size of $\angle BAD$ to the nearest minute.

(ii) Hence, find the length of BD correct to 2 decimal places.

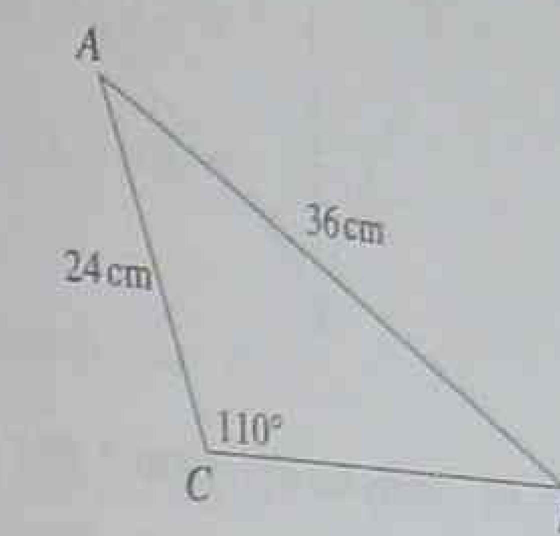
(L) The Area of a Triangle

26. Calculate the area of each triangle correct to 2 decimal places:

(i)



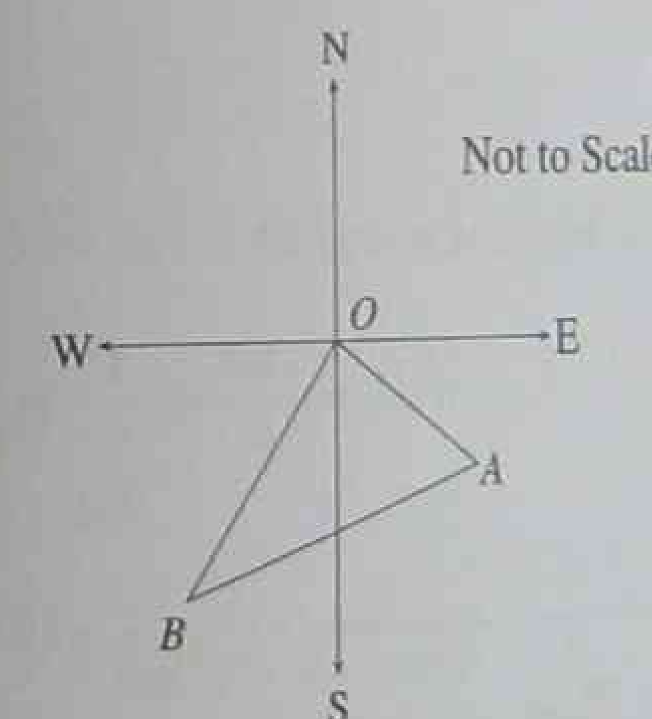
(ii)



(M) Miscellaneous

The following questions involve a number of the concepts covered in this topic. They are typical of the types of questions asked in exams and tend to be slightly more challenging.

27. The diagram below shows a ship sailing on a bearing of 142°T to port A, which is located 265 km away. At port A, the ship changes direction and travels at a bearing of 250°T to port B. Port B is at a bearing of 220° from O.



(i) Find the size of $\angle OBA$.

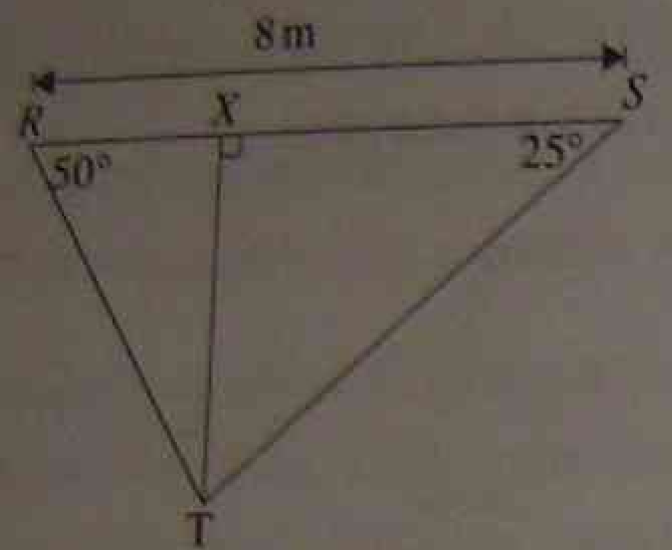
(ii) Find the distance of port B from port A expressing your answer correct to the nearest kilometre.

28. A person walking towards a building from a point A observes that the angle of elevation to the top of the building is 18°. At a point B, 50 metres towards the building from A, the angle of elevation is 25°. Find:

(i) The distance from A to the top of the building.

(ii) The height of the building correct to 2 decimal places.

29.



In the figure (not to scale) $\angle XRT = 50°$, $\angle XST = 25°$ and $RS = 8\,\text{m}$.
The foot of the perpendicular from T to RS is X.

(i) Apply the sine rule to ΔRST to write down an expression for the length RT.

(ii) Hence, find the length of TX correct to 1 decimal place.

30. Mary-Jane leaves her base camp, O, and walks in the direction 150°T until she reaches point A on the bank of the river. Unable to cross the river, she follows it on a bearing of 030°T and retires at point C. The bearing and distance of C from O are 120°T and 4 km respectively. Find the total distance walked by Mary-Jane. Express your answer in exact form.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $\sin 50° = \dfrac{21}{x}$ i.e. $x = \dfrac{21}{\sin 50°} = 27.4\,\text{cm}$ to 1 d.p. #

$\tan 50° = \dfrac{21}{y}$ i.e. $y = \dfrac{21}{\tan 50°} = 17.6\,\text{cm}$ to 1 d.p. #

(ii) $\sin 20° = \dfrac{x}{15}$ i.e. $x = 15\sin 20° = 5.1\,\text{cm}$ to 1 d.p. #

$\cos 20° = \dfrac{y}{15}$ i.e. $y = 15\cos 20° = 14.1\,\text{cm}$ to 1 d.p. #

2. (i) $\cos\theta = \dfrac{3.7}{11.2}$ i.e. $\theta = 71°$ to the nearest degree #

(ii) $\tan\theta = \dfrac{5}{8}$ i.e. $\theta = 32°$ to the nearest degree #

(iii) $BD = \sqrt{25^2 - 15^2} = 20\,\text{cm}$ (using pythagoras' theorem in ΔDBC)

$\therefore \sin\theta = \dfrac{20}{30} = \dfrac{2}{3}$ i.e. $\theta = 42°$ to the nearest degree #

3. (i) $AC^2 = 4^2 + 4^2$

$= 32$

$\therefore AC = \sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$ cm as required #

(ii) (a) $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{\frac{4}{4\sqrt{2}}} = \dfrac{4\sqrt{2}}{4} = \sqrt{2}$ #

(b) $\operatorname{cosec}(90° - \theta) = \dfrac{1}{\sin(90° - \theta)} = \dfrac{1}{\cos\theta} = \dfrac{1}{\frac{4}{4\sqrt{2}}} = \sqrt{2}$ #

4. (i) $\sin(90° - \theta) = \cos\theta = \dfrac{2}{3}$ #

(ii) $\sec^2\theta = \dfrac{1}{\cos^2\theta} = \dfrac{1}{\left(\frac{2}{3}\right)^2} = \dfrac{1}{\frac{4}{9}} = \dfrac{9}{4}$ #

(iii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{2}{3}}{\frac{3}{\sqrt{5}}} = \dfrac{2}{3} \times \dfrac{3}{\sqrt{5}} = \dfrac{2}{\sqrt{5}}$ #

(iv) $\cot(90^\circ-\theta)=\tan\theta=\dfrac{1}{\cot\theta}=\dfrac{1}{\frac{2}{\sqrt{5}}}=\dfrac{\sqrt{5}}{2}$ #

5. $LHS=\sqrt{\dfrac{1}{\tan^2\theta}-\dfrac{1}{\sec^2\theta}}=\sqrt{\dfrac{\sec^2\theta-\tan^2\theta}{\tan^2\theta\sec^2\theta}}$

$=\sqrt{\dfrac{1}{\tan^2\theta\sec^2\theta}}$ (note: $\sec^2\theta-\tan^2\theta=1$)

$=\dfrac{1}{\tan\theta\sec\theta}$

$=\dfrac{1}{\tan\theta}\cdot\dfrac{1}{\sec\theta}$

$=\cos\theta\cot\theta=RHS$ #

6. $LHS=\dfrac{2\cos\theta}{1-\cos\theta}-\dfrac{2\cos\theta}{1+\cos\theta}$

$=\dfrac{2\cos\theta(1+\cos\theta)-2\cos\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$=\dfrac{2\cos\theta+2\cos^2\theta-2\cos\theta+2\cos^2\theta}{1-\cos^2\theta}$

$=\dfrac{4\cos^2\theta}{\sin^2\theta}$ (note: $1-\cos^2\theta=\sin^2\theta$)

$=4\cot^2\theta=RHS$ #

7. $LHS=\dfrac{1-\cos\theta}{1+\sin\theta}+\dfrac{1+\cos\theta}{1-\sin\theta}$

$=\dfrac{(1-\sin\theta)(1-\cos\theta)+(1+\sin\theta)(1+\cos\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$=\dfrac{(1-\cos\theta-\sin\theta+\sin\theta\cos\theta)+(1+\cos\theta+\sin\theta+\sin\theta\cos\theta)}{(1-\sin^2\theta)}$

$=\dfrac{2+2\sin\theta\cos\theta}{\cos^2\theta}$ (note: $1-\sin^2\theta=\cos^2\theta$)

$=\dfrac{2}{\cos^2\theta}+\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}$

$=2\sec^2\theta+2\tan\theta=2(\tan\theta+\sec^2\theta)=RHS$ #

8. (i) $\sin(180^\circ-\theta)=\sin\theta=-0.725$ #

(ii) $\sin(360^\circ-\theta)=-\sin\theta=0.725$ #

(iii) $\sin(-\theta)=\sin(360^\circ-\theta)=0.725$ #

(iv) $\sin(180^\circ+\theta)=-\sin\theta=0.725$ #

9. (i) $\tan(360^\circ-\theta)=-\tan\theta=-\dfrac{2}{7}$ #

(ii) $\tan(180^\circ-\theta)=-\tan\theta=-\dfrac{2}{7}$ #

(iii) $\tan(90^\circ-\theta)=\cot\theta=\dfrac{1}{\tan\theta}=\dfrac{1}{\frac{2}{7}}=\dfrac{7}{2}$ #

(iv) $\tan(-\theta)=\tan(360^\circ-\theta)=-\dfrac{2}{7}$ #

10. $\tan A=2.245$

$A=66^\circ$ to the nearest degree

noting that tan is positive in the 1st and 3rd quadrants

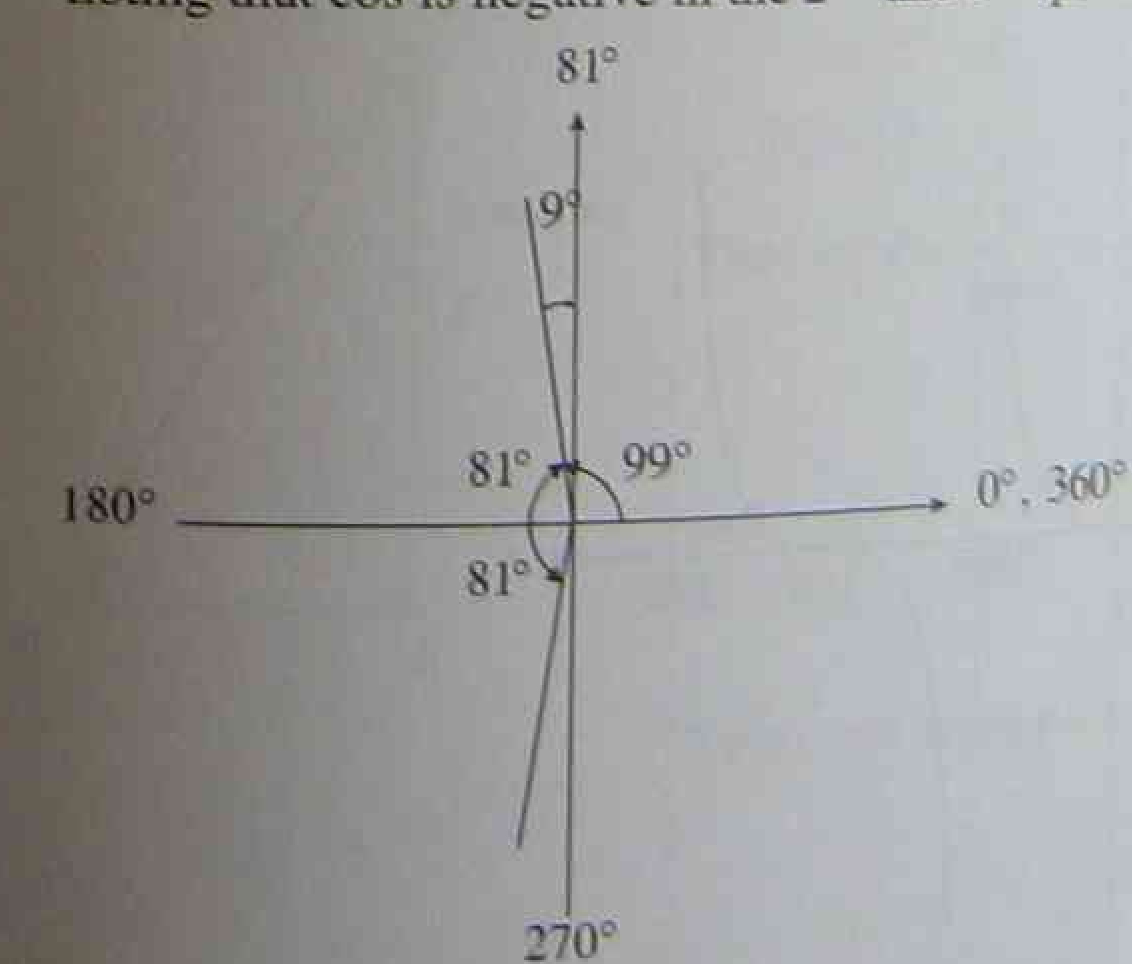
$\therefore\ A=66^\circ,\ 180^\circ+66^\circ$

$=66^\circ,\ 246^\circ$ #

11. $\cos\theta=-0.159$

$\theta=99^\circ$ to the nearest degree

noting that cos is negative in the 2nd and 3rd quadrants



note that the 'criss-cross' angle is 81°

$\therefore \theta = 99^\circ, (180^\circ + 81^\circ)$
$= 99^\circ, 261^\circ$ #

12. (i) $\sin\theta = 0.375$

$\theta = 22^\circ$ to the nearest degree

noting that sin is positive in the 1st and 2nd quadrants

$\therefore \theta = 22^\circ, 180^\circ - 22^\circ$
$= 22^\circ, 158^\circ$ #

(ii) (a) $\sin(202^\circ) = \sin(180^\circ + 22^\circ) = -\sin 22^\circ = -0.375$ #

(b) $\cos(68^\circ) = \cos(90^\circ - 22^\circ) = \sin 22^\circ = 0.375$ #

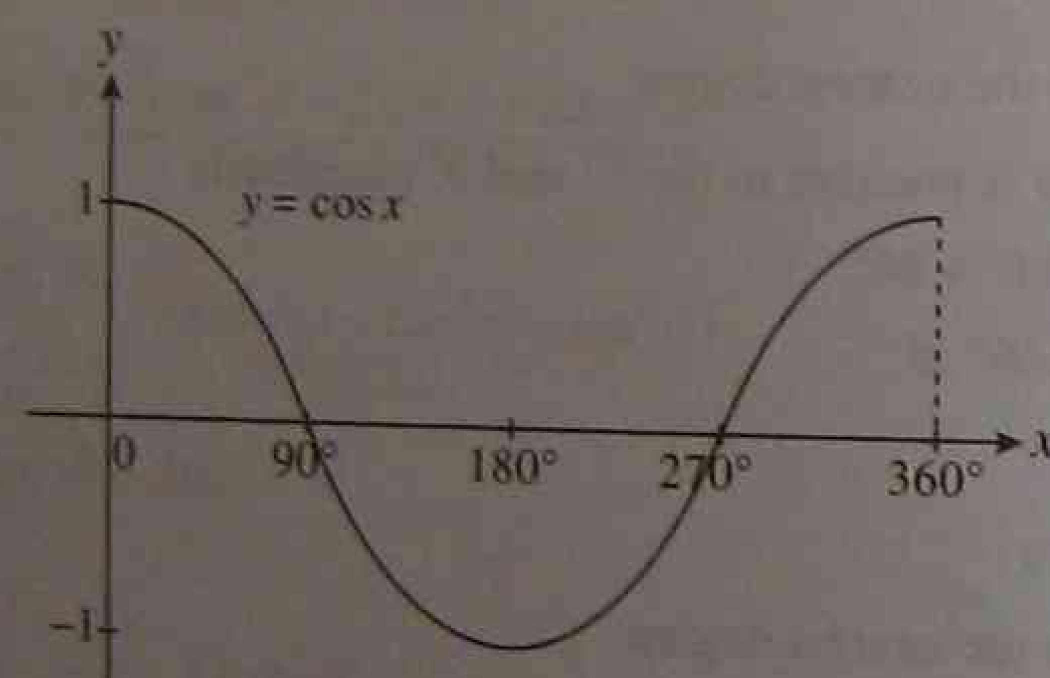
(c) $\sec(248^\circ) = \dfrac{1}{\cos(248^\circ)} = \dfrac{1}{\cos(180^\circ + 68^\circ)} = \dfrac{1}{\cos 68^\circ}$

$= -\dfrac{1}{0.375} = -\dfrac{1000}{375} = -\dfrac{8}{3}$ #

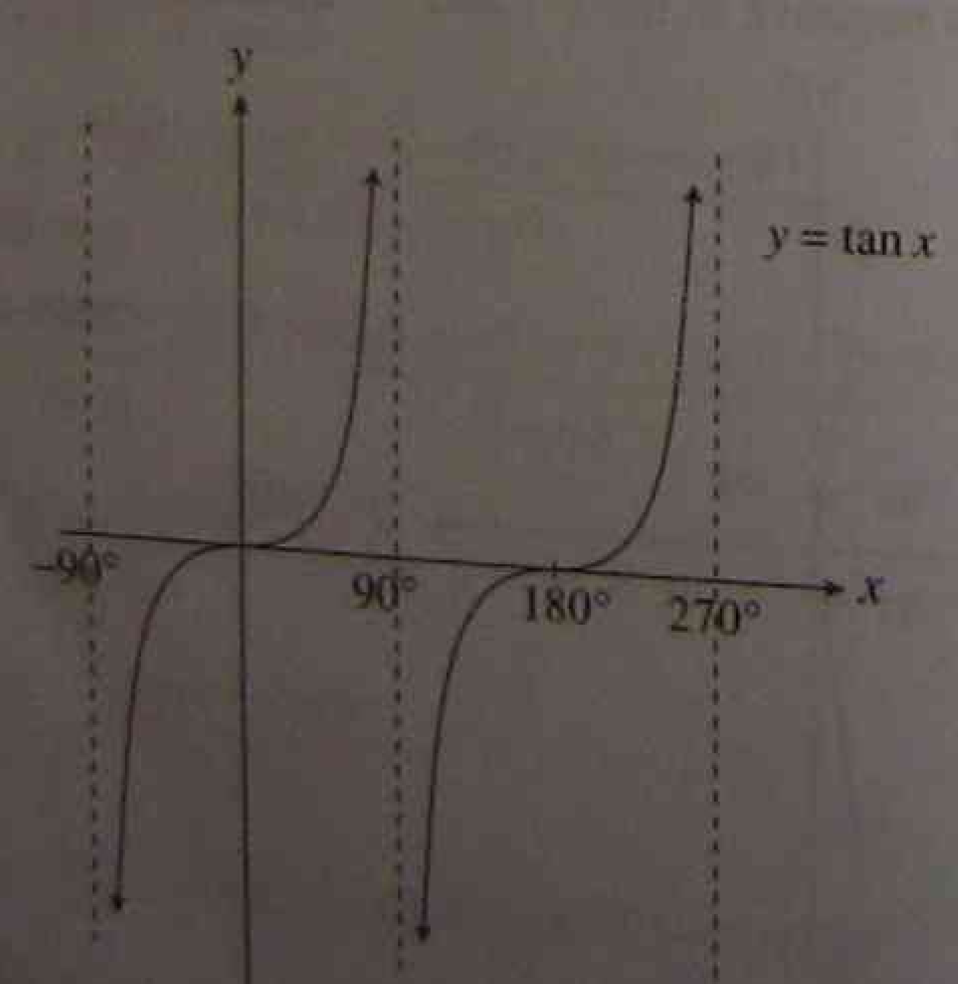
(d) $\operatorname{cosec}(338^\circ) = \dfrac{1}{\sin(338^\circ)} = \dfrac{1}{\sin(360^\circ - 22^\circ)} = \dfrac{1}{-\sin 22^\circ} = \dfrac{-1}{0.375} = \dfrac{-8}{3}$ #

Trigonometric Ratios

13.

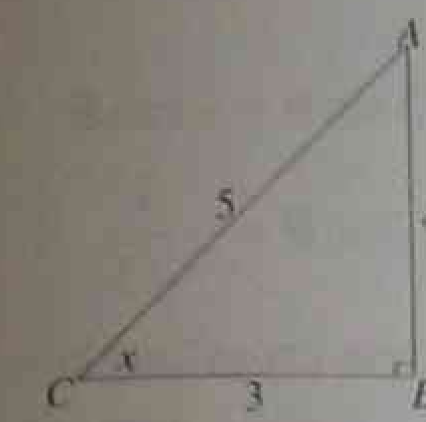


14.



15.

15.



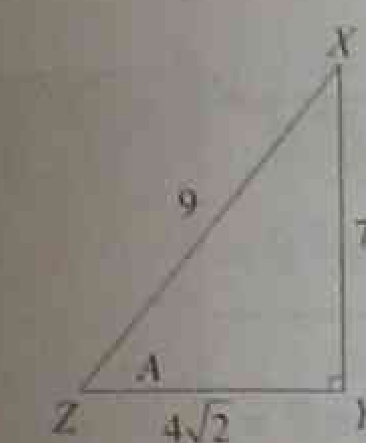
$AB + 3^2 = 5^2$

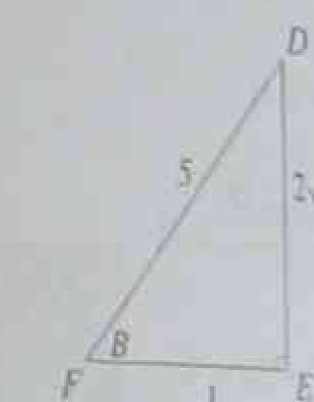
i.e. $AB^2 = 25 - 9 = 16$

$AB = 4$

(i) $\sin x = \dfrac{4}{5}$ (ii) $\tan x = \dfrac{4}{3}$ (iii) $\cot x = \dfrac{1}{\tan x} = \dfrac{1}{\frac{4}{3}} = \dfrac{3}{4}$ #

16.





$XZ^2 = XY^2 + YZ^2$

$9^2 = 7^2 + YZ^2$

$\therefore YZ = \sqrt{32} = 4\sqrt{2}$

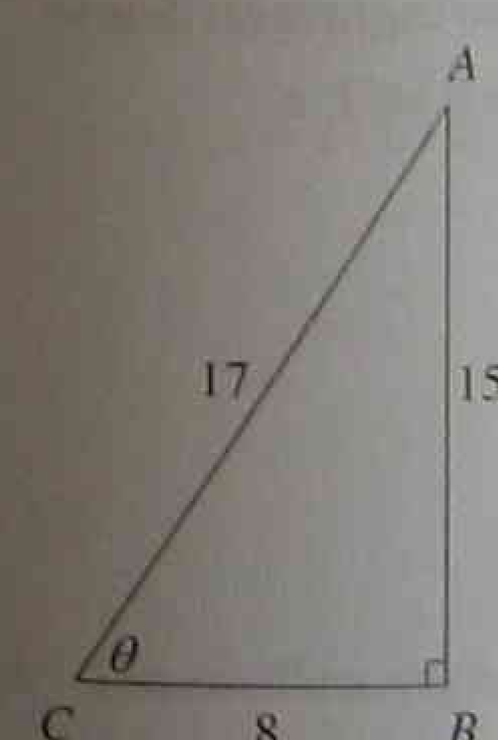
$DF^2 = DE^2 + EF^2$

$5^2 = DE^2 + 1^2$

$\therefore DE = \sqrt{24} = 2\sqrt{6}$

(i) $\cos A = \dfrac{4\sqrt{2}}{9}$ (ii) $\sin B = \dfrac{2\sqrt{6}}{5}$ (iii) $\sec A + \tan B = \dfrac{1}{\cos A} + \tan B = \dfrac{9}{4\sqrt{2}} + 2\sqrt{}$

17. Since tan is positive and sin negative $\therefore \theta$ is in the 3rd quadrant.



$AC^2 = AB^2 + BC^2$

$= 15^2 + 8^2$

$\therefore AC = 17$

Thus $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{\frac{-8}{17}} = -\dfrac{17}{8}$

and $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\frac{15}{8}} = \dfrac{8}{15}$ #

Note: cos is negative as θ is in the 3rd quadrant.

18. (i) $2\cos\theta = \sqrt{3}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = 30°, (360° - 30°)$

$= 30°, 330°$ #

(ii) $\sqrt{3}\sin\theta = -\cos\theta$

$\tan\theta = -\frac{1}{\sqrt{3}}$

'criss-cross' angle $= 30°$

$\theta = (180° - 30°), (360° - 30°)$

$= 150°, 330°$ #

(iii) $3\sin\theta = 2\tan\theta$

$3\sin\theta - 2\tan\theta = 0$

$3\sin\theta - \frac{2\sin\theta}{\cos\theta} = 0$

$\frac{3\sin\theta\cos\theta - 2\sin\theta}{\cos\theta} = 0$

i.e. $\sin\theta(3\cos\theta - 2) = 0$

i.e. $\sin\theta = 0 \quad \therefore \theta = 0°, 180°, 360°$

or $\cos\theta = \frac{2}{3} \quad \therefore \theta = 48°11', (360° - 48°11')$

$= 48°11', 311°49'$

$\therefore$ complete solutions for θ are

$\theta = 0°, 48°11', 180°, 311°49', 360°$ #

(iv) $\text{cosec}^2 = \frac{4}{3}$

$\frac{1}{\sin^2\theta} = \frac{4}{3}$

$\sin^2\theta = \frac{3}{4}$

$\sin\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

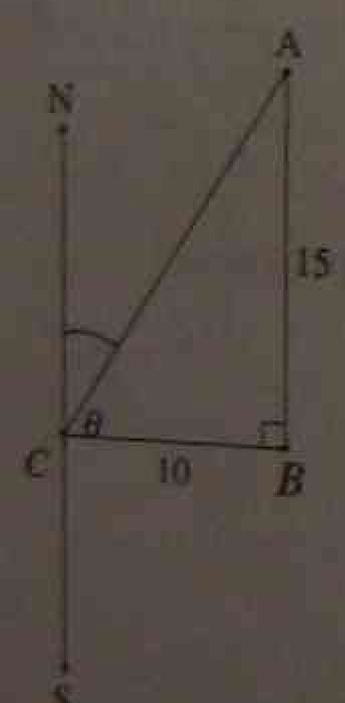
i.e. $\sin\theta = \frac{\sqrt{3}}{2} \quad \therefore \theta = 60°, 120°$

or $\sin\theta = -\frac{\sqrt{3}}{2} \quad \therefore \theta = 240°, 300°$

$\therefore$ complete solutions for θ are

$\theta = 60°, 120°, 240°, 300°$ #

19.



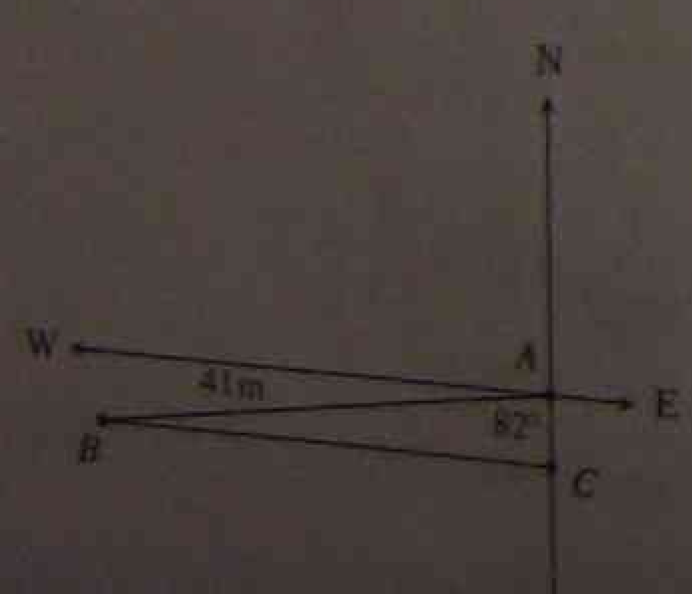
$\tan\theta = \frac{15}{10} = \frac{3}{2}$ i.e. $\theta = 56.31°$

bearing is $\angle NCA = 90 - \theta$

$= 34°$ to the nearest degree

$\therefore$ bearing of A from C is 034°T #

20.

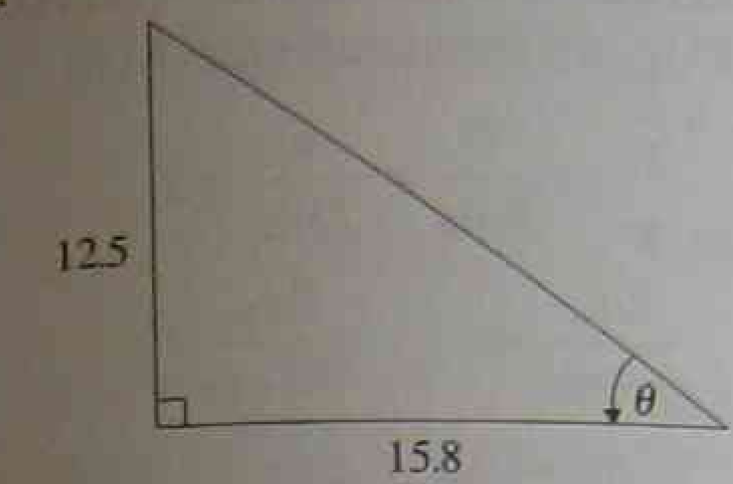


$\cos 82° = \frac{AC}{41}$

$\therefore AC = 41\cos 82° = 57.06$ correct to 2 d.p.

$\therefore B$ is located 5.71 m south of A #

21.

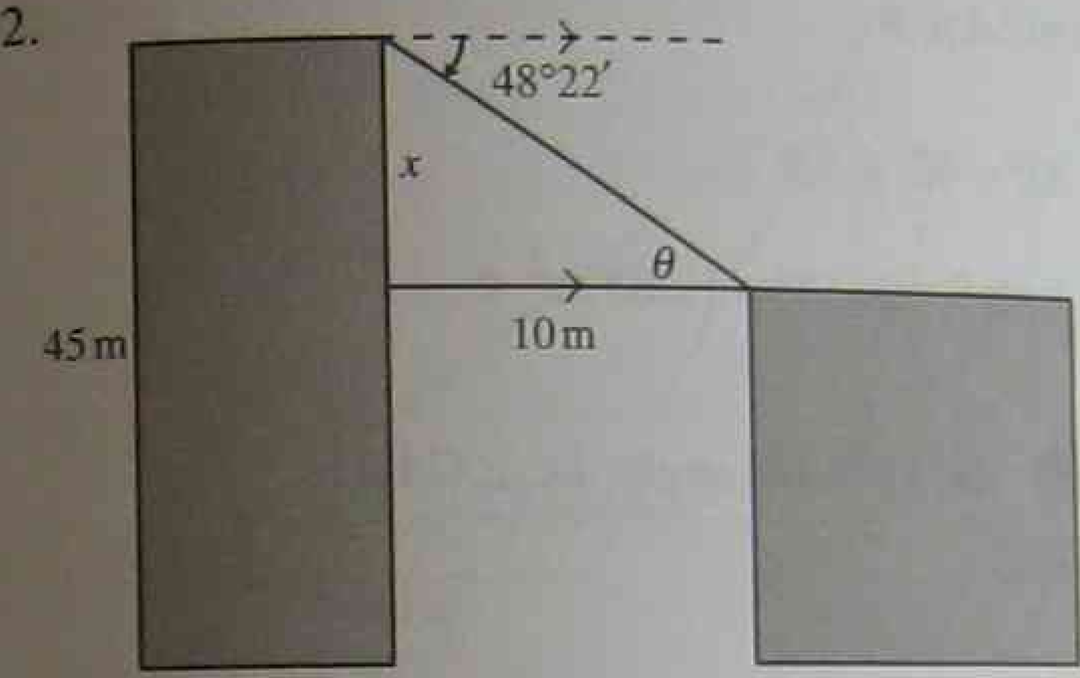


$\tan\theta = \frac{12.5}{15.8}$

$\therefore \theta = 38°21'$

$\therefore$ the angle of elevation of the sun is $38°21'$ #

22.



Looking at the diagram:

$\theta = 48°22'$ (alternate $\angle$'s equal between parallel lines)

$\tan\theta = \frac{x}{10}$

i.e. $x = 10\tan\theta = 10\tan(48°22') = 11.25\text{ m}$

$\therefore$ height of adjacent building $= 45 - 11.25 = 33.75\text{ m}$ #

23. (i) $x^2 = 10^2 + 8^2 - 2\times10\times8\times\cos120°$

$= 164 + 80$

$= 244$

i.e. $x = 2\sqrt{61}$ cm #

(ii) $\frac{x}{\sin120°} = \frac{9}{\sin45°}$

$x = \frac{9\sin120°}{\sin45°}$

$= \frac{9\times\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}}$

$= \frac{9\sqrt{3}}{2}\times\sqrt{2} = \frac{9\sqrt{6}}{2}$ #

24. The smallest angle is opposite the shortest side

i.e. $\angle ACB$

$\frac{b^2+c^2-a^2}{\ldots} = \frac{5000^2+3240^2-2980^2}{\ldots\times3240} = 0.8215....$

25. (i) To find $\angle BAD$ we use the cosine rule in ΔABC, note that $\angle BAC = \angle BAD$

$$\text{i.e. } \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{6^2+4^2-6^2}{2\times6\times4} = \frac{1}{3}$$

$$A = 70°32' \text{ to the nearest minute } \#$$

(ii) To find BD, we use the cosine rule in ΔABD

$$\text{i.e. } a^2 = b^2+c^2-2bc\cos A$$

$$BD^2 = 2^2+4^2-2\times2\times4\times\cos(70°32')$$

$$\therefore BD = 3.83\text{ m correct to 2 d.p. } \#$$

26. (i) Included angle $=180°-85°-40°=55°$

$$\text{Area} = \frac{1}{2}ab\sin c = \frac{1}{2}\times4\times6.5\times\sin55° = 10.65\text{ m}^2 \ \#$$

(ii) To find the area we require the included angle i.e. $\angle CAB$.

We first need to find $\angle ABC$:

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{36}{\sin110°} = \frac{24}{\sin B}$$

$$\text{i.e. } \sin B = \frac{24\times\sin110°}{36} = 0.6264...$$

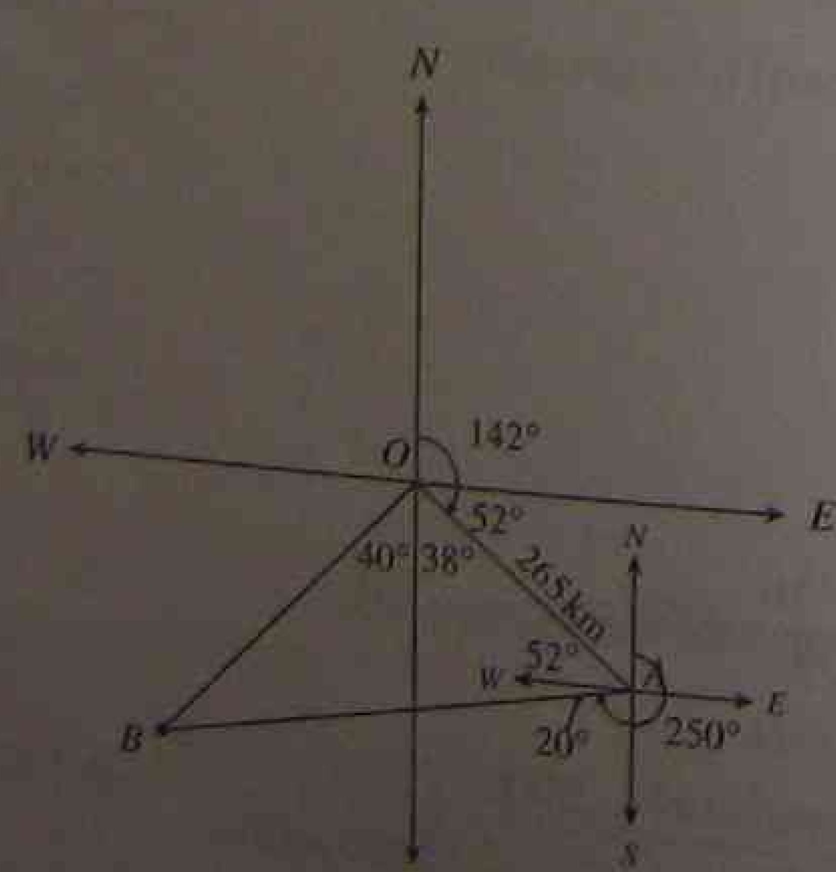
$$\therefore B = 38.789°...$$

$$\therefore \angle CAB = 180°-110°-b = 31.2104...$$

$$\text{Hence area} = \frac{1}{2}ab\sin C = \frac{1}{2}\times36\times24\times\sin(31.2104...)°$$

$$= 223.86\text{ cm}^2 \text{ correct to 2 d.p. } \#$$

27. (i)



From the diagram $\angle OBA = 180° - 78° - 72°$ ...

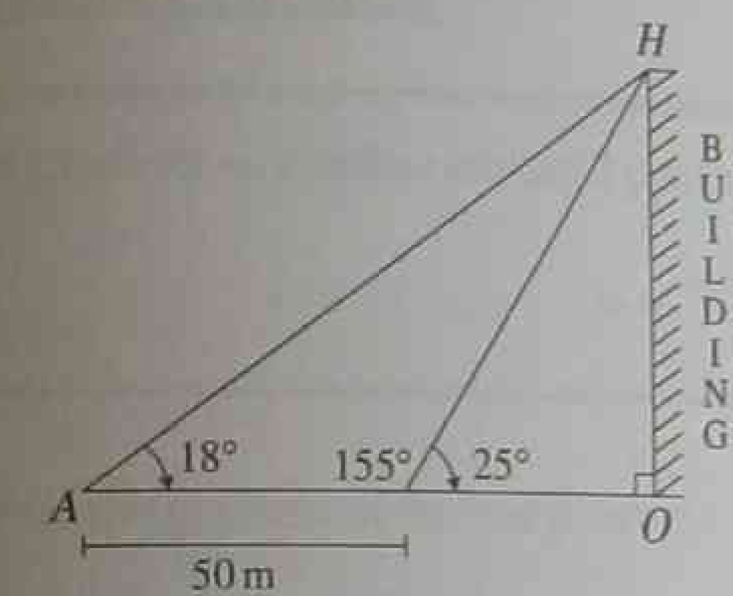
(ii) Using the sine rule in ΔOBA:

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AB}{\sin78°} = \frac{265}{\sin30°}$$

$$AB = \frac{265\times\sin78°}{\sin30°} = 518\text{ km to the nearest km } \#$$

28.



(i) $\angle AHB = 180°-155°-18° = 7°$ ($\angle$ sum of $\Delta = 180°$)

$\angle ABH = 180°-25° = 155°$ ($\angle$'s in a straight line)

Using the sine rule in ΔABH:

$$\text{i.e. } \frac{AH}{\sin155°} = \frac{50}{\sin7°}$$

$$AH = \frac{50\sin155°}{\sin7°} = 173.3898985$$

$$= 173.39\text{ m correct to 2 d.p. } \#$$

(ii) Looking at ΔAOH:

$$\sin18° = \frac{OH}{AH}$$

$$\text{i.e. } OH = 173.3898...\times\sin18°$$

$$= 53.580...$$

$$= 53.58\text{ m correct to 2 d.p. } \#$$

29. (i) $\angle RTS = 180°-50°-25° = 105°$ ($\angle$ sum of $\Delta = 180°$)

$$\text{now, } \frac{RT}{\sin25°} = \frac{8}{\sin105°}$$

$$\text{i.e. } RT = \frac{8\sin25°}{\sin105°} \ \#$$

25. (i) To find $\angle BAD$ we use the cosine rule in ΔABC, note that $\angle BAC = \angle BAD$

i.e. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{6^2+4^2-6^2}{2\times6\times4} = \dfrac{1}{3}$

$A = 70°32'$ to the nearest minute #

(ii) To find BD, we use the cosine rule in ΔABD

i.e. $a^2 = b^2+c^2-2bc\cos A$

$BD^2 = 2^2+4^2-2\times2\times4\times\cos(70°32')$

$\therefore BD = 3.83$ m correct to 2 d.p. #

26. (i) Included angle $= 180°-85°-40° = 55°$

Area $= \dfrac{1}{2}ab\sin c = \dfrac{1}{2}\times4\times6.5\times\sin 55° = 10.65\text{ m}^2$ #

(ii) To find the area we require the included angle i.e. $\angle CAB$.

We first need to find $\angle ABC$:

i.e. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{36}{\sin 110°} = \dfrac{24}{\sin B}$

i.e. $\sin B = \dfrac{24\times\sin 110°}{36} = 0.6264...$

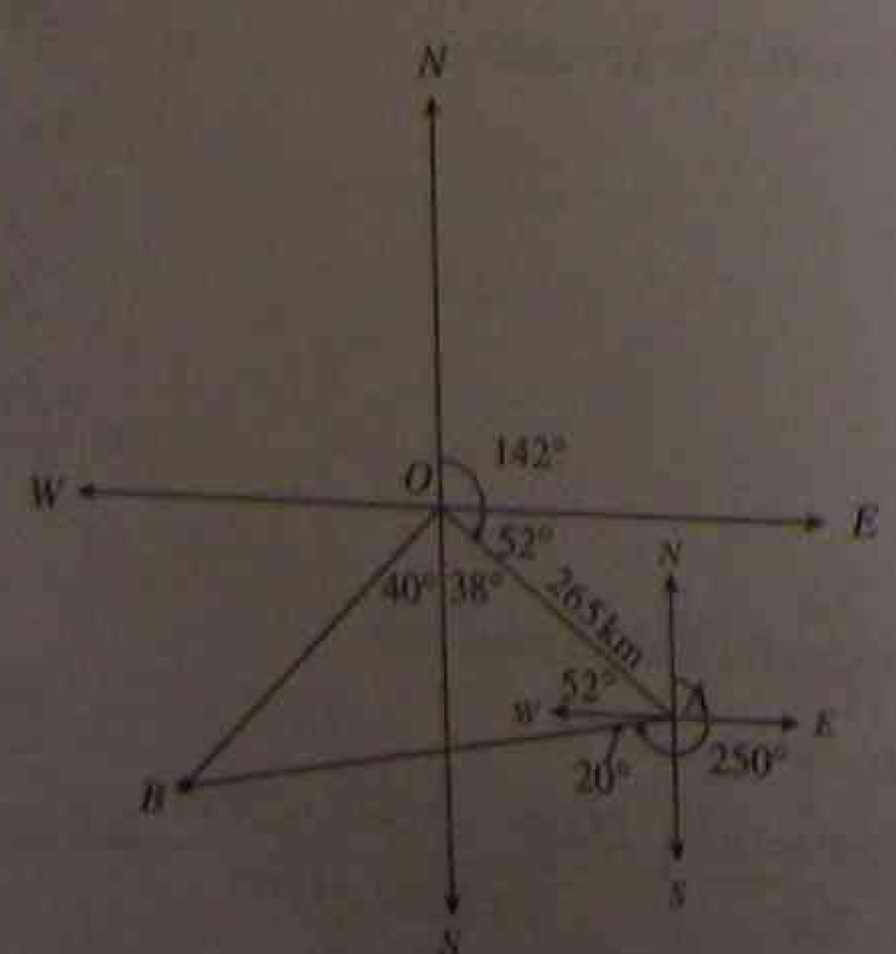
$\therefore B = 38.789°...$

$\therefore \angle CAB = 180°-110°-b = 31.2104...$

Hence area $= \dfrac{1}{2}ab\sin C = \dfrac{1}{2}\times36\times24\times\sin(31.2104...)°$

$= 223.86\text{ cm}^2$ correct to 2 d.p. #

27. (i)



From the diagram $\angle OBA = 180° ...$

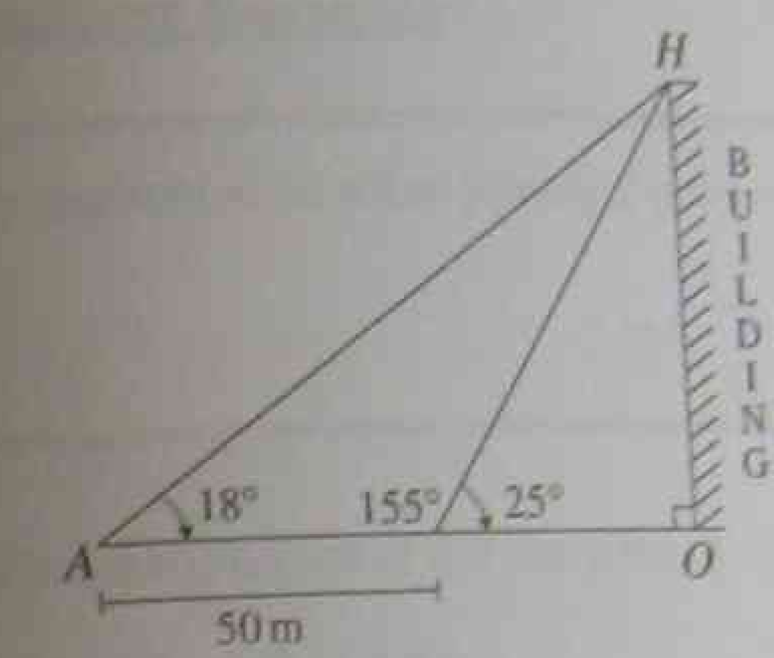
(ii) Using the sine rule in ΔOBA:

i.e. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{AB}{\sin 78°} = \dfrac{265}{\sin 30°}$

$AB = \dfrac{265\times\sin 78°}{\sin 30°} = 518$ km to the nearest km #

28.



(i) $\angle AHB = 180°-155°-18° = 7°$ ($\angle$ sum of $\Delta = 180°$)

$\angle ABH = 180°-25° = 155°$ ($\angle$'s in a straight line)

Using the sine rule in ΔABH:

i.e. $\dfrac{AH}{\sin 155°} = \dfrac{50}{\sin 7°}$

$AH = \dfrac{50\sin 155°}{\sin 7°} = 173.3898985$

$= 173.39$ m correct to 2 d.p. #

(ii) Looking at ΔAOH:

$\sin 18° = \dfrac{OH}{AH}$

i.e. $OH = 173.3898...\times\sin 18°$

$= 53.580...$

$= 53.58$ m correct to 2 d.p. #

29. (i) $\angle RTS = 180°-50°-25° = 105°$ ($\angle$ sum of $\Delta = 180°$)

now, $\dfrac{RT}{\sin 25°} = \dfrac{8}{\sin 105°}$

i.e. $RT = \dfrac{8\sin 25°}{\sin 105°}$ #

(ii) Looking at ΔRXT :

$$\sin 50^\circ = \frac{TX}{RT}$$

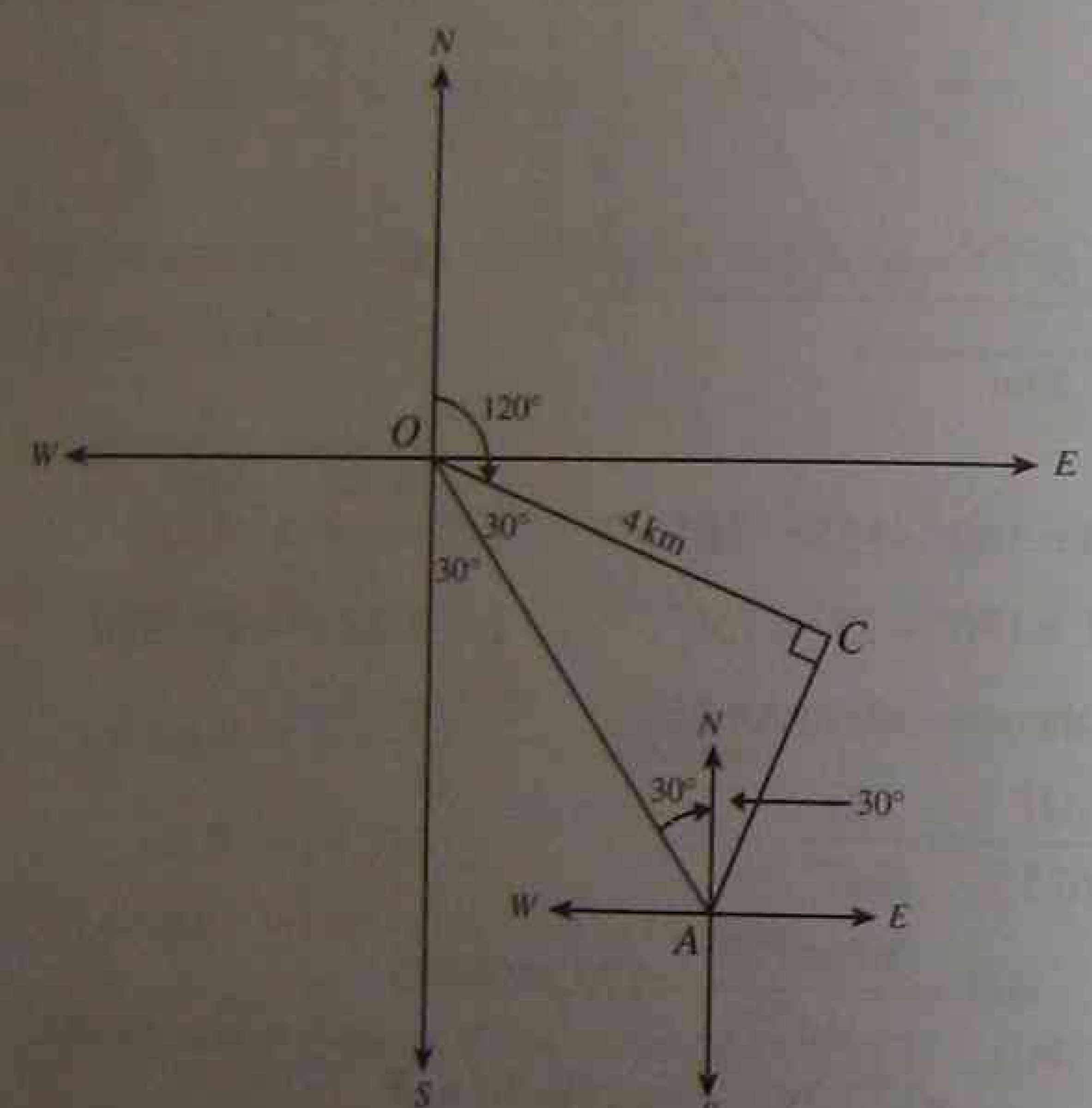
$$\therefore\ TX = RT\sin 50^\circ$$

$$= \frac{8\sin 25^\circ}{\sin 105^\circ} \times \sin 50^\circ$$

$$= 2.6813...$$

$= 2.7$ m correct to 1 d.p. #

30.



$\angle OCB = 180^\circ - 30^\circ - 60^\circ = 90^\circ$

now, $\tan 30^\circ = \frac{AC}{4}$ i.e. $AC = 4\tan 30^\circ = \frac{4}{\sqrt{3}}$

also $\cos 30^\circ = \frac{4}{OA}$ i.e. $OA = \frac{4}{\cos 30^\circ} = \frac{4}{\frac{\sqrt{3}}{2}} = \frac{8}{\sqrt{3}}$

$\therefore$ Total distance travelled $= \frac{4}{\sqrt{3}} + \frac{8}{\sqrt{3}} = \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$$= \frac{12\sqrt{3}}{3} = 4\sqrt{3} \text{ km } \#$$

TOPIC 4

LINEAR FUNCTIONS

Linear functions refer to the study of straight lines and their properties:

- Equations of stratight lines may be expressed as $ax + by + c = 0$ or $y = mx + b$.
- For $y = mx + b$, m is the gradient of the line and b is the y-intercept.

(A) The Gradient Formula

The gradient m, of a line joining 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1: Find the gradient of the line joining:

(i) $(1,-1)$ and $(4,8)$

(ii) $(0,0)$ and $(1,-1)$

(iii) $(3,-2)$ and $(-1,3)$

Solution 1:

(i) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{8-(-1)}{4-1} = \frac{9}{3} = 3$ #

(ii) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{-1-0}{1-0} = -1$ #

(iii) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{3-(-2)}{-1-3} = \frac{5}{-4} = -\frac{5}{4}$ #

(B) The Point Gradient Formula

The equation of a line passing through a point $A(x_1, y_1)$ and gradient m is given by:

$$(y - y_1) = m(x - x_1)$$

ing through:

(i) (1, 2) with gradient 1

(ii) (2, −5) with gradient $-\frac{1}{2}$

Solution 1:

(i) $(y-y_1)=m(x-x_1)$

$(y-2)=1(x-1)$

$y-2=x-1$

$y=x+1$ #

(ii) $(y-y_1)=m(x-x_1)$

$(y-(-5))=-\frac{1}{2}(x-2)$

$y+5=-\frac{1}{2}x+1$

$y=-\frac{1}{2}x-4$ #

Example 2: A line passes through the points $(-5, 3)$ and $(1, -5)$. Find:

(i) The gradient of the line.

(ii) Hence, find the equation of the line.

Solution 2:

(i) $m=\frac{y_2-y_1}{x_2-x_1}$

$m=\frac{-5-3}{1-(-5)}=\frac{-8}{6}=-\frac{4}{3}$ #

(ii) $(y-y_1)=m(x-x_1)$

$(y-3)=-\frac{4}{3}(x-(-5))$

$(y-3)=-\frac{4}{3}(x+5)$

$y-3=-\frac{4}{3}x-\frac{20}{3}$

$y=-\frac{4}{3}x-\frac{11}{3}$ #

C) The Two Point Formula

The equation of a line passing through 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Example 1: Find the equation of the line passing through the points:

(i) (1, 4) and (2, 7)

(ii) (−3, −4) and (2, −1)

Solution 1:

(i) $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$(y-4)=\frac{7-4}{2-1}(x-1)$

$y-4=\frac{3}{1}(x-1)$

$y-4=3x-3$

$y=3x+1$ #

(ii) $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$(y-(-4))=\frac{-1-(-4)}{2-(-3)}(x-(-3))$

$y+4=\frac{3}{5}(x+3)$

$y+4=\frac{3}{5}x+\frac{9}{5}$

$y=\frac{3}{5}x-\frac{11}{5}$ #

(D) Finding the Gradient and y-Intercept from the Equation of a Line Given in General Form

The general equation of a line is $ax+by+c=0$.

To find the gradient and y-intercept make y the subject of the formula and then read off the gradient and y-intercept

Example 1: Find the gradient and y-intercept of these lines:

(i) $12x+4y-28=0$

(ii) $2x-5y-20=0$

Solution 1:

(i) $12x+4y-28=0$

$4y=-12x+28$

$y=-3x+7$ which is of the form $y=mx+b$

i.e. $m=-3$ and $b=7$

$\therefore$ line has gradient -3 and y-intercept $(0, 7)$ #

(ii) $2x-5y-20=0$

$5y=2x-20$

$y=\frac{2}{5}x-4$ which is of the form $y=mx+b$

i.e. $m=\frac{2}{5}$ and $b=-4$

$\therefore$ line has gradient $\frac{2}{5}$ and y-intercept $(0, -4)$ #

(i) $(1, 2)$ with gradient 1

(ii) $(2, -5)$ with gradient $-\frac{1}{2}$

Solution 1:

(i) $(y-y_1)=m(x-x_1)$

$(y-2)=1(x-1)$

$y-2=x-1$

$y=x+1$ #

(ii) $(y-y_1)=m(x-x_1)$

$(y-(-5))=-\frac{1}{2}(x-2)$

$y+5=-\frac{1}{2}x+1$

$y=-\frac{1}{2}x-4$ #

Example 2: A line passes through the points $(-5, 3)$ and $(1, -5)$. Find:

(i) The gradient of the line.

(ii) Hence, find the equation of the line.

Solution 2:

(i) $m=\frac{y_2-y_1}{x_2-x_1}$

$m=\frac{-5-3}{1-(-5)}=\frac{-8}{6}=-\frac{4}{3}$ #

(ii) $(y-y_1)=m(x-x_1)$

$(y-3)=-\frac{4}{3}(x-(-5))$

$(y-3)=-\frac{4}{3}(x+5)$

$y-3=-\frac{4}{3}x-\frac{20}{3}$

$y=-\frac{4}{3}x-\frac{11}{3}$ #

C) The Two Point Formula

The equation of a line passing through 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Example 1: Find the equation of the line passing through the points:

(i) $(1, 4)$ and $(2, 7)$

(ii) $(-3, -4)$ and $(2, -1)$

Solution 1:

(i) $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$(y-4)=\frac{7-4}{2-1}(x-1)$

$y-4=\frac{3}{1}(x-1)$

$y-4=3x-3$

$y=3x+1$ #

(ii) $(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$(y-(-4))=\frac{-1-(-4)}{2-(-3)}(x-(-3))$

$y+4=\frac{3}{5}(x+3)$

$y+4=\frac{3}{5}x+\frac{9}{5}$

$y=\frac{3}{5}x-\frac{11}{5}$ #

(D) Finding the Gradient and y-Intercept from the Equation of a Line Given in General Form

The general equation of a line is $ax+by+c=0$.

To find the gradient and y-intercept make y the subject of the formula and then read off the gradient and y-intercept

Example 1: Find the gradient and y-intercept of these lines:

(i) $12x+4y-28=0$

(ii) $2x-5y-20=0$

Solution 1:

(i) $12x+4y-28=0$

$4y=-12x+28$

$y=-3x+7$ which is of the form $y=mx+b$

i.e. $m=-3$ and $b=7$

$\therefore$ line has gradient -3 and y-intercept $(0, 7)$ #

(ii) $2x-5y-20=0$

$5y=2x-20$

$y=\frac{2}{5}x-4$ which is of the form $y=mx+b$

i.e. $m=\frac{2}{5}$ and $b=-4$

$\therefore$ line has gradient $\frac{2}{5}$ and y-intercept $(0, -4)$ #

Example 2: Given $y=-\frac{1}{4}x+\frac{1}{2}$, find the general equation of the line.

Solution 2:

$$y=-\frac{1}{4}x+\frac{1}{2}$$
$$4y=-x+2$$
$$4y+x-2=0 \ \#$$

(E) The Distance Formula

The distance d, between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by :

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example 1: For each set of points, find the distance in surd form.

(i) (2, 2) and (1, −5)

(ii) (1, −3) and (3, 7)

(iii) (2, 4) and (5, −2)

Solution 1:

(i) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
$=\sqrt{(1-2)^2+(-5-2)^2}$
$=\sqrt{(-1)^2+(-7)^2}+=\sqrt{49+1}=\sqrt{50}$ units #

(ii) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
$=\sqrt{(3-1)^2+(7-(-3))^2}$
$=\sqrt{2^2+10^2}+=\sqrt{104}=2\sqrt{26}$ units #

(iii) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
$=\sqrt{(5-2)^2+(-2-4)^2}$
$=\sqrt{3^2+(-6)^2}=\sqrt{45}=3\sqrt{5}$ units #

Linear Functions

(F) The Midpoint Formula

The midpoint M, of the interval joining $A(x_1, y_1)$ to $B(x_2, y_2)$ is given by :

$$M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

Example 1: For each set of points find the mid-point:

(i) (4, 7) and (2, 1)

(ii) (−1, −5) and (−3, 2)

Solution 1:

(i) $M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$
$=\left(\frac{4+2}{2},\frac{7+1}{2}\right)$
$=(3,4)$ #

(ii) $M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$
$=\left(\frac{-1+-3}{2},\frac{-5+2}{2}\right)$
$=\left(-2,-\frac{3}{2}\right)$ #

(G) Parallel and Perpendicular Lines

- Two lines with gradients m_1 and m_2 respectively are parallel, if $m_1=m_2$.
- Two lines with gradients m_1 and m_2 respectively are perpendicular if

$$m_1m_2=-1 \quad \left(\text{or } m_2=\frac{-1}{m_1}\right)$$

Example 1: Find the equation of the line passing through the point (1, 3) and parallel to the line $2y-3x+1=0$.

Linear Functions

Solution 1:

$$2y-3x+1=0$$
$$2y=3x-1$$
$$y=\frac{3}{2}x-\frac{1}{2},\text{ which has gradient } m_1=\frac{3}{2}$$

Parallel lines have the same gradient i.e. $m_1=m_2$

$\therefore$ equation of line is :

$$(y-y_1)=m_2(x-x_1)$$
$$(y-3)=\frac{3}{2}(x-1) \quad \text{or} \quad y-3=\frac{3}{2}x-\frac{3}{2}$$
$$2y-6=3x-3 \quad \text{or} \quad y=\frac{3}{2}x+\frac{3}{2}$$
$$-3x+2y+3=0 \quad \text{or} \quad y=\frac{3}{2}(x+1) \quad \#$$

Example 2: Find the equation of the line passing through the point $(-1,\ 2)$ and perpendicular to the line $x-2y+2=0$.

Solution 2:

$$x-2y+2=0$$
$$2y=x+2$$
$$y=\frac{1}{2}x+1,\text{ which has gradient } m_1=\frac{1}{2}$$

Since lines are perpendicular, thus $m_2=-\frac{1}{m_1}=-2$

$\therefore$ equation of line is :

$$(y-y_1)=m_2(x-x_1)$$
$$(y-2)=-2(x-(-1))$$
$$y-2=-2x-2$$
$$y=-2x \quad \text{or} \quad y+2x=0 \quad \#$$

Example 3: Show that the line joining $A(-1,7)$ and $B(3,4)$ is parallel to $3x+4y=0$.

Solution 3:

$$3x+4y=0$$
$$4y=-3x$$
$$y=-\frac{3}{4}x,\text{ which has gradient } m_1=-\frac{3}{4}$$

Gradient of AB: $m_2=\frac{y_2-y_1}{x_2-x_1}=\frac{7-4}{-1-3}=\frac{3}{-4}=-\frac{3}{4}$

Since $m_1=m_2$ $\therefore$ the li...

Example 4: Show that the lines $x-3y+5=0$ and $2y+6x-10=0$ are perpendicular.

Solution 4:

$$x-3y+5=0$$
$$3y=x+5$$
$$y=\frac{1}{3}x+\frac{5}{3},\text{ which has gradient } m_1=\frac{1}{3}$$
$$2y+6x-10=0$$
$$2y=-6x+10$$
$$y=-3x+5 \text{ which has gradient } m_2=-3$$

Since $m_2=-\frac{1}{m_1}$ $\therefore$ the lines are perpendicular. #

(H) Intersection of Lines

To find the point of intersection of two lines algebraically, solve their equations simultaneously.

Example 1: Find the point of intersection of the following pairs of lines:

(i) $7x+y=0$ and $x-7y=-25$

(ii) $2x+y+18=0$ and $x-2y-1=0$

Solution 1:

(i)
$$7x+y=0 \quad \ldots\ldots (1)$$
$$x-7y=-25 \quad \ldots\ldots (2)$$
$$7\times(1):\ 49x+7y=0 \quad \ldots\ldots (3)$$
$$x-7y=-25 \quad \ldots\ldots (2)$$
$$(3)+(2):\ 50x+0=-25$$
$$x=-\frac{1}{2}$$

Substituting into (1) gives:

$$7\times-\frac{1}{2}+y=0$$
$$y=\frac{7}{2}$$

$\therefore$ point of intersection is $\left(-\frac{1}{2},\ \frac{7}{2}\right)$ #

(ii) $2x+y+18=0$ ...... (1)

$x-2y-1=0$ ...... (2)

$2\times(1):\ 4x+2y+36=0$ ...... (3)

$x-2y-1=0$ ...... (2)

$(3)+(2):\ 5x+0+35=0$

$5x=-35$

$x=-7$

Substituting into (1) gives:

$2\times-7+y+18=0$

$y=-4$

$\therefore$ point of intersection is $(-7,-4)$ #

(I) Sketching Straight Lines

There are a number of methods, but the simplest is:

- Substitute $x=0$ into the given equation, and solve it to find y-intercept.
- Substitute $y=0$ into the given equation, and solve it to find x-intercept.

Join the two points together to form the line.

Example 1: Sketch the following lines:

(i) $x-3y=3$ (ii) $2x+y+6=0$

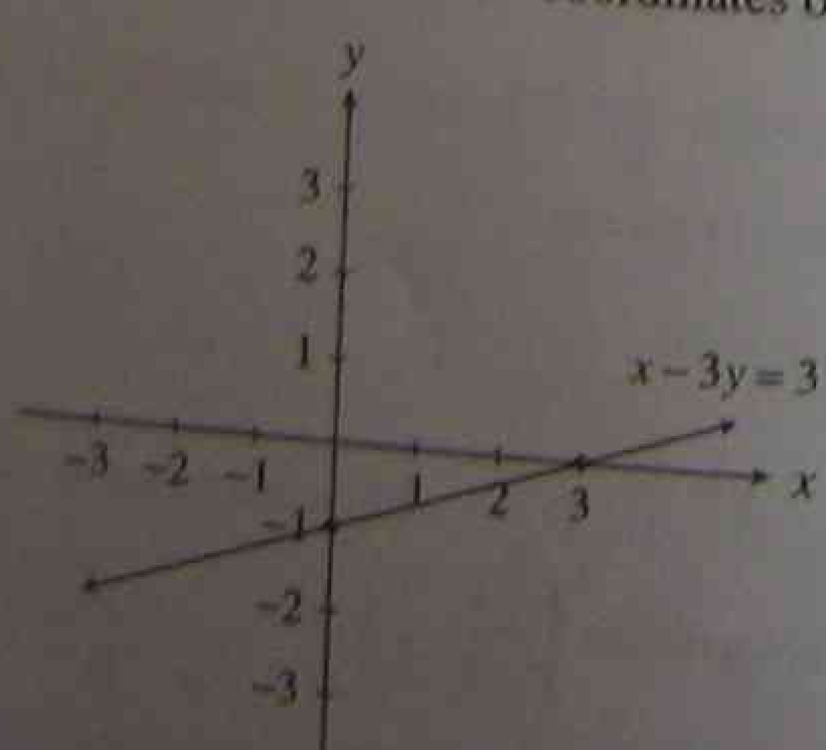
(iii) $x-2y-1=0$ (iv) $3x+4y-12=0$

Solution 1:

(i) $x-3y=3$

for $x=0$, $-3y=3$ i.e. $y=-1$ $\therefore$ coordinates of the point are $(0,-1)$

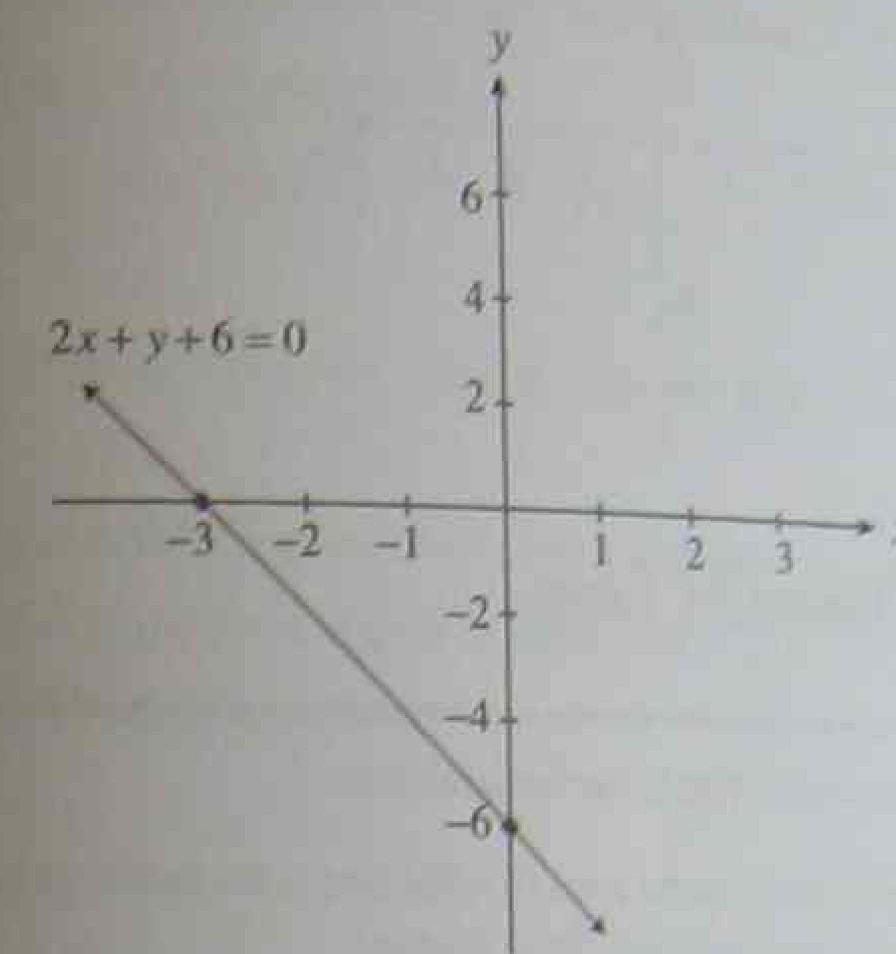
for $y=0$, $x=3$ $\therefore$ coordinates of the point are $(3,0)$



(ii) $2x+y+6=0$

for $x=0$, $y+6=0$ i.e. $y=-6$ $\therefore$ coordinates of the point are $(0,-6)$

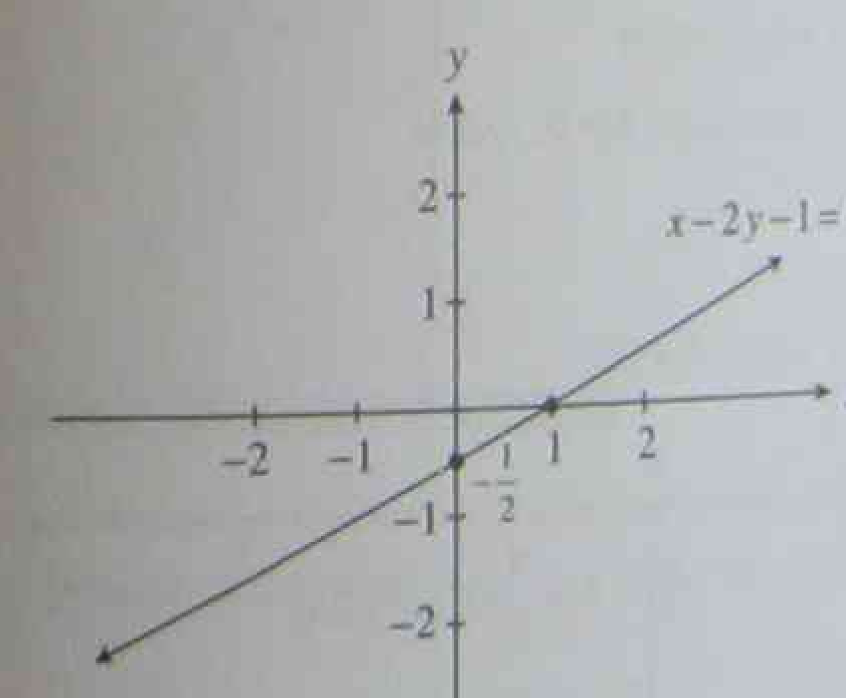
for $y=0$, $2x+6=0$ i.e. $x=-3$ $\therefore$ coordinates of the point are $(-3,0)$



(iii) $x-2y-1=0$

for $x=0$, $-2y-1=0$ i.e. $y=-\frac{1}{2}$ $\therefore$ coordinates of the point are $\left(0,-\frac{1}{2}\right)$

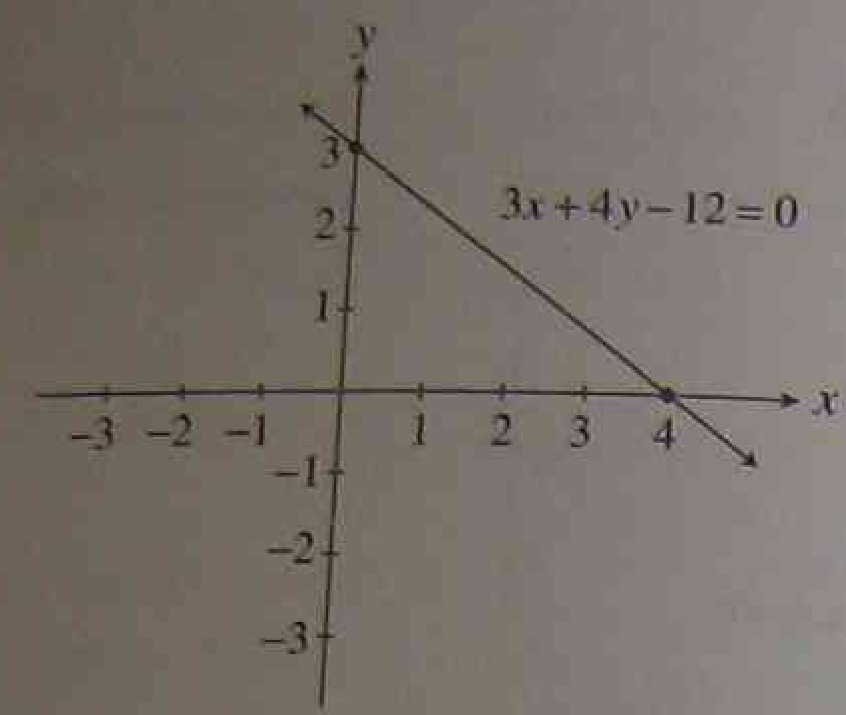
for $y=0$, $x-1=0$ i.e. $x=1$ $\therefore$ coordinates of the point are $(1,0)$



(iv) $3x+4y-12=0$

for $x=0$, $4y=12$ i.e. $y=3$ $\therefore$ coordinates of the point are $(0,3)$

for $y=0$, $3x=12$ i.e. $x=4$ $\therefore$ coordinates of the point are $(4,0)$



(J) Shading Regions Bound by Lines

To find the region bound by two linear inequalities:

- Sketch each line on the same axes (a solid line is included in the region and a "broken" line is not included in the region)
- Select a point (say $(0, 0)$) and substitute into each inequality to find which side of each line to shade.
- Shade the region satisfying both inequalities.

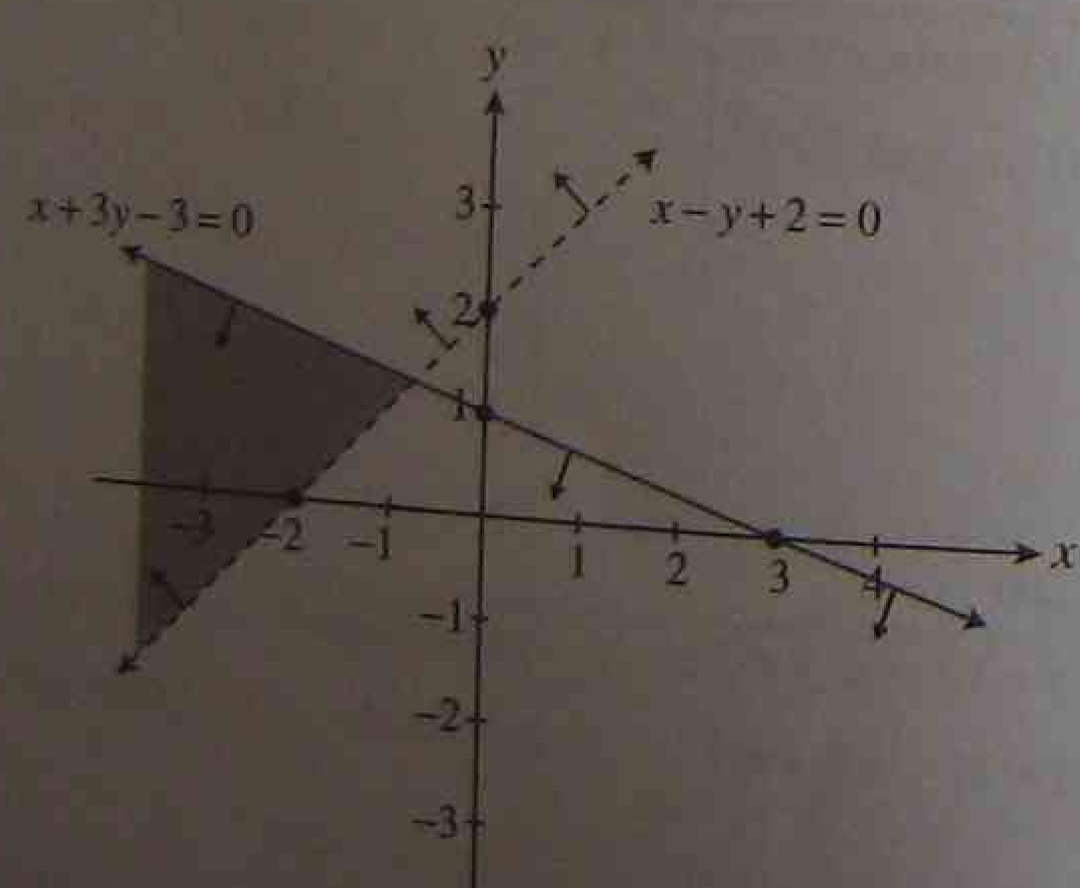
Example 1: Shade the region bound by the following linear inequalities:

(i) $x-y+2<0\,, x+3y-3\le 0$

(ii) $2x+3y-6\le 0\,, 3x-2y+3>0\,, y\ge 0$

Solution 1:

(i) $x-y+2<0,\ x+3y-3\le 0$



$x-y-2<0$

Substituting $(0, 0)$ gives $2<0$, which is false $\therefore$ region *excludes* $(0, 0)$

$x+3y-3\le 0$

Substituting $(0, 0)$ gives $-3\le 0$, which is true $\therefore$ region *includes* $(0, 0)$

Shade the common region. #

(ii) $2x+3y-6\le 0\,,\ 3x-2y+3>0\,,\ y\ge 0$

$2x+3y-6\le 0$

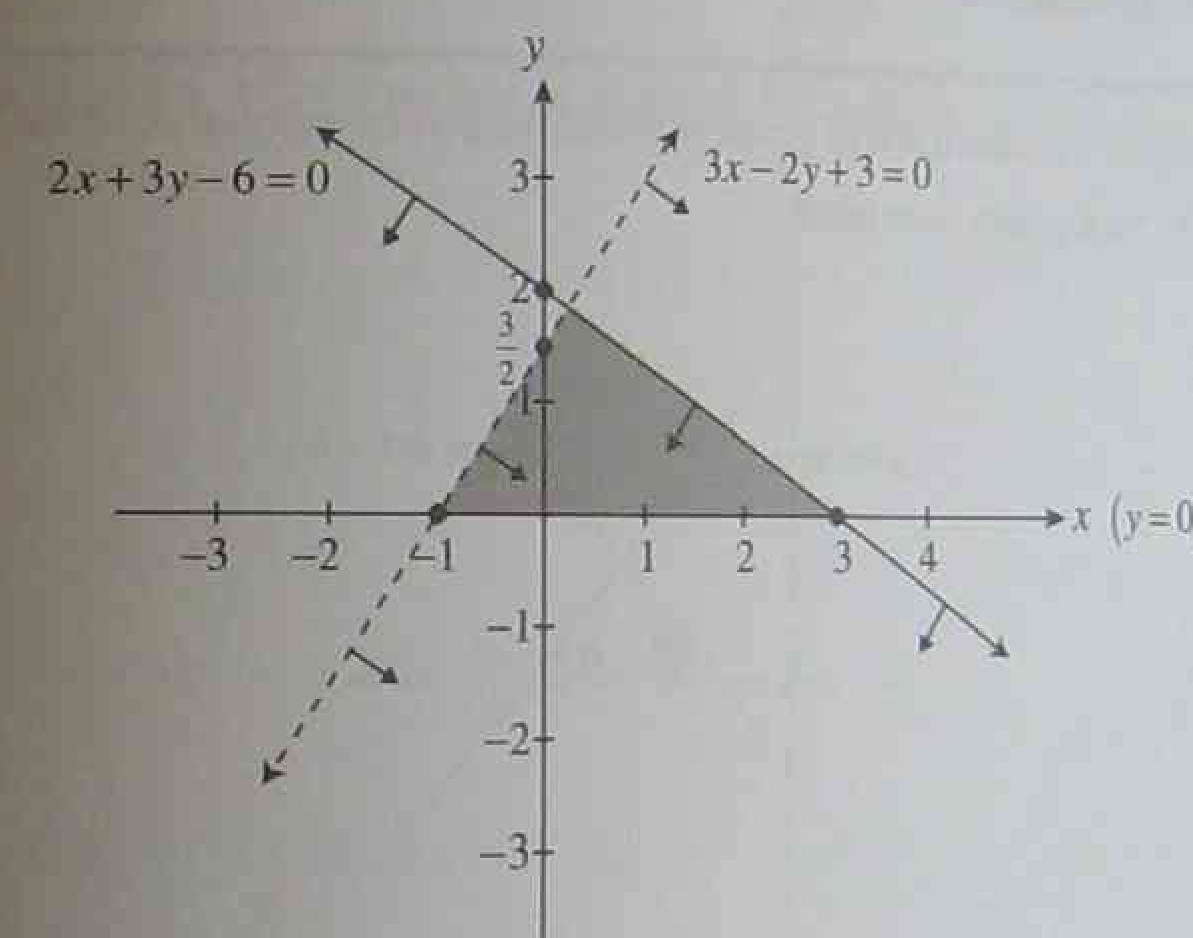
Substituting $(0, 0)$ gives $-6\le 0$, which is true $\therefore$ region *includes* $(0, 0)$

$3x-2y+3>0$

Substituting $(0, 0)$ gives $3>0$, which is true $\therefore$ region *includes* $(0, 0)$

$y\ge 0$ $\therefore$ the required region is above the x-axis.

Shade the common region. #



(K) The Perpendicular Distance Formula

The perpendicular distance of the point $A(x_1, y_1)$ to the line $ax+by+c=0$ is given by:

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

Example 1: Find the perpendicular distance from the point to the line. Express your answer in exact form.

(i) Point $(-1, 4)$ and line $x+4y-2=0$

(ii) Point $(3, 2)$ and line $x-3y+8=0$

Solution 1:

(i) $(-1, 4)$, $x+4y-2=0$

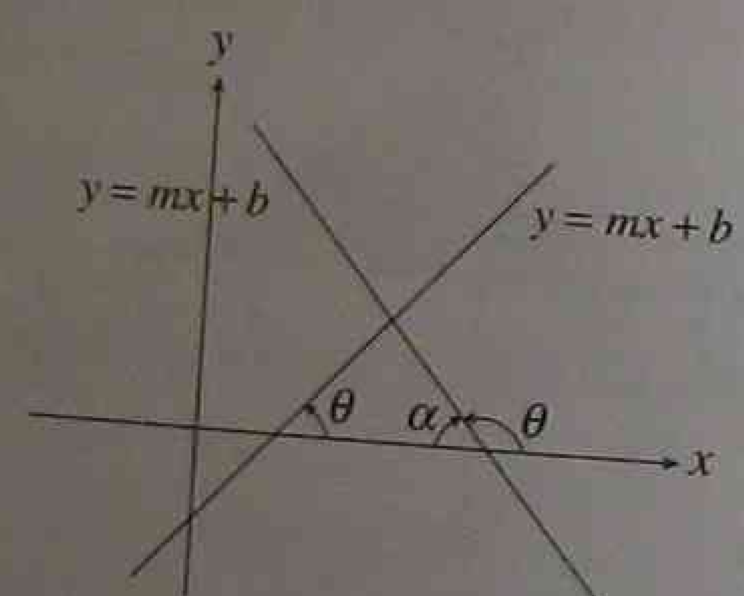
$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$
$$=\left|\frac{1\times-1+4\times4-2}{\sqrt{1^2+4^2}}\right|$$
$$=\left|\frac{-1+16-2}{\sqrt{17}}\right|$$
$$=\frac{13}{\sqrt{17}} \text{ units } \#$$

(ii) $(3, 2)$, $x-3y+8=0$

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$
$$=\left|\frac{1\times3+-3\times2+8}{\sqrt{1^2+(-3)^2}}\right|$$
$$=\left|\frac{3-6+8}{\sqrt{10}}\right|$$
$$=\frac{5}{\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}=\frac{5\sqrt{10}}{10}=\frac{\sqrt{10}}{2} \text{ units } \#$$

(L) Angle of Inclination

The angle θ between a line of gradient m and the positive direction of the x-axis is given by the formula: $m=\tan\theta$



Note θ may be acute $m>0$ or obtuse $m<0$. When θ is obtuse the angle of inclination is given by $\alpha=180°-\theta$.

Example 1: A line makes an angle of $45°$ with the positive direction of the x-axis and passes through the point $(-1, 1)$. Find the equation of the line.

Solution 1:

$m=\tan\theta$ i.e. $m=\tan 45°=1$

$\therefore$ line has gradient 1 and passes through the point $(-1, 1)$

i.e. $(y-y_1)=m(x-x_1)$

$$(y-1)=1(x-(-1))$$
$$y-1=x+1$$
$$y=x+2 \ \#$$

Example 2: Show that the line $2x-2y+5=0$ makes an angle of $45°$ with the positive direction of the x-axis.

Solution 2:

$$2x-2y+5=0$$
$$2y=2x+5$$
$$y=x+\frac{5}{2}, \text{ which has a gradient } m=1$$

$$\tan\theta=m$$
i.e. $\tan\theta=1$
$$\theta=45°$$

$\therefore$ the line $2x+2y+5=0$ makes an angle of $45°$ with the positive direciton of the x-axis. #

REVIEW EXERCISES

(A) The Gradient Formula

1. Find the gradient of the line joining:
   (i) $(2, 1)$ and $(3, 9)$
   (ii) $(-4, -1)$ and $(-2, 7)$
   (iii) $(2, 6)$ and $(4, -3)$

(B) The Point Gradient Formula

2. Find the equation of the straight line passing through:
   (i) $(-1, 3)$ with gradient -2
   (ii) $(4, -7)$ with gradient $\frac{2}{3}$
   (iii) $(1, 5)$ with gradient 4

3. A line passes through the points $(-3, 9)$ and $(7, 1)$. Find:
   (i) The gradient of the line.
   (ii) Hence, find the equation of the line.

(C) The Two Point Formula

4. Find the equation of the line through the points:
   (i) $(1,5)$ and $\left(\frac{1}{2}, -3\right)$
   (ii) $(-3,4)$ and $\left(\frac{1}{3}, 2\right)$

(D) Finding the Gradient and y-Intercept from the Equation of a Line Given in General Form

5. Express in general form:
   (i) $y+\frac{1}{4}x=3$
   (ii) $y=\frac{-2}{5}x-30$

6. Find the gradient and y-intercept of the following lines:
   (i) $15x-9y+39=0$
   (ii) $2x-12y+18=0$
   (iii) $\frac{1}{3}x+\frac{1}{2}y+1=0$

(E) The Distance Formula

7. For each set of points, find the distance in exact form:
   (i) $(2, 0)$ and $\left(0, \frac{3}{2}\right)$
   (ii) $(-2, 2)$ and $(2,7)$
   (iii) $(-8, -1)$ and $(-2, 7)$

(F) The Midpoint Formula

8. For each set of points, find the mid-point:
   (i) $(2, 0)$ and $\left(0, \frac{3}{2}\right)$
   (ii) $(-2, 2)$ and $(2, 7)$
   (iii) $(-8, -1)$ and $(-2, 7)$

(G) Parallel and Perpendicular Lines

9. Find the equation of the line perpendicular to the line joining the points $A(0, 4)$ and $B(-3, 0)$ and passing through the point $(0, 2)$.

10. Show that the line $2x-5y+10=0$ is perpendicular to $10x+4y-3=0$.

11. Find the equation of the line parallel to $4y=6x-1$ and passing through $(5, 4)$.

12. Determine whether the lines $3x+2y-5=0$ and $8y+12x+1=0$ are parallel or perpendicular to each other.

(H) Intersection of Lines

13. Find the point of intersection of the following pairs of lines:
    (i) $3x+2y-6=0$ and $x=4$

(ii) $x+y-3=0$ and $2x-3y+5=0$

14. Show that the line $x-y-1=0$ passes through the point of intersection of the lines $4x-3y-4=0$ and $3x+2y-3=0$.

(I) Sketching Straight Lines

15. Sketch the following lines:

(i) $3x+2y-6=0$

(ii) $x-2y-2=0$

(J) Shading Regions Bound by Lines

16. Shade the region bound by the following linear inequalities:

(i) $x-y-1\le 0$, $y<x$, $y\ge 0$

(ii) $3x-2y+12\ge 0$, $x<1$, $y\ge 2$

(K) The Perpendicular Distance Formula

17. Find the perpendicular distance from the point $(-2, 6)$ to the line $x+y+6=0$.

18. The line $3x+4y+32=0$ is a tangent to a circle centre $(2, -3)$. Find the diameter of the circle.

(L) Angle of Inclination

19. Find the angle of inclination (to the nearest minute) that the line $2x+3y+5=0$ makes with the x-axis.

20. A line makes an angle of 30° with the positive direction of the x-axis and passes through the point $(3, -1)$. Find the equation of the line.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{9-1}{3-2}=\dfrac{8}{1}=8$ #

(ii) $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{7-(-1)}{-2-(-4)}=\dfrac{8}{2}=4$ #

(iii) $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{-3-6}{4-2}=\dfrac{-9}{2}$ #

2. (i)
$$\begin{aligned}(y-y_1)&=m(x-x_1)\\(y-3)&=-2(x+1)\\y-3&=-2x-2\\y+2x-1&=0 \text{ \#}\end{aligned}$$

(ii)
$$\begin{aligned}(y-y_1)&=m(x-x_1)\\(y+7)&=\frac{2}{3}(x-4)\\3y+21&=2x-8\\3y-2x+29&=0 \text{ \#}\end{aligned}$$

(iii)
$$\begin{aligned}(y-y_1)&=m(x-x_1)\\(y-5)&=4(x-1)\\y-5&=4x-4\\y-4x-1&=0 \text{ \#}\end{aligned}$$

3. (i) $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{1-9}{7-(-3)}=\dfrac{-8}{10}=-\dfrac{4}{5}$ #

(ii)
$$\begin{aligned}(y-1)&=\frac{-4}{5}(x-7)\\5y-5&=-4x+28\\5y+4x-33&=0 \text{ \#}\end{aligned}$$

4. (i)
$$\begin{aligned}(y-y_1)&=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\\(y-5)&=\frac{-3-5}{\frac{1}{2}-1}(x-1)\\y-5&=\frac{-8}{-\frac{1}{2}}(x-1)\\y-5&=16(x-1)\\y-5&=16x-16\\y-16x+11&=0 \text{ \#}\end{aligned}$$

(ii)
$$\begin{aligned}(y-y_1)&=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\\(y-4)&=\frac{2-4}{\frac{1}{3}+3}(x+3)\\y-4&=\frac{-2}{\frac{10}{3}}(x+3)\\y-4&=\frac{-6}{10}(x+3)\\y-4&=\frac{-3}{5}(x+3)\\5y-20&=-3x-9\\5y+3x-11&=0 \text{ \#}\end{aligned}$$

5. (i) $y+\frac{1}{4}x=3$

$4y+x=12$

i.e. $x+4y-12=0$ #

(ii) $y=-\frac{2}{5}x-30$

$5y=-2x-150$

i.e. $2x+5y+150=0$ #

6. (i) $15x-9y+39=0$

$9y=15x+39$

$y=\frac{15}{9}x+\frac{39}{9}$

$=\frac{5}{3}x+\frac{13}{3}$

which has gradient $\frac{5}{3}$

and y-intercept $\frac{13}{3}$. #

(ii) $2x-12y+18=0$

$12y=2x+18$

$y=\frac{2}{12}x+\frac{18}{12}$

$=\frac{1}{6}x+\frac{3}{2}$

which has gradient $\frac{1}{6}$

and y-intercept $\frac{3}{2}$. #

(iii) $\frac{1}{3}x+\frac{1}{2}y+1=0$

$\frac{1}{2}y=-\frac{1}{3}x-1$

$y=-\frac{2}{3}x-2$

which has gradient $-\frac{2}{3}$ and y-intercept -2. #

7. (i) $d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$=\sqrt{\left(\frac{3}{2}-0\right)^2+(0-2)^2}=\sqrt{\frac{9}{4}+4}=\sqrt{\frac{25}{4}}=\frac{5}{2}$ units #

(ii) $d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$=\sqrt{(7-2)^2+(2+2)^2}=\sqrt{25+16}=\sqrt{41}$ units #

(iii) $d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}=\sqrt{(7+1)^2+(-2+8)^2}=\sqrt{64+36}=\sqrt{100}=10$ units #

8. (i) $M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$=\left(\frac{2+0}{2},\frac{0+\frac{3}{2}}{2}\right)=\left(1,\frac{3}{4}\right)$ #

Linear Functions

(ii) $M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$=\left(\frac{-2+2}{2},\frac{2+7}{2}\right)=\left(0,\frac{9}{2}\right)$ #

(iii) $M(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$=\left(\frac{-8-2}{2},\frac{-1+7}{2}\right)=(-5,3)$ #

9. Gradient of the line AB is given by:

$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{0-4}{-3-0}=\frac{-4}{-3}=\frac{4}{3}$

Since lines are perpendicular, $\therefore m_2=\frac{-1}{m_1}=-\frac{3}{4}$

$\therefore$ equation of perpendicular line passing through $(0,2)$ is given by:

$(y-y_1)=m(x-x_1)$

$(y-2)=-\frac{3}{4}(x-0)$

$y-2=-\frac{3}{4}x$

$4y-8=-3x$

i.e. $3x+4y-8=0$ #

10. $2x-5y+10=5$

$5y=2x+10$

$y=\frac{2}{5}x+2$, which has gradient $m_1=\frac{2}{5}$

$10x+4y-3=0$

$4y=-10x+3$

$y=\frac{-10}{4}x+\frac{3}{4}$

$=-\frac{5}{2}x+\frac{3}{4}$ which has gradient $m_2=\frac{-5}{2}$

$m_1\times m_2=\frac{2}{5}\times-\frac{5}{2}=-1$

$\therefore$ the lines are perpendicular to each other. #

Linear Functions

11. $4y = 6x - 1$

$y = \frac{6}{4}x - \frac{1}{4}$

$= \frac{3}{2}x - \frac{1}{4}$, which has gradient $m_1 = \frac{3}{2}$

for parallel lines $m_1 = m_2 = \frac{3}{2}$

$\therefore$ equation of line is given by:

$(y - y_1) = m(x - x_1)$

$(y-4) = \frac{3}{2}(x-5)$

$2y - 8 = 3x - 15$

$2y - 3x + 7 = 0$ #

12. $3x + 2y - 5 = 0$

$2y = -3x + 5$

$y = -\frac{3}{2}x + \frac{5}{2}$, which has gradient $m_1 = -\frac{3}{2}$

$8y + 12x + 1 = 0$

$8y = -12x - 1$

$y = -\frac{12}{8}x - \frac{1}{8}$

$= -\frac{3}{2}x - \frac{1}{8}$, which has gradient $m_2 = -\frac{3}{2}$

Since $m_1 = m_2$ $\therefore$ the lines are parallel to each other. #

13. (i) $3x + 2y - 6 = 0$, $x = 4$

$3\times4 + 2y - 6 = 0$

$12 + 2y - 6 = 0$

$2y = -6$

$y = -3$

$\therefore$ point of intersection is $(4, -3)$ #

(ii) $x + y - 3 = 0$ .... (1)

$2x - 3y + 5 = 0$ .... (2)

$3\times(1)$: $3x + 3y - 9 = 0$.....(3)

$(3)+(2)$: $5x + 0 - 4 = 0$

$5x = 4$

$x = \frac{4}{5}$

Linear Functions

Substituting into (1) gives:

$\frac{4}{5} + y - 3 = 0$

$y = 3 - \frac{4}{5} = \frac{11}{5}$

$\therefore$ point of intersection is $\left(\frac{4}{5}, \frac{11}{5}\right)$ #

14. $4x - 3y - 4 = 0$ ....(1)

$3x + 2y - 3 = 0$ ....(2)

$(1)\times3$: $12x - 9y - 12 = 0$ ....(3)

$(2)\times4$: $12x + 8y - 12 = 0$ ....(4)

$(3)-(4)$: $0 - 17y + 0 = 0$

$17y = 0$

i.e. $y = 0$

Substituting into (1) gives:

$4x - 0 - 4 = 0$

$4x = 4$

$x = 1$

$\therefore$ point of intersection is $(1, 0)$

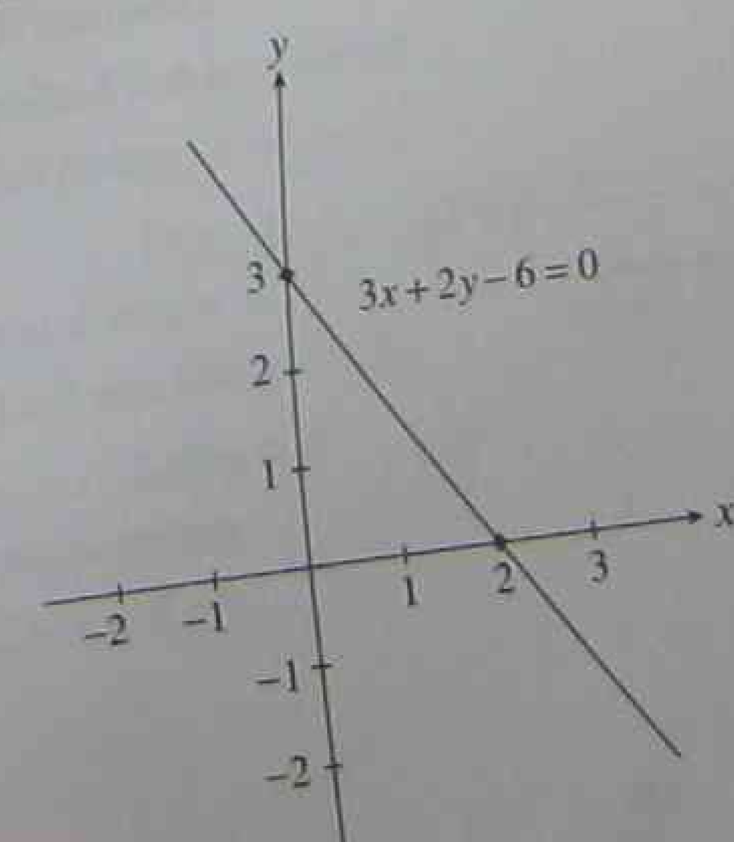
To show $(1,0)$ lies on $x - y - 1 = 0$, we substitute $x = 1$.

$1 - y - 1 = 0$

$y = 0$

$\therefore$ the point of intersection $(1, 0)$ also lies on the line $x - y - 1 = 0$. #

15. (i)

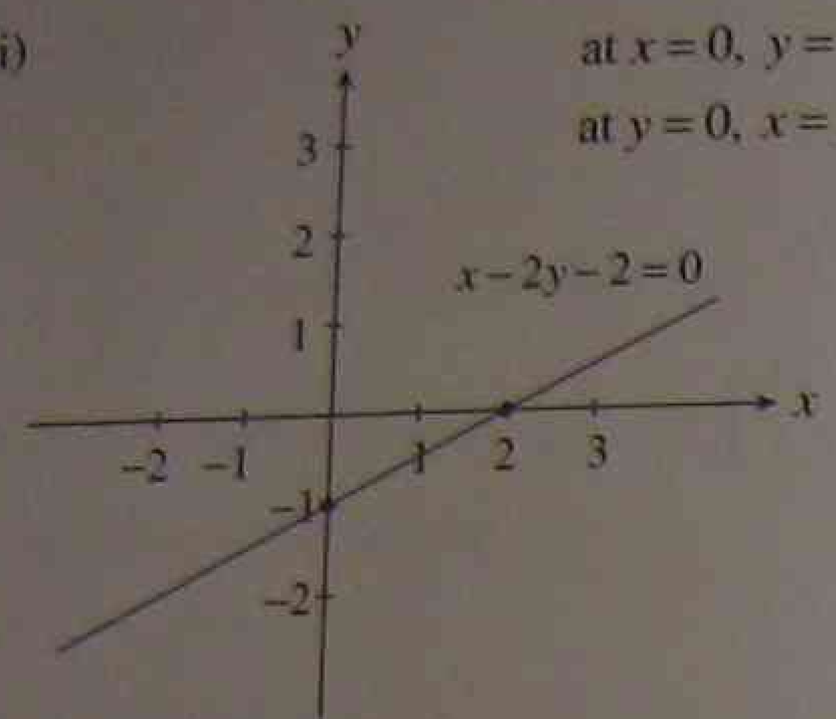


at $x = 0$, $y = 3$ (0, 3)

at $y = 0$, $x = 2$ (2, 0)

Linear Functions

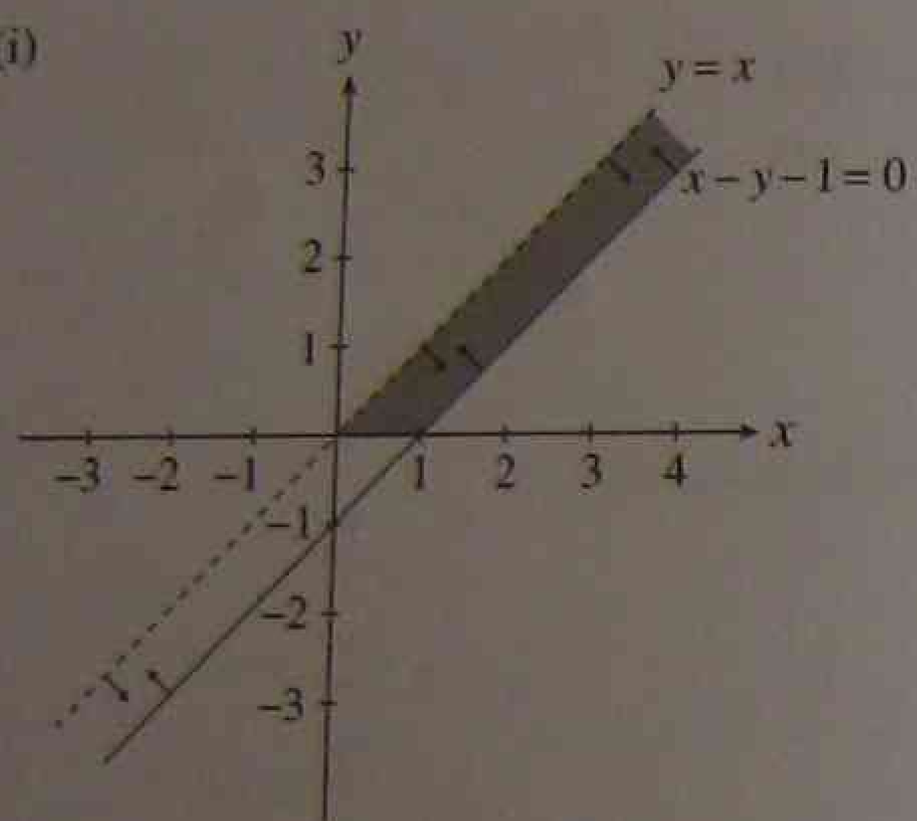
(ii)



at $x = 0,\ y = -1$ $\quad (0, -1)$

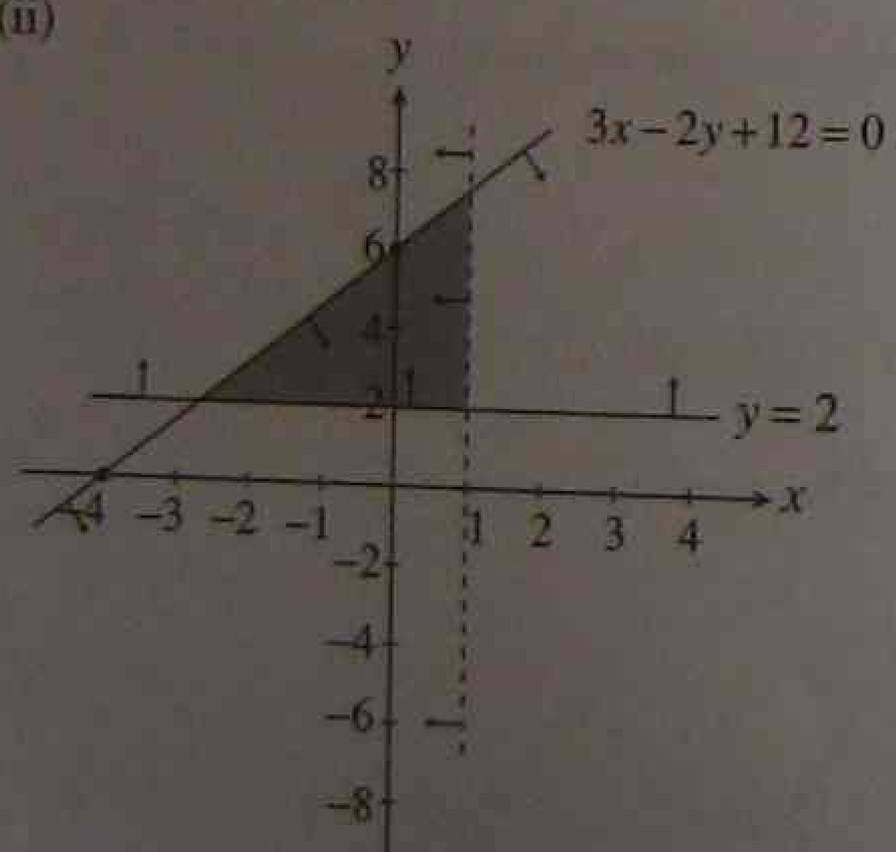
at $y = 0,\ x = 2$ $\quad (2, 0)$

16. (i)



- $x - y - 1 \le 0$
  Substitute (0, 0)
  i.e. $-1 \le 0$, which is true
  $\therefore$ region includes (0, 0).
- $y < x$
  Substitute (1, 0)
  (note that the line passes through (0, 0), so it can be used here)
  i.e. $0 < 1$, which is true.
  $\therefore$ region includes (1, 0).
- $y \ge 0$
  above $x-$axis.

(ii)



- $3x - 2y + 12 \ge 0$
  Substitute (0, 0)
  i.e. $12 \ge 0$, which is true
  $\therefore$ region includes (0, 0).
- $x < 1$
  region is to the left of the line $x = 1$.
- $y \ge 2$
  region is above the line $y = 2$.

17. $(-2, 6)$ $\quad x + y + 6 = 0$

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{1\times -2 + 1\times 6 + 6}{\sqrt{1^2 + 1^2}}\right| = \left|\frac{-2+6+6}{\sqrt{2}}\right| = \frac{10}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \text{ units } \#$$

18. If the line is a tangent to the circle , then the perpendicular distance from the centre of the circle to the line represents the radius of the circle.

i.e. $d = \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$ $\qquad (2, -3) \qquad 3x + 4y + 32 = 0$

$$= \left|\frac{3\times 2 + 4\times -3 + 32}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{6 - 12 + 32}{\sqrt{25}}\right| = \frac{26}{5} = 5\frac{1}{5} \text{ units}$$

$\therefore$ the diameter of the circle is $2\times 5\frac{1}{5} = 10\frac{2}{5}$ units #

19. $2x + 3y + 5 = 0$

$$3y = -2x - 5$$

$$y = -\frac{2}{3}x - \frac{5}{3}, \text{ which has a gradient of } -\frac{2}{3}$$

$\tan\theta = m \qquad$ i.e. $\tan\theta = -\dfrac{2}{3}$

$$\theta = 146°19'$$

Since m is negative, the angle of inclination is $(180° - \theta) = 33°41'$ to the nearest minute. #

20. $m = \tan\theta$ $\quad$ i.e. $\quad m = \tan 30° = \dfrac{1}{\sqrt{3}}$

$\therefore$ equation of line is given by:

$$(y - y_1) = m(x - x_1)$$

$$(y + 1) = \frac{1}{\sqrt{3}}(x - 3)$$

$$\sqrt{3}y + \sqrt{3} = x - 3\sqrt{3}$$

i.e. $\sqrt{3}y - x - 2\sqrt{3} = 0$ #

TOPIC 5

REAL FUNCTIONS

(A) Functional Notation

A curve $y = f(x)$ is a function, if for each value of x there is only **one** value of y.

The vertical line test
If it is possible to draw a vertical line to cut the curve in more than one point then $y = f(x)$ is **not** a function.

It is important to note that $y = f(x)$ may or may not be the same for all values of x.

Example 1: State whether each of the following curves are functions.

(i) $y = x^3$ (ii) $y = 2x-5$ (iii) $(x-1)^2 + y^2 = 9$

Solution 1:

(i) Since for each x-value there is only one y-value
$\therefore y = x^3$ is a function. #

(ii) Since for each x-value there is only one y-value
$\therefore y = 2x-5$ is a function. #

(iii) For $x = 1$, $y^2 = 9$ i.e. $y = \pm 3$
Since for $x = 1$ there are 2 values of y
$\therefore (x-1)^2 + y^2 = 9$ is not a function. #

Example 2: Let $f(x) = \begin{cases} 1-x^2 & \text{for } x \geq -1 \\ 2x+1 & \text{for } x < -1 \end{cases}$

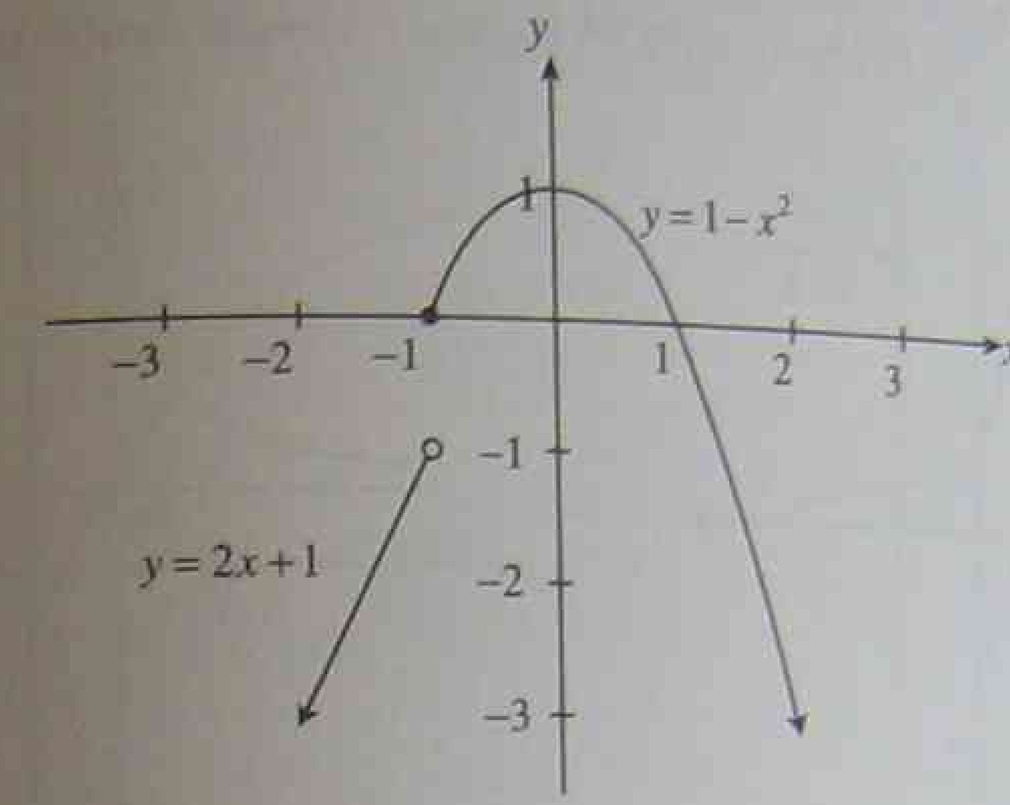
(i) Evaluate $f(-1) + f(0) - f(-2)$

(ii) Sketch the curve $y = f(x)$

Solution 2:

(i) $f(-1) + f(0) - f(-2)$
$= 1-(-1)^2 + 1-(0)^2 - (2\times -2+1)$
$= 0+1+3 = 4$ #

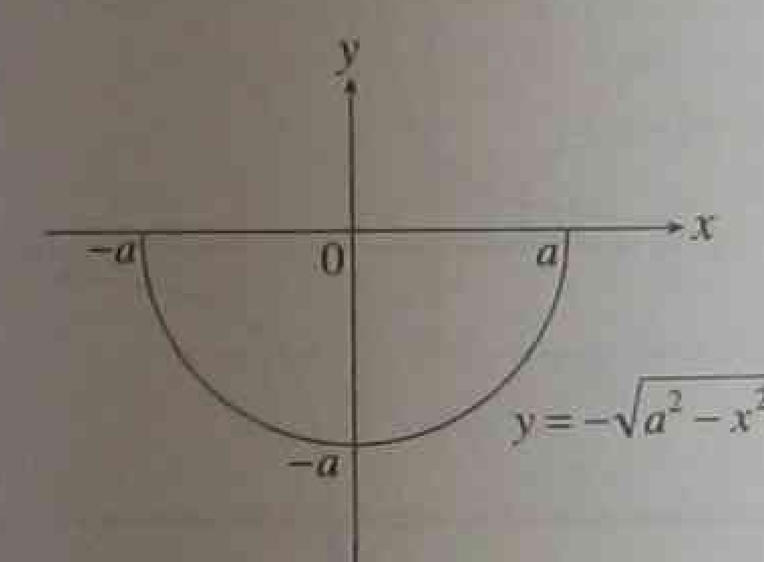
(ii)

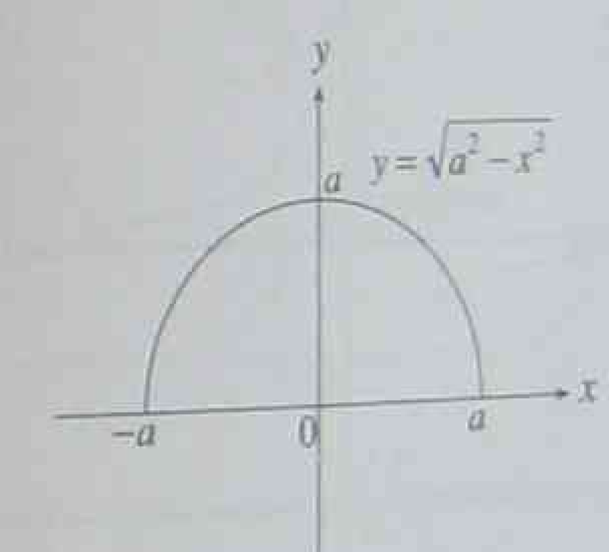


(B) Familiar Functions

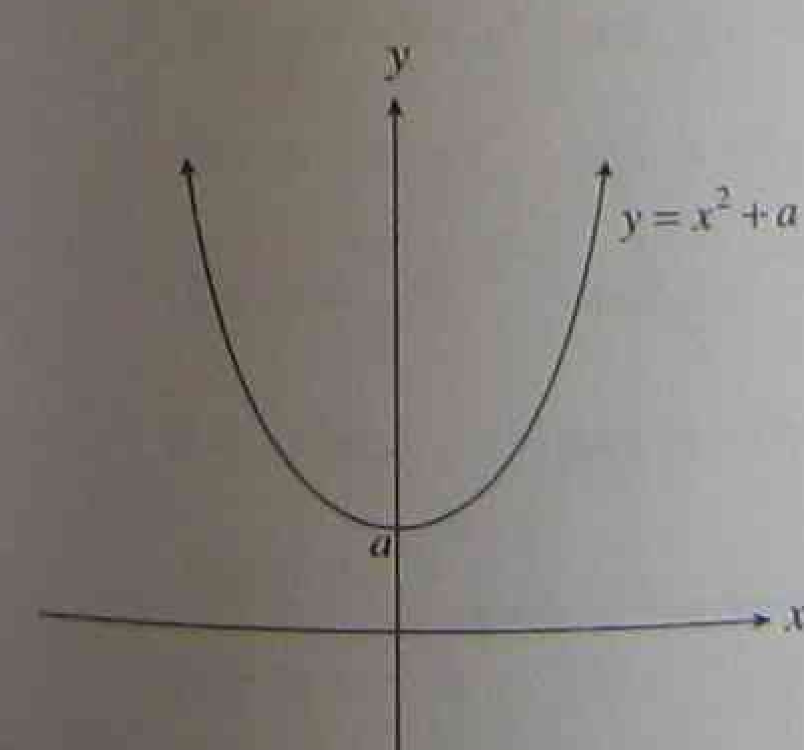
There are a number of graphs that need to be recognized and sketched automatically.

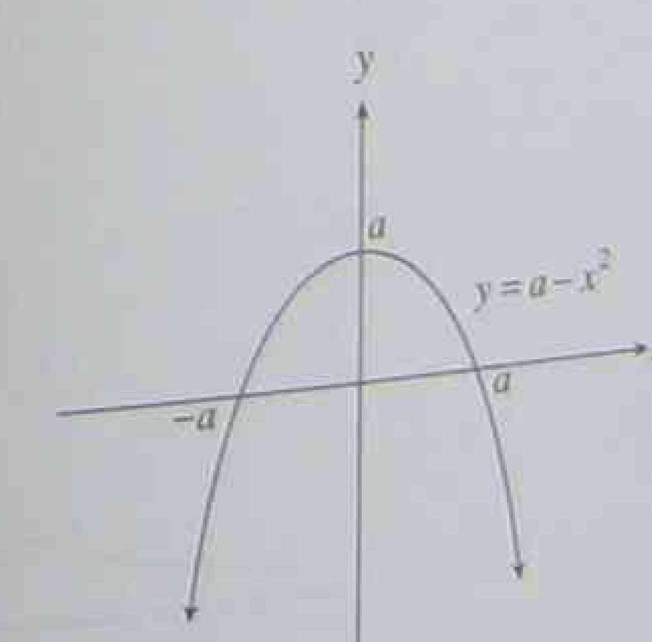
(1) **Semi-circle :** $y = -\sqrt{a^2 - x^2}$ and $y = \sqrt{a^2 - x^2}$





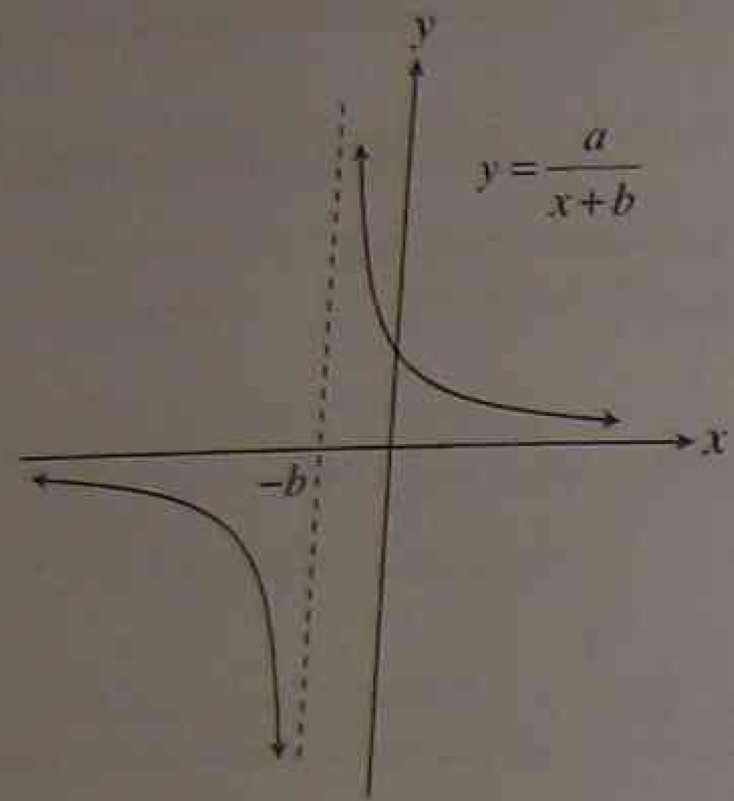
(2) **Parabola :** $y = x^2 + a$ $y = a - x^2$

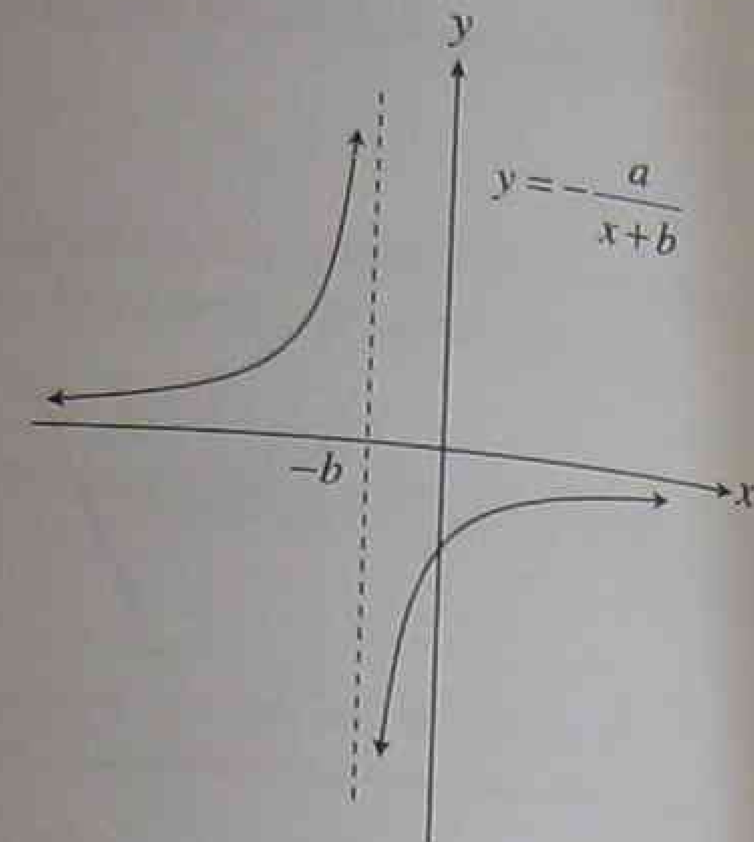




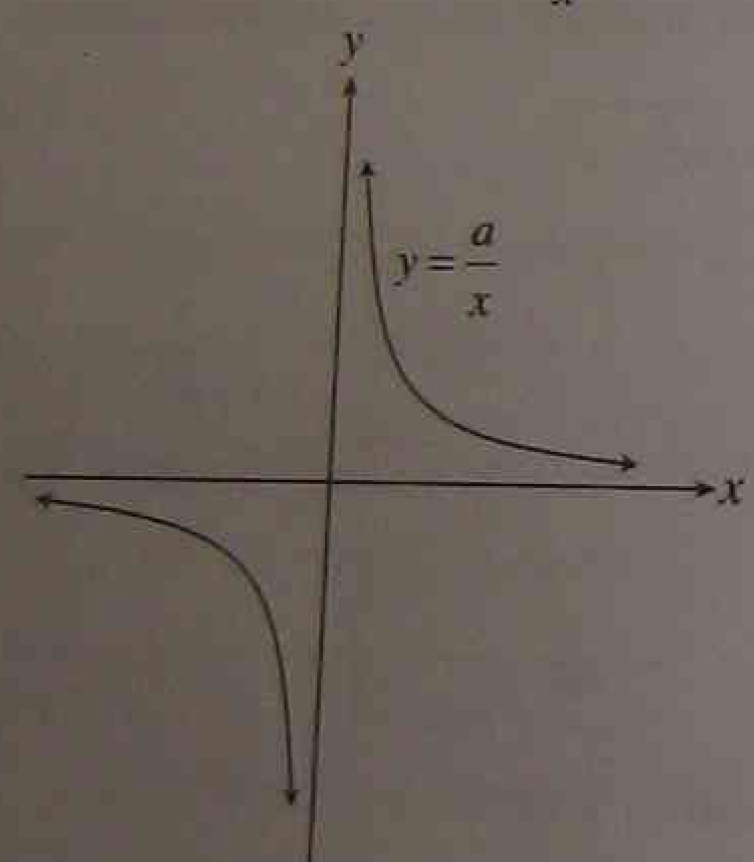
Real Functions

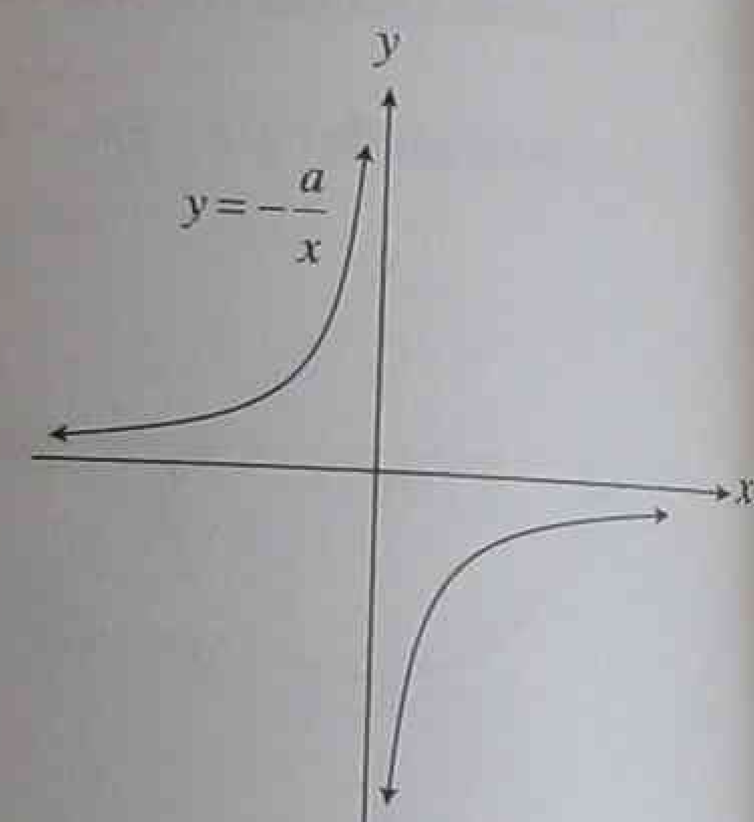
(3) **Hyperbola :** $y=\dfrac{a}{x+b}$ $\quad y=-\dfrac{a}{x+b}$



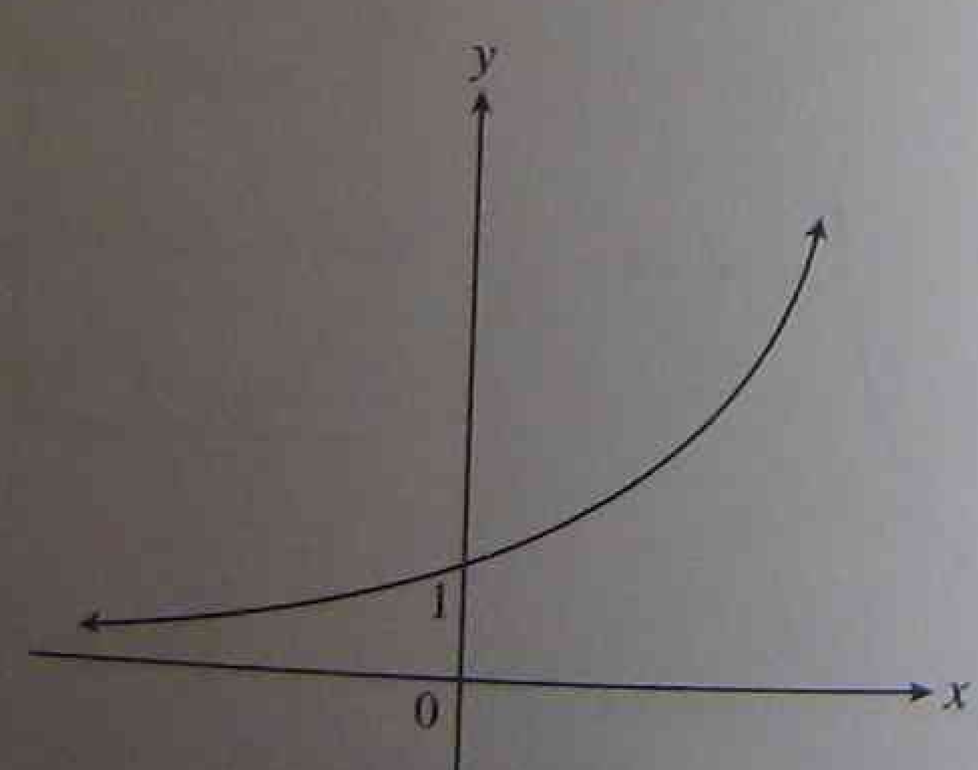


Note that for $b=0$, $y=\dfrac{a}{x}$ and $y=-\dfrac{a}{x}$

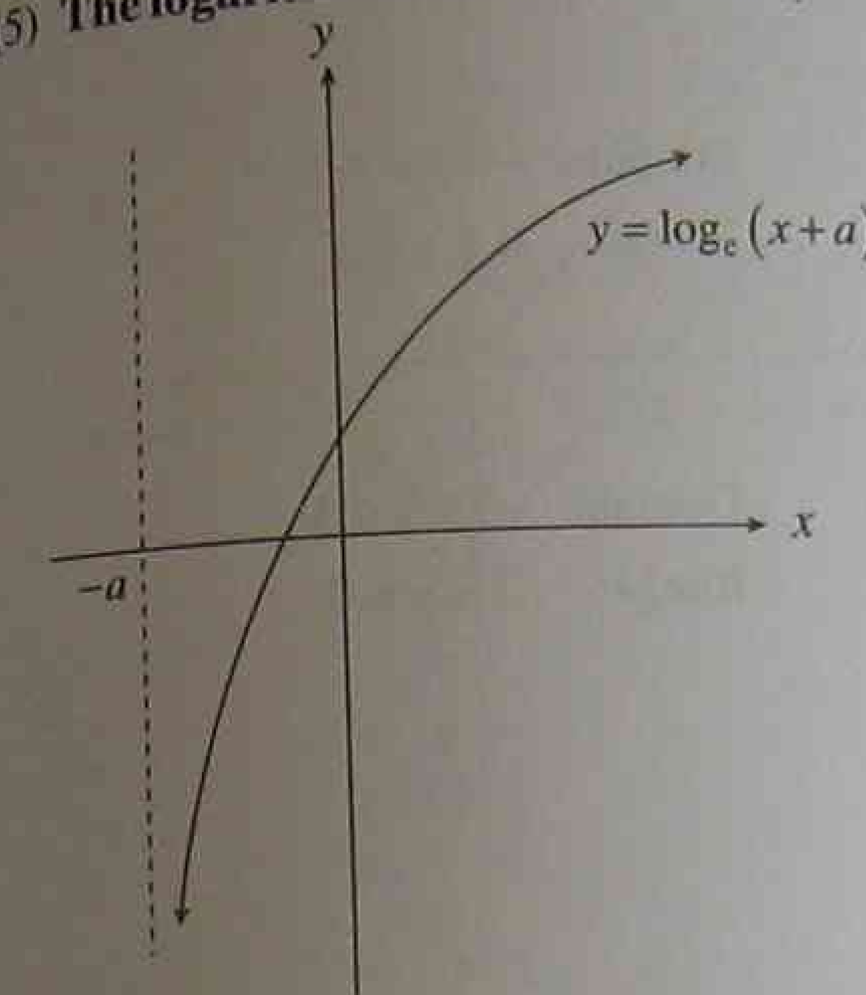


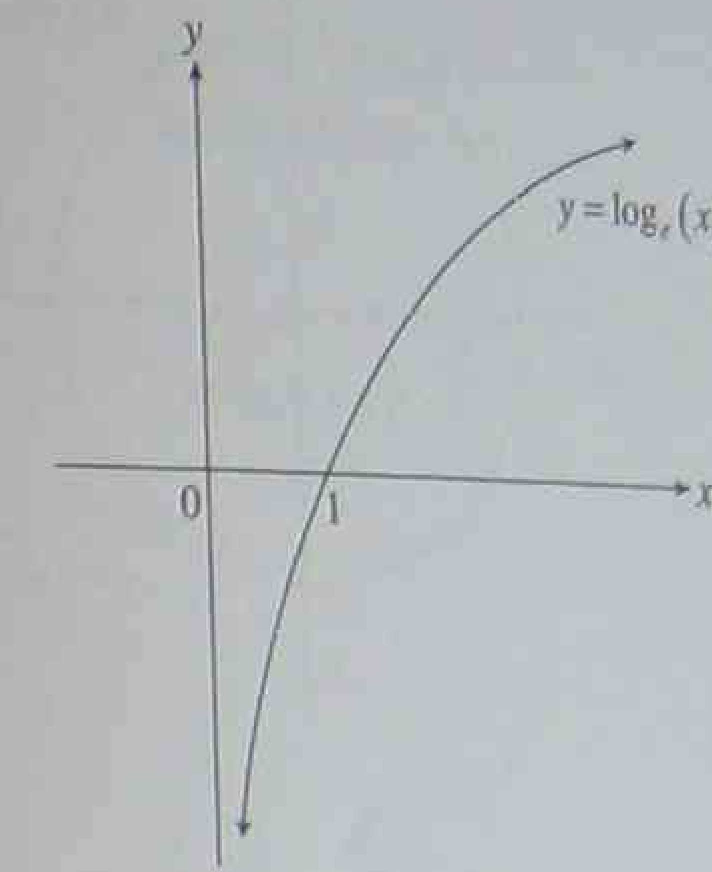


(4) **The exponential curve :** $y=a^x$

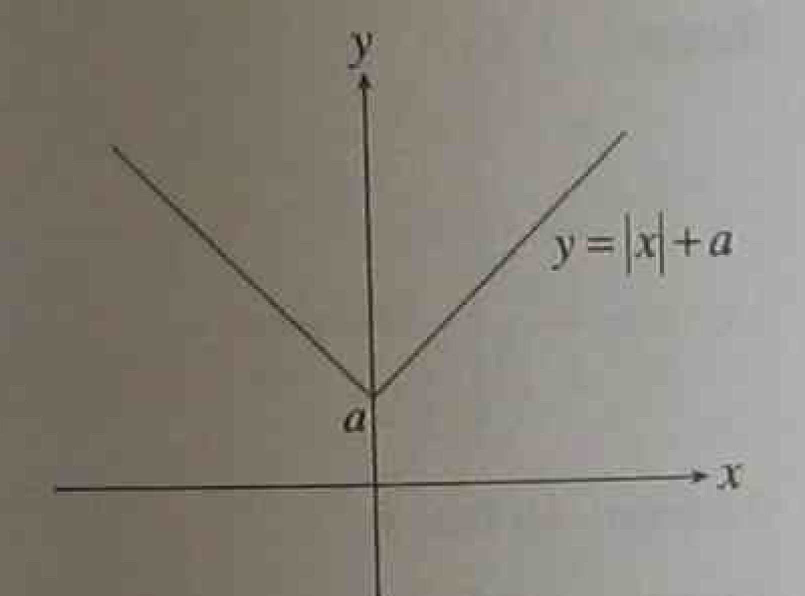


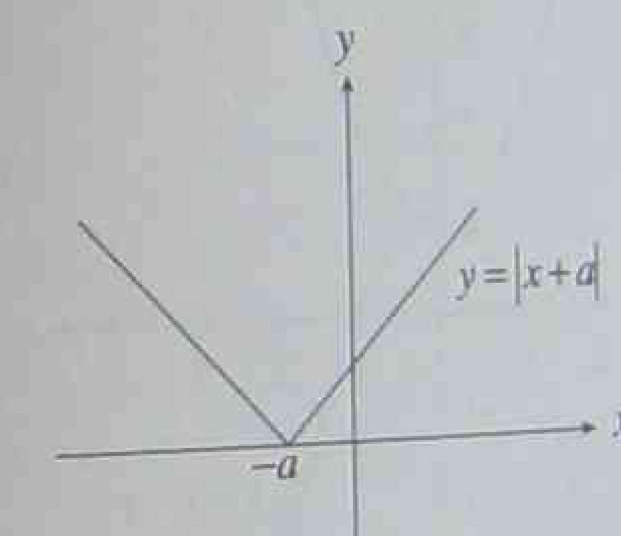
(5) **The logarithmic curve :** $y=\log_e(x+a), y=\log_e(x)$





(6) **The modulus function :** $y=|x|+a \quad y=|x+a|$





Example 1: Sketch the following functions and state their domain and range:

(i) $y=\sqrt{9-x^2}$ and $y=-\sqrt{9-x^2}$

(ii) $y=x^2+1$ and $y=1-x^2$

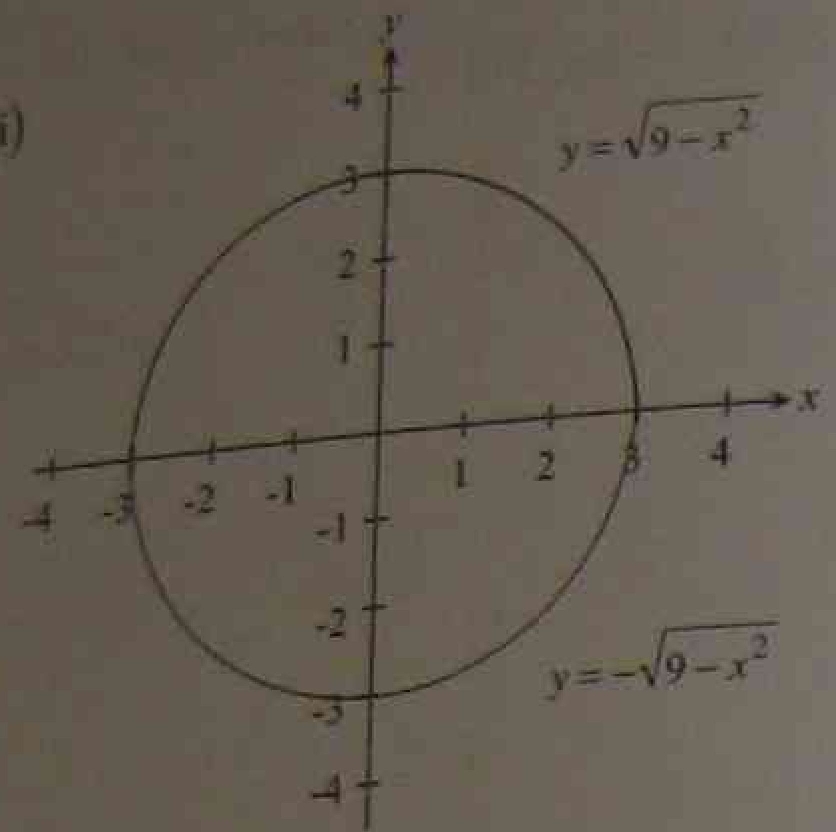
(iii) $y=\dfrac{1}{x}$ and $y=\dfrac{-1}{x}$

(iv) $y=|x|$ and $y=|x|-2$

(v) $y=3^x$

(vi) $y=x^3$ #

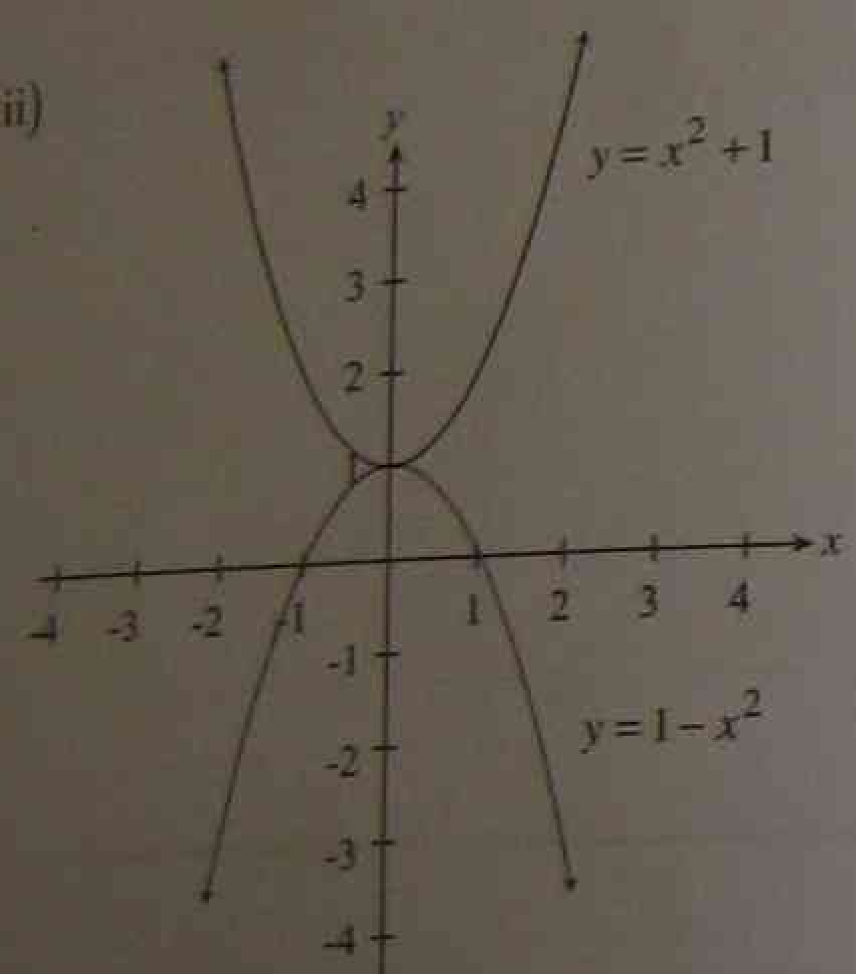
(i)



Domain: $-3 \le x \le 3$
Range: $0 \le y \le 3$

Domain: $-3 \le x \le 3$
Range: $-3 \le y \le 0$

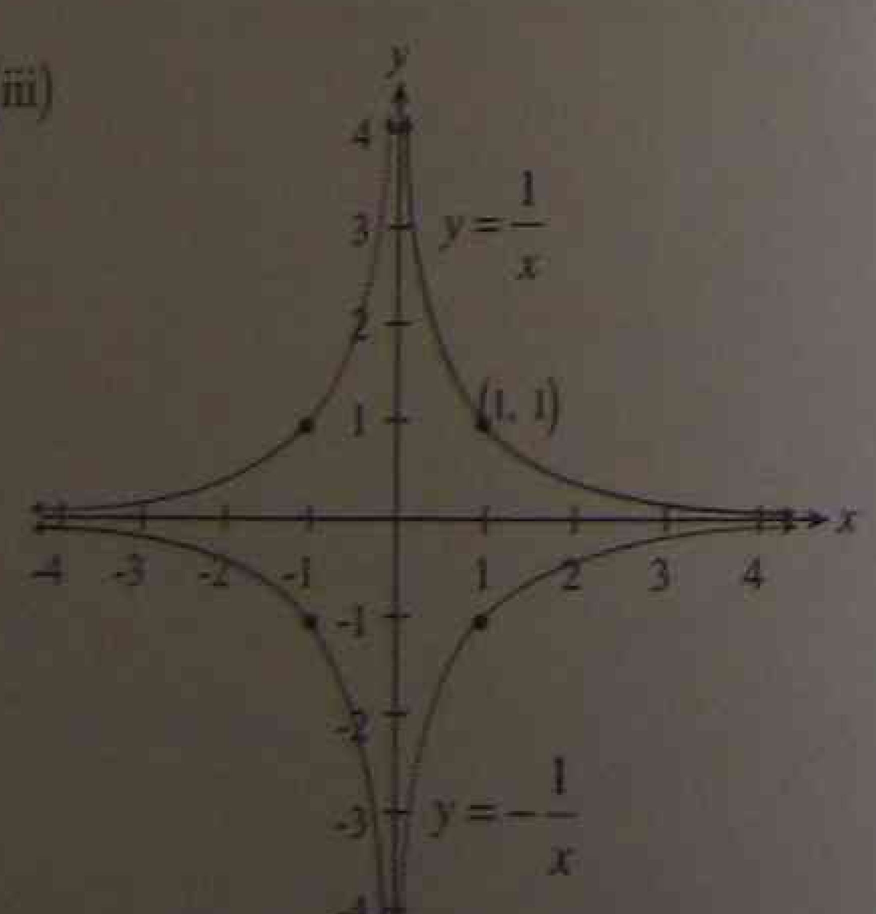
(ii)



Domain: all real x
Range: $1 \le y$

Domain: all real x
Range: $y \le 1$

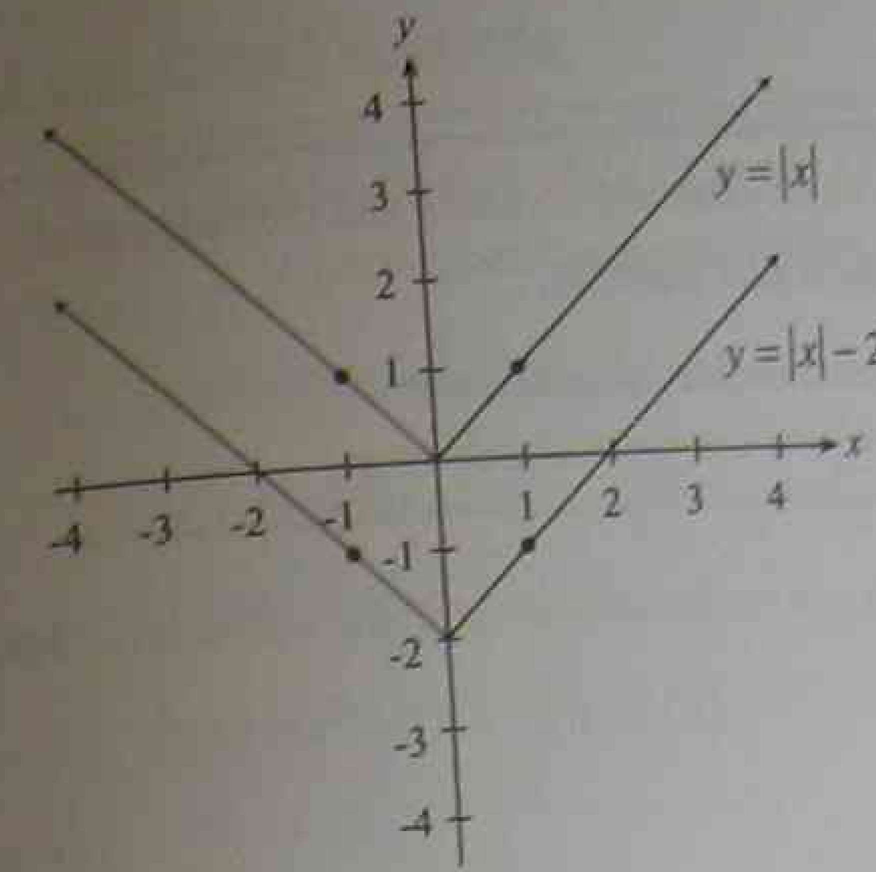
(iii)



Domain: all real x, where $x \ne 0$
Range: all real y, where $y \ne 0$

Domain: all real x, where $x \ne 0$
Range: all real y, where $y \ne 0$

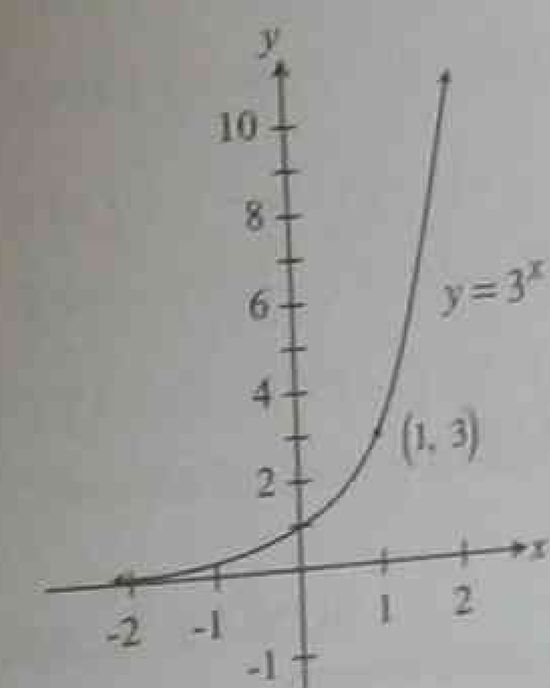
(iv)



Domain: all real x
Range: $y \ge 0$

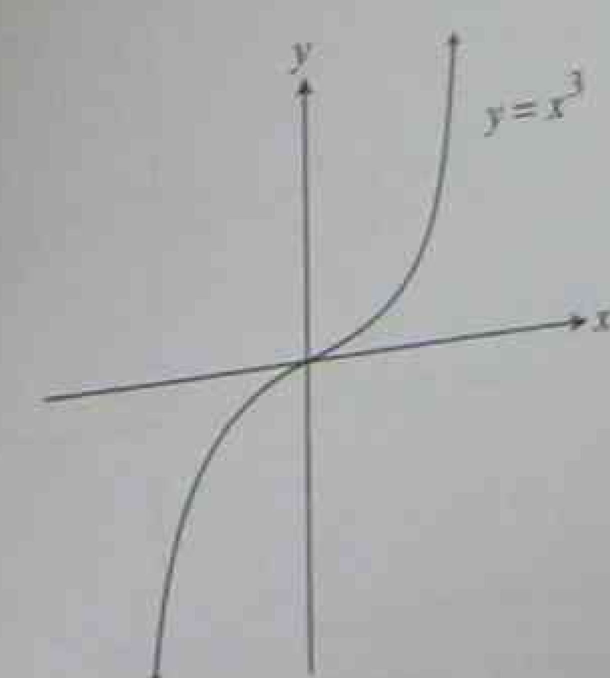
Domain: all real x
Range: $y \ge -2$

(v)



Domain: all real x
Range: $y > 0$

(vi)



Domain: all real x
Range: all real y

(C) Domain and Range

For $y = f(x)$, the domain relates to all possible x-values and the range to all possible y-values which satisfy that particular function.

The domain and range for the six most common functions are:

(1) **Semi-circle** $y = \sqrt{a^2 - x^2}$

Domain: $-a \le x \le a$

Range: $0 \le y \le a$

(2) **Hyperbola** $y = \dfrac{a}{x+b}$

Domain: all real x, $x \ne -b$

Range: all real y, $y \ne 0$

(3) **Parabola** $y = x^2 + a$

Domain: all real x

Range: $y \ge a$

(4) **Logarithmic Function** $y = \log_e(x+a)$

Domain: all real $x > -a$

Range: all real y

Note: a and b may be positive or negative integers.
The logarithmic function is covered in year 12.

(5) **Exponential Curve**: $y = a^x$

Domain : all real x

Range: $y > 0$

(6) **Modulus Function**: $y = |x| + a$

Domain: all real x

Range: $y \ge a$

Also $y = |x + a|$

Domain: all real x

Range: $y \ge 0$

Refer to the graph...

(D) Odd and Even Functions

- A functions is even if $f(-x) = f(x)$
  i.e. the function is symmetrical about the y-axis
- A function is odd if $f(-x) = -f(x)$
  i.e. the function is symmetrical about the origin.

Example 1: Determine whether these functions are odd or even or neither:

(i) $f(x) = x^2 + 3$ (ii) $f(x) = x^5 - x^3$ (iii) $f(x) = 2 - x$

Solution 1:

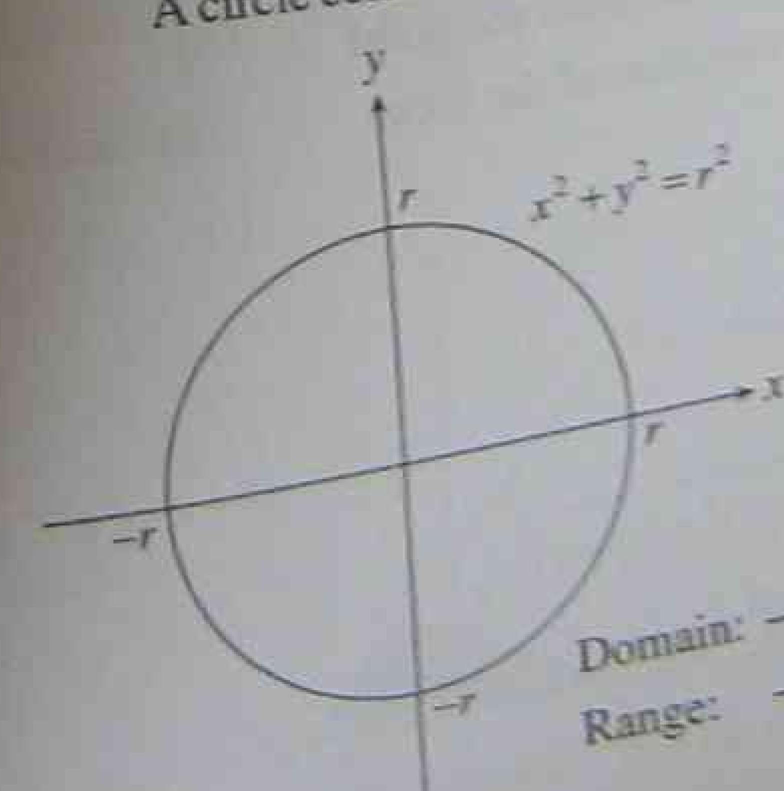
(i) $f(x) = x^2 + 3$, $f(-x) = (-x)^2 + 3 = x^2 + 3$
Since $f(x) = f(-x)$ $\therefore$ the function is even. #

(ii) $f(x) = x^5 - x^3$, $f(-x) = (-x)^5 - (-x)^3 = -x^5 + x^3 = x^3 - x^5$
Thus $-f(x) = f(-x)$ $\therefore$ the function is odd. #

(iii) $f(x) = 2 - x$, $f(-x) = 2 - (-x) = 2 + x$
Thus the function is neither odd or even. #

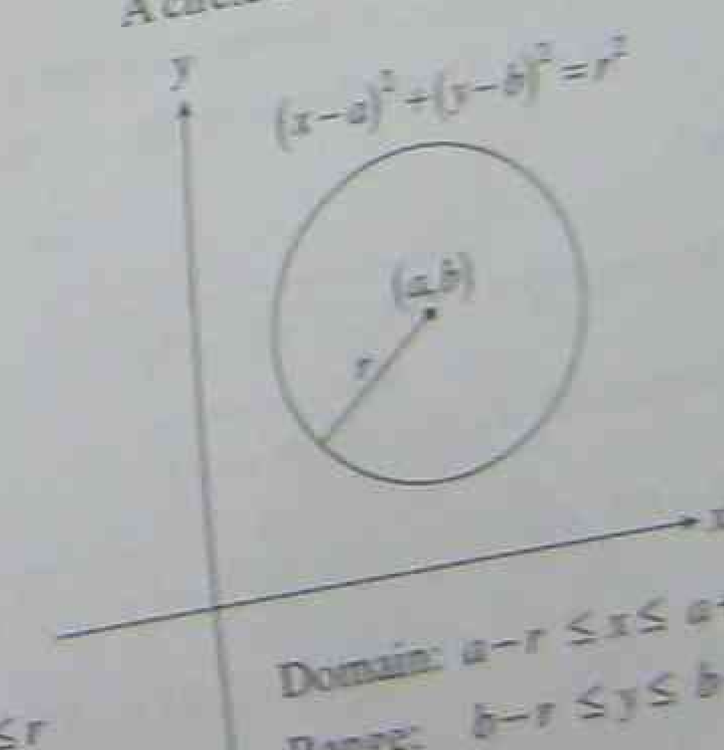
(E) Equation of a Circle

A circle centre the origin and radius r



Domain: $-r \le x \le r$
Range: $-r \le y \le r$

A circle centre (a,b) and radius r



Domain: $a - r \le x \le a + r$
Range: $b - r \le y \le b + r$

...le centre (1, -1) and radius 5 units.

Real Functions

Locus and the...

Solution 1:

$$(x-a)^2+(y-b)^2=r^2$$
$$(x-1)^2+(y-(-1))^2=5^2$$
$$(x-1)^2+(y+1)^2=25$$

Example 2: Find the radius and centre of the circle $x^2+y^2+12x+8y-29=0$

Solution 2:

To find the centre and radius we need to complete the square using the fact that $x^2+2ax=(x+a)^2-a^2$

i.e. $$x^2+y^2+12x+8y-29=0$$
$$x^2+12x+y^2+8y-29=0$$
$$(x+6)^2-36+(y+4)^2-16-29=0$$
i.e. $$(x+6)^2+(y+4)^2=81$$

$\therefore$ the circle has centre $(-6, -4)$ and radius 9 units. #

(F) Locus Problems

Locus problems involve the determination of a set of points which satisfy a given number of conditions.

Example 1: A point $P(x, y)$ moves such that it is equidistant from the point $Q(-3, 5)$ and the point $R(1, 1)$. Find the equation of the locus.

Solution 1:

Distance $PQ=\sqrt{(x+3)^2+(y-5)^2}$

Distance $PR=\sqrt{(x-1)^2+(y-1)^2}$

$PQ=PR$ i.e. $$\sqrt{(x+3)^2+(y-5)^2}=\sqrt{(x-1)^2+(y-1)^2}$$
$$(x+3)^2+(y-5)^2=(x-1)^2+(y-1)^2$$
$$x^2+6x+9+y^2-10y+25=x^2-2x+1+y^2-2y+1$$
$$8x-8y+32=0$$
$$x-y+4=0$$

$\therefore$ the locus of P is the straight line $x-y+4=0$.

Real Functions

Example 2: Find the locus of points $P(x, y)$ which are a distance 2 units from the point $A(-2, 3)$.

Solution 2:

$PA=\sqrt{(x+2)^2+(y-3)^2}$

now, $PA=2$ (given)

i.e. $\sqrt{(x+2)^2+(y-3)^2}=2$

$\therefore$ $(x+2)^2+(y-3)^2=4$

which is the equation of a circle centre $(-2, 3)$ radius 2 units. #

(G) Regions and Inequalities

The procedure for determining regions defined by inequalities is as follows:

1. Draw each line/curve on the same number plane.
2. Line is solid if included in the region and line is broken if not included in the region.
3. Shade appropriate region.

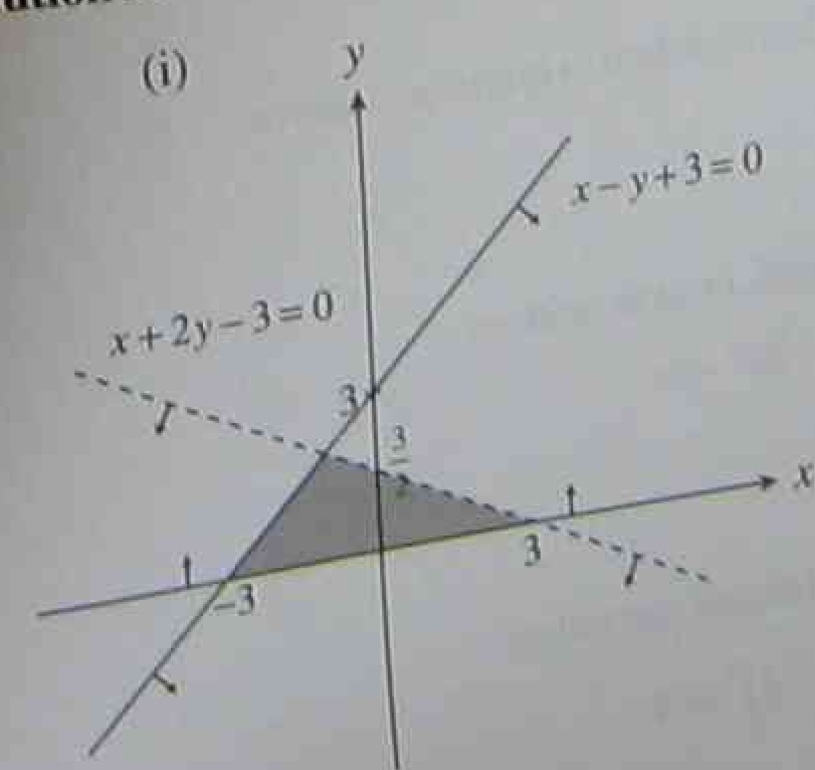
Example 1: Shade the region defined by the following inequalities

(i) $x-y+3\geq 0$, $x+2y-3<0$ and $y\geq 0$

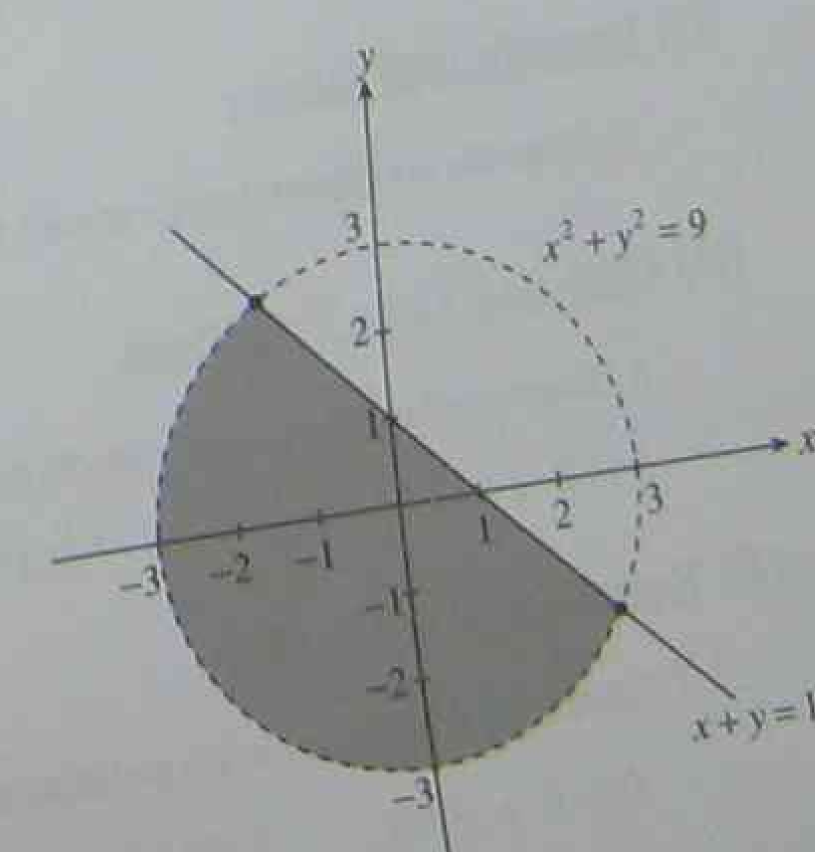
(ii) $x^2+y^2<9$ and $x+y\leq 1$

Solution 1:

(i)



(ii)



Real Functions

REVIEW EXERCISES

(A) Functional Notation

1. State whether the following curves are functions:

   (i) $y = x^2 + x$ (ii) $x = y^2$

2. Let $f(x) = \begin{cases} 5x+1 & \text{for } -5 \le x < 0 \\ 4-x^2 & \text{for } 0 \le x \le 5 \end{cases}$

   Find the value of $f(-5) - f(0) + f(5)$.

3. Let $f(x) = \begin{cases} x^2 & \text{when } x \le 0 \\ 3x-1 & \text{when } x > 0 \end{cases}$

   (i) Evaluate $f(-2) + f(2) - f(0)$.

   (ii) Sketch the curve $y = f(x)$.

(B) Familiar Function

4. Sketch the following curves, showing their essential features:

   (i) $y = 2^x$ (ii) $y = x^3$ (iii) $y = 9 - x^2$

   (iv) $-\sqrt{4-x^2}$ (v) $y = \dfrac{-5}{x+1}$ (vi) $y = |x| + 1$

(C) Domain and Range

5. State the domain and range of each function sketched in question 4 above.

(D) Odd and Even Functions

6. State whether the following functions are odd, even or neither:

   (i) $f(x) = 4x - x^3$ (ii) $f(x) = x^4 - 2x^2 - x$

(E) Equation of a Circle

7. Sketch the following circles and state their domain and range:

   (i) $x^2 + y^2 = 1$ (ii) $(x+1)^2 + (y-2)^2 = 4$

8. Find the equation of the circle centre (3, -1) and radius 9 units.

9. Find the centre and radius of th

(i) $x^2 + y^2 - 4x + 12y + 36 = 0$

(ii) $2x^2 + 4x + 2y^2 - 16y + 18 = 0$

(F) Locus Problems

10. Find the locus of points $P(x, y)$ which move such that they are equidistant from the points $A(2, -1)$ and $B(4, 8)$.

11. A point $P(x, y)$ moves such that it is equidistant from the lines $y = x$ and $y = -x$ respectively. Find the equation of the locus.

12. A point $P(x, y)$ moves so that its distance from $A(-3, 4)$ is twice its distance from the point $B(0, 4)$.

    (i) Show that the locus of points $P(x, y)$ is: $3x^2 + 3y^2 - 6x - 24y + 39 = 0$

    (ii) Describe the locus geometrically.

(G) Regions and Inequalities

13. Shade the region defined by the following inequalities:

    (i) $y \le 4x^2, \quad y \le 6 - 2x, \quad x \ge 0$

    (ii) $x^2 + y^2 \le 4, \quad |x| > 1$

    (iii) $y \ge x, \quad y \le x + 1, \quad y \le \sqrt{1-x^2}$

WORKED SOLUTIONS TO REVIEW EXERCISES

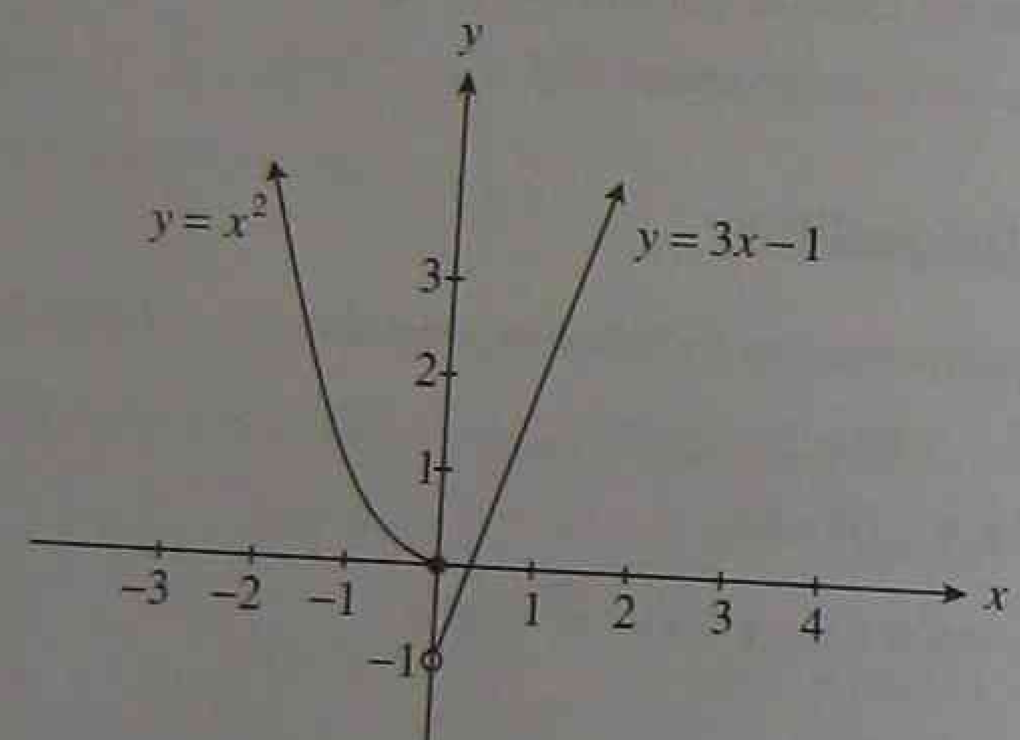
1. (i) Since each x-value has only one corresponding y-value
$\therefore\ y = x^2 + x$ is a function. #

(ii) For $x = 1$, $\quad y^2 = 1$ i.e. $y = \pm 1$
Since for $x = 1$ there are 2 values for y
$\therefore\ x = y^2$ is not a function. #

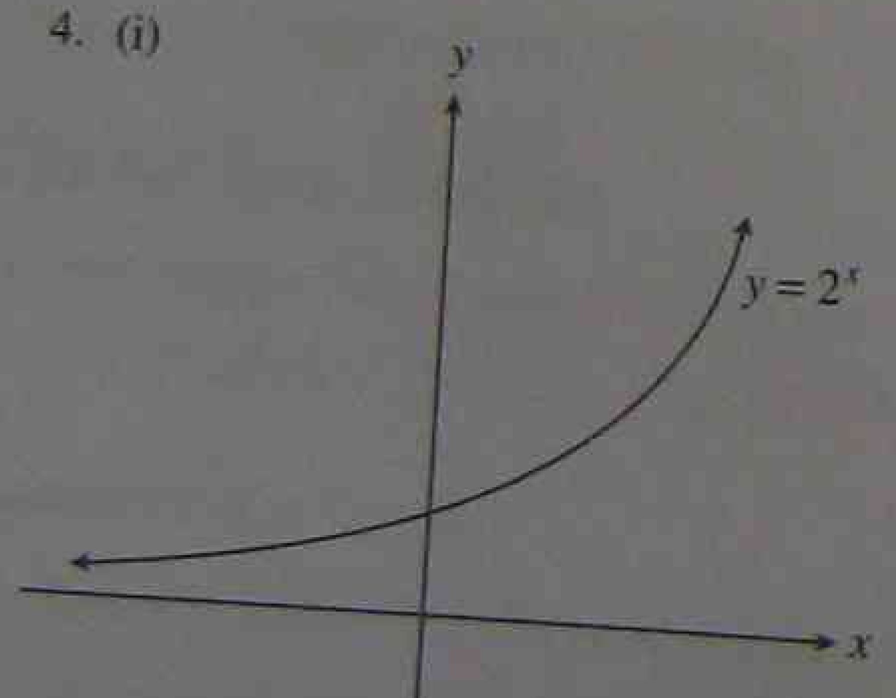
2. $f(-5) - f(0) + f(5) = (5\times -5 + 1) - (4 - 0) + (4 - 5^2)$
$= -24 - 4 - 21 = -49$ #

3. (i) $f(-2) + f(2) - f(10) = (-2)^2 + (3\times 2 - 1) - 0 = 4 + 5 = 9$ #

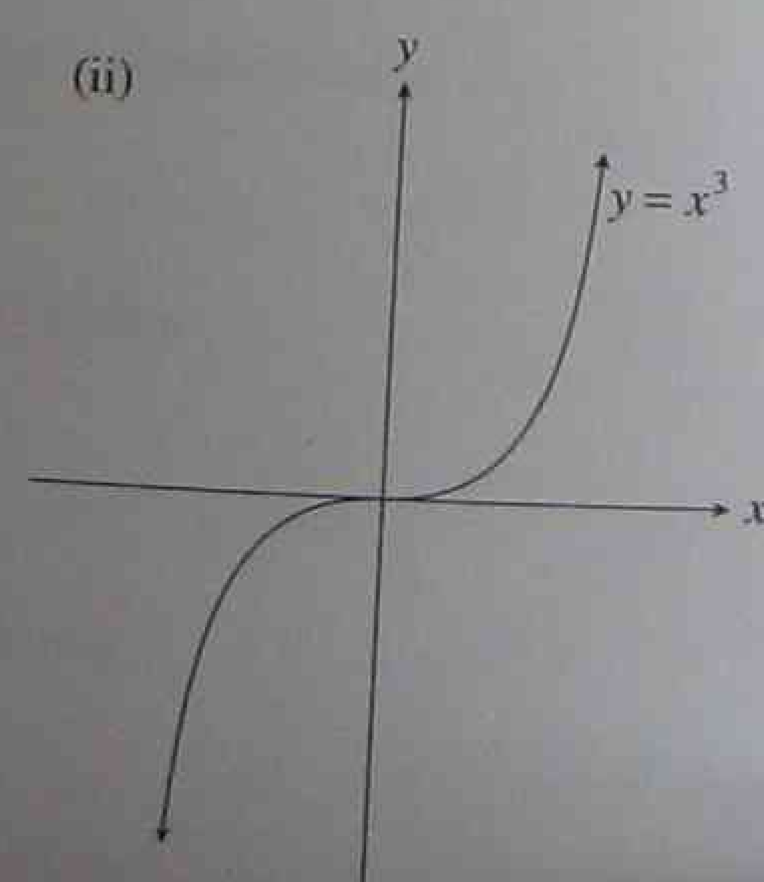
(ii)



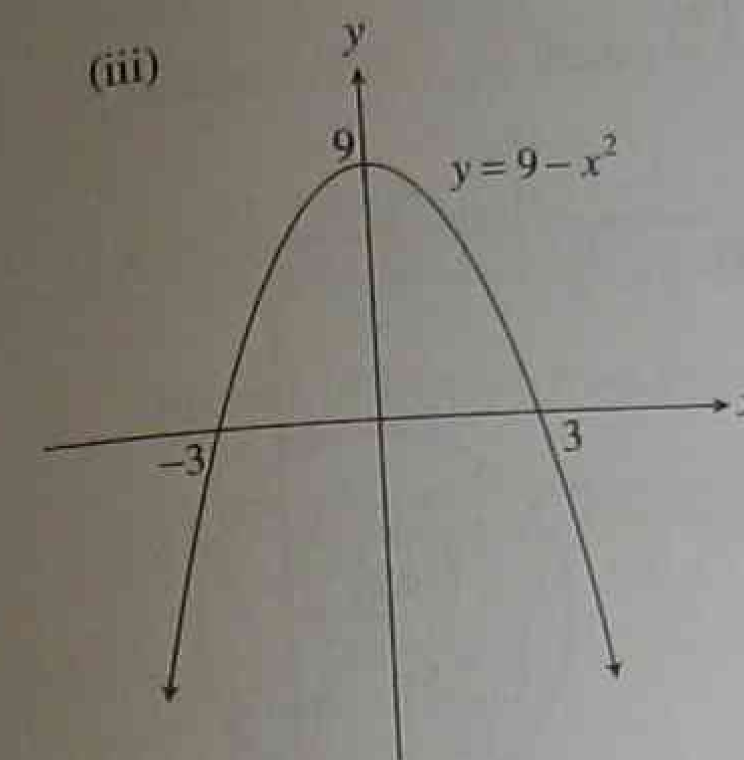
4. (i)



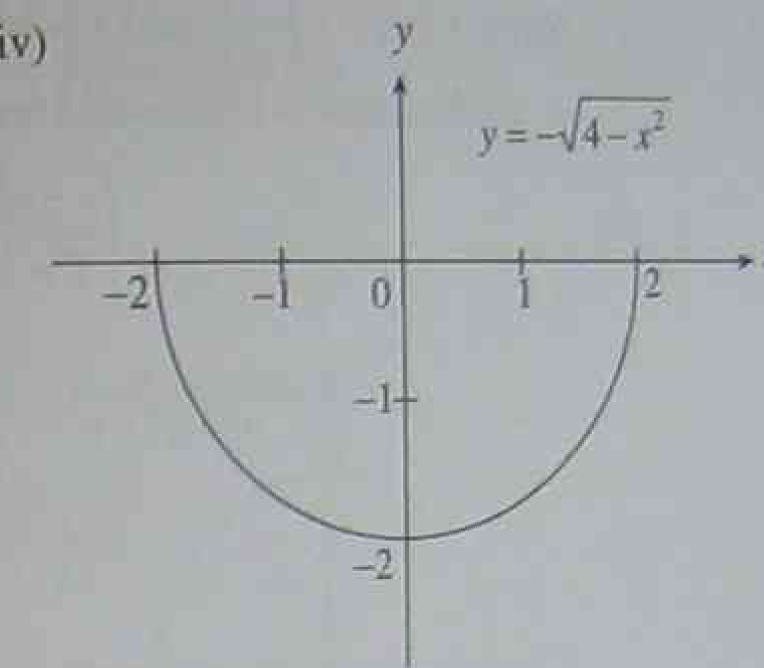
(ii)



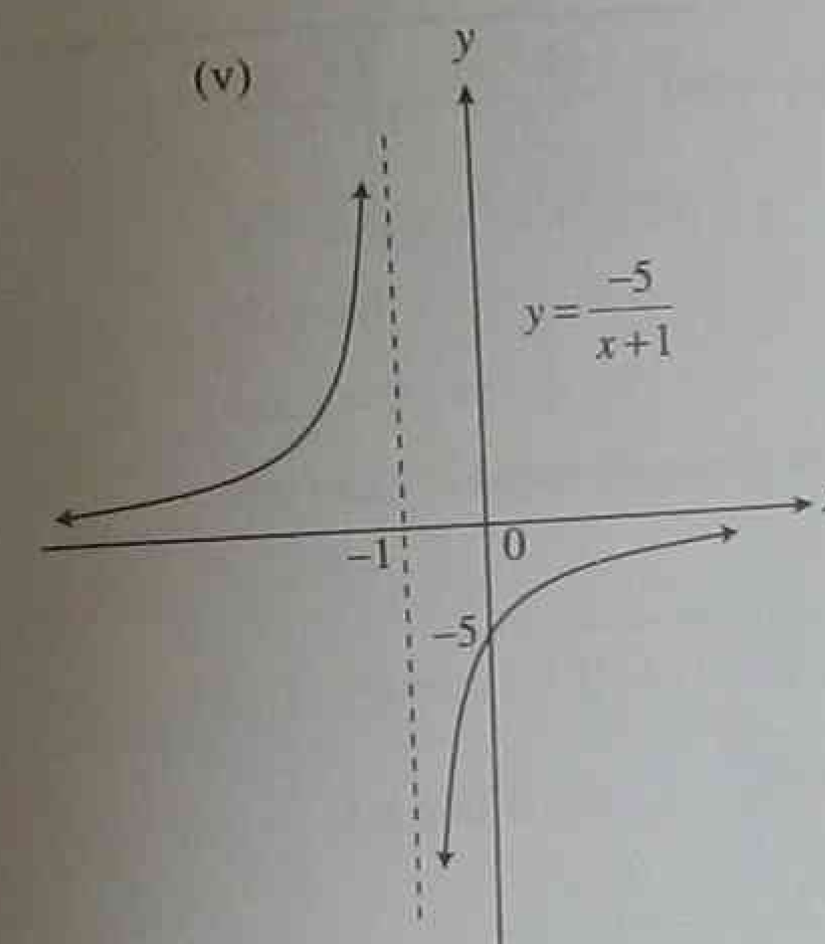
(iii)



(iv)



(v)



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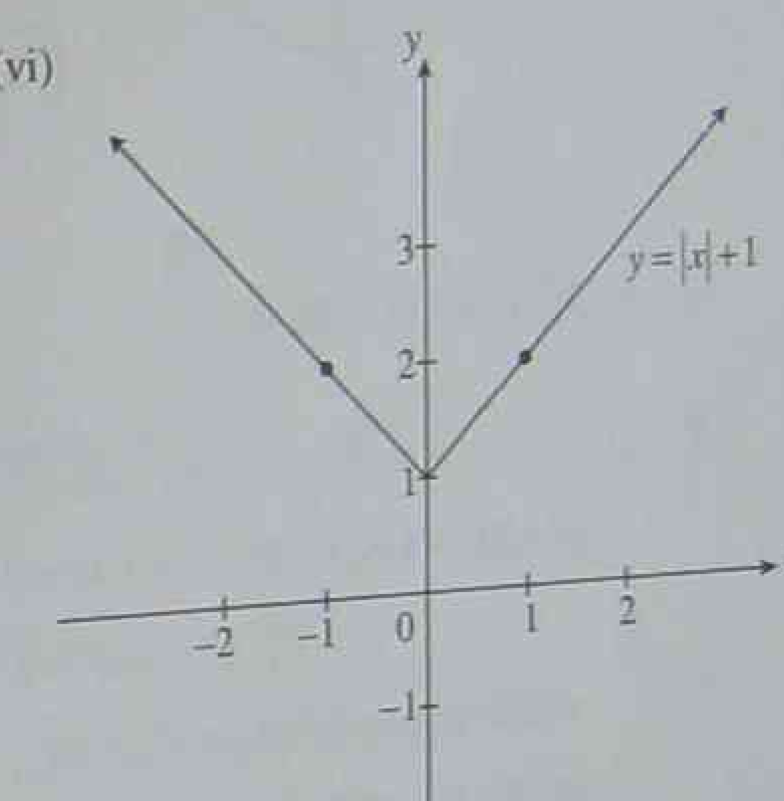


5. (i) Domain: all real x
Range: $y > 0$ #

(ii) Domain: all real x
Range: all real y #

(iii) Domain: all real x
Range: $y \le 9$ #

(iv) Domain: $-2 \le x \le 2$
Range: $-2 \le y \le 0$ #

(v) Domain: all real x, $x \ne -1$
Range: all real y, $y \ne 0$ #

(vi) Domain: all real x
Range: $y \ge 1$ #

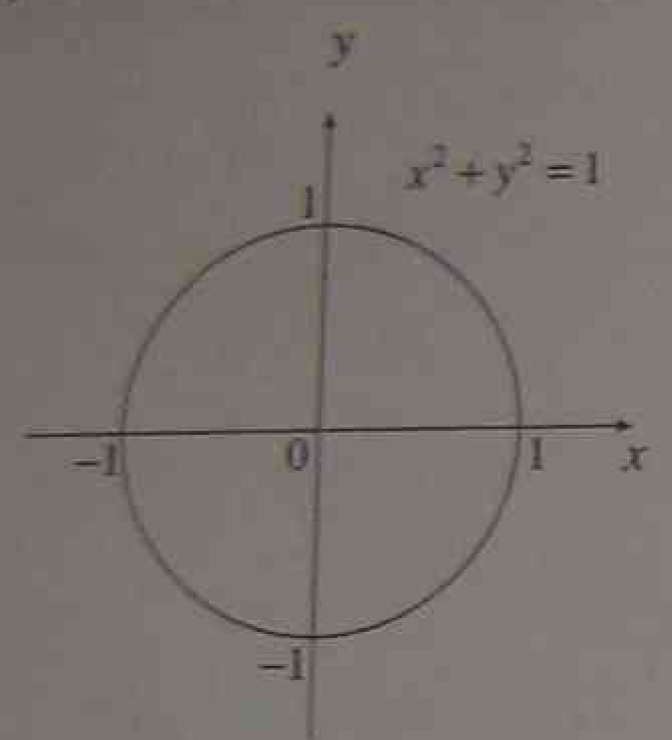
6. (i) $f(x) = 4x - x^3$, $f(-x) = -4x - (-x)^3 = -4x + x^3$
Since $-f(x) = f(-x)$ $\therefore$ the function is odd. #

(ii) $f(x) = x^4 - 2x^2 - x$, $\quad f(-x) = (-x)^4 - 2(-x)^2 - (-x) = x^4 - 2x^2 + x$

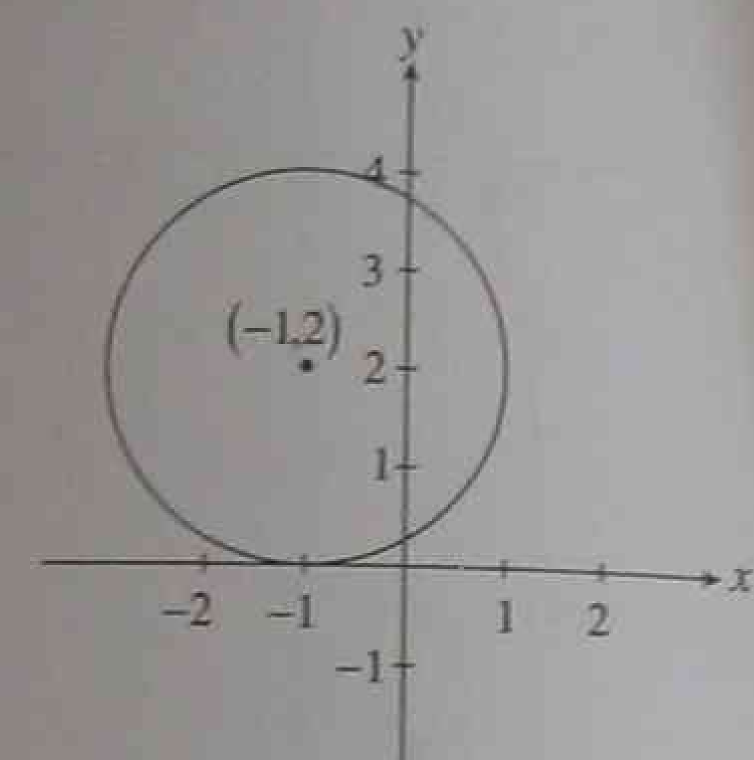
Since $f(x) \neq f(-x)$ and $f(-x) \neq -f(x)$

$\therefore$ the function is neither odd nor even.

7. (i)



(ii)



8. $(x-3)^2+(y+1)^2=81$ #

9. (i)

$$x^2+y^2-4x+12y+36=0$$
$$(x-2)^2-4+(y+6)^2-36+36=0$$
$$(x-2)^2+(y+6)^2=4$$

which is a circle centre $(2,-6)$

radius $=2$ units #

(ii)

$$2x^2+4x+2y^2-16y+18=0$$
$$x^2+2x+y^2-8y+9=0 \quad \text{(dividing by 2)}$$
$$(x+1)^2-1+(y-4)^2-16+9=0$$
$$(x+1)^2+(y-4)=8$$

which is a circle centre $(-1, 4)$

radius $=\sqrt{8}=2\sqrt{2}$ units #

10. $PA=\sqrt{(x-2)^2+(y+1)^2}$, $PB=\sqrt{(x-4)^2+(y-8)^2}$

$PA=PB$ i.e.

$$\sqrt{(x-2)^2+(y+1)^2}=\sqrt{(x-4)^2+(y-8)^2}$$
$$(x-2)^2+(y+1)^2=(x-4)^2+(y-8)^2$$
$$x^2-4x+4+y^2+2y+1=x^2-8x+16+y^2-16y+64$$

11. Distance of $P(x,y)$ from $y=x$ or $x-y=0$ is given by:

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|=\left|\frac{1\times x-1\times y+0}{\sqrt{1+(-1)^2}}\right|=\left|\frac{x-y}{\sqrt{2}}\right|$$

Distance of $P(x,y)$ from $y=-x$ or $x+y=0$ is given by:

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|=\left|\frac{1\times x+1\times y+0}{\sqrt{1^2+1^2}}\right|=\left|\frac{x+y}{\sqrt{2}}\right|$$

i.e. $\left|\frac{x-y}{\sqrt{2}}\right|=\left|\frac{x+y}{\sqrt{2}}\right|$

$$\frac{(x-y)^2}{2}=\frac{(x+y)^2}{2} \quad \left(\text{note } |a|=\sqrt{a^2}\right)$$
$$x^2-2xy+y^2=x^2+2xy+y^2$$
$$4xy=0$$

i.e. $x=0$ or $y=0$

i.e. the locus is either the y-axis or the x-axis. #

$\therefore$ the locus is either the y-axis or the x-axis. #

12.(i) $PA=\sqrt{(x+3)^2+(y-4)^2}$, $PB=\sqrt{x^2+(y-4)^2}$

$PA=2PB$ i.e.

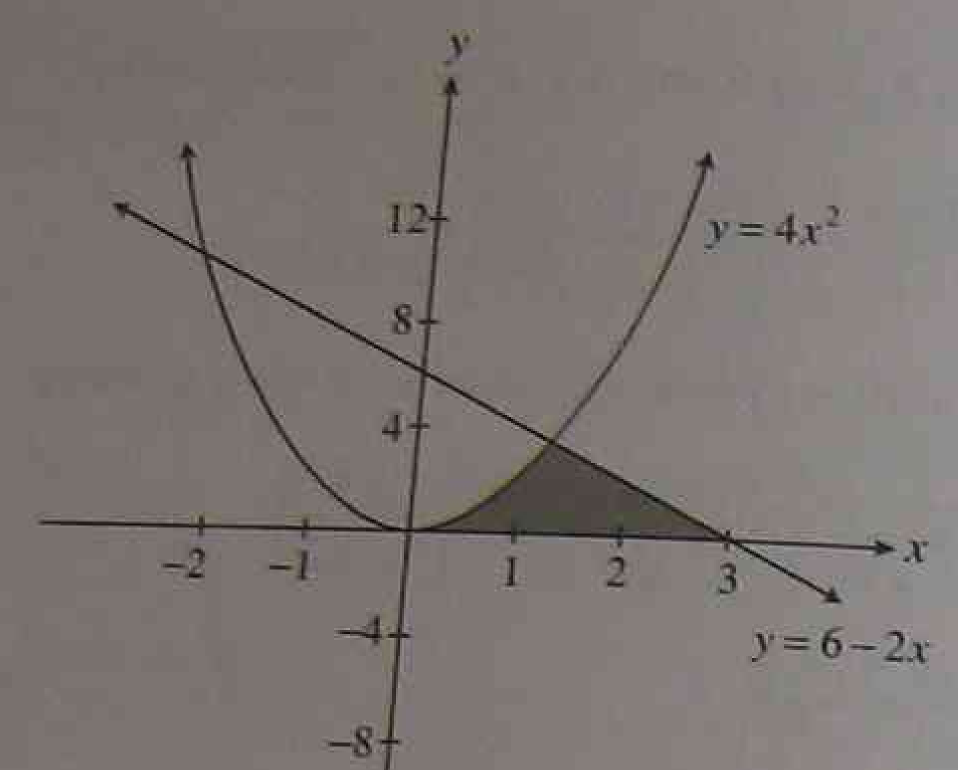
$$\sqrt{(x+3)^2+(y-4)^2}=2\sqrt{x^2+(y-4)^2}$$
$$(x+3)^2+(y-4)^2=4\left(x^2+(y-4)^2\right)$$
$$x^2+6x+9+y^2-8y+16=4x^2+4\left(y^2-8y+16\right)$$
$$3x^2+3y^2-6x-24y+39=0$$
$$x^2+y^2-2x-8y+13=0 \text{ \#}$$

(ii)

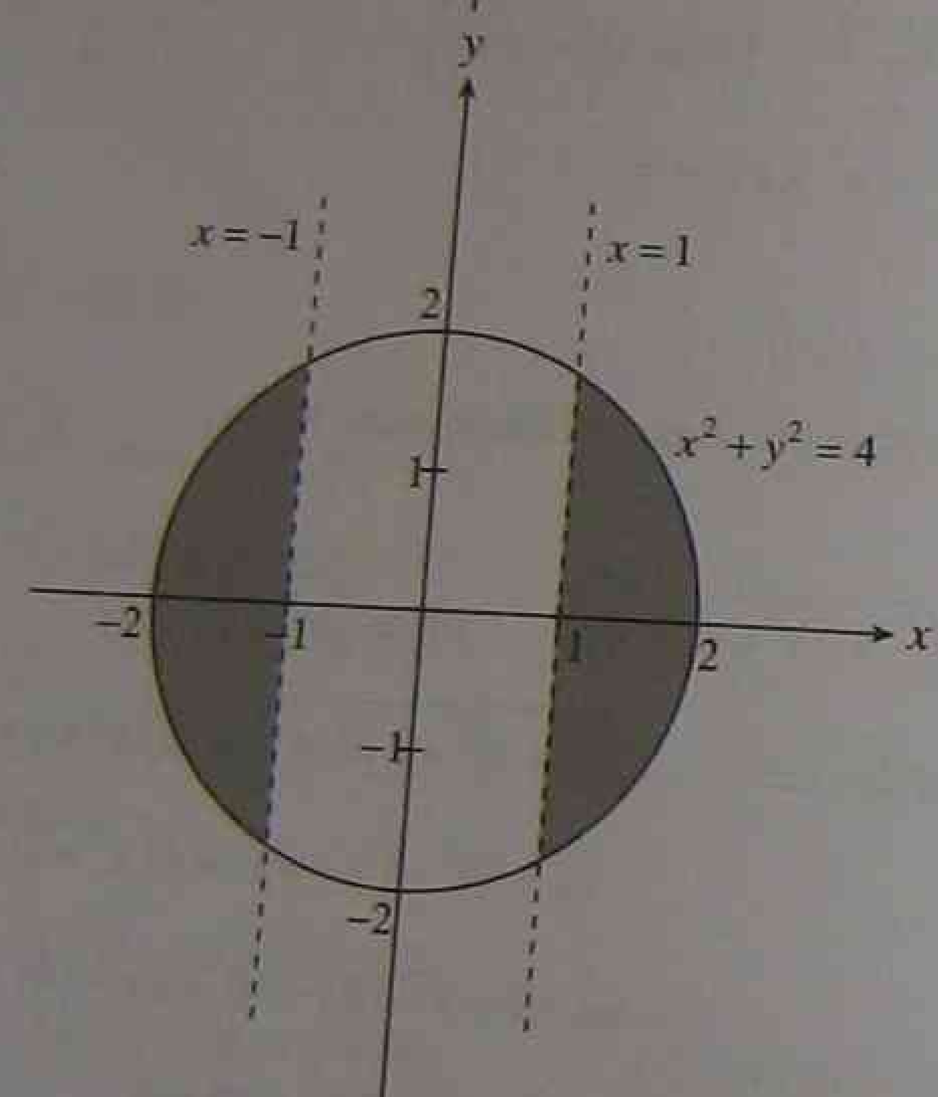
$$3x^2+3y^2-6x-24y+39=0$$
$$x^2+y^2-2x-8y+13=0$$
$$(x-1)^2-1+(y-4)^2-16+13=0$$
$$(x-1)^2+(y-4)^2=4$$

which is a circle centre $(1, 4)$ and radius $=2$ units #

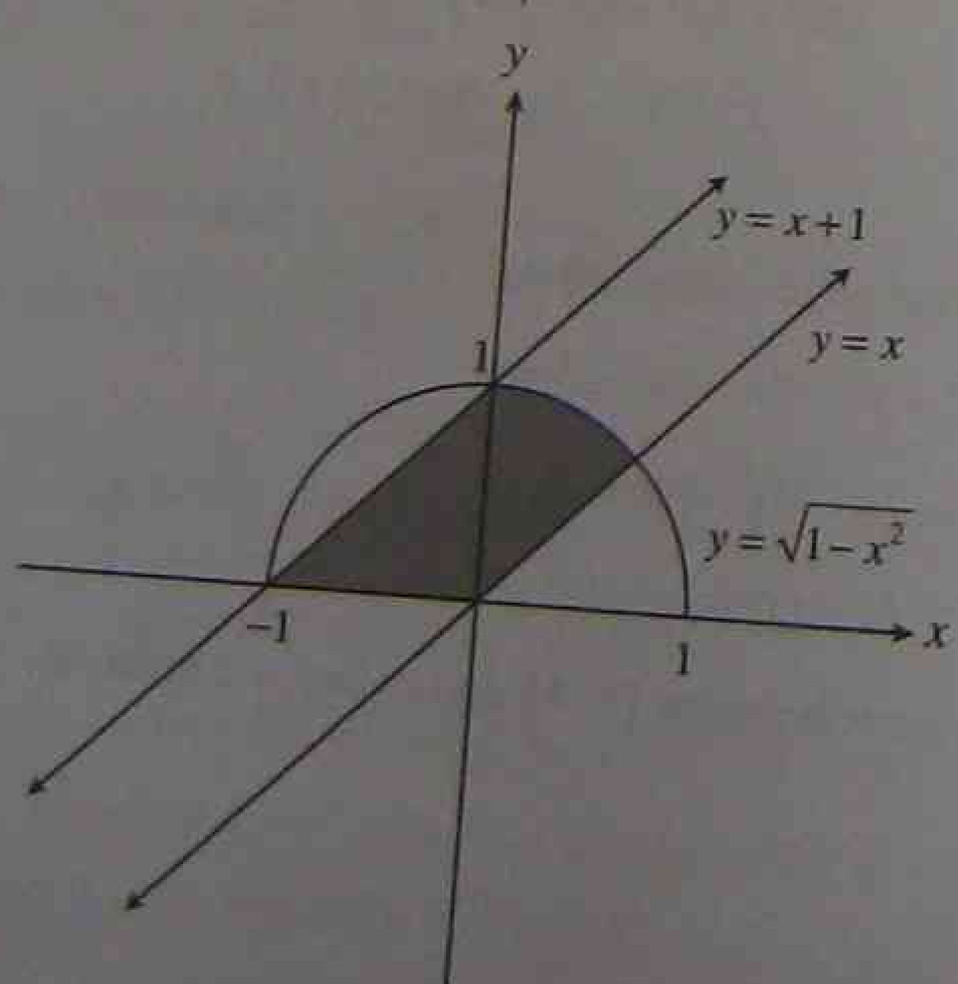
13. (i)



(ii)



(iii)



THE QUADRATIC POLYNOMIAL AND THE PARABOLA

(A) The Quadratic Polynomial

The general equation $y=ax^2+bx+c$, where a, b and c are constants and $a \neq 0$ a quadratic polynomial. The graph of the quadratic polynomial is called a parabola.

Example 1: Which of the following functions is a parabola:

(i) $y=3x^2-5x+9$ (ii) $y=x^2-x^3$

(iii) $y=4x-7$ (iv) $y=x^2-2$

Solution 1:

(i) $y=3x^2-5x+9$ is a parabola as it is of the form $y=ax^2+bx+c$. #

(ii) $y=x^2-x^3$ is not a parabola as it has an x^3 term. #

(iii) $y=4x-7$ is not a parabola as it does not have an x^2 term. #

(iv) $y=x^2-2$ is a parabola with $b=0$, $a=1$ and $c=-2$. #

(B) Completing the Square

Completing the square involves converting quadratic equations into perfect squares:

$$x^2+2bx=(x+b)^2-b^2$$

In completing the square, the following steps can be used:

1. Divide (if necessary) the quadratic equation by the coefficient of x^2, as the coefficient of x^2 must always be 1.
2. Form the perfect square by halving the coefficient of x (i.e. b), adding to b to x, and squaring (i.e. $(x+b)^2$).
3. Subtract b^2 from the perfect square.

Example 1: Complete the square and express in the form $(y-k)=(x-b)^2$.

(i) $y=x^2-8x+20$ (ii) $y=x^2-6x+5$

(iii) $y=2x^2-12x+2$ (iv) $y=3x^2+9x+6$

Solution 1:

(i) $y = x^2 - 8x + 20$
$= (x-4)^2 - 16 + 20$
$= (x-4)^2 + 4$
i.e. $(y-4) = (x-4)^2$ #

(ii) $y = x^2 - 6x + 5$
$= (x-3)^2 - 9 + 5$
$= (x-3)^2 - 4$
i.e. $(y+4) = (x-3)^2$ (note $k = -4$) #

(iii) $y = 2x^2 - 12x + 2$
i.e. $\frac{y}{2} = x^2 - 6x + 1$ (dividing throughout by 2)
$= (x-3)^2 - 9 + 1$
$= (x-3)^2 - 8$
i.e. $\frac{y}{2} + 8 = (x-3)^2$
$\frac{1}{2}(y+16) = (x-3)^2$ #

(iv) $y = 3x^2 + 9x + 6$
i.e. $\frac{y}{3} = x^2 + 3x + 2$ (dividing throughout by 3)
$= \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + 2 = \left(x+\frac{3}{2}\right)^2 - \frac{1}{4}$
$\frac{y}{3} + \frac{1}{4} = \left(x+\frac{3}{2}\right)^2$
$\frac{1}{3}\left(y+\frac{3}{4}\right) = \left(x+\frac{3}{2}\right)^2$ #

Example 2: By completing the square show that $y = x^2 - 4x + 14$ can not equal zero. Hence find the minimum value of y.

Solution 2:

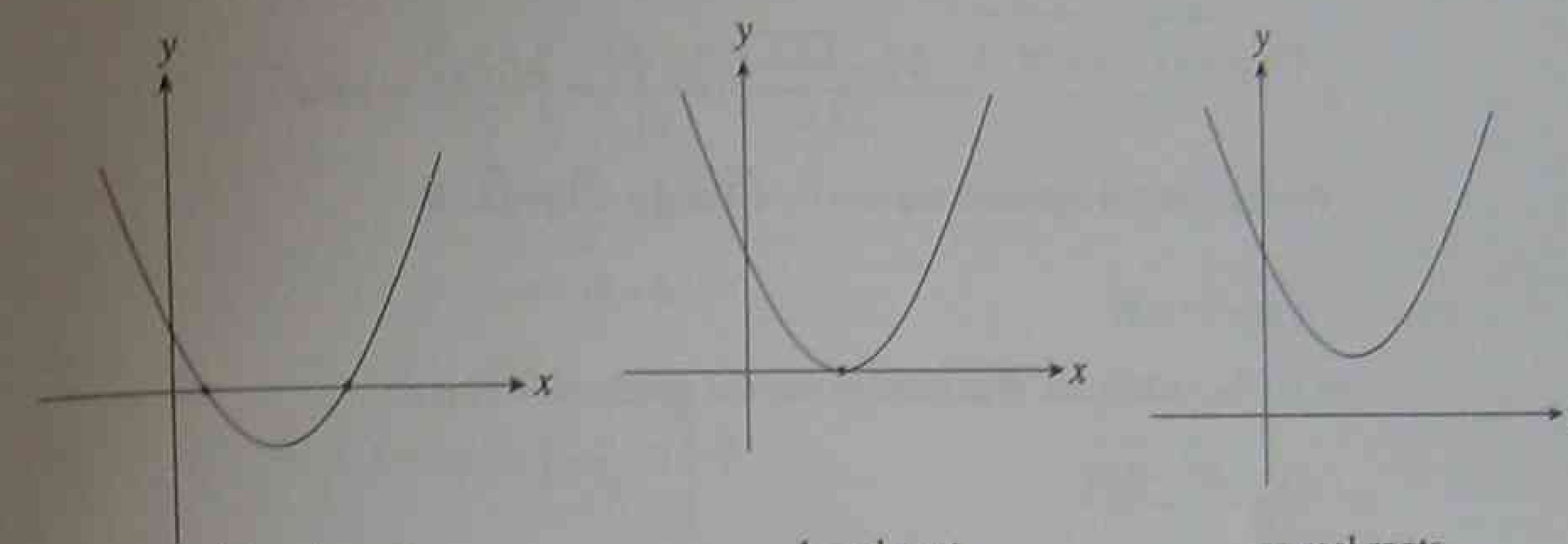
$y = x^2 - 4x + 14$
$= (x-2)^2 - 4 + 14$
$= (x-2)^2 + 10$

Since $(x-2)^2 \geq 0$ for all x $\therefore$ y is always positive and thus can not equal zero.

Hence, the minimum value of y occurs when $(x-2)^2 = 0$ i.e. $y = 10$. #

(C) Quadratic Roots

The ***roots*** of the quadratic equation $y = ax^2 + bx + c$ are the points where the curve crosses the x-axis (i.e. $y = 0$). There are 3 possible outcomes.



2 real roots | 1 real root | no real roots

To find the roots, we solve the equation $ax^2 + bx + c = 0$ using the following steps:

(i) Solve the equation by factorisation - if not possible, then go to (ii).

(ii) Solve the equation by the ***quadratic formula***.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1: Find the roots of the following quadratic equations:

(i) $x^2 - 6x + 5 = 0$ (ii) $2x^2 - 3x + 1 = 0$

(iii) $x^2 - 2x - 1 = 0$ (iv) $4x^2 + 12x + 9 = 0$

Solution 1:

(i) $x^2 - 6x + 5 = 0$
$(x-5)(x-1) = 0$
i.e. $x = 5, 1$ $\therefore$ the roots of the equation are $x = 1$ and $x = 5$ #

(ii) $2x^2 - 3x + 1 = 0$
$(2x-1)(x-1) = 0$
$\therefore$ $(2x-1) = 0$ or $(x-1) = 0$
$2x = 1$ $x = 1$
$x = \frac{1}{2}$ $\therefore$ the roots of the equation are $x = \frac{1}{2}$ and $x = 1$. #

(iii) $x^2-2x-1=0$

Cannot be factorised, thus need to use quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2\pm\sqrt{(-2)^2-4\times1\times-1}}{2}=\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

$\therefore$ the roots of the equation are $x=1-\sqrt{2}$ and $x=1+\sqrt{2}$. #

(iv) $4x^2+12x+7=0$

Can not be factorised, thus need to use the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-12\pm\sqrt{(12)^2-4\times4\times7}}{8}$$

$$=\frac{-12\pm\sqrt{144-112}}{8}=\frac{-12\pm\sqrt{32}}{8}=\frac{-12\pm4\sqrt{2}}{8}=\frac{-3\pm\sqrt{2}}{2}$$

$\therefore$ the roots of the equation are $x=\frac{-3-\sqrt{2}}{2}$ and $x=\frac{-3+\sqrt{2}}{2}$. #

(D) Axis of Symmetry

The parabola $y=ax^2+bx+c$ has its axis of symmetry at $x=-\frac{b}{2a}$.

The greatest or least value of y occurs at this point, which is also the y-coordinate of the vertex.

Example 1: Find the axis of symmetry and coordinates of the vertex for the following parabolas.

(i) $y=x^2-6x+5$ (ii) $y=4x^2+12x+9$

Solution 1:

(i) $y=x^2-6x+5$

Axis of symmetry: $x=-\frac{b}{2a}=\frac{-(-6)}{2\times1}=\frac{6}{2}=3$

y-coordinate of vertex occurs at $x=3$

i.e. $y=(3)^2-6\times3+5$

$=9-18+5=-4$

$\therefore$ Vertex is $(3,-4)$ #

(ii) $y=4x^2+12x+9$

Axis of symmetry: $x=\frac{-b}{2a}=\frac{-12}{2\times4}=\frac{-12}{8}=-\frac{3}{2}$

y-coordinate of the vertex occurs at $x=-\frac{3}{2}$

i.e. $y=4\left(-\frac{3}{2}\right)^2+12\left(-\frac{3}{2}\right)+9$

$=4\times\frac{9}{4}-18+9$

$=9-18+9$

$=0$

$\therefore$ Vertex is $\left(-\frac{3}{2},0\right)$ #

(E) Graphing Quadratic Functions

In graphing the parabola $y=ax^2+bx+c$, the following steps can be used:

1. If $a>0$ parabola is concave up.
   If $a<0$ parabola is concave down.
2. Identify the roots (the x-intercepts) of the parabola by solving $ax^2+bx+c=0$. This can be solved by factorisation or the quadratic formula.
3. Identify the y-intercept (i.e. when $x=0$).
4. Identify the coordinates of the vertex. The x-coordinate will be $x=\frac{-b}{2a}$ and the y-coordinate is found by substituting this value into the original curve.
5. Sketch the curve.

Example 1: Sketch the following parabolas, showing all essential features:

(i) $y=2x^2-3x-5$ (ii) $y=4x-x^2$ (iii) $y=-x^2+2x-5$

Solution 1:

(i) $y=2x^2-3x-5$

Step 1: $a>0$ $\therefore$ the parabola is concave up.

Step 2: Let $2x^2-3x-5=0$ to find the roots (x-intercepts).

i.e. $(2x-5)(x+1)=0$

$2x-5=0$ or $x+1=0$

$2x=5$ $\quad$ $x=-1$

$x=\frac{5}{2}$

Step 3: Let $x=0$ to find y-intercept.

i.e. $y=2\times0-3\times0-5=-5$

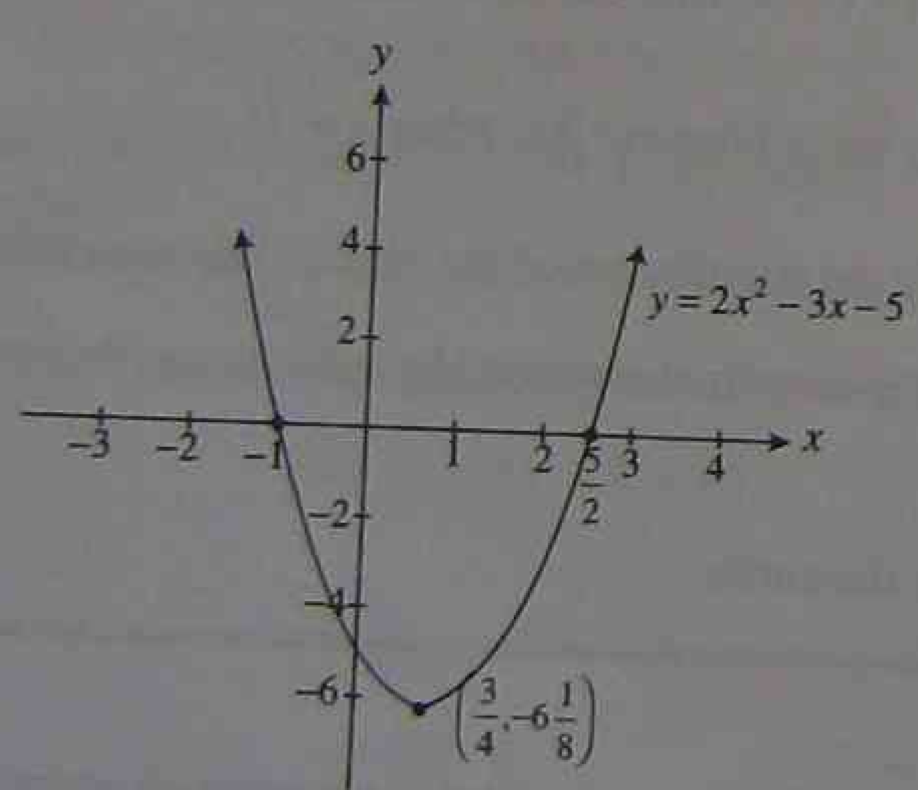
Step 4: x-coordinate of vertex is given by:

$x=\frac{-b}{2a}=\frac{3}{4}$

when $x=\frac{3}{4}$, $y=2\times\left(\frac{3}{4}\right)^2-3\times\frac{3}{4}-5=-6\frac{1}{8}$

$\therefore$ Vertex is $\left(\frac{3}{4},-6\frac{1}{8}\right)$

Step 5:



(ii) $y=4x-x^2$

Step 1: $a<0$ $\therefore$ parabola is concave down.

Step 2: Let $4x-x^2=0$ to find the roots (x-intercepts)

i.e. $x(4-x)=0$

$x=0$ or $4-x=0$

$x=4$

Step 3: Let $x=0$ to find the y-intercept.

i.e. $y=0-0=0$

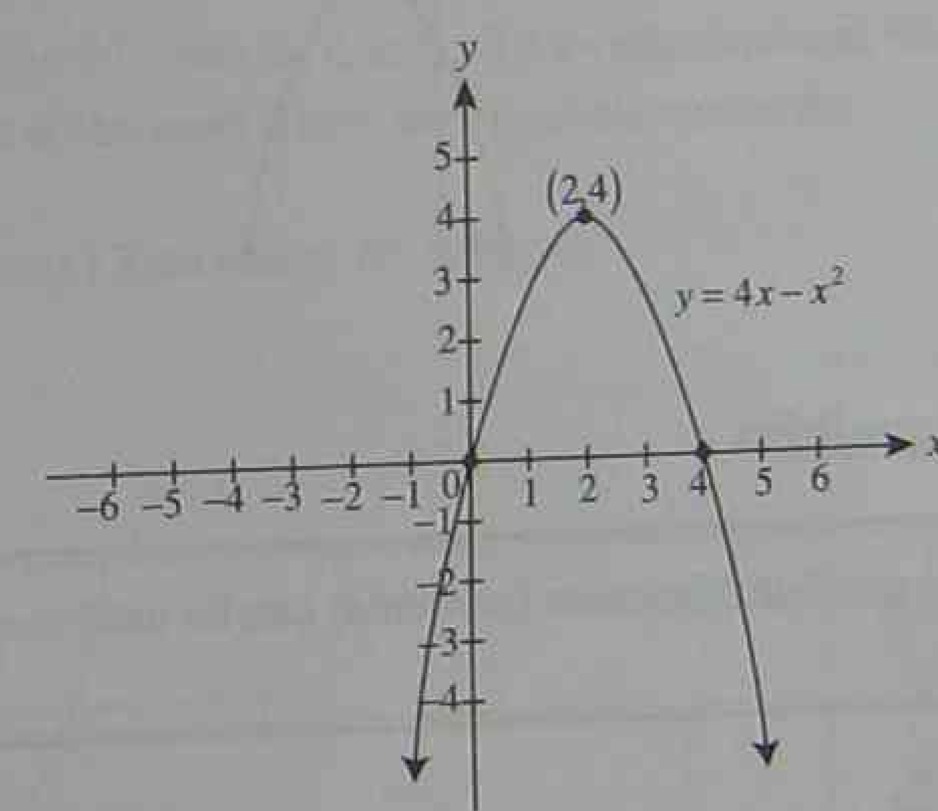
Step 4: x-coordinate of vertex is given by:

$x=\frac{-b}{2a}=\frac{-4}{-2}=2$

when $x=2$, $y=4\times2-2^2=8-4=4$

$\therefore$ Vertex is $(2, 4)$ #

Step 5:



(iii) $y=-x^2+2x-5$

Step 1: $a<0$ $\therefore$ parabola is concave down.

Step 2: Let $-x^2+2x-5=0$ to find the roots.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-2\pm\sqrt{4-4\times-1\times-5}}{-2}$$
$$=\frac{-2\pm\sqrt{-16}}{-2}$$

Since $\sqrt{-16}$ is not real, $\therefore$ there are no real roots

i.e. parabola doesn't cross the x-axis.

Step 3: Let $x=0$, $y=-0+0-5=-5$

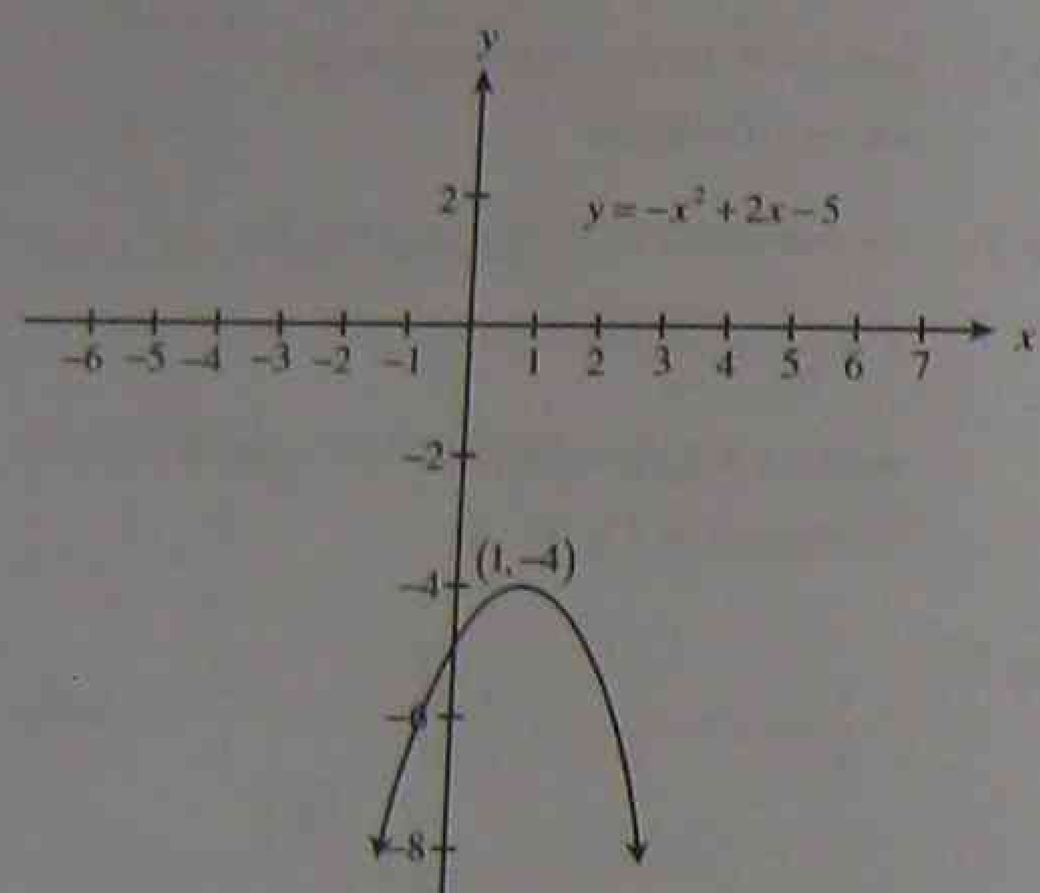
Step 4: x-coordinate of vertex is given by:

$x=\frac{-b}{2a}=\frac{-2}{-2}=1$

when $x=1$, $y=-(-1)^2+2\times1-5=-1+2-5=-4$

$\therefore$ vertex is $(1,-4)$ #

Step 5:



(F) Quadratic Inequalities

The graph of a quadratic function (parabola) can be used to solve quadratic inequalities.

xample 1: Solve:

(i) $x^2-x-2>0$ (ii) $-5x^2+6x-1\ge0$

lution 1:

(i) $x^2-x-2>0$

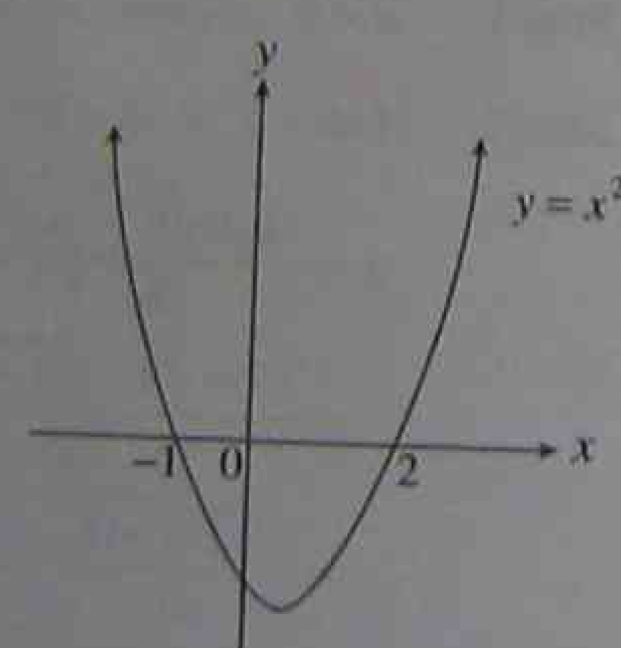
Let $y=x^2-x-2$

$=(x-2)(x+1)=0$

$\therefore\ x=2,-1$

from the graph $y>0$,

for $x>2$ and $x<-1$. #



(ii) $-5x^2+6x-1\ge0$

Let $y=-5x^2+6x-1$

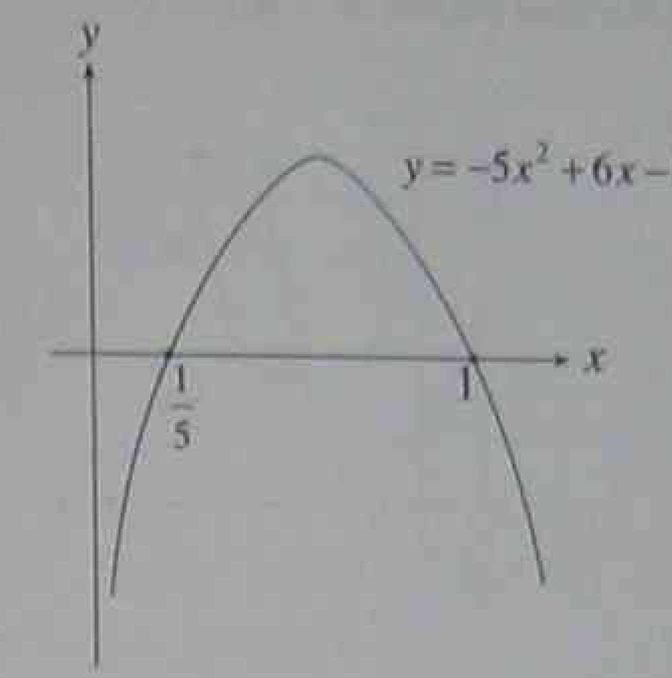
$=-(5x^2-6x+1)$

$=-(5x-1)(x-1)=0$

$\therefore\ x=\frac{1}{5},1$

from the graph $y\ge0$ for

$\frac{1}{5}\le x\le1$ #

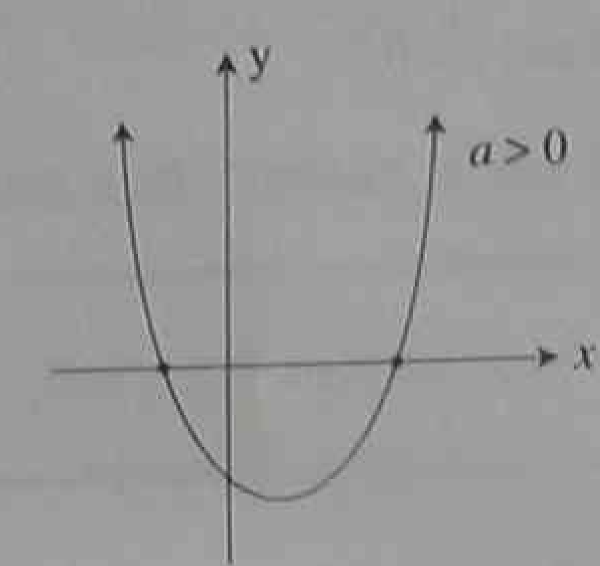


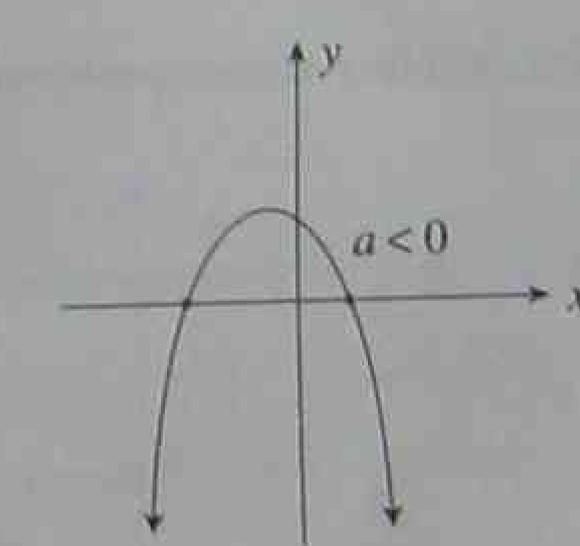
(G) The Discriminant, Positive Definite and Negative Definite

From before, the roots of $ax^2+bx+c=0$ are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

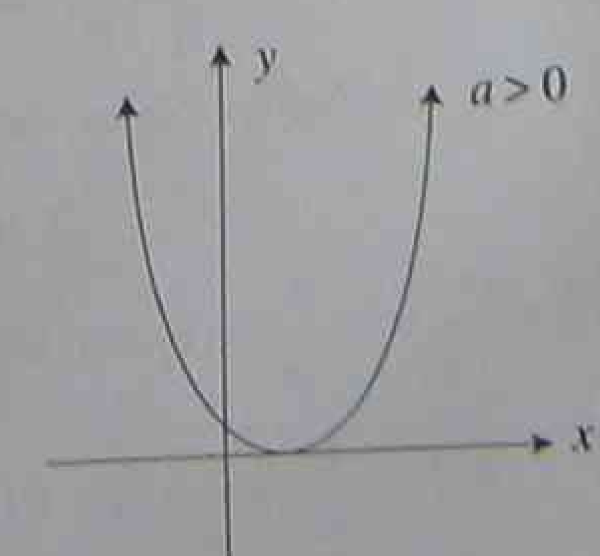
We denote the expression b^2-4ac by Δ and call it the **discriminant**. The discriminant determines the **nature of the roots**.There are 6 possible outcomes:

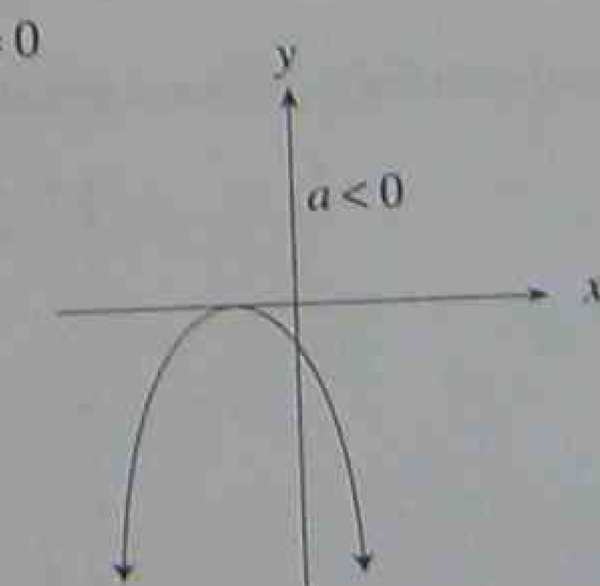
1. Distinct real roots / Two roots: $b^2-4ac>0$



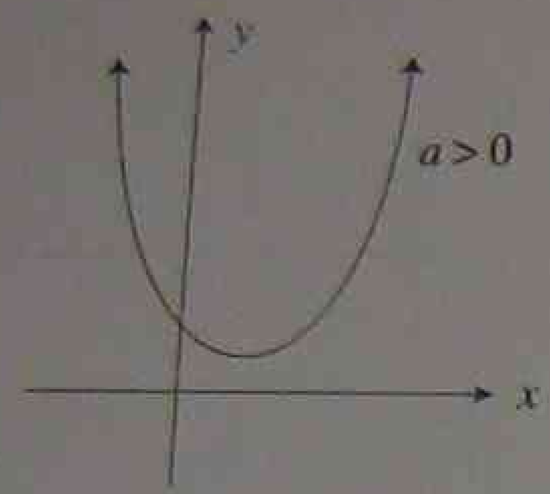


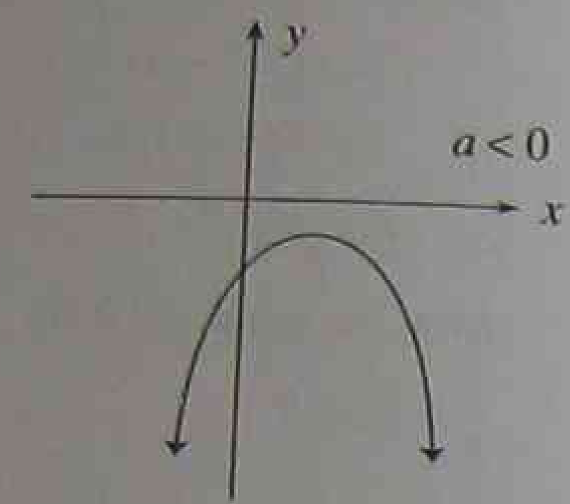
2. Equal roots / Exactly one root: $b^2-4ac=0$



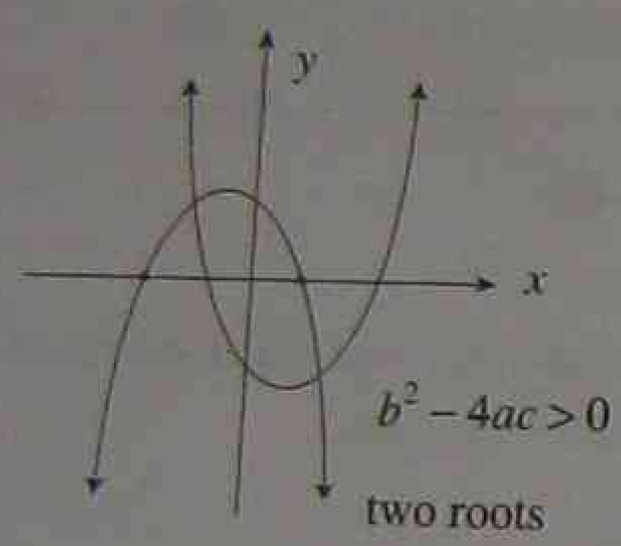


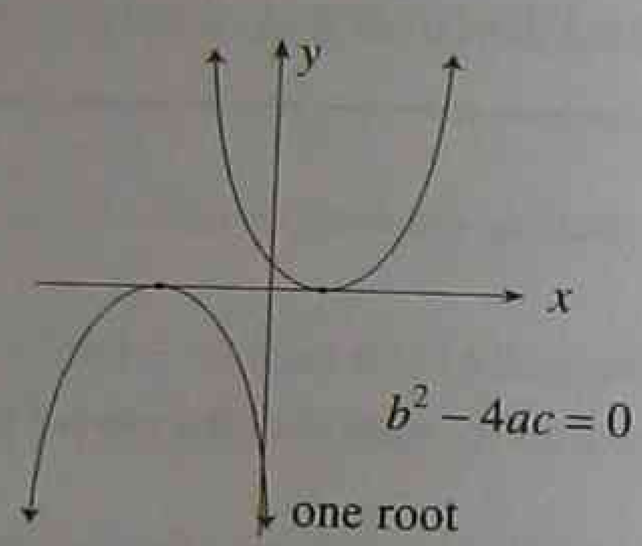
3. No real roots: $b^2-4ac<0$



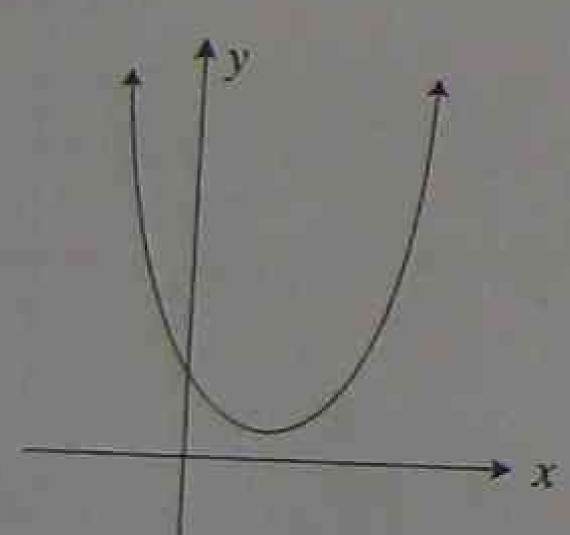


4. Real roots: $b^2-4ac\geq 0$

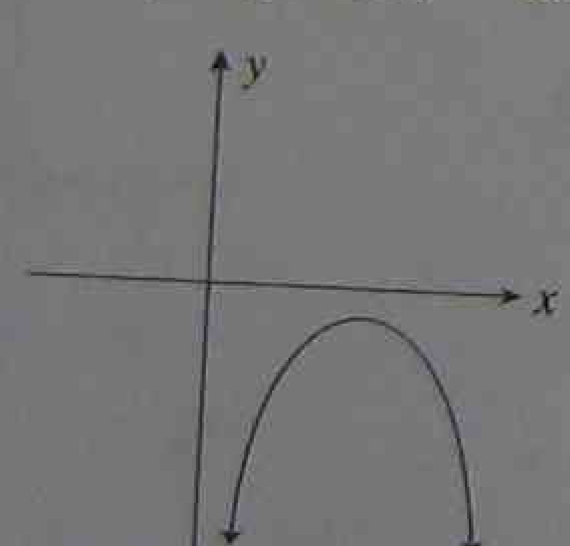




5. Positive definite / Always positive: $b^2-4ac<0$ and $a>0$



6. Negative definite / Always negative: $b^2-4ac<0$ and $a<0$



Example 1: For what values of k does the equation $3x^2-kx+3=0$ have real roots?

Solution 1:

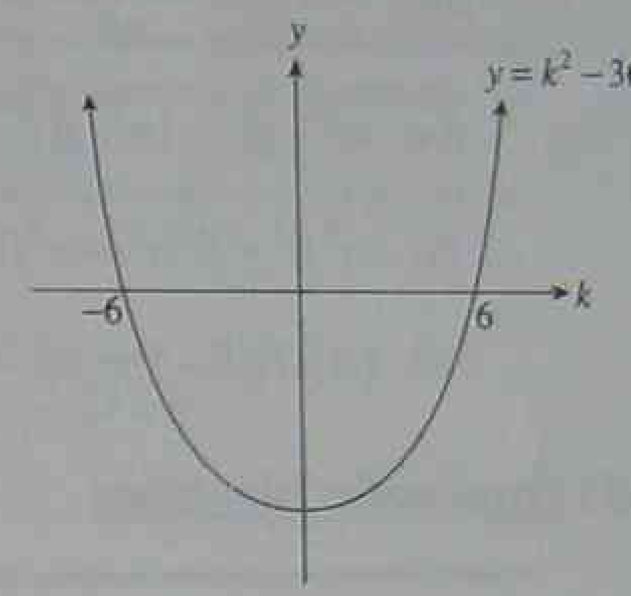
$$\Delta=b^2-4ac$$
$$=(-k)^2-4\times3\times3$$
$$=k^2-36$$

for real roots $b^2-4ac\geq 0$

i.e. $k^2-36\geq 0$

$(k-6)(k+6)\geq 0$

from the graph $k\geq 6$ or $k\leq -6$



Example 2: Find the set of values of k for which the equation $3x^2+kx+2$ is always positive.

Solution 2:

Always positive means that $a>0$ and $b^2-4ac<0$

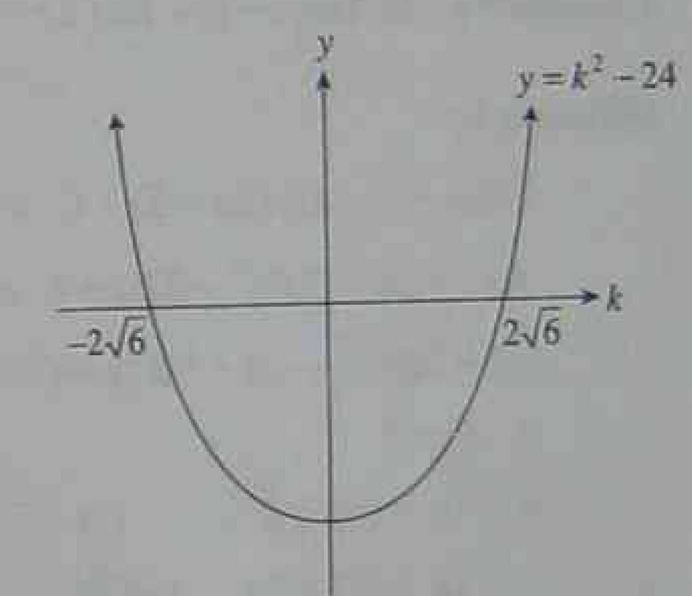
i.e. $k^2-4\times3\times2<0$

$k^2-24<0$

$\left(k-\sqrt{24}\right)\left(k+\sqrt{24}\right)<0$

$\left(k-2\sqrt{6}\right)\left(k+2\sqrt{6}\right)<0$

$\therefore\ -2\sqrt{6}<k<2\sqrt{6}$ #



(H) Sum and Product of the Roots

If α and β are roots of $ax^2+bx+c=0$, then: $\alpha+\beta=\dfrac{-b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

Example 1: Let α and β be the roots of the equation $x^2-3x-1=0$. Find the values of:

(i) $\alpha+\beta$

(ii) $\alpha\beta$

(iii) $\dfrac{5}{\alpha}+\dfrac{5}{\beta}$

(iv) $\alpha^2+\beta^2$

(v) $\alpha^2\beta^3+\beta^2\alpha^3$

(vi) $(\alpha-1)(\beta-1)$

Solution 1:

(i) $\alpha+\beta=-\dfrac{b}{a}=-\dfrac{(-3)}{1}=3$ #

(ii) $\alpha\beta=\dfrac{c}{a}=-\dfrac{1}{1}=-1$ #

(iii) $\frac{5}{\alpha}+\frac{5}{\beta}=\frac{5\beta+5\alpha}{\alpha\beta}=\frac{5(\alpha+\beta)}{\alpha\beta}=\frac{5\times3}{-1}=-15$ #

(iv) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=3^2-2\times-1=11$ #

(v) $\alpha^3\beta^3+\beta^3\alpha^3=\alpha^2\beta^2(\beta+\alpha)=(\alpha\beta)^2(\alpha+\beta)=(-1)^2\times3=3$ #

(vi) $(\alpha-1)(\beta-1)=\alpha\beta-\alpha-\beta+1=\alpha\beta-(\alpha+\beta)+1=-1-(3)+1=-3$ #

(I) Quadratic Identities

If $a_1x^2+b_1x+c_1=a_2x^2+b_2x+c_2$ for more than 2 values of x,
then $a_1=a_2$, $b_1=b_2$, $c_1=c_2$.

Example 1: If $Ax(x-1)+Bx(2x-3)+C=8x^2-11x+5$, find the values of A, B and C.

Solution 1:

$Ax(x-1)+Bx(2x-3)+C=8x^2-11x+5$

$Ax^2-Ax+2Bx^2-3Bx+C=8x^2-11x+5$

$(A+2B)x^2-(A+3B)x+C=8x^2-11x+5$

$\therefore\ C=5$

$A+2B=8$ ......(1)

$A+3B=11$ .....(2)

$(2)-(1): B=3$

Subtituting into (1) gives:

$A+2\times3=8$

$A=2$

$\therefore A=2, B=3, C=5$ #

Example 2: If $A(x-2)^2+Bx+C=3x^2-8$, find the values of A, B and C.

Solution 2:

$A(x-2)^2+Bx+C=3x^2-8$

$A(x^2-4x+4)+Bx+C=3x^2-8$

$Ax^2+(B-4A)x+4A+C=3x^2-8$

i.e. $A=3$

$B-4A=0$ i.e. $B-4\times3=0$ i.e. $B=12$

$4A+C=-8$ i.e. $4\times3+C=-8$ i.e. $C=-20$

$\therefore A=3, B=12$ and $C=-20$ #

(J) Equations Reducible to Quadratics

More difficult equations may be solved by first reducing the equation to a quadratic through a simple substitution.

Example 1: Solve the following equations for all values x:

(i) $4x^4-4x^2-3=0$ (ii) $9^x+12(3^x)-27=0$

Solution 1:

(i) $4x^4-4x^2-3=0$

Let $u=x^2$

i.e. $4u^2-4u-3=0$

$(2u+1)(2u-3)=0$

i.e. $2u+1=0$ or $2u-3=0$

$u=-\frac{1}{2}$ $u=\frac{3}{2}$

thus, $x^2=-\frac{1}{2}$ or $x^2=\frac{3}{2}$

has no real solutions. $x=\pm\sqrt{\frac{3}{2}}$

$\therefore$ the only solution for x are $x=\sqrt{\frac{3}{2}}$ and $x=-\sqrt{\frac{3}{2}}$ #

(ii) $9^x+12(3^x)+27=0$

$(3^2)^x+12(3^x)+27=0$

$(3^x)^2+12(3^x)+27=0$

Let $u=3^x$

i.e. $u^2+12u+27=0$

$(u-9)(u-3)=0$

i.e. $u-9=0$ or $u-3=0$

$u=9$ $u=3$

thus, $3^x=9$ $3^x=3$

$x=2$ $x=1$

$\therefore$ the solutions for x are $x=1$ and $x=2$ #

Quadratic Polynomials

REVIEW EXERCISES

(A) The Quadratic Polynomial

1. Which of the following functions is a parabola:

(i) $y = x^2 + 1$ (ii) $y = x(x^2 - 5)$ (iii) $y = 9 - x^2$

(B) Completing the Square

2. Complete the square and express in the form $(y-k) = (x-b)^2$.

(i) $y = x^2 - 10x + 5$ (ii) $y = x^2 - 14x + 49$

(ii) $y = 2x^2 - 36x + 40$ (iv) $y = 5x^2 - 10x + 15$

(C) Quadratic Roots

3. Find the roots of the following quadratic equations:

(i) $x^2 - 5x + 6 = 0$ (ii) $3x^2 - 6x + 1 = 0$

(iii) $2x^2 + 3x - 5 = 0$ (iv) $3 + 2x - 2x^2 = 0$

(D) Axis of Symmetry

4. Find the axis of symmetry and the coordinates of the vertex for the following parabolas:

(i) $x^2 + 4x$ (ii) $y = 1 - x - 4x^2$

(E) Graphing Quadratic Functions

5. Sketch the following parabolas, showing all essential features:

(i) $y = 9 - x^2$ (ii) $y = 4x^2 - 12x$

(iii) $y = x^2 - 8x + 16$ (iv) $y = x^2 - 4x - 10$

(F) Quadratic Inequalities

6. Solve for x:

(i) $x^2 + 6x + 5 > 0$ (ii) $9 + 6x - 3x^2 > 0$

(iii) $4x^2 - x - 2 \le 0$ (iv) $x^2 - 10x + 25 \ge 0$

(G) The Discriminant, Positive Definite and Negative Definite

7. For what values of k does the quadratic equation $(k+3)x^2 + 4x + k = 0$ have no real roots.

8. Show that the roots of the equation $x^2 - (k+1)x + (k-2) = 0$ are distinct and real for all values of k.

(H) Sum and Product of the Roots

9. Let α and β be the roots of the equation $3x^2 + 6x - 3 = 0$. Find the values of :

(i) $\alpha + \beta$ (ii) $\alpha\beta$ (iii) $\alpha^2 + \beta^2$

(iv) $\alpha^2\beta + \beta^2\alpha$ (v) $(\alpha+2)(\beta+2)$ (vi) $\alpha - \beta$

10. Given the equation $2x^2 + 4x - 3 = 0$ has two roots α and β, evaluate the following:

(i) $\alpha + \beta$ (ii) $\alpha\beta$ (iii) $\alpha^2 + \beta^2$

(iv) $\alpha - \beta$ (v) $\alpha^2 - \beta^2$ (vi) $\alpha^3 - \beta^3$

(I) Quadratic Identities

11. If $A(x+1)(x-2) + B(x+1) + C = 3x^2 + 11x - 6$, find the values of A, B and C.

12. If $Ax(x-5) + (Bx-1)(C+1) = 2x^2 + 15x - 5$, find the values of A, B and C.

(J) Equations Reducible to Quadratics

13. Solve the following equations for x:

(i) $x^6 - 7x^3 - 8 = 0$ (ii) $4^x - 3(2^x) + 2 = 0$

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $y=x^2+1$ is a parabola. #

(ii) $y=x\left(x^2-5\right)=x^3-5x$ which is not a parabola. #

(iii) $y=9-x^2$ is a parabola. #

2. (i)
$$\begin{aligned} y&=x^2-10x+5\\ &=(x-5)^2-25+5\\ &=(x-5)^2-20 \end{aligned}$$
$(y+20)=(x-5)^2$ #

(ii)
$$\begin{aligned} y&=x^2-14x+49\\ &=(x-7)^2-49+49\\ &=(x-7)^2 \end{aligned}$$ #

(iii)
$$\begin{aligned} y&=2x^2-36x+40\\ \frac{y}{2}&=x^2-18x+20\\ &=(x-9)^2-81+20\\ &=(x-9)^2-61 \end{aligned}$$
$\left(\frac{y}{2}+61\right)=(x-9)^2$

$\frac{1}{2}(y+122)=(x-9)^2$ #

(iv)
$$\begin{aligned} y&=5x^2-10x+15\\ \frac{y}{5}&=x^2-2x+3\\ &=(x-1)^2-1+3\\ &=(x-1)^2+2 \end{aligned}$$
$\left(\frac{y}{5}-2\right)=(x-1)^2$

$\frac{1}{5}(y-10)=(x-1)^2$ #

3. (i) $x^2-5x+6=0$

$(x-6)(x+1)=0$

$x=6,-1$ #

(ii) $3x^2-6x+1=0$
$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &=\frac{6\pm\sqrt{36-4\times3\times1}}{6}\\ &=\frac{6\pm\sqrt{24}}{6}=\frac{6\pm2\sqrt{6}}{6}\\ &=1\pm\frac{\sqrt{6}}{3} \end{aligned}$$
$\therefore$ roots of equation are:

$x=1+\frac{\sqrt{6}}{3}$ and $x=1-\frac{\sqrt{6}}{3}$ #

(iii) $2x^2+3x-5=0$

$(2x+5)(x-1)=0$

$2x+5=0$ or $x-1=0$

$2x=-5$ $\quad$ $x=1$

$x=-\frac{5}{2}$ #

(iv) $3+2x-2x^2=0$, multiplying throughout by -1 gives:

$2x^2-2x-3=0$, which can not be factorised.
$$\begin{aligned} \therefore x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4+4\times2\times3}}{4}\\ &=\frac{2\pm\sqrt{28}}{4}=\frac{2\pm2\sqrt{7}}{4}=\frac{1\pm\sqrt{7}}{2} \end{aligned}$$
$\therefore$ the roots of the equation are $x=\frac{1+\sqrt{7}}{2}$ and $x=\frac{1-\sqrt{7}}{2}$ #

4. (i) $y=x^2+4x$

$x=-\frac{b}{2a}=-\frac{4}{2}=-2$

when $x=-2$

$y=(-2)^2+4\times-2$

$=4-8=-4$

$\therefore$ vertex is $(-2,-4)$ #

(ii) $y=1-x-4x^2$

$=-4x^2-x+1$

$x=-\frac{b}{2a}=-\frac{(-1)}{2\times-4}=-\frac{1}{8}$

when $x=-\frac{1}{8}$

$y=1-\left(-\frac{1}{8}\right)-4\left(-\frac{1}{8}\right)^2$

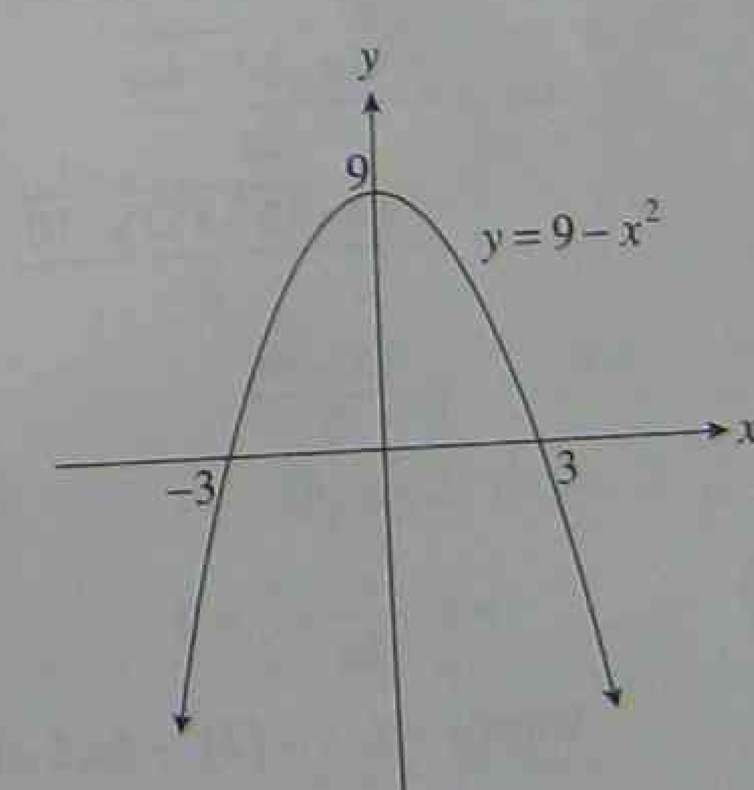
$=1+\frac{1}{8}-\frac{4}{64}=1\frac{1}{16}$

$\therefore$ vertex is $\left(-\frac{1}{8},1\frac{1}{16}\right)$ #

5. (i) $y=9-x^2=(3-x)(3+x)$

$a<0$ $\therefore$ concave down

at $x=0,\ y=9$ #



Quadratic Polynomials

(ii) $y=4x^2-12x=4x(x-3)$

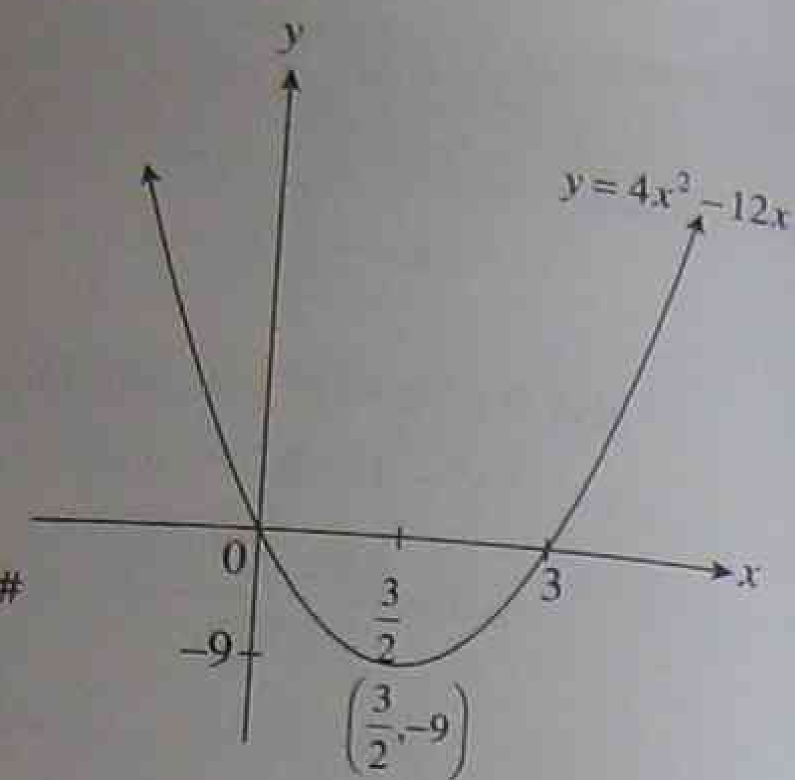
$a>0 \therefore$ concave up.

at $x=0, y=0$

at $y=0, x=0, 3$

$x=-\frac{b}{2a}=\frac{12}{8}=\frac{3}{2}$

when $x=\frac{3}{2}$, $y=4\times\frac{3}{2}\left(\frac{3}{2}-3\right)=-9$ #



(iii) $y=x^2-8x+16=(x-4)^2$

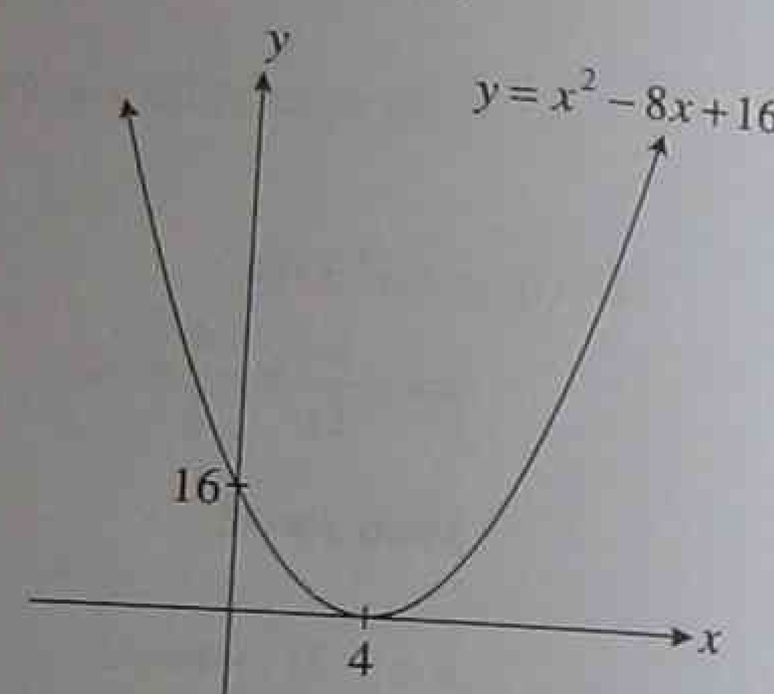
$a>0 \therefore$ concave up

when $x=0$, $y=16$

when $y=0$, $x=4$

$x=-\frac{b}{2a}=\frac{8}{2}=4$

$\therefore$ $(4,0)$ is the vertex.



(iv) $y=-x^2-4x-10$

$a>0 \therefore$ concave up

when $x=0$, $y=-10$

when $y=0$, $x^2-4x-10=0$

i.e. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

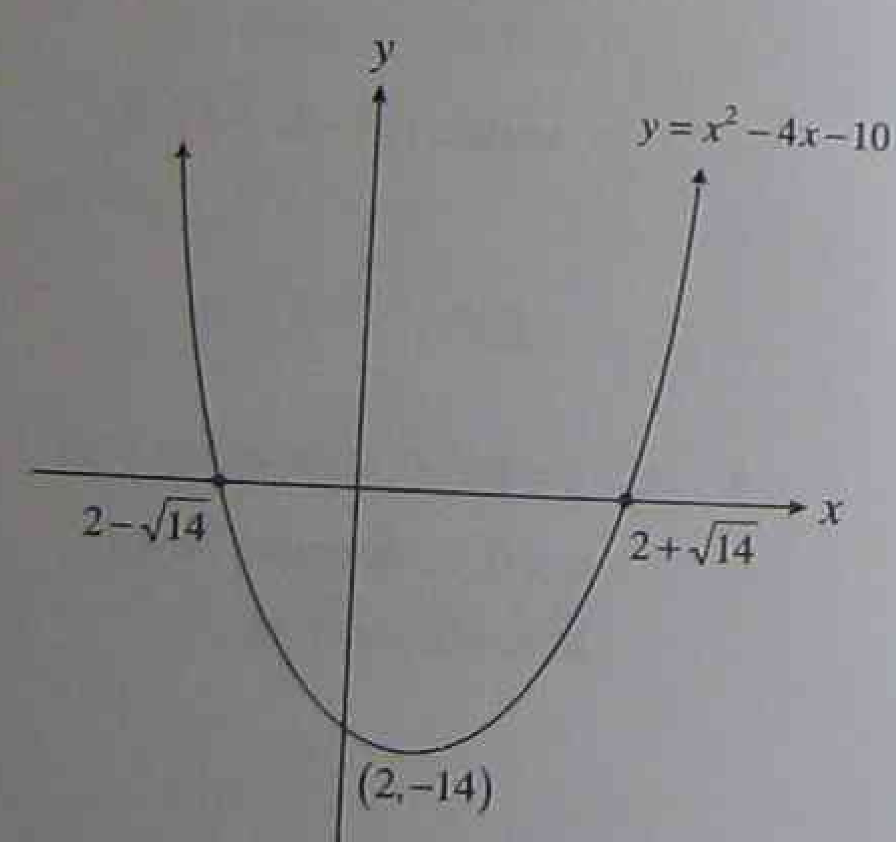
$=\frac{4\pm\sqrt{16-4\times1\times-10}}{2}$

$=\frac{4\pm\sqrt{56}}{2}$

$=2\pm\sqrt{14}$

$x=-\frac{b}{2a}=\frac{4}{2}=2$

when $x=2$, $y=(2)^2-4\times2-10=-14$ $\therefore$ vertex is $(2,-14)$ #



6. (i) $x^2+6x+5>0$

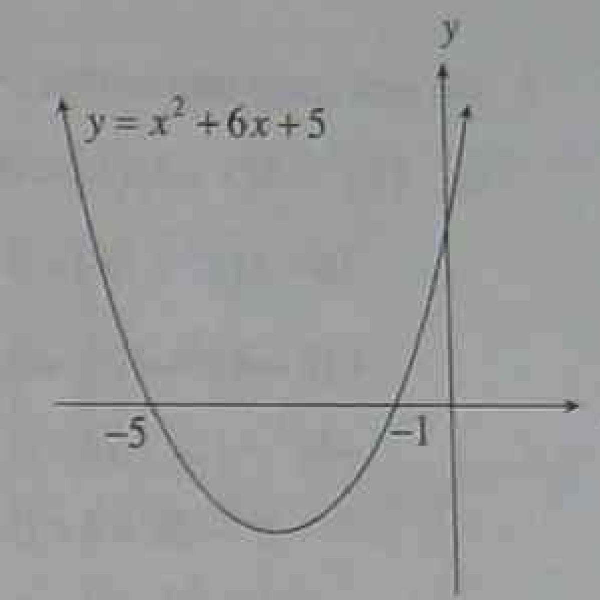
Let $y=x^2+6x+5$

$=(x+5)(x+1)=0$

$\therefore$ $x=-5, -1$

from the graph, $y>0$, for

$x>-1$ and $x<-5$ #



(ii) $9+6x-3x^2>0$

$3+2x-x^2>0$

$-x^2+2x+3>0$

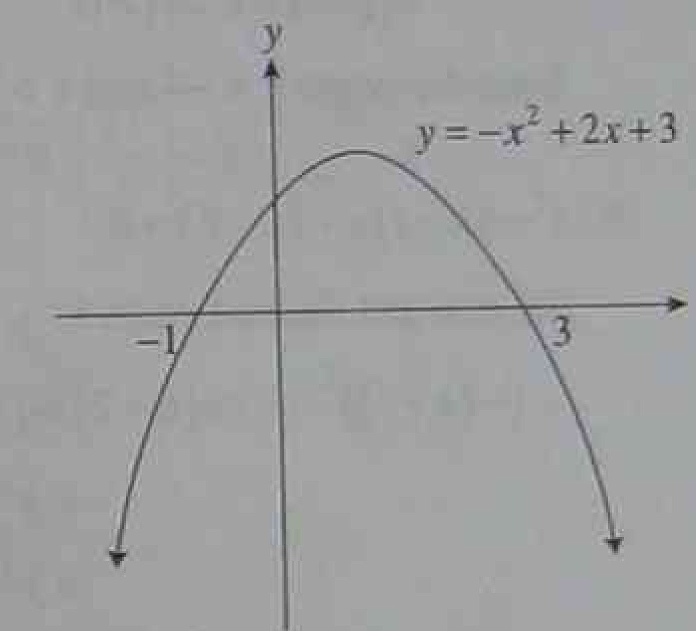
Let $y=-x^2+2x+3$

$=-(x^2-2x-3)$

$=-(x-3)(x+1)=0$

$\therefore$ $x=-1, 3$

from the graph $y>0$, for $-1<x<3$ #



(iii) $4x^2-x-2\le 0$

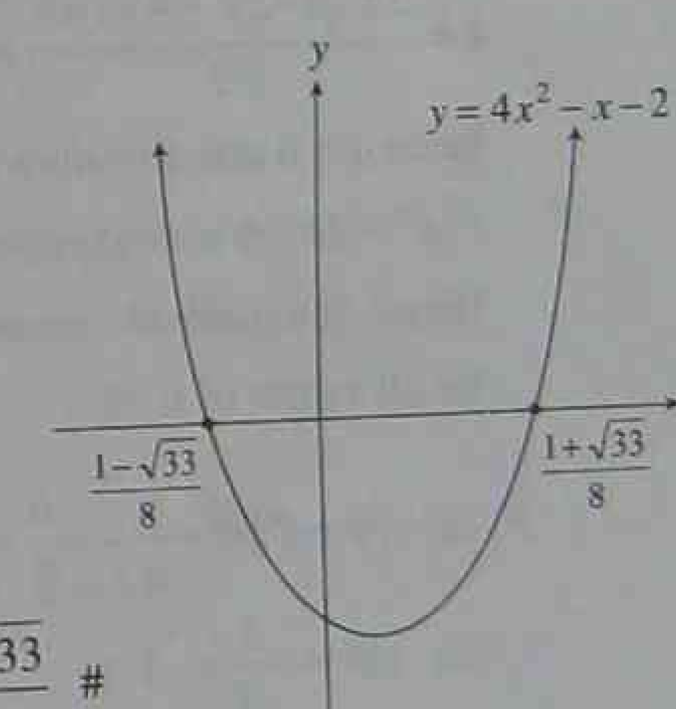
Let $y=4x^2-x-2=0$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{1\pm\sqrt{(-1)^2-4\times4\times-2}}{8}=\frac{1\pm\sqrt{33}}{8}$

$\therefore$ $x=\frac{1-\sqrt{33}}{8}$ and $x=\frac{1+\sqrt{33}}{8}$

from the graph $y\le 0$ for $\frac{1-\sqrt{33}}{8}\le x\le\frac{1+\sqrt{33}}{8}$ #



(iv) $x^2-10x+25\ge 0$

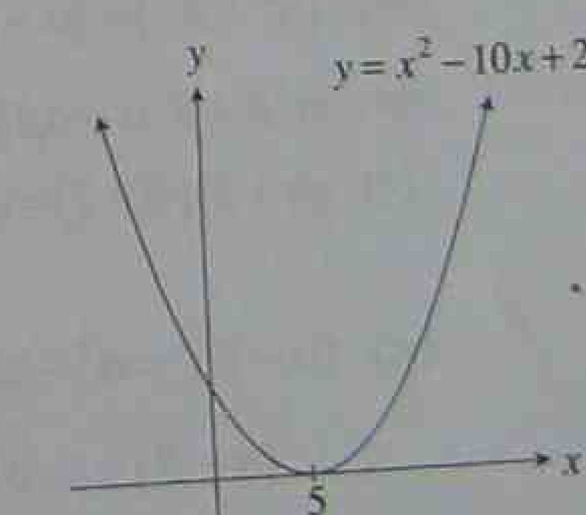
Let $y=x^2-10x+25=0$

$=(x-5)^2=0$

$\therefore$ $x=5$

from the graph $y\ge 0$

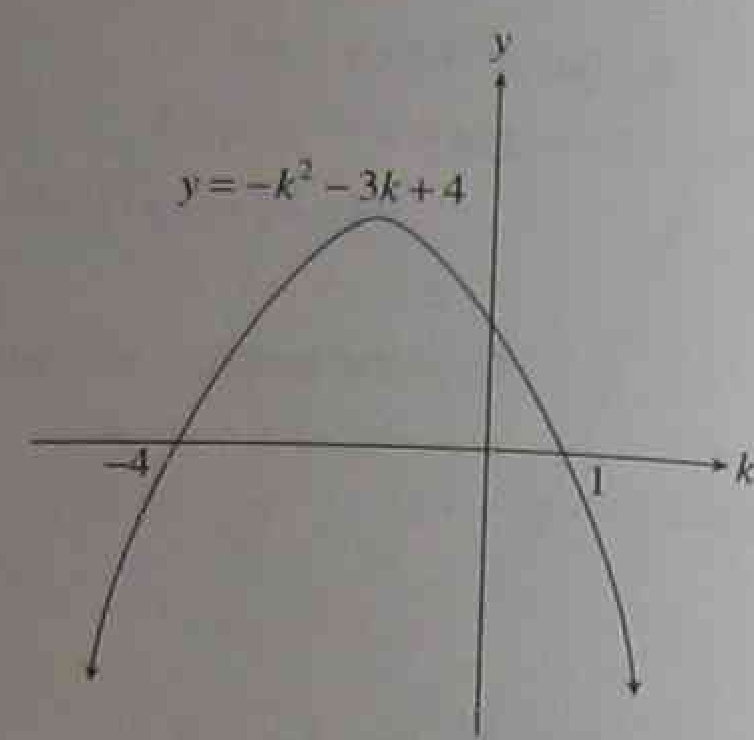
for all real values of x. #



7. No real roots means that $b^2-4ac<0$

i.e. $(4)^2-4(k+3)(k)<0$

$$16-4\left(k^2+3k\right)<0$$
$$16-4k^2-12k<0$$
$$-4k^2-12k+16<0$$
$$-k^2-3k+4<0$$
$$-\left(k^2+3k-4\right)<0$$
$$-(k+4)(k-1)<0$$



from the graph $k<-4$ and $k>1$. #

8. $x^2-(k+1)x+(k-2)=0$

For real and distinct roots $b^2-4ac>0$

$$\therefore\ (-(k+1))^2-4\times1(k-2)=(k+1)^2-4k+8$$
$$=k^2+2k+1-4k+8$$
$$=k^2-2k+9$$

Let $k^2-2k+9=0$

$$k=\frac{2\pm\sqrt{(-2)^2-4\times1\times9}}{2}=\frac{2\pm\sqrt{-32}}{2}\text{ which is undefined.}$$

Since $a>0$ and no values of k satisfy the equation $k^2-2k+9=0$

$\therefore\ k^2-2k+9$ is always positive.

Hence, the quadratic equation $x^2-(k+1)x+(k-2)$ has real and distinct roots for all values of k. #

9. (i) $(\alpha+\beta)=-\frac{b}{a}=-\frac{6}{3}=-2$ #

(ii) $\alpha\beta=-\frac{3}{3}=-1$ #

(iii) $\left(\alpha^2+\beta^2\right)=(\alpha+\beta)^2-2\alpha\beta=(-2)^2-2\times-1=4+2=6$ #

(iv) $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)=-1\times-2=2$ #

(v) $(\alpha+2)(\beta+2)=\alpha\beta+2\alpha+2\beta+4=\alpha\beta+2(\alpha+\beta)+4$

$$=-1+2\times-2+4=-1\ \#$$

(vi) $(\alpha-\beta)^2=\alpha^2-2\alpha\beta+\beta^2=\left(\alpha^2+\beta^2\right)-2\alpha\beta=6-2\times-1=8$

$\therefore\ (\alpha-\beta)=\pm\sqrt{8}=\pm2\sqrt{2}$ #

10. (i) $(\alpha+\beta)=-\frac{b}{a}=-\frac{4}{2}=-2$ #

(ii) $\alpha\beta=-\frac{3}{2}$ #

(iii) $\left(\alpha^2+\beta^2\right)=(\alpha+\beta)^2-2\alpha\beta=(-2)^2-2\times\left(-\frac{3}{2}\right)=4+3=7$ #

(iv) $(\alpha-\beta)^2=\alpha^2-2\alpha\beta+\beta^2=7-2\times-\frac{3}{2}=10\ \ \therefore(\alpha-\beta)=\pm\sqrt{10}$ #

(v) $\left(\alpha^2-\beta^2\right)=(\alpha-\beta)(\alpha+\beta)=\left(\pm\sqrt{10}\right)(-2)=\pm2\sqrt{10}$ #

(vi) $\left(\alpha^3-\beta^3\right)=(\alpha-\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)$

$$=\left(\pm\sqrt{10}\right)\left(7+-\frac{3}{2}\right)=\left(\pm\sqrt{10}\right)\left(\frac{11}{2}\right)=\pm\frac{11\sqrt{10}}{2}\ \#$$

11. $A(x+1)(x-2)+B(x+1)+C=3x^2+11x-6$

$$A\left(x^2-x-2\right)+Bx+B+C=3x^2+11x-6$$
$$Ax^2+(B-A)x+(B+C-2A)=3x^2+11x-6$$

$\therefore\ A=3$

$B-A=11$ i.e. $B-3=11$ i.e. $B=14$

$B+C-2A=-6$ i.e. $14+C-2\times3=-6$ i.e. $C=-14$

$\therefore\ A=3,\ B=14,\ C=-14.$ #

12. $Ax(x-5)+(Bx-1)(C+1)=2x^2+15x-5$

$$Ax^2-5Ax+BCx+Bx-C-1=2x^2+15x-5$$
$$Ax^2+x(BC-5A+B)-(C+1)=2x^2+15x-5$$

$\therefore\ A=2$

$C+1=5$ i.e. $C=4$

$BC-5A+B=15$ i.e. $4B-5\times2+B=15$ i.e. $B=5$

$\therefore\ A=2,\ B=5,\ C=4$ #

13. (i) $x^6-7x^3-8=0$

$$\left(x^3\right)^2-7x^3-8=0$$

Let $u=x^3$

i.e. $u^2-7u-8=0$

$$(u-8)(u+1)=0$$

i.e. $u-8=0$ or $u+1=0$

$u=8$ $\quad$ $u=-1$

thus, $x^3=8$ $\quad\quad$ $x^3=-1$

$x=2$ $\quad\quad$ $x=-1$

$\therefore$ the solutions for x are $x=-1$ and $x=2$. #

(ii) $4^x-3(2^x)+2=0$

$$(2^2)^x-3(2^x)+2=0$$

$$(2^x)^2-3(2^x)+2=0$$

Let $u=2^x$

i.e. $u^2-3u+2=0$

$$(u-2)(u-1)=0$$

i.e. $u-2=0$ or $u-1=0$

$u=2$ $\quad\quad$ $u=1$

thus, $2^x=2$ $\quad\quad$ $2^x=1$

$x=1$ $\quad\quad$ $x=0$

$\therefore$ the solutions for x are $x=0$ and $x=1$. #

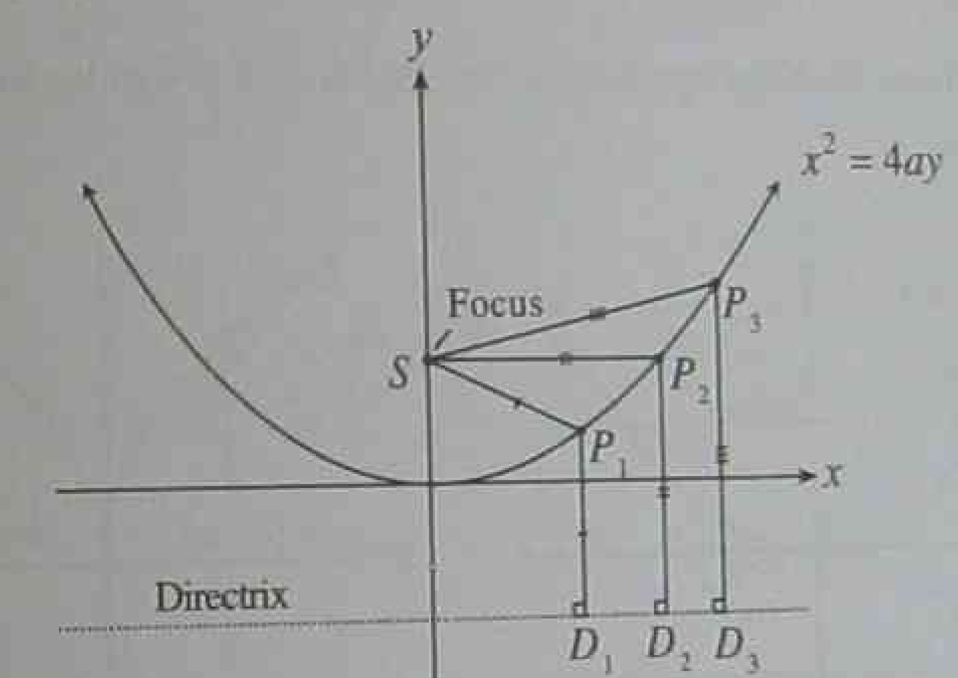
TOPIC 7

LOCUS AND THE PARABOLA

(A) The Parabola $x^2=4ay$

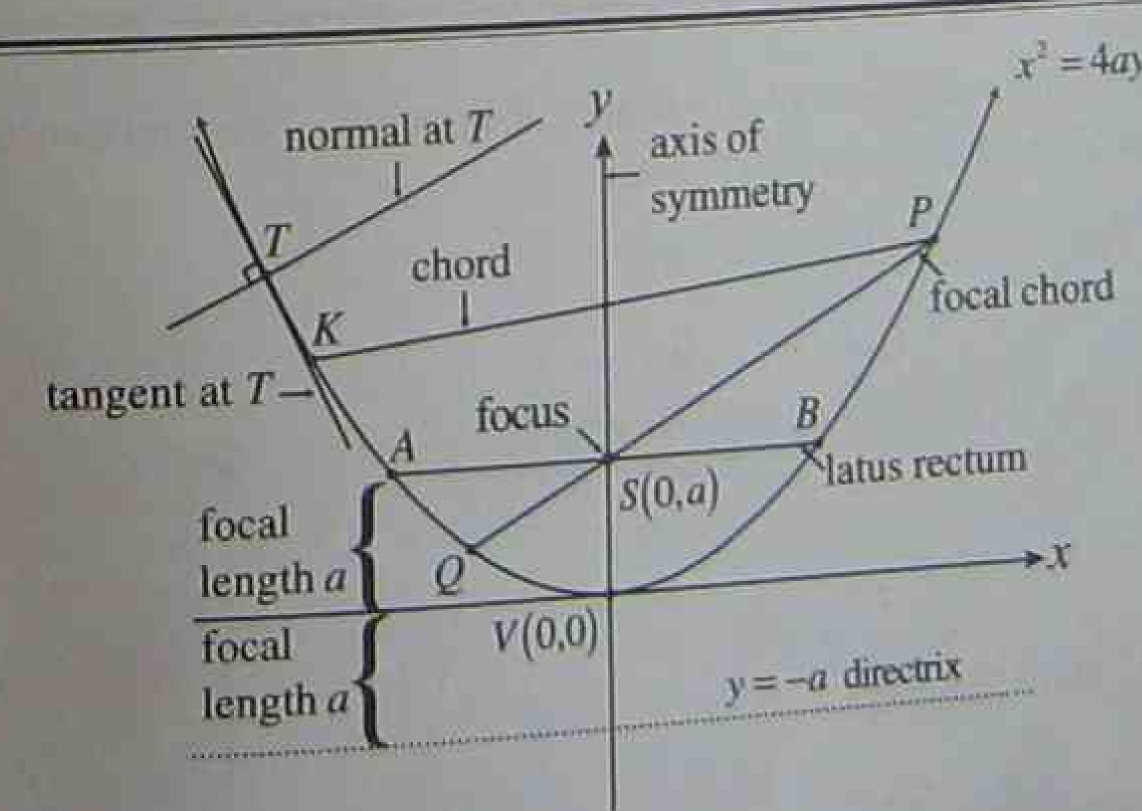
(i) Locus

A parabola is defined as a locus of a point which moves so that its distance from a fixed point (called the **focus**) is equal to its distance from a fixed straight line (called the **directrix**). This is illustrated in the diagram below. Note that for any point on the parabola, $PS=PD$.



$$P_1S=P_1D_1,\ P_2S=P_2D_2,\ P_3S=P_3D_3$$

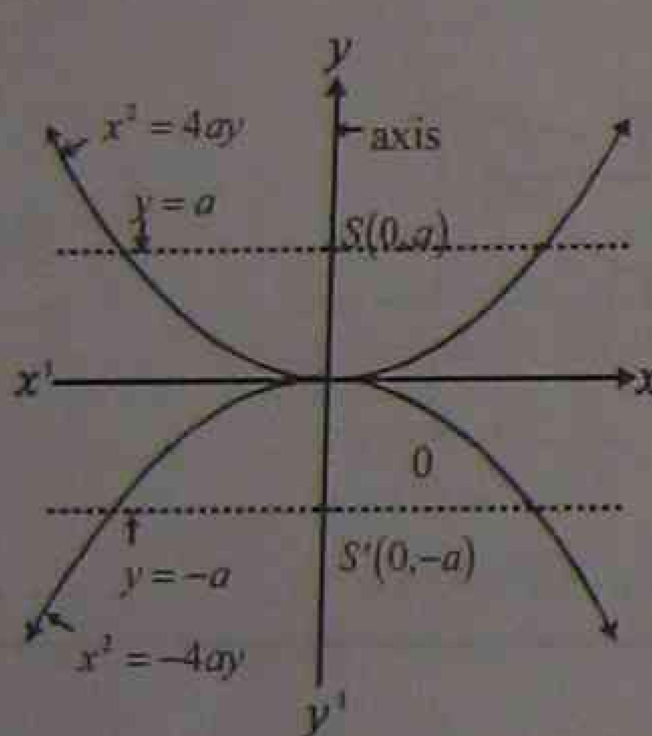
(ii) Features of the Parabola $x^2=4ay, a>0$

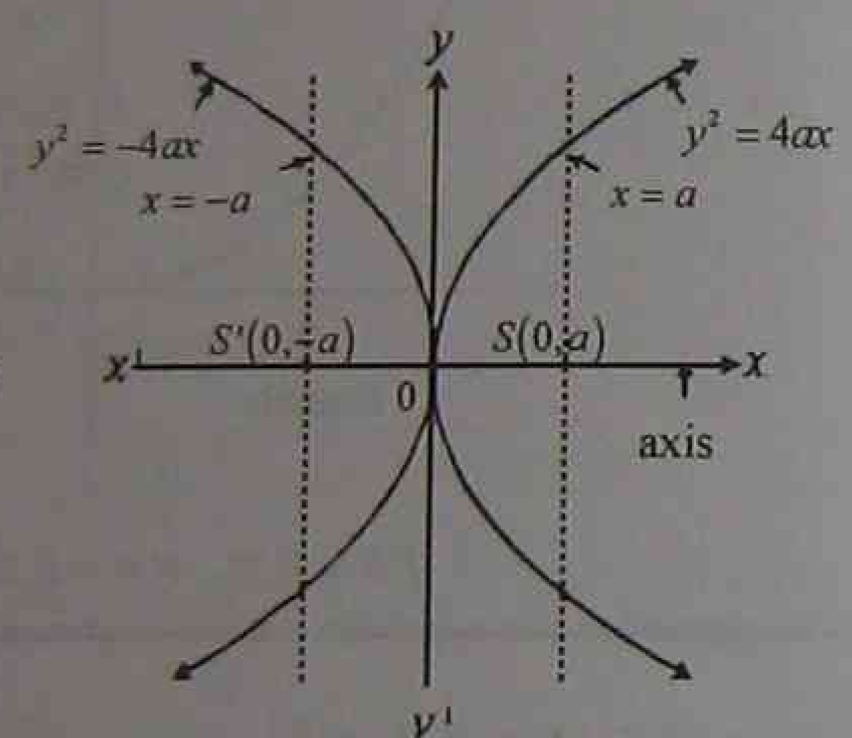


- **Vertex** V at the origin (0,0)
- **Focal Length** is 'a' units; determined by dividing the coefficient of y by 4 (i.e. $\frac{4a}{4}=a$).

- **Focus** S is at $(0, a)$ (i.e. a units above the vertex).
- **Directrix** has the equation $y = -a$ (i.e. a units below the vertex).
- **Focal Chord** is a chord joining two points on the parabola passing through the focus S.
- **Axis of Symmetry** of the parabola is the y-axis i.e. $x = 0$ (axis of symmetry always passes through the vertex of the parabola).
- **Latus Rectum** is determined as the focal chord (i.e. passing through S) which is perpendicular to the axis of the parabola.
- The **Tangent** is the straight line touching the parabola at T and the **normal** is the straight line perpendicular to the tangent.

Furthermore, the above principles may be applied to similar forms of the parabola.





Example 1: Graph the following parabolas indicating their vertex, focus and directrix. Also state their focal length and axis of symmetry:

(i) $x^2 = 2y$ (ii) $x^2 = -8y$

(iii) $y^2 = x$ (iv) $y^2 = -4x$

Solution 1:

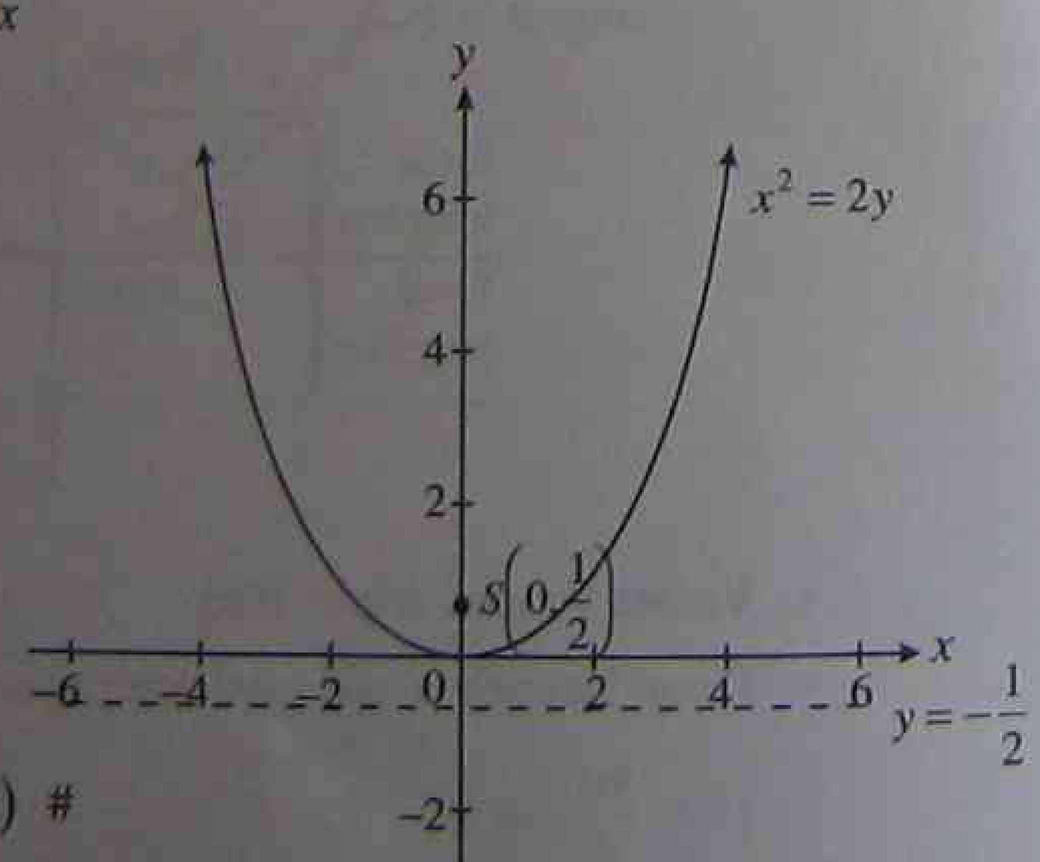
(i) $x^2 = 2y$

Focal length $= \frac{2}{4} = \frac{1}{2}$

$V(0, 0)$ $S\left(0, \frac{1}{2}\right)$

Directrix: $y = -\frac{1}{2}$

Axis of symmetry y-axis $(x = 0)$ #



(ii) $x^2 = -8y$

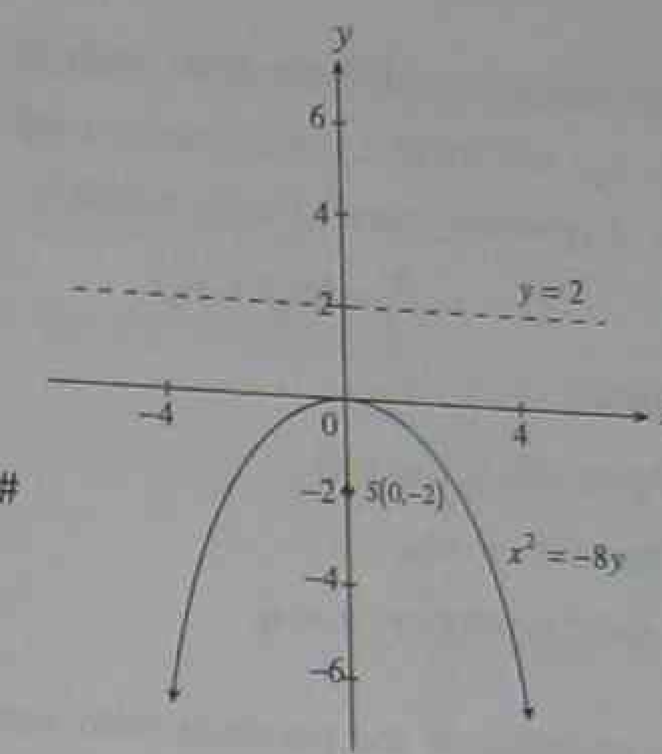
Focal length $= \frac{8}{4} = 2$

$V(0, 0)$

$S(0, -2)$

Directrix $y = 2$

Axis of symmetry y-axis $(x = 0)$ #



(iii) $y^2 = x$

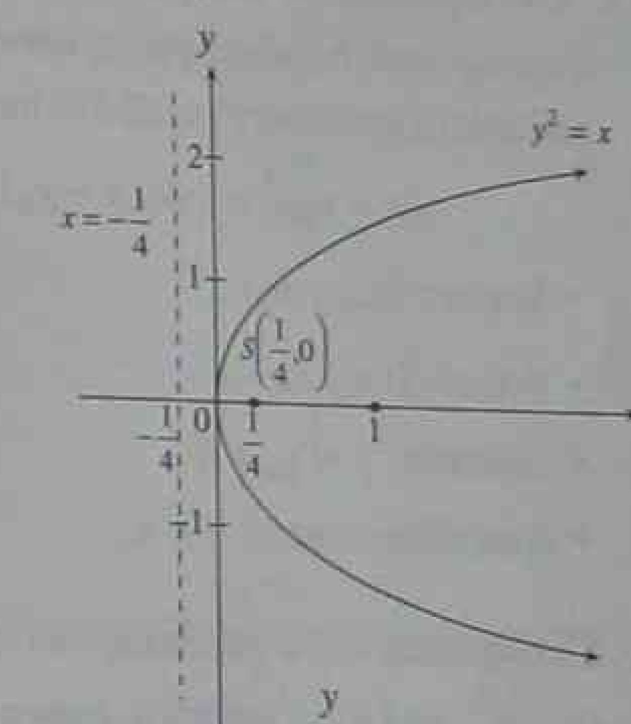
Focal length $= \frac{1}{4}$

$V(0, 0)$

$S\left(\frac{1}{4}, 0\right)$

Directrix $x = -\frac{1}{4}$

Axis of symmetry x-axis $(y = 0)$ #



(iv) $y^2 = -4x$

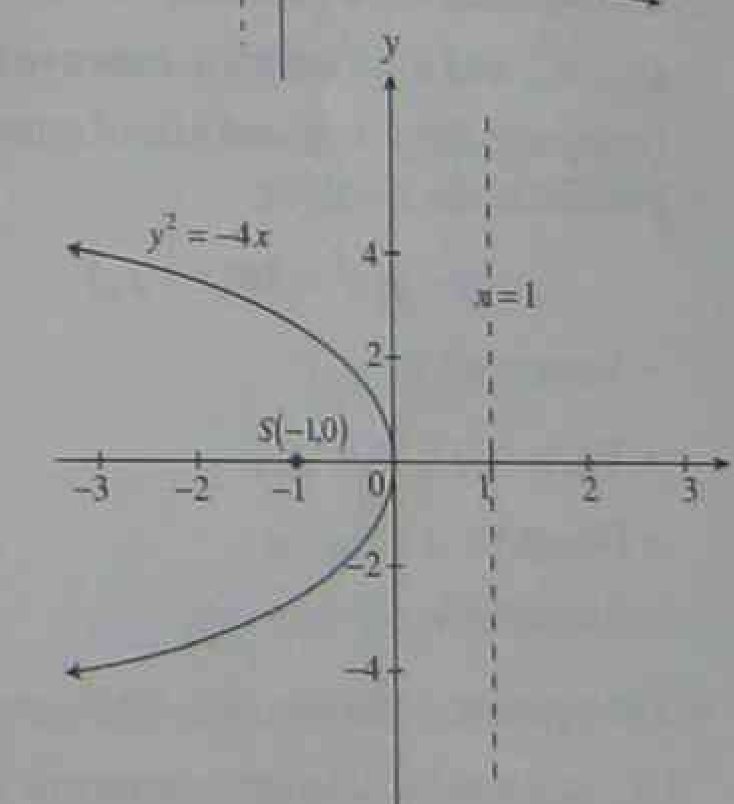
Focal length $= \frac{4}{4} = 1$

$V(0, 0)$

$S(-1, 0)$

Directrix $x = 1$

Axis of symmetry x-axis $(y = 0)$ #



(B) Change of Origin

Not all parabolas have their vertex at the origin.

The process of changing the origin (i.e. the vertex), however, has no effect on the basic characteristics of the curve itself. There are 4 types:

(iv) $(y-3)^2=-2(x+4)$

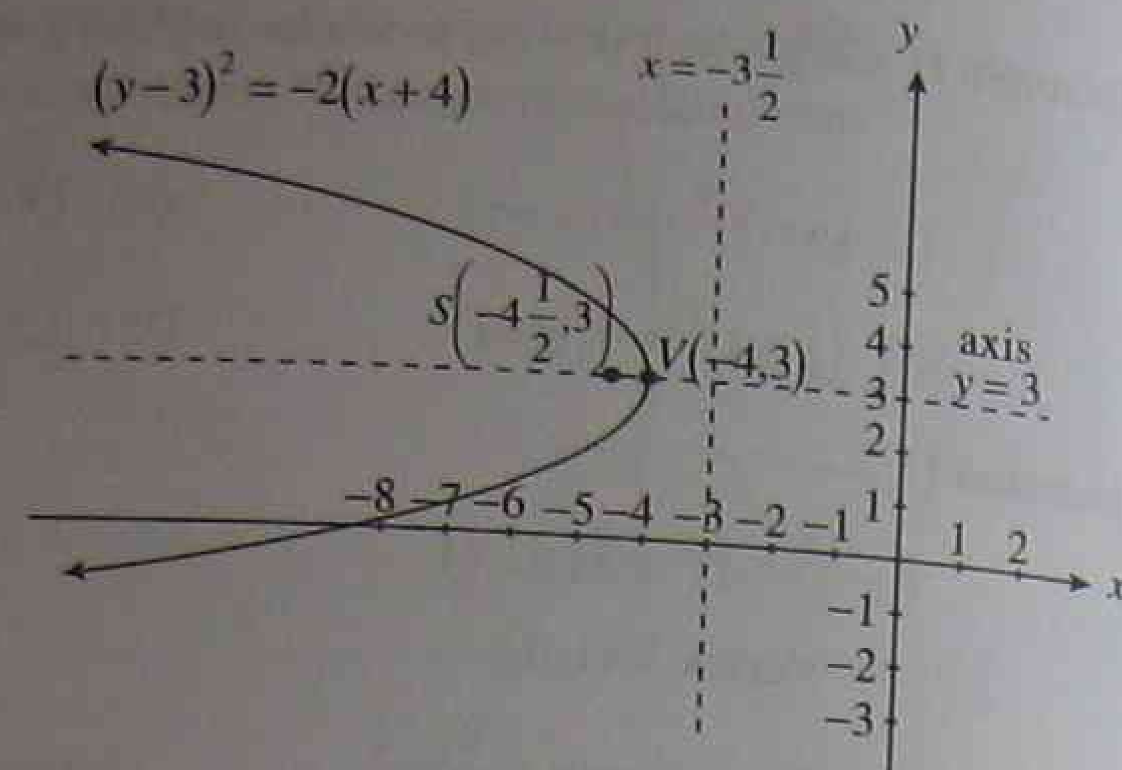
Vertex $V(-4, 3)$

Focal length $=\frac{2}{4}=\frac{1}{2}$

Focus: $S\left(-4\frac{1}{2}, 3\right)$

Directrix: $x=-3\frac{1}{2}$

Axis of symmetry: $y=3$ #



(C) Finding the Vertex, Focus, Directrix and Axis of Symmetry of $y=ax^2+bx+c$

By completing the square, the quadratic equation of the form $y=ax^2+bx+c$ can be rewritten as:

$$(x-x_0)^2=4a(y-y_0)$$

Example 1: Find the vertex, focus, directrix and axis of symmetry of the following parabolas:

(i) $y=x^2+4x+6$ (ii) $2y=6x-x^2+3$

Solution 1:

(i)
$$\begin{aligned} y&=x^2+4x+6\\ &=(x+2)^2-4+6\\ &=(x+2)^2+2 \end{aligned}$$

i.e. $(y-2)=(x+2)^2$

which has Vertex: $(-2, 2)$

Focal length: $\frac{1}{4}$

Focus: $\left(-2, 2\frac{1}{4}\right)$

Directrix: $y=1\frac{3}{4}$

Axis of symmetry: $x=-2$ #

(ii)
$$\begin{aligned} 2y&=6x-x^2+3\\ -2y&=x^2-6x-3\\ -2y&=(x-3)^2-9-3\\ -2y+12&=(x-3)^2\\ -2(y-6)&=(x-3)^2 \end{aligned}$$

which has Vertex: $(3, 6)$

Focus length $=\frac{2}{4}=\frac{1}{2}$

$\therefore$ Focus: $\left(3, 5\frac{1}{2}\right)$

Directrix: $y=6\frac{1}{2}$

Axis of symmetry: $x=3$ #

(D) Finding the Equation of a Parabola

Given any of the vertex, focus, directrix or axis of symmetry of a parabola, it is possible to find the equation of the parabola.

Example 1: Find the equation of the parabola with vertex $(1, 1)$, and focus $(1, 3)$.

Solution 1:

Vertex $(1, 1)$, focus $(1, 3)$ $\therefore$ the focus is above the vertex

i.e. parabola is concave up.

Also, focal length $=3-1=2$

$(x-1)^2=4\times 2(y-1)$

$\therefore$ $(x-1)^2=8(y-1)$ #

Example 2: The parabola ax^2+bx+c has its focus at $(3, 1)$ and its directrix at $y=5$:

(i) Find the coordinates of the vertex.

(ii) Hence, find the values of a, b and c. #

Solution 2:

(i) The vertex will have the same x-coordinate as the focus and its y-coordinate will lie half way between the focus and the directrix (i.e. $y=3$). Thus vertex has coordinates $(3, 3)$.

(ii) Focus is below the vertex $\therefore$ parabola is of the form:

$(x-x_0)^2=-4a(y-y_0)$ with vertex (x_0, y_0)

also, focus length $=3-1=2$

$$\therefore \quad (x-3)^2=-8(y-3)$$

$$x^2-6x+9=-8y+24$$

$$\therefore \quad y=-\frac{1}{8}x^2+\frac{3}{4}x+\frac{15}{8}$$

i.e. $a=-\frac{1}{8}$, $b=\frac{3}{4}$ and $c=\frac{15}{8}$ #

REVIEW EXERCISES

(A) The Parabola $x^2 = 4ay$

1. Graph the following parabolas indicating their vertex focus and directrix. Also state their focal length and axis of symmetry:

(i) $x^2 = 4y$ (ii) $x^2 = -2y$

(iii) $y^2 = 12x$ (iv) $y^2 = -x$

(B) Change of Origin

2. Graph the following parabolas indicating their vertex, focus and directrix. Also state their focal length and axis of symmetry:

(i) $(x+1)^2 = 2(y+1)$ (ii) $x^2 = -4(y+6)$

(iii) $(y+2)^2 = 8(x-2)$ (iv) $(y-3)^2 = -(x-1)$

(C) Finding the Vertex, Focus, Directrix and Axis of Symmetry of $y = ax^2 + bx + c$

3. Find the vertex, focus, directrix and axis of symmetry of the following parabolas:

(i) $4y = x^2 + 6x + 5$ (ii) $y = -x^2 + 4x - 6$

(iii) $y^2 = 16(4-x)$ (iv) $x = \frac{1}{4}y^2 - y + 3$

(D) Finding the Equation of a Parabola

4. Find the equation of the parabola vertex (-2, 3) and focus (0, 3).

5. The parabola $ax^2 + bx + c$ has its focus at (2, 5) and its directrix is the line $y = -7$. Find the values of a, b, and c.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $x^2 = 4y$

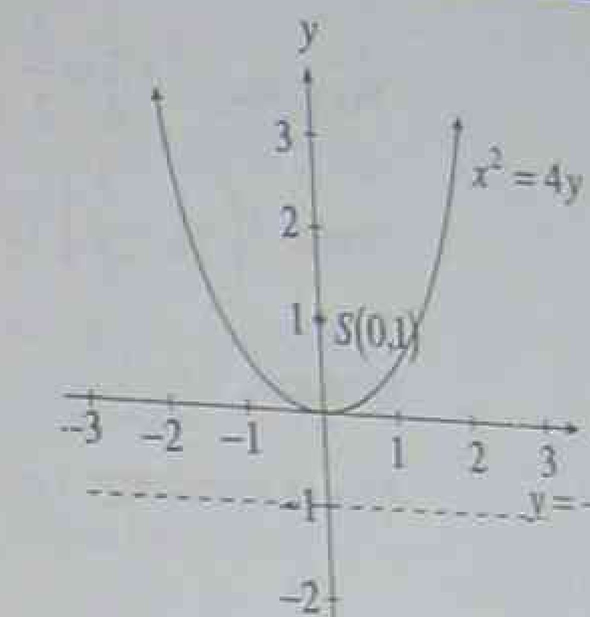
Focal length: $\frac{4}{4} = 1$

$V(0, 0)$

$S(0, 1)$

Directrix: $y = -1$

Axis of symmetry: y-axis $(x = 0)$ #



(ii) $x^2 = -2y$

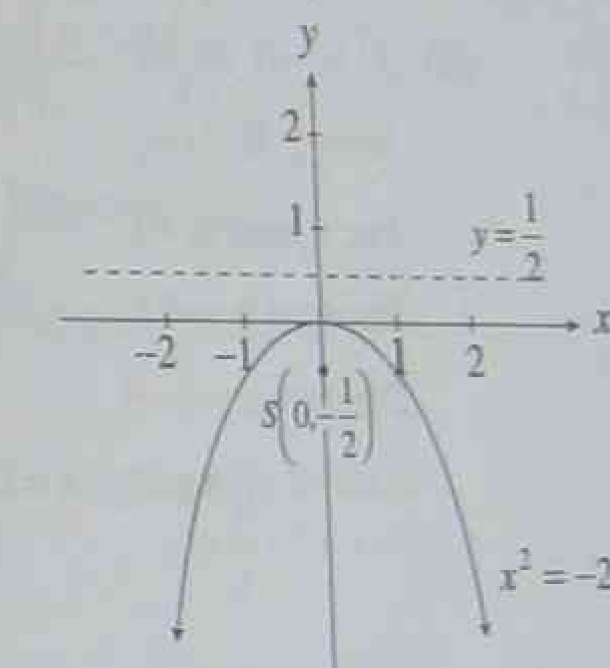
Focal length $= \frac{2}{4} = \frac{1}{2}$

$V(0, 0)$

$S\left(0, -\frac{1}{2}\right)$

Directrix: $y = \frac{1}{2}$

Axis of symmetry: y-axis $(x = 0)$ #



(iii) $y^2 = 12x$

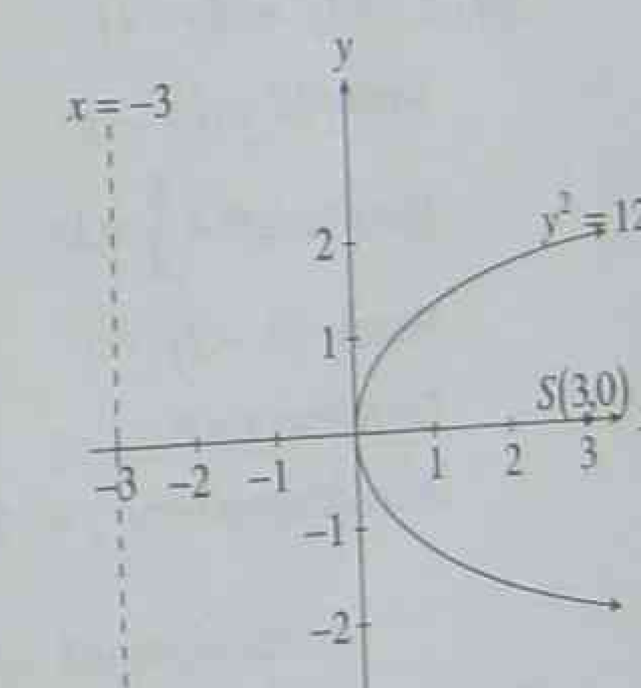
Focal length: $\frac{12}{4} = 3$

$V(0, 0)$

$S(3, 0)$

Directrix: $x = -3$

Axis of symmetry: x-axis $(y = 0)$ #



(iv) $y^2 = -x$

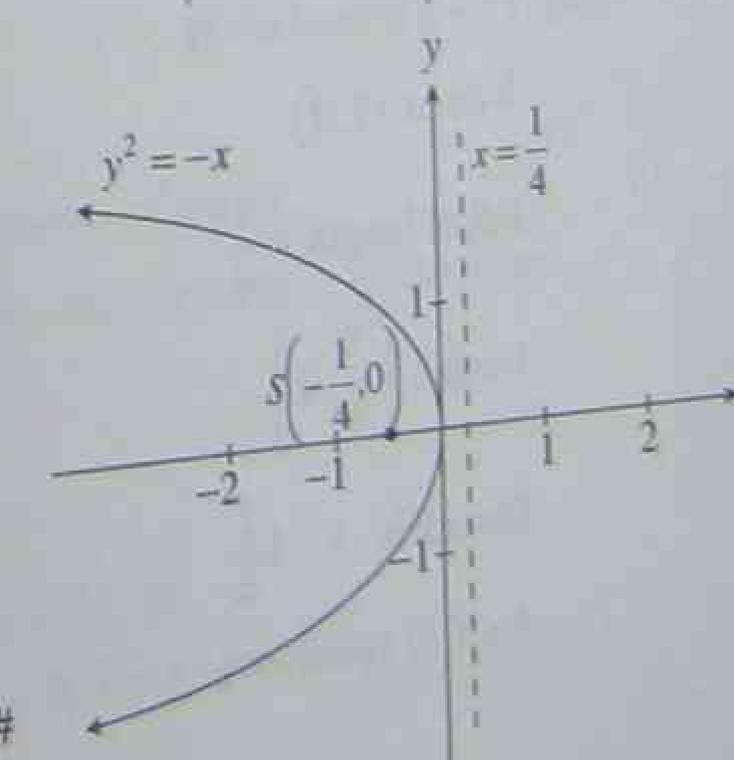
Focal length: $\frac{1}{4}$

$V(0, 0)$

$S\left(-\frac{1}{4}, 0\right)$

Directrix: $x = \frac{1}{4}$

Axis of symmetry: x-axis $(y = 0)$ #



5. Directrix $y=-7$ is parallel to x-axis, and the focus $(2,\ 5)$ is above it, $\therefore$ parabola of the form $(x-x_0)^2 = 4a(y-y_0)$.

Now, the vertex will have the same x-coordinate as the focus and its y-coordinate will lie half way between the focus $(2,\ 5)$ and the directrix $y=-7$. Thus vertex has coordinates $(2,\ -1)$.

Also, focal length $= 5-(-1) = 6$

$$\therefore\quad (x-x_0)^2 = 4a(y-y_0)$$
$$(x-2)^2 = 4\times 6(y+1)$$
$$x^2-4x+4 = 24y+24$$
$$\text{i.e.}\quad 24y = x^2-4x-20$$
$$y = \frac{1}{24}x^2 - \frac{1}{6}x - \frac{20}{24}$$
$$\therefore\ a=\frac{1}{24},\ b=-\frac{1}{6}\ \text{and}\ c=-\frac{20}{24}=-\frac{5}{6}\quad \#$$

SERIES

(A) Arithmetic Series

An arithmetic series is one where the difference, d, between successive terms is constant. If T_1, T_2 and T_3 are the first 3 terms of an arithmetic series, then:

$$T_2 - T_1 = T_3 - T_2 = d$$

and in general: $T_n - T_{n-1} = d$.

To find the n^{th} term, T_n, of an arithmetic series with common difference d, and first term a, we use:

$$T_n = a + (n-1)d$$

To find the sum to n terms, S_n, of a geometric series we use:

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \text{ or}$$

$$S_n = \frac{n}{2}\left[a+l\right] \text{ where } l = \text{last}\left(\text{or } n^{\text{th}}\right) \text{ term}$$

The second formula can only be used if the last term, l, is known.

At times, you may be given details about the sum to n terms, S_n, and the sum to $(n-1)$ terms S_{n-1}, and asked to find T_n. In this case we use:

$$S_n - S_{n-1} = T_n$$

Example 1: Consider the arithmetic sequence 32, 20, 8, ... find:

(i) The equation for T_n in its simplest form.

(ii) Hence, find the 20^{th} term.

(iii) The expression for S_n in its simplest form.

(iv) The sum of the first 20 terms.

(v) The sum of the next 20 terms.

Solution 1:

(i) $T_n = a + d(n-1)$
$\quad = 32 + (-12)(n-1)$
$\quad = 44 - 12n$ #

(ii) $T_{20} = 44 - 12\times 20 = -196$ #

Example 1: The first three terms of a geometric series are: 3, 12, 48.
Find the value of the twelfth term.

Solution 1:

Let $T_1 = 3, T_2 = 12, T_3 = 48$

$$r = \frac{T_2}{T_1} = \frac{12}{3} = 4$$

$$T_n = ar^{n-1}$$

$$T_{12} = 3(4)^{12-1} = 3\times 4^{11} \quad \#$$

Example 2: The third term of a geometric series is 2 and the sixth term is 128. Find:

(i) the common ratio, r, and the first term, a.

(ii) the sum of the first 8 terms.

Solution 2:

(i) $T_n = ar^{n-1}$

$T_3 = ar^2 = 2$ and $T_6 = ar^5 = 128$

now, $\dfrac{T_6}{T_3} = \dfrac{ar^5}{ar^2} = \dfrac{128}{2}$

i.e. $r^3 = 64$

$r = 4$

$\therefore$ substituting this into T_3, gives:

$$2 = a(4)^2$$

$$a = \frac{1}{8} \quad \#$$

(ii) $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$

$$S_8 = \frac{\frac{1}{8}\left(4^8 - 1\right)}{4-1}$$

$$= 2{,}730.625 = 2{,}730\frac{5}{8} \quad \#$$

Example 3: For the geometric sequence 4374, 1458, 486,...find:

(i) the 9[th] term. Express your answer as a simplified fraction.

(ii) the sum of the first 9 terms correct to 3 decimal places.

(iii) which term in the sequence is $\frac{2}{\ldots}$

Solution 3:

(i) $r = \dfrac{T_2}{T_1} = \dfrac{1458}{4374} = \dfrac{1}{3}$

$$T_n = ar^{n-1}$$

$$T_9 = 4374\left(\frac{1}{3}\right)^{9-1} = \frac{4374}{3^8} = \frac{4374}{6561} = \frac{2}{3} \quad \#$$

(ii) $S_n = \dfrac{a\left(1 - r^n\right)}{1-r}$

$$S_9 = \frac{4374\left(1 - \left(\frac{1}{3}\right)^9\right)}{1 - \frac{1}{3}}$$

$$= 6560.667 \quad \#$$

(iii) $T_n = ar^{n-1}$

Let $T_n = \dfrac{2}{729}$ and solve for n:

$$\frac{2}{729} = 4374\left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{729} = 2187\left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{3^6} = 3^7\left(\frac{1}{3}\right)^{n-1} \quad \left(\text{note } 729 = 3^6 \text{ and } 2187 = 3^7\right)$$

$$\frac{1}{3^{13}} = \frac{1}{3^{n-1}}$$

i.e. $3^{n-1} = 3^{13}$

$n - 1 = 13$

$n = 14 \quad \#$

Example 4: The first three terms of a sequence are $x-3$, $2x+1$, $5x-1$. Find the values c
if the sequence is geometric.

Solution 4:

If a sequence is geometric, then the terms must have a common ratio:

i.e. $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$

i.e. $\frac{2x+1}{x-3}=\frac{5x-1}{2x+1}$

$(2x+1)^2=(5x-1)(x-3)$

$4x^2+4x+1=5x^2-15x-x+3$

i.e. $x^2-20x+2=0$

Using the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{20\pm\sqrt{(20)^2-4\times1\times2}}{2}$$

$$=\frac{20\pm\sqrt{400-8}}{2}$$

$$=\frac{20\pm\sqrt{392}}{2}$$

$$=\frac{20\pm2\sqrt{98}}{2}=10\pm\sqrt{98}$$

$\therefore\ x=10+\sqrt{98}$ or $x=10-\sqrt{98}$ #

(C) Sigma (Σ) Notation

The summing up of an arithmetic or geometric progression may sometimes be expressed using the sigma (Σ) notation.

Example 1: Find the value of $\sum_{k=1}^{10}(3k+2)$.

Solution 1:

$T_1=3\times1+2=5,\ T_2=3\times2+2=8,\ T_3=3\times3+2=11$ and $T_{10}=3\times10+2=32$

$\therefore\ \sum_{k=1}^{10}(3k+2)=5+8+11+...+32$

which is an arithmetic progression with $a=5$, and $l=32$

$\therefore\ S_n=\frac{n}{2}(a+l)$

$S_{10}=\frac{10}{2}(5+32)=185$ #

Series

(D) Infinite Geometric Series

If $-1<r<1$, then the sum of a geometric series is said to have a ***Limiting sum*** (i.e. it tends to approach some finite number as n tends to infinity).

If $-1<r<1$, then $S_\infty=\frac{a}{1-r}$ and S_∞ has a ***Limiting sum***.

Example 1: Find the limiting sum of $4+2+1+\frac{1}{2}+...$

Solution 1:

This is an infinite geometric series with $a=4$ and $r=\frac{1}{2}$

$$S_\infty=\frac{a}{1-r}=\frac{4}{1-\frac{1}{2}}=\frac{4}{\frac{1}{2}}=8\ \#$$

Example 2: Find the value of x for which the series $1+3x+9x^2+27x^3+...$ has a limiting sum.

Solution 2:

This is an infinite geometric series with $a=1$ and $r=3x$. For the series to have a limiting sum $-1<r<1$.

i.e. $-1<3x<1$

i.e. $-\frac{1}{3}<x<\frac{1}{3}$ #

(E) Recurring Decimals Using Geometric Series

Recurring decimals may be expressed as an infinite geometric series and then as simplified fractions.

Example 1: Express $0.\dot{7}$ as an infinite geometric series and hence as a simplified fraction.

Solution 1:

$0.\dot{7}=0.7777...$

$=0.7+0.07+0.007+...$

This is an infinite geometric series with $a=0.7$ and $r=0.1$

Series

$$S_\infty = \frac{a}{1-r} = \frac{0.7}{1-0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

$$\therefore 0.\dot{7} = \frac{7}{9} \quad \#$$

Example 2: By considering $0.3\dot{5}$ as the sum of an infinite geometric series, express $0.3\dot{5}$ as a fraction in its simplest form.

Solution 2:

$$0.3\dot{5} = 0.35555\ldots = 0.3 + \underbrace{0.05 + 0.005 + 0.0005\ldots}$$

The part in brackets forms an infinite geometric series with $a = 0.05$, $r = 0.1$

$$\therefore 0.3\dot{5} = 0.3 + \frac{a}{1-r} = 0.3 + \frac{0.05}{1-0.1} = \frac{3}{10} + \frac{0.05}{0.9} = \frac{3}{10} + \frac{5}{90} = \frac{16}{45} \quad \#$$

REVIEW EXERCISES

(A) Arithmetic Series

1. Find which term in the arithmetic series: $5+11+17+\ldots$ is equal to 149.
2. Find the 20th term in the arithmetic progression: 90, 75, 60, 45, ...
3. Determine the number of terms in the following arithmetic series:

   $$21+16+11+\ldots+(-189)$$

   Hence find its sum.
4. Find the sum of the series $1\frac{1}{2}+4+6\frac{1}{2}+9+\ldots$ to 10 terms.

   Express your answer in exact form.
5. An arithmetic series has 32 as its fifth term and the difference between the ninth and tenth terms is -8.
   (i) Find the first term a.
   (ii) Calculate the sum of the first 15 terms.
6. The 12th term of an arithmetic series is 58 and the 3rd term is 4. Find:
   (i) The first term, a and the common difference, d.
   (ii) The sum of the first 12 terms.
7. The first term of an arithmetic series is 20, and the sixth term is double the fourth. Find the sum of the first 20 terms of the series.
8. $3x^3$, $8x^2$, $5x$ are three consecutive terms of an arithmetic series.
   (i) Show that $3x^3-16x^2+5x=0$
   (ii) What is the common difference(s), d?

(B) Geometric Series

9. The first three terms of a geometric progression are: 300, 150, 75... Find the value of the 10th term. Express your answer as a fraction in its simplest form.
10. The fourth term of a geometric series is 1 and the ninth term is 32. Find the sum of nine terms.
11. The third term of a geometric series is 625 and the ratio of the fourth term to the sixth is 25. Find ratio, r and hence the sum of the first 10 terms.
12. How many terms of the geometric progression $2+4+8+\ldots$ must be summed to obtain a value of 510.

Series

13. How many terms must be taken of the geometric series $5+10+20+\ldots$ such that their sum exceeds 1,500?

14. Consider the geometric series: $1+6+36+216+\ldots$

(i) How many terms are below 1,000,000?

(ii) Find the sum of these terms.

15. The sum of the first three terms of a geometric series is 105 and the first term is 5. Given all the terms in this series are positive, find:

(i) The common ratio, r.

(ii) The sum of the first 15 terms expressing your answer in scientific notation to 3 significant figures.

16. Find the value of x such that: $x+1,\ x+3,\ \frac{2x+13}{2}$ will give a set of three numbers in geometric progression.

(C) Sigma (Σ) Notation

17. Find $\sum_{k=1}^{15}(5k-10)$

18. Find $\sum_{k=1}^{10}\left(2^k+3\right)$

(D) Infinite Geometric Series

19. Find the limiting sum of: $-1-\frac{1}{8}-\frac{1}{64}\ldots$

20. Find the values of x for which the series: $\frac{1-2x}{2}+\frac{(1-2x)^2}{4}+\frac{(1-2x)^3}{8}+\ldots$ has a limiting sum.

(E) Recurring Decimals Using Geometric Series

21. Express the following decimals as infinite geometric series and hence as a fraction in its simplest form:

(i) $0.3\dot{4}$ (ii) $0.4\dot{7}\dot{5}$

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $T_n = a+(n-1)d$

$149 = 5+(n-1)6$

$= 5+6n-6$

$150 = 6n$

$n = 25$

$\therefore$ the 25^{th} term $=149$. #

2. $T_n = a+(n-1)d$

$T_{20} = 90+(20-1)\times -15$

$= 90-19\times 15$

$= -195$

$\therefore$ the 20^{th} term $= -195$ #

3. $T_n = a+(n-1)d$

$-189 = 21+(n-1)(-5)$

$-189 = 21-5n+5$

$5n = 215$

$n = 43$

$\therefore$ there are 43 terms in the series.

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{43} = \frac{43}{2}\left[2\times 21+(43-1)\times -5\right]$

$= \frac{43}{2}[42-210]$

$= \frac{43}{2}[168]$

$= 3612$ #

4. $S_n = \frac{n}{2}\left[2a+d(n-1)\right]$

$S_{10} = \frac{10}{2}\left[2\times 1\frac{1}{2}+2\frac{1}{2}(10-1)\right]$

$= 5\left[3+\frac{45}{2}\right]$

$= 5\left[\frac{6+45}{2}\right]$

$= \frac{255}{2}$ or $127\frac{1}{2}$ #

5. $T_n = a+(n-1)d,\ T_5 = a+4d,\ T_{10}-T_9 = -8$

(i) By definition: $T_n - T_{n-1} = d$ i.e. $d=-8$

thus, $T_5 = a+4d = 32$

i.e. $32 = a+4\times -8$

$\therefore\quad a = 64$ #

(ii) $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{15} = \frac{15}{2}\left[2\times 64+(15-1)\times -8\right] = 120$ #

6. $T_n = a+(n-1)d,\ T_{12} = a+11d,\ T_3 = a+2d$

(i) $T_{12} = a+11d = 58$

$T_3 = a+2d = 4$

$T_{12}-T_3:\ 9d = 54$

$d = 6$

Substituting this into T_{12}, gives:

$a+11\times 6=58$

$a=-8$ #

(ii) $S_n=\frac{n}{2}(a+l)$ where l is the last or n^{th} term

i.e. $S_{12}=\frac{12}{2}(-8+58)=300$ #

7. $a=20,\ T_n=a+d(n-1)=20+d(n-1)$

Now, $T_6=2T_4$ (given)

thus, $20+5d=2(20+3d)$

$20+5d=40+6d$

$\therefore\ d=-20$

$S_n=\frac{n}{2}[2a+d(n-1)]$

$S_{20}=\frac{20}{2}[2\times 20+-20(20-1)]=10[40-20\times 19]=-3{,}400$ #

8. (i) Let $T_1=3x^3,\ T_2=8x^2,\ T_3=5x$

If T_1, T_2, T_3 are three consecutive terms in an arithmetic series, then:

$T_2-T_1=T_3-T_2$

i.e. $8x^2-3x^3=5x-8x^2$

i.e. $3x^3-16x^2+5x=0$ #

(ii) $3x^3-16x^2+5x=0$

$x(3x^2-16x+5)=0$

$x(3x-1)(x-5)=0$

$\therefore\ x=0,\frac{1}{3},5$

Clearly, $x=0$ yields no series $\therefore\ x=\frac{1}{3},5$

for $x=\frac{1}{3}$: $d=5x-8x^2=5\times\frac{1}{3}-8\left(\frac{1}{3}\right)^2=\frac{7}{9}$

$x=5$: $d=5x-8x^2=5\times 5-8\times 25=-175$ #

9. $r=\frac{T_2}{T_1}=\frac{150}{300}=\frac{1}{2}$

$T_n=ar^{n-1}$

Series

$T_{10}=300\left(\frac{1}{2}\right)^{10-1}=\frac{75}{128}$ #

10. $T_n=ar^{n-1},\ T_4=ar^3=1,\ T_9=ar^8=32$

i.e. $ar^3=1$ and $ar^8=32$

$\therefore\ \frac{ar^8}{ar^3}=\frac{32}{1}$ (dividing)

$r^5=32$

$r=2$

Substituting into T_4 gives: $a=\frac{1}{8}$

$S_n=\frac{a(r^n-1)}{r-1}$

$S_9=\frac{\frac{1}{8}(2^9-1)}{2-1}=63.875$ #

11. $T_n=ar^{n-1},\ T_3=ar^2=625,\ \frac{T_4}{T_6}=\frac{ar^3}{ar^5}=25$ i.e. $\frac{1}{r^2}=25$

$\frac{1}{r^2}=25$

$r^2=\frac{1}{25}$

Substituting this into T_3, gives:

$a\left(\frac{1}{25}\right)=625$

$a=15{,}625$

Now, $S_n=\frac{a(1-r^n)}{1-r}\quad\therefore\ S_{10}=\frac{15{,}625\left(1-\left(\frac{1}{5}\right)^{10}\right)}{1-\frac{1}{5}}=19{,}531.248$ #

12. $S_n=\frac{a(r^n-1)}{r-1},\ a=2,\ r=\frac{4}{2}=2$

$\therefore\ S_{10}=\frac{2(2^n-1)}{2-1}$

$S_{10}=2(2^n-1)$

$256 = 2^n$

$2^8 = 2^n$ i.e. $n = 8$ #

13. $a = 5,\ r = \frac{10}{5} = 2,\ S_n = \frac{a(r^n - 1)}{r-1} > 1{,}500$

i.e. $\frac{5(2^n - 1)}{2-1} > 1{,}500$

$2^n - 1 > 300$

$2^n > 301$

Now, $2^8 = 256$ from question before and $2^9 = 512$

$\therefore n = 9$ i.e. 9 terms need to be taken for the sum to first exceed 1500 #

14. (i) $a = 1,\ r = 6,\ T_n = ar^{n-1} < 1{,}000{,}000$

i.e. $1 \times 6^{n-1} < 1{,}000{,}000$

Now, $6^7 = 279{,}936$ and $6^8 = 1{,}679{,}616$

$\therefore\ 6^{n-1} \le 6^7$

i.e. $n - 1 \le 7$

$n \le 8$

$\therefore$ there are 8 terms below $1{,}000{,}000$ #

(ii) $S_n = \frac{a(r^n - 1)}{r-1}$

$S_8 = \frac{1(6^8 - 1)}{6-1} = 335{,}923$ #

15. (i) $S_n = \frac{a(r^n - 1)}{r-1},\ a = 5$

i.e. $S_3 = \frac{5(r^3 - 1)}{r-1} = 105$

$\frac{5(r-1)(r^2 + r + 1)}{(r-1)} = 105$

$r^2 + r + 1 = 21$

$r^2 + r - 20 = 0$

$(r+5)(r-4) = 0$

$\therefore r = 4, -5$

Since the terms are positive, $\therefore\ r > 0$

i.e. $r = 4$ #

(ii) $S_{15} = \frac{5(4^{15} - 1)}{4-1} = 1{,}789{,}564{,}705 = 1.79 \times 10^9$ #

16. Let $T_1 = x + 1,\ T_2 = x + 3,\ T_3 = \frac{2x+13}{2}$.

If terms are in geometric progression,

$\therefore\ \frac{T_2}{T_1} = \frac{T_3}{T_2}$

i.e. $\frac{x+3}{x+1} = \frac{\frac{2x+13}{2}}{x+3}$

$\frac{x+3}{x+1} = \frac{2x+13}{2(x+3)}$

$2(x+3)^2 = (2x+13)(x+1)$

$2(x^2 + 6x + 9) = 2x^2 + 2x + 13x + 13$

$2x^2 + 12x + 18 = 2x^2 + 15x + 13$

i.e. $5 = 3x$

$x = \frac{5}{3}$ #

17. $\sum_{k=1}^{15}(5k - 10) = (5\times1 - 10) + (5\times2 - 10) + (5\times3 - 10) + \ldots + (5\times15 - 10)$

$= -5 + 0 + 5 + \ldots + 65$

which is an arithmetic series with $a = -5,\ l = 65,\ n = 15$

i.e. $S_n = \frac{n}{2}(a + l)$

$S_{15} = \frac{15}{2}(-5 + 65) = 450$ #

18. $\sum_{k=1}^{10}(2^k + 3) = (2^1 + 3) + (2^2 + 3) + (2^3 + 3) + \ldots + (2^{10} + 3)$

$= \underbrace{(2^1 + 2^2 + 2^3 + \ldots + 2^{10})} + (3 + 3 + 3 + \ldots + 3)$

The part in brackets is a geometric series with $a = 2,\ r = 2,\ n = 10$

$= \frac{2(2^{10} - 1)}{2-1} + 3 \times 10 = 2{,}076$ #

19. $S_\infty = \dfrac{a}{1-r}$, $a=-1$, $r = \dfrac{-\frac{1}{8}}{-1} = \dfrac{1}{8}$

$\therefore\ S_\infty = \dfrac{-1}{1-\frac{1}{8}} = -\dfrac{8}{7}$ #

20. If the series has a limiting sum then $-1 < r < 1$

Now, $r = \dfrac{\frac{(1-2x)^2}{4}}{\frac{1-2x}{2}} = \dfrac{(1-2x)^2}{4} \times \dfrac{2}{(1-2x)} = \dfrac{1-2x}{2}$

$\therefore\ -1 < \dfrac{1-2x}{2} < 1$

$-2 < 1-2x < 2$

$-3 < -2x < 1$

$\dfrac{3}{2} > x > -\dfrac{1}{2}$

Expressed more conventionally, for $-\dfrac{1}{2} < x < \dfrac{3}{2}$, the series has a limiting sum. #

21.(i) $0.3\dot{4} = 0.34444\ldots$

$= 0.3 + \underbrace{0.04 + 0.004 + 0.0004}+\ldots$

The part in brackets forms an infinite geometric series with $a = 0.04$, $r = 0.1$

i.e. $0.3\dot{4} = 0.3 + \dfrac{0.04}{1-0.1}$

$= 0.3 + \dfrac{0.04}{0.9} = \dfrac{3}{10} + \dfrac{4}{90} = \dfrac{31}{90}$ #

(ii) $0.4\dot{7}\dot{5} = 0.4757575\ldots$

$= 0.4 + \underbrace{0.075 + 0.00075 + 0.0000075 +}\ldots$

The part in brackets forms an infinite geometric series with $a = 0.075$, $r = 0.01$

i.e. $0.4\dot{7}\dot{5} = 0.4 + \dfrac{0.075}{1-0.01}$

$= 0.4 + \dfrac{0.075}{0.99} = \dfrac{4}{10} + \dfrac{75}{990} = \dfrac{157}{330}$ #

TOPIC 9

THE TANGENT AND THE DERIVATIVE OF A FUNCTION

(A) Limits

If $\lim\limits_{x\to c} f(x) = f(c)$, then as x approaches $f(c)$, $f(x)$ approaches (e).

In solving limits, follow these steps:

1. Ensure that the denominator (if applicable) will not equal zero if you input the relevant x value. If so, factorise the numerator to eliminate the zero occurring in the denominator.
2. If the question is $\lim\limits_{x\to\infty}$, divide by highest power, otherwise leave as is.
3. Input x value to find the limit.

Example 1: Find the following limits:

(i) $\lim\limits_{x\to 1} \dfrac{x+3}{x}$

(ii) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

(iii) $\lim\limits_{x\to\infty} \dfrac{2x^2-5}{5x^2-2x}$

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$

Solution 1:

(i) $\lim\limits_{x\to 1} \dfrac{x+3}{x} = \dfrac{4}{1} = 4$ #

(ii) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} \dfrac{(x-3)(x+3)}{(x-3)} = \lim\limits_{x\to 3}(x+3) = 6$ #

(iii) $\lim\limits_{x\to\infty} \dfrac{2x^2-5}{5x^2-2x} = \lim\limits_{x\to\infty} \dfrac{2-\frac{5}{x^2}}{5-\frac{2}{x}} = \dfrac{2-0}{5-0} = \dfrac{2}{5}$ #

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)} = 1^2+1+1 = 3$ #

(B) Gradient of a Secant

The gradient of a secant (PQ) through the two points $P(a,\ f(a))$ and $Q(x,\ f(x))$ on the curve $y=f(x)$ is given by:

$$m=\frac{f(x)-f(a)}{x-a}$$

Example 1: Find the gradient of the secant to the curve $y=x^2+x$ passing through the points $x=1$ and $x=2$.

Solution 1:

$$m=\frac{f(x)-f(a)}{x-a}=\frac{f(2)-f(1)}{2-1}=\frac{(2^2+2)-(1^2+1)}{1}=4$$

$\therefore$ the gradient of secant is 4. #

(C) Gradient of a Tangent from First Principles

By the gradient formula:

$$m=\frac{f(x)-f(a)}{x-a}$$

As $x\to a$, the point Q moves along the curve towards P and the gradient of the secant becomes the gradient of the tangent at P.

The gradient of the tangent is given by:

$$f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Example 1: Find, from first principles, the gradient of the tangent to the curve:

(i) $y=x^2-5x$ at $x=2$

(ii) $y=x^3-64$ at $x=4$

Solution 1:

(i) $f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to 2}\frac{(x^2-5x)-(2^2-5\times 2)}{}$

$=\lim_{x\to 2}\frac{x^2-5x+6}{x-2}$

$=\lim_{x\to 2}\frac{(x-2)(x-3)}{(x-2)}=-1$ #

(ii) $f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$=\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$

$=\lim_{x\to 4}\frac{(x^3-64)-(4^3-64)}{x-4}$

$=\lim_{x\to 4}\frac{x^3-64}{x-4}$

$=\lim_{x\to 4}\frac{(x-4)(x^2+4x+16)}{(x-4)}=4^2+4\times4+16=48$ #

(D) Differentiation

Notations: $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}(f(x))$, y'

(i) The Basic Rule

$\frac{d}{dx}(x^n)=nx^{n-1}$ (n may be positive or negative)

$\frac{d}{dx}(cx^n)=cnx^{n-1}$ (n may be positive or negative)

$\frac{d}{dx}(c)=0$ (where c is a constant)

Example 1: Differentiate the following with respect to x:

(i) x^2-4 (ii) x^4-x^{-3} (iii) $5x^3+x$

(iv) $4x^{-2}-2x^{-3}$ (v) $\frac{1}{3}(x+2)-\frac{1}{6}(x-1)$

Solution 1:

(i) $\frac{d}{dx}(x^2-4)=2x$ #

(ii) $\frac{d}{dx}\left(x^4 - x^{-3}\right) = 4x^3 - \left(-3x^{-4}\right) = 4x^3 + 3x^{-4}$ #

(iii) $\frac{d}{dx}\left(5x^3 + x\right) = 15x^2 + 1$ #

(iv) $\frac{d}{dx}\left(4x^{-2} - 2x^{-3}\right) = -8x^{-3} - \left(2\times -3x^{-4}\right)$

$= -8x^{-3} - \left(-6x^{-4}\right)$

$= -8x^{-3} + 6x^{-4}$ #

(v) $\frac{d}{dx}\left(\frac{1}{3}(x+2) - \frac{1}{6}(x-1)\right) = \frac{d}{dx}\left(\frac{1}{3}x + \frac{2}{3} - \frac{1}{6}x + \frac{1}{6}\right)$

$= \frac{d}{dx}\left(\frac{1}{6}x + \frac{5}{6}\right) = \frac{1}{6}$

(ii) The Derivatives of $x^{\frac{1}{n}}$, $x^{\frac{m}{n}}$, $\frac{1}{x}$ and $\sqrt{x}$

The derivatives of $x^{\frac{1}{n}}$, of $\frac{1}{x}$ and $\sqrt{x}$, all satisfy the basic rule: $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

$$\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \frac{1}{n}x^{\frac{1}{n}-1}$$

$$\frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{m}{n}x^{\frac{m}{n}-1}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$

Example 1: Differentiate the following with respect to x:

(i) $\frac{5}{x}$ (ii) $4\sqrt{x}$ (iii) $\sqrt[3]{x}$

(iv) $\sqrt{x}(1+x)$ (v) $\left(\frac{1}{2\sqrt{x}}\right)^3$

Solution 1:

(i) $\frac{d}{dx}\left(\frac{5}{x}\right) = \frac{d}{dx}\left(5x^{-1}\right) = 5\cdot -x^{-2} = -5x^{-2}$ #

(ii) $\frac{d}{dx}\left(4\sqrt{x}\right) = \frac{d}{dx}\left(4x^{\frac{1}{2}}\right) = 4\cdot\frac{1}{2}x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$ #

(iii) $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ #

(iv) $\frac{d}{dx}\left(\sqrt{x}(1+x)\right) = \frac{d}{dx}\left(\sqrt{x} + x\sqrt{x}\right)$

$= \frac{d}{dx}\left(x^{\frac{1}{2}} + x^1\cdot x^{\frac{1}{2}}\right)$

$= \frac{d}{dx}\left(x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}}$

(v) $\frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right)^3 = \frac{d}{dx}\left(\frac{1}{2x^{\frac{1}{2}}}\right)^3$

$= \frac{d}{dx}\left(\frac{1}{8x^{\frac{3}{2}}}\right)$

$= \frac{d}{dx}\left(\frac{1}{8}x^{-\frac{3}{2}}\right) = \frac{1}{8}\times -\frac{3}{2}x^{-\frac{5}{2}} = -\frac{3}{16}x^{-\frac{5}{2}}$

(iii) Chain Rule

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1}\cdot f'(x)$$

$$\frac{d}{dx}\left[f(x)\right]^{\frac{m}{n}} = \frac{m}{n}\left[f(x)\right]^{\frac{m}{n}-1}\cdot f'(x)$$

Example 1: Differentiate the following with respect to x:

(i) $(5-2x)^4$ (ii) $\frac{1}{2-x}$ (iii) $\sqrt{1-x^2}$

(iv) $\sqrt[3]{x^3 - 9x}$ (v) $\frac{3}{(1-3x)^5}$

Solution 1:

(i) $\frac{d}{dx}\left[(5-2x)^4\right]=4(5-2x)^3\cdot\frac{d}{dx}(5-2x)$

$=4(5-2x)^3\cdot -2=-8(5-2x)^3$ #

(ii) $\frac{d}{dx}\left(\frac{1}{2-x}\right)=\frac{d}{dx}\left((2-x)^{-1}\right)$

$=-(2-x)^{-2}\cdot -1=(2-x)^{-2}=\frac{1}{(2-x)^2}$ #

(iii) $\frac{d}{dx}\left(\sqrt{1-x^2}\right)=\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right)$

$=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot\frac{d}{dx}(1-x^2)$

$=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot -2x$

$=-x(1-x^2)^{-\frac{1}{2}}$ #

(iv) $\frac{d}{dx}\left(\sqrt[3]{x^3-9x}\right)=\frac{d}{dx}\left[(x^3-9x)^{\frac{1}{3}}\right]$

$=\frac{1}{3}(x^3-9x)^{-\frac{2}{3}}\cdot\frac{d}{dx}(x^3-9x)$

$=\frac{1}{3}(x^3-9x)^{-\frac{2}{3}}\cdot(3x^2-9)$

$=\frac{1}{3}(x^3-9x)^{-\frac{2}{3}}\cdot 3(x^2-3)$

$=(x^3-9x)^{-\frac{2}{3}}(x^2-3)$ #

(v) $\frac{d}{dx}\left(\frac{3}{(1-3x)^5}\right)=\frac{d}{dx}\left(3(1-3x)^{-5}\right)$

$=3\cdot -5(1-3x)^{-6}\cdot\frac{d}{dx}(1-3x)$

$=-15(1-3x)^{-6}\cdot -3$

$=45(1-3x)^{-6}$

$=\frac{45}{(1-3x)^6}$ #

(iv) Product Rule

$$\frac{d}{dx}\left(u(x)\cdot v(x)\right)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Note that the product rule is used when the question is of the form:

(function of x) $\times$ (function of x)

Example 1: Differentiate the following:

(i) $x^2(1-x)^2$ (ii) $(x+3)(x^2-4)$ (iii) $x\sqrt{1+x}$

Solution 1:

(i) $\frac{d}{dx}\left(x^2(1-x)^2\right)=x^2\cdot\frac{d}{dx}\left[(1-x)^2\right]+(1-x)^2\cdot\frac{d}{dx}(x^2)$

$=x^2\cdot 2(1-x)\cdot -1+(1-x)^2\cdot 2x$

$=-2x^2(1-x)+2x(1-x)^2$

$=2x(1-x)[1-x-x]=2x(1-x)(1-2x)$ #

(ii) $\frac{d}{dx}\left[(x+3)(x^2-4)\right]=(x+3)\cdot\frac{d}{dx}(x^2-4)+(x^2-4)\cdot\frac{d}{dx}(x+3)$

$=(x+3)\cdot 2x+(x^2-4)\cdot 1$

$=2x^2+6x+x^2-4=3x^2+6x-4$ #

(iii) $\frac{d}{dx}\left(x\sqrt{1+x}\right)=\frac{d}{dx}\left(x(1+x)^{\frac{1}{2}}\right)$

$=x\cdot\frac{d}{dx}\left[(1+x)^{\frac{1}{2}}\right]+(1+x)^{\frac{1}{2}}\cdot\frac{d}{dx}(x)$

$=x\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}+(1+x)^{\frac{1}{2}}$

$=\frac{x}{2\sqrt{1+x}}+\sqrt{1+x}$ #

(v) Quotient Rule

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Note that with the quotient rule, u must be the numerator and v must be the denominator.

Example 1: Differentiate the following:

(i) $\frac{2x}{x^2+1}$ (ii) $\frac{1-x^2}{1+x^2}$

Solution 1:

(i) $$\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)=\frac{\left(x^2+1\right)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$=\frac{\left(x^2+1\right)\cdot 2-2x\cdot 2x}{\left(x^2+1\right)^2}$$

$$=\frac{2x^2+2-4x^2}{\left(x^2+1\right)^2}$$

$$=\frac{2-2x^2}{\left(x^2+1\right)^2}$$

$$=\frac{2\left(1-x^2\right)}{\left(x^2+1\right)^2} \quad \#$$

(ii) $$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right)\cdot\frac{d}{dx}\left(1-x^2\right)-\left(1-x^2\right)\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right)\cdot -2x-\left(1-x^2\right)\cdot 2x}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2} \quad \#$$

REVIEW EXERCISES

(A) Limits

1. Find the following limits:

(i) $\lim_{x\to 1}\left(x^4-2x+5\right)$ (ii) $\lim_{x\to 4}\frac{x^3-64}{x^2-3x-4}$

(iii) $\lim_{x\to\infty}\frac{x^2+1}{2x^2}$ (iv) $\lim_{x\to -5}\frac{25-x^2}{\sqrt{x+5}}$

(B) Gradient of a Secant

2. Find the gradient of the secant to the curve $y=5x-2x^3$ passing through the points $x=3$ and $x=1$.

(C) Gradient of a Tangent from First Principles

3. Find from first principles the gradient of the tangent to the curve:

(i) $y=2x^3$ at $x=1$

(ii) $y=2x^2-3x$ at $x=2$

(D) Differentiation

(i) The Basic Rule

4. Differentiate the following:

(i) $2x^3+4x$ (ii) $4x^2-x^{-3}$

(iii) $\frac{1}{2}x^8$ (iv) x^5+3x^{-1}

(ii) The Derivatives of $x^{\frac{1}{n}}$, $x^{\frac{m}{n}}$, $\frac{1}{x}$ *and* $\sqrt{x}$

5. Differentiate:

(i) $\frac{2}{x^2}$ (ii) $2\sqrt{x}$ (iii) $\frac{5}{\sqrt{x}}$

(iv) $x^2\sqrt{x}$ (v) $\sqrt{x^3}$ (vi) $\frac{10}{x}+\sqrt{x}$

(iii) Chain Rule

6. Differentiate:

(i) $\dfrac{1}{(2-x)^2}$ (ii) $2\sqrt{x^2+4}$ (iii) $(5-2x)^3$

(iv) $\left(1+\sqrt{x}\right)^5$ (v) $\left(x^3-x\right)^5$ (vi) $\dfrac{1}{\sqrt{1-x}}$

(iv) Product Rule

7. Differentiate:

(i) $(1-x)\left(5+x^2\right)$ (ii) $2x(1+2x)^3$ (iii) $x^5(1-x)^5$

(iv) $\left(x^3-8\right)\left(2-x^2\right)$ (v) $2x\sqrt{5-x^2}$ (vi) $\dfrac{1}{2}x\sqrt{x+1}$

(v) Quotient Rule

8. Differentiate:

(i) $\dfrac{x-1}{x+1}$ (ii) $\dfrac{5-2x}{x^2}$ (iii) $\dfrac{x}{x^2-2}$

(iv) $\dfrac{x^2-2}{1+x}$ (v) $\dfrac{5x}{(x-1)^2}$ (vi) $\dfrac{(x-1)^2}{x}$

WORKED SOLUTIONS TO REVIEW EXERCISES

1.

(i) $\lim\limits_{x\to1}\left(x^4-2x+5\right)=1^4-2\times1+5=4$ #

(ii) $\lim\limits_{x\to4}\dfrac{x^3-64}{x^2-3x-4}=\lim\limits_{x\to4}\dfrac{(x-4)\left(x^2+4x+16\right)}{(x-4)(x+1)}=\dfrac{4^2+4x+16}{5}=\dfrac{48}{5}$ #

(iii) $\lim\limits_{x\to\infty}\dfrac{x^2+1}{2x^2}=\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x^2}}{2}=\dfrac{1}{2}$ #

(iv) $\lim\limits_{x\to-5}\dfrac{25-x^2}{\sqrt{x+5}}=\lim\limits_{x\to-5}\dfrac{(5-x)(5+x)}{\sqrt{5+x}}=\lim\limits_{x\to-5}(5-x)\sqrt{5+x}=0$ #

2. $m=\dfrac{f(x)-f(a)}{x-a}$, where m is the gradient of the secant

$=\dfrac{f(3)-f(1)}{3-1}=\dfrac{(15-54)-(5-2)}{2}=-\dfrac{42}{2}=-21$ #

3. (i) $f'(x)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$=\lim\limits_{x\to1}\dfrac{f(x)-f(1)}{x-1}$

$=\lim\limits_{x\to1}\dfrac{2x^3-2}{x-1}$

$=\lim\limits_{x\to1}\dfrac{2\left(x^3-1\right)}{x-1}=\lim\limits_{x\to1}\dfrac{2(x-1)\left(x^2+x+1\right)}{(x-1)}=\lim\limits_{x\to1}2\left(x^2+x+1\right)=6$ #

(ii) $f'(x)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

$=\lim\limits_{x\to2}\dfrac{f(x)-f(2)}{x-2}$

$=\lim\limits_{x\to2}\dfrac{\left(2x^2-3x\right)-(8-6)}{(x-2)}$

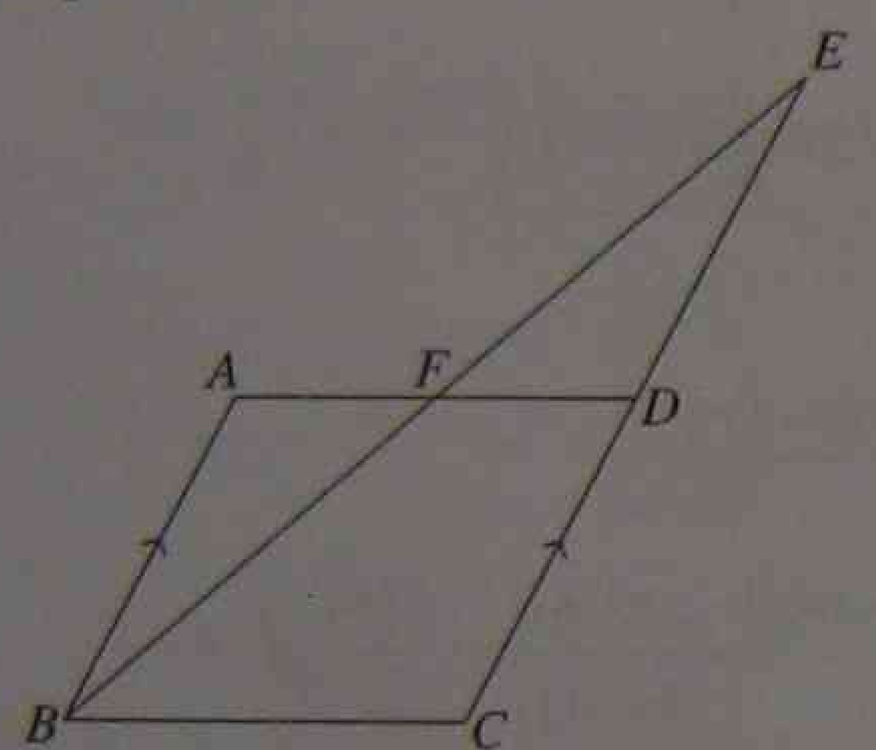
$=\lim\limits_{x\to2}\dfrac{2x^2-3x-2}{(x-2)}=\lim\limits_{x\to2}\dfrac{(2x+1)(x-2)}{(x-2)}=\lim\limits_{x\to2}(2x+1)=5$ #

APPLICATIONS OF GEOMETRICAL PROPERTIES

Many problems in geometry are not numerical but are theoretical in nature. The following exercises are based on problems involving deduction.

(A) Problems Involving Deduction

Example 1:



In the diagram:

$AB \parallel CE$

F is the mid-point of AD.

Show that $\Delta\, AFB \equiv \Delta\, FED$.

Solution 1:

$\angle AFB = \angle EFD$ (vertically opposite angles equal)

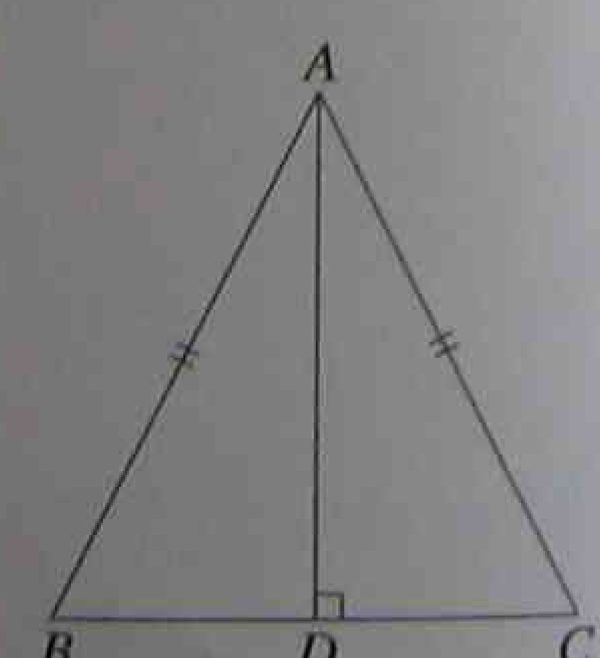
$\angle ABF = \angle DEF$ (alternate angles equal, $AB \parallel CE$)

$AF = FD$ (as F is the midpoint of AD)

$\therefore\ \Delta\, ABF \equiv \Delta\, EFD$ (AAS) #

Example 2: ABC is a triangle in which $AB = AC$. A perpendicular is drawn from A to meet BC at D. Show that $BD = DC$.

Solution 2:



$AB = AC$ (given)

Side AD is common.

$\angle ADC = \angle ADB = 90°$ ($AD \perp BC$, given)

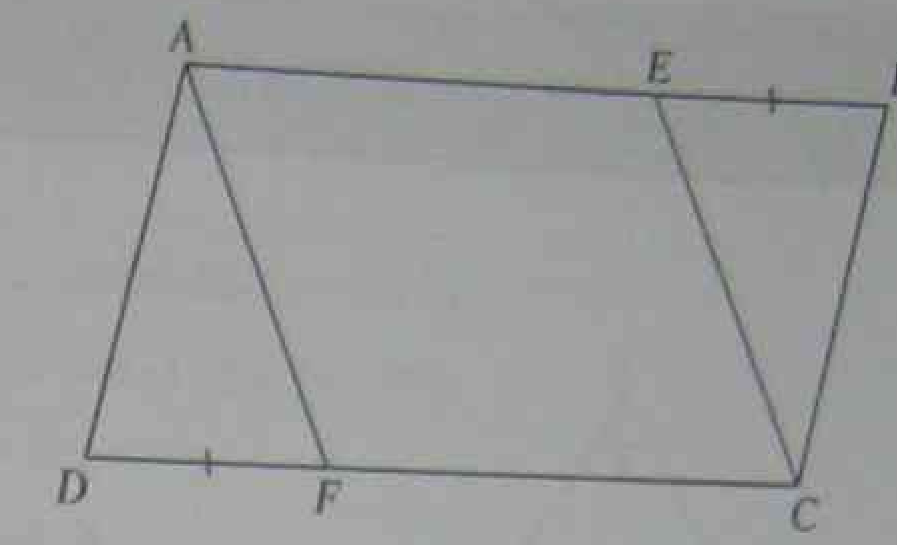
Thus, $\Delta\, ADB \equiv \Delta\, ADC$ (RHS)

$\therefore\ BD = DC$ (corresponding sides of $\equiv \Delta$'s)

Example 3: $ABCD$ is a parallelogram.

$BE = DF$

Prove that $AF \parallel EC$.



Solution 3:

$AB \parallel CD$ (opposite sides of a parallelogram parallel)

$AB = CD$ (opposite sides of a parallelogram equal)

$BE = DF$ (given)

Thus, $AB - BE = CD - DF$

i.e. $AE = CF$

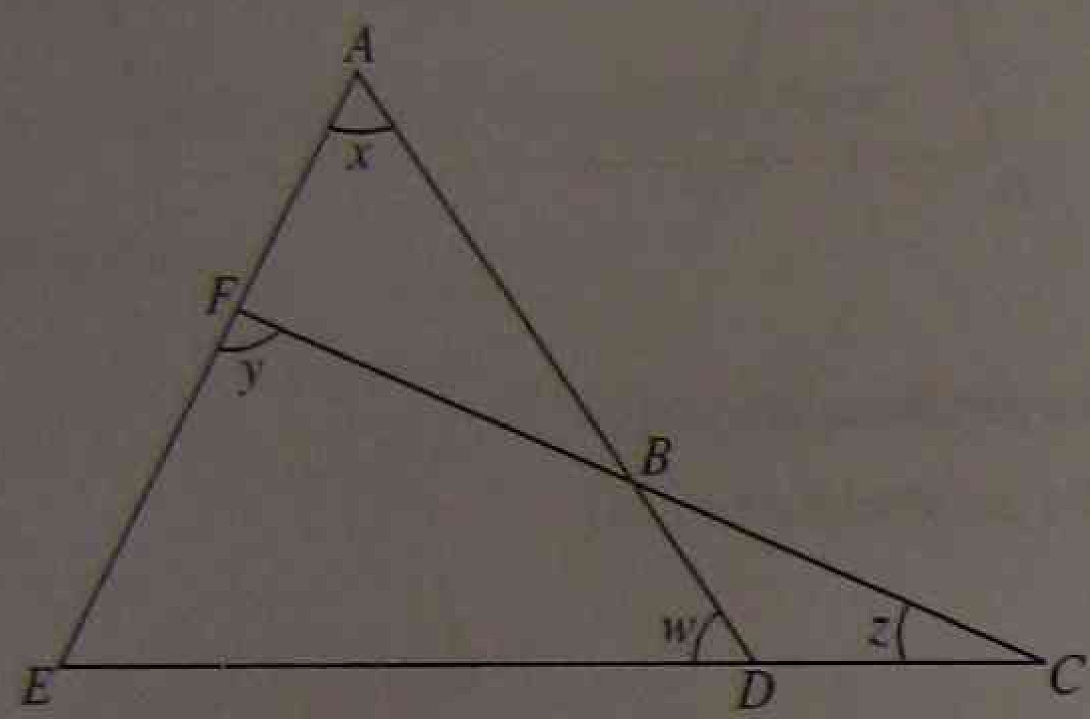
Since points E and F lie on AB and CD respectively, then $AE \parallel FC$.

Hence, $AECF$ is a parallelogram (one pair of opposite sides = and $\parallel$)

$\therefore$ AF $\parallel$ EC. (opposite sides of a parallelogram are $\parallel$)

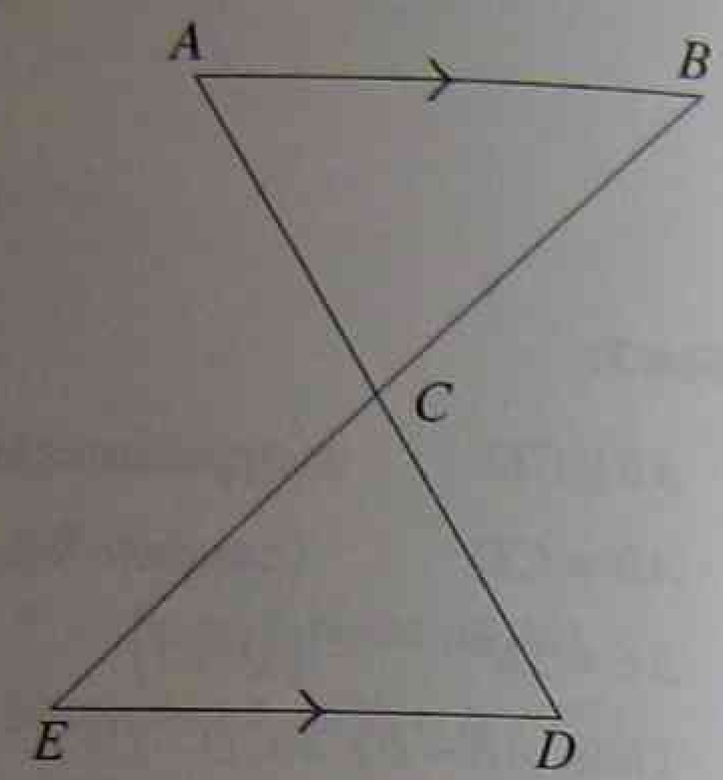
REVIEW EXERCISES

1.



Prove that $w° + x° = y° + z°$.

2.

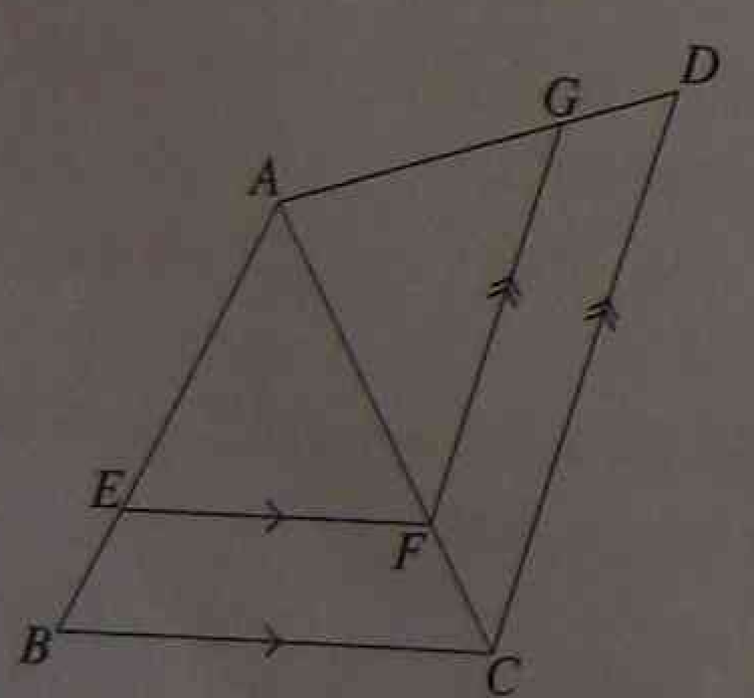


$AB \parallel ED$

C is the midpoint of AD.

Prove that C is also the midpoint of BE.

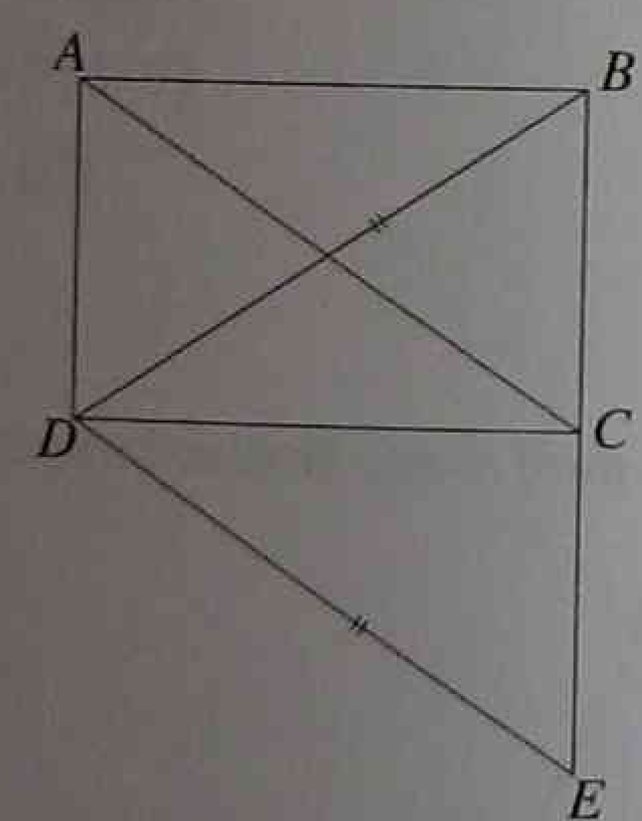
3.



$EF \parallel BC$, $FG \parallel CD$

Prove that $\dfrac{EF}{BC} = \dfrac{FG}{CD}$.

4.



$ABCD$ is a rectangle.

Points B, C, E are colinear.

$BD = DE$.

Prove ACED is a parallelogram.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $\angle AED = 180° - x° - w°$ ($\angle$ sum of $\Delta AED = 180°$)

   $\angle AED = 180° - y° - z°$ ($\angle$ sum of $\Delta FEC = 180°$)

   $\therefore\ 180° - x° - w° = 180° - y° - z°$

   i.e. $x° + w° = y° + z°$ as required. #

2. $\angle ACB = \angle ECD$ (vertically opposite $\angle$'s $=$)

   $\angle BAC = \angle CDE$ (alternate $\angle$'s $=$, $AB \parallel ED$)

   $AC = CD$ (C is the midpoint of AD, given)

   Thus, $\Delta ABC \equiv \Delta CED$ (AAS)

   $\therefore\ BC = CE$ (corresponding sides of $\equiv$ Δ's are $=$)

   and since $BC = CE$ $\therefore$ C is the midpoint of BE. #

3. In ΔABC: $\angle BAC = \angle EAF$ (common angle to ΔABC, ΔAEF)

   $\angle AEF = \angle ABC$ $\left\{\begin{array}{l}\text{corresponding } \angle\text{'s equal} \\ EF \parallel BC\end{array}\right\}$

   $\therefore\ \Delta AEF \mathrel{|||} \Delta ABC$ (AAA)

   thus, $\dfrac{EF}{BC} = \dfrac{AF}{AC}$ $\left\{\begin{array}{l}\text{corresponding sides of } ||| \ \Delta\text{'s are in } = \\ \text{proportion}\end{array}\right\}$

   In ΔACD: $\angle FAG = \angle CAD$ (common angle to ΔFAG, ΔCAD)

   $\angle AFG = \angle ACD$ $\left\{\begin{array}{l}\text{corresponding } \angle\text{'s equal} \\ FG \parallel CD\end{array}\right\}$

   $\therefore\ \Delta AGF \mathrel{|||} \Delta ADC$ (AAA)

   $\therefore\ \dfrac{FG}{CD} = \dfrac{AF}{AC}$

   Hence, combining the above results gives $\dfrac{EF}{BC} = \dfrac{FG}{CD}$ #

4. $\angle ADC = 90°$ (right angle in rectangle $ABCD$)

$\angle DCE = 180° - 90° = 90°$ ($\angle$'s in a straight line, B, C, E collinear)

$BD = AC$ (diagonals of a rectangle are $=$)

$BD = DE$ (given)

$\therefore DE = AC$

Looking at ΔADC and ΔDCE:

$\angle ADC = \angle DCE = 90°$ (proven above)

$DE = AC$ (proven above)

$\therefore \Delta ADC \equiv \Delta DCE$ (RHS)

thus, $CE = AD$ (corresponding sides of congruent Δ's $=$)

Hence, $ACED$ is a parallelogram (two pairs of opposite sides equal) #

TOPIC 11

COORDINATE METHODS IN GEOMETRY

The formulae and properties learned in Topic 4 may be applied to solve more complex geometrical problems.

Example 1: Consider the lines L_1 and L_2 with the equations $x + 2y + 3 = 0$ and $2x - y + 6 = 0$ respectively.

(i) Show that L_1 and L_2 are perpendicular to each other.

(ii) Prove that L_1 and L_2 intersect at $Q(-3, 0)$

(iii) Sketch L_1 and L_2 showing the above features and the coordinates of their points of intersection with the coordinate axes.

(iv) On your diagram, shade the region defined by the following inequalities: $x + 2y + 3 \geq 0$, $2x - y + 6 \geq 0$, $x \leq -1$.

(v) Show that the area of the shaded region is 5 units^2.

Solution 1:

(i) L_1: $x + 2y + 3 = 0$ $\quad$ L_2: $2x - y + 6 = 0$

$2y = -x - 3$ $\quad$ $y = 2x + 6$

$y = -\frac{1}{2}x - 3$

From the above equations:

L_1 has gradient $-\frac{1}{2}$ and L_2 gradient 2.

Since $m_1 \times m_2 = -\frac{1}{2} \times 2 = -1$

$\therefore$ L_1 and L_2 are perpendicular to each other. #

(ii) L_1: $x + 2y + 3 = 0$ ....(1)

L_2: $2x - y + 6 = 0$ ....(2)

$2 \times (1)$: $2x + 4y + 6 = 0$ ....(3)

$(3) - (2)$: $0 + 5y + 0 = 0$

$5y = 0$

$y = 0$

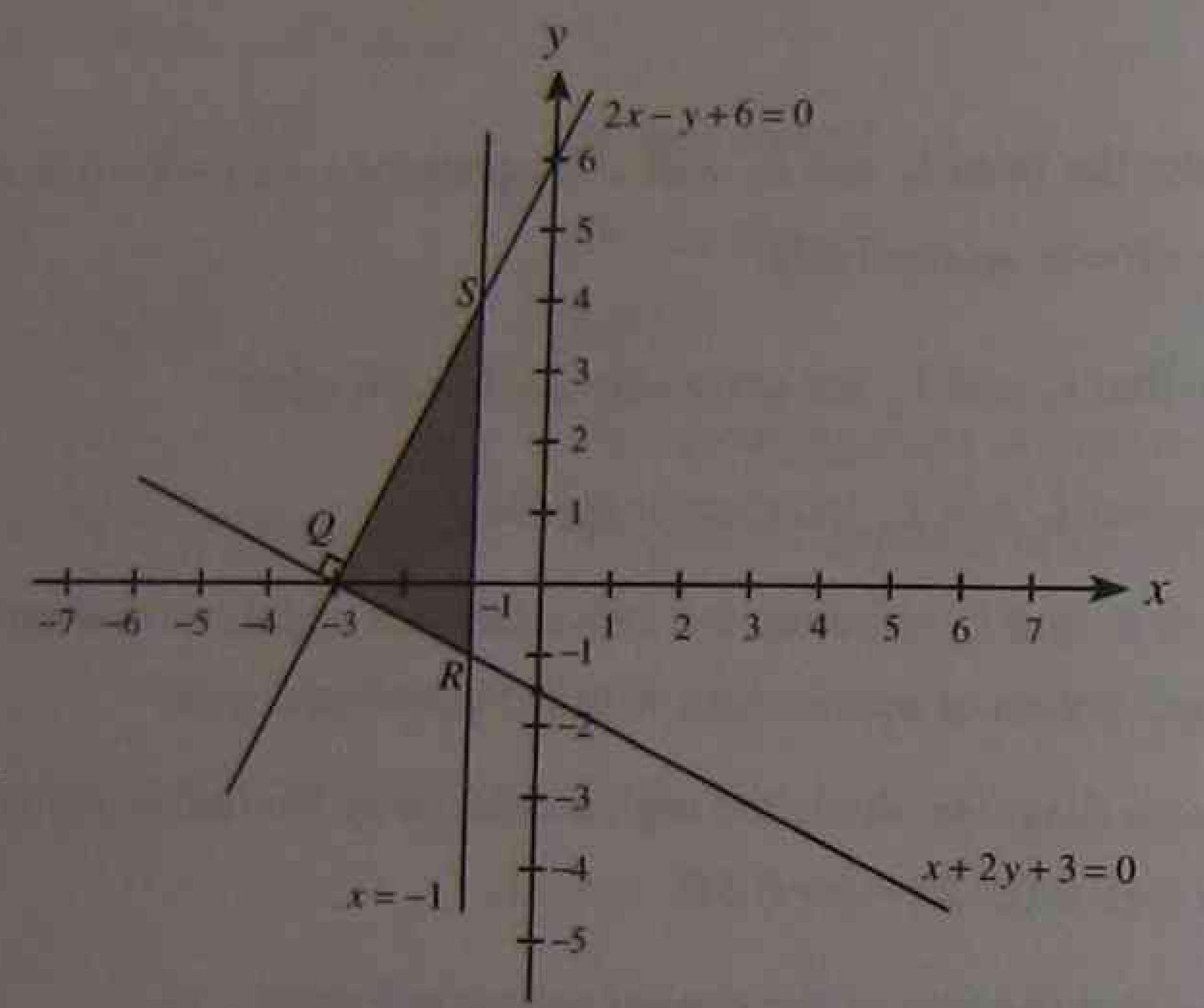
Substituting into (1) gives:

$x+0+3=0$

i.e. $x=-3$

$\therefore$ Q has coordinates $(-3, 0)$ #

(iii) and (iv)



(v) The points where $x=-1$ intersects the lines L_1 and L_2 need to be found (namely, R and S):

L_1: $x+2y+3=0$ at $x=-1$ | L_2: $2x-y+6=0$ at $x=-1$

$-1+2y+3=0$ | $-2-y+6=0$

$2y=-2$ | $y=4$

$y=-1$ | i.e. $S(-1, 4)$

i.e. $R(-1,-1)$

Thus, the distance $RS=5$ units $(4-(-1)=5)$

Height of $\Delta QSR=2$ units.

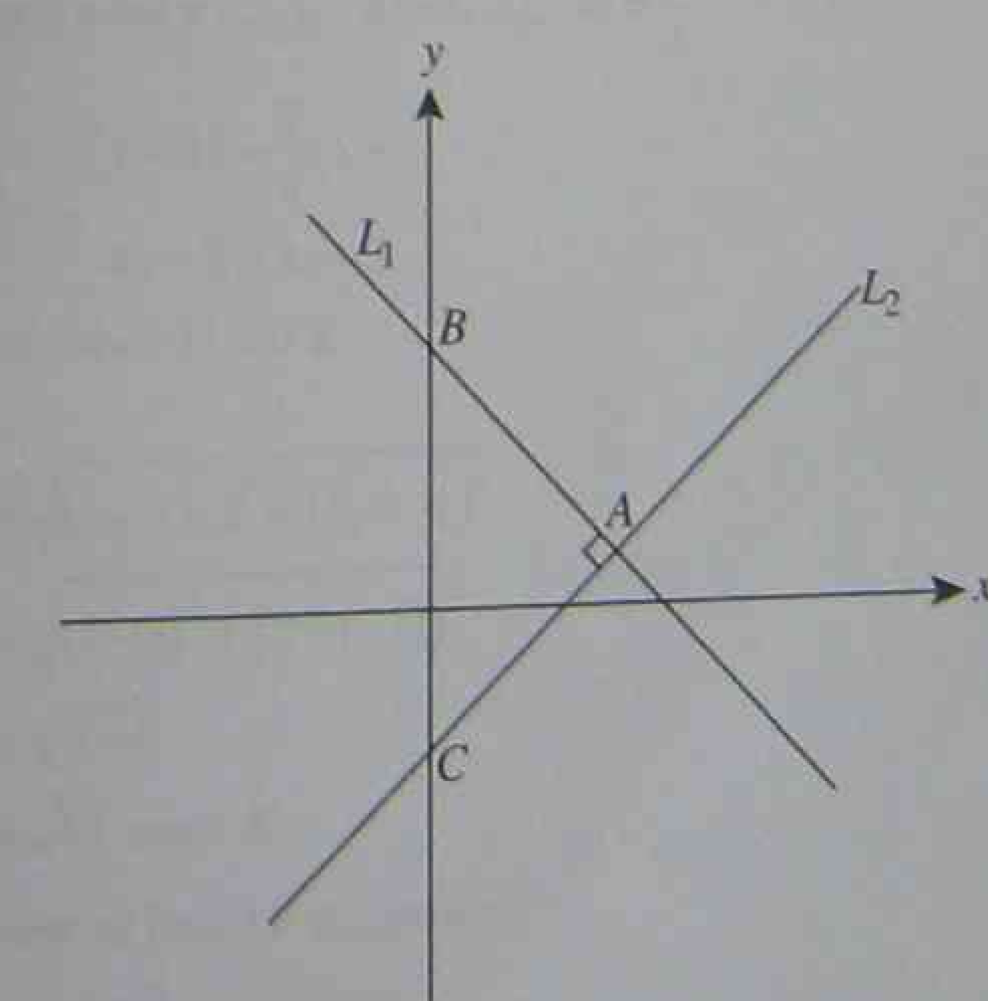
$\therefore$ Area of $\Delta QSR=\frac{1}{2}\times 5\times 2=5$ units^2 #

REVIEW EXERCISES

1. Given the points $A(-1, 2)$ and $B(3, 5)$ find:
   (i) The length of the interval joining A and B.
   (ii) The coordinates of the midpoint of the interval AB.
   (iii) The gradient of the line AB and hence, the angle of inclination of AB to the positive x-axis. (Answer to the nearest degree.)
   (iv) The equation of AB.

2. Given the points $A(-1, 3)$ and $B(2, 1)$ find:
   (i) The length of the interval AB.
   (ii) The equation of line AB.
   (iii) The perpendicular distance from the origin to the line AB.
   (iv) Hence, determine the equation of the circle centre the origin and with line AB as its tangent.

3. In the diagram below, the line L_1 has equation $x+2y-5=0$ and intersects the line L_2 at a right angle at point A with co-ordinates $(3, 1)$. Copy the diagram.



Coordinate Geometry

(i) Show that the equation of the line, L_2, is given by: $y-2x+5=0$

(ii) Find the coordinates of the points B and C.

(iii) Find the area of ΔABC.

(iv) On your diagram, shade the region satisfying the following inequalities:
$x\geq 0,\ \ 0\leq y\leq 1,\ \ y-2x+5\geq 0$

(v) Hence, find the area of the shaded region.

4. The coordinates of points A, B and C are $(-3, 1)$, $(1, 4)$ and $(2, 1)$ respectively.

(i) Find the length BC and the gradient of BC.

(ii) Show that the equation of the line l, perpendicular to BC passing through A is $3y=x+6$.

(iii) Find the length of AC.

(iv) Find the coordinates of a point $D(x,\ y)$ which lies on line l such that $BD=AC$.

(v) Find the coordinates of M, the midpoint of interval BC.

(vi) Show that M lies on the line l and also bisects the interval AD.

(vii) Determine the shape of quadrilateral $ABCD$. Give reasons.

(viii) Find the area of quadrilateral $ABCD$.

WORKED SOLUTION TO REVIEW EXERCISES

1. (i) $AB=\sqrt{(y_2-y_1)^2+x_2-x_1}$

$=\sqrt{(5-2)^2+(3+1)^2}$

$=\sqrt{9+16}=5$ units #

(ii) $M\ (x,\ y)=\left(\dfrac{x_1+x_2}{2},\ \dfrac{y_1+y_2}{2}\right)$

$=\left(\dfrac{-1+3}{2},\ \dfrac{2+5}{2}\right)=\left(1,\ \dfrac{7}{2}\right)$ #

(iii) $m=\dfrac{y_2-y_1}{x_2-x_1}$

$m_{AB}=\dfrac{5-2}{3-(-1)}=\dfrac{3}{4}$

Now, $\tan\theta=m$

i.e. $\tan\theta=\dfrac{3}{4}$

$\theta=37°$

$\therefore$ the angle of inclination of AB to the positive x-axis is 37° to the nearest degree. #

(iv) $(y-y_1)=m(x-x_1)$, $A(-1,\ 2)$, $m=\dfrac{3}{4}$

$(y-2)=\dfrac{3}{4}(x+1)$

$4y-8=3x+3$

i.e. $3x-4y+11=0$ #

2. (i) $\mathrm{AB}=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$=\sqrt{(1-3)^2+(2-(-1))^2}$

$=\sqrt{4+9}$

$=\sqrt{13}$ units #

(ii) Gradient of line AB is given by:

$m_{AB}=\dfrac{1-3}{2-(-1)}=-\dfrac{2}{3}$

of AB is given by:

$$(y-1)=-\frac{2}{3}(x-2)$$
$$3y-3=-2x+4$$
i.e. $3y+2x-7=0$ #

(iii) Perpendicular distance from $O(0, 0)$ to the line $2x+3y-7=0$ is given by:

$$d=\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$
$$=\left|\frac{0+0-7}{\sqrt{3^2+2^2}}\right|=\frac{7}{\sqrt{13}} \text{ units } \#$$

(iv) Centre of circle is $(0, 0)$, radius $=\frac{7}{\sqrt{13}}$ units.

Hence, equation of circle is $x^2+y^2=\left(\frac{7}{\sqrt{13}}\right)^2=\frac{49}{13}$ #

3. (i) L_1 has equation $x+2y-5=0$

i.e. $2y=5-x$
$$y=\frac{5}{2}-\frac{1}{2}x$$

i.e. gradient of $L_1=-\frac{1}{2}$ $\therefore$ gradient of $L_2=2$

Hence, equation of L_2 is given by:
$$(y-1)=2(x-3)$$
$$y-1=2x-6$$
i.e. $y-2x+5=0$

(ii) L_1: $x+2y-5=0$ crosses the y-axis at $x=0$

i.e. $0+2y-5=0$
$$2y=5$$
$$y=\frac{5}{2} \text{ thus, point } B=\left(0, \frac{5}{2}\right) \ \#$$

L_2: $y-2x+5=0$ crosses the y-axis at $x=0$

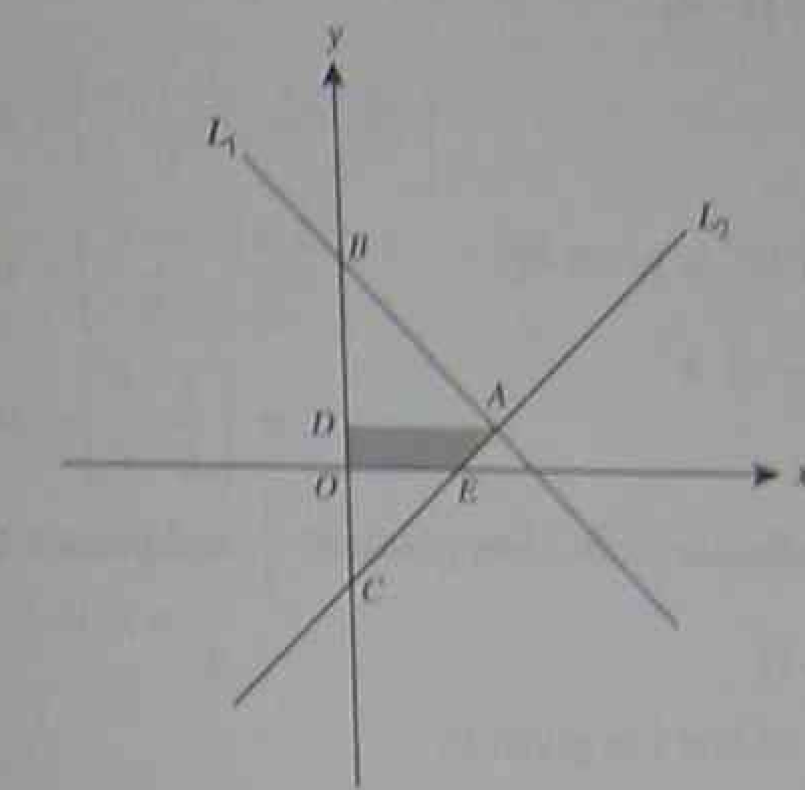
i.e. $y-0+5=0$
$$y=-5 \text{ thus, point } C(0, -5) \ \#$$

(iii) Distance $AC=\sqrt{(1+5)^2+3^2}=\sqrt{45}=3\sqrt{5}$ units

Distance $AB=\sqrt{3^2+\left(1-\frac{5}{2}\right)^2}=\sqrt{9+\frac{9}{4}}=\sqrt{\frac{45}{4}}=\frac{3\sqrt{5}}{2}$ units

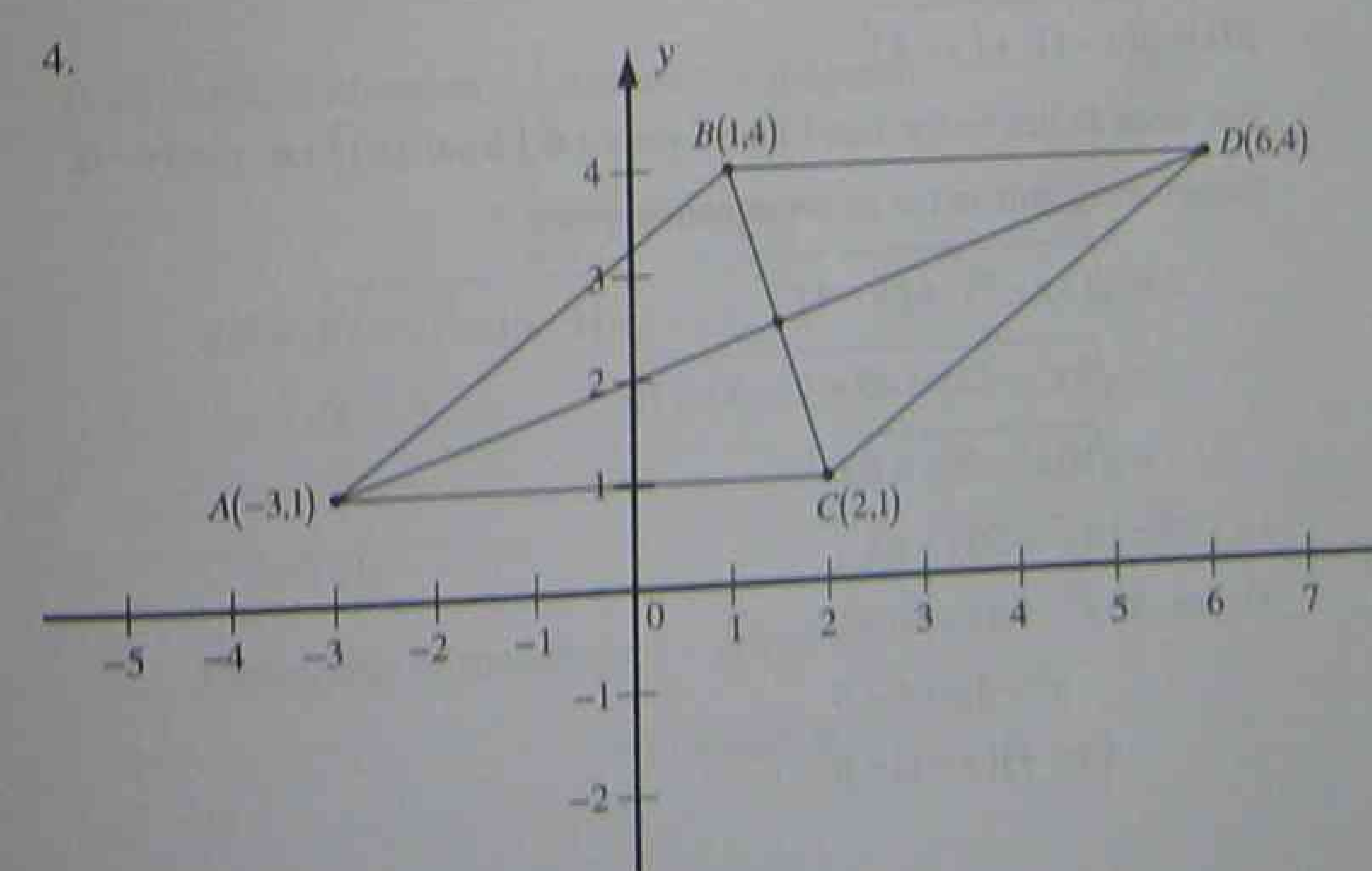
$\therefore$ Area of $\Delta ABC=\frac{1}{2}\times 3\sqrt{5}\times\frac{3\sqrt{5}}{2}=\frac{45}{4}$ units^2 #

(iv)



(v) Area of shaded region = Area of ΔABC – Area of ΔABD – Area of ΔOEC
$$=\frac{45}{4}-\frac{1}{2}\times 3\times\frac{3}{2}-\frac{1}{2}\times 5\times\frac{5}{2}$$
$$=\frac{45}{4}-\frac{9}{4}-\frac{25}{4}$$
$$=\frac{11}{4} \text{ units}^2 \ \#$$

4.



(i) The length of BC is given by:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(2-1)^2+(1-4)^2}$$
$$=\sqrt{1+9}$$
$$=\sqrt{10}\text{ units}$$

The gradient of BC is given by:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{1-4}{2-1}=-3\ \#$$

(ii) The line perpendicular to BC has gradient $\frac{1}{3}$ and passes through the point $A(-3, 1)$.

$\therefore$ the equation of line l is given by:

$$y-y_1=m(x-x_1)$$
$$y-1=\frac{1}{3}(x+3)$$
$$3y-3=x+3$$
$$3y=x+6\ \#$$

(iii) $$AC=\sqrt{(1-1)^2+(2+3)^2}$$
$$=\sqrt{25}$$
$$=5\text{ units}\ \#$$

(iv) $AC=5,\ D(x, y),\ B(1, 4),\ BD=5$

$$BD=\sqrt{(x-1)^2+(y-4)^2}$$

But since D lies on the line l then $3y=x+6$ [from (b)] i.e. $x=3y-6$.

Substituting this in the above equation gives:

$$5=\sqrt{(3y-7)^2+(y-4)^2}$$
$$=\sqrt{9y^2-42y+49+y^2-8y+16}$$
$$=\sqrt{10y^2-50y+65}$$

i.e. $25=10y^2-50y+65$

Hence, $10y^2-50y+40=0$
$$y^2-5y+4=0$$
$$(y-4)(y-1)=0$$

i.e. $y=4,\ y=1$

at $y=4,\ x=6$

$y=1,\ x=-3$ which is point A

$\therefore$ point D is given by $(6, 4)$ #

(v) $$\text{Midpoint}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$
$$M=\left(\frac{2+1}{2},\frac{1+4}{2}\right)$$
$$M=\left(\frac{3}{5},\frac{5}{2}\right)\ \#$$

(vi) $3y=x+6$ at $x=\frac{3}{2}$

$$3y=6+\frac{3}{2}$$
$$=\frac{15}{2}$$

$\therefore\ y=\frac{15}{6}=\frac{5}{2}$ as required.

Also, to show that M bisects interval AD, all we need to do is show that M is also the midpoint of AD:

i.e. midpoint of $AD=\left(\frac{-3+6}{2},\frac{1+4}{2}\right)=\left(\frac{3}{2},\frac{5}{2}\right)=M$ #

(vii) Quadrilateral $ABCD$ is a rhombus.

Reason: diagonals bisect each other at right angles. #

(viii) $$\text{Area of Rhombus}=\frac{1}{2}\times\text{product of diagonals}$$
$$=\frac{1}{2}\times BC\times AD$$

$$AD=\sqrt{(6+3)^2+(4-1)^2}$$
$$=\sqrt{81+9}$$
$$=\sqrt{90}$$
$$=3\sqrt{10}$$

Hence, area of rhombus $=\frac{1}{2}\times\sqrt{10}\times3\sqrt{10}$
$$=15\text{ units}^2\ \#$$

TOPIC 12

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

(A) The Significance of the First Derivative

The first derivative can be used to indicate for which values of x the curve is increasing or decreasing:

If $\frac{dy}{dx} > 0$ the curve is increasing

If $\frac{dy}{dx} < 0$ the curve is decreasing

Example 1: For what values of x is $y = 6x^2 + 3x + 5$ decreasing?

Solution 1:

$y = 6x^2 + 3x + 5$

$\frac{dy}{dx} = 12x + 3$, for the curve to be decreasing, $\frac{dy}{dx} < 0$

i.e. $12x + 3 < 0$

$12x < -3$

$x < -\frac{3}{12} = -\frac{1}{4}$

$\therefore$ the function is decreasing for $x < -\frac{1}{4}$ #

Example 2: For what values of x is the curve $y = 2x^3 - 3x^2$ increasing?

Solution 2:

$y = 2x^3 - 3x^2$

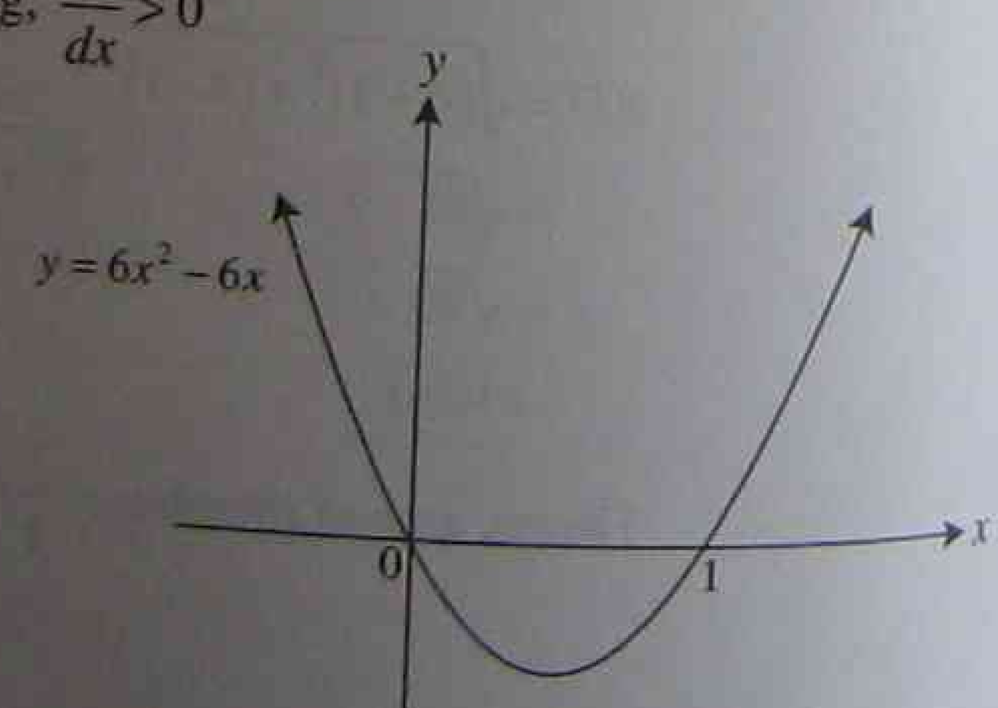
$\frac{dy}{dx} = 6x^2 - 6x$, for the curve to be increasing, $\frac{dy}{dx} > 0$

i.e. $6x^2 - 6x > 0$

$6x(x-1) > 0$

from the graph

$y > 0$ for $x < 0$ or $x > 1$ #



(B) Finding Stationary Points

Stationary points on a curve, $y = f(x)$, are found by following the steps below:

1. Find $f'(x)$ or $\frac{dy}{dx}$.
2. Solve $f'(x) = 0$, for value(s) of x.
3. Substitute x values back into the original curve $y = f(x)$ to find y-value(s).

Example 1: Find the coordinates of the stationary points on the curve $f(x) = x^3 - 6x^2 + 10$.

Solution 1:

$f(x) = x^3 - 6x^2 + 10$

$f'(x) = 3x^2 - 12x$, let $f'(x) = 0$ to find stationary points

i.e. $3x^2 - 12x = 0$

$3x(x-4) = 0$

$\therefore$ $x = 0$ or $x = 4$

at $x = 0$, $f(0) = 10$

at $x = 4$, $f(4) = 4^3 - 6(4)^2 + 10 = 64 - 96 + 10 = -22$

$\therefore$ the stationary points of $f(x)$ are $(0, 10)$ and $(4, -22)$

(C) Nature of Stationary Points

Stationary points are of 3 types:

1. maximum turning points
2. minimum turning points
3. horizontal points of inflection

To determine the nature of stationary points, there are two methods:

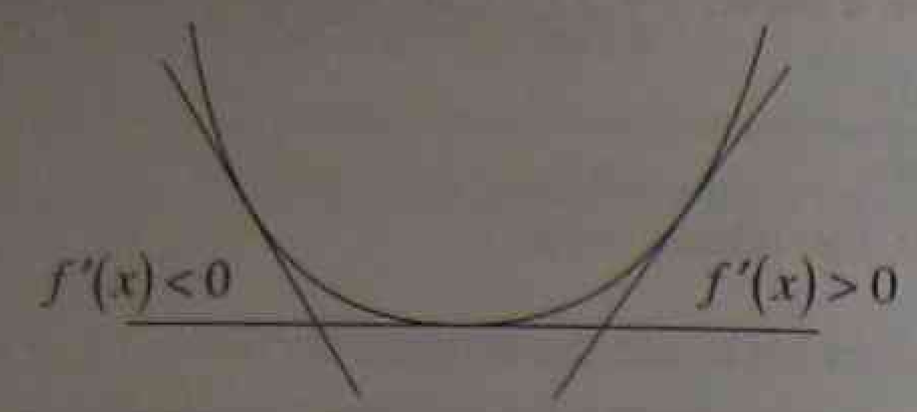
Method 1- uses the first derivative, and is explained below.

Method 2- uses the second derivative, and is explained in section (D).

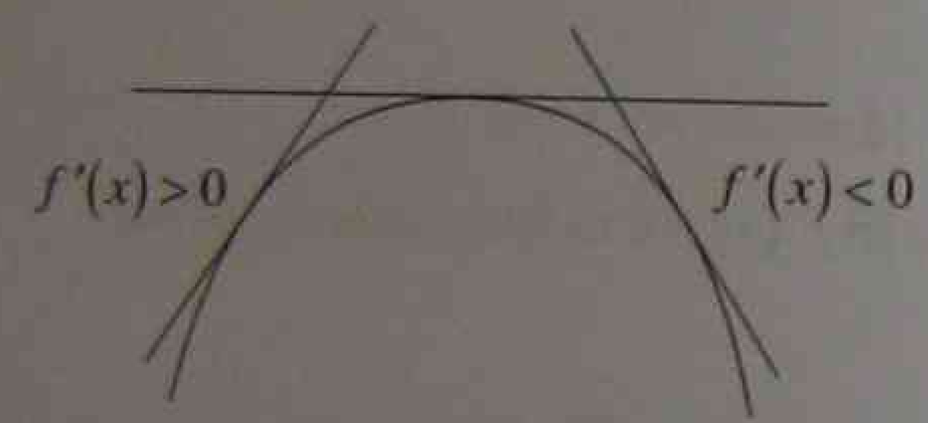
Method 1:

1. Find $f'(x)$.
2. Let $f'(x) = 0$ and solve for x to determine the stationary points.
3. Now determine the nature of the stationary points as follows:

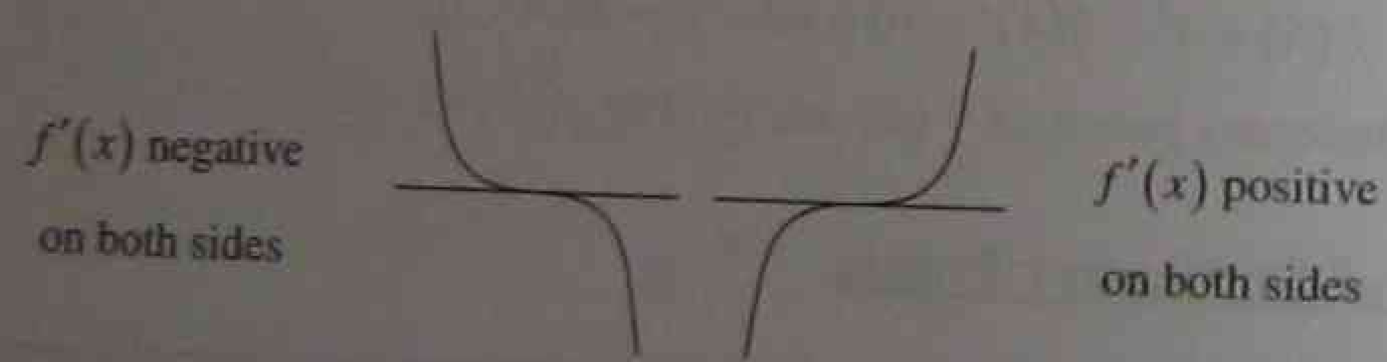
- If $f'(x)<0$ to the left of a stationary point, and $f'(x)>0$ to the right of that point, then the stationary point is a ***minimum***.



- If $f'(x)>0$ to the left of a stationary point, and $f'(x)<0$ to the right of that point, then the stationary point is a ***maximum***.



- If $f'(x)$ is negative or positive on both sides of a stationary point, then the point is a ***horizantal point of inflection***.



Example 1: Determine the coordinates and nature of the stationary points to the curve $f(x)=x^3-2x^2+x+4$.

Solution 1:

$f(x)=x^3-2x^2+x+4$

$f'(x)=3x^2-4x+1$, let $f'(x)=0$ to find stationary points

i.e. $3x^2-4x+1=0$

$(3x-1)(x-1)=0$

$\therefore\ x=\frac{1}{3}$ or $x=1$

To determine their nature, check the first derivative on either side:

x	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	1	$\frac{3}{2}$
$f'(x)$	+ve	0	-ve	-ve	0	+ve

$\therefore$ at $x=\frac{1}{3}$ there is maximum turning point,

and at $x=1$ there is a minimum turning point.

When $x=\frac{1}{3}$, $f\left(\frac{1}{3}\right)=\left(\frac{1}{3}\right)^3-2\left(\frac{1}{3}\right)^2+\frac{1}{3}+4=4\frac{4}{27}$

When $x=1$, $f(1)=1^3-2(1)^2+1+4=4$

$\therefore$ there is a maximum turning point at $\left(\frac{1}{3},4\frac{4}{27}\right)$ and a minimum point at $(1,4)$. #

(D) The Second Derivative and Its Applications

(i) Introduction

The second derivative is the derivative of the first derivative:

Notation : $\frac{d^2y}{dx^2}$, $f''(x)$, y''

Example 1: Find the second derivative of the following functions:

(i) $f(x)=2x^3-3x^2+x$ (ii) $f(x)=x(x-5)^2$

Solution 1:

(i) $f(x)=2x^3-3x^2+x$

$f'(x)=6x^2-6x+1$

$f''(x)=12x-6$ #

(ii) $f(x)=x(x-5)^2$

$=x(x^2-10x+25)$

$=x^3-10x^2+25x$

$f'(x)=3x^2-20x+25$

$f''(x)=6x-20$ #

(ii) Nature of Stationary Points

The second derivative can be used to determine the nature of stationary points:

Method 2:

1. If at $f'(x)=0$, $f''(x)>0$, stationary point is a ***minimum***.
2. If at $f'(x)=0$, $f''(x)<0$, stationary point is a ***maximum***.
3. If at $f'(x)=0$, $f''(x)=0$, then we must consider the sign of the first derivative on either side of the stationary point as in Method 1 part 3.

Example 1: Consider the function $f(x)=2x^2(x-3)^2$, Find any stationary points and determine their nature.

Solution 1:

$$f(x)=2x^2(x-3)^2$$

$$\begin{aligned} f'(x) &= 2x^2\cdot\frac{d}{dx}\left((x-3)^2\right)+(x-3)^2\cdot\frac{d}{dx}\left(2x^2\right) \\ &= 2x^2\cdot 2(x-3)+(x-3)^2\cdot 4x \\ &= 4x^2(x-3)+4x(x-3)^2 \\ &= 4x(x-3)[x+x-3] \\ &= 4x(x-3)(2x-3) \end{aligned}$$

Let $f'(x)=0$ to find stationary points

i.e. $4x(x-3)(2x-3)=0$

$\therefore\ x=0, 3, \frac{3}{2}$

$f(0)=0$, $f(3)=0$

$$f\left(\frac{3}{2}\right)=2\left(\frac{3}{2}\right)^2\left(\frac{3}{2}-3\right)^2=2\left(\frac{9}{4}\right)\left(-\frac{3}{2}\right)^2=2\left(\frac{9}{4}\right)\times\left(\frac{9}{4}\right)=10\frac{1}{8}$$

To determine the nature of the stationary points, find the second derivative:

$$\begin{aligned} f'(x) &= 4x^2(x-3)+4x(x-3)^2 \\ &= 4x^3-12x^2+4x\left(x^2-6x+9\right) \\ &= 4x^3-12x^2+4x^3-24x^2+36x \\ &= 8x^3-36x^2+36x \end{aligned}$$

$\therefore\ f''(x)=24x^2-72x+36$

Now, $f''(0)=36\ \therefore\ (0,0)$ is a minimum

$f''(3)=36\ \therefore\ (3,0)$ is a minimum

$f''\left(\frac{3}{2}\right)=-18\ \therefore\ \left(\frac{3}{2}, 10\frac{1}{8}\right)$ is a maximum #

Example 2: Find any stationary point(s) to the curve: $y=x^3+5$.

Solution 2:

$y=x^3+5$

$\frac{dy}{dx}=3x^2$, let $\frac{dy}{dx}=0$ to find stationary points

i.e. $3x^2=0$

$x=0$

To find nature of stationary point, use second derivative

i.e. $\frac{d^2y}{dx^2}=6x$ at $x=0$, $\frac{d^2y}{dx^2}=0$

Thus, need to check the sign of $\frac{dy}{dx}$ on either side of $x=0$ to determine the nature of the stationary point.

x	-1	0	1
$\frac{dy}{dx}$	+ve	0	+ve

since $\frac{dy}{dx}$ is positive on both sides of the stationary point

$\therefore\ x=0$ is a horizontal point of inflection.

when $x=0$, $y=0^3+5=5$

$\therefore\ (0,5)$ is a horizontal point of inflection. #

(iii) Points of Inflection

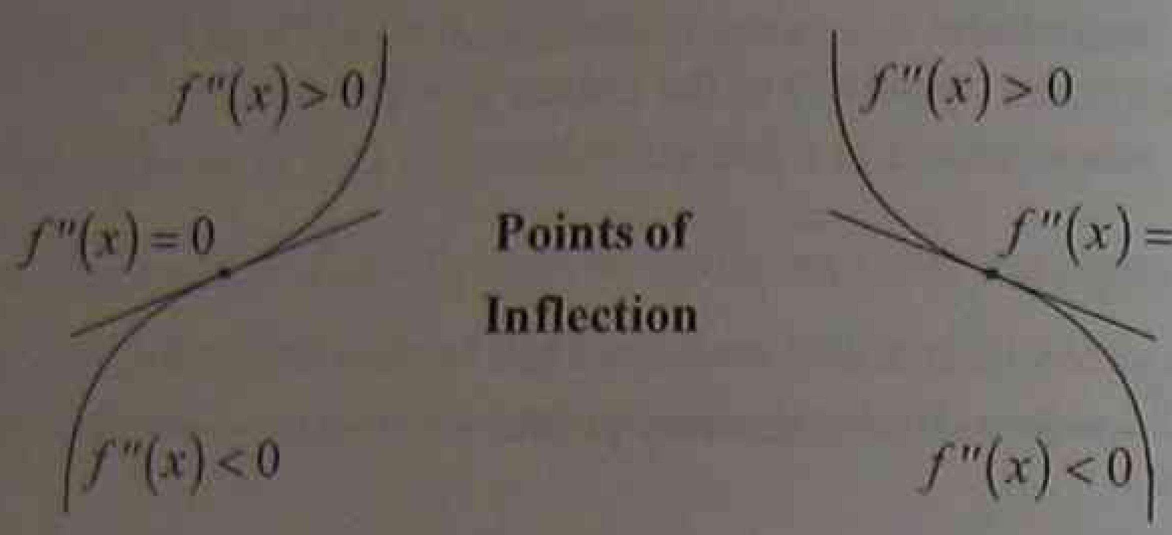
The second derivative can also be used to determine non-horizontal (or 'vertical') points of inflection.

Points of inflection can be found as follows:

1. Find $f''(x)$.
2. Solve $f''(x)=0$ to find x values(s).

3. Verify that a point of inflection exists at that point. This is done by showing that $f''(x)$ has opposite signs on either side of that point.

4. Substitute the x-values into $y = f(x)$ to find the y-value(s).



Example 1: Find the coordinates of any points of inflection to the function: $f(x)=x^3-3x^2$.

Solution 1:

$f(x)=x^3-3x^2$

$f'(x)=3x^2-6x$

$f''(x)=6x-6$, let $f''(x)=0$ to find possible points of inflection

i.e. $6x-6=0$

$6(x-1)=0$ i.e. $x=1$

When $x=1$, $f(1)=1^3-3(1)^2=-2$

To verify that a point of inflection exists at $x=1$, check whether $f''(x)$ changes sign in passing through $x=1$.

$\therefore$ the curve has a point of inflection at $(1,-2)$ #

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$f''(x)$	-ve	0	+ve

(E) Sketching Curves

All the information covered in sections (B) to (D) will be necessary in effectively answering curve sketching questions.

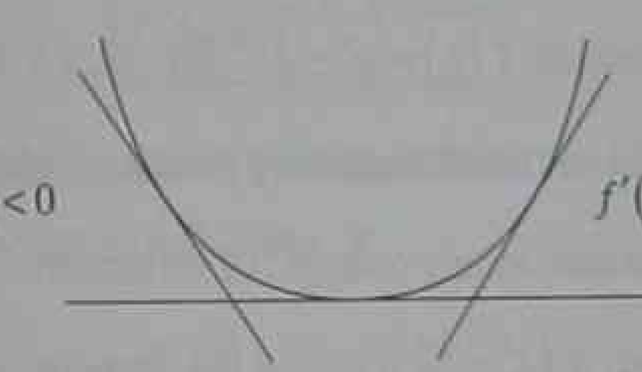
These have been summarised in one comprehensive step-by-step method set out below.

A Reliable Method for Sketching Curves

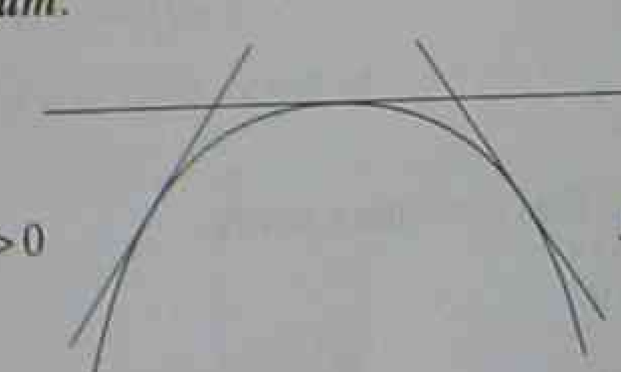
1. Find $f'(x)$.
2. Let $f'(x)=0$ and solve for x.
3. Find $f''(x)$ (the second derivative or the derivative of $f'(x)$)
4. Input the values from 2 (i.e. $f'(x)=0$) into $f''(x)$:

   - If $f''(x)>0$, stationary point is a ***minimum***
   - If $f''(x)<0$, stationary point is a ***maximum***
   - If $f''(x)=0$, need to examine the sign of $f'(x)$ on either side of the stationary point i.e.:

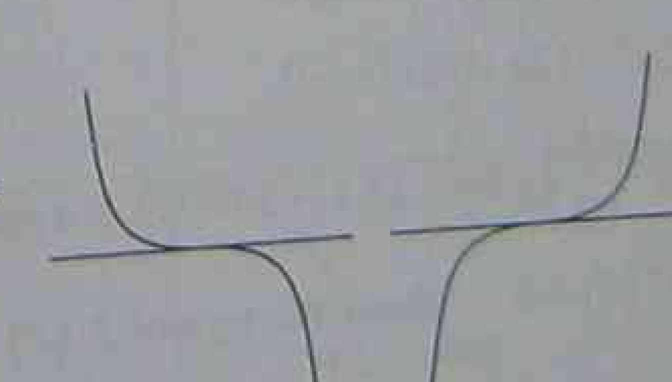
* If $f'(x)<0$ to the left of a stationary point, and $f'(x)>0$ to the right of that point, then the stationary point is a ***minimum***.



* If $f'(x)>0$ to the left of a stationary point, and $f'(x)<0$ to the right of that point, then the stationary point is a ***maximum***.



* If $f'(x)$ is negative or positive on either side of a stationary point, then the point is a ***horizontal point of inflection***.



5. Let $f''(x)=0$ and solve for x. This identifies possible points of inflection. To verify that a point of inflection exists, check that $f''(x)$ changes sign in passing through that point (i.e. $f''(x)$ has opposite signs on either side of that point).
6. Find the point(s) at which the curve cuts the y-axis (i.e. $x=0$). Finding the point(s) at which the curve cuts the x-axis should only be done if the function is simple.
7. Investigate domain and range and existence of any asymptotes.
8. Find the range at end points.
9. Sketch the curve showing all the above mentioned features.

Example 1: Consider the function: $f(x)=x^4-2x^3$ for $-1\le x\le 2.5$

(i) Find the coordinates of all stationary points and determine their nature.

(ii) Find the coordinates of any point(s) of inflection.

(iii) Sketch the curve: $f(x)=x^4-2x^3$ in the domain $-1\le x\le 2.5$, clearly showing all essential features.

Solution 1:

(i) $f(x)=x^4-2x^3$

$f'(x)=4x^3-6x^2$

Let $f'(x)=0$ to find stationary points.

i.e. $4x^3-6x^2=0$

$x^2(4x-6)=0$

$x=0$ or $x=\frac{6}{4}=1.5$

When $x=0$, $f(0)=0$

When $x=1.5$, $f(1.5)=-1.6875$

$f''(x)=12x^2-12x=12x(x-1)$

At $x=\frac{3}{2}$, $f''\left(\frac{3}{2}\right)>0$ $\therefore$ $(1.5,-1.6875)$ is a minimum

At $x=0$, $f''(0)=0$ $\therefore$ check the sign of $f'(x)$ on either side of that point:

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f'(x)$	-ve	0	-ve

$\therefore$ $(0,0)$ is a horizontal point of inflection. #

(ii) Let $f''(x)=0$ to find possible point(s) of inflection.

i.e. $12x(x-1)=0$

$x=0$ or $x=1$

When $x=1$, $f(1)=1^4-2(1)^3=-1$

From above, $x=0$ is a horizontal point of inflection, thus, only consider $x=1$.

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$f''(x)$	-ve	0	+ve

since $f''(x)$ changes sign in passing through $x=1$

$\therefore$ $(1,-1)$ is a point of inflection. #

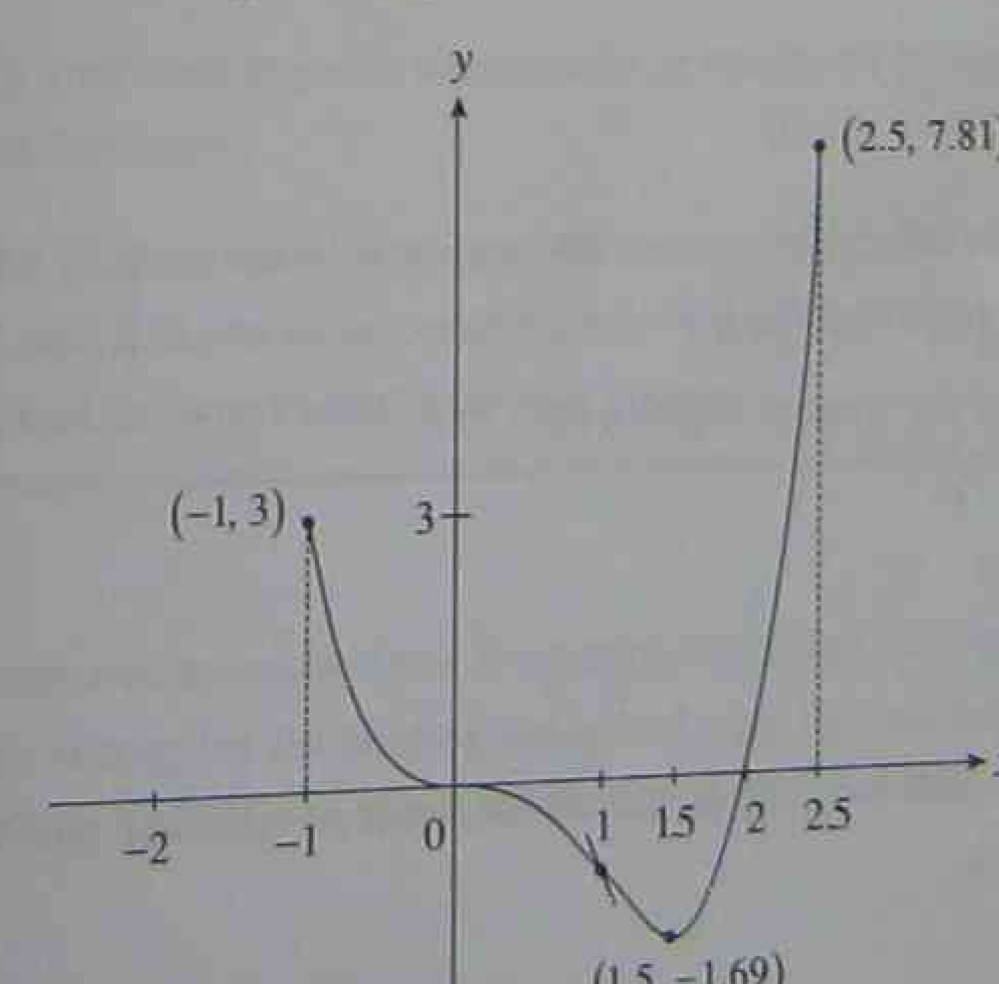
(iii) When $x=-1$, $f(-1)=(-1)^4-2(-1)^3=1+2=3$

When $x=2.5$, $f(2.5)=7.8125$

When $y=0$, i.e. $x^4-2x^3=0$

$x^3(x-2)=0$

$x=0$ and $x=2$ #



(F) Tangents and Normals to a Curve

The gradient of the tangent is equal to the derivative at the point of contact.

To find the equation of the tangent or normal to a curve at a point, $x = x_1$ say, follow these steps:

1. Find $f'(x)$.
2. Substitute $x = x_1$ into $f'(x)$ to find the gradient of tangent (m_t).
   i.e. $m_1 = f'(x_t)$

   To find the gradient of normal (m_n) we use: $m_n = \dfrac{-1}{m_t}$
3. Find the equation of the tangent or normal by using the point gradient formula: $(y - y_1) = m(x - x_1)$

   Note that y_1 is found by substituting $x = x_1$ into the original curve $y = f(x)$.

Example 1: Find the equation of the tangent to the curve $y = x^3 - 2x$ at the point where $x = 2$.

Solution 1:

$$y = x^3 - 2x$$
$$\frac{dy}{dx} = 3x^2 - 2$$
At $x = 2$, $\dfrac{dy}{dx} = 3(2)^2 - 2 = 10$

$\therefore$ gradient of tangent $= 10$

When $x = 2$, $y = (2)^3 - 2(2) = 4$

Thus, equation of the tangent is given by:

$$(y-4) = 10(x-2)$$
$$y - 4 = 10x - 20$$
$$y = 10x - 16$$

Example 2: Find the equation of the normal to the curve $y = (1+2x)^3$ at the point $(0, 1)$.

Solution 2:

$$y = (1+2x)^3$$
$$\frac{dy}{dx} = 3(1+2x)^2 \cdot 2 = 6(1+2x)^2$$
At $x = 0$, $\dfrac{dy}{dx} = 6(1+0)^2 = 6$

$\therefore$ gradient of the normal $= -\dfrac{1}{6}$

Thus, equation of the normal is given by:

$$(y-1) = -\frac{1}{6}(x-0)$$
$$y = -\frac{1}{6}x + 1 \ \#$$

(G) Maxima and Minima Problems

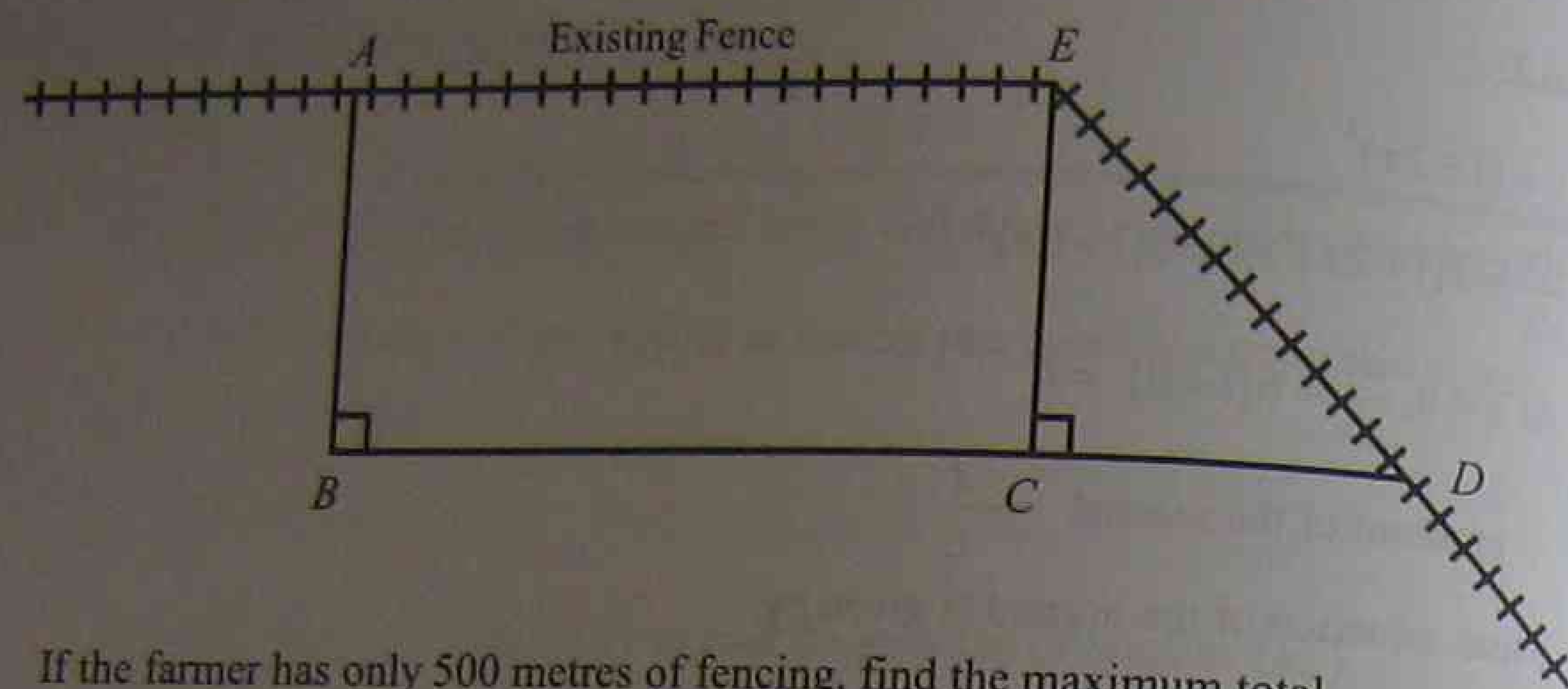
A step-by-step approach to solving maxima and minima problems is set out below:

1. Express the problem algebraically: two expressions are generally required.
2. Eliminate one of the variables such that the Area/Volume/Cost etc. to be minimised or maximised is in terms of only one variable. Keep the required expression in mind so that you know which variable needs to be eliminated.
3. Find the first derivative of the required expression, set it to zero and solve.
4. Verify your answer yields a maximum or minimum by using the second or first derivative.
5. Make a final comment/statement which directly answers the question. Make sure that you do not omit the relevant units. If none are given, then place the word 'units' after your answer.

Example 1: A farmer wishes to construct 2 separate enclosures, one rectangular and another triangular, for holding sheep and pigs. An existing fence forms part of the boundary as shown in the figure below.

Applications of Differentiation

Applications of



If the farmer has only 500 metres of fencing, find the maximum total area of holding enclosures, given that $AB = x$, $BC = y$ and $EC = CD$.

Solution 1:

Area of rectangle $= xy$

Area of triangle $= \frac{1}{2} \cdot x \cdot x = \frac{1}{2}x^2$

Perimeter of fencing $= 3x + y = 500$

i.e. $y = 500 - 3x$

$$\text{Total Area: } A = xy + \frac{1}{2}x^2$$
$$= x(500 - 3x) + \frac{1}{2}x^2$$
$$= 500x - 3x^2 + \frac{1}{2}x^2$$
$$= 500x - \frac{5}{2}x^2$$
$$\frac{dA}{dx} = 500 - 5x$$

A maximum or minimum will occur when $\frac{dA}{dx} = 0$

i.e. $500 - 5x = 0$

$500 = 5x$

$x = 100$

Now, $\frac{d^2A}{dx^2} = -5$, as $\frac{d^2A}{dx^2} < 0$ $\therefore$ $x = 100$ gives a maximum area

When $x = 100$, $A = 500(100) - \frac{5}{2}(100)^2$

$= 25{,}000 \text{ m}^2$

$\therefore$ the maximum area of holding enclosores $= 25{,}000 \text{ m}^2$ #

(H) Primitive Functions

Finding the primitive function is the reverse operation of differentiation (i.e. we know $f'(x)$ and we are trying to find $f(x)$).

If $f'(x) = x^n$, then $f(x) = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

If $f'(x) = (ax+b)^n$, then $f(x) = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad n \neq -1$

{The method involved is to **add** one to the power and **divide** by the new power and then add the constant C.}

Example 1: Find the primitive functions of:

(i) $3x^2 - 5$ (ii) $(2x+3)^3$ (iii) $\frac{1}{x^3}$

Solution 1:

(i) $f'(x) = 3x^2 - 5$, $\quad f(x) = \frac{3x^3}{3} - 5x + C = x^3 - 5x + C$ #

(ii) $f'(x) = (2x+3)^3$, $\quad f(x) = \frac{(2x+3)^4}{2 \times 4} + C = \frac{(2x+3)^4}{8} + C$ #

(iii) $f'(x) = \frac{1}{x^3} = x^{-3}$, $\quad f(x) = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$ #

Example 2: The curve $y = f(x)$ passes through the point $\left(1, \frac{2}{3}\right)$ and $f'(x) = \frac{(x-3)^2}{2}$.

Find the equation of the curve.

Solution 2:

$f'(x) = \frac{(x-3)^2}{2}$, i.e. $f(x) = \frac{(x-3)^3}{2 \times 3} + C = \frac{(x-3)^3}{6} + C$

At $x = 1$, $f(1) = \frac{(1-3)^3}{6} + C = \frac{2}{3}$

$-\frac{8}{6} + C = \frac{2}{3}$ i.e. $C = \frac{2}{3} + \frac{8}{6} = 2$

$\therefore$ $f(x) = \frac{(x-3)^2}{6} + 2$ #

REVIEW EXERCISES

(A) The Significance of the First Derivative

1. For what values of x is the curve $y = x^2 + 4x$ decreasing?

2. For what values of x is the function $f(x) = x^3 - \frac{7}{2}x^2 + 4x$ increasing?

(B) Finding Stationary Points

3. Find the coordinates of the stationary points on the curve: $y = 2x^2 - 8x^3$.

(C) Nature of Stationary Points

4. Find the coordinates and the nature of the stationary points to the curve: $y = 2x^3 + 9x^2 + 12x$. Use the first derivative.

(D) The Second Derivative and Its Applications

5. Find the second derivative of the following functions:
   (i) $f(x) = x^3 - 3x^2 + 2x + 7$ (ii) $f(x) = 3x^5 - 2x^4 - 4x^3 + x$

6. Consider the function: $f(x) = x^3 - 6x^2$. Find any stationary points and determine their nature using the second derivative.

7. Consider the function: $f(x) = x^4 - 6x^2 + 2x$. Find the coordinates of any points of inflection.

(E) Sketching Curves

8. Consider the function: $f(x) = x^3 - 3x$, for $-2 \le x \le 2$.
   (i) Find any stationary points and determine their nature.
   (ii) Find any points of inflection.
   (iii) Find the points where the function crosses the coordinate axes.
   (iv) Sketch the curve: $y = f(x)$ for $-2 \le x \le 2$, clearly showing all the above features.

9. For the curve: $y = x^3 - 6x^2 + 9x$.
   (i) Find the stationary points and determine their nature.
   (ii) Find any points of inflection.
   (iii) For what values of x is the curve decreasing?
   (iv) Sketch the curve in the domain $0 \le x \le 4$.

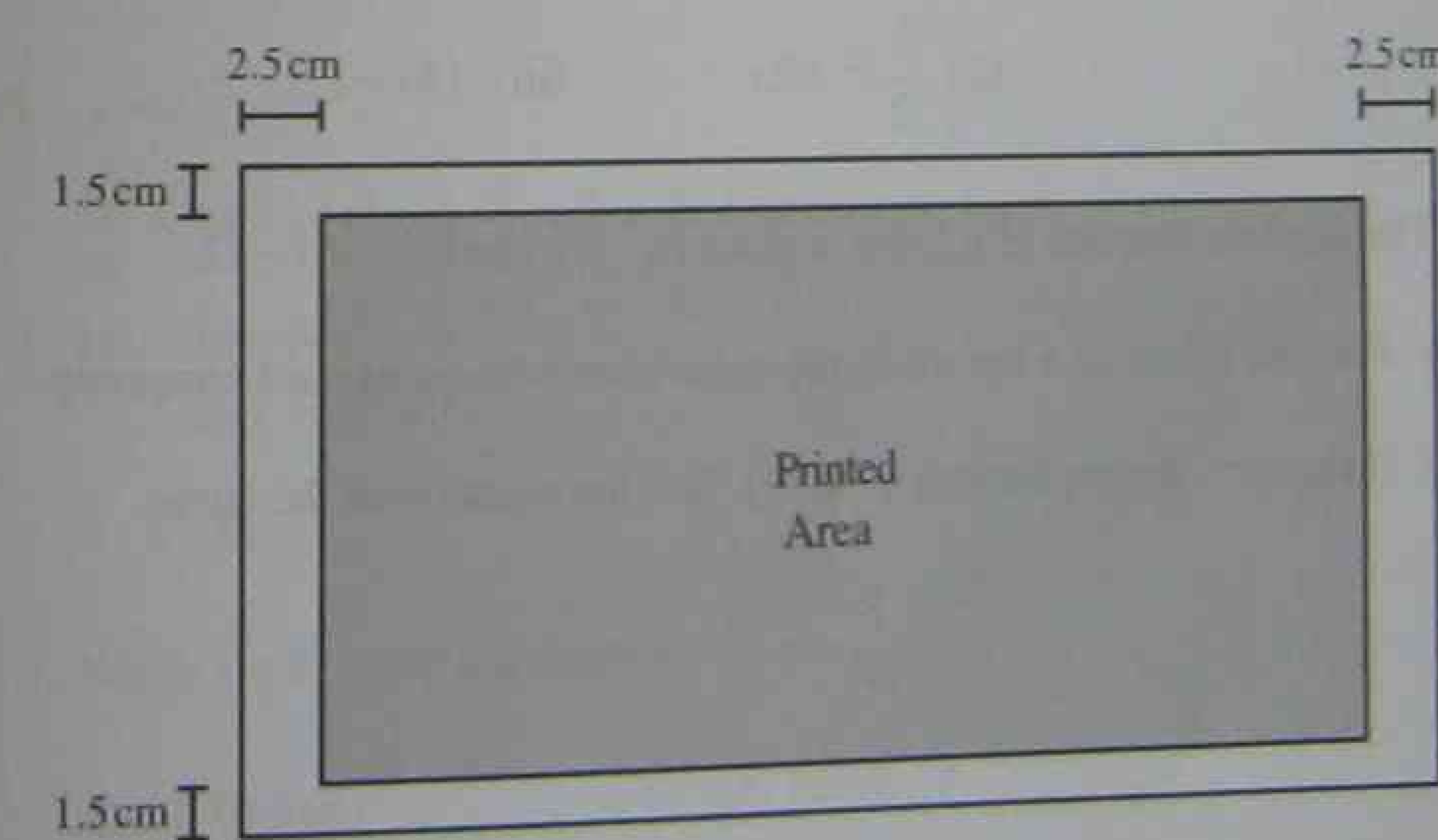
(F) Tangents and Normals to a Curve

10. Find the equation of the normal to the curve: $y = (1 - 3x)^4$ at the point $(0, 1)$.

11. Let $f(x) = x^3 + 2x^2 - 5x + 1$
    (i) Find the equation of the tangent to the curve $y = f(x)$ at $x = 0$.
    (ii) Find the x-coordinates of the points where the curve has a tangent parallel to the line whose equation is $y = 15x + 3$.

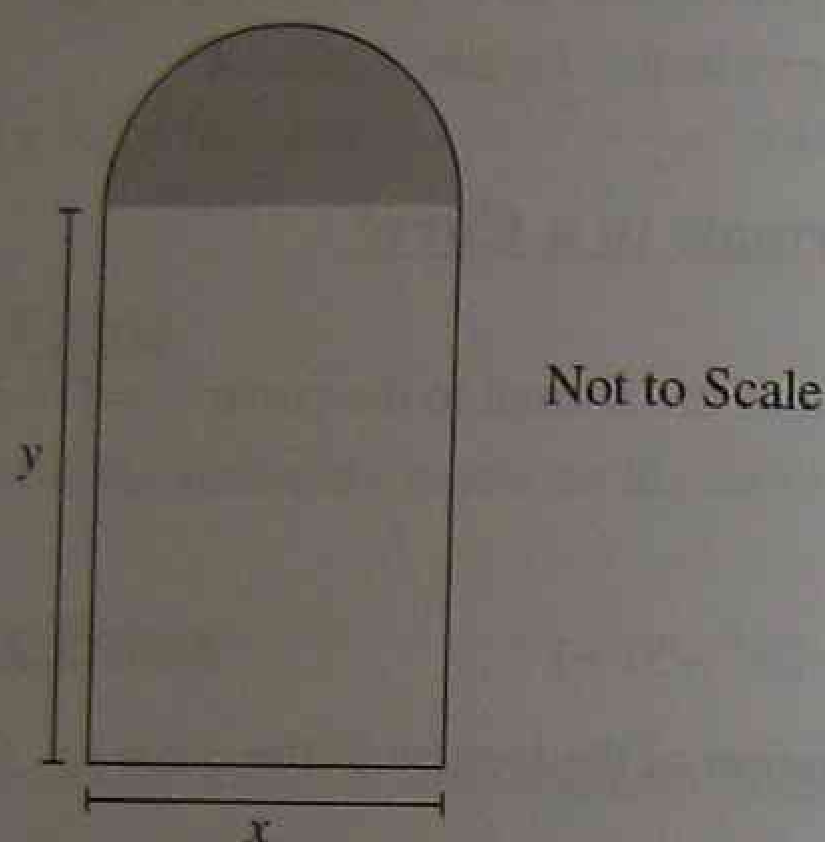
(G) Maxima and Minima Problems

12. The area of a rectangular sheet of paper is 600 cm. A margin of 1.5 cm is allowed at the top and bottom and 2.5 cm on each edge.



Find the exact dimensions of the printed area if is to be maximised?

13. A large bay window is to have the shape of a rectangle topped by a semi-circle as shown in the diagram below. The rectangle is of clear glass while the semi-circle is of tinted glass that transmits only half as much light per unit area as clear glass does.



If the total perimeter of the window is 20 m, find the dimensions x and y of the window admitting the most light.

(H) Primitive Functions

14. Find the primitive functions of:

(i) $x^3 - 1$ (ii) $\frac{1}{2}x^2 + 2x$ (iii) $(4x-7)^4$

15. The gradient function of a curve is given by: $f'(x) = 6x^2 - 6x - 12$.

(i) Find the values of x for which the curve rises with downward concavity.

(ii) If the curve passes through $(-1, 13)$, find the equation of the curve.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. $y = x^2 + 4x$

$\frac{dy}{dx} = 2x + 4$

Curve is decreasing when $\frac{dy}{dx} < 0$

i.e. $2x + 4 < 0$

$2x < -4$

$x < -2$ #

2. $f(x) = x^3 - \frac{7}{2}x^2 + 4x$

$f'(x) = 3x^2 - 7x + 4$

Function is increasing when $f'(x) > 0$

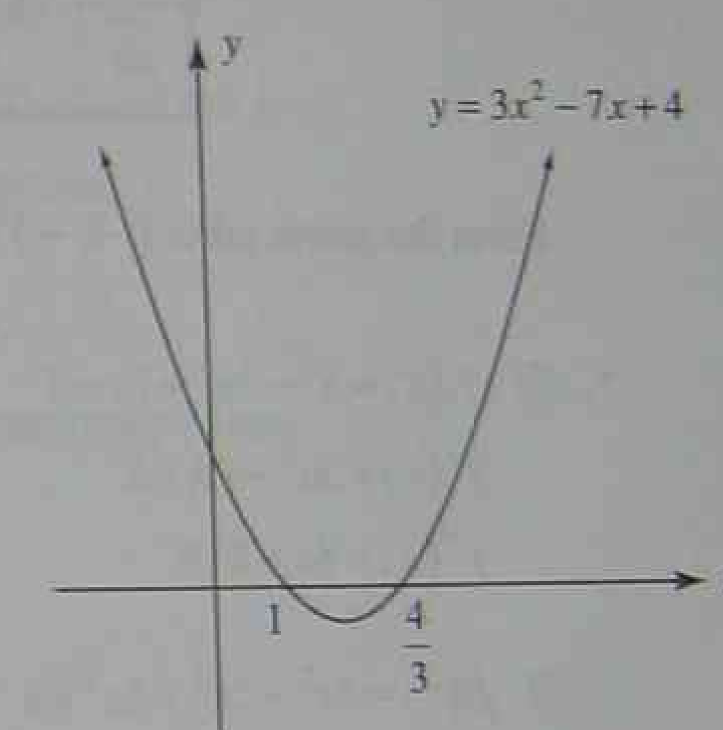
i.e. $3x^2 - 7x + 4 > 0$

$(3x-4)(x-1) > 0$

Let $y = (3x-4)(x-1)$

from the graph $y > 0$

for $x < 1$ or $x > \frac{4}{3}$ #



3. $y = 2x^2 - 8x^3$

$\frac{dy}{dx} = 4x - 24x^2$; let $\frac{dy}{dx} = 0$ to find stationary points:

i.e. $4x - 24x^2 = 0$

$4x(1-6x) = 0$

$\therefore\ x = 0, \frac{1}{6}$

When $x = 0$, $y = 0$ and when $x = \frac{1}{6}$, $y = \frac{1}{54}$

$\therefore$ $(0, 0)$ and $(\frac{1}{6}, \frac{1}{54})$ are stationary points. #

4. $y = 2x^3 + 9x^2 + 12x$

$\frac{dy}{dx} = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2)$; let $\frac{dy}{dx} = 0$ to find stationary points:

i.e. $6\left(x^2+3x+2\right)=0$

$6(x+2)(x+1)=0$

$\therefore\ x=-1,-2$

When $x=-1$, $y=2(-1)^3+9(-1)^2+12(-1)=-5$

When $x=-2$, $y=2(-2)^3+9(-2)^2+12(-2)=-4$

x	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-\frac{5}{2}$	-2	$-\frac{3}{2}$
$\frac{dy}{dx}$	-ve	0	+ve	+ve	0	-ve

From the above table $(-1,-5)$ is a minimum and $(-2,-4)$ is a maximum. #

5. (i) $f(x)=x^3-3x^2+2x+7$

$f'(x)=3x^2-6x+2$

$f''(x)=6x-6$ #

(ii) $f(x)=3x^5-2x^4-4x^3+x$

$f'(x)=15x^4-8x^3-12x^2+1$

$f''(x)=60x^3-24x^2-24x$ #

6. $f(x)=x^3-6x^2$

$f'(x)=3x^2-12x$; let $f'(x)=0$ to find stationary points:

i.e. $3x^2-12x=0$

$3x(x-4)=0$

$\therefore\ x=0,4$

When $x=0$, $f(0)=0$ and when $x=4$, $f(4)=-32$

$f''(x)=6x-12$

$f''(0)=-12<0$ $\therefore$ $(0,0)$ is a maximum

$f''(4)=12>0$ $\therefore$ $(4,-32)$ is a minimum #

7. $f(x)=x^4-6x^2+2x$

$f'(x)=4x^3-12x+2$

i.e. $12x^2-12=0$

$12\left(x^2-1\right)=0$

$x=-1,1$

When $x=-1$, $f(-1)=-7$ and when $x=1$, $f(1)=-3$

x	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$
$\frac{dy}{dx}$	+ve	0	-ve	-ve	0	+ve

Since $f''(x)$ changes sign in passing through $x=-1$ and $x=1$

$\therefore$ $(-1,-7)$ and $(1,-3)$ are points of inflection. #

8. (i) $f(x)=x^3-3x$

$f'(x)=3x^2-3$; let $f'(x)=0$ to find stationary points

i.e. $3x^2-3=0$

$3\left(x^2-1\right)=0$

$x=-1,1$

When $x=-1$, $f(-1)=2$ when $x=1$, $f(1)=-2$

$f''(x)=6x$

$f''(-1)=-6<0$ $\therefore$ $(-1,2)$ is a maximum turning point

$f''(1)=6>0$ $\therefore$ $(1,-2)$ is a minimum turning point. #

(ii) Let $f''(x)=0$ to find possible points of inflection.

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f''(x)$	-ve	0	+ve

i.e. $6x=0$

$x=0$

Since $f''(x)$ changes sign in passing through $x=0$

$\therefore$ $(0,0)$ is a point of inflection. #

(iii) At $x=0$, $f(0)=0$

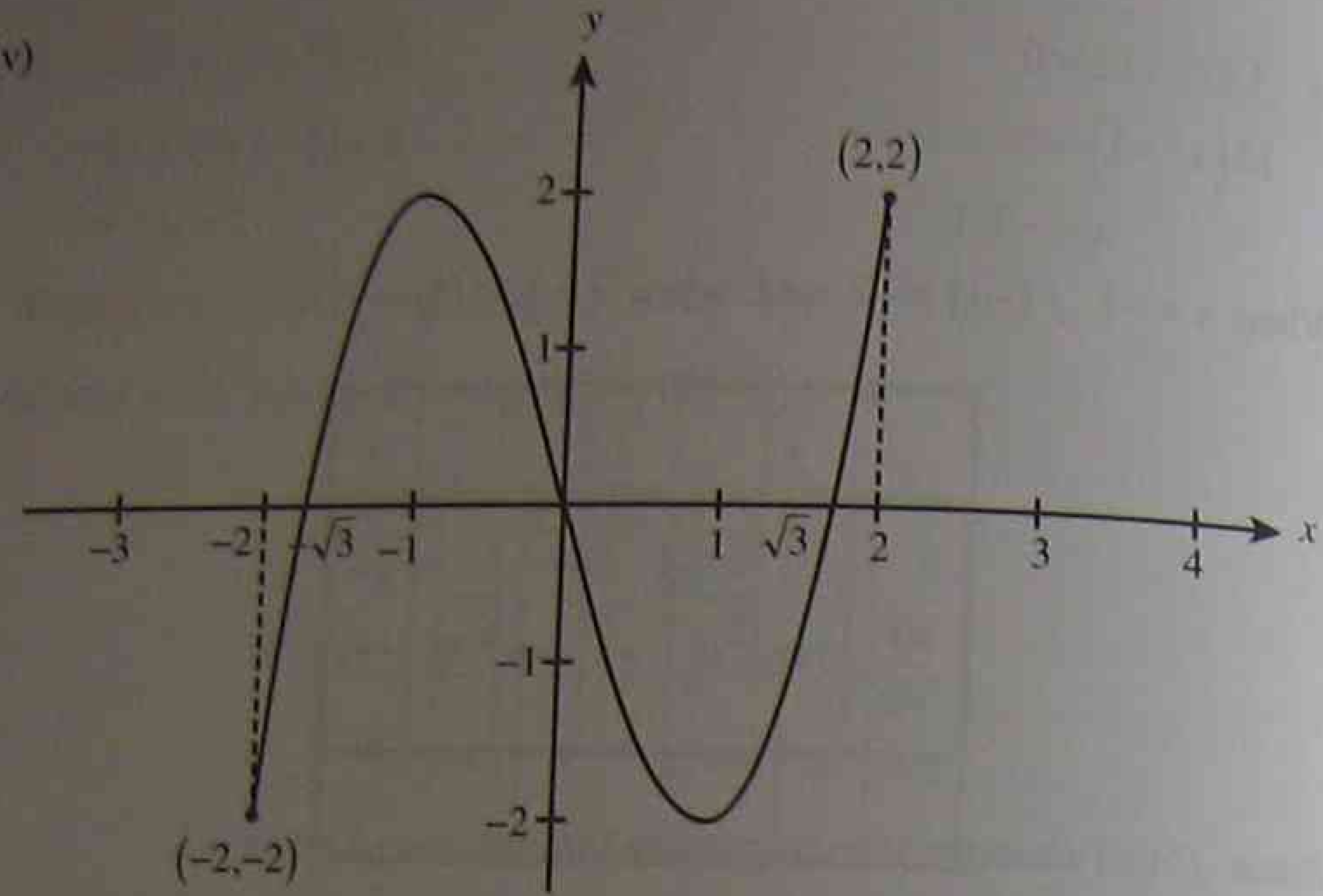
At $f(x)=0$, $x^3-3x=0$

$x\left(x^2-3\right)=0$

$x=0,-\sqrt{3},+\sqrt{3}$

i.e. $(0,0)$, $(-\sqrt{3},0)$ and $(\sqrt{3},0)$ #

(iv)



9. (i) $y = x^3 - 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 - 12x + 9$; let $\frac{dy}{dx} = 0$ to find stationary points

i.e. $3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x-3)(x-1) = 0$

$\therefore\ x = 1, 3$

When $x = 1,\ y = 1^3 - 6(1)^2 + 9(1) = 4$

When $x = 3,\ y = 3^3 - 6(3)^2 + 9(3) = 0$

$\frac{d^2y}{dx^2} = 6x - 12$

When $x = 1$, $\frac{d^2y}{dx^2} = -6 < 0$ $\therefore$ $(1, 4)$ is a maximum turning point

When $x = 3$, $\frac{d^2y}{dx^2} = 6 > 0$ $\therefore$ $(3, 0)$ is a minimum turning point #

(ii) Let $\frac{d^2y}{dx^2} = 0$ to find possible points of inflection

i.e. $6x - 12 = 0$

$x = 2$

When $x = 2,\ y = 2^3 - 6(2)^2 + 9(2) = 2$

x	$\frac{3}{2}$	2	$\frac{5}{2}$
$\frac{d^2y}{dx^2}$	-ve	0	+ve

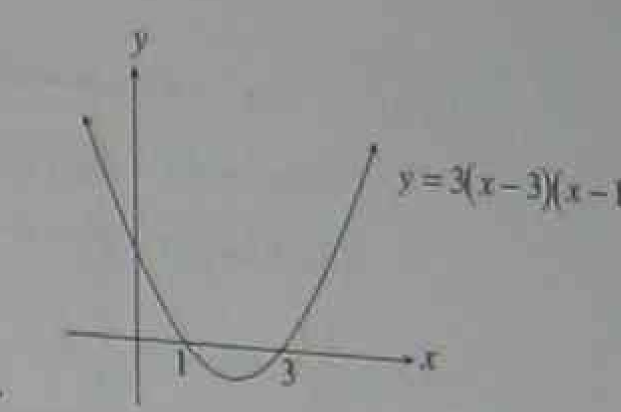
Since $\frac{d^2y}{dx^2}$ changes sign in passing through $x = 2$

$\therefore$ the point $(2, 2)$ is a point of inflection. #

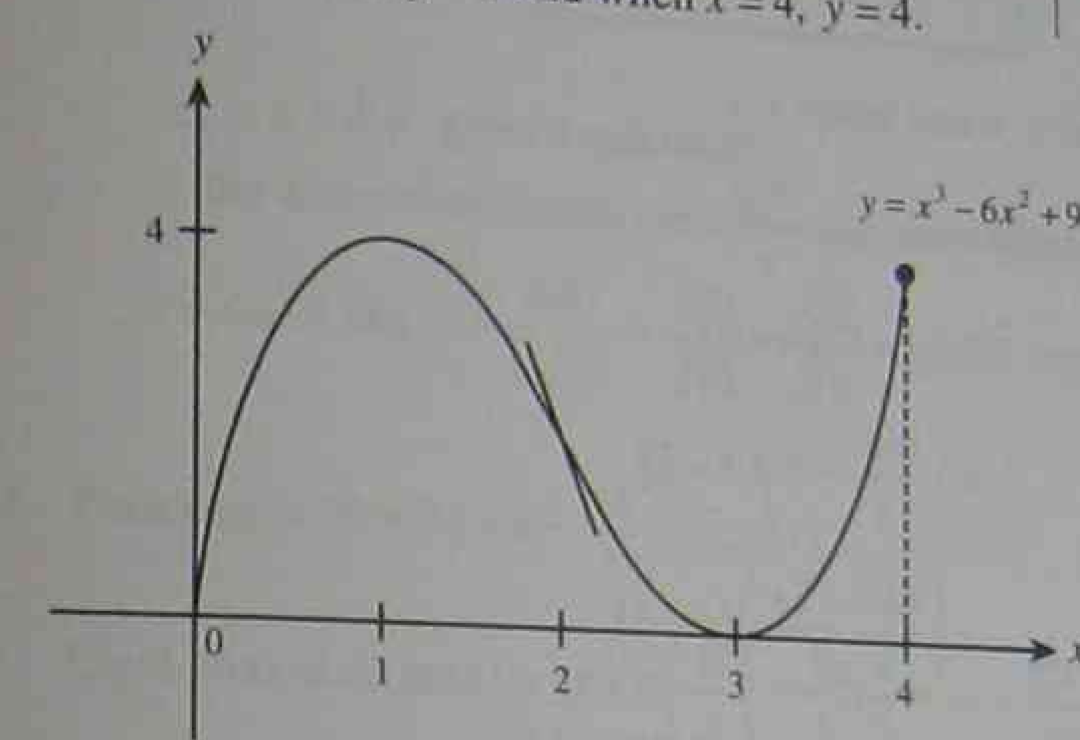
(iii) Curve is decreasing when $\frac{dy}{dx} < 0$

i.e. $3(x-3)(x-1) < 0$

from the graph $y < 0$ when $1 < x < 3$. #



(iv) When $x = 0,\ y = 0$ and when $x = 4,\ y = 4$.



10. $y = (1 - 3x)^4$

$\frac{dy}{dx} = 4(1-3x)^3 \cdot -3 = -12(1-3x)^3$

at $x = 0$, $\frac{dy}{dx} = -12$

$\therefore$ gradient of normal is $\frac{1}{12}$

$(y - 1) = \frac{1}{12}(x - 0)$

$y - 1 = \frac{1}{12}x$

$y = \frac{1}{12}x + 1$ #

11. (i) $f(x) = x^3 + 2x^2 - 5x + 1,\ f'(x) = 3x^2 + 4x - 5,\ f'(0) = -5$

$\therefore$ gradient of tangent $= -5$

$\therefore$ equation of tangent is given by:

at $x = 0$, $f(0) = 1$

$(y - 1) = -5(x - 0)$

$y = -5x + 1$ #

(ii) Gradient of $y = 15x + 3$ is 15. Let $f'(x) = 15$ and solve for x.

i.e. $3x^2+4x-5=15$

$$3x^2+4x-20=0$$

$$(3x+10)(x-2)=0$$

$$x=-\frac{10}{3} \quad \text{or} \quad x=2 \ \#$$

12. Let length of paper $= y$ and width $= x$

Area of paper: $xy=600$ i.e. $y=\frac{600}{x}$

Area of printed area: $A=(y-5)(x-3)$

$$=\left(\frac{600}{x}-5\right)(x-3)$$

$$=\left(\frac{600-5x}{x}\right)(x-3)$$

$$=\frac{600x-5x^2-1{,}800+15x}{x}$$

$$=\frac{615x-5x^2-1{,}800}{x}$$

$$\frac{dA}{dx}=\frac{x\cdot(615-10x)-\left(615x-5x^2-1{,}800\right)\cdot 1}{x^2}$$

$$=\frac{615x-10x^2-615x+5x^2+1{,}800}{x^2}$$

$$=\frac{1{,}800-5x^2}{x^2}$$

Let $\frac{dA}{dx}=0$ to find maximum or minimum.

i.e. $\frac{1{,}800-5x^2}{x^2}=0$

i.e. $1{,}800-5x^2=0$

$$5x^2=1{,}800$$

$$x^2=360$$

$$x=\pm\sqrt{360}=\pm 6\sqrt{10},$$

Since $x>0$

$\therefore x=6\sqrt{10}$.

Applications of Differentiation

x	$5\sqrt{10}$	$6\sqrt{10}$	$7\sqrt{10}$
$\frac{dA}{dx}$	2.2	0	-1.3

i.e. $x=6\sqrt{10}$ gives a maximum.

$\therefore$ the dimensions which yield the maximum printed area are:

$$x=6\sqrt{10}\text{ cm},\ y=\frac{600}{6\sqrt{10}}=\frac{100}{\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}=10\sqrt{10}\text{ cm. }\#$$

13. Perimeter $=20=2y+x+\frac{\pi x}{2}$ .....(1)

Light (based on area) $=xy+\pi\left(\frac{x}{2}\right)^2\times\frac{1}{2}\times\frac{1}{2}=xy+\frac{\pi x^2}{16}$ .....(2)

Now solving equation (1) for y we get:

$$20=2y+x+\frac{\pi x}{2}$$

$$40=4y+2x+\pi x$$

$$4y=40-2x-\pi x$$

$$y=10-\frac{x}{2}-\frac{\pi x}{4}$$

Substituting into equation (2) we get:

$$\text{Light}=L=10x-\frac{x^2}{2}-\frac{\pi x^2}{4}+\frac{\pi x^2}{16}$$

$$=10x-\frac{x^2}{2}-\frac{3\pi x^2}{16}$$

$$\frac{dL}{dx}=10-x-\frac{3\pi x}{8}=0 \quad \text{for maxima/minima}$$

i.e. $x\left(1+\frac{3\pi}{8}\right)=10$

$$x=\frac{10}{1+\frac{3\pi}{8}}=\frac{80}{8+3\pi}\text{ m}$$

$\frac{d^2L}{dx^2}=-1-\frac{3\pi}{8}<0 \quad \therefore x=\frac{80}{8+3\pi}$ yields the maximum light.

$$\therefore y=10-\frac{x}{2}-\frac{\pi x}{4}=10-\frac{40}{8+3\pi}-\frac{20\pi}{8+3\pi}$$

(B) The Definite Integral

The definite integral is one in which the integral has limits, which must be used to find its numerical value.

If the primitive function of $f(x)$ is $F(x)$ then,

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Example 1: Find the exact value of:

(i) $\int_0^2 (9-2x)^3\,dx$ (ii) $\int_1^4 3x - 5\sqrt{x}\,dx$

(iii) $\int_{-1}^3 \sqrt{x+1}\,dx$ (iv) $\int_1^2 \frac{x^4-1}{x^2}\,dx$

Solution 1:

(i) $\int_0^2 (9-2x)^3\,dx = \left[\frac{(9-2x)^4}{4\times-2}\right]_0^2$

$= \left[\frac{(9-2x)^4}{-8}\right]_0^2$

$= \left(\frac{5^4}{-8}\right) - \left(\frac{9^4}{-8}\right) = 742$ #

(ii) $\int_1^4 3x - 5\sqrt{x}\,dx = \int_1^4 3x - 5x^{\frac{1}{2}}\,dx = \left[\frac{3x^2}{2} - \frac{5x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= \left[\frac{3x^2}{2} - \frac{10}{3}x^{\frac{3}{2}}\right]_1^4$

$= \left(24 - \frac{80}{3}\right) - \left(\frac{3}{2} - \frac{10}{3}\right)$

Integration

(iii) $\int_{-1}^3 \sqrt{x+1}\,dx = \int_{-1}^3 (x+1)^{\frac{1}{2}}\,dx = \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^3$

$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^3$

$= \left(\frac{2}{3}\times 8\right) - \left(\frac{2}{3}\times 0\right) = \frac{16}{3}$ #

(iv) $\int_1^2 \frac{x^4-1}{x^2}\,dx = \int_1^2 x^2 - \frac{1}{x^2}\,dx = \int_1^2 x^2 - x^{-2}\,dx$

$= \left[\frac{x^3}{3} + x^{-1}\right]_1^2$

$= \left[\frac{x^3}{3} + \frac{1}{x}\right]_1^2$

$= \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}$ #

(C) Approximations to Definite Integrals

Sometimes it is very difficult if not impossible to evaluate an integral. In such cases the integral needs to be approximated using either the **Trapezoidal Rule** or **Simpson's Rule**.

(i) Method 1 - Trapezoidal Rule

The approximate area enclosed between $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n\right]$$

where h is the width of the strip

Example 1: Use the trapezoidal rule with 4 function values to find an approximation to $\int^4 2^{x+1}\,dx$.

Integration

Solution 1:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2) + y_3\right]$$

$$= \frac{1}{2}\left[2^2 + 2\left(2^3 + 2^4\right) + 2^5\right] \quad \text{note } h = 1$$

$$= \frac{1}{2}\left[4 + 2\times 24 + 32\right]$$

$$= 42 \ \#$$

Example 2: Find the approximate value of $\int_0^4 f(x)\,dx$ by the Trapezoidal rule, using the table below. Express your answer correct to two decimal places.

x	0	1	2	3	4
$f(x)$	0	1.42	5.66	12.73	22.63

Solution 2:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$= \frac{1}{2}\left[0 + 2(1.42 + 5.66 + 12.73) + 22.63\right]$$

$$= 31.125 = 31.13 \ \text{correct to 2 d.p.} \ \#$$

(ii) Method 2- Simpson's Rule

The approximate area enclosed between $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \ldots) + 4(y_1 + y_3 + \ldots)\right]$$

{A much easier way to remember the formula is ***FOTE*** *(Four Odd Two Even)}*

Example 1: Consider the function $y = \cot x$.

(i) Complete the table:

x	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
y					

(ii) Apply Simpson's rule with five function values to find an approximation to $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$ #

Solution 1:

(i)

x	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
y	1	0.668	0.414	0.199	0

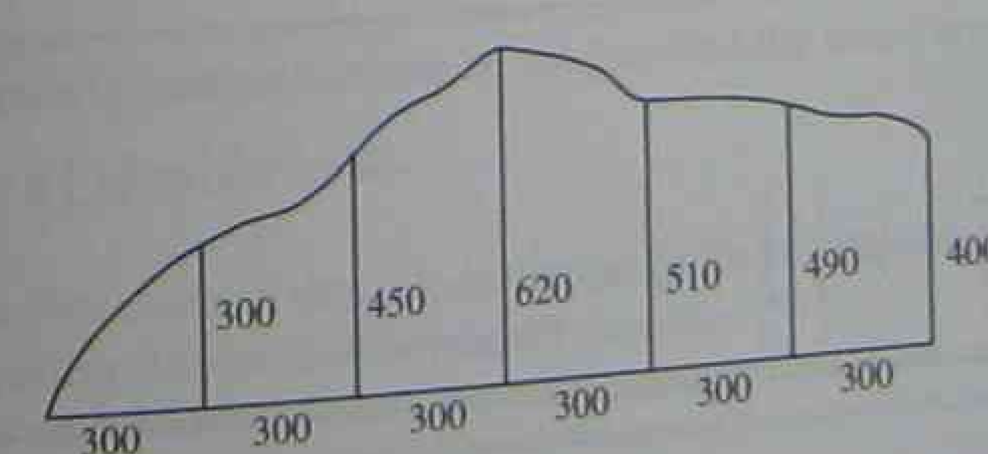
(note: $\cot x = \frac{\cos x}{\sin x}$)

(ii) $$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \ldots) + 4(y_1 + y_3 + \ldots)\right]$$

$$= \frac{\frac{\pi}{16}}{3}\left[1 + 0 + 2(0.414) + 4(0.668 + 0.199)\right] \quad \left(\text{note: } h = \frac{5\pi}{16} - \frac{\pi}{4} = \frac{\pi}{16}\right)$$

$$= \frac{\pi}{48}\left[5.296\right] = 0.346 \ \text{correct to 3 d.p.} \ \#$$

Example 2: Use Simpson's rule to find the approximate area of the following block of land. All measurements are in metres.



Solution 2:

h = common width of strip = 300 m

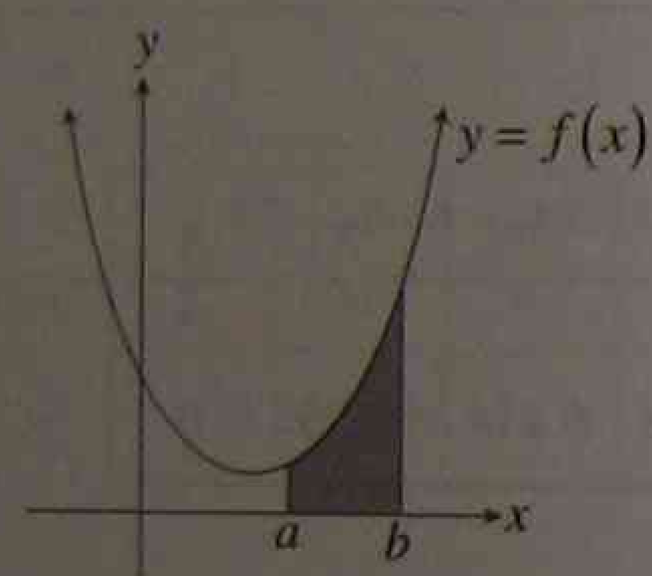
$$A \approx \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \ldots) + 4(y_1 + y_3 + \ldots)\right]$$

$$= \frac{300}{3}\left[0+400+2(450+510)+4(300+620+490)\right] = 796{,}000\ \text{m}^2\ \#$$

(D) Areas by Integration

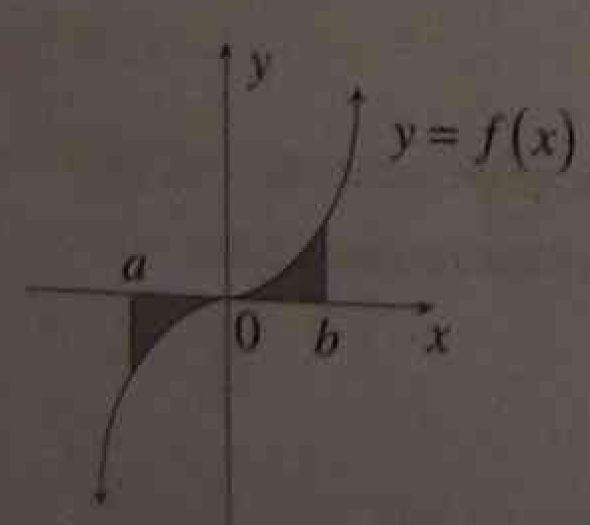
(i) Exact Area between a Curve and the x-axis

Integration can be used to find the *exact* area between the curve $y = f(x)$ and the **x-axis.**



$$A = \int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Areas below the x-axis are always negative. In these circumstances we take the ***absolute value*** of this area.



$$A = \left|\int_a^0 f(x)\,dx\right| + \int_0^b f(x)\,dx$$

Example 1: Find the area bounded by the curve $y = 5 - x^2$ and the lines $x = 0$ and $x = 2$.

Solution 1:

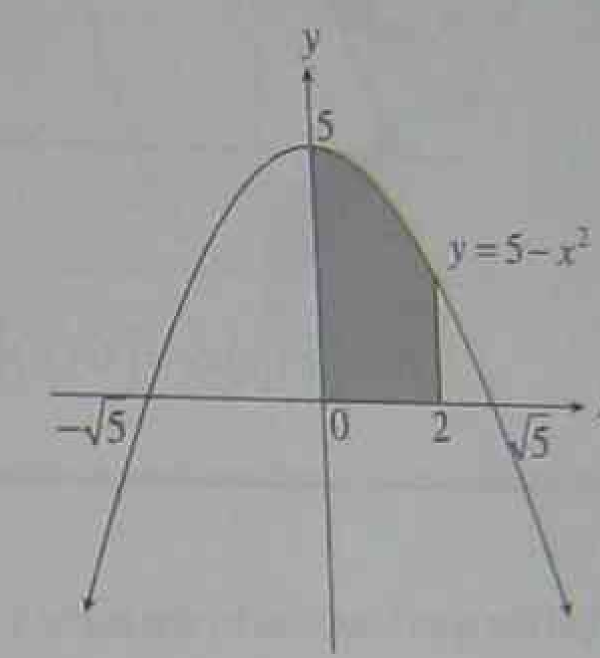
The area is all above the x-axis

$$A = \int_0^2 \left(5 - x^2\right) dx$$

$$= \left[5x - \frac{x^3}{3}\right]_0^2$$

$$= \left(10 - \frac{8}{3}\right) - (0)$$

$$= 7\frac{1}{3}\ \text{units}^2\ \#$$



Example 2: Find the area bounded by the curve $y = 2x^3$, the x-axis and the lines $x = -1$ and $x = 2$.

Solution 2:

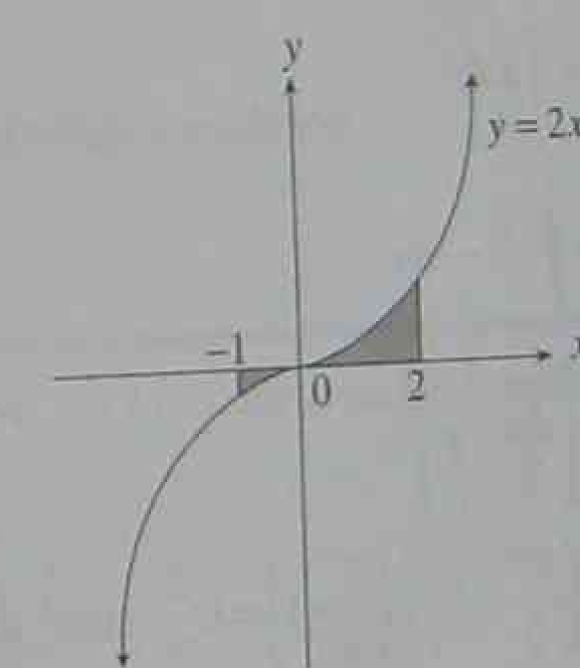
Part of the area is below the x-axis

$$\therefore\ A = \left|\int_{-1}^0 2x^3 dx\right| + \int_0^2 2x^3 dx$$

$$= \left|\left[\frac{2x^4}{4}\right]_{-1}^0\right| + \left[\frac{2x^4}{4}\right]_0^2$$

$$= \left|0 - \left(\frac{2}{4}\right)\right| + \left(\frac{2\times 16}{4} - 0\right)$$

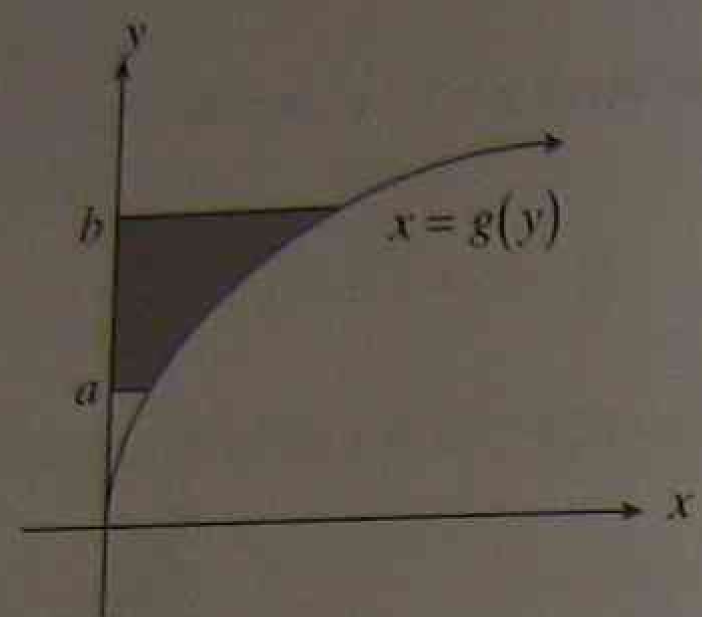
$$= \frac{2}{4} + 8 = 8\frac{1}{2}\ \text{units}^2\ \#$$



(ii) Exact Area between a Curve and the y-axis

Similarly, integration can be used to find the exact area between the curve $x = g(y)$ and the y-axis.

The first essential step is to rearrange the function to obtain x in terms of y. Also, the integral limits must be read off the y-axis.



$$A=\int_a^b g(y)\,dy=\left[G(y)\right]_a^b=G(b)-G(a)$$

Example 1: Find the area bounded by the curve $y=\sqrt{x+1}$ the y-axis and the lines $y=1$ and $y=3$.

Solution 1:

$y=\sqrt{x+1}$

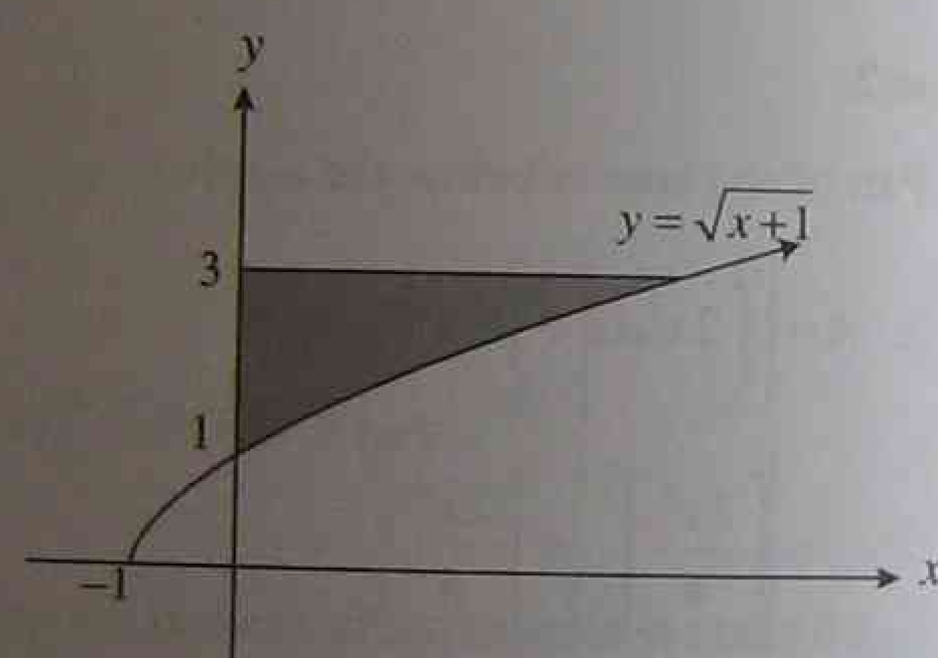
$y^2=x+1$ i.e. $x=y^2-1$

$$\therefore\ A=\int_1^3 x\,dy$$

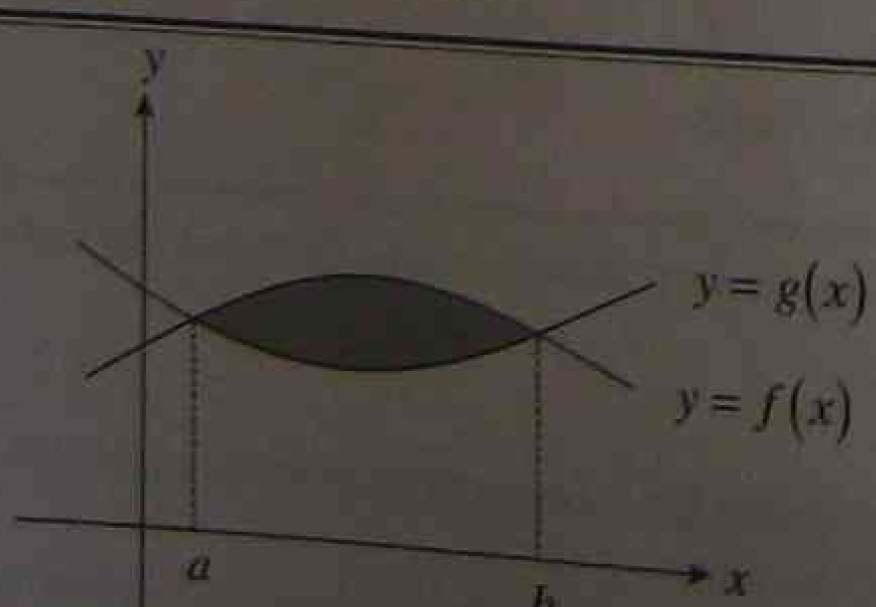
$$=\int_1^3 y^2-\ dy$$

$$=\left[\frac{y^3}{3}-y\right]_1^3$$

$$=\left(\frac{27}{3}-3\right)-\left(\frac{1}{3}-1\right)=6\frac{2}{3}\text{ units}^2\ \#$$



(iii) The Area Between Two Curves



The shaded area

$$=\int_a^b f(x)-g(x)\,dx$$

Example 1: Find the area bounded by the curves $y=x^2$ and $x=y^2$.

Solution 1:

To find the points of intersection

we solve $y=x^2$ and $x=y^2$

$x=y^2$

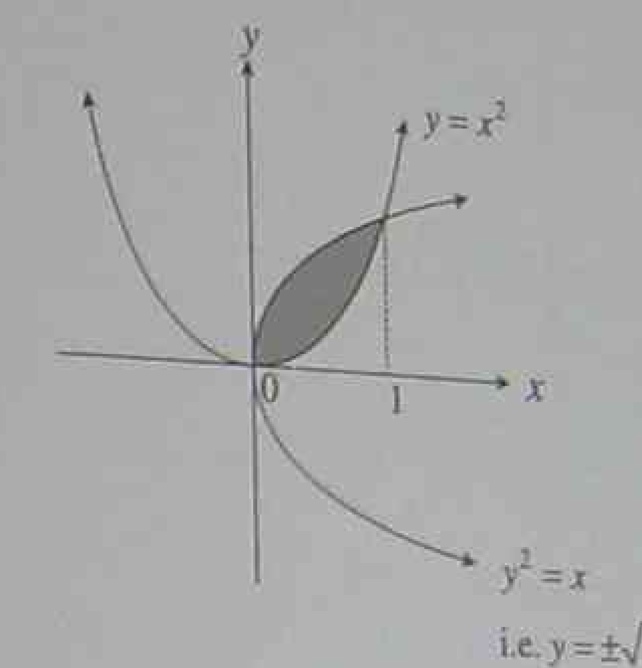
$=\left(x^2\right)^2$ (substituting $y=x^2$)

$=x^4$

i.e. $x-x^4=0$

$x\left(1-x^3\right)=0$

$\therefore\ x=0,1$



$$\therefore\ \text{Shaded Area}=\int_0^1\sqrt{x}-x^2dx=\int_0^1 x^{\frac{1}{2}}-x^2dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right]_0^1=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}\text{ units}^2\ \#$$

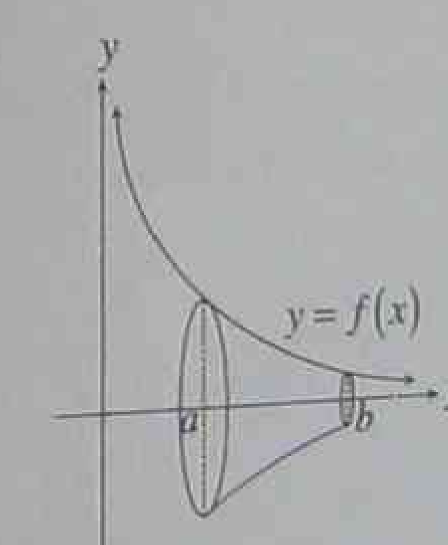
(E) Volumes by Integration

Integration can also be used to find volumes of solids of revolution

(i) Volumes of Revolution about the x-axis

The volume generated by rotating the curve $y=f(x)$ about the x-axis between $x=a$ and $x=b$ is given by:

$$V=\pi\int_a^b y^2\,dx$$



Example 1: Find the volume generated when the area bounded by the curve $y=\sqrt{1-x^2}$ and the x-axis is rotated about the x-axis.

Solution 1:

$$V=\pi\int_a^b y^2\,dx \qquad y=\sqrt{1-x^2}\text{, i.e. } y^2=1-x^2$$

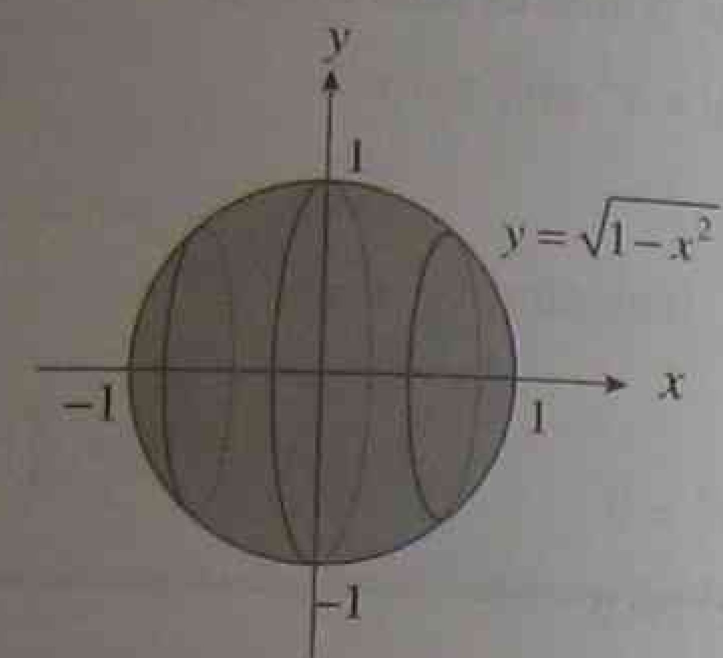
$$=\pi\int_{-1}^{1} y^2\,dx$$

$$=\pi\int_{-1}^{1}\left(1-x^2\right)dx$$

$$=\pi\left[x-\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\pi\left[\left(1-\frac{1}{3}\right)-\left(-1-\frac{-1}{3}\right)\right]$$

$$=\pi\left[\frac{2}{3}-\left(-\frac{2}{3}\right)\right]=\frac{4\pi}{3}\text{ units}^3\ \#$$

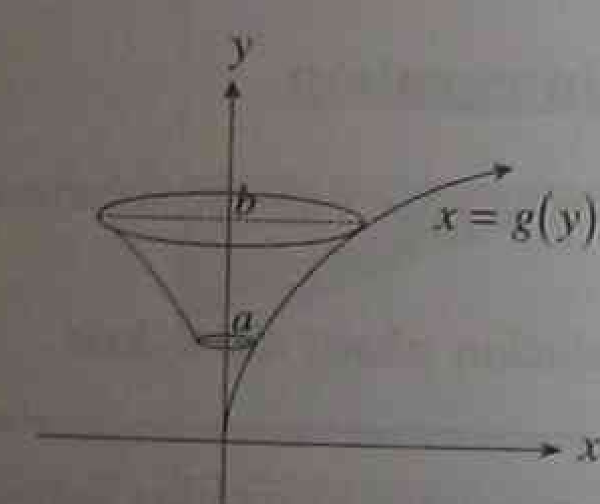


Integration

(ii) Volumes of Revolution about the y-axis

The volumes generated by rotating the curve $x=g(y)$ about the y-axis between $y=a$ and $y=b$ is given by:

$$V=\pi\int_a^b x^2\,dy$$



Example 1: Find the volume generated when the area bounded by the curve $y=9-x^2$ for $x\geq 0$, the y-axis and the line $y=3$ is rotated about the y-axis.

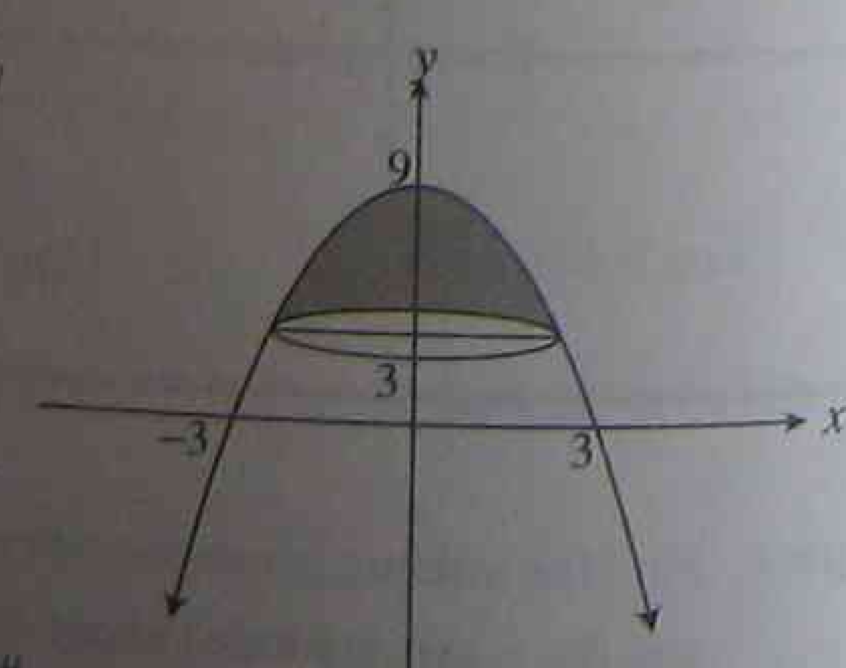
Solution 1:

$$V=\pi\int_a^b x^2\,dy \qquad y=9-x^2 \text{ i.e. } x^2=9-y$$

$$=\pi\int_3^9 9-y\,dy$$

$$=\pi\left[9y-\frac{y^2}{2}\right]_3^9$$

$$=\pi\left[\left(81-\frac{81}{2}\right)-\left(27-\frac{9}{2}\right)\right]=18\pi\text{ units}^3\ \#$$



REVIEW EXERCISES

(A) Indefinite Integrals

1. Find: (i) $\int 3x^{-\frac{3}{2}}dx$ (ii) $\int(3x-5)^{10}dx$

(iii) $\int 4x^3-\frac{5}{x^2}\,dx$ (iv) $\int\frac{1}{\sqrt{x-5}}\,dx$

(v) $\int\frac{1}{(2x+3)^2}\,dx$ (vi) $\int\left(1+\sqrt{x}\right)^2dx$

(B) Definite Integrals

2. Find the exact value of the following definite integrals:

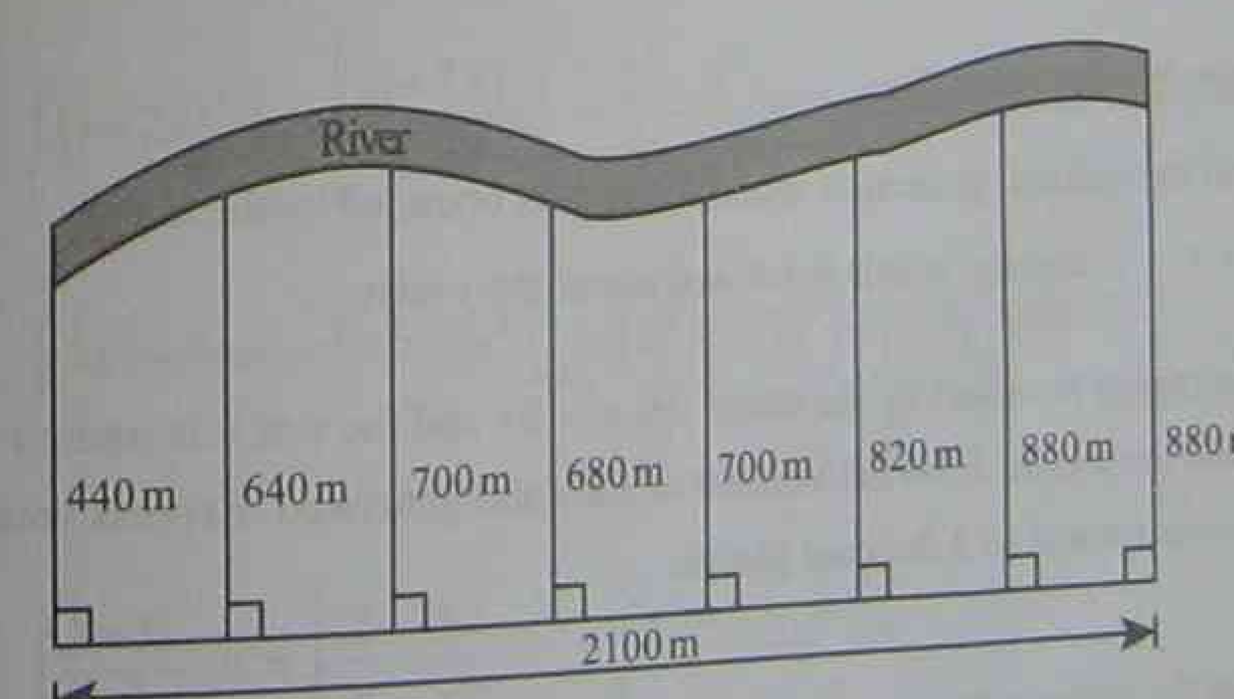
(i) $\int_0^1 x^3-x^4\,dx$ (ii) $\int_1^2(1+2x)^3\,dx$

(iii) $\int_{-1}^{2}3x^2+4x\,dx$ (iv) $\int_1^3\frac{2+5x}{x^3}\,dx$

(C) Approximations to Definite Integrals

3. Use the trapezoidal rule with 2 function values to approximate $\int_1^3 x\sqrt{x+1}\,dx$

4. A particular parcel of land in the country is bounded by a river as shown below.

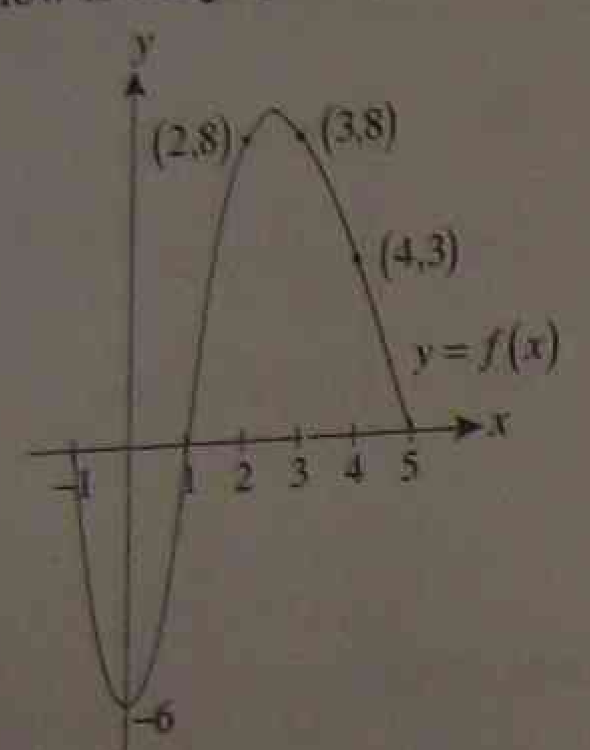


The perpendicular heights are measured at equal sub-intervals and are recorded as shown.

Use the trapezoidal rule to appoximate the area of the land.

5. Use the Simpson's rule with five function values to give an estimate to $\int_0^2 4^{-x}dx$

6. Below is the graph of $y=f(x)$ for $0-1\le x\le 5$



Use Simpson's rule to find the approximate area enclosed by the curve, the x-axis and the lines $x=-1$ and $x=5$.

(D) Areas by Integration

7. Find the area bounded by the parabola $y=x^2+1$, the x-axis and the lines $x=1$ to $x=3$.

8. Find the area enclosed by the curves $y=x^2$ and $y=x^3$.

9. Find the area bounded by the curve $y=x^2-4$ and the line $y=x+2$.

(E) Volumes by Integration

10. Find the volume generated when the region bounded by the curve $y=x^2-3$ and the x-axis is rotated about the y-axis.

11. The region bounded by the curve $y=x^2-5x$ and the x-axis is rotated about the x-axis. Find the volume of the solid generated. Express your answer correct to 2 decimal places.

12. Find the volume generated when the curve $y=x+\frac{1}{x}$ is rotated about the x-axis between $x=1$ and $x=2$.

13. Find the volume generated when the curve $y=\sqrt{x}+1$ is rotated about the y-axis between the ordinates

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $\int 3x^{-\frac{3}{2}}\,dx=\frac{3x^{-\frac{1}{2}}}{-\frac{1}{2}}+C=-\frac{6}{\sqrt{x}}+C$ #

(ii) $\int(3x-5)^{10}\,dx=\frac{(3x-5)^{11}}{11\times 3}+C=\frac{(3x-5)^{11}}{33}+C$ #

(iii) $\int 4x^3-\frac{5}{x^2}\,dx=\int 4x^3-5x^{-2}\,dx=\frac{4x^4}{4}-\frac{5x^{-1}}{-1}+C=x^4+\frac{5}{x}+C$ #

(iv) $\int\frac{1}{\sqrt{x-5}}\,dx=\int(x-5)^{-\frac{1}{2}}\,dx=\frac{(x-5)^{\frac{1}{2}}}{\frac{1}{2}}+C=2\sqrt{x-5}+C$ #

(v) $\int\frac{1}{(2x+3)^2}\,dx=\int(2x+3)^{-2}\,dx=\frac{(2x+3)^{-1}}{-1\times 2}+C=\frac{-1}{2(2x+3)}+C$ #

(vi) $\int\left(1+\sqrt{x}\right)^2\,dx=\int 1+2\sqrt{x}+x\,dx$

$=\int 1+2x^{\frac{1}{2}}+x\,dx=x+\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^2}{2}+C=x+\frac{4}{3}x^{\frac{3}{2}}+\frac{x^2}{2}+C$ #

2. (i) $\int_0^1 x^3-x^4\,dx=\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1=\left(\frac{1}{4}-\frac{1}{5}\right)-(0)=\frac{1}{20}$ #

(ii) $\int_1^2(1+2x)^3\,dx=\left[\frac{(1+2x)^4}{2\times 4}\right]_1^2=\frac{5^4}{8}-\frac{3^4}{8}=68$ #

(iii) $\int_{-1}^2 3x^2+4x\,dx=\left[x^3+2x^2\right]_{-1}^2=\left(2^3+2(2)^2\right)-\left((-1)^3+2(-1)^2\right)$

$=16-(-1+2)=15$ #

(iv) $\int_1^3\frac{2+5x}{x^3}\,dx=\int_1^3\frac{2}{x^3}+\frac{5x}{x^3}\,dx=\int_1^2 2x^{-3}+5x^{-2}\,dx$

$=\left[\frac{2x^{-2}}{-2}+\frac{5x^{-1}}{-1}\right]_1^2=\left(-\frac{1}{4}-\frac{5}{2}\right)-\left(-\frac{1}{1}-5\right)=3\frac{1}{4}$ #

3. $\int^b f(x)dx=\frac{h}{2}[y_0+y_n]=\frac{2}{2}\left[1\sqrt{1+1}+3\sqrt{3+1}\right]=\sqrt{2}+3\sqrt{4}=6+\sqrt{2}$ #

4. $A \approx \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \ldots)\right],\ h = \frac{2100}{7} = 300$

$= \frac{300}{2}\left[440 + 880 + 2(640 + 700 + 680 + 700 + 820 + 880)\right]$

$= 1{,}524{,}000\,\text{m}^2$ #

5. $\int_a^b 4^{-x}\,dx \approx \frac{h}{3}\left[y_0 + y_4 (y_1 + y_3 + \ldots)\right] + 2(y_2 + y_4 + \ldots),\ h = \frac{1}{2}$

$= \frac{\frac{1}{2}}{3}\left[4^0 + 4^{-2} + 4\left(4^{-\frac{1}{2}} + 4^{-\frac{3}{2}}\right) + 2\left(4^{-1}\right)\right]$

$= \frac{1}{6}\left[1 + \frac{1}{16} + 4\left(\frac{1}{2} + \frac{1}{8}\right) + \frac{1}{2}\right] = \frac{65}{96}$ #

6. $\int_{-1}^{5} f(x)\,dx = \left|\int_{-1}^{1} f(x)\,dx\right| + \int_{1}^{5} f(x)\,dx$

now $\int_{1}^{5} f(x)\,dx \approx \frac{1}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2y_2\right] = \frac{1}{3}\left[0 + 0 + 4(8+3) + 2(8)\right]$

$= \frac{1}{3}[44 + 16] = 20$

$\left|\int_{-1}^{1} f(x)\,dx\right| \approx \left|\frac{1}{3}[y_0 + 4y_1 + y_2]\right| = \left|\frac{1}{3}[0 + 4 \times -6 + 0]\right| = |-8| = 8$

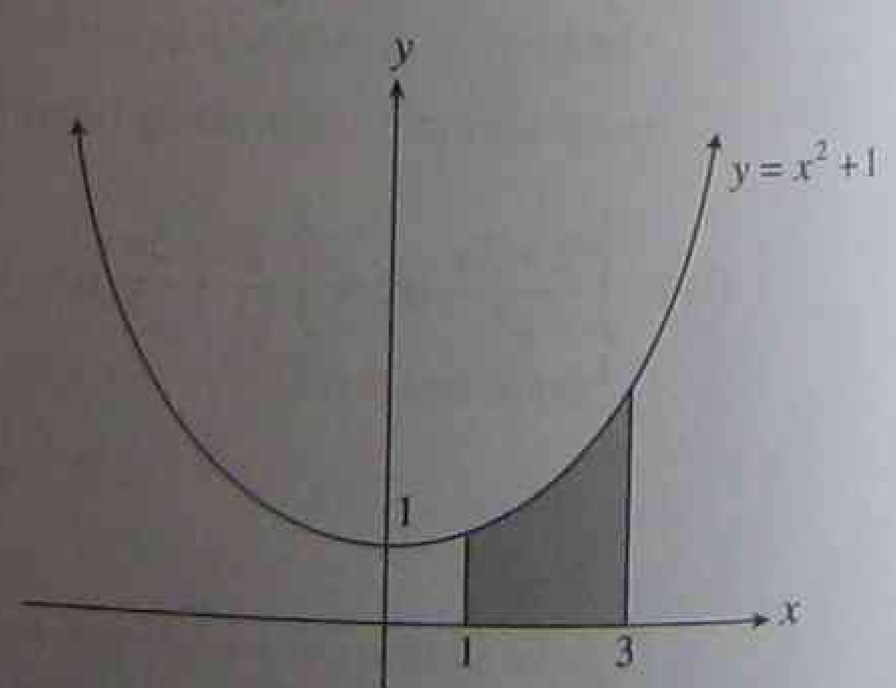
$\int_{-1}^{5} f(x)dx \approx 20 + 8 = 28\ \text{units}^2$ #

7. $A = \int_1^3 x^2 + 1\,dx$

$= \left[\frac{x^3}{3} + x\right]_1^3$

$= \left[\left(\frac{3^3}{3} + 3\right) - \left(\frac{1^3}{3} + 1\right)\right]$

$= 12 - 1\frac{1}{3} = 10\frac{2}{3}\ \text{units}^2$ #



8. To find the points of intersection we solve:

$x^3 = x^2$

$x^3 - x^2 = 0$

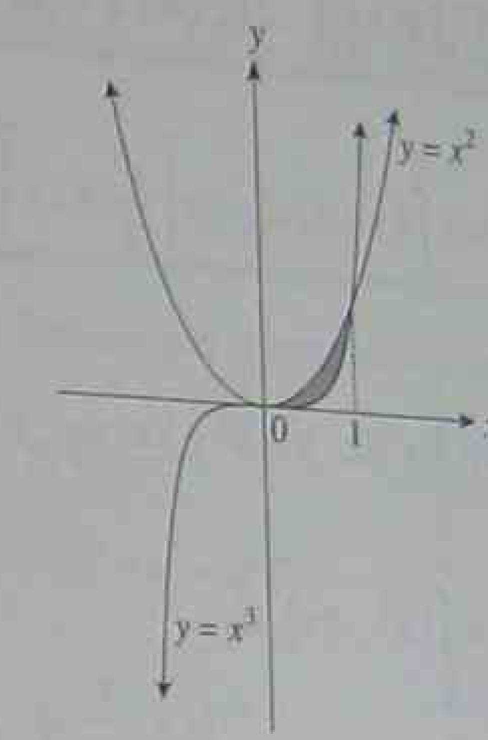
$x^2(x - 1) = 0$

i.e. $x = 0, 1$

$\therefore\ A = \int_0^1 x^2 - x^3\,dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= \left(\frac{1}{3} - \frac{1}{4}\right) - (0) = \frac{1}{12}\ \text{units}^2$



9. To find the points of intersection we solve:

$x^2 - 4 = x + 2$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

i.e. $x = -2, 3$

Shaded Area = Area 'A' + Area 'B' + Area 'C'

Area 'A' $= \int_2^3 (x+2) - (x^2 - 4)\,dx$

$= \int_2^3 x - x^2 + 6\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 6x\right]_2^3 = \left(\frac{9}{2} - \frac{27}{3} + 18\right) - \left(\frac{4}{2} - \frac{8}{3} + 12\right) = 2\frac{1}{6}\ \text{units}^2$ #

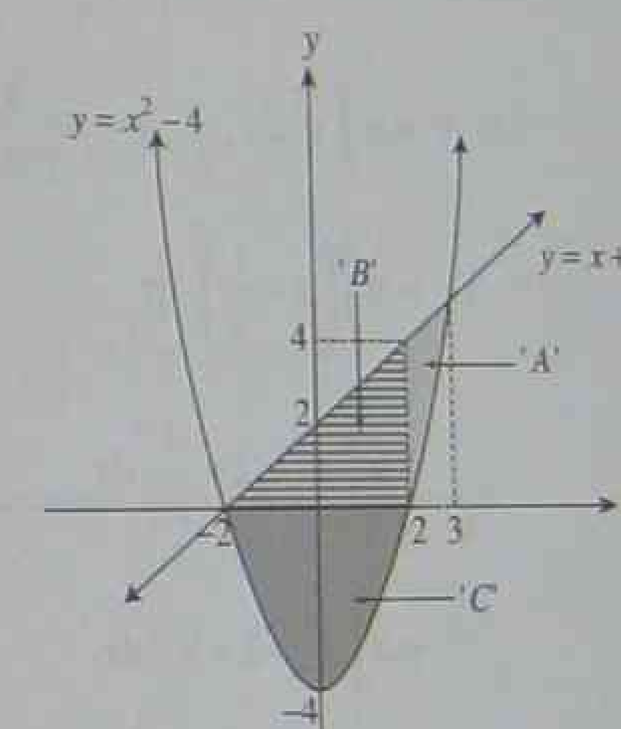
Area 'B' $= \frac{1}{2}bh = \frac{1}{2} \times 4 \times 4 = 8\ \text{units}^2$

Area 'C' $= \left|\int_{-2}^{2} (x^2 - 4)\,dx\right|$

$= \left|\left[\frac{x^3}{3} - 4x\right]_{-2}^{2}\right| = \left|\left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)\right|$

$= \left|\frac{16}{3} - 16\right| = 10^{\frac{2}{3}}\ \text{units}^2$

$\therefore$ Shaded Area $= 2\frac{1}{6} + 8 + 10\frac{2}{3} = 20\frac{5}{6}\ \text{units}^2$ #



10. $V=\pi\int_a^b x^2\,dy,\quad y=x^2-3\quad \text{i.e. } x^2=y+3$

$$=\pi\int_{-3}^{0}(y+3)\,dy=\pi\left[\frac{y^2}{2}+3y\right]_{-3}^{0}=\pi\left[(0+0)-\left(\frac{9}{2}-9\right)\right]=\frac{9\pi}{2}\text{ units}^3\ \#$$

11. $V=\pi\int_a^b y^2dx,\ y=x^2-5x\quad \text{i.e. } y^2=\left(x^2-5x\right)^2$

$$=\pi\int_0^5\left(x^2-5x\right)^2dx$$

$$=\pi\int_0^5 x^4-10x^3+25x^2dx=\pi\left[\frac{x^5}{5}-\frac{10x^4}{4}+\frac{25x^3}{3}\right]_0^5=327.25\text{ units}^3\ \#$$

12. $V=\pi\int y^2dx,\quad y=x+\frac{1}{x}\quad \therefore\ y^2=\left(x+\frac{1}{x}\right)^2$

$$\text{i.e.}=\pi\int_1^2\left(x+\frac{1}{x}\right)^2dx$$

$$=\pi\int_1^2 x^2+2+\frac{1}{x^2}\,dx$$

$$=\pi\int_1^2 x^2+2+x^{-2}dx$$

$$=\pi\left[\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}\right]_1^2$$

$$=\pi\left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^2=\pi\left[\left(\frac{8}{3}+4-\frac{1}{2}\right)-\left(\frac{1}{3}+2-1\right)\right]=\frac{29\pi}{6}\text{ units}^3\ \#$$

13. When $x=0,\ y=1$ and when $x=4,\ y=3$

$$V=\pi\int_1^3 x^2\,dy\quad \text{now } y=\sqrt{x}+1\quad \text{i.e. } \sqrt{x}=y-1$$

$$x=(y-1)^2$$

$$=\pi\int_1^3(y-1)^4dy\qquad x^2=(y-1)^4$$

$$=\pi\left[\frac{(y-1)^5}{5}\right]_1^3=\pi\left[\left(\frac{32}{5}\right)-(0)\right]=\frac{32\pi}{5}\text{ units}^3\ \#$$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

(A) Index Laws

These laws need to be learnt in both forward and reverse directions.

1. $x^0=1$	2. $\sqrt[n]{x}=x^{\frac{1}{n}}$
3. $x^{-a}=\frac{1}{x^a}$	4. $x^a\times x^b=x^{a+b}$
5. $\sqrt{x}=x^{\frac{1}{2}}$	6. $\frac{x^a}{x^b}=x^{a-b}$
7. $\left(x^a\right)^b=x^{ab}$	8. $x^{\frac{a}{b}}=\left(x^{\frac{1}{b}}\right)^a$
9. $\left(\frac{1}{x^a}\right)^b=x^{-ab}$	10. $(xy)^a=x^a\times y^a$
11. $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$	12. $\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$

Example 1: Simplify the following:

(i) $4^{\frac{1}{2}}$ (ii) $8^{\frac{2}{3}}$ (iii) $9^{-\frac{3}{2}}$

(iv) $3^{-2}+25^{-\frac{1}{2}}$ (v) $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$ (vi) $625^{\frac{1}{4}}+2^0$

Solution 1:

(i) $4^{\frac{1}{2}}=\sqrt{4}=2$ #

(ii) $8^{\frac{2}{3}}=\left(\sqrt[3]{8}\right)^2=(2)^2=4$ #

(iii) $9^{-\frac{3}{2}}=\frac{1}{9^{\frac{3}{2}}}=\frac{1}{\left(\sqrt{9}\right)^3}=\frac{1}{3^3}=\frac{1}{27}$ #

(iv) $3^{-2}+25^{-\frac{1}{2}}=\frac{1}{3^2}+\frac{1}{\sqrt{25}}=\frac{1}{9}+\frac{1}{5}=\frac{14}{45}$ #

(v) $\left(\frac{27}{8}\right)^{-\frac{2}{3}} = \frac{(27)^{-\frac{2}{3}}}{(8)^{-\frac{2}{3}}} = \frac{\frac{1}{(27)^{\frac{2}{3}}}}{\frac{1}{8^{\frac{2}{3}}}} = \frac{\frac{1}{\left(\sqrt[3]{27}\right)^2}}{\left(\sqrt[3]{8}\right)^2} = \frac{\frac{1}{9}}{\frac{1}{4}} = \frac{4}{9}$ #

(vi) $625^{\frac{1}{4}} + 2^0 = 5 + 1 = 6$ #

Example 2: Simplify the following:

(i) $x^3 \times x^2 \times x$ (ii) $\left(x^4y^2\right)^{\frac{1}{2}}$ (iii) $\frac{\left(x^3y^3\right)^{\frac{1}{2}}}{\left(x^2y^2\right)^{\frac{1}{3}}}$

(iv) $\left(\frac{x\sqrt{y}}{x^{-2}}\right)^4 \div \left(\frac{x^2z^3}{y}\right)^3$ (v) $\frac{250}{\left(25^n\right)^3 \times 125^{1-2n}}$

Solution 2:

(i) $x^3 \times x^2 \times x^1 = x^{3+2+1} = x^6$ #

(ii) $\left(x^4y^2\right)^{\frac{1}{2}} = \left(x^4\right)^{\frac{1}{2}} \times \left(y^2\right)^{\frac{1}{2}} = x^{4\times\frac{1}{2}} \times y^{2\times\frac{1}{2}} = x^2y$ #

(iii) $\frac{\left(x^3y^3\right)^{\frac{1}{2}}}{\left(x^2y^2\right)^{\frac{1}{3}}} = \frac{\left(x^3\right)^{\frac{1}{2}}\left(y^3\right)^{\frac{1}{2}}}{\left(x^2\right)^{\frac{1}{3}}\left(y^2\right)^{\frac{1}{3}}} = \frac{x^{\frac{3}{2}}y^{\frac{3}{2}}}{x^{\frac{2}{3}}y^{\frac{2}{3}}} = x^{\frac{3}{2}-\frac{2}{3}}y^{\frac{3}{2}-\frac{2}{3}} = x^{\frac{5}{6}}y^{\frac{5}{6}} = (xy)^{\frac{5}{6}}$ #

(iv) $\left(\frac{x\sqrt{y}}{z^{-2}}\right)^4 \div \left(\frac{x^2z^3}{y}\right)^3 = \frac{\left(xy^{\frac{1}{2}}\right)^4}{\left(z^{-2}\right)^4} \times \frac{(y)^3}{\left(x^2z^3\right)^3} = \frac{x^4y^2}{z^{-8}} \times \frac{y^3}{x^6z^9} = \frac{y^5}{x^2z}$ #

(v) $\frac{250}{\left(25^n\right)^3 \times 125^{1-2n}} = \frac{250}{\left(5^{2n}\right)^3 \times \left(5^3\right)^{1-2n}}$

$= \frac{250}{5^{6n} \times 5^{3-6n}} = \frac{250}{5^3} = \frac{250}{125} = 2$ #

Example 3: Solve:

(i) $x^5 = 161051$ (ii) $(1-x)^3 = 0.027$ (iii) $2^{x-4} = 32$

Solution 3:

(i) $x^5 = 161051$

$x = \sqrt[5]{161051} = (161051)^{\frac{1}{5}} = 11$ #

(ii) $(1-x)^3 = 0.027$

$(1-x) = \sqrt[3]{0.027} = (0.027)^{\frac{1}{3}} = 0.3$

$\therefore\ x = 1 - 0.3 = 0.7$ #

(iii) $2^{x-4} = 32$

$2^{x-4} = 2^5$

$\therefore\ x - 4 = 5$ i.e. $x = 9$ #

(B) Logarithmic Laws

These laws need to be learnt in both forward and reverse directions.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^n = n\log_a x$
4. $\log_a x + \log_a y = \log_a (xy)$
5. $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
6. If $\log_a b = c$ then $b = a^c$
7. Change of base: $\log_a x = \frac{\log_b x}{\log_b a}$
8. Extensions:

$\log_a x + \log_a y + \log_a z = \log_a (xyz)$

$\log_a x - \log_a y - \log_a z = \log_a \left(\frac{x}{yz}\right)$

Example 1: Simplify the following:

(i) $\log_2 8$ (ii) $\log_5 5\sqrt{125}$ (iii) $\log_5 100 - \log_5 4$

Solution 1:

(i) $\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3$ #

(ii) $\log_5 5\sqrt{125} = \log_5 5\sqrt{5^3} = \log_5\left(5\times 5^{\frac{3}{2}}\right) = \log_5 5^{\frac{5}{2}} = \frac{5}{2}\log_5 5 = \frac{5}{2}$ #

(iii) $\log_5 100 - \log_5 4 = \log_5\left(\frac{100}{4}\right) = \log_5 25 = \log_5 5^2 = 2\log_5 5 = 2$ #

(iv) $\frac{\log\left(x^3\right)+\log\left(x^2\right)}{\log x} = \frac{3\log x + 2\log x}{\log x} = \frac{5\log x}{\log x} = 5$ #

(v) $\frac{\log 64}{\log 4} = \frac{\log 2^6}{\log 2^2} = \frac{6\log 2}{2\log 2} = 3$ #

Example 2: Solve:

(i) $\log_3 x = 3$ (ii) $\log_x 64 = 2$ (iii) $2\log x = \log(x+2)$

Solution 2:

(i) $\log_3 x = 3$ i.e. $x = 3^3 = 27$ #

(ii) $\log_x 64 = 2$ i.e. $64 = x^2$ i.e. $x = 8,\ x \neq -8$ as $x > 0$

(iii) $2\log x = \log(x+2)$

$\log x^2 = \log(x+2)$

$\therefore\ x^2 = x+2$

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore\ x = -1,\ 2$

since $x > 0$ as log of a negative is undefined $\therefore\ x = 2$ is the only solution #

Example 3: If $\log_a b = 2.5$ and $\log_a c = 4$ then find:

(i) $\log_a b^2$ (ii) $\log_a\left(\frac{b}{c}\right)$

(iii) $\log_a\left(\sqrt{bc}\right)$ (iv) $\log_a\sqrt{\frac{b}{a}}$

Solution 3:

(i) $\log_a b^2 = 2\log_a b = 2\times 2.5 = 5$ #

(ii) $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c = 2.5 - 4 = -1.5$ #

Exponential & Log Functions

(iii) $\log_a\left(\sqrt{bc}\right) = \log_a\left(\sqrt{b}\sqrt{c}\right)$

$= \log_a\sqrt{b} + \log_a\sqrt{c}$

$= \log_a b^{\frac{1}{2}} + \log_a c^{\frac{1}{2}} = \frac{1}{2}\log_a b + \frac{1}{2}\log_a c = \frac{2.5}{2} + \frac{4}{2} = 3.25$ #

(iv) $\log_a\sqrt{\frac{b}{a}} = \log_a\left(\frac{b}{a}\right)^{\frac{1}{2}}$

$= \frac{1}{2}\log_a\left(\frac{b}{a}\right) = \frac{1}{2}\left[\log_a b - \log_a a\right] = \frac{1}{2}\left[2.5 - 1\right] = 0.75$ #

(C) Change of Base

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example 1: Find the exact value of:

(i) $\log_{16} 64$ (ii) $\log_8 27 - \log_2\sqrt{3}$

Solution 1:

(i) $\log_{16} 64 = \frac{\log_2 64}{\log_2 16} = \frac{\log_2 2^6}{\log_e 2^4} = \frac{6\log_2 2}{4\log_2 2} = \frac{3}{2}$ #

(ii) $\log_8 27 - 2\log_2\sqrt{3} = \frac{\log_2 27}{\log_2 8} - 2\log_2(3)^{\frac{1}{2}}$

$= \frac{\log_2 3^3}{\log_2 2^3} - \log_2 3$

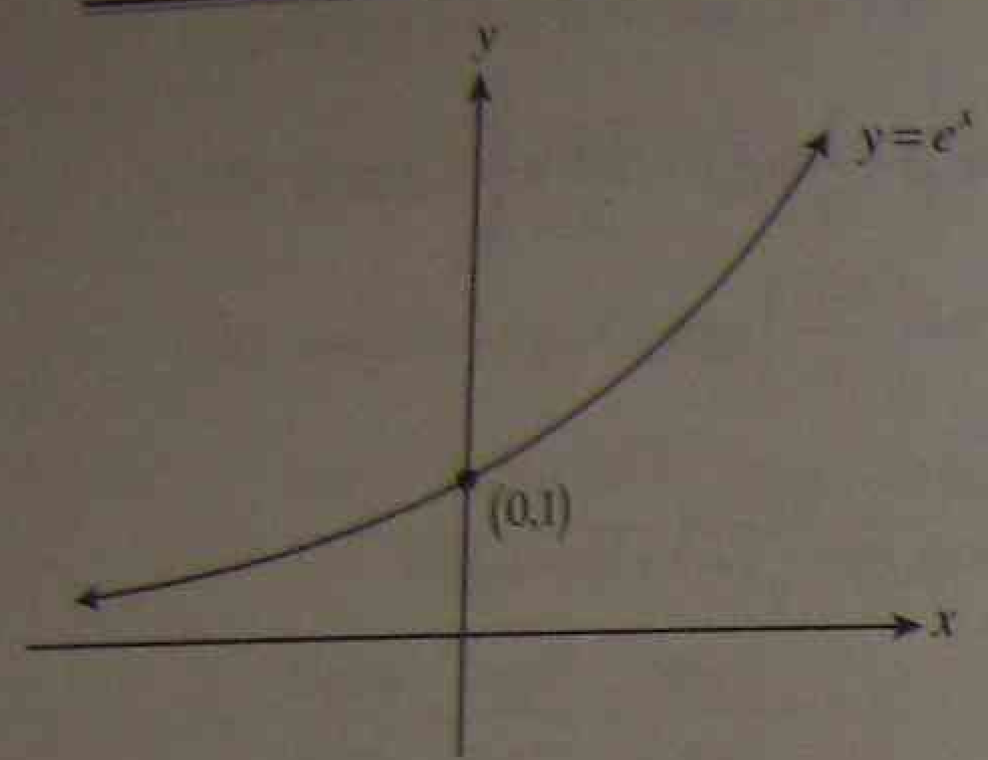
$= \frac{3\log_2 3}{3\log_2 2} - \log_2 3 = \log_2 3 - \log_2 3 = 0$ #

(D) Exponential Functions

The exponential function is represented by e^x where e is an irrational number $\left(\text{like } \pi \text{ or } \sqrt{2}\right)$ and has an approximate value of 2.718.

Exponential & Log Functions

(i) Graph of $y = e^x$

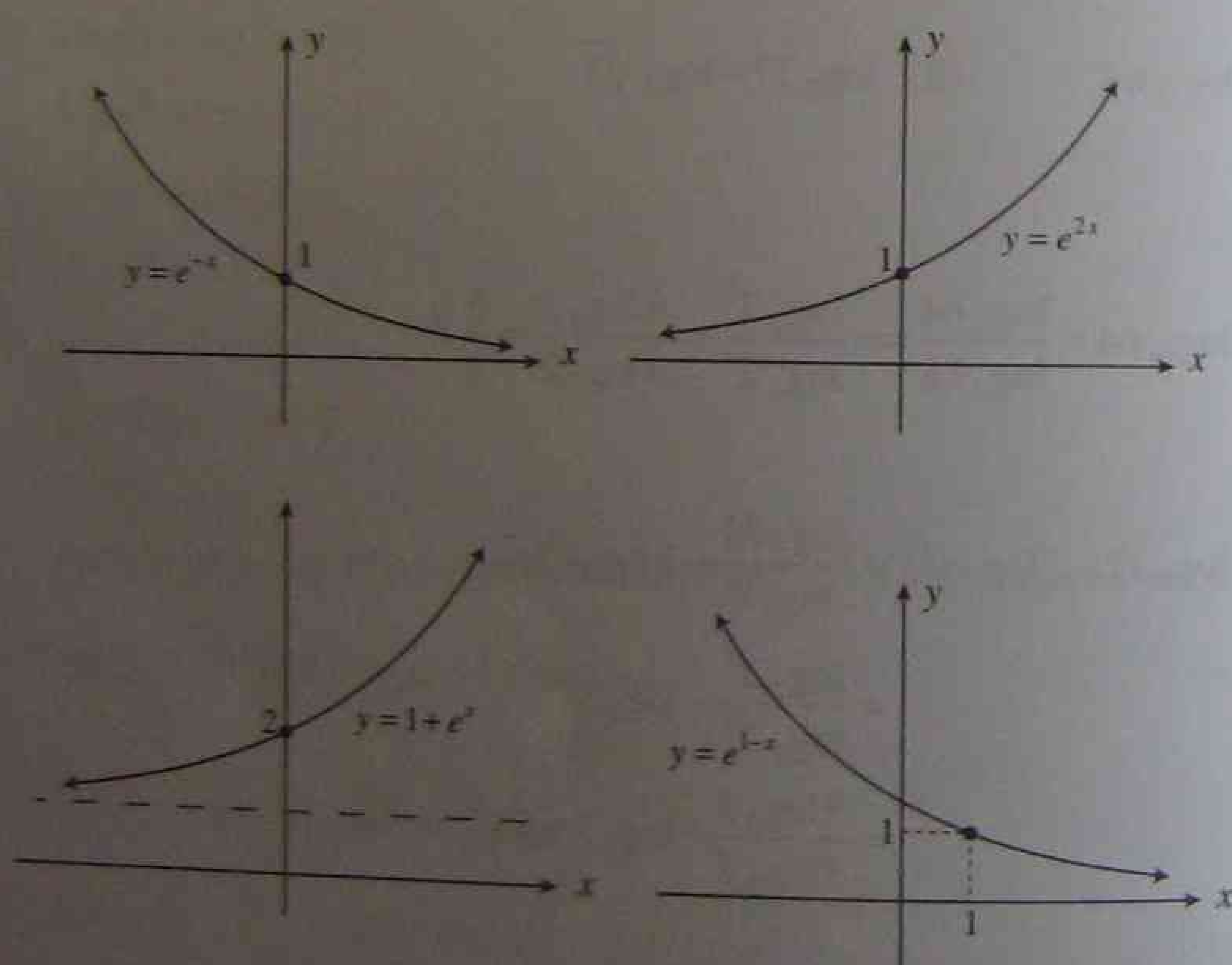


Properties: 1. Passes through $(0,1)$.

2. $e^x > 0$ for all x
3. Is always increasing.
4. Is always concave up.
5. $x \to \infty,\ e^x \to \infty$
6. $x \to -\infty,\ e^x \to 0$

(ii) Graph of Other Exponential Functions

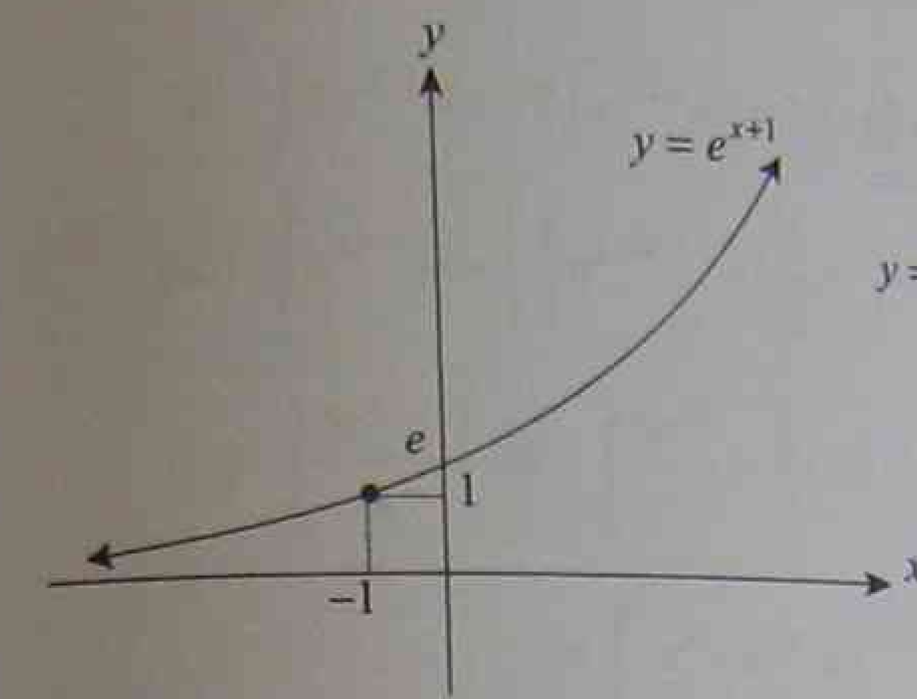
Students need to also be able to sketch variations on the basic curve $y = e^x$

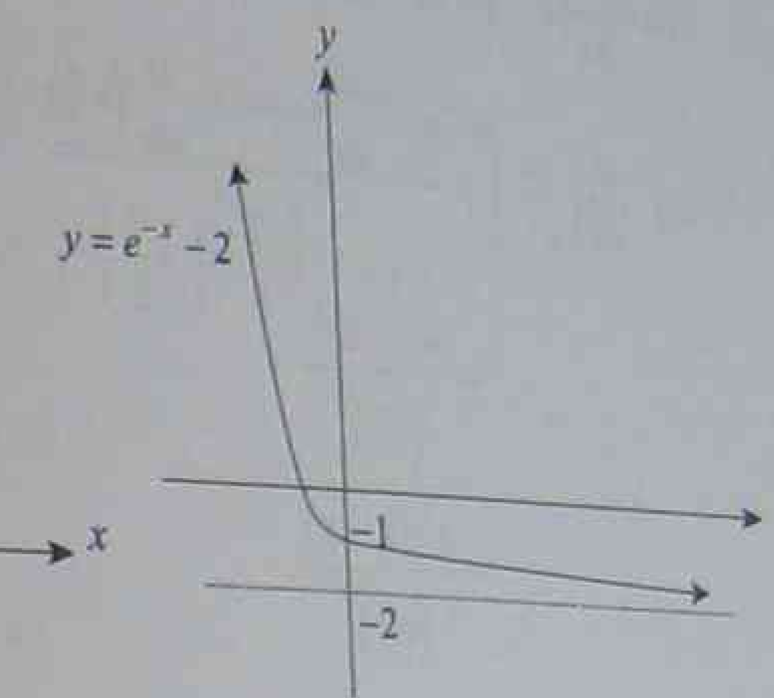


Example 1: Sketch the graphs of:

(i) $y = e^{x+1}$ (ii) $y = e^{-x} - 2$

Solution 1:





(iii) The Derivative of a^x

$$\text{If } y = a^x,\ \frac{dy}{dx} = \log_e a\ a^x$$

Example 1: Differentiate:

(i) 3^x (ii) 5^x

Solution 1:

(i) $\frac{d}{dx}\left(3^x\right) = \log_e 3.3^x$ #

(ii) $\frac{d}{dx} = \log_e 5.5^x$ #

(iv) The Derivative of e^x

$$\frac{d}{dx}\left(e^x\right) = e^x,\quad \frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

Example 1: Differentiate:

(i) e^{5x} (ii) $\frac{x}{e^x}$ (iii) x^2e^x

(iv) e^{1-2x} (v) e^{5-x^3} (vi) $\left(e^x + e^{-x}\right)^2$

Solution 1:

(i) $\frac{d}{dx}\left(e^{5x}\right)=5e^{5x}$ #

(ii) $\frac{d}{dx}\left(\frac{x}{e^{x}}\right)=\frac{e^{x}\cdot\frac{d}{dx}(x)-x\cdot\frac{d}{dx}\left(e^{x}\right)}{\left(e^{x}\right)^{2}}$

$$=\frac{e^{x}-xe^{x}}{\left(e^{x}\right)^{2}}$$

$$=\frac{e^{x}(1-x)}{\left(e^{x}\right)^{2}}$$

$$=\frac{1-x}{e^{x}} \ \#$$

(iii) $\frac{d}{dx}\left(x^{2}e^{x}\right)=x^{2}\cdot\frac{d}{dx}\left(e^{x}\right)+e^{x}\cdot\frac{d}{dx}\left(x^{2}\right)$

$$=x^{2}\cdot e^{x}+e^{x}\cdot \ x=e^{x}\left(x^{2}+ \ x\right)$$

(iv) $\frac{d}{dx}\left(e^{1-2x}\right)=e^{1-2x}\cdot\frac{d}{dx}(1-2x)=e^{1-2x}\cdot-2=-2e^{1-2x}$ #

(v) $\frac{d}{dx}\left(e^{5-x^{3}}\right)=e^{5-x^{3}}\cdot\frac{d}{dx}\left(5-x^{3}\right)=e^{5-x^{3}}\cdot-3x^{2}=-3x^{2}e^{5-x^{3}}$ #

(vi) $\frac{d}{dx}\left[\left(e^{x}+e^{-x}\right)^{2}\right]=2\left(e^{x}+e^{-x}\right)\cdot\frac{d}{dx}\left(e^{x}+e^{-x}\right)$

$$=\left(e^{x}+e^{-x}\right)\left(e^{x}-e^{-x}\right)$$

$$=\left(e^{2x}-e^{-2x}\right)$$

(v) The Integral of e^{x}

$$\int e^{x}dx=e^{x}+C,\ \int e^{ax+b}dx=\frac{1}{a}e^{ax+b}+C$$

Example 1: Find:

(i) $\int e^{3x}\,dx$ (ii) $\int 2e^{3-2x}\,dx$

(iii) $\int_{1}^{2} e^{2x-1}\,dx$ (iv) $\int_{0}^{1}\frac{5e^{2x}-1}{e^{x}}\,dx$

Solution 1:

(i) $\int e^{3x}dx=\frac{1}{3}e^{3x}+C$ #

(ii) $\int 2e^{3-2x}dx=2\int e^{3-2x}dx=2\times-\frac{1}{2}e^{3-2x}+C=-e^{3-2x}+C$ #

(iii) $\int_{1}^{2} e^{2x-1}dx=\left[\frac{1}{2}e^{2x-1}\right]_{1}^{2}=\left(\frac{1}{2}e^{3}\right)-\left(\frac{1}{2}e\right)=\frac{1}{2}\left(e^{3}-e\right)$ #

(iv) $\int_{0}^{1}\frac{5e^{2x}-1}{e^{x}}\,dx=\int_{0}^{1}\frac{5e^{2x}}{e^{x}}-\frac{1}{e^{x}}\,dx$

$$=\int_{0}^{1} e^{x}-e^{-x}\,dx$$

$$=\left[\ e^{x}+e^{-x}\right]_{0}^{1}$$

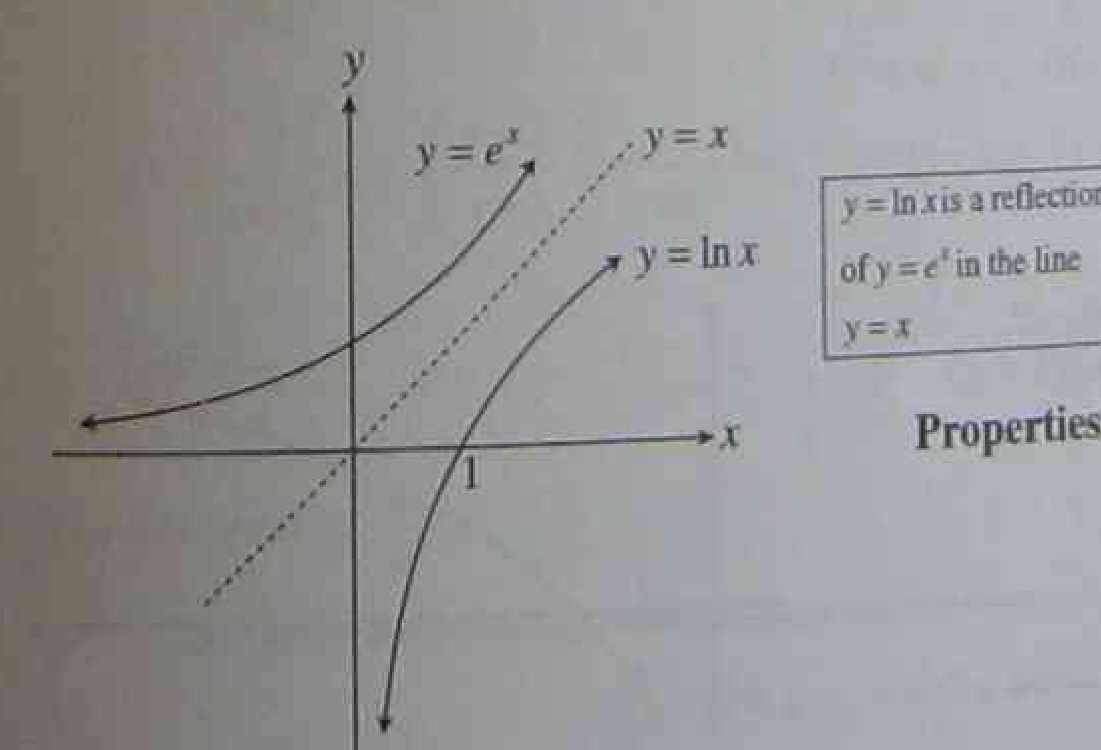
$$=\left(\ e+e^{-1}\right)-\left(\ +\ \right)=\ e+\frac{1}{e}-$$

(E) Logarithmic Functions

Logarithms to the base e are called ***natural logarithms***.

The natural logarithmic function is represented by $\log_{e}x$ or simply by $\ln x$.

(i) Graph of $y=\log_{e}x$



Properties: 1. Passes through $(1,0)$.

2. Is only defined for $x>0$

3. Is always increasing.

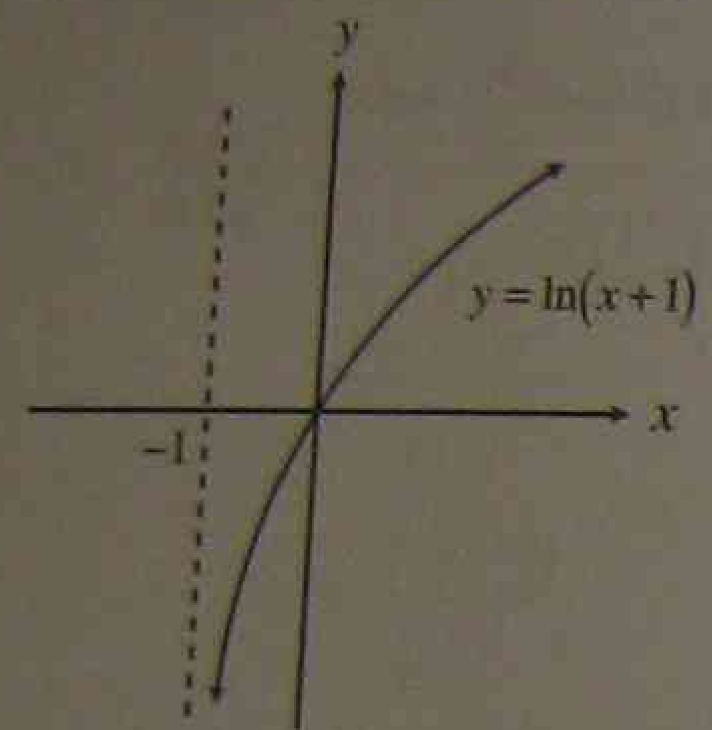
4. Is always concave down.

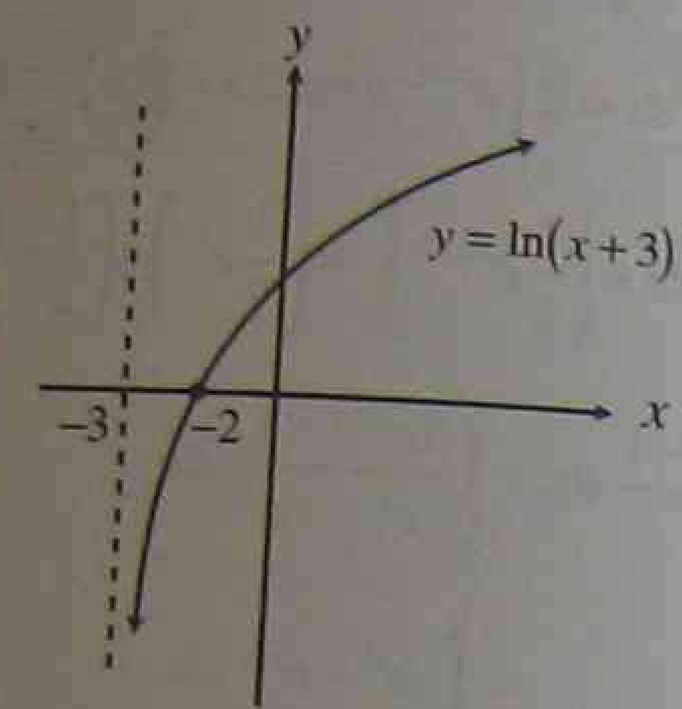
5. $x\to\infty$, $\ln x\to\infty$

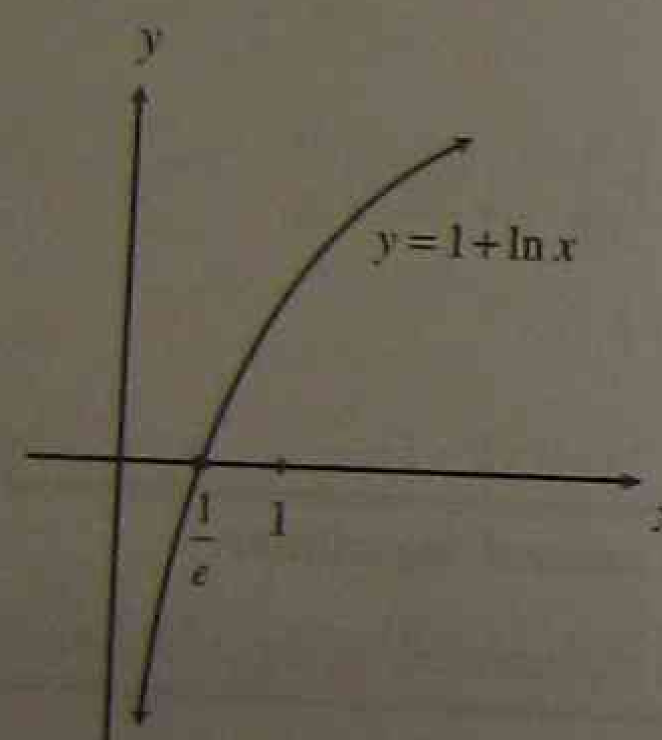
6. $x\to 0$, $\ln x\to-\infty$

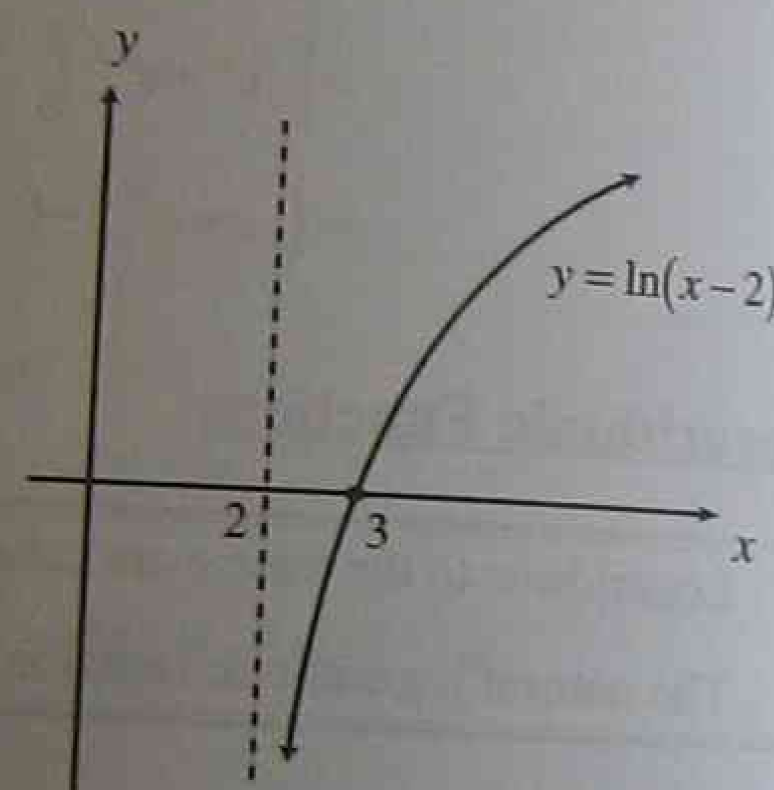
(ii) Graphs of Other Logarithmic Functions

Students need also to be able to sketch graphs with slight variation on the basic curve $y=\ln x$.





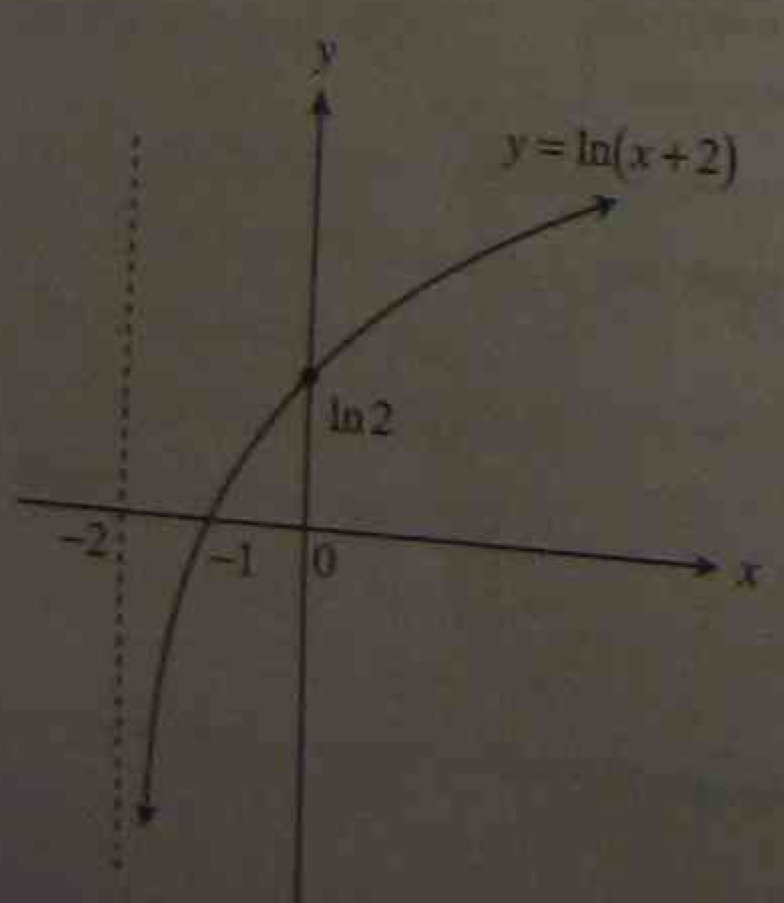


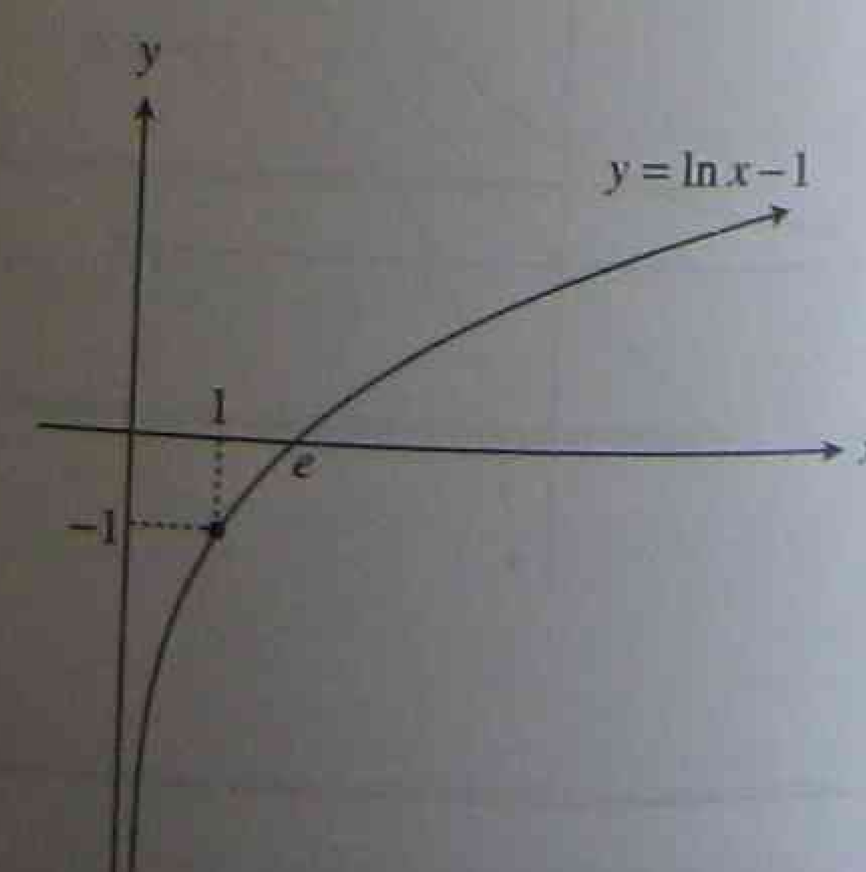


Example 1: Sketch the graphs of:

(i) $y=\ln(x+2)$ (ii) $y=\ln x-1$

Solution 1:





(iii) The Derivative of $\log_e x$

$$\frac{d}{dx}(\log_e x)=\frac{1}{x},\quad \frac{d}{dx}(\log_e f(x))=\frac{f'(x)}{f(x)}$$

Example 1: Differentiate:

(i) $\log_e 2x$ (ii) $\log_e(7x^2-3)$ (iii) $x\log_e x$

(iv) $\dfrac{\log_e x}{x}$ (v) $\log_e(x^2-2x+1)$ (vi) $\log_e\left(\dfrac{x+2}{x-1}\right)$

Solution 1:

(i) $\dfrac{d}{dx}(\log_e 2x)=\dfrac{\frac{d}{dx}(2x)}{2x}=\dfrac{2}{2x}=\dfrac{1}{x}$ #

(ii) $\dfrac{d}{dx}\left(\log_e\left(7x^2-3\right)\right)=\dfrac{\frac{d}{dx}\left(7x^2-3\right)}{7x^2-3}=\dfrac{14x}{7x^2-3}$ #

(iii) $\dfrac{d}{dx}(x\log_e x)=x\cdot\dfrac{d}{dx}(\log_e x)+\log_e x\cdot\dfrac{d}{dx}(x)=x\cdot\dfrac{1}{x}+\log_e x=1+\log_e x$ #

(iv) $\dfrac{d}{dx}\left(\dfrac{\log_e x}{x}\right)=\dfrac{x\cdot\frac{d}{dx}(\log_e x)-\log_e x\cdot\frac{d}{dx}(x)}{x^2}=\dfrac{x\cdot\frac{1}{x}-\log_e x\cdot 1}{x^2}=\dfrac{1-\log_e x}{x^2}$ #

(v) $\dfrac{d}{dx}\left(\log_e\left(x^2-2x+1\right)\right)=\dfrac{\frac{d}{dx}\left(x^2-2x+1\right)}{x^2-2x+1}=\dfrac{2x-2}{x^2-2x+1}=\dfrac{2(x-1)}{(x-1)^2}=\dfrac{2}{(x-1)}$ #

(vi) $\dfrac{d}{dx}\left[\log_e\left(\dfrac{x+2}{x-1}\right)\right]=\dfrac{d}{dx}\left[\log_e(x+2)-\log_e(x-1)\right]$

$=\dfrac{1}{x+2}-\dfrac{1}{x-1}=\dfrac{(x-1)-(x+2)}{(x+2)(x-1)}=\dfrac{-3}{(x+2)(x-1)}$ #

(iv) The Integral of $\frac{1}{x}$

$$\int\frac{1}{x}dx=\log_e x+C$$

$$\int\frac{f'(x)}{f(x)}=\log_e f(x)+C$$

$$\int\frac{dx}{ax+b}=\frac{1}{a}\log_e(ax+b)+C$$

Example 1: Find:

(i) $\int \frac{3}{x}\,dx$ (ii) $\int \frac{2x}{x^2-5}\,dx$ (iii) $\int \frac{x^2}{x^3+1}\,dx$

(iv) $\int_1^2 \left(\frac{1}{x}-\frac{2}{x-1}\right)dx$ (v) $\int_0^{\ln 2} \frac{2e^x}{e^x+1}\,dx$ (vi) $\int_0^{\frac{1}{\sqrt{3}}} \frac{3x}{1+3x^2}\,dx$

Solution 1:

(i) $\int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx = 3\log_e x + C$ #

(ii) $\int \frac{2x}{x^2-5}\,dx = \log_e\left(x^2-5\right)+C$ #

(iii) $\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}\,dx = \frac{1}{3}\log_e\left(x^3+1\right)+C$ #

(iv) $\int_1^2 \left(\frac{1}{x}-\frac{2}{x+1}\right)dx = \int_1^2 \frac{1}{x}\,dx - 2\int_1^2 \frac{1}{x+1}\,dx$

$$= \left[\log_e(x)\right]_1^2 - 2\left[\log_e(x+1)\right]_1^2$$
$$= \left(\log_e 2 - \log_e 1\right) - 2\left(\log_e 3 - \log_e 2\right)$$
$$= \log_e 2 - 2\log_e 3 + 2\log_e 2 = 3\log_e 2 - 2\log_e 3 \ \#$$

(v) $\int_0^{\ln 2} \frac{2e^x}{e^x+1}\,dx = 2\int_0^{\ln 2} \frac{e^x}{e^x+1}\,dx$

$$= 2\left[\ln\left(e^x+1\right)\right]_0^{\ln 2} \qquad [\text{note: } \log_e x = \ln x]$$
$$= 2\left[\ln\left(e^{\ln 2}+1\right)-\ln(2)\right] = 2[\ln 3 - \ln 2] = 2\ln\left(\frac{3}{2}\right) \ \#$$

(vi) $\int_0^{\frac{1}{\sqrt{3}}} \frac{3x}{1+3x^2}\,dx = \frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{6x}{1+3x^2}\,dx$

$$= \frac{1}{2}\left[\ln\left(1+3x^2\right)\right]_0^{\frac{1}{\sqrt{3}}}$$
$$= \frac{1}{2}\left[\ln\left(1+3\left(\frac{1}{\sqrt{3}}\right)^2\right)-\ln(1)\right] = \frac{1}{2}\ln 2 \ \#$$

(F) Applications of Differentiation and Integration

Differentiation and integration techniques can be applied to questions involving logarithmic and exponential functions.

Exponential & Log Functions

Example 1: Find the equation of the tangent to the curve $y=\ln\left(x^2\right)$ at $x=1$.

Solution 1:

$$y=\ln\left(x^2\right)$$
$$\frac{dy}{dx}=\frac{2x}{x^2}=\frac{2}{x}$$

at $x=1$, $\frac{dy}{dx}=\frac{2}{1}=2$ $\therefore$ gradient of tangent = 2

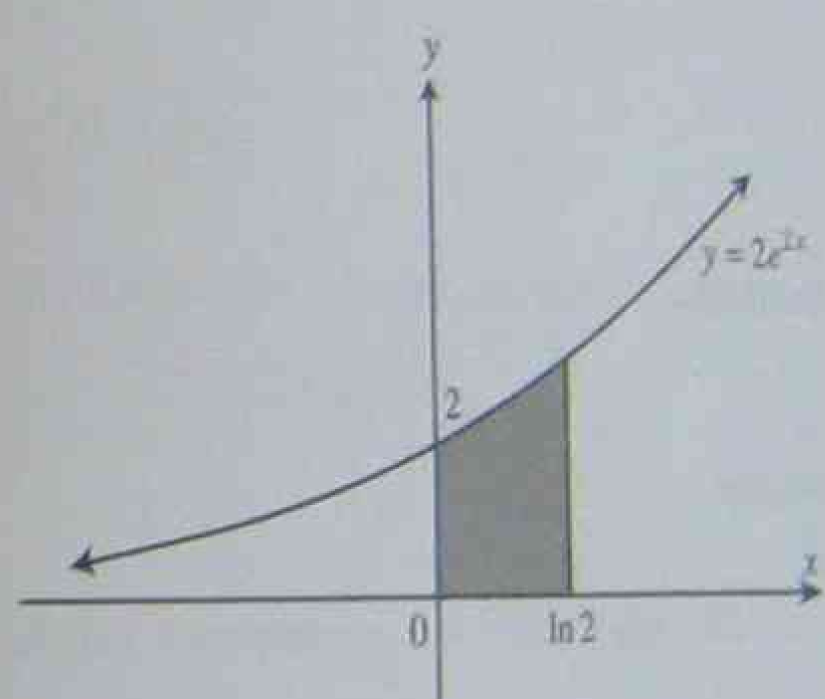
when $x=1$, $y=0$ $\therefore$ equation of tangent is given by:

$$\left(y-y_1\right)=m\left(x-x_1\right)$$
$$(y-0)=2(x-1)$$
$$y=2x-2 \ \#$$

Example 2: Find the area bounded by the curve $y=2e^{2x}$ and the lines and $x=\ln 2$.

Solution 2:

$$A=\int_0^{\ln 2} 2e^{2x}\,dx$$
$$=\left[\frac{2e^{2x}}{2}\right]_0^{\ln 2}$$
$$=\left[e^{2x}\right]_0^{\ln 2}$$
$$=e^{2\ln 2}-1=e^{\ln 4}-1=3 \text{ units}^2 \ \#$$



Example 3: Find the volume generated by rotating the curve $y=2e^{2x}$ about the x-axis between $x=0$ and $x=\ln 2$.

Solution 3:

$$V=\pi\int_a^b y^2\,dx=\pi\int_0^{\ln 2}\left(2e^{2x}\right)^2\,dx$$
$$=\pi\int_0^{\ln 2} 4e^{4x}\,dx$$
$$=\pi\left[e^{4x}\right]_0^{\ln 2}$$
$$=\pi\left[e^{4\ln 2}-1\right]=\pi\left[e^{\ln 16}-1\right]=\pi(16-1)=15\pi \text{ units}^3 \ \#$$

Exponential & Log Functions

(G) Other Applications

Example 1: Find $\frac{d}{dx}\log_e(\sin x)$, hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$.

Solution 1:

$$\frac{d}{dx}\log_e(\sin x)=\frac{\frac{d}{dx}(\sin x)}{\sin x}=\frac{\cos x}{\sin x}=\cot x \text{ integrating both sides, we get:}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d}{dx}\log_e(\sin x)\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot x\,dx$$

i.e. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot x\,dx=\left[\ln(\sin x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\ln\left(\sin\frac{\pi}{2}\right)-\ln\left(\sin\frac{\pi}{4}\right)$$

$$=\ln(1)-\ln\left(\frac{1}{\sqrt{2}}\right)=0-\left(\ln 1-\ln\sqrt{2}\right)=\ln\sqrt{2}=\frac{1}{2}\ln 2 \text{ #}$$

Example 2: Find the sum of the first 15 terms of the following series:

$$2e^{-1}+4e^{-2}+8e^{-3}+\ldots$$

Solution 2:

$2e^{-1}+4e^{-2}+8e^{-3}+\ldots$ is a geometric series with $a=2e^{-1}$, $r=\frac{4e^{-2}}{2e^{-1}}=2e^{-1}$

i.e. $S_n=\frac{a(1-r^n)}{(1-r)}$ $\quad\therefore\quad S_{15}=\frac{2e^{-1}\left(1-\left(2e^{-1}\right)^{15}\right)}{1-2e^{-1}}=2.7565$ #

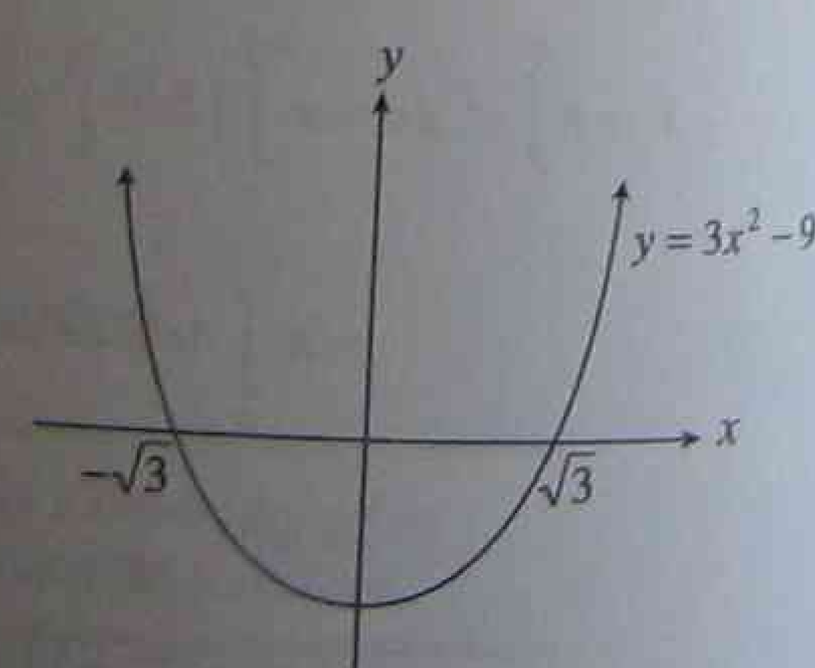
Example 3: State the domain of $g(x)=\log_e\left(3x^2-9\right)$.

Solution 3:

Since $\log_e x$ is defined for $x>0$

$\therefore\ 3x^2-9>0$ i.e. $3\left(x^2-3\right)>0$

$3\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)>0$



from the graph $y>0$ when $x>\sqrt{3}$ or $x<-\sqrt{3}$

$\therefore$ the domain of $g(x)$ is $x>\sqrt{3}$ or $x<-\sqrt{3}$ #

REVIEW EXERCISES

(A) Index Laws

1. Simplify:

(i) $27^{\frac{1}{3}}$ (ii) $16^{-\frac{3}{4}}$ (iii) $\frac{4^{-2}}{4^{-5}}$

(iv) $\left(\frac{x^2y}{z^{-1}}\right)^3\div\left(\frac{z^2x}{y^2}\right)^2$ (v) $\frac{64}{8^x\times 2^{2-3x}}$ (vi) $\left(\frac{x\sqrt{x}}{27}\right)^{\frac{1}{3}}$

2. Solve:

(i) $x^8=6561$ (ii) $\left(1+\frac{x}{2}\right)^4=256$ (iii) $2^{5-3x}=1$

(B) Logarithmic Laws

3. Simplify:

(i) $\frac{\log 27}{\log 9}$ (ii) $\log_4 96-\log_4 6$ (iii) $\log_3 27+\log_3 81$

(iv) $\log_3\left(\frac{1}{9}\right)$ (v) $\log\left(\frac{x^2}{\sqrt{y}}\right)+\log\left(\frac{1}{xy}\right)-\log\left(\frac{\sqrt{x}}{y}\right)$

4. Solve:

(i) $\log_8 x=-\frac{1}{3}$ (ii) $\log_x 125=3$

(iii) $2\log_7 5=\log_7 2x-\log_7 3$ #

5. If $\log_a 3=1.58$ and $\log_a 5=2.32$, find:

(i) $\log_a 15$ (ii) $\log_a 75$ (iii) $\log_a\sqrt{\frac{5}{3}}$

6. If $\log_6 3=0.613$, find:

(i) $\log_6 18$ (ii) $\log_6 54$ (iii) $\log_6 2$

(C) Change of Base

7. Find the exact value of $\log_{16} 8$.

8. Find correct to 3 decimal pluces the value of $\log_5 9$.

(D) Exponential Functions

9. Graph the curves: (i) $y = e^{3x}$ (ii) $y = 2 + e^{3x}$

10. Find: $\dfrac{d}{dx}\left(4^x\right)$

11. If $y = 7^x$, find $\dfrac{dy}{dx}$ when $x = 1$.

12. Differentiate the following:

(i) $\dfrac{1}{2}e^{-x}$ (ii) $4e^{\frac{x}{2}}$ (iii) $3xe^{-x}$

(iv) e^{5-2x} (v) $\dfrac{1}{\sqrt{e^x}}$ (vi) $\dfrac{e^{x^{\frac{1}{2}}}}{x}$

13. If $y = e^{kx}$ is a solution of the equation $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 7y = 0$. Find the values of k.

14. Find:

(i) $\displaystyle\int \frac{1}{e^{2x}}\,dx$ (ii) $\displaystyle\int e^{9x-1}\,dx$ (iii) $\displaystyle\int \sqrt{e^x}\,dx$

(iv) $\displaystyle\int_0^2 e^{2x+1}\,dx$ (v) $\displaystyle\int_0^1 \frac{e^{3x}+3}{e^x}\,dx$

(E) Logarithmic Functions

15. Sketch the curve $y = \ln(x-1)$.

16. Differentiate the following:

(i) $\log_e\left(4x^3\right)$ (ii) $\log_e(2x-7)$ (iii) $\log_e\left(e^{-x}+1\right)$

(iv) $x^2\log_e x$ (v) $\dfrac{1}{(1-\log_e x)^2}$ (vi) $\log_e\left(\dfrac{x^2-4}{x+1}\right)$

17. Find:

(i) $\dfrac{4}{2-x}dx$ (ii) $\displaystyle\int \frac{x}{1+2x^2}dx$ (iii) $\displaystyle\int \frac{1}{4x}dx$

(iv) $\displaystyle\int^{1} \frac{2x}{\ldots}dx$ $\displaystyle\int^{\ln 2} e^x \ldots$ $\displaystyle\int^{1} 2x+1 \ldots$

(F) Applications of Differentiation and Integration

18. Find the gradient of the normal to the curve $y = \log_e\left(3x^2 - 9\right)$ at $x = 2$.

19. Find the equation of the tangent to the curve $y = e^{-x}$ at $x = -1$.

20. Find the area bound by the curve $y = \dfrac{10}{x}$ and the lines $x = 1$ and $x = 2$.

21. Find the volume of the solid of revolution formed by rotating the curve $y = 1 + e^{-2x}$ about the x-axis between $x = 0$ and $x = \log_e 2$.

(G) Other Applications

22. Find $\dfrac{d}{dx}\left(e^{x^3}\right)$, hence evaluate $\displaystyle\int_0^1 x^2e^{x^3}\,dx$.

23. Find the sum of the first 15 terms for the following series:

$\log_e 3 + \log_e 27 + \log_e 243 + \ldots$

24. Consider the function $f(x) = xe^x$

(i) Find $f'(x)$ and $f''(x)$.

(ii) Find the coordinates of any stationary points.

(iii) Find the coordinates of any points of inflection.

(ii) Sketch the curve $f(x) = xe^x$

(v) Find the values of c for which $xe^x = c$ has two solutions.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$ #

(ii) $16^{-\frac{3}{4}} = \dfrac{1}{16^{\frac{3}{4}}} = \dfrac{1}{\left(16^{\frac{1}{4}}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ #

(iii) $\dfrac{4^{-2}}{4^{-5}} = 4^{-2-(-5)} = 4^3 = 64$ #

(iv) $\left(\dfrac{x^2y}{z^{-1}}\right)^3 \div \left(\dfrac{z^2x}{y^2}\right)^2 = \dfrac{x^6y^3}{z^{-3}} \times \dfrac{y^4}{z^4x^2} = \dfrac{x^4y^7}{z}$ #

(v) $\dfrac{64}{8^x \times 2^{2-3x}} = \dfrac{64}{2^{3x} \times 2^{2-3x}} = \dfrac{64}{2^2} = \dfrac{64}{4} = 16$ #

(vi) $\left(\dfrac{x\sqrt{x}}{27}\right)^{\frac{1}{3}} = \dfrac{\left(x \times x^{\frac{1}{2}}\right)^{\frac{1}{3}}}{(27)^{\frac{1}{3}}} = \dfrac{\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}}{3} = \dfrac{x^{\frac{1}{2}}}{3} = \dfrac{\sqrt{x}}{3}$ #

2. (i) $x^8 = 6561$ i.e. $x = (6561)^{\frac{1}{8}} = 3$ #

(ii) $\left(1 + \dfrac{x}{2}\right)^4 = 256$ i.e. $1 + \dfrac{x}{2} = (256)^{\frac{1}{4}}$ i.e. $1 + \dfrac{x}{2} = 4,\ x = 6$ #

(iii) $2^{5-3x} = 1$ i.e. $5 - 3x = 0, 3x = 5$ i.e. $x = \dfrac{5}{3}$ #

3. (i) $\dfrac{\log 27}{\log 9} = \dfrac{\log 3^3}{\log 3^2} = \dfrac{3\log 3}{2\log 3} = \dfrac{3}{2}$ #

(ii) $\log_4 96 - \log_4 6 = \log_4\left(\dfrac{96}{6}\right) = \log_4 16 = \log_4 4^2 = 2\log_4 4 = 2$ #

(iii) $\log_3 27 + \log_3 81 = \log_3 3^3 + \log_3 3^4 = 3\log_3 3 + 4\log_3 3 = 3 + 4 = 7$ #

(iv) $\log_3\left(\dfrac{1}{9}\right) = \log_3 1 - \log_3 9 = 0 - \log_3 3^2 = -2\log_3 3 = -2$ #

(v) $\log\left(\dfrac{x^2}{\sqrt{y}}\right) + \log\left(\dfrac{1}{xy}\right) - \log\left(\dfrac{\sqrt{x}}{y}\right) = \log\left[\dfrac{\dfrac{x^2}{\sqrt{y}} \times \dfrac{1}{xy}}{\dfrac{\sqrt{x}}{y}}\right] = \log\left[\dfrac{x}{y\sqrt{y}} \times \dfrac{y}{\sqrt{x}}\right]$

$= \log\sqrt{\dfrac{x}{y}} = \dfrac{1}{2}\log\left(\dfrac{x}{y}\right)$ #

Exponential & Log Functions

4. (i) $\log_8 x = -\dfrac{1}{3}$ i.e. $x = 8^{-\frac{1}{3}} = \dfrac{1}{8^{\frac{1}{3}}} = \dfrac{1}{2}$ #

(ii) $\log_x 125 = 3$ i.e. $x^3 = 125$ i.e. $x = 5$ #

(iii) $2\log_7 5 = \log_7 2x - \log_7 3$

$\log_7 5^2 = \log_7\left(\dfrac{2x}{3}\right)$

i.e. $25 = \dfrac{2x}{3}$

$75 = 2x$ i.e. $x = 37.5$ #

5. (i) $\log_a 15 = \log_a(3 \times 5) = \log_a 3 + \log_a 5 = 1.58 + 2.32 = 3.9$ #

(ii) $\log_a 75 = \log_a(15 \times 5) = \log_a 15 + \log_a 5 = 3.9 + 2.32 = 6.22$ #

(iii) $\log_a\sqrt{\dfrac{5}{3}} = \log_a\left(\dfrac{5}{3}\right)^{\frac{1}{2}} = \dfrac{1}{2}\log_a\left(\dfrac{5}{3}\right) = \dfrac{1}{2}[\log_a 5 - \log_a 3] = \dfrac{1}{2}[2.32 - 1.58] = 0.37$ #

6. (i) $\log_6 18 = \log_6(6 \times 3) = \log_6 6 + \log_6 3 = 1.613$ #

(ii) $\log_6 54 = \log_6(18 \times 3) = \log_6 18 + \log_6 3 = 1.613 + 0.613 = 2.226$ #

(iii) $\log_6 2$, now $\log_6 18 = \log_6(2 \times 9) = \log_6(2 \times 3^2)$

i.e. $\log_6(2) + \log_6(3^2) = 1.613$

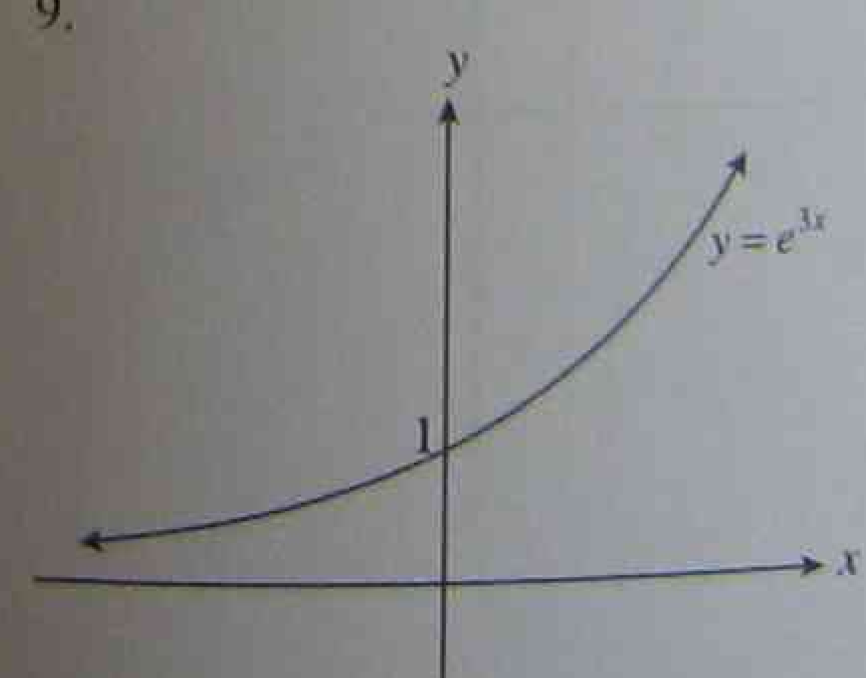
$\log_6 2 + 2\log_6 3 = 1.613$

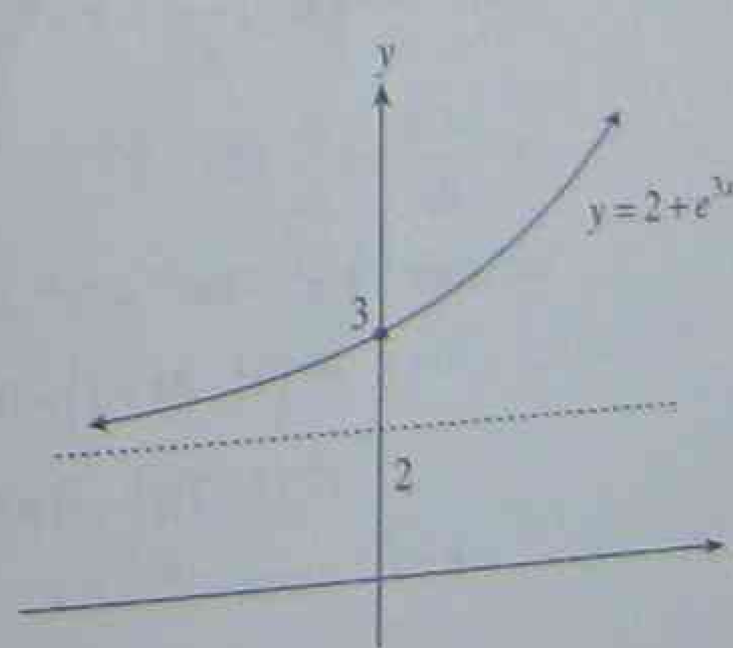
$\therefore\ \log_6 2 = 1.613 - 2 \times 0.613 = 0.387$ #

7. $\log_{16} 8 = \dfrac{\log_2 8}{\log_2 16} = \dfrac{\log_2 2^3}{\log_2 2^4} = \dfrac{3\log_2 2}{4\log_2 2} = \dfrac{3}{4}$ #

8. $\log_5 9 = \dfrac{\log_{10} 9}{\log_{10} 5} = 1.365$ correct to 3 d.p. #

9.





10. $\frac{d}{dx}\left(4^x\right)=\log_e 4\cdot 4^x$ #

11. $\frac{d}{dx}\left(7^x\right)=\log_e 7^x\cdot 7^x$ at $x=1$

$=7\log_e 7=13.62$ correct to 2 d.p. #

12. (i) $\frac{d}{dx}\left(\frac{1}{2}e^{-x}\right)=-\frac{1}{2}e^{-x}$ #

(ii) $\frac{d}{dx}\left(4e^{\frac{x}{2}}\right)=\frac{1}{2}\times 4e^{\frac{x}{2}}=2e^{\frac{x}{2}}$ #

(iii) $\frac{d}{dx}\left(3xe^{-x}\right)=3x\cdot -e^{-x}+e^{-x}\cdot 3=-3xe^{-x}+3e^{-x}=3e^{-x}(1-3x)$ #

(iv) $\frac{d}{dx}\left(e^{5-2x}\right)=-2e^{5-2x}$ #

(v) $\frac{d}{dx}\left(\frac{1}{\sqrt{e^x}}\right)=\frac{d}{dx}\left(\frac{1}{e^{\frac{x}{2}}}\right)=\frac{d}{dx}\left(e^{-\frac{x}{2}}\right)=-\frac{1}{2}e^{-\frac{x}{2}}$ #

(vi) $\frac{d}{dx}\left(\frac{e^{\frac{1}{x^2}}}{x}\right)=\frac{d}{dx}\left(\frac{e^{x^{-2}}}{x}\right)=\frac{x.\frac{d}{dx}\left(e^{x^{-2}}\right)-e^{x^{-2}}.\frac{d}{dx}(x)}{x^2}$

$$=\frac{x.-2x^{-3}.e^{x^{-2}}-e^{x^{-2}}}{x^2}$$

$$=\frac{e^{x^{-2}}\left(-\frac{2x}{x^3}-1\right)}{x^2}=\frac{-e^{x^{-2}}\left(\frac{2}{x^2}+1\right)}{x^2}=\frac{-e^{x^{-2}}\left(2+x^2\right)}{x^4}\ \#$$

13. $y=e^{kx},\ \frac{dy}{dx}=ke^{kx},\ \frac{d^2y}{dx^2}=k^2e^{kx}$

i.e. $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+7y=0$

becomes $k^2e^{kx}-8ke^{kx}+7e^{kx}=0$

$e^{kx}\left(k^2-8k+7\right)=0$

$e^{kx}(k-7)(k-1)=0$

$\therefore\ k=1,7$ note $e^{kx}\neq 0$ #

14. (i) $\int\frac{1}{e^{2x}}\,dx=\int e^{-2x}\,dx=-\frac{1}{2}e^{-2x}+C$ #

(ii) $\int e^{9x-1}\,dx=\int\frac{1}{9}e^{9x-1}+C$ #

(iii) $\int\sqrt{e^x}\,dx=\int e^{\frac{x}{2}}\,dx=\frac{e^{\frac{x}{2}}}{\frac{1}{2}}+C=2e^{\frac{x}{2}}+C$ #

(iv) $\int_0^2 e^{2x+1}\,dx=\left[\frac{1}{2}e^{2x+1}\right]_0^2=\frac{1}{2}\left(e^5-e^1\right)=\frac{e^5-e}{2}$ #

(v) $\int_0^1\frac{e^{3x}+3}{e^x}\,dx=\int_0^1\frac{e^{3x}}{e^x}+\frac{3}{e^x}\,dx$

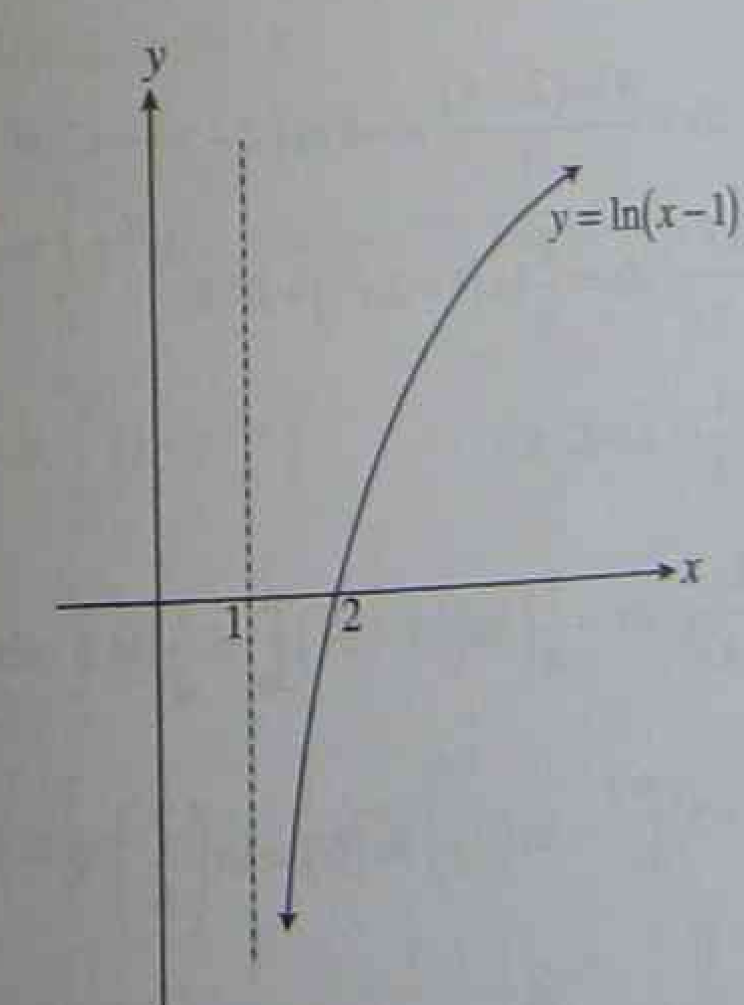
$$=\int_0^1 e^{2x}+3e^{-x}\,dx$$

$$=\left[\frac{1}{2}e^{2x}-3e^{-x}\right]_0^1$$

$$=\left(\frac{1}{2}e^2-3e^{-1}\right)-\left(\frac{1}{2}-3\right)$$

$$=\frac{e^2}{2}-\frac{3}{e}+\frac{5}{2}=\frac{e^3-6+5e}{2e}\ \#$$

15.



Exponential & Log Functions

16. (i) $\frac{d}{dx}\left[\log_e\left(4x^3\right)\right]=\frac{12x^2}{4x^3}=\frac{3}{x}$ #

(ii) $\frac{d}{dx}\left[\log_e(2x-7)\right]=\frac{2}{2x-7}$ #

(iii) $\frac{d}{dx}\left[\log_e\left(e^{-x}+1\right)\right]=\frac{-e^{-x}}{e^{-x}+1}$ #

(iv) $\frac{d}{dx}\left(x^2\log_e x\right)=x^2\cdot\frac{1}{x}+\log_e x\cdot 2x=x+2x\log_e x=x\left(1+2\log_e x\right)$ #

(v) $\frac{d}{dx}\left[\frac{1}{\left(1-\log_e x\right)^2}\right]=\frac{d}{dx}\left[\left(1-\log_e x\right)^{-2}\right]$

$=-2\left(1-\log_e x\right)^{-3}\frac{d}{dx}\left(1-\log_e x\right)$

$=\frac{-2}{\left(1-\log_e x\right)^3}\cdot-\frac{1}{x}=\frac{2}{x\left(1-\log_e x\right)^3}$ #

(vi) $\frac{d}{dx}\left[\log_e\left(\frac{x^2-4}{x+1}\right)\right]=\frac{d}{dx}\left[\log_e\left(x^2-4\right)-\log_e(x+1)\right]$

$=\frac{2x}{x^2-4}-\frac{1}{x+1}$

$=\frac{2x(x+1)-\left(x^2-4\right)}{\left(x^2-4\right)(x+1)}=\frac{2x^2+2x-x^2+4}{\left(x^2-4\right)(x+1)}=\frac{x^2+2x+4}{\left(x^2-4\right)(x+1)}$ #

17. (i) $\int\frac{4}{2-x}\,dx=4\int\frac{1}{2-x}\,dx=\frac{4\ln(2-x)}{-1}=-4\ln(2-x)+C$ #

(ii) $\int\frac{x}{1+2x^2}\,dx=\frac{1}{4}\int\frac{4x}{1+2x^2}\,dx=\frac{1}{4}\ln\left(1+2x^2\right)+C$ #

(iii) $\int\frac{1}{4x}\,dx=\frac{1}{4}\int\frac{1}{x}\,dx=\frac{1}{4}\ln x+C$ #

(iv) $\int_0^1\frac{2x}{1+4x^2}\,dx=\frac{1}{4}\int_0^1\frac{8x}{1+4x^2}\,dx=\frac{1}{4}\left[\ln\left(1+4x^2\right)\right]_0^1=\frac{1}{4}\left[\ln 5-\ln 1\right]=\frac{\ln 5}{4}$ #

(v) $\int_0^{\ln 2}\frac{e^x}{e^x+5}\,dx=\left[\ln\left(e^x+5\right)\right]_0^{\ln 2}=\ln(7)-\ln(6)=\ln\left(\frac{7}{6}\right)$ #

(vi) $\int_0^1\frac{2x+1}{x^2+x+5}\,dx=\left[\ln\left(x^2+x+5\right)\right]_0^1=\ln(7)-\ln(5)=\ln\left(\frac{7}{5}\right)$ #

18. $y=\log_e\left(3x^2-9\right),\ \frac{dy}{dx}=\frac{6x}{3x^2-9}$

at $x=2,\ \frac{dy}{dx}=\frac{6\times 2}{3(2)^2-9}=\frac{12}{12-9}=4$ i.e. gradient of tangent $=4$

$\therefore$ gradient of normal $=-\frac{1}{4}$

19. $y=e^{-x},\ \frac{dy}{dx}=-e^{-x}$ at $x=-1,\ \frac{dy}{dx}=-e$

i.e. gradient of tangent $=-e$, at $x=-1, y=e$

$\therefore$ equation of tangent:

$\left(y-y_1\right)=m\left(x-x_1\right)$

$(y-e)=-e(x+1)$

$y-e=-ex-e$

$y=-ex$ #

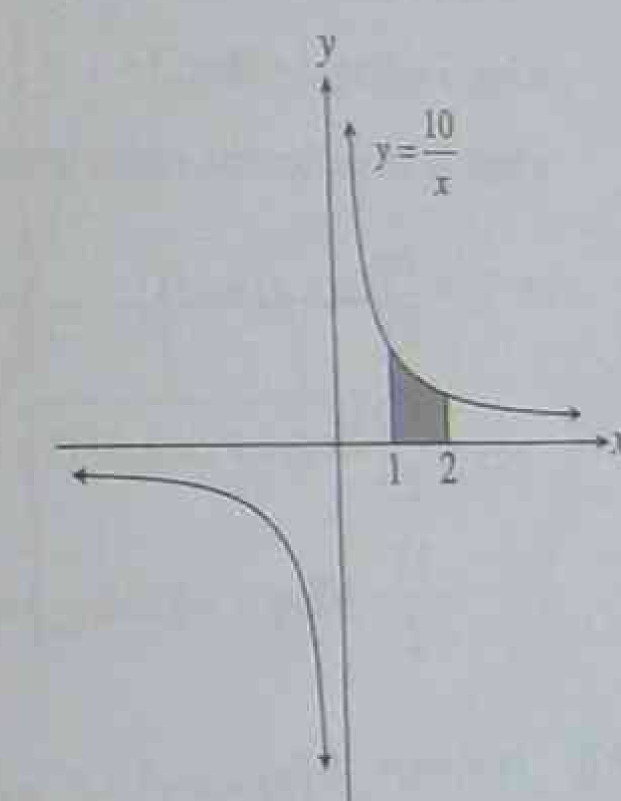
20. Area $=\int_1^2\frac{10}{x}\,dx$

$=10\int_1^2\frac{1}{x}\,dx$

$=10\left[\ln x\right]_1^2$

$=10(\ln 2-\ln 1)$

$=10\ln 2$ units^2 #



21. Volume $=\pi\int_a^b y^2\,dx,\ \ y=1+e^{-2x}$ i.e. $y^2=\left(1+e^{-2x}\right)^2$

$=\pi\int_0^{\ln 2}\left(1+e^{-2x}\right)^2$

$=\pi\int_0^{\ln 2}\left(1+2e^{-2x}+e^{-4x}\right)dx$

$=\pi\left[x-e^{-2x}-\frac{1}{4}e^{-4x}\right]_0^{\ln 2}$

$=\pi\left[\left(\ln 2-\frac{1}{4}-\frac{1}{4}\times\frac{1}{16}\right)-\left(0-1-\frac{1}{4}\right)\right]$

$=\pi\left[\ln 2+\frac{63}{64}\right]$units^3 #

22. $\frac{d}{dx}\left(e^{x^3}\right)=3x^2e^{x^3}$

Integrating both sides gives:

$$\int_0^1 \frac{d}{dx}\left(e^{x^3}\right)dx=\int_0^1 3x^2e^{x^3}$$

i.e $\int_0^1 3x^2e^{x^3}=\left[e^{x^3}\right]_0^1$

$\therefore \int_0^1 x^2e^{x^3}=\frac{1}{3}\left[e^{x^3}\right]_0^1=\frac{1}{3}(e-1)$ #

23. $\log_e 3+\log_e 27+\log_e 243+\ldots$

$=\log_e 3+\log_e 3^3+\log_e 3^5+\ldots$

$=\log_e 3+3\log_e 3+5\log_e 3+\ldots$

which is an arithmetic series with $a=\log_e 3, d=2\log_e 3$

i.e $S_n=\frac{n}{2}\left[2a+d(n-1)\right]$

$S_{15}=\frac{15}{2}\left[2\log_e 3+2\log_e 3(15-1)\right]$

$=\frac{15}{2}\times 30\log_e 3=225\log_e 3=247.188$ #

24. (i) $f(x)=xe^x, f'(x)=x\cdot e^x+e^x=e^x(x+1)$

$f''(x)=e^x\cdot 1+(x+1)\cdot e^x=e^x(1+x+1)=e^x(x+2)$ #

(ii) Let $f'(x)=0$ to find stationary points

i.e. $e^x(x+1)=0$

$\therefore x=-1$

at $x=-1$, $f''(-1)>0$ and $f(-1)=-e^{-1}=-\frac{1}{e}$

$\therefore \left(-1,-\frac{1}{e}\right)$ is a minimum turning point #

(iii) Let $f''(x)=0$ to find possible points of inflection

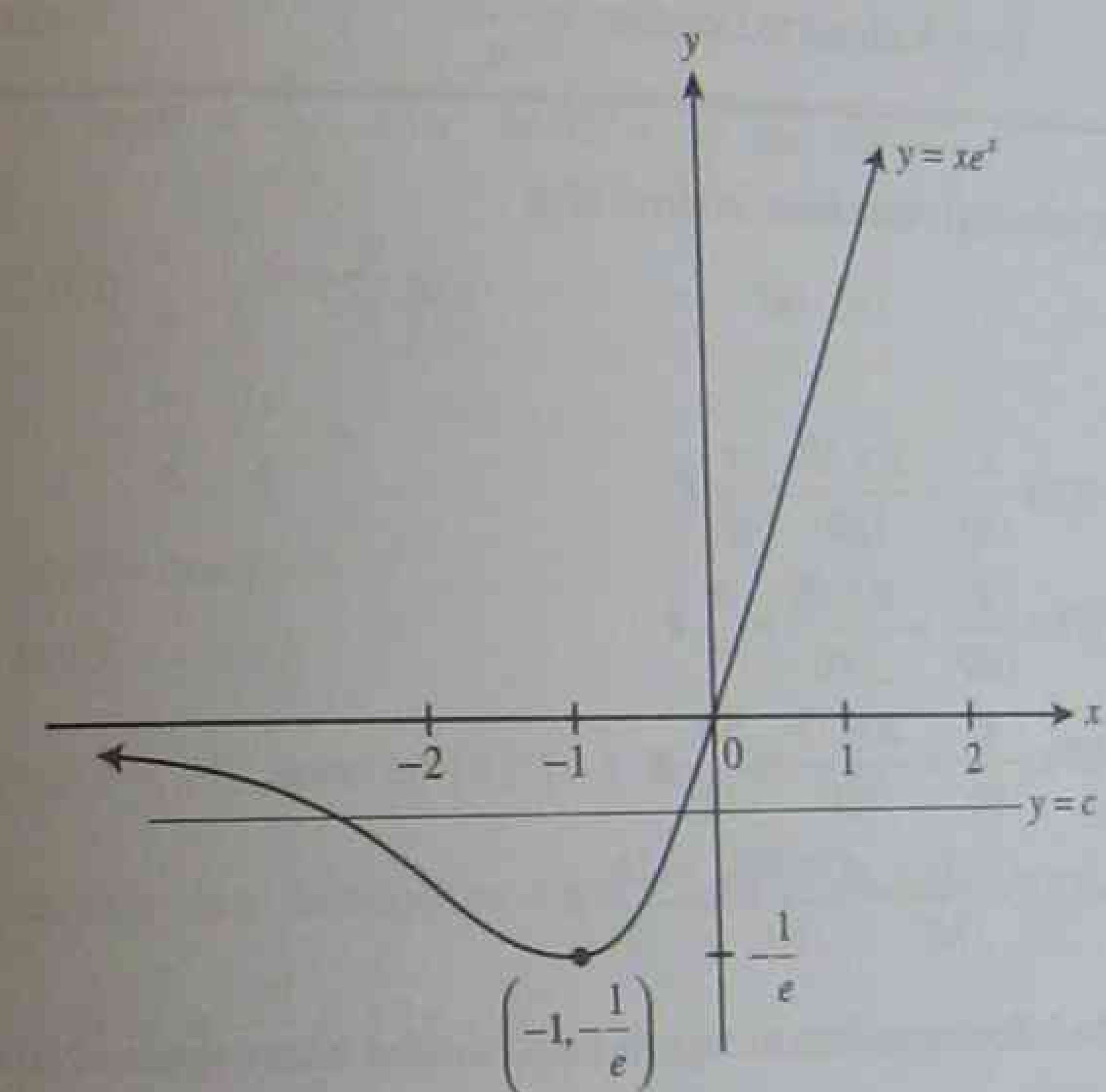
i.e. $e^x(x+2)=0$ i.e. $x=-2$ is a possible point of inflection:

$f''(-1.5)>0,\ f''(-2.5)<0$

Since $f''(x)$ changes sign in passing through $x=-2$

$\therefore \left(-2,-\frac{2}{e^2}\right)$ is a point of inflection. #

(iv)



(v) From the graph the lines $y=c$ intersects the curve $y=xe^x$ at 2 points (i.e. 2 solutions) for $-\frac{1}{e}<x<0$ #

TOPIC 15

TRIGONOMETRIC FUNCTIONS

(A) Radians and Degrees

Conversions :

From Degrees to Radians: $\times\dfrac{\pi}{180^\circ}$

From Radians to Degrees: $\times\dfrac{180^\circ}{\pi}$

Example 1: Convert each angle to radians, in terms of π :

(i) 30° (ii) 60° (iii) 75° (iv) 330°

Solution 1:

(i) $30^\circ = 30^\circ \times \dfrac{\pi}{180} = \dfrac{\pi\times 30}{180} = \dfrac{\pi}{6}$ #

(ii) $60^\circ = 60^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi\times 60}{180} = \dfrac{\pi}{3}$ #

(iii) $75^\circ = 75^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi\times 75}{180} = \dfrac{5\pi}{12}$ #

(iv) $330^\circ = 330^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi\times 330}{180} = \dfrac{11\pi}{6}$ #

Example 2: Convert the following radians to degrees (and minutes where applicable):

(i) $\dfrac{\pi}{4}$ (ii) $\dfrac{3\pi}{8}$ (iii) 2 (iv) 1

Solution 2:

(i) $\dfrac{\pi}{4} = \dfrac{\pi}{4}\times\dfrac{180^\circ}{\pi} = 45^\circ$ #

(ii) $\dfrac{3\pi}{8} = \dfrac{3\pi}{8}\times\dfrac{180^\circ}{\pi} = 67^\circ 30'$ #

(iii) $2 = 2\times\dfrac{180^\circ}{\pi} = 114^\circ 35'$ #

(iv) $1 = 1\times\dfrac{180^\circ}{\pi} = 57^\circ 18'$ #

(B) Solving Trigonometric Equations in Terms of Radians

Make the trigonometric ratio the subject of the equation and solve for all solutions in the given domain.

Example 1: Find all values of θ with $0 \le \theta \le 2\pi$, for which:

(i) $\sec\theta = \sqrt{2}$ (ii) $\sin\theta + \cos\theta = 0$

(iii) $1 - 2\sin 2\theta = 0$ (iv) $\cot^2\theta = \dfrac{1}{3}$

Solution 1:

(i) $\sec\theta = \sqrt{2}$

$\dfrac{1}{\cos\theta} = \sqrt{2}$

i.e. $\cos\theta = \dfrac{1}{\sqrt{2}}$, now $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$ and $45^\circ = \dfrac{\pi}{4}$

$\therefore\ \theta = \dfrac{\pi}{4}, \left(2\pi - \dfrac{\pi}{4}\right)$

$= \dfrac{\pi}{4}, \dfrac{7\pi}{4}$ #

(ii) $\sin\theta + \cos\theta = 0$

$\sin\theta = -\cos\theta$

$\tan\theta = -1$, now $\tan(45^\circ) = 1$ and $45^\circ = \dfrac{\pi}{4}$

noting tan is negative in the 2^{nd} and 4^{th} quadrants:

$\therefore\ \theta = \left(\pi - \dfrac{\pi}{4}\right), \left(2\pi - \dfrac{\pi}{4}\right)$

$= \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ #

(iii) $1 - 2\sin 2\theta = 0$

$2\sin 2\theta = 1$

$\sin 2\theta = \dfrac{1}{2}$, now $\sin 30^\circ = \dfrac{1}{2}$ and $30^\circ = \dfrac{\pi}{6}$

noting $0 \le 2\theta \le 4\pi$, gives:

$2\theta = \dfrac{\pi}{6}, \left(\pi - \dfrac{\pi}{6}\right), \left(2\pi + \dfrac{\pi}{6}\right), 2\pi + \left(\pi - \dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$

$\therefore\ \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$ #

Trigonometric Functions

(iv) $\cot^2\theta = \dfrac{1}{3}$

$\cot\theta = \pm\dfrac{1}{\sqrt{3}}$, noting $\cot\theta = \dfrac{1}{\tan\theta}$ gives:

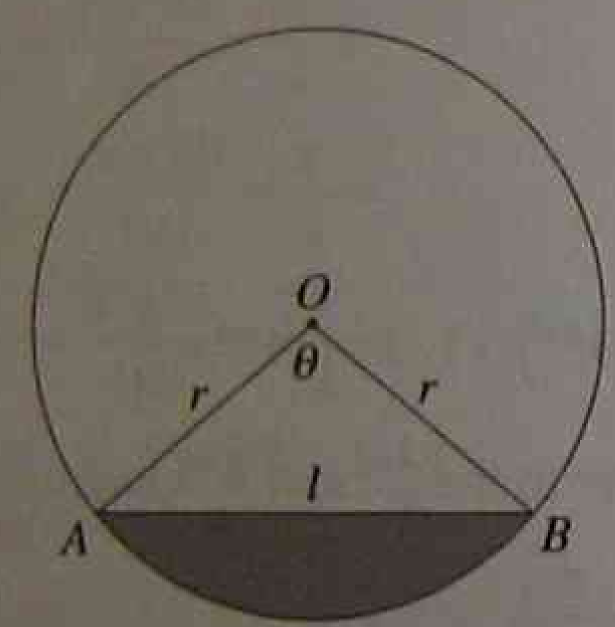
$\tan\theta = \pm\sqrt{3}$, noting $\tan 60^\circ = \sqrt{3}$ and $60^\circ = \dfrac{\pi}{3}$

$$\therefore\ \theta = \frac{\pi}{3}, \left(\pi - \frac{\pi}{3}\right), \left(\pi + \frac{\pi}{3}\right), \left(2\pi - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\ \#$$

(C) Arc Length, Area of a Sector and Area of a Segment

The use of radians gives a simple method for determining the length of an arc, area of a sector and area of a segment.

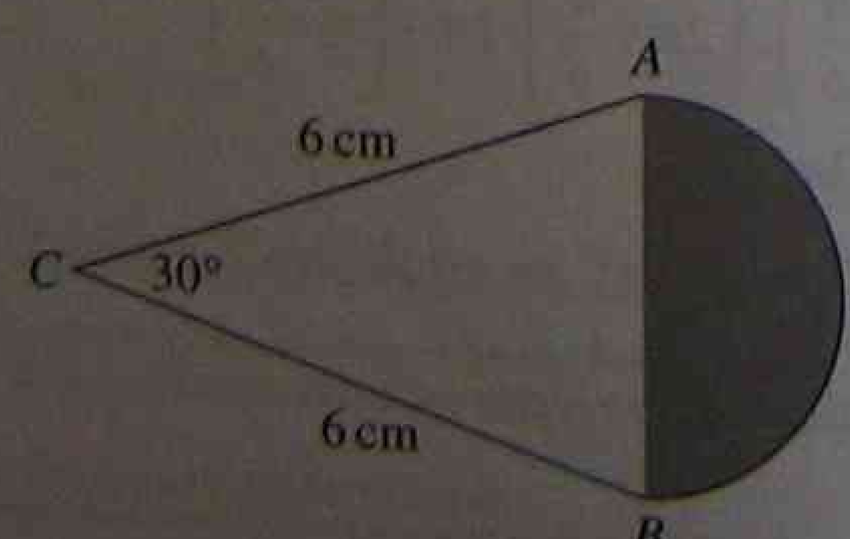


Length of Arc AB $= l = r\theta_{rad.}$ units

Area of sector AOB $= \dfrac{1}{2}r^2\theta_{rad.}$ units^2

Area of segment $= \dfrac{1}{2}r^2(\theta_{rad.} - \sin\theta)$ units^2 (shaded region)

Example 1: ABC is a sector of a circle with center C and radius 6 cm. $\angle ACB = 30^\circ$.



(i) Find the length of the arc AB.

(ii) Find the area of the sector ABC.

(iii) Find the area of the minor segment (i.e. the shaded region).

Solution 1:

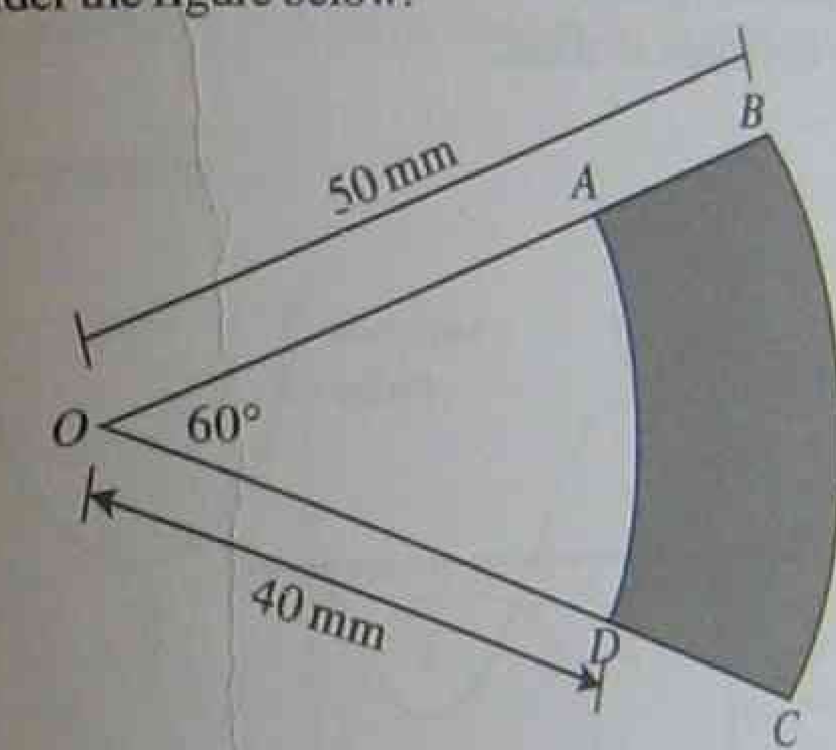
(i) Arc length, $l = r\theta_{rad.} = 6\times 30^\circ \times \dfrac{\pi}{180^\circ} = 6\times\dfrac{\pi}{6} = \pi$ cm #

(ii) Area of sector, $A = \dfrac{1}{2}r^2\theta_{rad.} = \dfrac{1}{2}\times 6^2 \times \dfrac{\pi}{6} = 3\pi$ cm^2 #

(iii) Area of minor segment $= \dfrac{1}{2}r^2(\theta_{rad.} - \sin\theta) = \dfrac{1}{2}(6)^2\left(\dfrac{\pi}{6} - \sin\dfrac{\pi}{6}\right)$

$=$ cm^2 #

Example 2: Consider the figure below:



Find: (i) The perimeter of the shaded region.

(ii) The area of the shaded region.

Express your answer in terms of π

Solution 2:

(i) Arc length $AD = r\theta_{rad.} = 40\times 60^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{40\pi}{3}$ mm

Arc length of $BC = r\theta_{rad.} = 50\times 60^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{50\pi}{3}$ mm

$AB = CD = 10$ mm

$\therefore$ perimeter of shaded area region $= 2\times 10 + \dfrac{40\pi}{3} + \dfrac{50\pi}{3}$

$= 20 + \dfrac{90\pi}{3} = (20 + 30\pi)$ mm #

(ii) Area of sector $OAD = \dfrac{1}{2}r^2\theta_{rad.} = \dfrac{1}{2}(40)^2\times\dfrac{\pi}{3} = \dfrac{800\pi}{3}$ mm^2

(iv) $\cot^2\theta = \frac{1}{3}$

$\cot\theta = \pm\frac{1}{\sqrt{3}}$, noting $\cot\theta = \frac{1}{\tan\theta}$ gives:

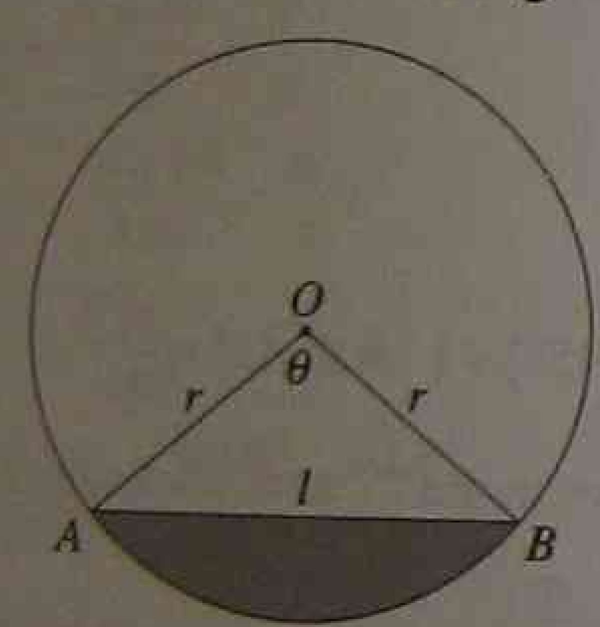
$\tan\theta = \pm\sqrt{3}$, noting $\tan 60° = \sqrt{3}$ and $60° = \frac{\pi}{3}$

$$\therefore\ \theta = \frac{\pi}{3}, \left(\pi - \frac{\pi}{3}\right), \left(\pi + \frac{\pi}{3}\right), \left(2\pi - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\ \#$$

(C) Arc Length, Area of a Sector and Area of a Segment

The use of radians gives a simple method for determining the length of an arc, area of a sector and area of a segment.

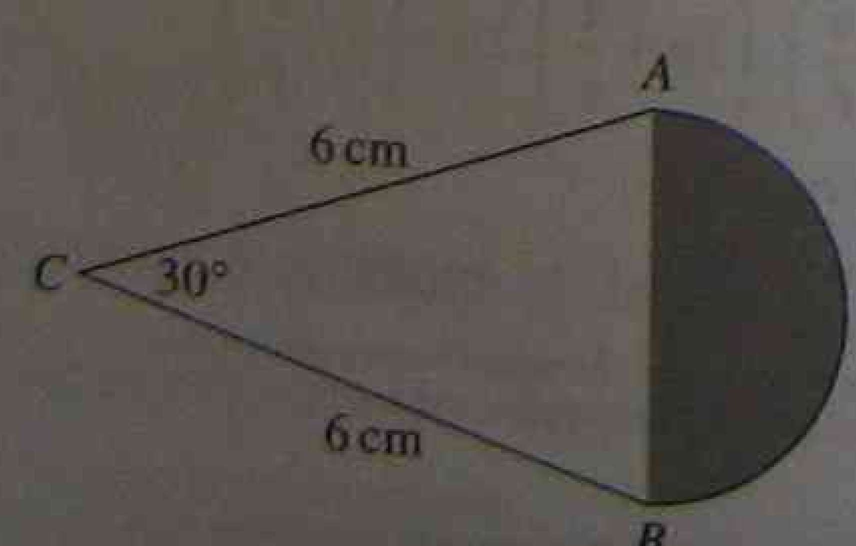


Length of Arc AB $= l = r\theta_{\text{rad.}}$ units

Area of sector AOB $= \frac{1}{2}r^2\theta_{\text{rad.}}$ units^2

Area of segment $= \frac{1}{2}r^2\left(\theta_{\text{rad.}} - \sin\theta\right)$ units^2 (shaded region)

Example 1: ABC is a sector of a circle with center C and radius 6 cm. $\angle ACB = 30°$.



(i) Find the length of the arc AB.

(ii) Find the area of the sector ABC.

(iii) Find the area of the minor segment (i.e. the shaded region).

Solution 1:

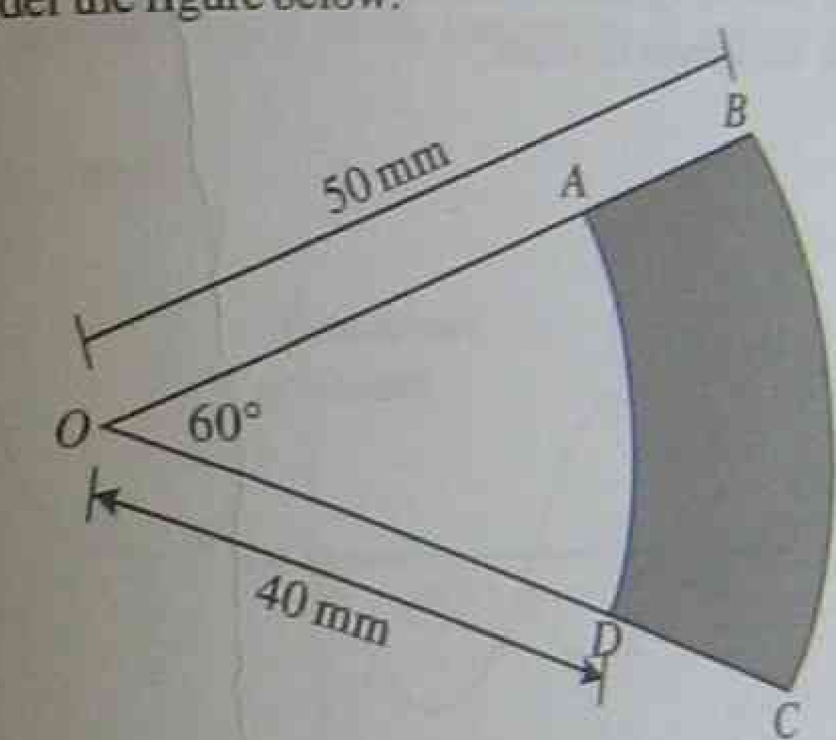
(i) Arc length, $l = r\theta_{\text{rad.}} = 6\times 30°\times\frac{\pi}{180°} = 6\times\frac{\pi}{6} = \pi$ cm #

(ii) Area of sector, $A = \frac{1}{2}r^2\theta_{\text{rad.}} = \frac{1}{2}\times 6^2\times\frac{\pi}{6} = 3\pi$ cm^2 #

(iii) Area of minor segment $= \frac{1}{2}r^2\left(\theta_{\text{rad.}} - \sin\theta\right) = \frac{1}{2}(6)^2\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$

$=$ cm^2 #

Example 2: Consider the figure below:



Find: (i) The perimeter of the shaded region.

(ii) The area of the shaded region.

Express your answer in terms of π

Solution 2:

(i) Arc length $AD = r\theta_{\text{rad.}} = 40\times 60°\times\frac{\pi}{180°} = \frac{40\pi}{3}$ mm

Arc length of $BC = r\theta_{\text{rad.}} = 50\times 60°\times\frac{\pi}{180°} = \frac{50\pi}{3}$ mm

$AB = CD = 10$ mm

$\therefore$ perimeter of shaded area region $= 2\times 10 + \frac{40\pi}{3} + \frac{50\pi}{3}$

$= 20 + \frac{90\pi}{3} = (20 + 30\pi)$ mm #

(ii) Area of sector $OAD = \frac{1}{2}r^2\theta_{\text{rad.}} = \frac{1}{2}(40)^2\times\frac{\pi}{3} = \frac{800\pi}{3}$ mm^2

$$\text{Area of sector } OBC = \frac{1}{2}r^2\theta_{\text{rad.}} = \frac{1}{2}(50)^2 \times \frac{\pi}{3} = \frac{1250\pi}{3}\text{ mm}^2$$

$$\therefore \text{ Area shaded region} = \frac{1250\pi}{3} - \frac{800\pi}{3} = \frac{450\pi}{3} = 150\pi \text{ mm}^2 \ \#$$

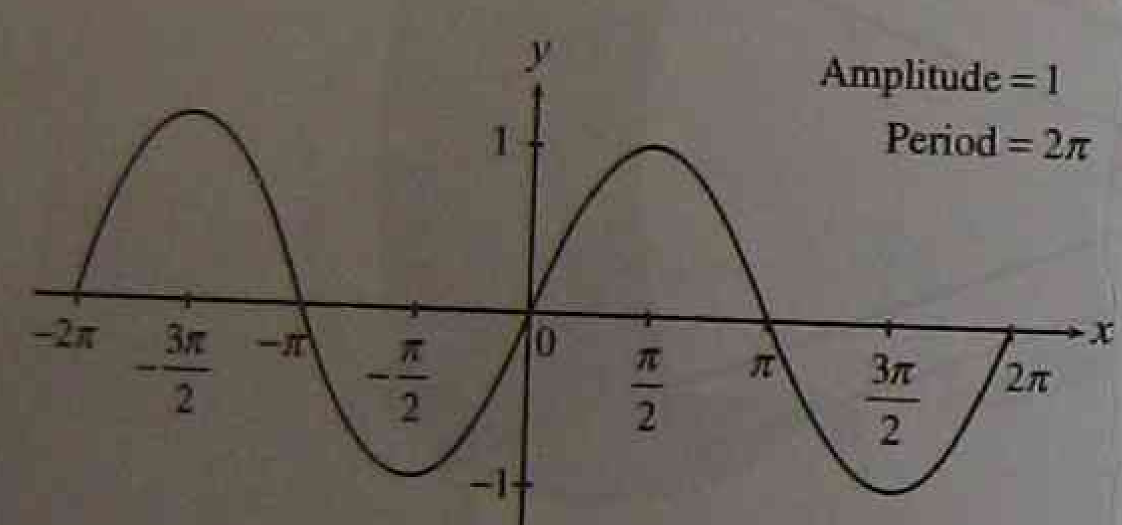
(D) Graphs of Trigonometric Functions

The graphs of trigonometric functions are periodic, i.e. they repeat themselves after a given interval called the period.

The amplitude is always measured as the distance from the middle value to the maximum value of the range.

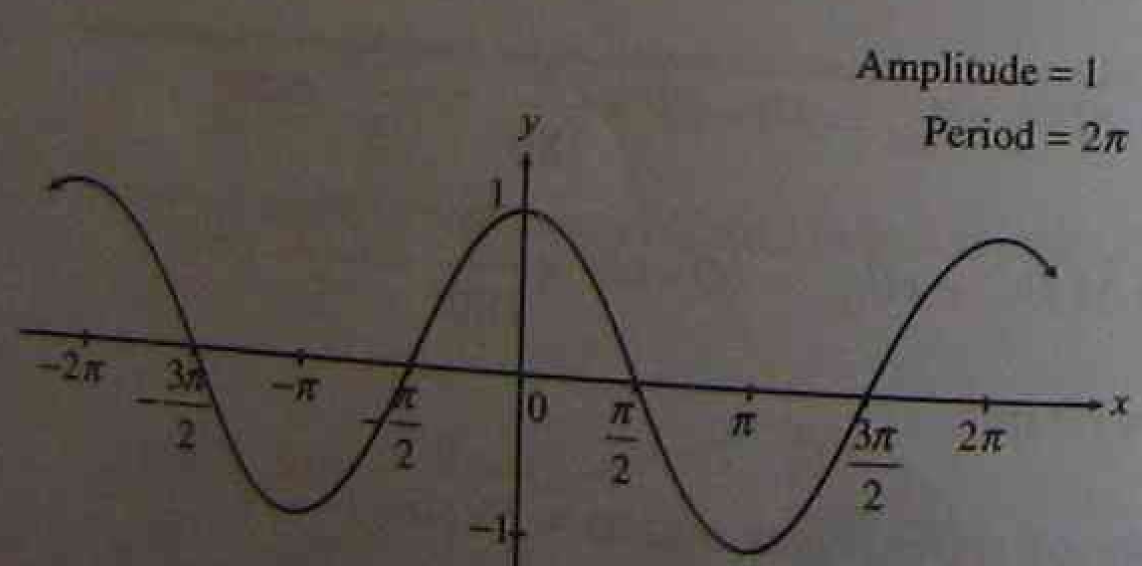
In this course, students need to be able to sketch the basic trigonometric functions and any slight variations to them.

(i) $y = \sin x$



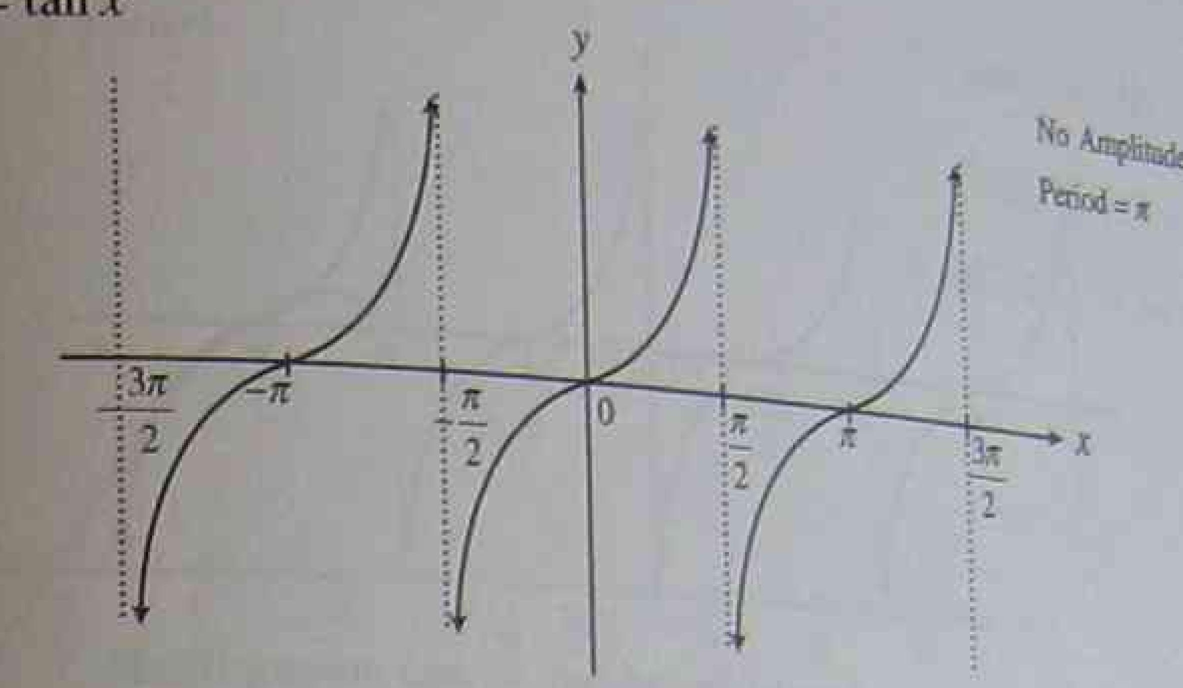
In general, $y = A\sin nx$ has period $\dfrac{2\pi}{n}$ and amplitude A.

(ii) $y = \cos x$



In general, $y = A\cos nx$ has period $\dfrac{2\pi}{n}$ and amplitude A.

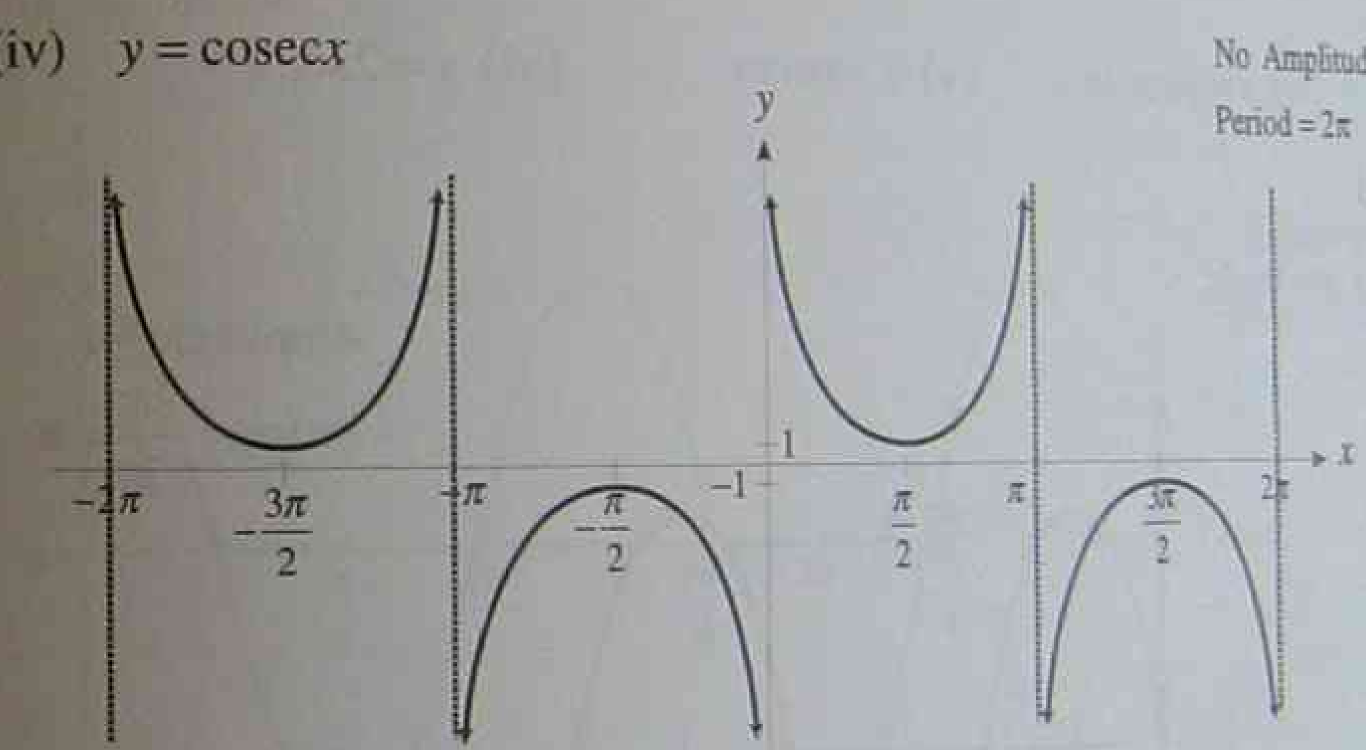
(iii) $y = \tan x$



Note that $y = A\tan nx$ has no amplitude as there is no maximum value, i.e. it is indefinite.

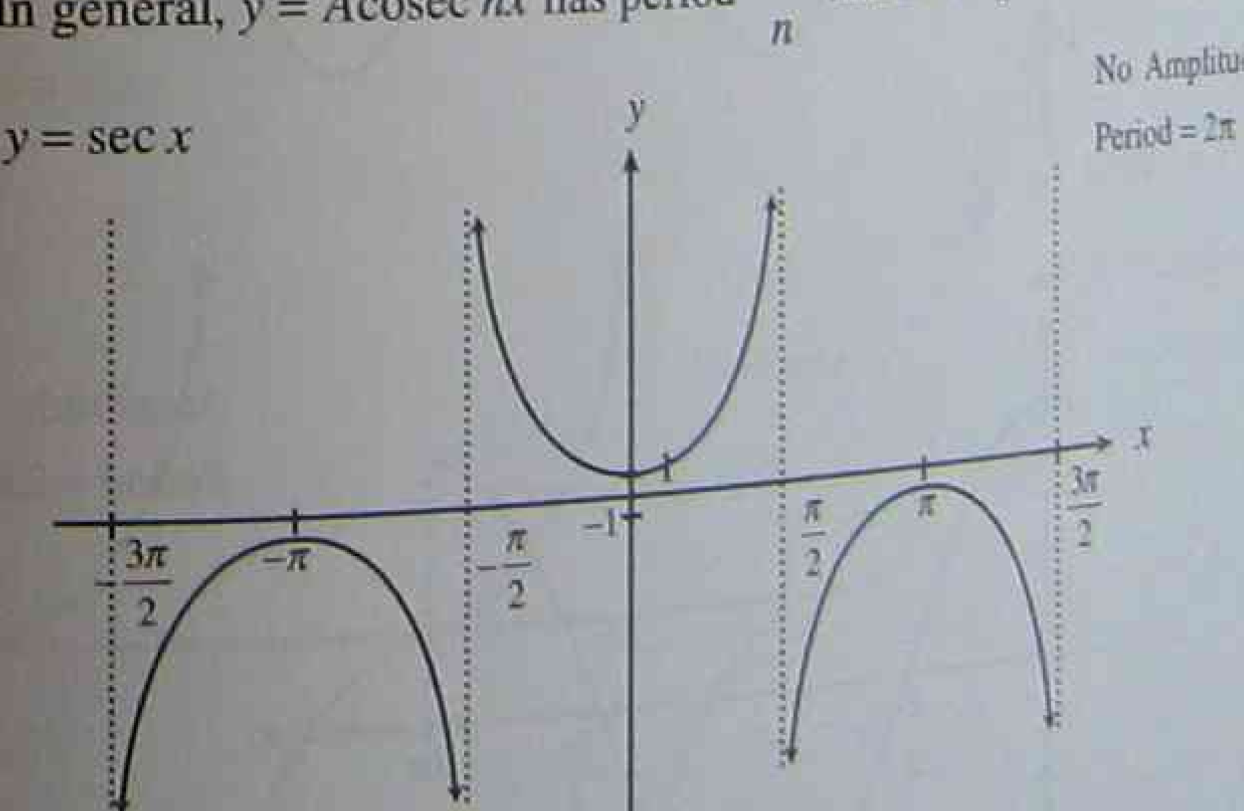
In general, $y = A\tan nx$ has period $\dfrac{\pi}{n}$ and no amplitude.

(iv) $y = \operatorname{cosec} x$



In general, $y = A\operatorname{cosec} nx$ has period $\dfrac{2\pi}{n}$ and no amplitude.

(v) $y = \sec x$

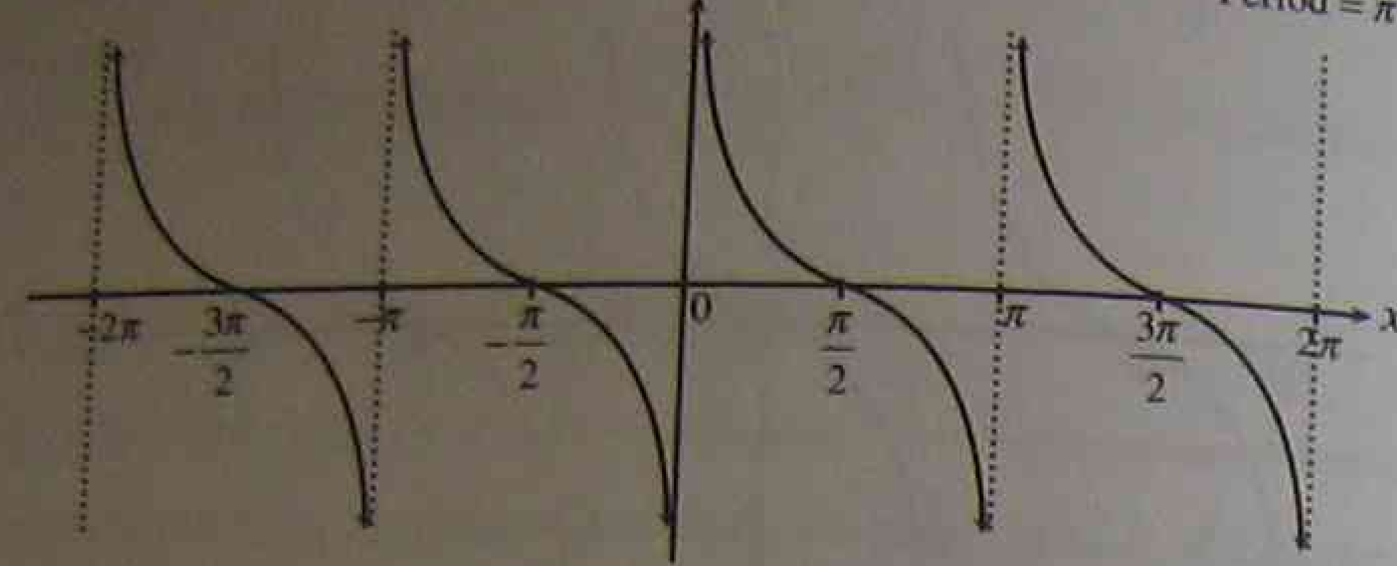


In general, $y = A\sec nx$ has period $\dfrac{2\pi}{n}$ and no amplitude.

(vi) $y = \cot x$

No Amplitude

Period $= \pi$



In general, $y = A\cot nx$ has period $\dfrac{\pi}{n}$ and no amplitude.

Example 1: Graph the following functions for $0 \le x \le 2\pi$:

(i) $y = \sin 2x$ (ii) $y = 4\cos x$ (iii) $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$

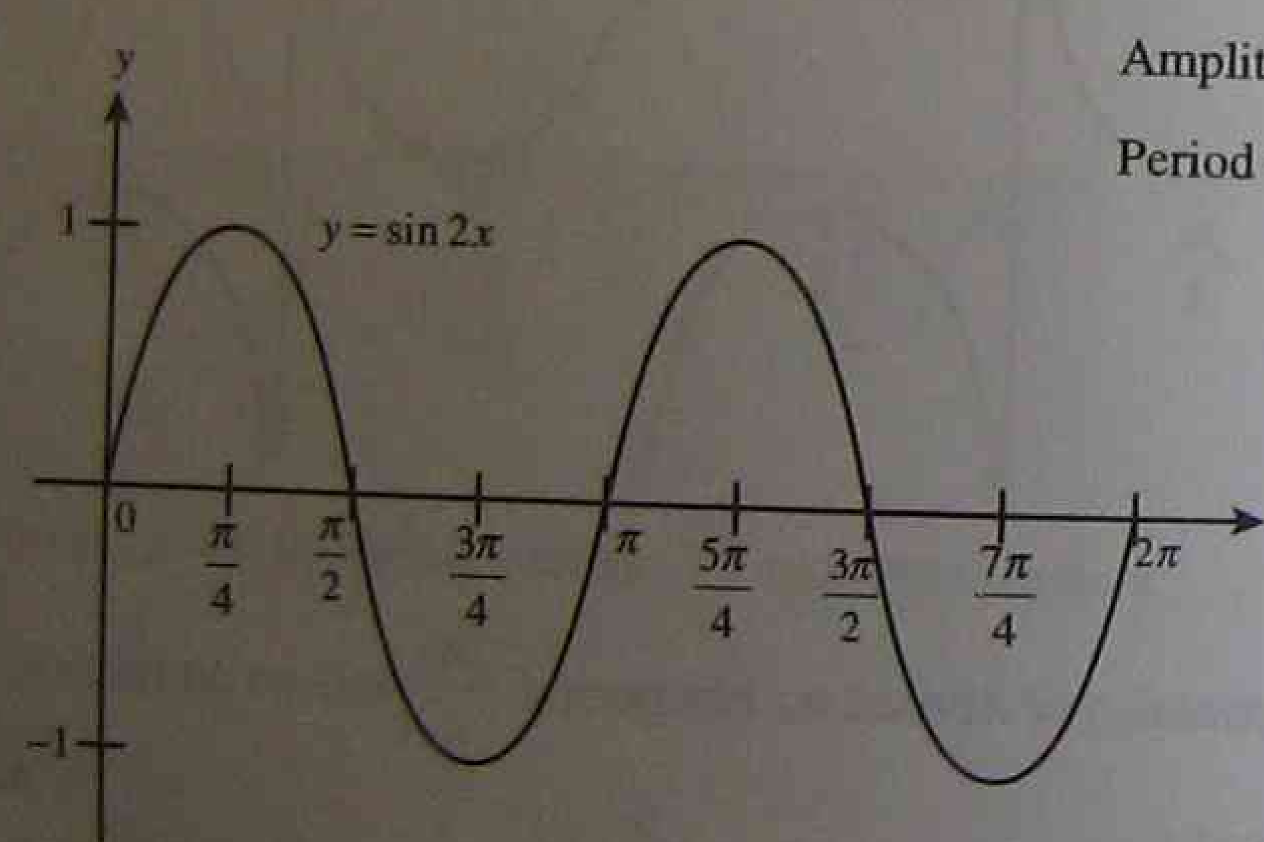
(iv) $y = 2\cos 2x$ (v) $y = \tan \pi x$ (vi) $y = 2\sec x$

Solution 1:

(i) $y = \sin 2x$

Amplitude $= 1$

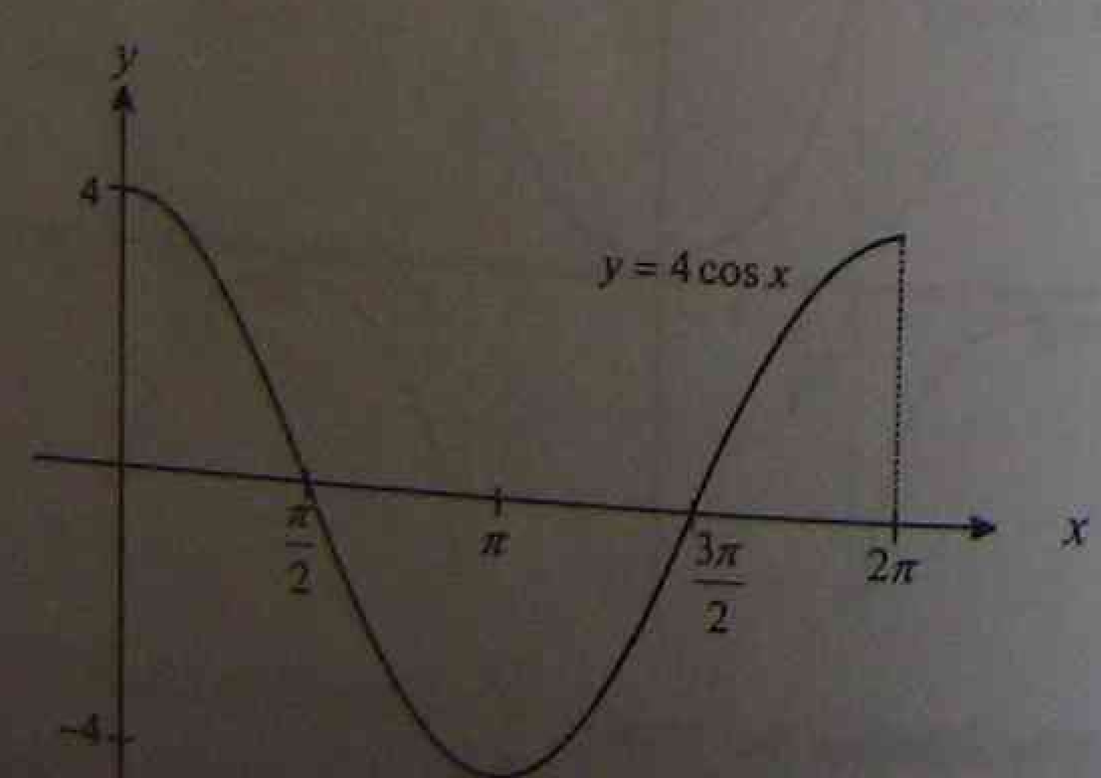
Period $= \dfrac{2\pi}{2} = \pi$



(ii) $y = 4\cos x$

Amplitude $= 4$

Period $= 2\pi$

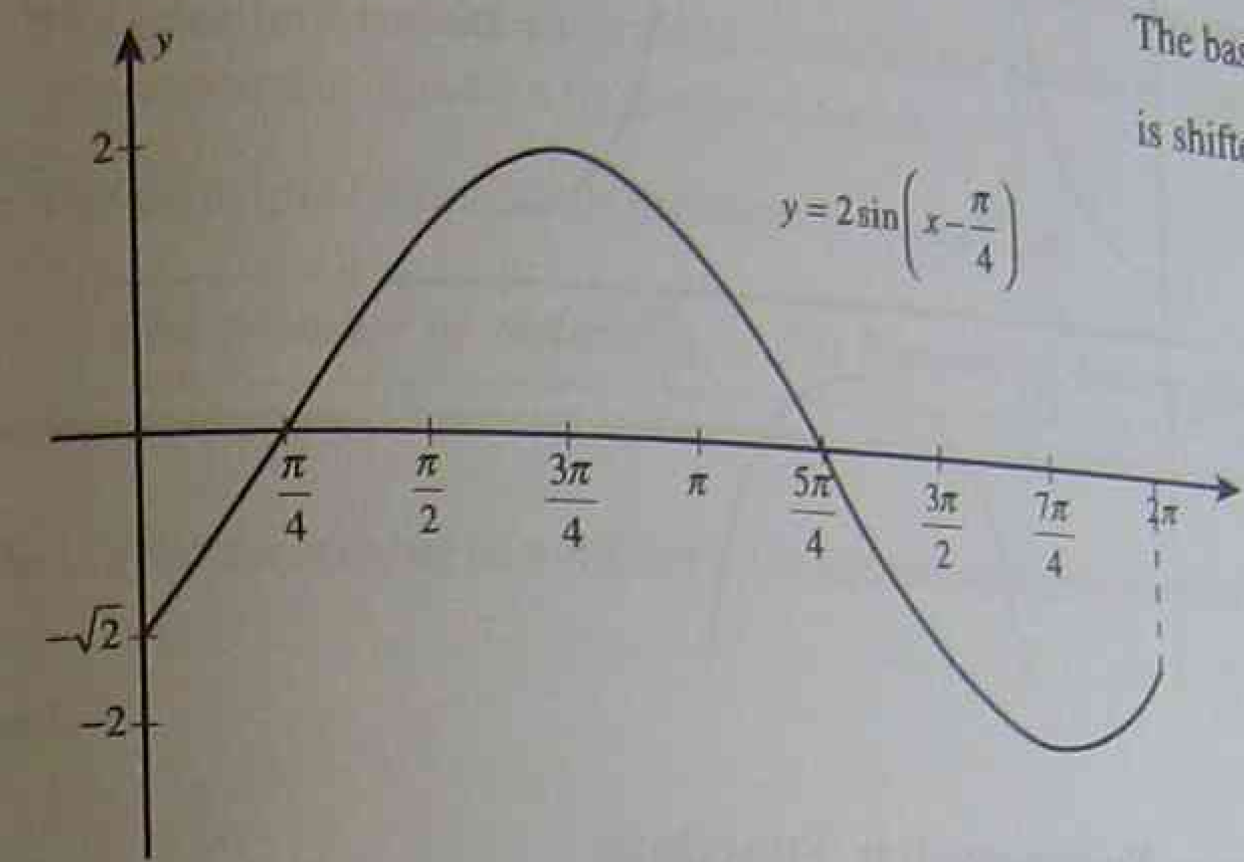


(iii) $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$

The basic curve $y = \sin x$ is shifted $\dfrac{\pi}{4}$ to the right.

Amplitude $= 1$

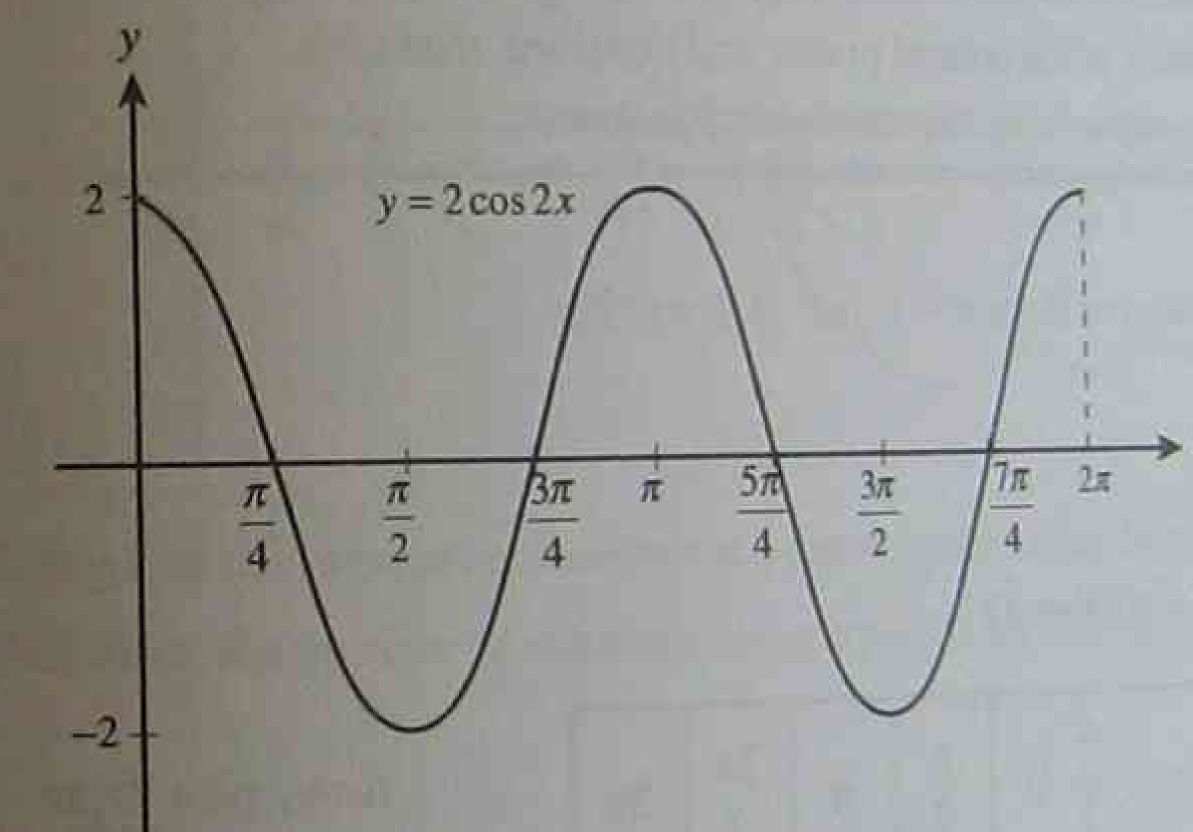
Period $= \dfrac{2\pi}{2} = \pi$



(iv) $y = 2\cos 2x$

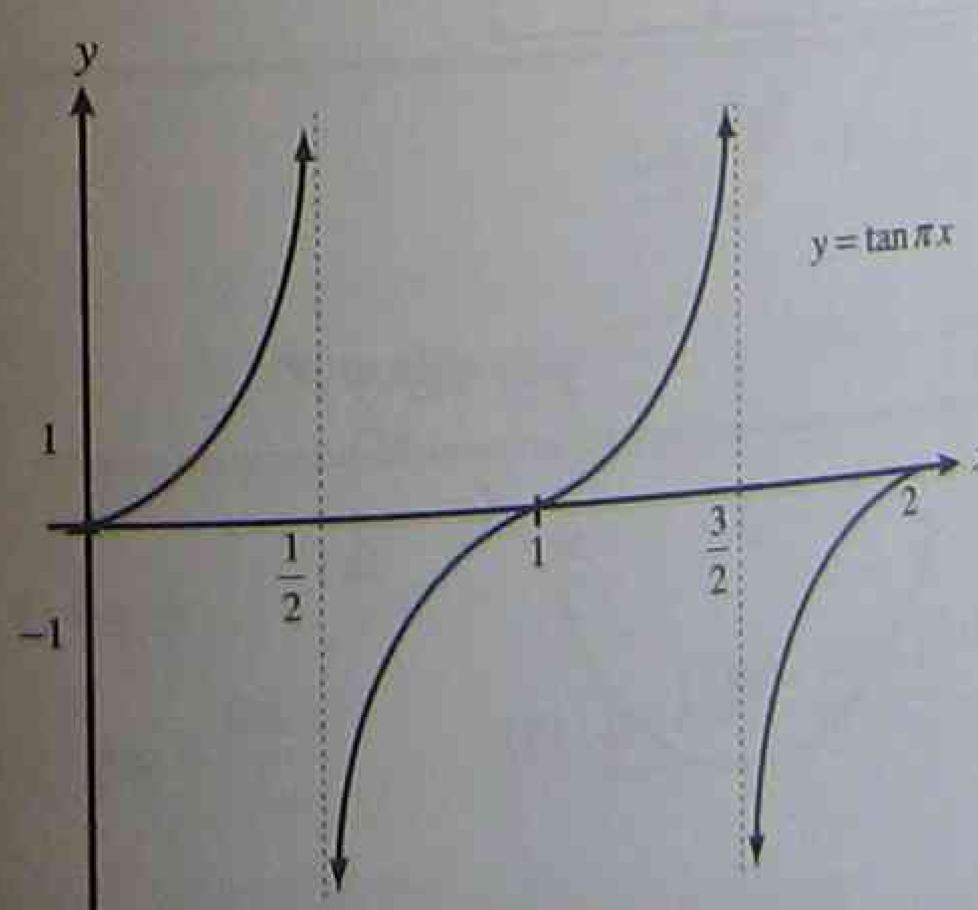
Amplitude $= 2$

Period $= \dfrac{2\pi}{2} = \pi$



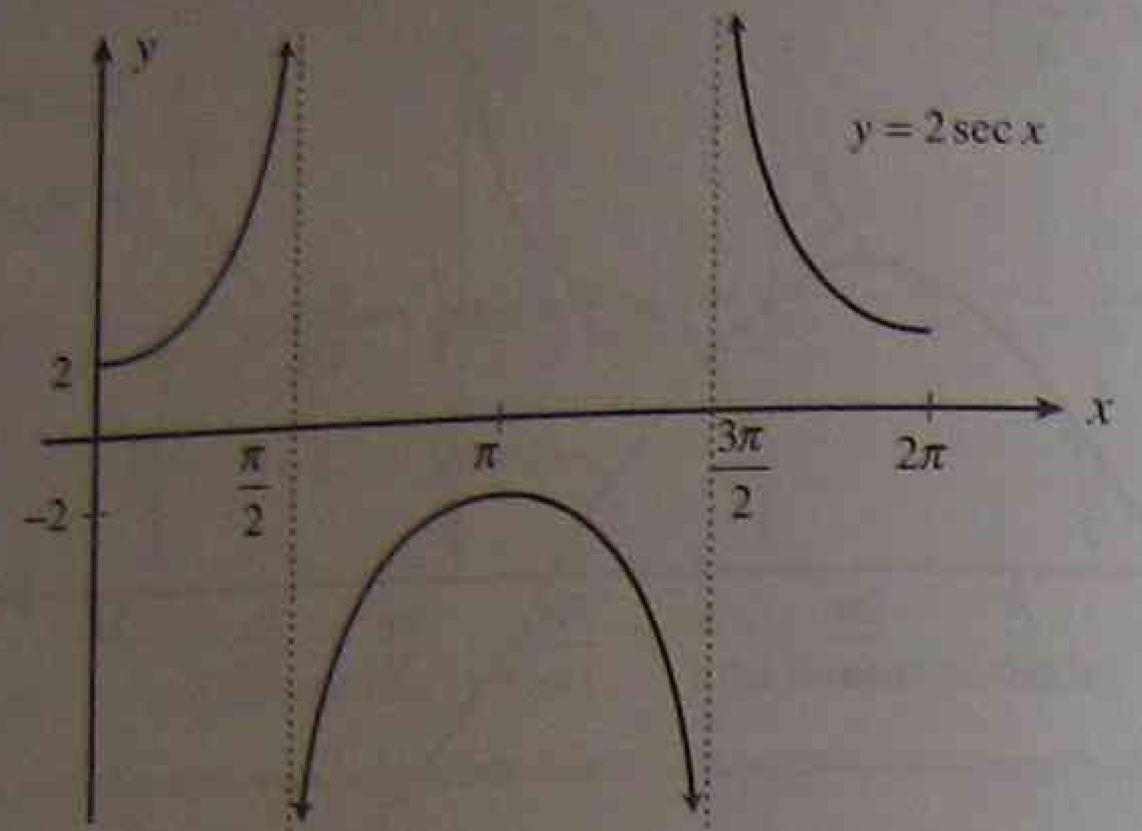
(v) $y = \tan \pi x$

Period $= \dfrac{\pi}{\pi} = 1$



(iv) $y = 2\sec x$

Period $= 2\pi$



(E) Sketching Harder Trigonometric Functions

The best method to use in sketching these graphs is to set up a table of values at the critical points and combine that with knowledge of the basic trigonometric functions.

Example 1: Sketch the curve $y = 2\cos x - 1$ for $0 \le x \le 2\pi$

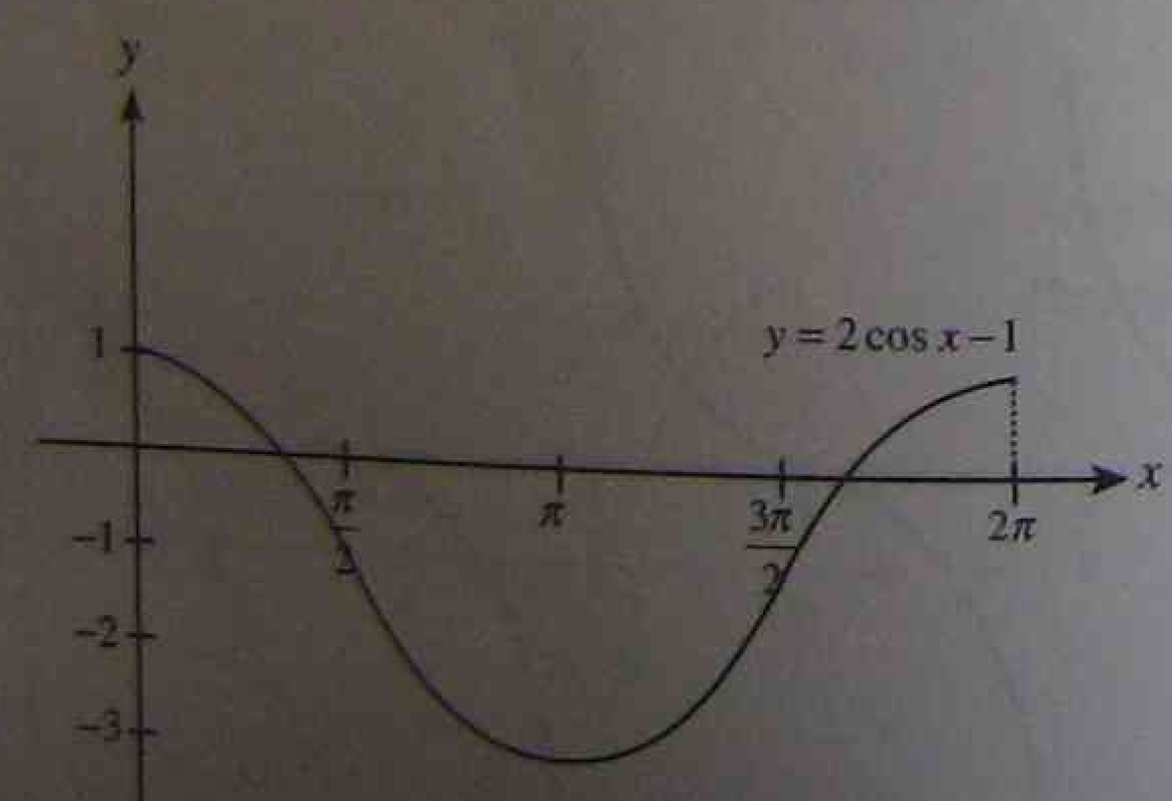
Solution 1:

$y = 2\cos x - 1$

Amplitude $= 2$, Period $= 2\pi$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$2\cos x - 1$	1	-1	-3	-1	1

now, plot these points and form the curve.



(F) Finding the Number of Solutions by Graphical Means

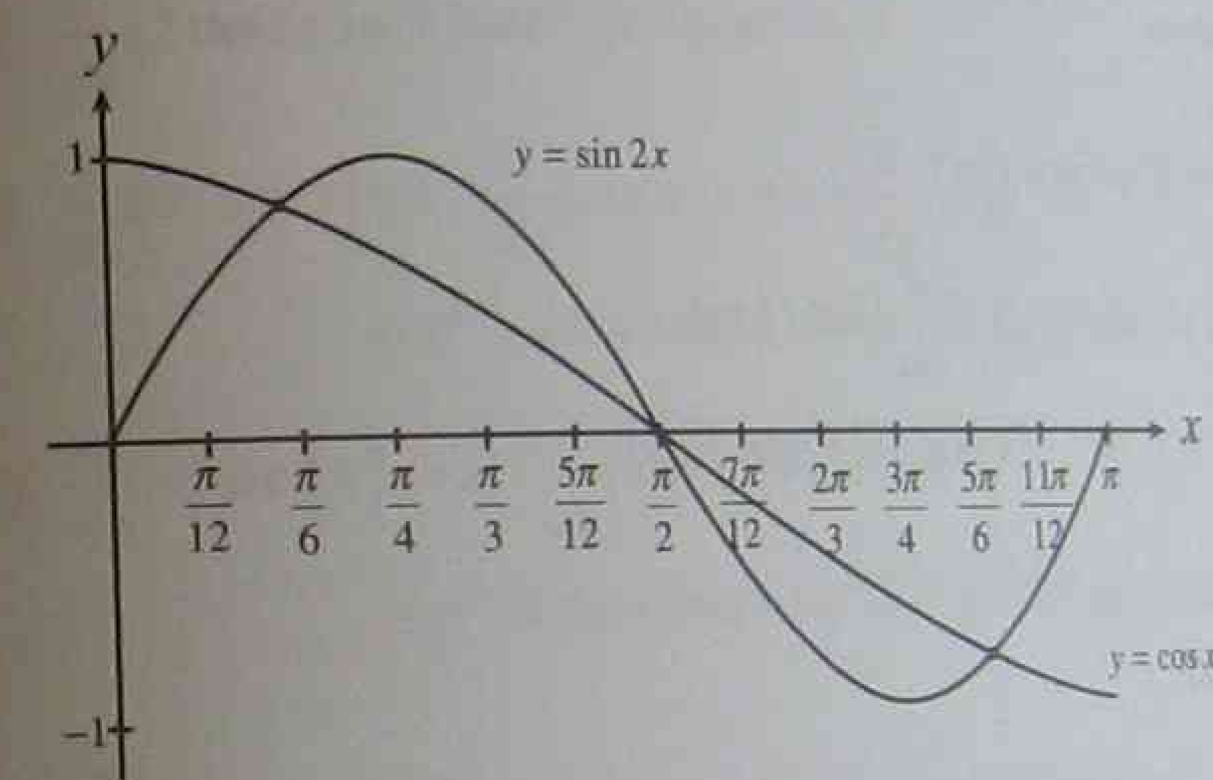
The questions require students to determine the number of solutions to a trigonometric equation by graphical means.

The principle behind these problems is the fact that:

The Number of Solutions = The Number of Intersection Points

Example 1: Find the number of solutions to the equation $\sin 2x = \cos x$ for $0 \le x \le \pi$.

Solution 1:



Notice that the two curves intersect at 3 distinct points over the required domain. Therefore, the number of solutions to the equation $\sin 2x = \cos x$ is 3. #

(G) $\lim\limits_{x\to 0} \frac{\sin x}{x}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Note also: $\lim\limits_{x\to 0} \frac{\sin x^n}{x^n} = 1$ and $\lim\limits_{x\to 0} \frac{\sin ax}{ax} = 1$

Example 1: Find:

(i) $\lim\limits_{x\to 0} = \frac{\sin x}{5x}$ (ii) $\lim\limits_{x\to 0} \frac{(2\sin x \cos x)^2}{x^2}$ (iii) $\lim\limits_{x\to 0} \frac{2\sin 3x}{x}$

Trigonometric Functions

Solution 1:

(i) $\lim_{x\to 0}\frac{\sin x}{5x}=\lim_{x\to 0}\frac{1}{5}\times\frac{\sin x}{x}=\frac{1}{5}\times 1=\frac{1}{5}$ #

(ii) $\lim_{x\to 0}\frac{(2\sin x\cos x)^2}{x^2}=\lim_{x\to 0}\frac{4\sin^2 x\cos^2 x}{x^2}=\lim_{x\to 0}4\times\frac{\sin^2 x}{x^2}\times\cos^2 x=4\times 1\times 1=4$ #

(iii) $\lim_{x\to 0}\frac{2\sin 3x}{x}=\lim_{x\to 0}6\times\frac{\sin 3x}{3x}=6\times 1=6$ #

(H) Differentiation of Trigonometric Functions

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\sin f(x))=f'(x)\cos f(x)$$

$$\frac{d}{dx}(\sin^n f(x))=n\sin^{n-1}f(x)\frac{d}{dx}(\sin f(x))$$
$$=n\sin^{n-1}f(x)\times f'(x)\cos f(x)$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\cos f(x))=-f'(x)\sin f(x)$$

$$\frac{d}{dx}(\cos^n f(x))=n\cos^{n-1}f(x)\frac{d}{dx}(\cos f(x))$$
$$=n\cos^{n-1}f(x)\times -f'(x)\sin f(x)$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\tan f(x))=f'(x)\sec^2 f(x)$$

$$\frac{d}{dx}(\tan^n f(x))=n\tan^{n-1}f(x)\frac{d}{dx}(\tan f(x))$$
$$=n\tan^{n-1}f(x)\times f'(x)\sec^2 f(x)$$

Example 1: Differentiate the following:

(i) $\sin 2x$ (ii) $\sin^3 x$ (iii) $\sqrt{\cos x}$

(iv) $2\tan 3x$ (v) $x^3\cos x$ (vi) $e^x\sin^2 x$

(vii) $(\cos x+\sin x)^2$ (viii) $\frac{\tan x}{x}$ (ix) $\sin^3 3x$

Solution 1:

(i) $\frac{d}{dx}(\sin 2x)=2\cos 2x$ #

(ii) $\frac{d}{dx}(\sin^3 x)=3\sin^2\cdot\frac{d}{dx}(\sin x)=3\sin^2 x\cos x$ #

(iii) $\frac{d}{dx}(\sqrt{\cos x})=\frac{d}{dx}\left[(\cos x)^{\frac{1}{2}}\right]=\frac{1}{2}(\cos x)^{-\frac{1}{2}}\cdot\frac{d}{dx}(\cos x)$

$=\frac{1}{2\sqrt{\cos x}}\cdot-\quad x=\frac{-\sin x}{2\sqrt{\cos x}}$

(iv) $\frac{d}{dx}(2\tan 3x)=2.3\sec^2 3x=6\sec^2 3x$ #

(v) $\frac{d}{dx}(x^3\cos x)=x^3\cdot\frac{d}{dx}(\cos x)+\cos x.\frac{d}{dx}(x^3)$

$=x^3\cdot-\sin x+\cos x\cdot 3x^2=-x^3\sin x+3x^2\cos x$ #

(vi) $\frac{d}{dx}(e^x\sin^2 x)=e^x\cdot\frac{d}{dx}(\sin^2 x)+\sin^2 xe^x$

$=e^x\cdot 2\sin x\cdot\cos x+e^x\sin^2 x=2e^x\sin x\cos x+e^x\sin^2 x$ #

(vii) $\frac{d}{dx}(\cos x+\sin x)^2=2(\cos x+\sin x)^1\cdot\frac{d}{dx}(\cos x+\sin x)$

$=2(\cos x+\sin x)(\cos x-\sin x)=2(\cos^2 x-\sin^2 x)$ #

(viii) $\frac{d}{dx}\left(\frac{\tan x}{x}\right)=\frac{x\cdot\frac{d}{dx}(\tan x)-\tan x\cdot\frac{d}{dx}(x)}{x^2}=\frac{x\sec^2 x-\tan x}{x^2}$ #

(ix) $\frac{d}{dx}(\sin^3 3x)=3\sin^2 3x\cdot\frac{d}{dx}(\sin 3x)=3\sin^2 3x\cdot 3\cos 3x=9\sin^2 3x\cos 3x$ #

(I) Integration of Trigonometric Functions

$$\int\sin x\,dx=-\cos x+C$$

$$\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$$

$$\int\cos x\,dx=\sin x+C$$

$$\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

Example 1: Find:

(i) $\int \sin\left(\frac{x}{2}\right) dx$ (ii) $\int 2\cos(3x+1)\,dx$ (iii) $\int \frac{\sec^2 x}{2}\, dx$

(iv) $\int_0^{\frac{\pi}{4}} \sec^2 3x \, dx$ (v) $\int_{\frac{\pi}{3}}^{\pi} 3\cos\left(\frac{x}{2}\right) dx$

Solution 1:

(i) $\int \sin\left(\frac{x}{2}\right) dx = -\frac{1}{\frac{1}{2}}\cdot\cos\left(\frac{x}{2}\right) + C = -2\cos\left(\frac{x}{2}\right) + C$ #

(ii) $\int 2\cos(3x+1)\, dx = 2.\frac{1}{3}\cdot\sin(3x+1) + C = \frac{2}{3}\sin(3x+1) + C$ #

(iii) $\int \frac{\sec^2 x}{2} dx = \frac{1}{2}\int \sec^2 x \, dx = \frac{1}{2}\tan x + C$ #

(iv) $\int_0^{\frac{\pi}{4}} \sec^2 3x \, dx = \left[\frac{1}{3}\tan 3x\right]_0^{\frac{\pi}{4}} = \frac{1}{3}\left[\tan\frac{3\pi}{4} - \tan 0\right] = \frac{1}{3}(-1-0) = -\frac{1}{3}$ #

(v) $\int_{\frac{\pi}{3}}^{\pi} 3\cos\left(\frac{x}{2}\right) dx = 3\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{x}{2}\right) dx$

$$= 3\left[\frac{1}{\frac{1}{2}}\sin\left(\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\pi}$$

$$= 6\left[\sin\left(\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\pi} = 6\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right] = 6\left(1 - \frac{1}{2}\right) = 3 \text{ \#}$$

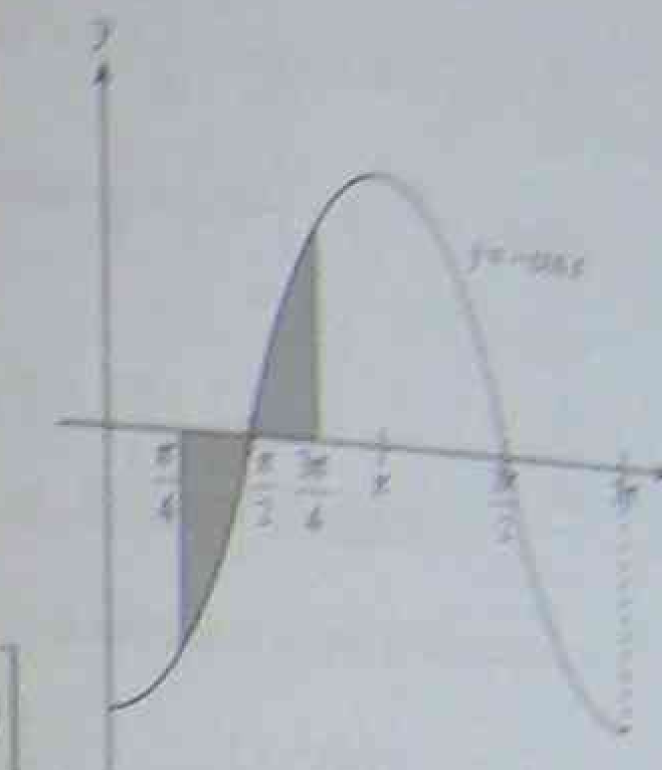
(J) Applications of Differentiation and Integration

Differentiation and integration techniques can be applied to questions involving trigonometric functions.

Example 1: Find the area bounded by the curve $y = -\cos x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Express your answer in surd form.

Solution 1:

From the graph, part of the curve is below the x-axis:



$$\therefore \text{ Area } = \left|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\cos x \, dx\right| + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\cos x \, dx$$

$$= \left|\left[-\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right| + \left[-\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$$

$$= \left|\left[-1 - \left(-\frac{1}{\sqrt{2}}\right)\right]\right| + \left[-\frac{1}{\sqrt{2}} - (-1)\right]$$

$$= \left|-1 + \frac{1}{\sqrt{2}}\right| + 1 - \frac{1}{\sqrt{2}} \quad \left(\text{noting: } -1 + \frac{1}{\sqrt{2}} \text{ is negative}\right)$$

$$= -\left(-1 + \frac{1}{\sqrt{2}}\right) + 1 - \frac{1}{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}$$

$$= 2 - \frac{2}{\sqrt{2}}$$

$$= 2 - \sqrt{2} \text{ units}^2$$

Example 2: Find the volume generated when the region bounded by the x-axis and the curve $y = 2\sec 2x$ from $x = 0$ to $x = \frac{\pi}{8}$ is rotated about the x-axis to form a solid.

Solution 2:

$$V = \int_a^b \pi y^2 dx, \qquad y = 2\sec 2x \quad \therefore \quad y^2 = 4\sec^2 2x$$

$$= \pi \int_0^{\frac{\pi}{8}} 4\sec^2 2x \; dx$$

$$= 4\pi \int_0^{\frac{\pi}{8}} \sec^2 2x \, dx = 4\pi\left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}} = 2\pi\left[\tan\frac{\pi}{4} - \tan\theta\right] = 2\pi \text{ units}^3$$

Example 3: Find the equation of the tangent to the curve $y = x\tan x$ at $x = \frac{\pi}{4}$.

Solution 3:

$$y = x\tan x$$

$$\frac{dy}{dx} = x \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(x) = x\sec^2 x + \tan x$$

when $x = \frac{\pi}{4}$: $y = \frac{\pi}{4}\tan\frac{\pi}{4} = \frac{\pi}{4}$ and

$$\frac{dy}{dx} = \frac{\pi}{4}\cdot\left(\frac{1}{\cos\frac{\pi}{4}}\right)^2 + \tan\frac{\pi}{4} = \frac{\pi}{4}\left(\sqrt{2}\right)^2 + 1 = \frac{\pi}{2} + 1$$

$\therefore$ equation of the curve is given by:

$$(y - y_1) = m(x - x_1)$$

$$\left(y - \frac{\pi}{4}\right) = \left(\frac{\pi}{2} + 1\right)\left(x - \frac{\pi}{4}\right)$$

$$y - \frac{\pi}{4} = \left(\frac{\pi}{2} + 1\right)x - \frac{\pi^2}{8} - \frac{\pi}{4}$$

$$\therefore \quad y = \left(\frac{\pi}{2} + 1\right)x - \frac{\pi^2}{8}$$

REVIEW EXERCISES

(A) Radians and Degrees

1. Convert the follwing degrees to radians in term of π:

   (i) $45°$ (ii) $150°$ (iii) $240°$ (iv) $300°$

2. Convert the following radian measures to degrees:

   (i) π (ii) $\frac{2\pi}{3}$ (iii) $\frac{\pi}{12}$ (iv) $300°$

(B) Solving Trigonometric Equations in Terms of Radians

3. Find all values of θ for $0 \le \theta \le 2\pi$:

   (i) $2\cos\theta = \sqrt{3}$ (ii) $2\sin\left(\theta - \frac{\pi}{4}\right)$

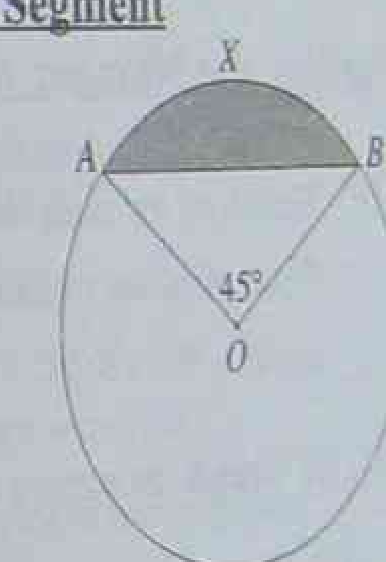
   (iii) $\sec^2\left(\frac{\theta}{2}\right) = 2$ (iv) $\sin 2\theta = \cos 2\theta$

(C) Arc Length, Area of a Sector and Area of a Segment

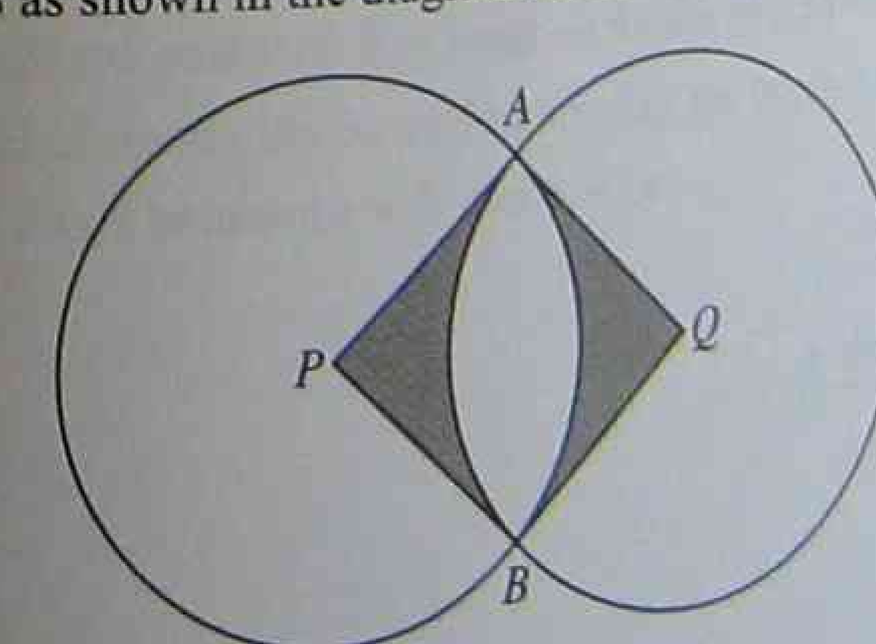
4. Given that the arc $AXB = 2\pi$ cm and subtends an angle of $45°$ at O. Find:

   (i) The radius of the circle.

   (ii) The area of the shaded region.



5. Two circles centres P and Q respectively and radii 3 cm intersect each other at A and B as shown in the diagram above. Given AB = PQ:



   (i) Find the area of the sector PAB.

   (ii) Hence find the area of the shaded region.

6. The sector OAB of a circle centre O and radius r has an area of $\frac{3\pi}{4}$ cm^2.
If the arc AB subtends an angle of $30°$ at O, find the length of the arc AB and the radius of the circle r.

(D) Graphs of Trigonometric Functions

7. Graph the following functions for $0 \le x \le 2\pi$:

(i) $y = 2\sin x$ (ii) $y = \tan\left(x + \frac{\pi}{2}\right)$

(iii) $y = \frac{1}{2}\sin 2x$ (iv) $y = 2\cos\left(\frac{x}{2}\right)$

(E) Sketching Harder Trigonometric Functions

8. Sketch the curve $y = 1 + 3\sin 2x$ for $0 \le x \le 2\pi$.

9. On the same diagram sketch the graphs of $y = \sin x$ and $y = \cos x$ and hence the curve $y = \sin x + \cos x$ over the domain $0 \le x \le 2\pi$.

(F) Finding the Number of Solutions by Graphical Means

10. Sketch on the same diagrams the curves $y = \cos 2x$ and $y = \sin x$ for $0 \le x \le 2\pi$. Hence, find the number of solutions to the equation $\cos 2x = \sin x$ over the domain $0 \le x \le 2\pi$.

11. (i) Sketch the curve $y = 1 - 2\sin \pi x$ for $0 \le x \le 2$.

(ii) On the same set of axes sketch the line $y = x - 1$.

(iii) Hence write down the number of solutions to the equation $1 - 2\sin \pi x = x - 1$ for $0 \le x \le 2$.

(G) $\lim\limits_{x \to 0} \frac{\sin x}{x}$

12. Find $\lim\limits_{x \to 0} \frac{\sin 5x \sin x}{x^2}$

(H) Differentiation of Trigonometric Functions

13. Differentiate the following:

(i) $3\cos 2x$ (ii) $\frac{\sin x}{x}$ (iii) $\tan^3(2x+5)$

(iv) $(x^2+5)\tan 2x$ (v) $\cos x^3$ (vi) $\sqrt{e^{-x}\sin x}$

(I) Integration of Trigonometric Functions

14. Find:

(i) $\int \sin 2x - x \, dx$ (ii) $\int \frac{\cos 5x}{5} \, dx$ (iii) $\int \tan^2\left(\frac{x}{2}\right) dx$

(iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x \, dx$ (v) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$ (vi) $\int_{\frac{1}{3}}^{\frac{1}{2}} \sin \pi x \, dx$

(J) Applications of Differentiation and Integration

15. Find the equation of normal to the curve $y = \sin^2 x$ at $x = \frac{\pi}{3}$

16. Find the area bounded by the curve $y = \sin 2x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

17. Find the volume of the solid of revolution formed when the curve $y = 2\sqrt{\cos x}$ is rotated about the x-axis between the ordinates $x = 0$ and $x = \frac{\pi}{2}$.

18. Apply Simpson's rule with five function values to find the volume of the solid of revolution formed by rotating the curve $y = \sin x + \cos x$ about the x-axis between $x = 0$ and $x = 2\pi$.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $45° = 45° \times \dfrac{\pi}{180°} = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$ #

(ii) $150° = 150° \times \dfrac{\pi}{180°} = \dfrac{150\pi}{180} = \dfrac{5\pi}{6}$ #

(iii) $240° = 240° \times \dfrac{\pi}{180°} = \dfrac{4\pi}{3}$ #

(iv) $300° = 300° \times \dfrac{\pi}{180°} = \dfrac{300\pi}{180} = \dfrac{5\pi}{3}$ #

2. (i) $\pi = \pi \times \dfrac{180°}{\pi} = 180°$ #

(ii) $\dfrac{2\pi}{3} = \dfrac{2\pi}{3} \times \dfrac{180°}{\pi} = \dfrac{360°}{3} = 120°$ #

(iii) $\dfrac{\pi}{12} = \dfrac{\pi}{12} \times \dfrac{180°}{\pi} = \dfrac{180°}{12} = 15°$ #

(iv) $3 = 3 \times \dfrac{180°}{\pi} = 171°53'$ #

3. (i) $2\cos\theta = \sqrt{3}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$, noting $\cos 30° = \dfrac{\sqrt{3}}{2}$

$\therefore\ \theta = \dfrac{\pi}{6}, \dfrac{11\pi}{6}$ #

(ii) $2\sin\left(\theta - \dfrac{\pi}{4}\right) = 1$

$\sin\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{1}{2}$, noting $\sin 30° = \dfrac{1}{2}$

$\therefore\ \theta - \dfrac{\pi}{4} = \dfrac{\pi}{6}, \left(\pi - \dfrac{\pi}{6}\right)$

$= \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore\ \theta = \left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right), \left(\dfrac{5\pi}{6} + \dfrac{\pi}{4}\right)$

$= \dfrac{5\pi}{12}, \dfrac{13\pi}{12}$ #

(iii) $\sec^2\left(\dfrac{\theta}{2}\right) = 2$

$\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1}{2}$

$\cos\left(\dfrac{\theta}{2}\right) = \pm\dfrac{1}{\sqrt{2}}$, noting $\cos 45° = \dfrac{1}{\sqrt{2}}$

$\therefore\ \cos\left(\dfrac{\theta}{2}\right) = \dfrac{1}{\sqrt{2}}$ or $\cos\left(\dfrac{\theta}{2}\right) = -\dfrac{1}{\sqrt{2}}$

since $0 \le \theta \le 2\pi\ \therefore\ 0 \le \dfrac{\theta}{2} \le \pi$

i.e. $\dfrac{\theta}{2} = \dfrac{\pi}{4}, \left(\pi - \dfrac{\pi}{4}\right)$

$\therefore\ \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

$\therefore\ \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ #

(iv) $\sin 2\theta = \cos 2\theta$

$\tan 2\theta = 1$, noting $\tan 45° = 1$

and since $0 \le \theta \le 2\pi$

$0 \le 2\theta \le 4\pi$

i.e. $2\theta = \dfrac{\pi}{4}, \left(\pi + \dfrac{\pi}{4}\right), \left(2\pi + \dfrac{\pi}{4}\right), \left(3\pi + \dfrac{\pi}{4}\right)$

$= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$

$\therefore\ \theta = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$ #

4. (i) $l = r\theta_{\text{rad.}}$

$2\pi = r \times 45° \times \dfrac{\pi}{180°}$

$= r \times \dfrac{\pi}{4}$

$\therefore\ r = 8$ #

(ii) Area $= \dfrac{1}{2}r^2(\theta_{\text{rad.}} - \sin\theta)$

$= \dfrac{1}{2}(8)^2\left(\dfrac{\pi}{4} - \sin\dfrac{\pi}{4}\right)$

$= \dfrac{1}{2} \times 64\left(\dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}\right)$

$= 32\left(\dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}\right)$

$= 2.51\ \text{cm}^2$ correct to 2 d.p. #

5. (i) Quadrilateral $AQBP$ is a square since all sides and diagnals are equal in length. $\therefore\ \angle APB = 90°$

Area of sector $PAB = \dfrac{1}{2}r^2\theta_{\text{rad.}} = \dfrac{1}{2}(3)^2 90° \times \dfrac{\pi}{180°} = \dfrac{9\pi}{4}\ \text{cm}^2$ #

(ii) Area of shaded region $= 2 \times$ Area of minor segment AB

$= 2 \times \dfrac{1}{2}r^2(\theta_{\text{rad.}} - \sin\theta)$

$= (3)^2\left(\dfrac{\pi}{2} - \sin\dfrac{\pi}{2}\right)$

$= 9\left(\dfrac{\pi}{2} - 1\right)\ \text{cm}^2$ #

6. Area of sector $= \frac{1}{2}r^2\theta_{rad} = \frac{1}{2}r^2 \times 30° \times \frac{\pi}{180°} = \frac{1}{2}r^2 \times \frac{\pi}{6} = \frac{\pi r^2}{12}$

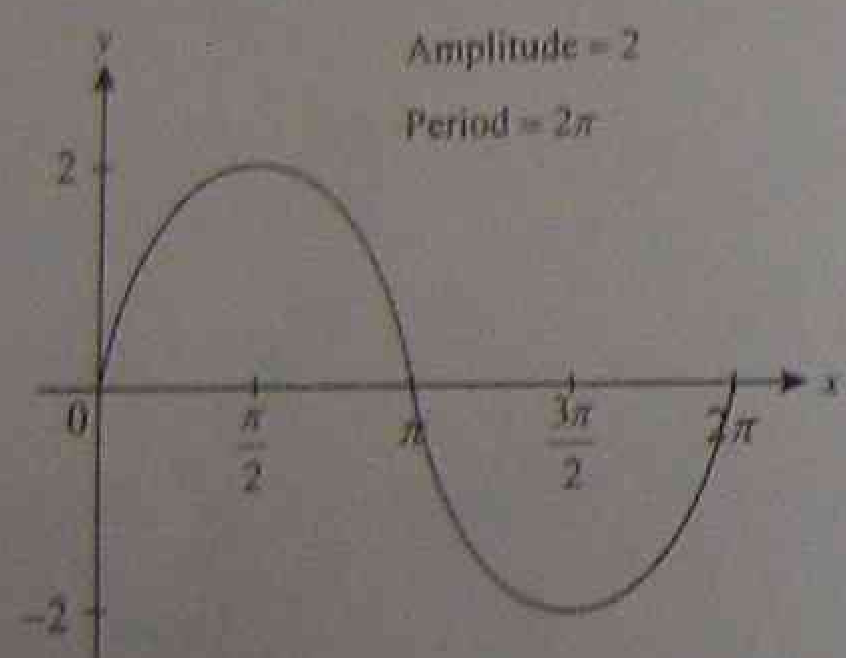
now $\frac{\pi r^2}{12} = \frac{3\pi}{4}$

i.e. $r^2 = \frac{36}{4} = 9$

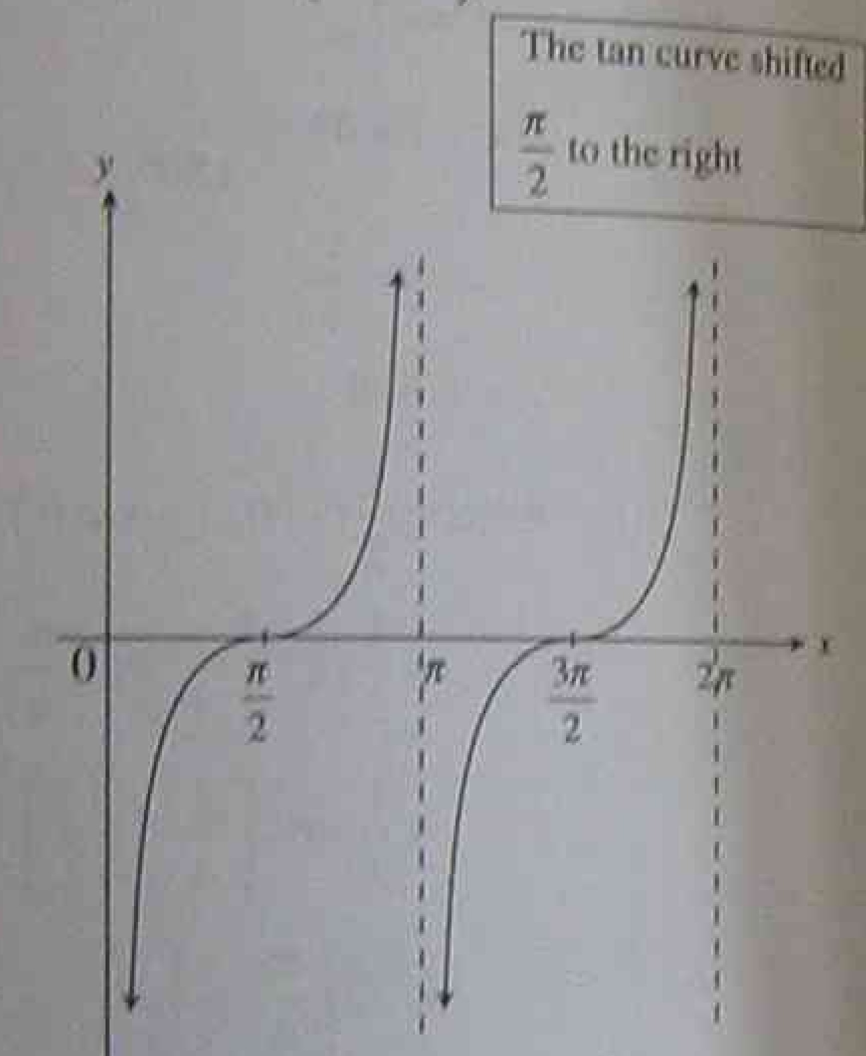
$r = 3$ cm (since $r > 0$)

Arc AB: $l = r\theta_{rad} = 3 \times \frac{\pi}{6} = \frac{\pi}{2}$ cm #

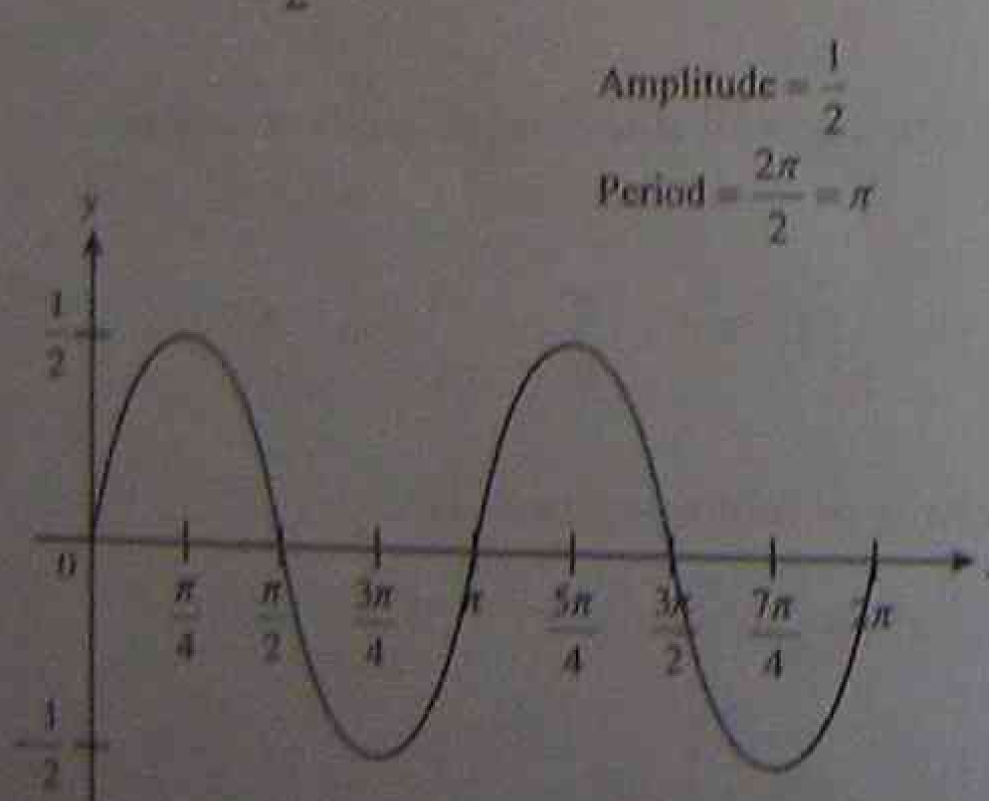
7. (i) $y = 2\sin x$



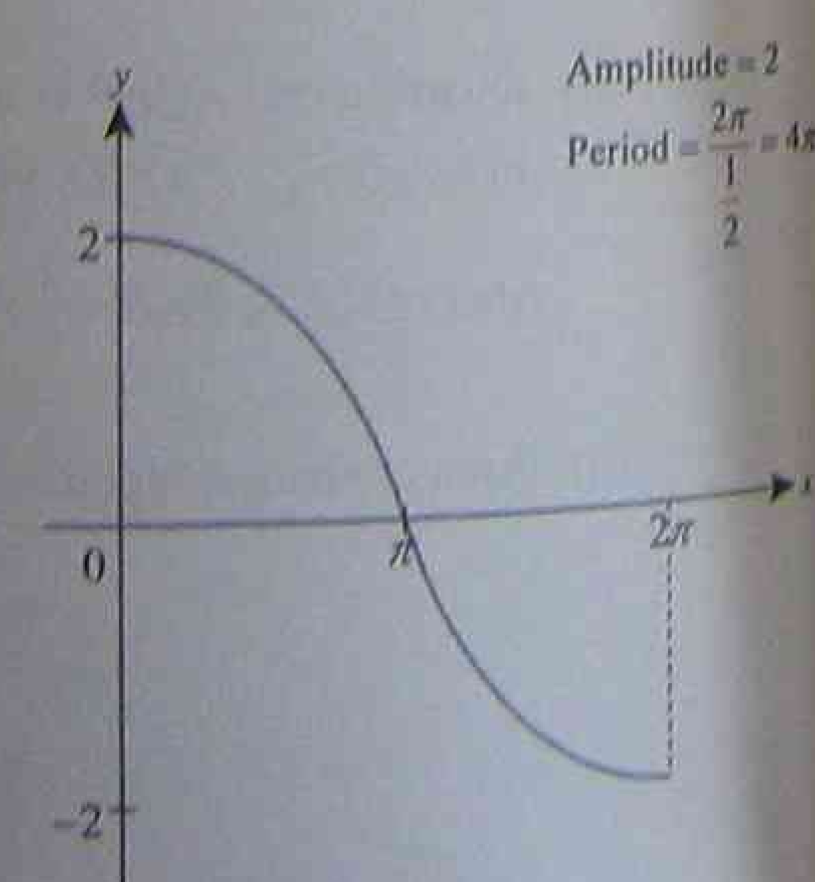
(ii) $y = \tan\left(x + \frac{\pi}{2}\right)$



(iii) $y = \frac{1}{2}\sin 2x$



(iv) $y = 2\cos\left(\frac{x}{2}\right)$



8. $y = 1 + 3\sin 2x$

Amplitude $= 3$, Period $= \frac{2\pi}{2} = \pi$

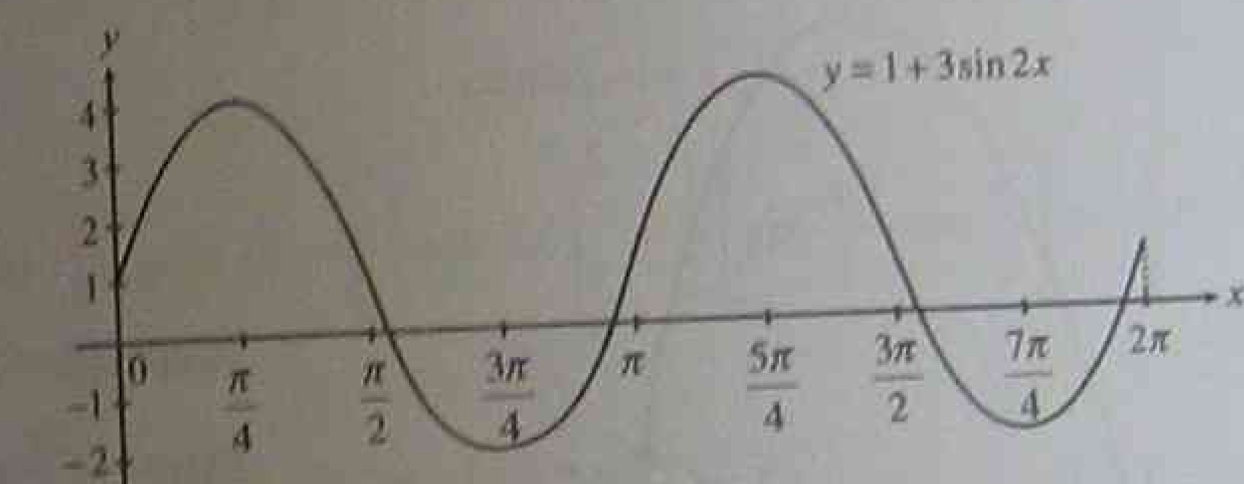
Now, set up a table of values at the critical points:

i.e. $2x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

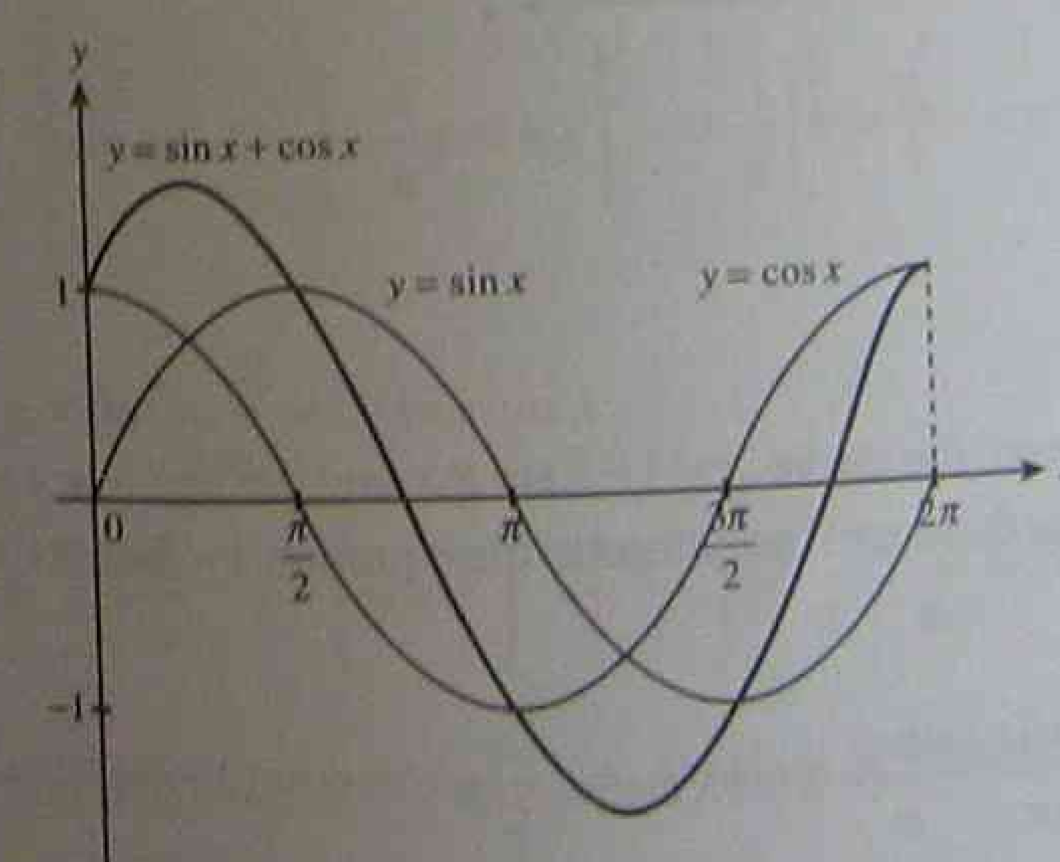
$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$1+3\sin 2x$	1	4	1	-2	1

now, plot these points and form the curve.



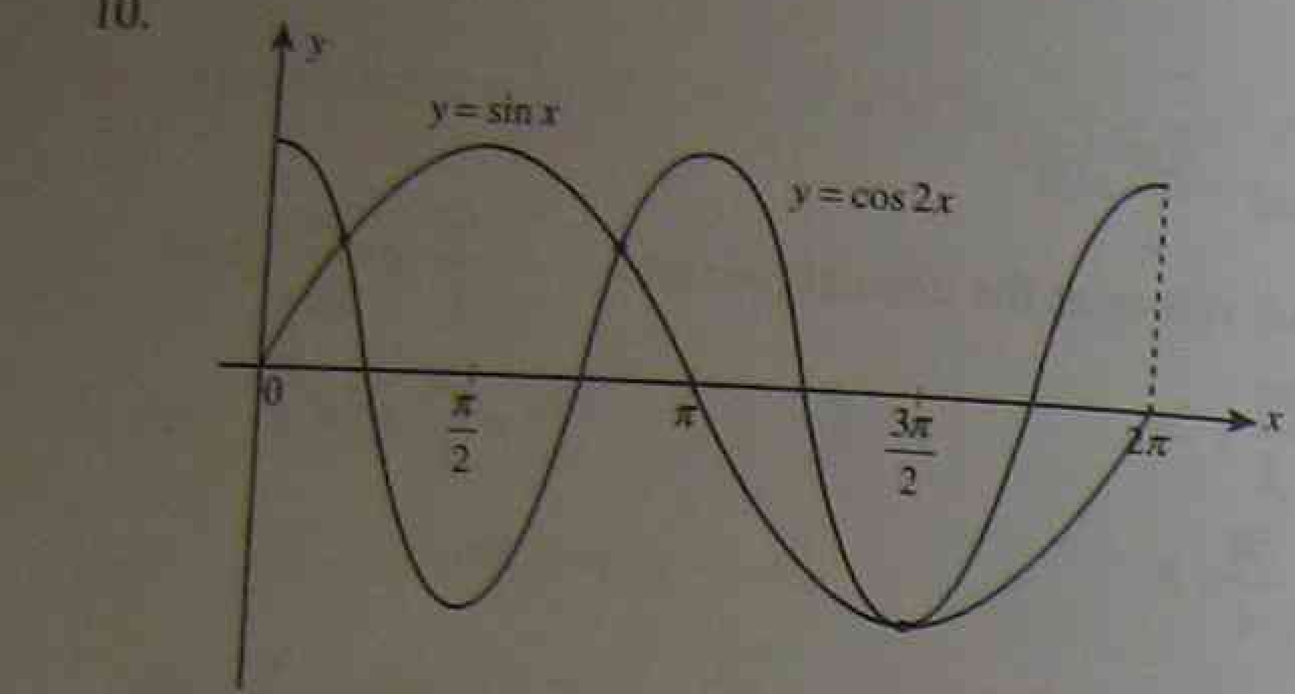
9.



The graph of $y = \sin x + \cos x$ was done by adding the x-values of the individual curves $y = \sin x$ and $y = \cos x$.

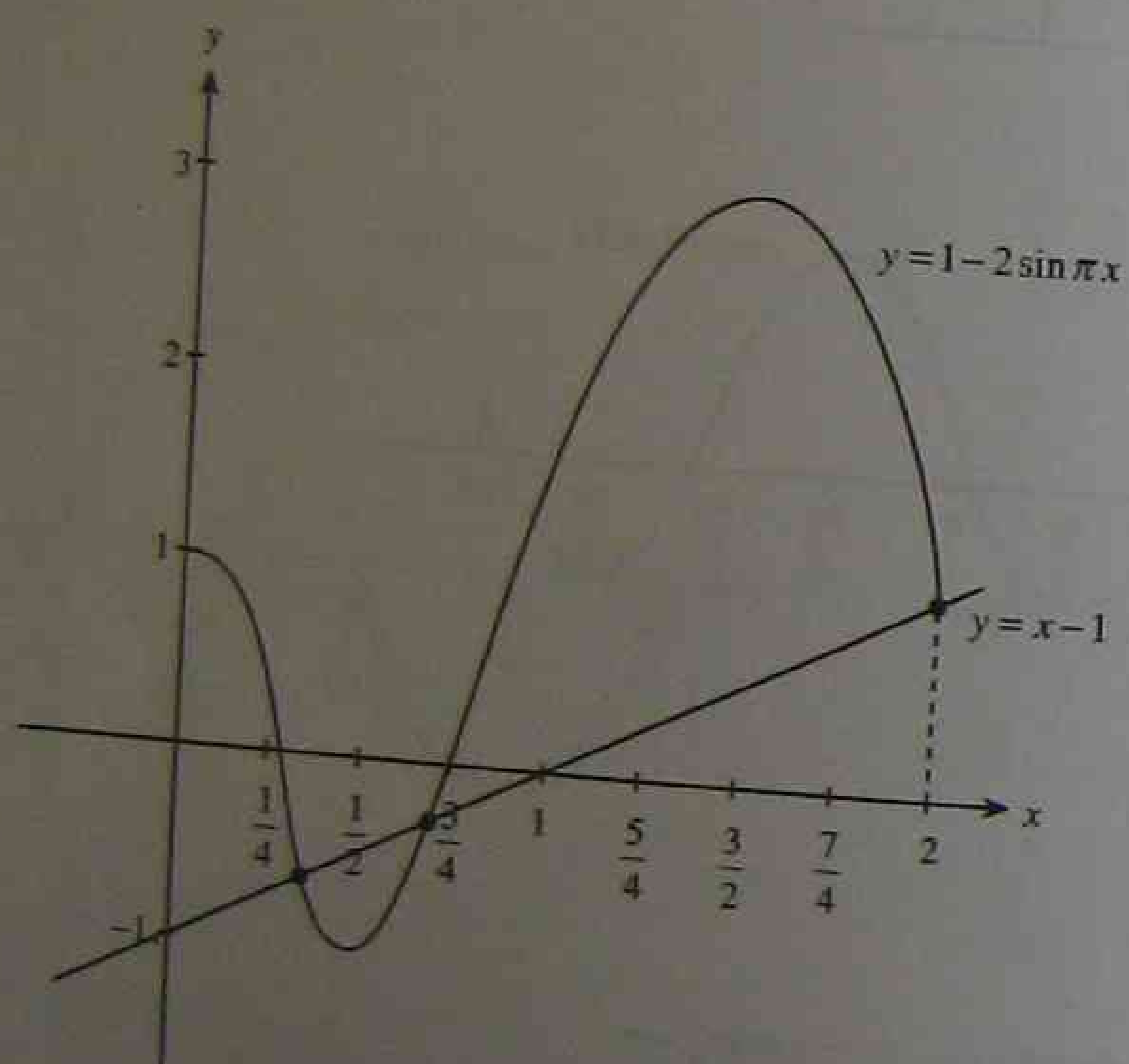
This is best done by using a compass.

10.



From the diagram above, the curves intersect at 3 distinct points.
$\therefore \cos 2x = \sin x$ has 3 solutions in the domain $0 \le x \le 2\pi$. #

11. (i) & (ii)



Amplitude = 2

Period $= \frac{2\pi}{2} = \pi$

(iii) From the graph above, the curve $y = 1 - 2\sin \pi x$ and the line $y = x - 1$ intersect at 3 points. Therefore there are 3 solutions to the equation $1 - 2\sin \pi x = x - 1$ for $0 \le x \le 2$.

12. $\lim_{x \to 0} \frac{\sin 5x \sin x}{x^2} = \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot \frac{\sin x}{x} \cdot 5 = 1 \times 1 \times 5 = 5$ #

13. (i) $\frac{d}{dx}(3\cos 2x) = -3 \cdot 2\sin 2x = -6\sin 2x$ #

(ii) $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(x)}{x^2} = \frac{x\cos x - \sin x}{x^2}$ #

(iii) $\frac{d}{dx}\left(\tan^3(2x+5)\right) = 3\tan^2(2x+5) \cdot \frac{d}{dx}\left(\tan(2x+5)\right)$

$= 3\tan^2(2x+5) \cdot 2\sec^2(2x+5) = 6\tan^2(2x+5)\sec^2(2x+5)$ #

(iv) $\frac{d}{dx}\left((x^2+5)\tan 2x\right) = (x^2+5) \cdot \frac{d}{dx}(\tan 2x) + \tan 2x \cdot \frac{d}{dx}(x^2+5)$

$= (x^2+5) \cdot 2\sec^2 2x + \tan 2x \cdot 2x$

$= 2(x^2+5)\sec^2 2x + 2x\tan 2x$ #

14. (i) $\int \sin 2x - x \, dx = -\frac{1}{2}\cos 2x - \frac{x^2}{2} + C$ #

(ii) $\int \frac{\cos 5x}{5} dx = \frac{1}{5}\int \cos 5x \, dx = \frac{1}{5} \cdot \frac{1}{5}\sin 5x + C = \frac{1}{25}\sin 5x + C$ #

(iii) $\int \tan^2\left(\frac{x}{2}\right) dx = \int \sec^2\left(\frac{x}{2}\right) - 1 \, dx = \frac{1}{\frac{1}{2}}\tan\left(\frac{x}{2}\right) - x + C = 2\tan\left(\frac{x}{2}\right) - x + C$ #

(iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right] = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}$ #

(v) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx = \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}} = \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan 0\right] = \frac{1}{2}(1-0) = \frac{1}{2}$ #

(vi) $\int_{\frac{1}{3}}^{\frac{1}{2}} \sin \pi x \, dx = \left[-\frac{1}{\pi}\cos \pi x\right]_{\frac{1}{3}}^{\frac{1}{2}} = -\frac{1}{\pi}\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right] = -\frac{1}{\pi}\left[0 - \frac{1}{2}\right] = \frac{1}{2\pi}$ #

15. $y = \sin^2 x$, $\frac{dy}{dx} = 2\sin x \cos x$

at $x = \frac{\pi}{3}$, $y = \left(\sin\frac{\pi}{3}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$, $\frac{dy}{dx} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$

gradient of tangent $= \frac{\sqrt{3}}{2}$ $\therefore$ gradient of normal $= -\frac{2}{\sqrt{3}}$.

equation of normal is given by:

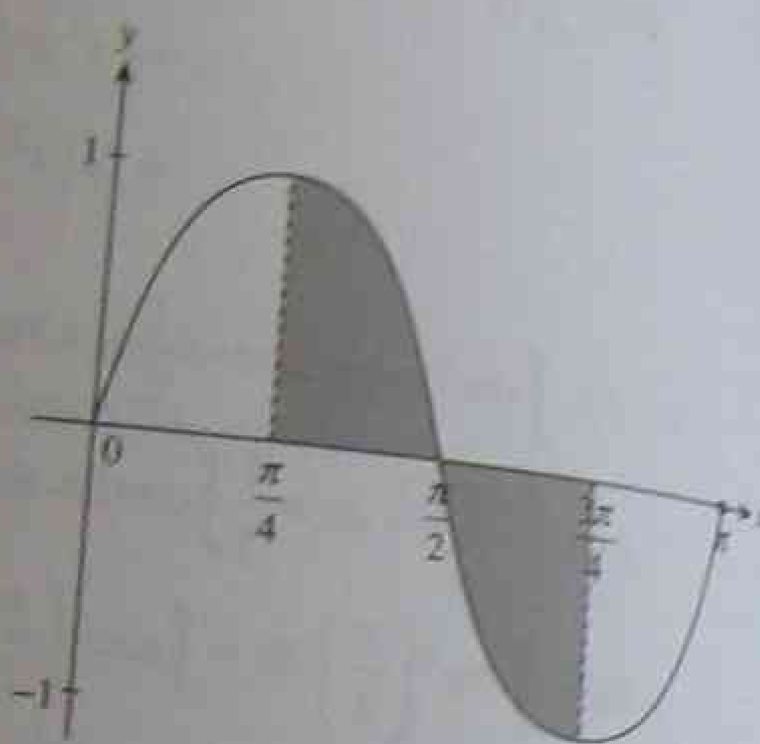
$(y - y_1) = m(x - x_1)$

$\left(y - \frac{3}{4}\right) = -\frac{2}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)$

$$y-\frac{3}{4}=-\frac{2}{\sqrt{3}}x+\frac{2\pi}{3\sqrt{3}}$$
$$y=-\frac{2}{\sqrt{3}}x+\frac{2\pi}{3\sqrt{3}}+\frac{3}{4}\quad\#$$

16. Shaded Area

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin 2x\,dx+\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\sin 2x\,dx\right|$$
$$=\left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}+\left|\left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\right|$$
$$=-\frac{1}{2}\left[\cos\pi-\cos\frac{\pi}{2}\right]+\left|-\frac{1}{2}\left[\cos\frac{3\pi}{2}-\cos\pi\right]\right|$$
$$=-\frac{1}{2}(-1-0)+\left|-\frac{1}{2}(0-(-1))\right|=\frac{1}{2}+\frac{1}{2}=1\text{ units}^2\quad\#$$



17. Volume $=\pi\int_0^{\frac{\pi}{2}}y^2dx,\quad y=2\sqrt{\cos x},\quad y^2=4\cos x$

$$=\pi\int_0^{\frac{\pi}{2}}4\cos x\,dx=\pi\left[4\sin x\right]_0^{\frac{\pi}{2}}=4\pi(1-0)=4\pi\text{ units}^3\quad\#$$

18. Volume $=\pi\int_0^{2\pi}y^2dx,\quad y=(\sin x+\cos x),\quad y^2=(\sin x+\cos x)^2$

$$=\pi\int_0^{2\pi}(\sin x+\cos x)^2\,dx$$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$(\sin x+\cos x)^2$	1	1	1	1	1

Now, $\int_0^{2\pi}(\sin x+\cos x)^2\approx\frac{h}{3}\left[y_0+y_n+4(y_1+y_3+\ldots)+2(y_2+y_4+\ldots)\right]$

∴ $\int_0^{2\pi}(\sin x+\cos x)^2\approx\frac{\frac{\pi}{2}}{3}\left[y_0+y_4+4(y_1+y_3)+2y_2\right]$

$$=\frac{\pi}{6}\left[1+1+4(1+1)+2\times1\right]=2\pi$$

∴ Volume $=\pi\times2\pi=2\pi^2$ units^3 #

TOPIC 16

APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

(A) Rates of Change

If a physical quantity Q (e.g. population, volume etc.) changes over time t, then the **rate** at which Q changes is given by $\frac{dQ}{dt}$

If $\frac{dQ}{dt}=f(t)$, then

$$Q=\int f(t)\,dt=F(t)+C$$

The value of C is determined by using the data given in the question.

Example 1: A block of ice is melting so that after t seconds its volume, Vcm^3, at time t seconds is given by:

$$V=60t-15t^2+120$$

(i) Find the initial volume of the block of ice.

(ii) Find at what time will the block have completely melted.

(iii) Find the rate at which the block is melting at 3 seconds.

Solution 1:

(i) $V=60t-15t^2+120$

at $t=0$, $V=120\text{ cm}^3$ #

(ii) The block would have completely melted when $V=0$

i.e. $60t-15t^2+120=0$

$15(-t^2+4t+8)=0$

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-4\times-1\times8}}{-2}=\frac{-4\pm\sqrt{48}}{-2}=2\pm2\sqrt{3}$$

∴ the block would have melted after $t=2+2\sqrt{3}$ seconds. #

(iii) $V=60t-15t^2-120$

$\frac{dV}{dt}=60-30t$

at $t=3$, $\frac{dV}{dt}=60-90=-30\text{ cm}^3/\text{sec}$.

∴ the block is melting at the rate of 30 cm^3/sec. at 3 seconds #

Example 2: The rate in litres per year at which the volume of water is changing in a dam is given by: $\frac{dV}{dt} = 2e^t - 60$

(i) Find the volume (to the nearest litre) in the dam after 2 years, if initially the dam has 1000 litres in it.

(ii) Find the minimum volume of water in the dam to the nearest litre.

Solution 2:

(i) $\frac{dV}{dt} = 2e^t - 60$

$V = \int 2e^t - 60,\ dt = 2e^t - 60t + C$

at $t = 0$, $V = 1000$ i.e. $2\times 1 - 0 + C = 1000$ i.e. $C = 998$

$\therefore\ V = 2e^t - 60t + 998$

at $t = 2$, $V = 892.778 = 893$ litres #

(ii) Minimum volume occurs at $\frac{dV}{dt} = 0$

i.e. $2e^t - 60 = 0$

$2e^t = 60$

$e^t = 30$ i.e. $t = \ln(30)$

$\therefore$ minimum volume $= 2e^{\ln 30} - 60\times \ln 30 + 998 = 854$ litres to the nearest litre #

(B) Understanding Rates of Change

- If a physical quantity Q is increasing at a decreasing rate, then:

$$\frac{dQ}{dt} > 0 \text{ and } \frac{d^2Q}{dt^2} < 0$$

- If a physical quantity Q is increasing at an increasing rate, then:

$$\frac{dQ}{dt} > 0 \text{ and } \frac{d^2Q}{dt^2} > 0$$

- If a physical quantity is decreasing at an increasing rate, then:

$$\frac{dQ}{dt} < 0 \text{ and } \frac{d^2Q}{dt^2} < 0$$

- If a physical quantity Q is decreasing at a decreasing rate, then:

$$\frac{dQ}{dt} < 0 \text{ and } \frac{d^2Q}{dt^2} > 0$$

Example 1: If the population P of a particular species of whales is still decreasing, but continued conservation efforts to protect the species appears to be working.

What does this statement suggest about the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

Solution 1:

The population of whales is still decreasing i.e. $\frac{dP}{dt} < 0$ but due to conservation efforts, the rate of decrease is being slowed i.e. the population of whales P is decreasing at a decreasing rate i.e. $\frac{d^2P}{dt^2} > 0$. #

(C) Exponential Growth and Decay

If the rate of change of some physical quantity Q (eg. a population, volume) is proportional to itself, then it is increasing (or decreasing) exponentially.

This can be expressed mathematically as follows:

- If the quantity present increases at a rate proportional to the amount present at a given time, then:

$$\frac{dQ}{dt} = kQ \text{ and } Q = Q_0 e^{kt}$$

- If the quantity present decreases at a rate proportional to the amount present at a given time, then:

$$\frac{dQ}{dt} = -kQ \text{ and } Q = Q_0 e^{-kt}$$

Where: Q is the quantity at time t
Q_0 is the quantity present at $t = 0$
k is a constant-the growth rate (or decay rate)

Example 1: The population of a particular city grows at a rate proportional to the population present.

The rate of change is given by $\frac{dN}{dt} = kN$

Where k is a constant, t is the time in years and N is the population present at time t.

(i) Show that $N = N_0 e^{kt}$ is a solution to this equation.

(ii) If the population doubles every 60 years, how long does it take for the population to triple in size?

Solution 1:

(i) $N = N_0 e^{kt}$

$$\frac{dN}{dt} = k \cdot N_0 e^{kt} = kN \text{ as required} \quad \#$$

(ii) $N = N_0 e^{kt}$, the population doubles when $N = 2N_0$

i.e. $2N_0 = N_0 e^{kt}$

i.e. $2 = e^{60k}$ i.e. $k = \dfrac{\ln 2}{60} = 0.011552453$

the population trebles when $N = 3N_0$

i.e. $3N_0 = N_0 e^{kt}$

i.e. $3 = e^{kt}$ i.e. $t = \dfrac{\ln 3}{k} = 95.1$ years #

Example 2: The number at whales in a particular region on 1[st] January 1995 is 6,200. If the population of whales W at time t years varies according to the following equation:

$$W = 6200e^{-0.015t}$$

(i) Find the month and the year at which the population of whales is expected to have halved.

(ii) At what rate is the number of whales decreasing when $t = 10$.

Solution 2:

(i) Let $W = 3{,}100$ (half the initial whale population)

i.e. $3{,}100 = 6{,}200e^{-0.015t}$

$0.5 = e^{-0.015t}$

$t = 46.2$ years.

i.e. 46 years and 2.5 months

Thus the month and year at which the whale population is expected to have halved is March 2041. #

(ii) $W = 6{,}200e^{-0.015t}$

$$\frac{dW}{dt} = -93e^{-0.015t} \quad \text{at } t = 10$$

$$\frac{dW}{dt} = -80.05 \approx -80 \text{ whales/year}$$

i.e. at $t = 10$ the whale population is expected to be decreasing at the rate of 80 whales per year. #

(D) Displacement, Velocity and Acceleration

(i) Displacement

The displacement of a particle represents its position relative to the origin.

It is denoted by x and always expressed as a function of time i.e. $x = f(t)$.

(ii) Velocity

Velocity, v, is the rate at which displacement changes:

$$v = \frac{dx}{dt} = \dot{x}$$

i.e. find the first derivative of x.

(iii) Acceleration

Acceleration, a, is the rate at which the velocity changes:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$$

i.e. find the first derivative of v or find the second derivative of x.

Example 1: A particle moves from O in such a way that its displacement x m from the origin after t seconds is given by:

$$x = 2t^3 - 3t^2$$

Find:

(i) the displacement of the particle at $t = 2$.

(ii) the initial velocity and acceleration of the particle. In which direction is the particle moving initially?

(iii) the time when the particle is next at rest.

(iv) the time when the particle is next at the origin.

Solution 1:

(i) $x = 2t^3 - 3t^2$

at $t = 2$, $x = 2(2)^3 - 3(2)^2 = 16 - 12 = 4$ m to the right of O. #

(ii) $x = 2t^3 - 3t^2$

$$v = \frac{dx}{dt} = 6t^2 - 6t, \text{ at } t = 0, \quad v = 0$$

$$a = \frac{dv}{dt} = 12t - 6, \text{ at } t = 0, \quad a = -6$$

since acceleration is negative and velocity is zero, then the particle moves initially to the left of O. #

(iii) Particle is at rest when $v = 0$

i.e. $6t^2 - 6t = 0$

$6t(t-1) = 0$

i.e. $t = 0, 1$

$\therefore$ the particle is next at rest at $t = 1$ second. #

(iv) Particle is next at the origin at $x = 0$

i.e. $2t^3 - 3t^2 = 0$

$t^2(2t-3) = 0$

i.e. $t = 0, \frac{3}{2}$

$\therefore$ the particle next returns to the origin at $t = \frac{3}{2}$ seconds. #

Example 2: A point P moves in a straight line, such that its displacement from the origin at time t seconds is given by:

$$x = 3e^{-5t} + 3t - 2$$

(i) Find the initial displacement, velocity and acceleration of the particle.
(ii) Find the time when the particle is stationary.
(iii) Find an expression for the acceleration of the particle in terms of its velocity.
(iv) Describe the motion of P as $t \to \infty$ in terms of its displacement, velocity and acceleration.

Solution 2:

(i) $x = 3e^{-5t} + 3t - 2$, at $t = 0$, $x = 1$ #

$v = \frac{dx}{dt} = -15e^{-5t} + 3$, at $t = 0$, $v = -12$ #

$a = \frac{dv}{dt} = 75e^{-5t}$, at $t = 0$, $a = 75$ #

(ii) Particle is stationary when $v = 0$

i.e. $-15e^{-5t} + 3 = 0$

$e^{-5t} = \frac{3}{15} = 0.2$ i.e. $t = \frac{\ln(0.2)}{-5} = 0.322$

at $t = 0.322$, $x = -0.434$, $a = 15$ #

(iv) As $t \to \infty$, the particle's acceleration approaches zero and its velocity approaches a constant speed of 3 m/sec as the particle continues to move in a positive/forward direction. #

(E) Using Integration

In some cases the velocity or acceleration functions may be given, requiring students to find v and x.

In these cases, integration is used:

$v = \int a\, dt + C_1$ integrate acceleration to find the velocity

$x = \int v\, dt + C_2$ integrate velocity to find the displacement

Note: C_1 and C_2 are constants which must be calculated from information provided in the question.

Example 1: A particle moves such that its velocity, v m/s, at any time t is given by:

$$v = 1 + 4e^{-2t}$$

Initially the particle is at $x = 8$.

(i) Find the acceleration of the particle at $t = 0$.
(ii) Find an expression for the displacement x at time t.
(iii) Find the distance travelled by the particle in the first second.

Solution 1:

(i) $v = 1 + 4e^{-2t}$

$a = \frac{dv}{dt} = -8e^{-2t}$ $\therefore$ at $t = 0$, $a = -8$ m/s^2 #

(ii) $v = 1 + 4e^{-2t}$

$x = \int v\, dt = \int 1 + 4e^{-2t} dt = t - 2e^{-2t} + C$

at $t = 0, x = 8$

i.e. $8 = 0 - 2\times 1 + C$ i.e. $C = 10$

$\therefore$ $x = t - 2e^{-2t} + 10$ #

(iii) At $t = 1$, $x = 1 - 2e^{-2} + 10 = 11 - 2e^{-2} = 10.73$ m

$\therefore$ distance travelled in the first second $= 10.73 - 8 = 2.73$ m #

Example 2: A particle is moving in a straight line such that its acceleration, a cm/s^2, after t seconds is given by:

$$a = 4t - 8$$

Initially the particle is at rest at $x = 2$. Find:

(i) v as a function of t.
(ii) x as a function of t.
(iii) when the particle is stationary again.

Solution 2:

(i) $v = \int a\,dt = \int 4t - 8\,dt = 2t^2 - 8t + C$

at $t = 0, v = 0$ i.e. $C = 0$

$\therefore \quad v = 2t^2 - 8t$ #

(ii) $x = \int v\,dt = \int 2t^2 - 8t\,dt = \dfrac{2t^3}{3} - 4t^2 + C$

at $t = 0, x = 2$ i.e. $2 = 0 - 0 + C$ i.e. $C = 2$

$\therefore \quad x = \dfrac{2t^3}{3} - 4t^2 + 2$ #

(iii) Particle is stationary when $v = 0$

i.e. $2t^2 - 8t = 0$

$2t(t-4) = 0$

i.e. $t = 0, 4$

$\therefore$ particle is stationary again at $t = 4$ seconds. #

(F) Maximum Values of Displacement and Velocity

Maximum value for **displacement** occurs when

velocity = 0, i.e. $v = \dfrac{dx}{dt} = 0$

Maximum value for **velocity** occurs when

acceleration = 0, i.e. $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = 0$

Example 1: The displacement of a particle x cm after t seconds is given by:

$$x = 24t - 3t^2 - t^3$$

Find the maximum displacement.

Solution 1:

$x = 24t - 3t^2 - t^3$

$\dfrac{dx}{dt} = 24 - 6t - 3t^2$, let $\dfrac{dx}{dt} = 0$ to find maximum displacement

i.e. $24 - 6t - 3t^2 = 0$

$-3(t^2 + 2t - 8) = 0$

$-3(t+4)(t-2) = 0$

$\therefore \quad t = 2$ (note $t \neq -4$ as $t > 0$)

$\therefore$ maximum displacement occurs at $t = 2$

i.e. $x = 24 \times 2 - 3(2)^2 - (2)^3 = 48 - 12 - 8 = 28$ cm to the right. #

(G) Distance vs Displacement and Speed vs Velocity

Distance and **speed** are 'scalar' measures i.e. their direction is unimportant, only their magnitude significant:

i.e. speed $= |v|$, distance $= |x|$

Displacement and **velocity** are 'vectors' i.e. their directions and magnitude are significant.

- negative v means particle moving to the left.
- positive v means particle moving to the right.
- negative x means particle located to the left of origin (O).
- positive x means particle located to the right of origin (O).

Example 1: A particle moves along a straight line so that its velocity v m/s at time t seconds is given by:

$$v = 3t^2 - 4t$$

(i) In which direction is the particle moving at $t = 1$?

(ii) After which time does the particle change direction?

(iii) If the particle is at point O at time $t = 1$, find an expression for the displacement x of the particle as a function of time.

(iv) Find the total distance travelled by the particle during the first 3 seconds. #

Solution 1:

(i) $v = 3t^2 - 4t$

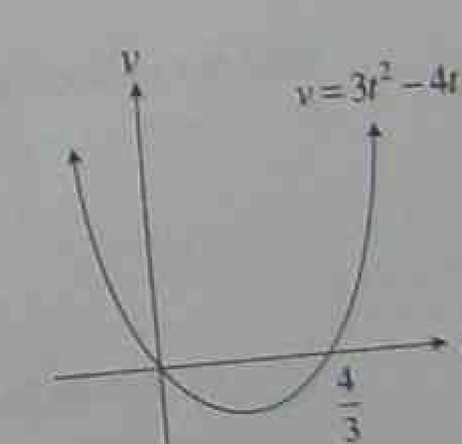
at $t = 1$, $v = 3 - 4 = -1$

$\therefore$ the particle is moving to the left at $t = 1$ #

(ii) $v = 3t^2 - 4t$

$= t(3t - 4)$

from the graph; the particle changes direction at $t = \dfrac{4}{3}$ seconds and begins to move to the right #



(iii) $x = \int v\,dt$

$= \int 3t^2 - 4t\,dt = t^3 - 2t^2 + C$

at $t = 1, x = 0$ i.e. $0 = 1 - 2 + C$ i.e. $C = 1$

$\therefore \quad x = t^3 - 2t^2 + 1$ #

(iv) The particle moves to the left from $t = 0$ until $t = \frac{4}{3}$ and then moves to the right until $t = 3$.

t	0	1	$\frac{4}{3}$	3
s	1	0	$-\frac{5}{27}$	10

The motion of the particle is as follows:

$\therefore$ Total distance $= \left(1 + \frac{5}{27}\right) + \left(10 + \frac{5}{27}\right) = 11\frac{10}{27}$ m #

Review Exercises

(A) Rates of Change

1. An open water container is being emptied whilst it is raining and the number of litres of water, V, in the container at time t minutes is given by:

$$V = 3t - t^3 + 300$$

(i) How many litres were there in the tank initially?

(ii) At what time will the container be completely empty?

(iii) What is the maximum number of litres in the container?

(iv) At what rate is the water draining out of the container when $t = 3$?

2. The rate of fall of the price $P of a stock after t weeks is given by:

$$\frac{dP}{dt} = \frac{-20}{(t+1)^2}$$

(i) If the price of the stock was initially $15, find the price of the stock after 4 weeks.

(ii) What is lowest price the stock can reach?

(iii) On which day will the price fall to $10?

(B) Understanding Rates of Change

3. The population, N, of a particular town is increasing at a decreasing rate over time.

(i) What does this statement suggest about the sign of $\frac{dN}{dt}$ and $\frac{d^2N}{dt^2}$?

(ii) Sketch the graph of N against time t.

(C) Exponential Growth and Decay

4. The rate at which the amount, Q, of a carbon 14 isotope decays over time is given by:

$$\frac{dQ}{dt} = -kQ \quad \text{where } k \text{ is a constant}$$

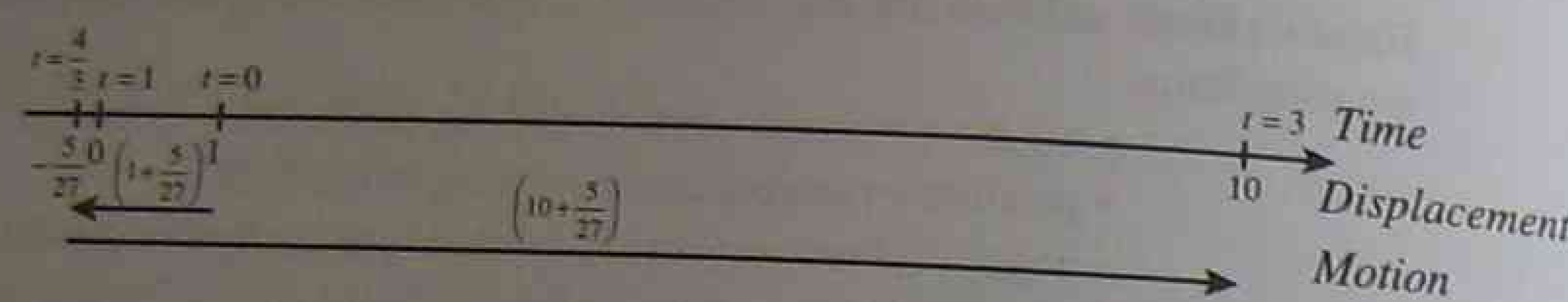
(i) Show that $Q = Q_0 e^{-kt}$, where Q_0 is the amount of carbon 14 present initially, satisfies the given equation.

(ii) A piece of wood found in an archaeological dig is compared with a piece of wood, and it is determined that the former piece contains 50% of its amount of carbon 14. Given that the half-life of carbon 14 is approxim years, estimate the age of the piece of wood found at the archaeological

(iv) The particle moves to the left from $t=1$ until $t=\frac{4}{3}$ and then moves to the right until $t=3$.

t	0	1	$\frac{4}{3}$	3
x	1	0	$-\frac{5}{27}$	10

The motion of the particle is as follows:



$\therefore$ Total distance $=\left(1+\frac{5}{27}\right)+\left(10+\frac{5}{27}\right)=11\frac{10}{27}$ m #

REVIEW EXERCISES

(A) Rates of Change

1. An open water container is being emptied whilst it is raining and the number of litres of water, V, in the container at time t minutes is given by:

$$V=5t-t^2+300$$

(i) How many litres were there in the tank initially?

(ii) At what time will the container be completely empty?

(iii) What is the maximum number of litres in the container?

(iv) At what rate is the water draining out of the container when $t=5$?

2. The rate of fall of the price $\$P$ of a stock after t weeks is given by:

$$\frac{dP}{dt}=\frac{-20}{(t+1)^3}$$

(i) If the price of the stock was initially \$15, find the price of the stock after 4 weeks.

(ii) What is lowest price the stock can reach?

(iii) On which day will the price fall to \$10?

(B) Understanding Rates of Change

3. The population, N, of a particular town is increasing at a decreasing rate over time t.

(i) What does this statement suggest about the sign of $\frac{dN}{dt}$ and $\frac{d^2N}{dt^2}$?

(ii) Sketch the graph of N against time t.

(C) Exponential Growth and Decay

4. The rate at which the amount, Q, of a carbon 14 isotope decays over time is given by:

$$\frac{dQ}{dt}=-kQ \quad \text{where } k \text{ is a constant}$$

(i) Show that $Q=Q_0e^{-kt}$, where Q_0 is the amount of carbon 14 present initially, satisfies the given equation.

(ii) A piece of wood found in an archaeological dig is compared with a piece of new wood, and it is determined that the former piece contains 70% of its original amount of carbon-14. Given that the half-life of carbon-14 is approximately 3760 years, estimate the age of the piece of wood found at the archeological dig.

5. The rate of 'flu inflection' in a population of a city is proportional to the number of infected individuals. That is the number of infected people F after t weeks satisfies the following equation.

$$F = F_0 e^{kt} \quad \text{where } k \text{ is a constant}$$

(i) If after 3 weeks, there is twice the number of infections to begin with, find the value of k correct to 2 decimal places.

(ii) If there were a thousand cases of flu infection originally, how many are there after 7 weeks? Express your answer correct to 3 significant places.

6. The percentage $R(t)$ of listeners who respond to a radio ad campaign within t days is modelled by the equation:

$$R(t) = 0.13\left(1 - e^{-0.3t}\right)$$

(i) According to the model what percentage of listeners are expected to respond within 5 days.

(ii) After how many days will 9% of the listeners have responded?

(iii) What is the practical significance of the number 0.13 in the above formula?

(D) Displacement, Velocity and Acceleration

7. A particle P moves in a straight line so that its position, $X(t)$ cm at time t seconds is given by:

$$X(t) = 4 - \ln(1+t)^2$$

(i) Find the initial displacement of P.

(ii) Find an expression for the velocity $V(t)$ and the acceleration $A(t)$ of P at any time t.

(iii) At what time is P at the origin?

(iv) Describe the motion of the particle.

8. The displacement, x m, of a particle moving along the x axis is given by:

$$x = e^t + 10e^{-2t}$$

Find:

(i) the initial displacement of the particle.

(ii) the initial velocity and acceleration of the particle.

(iii) when the particle comes to rest.

(E) Using Integration

9. Let the acceleration of a particle in m/s^2 be given by:

$$\frac{d^2x}{dt^2} = 8\sin 3t$$

Find the initial velocity and displacement of the particle if the particle is at $x = 4$ after π seconds travelling with velocity 3 m/s.

10. Let the acceleration of a particle P after t minutes be given by $a = \dfrac{24t}{\left(3t^2 - 1\right)}$

Given its velocity is $\log_e 256$ m/s after 1 minute:

(i) Prove that the particle's velocity after t minutes is given by:

$$v = 4\log_e\left(6t^2 - 2\right)$$

(ii) Hence find the time(s) when the particle is stationary.

(F) Maximum Values of Displacement and Velocity

11. The height of a certain ball shot out of a circus canon is given by:

$$h(t) = 2 + 18t - 5t^2$$

where h is in metres and t in seconds since the ball's launch.

(i) At what time does the ball reach its maximum height?

(ii) What is the maximum height it achieves?

12. The displacement in metres at time t seconds of a particle P moving along the x-axis is given by:

$$x = 2\cos 2t$$

Determine the maximum displacement and velocity of the particle.

13. The velocity of a car, v km/h driving from town A to town B, t hours after leaving town A is given by:

$$v = 36\left(25t - 75t^2\right)$$

Find:

(i) the time taken to travel between town A and town B.

(ii) the maximum velocity attained during this journey.

(iii) the distance between the two towns.

Applications of Calculus

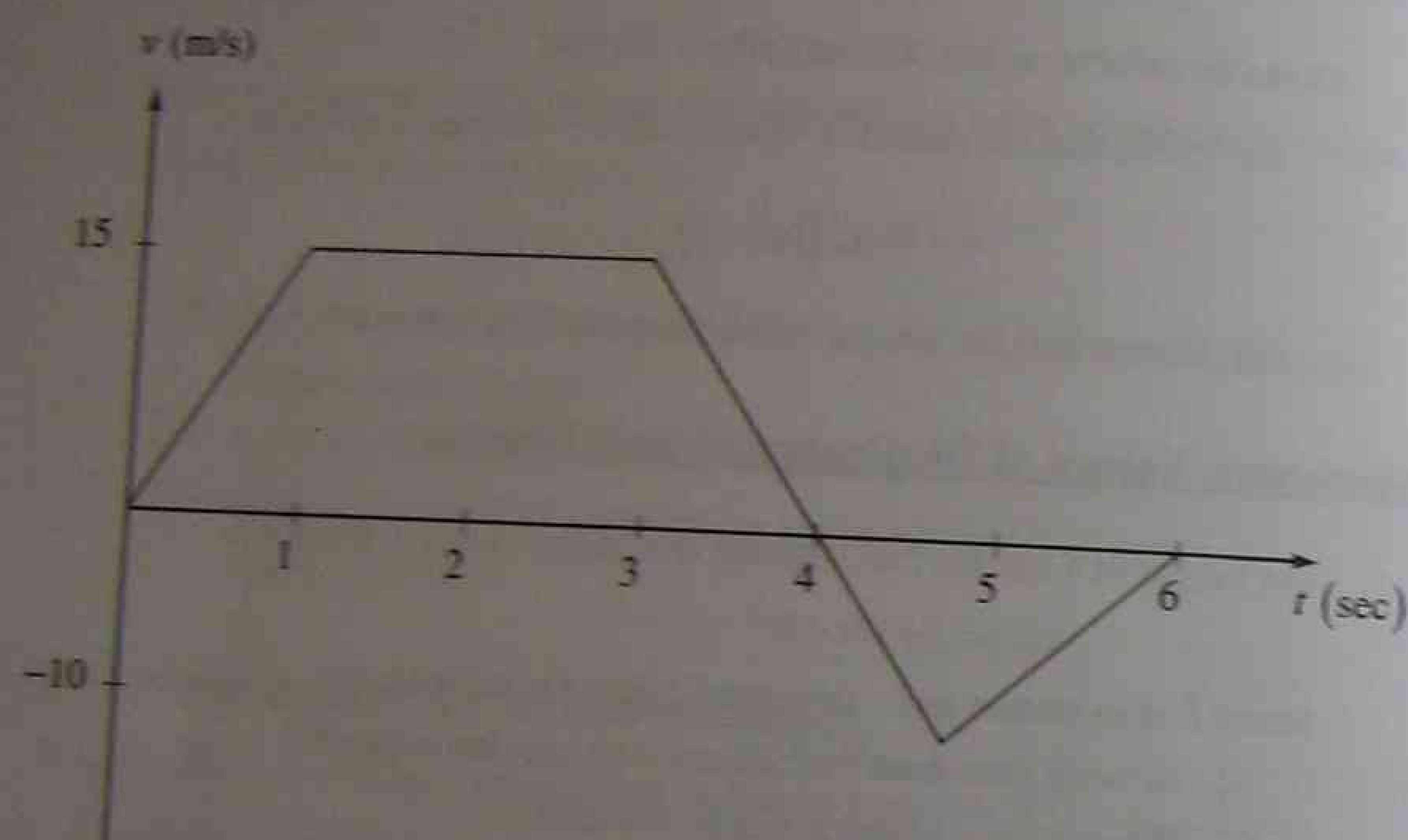
(G) Distance vs Displacement and Speed vs Velocity

14. The displacement in metres at time t seconds of a particle P moving along the x-axis is given by:

$$x = 2\sin(t-1) - t$$

Find the maximum velocity and speed of the particle.

15. The graph shown below represents the velocity (v m/s) of a particle after t seconds. The particle is moving in a staight line from rest:



(i) What is the velocity of the particle after 2 seconds?
(ii) When is the particle at rest?
(iii) At what time does the particle change direction?
(iv) Find the distance travelled by the particle in the first 4 seconds.
(v) What is the displacement of the particle after 6 seconds?

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $V = 5t - t^2 + 300$, at $t = 0, V = 300$ litres. #

(ii) Container completely empty at $V = 0$

i.e. $5t - t^2 + 300 = 0$

$-(t^2 - 5t - 300) = 0$

$-(t-20)(t+15) = 0$

$\therefore\ t = 20$ minutes as $t > 0$ #

(iii) Maximum occurs when $\dfrac{dV}{dt} = 0$

i.e. $\dfrac{dV}{dt} = 5 - 2t = 0$ i.e. $t = \dfrac{5}{2}$

$\therefore$ maximum volume $= 5\left(\dfrac{5}{2}\right) - \left(\dfrac{5}{2}\right)^2 + 300 = 306\dfrac{1}{4}$ litres #

(iv) $\dfrac{dV}{dt} = 5 - 2t$, at $t = 5$ $\dfrac{dV}{dt} = 5 - 2\times5 = -5$ litres/min

$\therefore$ water is draining out of the container at 5 litres/min #

2. (i) $P = \displaystyle\int \frac{dP}{dt}\,dt = \int \frac{-20}{(t+1)^3}\,dt = -20\int (t+1)^{-3}\,dt = \frac{-20\times(t+1)^{-2}}{-2} + C$

$= \dfrac{10}{(t+1)^2} + C$

at $t = 0, P = 15$, i.e. $15 = \dfrac{10}{1^2} + C$ i.e. $C = 5$

$\therefore\ P = \dfrac{10}{(t+1)^2} + 5$

at $t = 4$, $P = \dfrac{10}{5^2} + 5 = \$5.40$ #

(ii) $P = \dfrac{10}{(t+1)^2} + 5$ as $t \to \infty$, $P \to \$5$

$\therefore$ the lowest price the stock can reach is \$5 #

(iii) $10 = \dfrac{10}{(t+1)^2} + 5$

$5 = \dfrac{10}{(t+1)^2}$

$(t+1)^2 = 2$ i.e. $t+1 = \sqrt{2}$ i.e. $t = \sqrt{2} - 1$ weeks

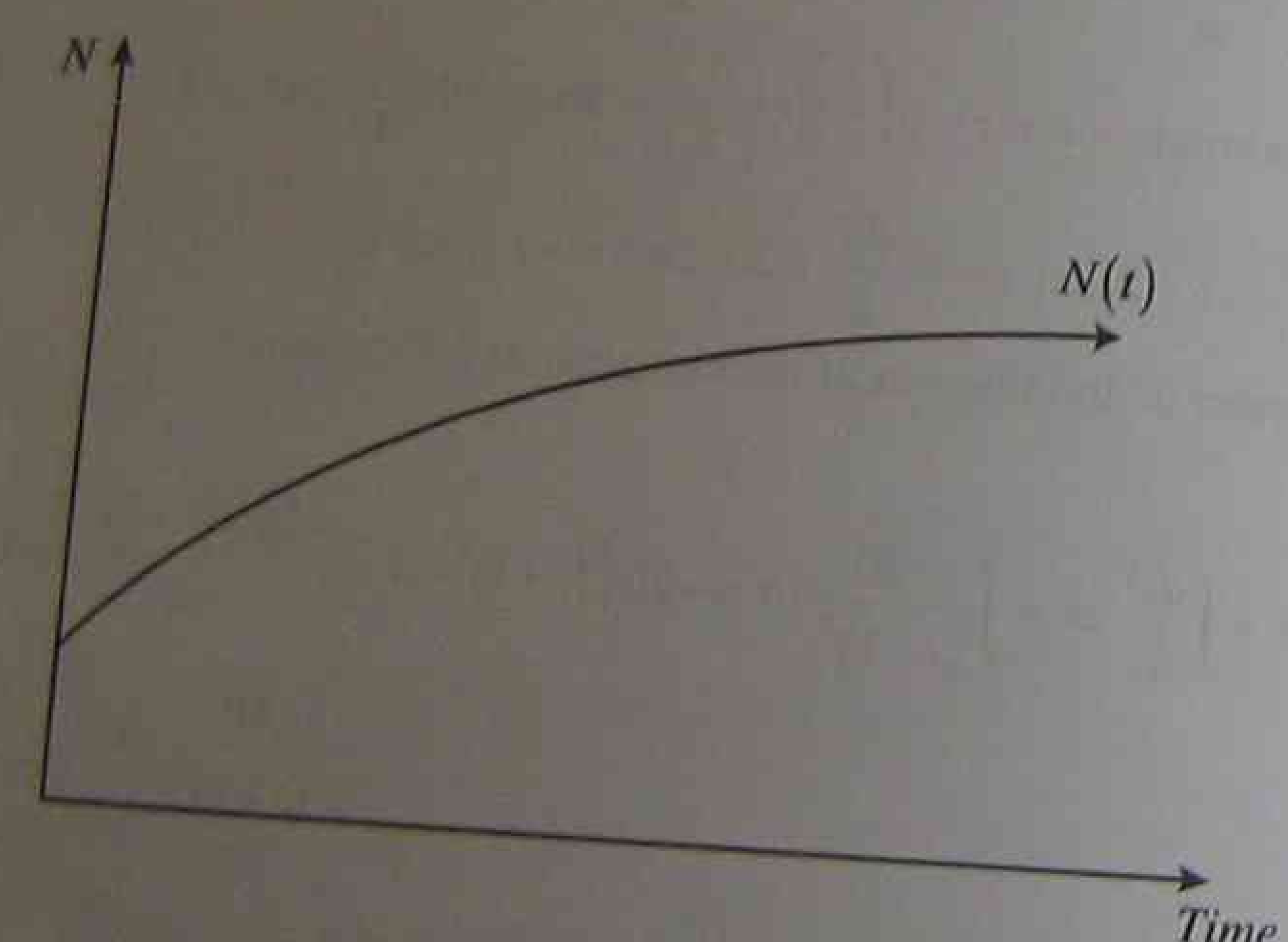
$\text{days} = \left(\sqrt{2} - 1\right) \times 7 = 2.899..$

$\therefore$ on the 3rd day the price will fall to \$10 #

(A) <u>Understanding Rates of Change</u>

3. (i) $\dfrac{dN}{dt} > 0$ and $\dfrac{d^2N}{dt^2} < 0$ #

(ii)



4. (i) $Q = Q_0 e^{-kt}$

$\dfrac{dQ}{dt} = Q_0 \cdot -ke^{-kt} = -kQ_0 e^{-kt} = -kQ$ as required #

(ii) The half life of carbon 14 is 3760 years.

i.e. $\dfrac{1}{2} Q_0 = Q_0 e^{-3760k}$

$\dfrac{1}{2} = e^{-3760k}$

$k = 0.000184347$

thus $0.7 = e^{-kt}$

5. (i) $F = F_0 e^{kt}$

$2F_0 = F_0 e^{3k}$

i.e. $2 = e^{3k}$, $k = \dfrac{\ln 2}{3} = 0.23104906 = 0.23$ correct to 2 d.p. #

(ii) $F_0 = 1000$, $t = 7$, $k = 0.23$

$F = 1000\, e^{7 \times 0.23} = 5003$ cases $= 5000$ cases correct to 3 s.f. #

6. (i) $R(t) = 0.13\left(1 - e^{-0.3t}\right)$

$R(5) = 0.13\left(1 - e^{-1.5}\right) = 0.10099... = 10.1\%$ to 1 d.p. #

(ii) $0.09 = 0.13\left(1 - e^{-0.3t}\right)$

$e^{-0.3t} = 0.307692307, t = \dfrac{\ln(0.3076...)}{-0.3} = 3.93 \approx 4$ days

i.e. after 4 days 9% of the listeners would have responded #

(iii) The practical significance of 0.13 is that according to this model it is expected that a maximum of 13% of listeners would respond to a particular ad campaign as time extends indefinitely i.e. $\lim\limits_{t \to \infty} R(t) = 0.13$. #

7. (i) At $t = 0$, $X(0) = 4 - \ln 1^2 = 4$ cm to the right #

(ii) $X(t) = 4 - \ln(1+t)^2 = 4 - 2\ln(1+t)$

$V(t) = X'(t) = \dfrac{d}{dt}\left(4 - 2\ln(1+t)\right) = \dfrac{-2}{1+t}$

$A(t) = V'(t) = \dfrac{d}{dt}\left(-2(1+t)^{-1}\right) = -2.-1(1+t)^{-2} = \dfrac{2}{(1+t)^2}$ #

(iii) P is at the origin when $X(t) = 0$

i.e. $0 = 4 - 2\ln(1+t)$

$2\ln(1+t) = 4$

$\ln(1+t) = 2$

i.e. $t = e^2 - 1$ seconds #

(iv) The particle is initially at a point 4 cm to the right of the origin and then travels left towards and past the origin with ever decreasing velocity. The particle continues to travel in the negative direction getting further and further away from the origin. #

8. (i) At $t = 0$, $x = e^0 + 10e^0 = 1 + 10 = 11$ m to the right #

(ii) $x = e^t + 10e^{-2t}$

$v = \dot{x} = e^t - 20e^{-2t}$, at $t = 0, v = e^0 - 20e^0 = -19$ m/s

$a = \ddot{x} = e^t + 40e^{-2t}$, at $t = 0, a = e^0 + 40e^0 = 41$ m/s^2 #

(iii) Particle comes to rest when $v = 0$

i.e. $e^t - 20e^{-2t} = 0$

$e^t = 20e^{-2t}$

$e^{3t} = 20$

i.e. $t = \dfrac{\ln(20)}{3} = 0.998... \approx 1$ second #

9. Velocity: $v = \int 8\sin 3t \, dt = -\dfrac{8}{3}\cos 3t + C$

at $t = \pi, v = 3$

i.e. $3 = -\dfrac{8}{3}\cos 3\pi + C$

i.e. $C = 3 - \dfrac{8}{3} = \dfrac{1}{3}$

$\therefore$ $v = \dfrac{1}{3} - \dfrac{8}{3}\cos 3t$, at $t = 0$, $v = \dfrac{1}{3} - \dfrac{8}{3} \times 1 = -\dfrac{7}{3}$ m/s #

Displacement: $x = \int v \, dt = \int \dfrac{1}{3} - \dfrac{8}{3}\cos 3t \, dt = \dfrac{1}{3}t - \dfrac{8}{9}\sin 3t + C$

at $t = \pi, x = 4$

i.e. $4 = \dfrac{\pi}{3} - \dfrac{8}{9}\sin 3\pi + C$

$= \dfrac{\pi}{3} + C$

i.e. $C = 4 - \dfrac{\pi}{3}$

$\therefore x = \dfrac{1}{3}t - \dfrac{8}{9}\sin 3t + \left(4 - \dfrac{\pi}{3}\right)$

at $t = 0$, $x = \left(4 - \dfrac{\pi}{3}\right)$ m #

10. (i) $a = \dfrac{24t}{\left(3t^2 - 1\right)}$

Applications of Calculus

$v = \int \dfrac{24t}{3t^2 - 1} dt = 4\int \dfrac{6t}{3t^2 - 1} dt = 4\log_e\left(3t^2 - 1\right) + C$

at $t = 1, v = \log_e 256$

i.e. $\log_e 256 = 4\log_e(3 - 1) + C$

$\log_e 2^8 = 4\log_e 2 + C$

$8\log_e 2 = 4\log_e 2 + C$

$\therefore$ $C = 8\log_e 2 - 4\log_e 2 = 4\log_e 2$

$\therefore$ $v = 4\log_e\left(3t^2 - 1\right) + 4\log_e 2 = 4\log_e\left[2\left(3t^2 - 1\right)\right] = 4\log_e\left(6t^2 - 2\right)$ #

(ii) Particle is stationary when $v = 0$

i.e. $4\log_e\left(6t^2 - 2\right) = 0$

i.e. $6t^2 - 2 = 1$

$6t^2 = 3$

$t^2 = \dfrac{1}{2}$

$\therefore$ $t = \dfrac{1}{\sqrt{2}}$ minutes (since $t > 0$) #

11. (i) Maximum height attained when $\dfrac{dh(t)}{dt} = 0$

i.e. $h(t) = 2 + 18t - 5t^2$

$\dfrac{dh(t)}{dt} = 18 - 10t = 0$

i.e. $18 = 10t$

$t = 1.8$ seconds #

(ii) Maximum height $= 2 + 18 \times (1.8) - 5 \times (1.8)^2 = 18.2$ metres #

12. $x = 2\cos 2t$,

now $-1 \le \cos 2t \le 1$ $\therefore$ maximum displacement occurs when $\cos 2t = 1$

i.e. $x = 2 \times 1 = 2$ metres to the right #

$v = \dot{x} = \dfrac{d}{dt}(2\cos 2t) = 2 \cdot -2\sin 2t = -4\sin 2t$

now $-1 \le \sin 2t \le 1$ $\therefore$ maximum velocity occurs when $\sin 2t = 1$

i.e. $v = -4 \times 1 = -4$ m/s #

13. (i) Duration of journey occurs when $v = 0$

i.e. $36\left(25t - 75t^2\right) = 0$

i.e. $25t(1 - 3t) = 0$

(ii) $x = e^t + 10e^{-2t}$

$v = \dot{x} = e^t - 20e^{-2t}$, at $t = 0, v = e^0 - 20e^0 = -19$ m/s

$a = \ddot{x} = e^t + 40e^{-2t}$, at $t = 0, a = e^0 + 40e^0 = 41$ m/s^2 #

(iii) Particle comes to rest when $v = 0$

i.e. $e^t - 20e^{-2t} = 0$

$e^t = 20e^{-2t}$

$e^{3t} = 20$

i.e. $t = \dfrac{\ln(20)}{3} = 0.998... \approx 1$ second #

9. Velocity: $v = \int 8\sin 3t \, dt = -\dfrac{8}{3}\cos 3t + C$

at $t = \pi, v = 3$

i.e. $3 = -\dfrac{8}{3}\cos 3\pi + C$

i.e. $C = 3 - \dfrac{8}{3} = \dfrac{1}{3}$

$\therefore$ $v = \dfrac{1}{3} - \dfrac{8}{3}\cos 3t$, at $t = 0$, $v = \dfrac{1}{3} - \dfrac{8}{3} \times 1 = -\dfrac{7}{3}$ m/s #

Displacement: $x = \int v \, dt = \int \dfrac{1}{3} - \dfrac{8}{3}\cos 3t \, dt = \dfrac{1}{3}t - \dfrac{8}{9}\sin 3t + C$

at $t = \pi, x = 4$

i.e. $4 = \dfrac{\pi}{3} - \dfrac{8}{9}\sin 3\pi + C$

$= \dfrac{\pi}{3} + C$

i.e. $C = 4 - \dfrac{\pi}{3}$

$\therefore x = \dfrac{1}{3}t - \dfrac{8}{9}\sin 3t + \left(4 - \dfrac{\pi}{3}\right)$

at $t = 0$, $x = \left(4 - \dfrac{\pi}{3}\right)$ m #

10. (i) $a = \dfrac{24t}{(3t^2 - 1)}$

$v = \int \dfrac{24t}{3t^2 - 1} dt = 4\int \dfrac{6t}{3t^2 - 1} dt = 4\log_e(3t^2 - 1) + C$

at $t = 1, v = \log_e 256$

i.e. $\log_e 256 = 4\log_e(3 - 1) + C$

$\log_e 2^8 = 4\log_e 2 + C$

$8\log_e 2 = 4\log_e 2 + C$

$\therefore$ $C = 8\log_e 2 - 4\log_e 2 = 4\log_e 2$

$\therefore$ $v = 4\log_e(3t^2 - 1) + 4\log_e 2 = 4\log_e\left[2(3t^2 - 1)\right] = 4\log_e(6t^2 - 2)$ #

(ii) Particle is stationary when $v = 0$

i.e. $4\log_e(6t^2 - 2) = 0$

i.e. $6t^2 - 2 = 1$

$6t^2 = 3$

$t^2 = \dfrac{1}{2}$

$\therefore$ $t = \dfrac{1}{\sqrt{2}}$ minutes (since $t > 0$) #

11. (i) Maximum height attained when $\dfrac{dh(t)}{dt} = 0$

i.e. $h(t) = 2 + 18t - 5t^2$

$\dfrac{dh(t)}{dt} = 18 - 10t = 0$

i.e. $18 = 10t$

$t = 1.8$ seconds #

(ii) Maximum height $= 2 + 18 \times (1.8) - 5 \times (1.8)^2 = 18.2$ metres #

12. $x = 2\cos 2t$,

now $-1 \le \cos 2t \le 1$ $\therefore$ maximum displacement occurs when $\cos 2t = 1$

i.e. $x = 2 \times 1 = 2$ metres to the right #

$v = \dot{x} = \dfrac{d}{dt}(2\cos 2t) = 2 \cdot -2\sin 2t = -4\sin 2t$

now $-1 \le \sin 2t \le 1$ $\therefore$ maximum velocity occurs when $\sin 2t = 1$

i.e. $v = -4 \times 1 = -4$ m/s #

13. (i) Duration of journey occurs when $v = 0$

i.e. $36(25t - 75t^2) = 0$

i.e. $25t(1 - 3t) = 0$

$t = 0, \frac{1}{3}$ hrs

i.e. time taken to travel from town A to town B is 20 min. #

(ii) Maximum velocity occurs when $\frac{dv}{dt} = 0$

i.e. $v = 36(25t - 75t^2)$

$\frac{dv}{dt} = 36(25 - 150t) = 0$

i.e. $25 = 150t$

$t = \frac{1}{6}$ hrs

$\therefore$ Maximum velocity $= 36\left(25\times\frac{1}{6} - 75\times\left(\frac{1}{6}\right)^2\right) = 75$ km/hr #

(iii) $x = \int v\,dt = 36\int 25t - 75t^2\,dt = 36\left(\frac{25t^2}{2} - 25t^3\right) + C$

at $t = 0, x = 0 \quad \therefore C = 0$

$\therefore x = 36\left(\frac{25t^2}{2} - 25t^3\right)$

at $t = \frac{1}{3}, x = 18\times 25\left(\frac{1}{3}\right)^2 - 36\times 25\left(\frac{1}{3}\right)^3 = 16\frac{2}{3}$ km #

14. $x = 2\sin(t-1) - t$

$v = \dot{x} = \frac{d}{dt}(2\sin(t-1) - t) = 2\cos(t-1) - 1$

now, $-1 \le \cos(t-1) \le 1$

$\therefore$ maximum velocity occurs when $\cos(t-1) = 1$ i.e. $v = 2\times 1 - 1 = 1$ m/s

maximum speed occurs when $|v|$ is maximum i.e. when $\cos(t-1) = -1$

$\therefore$ maximum speed $= |2\times -1 - 1| = |-3| = 3$ m/s #

15. (i) 15 m/s #

(ii) $t = 0, 4, 6$ seconds #

(iii) After $t = 4$ seconds i.e. when velocity changes from a positive value to a negative value. #

(iv) Distance, is area under the curve from $0 \le t \le 4$

i.e. distance $= \frac{1}{2}\times 15 + 2\times 15 + \frac{1}{2}\times 15 = 3\times 15 = 45$ metres #

... $= 45 - \left(\frac{2}{...}\times 10\right) = 45 - 10 = 35$

Applications of Calculus

TOPIC 17

(A) The Probability of an Event

The probability of an event can be defined as the likelihood of an event occurring.

Thus in a trial with n equally likely out comes, the probability that a particular event A occurs is given by:

$$P(A) = \frac{\text{the number of ways } A \text{ can occur}}{\text{the total possible outcomes } (n)}$$

Note : All probabilities must lie between 0 and 1 (i.e. $0 \le P(A) \le 1$)

Example 1: A card is drawn from a normal deck of 52 cards. Find the probability of:

(i) drawing an Ace.

(ii) drawing a diamond.

(iii) drawing a red card.

Solution 1:

(i) There are 4 Aces $\therefore P(\text{of drawing an Ace}) = \frac{4}{52} = \frac{1}{13}$ #

(ii) There are 13 diamonds $\therefore P(\text{of drawing a diamond}) = \frac{13}{52} = \frac{1}{4}$ #

(iii) There are 26 red cards $\therefore P(\text{of drawing a red card}) = \frac{26}{52} = \frac{1}{2}$ #

Example 2: An ordinary die is thrown. Find the probability of getting:

(i) a number less than 4.

(ii) an odd number.

(iii) an odd number less than 4.

Solution 2:

(i) There are 3 numbers less than 4 – namely 1, 2 and 3

$\therefore P(\text{number less than } 4) = \frac{3}{6} = \frac{1}{2}$ #

(ii) There are 3 odd numbers – namely 1, 3 and 5

$\therefore P(\text{number is odd}) = \frac{3}{6} = \frac{1}{2}$ #

$t=0, \frac{1}{3}$ hrs

i.e. time taken to travel from town A to town B is 20 min. #

(ii) Maximum velocity occurs when $\frac{dv}{dt}=0$

i.e. $v=36\left(25t-75t^2\right)$

$\frac{dv}{dt}=36(25-150t)=0$

i.e. $25=150t$

$t=\frac{1}{6}$ hrs

∴ Maximum velocity $=36\left(25\times\frac{1}{6}-75\times\left(\frac{1}{6}\right)^2\right)=75$ km/hr #

(iii) $x=\int v\,dt=36\int 25t-75t^2\,dt=36\left(\frac{25t^2}{2}-25t^3\right)+C$

at $t=0, x=0$ ∴ $C=0$

∴ $x=36\left(\frac{25t^2}{2}-25t^3\right)$

at $t=\frac{1}{3}, x=18\times25\left(\frac{1}{3}\right)^2-36\times25\left(\frac{1}{3}\right)^3=16\frac{2}{3}$ km #

14. $x=2\sin(t-1)-t$

$v=\dot{x}=\frac{d}{dt}\left(2\sin(t-1)-t\right)=2\cos(t-1)-1$

now, $-1\le\cos(t-1)\le1$

∴ maximum velocity occurs when $\cos(t-1)=1$ i.e. $v=2\times1-1=1$ m/s

maximum speed occurs when $|v|$ is maximum i.e. when $\cos(t-1)=-1$

∴ maximum speed $=|2\times-1-1|=|-3|=3$ m/s #

15. (i) 15 m/s #

(ii) $t=0, 4, 6$ seconds #

(iii) After $t=4$ seconds i.e. when velocity changes from a positive value to a negative value. #

(iv) Distance, is area under the curve from $0\le t\le4$

i.e. distance $=\frac{1}{2}\times15+2\times15+\frac{1}{2}\times15=3\times15=45$ metres #

(v) Displacement $=45-(\text{area from } t=4 \text{ to } t=6)=45-\left(\frac{2}{2}\times10\right)=45-10=35$

∴ particle is located 35 m to the right aft...

PROBABILITY

(A) The Probability of an Event

The probability of an event can be defined as the likelihood of an event occurring.

Thus in a trial with n equally likely out comes, the probability that a particular event A occurs is given by:

$$P(A)=\frac{\text{the number of ways } A \text{ can occur}}{\text{the total possible outcomes } (n)}$$

Note : All probabilities must lie between 0 and 1 (i.e. $0\le P(A)\le1$)

Example 1: A card is drawn from a normal deck of 52 cards. Find the probability of:

(i) drawing an Ace.

(ii) drawing a diamond.

(iii) drawing a red card.

Solution 1:

(i) There are 4 Aces ∴ $P(\text{of drawing an Ace})=\frac{4}{52}=\frac{1}{13}$ #

(ii) There are 13 diamonds ∴ $P(\text{of drawing a diamond})=\frac{13}{52}=\frac{1}{4}$ #

(iii) There are 26 red cards ∴ $P(\text{of drawing a red card})=\frac{26}{52}=\frac{1}{2}$ #

Example 2: An ordinary die is thrown. Find the probability of getting:

(i) a number less than 4.

(ii) an odd number.

(iii) an odd number less than 4.

Solution 2:

(i) There are 3 numbers less than 4 – namely 1, 2 and 3

∴ $P(\text{number less than } 4)=\frac{3}{6}=\frac{1}{2}$ #

(ii) There are 3 odd numbers – namely 1, 3 and 5

∴ $P(\text{number is odd})=\frac{3}{6}=\frac{1}{2}$ #

(iii) There are 2 odd numbers less than 4 – namely 1 and 3

$\therefore P(\text{odd number less than } 4) = \frac{2}{6} = \frac{1}{3}$ #

(B) The Probability an Event Does Not Occur

If $P(A)$ is the probability that event A occurs and $P(\overline{A})$ is the probability the event does not occur, then:

$$P(\overline{A}) = 1 - P(A)$$

Example 1: A bag contains 5 red, 4 white, 3 blue, 2 green and 1 black marble. One marble is drawn from the bag. Find the probability:

(i) it is white.

(ii) it is not white.

(iii) it is not black.

Solution 1:

(i) There are 4 white marbles out of 15 $\therefore$ $P(\text{marble is white}) = \frac{4}{15}$ #

(ii) $P(\text{marble is not white}) = 1 - P(\text{marble is white}) = 1 - \frac{4}{15} = \frac{11}{15}$ #

(iii) $P(\text{marble is not black}) = 1 - P(\text{marble is black}) = 1 - \frac{1}{15} = \frac{14}{15}$ #

Example 2: One card is selected at random from a pack of 52 cards. Find the probability that the card is:

(i) not a king.

(ii) not a red king.

(iii) not the king of hearts.

Solution 2:

(i) $P(\text{not a king}) = 1 - P(\text{it is a king}) = 1 - \frac{4}{52} = \frac{12}{13}$ #

(ii) $P(\text{not a red king}) = 1 - P(\text{it is a red king}) = 1 - \frac{2}{52} = 1 - \frac{1}{26} = \frac{25}{26}$ #

(iii) $P(\text{not the king of hearts}) = 1 - P(\text{it is the king of hearts}) = 1 - \frac{1}{52} = \frac{51}{52}$ #

Probability

(C) The 'Product' Theorem of Probability (The 'AND' Rule)

To find the probability that events A and B will both occur, we simply multiply together the probabilities of A and B, i.e.

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example 1: A bag A contains 5 black marbles and 3 white marbles and a bag B contains 2 black marbles and 2 white marbles.

Two marbles are chosen at random from each bag. Find the probability that:

(i) both marbles are white.

(ii) the marbles are different colours.

Solution 1:

(i) $P(\text{both marbles are white})$

$= P(\text{marble selected from bag A is white}) \times P(\text{marble selected from bag B is white})$

$= \frac{3}{8} \times \frac{2}{4} = \frac{3}{16}$ #

(ii) $P(\text{marbles are different colours})$

$= 1 - P(\text{marbles are the same colour})$

$= 1 - P(\text{both marbles are white}) - P(\text{both marbles are black})$

$= 1 - \frac{3}{16} - \left(\frac{5}{8} \times \frac{2}{4}\right)$

$= 1 - \frac{3}{16} - \frac{5}{16}$

$= \frac{1}{2}$ #

Example 2: The probability of relief from a cold with medicine X is $\frac{4}{5}$ whilst with cold medicine Y is $\frac{3}{4}$. Cold sufferer A takes X and cold sufferer B takes Y.

Find the probability that:

(i) both are cured.

(ii) only one is cured.

Probability

Solution 2:

(i) $P(\text{both are cured})$

$= P(\text{A is cured}) \times P(\text{B is cured})$

$= \frac{4}{5} \times \frac{3}{4} = \frac{3}{5}$ #

(ii) $P(\text{only one is cured})$

$= P(\text{A is cured and B is not cured}) + P(\text{B is cured and A is not cured})$

$= P(\text{A is cured}) \times P(\text{B is not cured}) + P(\text{B is cured}) \times P(\text{A is not cured})$

$= \frac{4}{5} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{5}$

$= \frac{4}{20} + \frac{3}{20} = \frac{7}{20}$ #

(D) The 'Addition' Theorem of Probability (The 'OR' Rule)

Two events (say A and B) are ***mutually exclusive*** if they cannot occur at the same time.

If two events are ***mutually exclusive***, then:

$$P(A \text{ or } B) = P(A) + P(B)$$

If two events are **not mutually exclusive**, then:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

Example 1: A pair of dice are thrown together at random. Find the probability that:

(i) the sum of two numbers thrown is either 11 or 12.

(ii) a 2 or 3 is thrown.

Solution 1:

There 36 equally likely outcomes when two dice are tossed. These are listed below:

11	21*	31*	41	51	61
12*	22*	32*	42*	52*	62*
13	23*	33*	43*	53*	63*
14	24*	34*	44	54	64
15	25*	35*	45	55	<u>65</u>
16	26*	36*	46	<u>56</u>	<u>66</u>

(i) $P(\text{the sum of the 2 numbers is either 11 or 12})$

$= P(\text{sum of the 2 numbers is 11}) + P(\text{sum of the 2 numbers is 12})$

$= \frac{3}{36}$

$= \frac{1}{12}$ # (see events underlined in the above table)

(ii) $P(\text{a 2 or a 3 is thrown})$

$= P(\text{a 2 is thrown}) + P(\text{a 3 is thrown}) - P(\text{a 2 and a 3 is thrown})$

$= \frac{11}{36} + \frac{11}{36} - \frac{2}{36}$

$= \frac{20}{36}$

$= \frac{5}{9}$ # (see events indicated with an *)

Example 2: A class of 30 pupils contains 18 pupils who are basketball fans, 14 who are football fans and 6 pupils who are neither. If one pupil is chosen at random, find the probability that this pupil is:

(i) either a basketball or a football fan.

(ii) both a basketball and a football fan.

Solution 2:

(i) $P(\text{either a basketball or a football fan})$

$= 1 - P(\text{neither a basketball or a football fan})$

$= 1 - \frac{6}{30}$

$= \frac{24}{30} = \frac{4}{5}$ #

(ii) $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{AB})$

i.e. $P(\text{either a basketball or a football fan})$

$= P(\text{a basketball fan}) + P(\text{football fan}) - P(\text{both basketball and football fan})$

i.e. $\frac{4}{5} = \frac{18}{30} + \frac{14}{30} - P(\text{both basketball and football fan})$

$\therefore \quad P(\text{both basketball and football fan}) = \frac{32}{30} - \frac{4}{5} = \frac{8}{30} = \frac{4}{15}$ #

Probability

(E) Tree Diagrams

Tree diagrams can be used in two and three stage experiments to find all possible outcomes of an experiment.

The following rules apply to tree diagrams.

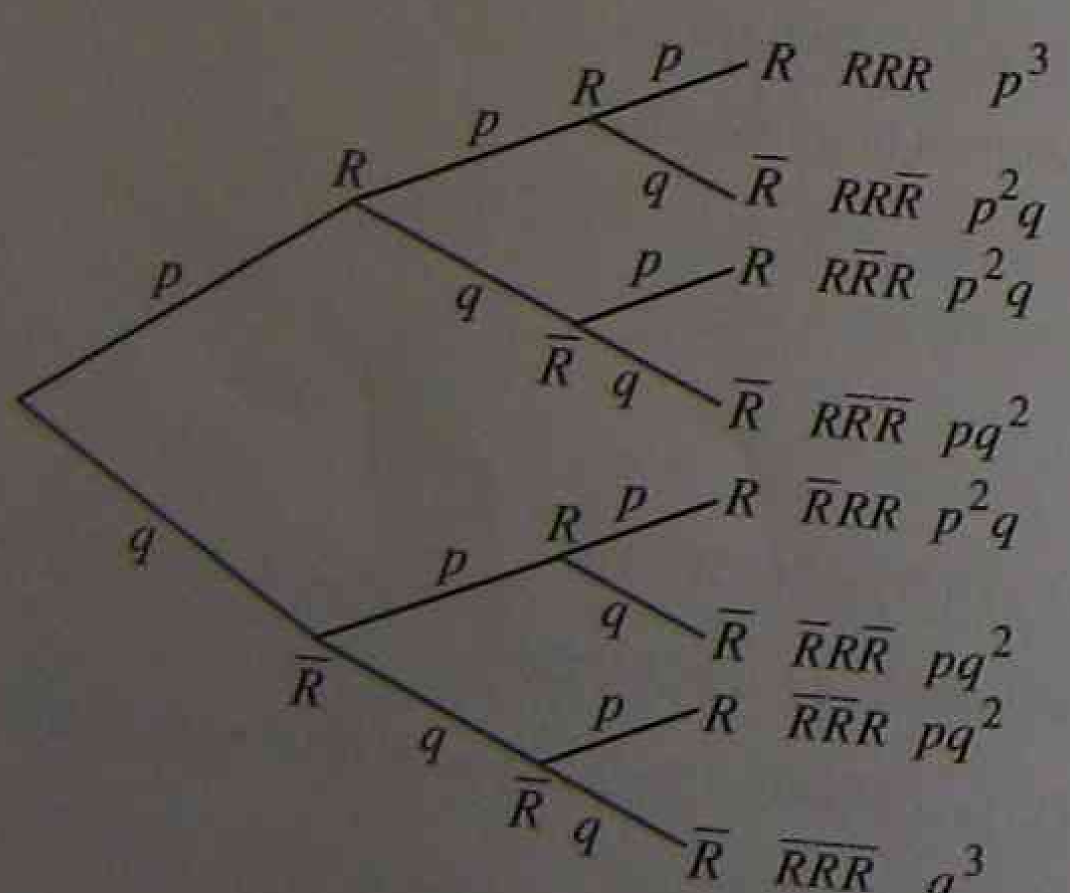
1. ***Multiply along the branches*** to determine the probability of each outcome.
2. Probabilities between the branches sum to 1.
3. With ***Independent Events*** (eg. tossing a coin), the probabilities between the branches remain constant.
4. With ***Dependant Events*** (eg. selecting marbles from a bag without replacement), the probabilities between the branches vary.

Example 1: The probability of rain in a particular city in the month of May is 0.4. Find the probability that over an interval of 3 days there is exactly 2 days of rain.

Solution 1:

Let $p=(0.4)$ and $q=(0.6)$ be respectively the probabilities for rain (R) and no rain $(\overline{R})$.

The tree diagram is as follows:



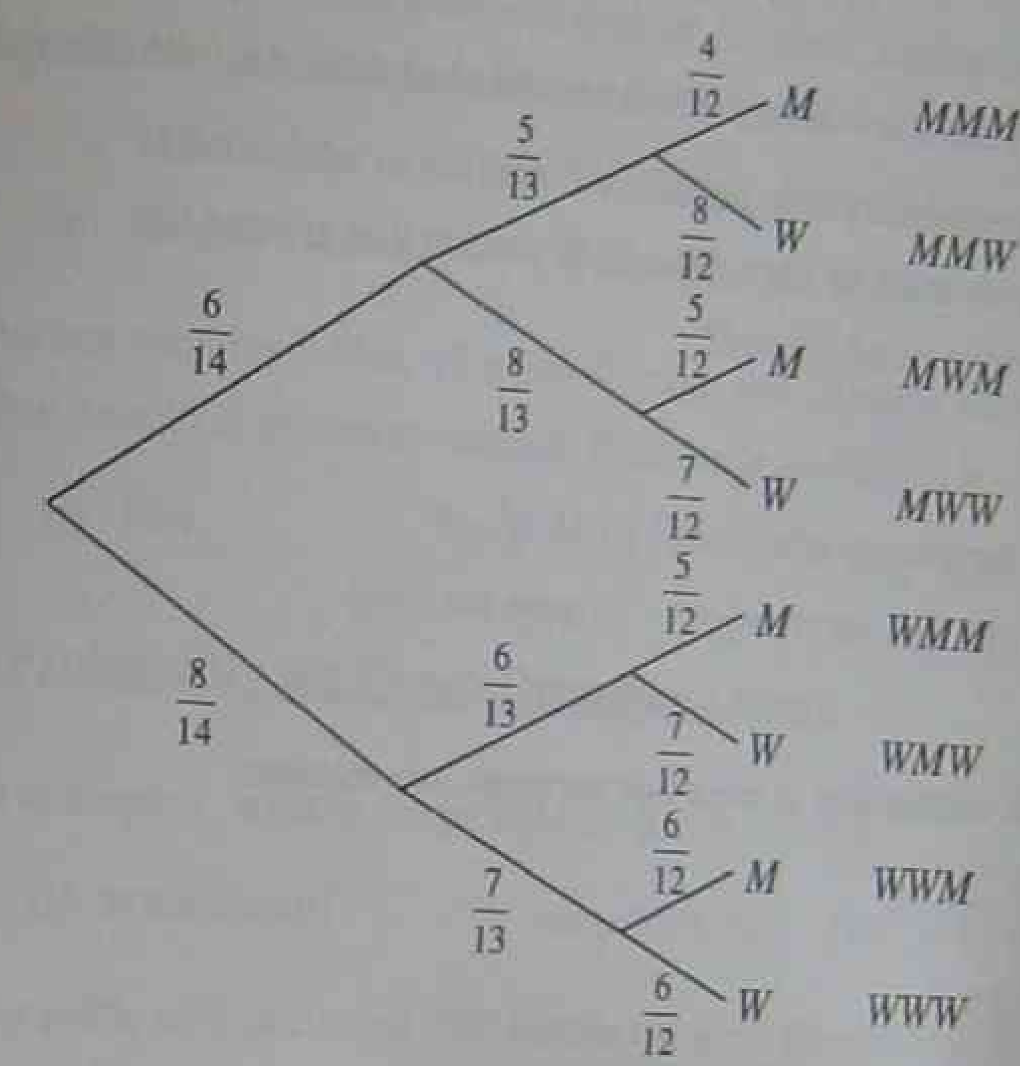
Required Probability $=P\left(RR\overline{R}\right)+P\left(R\overline{R}R\right)+P\left(\overline{R}RR\right)=3p^2q=\dfrac{36}{125}$ #

Example 2: From a group of 6 men and 8 women a committee of 3 is chosen at random. Find the probability that the committee will consist of:

(i) exactly one man.

(ii) a majority of women.

Solution 2:



(i) $P(\text{exactly one man})=P(MWW)+P(WWM)+P(WMW)$

$$=\left(\frac{6}{14}\times\frac{8}{13}\times\frac{7}{12}\right)+\left(\frac{8}{14}\times\frac{7}{13}\times\frac{6}{12}\right)+\left(\frac{8}{14}+\frac{6}{13}\times\frac{7}{12}\right)$$

$$=\frac{2}{13}+\frac{2}{13}+\frac{2}{13}=\frac{6}{13} \quad \#$$

(ii) $P(\text{a majority of women})=P(MWW)+P(WMW)+P(WWM)+P(WWW)$

$$=\frac{2}{13}+\frac{2}{13}+\frac{2}{13}+\frac{2}{13}=\frac{8}{13} \quad \#$$

(F) The Probability of 'At Least One'

$P(\text{at least one})=1-P(\text{event does not occur})$

Example 1: In the previous example what is the probability of at least one woman on the committee?

Solution 1:

$$
\begin{aligned}
P(\text{at least one woman}) &= 1 - P(\text{no women on the committee}) \\
&= 1 - P(MMM) \\
&= 1 - \frac{6}{14} \times \frac{5}{13} \times \frac{4}{12} = \frac{86}{91}
\end{aligned}
$$

Example 2: The probability that a new born fish will survive to adulthood is 0.05. If n fish are born:

(i) what is the probability than no fish will survive to adulthood?

(ii) how many fish must be born to be 95% certain that at least one fish will survive to adulthood?

Solution 2:

(i) $P(\text{no fish survives to adulthood}) = (0.95)^n$ #

(ii) $P(\text{at least one fish survives}) = 1 - P(\text{none survive})$

$$= 1 - (0.95)^n$$

to be 95% certain that at least one survives, we require:

$1 - (0.95)^n > 0.95$

$0.05 > (0.95)^n$

$\ln(0.05) > \ln(0.95)^n$

$\ln(0.05) > n\ln(0.95)$

$\therefore n > \dfrac{\ln(0.05)}{\ln(0.95)} = 58.403...$ (not the change of sign as $\ln(0.95)$ is negative).

$\therefore$ 59 fish must be born to be 95% certain that at least one will survive to adulthood #

REVIEW EXERCISES

(A) The Probability of an Event

1. A card is drawn from a normal deck of 52 cards. Taking the Ace to be the highest card,find the probability that the card is:

   (i) a club (ii) black (iii) a jack

   (iv) king of diamonds (v) less than 10

2. An urn contains 9 blue, 12 white, 7 green and 2 yellow marbles. One marble is chosen at random. Find the probability it is:

   (i) blue (ii) green (iii) blue or white

(B) The Probability an Event Does Not Occur

3. In example 1, what is the probability that the card is:

   (i) not a queen? (ii) not a diamond? (iii) not a picture card?

4. In a raffle with one prize, 500 tickets are sold. Brenda buys 10 tickets and Kelly buys 5 tickets. What is the probability that:

   (i) Brenda does not win a prize? (ii) Kelly does not win a prize?

5. A pair of dice are thrown and the numbers on the upper most faces are added. Find the probability that the sum is more than 3.

(C) The 'Product' Theorem of Probability (The 'AND' Rule)

6. From a box containing 5 white, 3 red and 2 black jellybeans, three jellybeans are chosen at random and eaten. Find the probability that:

   (i) all jellybeans were red (ii) none of the jellybeans were white

7. A bag contains 4 white balls and 3 black balls. Two balls are drawn from the bag in succession. Find the probability that both balls are white if:

   (i) The first ball drawn is put in the bag before the second ball is drawn.

   (ii) The first ball drawn is not put back in the bag before the second ball is drawn.

8. Two cards are chosen from the five cards shown below without replacement.

−2	−1	0	1	2

Find the probability that the product of the two numbers is **not** negative.

(D) The 'Addition' Theorem of Probability (The 'OR' Rule)

9. In a group of 60 students, there are 20 who take French, 15 who take Japanese and 5 who take both languages. If one student is chosen at random, what is the probability that this student:

   (i) takes French or Japanese? (ii) takes neither French nor Japanese?

10. In a small country town of 2000 people. 1200 people are the descendants of clan A, 950 are the descendants of clan B, while 450 are not descendants of either clan. If one person is chosen at random, find the probability that this person is a descendant of:

   (i) both clan A and B. (ii) clan B only.

(E) Tree Diagrams

11. A biased coin has the probability of $\frac{3}{5}$ of showing a head and $\frac{2}{5}$ of showing a tail. This coin is tossed three times. Find the probability of getting:

   (i) three heads or three tails. (ii) no tails.

12. A family consists of 4 children: Linda, Mary, Nancy and Kent. Two children are selected at random to mow the lawn. Find the probability of choosing:

   (i) two girls. (ii) Kent but not Linda.

(F) The Probability of 'At Least One'

13. In a box of 12 batteries, 3 are defective and 9 are good. If 3 batteries are chosen at random find the probability of at least one good battery.

14. The probability that a seed if planted will grow into a healthy tree is $\frac{3}{4}$. If n seeds are planted:

   (i) find the probability that no seeds will grow into healthy trees.

   (ii) how many seeds must be planted to be 99% certain that at least one tree will grow to be healthy?

Probability

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $P(\text{a club}) = \frac{13}{52} = \frac{1}{4}$ #

   (ii) $P(\text{a black card}) = \frac{26}{52} = \frac{1}{2}$ #

   (iii) $P(\text{a jack}) = \frac{4}{52} = \frac{1}{13}$ #

   (iv) $P(\text{king of diamonds}) = \frac{1}{52}$ #

   (v) $P(\text{card is less than } 10) = \frac{4\times 8}{52} = \frac{32}{52} = \frac{8}{13}$ #

2. (i) $P(\text{blue}) = \frac{9}{30} = \frac{3}{10}$ #

   (ii) $P(\text{green}) = \frac{7}{30}$ #

   (iii) $P(\text{blue or white}) = \frac{21}{30} = \frac{7}{10}$ #

3. (i) $P(\text{not a queen}) = 1 - P(\text{card is a queen}) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$ #

   (ii) $P(\text{not a diamond}) = 1 - P(\text{card is a diamond}) = 1 - \frac{13}{52} = 1 - \frac{1}{4} = \frac{3}{4}$ #

   (iii) $P(\text{not a picture card}) = 1 - P(\text{card is a picture card})$

   $= 1 - \frac{12}{52} = 1 - \frac{3}{13} = \frac{10}{13}$ #

4. (i) $P(\text{Brenda does not win}) = 1 - P(\text{Brenda wins}) = 1 - \frac{10}{500} = 1 - \frac{1}{50} = \frac{49}{50}$ #

   (ii) $P(\text{Kelly does not win}) = 1 - P(\text{Kelly wins}) = 1 - \frac{5}{500} = 1 - \frac{1}{100} = \frac{99}{100}$ #

5. $P(\text{sum is more than } 3) = 1 - P(\text{sum is less than or equal to } 3)$

   now, $P(\text{sum is less than or equal to } 3) = \frac{3}{36} = \frac{1}{12}$ $\{(1,2),(2,1),(1,1)\}$

   $\therefore$ $P(\text{sum is more than } 3) = 1 - \frac{1}{12} = \frac{11}{12}$ #

6.

(i) $P(\text{all 3 jellybeans were red}) = P(1^{st}\text{ was red}) \times P(2^{nd}\text{ was red}) \times P(3^{rd}\text{ was red})$

$$= \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8}$$

$$= \frac{1}{120} \quad \#$$

(ii) $P(\text{none of the 3 jellybeans were white}) = P(\text{all 3 jellybeans were red or black})$

$$= \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8}$$

$$= \frac{1}{12} \quad \#$$

7. (i) $P(\text{both balls are white}) = P(1^{st}\text{ is white}) \times P(2^{nd}\text{ is white}) = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$ #

(ii) $P(\text{both balls are white without replacement}) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$ #

8. $P(\text{product of the numbers is not negative})$

$= 1 - P(\text{product of the numbers is negative})$

now,

$P(\text{product of the numbers is negative})$

$= P(1^{st}\text{ number is positive}) \times P(2^{nd}\text{ number is negative})$

$\quad + P(1^{st}\text{ number is negative}) \times P(2^{nd}\text{ number is positive})$

$$= \frac{2}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{2}{4} = \frac{2}{5}$$

$\therefore P(\text{product of the numbers is not negative}) = 1 - \frac{2}{5} = \frac{3}{5}$ #

9. (i) $P(\text{takes French or Japenese}) = P(\text{takes French}) + P(\text{takes Japanese})$

$\quad - P(\text{takes both French and Japanese})$

$$= \frac{20}{60} + \frac{15}{60} - \frac{5}{20} = \frac{30}{60} = \frac{1}{2} \quad \#$$

(ii) $P(\text{takes neither French or Japanese}) = 1 - P(\text{takes French or Japanese})$

$$= 1 - \frac{1}{2} = \frac{1}{2} \quad \#$$

10. (i) Let x be the number of people that are descendants of both clans A and B

$\therefore \quad 450 + 1200 + 950 - x = 2000$

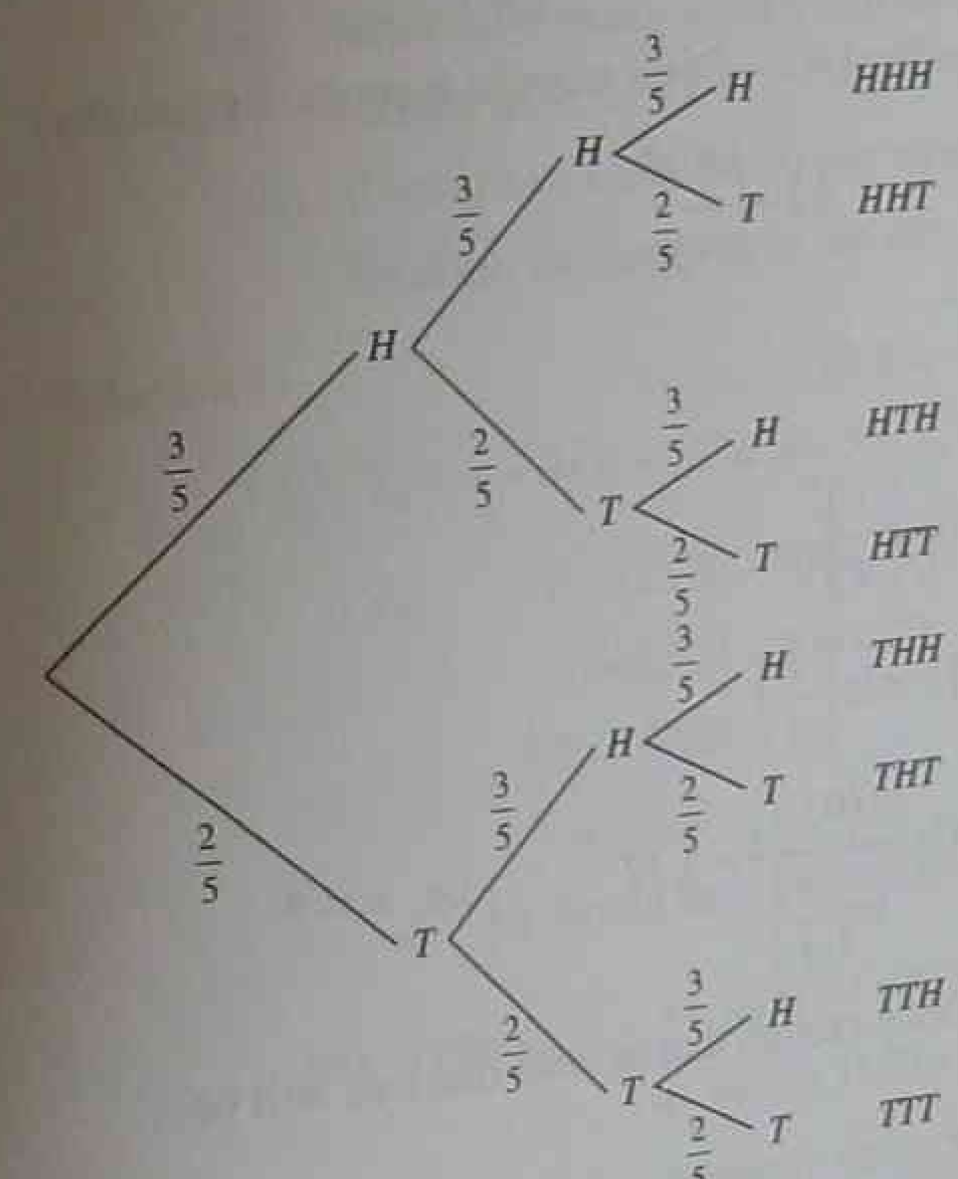
i.e. $\quad x = 600$

$\therefore \quad P(\text{descendant of both clan A and B}) = \frac{600}{2000} = \frac{3}{10}$ #

(ii) Number of descendant of clan B only $= 950 - 600 = 350$

$\therefore \quad P(\text{descendant of clan B only}) = \frac{350}{2000} = \frac{7}{40}$ #

11.



(i) $P(3\text{ heads or 3 tails})$

$= P(3\text{ heads}) + P(3\text{ tails})$

$$= \left(\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}\right)$$

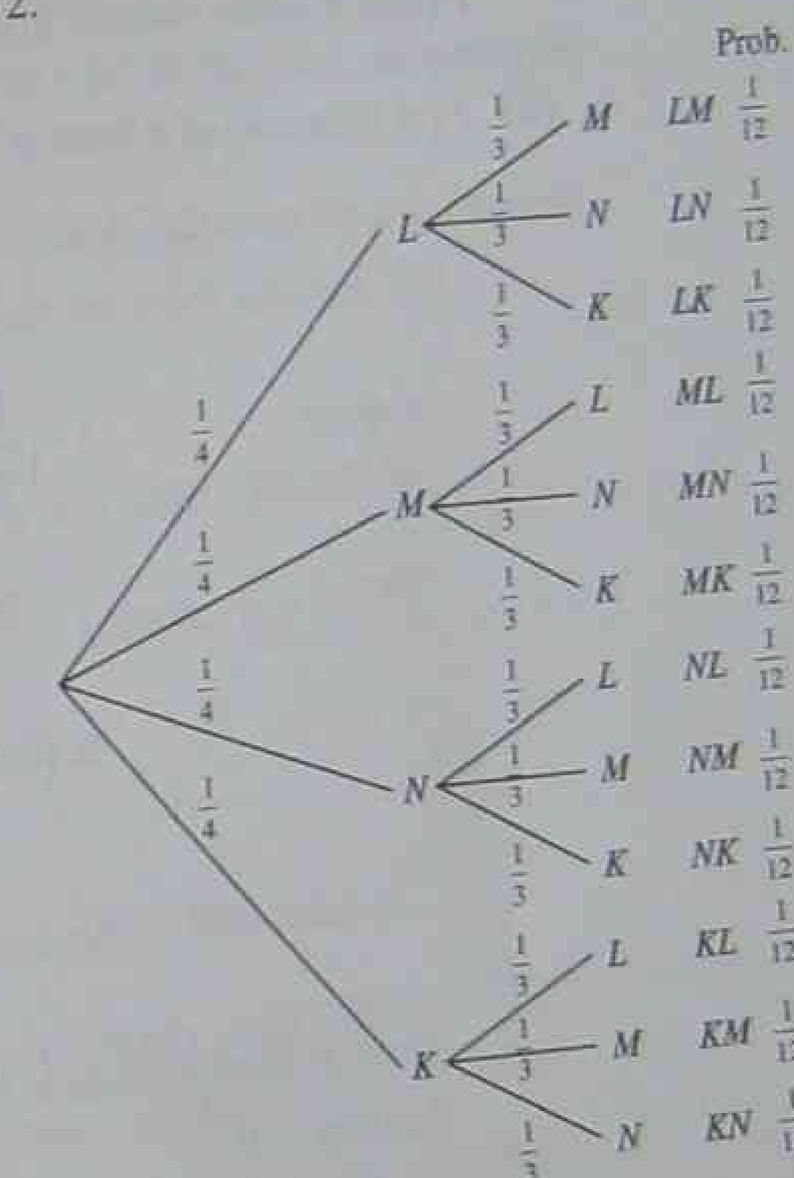
$$= \frac{7}{25} \quad \#$$

(ii) $P(\text{No tails}) = P(3\text{ heads})$

$$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$$

$$= \frac{27}{125} \quad \#$$

12.



(i) $P(\text{two girls})$

$= P(LM) + P(LN) + P(ML)$

$\quad + P(MN) + P(NL) + P(NM)$

$$= 6 \times \frac{1}{12}$$

$$= \frac{1}{2} \quad \#$$

(ii) $P(\text{Kent but not Linda}) = P(MK) + P(NK) + P(KM) + P(KN)$

$$= 4 \times \frac{1}{12}$$

$$= \frac{1}{3} \quad \#$$

13. $P(\text{at least one good batteries}) = 1 - P(\text{all batteries defective})$

$$= 1 - \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10}$$

$$= \frac{210}{220} \#$$

14. (i) $P(\text{that one seed does not grow into a healthy tree}) = \frac{1}{4}$

$P(\text{that } n \text{ seeds will not grow into healthy trees}) = \left(\frac{1}{4}\right)^n$ #

(ii) $P(\text{at least one of } n \text{ trees is healthy}) = 1 - P(\text{no trees in } n \text{ seeds are healthy})$

$$= 1 - \left(\frac{1}{4}\right)^n$$

$\therefore$ require $1 - \left(\frac{1}{4}\right)^n > 0.99$

$$0.01 > \left(\frac{1}{4}\right)^n$$

$$\ln(0.01) > n \ln\left(\frac{1}{4}\right)$$

$$n > \frac{\ln(0.01)}{\ln\left(\frac{1}{4}\right)} = 3.32 \qquad \text{i.e. } n = 4$$

$\therefore$ need to plant 4 seeds to be 99% certain that at least one tree will be healthy #

TOPIC 18

SERIES APPLICATIONS

(A) Applications of Arithmetic and Geometric Series

Many practical problems can be reduced to arithmetic or geometric series

Example 1: On a certain date a farmer has 3,600 hens. One week later he sells 30 hens and continues to do this each week.

(i) After how many weeks will he have sold all his hens, assuming no further increases?

(ii) If each hen costs \$4.00 per week to feed, calculate the total cost of feeding the hens up to the day when his stock reaches 750.

Solution 1:

(i) $T_n = a + d(n-1),\ d = -30,\ T_n = 0,\ a = 3600 - 30 = 3570$

i.e. $T_n = 3570 - 30(n-1)$ (check: $T_1 = 3570$)

$$0 = 3570 - 30n + 30$$

$$30n = 3600$$

$$n = 120$$

$\therefore$ he would have sold all his hens after 120 weeks #

(ii) Total Cost $= 4 \times 3600 + 4 \times 3570 + 4 \times 3540 + \ldots + 4 \times 750$

$$= 4(3600 + 3570 + 3540 + \ldots + 750)$$

which is an arithmetic series with $a = 3600,\ d = -30,\ l = 750$ to find the number of terms solve:

$$T_n = a + d(n-1) \quad \text{for } T_n = 750$$

i.e. $750 = 3600 - 30(n-1)$

$$= 3630 - 30n$$

$$30n = 2880$$

$$n = 96$$

$\therefore$ $S_n = \frac{n}{2}(a + l)$

$$= \frac{96}{2}(3600 + 750) = 208,800$$

$\therefore$ total cost of feeding $= 4 \times 208,800 = \$835,200$ #

Example 2: A tennis ball is released from the top of a building 25 m high onto a flat surface. Assuming no air resistance and that the ball bounces back to half the height it has fallen after each successive bounce, find:

(i) the height the ball bounces back to after the 4^{th} bounce.

(ii) at which bounce will the height reached by the ball first be below 20 cm.

Solution 2:

The bouncing pattern of the ball follows a geometric series:

i.e. $T_n = ar^{n-1}$ where $r = 0.5$ and $a = 12.5$

i.e. $T_n = 12.5(0.5)^{n-1}$

(note that after 1 bounce the height of the ball $= 25 \times 0.5 = 12.5$ cm)

(i) $T_4 = 12.5(0.5)^3 = 1.5625$ m #

(ii) $0.2 = 12.5(0.5)^{n-1}$

$0.016 = (0.5)^{n-1}$

$\ln(0.016) = (n-1)\ln(0.5)$

$n = 1 + \dfrac{\ln(0.016)}{\ln(0.5)} = 6.966$

$\therefore$ the first time the ball will bounce back to a height less than 20 cm will be on its 7^{th} bounce. #

(B) Compound Interest

The amount to which an initial amount of money P accumulates to, if invested at r% per period for a duration of n periods is:

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

The compound interest earned over the period will be the difference between the accumulated amount A_n and the initial investment P.

Compound Interest Earned $= A_n - P$

Example 1: Kathy invests \$10,000 at a rate of 6% p.a. for five years. If the interest is compounded annually, find:

(i) the accumulated value at the end of five years.

(ii) the compound interest earned over the period.

Solution 1:

(i) $A_n = P\left(1 + \dfrac{r}{100}\right)^n$, $P = 10{,}000$, $r = 6$, $n = 5$

$= 10000\left(1 + \dfrac{6}{100}\right)^5 = 10000(1.06)^5 = \$13{,}382.26$ #

(ii) $C.I. = A_n - P = 13{,}382.26 - 10{,}000 = \$3{,}382.26$ #

Example 2: If \$10,000 is invested in an account paying an annual rate of 6 % compounded monthly, how many full years and months will it be until the balance is greater than \$100,000?

Solution 2:

$A_n = P\left(1 + \dfrac{r}{100}\right)^n$, $r = \dfrac{6}{12} = 0.5$, $P = 10{,}000$, $A_n = 100{,}000$

i.e. $10{,}000\left(1 + \dfrac{0.5}{100}\right)^n = 100{,}000$

$10000(1.005)^n = 100{,}000$

$(1.005)^n = 10$

$n\ln(1.005) = \ln 10$

$n = \dfrac{\ln(10)}{\ln(1.005)} = 461.67$ months

i.e. after 38 years and 6 months the balance will have first exceeded \$100,000. #

(C) Superannuation

In superannuation questions, a fixed amount (or contribution) is invested each period over a number of years.

The accumulated values of each contribution is determined (using the previous formula for A_n) and summed to calculate the total investment at the end of the period.

$$\text{Total Investment} = P\left(1 + \frac{r}{100}\right)\left[\frac{\left(1 + \frac{r}{100}\right)^n - 1}{\frac{r}{100}}\right]$$

where: P = value of each contribution.
r = interest earned per period.

Example 1: Jack invests \$1,500 at the beginning of each year into an account earning 5% p.a. compounded annually. How much will this investment be after 7 years?

Solution 1:

- The first \$1,500 investment earns interest for 7 years:

  i.e. $A_7 = 1,500\left(1+\frac{5}{100}\right)^7 = 1,500(1.05)^7$

- The second \$1,500 invested earns interest for 6 years:

  i.e. $A_6 = 1500(1.05)^6$

  .

  .

  .

- The last \$1,500 invested earns interest for 1 year:

  i.e. $A_1 = 1500(1.05)^1$

$\therefore$ **Total value of investment**

$= 1500(1.05) + 1500(1.05)^2 + \ldots + 1500(1.05)^6 + 1500(1.05)^7$

which is a geometric series with $a = 1500(1.05)$, $r = 1.05$, $n = 7$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\therefore \text{ Total} = 1500(1.05)\left(\frac{(1.05)^7 - 1}{1.05 - 1}\right) = \$12,823.66 \quad \#$$

Example 2: Eve invests \$100 at the beginning of each month into a superannuation fund. If the fund pays interest at 6% per annum compounded monthly, what will Eve's superannuation be worth at the end of 35 years?

Solution 2:

The calculation must be based on **Months**

the monthly contribution = \$100

the monthly interest rate $= \frac{6}{12} = 0.5\%$

the number of months $= 35 \times 12 = 420$

$$\therefore \text{ Total} = 100\left(1+\frac{0.5}{100}\right)^{420} + 100\left(1+\frac{0.5}{100}\right)^{419} + \ldots + 100\left(1+\frac{0.5}{100}\right)^2 + 100\left(1+\frac{0.5}{100}\right)^1$$

$$= 100(1.005)^1 + 100(1.005)^2 + \ldots + 100(1.005)^{420}$$

which is a geometric series with $a = 100(1.005)$, $r = 1.005$, $n = 420$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\therefore \text{ Total} = 100(1.005)\left[\frac{(1.005)^{420} - 1}{1.005 - 1}\right] = \$143,183.39 \quad \#$$

(D) Loan Repayments

In loan repayments, a certain amount is borrowed and regular instalments are made at the end of each period untill the loan has been repaid. This means that the amount owing at maturity reduces to zero.

Students are typically required to determine the value of each periodic payment.

$$A_n = X\left(1+\frac{r}{100}\right)^n - M\frac{\left(1+\frac{r}{100}\right)^n - 1}{\left(\frac{r}{100}\right)} = 0$$

where: X = amount borrowed.

r = interest charged per period.

M = periodic payment.

A_n = amount owing after n periods ($A_n = 0$ at maturity).

Example 1: A loan of \$15,000 is to be repaid by equal monthly instalments, repayments commencing at the end of the first month of the loan. The interest is calculated at the rate of 9.6% per annum compounded monthly. If the monthly instalment is M dollars, prove that:

(i) The amount owing at the end of the 2nd month is $\$15,240.96 - 2.008M$

(ii) If the loan including interest is exactly repaid at the end of 5 years find the monthly repayment M.

Solution 1:

The calculation must be based on months

monthly interest $= \frac{9.6}{12}\% = 0.8\%$

number of months $= 5 \times 12 = 60$

Let A_n be the amount owing after n months.

Series Applications

(i) The amount owing after 1 month will be the accumulated value of the $15,000 plus interest less one instalment:

i.e. $A_1 = 15{,}000\left(1+\frac{0.8}{100}\right)^1 - M$

$= 15{,}000(1.008)^1 - M$

$A_2 = A_1 \times (1.008) - M$

$= (15{,}000(1.008) - M) \times 1.008 - M$

$= 15{,}000(1.008)^2 - M(1 + 1.008)$

$= 15{,}240.96 - 2.008M$ as required #

(ii) Now, generalising the above result:

$$A_{60} = 15000(1.008)^{60} - M\left(1 + 1.008 + (1.008)^2 + \ldots + (1.008)^{59}\right)$$

now, the part in brackets is a geometric series with $a = 1$, $r = 1.008$, $n = 60$

$$\therefore A_{60} = 15000(1.008)^{60} - M\frac{a\left(r^n - 1\right)}{r-1}$$

$$= 15000(1.008)^{60} - M\frac{1\left((1.008)^{60} - 1\right)}{1.008 - 1}$$

$$= 24.194.86402 - 76.62386683M$$

since the loan is repaid in 60 months $\therefore A_{60} = 0$

i.e. $M = \$315.76$ per month #

REVIEW EXERCISES

(A) Applications of Arithmetic and Geometric Series

1. An office worker is employed at an initial salary of $23,000 per annum. After each year this salary is increased by $1,500.
   (i) What is the salary earned in the 15th year of service?
   (ii) What is the total salary earned for the first 15 years?

2. The membership for a newly opened health club has an average increase of 60 new members a month. If it had 100 members initially and the cost of membership is $30/month find the total takings for the first year of its operation.

3. A plant grows 12 cm in the first month that it is planted and in subsequent months grows an additional $\frac{2}{3}$ of the length grown in the preceeding month. If a plant has just been planted, find:
   (i) the amount the plant has grown in the 6th month.
   (ii) the height of the plant after 6 months.

4. A hospital patient receives 10mg dose of medicine at the beginning of each day. Prior to the next dose, the amount of medicine in the patient's body reduces to $\frac{3}{5}$ of the amount present. Find:
   (i) the amount of medicine in the body on the 7th day, just after the next dose is given. Give your answer correct to one decimal place.
   (ii) Does the medicine present in the patient's body ever exceed 25 mg?

(B) Compound Interest

5. Find the compound interest earned if $3,000 is invested for 4 years at:
   (i) 7% p.a. compounded annually.
   (ii) 6% p.a. compounded quarterly.
   (iii) 5.4% p.a. compounded monthly.

6. A car valued at $40,000 is depreciated at the rate of 20% p.a. Find the depreciated value of the car after 4 years.

7. If a property doubled in price over a period of eight years, find the implied rate of annual compound interest earned on the property.

8. Find the difference between the compound interest and simple interest earned on \$10,000 at 5% p.a. compounded annually for 3 years.

(C) Superannuation

9. When Adrian was born his mother deposited \$1,000 into a Trust Account earning 5.5% p.a. compounded annually. On each subsequent birthday up to and including his seventeenth birthday she deposited \$500 into this account.

   (i) Show that the balance of the account on Adrian's 3rd birthday is given by:

   $$1000(1.055)^3+500\left(1+1.055+1.055^2\right)$$

   (ii) Hence, calculate the total amount that was in the account on Adrian's eighteenth birthday.

10. Mary invests \$2,500 at the beginning of each year into a superannuation fund for 25 years. The fund earns interest at 6.00% p.a. compounded half yearly.

    (i) Show that Mary's initial deposit accumulates to \$10,959.77 after 25 years.

    (ii) What will Mary's superannuation be worth at the end of 25 years?

(D) Loan Repayments

11. Mr Carter borrows \$210,000 to purchase a unit at 6.60% p.a. compounded monthly for a term of 25 years and agrees to repay the loan in equal monthly instalments.

    (i) Calculate the value of each monthly instalment.

    (ii) Calculate the equivalent simple monthly interest rate charged on the loan, correct to 2 decimal places.

12. Jackie borrows \$3,000 from her friend Julie to buy a new computer. The interest is calculated at the rate of 4% per annum compounded quarterly.

    Jackie intends to repay the loan with interest in two equal instalments of \$M. Find M, if the first instalment is in six months time and the second is in twelve months time.

WORKED SOLUTIONS TO REVIEW EXERCISES

1. (i) $T_n=a+d(n-1),\ d=1500,\ n=15,\ a=23{,}000$

   i.e. $T_{15}=23{,}000+1500(15-1)=\$44{,}000$ #

   (ii) Total salary earned for the first 15 years $=23{,}000+24{,}500+\ldots+44{,}000$

   $S_n=\frac{n}{2}(a+l)$ i.e. $S_{15}=\frac{15}{2}(23{,}000+44{,}000)=\$502{,}500$ #

2. Memberships $=100+160+220+280+\ldots$

   which is an arithmetic series with $a=100$, $d=60$, $n=12$

   $\therefore\ S_n=\frac{n}{2}\left[2a+d(n-1)\right]$

   i.e. $S_{12}=\frac{12}{2}\left[2\times100+60(12-1)\right]=5{,}160$

   $\therefore$ Total takings $=5{,}160\times30=\$154{,}800$ #

3. (i) Amount grown each month $=12,\ 12\times\frac{2}{3},\ 12\left(\frac{2}{3}\right)^2,\ 12\left(\frac{2}{3}\right)^3,\ldots$

   which is a geometric series with $a=12,\ r=\frac{2}{3},\ n=6$

   i.e. $T_n=ar^{n-1}\quad\therefore\ T_6=12\left(\frac{2}{3}\right)^5=1\frac{47}{81}$ cm #

   (ii) Height of plant $=12+12\times\frac{2}{3}+12\left(\frac{2}{3}\right)^2+\ldots+12\left(\frac{2}{3}\right)^5$

   now, $S_n=\dfrac{a\left(1-r^n\right)}{1-r}\quad\therefore\ S_6=\dfrac{12\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}}=32.8$ cm to 1 d.p. #

4. (i) Amount of medicine in the body on 1st day $=10$ mg

   Amount of medicine in the body on 2nd day $=10+10\left(\frac{3}{5}\right)$ mg

   Amount of medicine in the body on 3rd day $=\left[10+10\left(\frac{3}{5}\right)\right]\left(\frac{3}{5}\right)+10$

   $=10\left(1+\frac{3}{5}+\left(\frac{3}{5}\right)^2\right)$ mg

Generalising this result:

The amount of medicing in the body on 7^{th} day $= 10\left(1+\frac{3}{5}+\left(\frac{3}{5}\right)^2+\ldots+\left(\frac{3}{5}\right)^6\right)$

The part in brackets is a geometric series with $a=1,\ r=\frac{3}{5},\ n=7$

i.e. $S_n = \dfrac{a(1-r^n)}{1-r}$ $\therefore\ S_7 = \dfrac{1\left(1-\left(\frac{3}{5}\right)^7\right)}{1-\frac{3}{5}} = 2.430016$

$\therefore$ Total amount of medicine on 7^{th} day $= 10\times 2.430016 = 24.3$ mg to 1d.p. #

(ii) From (i), total medicine present $= 10+10\left(\frac{3}{5}\right)+10\left(\frac{3}{5}\right)^2+\ldots$

which is an infinite geometric series with $a=10,\ r=\frac{3}{5}$

i.e. $S_\infty = \dfrac{10}{1-\frac{3}{5}} = \dfrac{10}{\frac{2}{5}} = 25$

$\therefore$ the maximum amount of medicine in the patient's body is 25 mg as time increases indefinitely, and thus never exceeds 25 mg #

5. $A_n = P\left(1+\frac{r}{100}\right)^n,\ C.I. = A_n - P$

(i) $A_n = 3000\left(1+\frac{7}{100}\right)^4 = \$3{,}932.39$

$C.I. = A_n - P = 3{,}932.39 - 3000 = \$932.39$ #

(ii) $A_n = 3000\left(1+\frac{1.5}{100}\right)^{16} = \$3{,}806.96$ $\left(\text{note: } r = \frac{6}{4} = 1.5\right)$

$C.I. = A_n - P = 3{,}806.96 - 3000 = \$806.96$ #

(iii) $A_n = 3000\left(1+\frac{0.45}{100}\right)^{48} = \$3{,}721.50$ $\left(\text{note: } r = \frac{5.4}{12} = 0.45\right)$

$C.I. = A_n - P = 3{,}721.50 - 3000 = \$721.50$ #

6. Depreciated value $= 40{,}000\left(1-\frac{20}{100}\right)^4 = 40000(0.8)^4 = \$16{,}384$ #

7. $P\left(1+\frac{r}{100}\right)^8 = 2P$

i.e. $\left(1+\frac{r}{100}\right)^8 = 2$

i.e. $\ln\left(1+\frac{r}{100}\right)^8 = \ln 2$

$8\ln\left(1+\frac{r}{100}\right) = \ln 2$

$1+\frac{r}{100} = e^{\frac{\ln 2}{8}}$

i.e. $r = 9.05\%$ p.a. compounded annually #

8. $A_n = 10{,}000\left(1+\frac{5}{100}\right)^3 = 10000(1.05)^3 = \$11{,}576.25$

$C.I. = A_n - P = 11{,}576.25 - 10{,}000 = \$1{,}576.25$

Simple Interest $= \dfrac{Prn}{100} = \dfrac{10000\times 5\times 3}{100} = \$1{,}500$

$\therefore\ C.I. - S.I. = 1{,}576.25 - 1{,}500 = \$76.25$ #

9. (i) Let A_n be the account balance on the n^{th} birthday:

i.e. $A_1 = 1000(1.055)+500$

$A_2 = A_1(1.055)+500$
$= [100(1.055)+500](1.055)+500$
$= 1000(1.055)^2 + 500(1+1.055)$

$A_3 = A_2(1.055)+500$
$= [100(1.055)^2+500(1+1.055)](1.055)+500$
$= 1000(1.055)^3+500(1+1.055+1.055^2)$ #

(ii) Extending the result of (i), the account balance on the 17^{th} birthday is:

$A_{17} = 1000(1.055)^{17} + 500(1+1.055+\ldots+1.055^{16})$

Since no deposit is made on the eighteenth birthday

$\therefore$ the account balance on the eighteenth birthday is simply the account balance on the seventeenth birthday accumulated for one year.

Series

i.e. $A_{18} = A_{17} \times (1.055)$

$$= \left[1000(1.055)^{17} + 500\left(1 + 1.055 + ... + 1.055^{16}\right)\right](1.055)$$

$$= 1000(1.055)^{18} + 500(1.055)\left(1 + 1.055 + ... + = 1.005^{16}\right)$$

The part in brackets is a geometric series with $a = 1$, $r = 1.055$, $n = 18$

$$\therefore \quad A_{18} = 1000(1.055)^{18} + 500(1.055) \cdot \frac{1\left((1.055)^{17} - 1\right)}{(1.055 - 1)}$$

$$= \$16,862.07 \quad \#$$

10. The calculation must be based on half years

annual contribution = \$2,500

half- yearly interest rate $= \frac{6}{2} = 3\%$

Number of half years $= 2 \times 25 = 50$

(i) Initial deposit of \$2,500 accumulates for 50 half years at 3% per half year:

i.e. $2500(1.03)^{50} = \$10,959.77$ #

(ii) 1st deposit accumulates to $2500(1.03)^{50}$ (50 half- years)

2nd deposit accumulates to $2500(1.03)^{48}$ (48 half-years)

.

.

.

25th deposit accumulates to $2500(1.03)^{2}$ (2 half-years)

$$\therefore \text{ Total value} = 2500(1.03)^2 + 2500(1.03)^4 + ... + 2500(1.03)^{48} + 2500(1.03)^{50}$$

$$= 2500(1.03)^2\left[1 + (1.03)^2 + ... + (1.03)^{46} + (1.03)^{48}\right]$$

which is a geometric series with $a = 1$, $r = (1.03)^2$, $n = 25$

$$\text{Total} = 2500(1.03)^2 \cdot \left(\frac{\left[(1.03)^2\right]^{25} - 1}{(1.03)^2 - 1}\right) = \$147,372.16 \quad \#$$

11. (i) Let A_n be the amount owing after n months and M be the amount of each monthly instalment

monthly interest $= \frac{0.066}{12} = 0.0055$

$$A_1 = 210,000(1.0055) - \text{M}$$

$$A_2 = A_1(1.0055) - \text{M}$$

$$= 210,000(1.0055)^2 - \text{M}(1.005) - \text{M}$$

$$= 210,000(1.0055)^2 - \text{M}(1 + 1.0055)$$

Generalising this result gives:

$$A_{300} = 210,000(1.0055)^{300} - \text{M}\left(1 + 1.0055 + ... + 1.0055^{299}\right)$$

$$= 210,000(1.0055)^{300} - \text{M}\left(\frac{1.0055^{300} - 1}{1.0055 - 1}\right)$$

$$= 1,088,533.438 - 760.6350116\text{M}$$

But $A_{300} = 0$ i.e. $\text{M} = \$1,431.09$ #

(ii) $S = \frac{Prn}{100}$

But $S = \text{M} \times 300 - 210,000$

$= 1,431.09 \times 300 - 210,000$

$= \$219,329$

$$\therefore \quad 219,329 = 210,000 \times \frac{r}{100} \times 300$$

$r = 0.3481\% \approx 0.35\%$ interest per month correct to 2 d.p. #

12. $r = \frac{4\%}{4} = 1\%$ per quarter, 6 months = 2 quarters

$\therefore$ total amount owed in 6 month's time $= 3000(1.01)^2 - \text{M}$

and total amount owed in 12 month's time $= \left[3000(1.01)^2 - \text{M}\right](1.01)^2 - \text{M}$

$$= 3000(1.01)^4 - \text{M}\left(1 + (1.01)^2\right)$$

Since the loan is repaid in 12 months:

$$\therefore \quad 3000(1.01)^4 - \text{M}\left(1 + (1.01)^2\right) = 0$$

$$\text{i.e.} \quad \text{M} = \frac{3000(1.01)^4}{1 + (1.01)^2} = \$1,545.37 \quad \#$$

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Contents

Danny

FINANCIAL MATHEMATICS

Multiple Choice Questions

Earning Money

$180

1 In working a normal 40 hour week at \$4.50 per hour and 5 hours at time-and-a-half, a person's gross wage should be

(A) \$185 (B) \$180 (C) \$202.50 (D) \$213.75

2 A couple's combined weekly wage is \$985.70. During the coming month, they will receive four weeks pay and they have to pay the following bills:

Car registration	\$378.50
Water	\$98.75
Telephone	\$67.40
Rent	\$300 per week (4 weeks)

The amount of their monthly pay left will be

(A) $985.70 \times 4 - 378.50 + 98.75 + 67.40 - 300 \times 4$

(B) $985.70 - \dfrac{378.50 - 98.75 - 67.40}{4} - 300$

(C) $685.70 \times 4 - 378.50 - 98.75 - 67.40$

(D) $985.70 - \dfrac{378.50 + 98.75 + 67.40 + 300}{4}$

3 Before tax, a person earns \$485 for a working week that included 36 hours at normal time and 2 hours overtime at time-and-a-half. The hourly wage rate was

(A) \$13.47 (B) \$12.76 (C) \$12.44 (D) \$12.13

4 A solicitor quotes the conveyancing charge on the purchase of a home is "\$615 plus \$2.20 for each \$500 or part thereof over \$56 000". The charge for the sale of a \$145 000 property would be

(A) \$1006.60 (B) \$810.80 (C) \$391.60 (D) \$1117.20

5 An author earns \$1.50 for every book sold. The retail price is \$28.95. What percentage of the retail price is the royalty?

(A) $\frac{1.50}{28.95} \times 100$ (B) $\frac{1.50}{27.45} \times 100$

(C) $\frac{1.50}{100} \times 28.95$ (D) $\frac{27.45}{28.95} \times 100$

6 A taxi driver charges \$2.05 flag fall plus \$1.25 per kilometre. The charge for a 19 km journey would be

(A) \$23.75 (B) \$25.80 (C) \$38.95 (D) \$40.20

7 Andre earns \$32 652 p.a. and, for his 4 week holiday, is paid a 17 ½ % bonus on top of his normal gross weekly pay. How much per week is Andre paid for his holidays? (Take the year to be $52^1/_7$ weeks.)

(A) \$735.79 (B) \$627.92 (C) \$737.81 (D) \$1428.53

8 Andy has an annual salary of \$29 650 but pays 10% of it to a superannuation fund. He pays tax on the remainder at a rate of \$3060 + 34 cents for every dollar over \$20 700. His fortnightly net pay, taking $52^1/_7$ weeks in a year is closest to

(A) \$196 (B) \$830 (C) \$1026 (D) \$1140

9 A water bill is set out as follows.

WILLIE-WASSER PTY. LTD'

Charges	\$
Water service	20.00
Sewerage service	72.60
Water usage ☐ kilolitres @ \$0.90 per kL	☐

If the total bill is \$169.10, how much water was used?

(A) 76.5 kL (B) 188 kL (C) 85 kL (D) 69 kL

Investing Money

10 If \$8000 were invested for one year, the amount of interest earned would be

INVESTMENT ACCOUNT	
\$0 - \$1999	0.00%
\$2000 - \$4999	8.50%
\$5000 - \$9999	9.50%
\$10 000 - \$19 999	10.00%
\$20 000 - \$49 999	10.50%
\$50 000 +	11.00%

(A) \$760 (B) \$7.60

(C) \$950 (D) \$95

11 If \$50 000 were invested at 8.5% p.a. compound interest for 7 years, the amount of interest earned would be about

(A) \$29 750 (B) \$88 507 (C) \$379 750 (D) \$38 507

12 A bank offers an interest rate of 5% p.a. with the interest compounded monthly. The monthly interest rate is closest to

(A) 6% (B) 0.00417% (C) 0.6% (D) 0.417%

13 A gold ring cost \$2000 in 1980. If its value increased by 2.5% p.a., its value 20 years later would have been

(A) \$(2000 x 1.25 x 20)
(B) $\$(0.025 \times 2000)^{20}$
(C) $\$(1.025)^{20} \times 2000$
(D) $\$2000 \times (0.0125)^{20}$

14 Simone has the choice of investing in one of the following accounts.

Account X: 6% p.a., interest compounded annually
Account Y: 5.75% p.a., interest compounded quarterly
Account Z: 6.2% p.a., flat rate

Which account will pay the most interest for an investment term of 10 years?

(A) Account X
(B) Account Y
(C) Account Z
(D) Two accounts return the same amount.

15 A company invested $3800 for 7 years. During that time, $650 interest was paid. The simple interest rate was

(A) 17% p.a. (B) 12% p.a. (C) 2.4% p.a. (D) 1.2% p.a.

16 When $5000 is deposited for 20 years at 7% p.a. interest, compounded quarterly, the interest earned is

(A) $2000 (B) $4837 (C) $15 032 (D) $7074

17 Julia inherited $100 000 and invested it. The balance in the account after a given number of years is shown in this table.

Term (Years)	Balance $
5	128 400.34
10	164 866.48
15	211 689.12
20	271 809.56
25	349 004.41
30	448 122.85

The figures show that her account earned

(A) 11.6% p.a., flat interest
(B) 5% p.a., compounded daily
(C) 5% p.a., compounded monthly
(D) 5.7 % p.a., compounded yearly

18 Two thousand dollars is deposited in an investment account which pays interest of 7.5% p.a., compounded annually.

The initial deposit will have earned a total of $5000 interest after

(A) 3 years (B) 4 years (C) 13 years (D) 18 years

19 If interest is compounded biannually, the interest rate per annum which will double your money in 10 years is

(A) 10% (B) 7.1% (C) 3.6% (D) 14.2%

Taxation

20 Before tax, a person earns $452 for a 38 ½ hour week. If they pay 38.5 cents in the dollar marginal tax, how much would they take home per hour when they work overtime at time-and-a-half?

(A) $4.52 (B) $7.22 (C) $10.83 (D) $11.74

Use the following table to answer Questions 21, 22 and 23.

Taxable Income	Tax
$1 - $5400	NIL
$5401 - $20 700	NIL plus 20 cents for each dollar over $5400
$20 701 - $36 000	$3060.00 plus 35.5 cents for each dollar over $20 700
$36 001 - $38 000	$8491.50 plus 38.5 cents for each dollar over $36 000
$38 001 - $50 000	$9261.50 plus 44.125 cents for each dollar over $38 000
$50 001 and over	$14 556.50 plus 47 cents for each dollar over $50 000

21 A person earning $5500 per year would pay

(A) $100 tax (B) $20 tax (C) no tax (D) $1100 tax.

22 A person paying $10 000 per year in income tax would have a taxable income of about

(A) $39 670 (B) $920 (C) $2000 (D) $48 000

23 Mardi had a total income of $42 368 in the last financial year. Her allowable deductions totalled $5296. Use the tax table above to calculate the amount of tax she had to pay.

(A) $18 782.40 (B) $10 427.12 (C) $10 820.24 (D) $8904.22

24 The health care levy of 1.25% of taxable income on an income of $23 560 would be

(A) $2945 (B) $294.50 (C) $1884.80 (D) $188.48

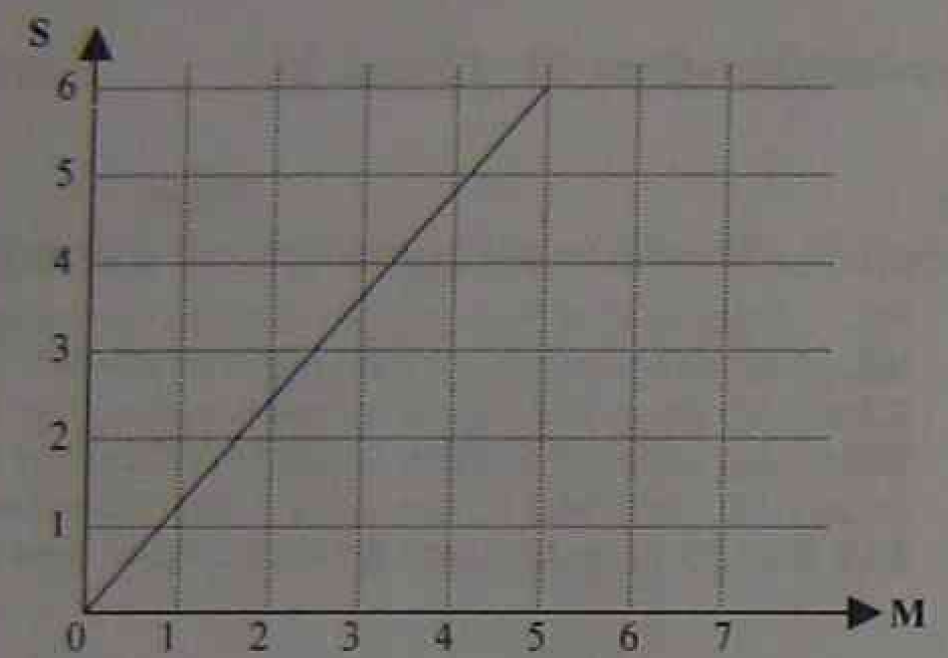
25 A surfboard is priced at $890 including 10% GST. The price of the board without GST is closest to

(A) $801 (B) $880 (C) $809 (D) $979

26 Paul has a taxable income of \$8000. He is taxed 20 cents for each dollar over \$5400. This tax rate is equivalent to a flat tax rate of

(A) 20% (B) 5.2% (C) 6.5% (D) 32%

27 A flat tax rate is added to the marked price (M) of goods, to give the selling price (S), as shown in this graph.



The tax rate is

(A) 45% (B) 1.2% (C) 60% (D) 20%

28 Upon the introduction of the GST in 2000, the following tax table replaced the one in Questions 21 to 23.

Income (\$)	Tax payable
Up to \$6000	Nil
\$6000 - \$20 000	17 cents for every dollar over \$6000
\$20 001 - \$50 000	\$2380 + 30 cents for every dollar over \$20 000
\$50 001 - \$60 000	\$11 380 + 42 cents for every dollar over \$50 000
\$60 001 +	\$15 580 + 47 cents for every dollar over \$60 000

Megan paid \$5950 tax. Her taxable income was

(A) \$23 570 (B) \$21 071 (C) \$31 900 (D) \$39 833

29 For each of the financial years 1999-2000 and 2000-2001, Alexei had an annual taxable income of \$15 500. When the new tax rates were introduced, he paid

(A) \$405 less income tax each year
(B) \$465 less income tax each year
(C) \$1623.50 less tax each year
(D) \$2030 less tax each year.

Credit and Borrowing

30 A credit card company charges 0.05753% interest per day on the amount owing. The interest charged in four weeks on a balance of \$900 would be

(A) \$14.50 (B) \$20.70 (C) \$51.78 (D) \$207

31 A car advertised for \$6500 is sold for \$500 deposit and \$45 per week for 3 years. The flat interest rate is about

(A) 2.8% p.a. (B) 5.7% p.a. (C) 8.7% p.a. (D) 16% p.a.

32 Mitchell spent \$220 on clothes using his credit card. He is charged 23% p.a. with interest compounded daily. After 30 days, he will owe

(A) \$389 (B) \$277 (C) \$270.60 (D) \$224.19

33 Marianne wants to borrow \$3600 for a sound system. She wants to pay off the debt in four years in monthly instalments. She is offered a flat rate of 6% p.a. interest from a finance company. A bank offered 8% p.a., the monthly repayment being \$87.88. Which offer is cheaper and by how much?

(A) Compound by about \$246
(B) Compound by about \$288
(C) Flat by about \$246
(D) Flat by about \$288

34 A credit union is lending Pierre \$10 000 to buy a car. The loan is for 5 years. Interest is compounded monthly.

The table shows that, after n months, the amount owing (A), is found by adding the interest (I) and subtracting the monthly repayment (R).

N	A (\$)	A + I (\$)	A + I − R (\$)
1	10 000.00	10 075.00	9867.42
2	9867.42	9941.43	9733.85
3	9733.85		
4			
5			
6			

The amount owing after the 6^{th} repayment has been made is

(A) \$9189.46 (B) \$8903.53 (C) \$9198.85 (D) \$9213.04

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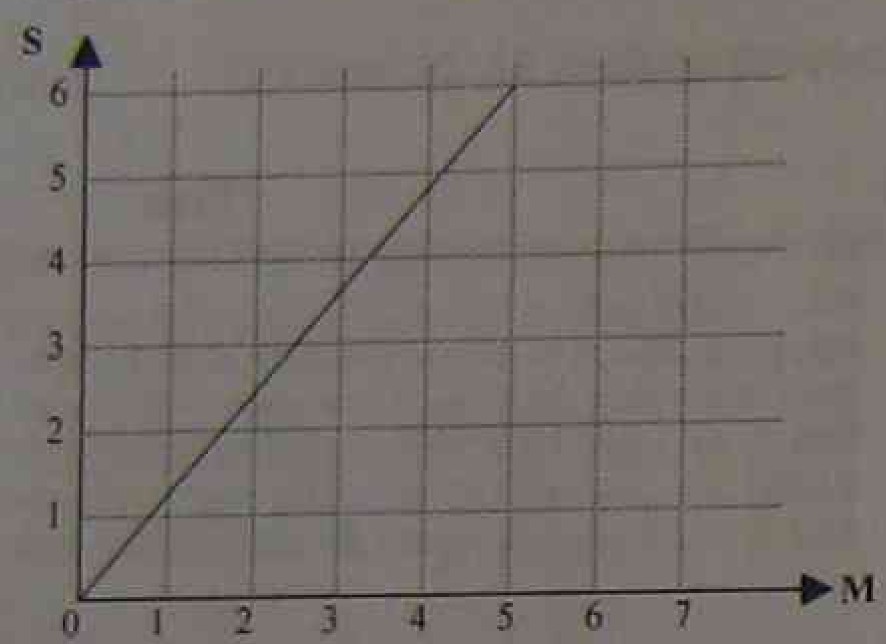

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The table shows that, after n months, the amount owing (A), is found by adding the interest (I) and subtracting the monthly repayment (R).

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1	10 000.00	10 075.00	9867.42
2	9867.42	9941.43	9733.85
3	9733.85		
4			
5			
6			

The amount owing after the 6th repayment has been made is

(A) $9189.46 (B) $8903.53 (C) $9198.85 (D) $9213.04

35 To repay a $9 000 loan in 6 years with reducible interest at a rate of 15% p.a., the monthly repayment would have to be about

(A) $8.10 (B) $21.15 (C) $190.35 (D) $136.25

36 The table below gives values of $\dfrac{(1+r)^n - 1}{r(1+r)^n}$ for different values of r and n.

n	5%	6%	7%	8%	9%	10%	11%	12%
1	0.95238	0.94340	0.93458	0.92593	0.91743	0.90909	0.90090	0.89286
2	1.85941	1.83339	1.80802	1.78326	1.75911	1.73554	1.71252	1.69005
3	2.72325	2.67301	2.62432	2.57710	2.53129	2.48685	2.44371	2.40183
4	3.54595	3.46511	3.38721	3.31213	3.23972	3.16987	3.10245	3.03735
5	4.32948	4.21236	4.10020	3.99271	3.88965	3.79079	3.69590	3.60478
6	5.07569	4.91732	4.76654	4.62288	4.48592	4.35526	4.23054	4.11141
7	5.78637	5.58238	5.38929	5.20637	5.03295	4.86842	4.71220	4.56376
8	6.46321	6.20979	5.97130	5.74664	5.53482	5.33493	5.14612	4.96764
9	7.10782	6.80169	6.51523	6.24689	5.99525	5.75902	5.53705	5.32825
10	7.72173	7.36009	7.02358	6.71008	6.41766	6.14457	5.88923	5.65022

The difference in the total repayments on a $100 000 loan if the terms are 10 years at 6% p.a. instead of 6 years at 10% p.a. is

(A) $9374 (B) $1897 (C) $3005 (D) $75 121

37 How much money, to the nearest dollar, will Janine have after 30 years if she puts $1 into an account every day and interest is compounded daily at 9% p.a.?

(A) $11 944 (B) $129 155 (C) $56 308 (D) $53 834

38 Andre intends to pay $200 per month for 20 years towards an annuity. If the interest rate remained at 10% p.a., compounded monthly, the present value of the annuity would be

(A) $1702.71 (B) $48 000 (C) $58 622.45 (D) $20 724.92

39 Use the present value formula for annuities to find the monthly repayment on a loan of $40 000 if interest is 12% p.a., compounded monthly and the loan is to be repaid in 6 years.

(A) $782 (B) $1093 (C) $556 (D) $1137

Depreciation

40 The computer hardware used to write this book cost $2400. If it depreciates at 25% of its previous year's value per year for the first three years, its value after three years would be about

(A) $600 (B) $37.50 (C) $1013 (D) $759

41 A company has spent $100 000 on new machinery. The depreciation can be claimed as a tax deduction. The taxation department allows either a straight line depreciation of 10% p.a. or declining balance depreciation of 12% p.a. Which of these statements is true?

(A) Both methods give the same value for the equipment after two years.
(B) Straight line depreciation gives a greater deduction for the second year.
(C) Under the declining balance method, the machinery will be valueless after 10 years.
(D) The declining balance method will return less in total tax deductions in the first ten years.

42 A computer costing $2500 is valued at $100 seven years later. Using the declining balance method of depreciation, the depreciation rate per annum is closest to

(A) 14% (B) 37% (C) 40% (D) 58%

43 The expected life of a piece of hospital equipment is seven years. To 'write it off' after seven years, the flat rate of depreciation would be closest to

(A) 7% p.a. (B) 14% p.a. (C) 17% p.a. (D) 21% p.a.

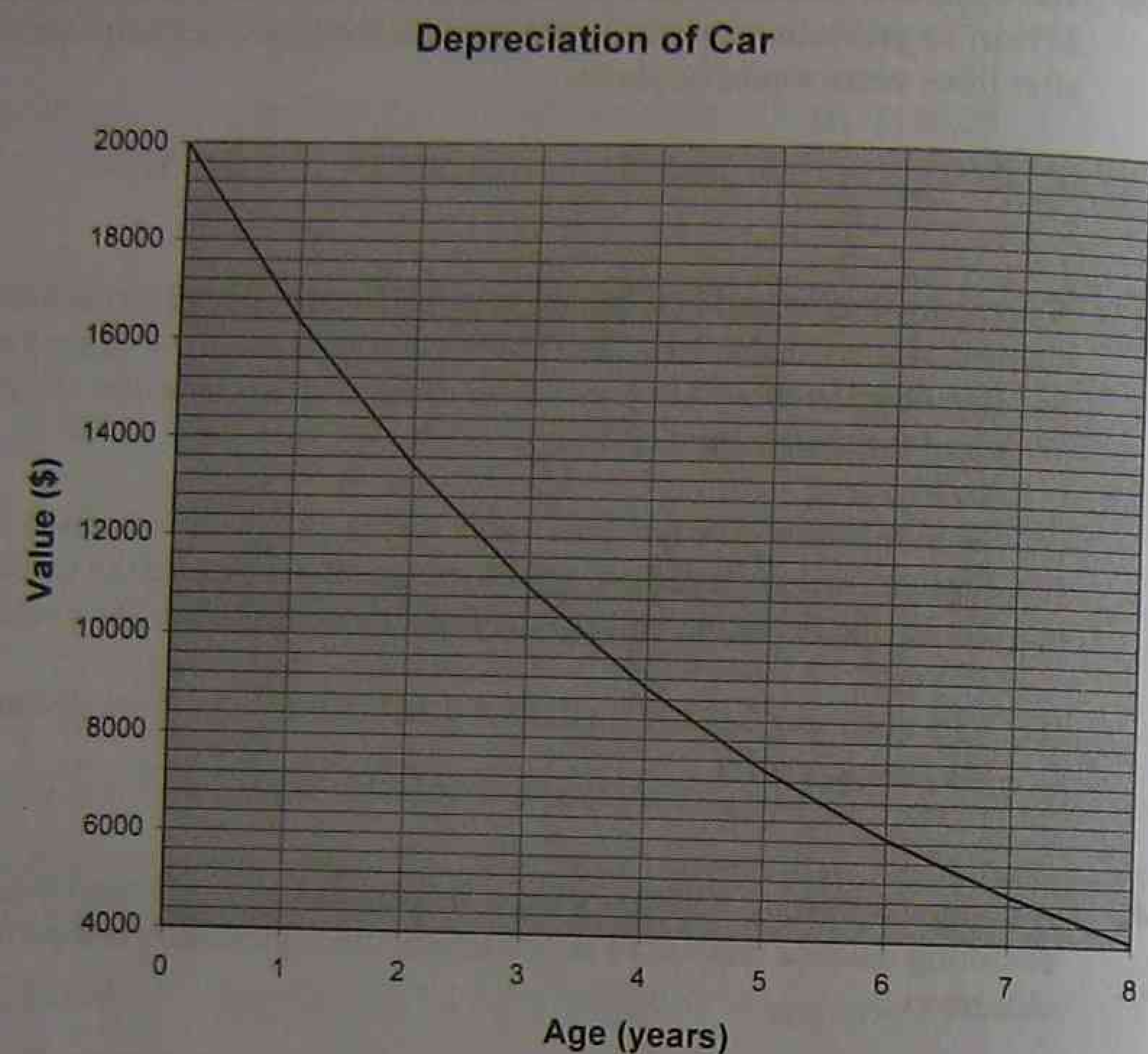
44 After 15 years, a camera is valued at $30. If it originally cost $630 and had depreciated at a flat rate, the annual depreciation was closest to

(A) 4% (B) 6% (C) 6.3% (D) 6.6%

45 After 10 years of depreciating at 20% p.a. on a declining balance scale, the value of a car is $400. Its original value was closest to

(A) $2480 (B) $3725 (C) $4000 (D) $5000

46 The history of depreciation of a car is shown in this graph. The value ($) is plotted against the age of the car in years.



The depreciation during the first 8 years is equivalent to a flat rate of

(A) 10% p.a. (B) 15% p.a. (C) 18% p.a. (D) 25% p.a.

47 The table shows the value of an item after a given number of years.

Number of years	Value ($)
1	12 000
2	9 500
3	7 700
4	6 150

The depreciation rate is closest to

(A) 20% p.a. straight line.
(B) 20% p.a. declining balance.
(C) 25% p.a. straight line.
(D) 25% p.a. declining balance.

Free Response Questions

Earning Money

1 The budget for a school musical production allows for the following expenditure.

Hire of costumes	1200
Materials for costumes	300
Make up	200
Set construction	500
Printing costs	100
Advertising	100
Violin repairs	80
Hire of cello	120

(a) Calculate the total expenditure.

(b) The theatre will seat 90 people per performance. If all 6 performances are sold out, what ticket price will allow the show to 'break even'?

(c) Suggest a suitable price to charge for tickets, giving reasons for your answer.

2 A parcel courier charges $3.50 plus $2.80 per kilogram or part thereof.

(a) Calculate the cost of sending a 3.4 kg parcel.

(b) A company has two 3.4 kg parcels to send to the same place. Calculate the saving, if any, of combining the two parcels into one.

(c) What is the weight of the heaviest parcel that could be sent for $20?

3 A wage earner earns $1240.44 before tax each fortnight.

(a) Using 365 days in a year, calculate the person's annual gross income.

(b) The person must pay $6082 per annum income tax. What is the person's take-home pay each fortnight?

(c) If the person works a 35 hour week, what is their hourly rate of pay before tax?

(d) From their pay packet, the person budgets $440 for food, $150 for other household expenses and $72 for superannuation. How much 'disposable income' is left?

4 A labourer earns \$30.50 per hour before deductions.

(a) If he works 25 hours per week on average, for much would he gross in a year (52 weeks)?

(b) When working in confined spaces, there is a site allowance of 5%. What would his new hourly rate be?

(c) For work on Saturdays, he is paid at 'time-and-a-half'. What would his hourly rate be on Saturdays?

(d) As this is a casual pay rate, the labourer is not entitled to pay for annual leave. If he takes 4 weeks holiday each year and does not work on public holidays or weekends, he finds that he only works for 200 days on average each year. If he works for 7 hours each day, how much would he expect to gross each year?

5 A shop assistant earns \$630 per week. For 4 weeks annual leave, a 17 ½ % loading applies. What is her weekly wage for each of the weeks of her holiday?

6 A singer is paid 10% of all retail sales of a CD as a royalty.

(a) If the CD sells for \$9.95, how much would the singer earn from the sale of 15 670 copies?

(b) The recording company offers the singer 15% of wholesale returns, i.e. the amount that the company receives from shops. If the company gives the shops a 35% discount of the retail price, how much difference will this make on sales of 15 670 CDs?

7 A real estate agent earns a commission on sales based on the following table.

Sales (\$)	Commission rate
Up to \$100 000	5%
On the next \$100 000	3%
On the next \$150 000	2.5%
Thereafter	2%

(a) What would the agent earn on a sale of a

(i) \$250 000 property? (ii) \$530 000

(b) What selling price would earn the agent \$12 580?

Investing Money

8 A house sold for \$64 000 in 1983. Ten years later, the owners decided to sell.

(a) Over the ten year period of ownership, inflation averaged about 7% p.a. What should the house be worth in 1993 if its value had increased according to the inflation rate?

(b) The house sold at auction for \$145 000. How much more is this than the expected value found in part (a)?

(c) The owners have to pay Capital Gains Tax at the rate of 47 ½ % on the profit (the answer to Part (b) of this question). How much tax would they have to pay?

(d) For the next 7 years, inflation averaged only 2% p.a. What would the expected value of the house be in the year 2000?

9 Profits for a company are divided between two partners in the ratio 5 : 8. What amount would the partner with the higher share receive from total profits of \$28 951?

10 If an investment purchase of \$23 000 increases in value by 5% p.a. each year, what will be its value after 19 years? (Nearest dollar)

11 A bank account earns 12% p.a. interest, calculated monthly. If \$3000 was left untouched in an account,

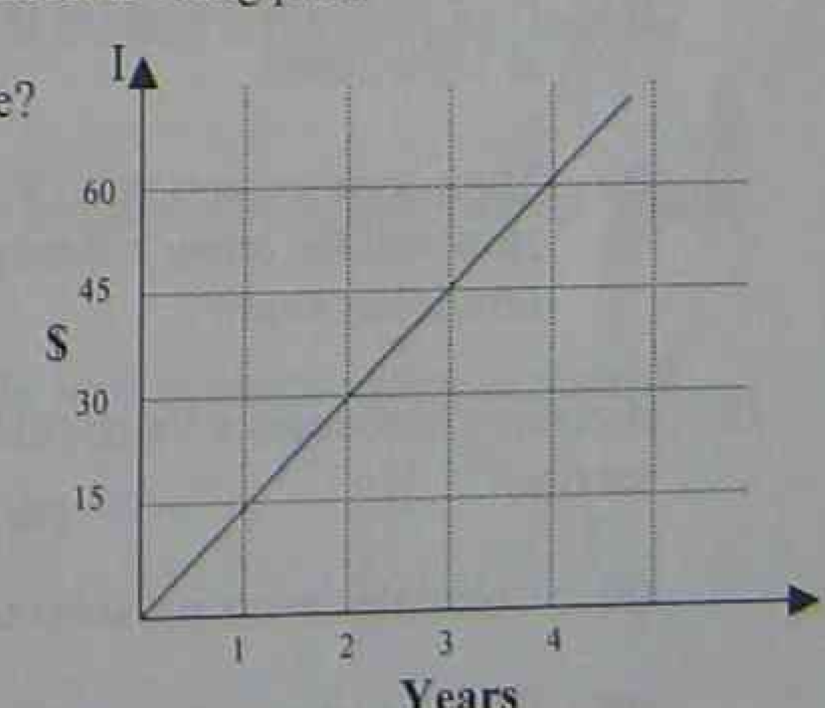
(a) what would be balance the after 2 years, correct to the nearest cent?

(b) how much interest would be earned?

12 This graph shows how much interest (I) an investment is yielding over a period of time (T). A flat rate of interest is being paid.

(a) What is the equation of the line?

(b) If \$500 was invested, what is the interest rate?



13 **Income Tax 1999 - 2000**

Income ($)	Tax payable
$1 - $5400	Nil
$5401 - $20 700	Nil plus 20 cents for each dollar over $5400
$20 701 - $36 000	$3060.00 plus 35.5 cents for each dollar over $20 700
$36 001 - $38 000	$8491.50 plus 38.5 cents for each dollar over $36 000
$38 001 - $50 000	$9261.50 plus 44.125 cents for each dollar over $38 000
$50 001 and over	$14 556.50 plus 47 cents for each dollar over $50 000

Income Tax 2000 - 2001

Income ($)	Tax payable
Up to $6000	Nil
$6000 - $20 000	17 cents for every dollar over $6000
$20 001 - $50 000	$2380 + 30 cents for every dollar over $20 000
$50 001 - $60 000	$11 380 + 42 cents for every dollar over $50 000
$60 001 +	$15 580 + 47 cents for every dollar over $60 000

The tables above show how the income tax rates in Australia changed with the introduction of the Goods and Services Tax.

(a) Calculate the income tax to be paid on a taxable income of $30 000 under each system.

(b) Calculate the savings for someone on an annual taxable income of $100 000.

(c) If a family expected to pay $80 per fortnight extra because of the goods and services tax, how much would they save / lose per fortnight if the sole breadwinner had a taxable income of $35 000.

14 A bus driver earned $30 000 before tax in 1999-2000 financial year and expects to gross the same amount in 2000-2001.

(a) How much tax would the person expect to pay in 2000-2001 compared to 1999-2000?

(b) The person expects the GST to add $40 per week to the family expenses. How will the driver's family budget be affected by the change in the taxation system?

15 A country introduces a Value Added Tax (VAT) of 12½ % on all goods and services.

(a) Calculate the tax payable on a dinner costing £86.

(b) Calculate the total cost of the dinner.

Credit and Borrowing

16 The following table shows the balance owing on a credit card each day for a week. Interest charged on the final balance each day is 0.04095%.

DATE	DEBIT ($)	BALANCE ($)
27/7	120.80	120.80
28/7	40.00	160.80
29/7		160.80
30/7	148.60	309.40
31/7	94.50	401.90

(a) Find the interest charged each day and the total interest payable for a week.

(b) If these were the only purchases for the month, what would the balance be for 1st August?

(c) What yearly interest rate is being charged?

17 A car is priced at $10 960. Damien buys the car and pays $1300 deposit. He pays the rest over a period of 4 years at 12% p.a. simple interest.

(a) How much is to be borrowed?

(b) How much simple interest is Damien charged over the four years?

(c) Calculate the monthly repayment needed to pay the car off.

18 A sailor buys a boat which is advertised at $10 000 cash. On hire purchase, the repayments are $300 per month for three years after a 10% deposit has been paid.

(a) What is the total amount paid for the boat under this hire purchase agreement?

(b) After the deposit was paid, how much was borrowed from the finance company?

(c) How much more was paid for the boat by choosing hire purchase?

(d) What flat rate of interest was charged?

19 Gemma is saving for a house.

(a) She deposits $12 000 into an account every six months. How much will this annuity be worth after 5 years if the interest rate is 7% p.a., compounded biannually?(Hint: take care with the value of n.)

(b) If the interest rate remained at 7% p.a. (interest compounded biannually), how much would Gemma have to deposit every six months so that her future annuity would be $175 000?

(c) Unfortunately, Gemma can only afford $12 000 every six months. The interest rate only stayed at 7% for the first two years, compounded biannually. What is the annuity worth after two years?

(d) She left the annuity in the account to earn interest at 5% p.a. for the next 3 years, interest compounded biannually. How much is in the account at the end of this time if she makes no more contributions?

(e) For the 3 years, Gemma deposited $12 000 every six months into another account which earned only 5% p.a. How much money did Gemma lose on her 5 years of contributions from part (i) because of the fall in interest rates?

20 An engineering company needs to replace machinery in 5 years time. The expected replacement cost is $300 000.

How much money should the company set aside each month in an investment account to cover this cost if the interest rate is 7.25% p.a. with interest compounded monthly?

21 On the day of a child's birth, parents were encouraged to invest $5 each month.

(a) If the interest rate at 6% p.a., compounded monthly, how much would be in the account on the child's 21st birthday?

(b) How much money was invested?

(c) (i) How much money should the parents have invested so that the child receives $10 000 on her 21st birthday?

(ii) What is the present value of the $10 000 annuity?

22 To pay for house extensions, a couple need to borrow $40 000.

(a) What are the monthly instalments if the loan is taken with interest charged at 15% p.a. reducible over

(i) 1 year (ii) 3 years (iii) 10 years (iv) 20 years?

(b) How much will be paid back in instalments if the loan is repaid in

(i) 1 year (ii) 3 years (iii) 10 years (iv) 20 years?

(c) If the couple have no other debts and can save approximately $700 per month, suggest the number of years that they can expect to take to repay the loan.

23 A single-income family has saved $24 000 towards the cost of a home unit which they are able to purchase for $110 000.

(a) How much will they have to borrow (at least)?

(b) The interest rate on the loan is 7.30% p.a. and they are advised to take the loan over 25 years.

(i) How much will each monthly instalment be?

(ii) How much should the family set aside each week for mortgage payments?

(iii) How much money will have been paid in instalments after one year?

(iv) How much money will have been paid in instalments after 25 years?

(v) How much interest will have been paid in 25 years?

(c) The family sells their car and buys an older model. The deal gives them an extra $6000 towards the unit. The amount borrowed is now $80 000.

(i) Calculate the monthly repayment for a 25 year loan.

(ii) Calculate the total repayments over the 25 years.

(iii) Calculate the amount of interest paid over the 25 years.

(iv) How much did they save in interest payments by having an extra $6000 deposit?

24 A new car depreciates 20% of its previous year's value each year. If the car was initially worth \$45 000, how old will it be before it is worth less than \$10 000?

25 A guitar, originally costing \$5000, is depreciating at 12½ % p.a. Find the salvage value after 5 years using

(a) the declining balance method

(b) the straight line method.

26 A manufacturing company spent \$1 million on equipment 10 years ago. Its current value is \$350 000.

(a) Use the straight line method to calculate by how much the equipment has depreciated each year.

(b) At this rate, how long will it take before the equipment is worthless?

(c) Using the declining balance method, find by what percentage of its previous year's value the equipment depreciated.

27 A laser printer has a serviceable life of 5 years. The tax office allows a maximum deduction of 24% p.a. using a declining balance model.

(a) Complete this table to show the value of the printer under a declining balance model.

Age (years)	Value (\$)	Depreciation (\$)
0	3200	768
1		
2		
3		
4		
5		

(b) Complete this table to show the value using straight line depreciation.

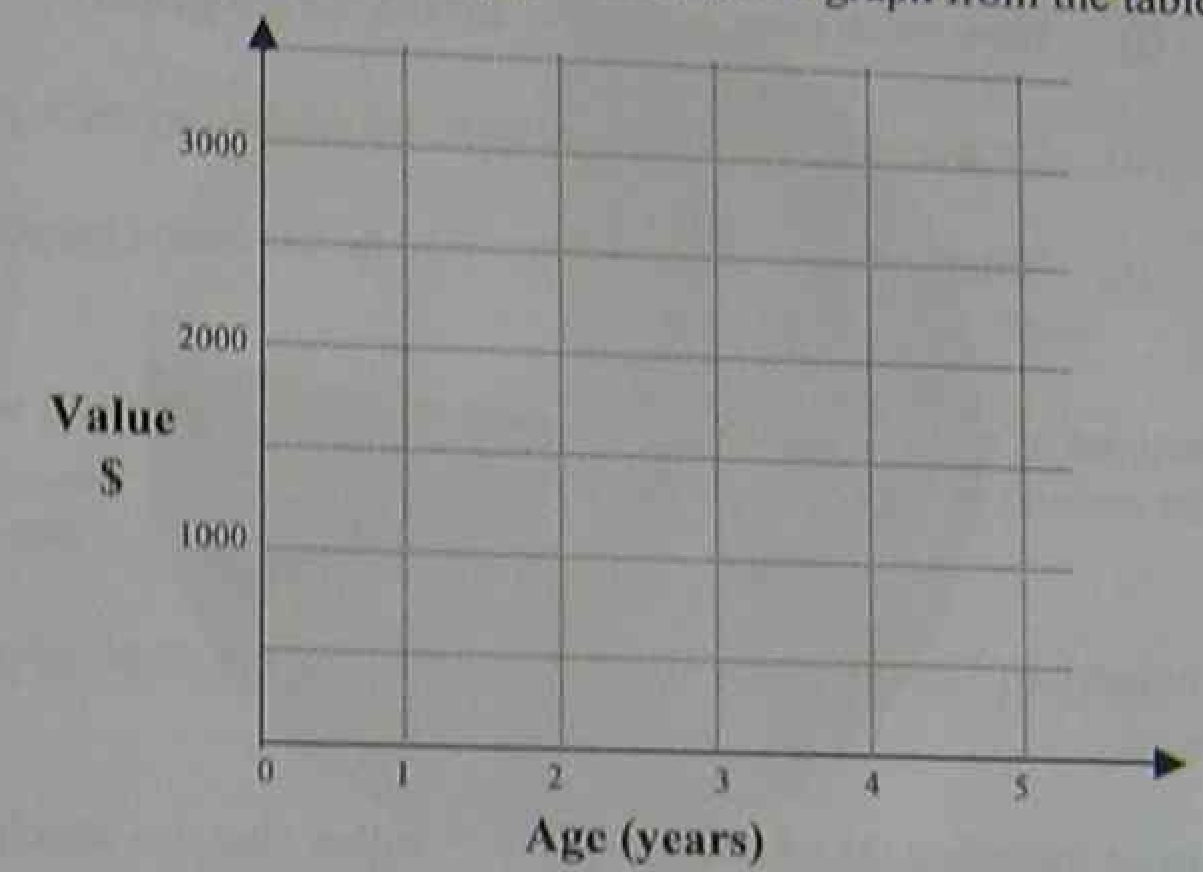
Age (years)	Value (\$)	Depreciation (\$)
0	3200	640
1		
2		
3		
4		
5		

(c) Discuss the financial advantages of each method.

28 The data gives the value of a photocopier after a given number of years.

Years	Value (\$)
0	3500
1	2450
2	1715
3	1200
4	840
5	590

(a) Copy these axes on grid paper and draw a graph from the table of values.



(b) Determine the annual rate of depreciation that has been used.

(c) If the depreciation were tax deductible, how much could the owners claim for depreciation which occurred in the second year of the photocopier's life?

(d) On the same axes, graph the value of the photocopier if a straight line method of depreciation had been used and the equipment written off after 5 years.

(e) Use your graph to determine when the photocopier would have had the same value under both methods.

(f) If the trend shown in the table continues, what will be the value of the photocopier after 7 years?

29 Use the formula $S = V_0(1 - r)^n$ to determine the salvage value S after 6 years of an item with an initial value of \$55 000 which depreciates at a rate of 35% p.a.

Miscellaneous Questions

30 A car is retailing for $22 000 and is expected to have a salvage value of $5000 in 10 years time.

(a) Use the declining balance method to find the rate of depreciation.

(b) To buy the car, Andre has to borrow $20 000 and pay off the loan in 10 years. His instalments are $287 per month.

(i) How much does Andre pay for the car altogether?

(ii) On average, how much interest did Andre pay each year?

(iii) What was the simple interest rate he has been charged?

31 An amount of money doubles in 3 years. At this rate, how long would it take for the amount to become 8 times the original amount?

32 The country of Mesopotamia imposes a fixed rate tax on all goods and services.

The price including tax of a ring is $275. Another ring, on which $50 tax is paid, has a pre-tax price which is double the pre-tax price of the first ring.

(a) Find the rate of tax.

(b) Find the pre-tax prices of the two rings.

33 In the year 32 AD, Judas invested 30 pieces of silver (say $30) in an account which paid only 0.5% p.a. interest, compounded annually. He was too embarrassed to use the account and it has only just been discovered, having been in existence for 1965 years.

(a) What do you calculate the balance in the account to be?

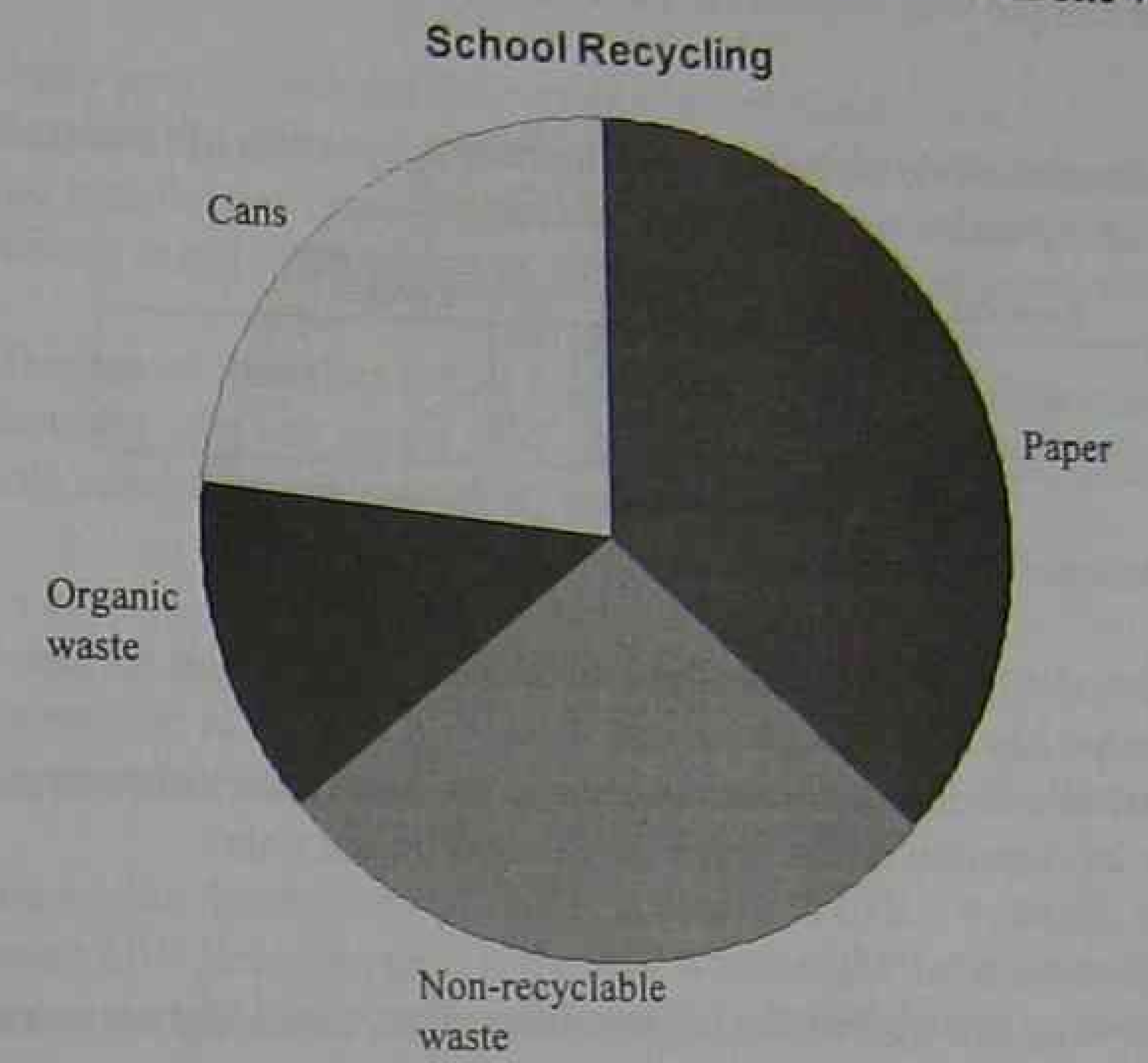
(b) The bank charged a fee of $25 per year for maintaining the account, the fee being subtracted from the balance in the account at the end of each year of the term of investment. What does the bank claim the balance in the account to be?

DATA ANALYSIS

Multiple Choice Questions

Statistics and Society

1 This graph shows the type of waste produced by a school in one week.



The school had 280 kg of rubbish in that week. The weight of recyclable paper was about

(A) 130 kg (B) 120 kg (C) 110 kg (D) 100 kg

2 In Mathematics, a statistic is

(A) either the mean, the median or the mode.
(B) a function of sample values.
(C) a category used in sampling.
(D) a term used for someone involved in a car accident.

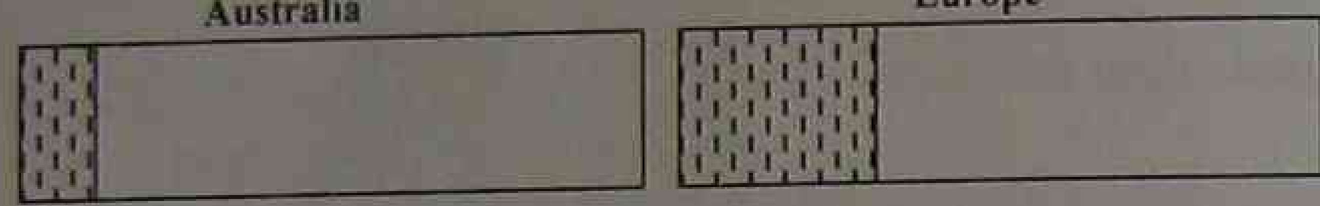
3 One advantage of a census over a survey is

(A) lower cost.
(B) questions are answered more honestly.
(C) small categories are not ignored.
(D) ease of administration.

4 A market researcher asked 15 people in Martin Place, Sydney, how they were going to vote at the next election. Three-quarters of the people stated that they would vote for one particular party. It is therefore reasonable to conclude that

(A) the survey was not properly constructed.
(B) one party is heading for a landslide win at the next election.
(C) the person asking the question caused people to answer in a particular way.
(D) the sample was random rather than stratified.

5 The following divided bar graphs show the proportion of land under cultivation (shaded) in Australia and Europe.



It is reasonable to conclude that

(A) Australia has less land under cultivation than Europe
(B) Europe has more desert country than Australia
(C) Australia should increase the area of its land under cultivation
(D) the information in the graph gives insufficient data.

6 The following table gives the highest temperature and highest rainfall recorded in each state and the date of the recordings.

State	Highest Temp. °C	Highest yearly rain (mm)
NSW	52.8 in 1877	4540 in 1950
Victoria	50.8 in 1906	3342 in 1917
Queensland	53.1 in 1889	11 251 in 1979
South Australia	50.7 in 1960	1851 in 1917
Western Australia	50.7 in 1906	2154 in 1973
Tasmania	40.8 in 1945	4505 in 1948
Northern Territory	48.3 in 1960	2966 in 1973

From these figures, it is reasonable to conclude that

(A) the world is getting hotter.
(B) the world is getting wetter.
(C) greenhouse gasses have no effect on weather.
(D) these records provide no evidence of climatic change.

Data Collection and Sampling

7 One of the first attempts to predict the result of an election was made in 1932 in the USA. A large number of voters were telephoned and asked whether they were going to vote for the Republican presidential candidate or the Democrat's candidate. An overwhelming majority said they would vote Republican but Franklin. D. Roosevelt, the Democrat, won comfortably. The explanation for this discrepancy was that

(A) many people changed their minds before the election.
(B) the pollsters did not ask the question correctly in the telephone poll.
(C) the sample was biased because only rich people owned telephones in 1932.
(D) voting is not compulsory in America so the polling method was flawed.

8 The Bureau of Statistics wants to get an accurate idea of the number of people in Australia who can play a tuba. The method which would provide the most reliable results for a reasonable cost would be

(A) a census.
(B) random sampling.
(C) stratified sampling.
(D) systematic sampling.

9 To determine the number of fish in a pond, 20 fish are caught, each one being replaced after being tagged. If the one was caught three times and all others were caught once only, researchers should conclude that

(A) there are about 60 fish in the pond.
(B) there are about 6 or 7 fish in the pond.
(C) there are probably more than 20 fish in the pond.
(D) it was not necessary to catch so many fish.

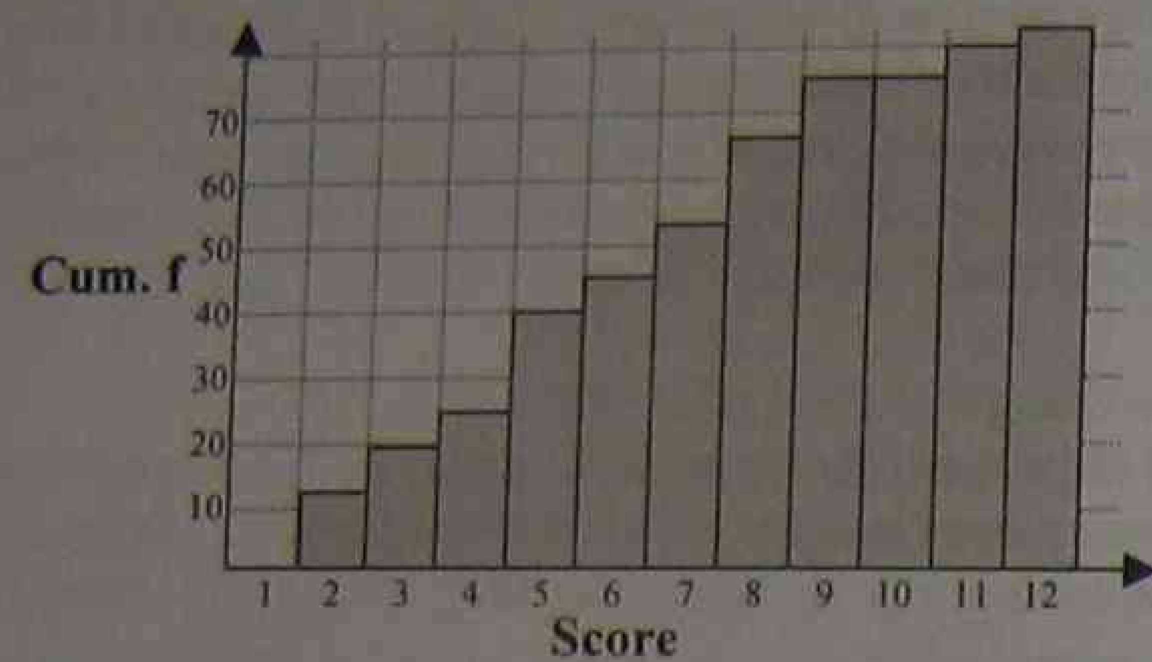
10 The heights of the boys in a school were measured and recorded. The data would be

(A) quantitative discrete.
(B) quantitative continuous.
(C) qualitative categorical.
(D) categorical and quantitative

11 The Convict Muster of 1828 collected information from all the convicts in NSW at that time. It was therefore an example of

(A) stratified sampling.
(B) a census.
(C) systematic sampling.
(D) random sampling.

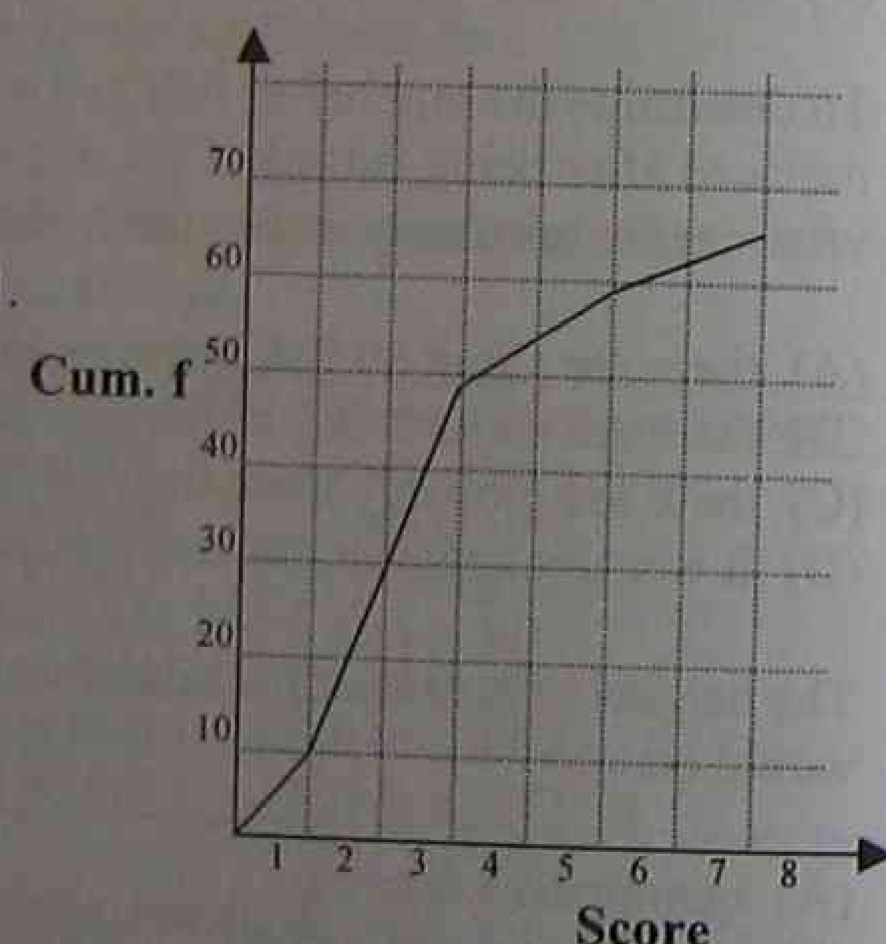
12 Which statement is true about this cumulative frequency histogram?



(A) There were 75 scores of 10.
(B) The median score was 6.
(C) There were more scores of 3 than scores of 2.
(D) The modal score was 12.

13 Which statement is true about this cumulative frequency polygon?

(A) The highest score was 8.
(B) The range is 65.
(C) The interquartile range is 1.
(D) The median is 4.



14 In the stem and leaf plot shown, the missing number could be

1	0 2 5
2	0 1 1 4 7 9
3	0 1 1 3 3 3 4 6 7 8
4	1 3 3 □ 5 5 6 8 9
5	3 7

(A) 4
(B) 3 or 4
(C) 4 or 5
(D) 3, 4 or 5

15 A political party published the following graph to show the success of its farming policy.

Which of the following statements is true?

(A) If the heights of the cows were representing farm production, then farm production under the present government had more than doubled.
(B) The pictures suggest that farm production had increased by about 50%
(C) The use of the cow drawing tends to relate farm production to a volume rather than a linear measurement.
(D) The political party is clearly using a graph to tell a lie.

16 The following scores were obtained in a class spelling test.

5, 8, 4, 7, 8, 6, 8, 7, 6, 4, 6, 9, 7, 9, 5, 5.

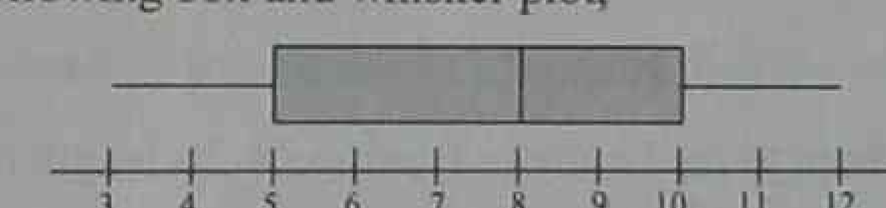
The upper quartile is

(A) 4 (B) 7 (C) 8 (D) 9

17 For the scores in the previous questions, the median would be

(A) 5 (B) 5.5 (C) 6 (D) 6.5

18 For the following box and whisker plot,



the interquartile range of the sample would be

(A) 3 (B) 5 (C) 8 (D) 9

19 The frequency of various car models passing a point was recorded. A type of graph not suitable for displaying the data is a

(A) column graph.
(B) sector graph.
(C) radar chart.
(D) divided bar graph.

20 A statistician calculates the standard deviation of a sample of 7 scores to be 13.8. From this, the population standard deviation would be closest to

(A) 15.8
(B) 13.8
(C) 12.1
(D) 1.97

21 The average number of people attending each of the last four lectures was 20. In order to raise this average to 30, how many people must attend the next lecture?

(A) 110 (B) 25 (C) 50 (D) 70

22 Some parts of the Snowy Mountains in NSW have a median rainfall of 3200 mm per year. Which statement below agrees with this fact?

(A) If a bucket were left outside for a year in this area, it would collect about 3.2 L of water.
(B) If you stayed at this place for a year, 320 cm of rain would probably fall.
(C) In a large number of years, half the years would have more than 3.2 m of rainfall.
(D) In most years, 3200 mm of rain falls in this area.

23 The lengths of babies born to a family were 48 cm, 49 cm, 50 cm, 48 cm, 60 cm and 48 cm. The mean length was

(A) 48 cm (B) 49 cm (C) 52 cm (D) 50 cm

24 Another baby is born to the family in Question 4. Its length is 33 cm. Which statistic will change most?

(A) Mean (B) Median (C) Mode (D) Range

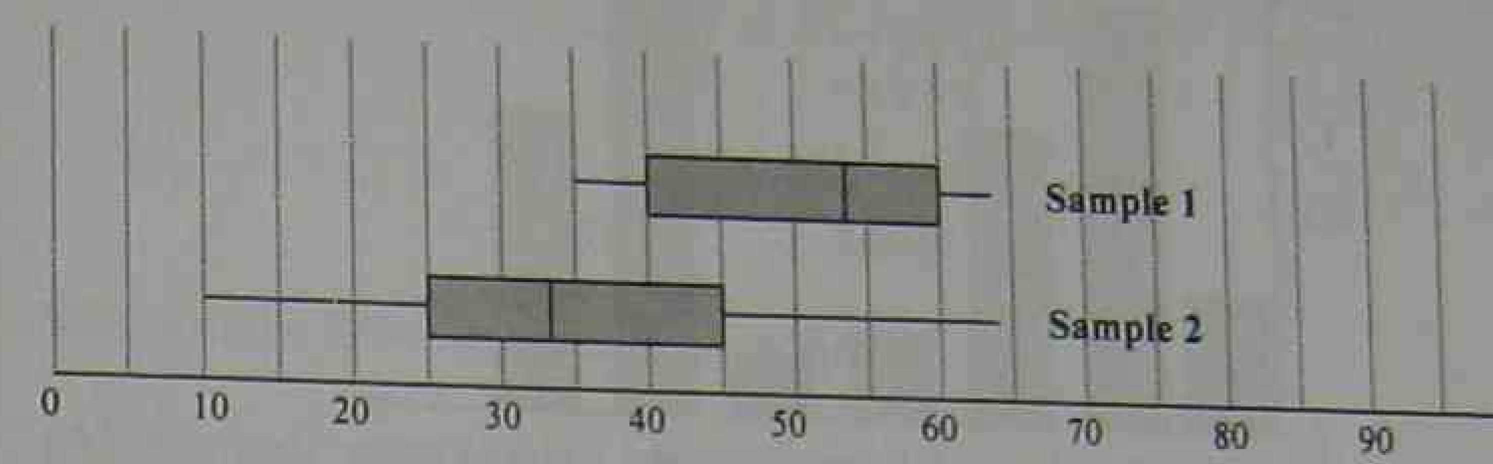
25 Find the standard deviation for the scores.

2 5 6 6 4 1

(A) 4 (B) 2.1 (C) 1.9 (D) 3.7

Interpreting Sets of Data

26 Which statement is true about the two samples illustrated by these box and whisker plots?



(A) Sample 1 has a greater interquartile range than Sample 2.
(B) Sample 1 has a greater range than Sample 2.
(C) The highest score in Sample 1 is more than the highest score in Sample 2.
(D) The median of Sample 1 is greater than the median of Sample 2.

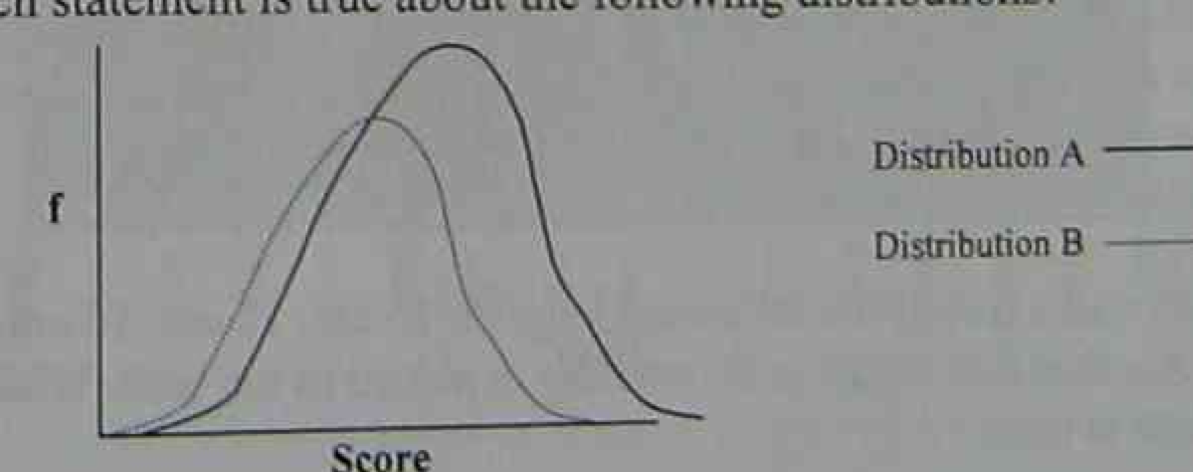
27 An electoral roll revealed these statistics of a small town.

	Adults	Youths	Children
Male	795	450	128
Female	687	650	117

A survey is to be conducted about the facilities that youths would like to see in the town. A sample of 100 youths is to be taken. The number of boys included in a representative stratified survey would be

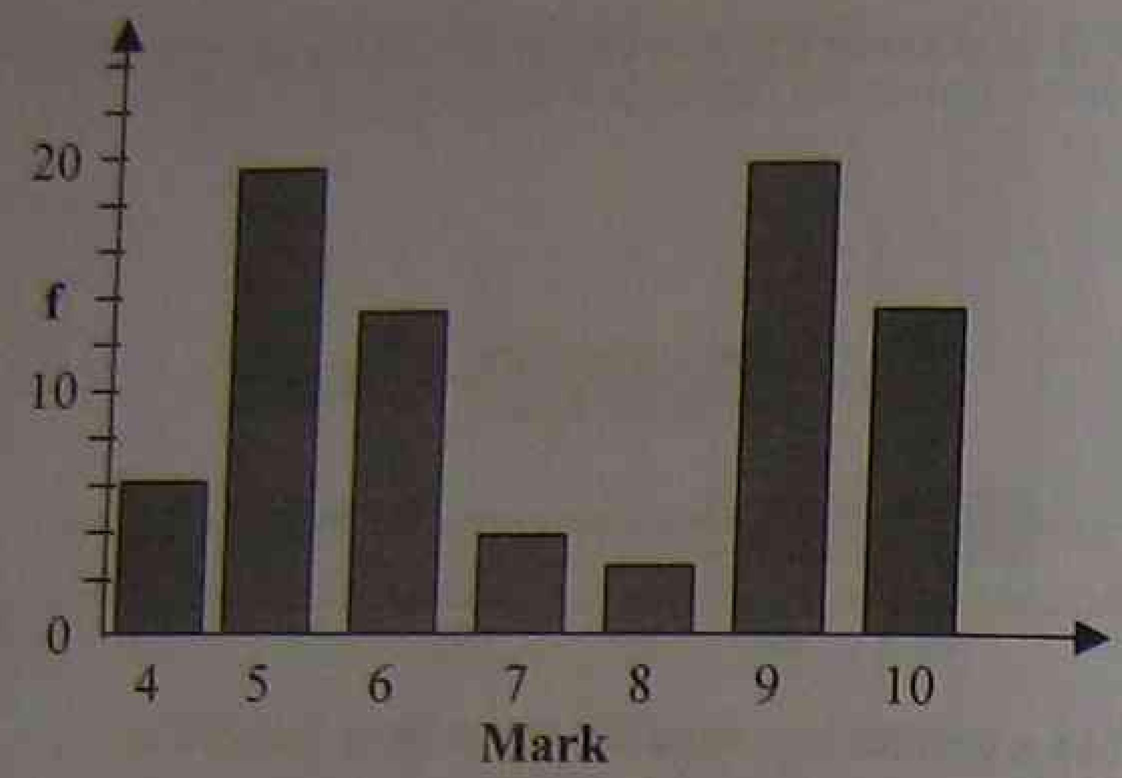
(A) 65 (B) 59 (C) 45 (D) 41

28 Which statement is true about the following distributions?



(A) The median of A is greater than the median of B.
(B) The mode of A is less than the mode of B.
(C) The mean of A is less than the mean of B.
(D) The range of A is less than the range of B.

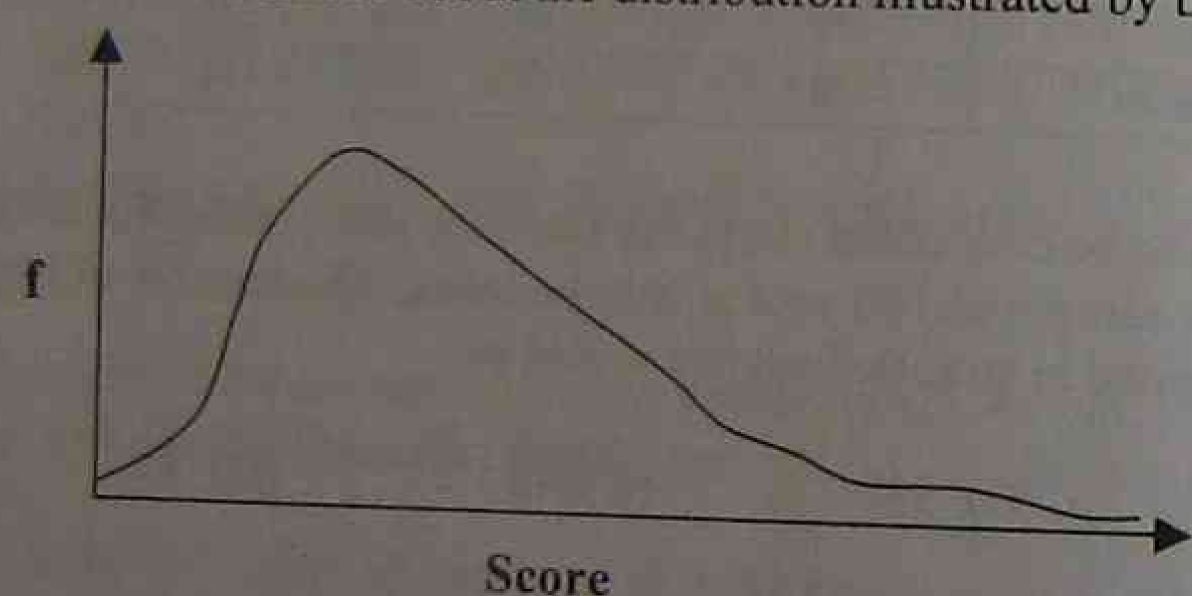
29 The results of a class test were graphed as shown below.



The distribution is best described as

(A) skewed.
(B) bimodal.
(C) normal.
(D) uniform.

30 Which statement is true about the distribution illustrated by the graph below?



(A) The distribution is positively skewed.
(B) The distribution is symmetrical.
(C) The mean of the distribution is less than the mode.
(D) The distribution is uniform.

31 A tank contains fish of the following lengths: 15 cm, 18 cm, 16 cm, 16 cm, 20 cm. Another fish which is 21 cm long is placed in the tank. Which of these statements is true?

(A) The mean length is increased by more than 10%.
(B) The median remains the same.
(C) The standard deviation increases.
(D) The mode changes.

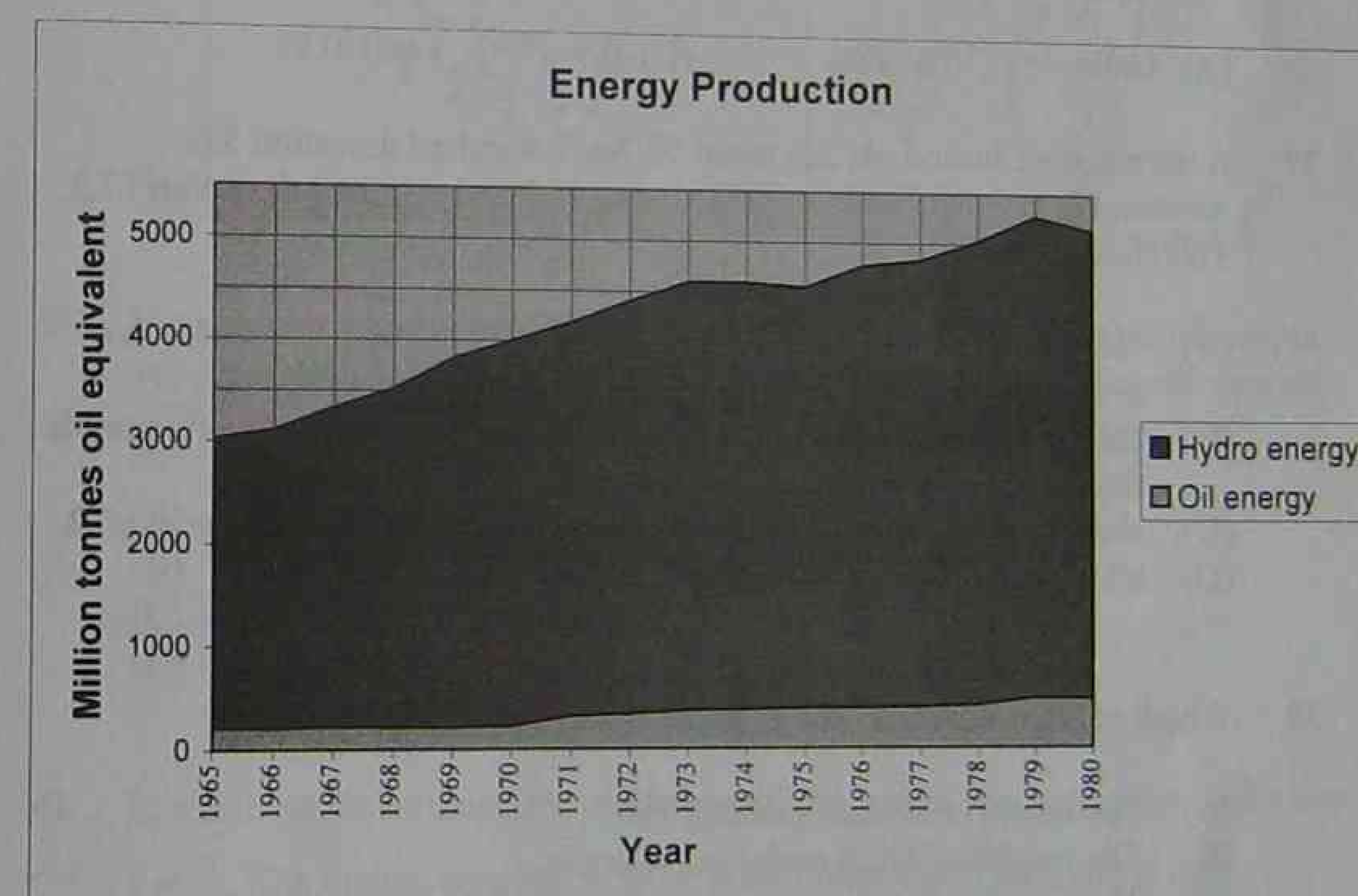
32 The marks on an examination were recorded for boys and girls in a back-to-back stem and leaf plot, as shown.

0	5	
98110	4	03
766440	3	011588
831	2	02569
6	1	25
	0	8
GIRLS		BOYS

Which of the following statements is true?

(A) The girls' marks have a higher mean and a higher range.
(B) The girls' marks have a higher median and a higher standard deviation.
(C) The girls' marks have a higher mean but a lower range.
(D) The girls' mean is lower than the boys' median.

33 The following area chart shows energy production in Russia over several decades.



How many million tonnes of oil would have been required to produce the 1976 hydro energy output?

(A) 400 (B) 4170 (C) 4370 (D) 4770

The Normal Distribution

34 A manufacturer makes circular lids with a mean radius of 76 mm. The standard deviation of a large sample was found to be 1.5 mm. In a batch of 1000 lids, how many would be expected to have a radius less than 74.5 mm?

(A) 320 (B) 200 (C) 160 (D) 100

35 In a normal distribution, a z score of 2.2 would be in the

(A) tenth decile.
(B) first decile.
(C) in the middle decile.
(D) close to the upper quartile.

36 A reel is labelled as having 200 m of cotton. It is known that, in reality, the length of cotton is normally distributed with mean 205.65 m and the standard deviation is 8.23 m. If reels are selected from a batch, the probability of the length lying between 189.19 m and 222.11 m is

(A) 100% (B) 98% (C) 0.9 (D) 0.95

37 A normal distribution, A, has mean 50.7 and standard deviation 5.0. Another normal distribution, B, has mean 45.7 and standard deviation 17.3. Therefore

(A) the scores in a sample from A would total more than the scores in a sample from B.
(B) the range of a sample from A would be less than the range of a sample from B.
(C) about 84% of the scores from A would be greater than the mean of B.
(D) all of the above would be true.

38 Which of these could be treated as a normal distribution?

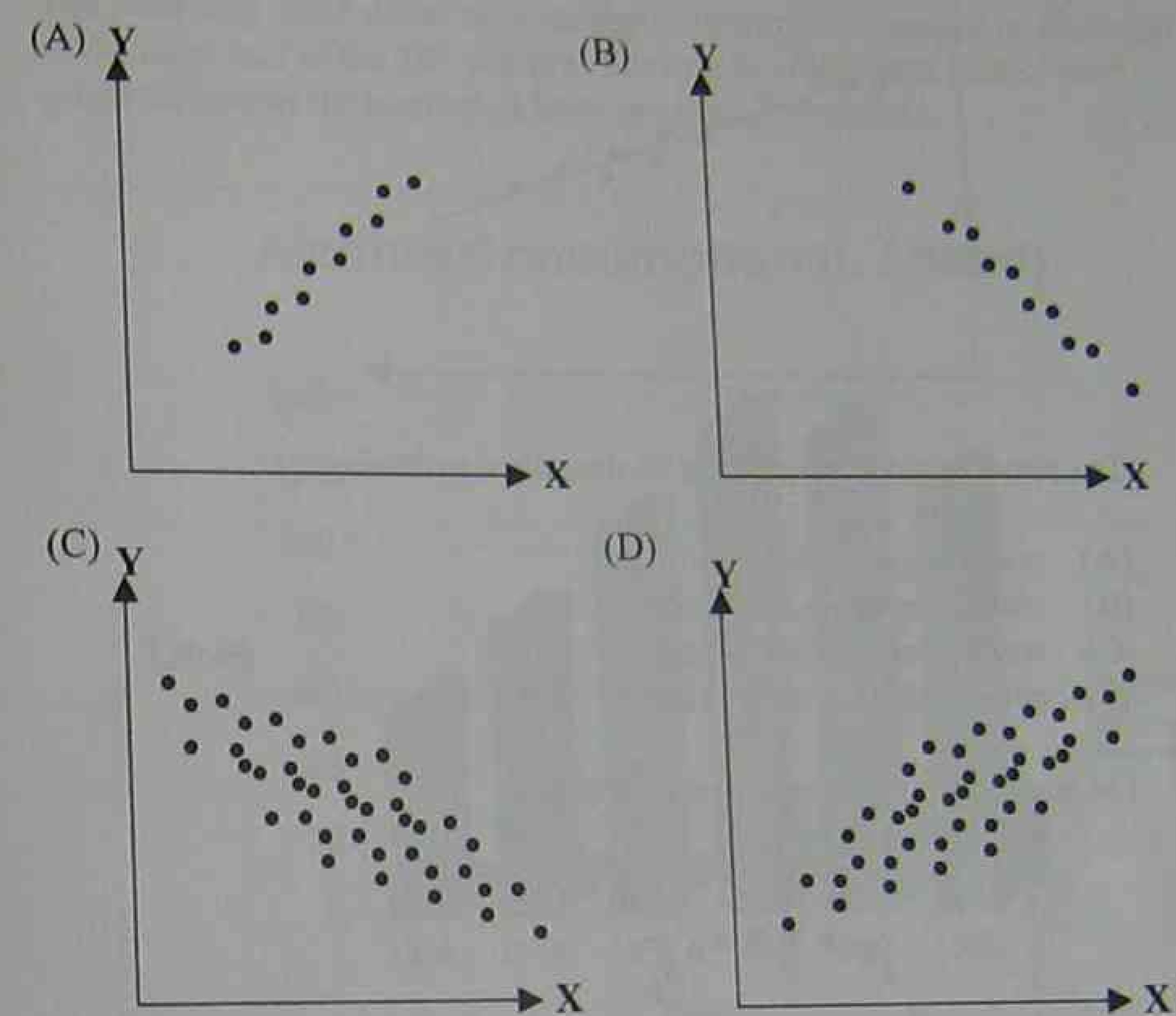
(A) The scores on a class spelling test.
(B) The heights of adult males in Australia.
(C) The time taken for someone to be served in a restaurant.
(D) The taxable income of Australians.

39 If hand spans are normally distributed with mean 22 cm and standard deviation 2.5 cm, the percentage of the population with a span less than 27 cm is

(A) 97 ½ % (B) 97% (C) 5% (D) 2.5%

Correlation

40 Two variables are found to have a weak negative correlation. The scatterplot which best represents a sample of results would be



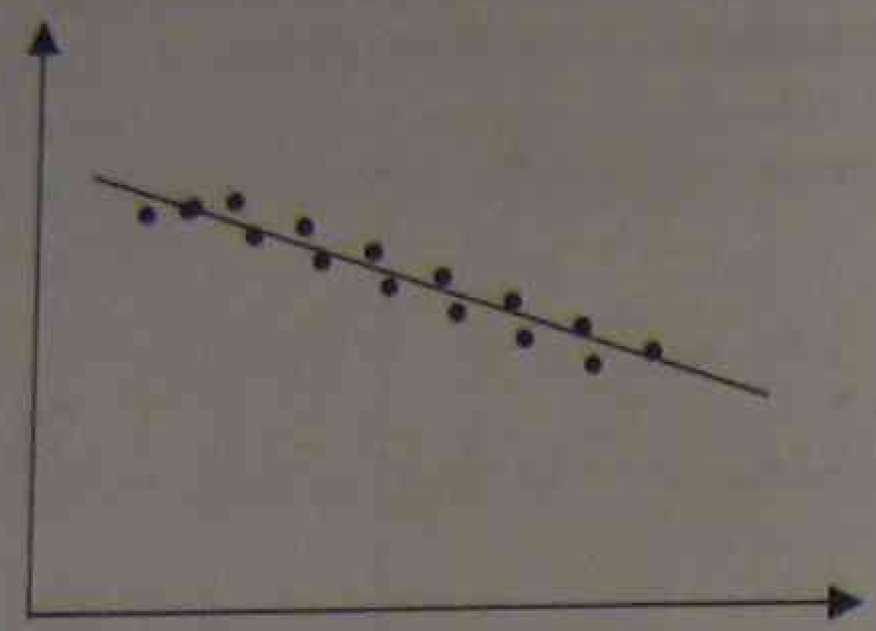
41 In analysing the results of an experiment with a very large number of sample values, a statistician found that two variables had a correlation coefficient of 0.23. On the basis of this experiment, which of these statements is true?

(A) One variable is dependent on the other.
(B) There is hardly any correlation between the variables.
(C) There is a weak negative correlation evident.
(D) The variables have about 23% of scores in common.

42 In a science experiment, the scatterplot of the results was very close to the line $y = \frac{x}{2}$. The logical conclusion to be drawn from the experiment is that

(A) one variable is directly dependent on the other.
(B) the variables have a correlation coefficient a little less than 1.
(C) half the results have the same value for both x and y.
(D) all three of the above statements are true.

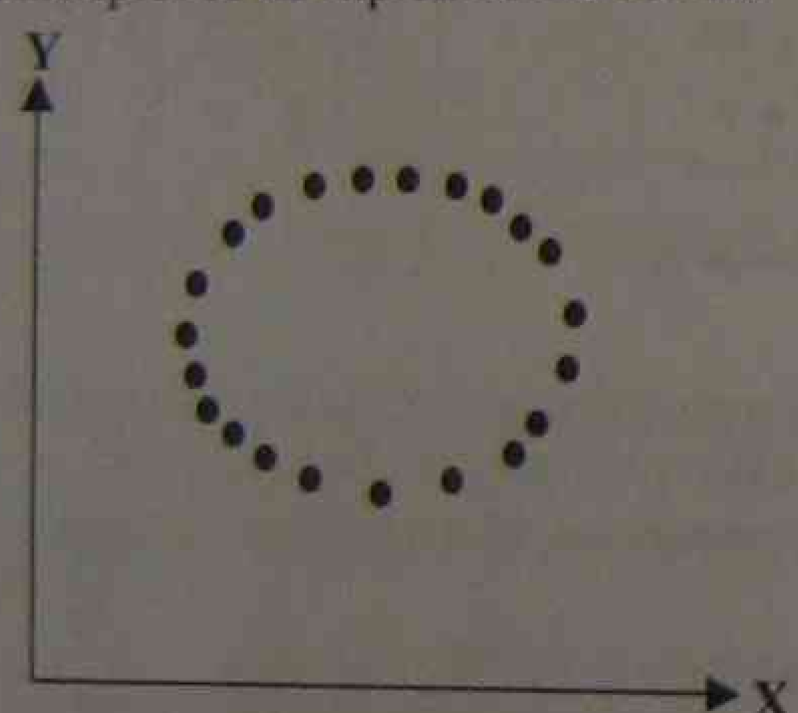
43 The results of an experiment were graphed, as shown, and a median regression line drawn.



The two variables would best be described as having

(A) weak negative correlation.
(B) strong negative correlation.
(C) weak positive correlation.
(D) strong positive correlation.

44 The scatterplot of an experiment is shown.



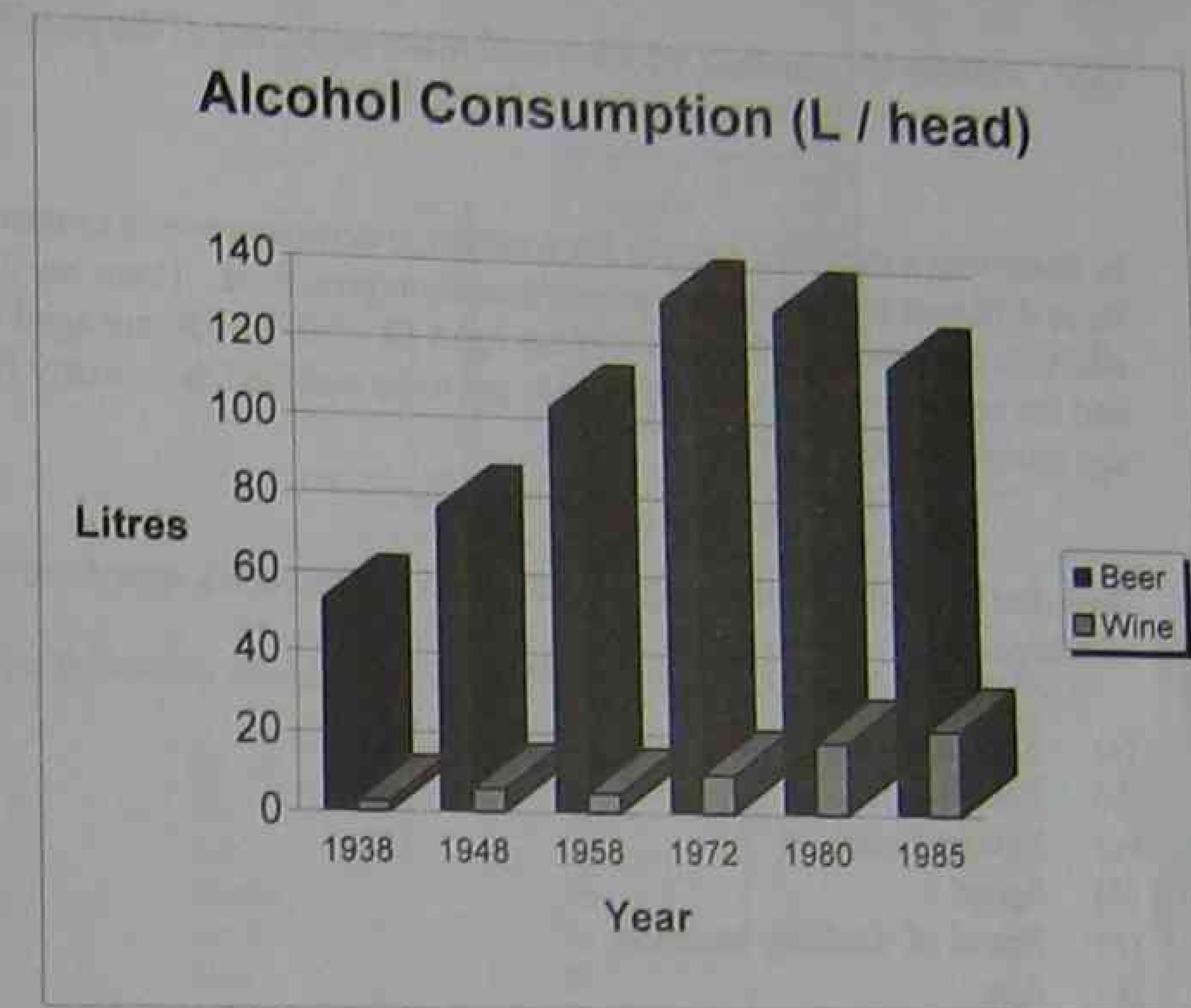
The best description of the relationship between variables X and Y is that

(A) they are strongly correlated but unrelated.
(B) they are strongly correlated and related.
(C) they are weakly correlated and unrelated.
(D) they are weakly correlated but related.

Free Response Questions

Statistics and Society

1 The following chart shows how alcohol consumption changed in Australia in the latter half of the 20th century. The height of the front face of each prism represents the number of litres per head consumed.



(a) How many litres of beer per head were consumed in in 1972?

(b) What trends can be noted about the amount and type of alcohol consumed by Australians over this period?

(c) Wine on average has 5 times the concentration of alcohol per litre of beer. In which year was the most pure alcohol consumed?

2 A researcher approaches people in a suburban shopping centre and asks them a question related to their health. List four problems that could damage the validity of the results obtained from the research.

3 Choose the most suitable type of graph for illustrating

(a) annual company profits (b) ethnic make-up of a town.

Data Collection and Sampling

4 A market research company wants to find out what cereal people eat for breakfast. They send out field workers to ask the question "What cereal do you eat for breakfast?"

(a) List three responses which could cause the field workers trouble.

(b) Rewrite the question which would make recording of the possible responses efficient, accurate and unambiguous.

5 In designing a stratified sample for a survey, a company wants to ensure that three different age groups are proportionally represented. There are 20 500 adults in the town. Of those, 6 850 are aged 18 – 30, 11 750 are aged 31 – 60 and the rest are over 60. If 500 people are to be sampled, how many from each age group are required?

6 Describe each of the following as quantitative continuous, quantitative discrete or categorical.

(a) Shoe sizes
(b) Foot lengths
(c) Favourite colour
(d) Speed
(e) Brand of washing machine
(f) Age

7 Use your calculator to generate a table of 50 single-digit random numbers.

(a) Complete a frequency distribution table for the numbers.

(b) Compare the frequencies obtained with the expected frequency of each number.

8 You are to be employed to ring people at random and ask them questions.

(a) Why might people object to giving information about how they vote?

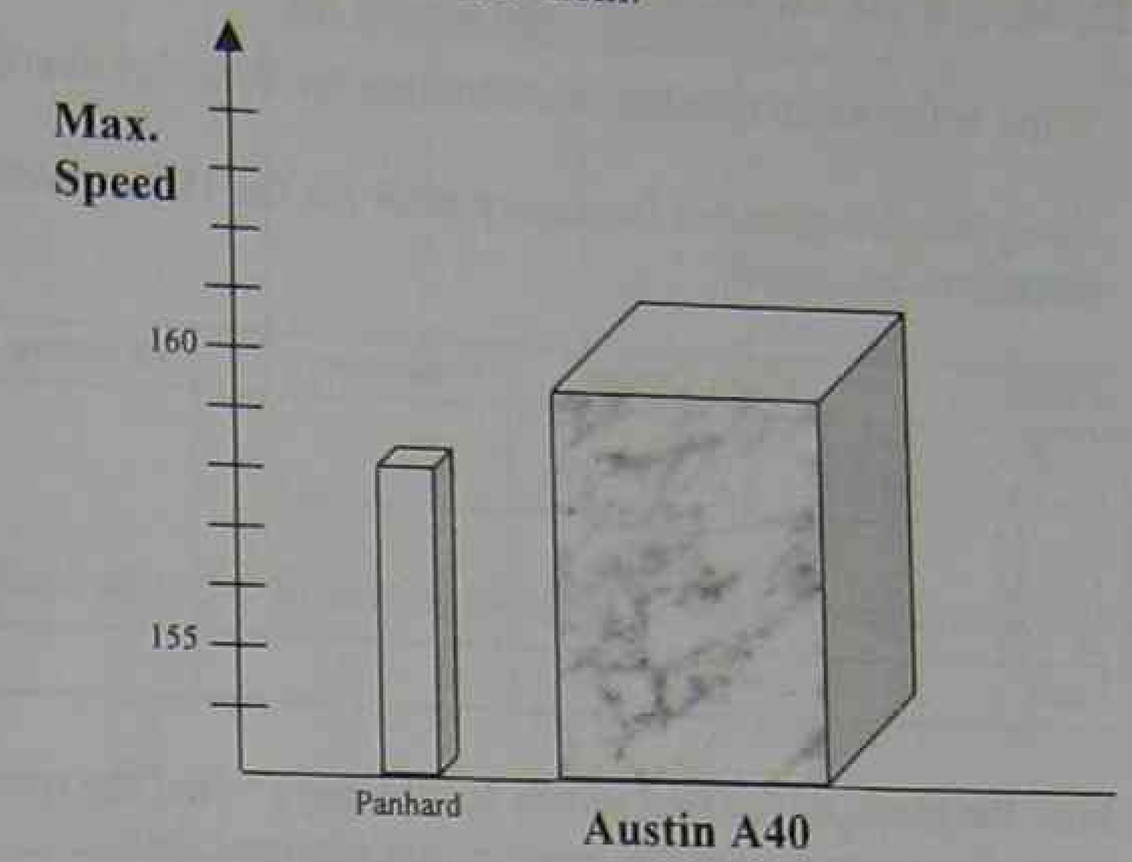
(b) Why might people refuse to give information about when they are home?

(c) Why might people not want to divulge their income?

(d) Why might people not want to answer any questions?

Displaying Single Data Sets

9 In this column graph below, there are three attempts to exaggerate the difference between the scores. List them.



10 The following data relates to the share price of a public company.

Month	Average price (cents)
Jan	87
Feb	90
March	89
April	86
May	83
June	85
July	88
Aug	90
Sept	93
Oct	90
Nov	89
Dec	90

(a) Choosing a suitable scale for your axes, draw a line graph to support the Chairman's statement that the share price was steady throughout the year.

(b) On another set of axes, show how a "displaced zero" can cause a line graph to support the claim that the share price soared in September.

(c) Find the mean and standard deviation for the data.

(d) What percentage of scores lie within one standard deviation of the mean?

11 A restaurateur noted the number of customers in the restaurant each evening for 21 consecutive days. The restaurant is closed on Mondays.

15, 22, 24, 0, 8, 10, 16, 18, 24, 25, 0, 7, 20, 17, 17, 20, 23, 0, 5, 15, 16.

(a) What is the mean number of customers for Saturday evenings?

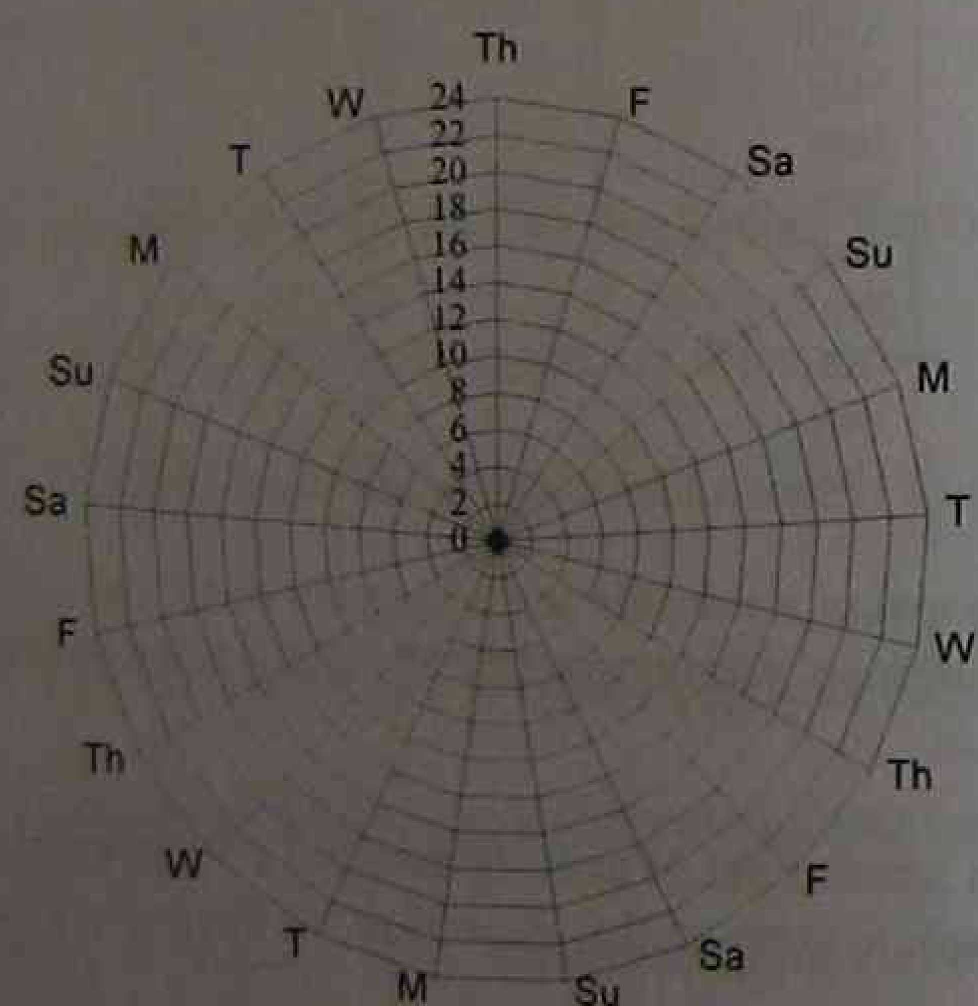
(b) Complete this grouped frequency table for the 18 days when the restaurant was open.

Class	Class Centre	Frequency	f x centre
1 – 5			
6 – 10			
11 – 15			
16 – 20			
21 – 25			
Total			

(c) Find the mean of the raw scores for the days when the restaurant was open. Compare it with the mean of the grouped data.

(d) List the graph types which would be suitable for displaying the data in the table.

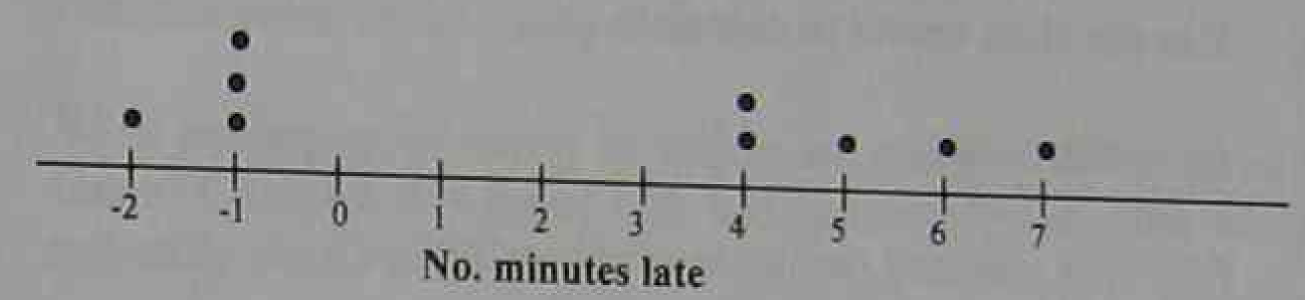
(e) Complete this radar chart for the 21 days.



12 An investigation was carried out to determine the punctuality of a local bus service. The number of minutes a bus was late was recorded as shown in this table.

No. minutes late	Frequency
-2	1
-1	3
0	12
1	12
2	6
3	4
4	2
5	1
6	1
7	1

(a) Complete the following dot plot of the information in the table.



(b) What is meant by the negative values for time?

(c) Find
(i) the median number of minutes late.
(ii) the most likely number of minutes late.
(iii) the mean number of minutes late.

(d) Which measure of central tendency gives the best indicator of punctuality?

(e) What percentage of arrivals were on time or better?

13 The following numbers were produced using a random number generator.

3, 0, 9, 8, 5, 4, 2, 9, 1, 7, 5, 7, 9, 4, 5, 6, 8, 6, 6, 2, 9, 7, 5, 0, 5, 0, 3, 7, 9, 8, 8, 0, 9, 1, 6, 7.

(a) Complete a frequency distribution table and add a cumulative frequency column.

(b) Draw a cumulative frequency histogram and polygon.

(c) Complete a five number summary for the data.

(d) Draw a box and whisker plot to illustrate the data.

14 The heights of children in a large school were measured and results collected into the table shown below.

Heights (cm)	Frequency	Class centre
121 – 130	75	
131 – 140	208	
141 – 150	315	
151 – 160	497	
161 – 170	358	
171 – 180	201	
181 – 190	50	

(a) Find the centre of each class.

(b) Use the class centre to calculate

(i) the mean.

(ii) the standard deviation of this sample.

(c) What percentage of this sample lies within one standard deviation of the mean?

15 If $x_1 + x_2 + x_3 + x_4 + \_\_ \, x_9 = 45$, what is the mean?

16 If $\bar{x}$ is the mean of a sample, what is the value of $\Sigma(x - \bar{x})$?

17 The mean of three scores is 7 and the range is 6. What are the possible values of the scores if all the scores are positive whole numbers?

18 Three people are asked to comment on the average weekly wage in Australia.

(a) Union officials want to make the point that wages are low. They say that the average person earns $420 per week but mean that an ordinary worker earns $420 per week. Which statistic are they calling an average?

(b) The CEO of a large company wants to claim that wages are high and says that the average earnings are $950 per week. Which statistic are they calling an average?

(c) The Bureau of Statistics produces a figure which it considers best reflects the centre of the distribution but news media call it an average. What is the correct name of this statistic?

19 The following marks were scored by students undertaking a final examination for a university course.

56, 60, 49, 67, 89, 67, 80, 54, 56, 39, 57, 69, 77, 88, 67, 65, 53, 56, 71, 79, 48, 58, 64, 83, 51, 45, 91, 68, 55, 74, 68, 47, 67, 45, 65, 70, 73, 65, 74, 78, 47, 58, 56, 74, 83, 68, 63, 82, 74, 57, 52, 69, 60, 76, 88, 54, 67, 66, 43, 61.

(a) Find the mean and standard deviation of the scores.

(b) A credit is given to students with a mark from 75 to 84. How many credits would be given?

(c) What is the probability that a student chosen at random has scored 50 marks or better?

(d) Find the value of the median score.

(e) What percentage of scores lie within one standard deviation of the mean? Compare this figure with the figure expected if the scores were normally distributed.

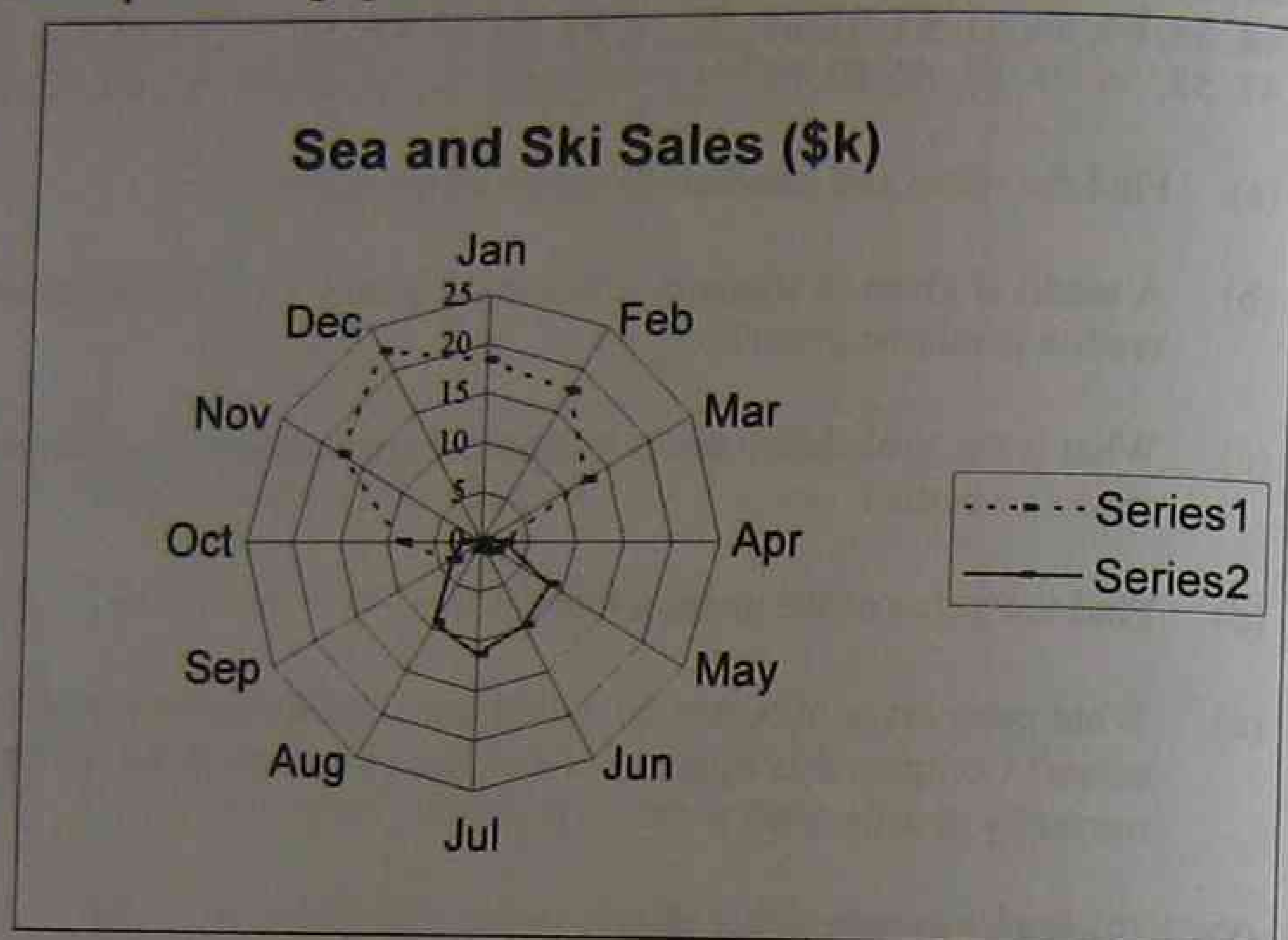
(f) Copy and complete this table.

Class	Class centre	f	Cum. f
30 – 39			
40 – 49			
50 – 59			
60 – 69			
70 – 79			
80 – 89			
90 – 99			
Total			

(g) Find the mean using grouped data and compare it with the true mean.

(h) Draw a cumulative frequency histogram and polygon from the grouped data and show how the median and interquartile range may be estimated. Compare the estimate with the true median.

20 A shop sells surfing and snow skiing equipment. Its net takings over a twelve month period are graphed on this radar chart.



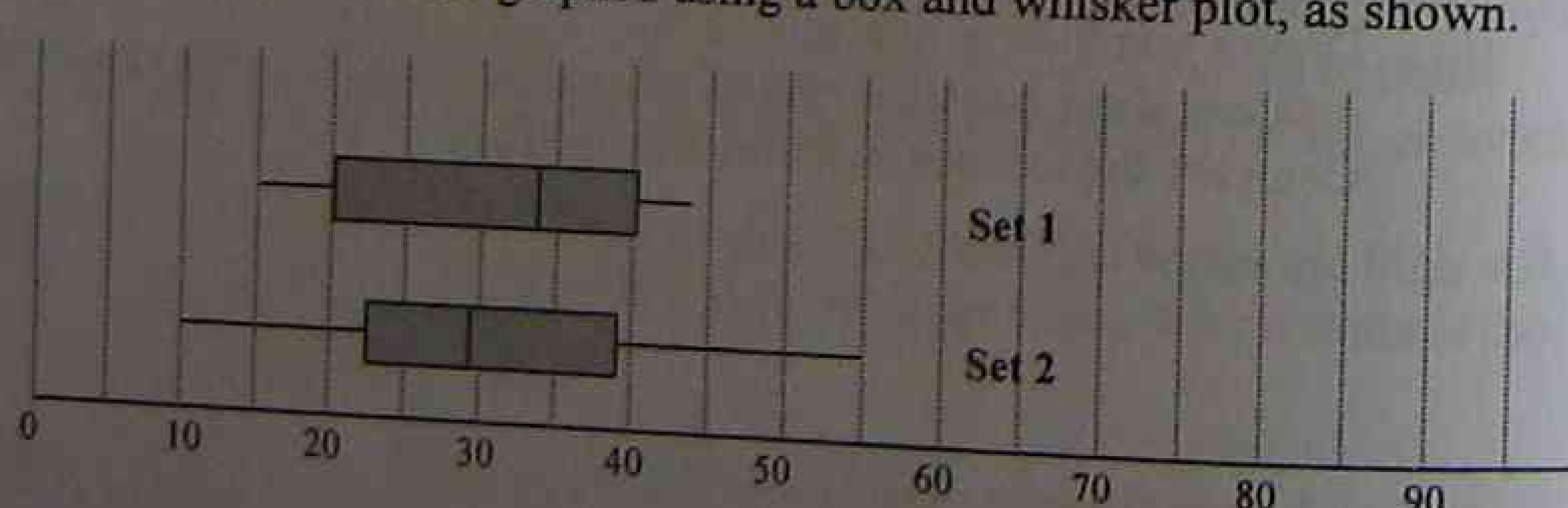
(a) Which series on the graph represents snow related sales? How can you tell?

(b) Which side of the business has the higher gross takings for the year?

(c) In which month was the highest gross sales figure recorded for the business?

(d) Construct a double column graph to illustrate the same data.

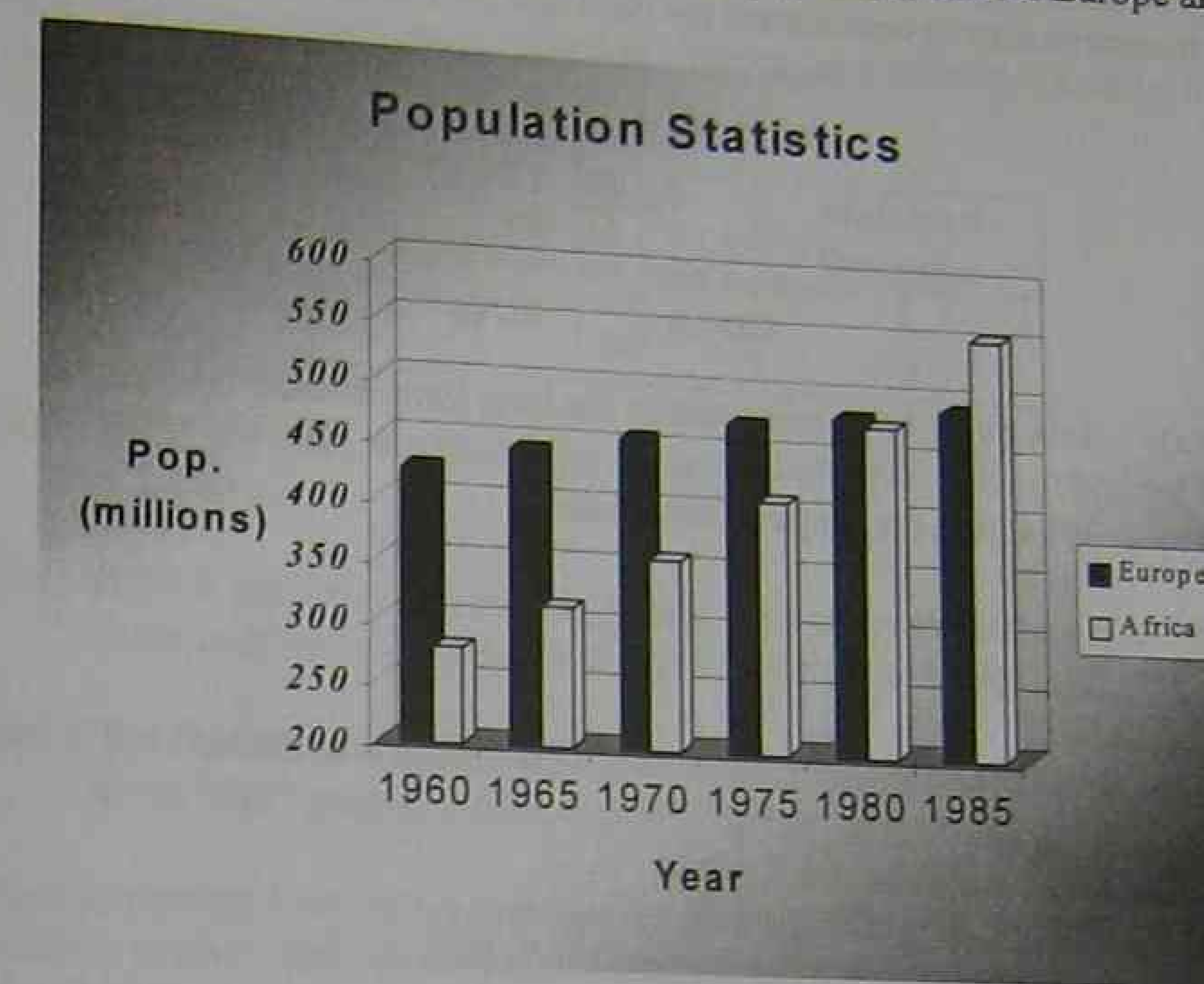
21 Two sets of scores were graphed using a box and whisker plot, as shown.



(a) Determine the five number summary for each set.

(b) Which graph is more affected by outliers?

22 The following graph was generated from the population data of Europe and Africa.



(a) How many more people were there in Europe than in Africa in 1970?

(b) Find the average rate of population growth over the 25 years for
(i) Europe
(ii) Africa

23 In the Australian population, blood groups are distributed as shown in the following table.

Type	O	A	B	AB
Rh positive	40%	31%	8%	2%
Rh negative	9%	7%	2%	1%

(a) What percentage of the Australian population has blood type

(i) O
(ii) A

(b) What percentage of the Australian population has Rh negative blood?

(c) A person with blood group B can receive blood other people, provided they are group B or group O. What percentage of the population can give blood to a group B person?

24 The following stem and leaf plot shows a comparison of long jump measurements in centimetres for boys and girls in an athletics carnival.

GIRLS		BOYS
	54	3
	53	0 2 4 8
2	52	1 1 5 6 X
9 7 4 1 1 0	51	6 8
6 4 2 2 1	50	

(a) How many boys competed in the long jump event?

(b) What was the range of scores for

(i) boys?
(ii) girls?

(c) How much longer than the best girl's jump was the best boy's jump?

(d) What values could X take?

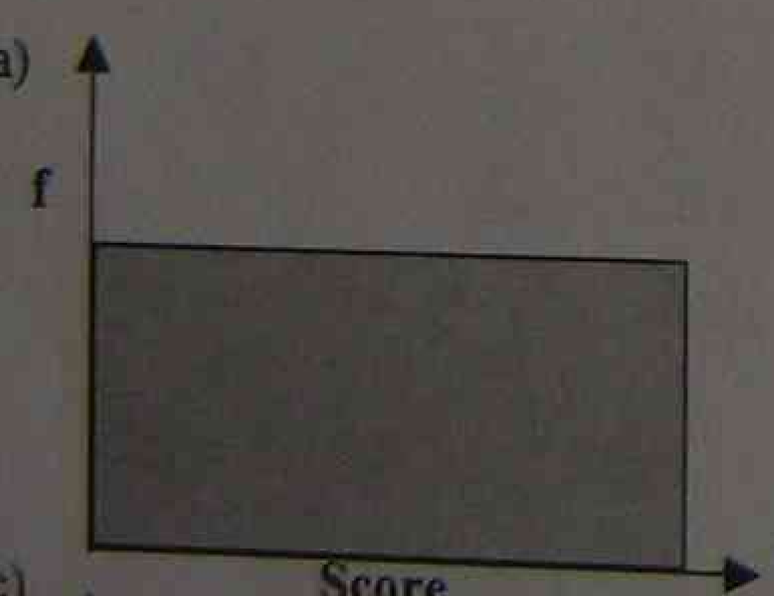
(e) If X = 9, calculate for the boys

(i) the mean
(ii) the mode
(iii) the median

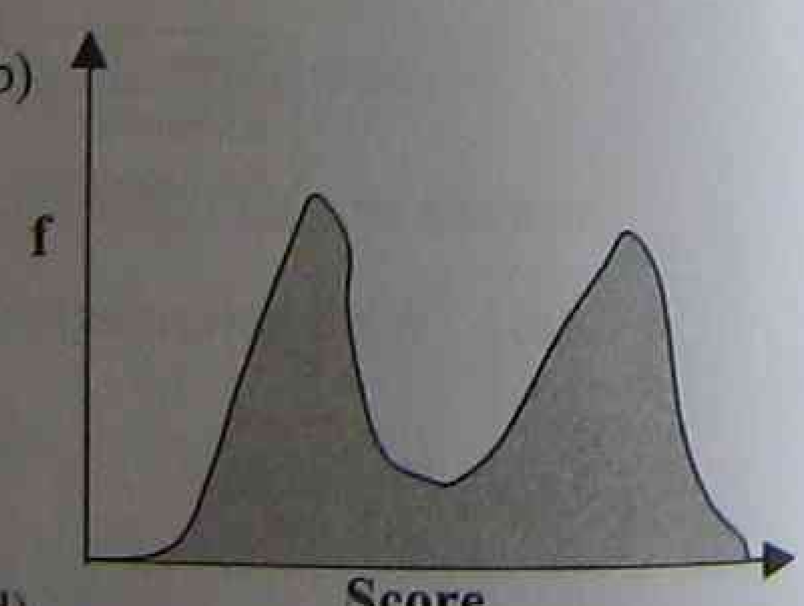
(f) On average, how much further did the boys jump than the girls?

25 For each of these frequency distribution graphs, choose terms to describe the main features. Suggest a situation where such a graph might arise.

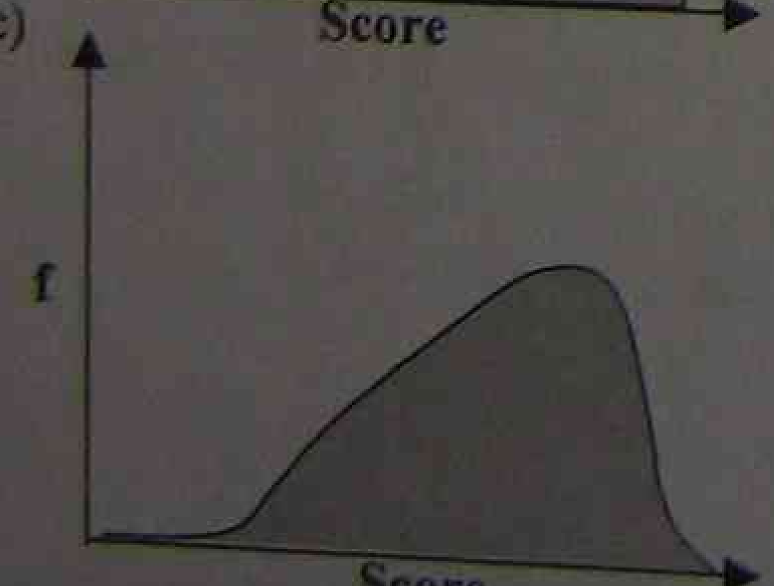
(a)



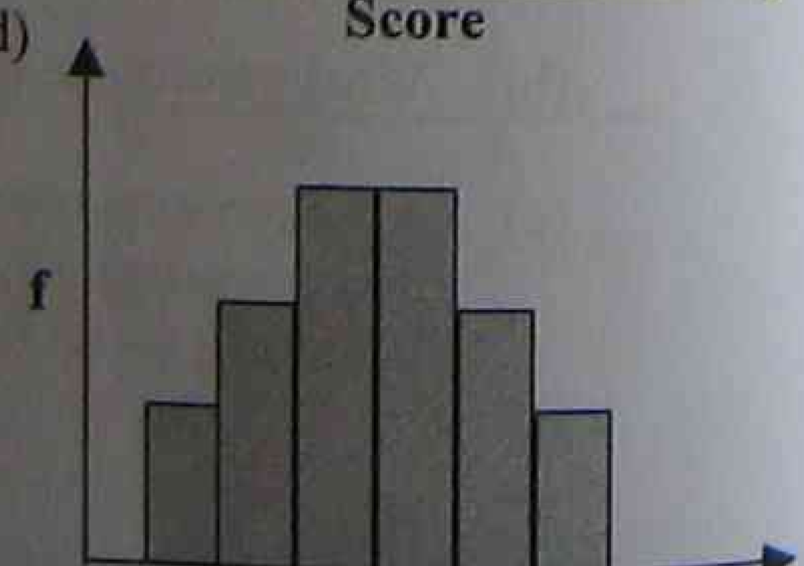
(b)



(c)



(d)



The Normal Distribution

26 This table gives means and standard deviations for a History test and a Maths test.

Subject	Mean	Standard Deviation
History	64	5
Maths	54	12

(a) Using z-scores, determine the mark in Maths which is equivalent to 74 in History?

(b) If 1000 people sat for the Maths test, how many could be expected to score over 90, assuming the scores are normally distributed?

(c) The History teachers have determined that 16% of candidates will be given "fail" grades. What mark will be the "pass" mark?

27 An engineering firm makes pistons with diameter 75 mm. Sampling of a large number gives a standard deviation of 0.001 mm. Assume a normal distribution.

(a) Within what range of diameters should 95% of the output of the firm lie?

(b) What percentage error would result from a piston whose size was 3 standard deviations above the mean?

(c) What would be the median piston diameter?

(d) In a batch of 1000 pistons, how many would be expected to be undersize by more than 0.002 mm?

28 Standardise these two sets of scores and determine which candidate has the highest total of z-scores.

Subject	Albert's Mark	Bernie's Mark	Mean	s.d.
English	55	60	50	5
Maths	71	51	55	8
Science	66	63	60	12
History	80	85	60	10
Geography	67	61	55	12
French	45	50	50	5

Correlation

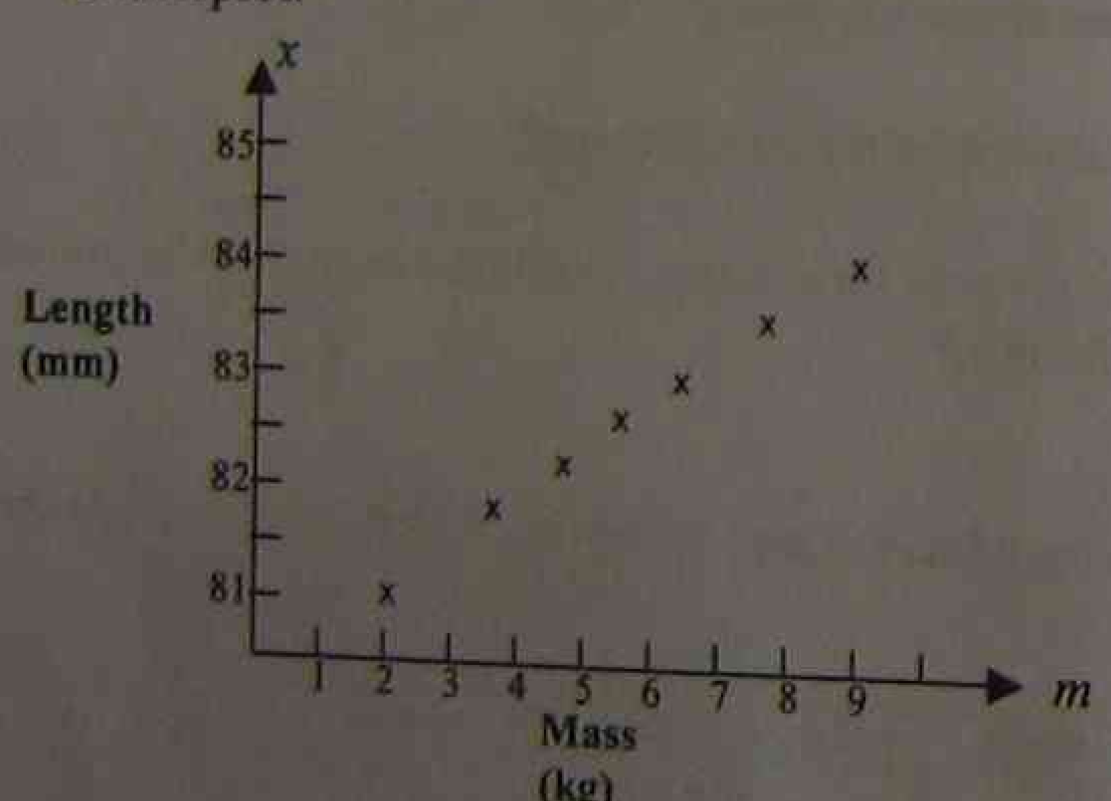
29 The marks for a class were recorded for their English and Maths results.

Name	English / 10	Maths / 10
Angela	5	6
Brian	6	7
Colin	4	6
Duc	9	4
Ellen	4	5
Faoud	7	6
Guan	5	8
Inigo	5	9
Jeffrey	7	6
Karen	5	5
Lisa	8	9
Michael	10	9
Neil	6	5
Owen	6	6
Peter	8	6
Sandra	5	3
Trevor	3	5

(a) Complete a scatterplot of these results.

(b) Describe the extent and type of correlation you observe.

30 A science experiment measured the length x mm of a piece of piano wire when a mass m kg was attached to one end. The results were graphed in this scatterplot.



(a) Draw the median regression line for these plots.

(b) Determine the gradient of the line.

(c) Determine the length of the wire when no mass was attached.

(d) What mass would make the wire 85.5 mm long?

Miscellaneous Questions

31 A random sample of 40-year-old male drivers were asked the question: "In how many accidents in the past five years have you been involved as a driver". The table below shows the results of the survey.

Number of accidents (x)	Number of drivers (f)
0	10
1	12
2	11
3	6
4	4
5	2

(a) How many drivers were interviewed in the survey?

(b) Draw a frequency polygon to illustrate the data.

(c) How many drivers had had fewer than 3 accidents in the past five years?

(d) Find the median number of accidents.

(e) Calculate

(i) the mean number of accidents per driver, correct to one decimal place.

(ii) the standard deviation, correct to one decimal place.

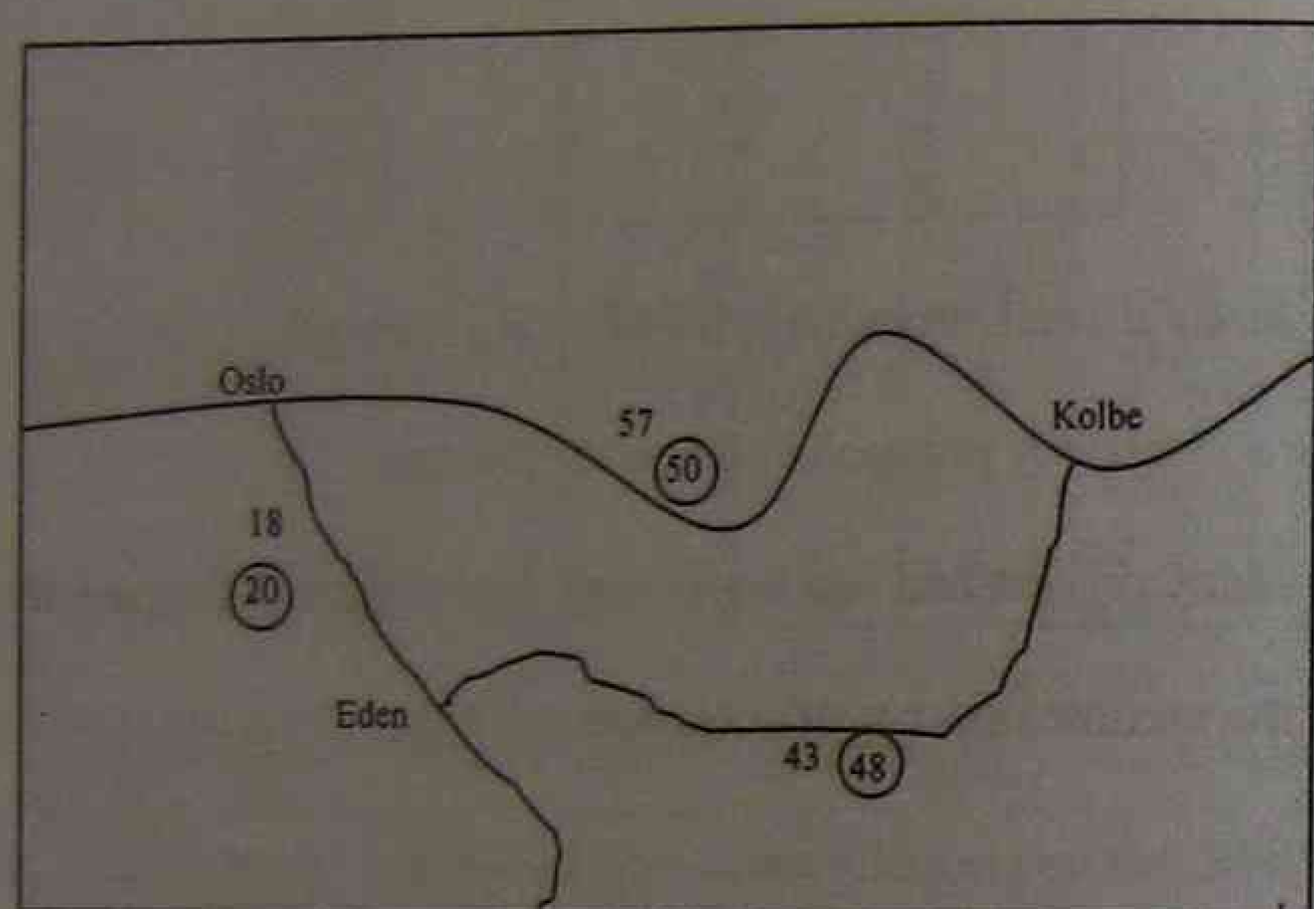
(f) Is it reasonable to treat these scores as though they were sampled from a normal distribution? Give reasons for your answer.

MEASUREMENT

Multiple Choice Questions

Units of Measurement

1 This road map shows distances between towns in kilometres and driving times (circled) in minutes.



The map indicates that, to drive from Oslo to Kolbe directly is

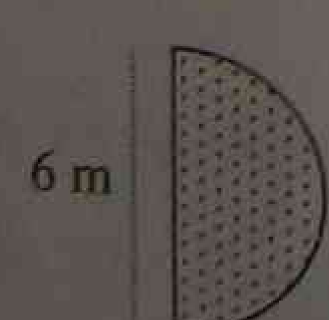
(A) shorter but takes more time than via Eden.
(B) shorter and takes less time than via Eden.
(C) longer and takes more time than via Eden.
(D) shorter and takes less time than via Eden.

2 In winning the gold medal for the 100m sprint in about 10 seconds, the sprinter would have averaged about

(A) 10 km/h (B) 20 km/h (C) 25 km/h (D) 35 km/h

3 Plant food is distributed at the rate of 100 g per m^2. The amount required for this lawn would be closest to

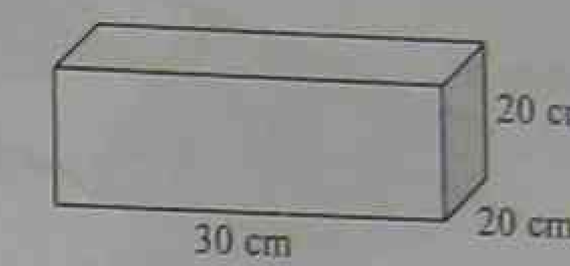
(A) 1410 g
(B) 2830 g
(C) 5660 g
(D) 11 310 g



4 A car travels 380 km on 32 L of petrol. Its average petrol consumption is

(A) 0.08 km/L (B) 8 km/L
(C) 8.4 L/100km (D) 11.8 L/100km

5 Water is pouring into this fish tank at the rate of 4 L per minute. How long would it take to fill the tank?



(A) 50 hours
(B) 3 minutes
(C) 3 hours
(D) 50 minutes

6 A length written down as 100.0 m

(A) is quoted to the nearest metre.
(B) indicates at most 50 cm error.
(C) indicates no more than 1% error.
(D) indicates no more than 50 mm error.

7 In calculating the area of a rectangular garden which has been found to measure 4.6 m x 3.8 m, the calculator gives the answer 17.48 so the sensible order of accuracy to write down would be

(A) 17.48 m^2 (B) 17.5 m^2 (C) 18 m^2 (D) 20 m^2

8 A farmer has two rectangular fields, one measuring 30 m x 40 m and the other 25 m x 60 m. The ratio of their areas is

(A) 2 : 3 (B) 3 : 4 (C) 4 : 5 (D) 5 : 6

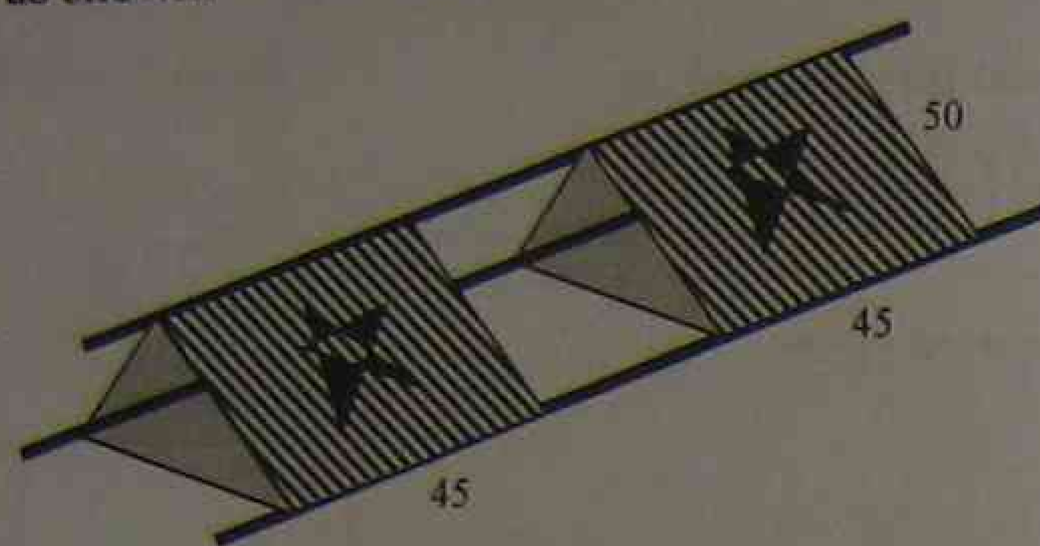
9 If two people shared a $17 450 gambling win in the ratio 17 : 8

(A) the smaller share would be $8211.76.
(B) the larger share would be $11 866.
(C) the smaller share would be $1396.
(D) the larger share would be $9238.24.

10 A cleaning solution is mixed 1 part concentrate to 10 parts of water. To make up 20 litres of solution, we would need

(A) 2000 mL of concentrate.
(B) 1818 mL of concentrate.
(C) 18 litres of water.
(D) 22 litres of water.

11 A box kite is made as shown by wrapping two pieces of cloth around three sticks. The sticks are an equal distance apart. Measurements in centimetres are as shown.

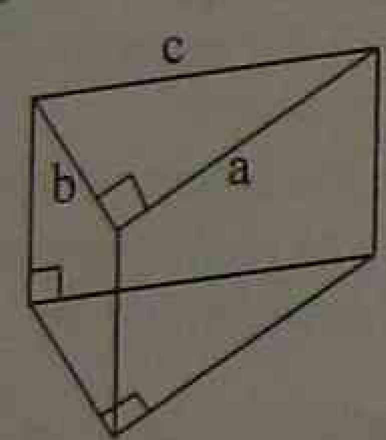


The surface area of the outside of the cloth faces totals

(A) 2250 cm² (B) 4500 cm² (C) 6750 cm² (D) 13 500 cm²

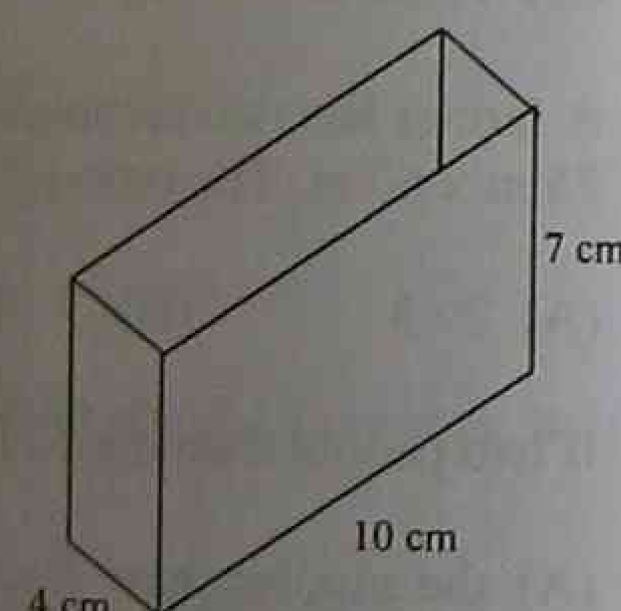
12 The volume of this solid is

(A) ½ abm
(B) abm
(C) b + a + m
(D) 2a + 2b + 2c + 3m



13 A box which would hold twice as much as the one drawn would have dimensions

(A) 10 cm, 14 cm, 4 cm
(B) 20 cm, 14 cm, 8 cm
(C) 10 cm, 7 cm, 2 cm
(D) 10 cm, 14 cm, 8 cm



14 The box in Question 13 has no lid. The outside surface area of the box is

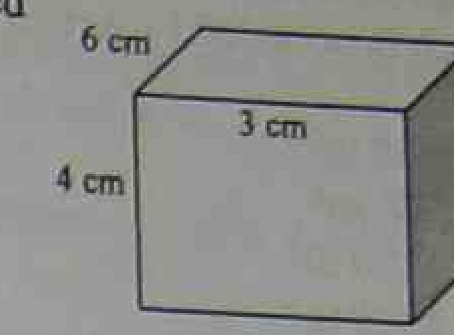
(A) 236 cm² (B) 112 cm² (C) 96 cm² (D) 15 cm²

15 A rectangular veranda 1.1 m x 5.5 m is to be tiled using 50 mm x 25 mm rectangular tiles with no gaps between tiles. The number of tiles required would be about

(A) 484 (B) 4840 (C) 48 400 (D) 484 000

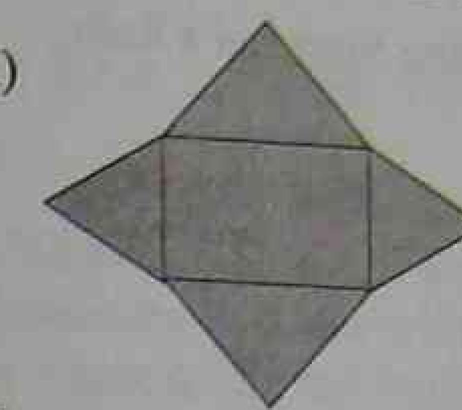
16 This box is to be covered in 2 cm wide tape. How many lineal metres of tape are required if there is to be no overlapping?

(A) 36 cm
(B) 72 cm
(C) 216 cm
(D) 54 cm

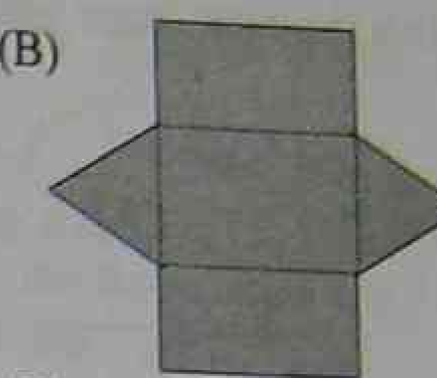


17 Which of the nets shown could possibly be folded to form a triangular prism?

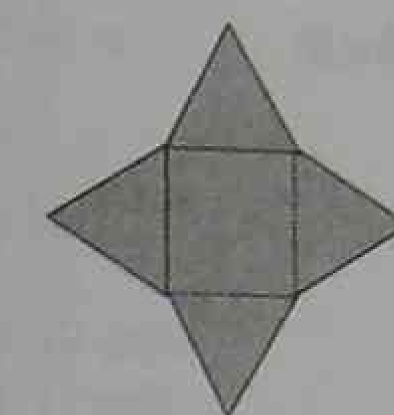
(A)



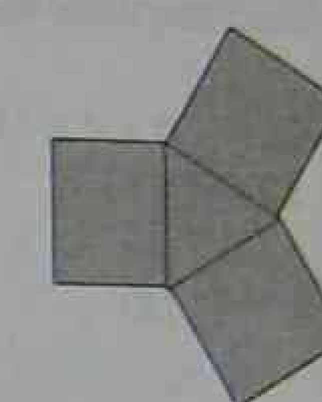
(B)



(C)

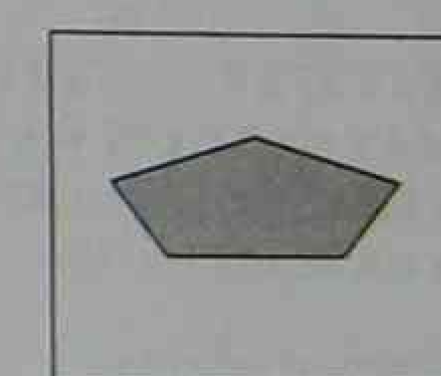


(D)



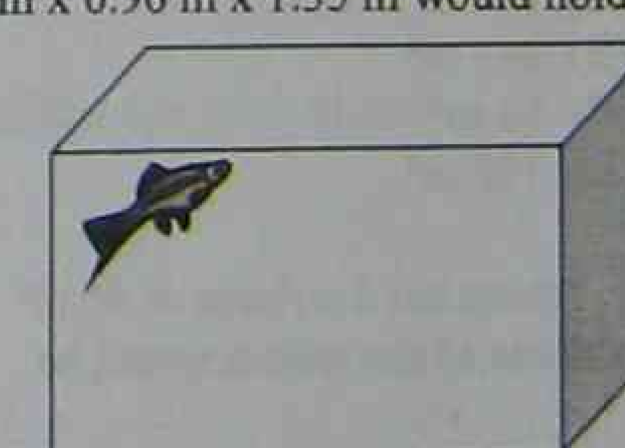
18 The shape that would be formed by extruding rubber through a hole of this shape would be called a

(A) pentagonal prism.
(B) hexagonal prism.
(C) cylinder.
(D) rectangular prism.



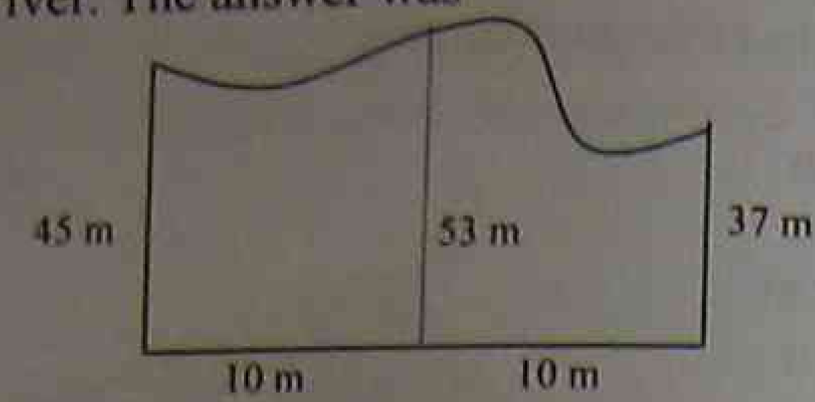
19 A fish tank as shown measuring 0.78 m x 0.96 m x 1.35 m would hold

(A) 101 088 L of water.
(B) 10 108.8 L of water.
(C) 1 011 L of water.
(D) 101 L of water.

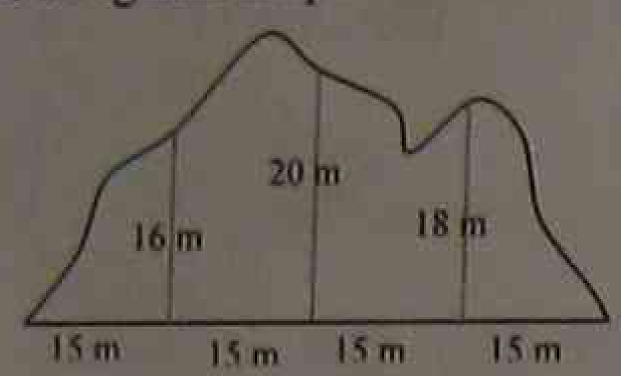


20 Simpson's Rule was used to find the approximate area of this block of land which is bounded by a river. The answer was

(A) 530 m^2
(B) 980 m^2
(C) 1350 m^2
(D) 8325 m^2



21 The area of this irregular shape was found using Simpson's Rule.

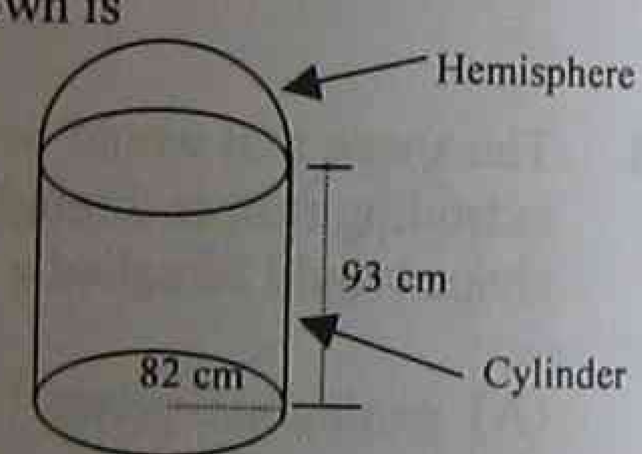


The correct method involves the following working:

(A) $5(4 \times 16 + 20 + 20 + 4 \times 18)$
(B) $^{15}/_3(16 + 4 \times 20 + 18)$
(C) $^{15}/_4(4 \times 16 + 20 + 20 + 4 \times 18)$
(D) $30(16 + 4 \times 20 + 18)$

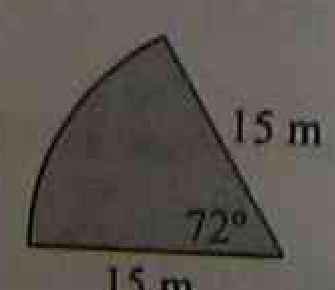
22 The area of the curved surface of the solid shown is

(A) $2 \times \pi \times 82 \times 93 + \frac{1}{2} \times \pi \times 82^2$
(B) $\frac{1}{2} \times 4 \times \pi \times 82^2 \times 93 + \pi \times 82^2 \times 93$
(C) $\pi \times 82^2 \times 93 + 4 \times \pi \times 82^2$
(D) $2 \times \pi \times 82 \times (93 + 82)$



23 A garden sprinkler can water the area shaded. The coverage area would be

(A) 107 m^2
(B) 141 m^2
(C) 63 m^2
(D) 102 m^2



24 A sphere has a volume of 46 m^3. If the sphere's diameter is doubled, the volume of the sphere would be

(A) 92 m^3 (B) 138 m^3 (C) 184 m^3 (D) 368 m^3

Similarity

25 A map scale is 2 cm : 400 m. Two points on the map are 6.3 cm apart. The distance between the two places in reality is closest to

(A) 126 km (B) 1 km (C) 630 km (D) 1.3 km

26 A house plan shows a scale of 1 : 50. A wall on this plan measures 50 mm. How long is the real wall?

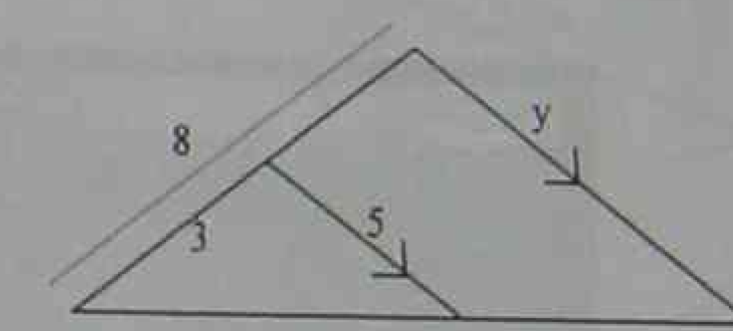
(A) 2.5 m (B) 1.0 m (C) 250 mm (D) 2.0 m

27 A 3D model is built of a town, using a scale of 1 : 100. One building is represented by a box with a rectangular floor measuring 55 mm x 80 mm. The floor area of the real building would be

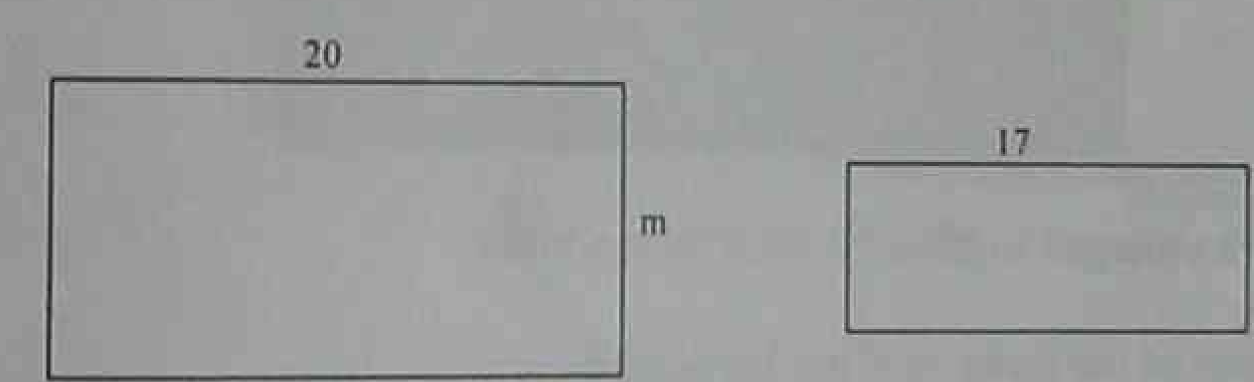
(A) 44 m^2 (B) 440 m^2 (C) 4400 mm^2 (D) 440 000 mm^2

28 The length of the side marked y is

(A) 10 units
(B) 11 units
(C) 12 ½ units
(D) 13 ⅓ units



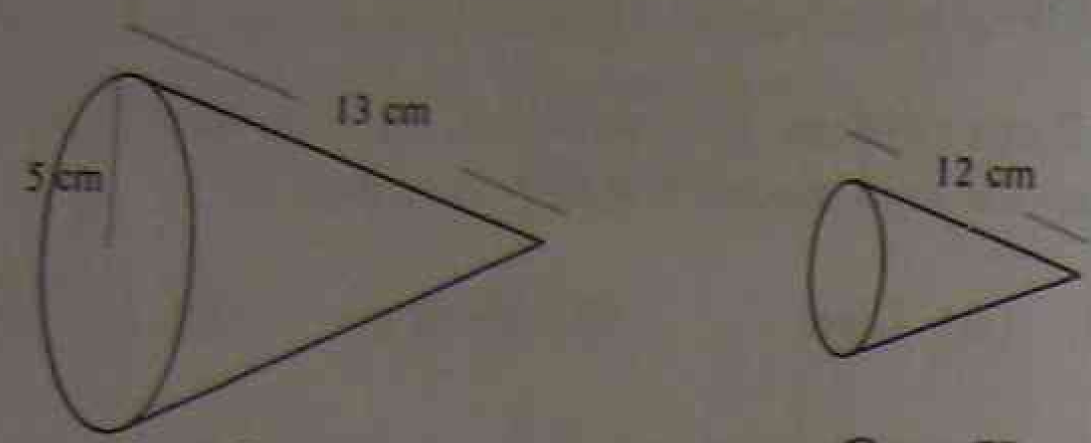
29



These two rectangles will be similar if the side marked m is

(A) $3 \times 20 \div 17$
(B) $17 \times 3 \div 20$
(C) $17 \div 3 \times 20$
(D) $20 \times 17 \div 3$

30



These two cones will be similar if the radius of the base of Cone II is

(A) $\dfrac{13 \times 12}{5}$ cm

(B) $\dfrac{5 \times 12}{13}$ cm

(C) $12 \div (13 - 5)$ cm

(D) $\dfrac{13 \times 5}{12}$ cm

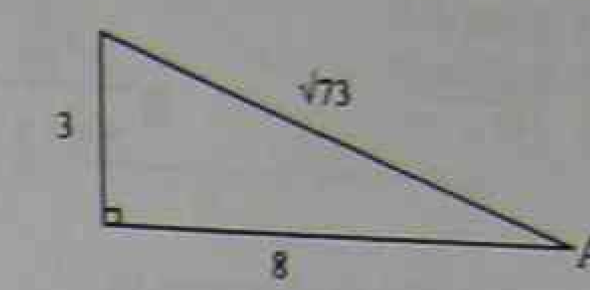
31 I have just taken a beautiful holiday photograph which measures 13 cm tall by 18 cm wide.

If I get it enlarged to fill a 52 cm x 54 cm frame

(A) part of the frame will not be needed.
(B) 13 cm of the photo will have to be trimmed from one side.
(C) 8 cm will have to be trimmed from the top.
(D) a total of 18 cm will have to be trimmed from the sides.

Right-Angled Triangles

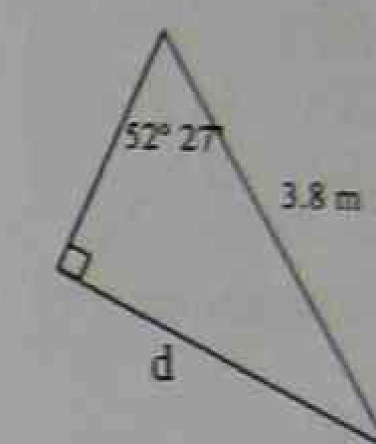
32 In this triangle, the value of cos A is



(A) $\dfrac{3}{8}$ (B) $\dfrac{8}{3}$

(C) $\dfrac{3}{\sqrt{73}}$ (D) $\dfrac{8}{\sqrt{73}}$

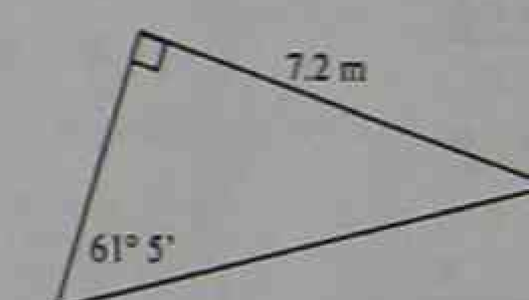
33 In this triangle, the length of the side marked d is



(A) 3.0 m (B) 2.3 m

(C) 4.9 m (D) 0.34 m

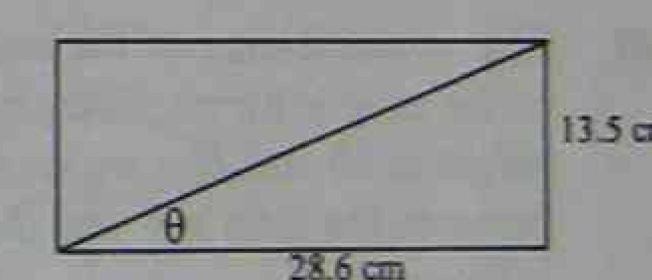
34 The length of the hypotenuse of this triangle is



(A) 14.9 m (B) 13.0 m

(C) 8.2 m (D) 6.3 m

35 In this rectangle, the angle between the base and the diagonal, marked θ, is



(A) 45° (B) 25°

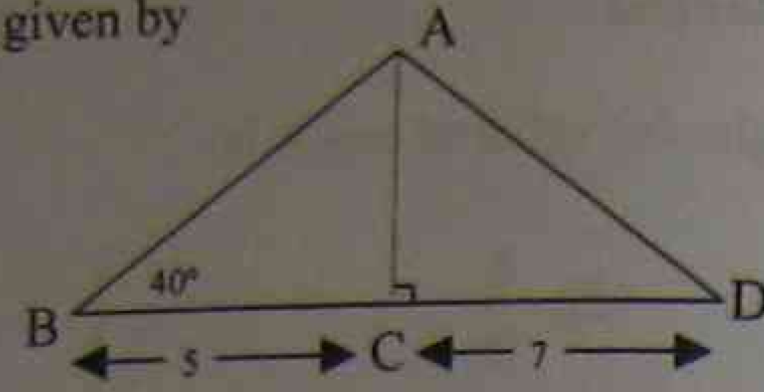
(C) 65° (D) 28°

36 Town A is 40 km due east of Orange.
Town B is due north of Town A on a bearing of 046° from Orange.
The distance from Town B to Orange is

(A) 27.8 km (B) 28.8 km (C) 55.6 km (D) 57.6 km

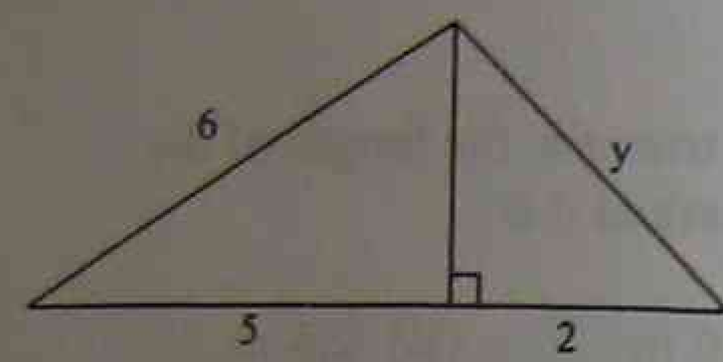
37 The height of ΔABD is given by

(A) $5 \cos 40°$
(B) $5 \tan 40°$
(C) $5 \sin 40°$
(D) $\sqrt{5^2 + 7^2}$



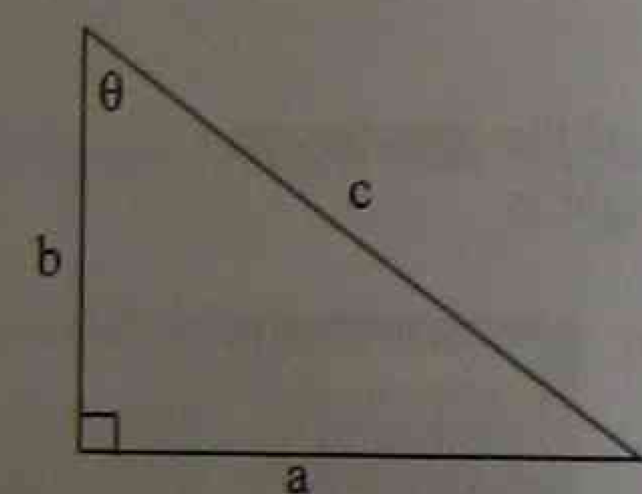
38 The value of y in this case is

(A) $\sqrt{13}$
(B) $\sqrt{15}$
(C) $\sqrt{65}$
(D) $\sqrt{85}$

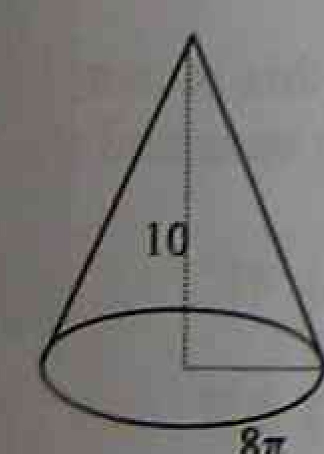


39 In the diagram, $\sin \theta =$

(A) $\dfrac{a}{a+b}$
(B) $\dfrac{a+b}{a}$
(C) $\dfrac{b-a}{c}$
(D) $\dfrac{a}{\sqrt{a^2+b^2}}$



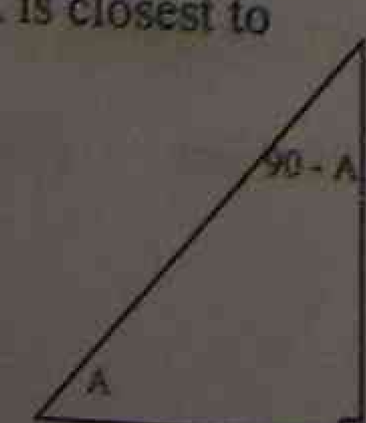
40 The circumference of the base of the cone is 8π cm. The perpendicular height is 10 cm. The angle made between the slant side and the horizontal is closest to



(A) 68° (B) 39° (C) 22° (D) 66°

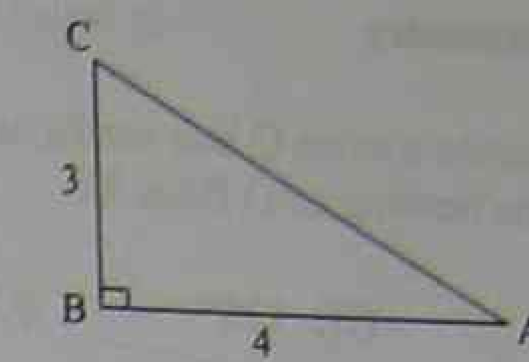
41 If $\sin (90 - A) = 0.5235$, then A is closest to

(A) 32°
(B) 59°
(C) 52°
(D) 35°



42 If $\tan A = {}^3/_4$, then $\cos C =$

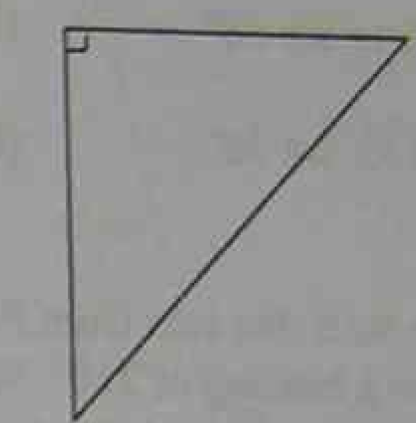
(A) ${}^4/_3$
(B) ${}^4/_5$
(C) ${}^3/_5$
(D) ${}^3/_7$



43 If $\cos B = {}^x/_y$ then $\tan B =$

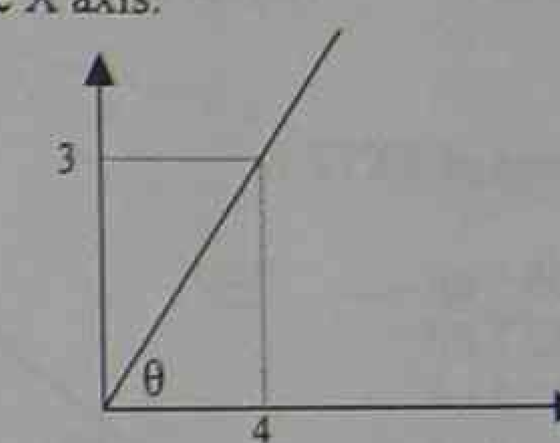
(A) $\dfrac{y}{x+y}$ (B) $\dfrac{x}{y-x}$

(C) $\dfrac{y}{\sqrt{y^2-x^2}}$ (D) $\dfrac{\sqrt{y^2-x^2}}{x}$



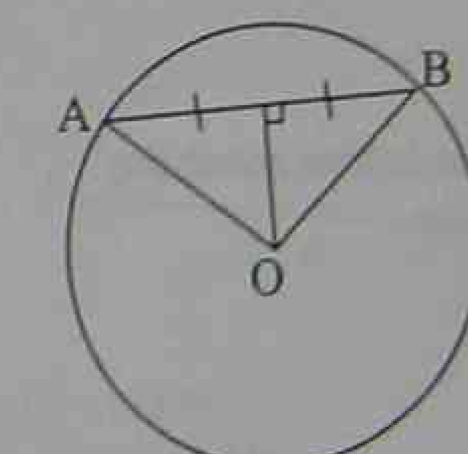
44 The line l makes and angle θ° with the X axis. To the nearest degree, θ is equal to

(A) 43
(B) 25
(C) 53
(D) 37



45 O is the centre of the circle which has radius 8 cm. If $\angle OBA = 60°$, then the length of the chord AB would be

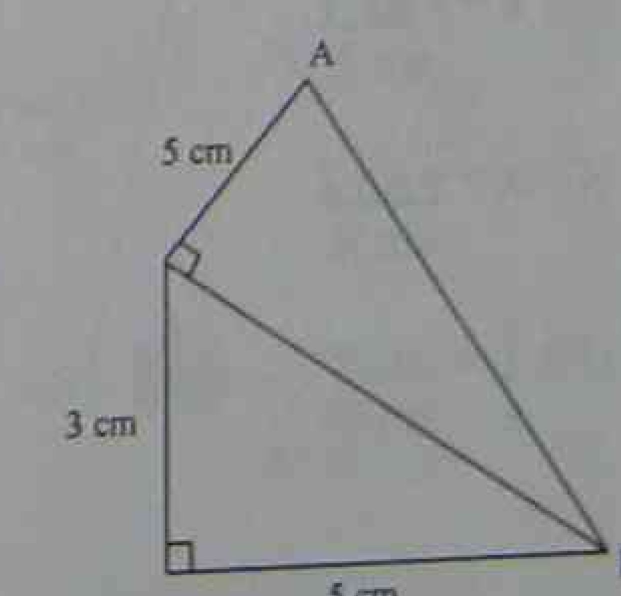
(A) 8 cm
(B) 16 cm
(C) 4 cm
(D) 14 cm



46 In the diagram, the length of side AB is closest to

(A) 6.0 cm (B) 7.7 cm

(C) 6.4 cm (D) 5.8 cm



Further Trigonometry

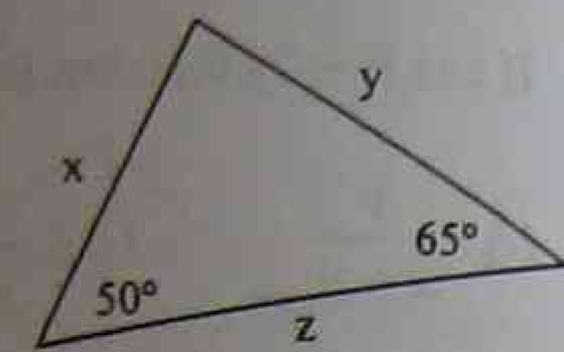
47 A hiker leaves a town O and walks on a bearing of 235° for 15 km to town X. What is the bearing of O from X?

(A) 055° (B) 125° (C) 215° (D) 035°

48 The area of this triangle is

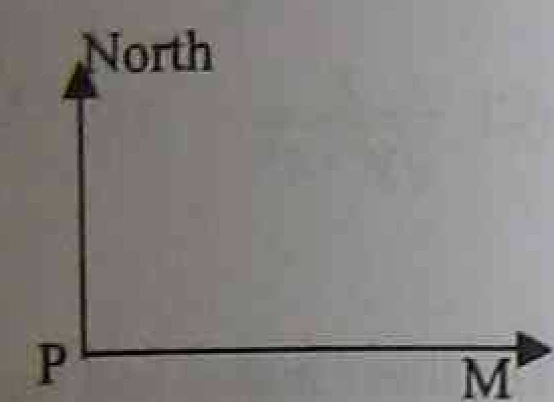
(A) ½ zy sin 50° (B) ½ zx sin 65°

(C) ½ xy sin 50° (D) ½ xy sin 65°



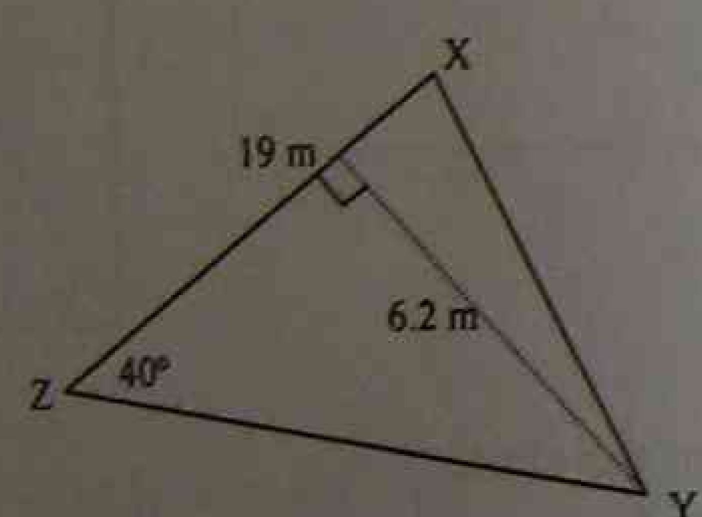
49 A ship sails due east from P to M. It then sails on a bearing of 210°. What is its bearing from P when this ship is closest to P?

(A) 030° (B) 120°
(C) 150° (D) 210°



50 The area of ΔXYZ is

(A) 58.9 m²
(B) 42.7 m²
(C) 37.9 m²
(D) 66.5 m²



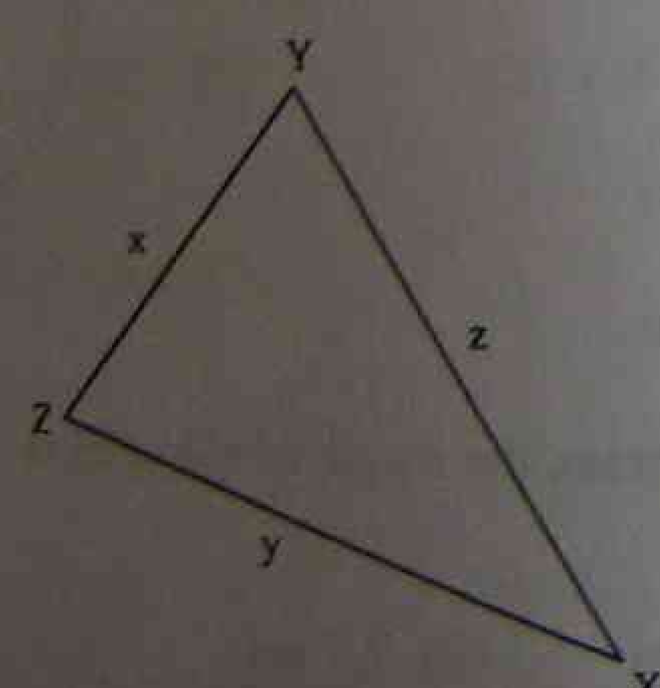
51 A statement that is true about the triangle shown is

(A) $y = \frac{x \sin Y}{\sin X}$

(B) $z = y\frac{\sin Y}{\sin Z}$

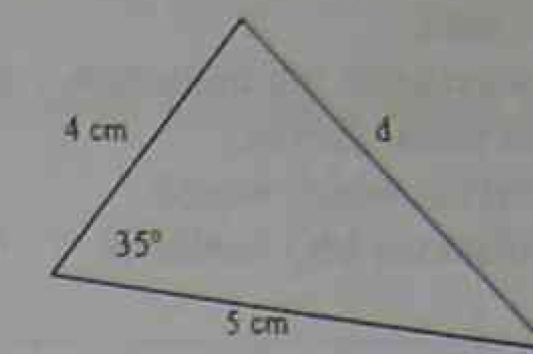
(C) $x = \frac{y \sin Z}{\sin X}$

(D) $\frac{y}{x} = \frac{\sin X}{\sin Y}$



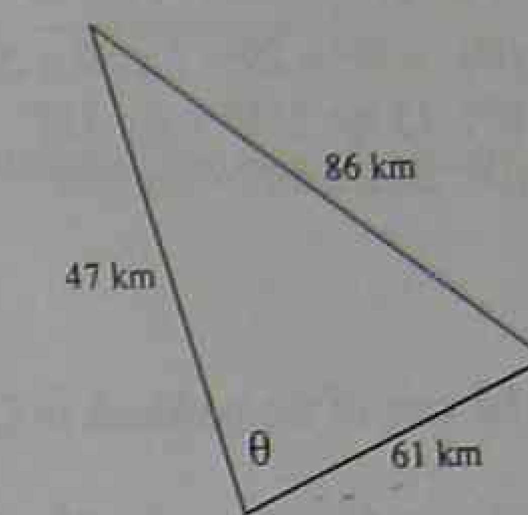
52 The length of the side marked d is closest to

(A) 6.0 cm
(B) 2.9 cm
(C) 8.2 cm
(D) 9.4 cm



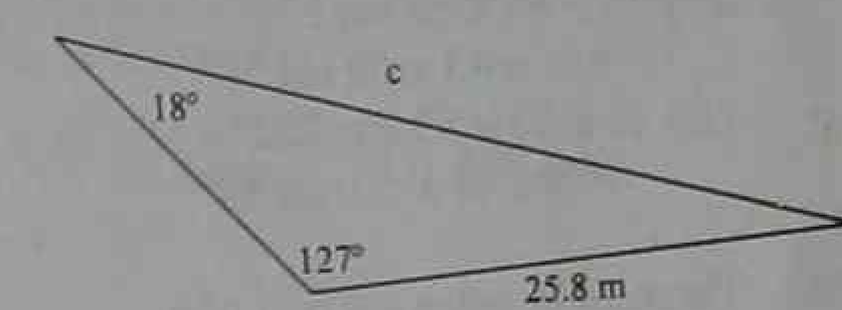
53 The size of the angle marked θ is closest to

(A) 75°
(B) 105°
(C) 43°
(D) 32°



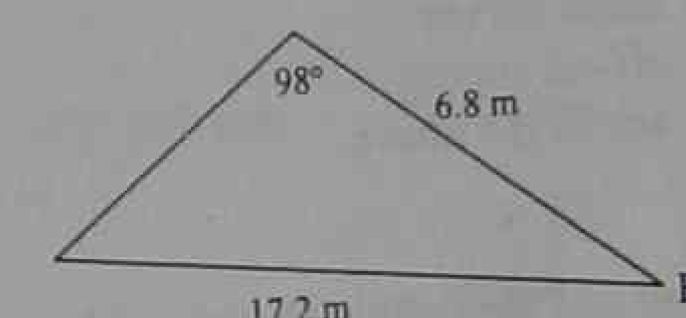
54 The length of the side marked c is closest to

(A) 67 m
(B) 10 m
(C) 39 m
(D) 31 m



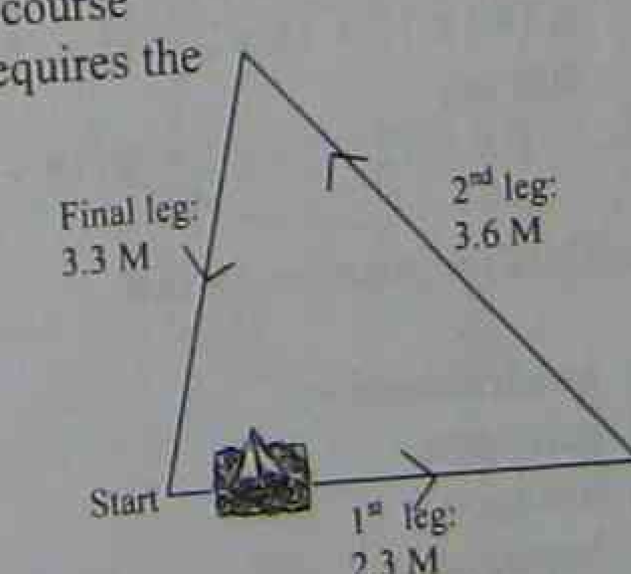
55 The size of angle B is closest to

(A) 23°
(B) 59°
(C) 3°
(D) 87°

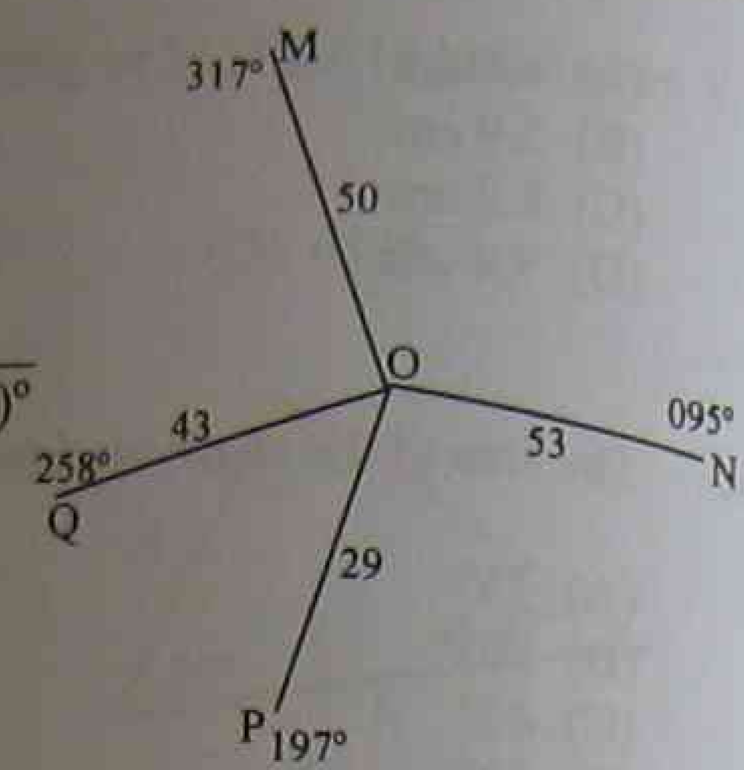


56 A yacht race is held over a triangular course as shown. If the first leg of the race requires the yachts to sail due east, on what bearing do they sail the final leg?

(A) 102°
(B) 168°
(C) 192°
(D) 258°



57 A radial survey of a paddock is drawn here. All measurements are in metres. From this radial survey, the expression which would give the distance MQ would be

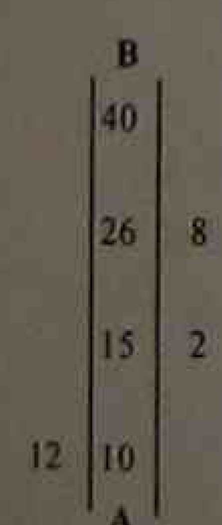


(A) $\sqrt{50^2 + 43^2 - 2 \times 50 \times 43 \cos(317 - 258)^\circ}$
(B) $\sqrt{50^2 + 29^2 - 2 \times 50 \times 29 \cos 20^\circ}$
(C) $43 \sin 258^\circ \div \sin 317^\circ$
(D) $50 \sin 59^\circ \div \sin 258^\circ$

58 The area of the paddock in Question 57 would be closest to

(A) $\frac{1}{2} \times 50 \times 53 + \frac{1}{2} \times 53 \times 29 + \frac{1}{2} \times 29 \times 43 + \frac{1}{2} \times 43 \times 50$
(B) $\frac{1}{2} \times 50 \times 53 \sin 222^\circ + \frac{1}{2} \times 53 \times 29 \sin 102^\circ + \frac{1}{2} \times 29 \times 43 \sin 61^\circ + \frac{1}{2} \times 43 \times 50 \sin 59^\circ$
(C) $\frac{1}{2} \times 50 \times 53 \sin 138^\circ + \frac{1}{2} \times 53 \times 29 \sin 102^\circ + \frac{1}{2} \times 29 \times 43 \sin 61^\circ + \frac{1}{2} \times 43 \times 50 \sin 59^\circ$
(D) $\frac{1}{2} \times 50 \times 53 \cos 222^\circ + \frac{1}{2} \times 53 \times 29 \cos 102^\circ + \frac{1}{2} \times 29 \times 43 \cos 61^\circ + \frac{1}{2} \times 43 \times 50 \cos 59^\circ$

59 These field notes are part of a



(A) plane table survey.
(B) radial survey.
(C) off-set survey.
(D) any of the above.

60 The area of the field in Question 59 is

(A) $\frac{1}{2} \times 91 \times (12 + 8 + 2)$
(B) $\frac{1}{2} [40 \times 12 + 15 \times 2 + 11 \times (8 + 2) + 14 \times 8]$
(C) 800 m²
(D) $\frac{1}{2} [10 \times 12 + 15 \times 2 + 26 (8 + 2) + 40 \times 8 + 30 \times 12]$

61 The field in Question 58 is a

(A) quadrilateral.
(B) pentagon.
(C) hexagon.
(D) heptagon.

Spherical Geometry

62 The time is 10 pm Monday in City X when it is 3 am Tuesday in City Y. City X is therefore

(A) 5° of latitude north of City Y.
(B) 75° of longitude east of City Y.
(C) 5° of longitude west of City Y.
(D) 75° of longitude west of City Y.

63 The latitude of a town is 45°S. The circumference of the earth is about 40 000 km. The town is therefore

(A) about 10 000 km from the Equator.
(B) about 10 000 km from Greenwich.
(C) about 5 000 km from the South Pole.
(D) about 5 000 km from Greenwich.

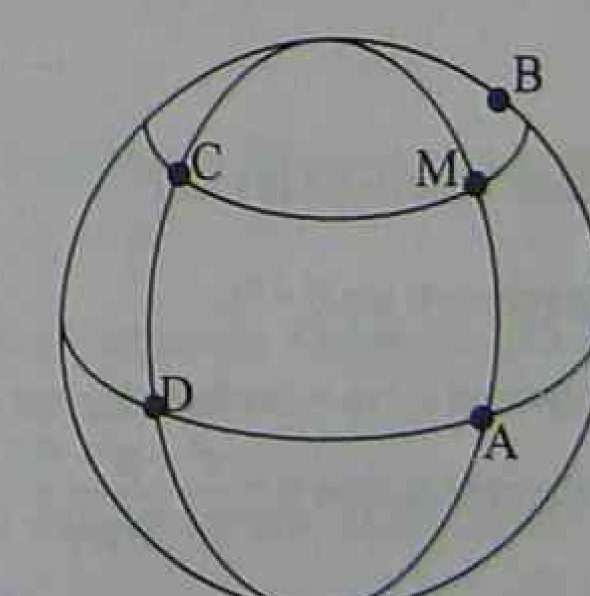
64 A 100 cm pendulum swings through an angle of 30°. The distance through which the tip of the pendulum moves is closest to

(A) 31.4 cm (B) 52.4 cm (C) 62.8 cm (D) 188 cm

65 Town A is at 32°N 58°W whereas Town B is at 27°S 58°W. If the radius of the earth is 6400 km, the shortest distance on the surface of the earth between the two towns is closest to

(A) 6590 km (B) 1320 km (C) 560 km (D) 89 km

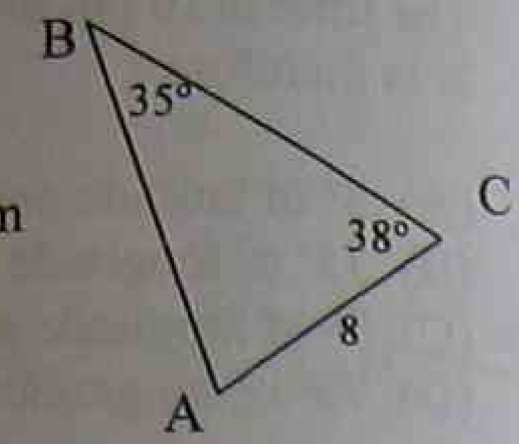
66 M is located at 42°N 59°W. The town with position 46°N 37°E could be



(A) A
(B) B
(C) C
(D) D

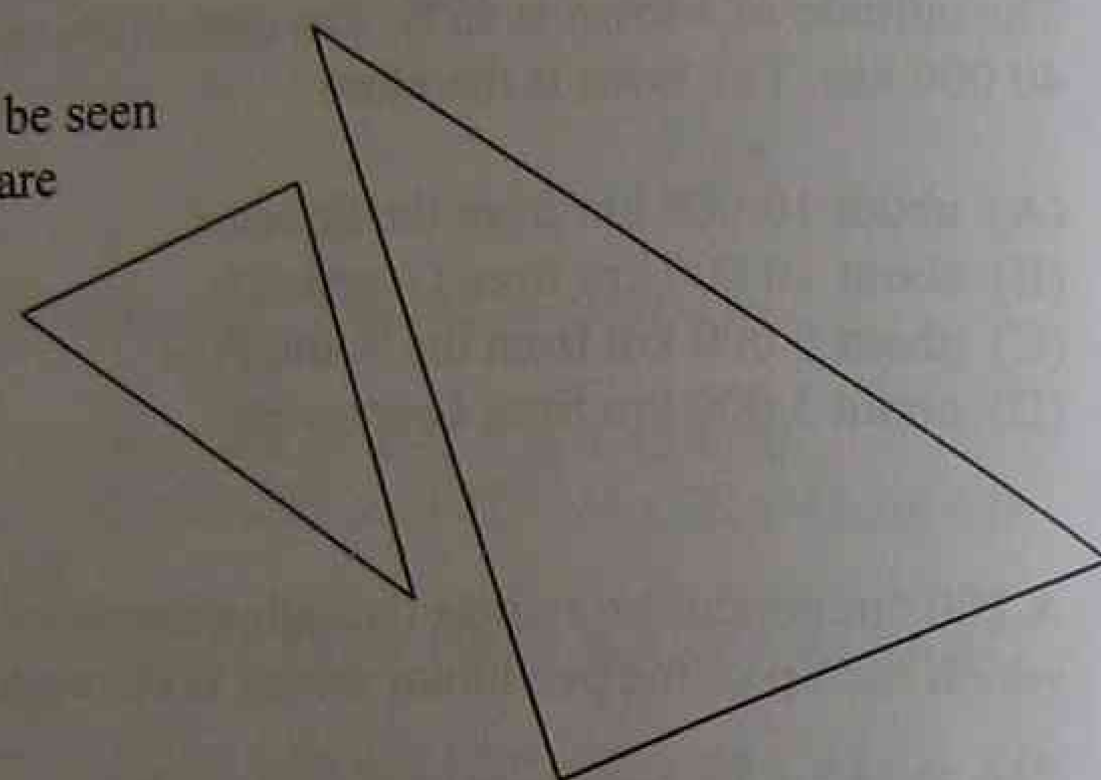
Miscellaneous Questions

67 To calculate the length of BC, it is best to use the



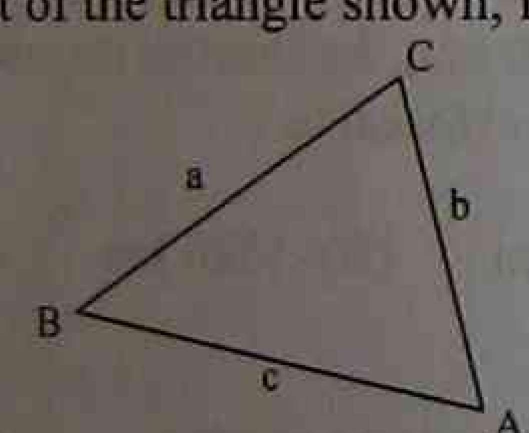
(A) sine rule
(B) cosine rule
(C) tan ratio
(D) Pythagoras' Theorem

68 By measurement, it can be seen that these two triangles are



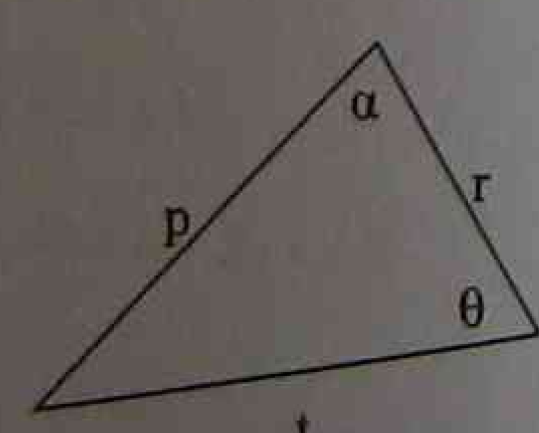
(A) congruent
(B) similar
(C) isosceles
(D) parallel.

69 To calculate the perpendicular height of the triangle shown, it is best to use



(A) b sin A
(B) $a^2 = b^2 + c^2 - 2bc \cos A$
(C) ½ cb sin A
(D) $\cos B = {}^{b}/_{c}$

70 Which statement is true?



Statement 1: $\sin \theta = {}^{p}/_{r}$

Statement 2: $rp = \sin \theta \sin \alpha$

Statement 3: $\cos \theta = \dfrac{r^2 + p^2 - t^2}{2rp}$

(A) Statement 1 only.
(B) Statement 2 only.
(C) Statement 3 only.
(D) None are true.

Free Response Questions

Units of Measurement

1 Express Economy Service charges according to this table for short deliveries.

WEIGHT	CENTS/KG
0 – 100 kg	103
101 – 250	52
251 – 500	48
501 – 750	35
751 – 4000	25
4001 – 8000	21
8000 +	17

(a) Calculate the cost of sending a package which weighs 350 kg.

(b) If the freight bill on a parcel showed \$20.60, how much did the parcel weigh?

2 A fuel tank holds 90L of fuel. The vehicle averages 8.6L/100 km.

(a) How far could the vehicle travel on a full tank?

(b) How many litres of fuel would it require for a 420 km trip?

3 A bushwalker averages about 2 km/h on a long hike.

(a) How much time should the person allow for a 15 km walk?

(b) How fast on average would the person have to walk if they set out at 1 pm and wanted to finish by sunset at 6:30 pm?

4 This is part of a train timetable.

Mount Victoria	8:06 am
Katoomba	8:24 am
Penrith	9:32 am
Parramatta	9:57 am
Strathfield	10:09 am
Sydney	10:23 am

(a) How long does the 8:06 am train from Mount Victoria take to get to Sydney?

(b) If the distance travelled from Mount Victoria to Sydney is 126.74 km, what was the average speed of the train in km/h?

5 How long would it take to fly 630 M at a constant speed of 280 knots?

6 A 20 litre container weighs 450 g when empty. What would be the total weight if it were full of water?

7 Write each of the following as approximations with the accuracy as indicated.

(a) 4999 m correct to the nearest kilometre.
(b) 100.005 correct to four significant figures.
(c) 0.0005 mm correct to the nearest centimetre.
(d) 12345 to correct two significant figures.

8 Find the maximum percentage error if a measurement is given as

(a) 50 km to the nearest kilometre.
(b) 2000 mL to the nearest millilitre.
(c) 2 L.
(d) 0.0005 mm.

9 A person records a measurement of a kitchen bench as 1556 mm when the correct length was 1554 mm. List three possible reasons for the error.

10 Write these numbers in scientific notation.

(a) 200 000
(b) $^1/_{400\,000}$
(c) 0.000 000 76

11 Write these numbers in the usual decimal form.

(a) 3.4×10^5
(b) 8.61×10^{-3}
(c) 5.978×10^8

12 A photographer needs to mix developer with water in the ratio 5 : 1.

(a) How much solution could be mixed from a 2 L bottle of concentrated developer?

(b) The developing tank can hold 1500 mL of solution. How much water and how much developer is required to be mixed?

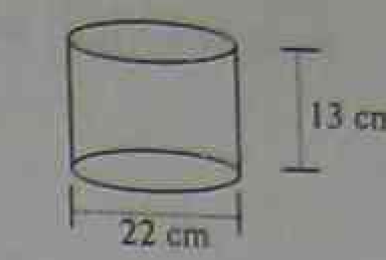
13 Lauren and Derek are in a partnership. Lauren has 8 shares and Derek has 7 shares. If the partnership makes a profit of $35 690, how much should each person get?

14 A fish and chips shop advertises 6 onion rings for a dollar. If I want 10 onion rings, how much should I expect to pay if I am charged at the same rate as for six?

15 The speed of light is 3×10^5 km / s. How far would light travel in one year?

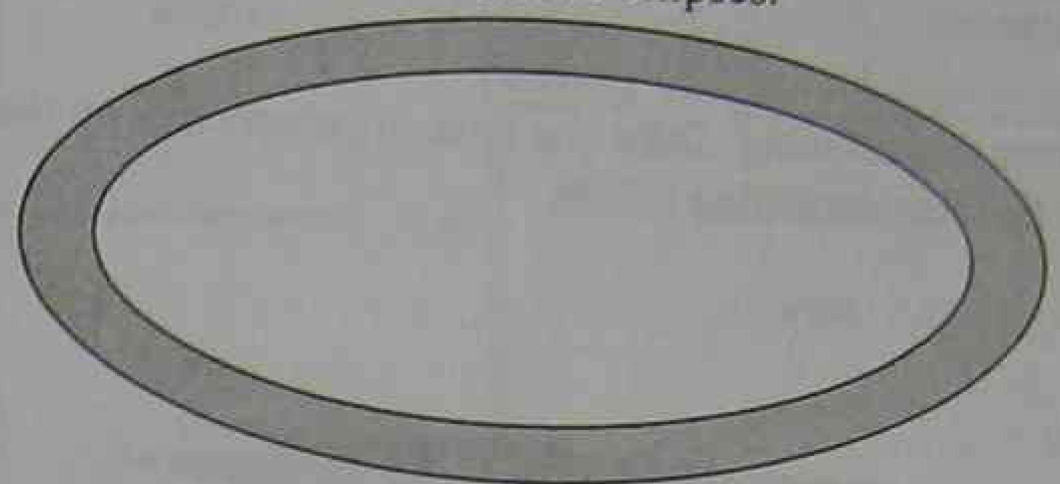
Area and Volume

16 (a) What is the total surface area of this closed cylindrical can?

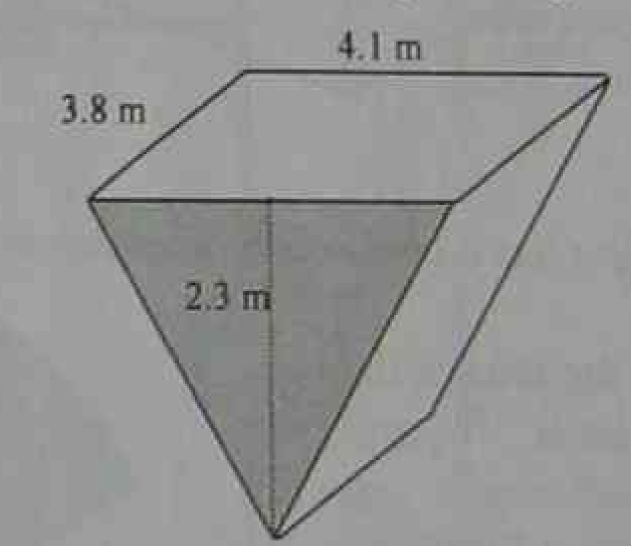


(b) How many curved surfaces for a batch of such cans could be cut from a sheet 100 cm x 70 cm?

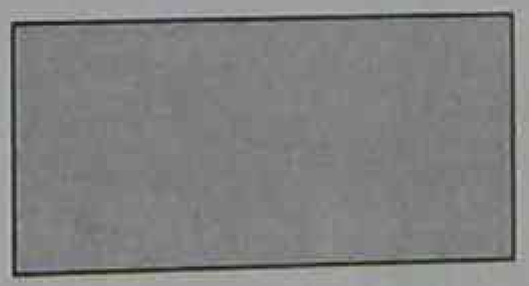
17 Take the appropriate measurements and determine the size of the shaded area in square centimetres between these two ellipses.



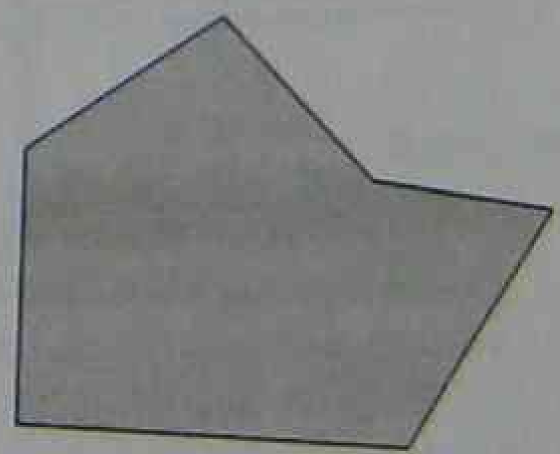
18 This shape represents a railway bulk transport wagon hopper. Calculate its capacity.



19 The diagram is a 1 : 100 scale plan for a theatre stage. Take the necessary measurements to find the area of the actual stage.



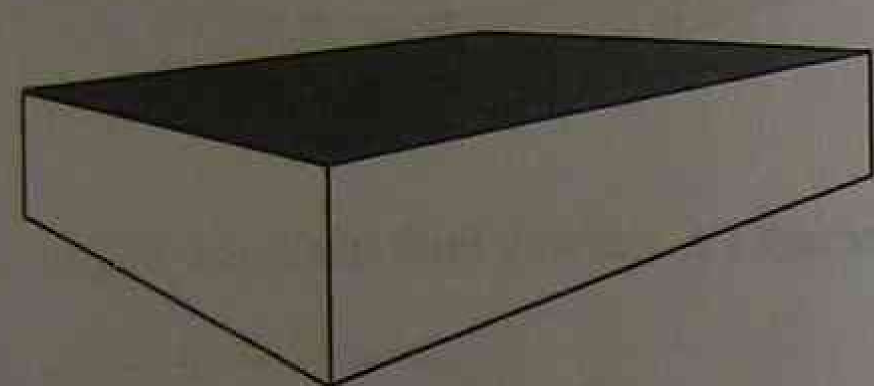
20 The diagram is a 1 : 150 000 scale drawing of a farm paddock. Take the necessary measurements to determine the area of the paddock to the nearest hectare.



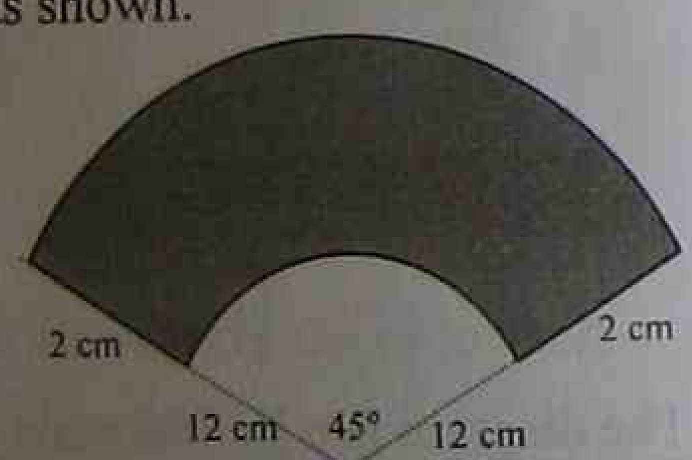
21 Copy and complete this perspective drawing of a rectangular prism.

Mark the vanishing points LV and RV.

22 Trace over this drawing. Draw the lines of perspective on the sketch. Use them to find the vanishing points.



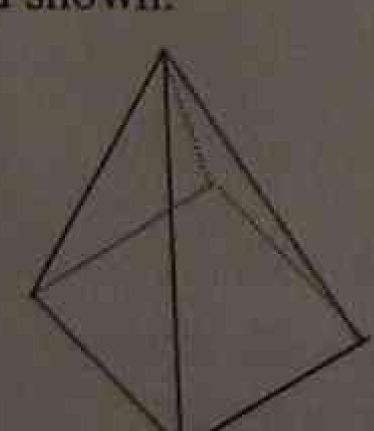
23 A rubber moulding has a cross section as shown.



(a) Show that the area of the cross section is $^{13\pi}/_2$ cm^2.

(b) What volume of rubber is needed to make a 10 m length of the moulding?

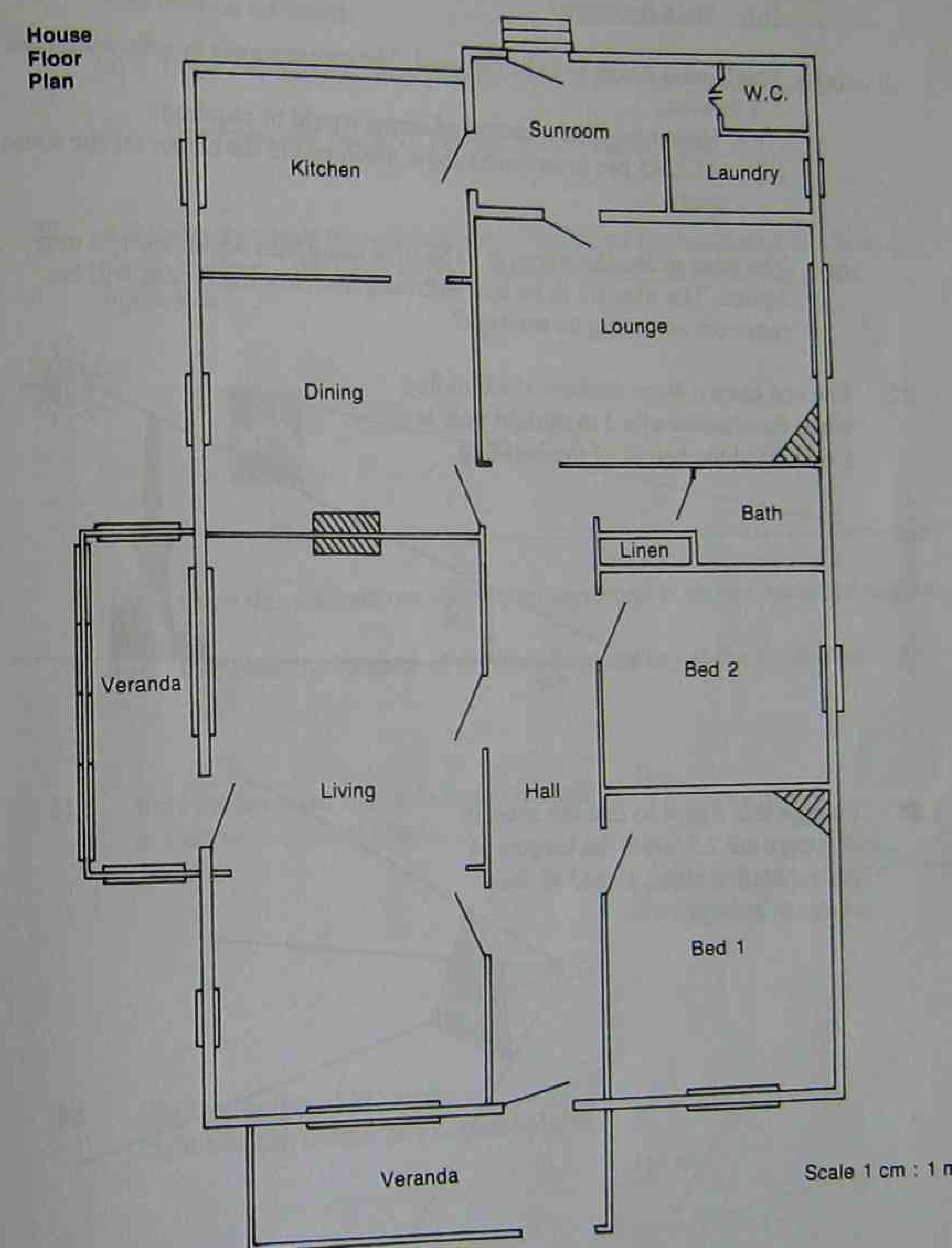
24 Sketch a net of the solid shown.



25 A 238 mm metal bar has a circular cross section, with diameter measured as 25 mm. Between what limits would the volume lie?

Similarity

26 Use this House Floor Plan to answer the questions on the following page.



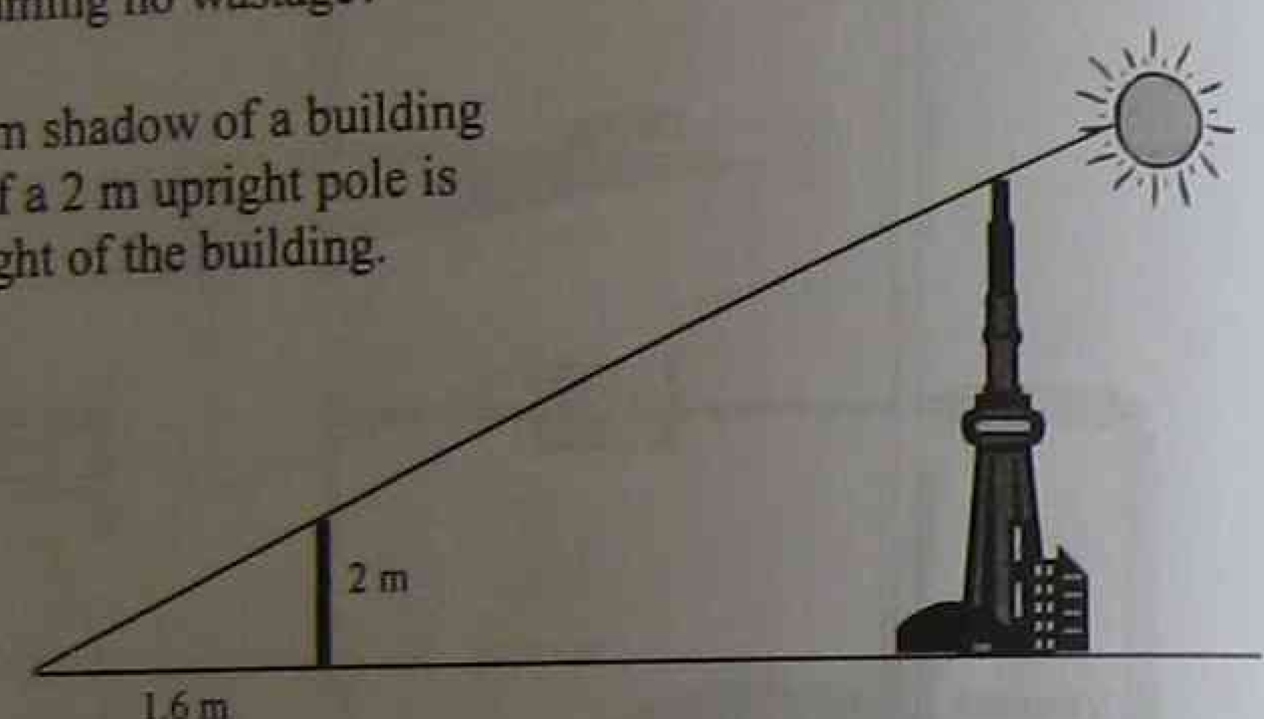
(a) How long is the Hall?

(b) What are the dimensions of the
  (i) Lounge Room?
  (ii) Living Room?
  (iii) Main Bedroom?

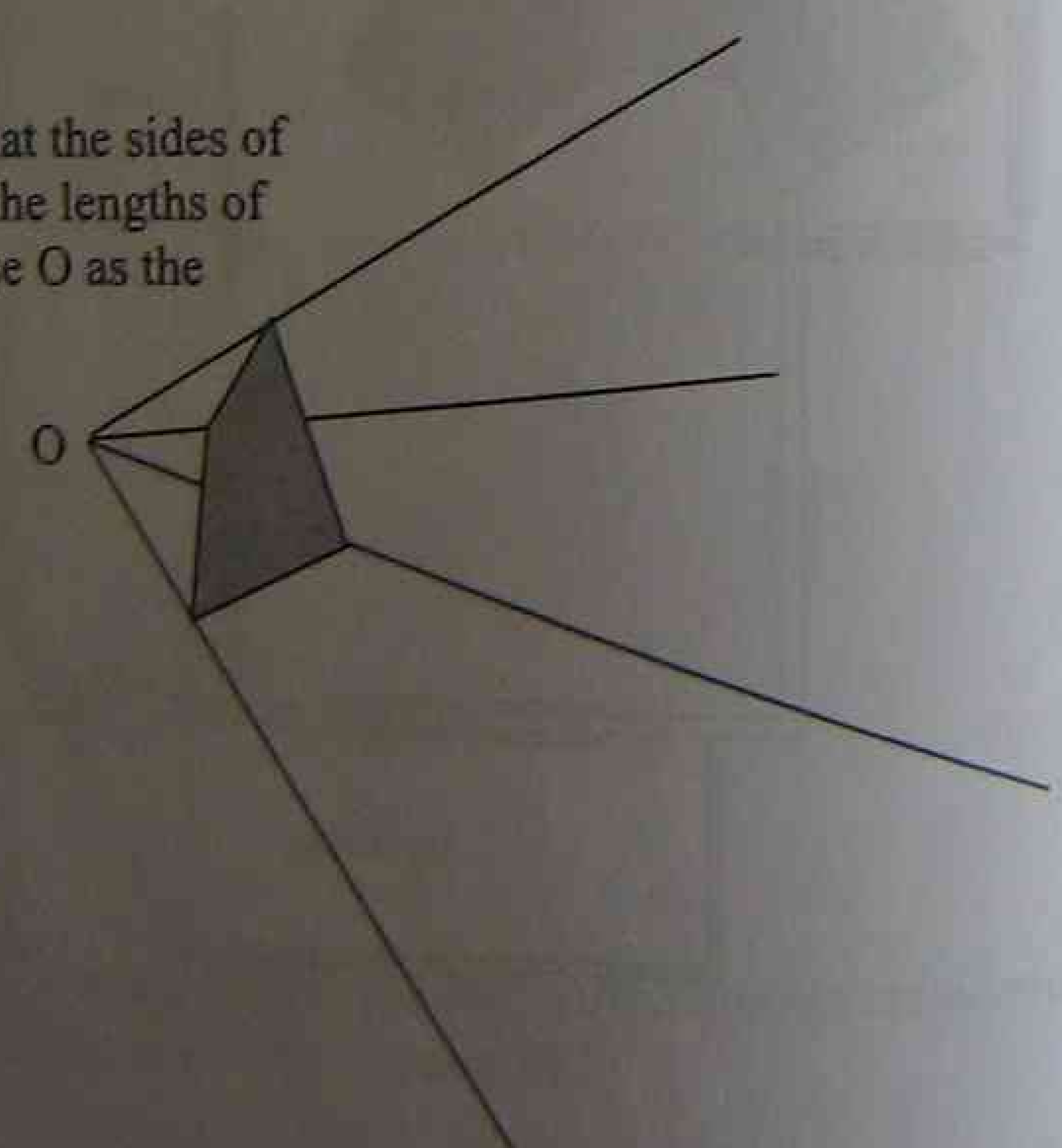
(c) The Dining Room is to be carpeted. The carpet comes in rolls which are 3.7 m wide.
  (i) How many lineal metres of carpet would be required?
  (ii) At $245 per lineal metre, how much would the carpet for the room cost?

(d) The floor of the Bathroom is to be tiled using tiles which are 150 mm square. The tiles are to be laid without gaps. How many tiles will be required, assuming no wastage?

27 The sun casts a 40 m shadow of a building when the shadow of a 2 m upright pole is 1.6 m. Find the height of the building.

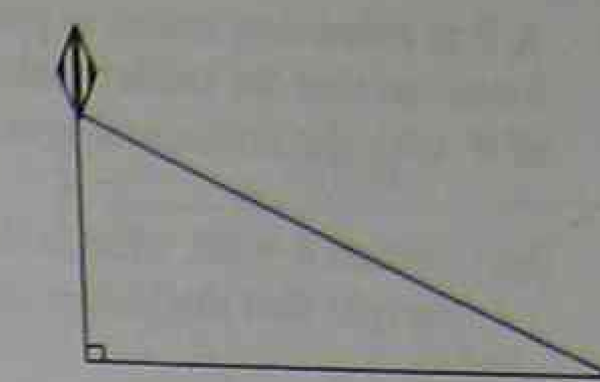


28 Enlarge this figure so that the sides of its image are 2.5 times the lengths of corresponding sides. Use O as the centre of enlargement.



Right-angled Triangles

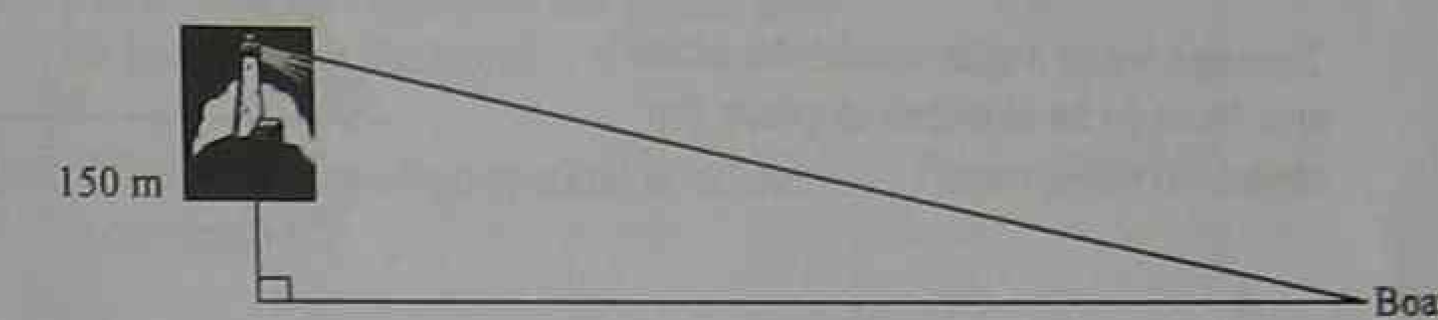
29 Children flying a kite on level ground measure the angle of elevation from the ground to be 38°. Another child standing directly under the kite is 92 m from the child holding the string.



(a) Copy this diagram onto your own paper and put the information on it.

(b) Determine the height of the kite to the nearest metre.

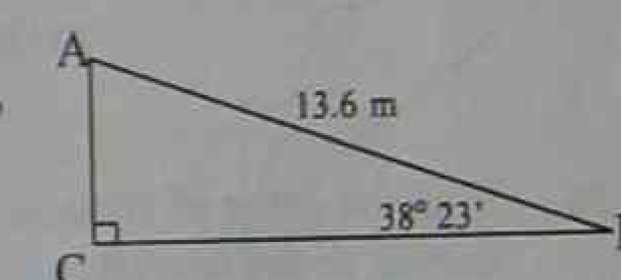
30 A lighthouse stands on a vertical cliff. The top of the lighthouse is 150 m above sea level. The angle of depression to the boat from the top of the lighthouse is 3°.



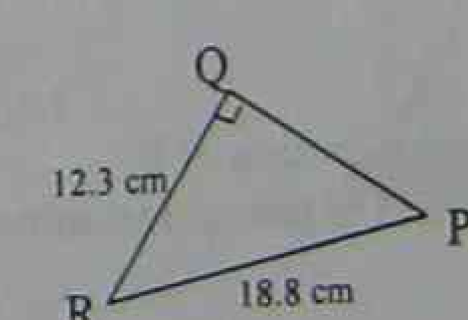
(a) Copy this diagram onto your own paper and mark the information given.

(b) Calculate the distance of the boat from the top of the lighthouse.

31 Find the length of BC in this right triangle, correct to 2 decimal places.



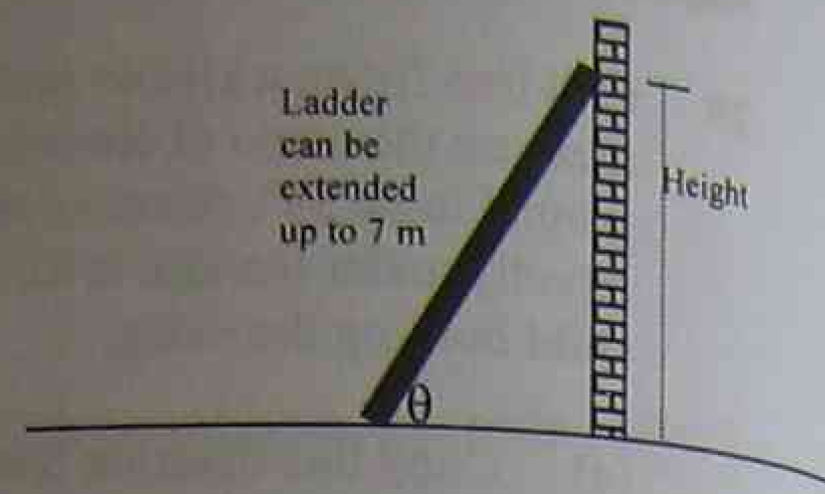
32 Find the length of PQ in this right triangle, correct to 2 decimal places.



33 A 7 m extension ladder is placed against a wall so that the ladder makes an angle of θ° with the horizontal ground.

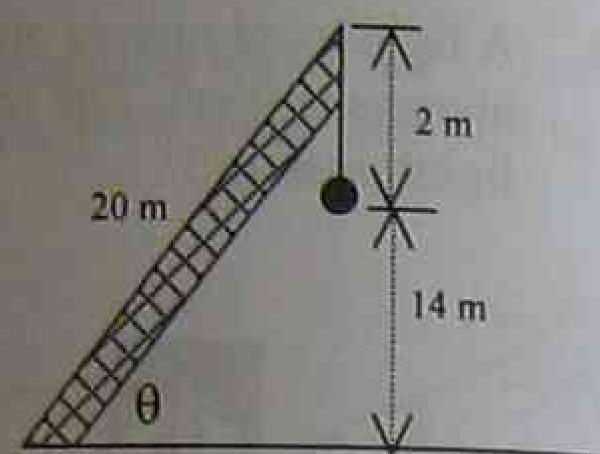
(a) When θ = 60, what is the maximum height that the ladder can reach?

(b) The ladder is fully extended and is at an angle of 75° with horizontal. It is then retracted to be only 5 m long. What angle will it now make with horizontal if the place of its foot is not changed?

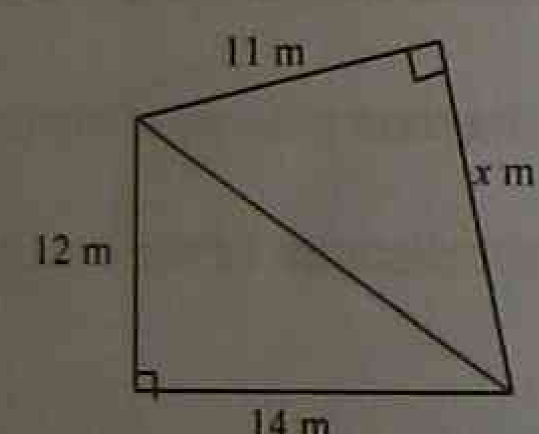


34 The drawing shows the 20 m arm of a crane. A cable is extended 2 m from the end of the crane's arm and is holding a load which is suspended 14 m above the ground.

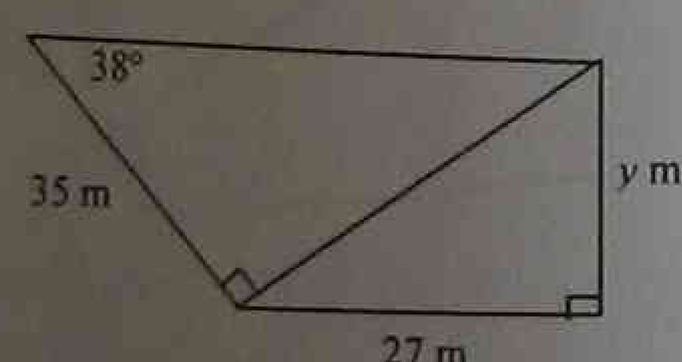
Through what angle would the crane's arm have to be lowered to place the object on the ground?



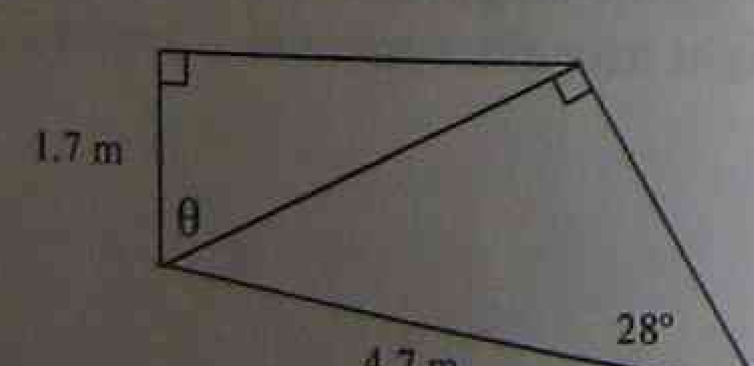
35 Find the length of the side marked x m.



36 Find the value of y, correct to the nearest tenth of a metre.



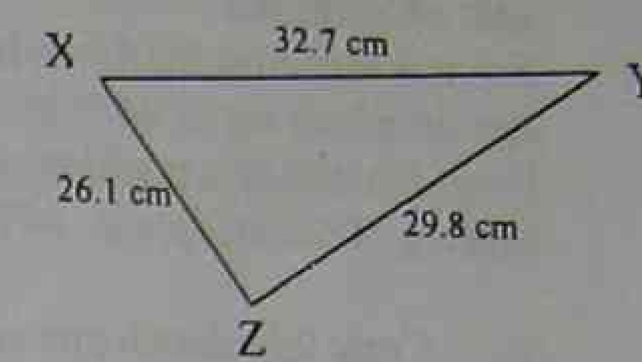
37 Find the value of θ correct to the nearest minute.



Further Trigonometry

38 (a) Find the size of the angle XYZ, correct to the nearest minute.

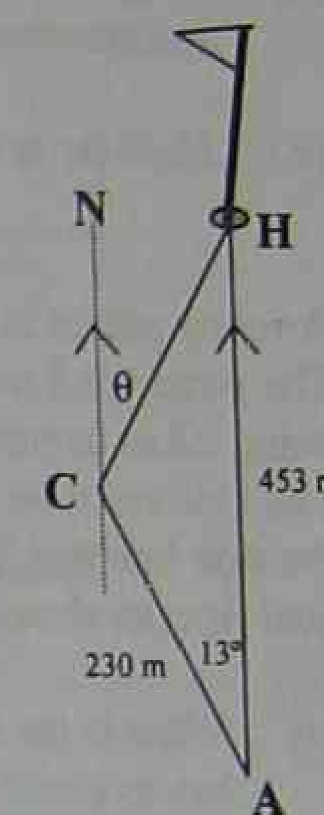
(b) Find the area of the triangle.



39 A golfer attempts to hit a ball down the fairway from A to the hole H due north of A. The distance from A to H is given as 453 m. Unfortunately, the golfer hits the ball at an angle of 13° as shown, to a point C which is 230 m from A.

(a) How much further does the golfer have to hit to reach the hole?

(b) What is the bearing, marked θ, of the hole from C?



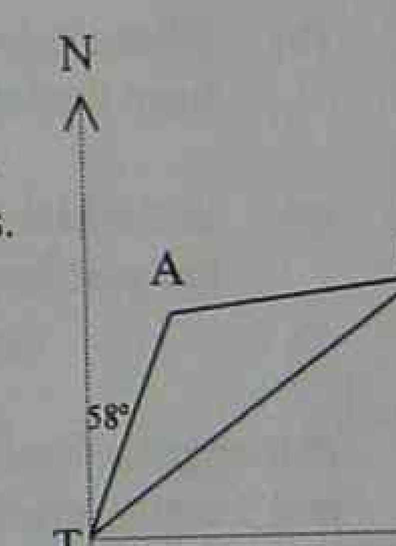
40 Two ships sail from Tahiti (T), Ship A heading on a bearing of 058° and Ship B on a course of 074°. Ship A sails at 8 knots while Ship B sails at 12 knots.

(a) Copy this sketch and enter the details given in the question.

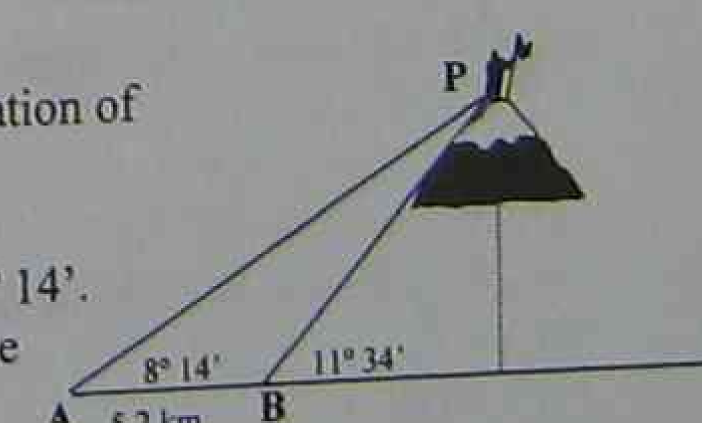
(b) How far apart are the two ships after 2 hours?

(c) Find the size of angle ABT and hence, the size of angle TAB.

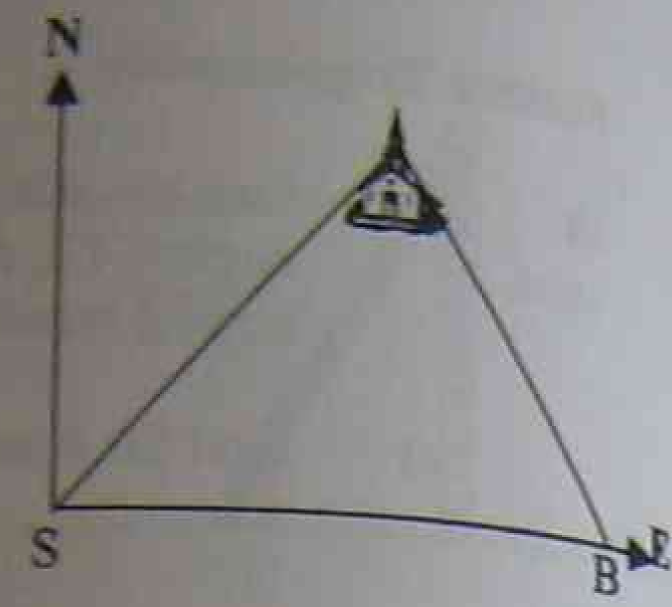
(d) What is the bearing of Ship B from Ship A after they have sailed for 2 hours?



41 A mountain peak P has an angle of elevation of 11° 34' from a point B. From a point A which is 5.2 km further away at the same altitude, its angle of elevation is 8° 14'. Use the sine rule to calculate the distance from A to the top of the peak?



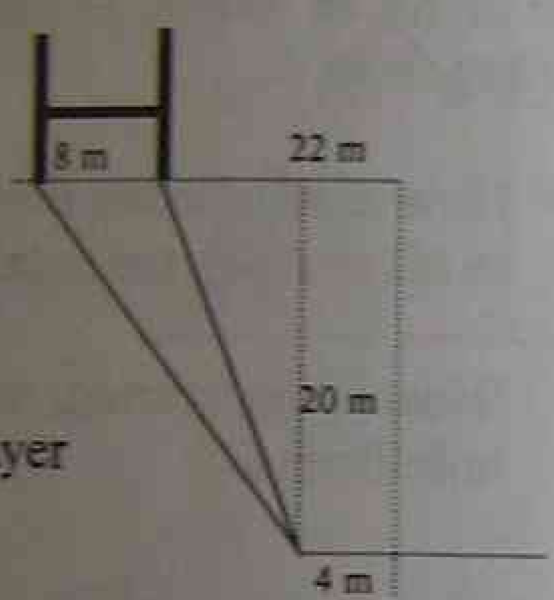
42 Two surveyors, Siegfried and Brunhilde, sight a church steeple. Brunhilde is 2 km east of Siegfried.
From where Siegfried stands, the steeple lies on a bearing of 047° 15'.
From Brunhilde's position, the bearing of the steeple is 347° 40'.



(a) Copy this sketch and add the information from the question.

(b) How far is each person from the church?

43 A rugby player is trying the kick a goal. The posts are 8 m apart, the near post being 22 m from the side line. The kick must be taken 4 m from the side line and 20 m from the goal line, as shown (not to scale).

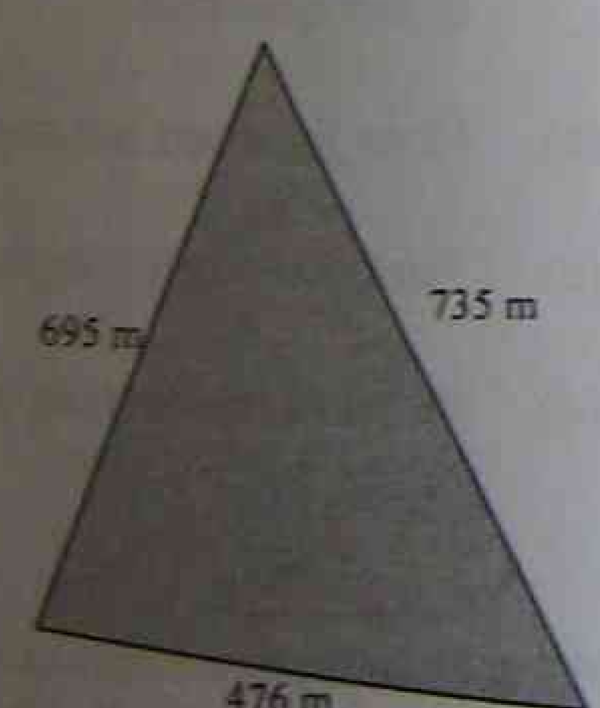


(a) What is the shortest distance the player has to kick?

(b) What angle do the posts subtend from the kicking position?

(c) If the kick were to be taken 8 m from the side line and 20 m out from the goal line, what angle would the goal posts subtend?

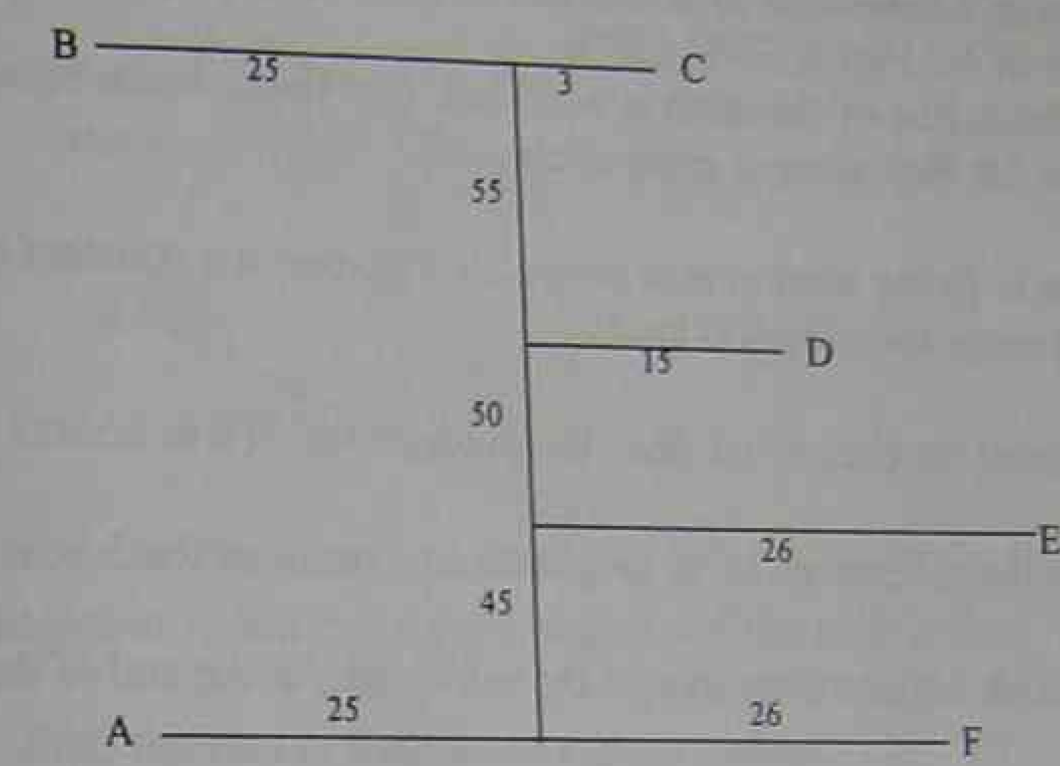
44 A triangular farm allotment was surveyed and the measurements were found to be as shown on the diagram (not to scale).



(a) Find the size of one angle.

(b) Find the area of the allotment to the nearest hectare.

45 A survey has been made of a block of land. The field notes, not to scale, are shown. All measurements are in metres.



(a) What is the mathematical shape of the block of land?

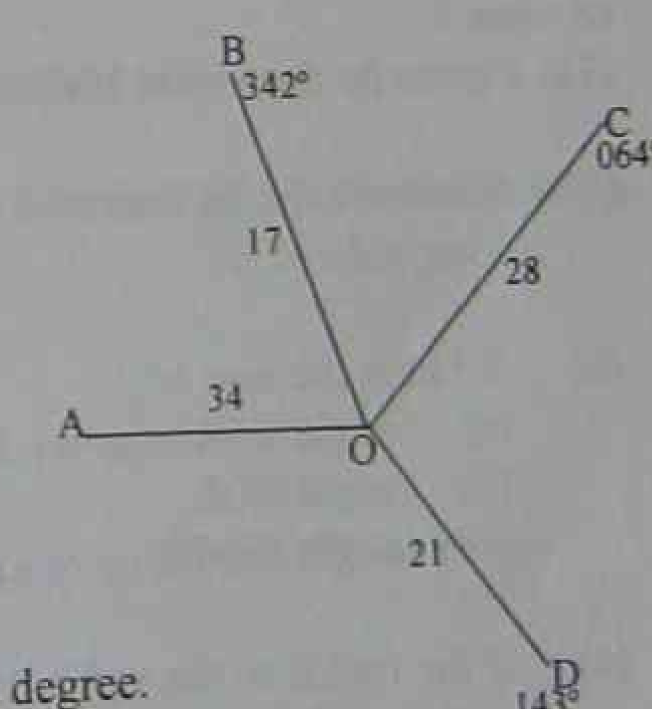
(b) Find the area of the block of land.

(c) How many blocks of this size would make 1 ha?

(d) Find the length of the boundary in metres.

(e) Find angle BCD to the nearest minute.

46 A radial survey of a small municipality is drawn here. All measurements are in kilometres.



(a) Find the size of angle BOC.

(b) Find the length of BC

(c) (i) If it is known that the distance from A to D is 58 km, calculate angle AOD, correct to the nearest degree.

(ii) Find the bearing of A from O.

(d) Find the area of the portion bounded by OC, OD and CD.

Spherical Geometry

47 The position of Adelaide is 35°S 138°E.
The position of Mt Fuji is 35°N 138°E.
Given that the radius of the earth is 6400 km, find the distance between Adelaide and Mt Fuji along a great circle.

48 An aeroplane is flying west to east above the Equator at a constant altitude of 200 m. Its average speed is 400 knots.

(a) How many nautical miles does the plane travel in five hours?

(b) What is the difference in its longitude as a result of the 5 hour flight?

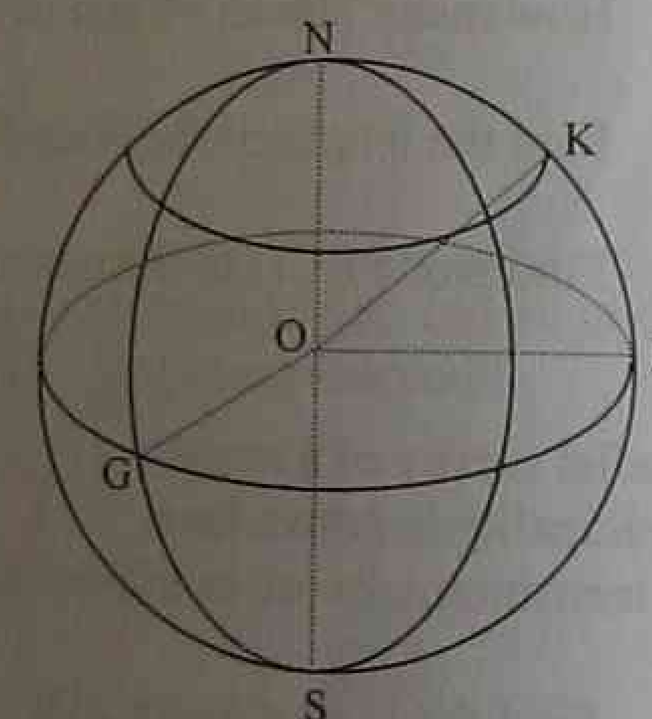
(c) What clock adjustments should the pilot make at the end of the flight?

49 A town is located at 58°N 140°E.

(a) Describe one of the great circles on which it lies.

(b) Describe one of the small circles on which it lies.

50 O is the centre of the earth.
N, S are the North and South Poles.
E is a town on the Equator.
K is a town at 40°N 127°E, north of town E.
G is a town on the Prime Meridian.



(a) What are the latitude and longitude of E?

(b) What is the size of
(i) angle KOE
(ii) angle SOE
(iii) angle EOG?

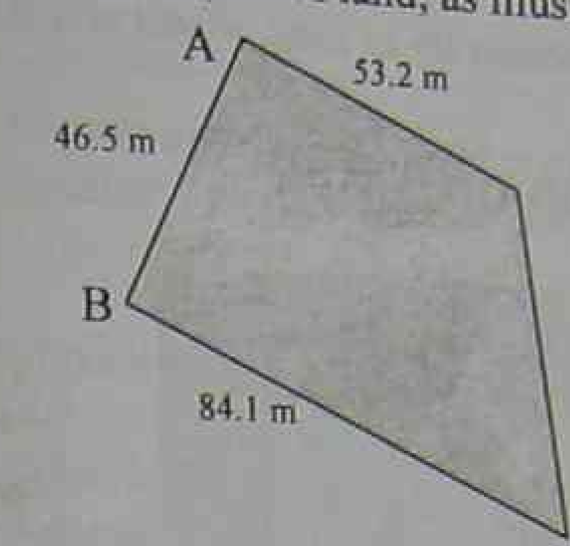
(c) If the radius of the earth is 6400 km, what is the distance on the surface of the earth between
(i) K and E
(ii) K and N?

(d) If a tunnel were to be dug from K to E, what would be its minimum length?

(e) What is the real time difference between K and G?

Miscellaneous Questions

51 A farmer buys a plot of land, as illustrated in the sketch below (not to scale).



The angles at A and B are right angles and the land is flat.

(a) Find the area of the plot in square metres.

(b) Find the length of the boundary.

(c) If fencing costs $15.86 per metre, what would it cost to fence the side AB?

52 A boat is travelling at 6 knots on a bearing of 273°.

(a) How far does it travel in 3 hours?

(b) After 3 hours, it changes direction and heads due south.

(i) Draw a diagram to represent this information.

(ii) How many nautical miles will it need to travel before it is due west of its original position?

53 A shower delivers water at the rate of 2L per minute.

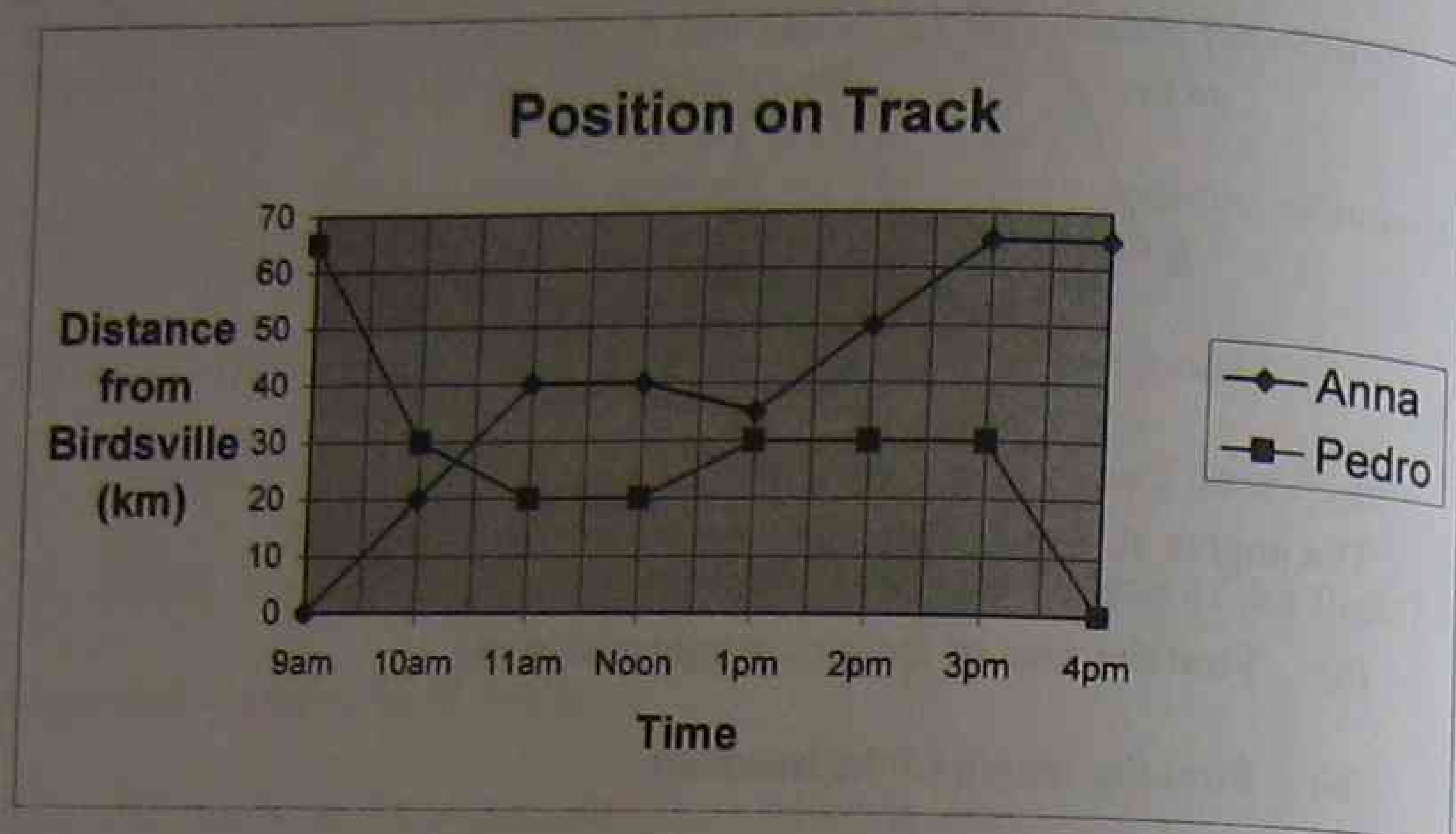
(a) At this rate, how long would it take to empty a 400L tank?

(b) What is the capacity of the tank in m^3?

(c) If the tank is a 2m tall cylinder, what is the radius of its base?

(d) If both the radius and height are increased by 10%, by what factor would the volume of the tank be increased?

54 Pedro and Anna set off from opposite ends of a 4 wheel drive track. Their distance from the Birdsville end of the track at any time is shown in the graph below.



(a) How far is Pedro from his starting point at 12 noon?

(b) How long is the track?

(c) What was Anna's average speed for the first two hours?

(d) When was Pedro stopped?

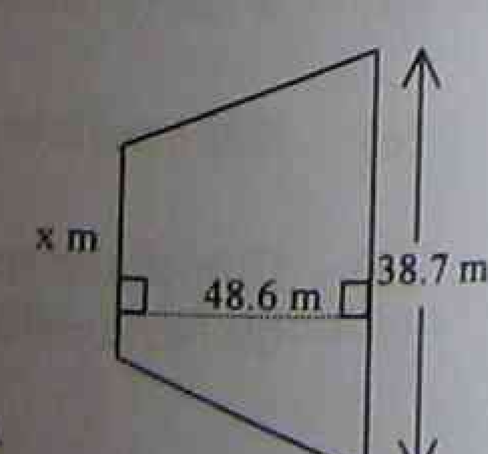
(e) At what time did Anna and Pedro pass each other?

(f) Fill in the times in this report:
"At ...., Anna stopped and called Pedro on CB radio and suggested that they have lunch together at a picnic spot she had seen just after they passed each other. After a break for morning tea, they both turned around their vehicles. Anna arrived at the picnic spot at .... Pedro then called on the radio to say that he would not have time and was turning around again. Pedro became bogged and was not pulled free until ...."

(g) How far did Pedro drive between 9 a.m. and 4 p.m.?

55 (a) Write down an expression for the area of this block of land.

(b) The area of the block is advertised as 1.01 ha. What is the length of the boundary marked x? Give your answer correct to two decimal places.



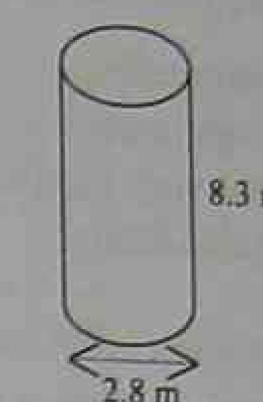
56 A cake of soap in the shape of a rectangular prism measures 6 cm x 2 cm x 10 cm.

(a) If it is totally used in 40 washes, how much soap is used each wash?

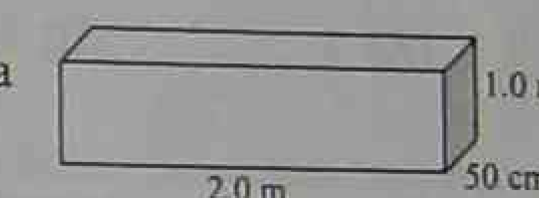
(b) How many washes would result in the remaining soap having a volume equal to a block with each dimension above halved?

57 This cylindrical storage bin is full of grain.

(a) What is the bin's capacity in m^3, correct to 3 significant figures?

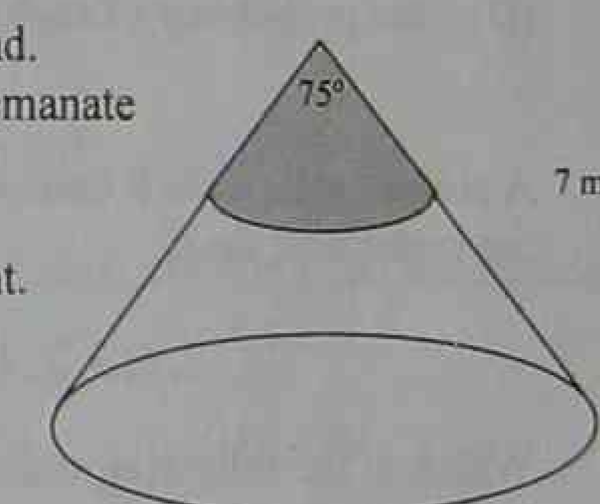


(b) Grain is released from the bin into crates for transportation. How many of these crates would be required to carry the contents of a full bin?



(c) Grain is released at the rate of 0.5 m^3/minute. How long does it take to empty a full bin?

58 A street lamp stands 7 m above level ground. It has a metal shade, which allows rays to emanate at an angle of 75°, as shown.



(a) Find the diameter of the circle of light.

(b) What is the area of the ground lit by the lamp?

59 The metal in a shot (for shot put) has a density of 7.8g/cm^3. Find the mass of a shot of diameter 10 cm made from the same material.

PROBABILITY

Multiple Choice Questions

The Language of Chance

1 An event has a probability of occurring equal to 0.0001. The chance of it occurring is best described as

(A) impossible.
(B) improbable.
(C) not very likely.
(D) quite likely.

2 How many different ways can the letters of the word BOOK be arranged in a line?

(A) 3 (B) 6 (C) 12 (D) 24

3 Three events are known to be equally likely. It is therefore true that

(A) the probability of each event is equal to ⅓.
(B) the probability of each event is less than ⅓.
(C) the probability of none of the events occurring is zero.
(D) the probability of each event is not more than ⅓.

4 A student rolls a die 8 times and records the following numbers showing on the uppermost face:

3, 6, 3, 2, 4, 4, 6, 4

Which of the following statements is true?

(A) The die has 8 faces.
(B) The die has four faces.
(C) The die is not necessarily cubic.
(D) The die could not be dodecahedral.

5 A coin has been tossed 10 times and each time, a Head has appeared. The chance of the 11^{th} toss being a Head is

(A) $(½)^{11}$ (B) ½ (C) 10 x ½ x ½ (D) $1 - (½)^{10}$

Relative Frequency and Probability

6 A group of people were asked for their shoe sizes. The results are shown in the table below.

Size	6	7	8	9	10	11	12
No. People	4	6	12	18	9	4	2

A person is chosen at random. What is the probability that this person has a shoe size greater than 9?

(A) $\frac{3}{11}$ (B) $\frac{23}{55}$ (C) $\frac{3}{7}$ (D) $\frac{15}{63}$

7 A biased coin was tossed 40 times and the following table results.

Result	Frequency
Head	8
Tail	32

If the same coin is tossed a very large number of times, the closest estimate of the probability of getting a Tail on the 20 000^{th} toss is

(A) 0.0016 (B) ½ (C) ¼ (D) ¾

8 Which of the following numbers could not be the answer to a probability calculation?

(A) $\sqrt{2} - 1$ (B) 10^{-7} (C) $\sin 20°$ (D) $3 - \pi$

9 The probability of an event is ¾. This is the same as saying that the chance of the event occurring is

(A) 34% (B) 25% (C) 0.34 (D) 75%

10 An ordinary cubic die is rolled. The probability of the number uppermost being a prime is

(A) ½ (B) ⅓ (C) ⅔ (D) 0

11 Lottery tickets are numbered 1 to 100 000. The probability of a five-digit number winning is closest to

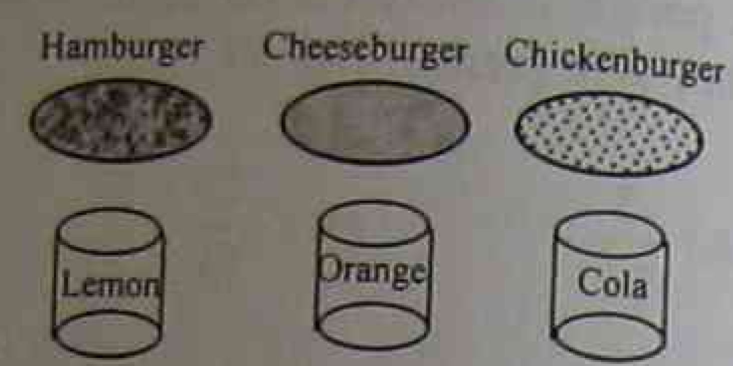
(A) 75% (B) 83% (C) 90% (D) 93%

Multi-stage Events

12 A gambler plays a game whereby two cards are drawn at random from a standard pack of 52 cards. The first card is the king of diamonds. Out of the following, which is the most likely result for the second card drawn?

(A) Another king (B) An ace
(C) Another diamond (D) Another picture card.

13 Three people are fed with a burger and a drink each. There are three different burgers and three different drinks. What is the probability of a person choosing at random a cola and a hamburger?



(A) $\frac{1}{3}$ (B) $\frac{1}{20}$ (C) $\frac{1}{10}$ (D) $\frac{1}{9}$

14 Two dice are rolled and the sum is noted. The probability that this sum is 8 is closest to

(A) 0.06 (B) 0.08 (C) 0.14 (D) 0.17

15 From five students each of whose names has been written on a card in a hat, two are to be chosen at random for a job. The probability that one particular student is chosen is

(A) $\frac{1}{25}$ (B) $\frac{2}{5}$ (C) $\frac{1}{20}$ (D) $\frac{9}{25}$

16 My pocket contains a 5 cent coin, a 20 cent coin and a \$2 coin. If I take out two coins together at random, the probability of the total value of the coins being \$2.05 is

(A) $\frac{1}{2} \times \frac{1}{3}$ (B) $\frac{1}{3} \times \frac{1}{3}$ (C) $\frac{1}{3} \times \frac{1}{3} \times 2$ (D) $\frac{1}{2} \times \frac{1}{3} \times 2$

17 A cricketer can, on average, catch 1 in 3 chances. If two balls are hit to him, the probability that he will catch one and drop one is

(A) $\frac{2}{3} \times \frac{2}{3}$
(B) $\frac{1}{3} \times \frac{1}{3}$
(C) $\frac{1}{3} \times \frac{2}{3}$
(D) $\frac{1}{3} \times \frac{1}{2}$

18 There are 20 tickets in a raffle. Alan bought 3 tickets. The probability of Alan not winning first prize with any of his tickets is

(A) $\frac{3}{20}$ (B) $\frac{3}{20} \times \frac{2}{19} \times \frac{1}{18}$

(C) $\frac{17}{20}$ (D) $\left[\frac{19}{20}\right]^3$

19 The probability of a man aged 102 dying in the coming 12 months is 0.21.
The probability of a man aged 103 dying in the coming 12 months is 0.25.

The probability of a 102 year old man being alive in two years time is therefore closest to

(A) 0.46 (B) 0.05 (C) 0.54 (D) 0.59

20 To decide who has first turn in a game, Alana and Bruce toss a coin in turn until one of them tosses a tail. If Alana tosses first, the probability, before tossing begins, that she will win with her second toss is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{8}$ (D) $\frac{1}{4}$

21 The number of different 3 letter arrangements, for car number plates, that can be formed from the letters of the alphabet is

(A) 26^3 (B) $26 \times 25 \times 24$ (C) $26^3 \div 6$ (D) $26 \times 25 \times 24 \div 6$

22 Six people at a meeting are to be seated at a round table. If the chair of the meeting must sit on one particular chair, the number of different ways the meeting may be seated is

(A) 24 (B) 25 (C) 120 (D) 720

23 At a party, 6 boys are to be matched with 6 girls by drawing a girl's name from one hat and a boy's name from another hat. The probability that Cathy will be matched with Con is

(A) $^1/_{36}$ (B) $^1/_{24}$ (C) $^1/_{12}$ (D) $^1/_6$

24 In NSW in 1985, there were 2941 road deaths. Of these, 33% recorded a blood alcohol level above 0.005%. It can therefore be concluded that

(A) about 970 died because of drink driving.
(B) alcohol accounts for more than half the deaths on NSW roads in 1985.
(C) alcohol was not a factor in 67% of the fatal accidents.
(D) none of the above statements is true.

25 In the Australian population, 31% of people have Type A blood. If there are 3.1 million people in Sydney, the number with Type A blood is closest to

(A) 100 000
(B) 1 million
(C) 961 000
(D) 96 100

26 A coin is biased so it shows heads to tails in the ratio 4 : 3 when it is tossed. Out of 1000 tosses, it is expected to show

(A) 571 heads (B) 500 heads (C) 750 heads (D) 667 heads

27 Betty and Jim are going to play a board game which requires each player to throw a "six" on the die before starting. By being a gentleman and allowing Betty to roll the die first,

(A) Jim had increased his chances of winning.

(B) Betty gets no advantage.

(C) Jim's chance of starting first on his second throw is $\frac{1}{36}$.

(D) Jim will have a higher probability of starting first if Betty throws "four" on her first throw.

28 A game called Keno is played in which a coin is tossed 20 times. A player can bet on Heads or Tails appearing more often. If the result is 10 Heads, 10 Tails, the house wins. If the probability of 10 Heads and 10 Tails is 0.176, the probability of more Heads appearing than Tails is

(A) 0.824 (B) 0.412 (C) 0.088 (D) 0.5

29 A raffle has 8 tickets numbered 1 to 8 and 8 prizes. The tickets are drawn one at a time without replacement. The probability that the tickets will be drawn in the order 1, 2, 3, 4, 5, 6, 7, 8 is

(A) 2.5×10^{-5}

(B) $8 \times \frac{1}{8}$

(C) $\frac{1}{8}$

(D) $(\frac{1}{8})^8$

30 The slots of a roulette wheel are numbered 1, 2, 3, 4, ..., 36, 37. The probability of the marble landing on a multiple of 3 is

(A) $\frac{1}{3}$ (B) $\frac{6}{19}$ (C) $\frac{11}{38}$ (D) $\frac{12}{37}$

31 Jo bought 20 tickets in a raffle. If 1000 tickets were sold altogether, Jo's chance of winning none of 1st, 2nd or 3rd prize is

(A) $\frac{980}{1000} \times \frac{980}{999} \times \frac{980}{998}$

(B) $\frac{980}{1000} \times \frac{979}{999} \times \frac{978}{998}$

(C) $\frac{20}{1000} \times \frac{19}{999} \times \frac{18}{998}$

(D) $\frac{980}{1000} \times \frac{980}{1000} \times \frac{980}{1000}$

32 If a standard cubic die is rolled 60 times, it is most likely that the number '4' would be shown

(A) 4 times (B) 6 times (C) 10 times (D) 24 times.

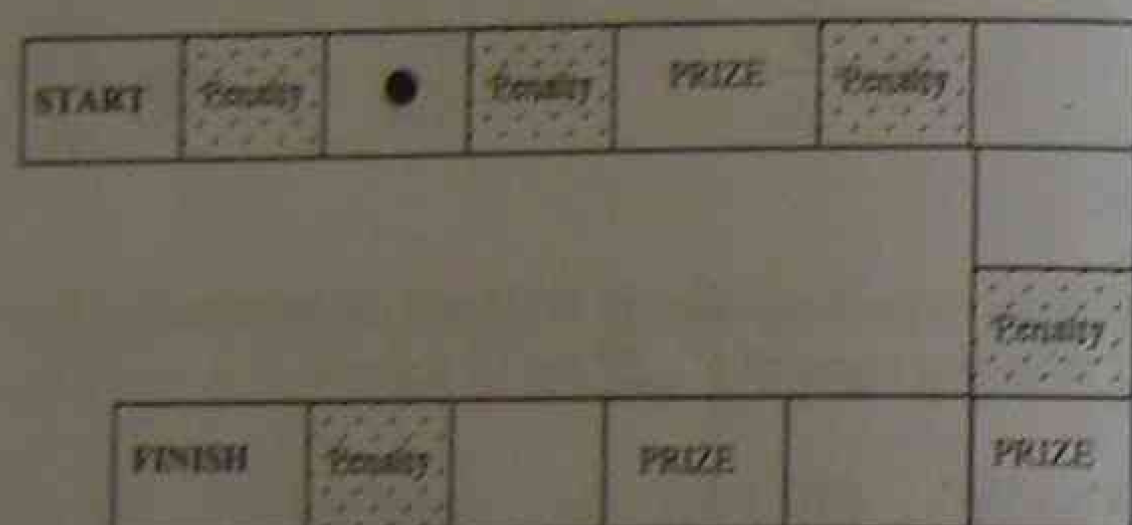
33 This table shows the interests of a group of students, e.g. 5 students play piano and hockey.

	Piano	Guitar
Hockey	5	12
Basketball	6	20

A student is selected at random. The probability that this person plays guitar and basketball is closest to

(A) 0.56 (B) 0.16 (C) 0.47 (D) 0.63

34 A player is about to throw two dice in this board game. The player has moved two squares from the start.



The probability that the player lands on a penalty square is

(A) $^5/_{12}$ (B) $^1/_3$ (C) $^1/_9$ (D) $^1/_4$

35 A club is running five raffles, each with 100 tickets for sale at $1 each. There is one prize worth $80 in each raffle. Out of the following choices, the tactic which provides the highest expected return is to

(A) buy five tickets in one raffle.
(B) buy one ticket in each of five raffles.
(C) buy no tickets.
(D) buy any 5 tickets from any 5 raffles.

36 A 'Two-up' game costs $2 per toss to play. Two coins are tossed. Players can bet on Double Head or Double Tail. If they win, they get all the money bet on the game. If 'one head, one tail appears', the bets jackpot to the next throw. The percentage return is

(A) 50% (B) 80% (C) 90% (D) 100%

Free Response Questions

The Language of Chance

1 How many different four-digit counting numbers are there?

2 A day of the week is randomly chosen and its first letter noted.

(a) List the outcomes.

(b) List the outcomes which are equally likely.

3 Describe each of these events as either impossible, improbable or unlikely.

(a) A coin is tossed and comes to rest standing on its edge.

(b) A standard cubic die is rolled and the number 7 is shown.

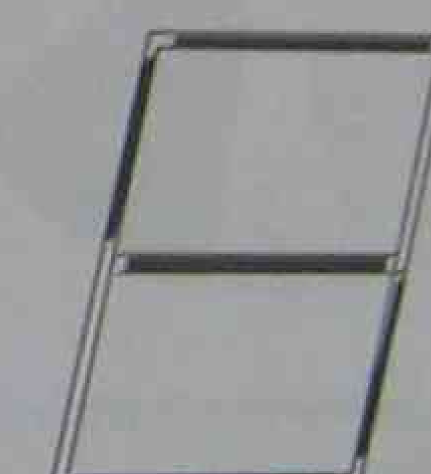
(c) A pregnant woman has twins.

(d) A person driving to work gets a punctured tyre.

(e) A piece of space junk lands on your house.

4 The scoring signs for some sports are similar to the display on digital watches.

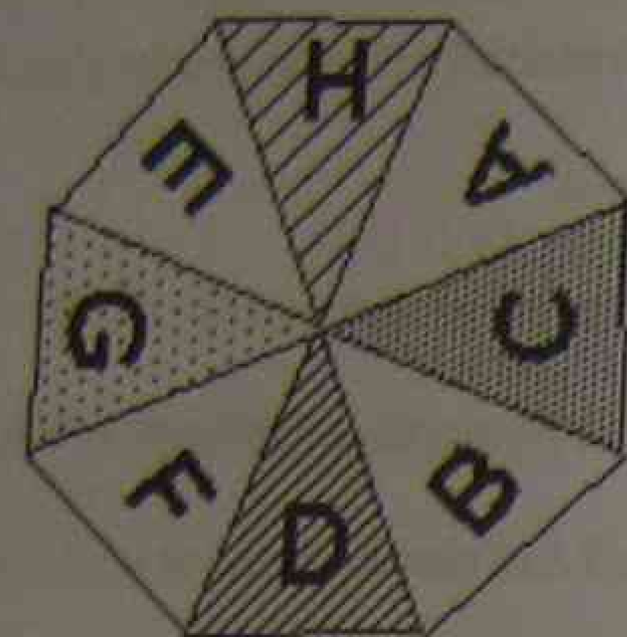
e.g. 5 is shown



How many different symbols could, be formed using this sign?

5 A baseballer playing in the backyard hits the ball straight through the open window of the neighbour's house which is 25 m away. Every other window was closed. He is relieved at not breaking a window and exclaims that the chance of the event having happened was a thousand to one. Another player disagrees, saying the event was certain.
Who is right and why?

6 Letters of the alphabet are painted on the sections of an old umbrella. The umbrella is then spun and lands on its side, indicating a letter. The winner is the first person to make a word from the letters accumulated.



(a) What is the probability that the letter on the ground will be a vowel?

(b) A player has the letters A and C. What is the probability that the player spins a letter that allows the formation of a three letter word?

7 A regular pack of 52 cards is shuffled and one card selected at random. What is the probability that the card selected is

(a) the Ace of Hearts?
(b) a queen?
(c) a black 10?
(d) a spade?
(e) either a 4 or a 5?
(f) a picture card?

8 A survey of 1200 people with television sets produced the following results for 8:30 pm on a Tuesday night.

Program	No. Viewers	Relative Frequency
Nature of the World	96	
Gala Performance	393	
First Love	241	
Midweek Movie	145	
Not watching TV		
TOTAL		

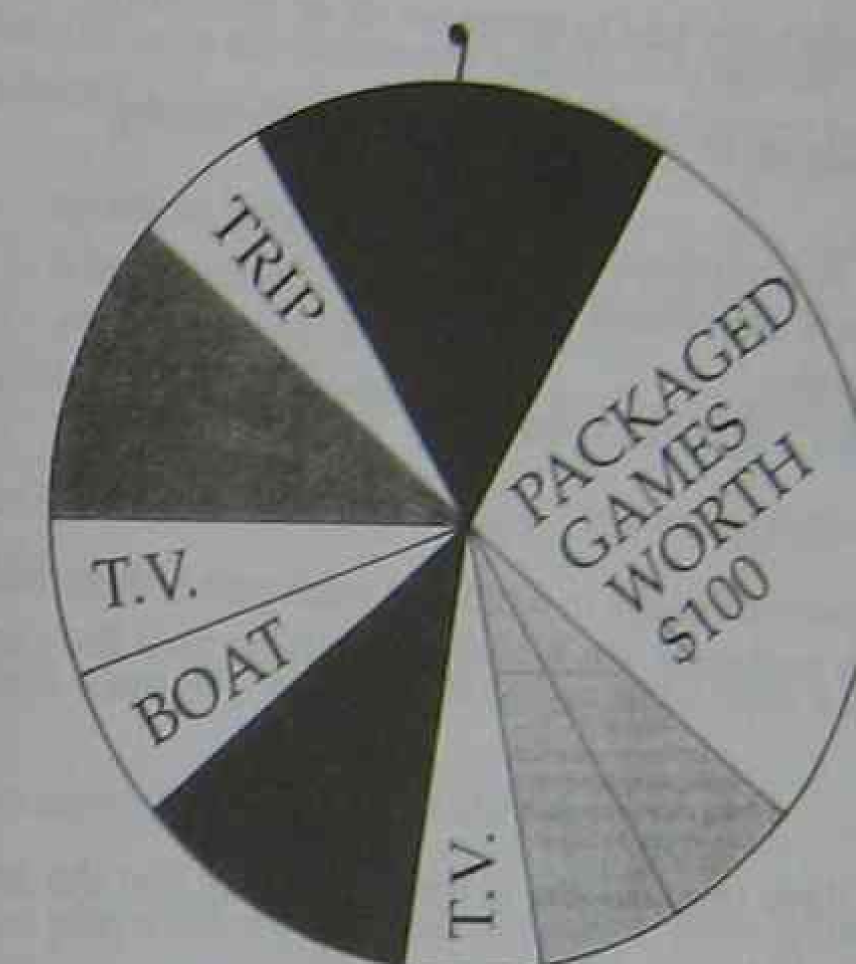
(a) Complete this table, giving answers correct to 3 decimal places.

(b) If this was a representative sample of a population of 2.4 million people, how many people were watching Gala Performance at that time?

9 Imagine that you are playing Poker and the first four cards you receive are the 7, 6, 5, and 4 of Spades. What is the probability that your fifth card will give you

(a) a straight flush (five cards in sequence from the same suit)?
(b) a straight (five cards in sequence but not from the same suit)?
(c) a flush (five cards, not in sequence but from the same suit)?
(d) a pair (two cards with the same number)?

10 A person is selected from the audience of a TV game show to spin the wheel shown. If the pointer stops at a shaded sector, the player does not win a prize. There are 18 sectors of equal size.



What is the probability of the person winning

(a) the boat?
(b) the TV?
(c) the trip?
(d) the games package?
(e) nothing?
(f) a prize of some sort?
(g) either a boat or the TV?

Multi-stage Events

11 A normal pack of 52 playing cards is shuffled.

(a) A card is selected at random. Find the probability of choosing the 6 of Hearts.

(b) Two cards are selected at random without replacement. What is the probability that one of them is the 6 of Hearts?

12 There are 15 milk chocolates, 10 dark chocolates and 5 candies in a box. Two items from the box are selected at random (without replacement).

(a) Draw a probability tree to represent these selections.

(b) Use the diagram to find the probability of selecting

(i) 2 milk chocolates.
(ii) no milk chocolates.
(iii) 2 candies.
(iv) at least one dark chocolate.

13 A man has four pairs of socks in a drawer, each pair a different colour. If he selects two socks randomly, what is the probability that they are a matching pair?

14 A coin is tossed three times. What is the probability that the same face appears all three times?

15 Tiles with letters printed on them, similar to those used in the board game "Scabble", are placed in a bag.

(a) If there is one tile for each letter of the alphabet, what is the probability that a letter drawn at random is a vowel {a, e, i, o, u}?

(b) If two letters are drawn together, what is the probability that both are vowels?

(c) A letter is drawn, the tile replaced and then a second tile drawn. The result of each draw is recorded as "vowel" or "not vowel".
(i) Draw a probability tree to represent the outcomes.
(ii) What is the probability that only one of the letters is a vowel?

16 A bag contains 5 red balls and 2 yellow balls.

(a) A ball is selected at random. What is the probability that it is

(i) red?
(ii) yellow ?

(b) A second ball is then taken from the bag without the first ball being replaced. If the first ball was yellow, what is the probability that the second ball is also yellow?

(c) Construct a probability tree and write out the sample space and associated probabilities for the experiment described in (b).

(d) If the red balls are numbered 1, 2, 2, 3, 3, and the yellow balls are numbered 1, 2, find the probability that the first ball drawn is

(i) a yellow ball marked '1'.
(ii) a ball marked '1', regardless of colour.
(iii) a ball marked '3'.
(iv) either a ball marked '2' or a yellow ball marked '1'.

17 A man has 3 suits, 9 shirts and 20 ties. How many different combinations does this provide?

18 A basketball team of 5 players is to be selected from a squad of 8 players.

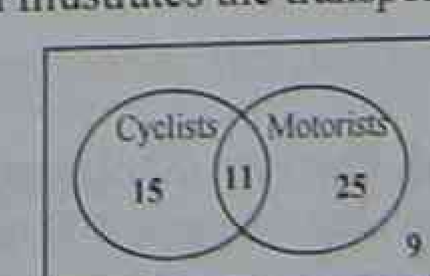
(a) How many different teams are possible?

(b) If the players were selected at random, what is the probability of a particular player being selected in the team?

19 (a) How many different three-letter arrangements (like BGA) may be formed from the letters in the word BARGE?

(b) How many of these arrangements are in alphabetical order?

20 This Venn diagram illustrates the transport choices in a small community.



If a person were selected at random, what is the probability that

(a) the person is a cyclist
(b) the person does not drive a car?

21 Assuming that the probability of a baby being male is 0.513, construct a probability tree and hence determine, correct to three decimal places, the probability that, in a family of three children

(a) all three will be girls

(b) two of the children will be girls.

22 A sports promoter reads that Sydney has about 120 wet days each year and reasons that the chance of rain for a cricket match on 30th April is ⅓.

(a) Is this reasoning correct? Explain.

(b) The match is to be held on Saturday and Sunday. The promoter reasons "If the chance of rain on Saturday is ⅓, then the chance of both days being dry is about 44%." Is this correct? Explain your answer.

23 In a game of Housie, each player gets a card with 15 numbers on it, like the one illustrated below.

	12		38		53	62	70	
3		26		44		66		88
9		24		47			74	90

A caller draws numbers at random from a bag and the players who have that number on their cards cover the number with a disc. The first player to have all numbers covered is the winner.

(a) If you play the game once with 19 other people, what is the probability that you will win?

(b) It costs $1 to 'buy' a card. There are 20 players and the winner receives all takings less 5%. If you play the game 100 times, what percentage of your playing costs would you expect to have at the end?

(c) You decide to buy two cards to improve your chances of winning. If you play 100 games with two cards each game, what percentage of your playing costs would you expect to have at the end?

24 The card game 'Twenty-one' appears in casinos as 'Blackjack'. All players, including the dealer, receive one card and may buy more cards in an effort to get as close as possible to a total of 21 points on the cards. Picture cards count as 10 and any Ace may be used as 1 or 11. Ace plus Jack total 21, for example. To win, players must get closer to 21 than the dealer. The dealer wins in the case of a tie. A total over 21 is a 'bust' and the player loses. To avoid complications, ignore the fact that one card is 'burnt' (shown by the dealer) at the beginning of the game.

(a) A player is dealt an Ace as the first card.

(i) What is the probability of the player having a total of 21 when the next card is drawn?

(ii) The player's second card is a '9'. This can be used to make a total of either 10 or 20. What is the probability that the next card will give the player a total of 21?

(iii) Should the player 'sit' or buy a third card?

(b) A player has two cards, a '9' and a '5' (totalling 14) and has to decide whether or not to buy another card.

(i) Assuming that an Ace would be used as a '1' (as 11 would bust the player), what is the average value of the cards that the player could receive?

(ii) Should the player ask for another card?

(iii) If the player buys a third card and it is an Ace, should the player risk a fourth card?

(c) What total on two cards is the highest for which buying another card gives a better than 50% chance of not busting?

25 A couple decide to keep having children until they get a daughter. Assume that the probabilities of son and daughter at birth are equal for this exercise.

(a) What is the probability that they will have only one child?

(b) What is the probability of their having four children?

(c) What number of children will make it certain that they have a daughter?

(d) If all couples took this course, what effect would it have on the balance of boys and girls in the community?

26 The following table illustrates the characteristics of women in a small town.

	Single	Married
0 – 18 years	57	3
19 years or older	17	76

(a) How many women are there in the town?

(b) If a woman were selected at random, what is the probability that she is married?

(c) If a married woman were selected randomly, what is the probability that she is 18 years old or younger?

(d) If a woman 19 years or older were selected, what is the probability that she is single?

27 A rare disease is known to be carried by one person in 500 000 in a community. A test has been developed to identify carriers and have them sterilised to prevent the disease from being transmitted to future generations. The test has a 98% accuracy rate.

(a) Complete this table for a representative sample of 1 million people.

	Carriers	Non-carriers	TOTAL
Positive test			
Negative test			
TOTAL			1 000 000

(b) If 1 million people take the test, how many could be expected to record a false result?

(c) If 200 disease carriers take the test, how many will get a (correct) positive result?

(d) What is the probability of a person actually carrying the disease if their test result is positive, forcing them to undergo sterilisation?

(e) As an extra precaution, anyone who tests positive takes the test again.

(i) Find the probability that a person who recorded a false positive result in the first test will score another positive result in the second test.

(ii) Out of the 1 000 000 tested, how many people would have two false positive results?

ALGEBRAIC MODELLING

Multiple Choice Questions

Basic Algebraic Skills

1 The adult dose of a medicine is 150 mg. Calculate the dose for a 7 year old child using Young's Rule:

$$\text{Child's dose} = \frac{(\text{age of child in years}) \times (\text{adult dose})}{(\text{age of child}) + 12}$$

(A) 55 mg (B) 162 mg (C) 1051 mg (D) 55.26 g

2 A solution to the equation $v^2 = u^2 + 2as$ is

(A) $v = 3, u = 1, a = 2, s = -2$
(B) $v = -3, u = 1, a = 2, s = 2$
(C) $v = -3, u = -1, a = -2, s = 2$
(D) $v = 3, u = -1, a = -2, s = 2$

3 The bend allowance B in sheet metal is given by the formula

$$B = 2\pi(r + \frac{T}{2}) \times \frac{A}{360}$$

where T is the thickness in mm, A is the angle in degrees and R is the radius of curvature in mm.
The bend allowance for a 1.3 mm thick sheet being bent at an angle of 45° with a radius of curvature of 20 mm would be about

(A) 16.2 (B) 15.8 (C) 125.7 (D) 20.5

4 If $y = 5 - 2x$, the value of x when $y = 6$ is

(A) 1 (B) –1 (C) 0.5 (D) -0.5

5 If $m = \frac{3(y-1)^2}{2x - y}$ then the value of m when $x = 3$ and $y = -1$ is

(A) –12
(B) 0
(C) 2
(D) 1.71, correct to 2 decimal places

Modelling Linear Relationships

6 M varies directly as P. If M = 24 when P = 10, then when P = 25, M =

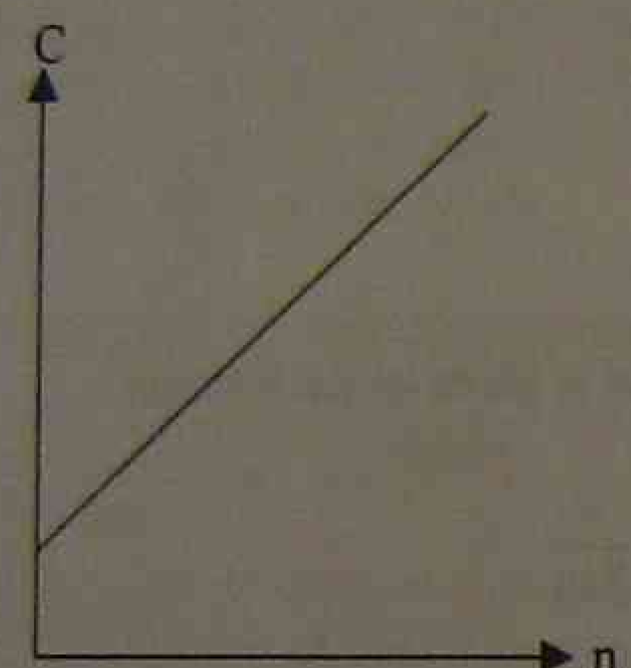
(A) 9.6 (B) 60 (C) 2.4 (D) 10.4

7 If $A 500 is worth 37 625 Yen, then paying 10 000 Yen for an amplifier would be the same as paying

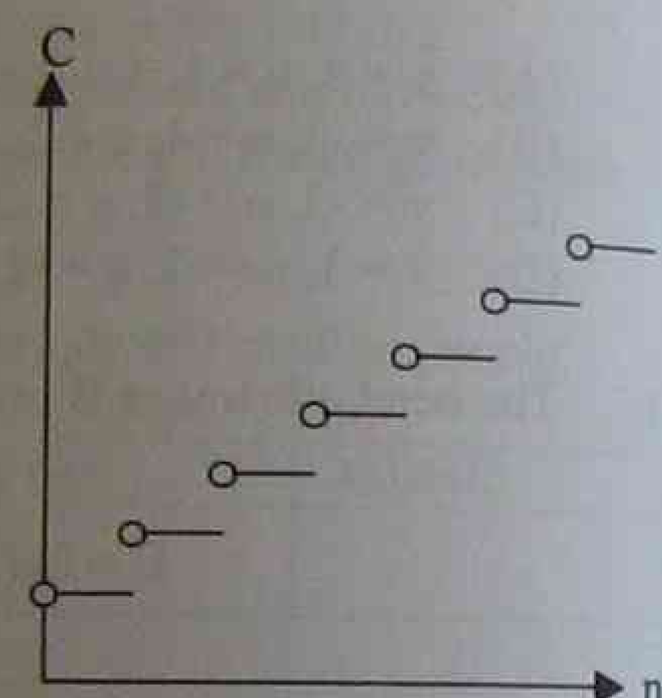
(A) $A 1881 (B) $A 7525 (C) $A 133 (D) $A 200

8 Elli's family want to celebrate her birthday at a restaurant. They need to pay $500 for entertainment and an extra $28 per person for food and drinks. Which graph best illustrates the cost C against number of people n?

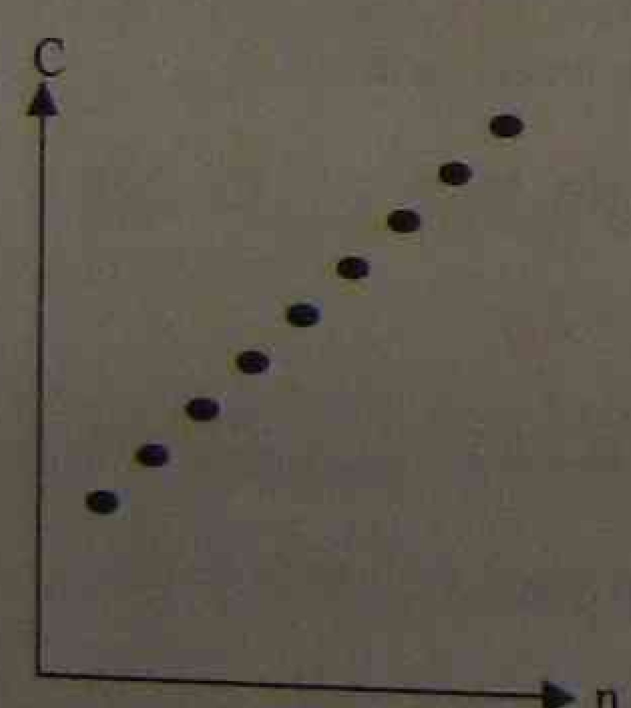
(A)



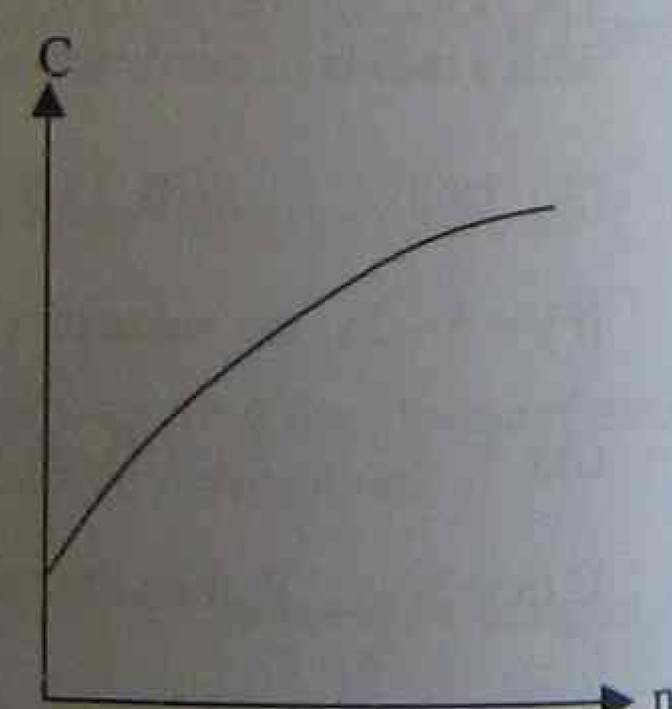
(B)



(C)



(D)

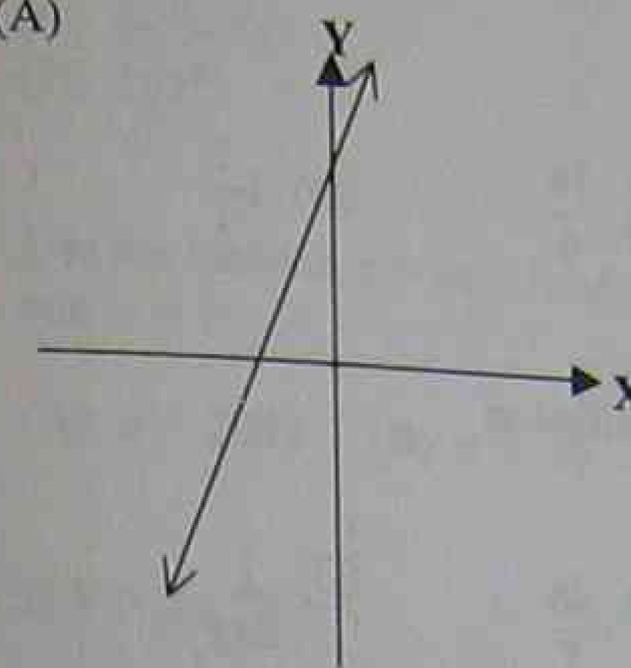


9 The result of reducing 3(x + 5) – 5(3 – x) to simplest form is

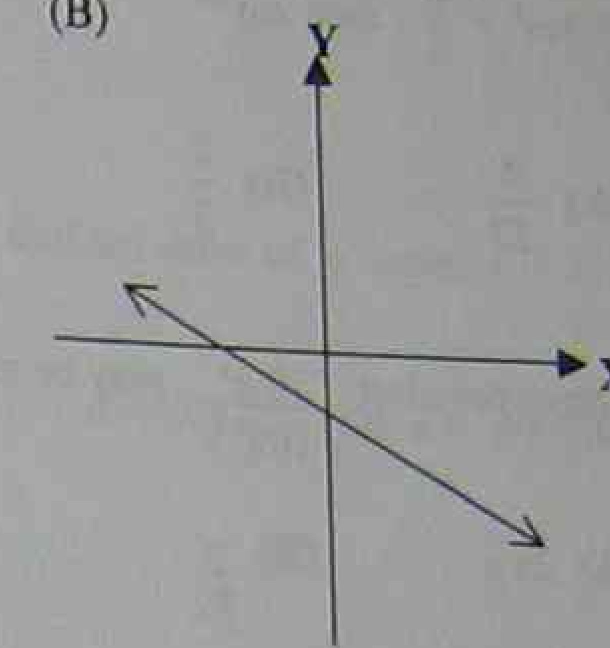
(A) 0 (B) –2x (C) 30 + 2x (D) 8x

10 Which of these graphs represents an increasing function with a positive Y intercept?

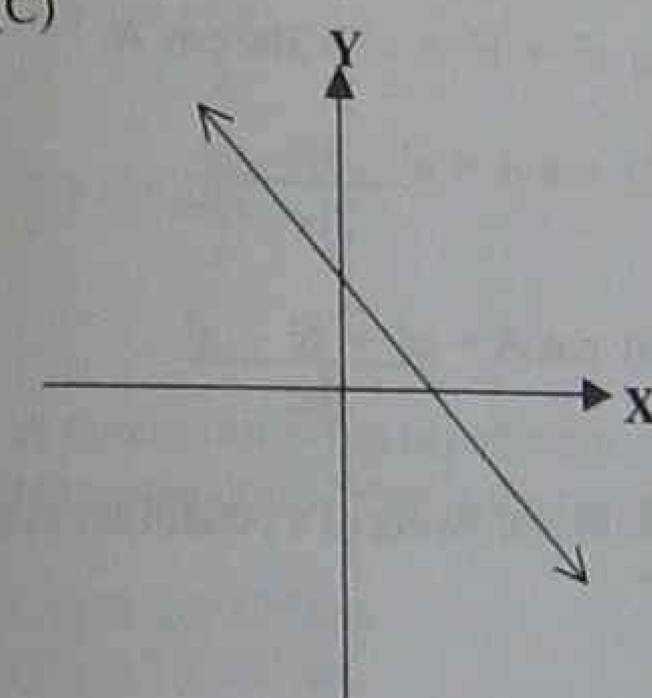
(A)



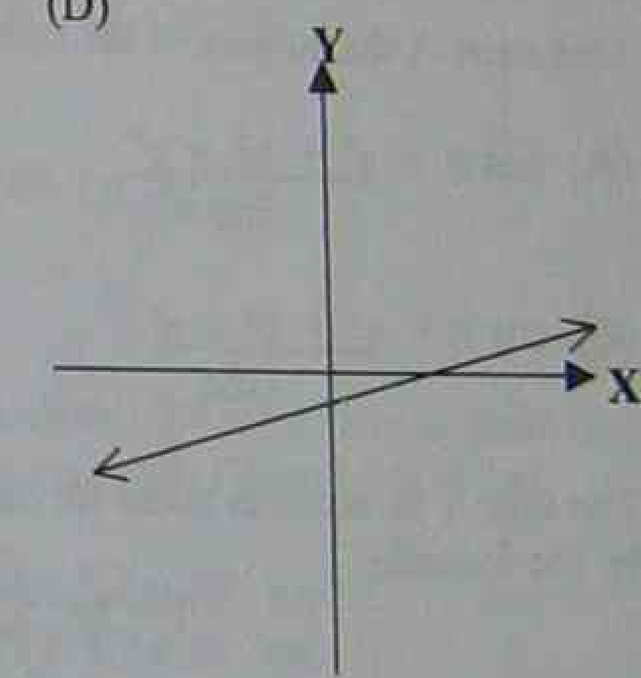
(B)



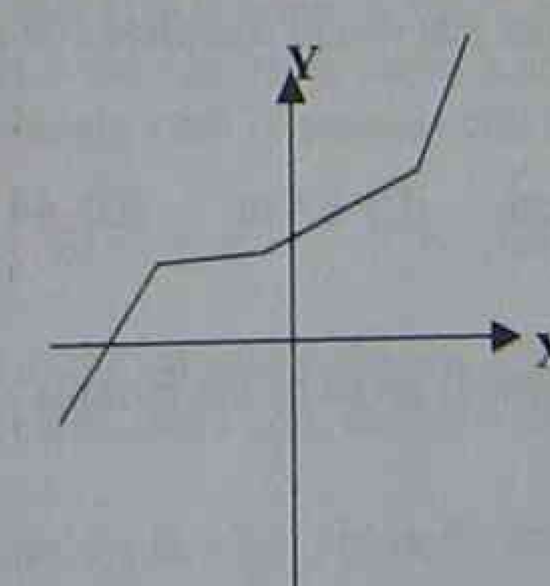
(C)



(D)



11 A function is illustrated on the axes below.



The best description of the function is that it is

(A) increasing linear.
(B) a step function.
(C) a piecemeal linear function.
(D) non-linear.

Algebraic Skills and Techniques (HSC course)

12 If $3m^3 = \frac{8}{9}$, then $2m =$

(A) $\frac{8}{27}$ (B) $\frac{2}{3}$ (C) $\frac{16}{9}$ (D) $1\frac{1}{3}$

13 The expression $\frac{20x^3}{10x^2y}$ may be simplified to

(A) $2xy$ (B) $\frac{y}{2x}$ (C) $\frac{2x}{y}$ (D) $\frac{1}{2xy}$

14 Make cos A the subject of the formula $a^2 = b^2 + c^2 - 2bc \cos A$.

(A) $\cos A = \frac{a^2 + b^2 + c^2}{2bc}$ (B) $\cos A = \frac{a^2 - b^2 - c^2}{2bc}$

(C) $\cos A = \frac{c^2 + b^2 - a^2}{-2bc}$ (D) $\cos A = \frac{c^2 + b^2 - a^2}{2bc}$

15 The time T in seconds taken for one complete swing of a pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum in metres and $g = 9.8\ m/s^2$.
The length of the pendulum that would complete 100 swings per minute would be closest to

(A) 9 cm (B) 69 cm (C) 2.4 m (D) 44 cm

16 The eccentricity of an ellipse is given by the formula

$$e^2 = 1 - \frac{b^2}{a^2}.$$

When b = 5.2 cm and e = 0.64, a is equal to

(A) 74.1 cm (B) 8.67 cm (C) 6.8 cm (D) 5.3 cm

17 When $3x^2(x^2 - 1) - 4x^2$ is expanded and like terms collected, the answer is

(A) $-4x^2$
(B) $3x^4 + 7x^2$
(C) $3x^4 - 7x^2$
(D) $10x^2$

18 Use the formula $s = ut + \frac{1}{2}at^2$ to find the value of a when $t = 2$, $s = 40$ and $u = 10$.

(A) $a = 2080$ (B) $a = 10$ (C) $a = 0.4$ (D) $a = 80\,020$

19 If $V = \frac{2\pi r}{t}$ then

(A) $r = \frac{tV\pi}{2}$ (B) $Vt - 2\pi = r$

(C) $t = \frac{2\pi V}{r}$ (D) $t = \frac{2\pi r}{V}$

20 A comet following in a circular path travels 2.6×10^{15} km to make one complete orbit. What is the radius of the orbit? (Hint: $r = C \div 2\pi$)

(A) 4.1×10^{14} km (B) 2.1×10^{14} km
(C) 4.1×10^{7} km (D) 8.2×10^{7} km

21 The height above ground level, h metres, of a piece of play dough dropped from an altitude of 100 m is given by the equation $h = 100 - \frac{1}{2}gt^2$ where $g = 9.8\ m/s^2$ and t is the number of seconds after being released. If it falls onto solid ground, its height after 10 seconds will be

(A) zero (B) –390 m (C) –2301 m (D) 590 m

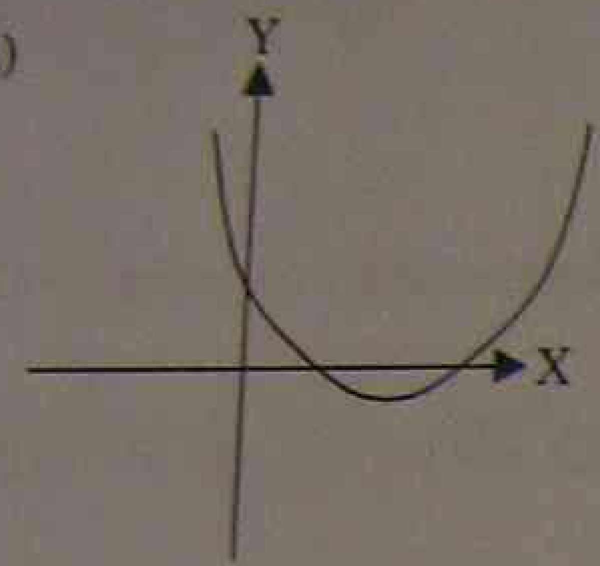
22 If h is the height of the eye above the sea, d the distance to the horizon and R the radius of the earth (6400 km), then $h^2 + 2hR = d^2$.

For a person whose eye is 1.7 m above the sea, the distance to the horizon would be about

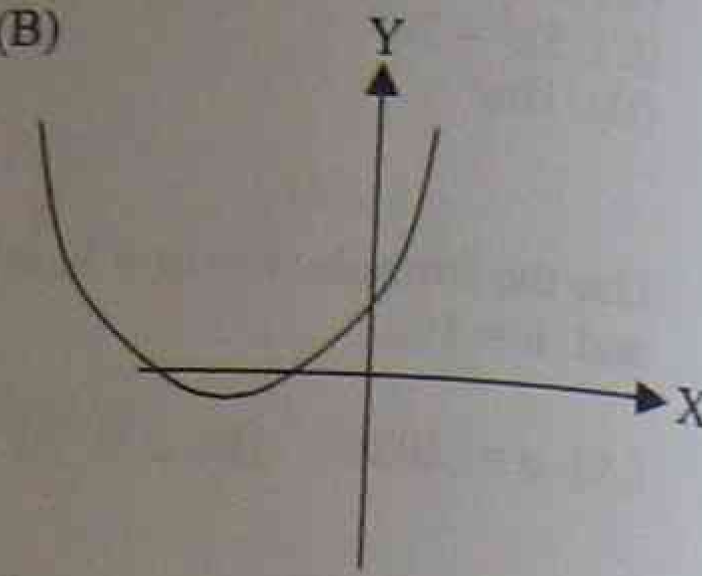
(A) 21.8 km (B) 0.148 km (C) 4.6 km (D) 147.5 km

23 Which of these graphs could represent $y = (x - 5)(x - 2)$?

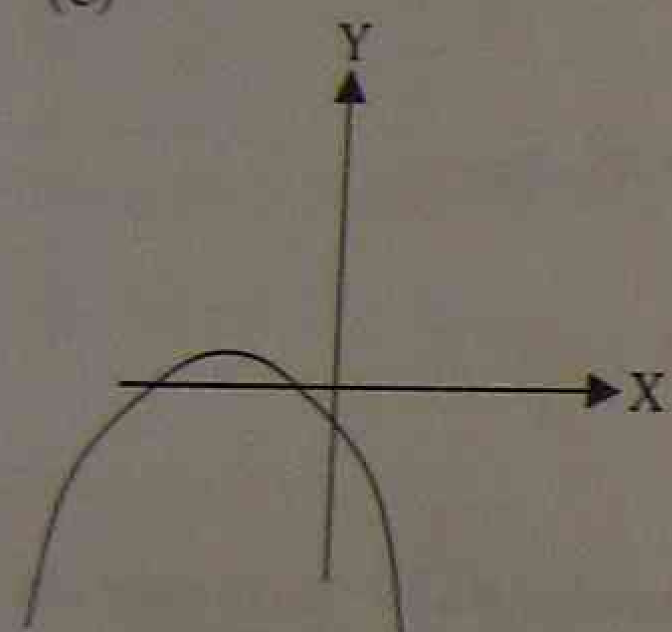
(A)



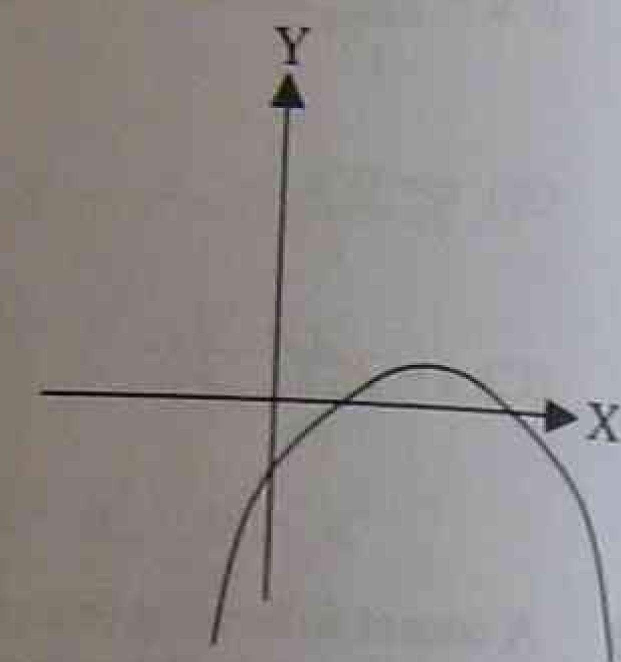
(B)



(C)



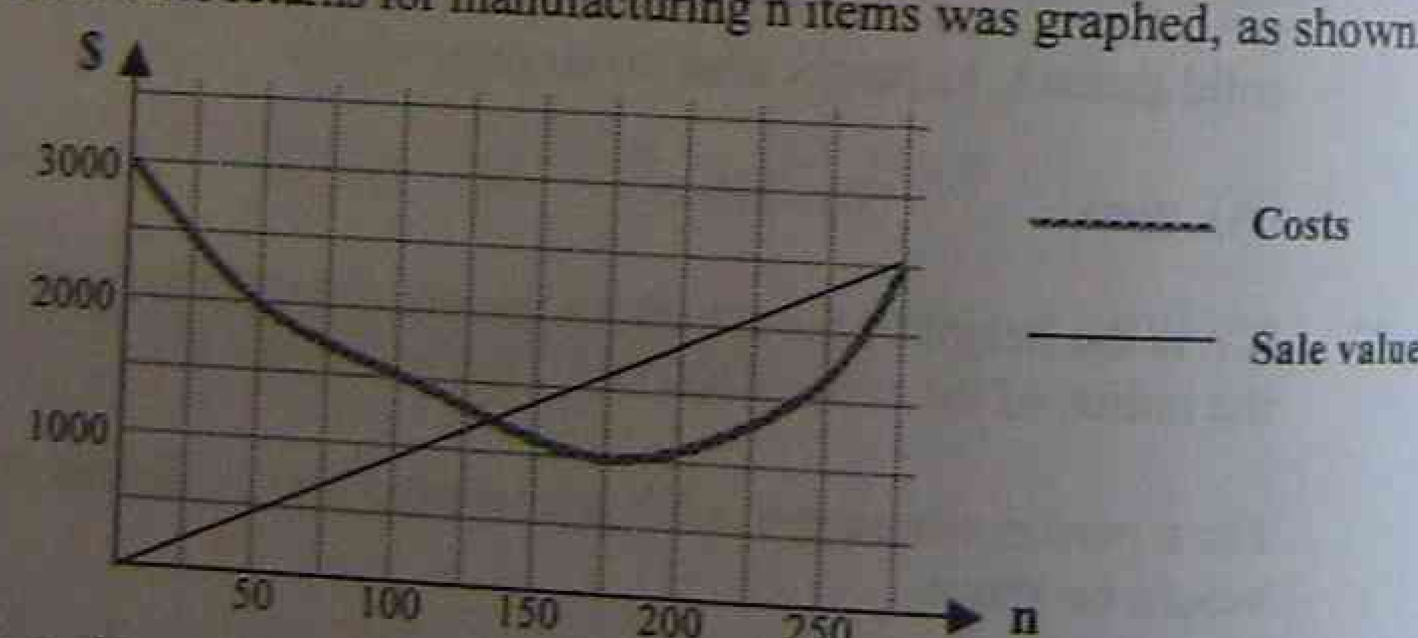
(D)



24 Variable A is inversely proportional to Variable B. When $A = 5.6$, $B = 8.4$ so when $A = 3.5$, B is closest to

(A) 2 (B) 4 (C) 5 (D) 13

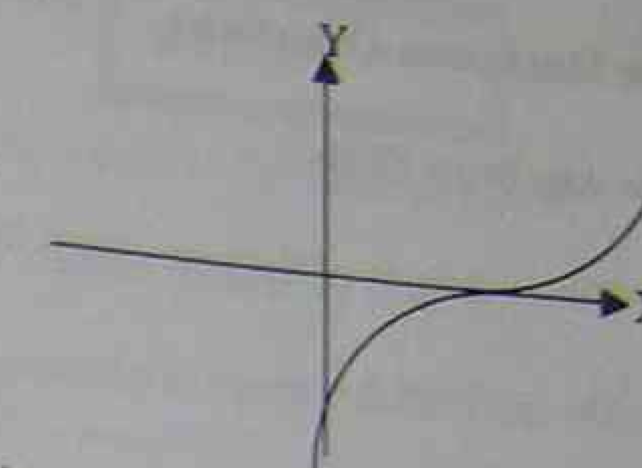
25 The costs and sales returns for manufacturing n items was graphed, as shown.



The number of items that should be made to give the greatest profit would be about

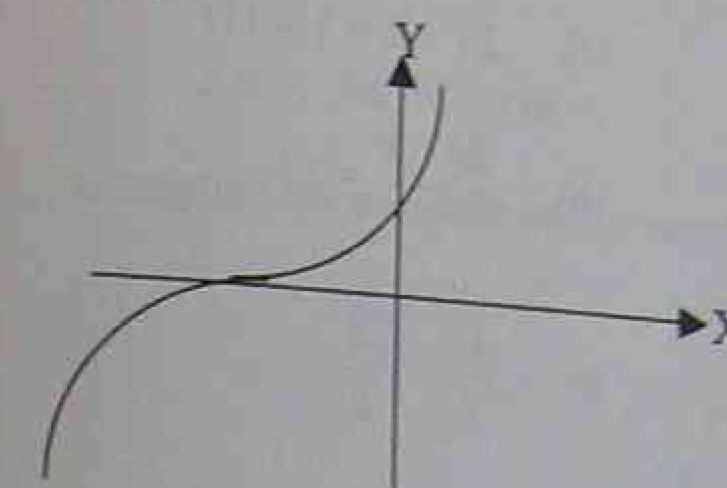
(A) 130 (B) 200 (C) 275 (D) 3000

26 This graph represents the equation $y = (x - 1)^3$.

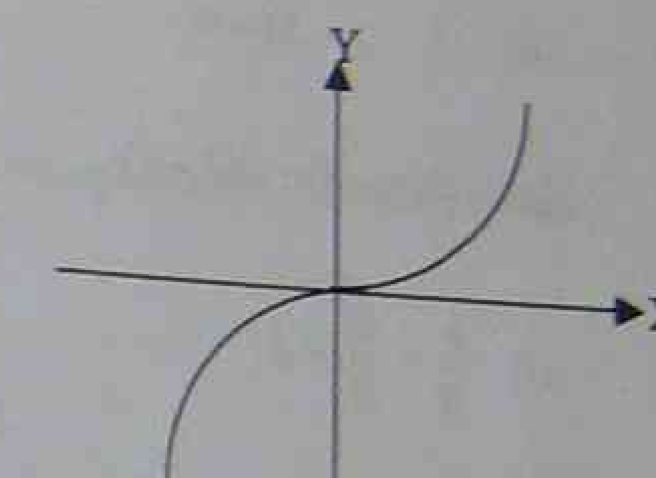


Which graph below best represents $y = (x - 1)^3 + 2$?

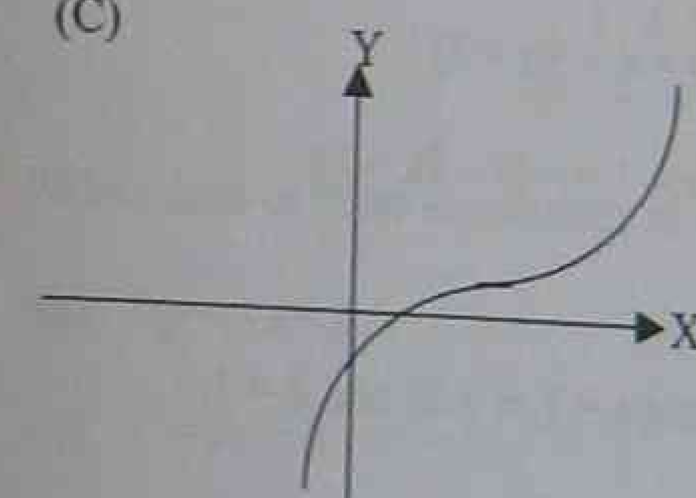
(A)



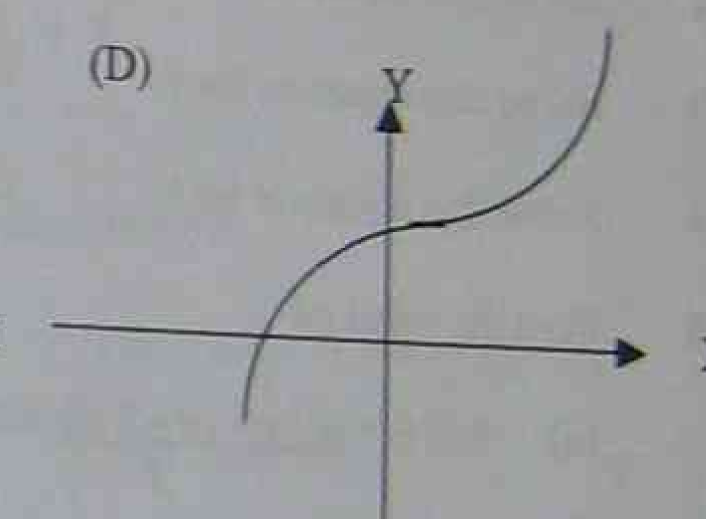
(B)



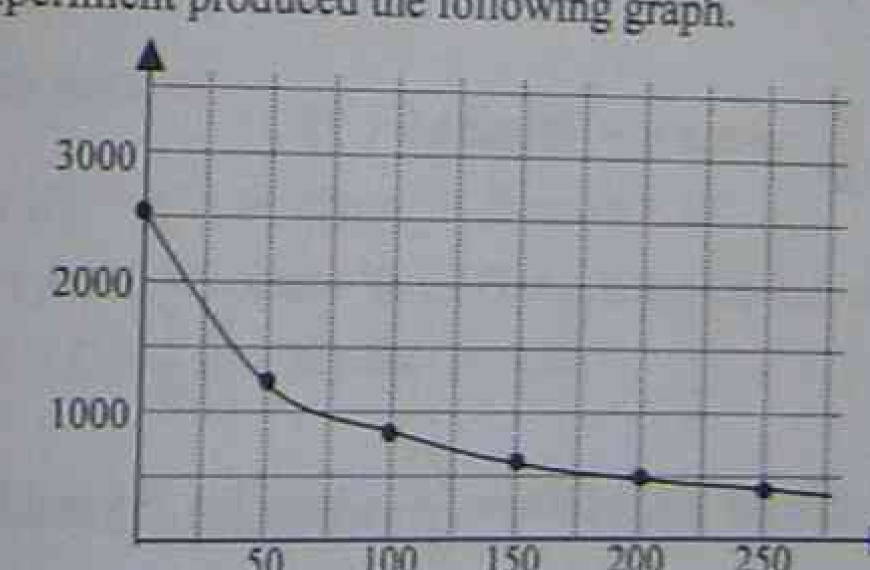
(C)



(D)



27 An experiment produced the following graph.



The function relating the variables is most likely

(A) linear (B) quadratic (C) cubic (D) hyperbolic

Free Response Questions

Basic Algebraic Skills

1 Find the value of the pronumeral in each of these equations.

(a) $-4 = a + 8$ (b) $3b - 7 = -3$

(c) $5 - 3c = 45$ (d) $\frac{d}{4} + 11 = -2$

(e) $6 - (e - 5) = 9$ (f) $2f + 5 = 1 - 11f$

(g) $5(g - 8) = 2(3 - g)$ (h) $\frac{2h-1}{5} = 17$

(i) $\frac{b}{2} - \frac{b}{5} = 4$ (j) $\frac{k}{7} = \frac{2k}{3}$

2 Solve this equation for n: $2(n - 1) = 3(n + 1)$

3 Solve this equation for r: $5r + 8 = 5(r + 8)$

4 Solve this equation for k: $2k - 1 = -(3 - 2k - 2)$

5 Find the value of

(a) S if $S = \frac{n[2a + (n - 1)d]}{2}$ where a = 7, n = 23 and d = 3.

(b) D if $D = \frac{yA}{y+12}$ where y = 3 and A = 20

(c) B if $B = \frac{m}{h^2}$ where m = 70 and h = 1.75

(d) A to 2 decimal places if $A = P(1 + r)^n$ where P = 1600, r = 0.0075 and n = 240

(e) k to 2 significant figures if $k = \frac{T^3}{R^2}$ where T = 365 and R = 1.48×10^8

6 How many metres of fencing would be required for a circular cricket ground with radius 62 metres?

7 This answer is from a calculator screen. 3.6215 -05

Write this answer as a basic numeral, correct to five decimal places.

8 (a) Write the next number in this number pattern: 1, 2, 3, ...

(b) State the rule which would generate this number pattern and write the next three terms: 3, 7, 11, 15, ...

(c) Find a rule which produces this number pattern and find the terms X and Y:

2, 6, X , 54, 162, 486, Y

9 Complete the next line of this number triangle (called Pascal's Triangle).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
.. ..

10 Write each of these expressions in simplest form.

(a) $3(p - 1) + 2p$ (b) $x(x - 1) + x(1 - x)$

(c) $y^2 - 2y + 8 - y^2 + 3y - 7$ (d) $4z^2 . z^3$

(e) $(2a^3)^2$ (f) $\frac{15x}{27y^2} \times \frac{3y}{5x^2}$

11 Find the value of

(a) c if $c^2 = a^2 + b^2$ where c = 26 and a = 10

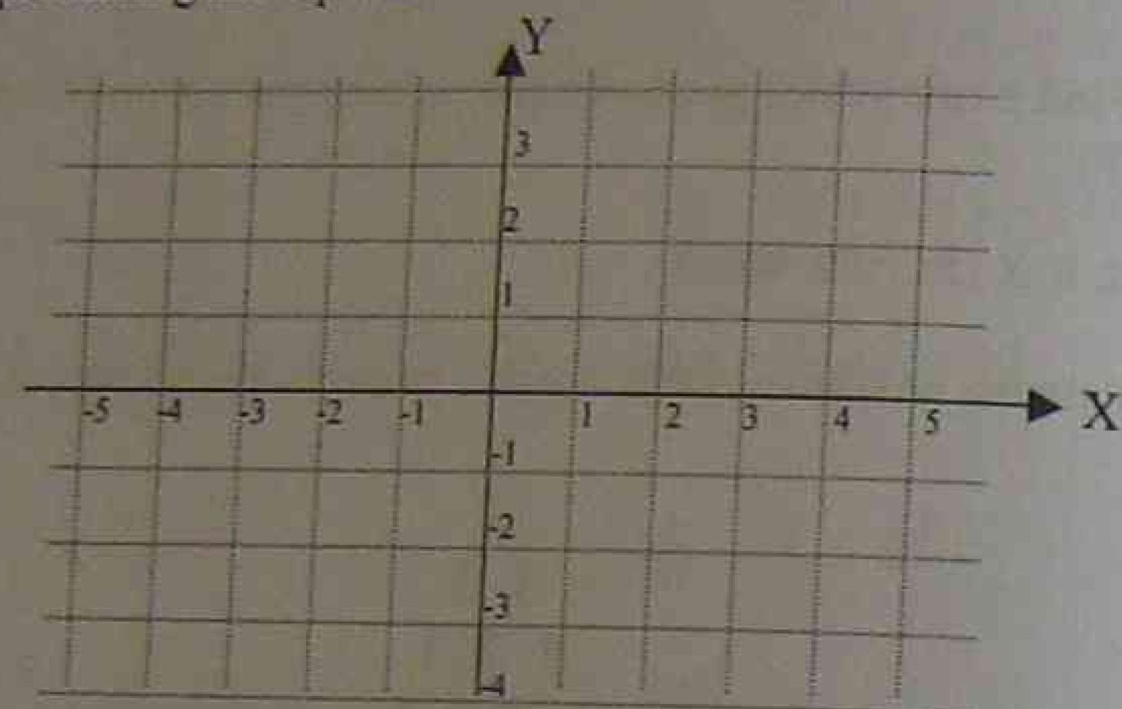
(b) F if $C = \frac{5}{9}(F - 32)$ and C = 120

(c) r to one decimal place if $S = 4\pi r^2$ and S = 160 000

12 (a) Complete this table of values for the equation $y = \frac{x}{2} + 1$.

X	-2	0	1	4
Y				

(b) Copy these axes. Plot the points from the table and draw the line representing the equation.

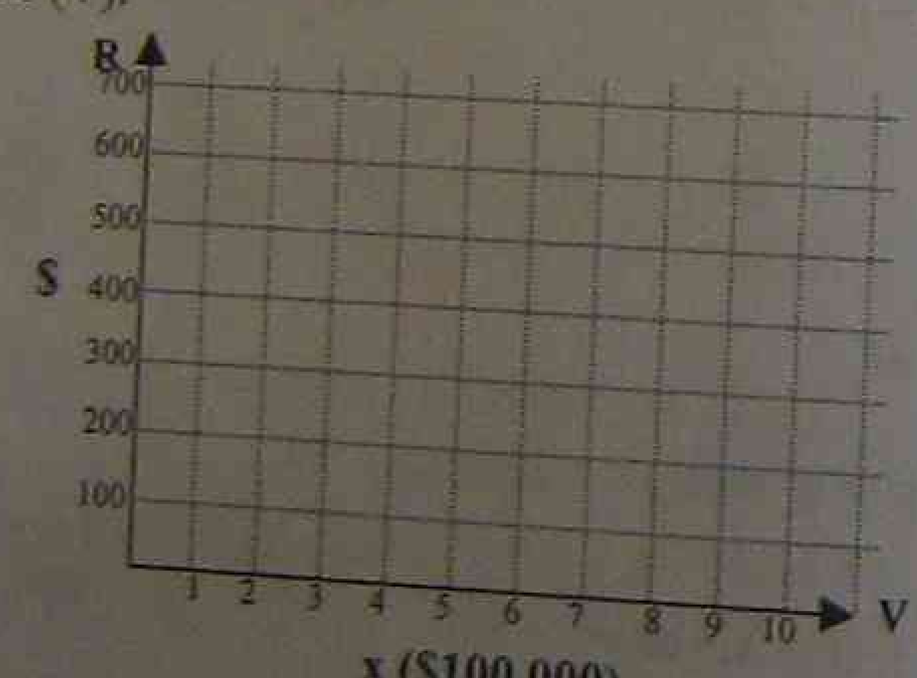


(c) Find the gradient and Y-intercept of the line.

13 A municipal council charges a rate with a basic service fee of \$240 plus 0.043% of the unimproved value of the land.

(a) Find the amount to be paid on a property valued at \$400 000.

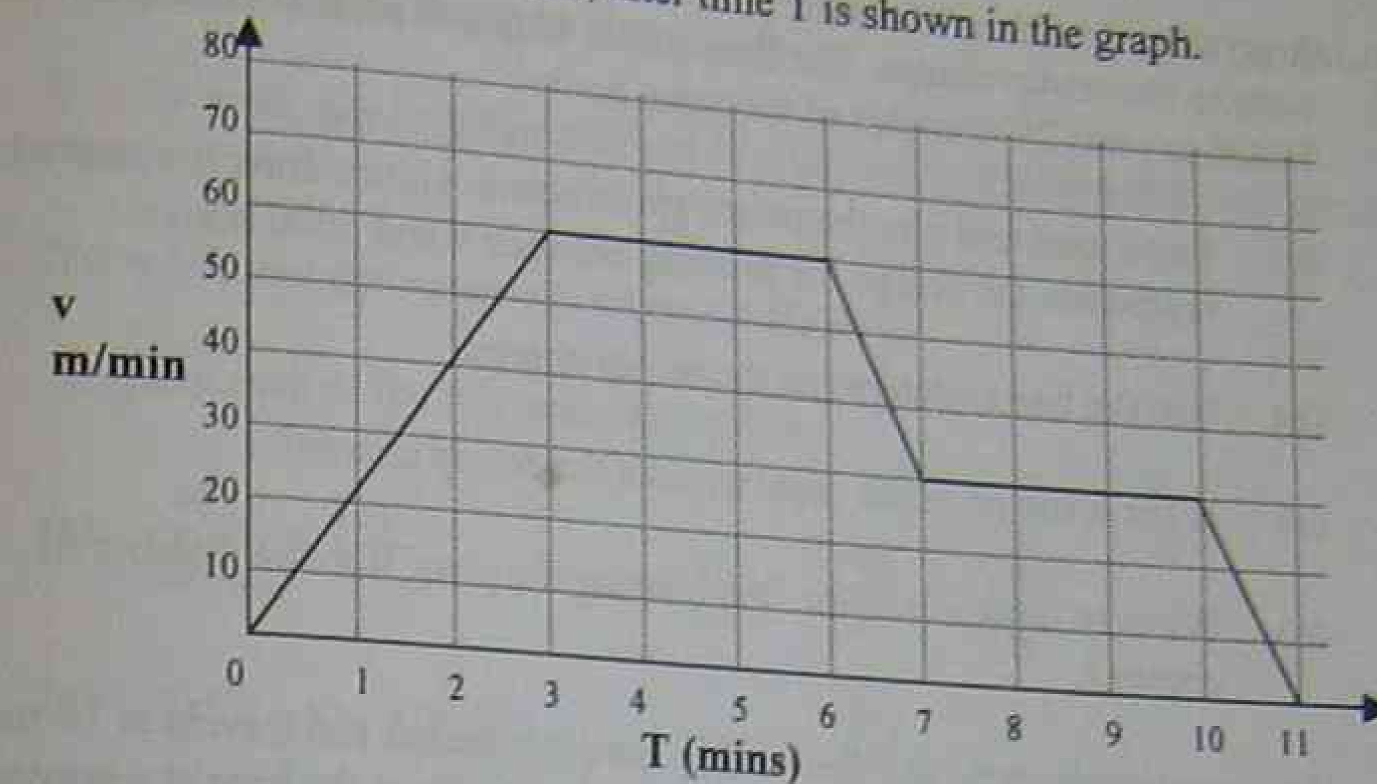
(b) Copy these axes. Graph the amount to be paid (R) against property value (V).



(c) Someone was paying \$530 to the council. Use the graph to find the value of the property.

(d) Find the equation of the line and use algebra to find the exact value for part (c).

14 The velocity v of a skater, Jo, after time T is shown in the graph.



(a) What was Jo's maximum velocity?

(b) What is her average velocity in the first three minutes?

(c) The area between the graph and the T axis will give the total distance travelled. Find that distance.

15 Americans coming to Australia are offered \$1 Australian for every 62.35 cents US that they wish to exchange.

(a) How much Australian money would an American get for \$300 US?

(b) What is the value in American dollars of a camera selling here for \$455 Australian?

16 An estate agent charges commission according to the following table.

Value of Sale	Agent's Fee
Up to and including \$50 000	5%
On the next \$100 000	3%
On the next \$200 000	2.5%
Thereafter	2%

(a) Draw a graph of the agent's fee against sale values from \$0 to \$500 000.

(b) Use the graph to find the sale value that would return the agent a fee of \$15 000.

(c) Use algebra to find the exact answer to part (b).

17 A printing firm determines the price for printing books by adding a price per copy to the set-up charge. The firm quotes \$3200 to print 1000 copies and \$4200 to print 2000 copies of the same book.

(a) Draw axes and plot the points given above. Hence draw the line which represents the charge for printing between 0 and 5000 books.

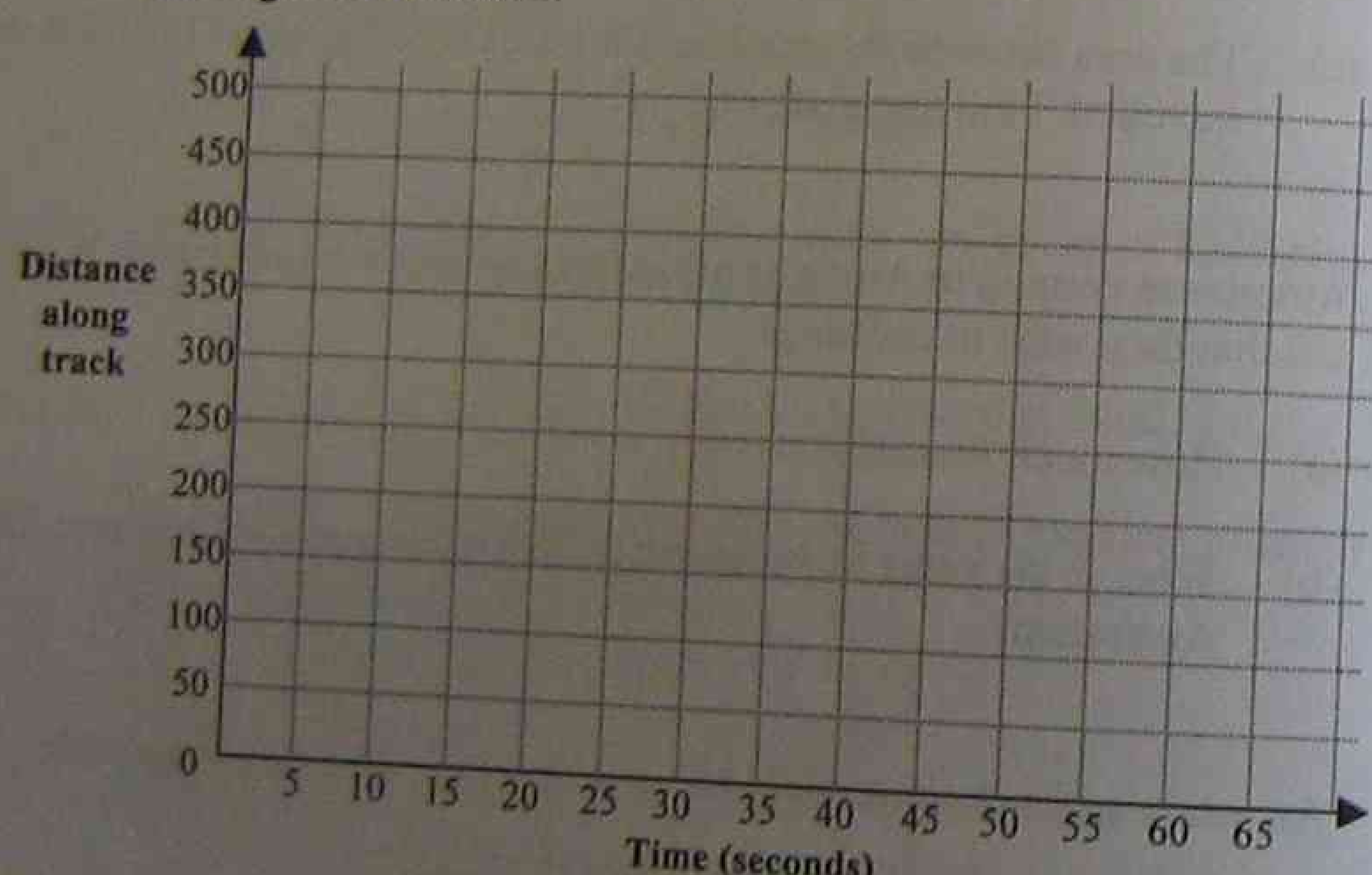
(b) Use the line to determine the set-up charge.

(c) What is the price per copy without the set-up charge?

(d) Write an equation of the line relating the price P to the number of copies n.

18 At a greyhound track, the mechanical hare is released and travels at 10 m/s. Five seconds later, a greyhound is released and chases the hare at a constant speed of 12 m/s.

(a) Copy these axes and draw lines to represent the position of the hare and the dog after t seconds.



(b) Use the graph to determine the time when the dog catches the hare.

(c) How far does the dog have to run before it catches the hare?

(d) How fast would the hare have to "run" if the dog is not to catch it before the finish, 500 m along the track?

(e) If the hare cannot be set to run any faster than 10 m/s, how long must the starters wait before releasing the dog if the dog is not to catch the hare before the finish line?

Algebraic Skills and Techniques (HSC course)

19 If $(3.1)^x = 40$, use your calculator to find the value of x correct to one decimal place.

20 If $d = 5t^2$, find the values of t when $d = 8000$.

21 Simplify each of the following expressions.

(a) $\dfrac{48xy}{25x} \times \dfrac{15y^2}{28x}$ (b) $\dfrac{14ab}{15c} \div \dfrac{35abc}{36}$

(c) $2x^2(2 - x) - x(x^2 - 5)$ (d) $\sqrt{4x^8}$

22 The volume of a sphere is given by the formula $V = \dfrac{4}{3}\pi r^3$ where r is the radius.

(a) Write the formula with r as the subject.

(b) Find the value of V, correct to 3 decimal places, when r = 17.3 m

(c) What is the radius in centimetres of a spherical balloon with a volume of 1 m^3?

23 Study this piece of algebra and locate the error.

Given:	$x = 1$
Take 1 from both sides	$x - 1 = 0$
Add $(x - 1)$ to both sides.	$2x - 2 = x - 1$
Take 2 out as a factor.	$2(x - 1) = x - 1$
Divide both sides by $(x - 1)$	$2 = 1$

24 Write $(9.6 \times 10^{-7}) \div (2.4 \times 10^{-9})$ in scientific notation.

25 A farad is a unit of capacitance (storage of electrical charge). The prefix 'nano' means 10^{-9}. Write 2469 nanofarads in farads using scientific notation.

26 Einstein's famous equation $E = mc^2$ states that matter of mass m kg may be converted to energy E in joules through a nuclear reaction, c being the speed of light (3×10^8 m/s).

Find the amount of mass required to produce 10 million kilojoules, roughly the amount of energy required to lift a family car 1 km above the earth.

27 The length of a pendulum varies inversely with the square of its period. If the length l is 0.089 m when the period T = 0.6 seconds, find l when T = 1.

28 The population of Latin America was taken in a census every five years. The results are shown in this table.

Year	1960	1965	1970	1975	1980
Population (millions)	217	249	284	322	362

(a) By what factor did the population increase between 1960 and 1965?

(b) Construct a line graph from this table

(c) On the same graph, construct another line graph to illustrate what the population would have been if the population had continued to increase by the same factor as your answer to part (a) for the later five year periods.

(d) Which model, linear, quadratic or exponential, would most closely fit this data?

29 The hands of a clock are superimposed at 12:00. At what time are they next superimposed? Complete these algebraic steps to determine the answer.

(a) At roughly what time would you expect the hands to be superimposed?

(b) Through how many degrees does the minute hand move in

(i) 1 minute? (ii) x minutes?

(c) Through what angle does the hour hand move in

(i) 1 minute? (ii) x minutes?

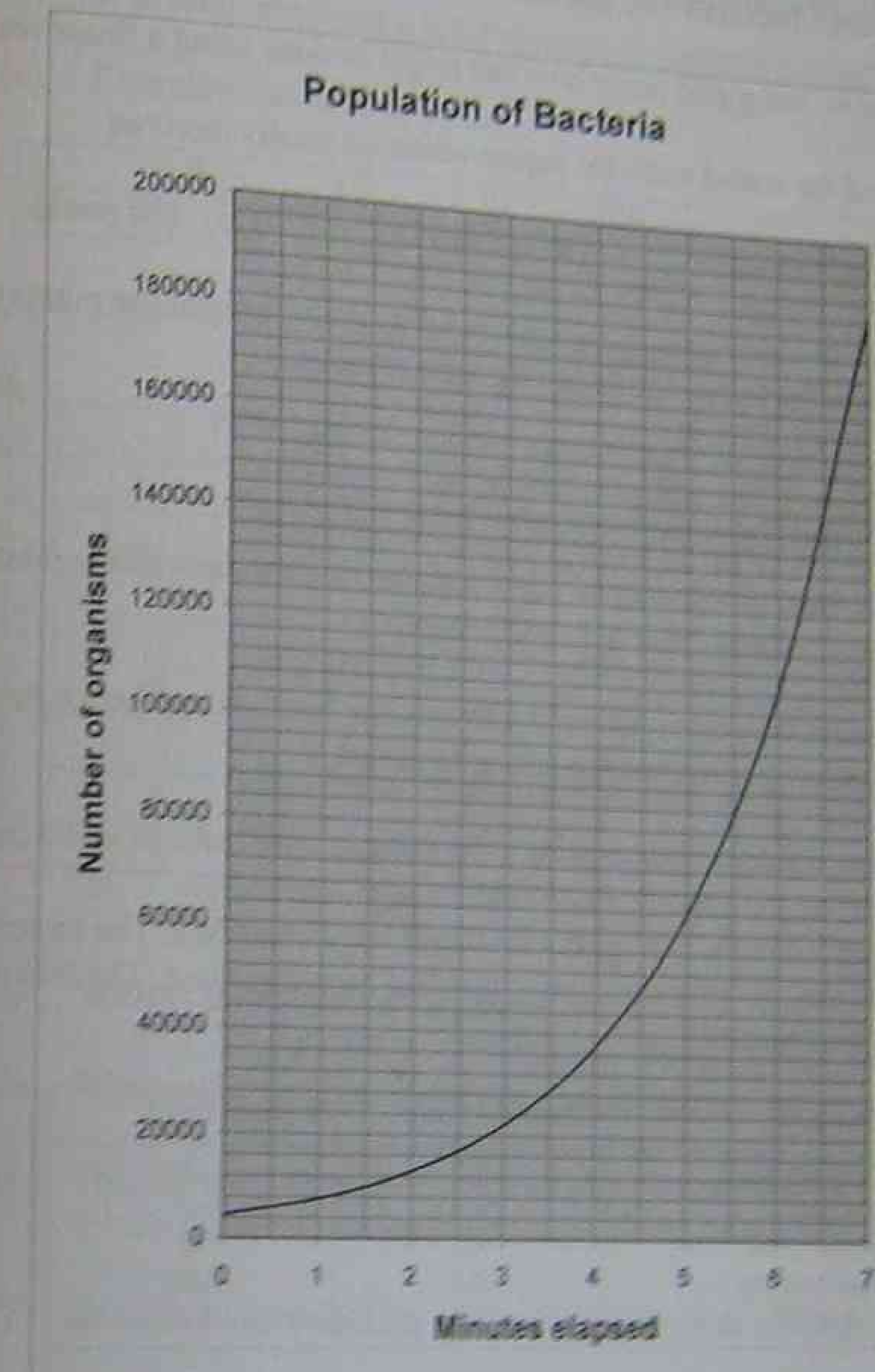
(d) Write an equation for the angle between the hands x minutes after 1:00.

(e) Solve the equation and give the time to the nearest second.

30 The cost V in dollars of making n items in a factory is found to follow a curve with equation $V = \frac{1000}{n+5}$. The sale value of n objects is given by the equation $V = 0.13 \times n$.

By graphing the two curves, determine the minimum number of objects that will allow the manufacturer to 'break even'.

31 This graph shows the number of bacteria in a culture after a certain time has elapsed. Initially, there were 4500 organisms present. After 1 minute, there were 7560 organisms present.



(a) Complete this table of values from the graph.

Time	0	1	2	3	4	5	6	7
Population	4500							

(b) Estimate the length of time it is taking for the population to double.

(c) What will be the population after 9 minutes has elapsed?

32 An examiner has marked a class set of papers. The highest mark was 185 and the lowest was 42. The examiner wants to rescale the marks so that the highest mark becomes 100 and the lowest becomes 30.

(a) Show, using grid paper how this might be done using a linear rescaling.

(b) Find the scaled mark for papers which originally received

(i) 60 marks (ii) 100 marks (iii) 150 marks.

33 The pressure inside a pump was measured as the volume of the pump cylinder was reduced by force. The following table resulted.

Volume (cm^3)	126	112	98	85	79
Pressure (kP)	60	67.5	77	89	95.7

(a) Plot these results on grid paper and describe the most likely relationship between the volume and the pressure.

(b) Explain why these points fit a hyperbolic function and hence determine a suitable equation to connect the variables.

(c) What would the pressure be if the volume were reduced to 60 cm^3?

34 In an experiment, various spherical objects were immersed. The experimenter then measured the volume of water that had been displaced. The following results were obtained.

Object	Diameter (cm)	Volume (mL)
Ping Pong ball	3.5	22
Marble	1.1	0.7
Billiard ball	4.5	46
Ball bearing	0.5	0.07
Toy ball	2.2	5.4

(a) Draw axes with suitable scales and plot these points.

(b) Interpolate points to complete a smooth curve.

(c) Describe the shape of the curve.

(d) Use the curve to find the volume displaced by a ball of diameter 3.0 cm.

(e) What diameter did a ball have, according to this graph, if the volume of water displaced was 30 mL?

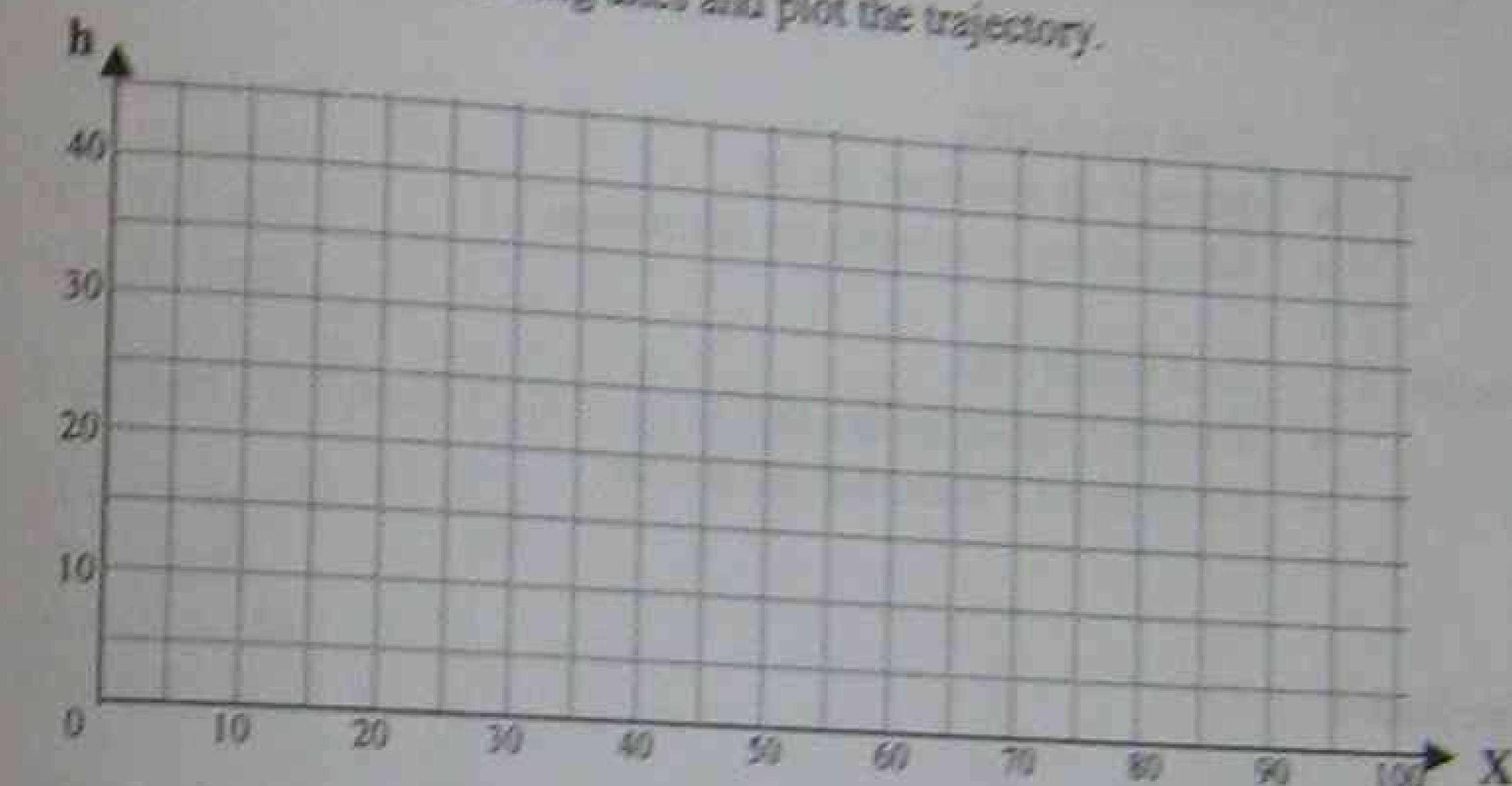
(f) Suggest an equation for the curve.

35 A cannonball is fired from a gun and travels on a trajectory with equation $h = x - \dfrac{x^2}{100}$ where h is the height above the horizontal plane on which the gun sits (i.e. ground level) and x is the distance in metres travelled horizontally.

(a) Complete this table of values.

x	0	10	20	40	60	80
h						

(b) Copy the following axes and plot the trajectory.



(c) What is the maximum height reached?

(d) What is the value of X when the cannonball hits the ground?

(e) The cannon is firing at the side of a hill, represented by the line $h = x - 60$. Draw the side of the hill on the previous graph.

(f) At what point would the cannonball hit the side of the hill?

36 Explain why the interpretation of the algebraic result would be faulty in each of the following cases.

(a) The cost of buying a given number of 50 cent stamps is plotted and the points joined to form a smooth curve.

(b) The value of shares is graphed for a week and a line of best fit is extended to find the expected value in two months time.

Sample Exam Paper 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 ½ hours
- Write using black or blue pen
- Calculators may be used
- A Formulae Sheet is provided at the back of this paper.

Section 1 Total marks **(22)**
- Attempt all questions 1 – 22
- Allow about 30 minutes for this section

Section 2 Total marks **(78)**
- Attempt Questions 23 – 28
- Allow about 2 hours for this section.

Section 1

Total marks (22)
Attempt Questions 1 – 22
Allow about 30 minutes for this section.

Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample ¼ of $320 is (A) $40 (B) $60 (C) $80 (D) $14

(A)○ (B)○ (C)● (D)○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ○ (C)● (D)○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

(A)● (B)● *Correct* (C)○ (D)○

1 A camel rider passes a signpost at 10:36 am which reads "Cairo 27 km". At 10:52 am on the same day, the rider passes a signpost that reads "Cairo 39 km".

If the rider did not change direction, the average speed of the camel between the signposts was

(A) 160 km/h
(B) 80 km/h
(C) 75 km/h
(D) 45 km/h

2 The lines y = 5 and y = –2(3 – x) intersect at the point

(A) (2, 5)
(B) (-2, 5)
(C) (5 ½ , 5)
(D) (½ , 5)

3 The probability of a 17 year old Australian male dying within 12 months is 0.00112. The probability of a 17 year old Australian female dying within 12 months is 0.00045. Which of the following statements is true?

(A) A 17 year old male is about 2 ½ times more likely to die in the next 12 months than a female of the same age.
(B) Out of every 100 000 Australian boys aged 17 years, 112 will definitely die within 12 months.
(C) Out of every 100 000 Australian girls aged 17 years, 45 will definitely die within 12 months.
(D) All of the statements above are true.

4 Vitebsk has position 55°N 30°E. Resistencia has position 25°S 60°W. When it is 6 am in Vitebsk, the time in Resistencia is

(A) midnight
(B) 4 am
(C) 8 am
(D) noon

5 If $5 400 were invested at 6% p.a. compound interest for 7 years, the amount of interest earned would be about

(A) $2720
(B) $8120
(C) $2268
(D) $7668

6 On the basis of readings taken over 140 years, the average rainfall for Sydney has been calculated to be 1218 mm per year. If 1500 mm fell this year, the new average would be

(A) 1218 mm
(B) 1220 mm
(C) 1228 mm
(D) 1359 mm

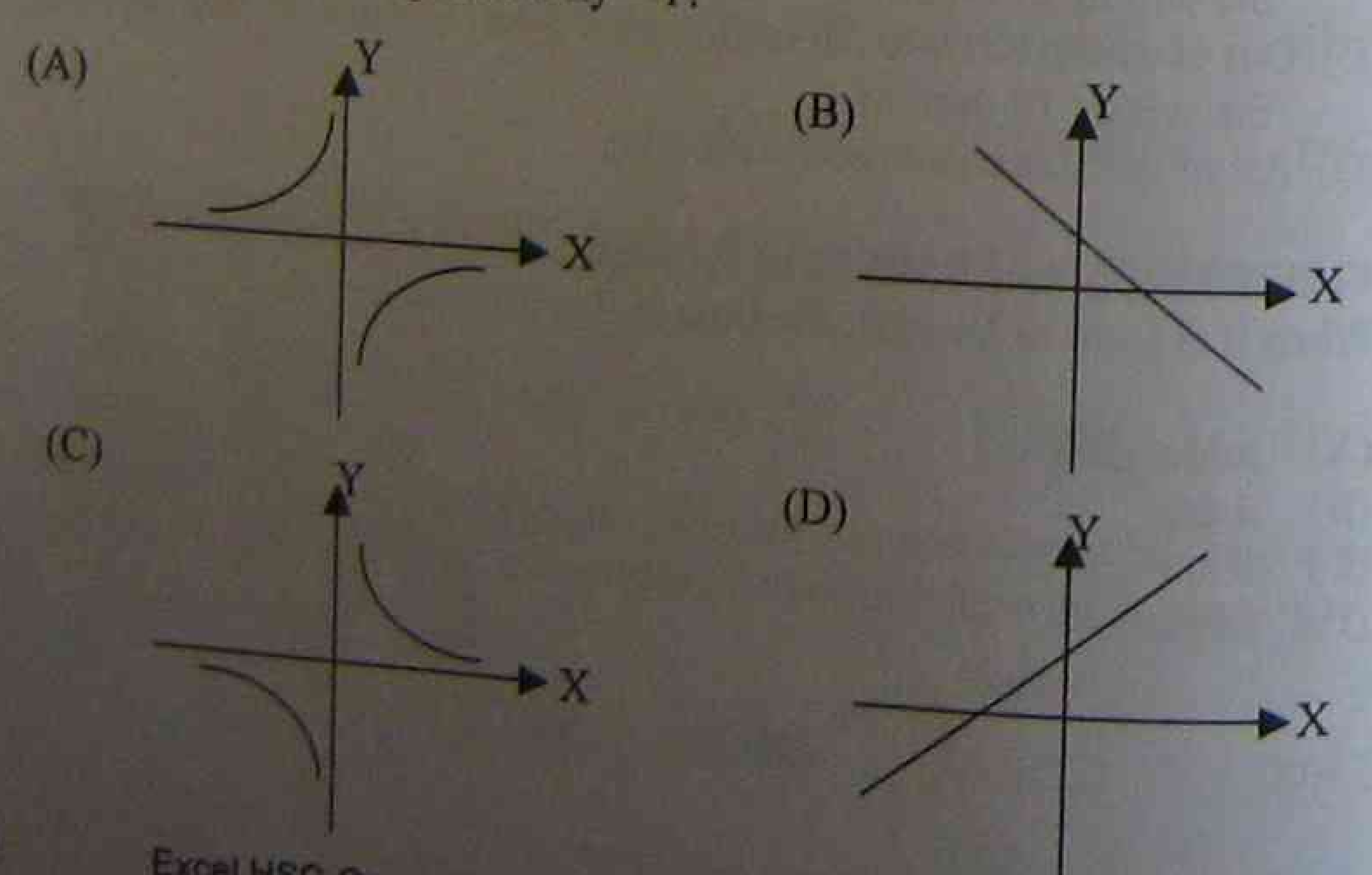
7 A 24 ha site is to have 330 homes built on it. The site area per house, on average, will be closest to a rectangular block of land with dimensions

(A) 13.75 m x 13.75 m
(B) 72.27 m x 72.27 m
(C) 36 m x 20 m
(D) 24 m x 330 m

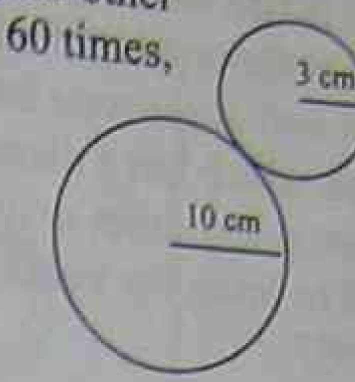
8 The exact value of $(8^{1/3} - 32^{1/5})^{3/4}$ is

(A) –102
(B) 0
(C) 102
(D) none of these

9 Which graph best represents xy = 1?



10 The two wheels are in contact, one driving the other without slipping. If the larger wheel turns 60 times, the smaller wheel turns



(A) 200 times.
(B) 200π times.
(C) 180 times.
(D) 180π times.

11 According to a road test report, a Holden Statesman holds 80L of petrol in its tank and uses 15.9 L /100km of highway driving. The maximum distance it could travel without refuelling is about

(A) $\dfrac{100}{80 \times 15.9}$ km

(B) $\dfrac{80 \times 100}{15.9}$ km

(C) $\dfrac{80 \times 15.9}{100}$ km

(D) $\dfrac{100 \times 15.9}{80}$ km

12 A nursery is producing small plants for sale. The mean height is 12.5 cm and the standard deviation is 2.5 cm. If a plant is shorter than 7.5 cm, it is rejected. What percentage of plants will be available for sale?

(A) 50%
(B) 95%
(C) 97.5%
(D) 99.7%

13 The volume of a cone is directly proportional to the square of the radius of its base. The volume is 283 cm^3 when the radius is 3 cm. If the height remains unchanged but the radius of the base increases to 6 cm, the new volume will be closest to

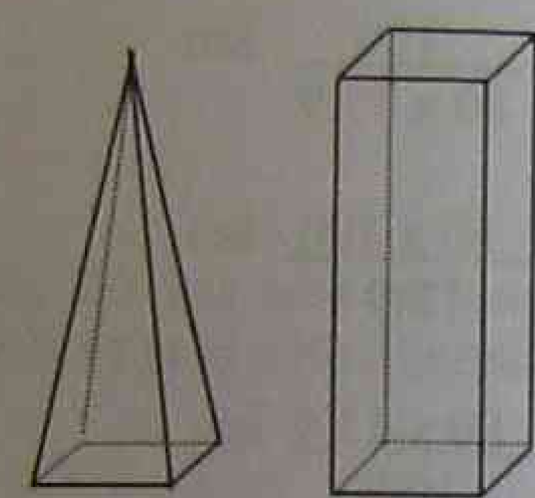
(A) 314 cm^3
(B) 566 cm^3
(C) 1132 cm^3
(D) 2264 cm^3

14 A bucket is tied to the end of a rope and rests at the bottom of a 20 m well. The other end of the rope is attached to a winch which has a diameter of 7 cm. How many times must the winch be turned to bring the bucket to the top of the well?

(A) 86
(B) 91
(C) 286
(D) 898

15 These two solids have exactly the same perpendicular height and base area.

The ratio of the volume of Solid I to that of Solid II is



Solid I Solid II

(A) 2 : 1
(B) 1 : 2
(C) 3 : 1
(D) 1 : 3

16 The results are illustrated for two tests, Test X and Test Y.

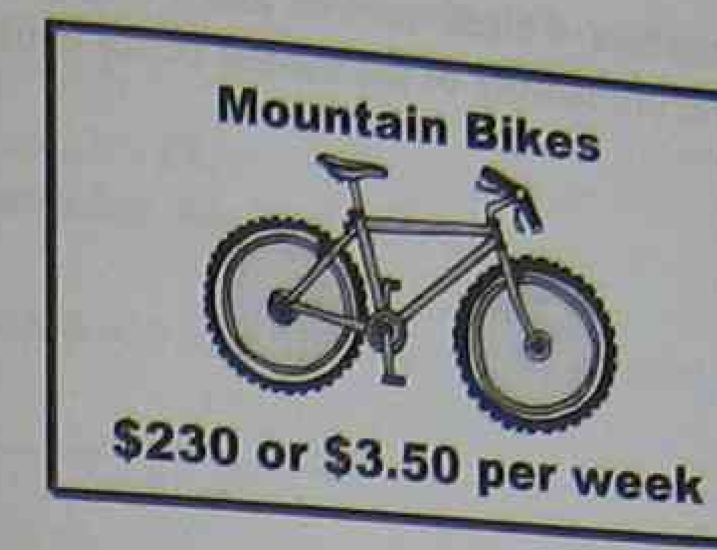
Test X

Test Y

Which of these statements is true?
Statement 1: The range of Test Y is greater than the range of Test X.
Statement 2: The median of Test Y is greater than the median of Test X.

(A) 1 only
(B) 2 only
(C) Both 1 and 2
(D) Neither 1 nor 2.

17



The shop charges 15% extra for time payment. If someone bought the bike at $3.50 per week,

(A) they would own the bike after about 66 weeks.
(B) they would probably be asked to make more than 85 payments.
(C) they would have to pay a total of $245 for the bike.
(D) they would own the bike in about a year if they paid $82 deposit.

18 The future value (FV) of an item is found using the formula

$$FV = PV\,(1 + r)^n$$

where PV is the present value, r is the percentage growth rate per annum and n in the number of years.
Therefore, if a house selling for $400 000 appreciates at 3% p.a. for 10 years, its value then will be about

(A) $537 500
(B) $5 514 300
(C) $520 000
(D) $1 048 500

19 A levy of 1.25% of taxable income is charged for medical cover. On a taxable income of $65 780, the levy would be

(A) $82.23
(B) $526.24
(C) $822.25
(D) $8222.50

20 A bag contains 5 red marbles, 4 black marbles and 3 white marbles. A marble is drawn at random. The probability of the marble being white is

(A) $\frac{1}{3}$

(B) $\frac{3}{10}$

(C) $\frac{3}{4}$

(D) $\frac{1}{4}$

21 $3a + 3ab^2$ can also be written as

(A) $3a(b^2 + 1)$
(B) $9ab^2$
(C) $6a^2b^2$
(D) $6ab^2$

22 The chance of having the winning ticket in a raffle is 1 in 4000. The only prize is \$10 000 and tickets are \$5 each. The expected financial return as a percentage of the "investment" of buying one ticket is

(A) 20%
(B) 2%
(C) 50%
(D) 5%

End of Section 1.

Section II

Total marks (78)
Attempt Questions 23 – 28
Allow about 2 hours for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 23 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The table shows the repayments each fortnight to repay a loan in a certain time. The total interest charged is also shown.

TOTAL AMOUNT \$	12 Months \$	Total Interest \$	18 Months \$	Total Interest \$	24 Months \$	Total Interest \$	30 Months \$	Total Interest \$
Up to 1200	51	121						
1400	60	141						
1600	68	161						
1800	76	181	53	267				
2000	85	201	59	297				
2200	93	221	65	326				
2400	102	241	71	356	56	474		
2600	110	261	77	386	60	514		
2800	119	281	83	416	65	553	54	695
3000	127	301	89	445	69	593	58	745
3200	136	321	94	475	74	632	62	794
3400	144	342	100	505	79	672	66	844
3600	152	362	106	534	83	711	69	894
3800	161	382	112	564	88	751	73	943
4000	169	402	118	594	92	791	77	993
4200	178	422	124	623	97	830	81	1043
4400	185	442	130	653	102	870	85	1092
4600	195	462	136	683	106	909	89	1142
4800	203	482	141	712	111	949	92	1191
5000	212	502	147	742	115	989	95	1241
5200	220	522	153	772	120	1029	100	1291
5400	228	542	159	801	125	1068	104	1340
5600	237	563	165	831	129	1107	108	1390
5800	245	583	171	861	134	1146	112	1440
6000	254	603	177	890	138	1186	115	1489

James and Gail want to borrow \$4000 for a computer. The maximum that they can afford to repay is \$140 each fortnight.

(i) From the table, what is the highest instalment each fortnight that they can afford on a loan of \$4000? 1

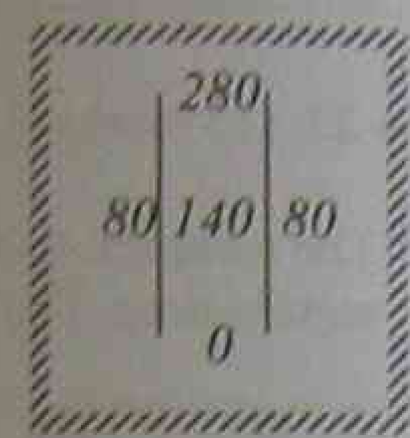
(ii) At this rate, how long would it take to repay the loan? 1

(iii) From the table, how much interest will they be required to pay? 1

(iv) If they had decided to repay the loan at \$77 each fortnight, how much extra interest would they have had to pay? 1

(v) Draw a graph of instalment against number of months and determine, giving reasons, how long it would take to pay off the loan if the couple paid \$140 per fortnight. 2

(b) A surveyor records the following measurements from an offset survey.



(i) What is the geometrical shape formed by the boundary? 1

(ii) What is the area of this field in hectares? 2

(iii) Calculate the perimeter of the field, correct to the nearest metre. 2

(c) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is the equation of an ellipse.

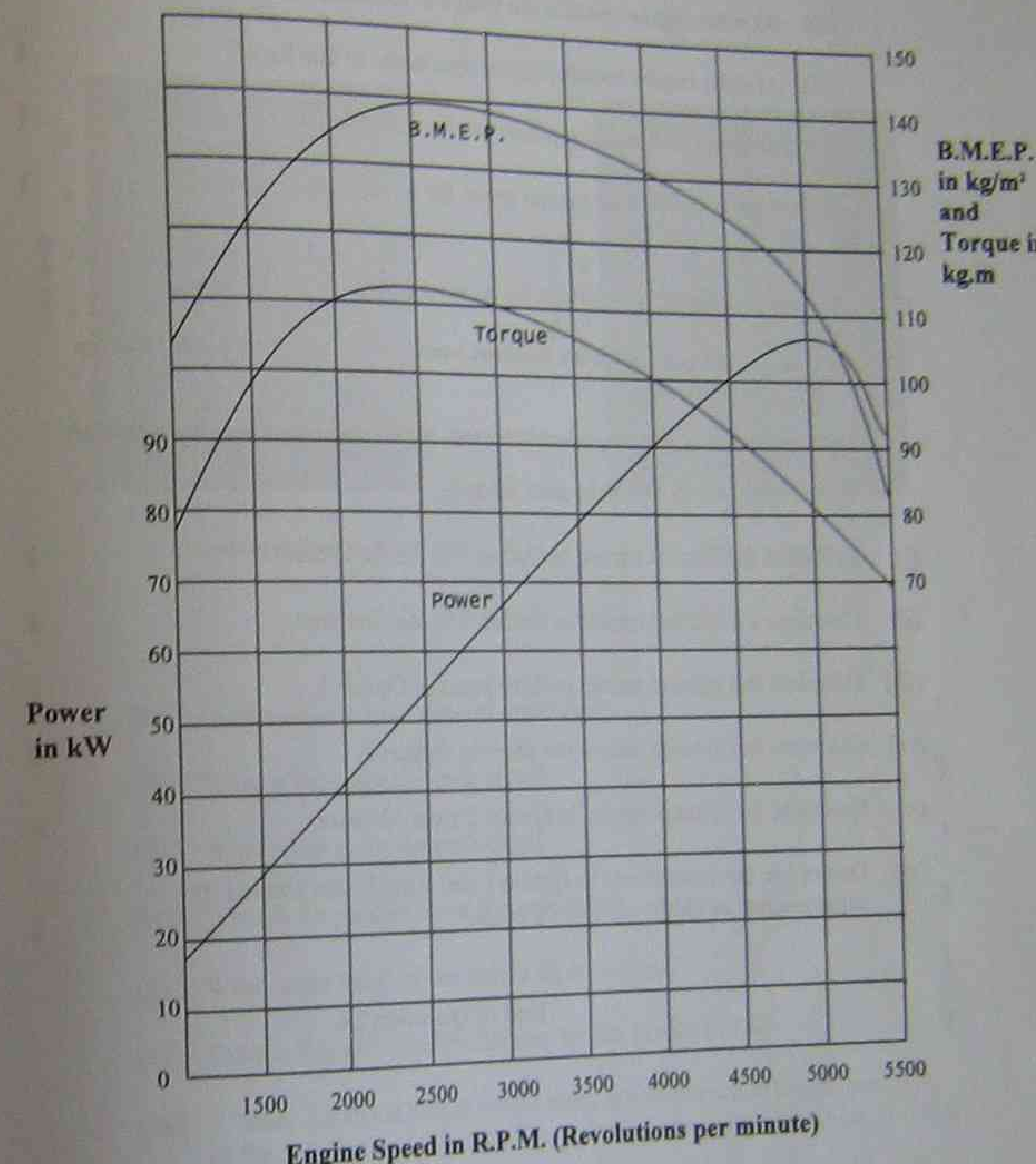
(i) Find the value of x when y = 0. 1

(ii) Calculate the area of the ellipse. 1

End of Question 23.

Question 24 (13 marks) Use a SEPARATE writing booklet.

(a) Refer to the graph below.



This graph comes from a dynamometer test of a car engine. The line labelled 'B.M.E.P.' represents the brake mean efficiency pressure. The other lines represent torque and power, as labelled.

Questions relating to the graph begin on the next page.

(i) What power is the engine producing when it is running at 3500 R.P.M.? 1

(ii) At what engine speed is the torque at its maximum? 1

(iii) At what engine speeds is the torque equal to 100 kg.m? 1

(iv) What is the maximum power produced by the engine? 1

(v) What is the slowest engine speed for which data is provided? 1

(b) A person is offered two options for an investment.

Option 1: 8% p.a. flat with interest sent to the investor each year by cheque.

Option 2: 6% p.a. compound interest, interest compounded annually.

The person invests $1000 in each scheme.

(i) Calculate the interest earned in Option 1 in the first year. 1

(ii) Calculate the interest earned in Option 2 in the first year. 1

(iii) Calculate the interest earned in three years in Option 1. 1

(iv) Calculate the balance after three years in Option 2. 2

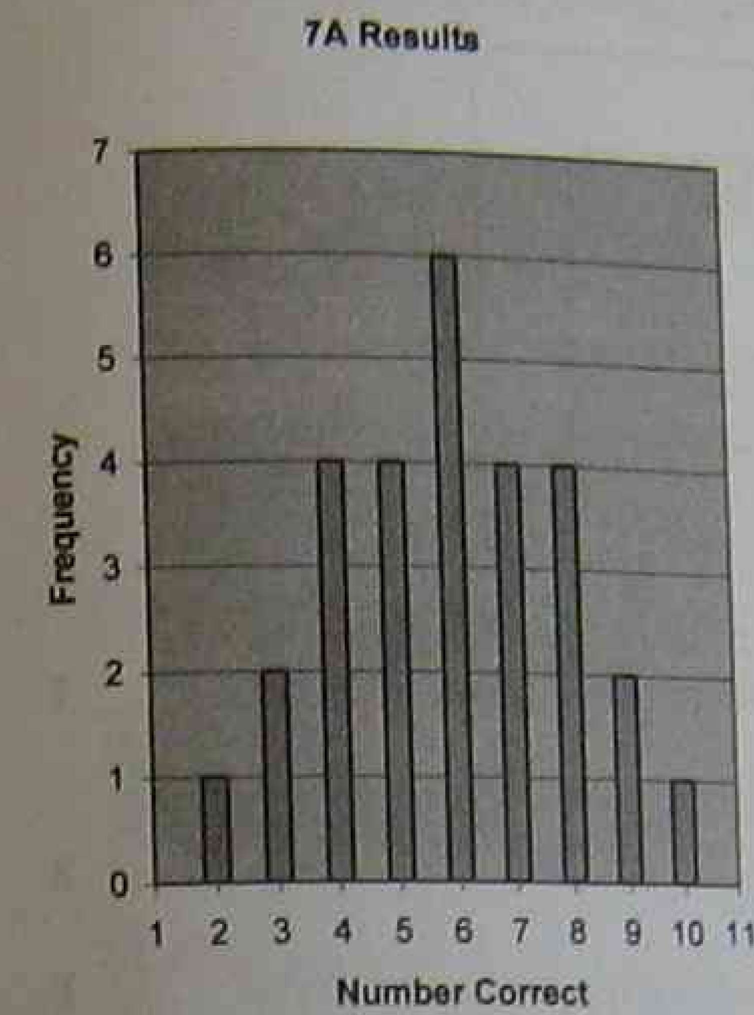
(v) Calculate the interest earned in Option 2 over 10 years. 1

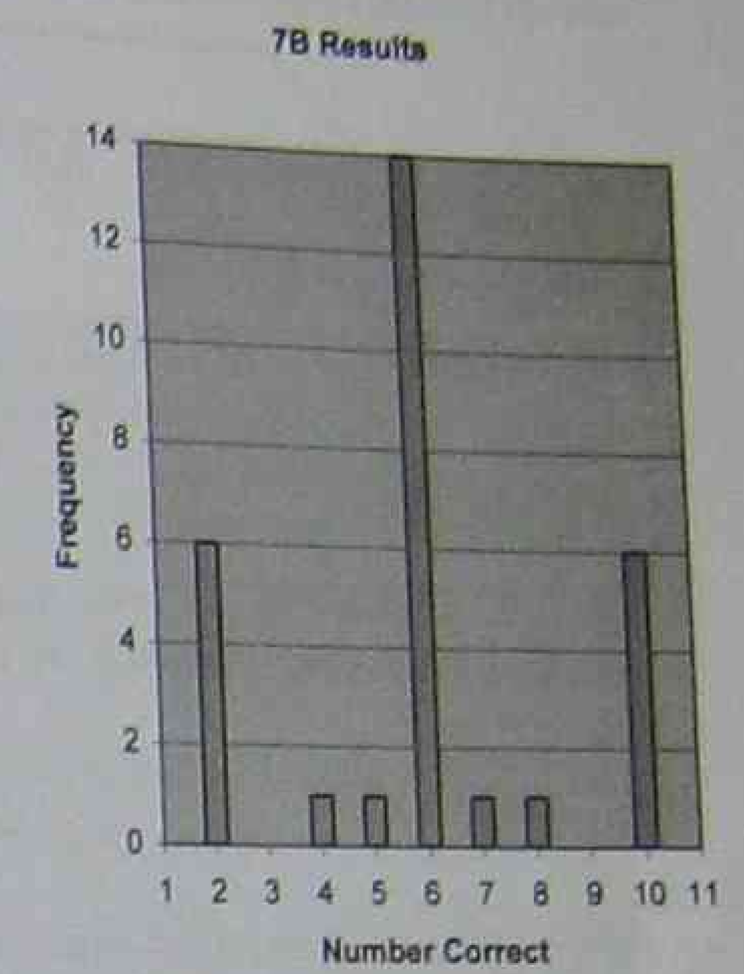
(vi) Determine the interest rate in Option 1 that would have yielded the same interest as Option 2 over 10 years. 2

End of Question 24.

Question 25 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Two classes were given a spelling test. The results were graphed as shown.





(i) How many students were in each class? 1

(ii) What is the mean for each class? 1

(iii) What is the mode for each class? 1

(iv) What is the median score for each class? 1

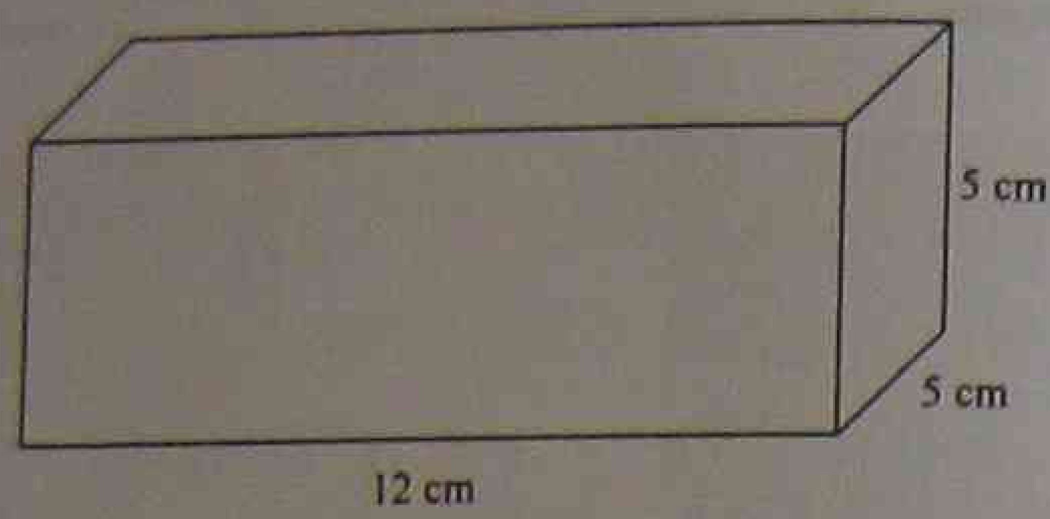
(v) What is the range of the scores for each class? 1

(vi) Draw a box and whisker diagram for the results in 7A. 2

(vii) Choose a statistic which would show the difference in the shapes of the distributions. Find the value of this statistic for each class. 2

Question 25 continues on the next page.

(b) The diagram shows a box in the shape of a rectangular prism with the top open.



(i) How many cubes of side length 2 cm could be packed into this box, assuming none protrude out of the box? 1

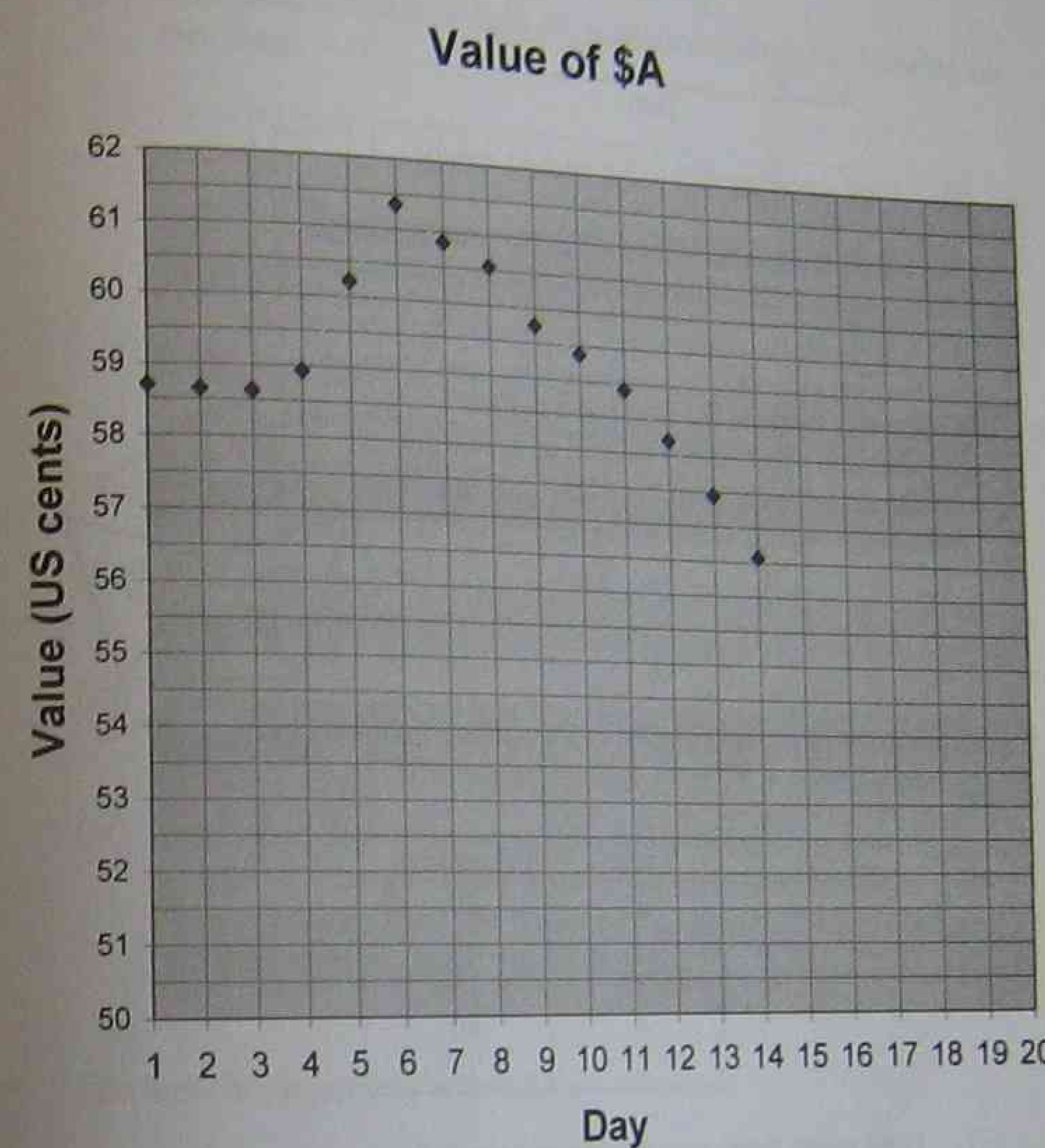
(ii) If the maximum number of cubes of side 5 cm were placed in the box, how much air space would be left? 2

(iii) If the box were filled with water, how many litres would it hold? 1

End of Question 25.

Question 26 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Gillian is planning a trip to America. She plotted the value of the Australian dollar at the close of trading each day for two weeks and recorded the results.



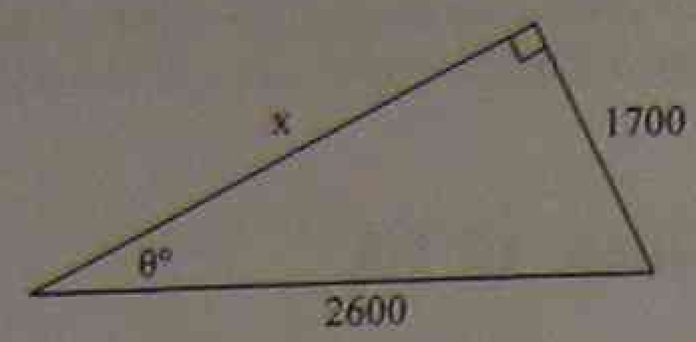
(i) What was the value of the Australian dollar when she started recording? 1

(ii) What was the average value of the $A for the first week? 1

(iii) What day would have been best for her to buy American dollar travellers' cheques? 1

(iv) Draw a line of best fit through the recordings taken from Day 6 to Day 14 and predict the value of $A1 if this downward trend had continued. 2

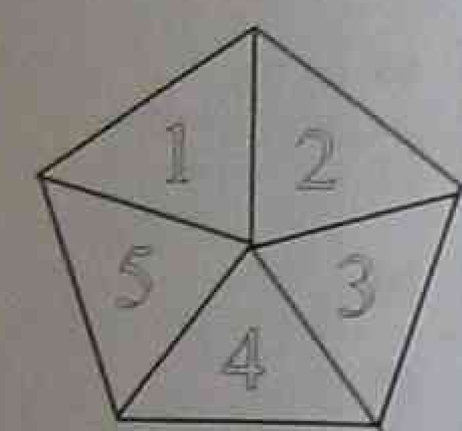
(b) A carpenter is given this diagram to represent a roof truss. To construct the truss, he needs to find the angle marked θ.



(i) Find the value of θ to the nearest minute. 2

(ii) Find the length of the side of the truss marked x. 2

(c) A spinner is made in the shape of a regular pentagon and numbers are painted of its sectors, as shown.



When spun the number closest to the surface (in the diagram, a '4') is recorded.

Event	Frequency
1	20
2	30
3	15
4	40
5	25

(i) How many times was the pentagon spun? 1

(ii) What is the experimental probability of spinning an even number? 1

(iii) Based on this experiment, out of 2000 spins, what would be the expected number of times that 3 would be recorded? Answer to the nearest whole number. 1

(iv) From the results of this experiment, can you conclude that the spinner is unbiased? Give reasons for your answer. 1

End of Question 26.

Question 27 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) Two dice are rolled and then a coin is tossed. The total on the dice is recorded, along with the result of the coin toss.

(i) Complete this table to show the possible outcomes for throwing the two dice. 1

Sum of faces	1	2	3	4	5	6
1			4			
2						
3						
4					9	
5						
6		8				

(ii) Find the probability of getting a total of 7 on the dice and a Head from the coin toss. 1

(iii) Find the probability that at least one of the dice shows 6 and the coin shows Tail. 1

(iv) A game is 'fair' if the expected return is equal to the cost of playing. A gambler charges \$1 per round in a game where the player wins if the coin shows Head and the sum of the dice is 3. What should the winner receive to make this a fair game? 1

(b) The line l has equation $y = 2x + 5$.

(i) Where does l cut the Y axis? 1

(ii) The line m intersects l at the point $(7, k)$. Find the value of k. 1

(iii) Another line has equation $ax + 4y = 17$. Make y the subject of the equation. 1

(iv) If the gradient of the line in (iii) is 2, find the value of a. 1

Question 27 continues on the next page.

(c) A young couple plan to buy a house and land package. The package that they have chosen is currently available for $240 000. The couple needs another $150 000.

(i) The couple decide to invest $28 000 at the beginning of each year for the next five years in a fixed deposit account which pays 8% p.a. interest, compounded annually. What will their annuity be worth at the end of the fifth year? **2**

(ii) Over the five years, inflation increases the price of the package by 2% p.a. What will the package cost by then? **2**

(iii) Show that they would have been able to afford the package if inflation had been 1.16% p.a. over the five years. **1**

End of Question 27.

Question 28 (13 marks) Use a SEPARATE writing booklet.

Marks

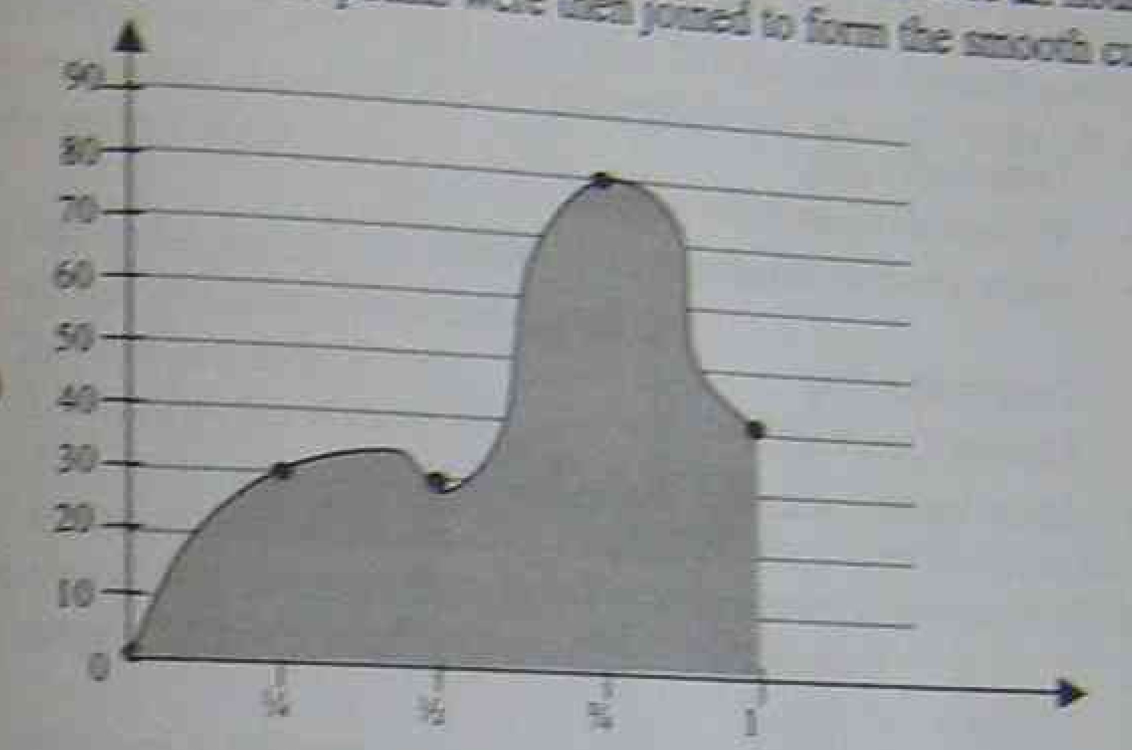
(a) The radius of a sphere was measured and recorded as 83 mm.

(i) Within what limits could the real radius lie? **1**

(ii) Within what limits would the volume lie? **2**

(b) The speed of a train was recorded every fifteen minutes for an hour and the results graphed. The points were then joined to form the smooth curve shown.

Speed (km/h)



Time (hours)

(i) What was the maximum speed recorded? **1**

(ii) During which 15 minutes period did the train's speed increase most? **1**

(iii) The area under the curve (shaded) represents the distance travelled by the train in the hour. Join the points plotted with straight lines and find the area of the polygon. **3**

(iv) Use Simpson's Rule (two applications) to obtain another approximation for the area. **2**

(c) A particle is travelling at a velocity v which varies inversely with time t. After 20 seconds, its velocity is 50 m/s.

(i) Find its velocity after 6 seconds. **1**

(ii) When is the velocity 25 m/s? **1**

(iii) What value will its velocity approach as time goes on? **1**

End of paper.

Sample Exam Paper 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 ½ hours
- Write using black or blue pen
- Calculators may be used
- A Formulae Sheet is provided at the back of this paper.

Section 1 Total marks **(22)**

- Attempt all questions 1 – 22
- Allow about 30 minutes for this section

Section 2 Total marks **(78)**

- Attempt Questions 23 – 28
- Allow about 2 hours for this section.

Section 1

Total marks (22)
Attempt Questions 1 – 22
Allow about 30 minutes for this section.

Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample ¼ of \$320 is (A) \$40 (B) \$60 (C) \$80 (D) \$14

(A) (B) (C) (D)

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) (B) (C) (D)

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

Correct

(A) (B) (C) (D)

1 Body Mass Index (BMI) is calculated using the formula

$$\boxed{\text{BMI} = \frac{\text{Mass in kg}}{(\text{Height in m})^2}}$$

The BMI of a 181 cm adult with a mass of 70 kg is

(A) 0.0021
(B) 2.137
(C) 21.37
(D) 1496

2 On average out of every 1000 live births, 513 are male. Assuming that there are few deaths, out of 400 million Chinese people under 20 years old, one would expect about

(A) 487 to be female.
(B) 10 million more males than females.
(C) 205 million to be female.
(D) 5.2 million more males than females.

3 Five rectangular shelves, each 900 mm x 300 mm, are to be cut in the most economical way from a sheet of plywood measuring 1200 mm x 1200 mm. The piece of plywood not used would have dimensions

(A) 900 mm x 100 mm
(B) 300 mm x 300 mm
(C) 1200 mm x 75 mm
(D) 1200 mm x 300 mm

4 A car travels at an average of 75 km/h for three hours. The distance covered is

(A) 25 km
(B) 75 km
(C) 215 km
(D) 225 km

5 Bulk freight is charged at \$138 per cubic metre. The price of sending a bale of wool measuring 180 cm x 180 cm x 120 cm would be

(A) \$536.50
(B) \$388.80
(C) \$38.90
(D) \$5

6 When $5(2a + 3b) - 6(4a - 7b)$ is expanded and simplified, the answer is

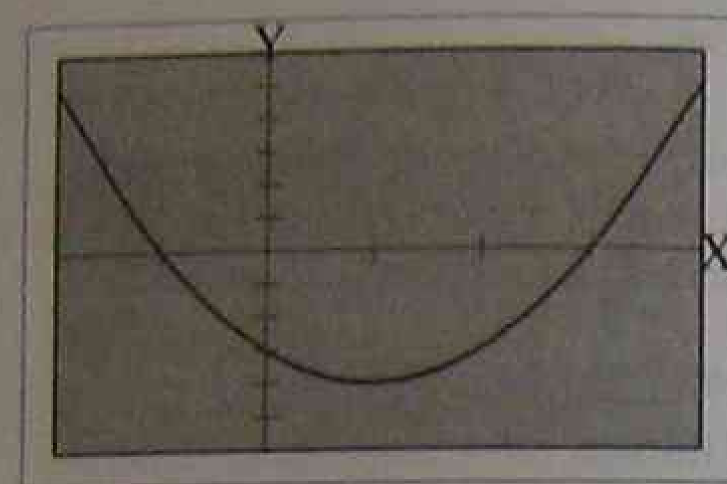
(A) $-14a - 10b$
(B) $57b - 14a$
(C) $16b - 14a$
(D) $-14a - 27b$

7 If $US 0.96 is worth $A 1.26, then

(A) $US 1.21 is worth $A 1
(B) $A 1 is worth $US 1.31
(C) $US 1 is worth $A 0.76
(D) $US 0.76 is worth $A 1

8 An equation which might generate this curve is

(A) $y = (x + 1)(x - 3)$
(B) $y = (4 - x)(4 + x)$
(C) $y = (x - 4)(x - 4)$
(D) $y = (x - 1)(x + 3)$



9 Ron opens a bank account that pays 8% p.a. compound interest, calculated every six months. If he invests $20 000, the balance after one year would be closest to

(A) $21 600
(B) $20 800
(C) $23 328
(D) $21 632

10 The intensity (I) of light varies inversely as the square of the distance (d) from the source. i.e.

$$I = k/_{d^2}$$

If the distance from the light source is changed from 18 m to 6 m, the intensity

(A) increases about 3 times.
(B) increases about 6 times.
(C) increases about 9 times.
(D) increases about 27 times.

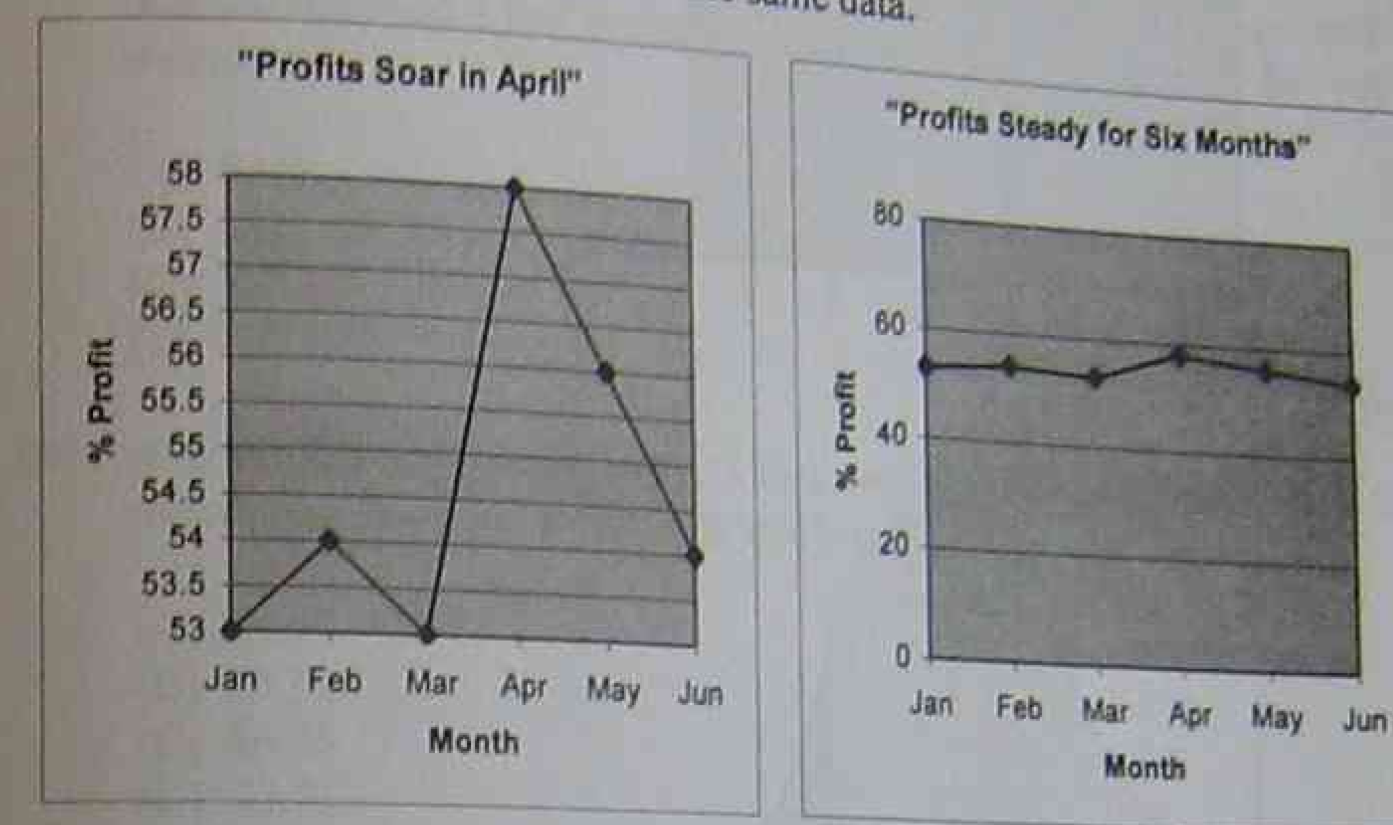
11 A ship sails for 207 M due east from Sydney, then turns north and sails another 128 M. Its bearing from Sydney would then be

(A) 032°
(B) 058°
(C) 052°
(D) 038°

12 The position of Sydney is 34°S 151°E. The position of Rabut is 34°N 7°W. In real time, (ignoring time zoning), Rabut is

(A) 10 hours 32 minutes behind Sydney
(B) 10 hours 32 minutes ahead of Sydney
(C) 9 hours 36 minutes behind Sydney
(D) 9 hours 36 minutes ahead of Sydney.

13 The two graphs below relate to the same data.



Which of the following statements is true?

(A) The graph on the left gives a more accurate representation of the data than the graph on the right.
(B) The graph on the right gives a more accurate representation of the data than the graph on the left.
(C) The graph on the left tends to exaggerate differences more than the graph on the right.
(D) The graph on the right uses the space available more effectively.

14 Five children call out their ages: 4, 7, 3, 8, 8. The median age is

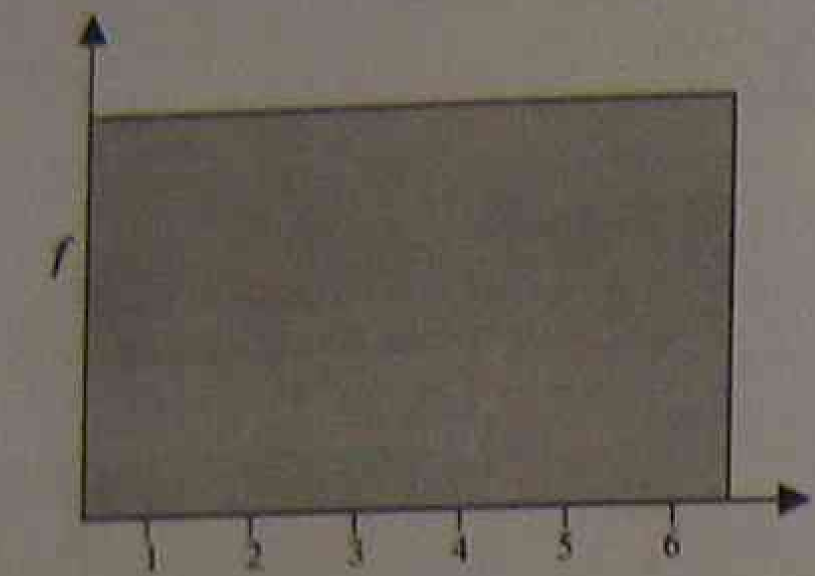
(A) 3
(B) 5
(C) 8
(D) 7

15 The results of the HSC and the teacher's estimates of what the students would score have a correlation coefficient equal to 0.93. This means that

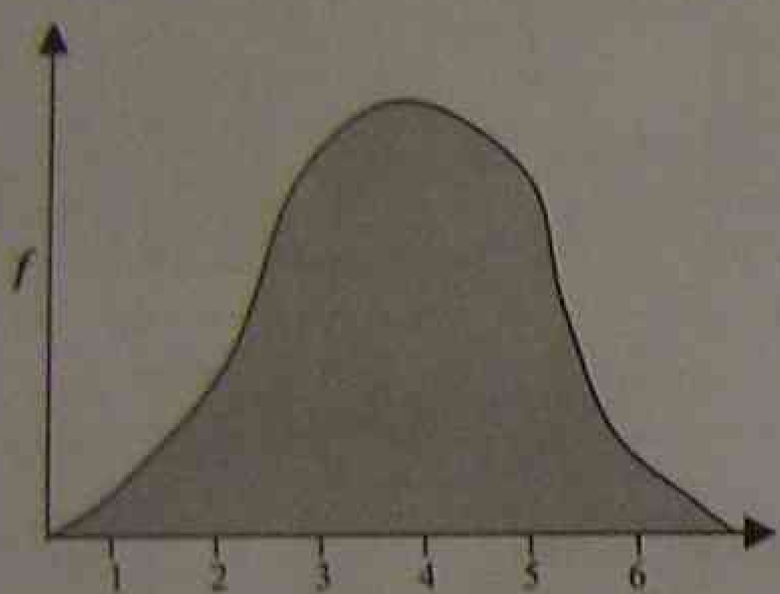
(A) 93% of the students in the class scored what the teacher predicted.
(B) the gradient of the line of regression in 0.93.
(C) the points on a scatter graph would lie close to a line with positive gradient.
(D) 7% of the students scored less than the mark predicted by the teacher.

16 A standard cubic die is rolled a large number of times and the number on the uppermost face recorded. The shape of the frequency distribution would most likely be closest to which of these?

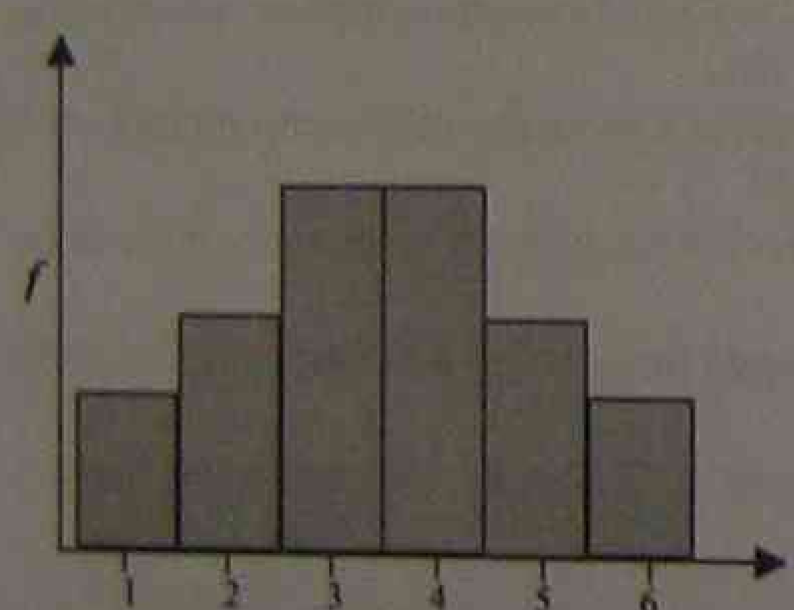
(A)



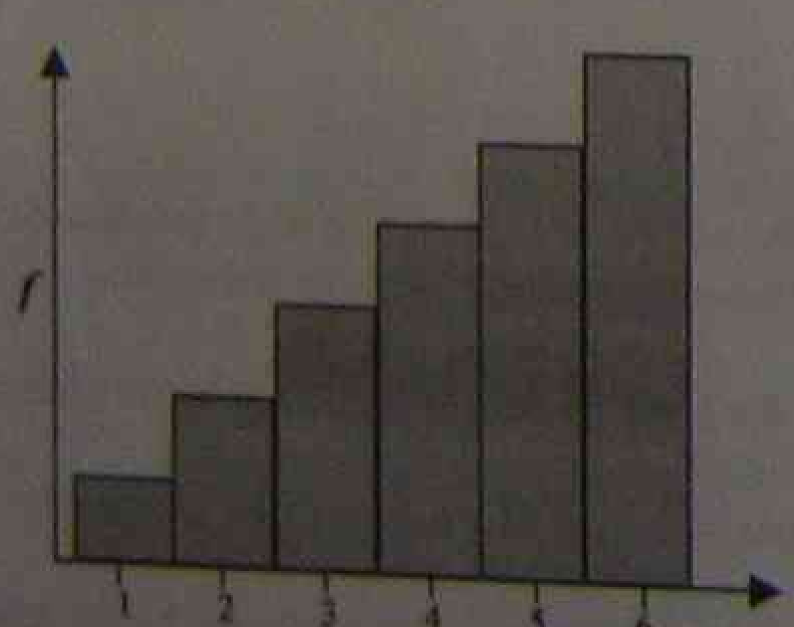
(B)



(C)



(D)



17 A house purchased in January 1987 for \$240 000 sold in January 2000 for \$400 000. The rate of compound appreciation that operated per annum was

(A) 66.6%
(B) 12.8 %
(C) 4%
(D) 1.6%

18 A person working for \$5.80 per hour works 38 hours at normal time and 2 hours at time-and-a-half during the same week. Their total pay for the week would be

(A) \$348
(B) \$237.80
(C) \$232
(D) \$220.40

19 A sandwich hand receives \$61.75 for an eight hour shift followed by an hour overtime at time-and-a-half. The hourly pay rate at normal time is

(A) \$4.57
(B) \$6.50
(C) \$6.86
(D) \$7.71

20 A computer loses 20% of its previous year's value each year. If it originally cost \$2000, after 5 years it would be worth

(A) nothing
(B) \$1600
(C) \$819
(D) \$655

21 In a normal distribution with mean 60 and standard deviation 12, a score of 48 would have a z-score of

(A) −1.0
(B) 1.0
(C) 12
(D) −12

22 A person paying \$200 deposit and \$20 per week for 52 weeks is buying an item available for cash for \$900. The effective rate of simple interest is about

(A) 24% p.a.
(B) 38% p.a.
(C) 49% p.a.
(D) 138% p.a.

End of Section 1.

Section II

Total marks (78)
Attempt Questions 23 – 28
Allow about 2 hours for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 23 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use this Loan Repayments Table to answer the questions that follow.

LOAN REPAYMENTS TABLE
Figures in table show the monthly instalments to be paid on $1000 loan.

Years	5%	6%	7%	8%	9%	10%
1	85.61	86.07	86.53	86.99	87.45	87.92
2	43.87	44.32	44.77	45.23	45.68	46.14
3	29.97	30.42	30.88	31.34	31.80	32.27
4	23.03	23.49	23.95	24.41	24.89	25.36
5	18.87	19.33	19.80	20.28	20.76	21.25
6	16.10	16.57	17.05	17.53	18.03	18.53
7	14.13	14.61	15.09	15.59	16.09	16.60
8	12.66	13.14	13.63	14.14	14.65	15.17
9	11.52	12.01	12.51	13.02	13.54	14.08
10	10.61	11.10	11.61	12.13	12.67	13.22

(i) How much per $1000 borrowed would have to be paid in monthly instalments to repay a loan in 10 years if the interest rate were 7% p.a.? **1**

(ii) A home buyer can afford to pay $900 per month in home loan repayments. How much could they borrow at 7% p.a. interest if they want to repay the loan in 10 years? **1**

(iii) A home buyer has a loan of $50 000 at 10% p.a. and is repaying the loan at a rate which will clear the debt in 10 years. How much faster would the loan be repaid with the same instalments if interest rates fell to 6% p.a.? **2**

(iv) Another couple are paying $1365.30 per month on a $90 000 loan for which the interest is 10% p.a. How much would they have to pay per month to repay the loan in half the time? **2**

Question 23 continues on the next page.

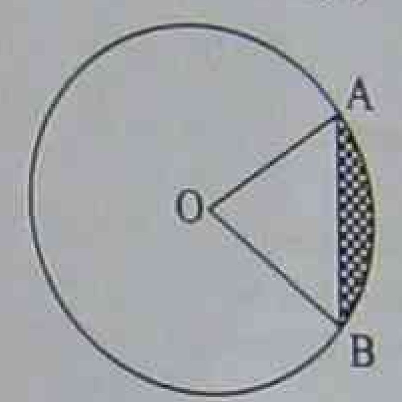
(b) Marbles numbered 1 – 1000 are placed in a bag.

(i) How many of the marbles in the bag have a number which includes the digit '4'? **2**

(ii) If a marble is drawn,what is the probability that the sum of its digits is 2? **1**

(iii) What is the probability that the number drawn will have two digits? **1**

(c) The radius of the circle is 8 cm and ∠ AOB = 80°.



(i) Find the area of the sector AOB, correct to the nearest cm². **1**

(ii) Find the area of ΔAOB, correct to the nearest cm². **1**

(iii) Find the area of the segment shaded, correct to the nearest cm². **1**

End of Question 23.

Question 24 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) $120 000 is to be invested in this account.

BUILD-UP ACCOUNT

$1 - $1999	7.75% p.a.
$2000 - $9999	9.75% p.a.
$10 000 - $49 999	10.25% p.a.
$50 000 and over	11.25% p.a.

Interest calculated on the minimum monthly balance and paid monthly.

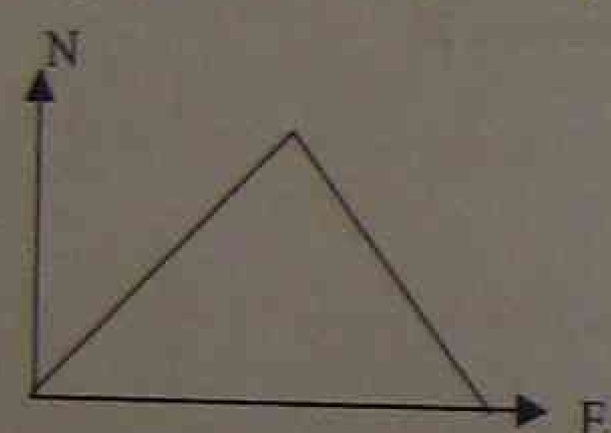
BOSBANK

(i) Calculate the interest to be paid for one month. **1**

(ii) Interest is paid by cheque to the investor each month. The investor does not reinvest the cheque.
How much would the investment plus interest be worth after one year? **2**

(b) Truong and Bao start walking from the same place but Truong walks on a bearing of 050° at 5 km/h. Bao walks east at 3.5 km/h.

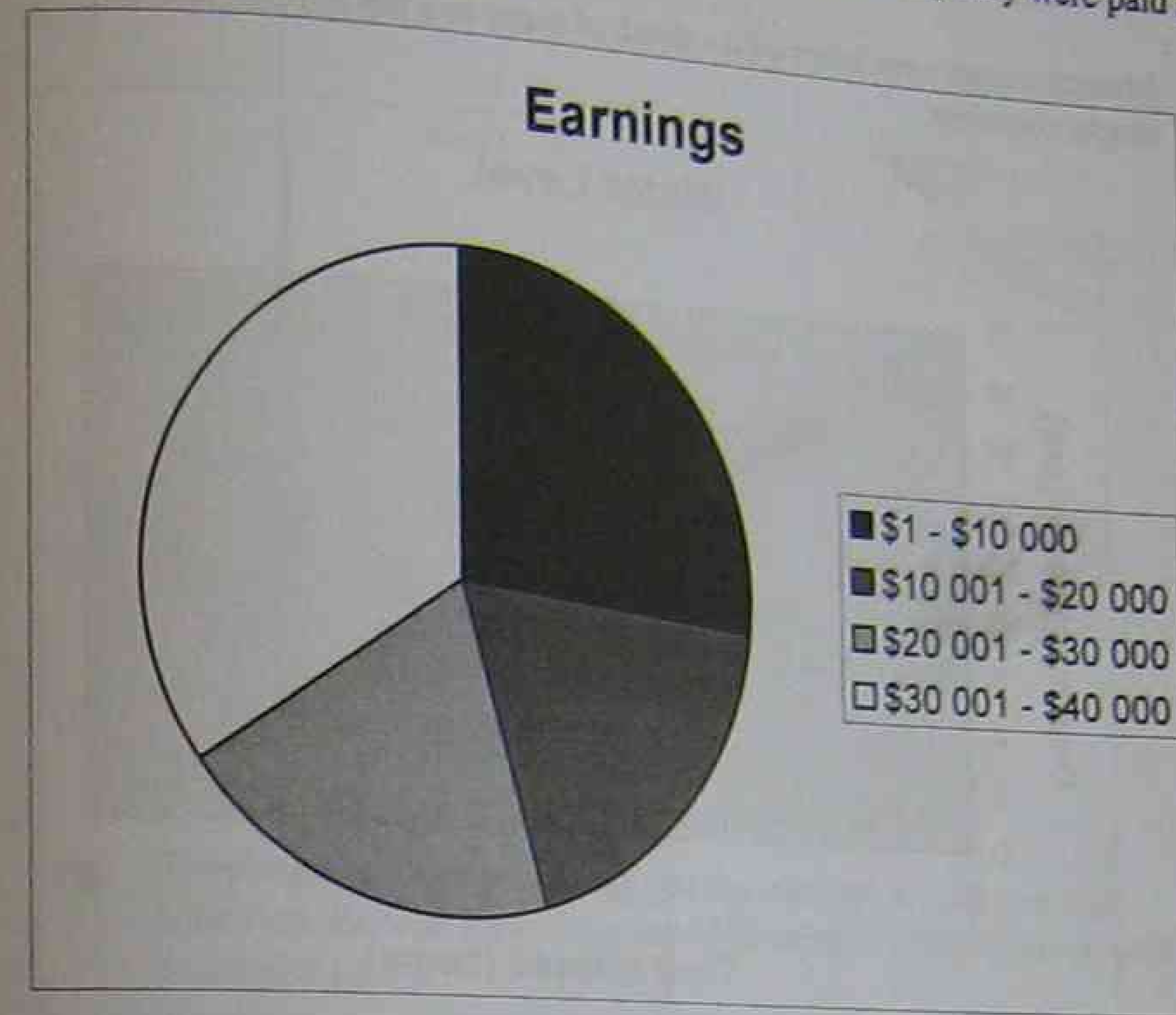
(i) Copy this diagram. Show the angle given and the paths which correspond to the names and the people. **1**



(ii) How far apart are Truong and Bao after 3 hours? **2**

Question 24 continues on the next page.

(c) A research company asked casual workers how much money they were paid in a typical week of employment.



The results for a sample of 180 of these workers are shown in this sector graph.

(i) What percentage of people surveyed were paid $10 000 or less? **1**

(ii) A person is chosen randomly from this sample. What is the probability that this person is in the $30 001 – $40 000 group? **1**

(iii) Copy and complete this frequency table. **2**

Earnings	Frequency	Class Centre
$1 - $10 000		$5 000.50
$10 001 - $20 000		
$20 001 - $30 000		
$30 001 - $40 000		

(iv) Calculate the mean amount of money paid to each person in the sample. **1**

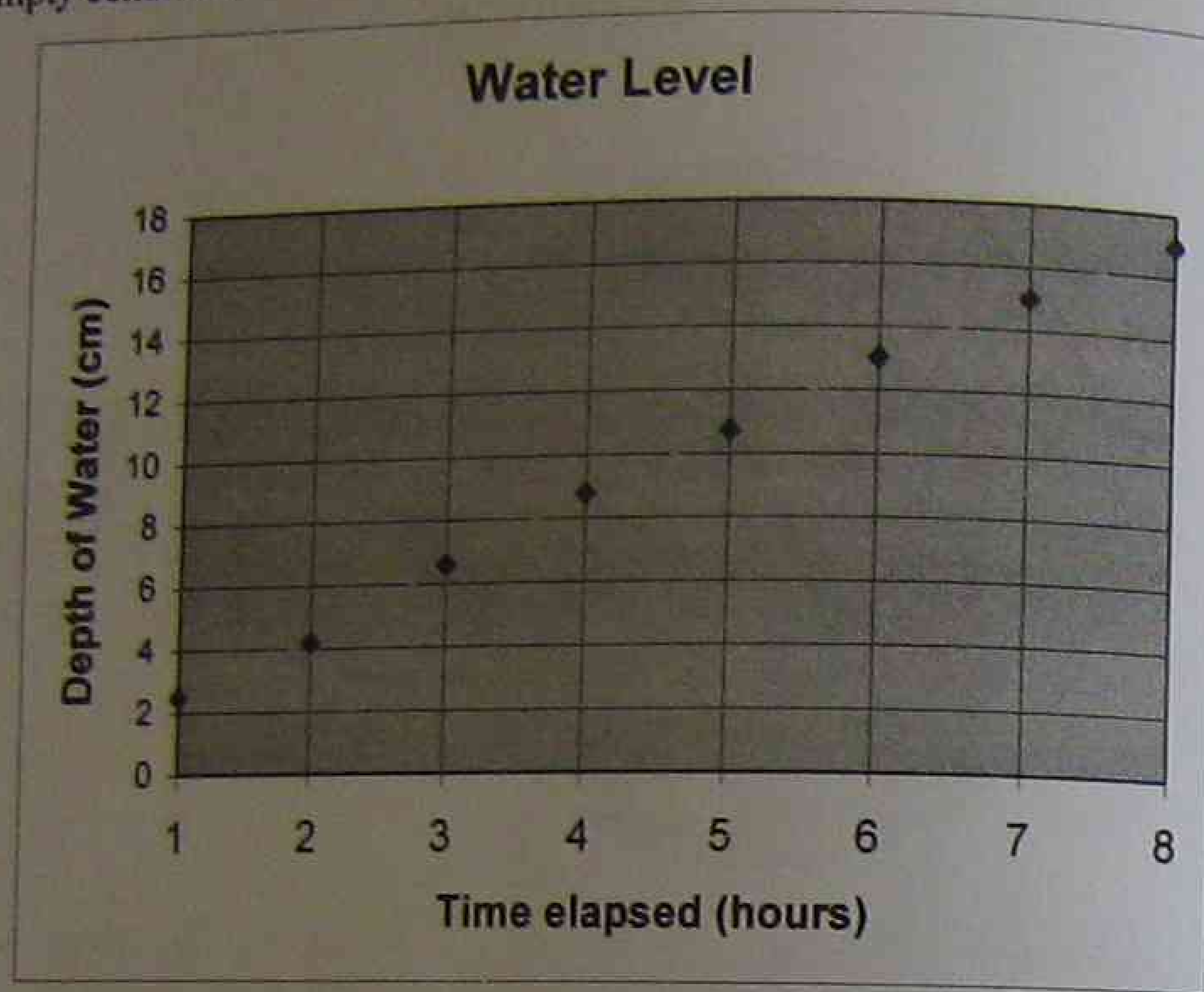
(v) Discuss the accuracy of the mean calculated in part (iv). **1**

(vi) Use the information in the table in part (iii) to calculate the standard deviation for the sample to 2 decimal places. **1**

End of Question 24.

Question 25 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Measurements were taken of the depth of water as a tap dripped into an empty container.



(i) What depth of water was in the container when the first reading was taken? 1

(ii) Draw the line of best fit through the points. 1

(iii) What is the gradient of the line of best fit? 1

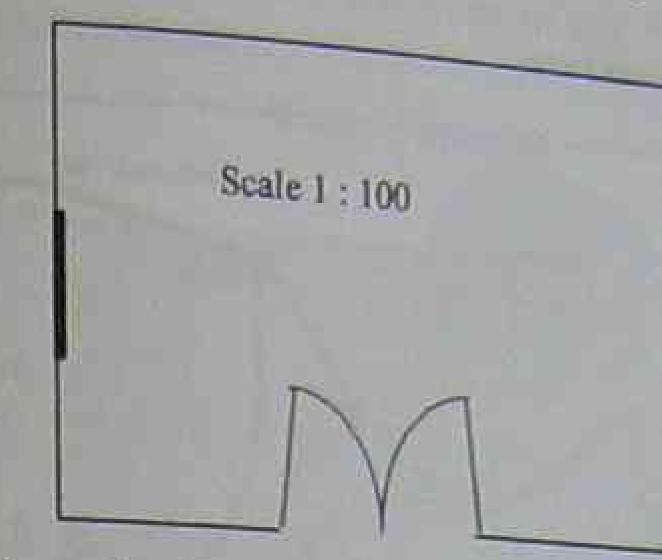
(iv) Write a possible equation of the line of best fit. 1

(v) What depth of water, according to your line of best fit, would be in the container after 20 hours? 1

(vi) Give the name of a possible geometrical shape for the container. 1

Question 25 continues on the next page.

(b) The following scale drawing shows the floor plan of a room with a 2900 mm ceiling.



(i) What are the dimensions of the room? 1

(ii) How much would it cost to carpet this room in 4 metre wide carpet which costs $285 per lineal metre? 1

(iii) Skirting boards are to be fitted (not across the doorway). Write down the lengths of the pieces required. 1

(iv) The doorway is 2000 mm high and the window is 1200 mm high. Calculate the total area which would be painted. (This includes walls and ceiling.) 2

(c) Four people go to the theatre together and buy four tickets which correspond to four seats in the same row.

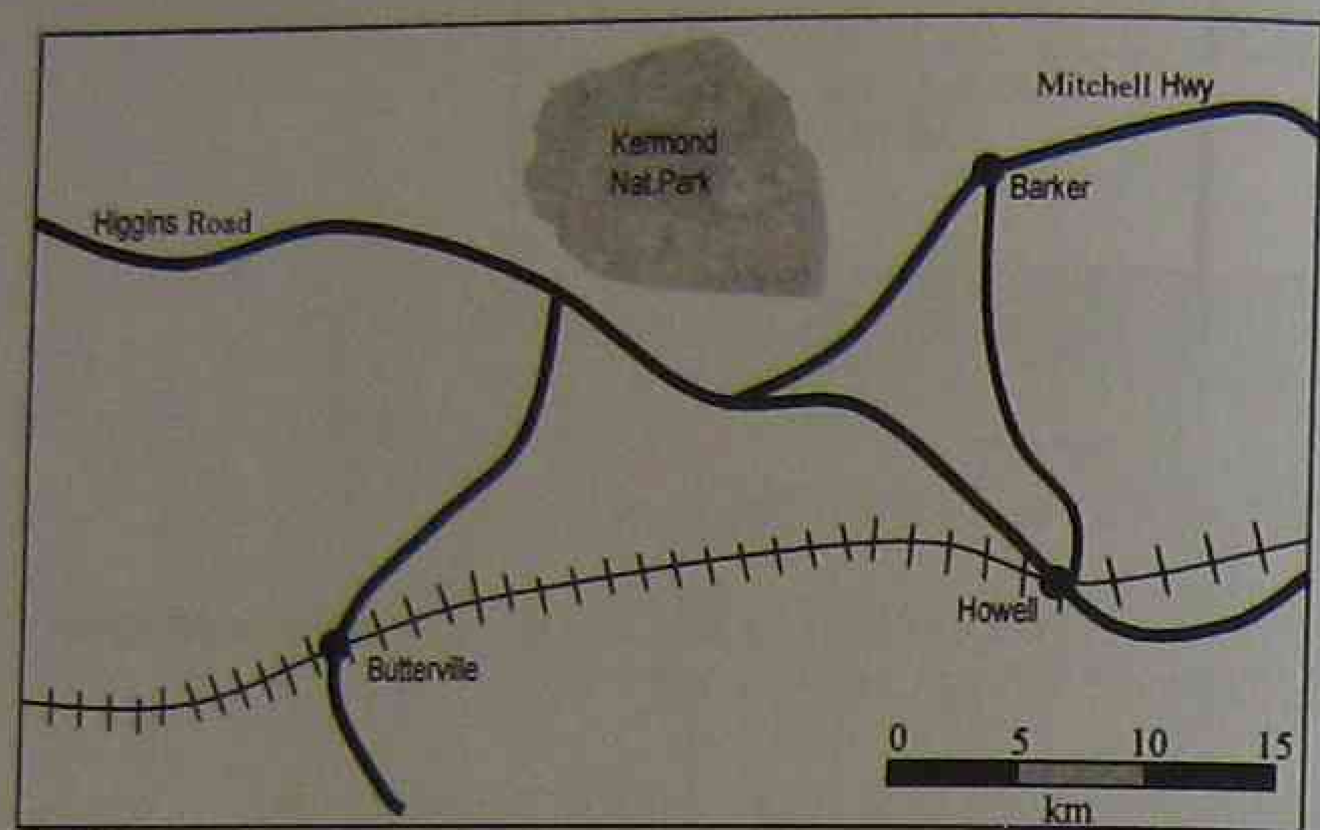
(i) How many different seating arrangements are there of the four people? 1

(ii) If two people want to sit together, how many possible seating arrangements are there that will allow them to do so? 1

End of Question 25.

Question 26 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) Answer the questions about this map.



(i) How many kilometres does 1 cm represent on this map? **1**

(ii) Write the scale of the map in the form 1 : *x*. **1**

(iii) What is the shortest distance between the towns of Butterville and Howell? **1**

(iv) What area is covered by the map? **1**

(b) A cannonball is fired and achieves a height of h metres after t seconds, according to the equation

$$h = 30t - 5t^2$$

(i) Copy and complete this table, showing the height of the cannonball at different times in its flight. **2**

t	0	1	2	3	4	5	6
h							

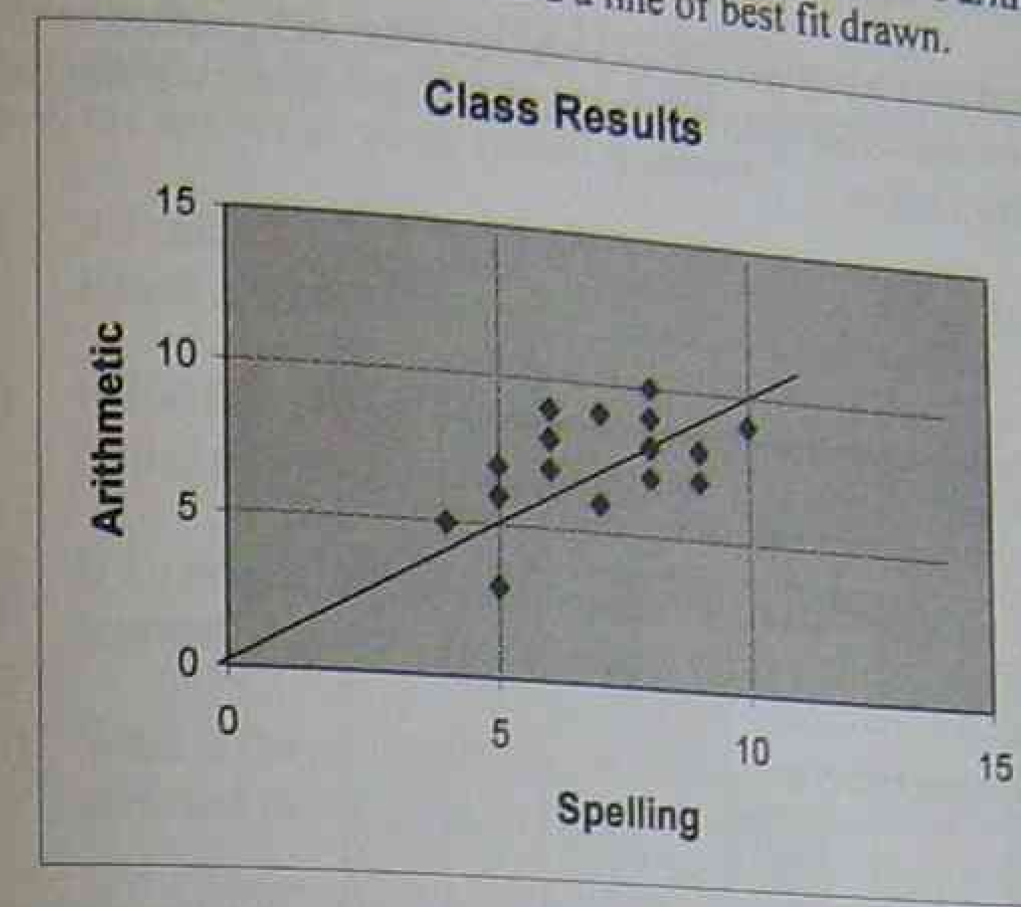
(ii) On grid paper, draw a graph to show the height of the cannonball during the first 6 seconds. **2**

(iii) What is the maximum height reached? **1**

(iv) After how many seconds does the cannonball hit the ground? **1**

Question 26 continues on the next page.

(c) Results for a class in two tests, one in spelling and one in arithmetic, were plotted on a graph, as shown, and a line of best fit drawn.



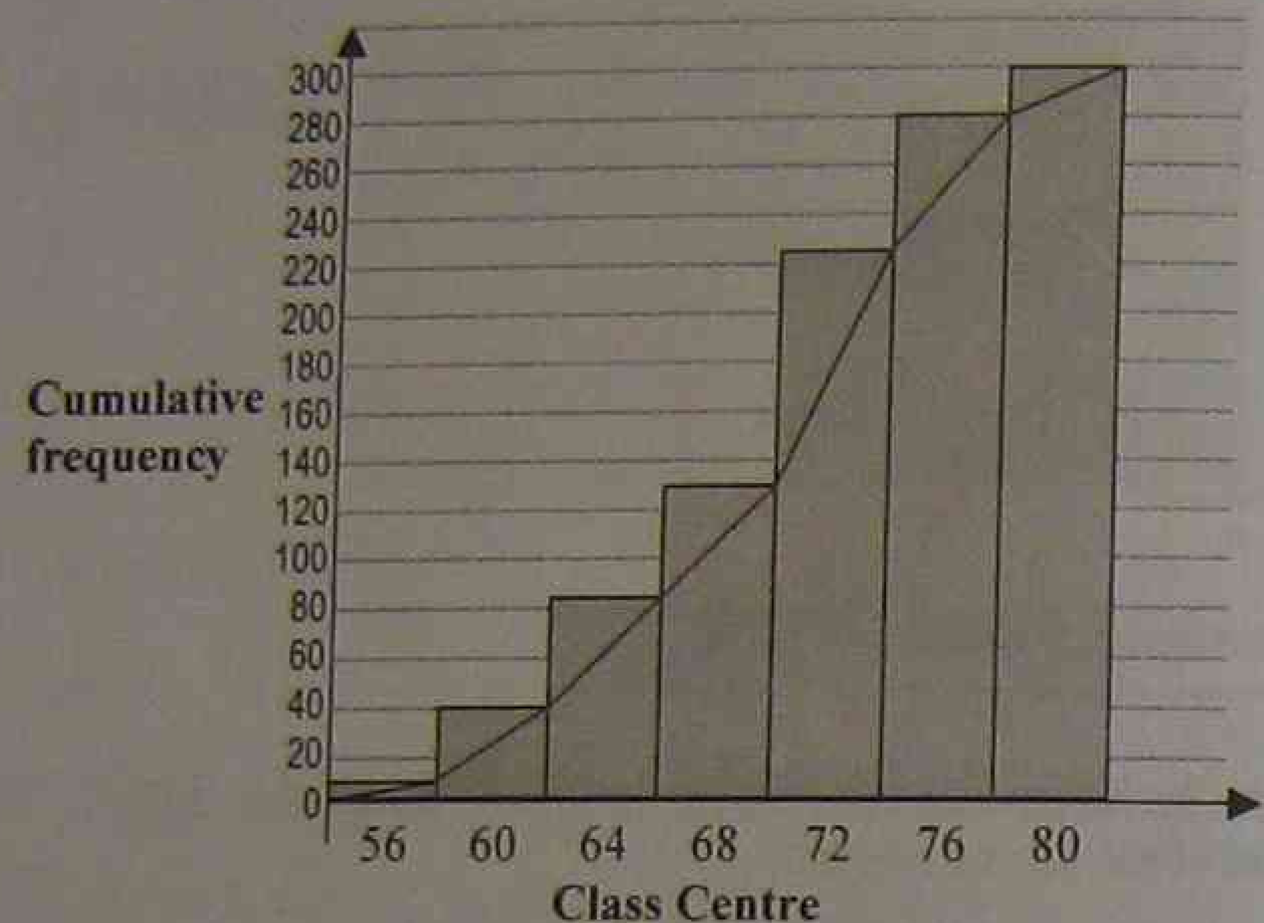
(i) If s is the spelling score and m is the arithmetic score, what is the equation of the line of best fit? **1**

(ii) Describe the extent of correlation between the variables. **2**

End of Question 26.

Question 27 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The speed in km/h of cars on a main road was recorded and graphed with class centres 56, 60, 64 and so on up to 80. The cumulative frequency histogram of the results is shown below.



(i) How many cars passed the checkpoint during the period? 1

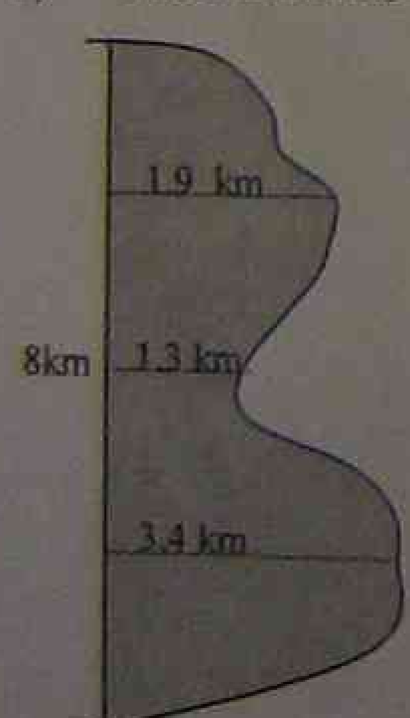
(ii) What is the greatest possible speed that was recorded? 1

(iii) Use the graph to find the median speed. 1

(iv) Determine the interquartile range. 2

(v) What score is on the 4^{th} decile? 1

b) Measurements were taken 2 km apart across the area of a community.



(i) Use Simpson's Rule to approximate the area of this community to the nearest km². 3

(ii) Find the area in hectares. 1

(iii) If 1253 people lived in the community in 2000, what was the population density per km²? 1

(iv) If the population increases at 5% p.a., use the formula $A = P(1 + r)^n$ to estimate the population in 2020. 1

(v) Draw a graph showing this increasing population for the years 2000 to 2020 in 4 year intervals. 1

End of Question 27.

Question 28 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) A disease is known to occur in a 1 in 100 000 head of population. A screening test for the disease has an 80% accuracy rate, i.e. in 80% of cases, the answer given by the test is correct.

(i) If 1 million people take the test, how many could be expected to actually have the disease? 1

(ii) If 100 000 without the disease take the test, how many will get a (false) positive result? 1

(iii) If 10 people with the disease take the test, how many will get a (correct) positive result? 1

(iv) What is the probability of a person actually having the disease if their test result is positive? 2

(b) Until recently, the following table applied for personal income tax for Australian residents.

Taxable income	Tax on this income
\$1 - \$5400	Nil
\$5401 - \$20 700	20 cents for each \$1 over \$5400
\$20 701 - \$38 000	\$3060 + 34 cents for each \$1 over \$20 700
\$38 001 - \$50 000	\$8942 + 43 cents for each \$1 over \$38 000
\$50 001 and over	\$14 102 + 47 cents for each \$1 over \$50 000

(i) Find the tax payable by a person with a taxable income of \$45 000. 2

(ii) A person with a taxable income of \$20 500 is concerned that a wage increase will put them into a higher 'tax bracket'. If their new take-home pay total \$20 800, how much extra will they pay in tax? 2

(iii) A person brings \$52 140 after tax for working 240 days in a year. If the person works 8 hours per day, what is their hourly rate of pay? 2

(iv) If a person has a before-tax income of \$50 000, what percentage of their taxable income will they pay in income tax? 2

End of paper.

Sample Exam Paper 3

General Instructions

- Reading time – 5 minutes
- Working time – 2 ½ hours
- Write using black or blue pen
- Calculators may be used
- A Formulae Sheet is provided at the back of this paper.

Section 1 Total marks **(22)**

- Attempt all questions 1 – 22
- Allow about 30 minutes for this section

Section 2 Total marks **(78)**

- Attempt Questions 23 – 28
- Allow about 2 hours for this section.

Section 1

Total marks (22)
Attempt Questions 1 – 22
Allow about 30 minutes for this section.

Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample ¼ of $320 is (A) $40 (B) $60 (C) $80 (D) $14

(A) (B) (C) (D)

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) (B) (C) (D)

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

Correct

(A) (B) (C) (D)

USE THIS TABLE TO ANSWER QUESTIONS 1 AND 2.

PARCEL DELIVERY CHARGES

Weight	Zone 1 $	Zone 2 $	Zone 3 $	Zone 4 $
Up to 500 g	16.00	17.00	18.00	19.50
Over 500 g up to 1 kg	19.50	21.50	23.50	26.50
Over 1 kg up to 1.5 kg	32.00	2600	29.00	33.50
Over 1.5 kg up to 2 kg	26.50	30.50	34.50	40.50
Each additional 500 g	3.50	4.50	5.50	7.00

Express Courier Pick-up an additional $3.40 per article.

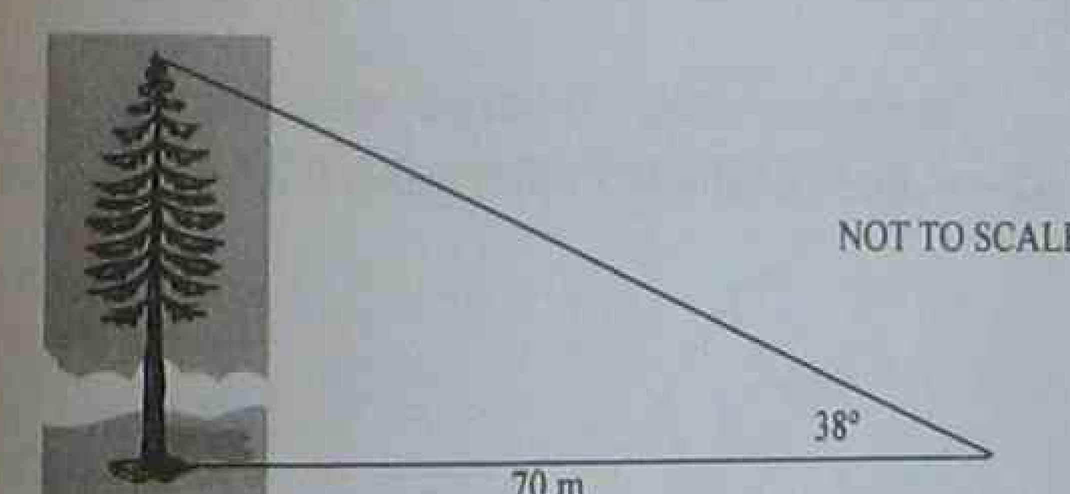
1 What is the cost of sending a 4 kg parcel to Zone 3?

(A) $35.00
(B) $48.50
(C) $40.00
(D) $56.50

2 A parcel weighing 1.25 kg is to be sent to Zone 2. What is the cost of sending the parcel with Express Courier Pick-up?

(A) $29.40
(B) $26.40
(C) $30.50
(D) $30.40

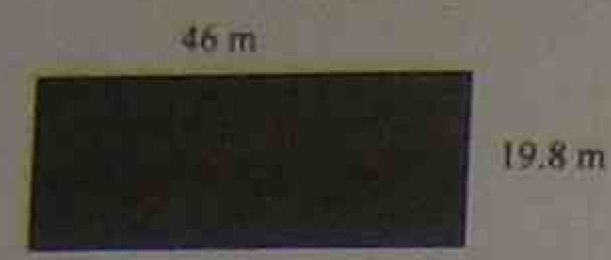
3 Foresters wish to determine the height of a fir tree.



By taking a sighting with a clinometer and measuring a base line as shown, the height of the tree would be given by

(A) $70 \div \tan 38°$
(B) $70 \tan 38°$
(C) $\tan 38° \div 70$
(D) $70 \sin 38°$

4 A house with floor area 130 m^2 is to be built on this block of land.



The percentage site coverage is about

(A) 14%
(B) 18.5%
(C) 7%
(D) 5%

5 $(-10a) \times (-4a) \div (5a^2) =$

(A) $-8a$
(B) 8
(C) $8a$
(D) -8

6 A bushwalker can average about 3.5 km/h. If the person started a 12 km walk at 2:05 pm, they should finish it at about

(A) 5:48 pm
(B) 5:55 pm
(C) 3:45 pm
(D) 5:30 pm

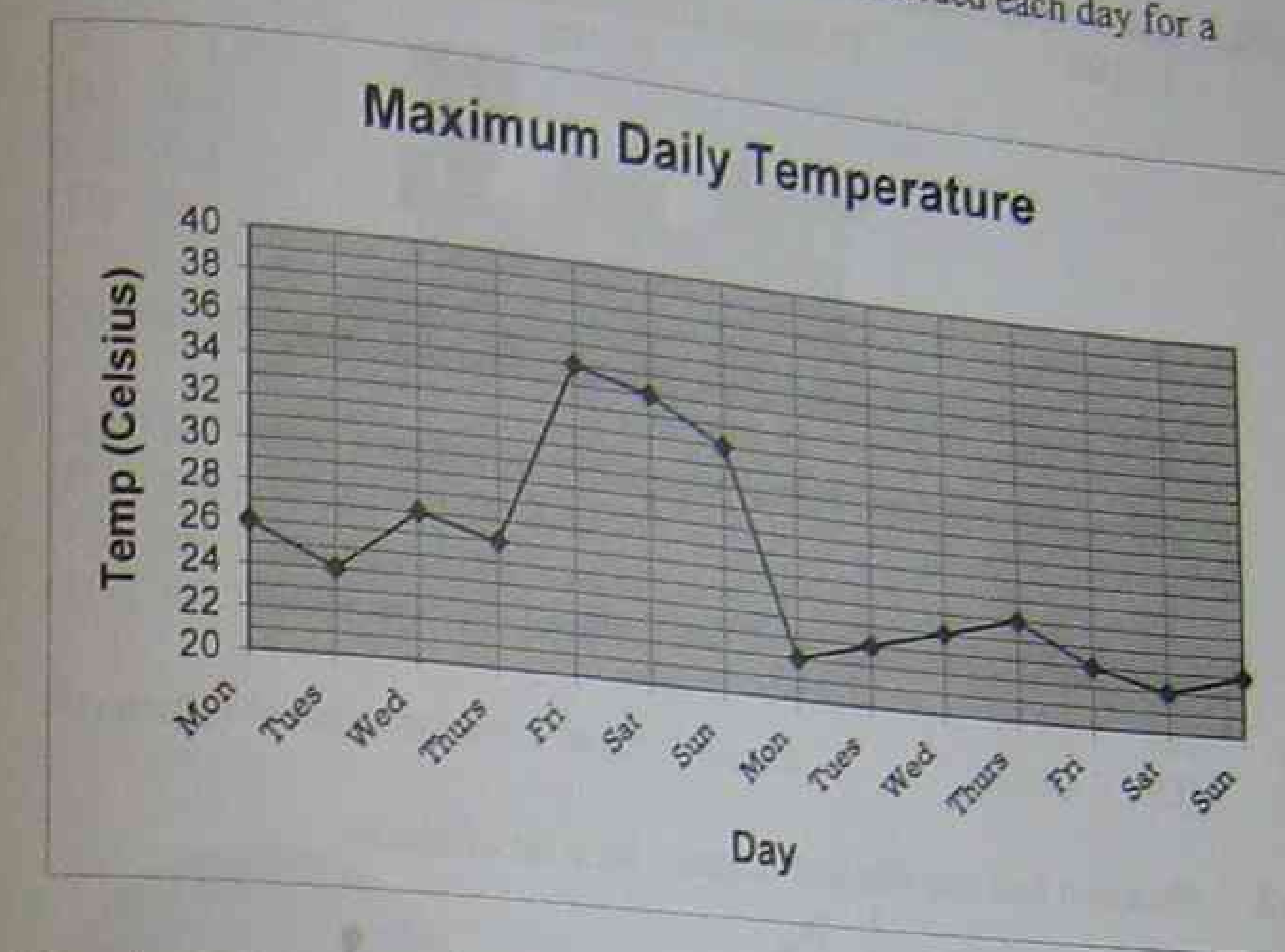
7 If $3(x + 1) = 4(2 - x)$, then the value of x which makes this equation true is

(A) $1\frac{4}{7}$
(B) 11
(C) $\frac{5}{7}$
(D) 5

8 After five years, the value of Ethel's car is \$11 500. If it has depreciated at 20% p.a., about how much was it worth 5 years earlier?

(A) \$12 650
(B) \$13 800
(C) \$28 600
(D) \$35 000

9 This line graph shows the maximum temperature recorded each day for a fortnight.

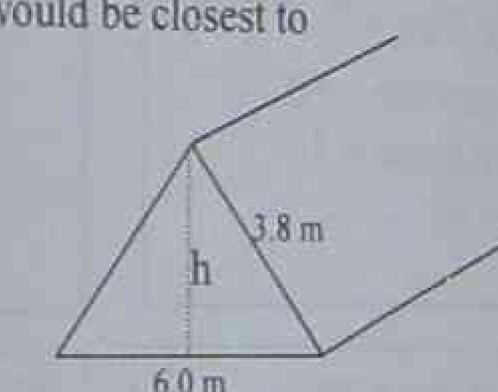


What is the difference between the average maximum temperature for the first week and the average maximum temperature for the second week?

(A) 12° C
(B) About 6° C
(C) About 4° C
(D) 27° C

10 This diagram shows the end of a gabled garage roof. The height (h) of the roof would be closest to

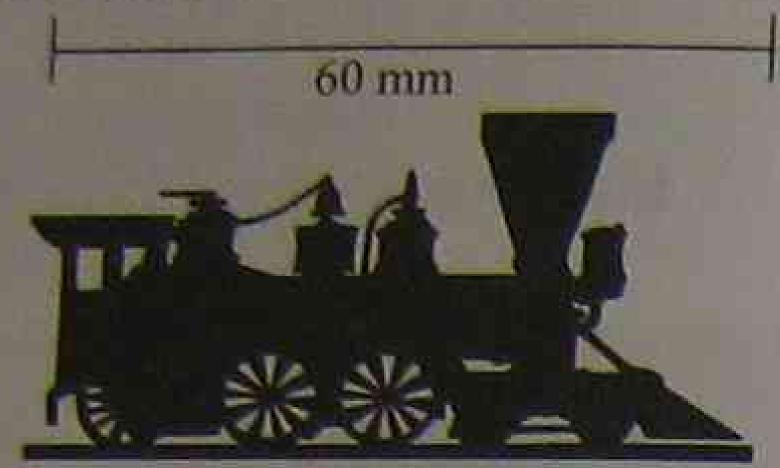
(A) 2.7 m
(B) 5.4 m
(C) 4.6 m
(D) 2.3 m



11 City X has position 21°N 10°W. If it is 3 pm in City X when it is 1 pm in City Y, City Y could have position

(A) 9°S 10°W
(B) 51°N 10°W
(C) 21°N 20°E
(D) 21°N 40°W

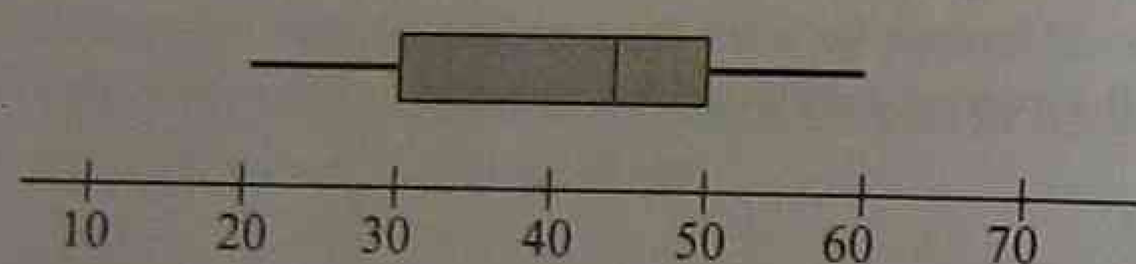
12 The overall length of this model locomotive is 60 mm. The scale used was 1 : 180.



The overall length of the real locomotive would be

(A) 3 m
(B) 30 m
(C) 10.8 m
(D) 1.080 m

13 Here is a box and whisker diagram for a set of scores.



The five number summary for the distribution is

(A)
range = 40
median = 43
upper quartile = 50
lower quartile = 30
mean = 40

(B)
range = 40
mean = 43
upper quartile = 50
lower quartile = 30
median = 40

(C)
lowest score = 20
median = 43
lower quartile = 30
upper quartile = 50
highest score = 60

(D)
lower quartile = 20
standard deviation = 20
median = 43
upper quartile = 60
range = 40

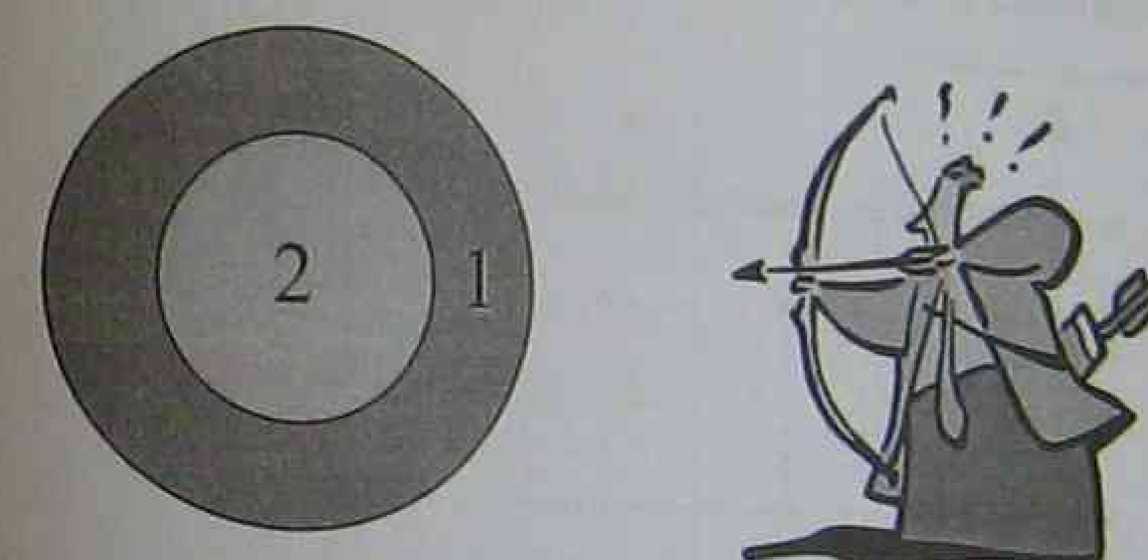
14 An event happens on 90% of days. The chance of the event NOT happening on all of four randomly picked days is closest to

(A) 0.1
(B) 0.4
(C) 0.5
(D) 0

15 On the number plane, the points satisfying the equation $y = 5$

(A) lie on the Y axis
(B) are 5 units from the X axis
(C) are 5 units from the Y axis
(D) are 5 units from the Origin.

16 An archer is shooting at the target as shown.



On average, the archer misses the target altogether one arrow in ten, scores 2 with probability 0.2 and scores 1 otherwise. With ten arrows, the archer would be expected to score

(A) 3
(B) 5
(C) 11
(D) 12

17 A painter charges \$18.50 /h. He charges 70% extra as an allowance for heights. The charge for a high job, which took 6 ¼ hours, would be

(A) \$(18.5 x 0.7 x 6.25)
(B) \$(18.5 x 1.7 x 6.25)
(C) \$(18.50 x 0.70 + 18.5 x 6.15)
(D) \$(18.50 x 0.7 x 0.25 + 18.50 x 6)

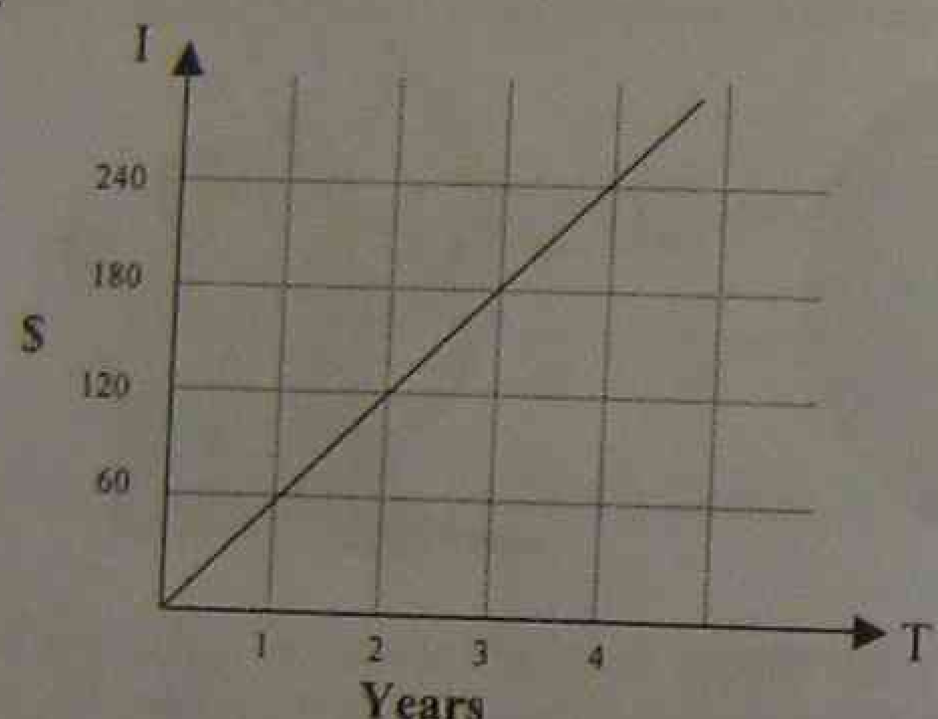
18 This stem and leaf display shows the number of runs Ali and Barbara scored during a number of cricket matches.

Ali		Barbara
6 4 1 0	0	2 9
9 2 1	1	3 6 7 8
7 1	2	0 4 5 9
5 2	3	4
7 1	4	5

Which of these statements is true?

(A) Barbara's median score is greater than Ali's.
(B) Ali and Barbara played the same number of matches.
(C) Barbara's range is greater than Ali's.
(D) Barbara's mean score is 21.

19 If $1000 is invested in an account, the interest paid is as indicated by this graph.



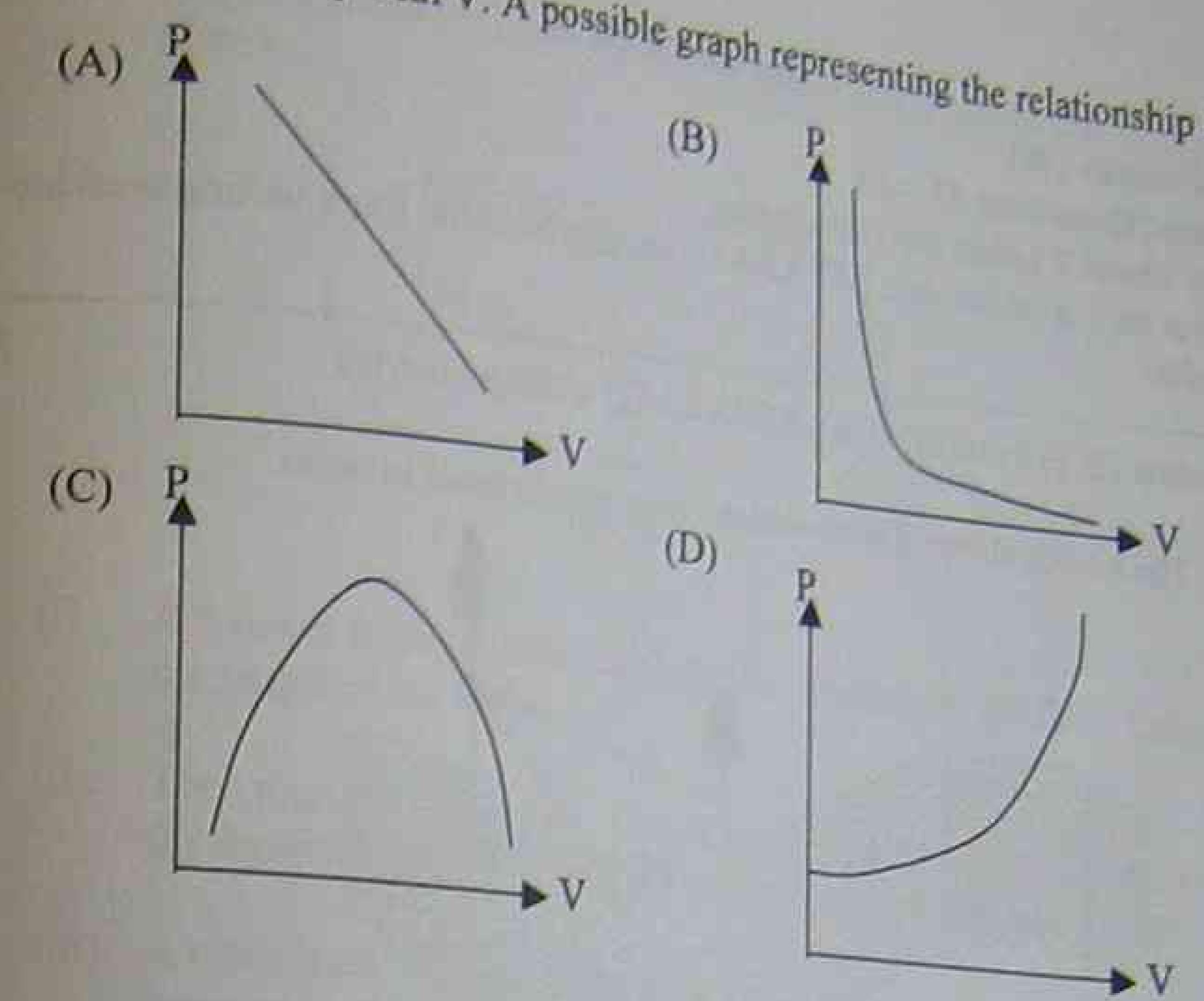
The annual flat rate of interest is

(A) 0.60%
(B) 6%
(C) 60%
(D) 1.6667%

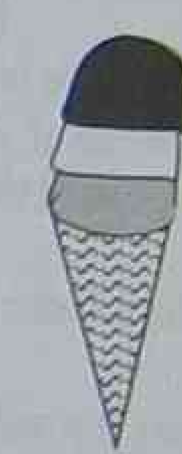
20 $\frac{9y^2}{5} \times 15y$ simplifies to

(A) $27y^2$
(B) $91y^3$
(C) $(3y)^3$
(D) $9y^3$

21 P varies directly with V. A possible graph representing the relationship is



22 A restaurant offers Triple Treat ice creams, as illustrated.



Each cone has three favours of ice cream, one on top of the other.
There are four favours available: chocolate, vanilla, strawberry and mango.
The number of possible combinations of favours, without considering the order on the cone, is

(A) 4
(B) 12
(C) 24
(D) 64

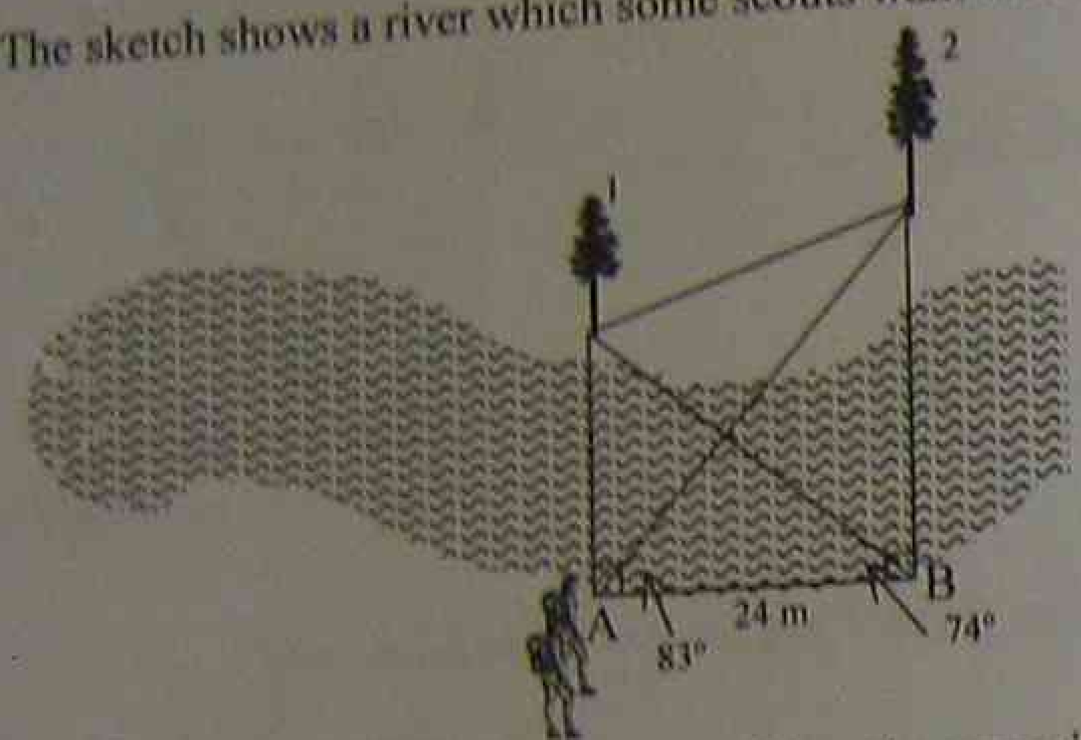
End of Section 1.

Section II

Total marks (78)
Attempt Questions 23 – 28
Allow about 2 hours for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 23 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The sketch shows a river which some scouts want to cross.



The scouts take a sighting onto a tree (1) on the opposite bank. They then measure a base line AB of 24 m and take an angle measurement of 74° from the end of the baseline, as shown.

(i) Calculate the distance across the river to the tree, to the nearest metre. **2**

(ii) Another tree (2) is sighted at an angle of 83° as shown. Find the distance from A to Tree 2. **2**

(iii) Show the distance between the trees is about 114 m. **2**

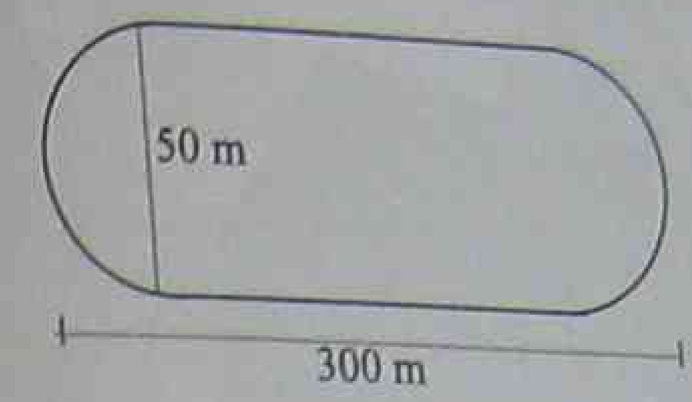
(b) During the 1960s, 20-year-old Australian men were conscripted into the army. Marbles were drawn from a bag and all men born on the date shown on the marble were drafted.

(i) If every possible birthday must be represented, how many marbles would be required? **1**

(ii) If 20 marbles were drawn, what is the probability of a person born on 10th July being drafted, correct to three decimal places? **1**

(iii) In which months would you expect more conscripts to be born? Explain your answer. **1**

(c) A sports ground is made up of a rectangle with semicircular ends, as shown.



(i) A fence is to be constructed along the boundary of the field. Calculate the length of the fence. **1**

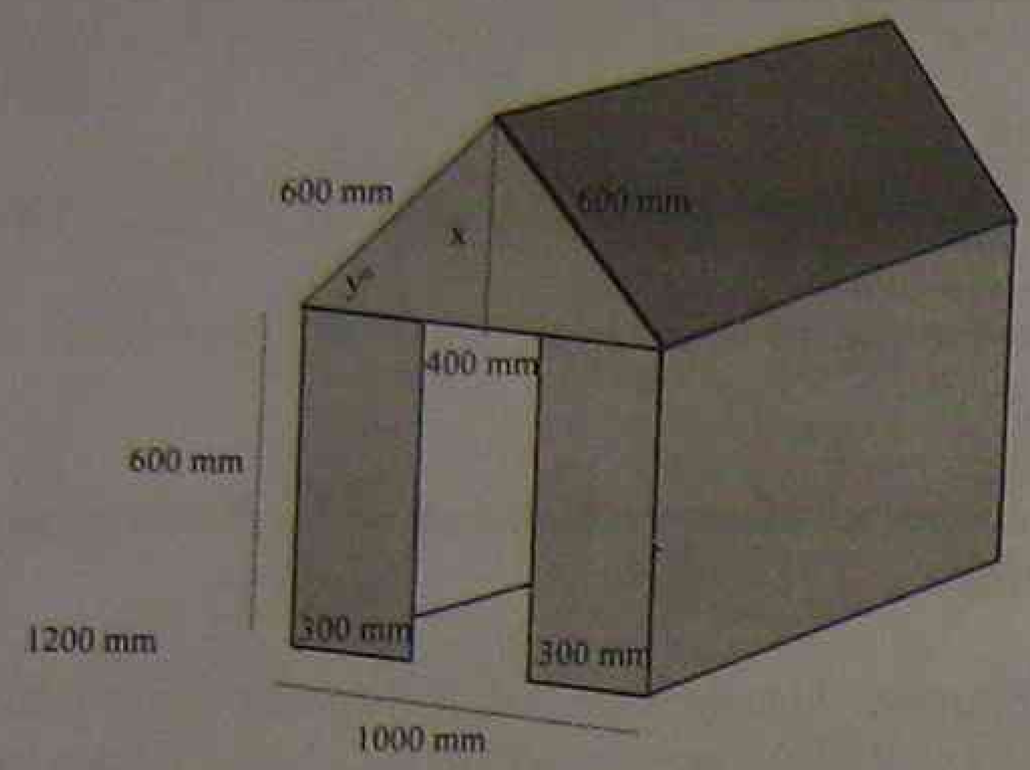
(ii) The ground is to returfed. Use the appropriate formulae for the areas of the figures involved to find the area correct to the nearest square metre. **1**

(iii) A cycle track 1 m wide is to be laid using the existing boundary as the inner edge. Find the area of the track, correct to the nearest square metre, assuming that the track is flat. **2**

End of Question 23.

Question 24 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) This is a sketch of a dog kennel for Fido (not to scale).



(i) What 2D shapes make up the roof? **1**

(ii) Make a scale drawing of the front of the kennel, using a scale of 1 : 10. **2**

(iii) Assuming symmetry, find the value of x to the nearest millimetre. **2**

(iv) Find the angle marked y°, correct to the nearest minute. **2**

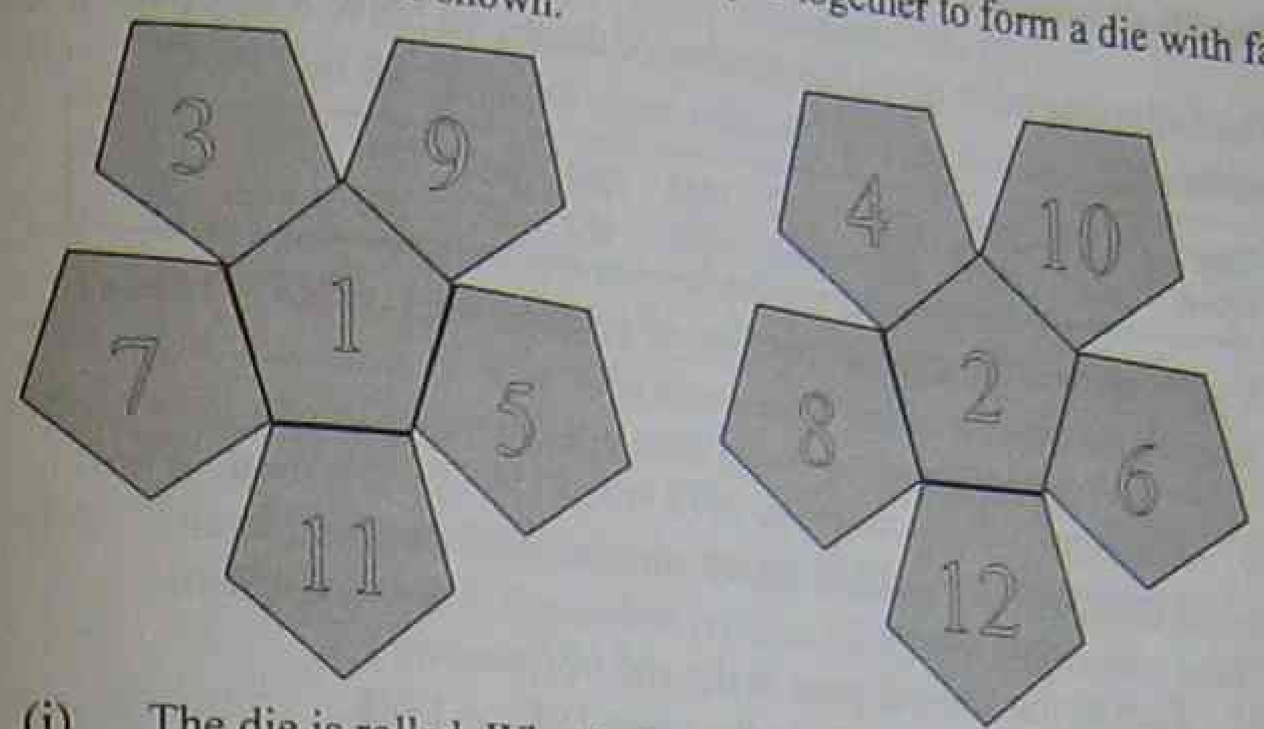
(b) Helen buys $1500 worth of goods on a credit card which charges 0.03821% compound interest per day on the outstanding balance.

(i) What is the annual interest rate being charged, correct to nearest whole percentage? **1**

(ii) If this purchase is the only amount debited to the card and there are no payments, how much interest will accrue in 25 days? **2**

Question 24 continues on the next page.

(c) These two nets are to be folded and taped together to form a die with faces numbered 1 to 12 as shown.



(i) The die is rolled. What is the probability that the uppermost face will show a number which is divisible by 3? **1**

(ii) If two such dice were made and rolled, what is the probability of the total of the uppermost faces being 12? **1**

(iii) A professional gambler finds it difficult to attract players by offering $5 return for a $1 bet if the player can throw doubles using the usual cubic dice, i.e. (1, 1), (2, 2) ... (6, 6). If he uses these 12 faced dodecahedral dice, what is the most he can offer as a prize for a $1 bet and still 'break even'? **1**

End of Question 24.

Question 25 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) The following table gives the number of deaths on NSW roads over a ten year period.

Year	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993
Killed	1037	1067	1029	959	1037	960	797	663	649	581

(i) What was the average number of people killed on NSW roads in the ten years? **1**

(ii) The road toll was worst in 1978 when 1384 people died. What percentage of the 1978 figure was the 1993 death toll? **1**

(iii) Turn to the tear-off page at the end of Question 25. Complete a line graph for the years 1984 to 1988. **1**

(iv) Draw a line of best fit. **1**

(v) From your line of best fit, how many people would have been expected to die on NSW roads in 1993? **1**

(b) The mass of a loaf of bread was found to be distributed normally with a mean of 450 g. A sample gave a standard deviation of $\sigma_n = 11.4$ g.

(i) If there were 20 loaves in the sample, use the formula $\sigma_{n-1} = \frac{n}{n-1}\sigma_n$ to find σ_{n-1}, the standard deviation of the distribution. **1**

(ii) What percentage of randomly selected loaves would you expect to have a mass less than 426 g? **1**

(iii) What is the probability that a loaf picked at random will have a mass between 438 g and 462 g? **1**

Question 25 continues on the next page.

(c) A publisher reports that 10 348 copies of a textbook have been sold. The retail value of the book is $14.95 per copy, including 10% GST.

(i) Show that the price per copy without GST, to the nearest cent is $13.59. **1**

(ii) What is the total return to booksellers without GST? **1**

(iii) The publisher gives a 30% discount to booksellers, based on the pre-GST price. What was the total return to the publisher? **1**

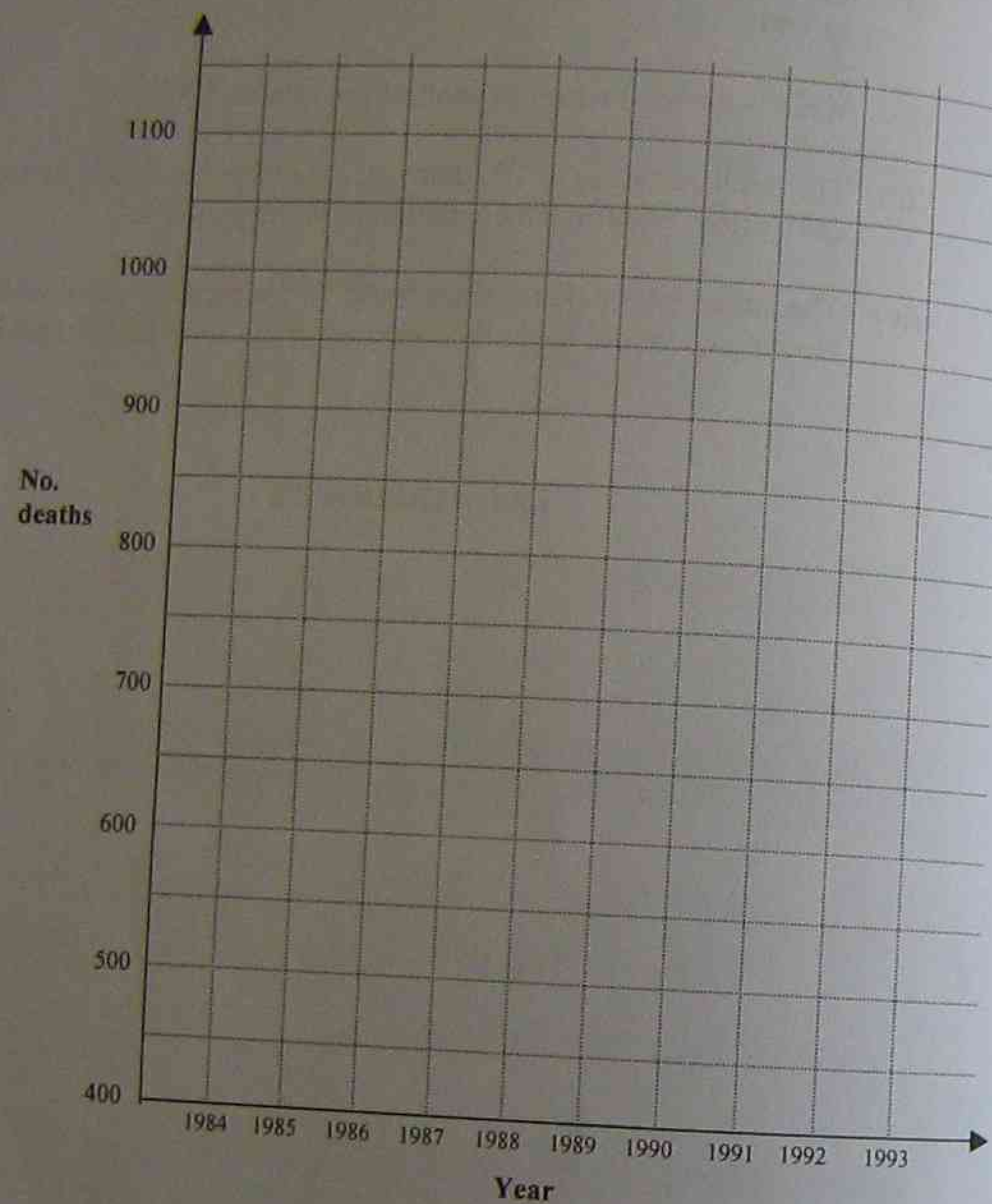
(iv) The author gets 12% royalties based on the pre-GST retail price. How much would the author receive for the 10 348 books sold? **2**

End of Question 25.

Tear-off page to be handed in with your written solutions.

Tear along this dotted line

Question 25 (a) Part (iii) and (iv)



Question 26 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) Three students, Anastasia, Belinda and Constance, are being considered for the prize for first in their year. Staff members disagree about the fairest method to compare the students' performance.

Student	Subject	Mark	Class mean	Class s.d.
Anastasia	English	76	65	12
	Maths	83	63	10
	Science	88	70	10
	Geography	80	70	10
Belinda	English	79	65	12
	Maths	82	63	10
	Science	80	70	10
	Commerce	90	65	20
Constance	English	75	65	12
	Maths	70	63	10
	History	85	60	10
	French	95	85	5

(i) Mr. Xenakis believes that the girls should be compared by adding the raw marks that each scored on the subjects which all attempted, i.e. English and Maths. Calculate the total of English and Maths for each student. Who does Mr. Xenakis believe should receive first prize? **1**

(ii) Mrs. Young believes that the marks should simply be added for each girl and the prize given to the girl with the highest total. Calculate the total mark for each student. To whom does Mrs. Young give first prize? **1**

(iii) Ms Zafiriou suggests that all marks should be converted to z scores and the prize awarded to the student with the highest total z scores. Calculate the total of the z scores for each student. To whom does Ms Zafiriou award first prize? **2**

(iv) Consider the three methods. Which of the three, if any, is fairest? Give reasons for your answer. **1**

(b) A new machine in a factory costs 3 million dollars to buy and is expected to be obsolete in 10 years.

(i) If a flat rate depreciation was applied, how much could be claimed as a tax deduction each year? **1**

(ii) The Taxation Department allows 24% of the machine's value in the previous year to be claimed. How much could be claimed at the end of the second year? 1

(c)

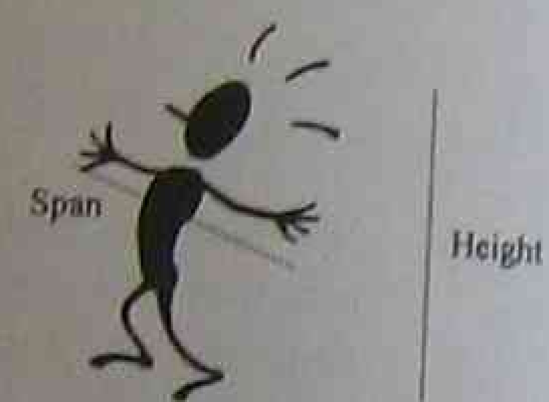
Score	Frequency
5	3
6	k
7	4
8	9

(i) Write an expression for the mean of this sample. 2

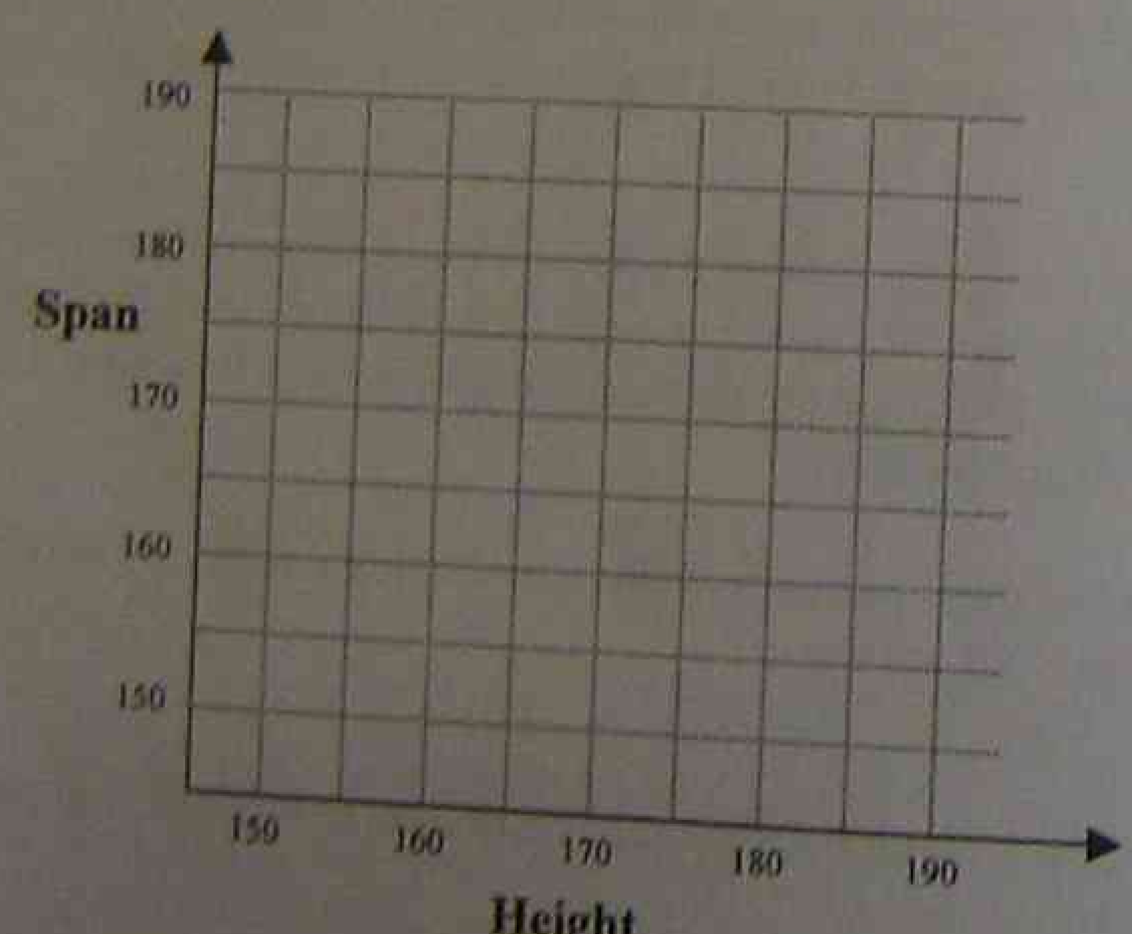
(ii) Find the value of k if sum of the scores is 133. 2

(d) Five students measured their heights and spans in centimetres and recorded the results as follows.

Name	Height	Span
Alana	153	149
Ben	181	180
Colin	172	173
Duc	160	162
Enrico	190	187



(i) Copy these axes and plot points from the table above. 1



(ii) Comment on the correlation of the heights and spans of these students. 1

End of Question 26.

Question 27 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) (i) Find the value of x if $2x + 3 = -6$ 2

(ii) Find the value of y if $\frac{y}{5} = \frac{y+2}{3}$ 2

(b) Gemma borrowed \$10 000 to buy a car. She took the loan for 5 years at 11% p.a., interest being reducible. Her repayment (R) is \$217.40, paid at the end of each month.

(i) Complete this table. 3

Month (N)	Principal (P)	Interest (I)	P + I	P + I – R
1	\$10000	\$91.67	\$10091.67	\$9874.27
2	\$9874.27			
3				

(ii) How much money did Gemma pay back on the loan? 1

(iii) What was the total interest paid? 1

(iv) What annual simple interest rate is equivalent to the reducible loan Gemma had? 1

(c) The effective interest rate for a loan is given by

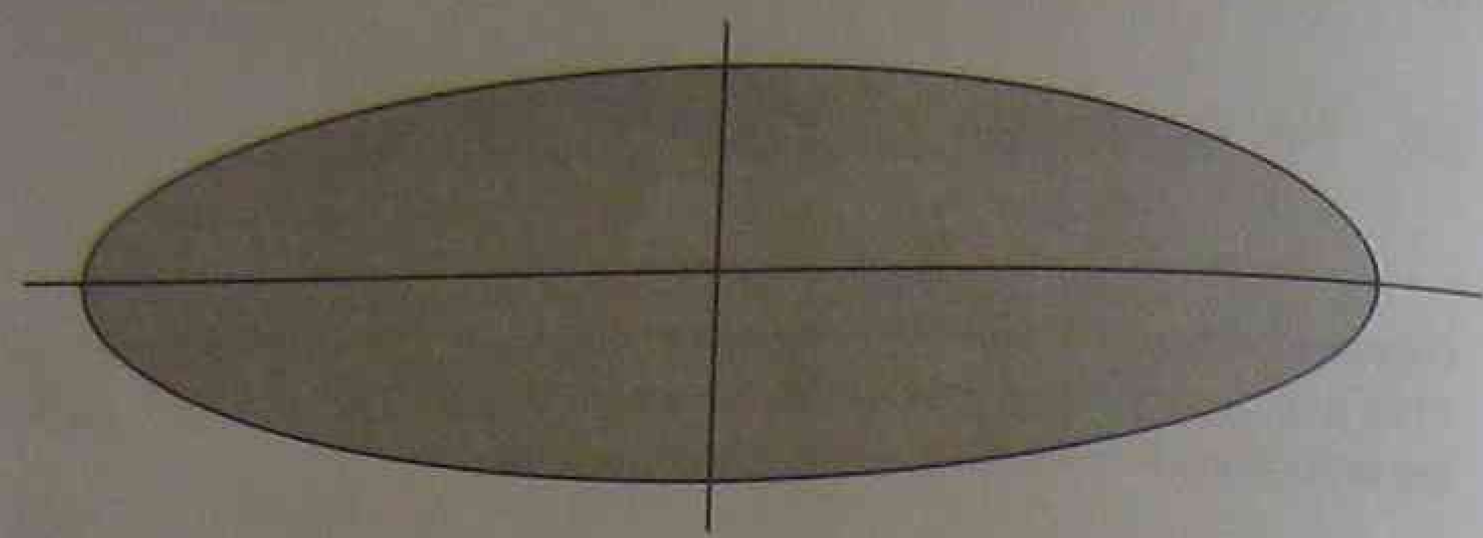
$$E = \frac{(1+r)^n - 1}{n}$$

where r is the given interest rate expressed as a decimal
n is the term of the loan.

Find E when r = 8 ¼ % p.a. and the term is 4 years. 1

Question 27 continues on the next page.

(d) Take the appropriate measurements and determine the area of this ellipse in square centimetres, correct to one decimal place. **2**



End of Question 27.

Question 28 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) The drawing shows the side elevation of a container for dog biscuits (not to scale) in the shape of a fire hydrant. It consists of a cylindrical can and a hemispherical lid. Dimensions are shown in millimetres.

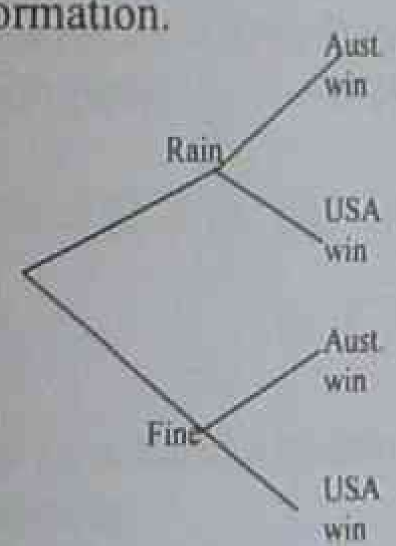
(i) What volume of water in mL would the cylinder hold? **2**

(ii) Calculate the surface area of the lid in square centimetres. **2**

(iii) What is the overall height of the container? **1**

(b) The chance of rain tomorrow is ⅓.
The chance of Australia beating USA tomorrow in basketball is 0.4.

(i) Copy and complete this probability tree diagram for the above information. **2**



(ii) What is the probability that USA will win and it will be a fine day? **1**

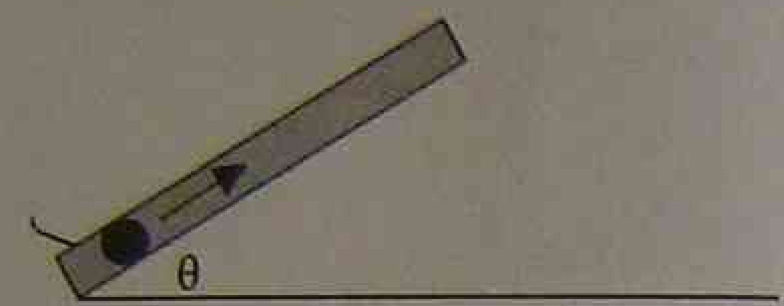
(iii) A punter places a $10 bet on Australia winning. The bookmaker will pay $15 for the win. What is the punter's financial expectation? **2**

Question 28 continues on the next page.

(c) The horizontal distance travelled by a cannonball before hitting the ground is called the range. The range of a cannonball, x metres, is given by the formula

$$x = \frac{2u^2 \sin\theta \cos\theta}{g}$$

where u is the initial velocity as the ball leaves the cannon, θ is the angle from horizontal of the barrel of the cannon in degrees and g is acceleration due to gravity, taken to be $9.8\ m/s^2$.



(i) How far away (on flat land) would a cannonball fall if the barrel were inclined at an angle of 30° and the initial velocity is 200 m/s? Give your answer correct to the nearest metre. 1

(ii) What would the range be if the cannon had an angle of 90° with the horizontal? 1

(iii) Show that the range when the angle is 60° is the same as for an angle of 30°. 1

End of paper.

Sample Exam Paper 4

General Instructions

- Reading time – 5 minutes
- Working time – 2 ½ hours
- Write using black or blue pen
- Calculators may be used
- A Formulae Sheet is provided at the back of this paper.

Section 1 Total marks **(22)**
- Attempt all questions 1 – 22
- Allow about 30 minutes for this section

Section 2 Total marks **(78)**
- Attempt Questions 23 – 28
- Allow about 2 hours for this section.

Section 1

Total marks (22)
Attempt Questions 1 – 22
Allow about 30 minutes for this section.

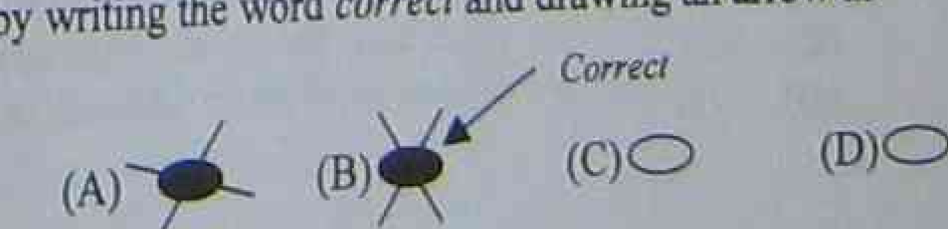
Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample ¼ of $320 is (A) $40 (B) $60 (C) $80 (D) $14

(A) ○ (B) ○ (C) ● (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

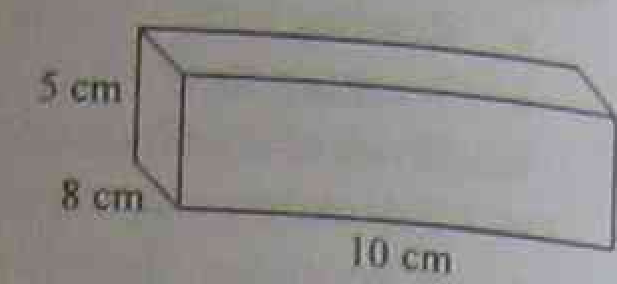
If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



1 If $\frac{5a}{2} = \frac{a+1}{4}$, then $a =$

(A) $\frac{1}{4}$ (B) $\frac{1}{9}$ (C) $\frac{2}{3}$ (D) $\frac{1}{18}$

2 Bellenden Ker in Queensland once recorded a 24 hour rainfall of 960 mm. If it had rained at a constant rate throughout that period, how long would it have taken to fill a container like this, left out in the open?



(A) 400 minutes
(B) 75 minutes
(C) About 2 ½ minutes
(D) Just over 1 minute

3 A poker player has three aces, a king and a queen. If he throws out the king and queen, the probability of one of the two replacement cards being the other ace is

(A) $\frac{1}{4}$ (B) $\frac{1}{52}$ (C) $\frac{1}{47}$ (D) $\frac{2}{47}$

4 A car costs $12 000. A deposit of 30% is left. Interest is charged on the balance at a flat rate of 9% p.a. for 2 years. Twenty-four equal instalments will pay the car off. How much is each instalment?

(A) $415.84
(B) $396.03
(C) $413.00
(D) $381.50

5 If a street directory has a scale of 1 : 26 500, then two places which are 2 km apart would, on the map, be

(A) 13.25 cm apart.
(B) 1.3 cm apart.
(C) 75 cm apart.
(D) 75 mm apart.

6 The answer to $\frac{(38.21)^2 - \sqrt[3]{9.781}}{13 - 4 \times 2.97}$ is closest to

(A) 1302
(B) 478
(C) 321
(D) 100

7 If $\sin 2x° = 0.32$ and x is acute, then x is about

(A) 0.53
(B) 18.66
(C) 9.3
(D) 53

8 A class has Mr Newton for Maths and Ms Curie for Science. Mr Newton's predictions of his class's HSC results have a correlation coefficient of 0.95 with the actual results. Ms Curie's predictions have a correlation coefficient of 0.73 with the class's HSC Science results. The difference indicates that

(A) Mr Newton is a better teacher than Ms Curie.
(B) the class did better overall in Maths than in Science.
(C) Ms Curie was not a good judge of her class's ability.
(D) none of the above are true.

9 The radius of the earth is about 6400 km. Its surface area is two-thirds water. The land area is therefore about

(A) 43 million km^2
(B) 129 million km^2
(C) 172 million km^2
(D) 366 000 million km^2

10 These scores were obtained by a class in a test.

3, 5, 0, 1, 2, 6, 2, 4, 5, 2.

The percentage of the scores that are within 2 standard deviations of the sample mean is

(A) 68%
(B) 70%
(C) 95%
(D) 100%

11 An aeroplane is to fly from the east coast of Australia to a town on the west coast on the same latitude. To fly the shortest distance, the navigator would choose a course which

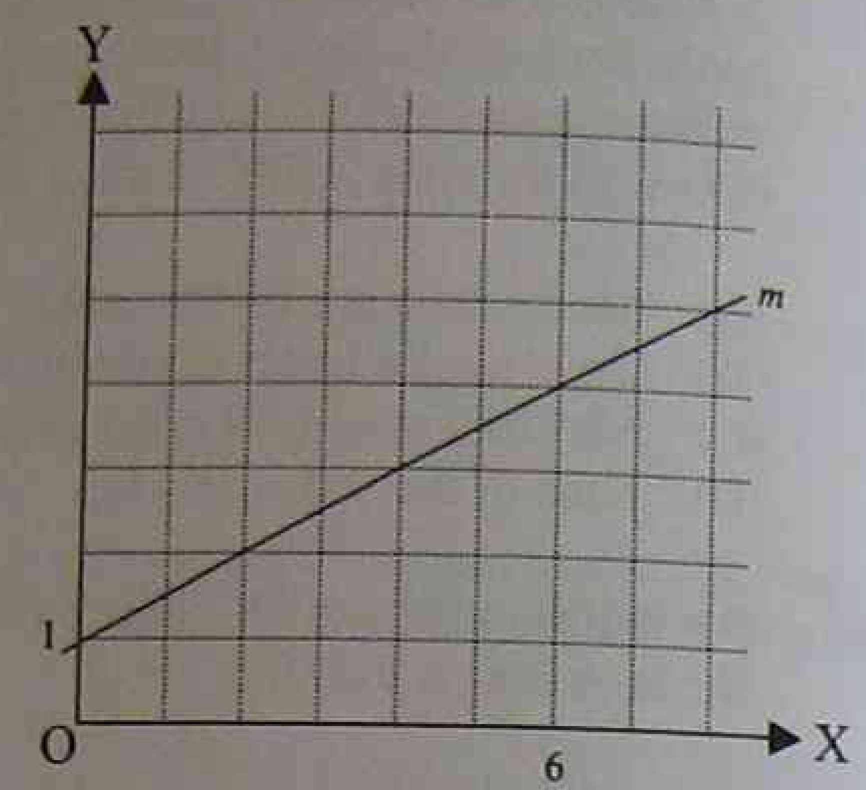
(A) was always south of the latitude on which the towns lie.
(B) was always north of the latitude on which the towns lie.
(C) approximated a straight line on the road map.
(D) stayed as close as possible to the latitude of the towns.

12 Sydney and Perth have a time difference of 2 hours, except in summer when Sydney adopts Daylight Saving and Perth does not. In January, when it is 8 pm in Sydney, the time in Perth is

(A) 11 pm.
(B) 9 pm.
(C) 5 pm.
(D) 7 pm.

13 The equation of the line *m* is

(A) y = ½ x + 1
(B) 3y = 6x
(C) 3y = 6x + 1
(D) 6y = x



14 The volume of a sphere varies directly with the cube of its radius. The volume is 36π cm^3 when the radius is 3 cm. When the volume is 288π cm^3, the radius is

(A) 6 cm
(B) 8 cm
(C) 6π cm
(D) 8π cm

15

Bird type	Cost per bird ($)
Galah	23
Canary	35
*	Cost of delivery: $20 per bird

The cost in dollars of buying g galahs and c canaries and having them delivered is

(A) 23g + 35c + 20
(B) 23g + 35c + 20g + c
(C) 23g + 35c +20 x 58
(D) 23g + 35c + 20(g + c)

16 The formula $\Sigma(x - \bar{x})$

(A) gives the mean of a sample.
(B) gives *n* times the mean of a sample.
(C) is equal to zero.
(D) gives the difference of each score from the mean.

17 Pauline works these hours.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
6	5	8	10	12	5

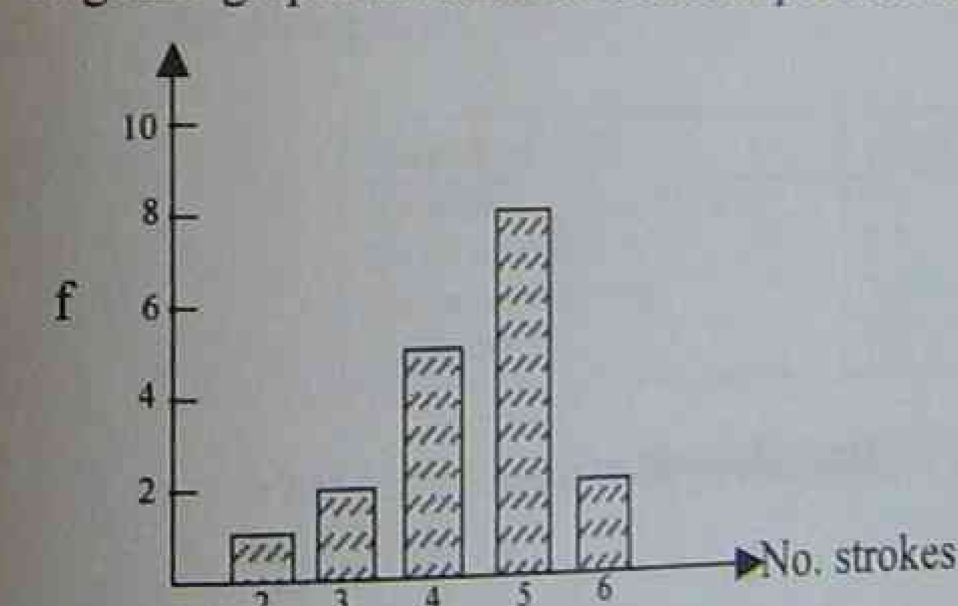
The normal time hourly rate is $6.65 /h. Any time over 8 hours on a week day is paid at time-and-a-half. Saturday work is paid at double time. Her gross weekly wage would be

(A) $305.90
(B) $339.15
(C) $342.48
(D) $359.10

18 If $7000 is invested for 10 years at 6.3% p.a. with interest compounding every 6 months, the balance to the nearest dollar will be

(A) $12 895
(B) $13 016
(C) $7221
(D) $9545

19 A golfer graphs the number of strokes per hole required for a round.



The median number of strokes taken was

(A) 4
(B) 5
(C) 6
(D) 8

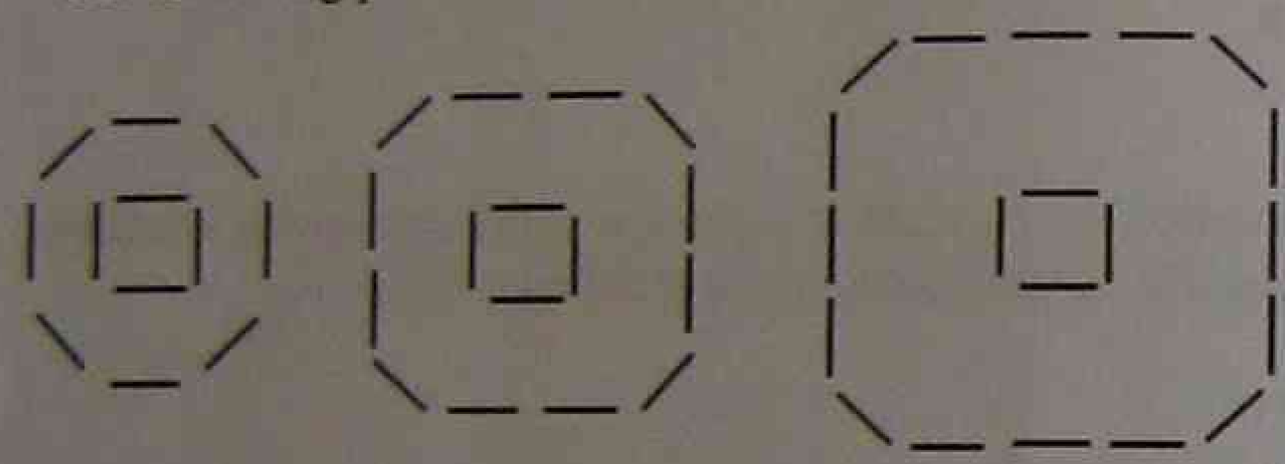
20 A security officer has 3 locks and 3 keys. If one key is placed alongside each lock, the probability that all the keys match the lock with which they have been placed is closest to

(A) 33%
(B) 30%
(C) 17%
(D) 4%

21 A telephone salesperson is paid 15% commission on all sales up to \$4000 and 22% of amounts over \$4000. If her commission was \$897, the sales totalled

(A) \$5350
(B) \$4077.27
(C) \$1480
(D) \$134.55

22 The following pattern was built from matchsticks.



Pattern for n = 1 **Pattern for n = 2** **Pattern for n = 3**

The number of matchsticks in the nth pattern could be

(A) $-6n^2 + 22n - 4$
(B) $4(n + 2)$
(C) $4n(4 - n)$
(D) $2(n^2 - n + 6)$

End of Section 1.

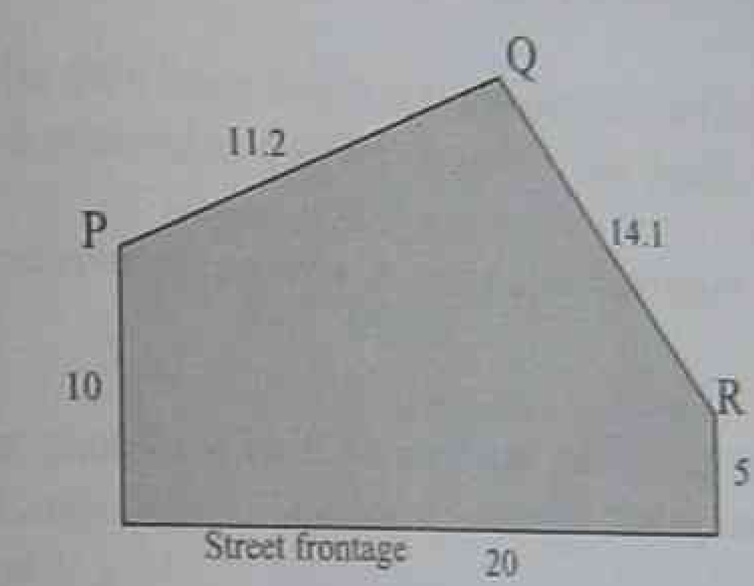
Section II

Total marks (78)
Attempt Questions 23 – 28
Allow about 2 hours for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 23 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows a site plan for a block of land, NOT DRAWN TO SCALE. Side lengths are shown in metres. The side fences are at right angles to the street frontage.



(i) A fence is to be built on the four boundaries other than the street frontage. At \$35 per metre, how much would the fence cost? **1**

(ii) The agent selling the property charges commission using the following scale.

Value of sale	Agent's fee
Up to and including \$15 000	5%
On the next \$45 000	3%
On the next \$40 000	2 ½ %
Thereafter	2%

What commission would be charged if the property sold for \$345 000? **2**

(iii) Show that the distance PR is equal to 20.6 m. **2**

(iv) Find the size of the angle in the fence at Q, correct to the nearest minute. **2**

(v) Find the area of the property to the nearest square metre. **2**

(b) Two standard cubic dice are rolled.

(i) List the outcomes that satisfy the event "The total on the dice is 6". **1**

(ii) If the dice were rolled 300 times, how many times would you expect 'doubles' (the same number on both dice)? **2**

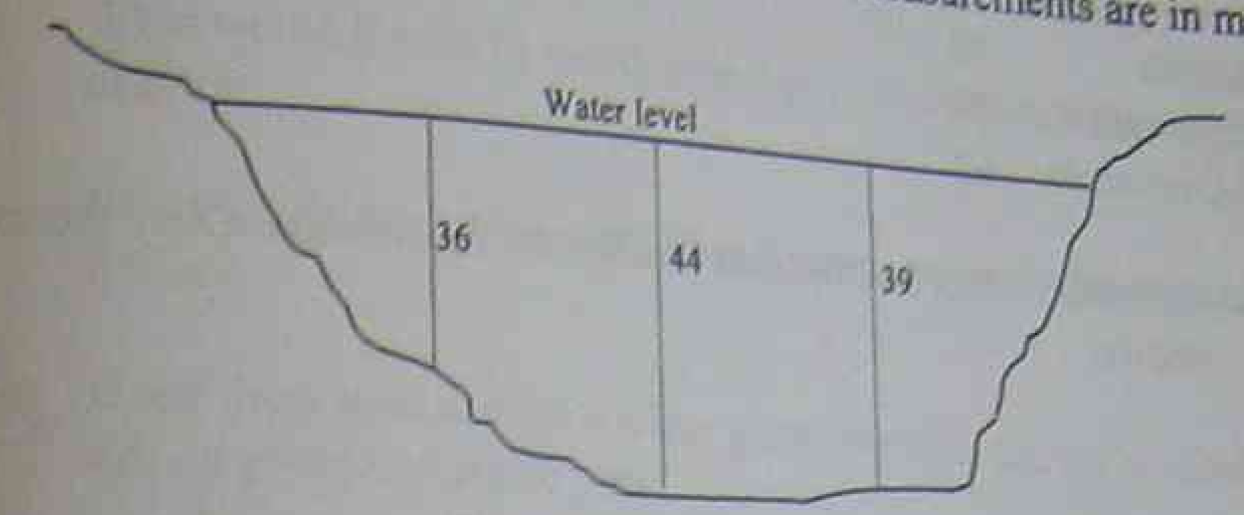
(iii) What is the most likely total on the dice? **1**

End of Question 23.

Question 24 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) The depth of water in a storage dam was measured at 20 m intervals across the dam wall, as shown (NOT TO SCALE). All measurements are in metres.

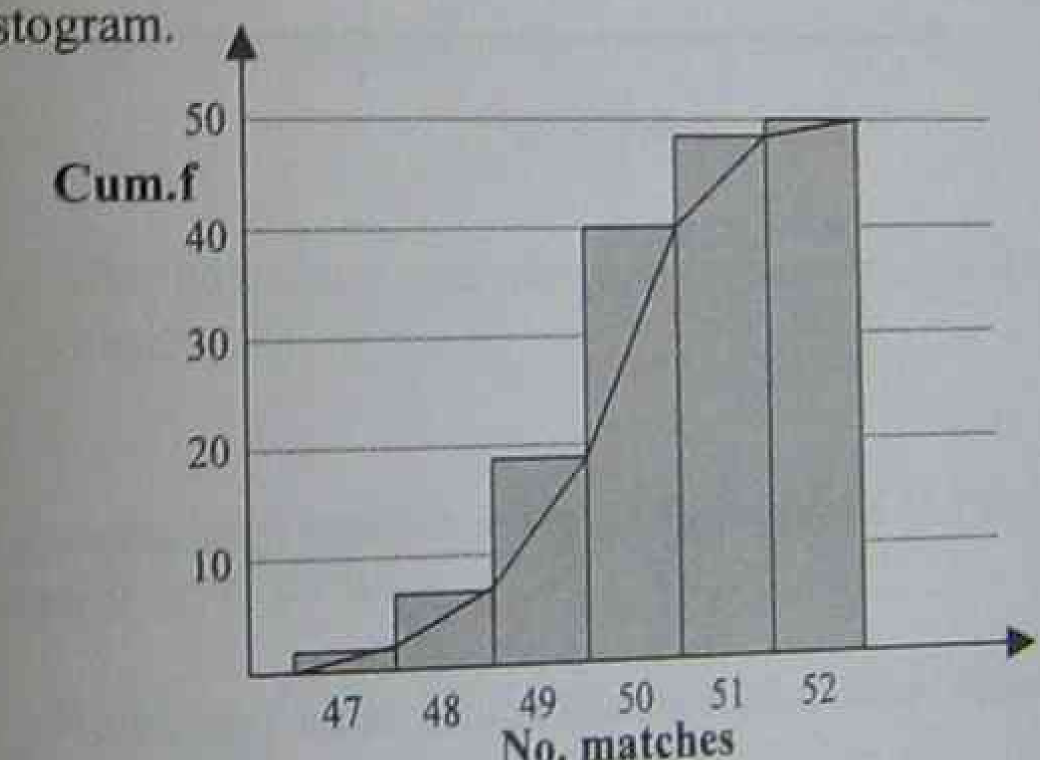


(i) Use Simpson's Rule to find the area of the cross-section. **3**

(ii) If the dam has a uniform cross-section, what volume of water would be contained in a 400 m length of the dam? **1**

(iii) Convert your answer in part (ii) to kilolitres. **1**

(b) A random sample of matchboxes was taken and the number of matches recorded in each box. The results are illustrated on this cumulative frequency histogram.



(i) If the company making the matches claimed that at least 50% of boxes contained no fewer than 50 matches, would this be a reasonable claim? Give reasons for your answer. **1**

(ii) From the graph, find the interquartile range. **2**

(iii) Calculate the median number of matches in the sample. **1**

(iv) If a box were purchased from a similar batch, what is the probability that it contains exactly 50 matches? **1**

(c) Here is a five number summary of a set of scores.

Lowest score	20
Lower quartile	30
Median	50
Upper quartile	60
Highest score	63

(i) Construct a box and whisker plot for the distribution. Copy the axis shown. **2**

(ii) Find the range of the distribution. **1**

End of Question 24.

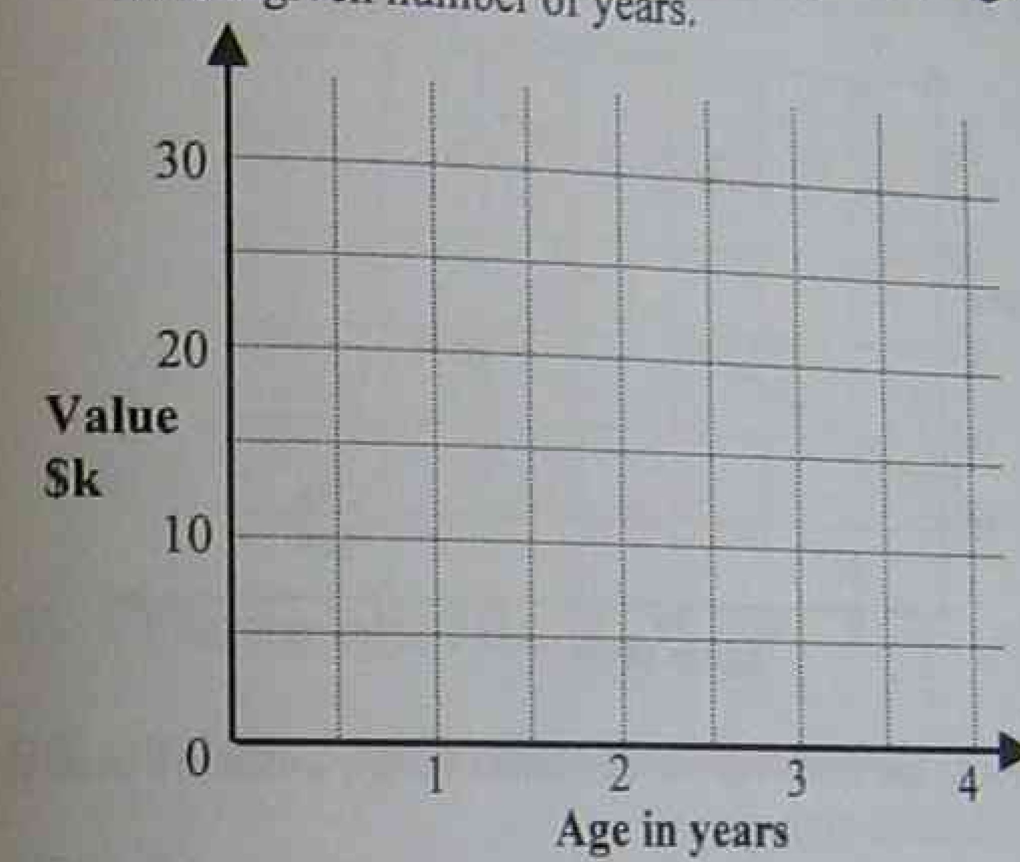
Question 25 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) A car depreciates at a rate of 28% for the first four years. It sold new for $28 695.

(i) What would the car be worth after one year, correct to the nearest dollar? **1**

(ii) What would the car be worth after 2 years, correct to the nearest dollar? **1**

(iii) Copy these axes and draw a line graph showing the car's value after a given number of years. **2**



(iv) After how many years would be car be worth one third of its original value? **1**

(b) The geographical position of Phnom Penh is 12°N 105°E and that of New Orleans is 30°N 90°W.

(i) How many hours behind Phnom Penh is New Orleans? **1**

(ii) What is the latitude and longitude of the place on the globe which is directly opposite New Orleans? **1**

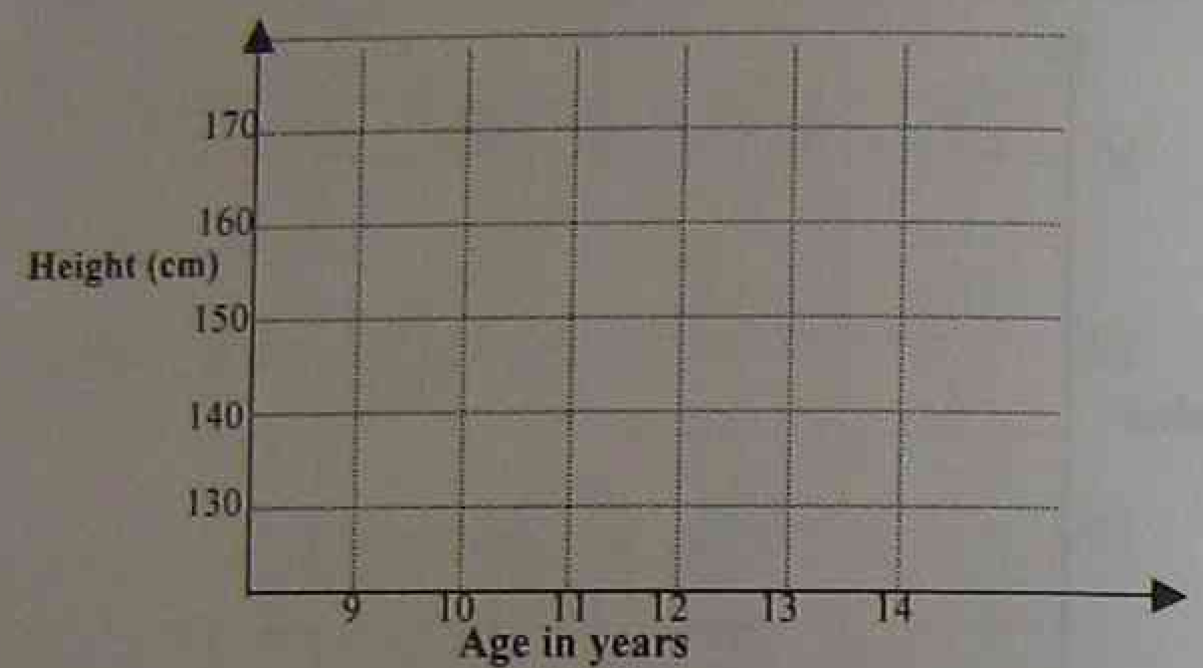
(iii) If the circumference of the earth is 40 000 km, how far is New Orleans from the Equator? **1**

Question 25 continues on the next page.

(c) Xavier decided to measure his height on his birthday each year and record it on a graph. After five recordings, he had this data.

Age in years	Height in cm.
9	130
10	135
11	140
12	148
13	157

(i) Copy these axes and plot points to illustrate the information above. **2**



(ii) Rather than join the dots, Xavier decided to draw a line of best fit. Draw such a line on your graph. **1**

(iii) By extending the line of best fit, estimate Xavier's height when he turns 21. **1**

(iv) How realistic is the answer to part (iii)? Explain your answer. **1**

End of Question 25.

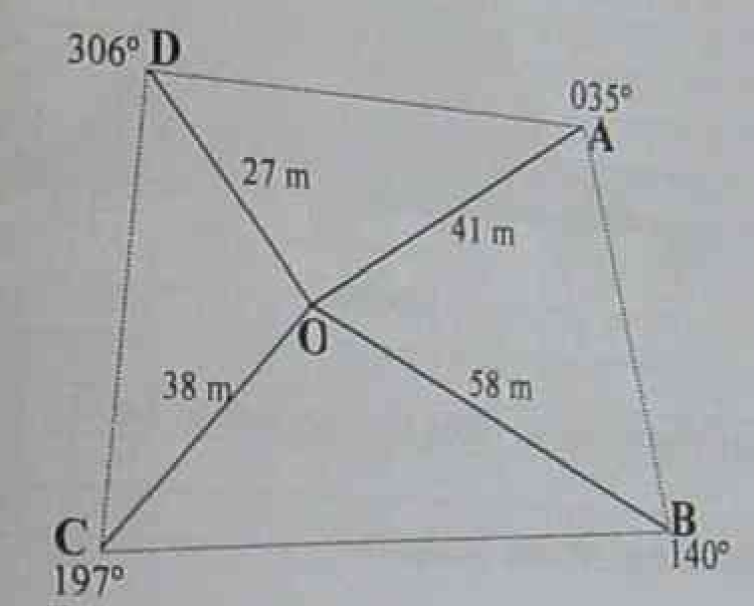
Question 26 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) (i) What is the simple interest on a loan of $12 000 for 6 years at 7 ½ % p.a.? **1**

(ii) John paid instalments totalling $3400 to repay a $2800 loan. Interest was charged at a flat rate for three years. What was the annual interest rate? **2**

(b) A radial survey was conducted in a paddock and the results recorded as shown on this diagram, not to scale.



(i) Find the angle DOA. **1**

(ii) Calculate the length of the boundary DA. **2**

(iii) What is the area of ΔAOB? **2**

(c) Police are interviewing a suspect whom they believe with a probability of 80% is lying. The suspect elects to take a lie detector test which is accurate on 90% of cases.

(i) Copy and complete this table, showing the likely results of 100 tests with the same probabilities. **3**

	Lying	Telling truth	Total
Test correct			
Test wrong			
Total			100

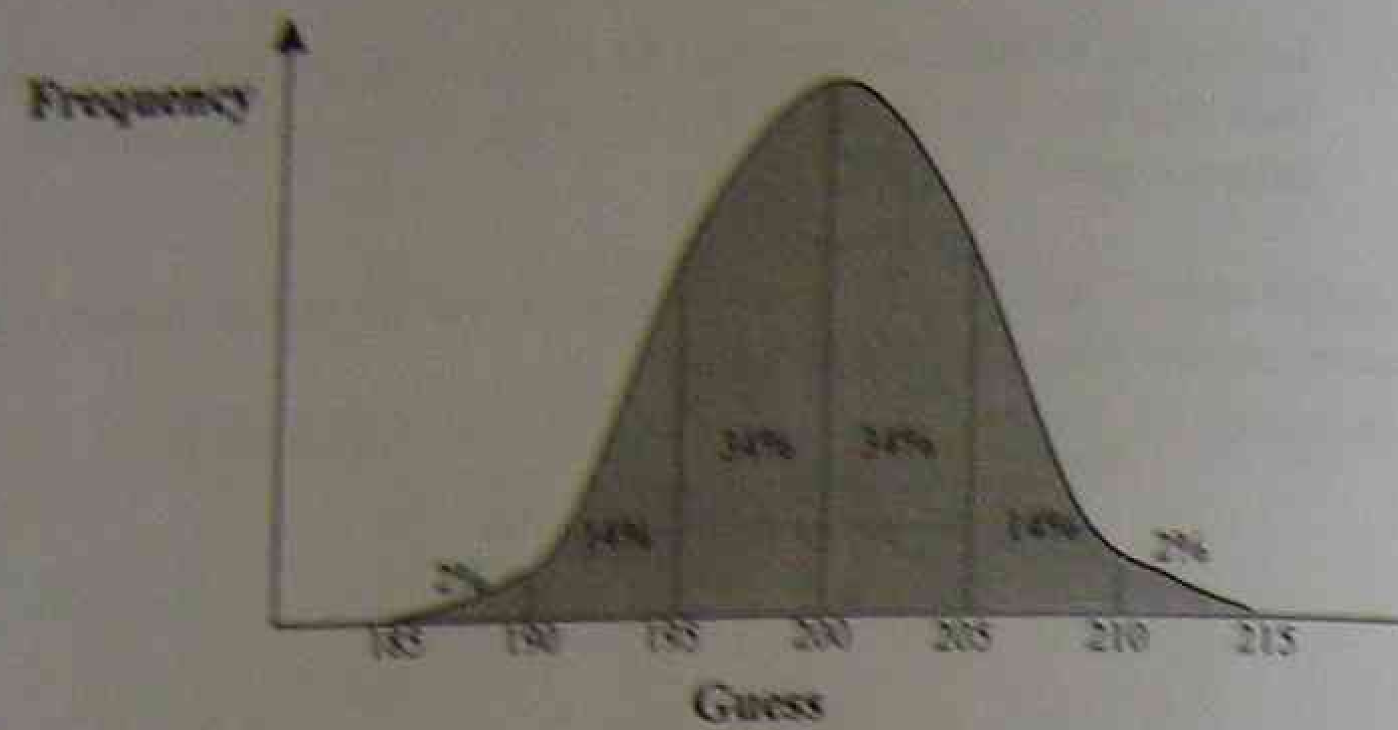
(ii) If a test result shows that a person is telling the truth, what is the probability that they are actually lying? **1**

(iii) If a test result shows that a person is lying, what is the probability that they are actually telling the truth? **1**

End of Question 26.

Question 27 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) In a guessing competition, 3780 contestants were asked to guess how many lollies there were in a jar. The results are recorded in this graph.



(i) What kind of distribution does this appear to be? **1**

(ii) What was the mean of the guesses? **1**

(iii) The standard deviation was 5. What percentage of the guesses was within 2 standard deviations of the mean? **1**

(iv) Approximately how many people thought that there were more than 205 lollies in the jar? **2**

(b) An engineer wishes to find the positive value of x which is a solution to the equation $x^2 + 3x = 7$.

(i) Show that the solution lies between 1 and 2. **1**

(ii) Determine whether the solution is greater than or less than 1.5 and hence find an approximation, correct to one decimal place for the solution. **2**

Question 27 continues on the next page.

(c) A wealthy investor buys a painting for \$150 000. After 10 years, he sells the painting for \$240 000.

(i) The table below shows the value of \$1 after a given number of years of investment at a given rate of compound interest. Use this table to estimate the average rate of appreciation applied to this investment per annum. **2**

Term (Years)	Rate											
	3%	3.10%	3.20%	3.30%	3.40%	3.50%	3.60%	3.70%	3.80%	3.90%	4.00%	4.10%
1	1.0300	1.0310	1.0320	1.0330	1.0340	1.0350	1.0360	1.0370	1.0380	1.0390	1.0400	1.0410
2	1.0609	1.0630	1.0650	1.0671	1.0692	1.0712	1.0733	1.0754	1.0774	1.0795	1.0816	1.0837
3	1.0927	1.0959	1.0991	1.1023	1.1055	1.1087	1.1119	1.1152	1.1184	1.1216	1.1249	1.1281
4	1.1255	1.1299	1.1343	1.1387	1.1431	1.1475	1.1520	1.1564	1.1609	1.1654	1.1699	1.1744
5	1.1593	1.1649	1.1706	1.1763	1.1820	1.1877	1.1934	1.1992	1.2050	1.2108	1.2167	1.2225
6	1.1941	1.2010	1.2080	1.2151	1.2221	1.2293	1.2364	1.2436	1.2508	1.2580	1.2653	1.2726
7	1.2299	1.2383	1.2467	1.2552	1.2637	1.2723	1.2809	1.2896	1.2983	1.3071	1.3159	1.3248
8	1.2668	1.2766	1.2866	1.2966	1.3067	1.3168	1.3270	1.3373	1.3477	1.3581	1.3686	1.3791
9	1.3048	1.3162	1.3278	1.3394	1.3511	1.3629	1.3748	1.3868	1.3989	1.4110	1.4233	1.4357
10	1.3439	1.3570	1.3702	1.3836	1.3970	1.4106	1.4243	1.4381	1.4520	1.4661	1.4802	1.4945
11	1.3842	1.3991	1.4141	1.4292	1.4445	1.4600	1.4756	1.4913	1.5072	1.5232	1.5395	1.5558
12	1.4258	1.4425	1.4593	1.4764	1.4936	1.5111	1.5287	1.5465	1.5645	1.5827	1.6010	1.6196
13	1.4685	1.4872	1.5060	1.5251	1.5444	1.5640	1.5837	1.6037	1.6239	1.6444	1.6651	1.6860

Term (Years)	4.20%	4.30%	4.40%	4.50%	4.60%	4.70%	4.80%	4.90%	5.00%	5.10%	5.20%	5.30%	5.40%
1	1.0420	1.0430	1.0440	1.0450	1.0460	1.0470	1.0480	1.0490	1.0500	1.0510	1.0520	1.0530	1.0540
2	1.0858	1.0878	1.0899	1.0920	1.0941	1.0962	1.0983	1.1004	1.1025	1.1046	1.1067	1.1088	1.1109
3	1.1314	1.1346	1.1379	1.1412	1.1444	1.1477	1.1510	1.1543	1.1576	1.1609	1.1643	1.1676	1.1709
4	1.1789	1.1834	1.1880	1.1925	1.1971	1.2017	1.2063	1.2109	1.2155	1.2201	1.2248	1.2295	1.2341
5	1.2284	1.2343	1.2402	1.2462	1.2522	1.2582	1.2642	1.2702	1.2763	1.2824	1.2885	1.2946	1.3008
6	1.2800	1.2874	1.2948	1.3023	1.3098	1.3173	1.3249	1.3325	1.3401	1.3478	1.3555	1.3632	1.3710
7	1.3337	1.3427	1.3518	1.3609	1.3700	1.3792	1.3884	1.3977	1.4071	1.4165	1.4260	1.4355	1.4451
8	1.3898	1.4005	1.4113	1.4221	1.4330	1.4440	1.4551	1.4662	1.4775	1.4887	1.5001	1.5116	1.5231
9	1.4481	1.4607	1.4733	1.4861	1.4989	1.5119	1.5249	1.5381	1.5513	1.5647	1.5781	1.5917	1.6053
10	1.5090	1.5235	1.5382	1.5530	1.5679	1.5829	1.5981	1.6134	1.6289	1.6445	1.6602	1.6760	1.6920
11	1.5723	1.5890	1.6059	1.6229	1.6400	1.6573	1.6748	1.6925	1.7103	1.7283	1.7465	1.7649	1.7834
12	1.6384	1.6573	1.6765	1.6959	1.7155	1.7352	1.7552	1.7754	1.7959	1.8165	1.8373	1.8584	1.8797
13	1.7072	1.7286	1.7503	1.7722	1.7944	1.8168	1.8395	1.8624	1.8856	1.9091	1.9329	1.9569	1.9812

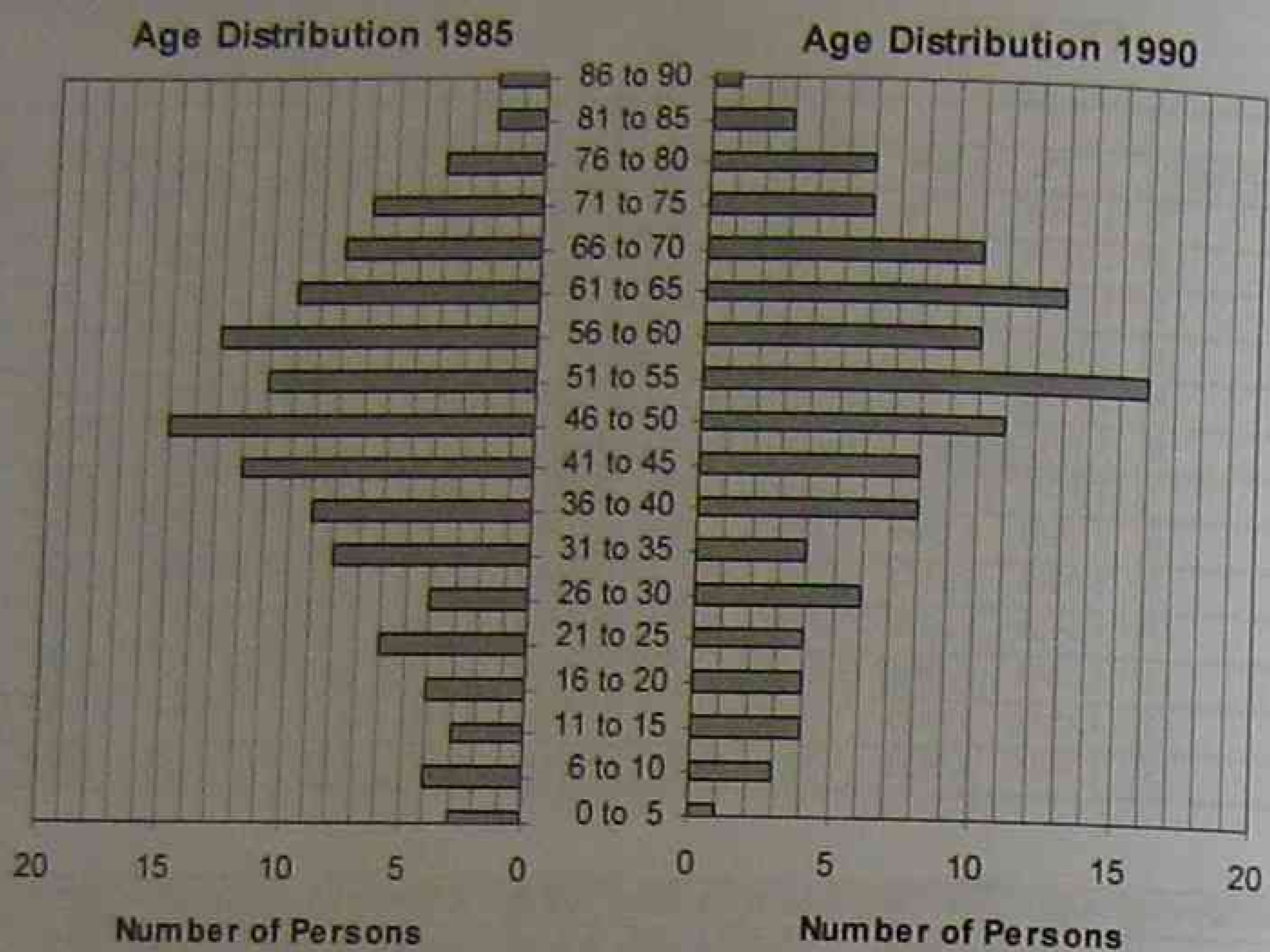
(ii) The inflation rate over the same period was 4% p.a. which we may take to be constant. What value would the \$150 000 ten years later? **1**

(iii) The investor had to pay 47 ½ % Capital Gains Tax on the profit after inflation had been accounted for. What tax is payable? **2**

End of Question 27.

Question 28 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The census in 1985, then in 1990, recorded the ages in years of the people living in a small country town. The results are illustrated in this graph.



(i) How many people were in the town in 1985? **2**

(ii) No one aged 50 or over left or came to live in the town between the two censuses. How many people who were over 80 in 1985 were still alive in 1990? **1**

(iii) There were 118 people in the town in 1990. In what group was the median age of the town in 1990? **1**

(iv) What was the modal age group in 1985? **1**

(v) From these graphs, is the population likely to decline, stay steady or grow? Give reasons for your answer. **2**

Question 28 continues on the next page.

(b) A teacher, Mr. Cardinal, decides to set aside \$200 every pay for deposit in an investment fund. The teacher is paid fortnightly. The fund pays 10% p.a. and this is compounded fortnightly.

(i) How much will the annuity be worth to the nearest dollar after 3 years of deposits, assuming 26 pays in each of those years? **2**

(ii) The teacher hoped to have \$200 000 as a retirement lump sum at the end of 7 years. Using 26 pay periods per year, how much would the teacher have to invest each fortnight to achieve his target? **2**

(iii) At the same time as the teacher starts investing, his wife, Mrs. Cardinal, establishes a term deposit account with a deposit of \$1500. The account pays 8% p.a. and operates for 6 months, then can be 'rolled over' at the same interest rate with deposits or withdrawals made at that time. If Mrs. Cardinal deposits an extra \$1500 in the account every six months and leaves the interest in the account, how much will the account be worth after three years? **2**

End of paper.

Sample Exam Paper 5

General Instructions

- Reading time – 5 minutes
- Working time – 2 ½ hours
- Write using black or blue pen
- Calculators may be used
- A Formulae Sheet is provided at the back of this paper.

Section 1 Total marks **(22)**
- Attempt all questions 1 – 22
- Allow about 30 minutes for this section

Section 2 Total marks **(78)**
- Attempt Questions 23 – 28
- Allow about 2 hours for this section.

Section 1

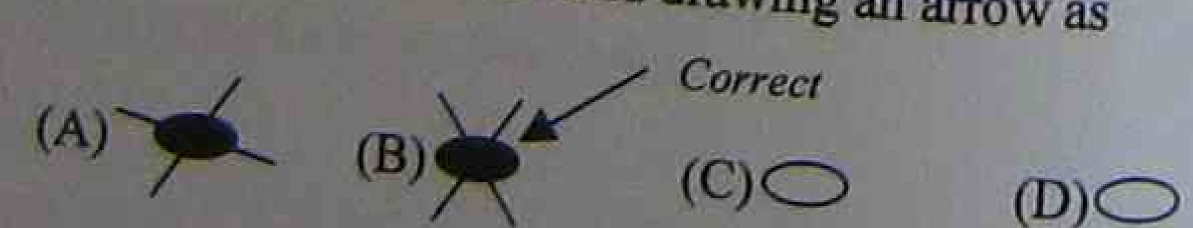
Total marks (22)
Attempt Questions 1 – 22
Allow about 30 minutes for this section.

Use the multiple choice answer sheet.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample ¼ of $320 is (A) $40 (B) $60 (C) $80 (D) $14

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:



1 A room measures 4.5 m x 5.8 m and has a ceiling 3.1 m above the floor. If paint covers 14 m^2 per litre per coat, the amount required for two coats on the walls (assuming there are no windows or doors) is

(A) about 9.1 L
(B) about 5.8 L
(C) about 4.6 L
(D) about 2.9 L

2 Alex and Jo would like to travel around Australia's coastline, a distance of about 36 000 km. Alex has a small Toyota which uses 6.4L/100 km of highway driving. Jo has a bigger car, a Ford that uses 7.2L/100 km. Petrol will cost about 90 cents per litre. How much more would the petrol for the trip cost, to the nearest dollar, if they take Jo's car?

(A) $259
(B) $360
(C) $405
(D) $2592

3 Make a the subject of the formula $v^2 = u^2 + 2as$

(A) $a = \frac{v^2 + u^2}{2s}$

(B) $a = \frac{v^2 - u^2}{2s}$

(C) $a = \frac{v - u}{\sqrt{2s}}$

(D) $a = \frac{v - u}{2s}$

4 A house is to be built on a rectangular concrete slab that is 150 mm thick and has an area of 120 m^2. The amount of ready-mixed concrete required for this slab is nearest to

(A) 1.8 m^3
(B) 18 m^3
(C) 180 m^3
(D) 18 000 m^3

5 A coin is tossed 4 times. The probability of getting "Heads" twice and "Tails" twice is

(A) 0.5
(B) 0.25
(C) 0.0625
(D) 0.375

6 Real estate values have increased at an average rate of 7% p.a. over the past 5 years. A property worth $150 000 five years ago would now be worth about

(A) $106 500
(B) $196 620
(C) $210 000
(D) $255 000

7 If 1 A.U. = 149 492 000 km, then 3.629 A.U. is closest to

(A) 5.43×10^8 km
(B) 4.12×10^7 km
(C) 2.4×10^{-8} km
(D) 1.49×10^7 km

8 If 8 M is equal to 14.816 km, then 2.3 km is equal to about

(A) 1.24 M
(B) 4.3 M
(C) 51.5 M
(D) 0.76 M

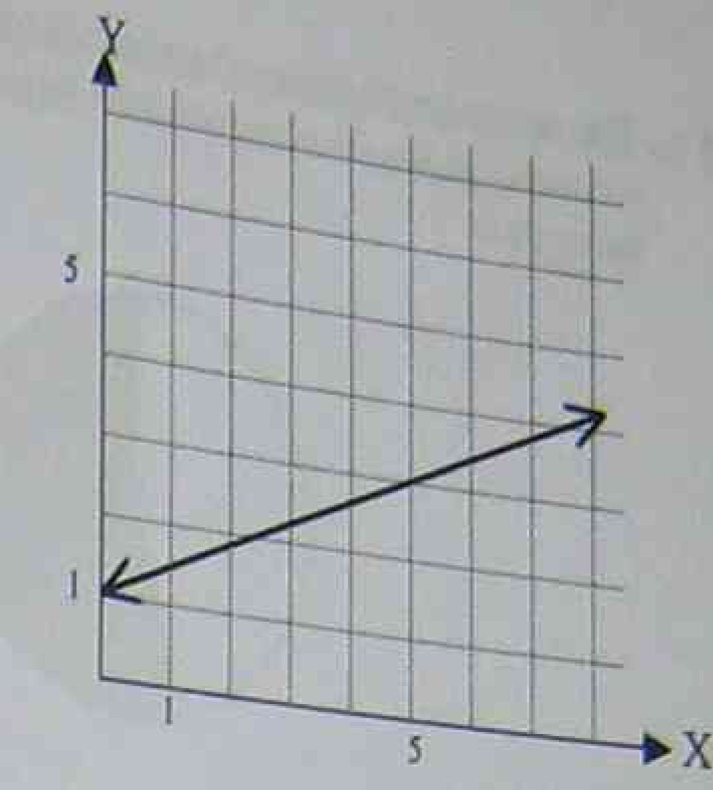
9 A train leaves Springwood at 6:46 am and reaches Central, 79.7 km away, at 7:51 am. The average speed of the train was about

(A) 53.1 km/h
(B) 73.6 km/h
(C) 75.9 km/h
(D) 79.3 km/h

10 The city of Laurencia is located 190° east of Greenwich. Therefore

(A) if it is Tuesday in Greenwich, it is Wednesday in Laurencia
(B) if it is Tuesday in Greenwich, it is Monday in Laurencia.
(C) its longitude is 190°E
(D) its longitude is 170°W

11 The gradient of the line is



(A) $\frac{5}{2}$ (B) 5

(C) $\frac{2}{5}$ (D) $\frac{1}{5}$

12 If $\frac{x+1}{3} + \frac{x-7}{4} = 2$, then

(A) $3(x+1) + 4(x-7) = 2$
(B) $4(x+1) + 3(x-7) = 2$
(C) $3(x+1) + 4(x-7) = 24$
(D) $4(x+1) + 3(x-7) = 24$

13 A survey collected information from 22 people. The results are shown in this table.

Score	Frequency
22	1
25	2
28	8
30	7
35	3
50	1

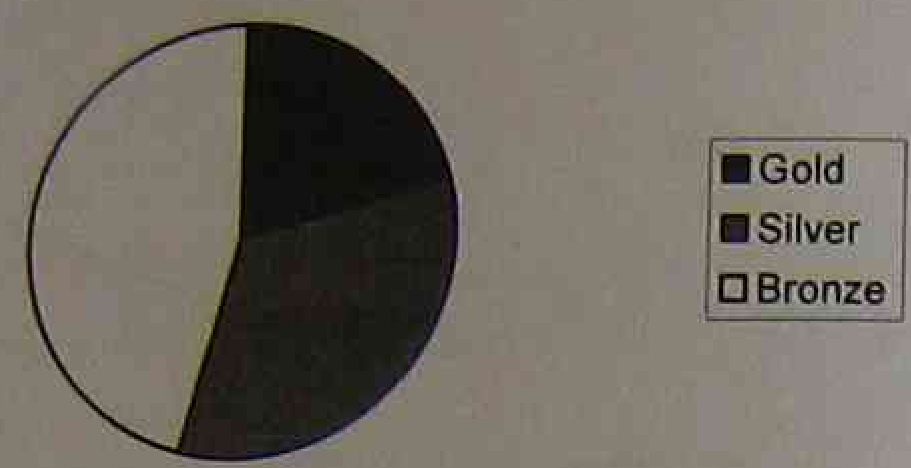
From this sample, the standard deviation of the entire population would be closest to

(A) 5.31
(B) 5.44
(C) 28.00
(D) 30.05

14 A technician is offered a salary of $58 000. At the moment she is paid $27 /h for a 40 hour week, working 52 weeks per year. The new job will pay

(A) $1840 more than the old one.
(B) $1650 more than the old one.
(C) $56 350 more than the old one
(D) $56 920 more than the old one.

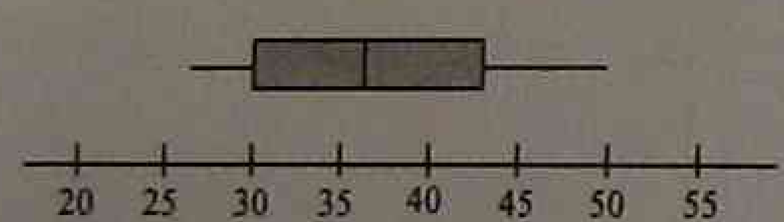
15 The sector graph shows the number of gold, silver and bronze medals won by a country at the Commonwealth Games. A bar graph is to be drawn for the same data.



Which statement could be wrong?

(A) A 36 cm bar would have 48 cm allocated to silver medals
(B) More than half the bar would be for bronze medals
(C) The silver and gold parts of the bar together would be longer than the bronze section.
(D) The part of the bar representing silver medals would be between the gold and bronze sections.

16



Statement 1: The interquartile range is about 23
Statement 2: The mean is about 37.

Which statement, if any, is true as shown by the box and whisker?

(A) 1 only
(B) 2 only
(C) Both 1 and 2
(D) Neither

17 If $s = ut + \frac{1}{2}at^2$ and $s = 12$, $u = 5$ and $t = 1$, then $a =$

(A) $3\frac{1}{4}$
(B) 7
(C) $6\frac{1}{2}$
(D) 14

18 A poker hand consists of 5 cards drawn without replacement from a pack of 52 cards which are all unique. The number of possible hands is

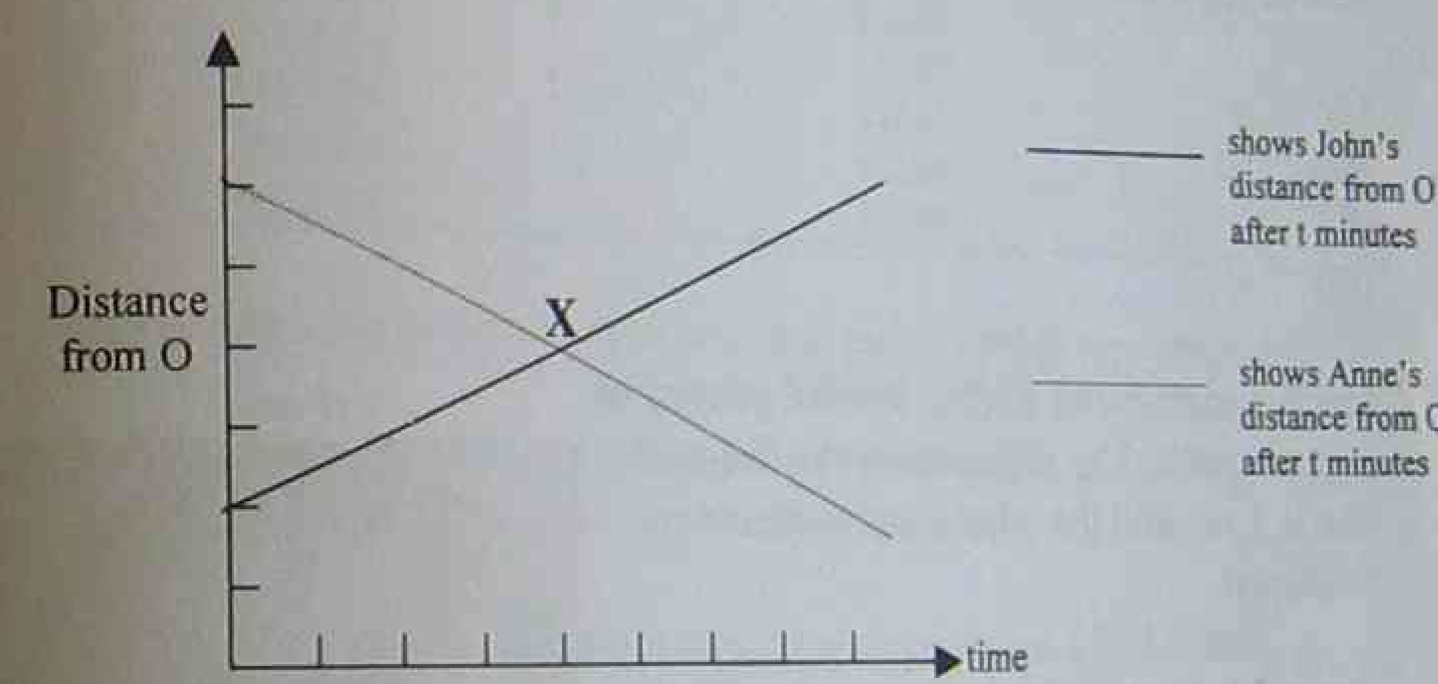
(A) $\dfrac{52 \times 51 \times 50 \times 49 \times 48}{5}$

(B) $52 \times 51 \times 50 \times 49 \times 48$

(C) $\dfrac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2}$

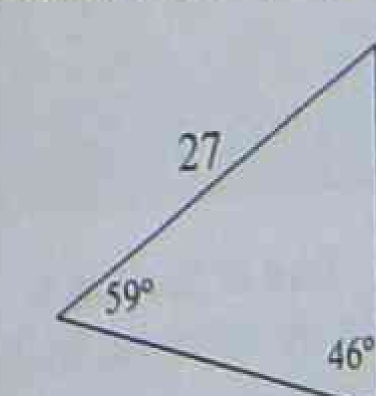
(D) $\dfrac{52 \times 51 \times 50 \times 49 \times 48}{13 \times 4}$

19 This graph shows the distance of two people from a point O after a given time has elapsed.



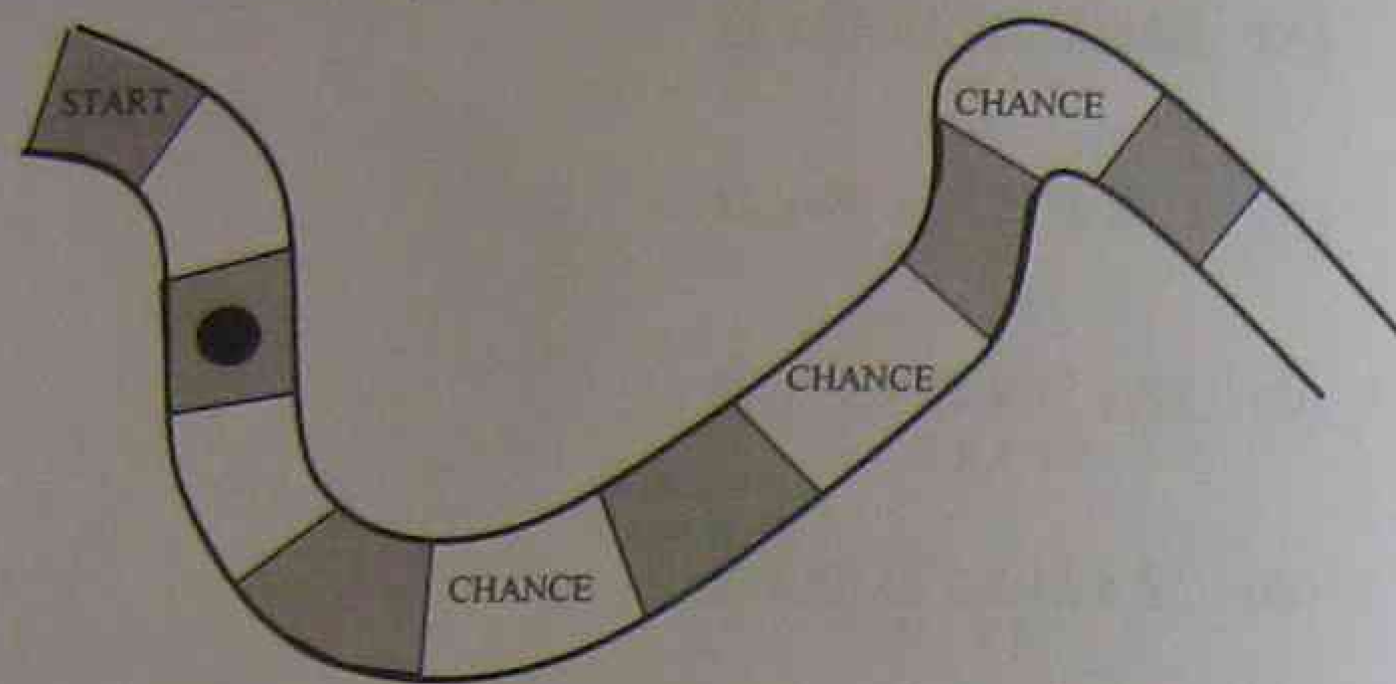
(A) Anne and John meet at X.
(B) Anne is moving away from her starting point whilst John is moving closer to it.
(C) Anne's average speed is the same as John's.
(D) Anne and John are going in opposite directions.

20 In the triangle shown , the side marked y is given by the expression



(A) $\dfrac{27 \sin 46^\circ}{\sin 59^\circ}$
(B) $\dfrac{27 \sin 59^\circ}{\sin 46^\circ}$
(C) $27 \sin 46^\circ$
(D) $27 \sin 59^\circ$

21 The diagram below shows part of a board game. The player has a black marker and moves according to the result of rolling a single cubic die which is numbered in the usual way.



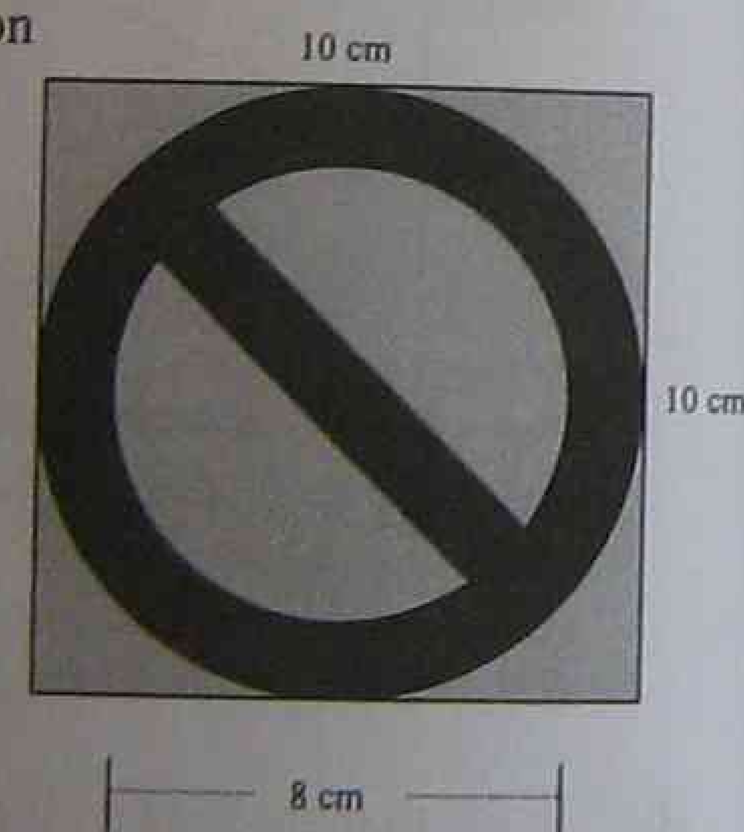
The probability of the player landing on "Chance" as the result of the next roll of the dice is

(A) $^1/_6$
(B) $^2/_3$
(C) $^1/_3$
(D) $^1/_2$

22 The diagram shows a rebus symbol painted on a grey square. The thickness of the diagonal line is 1 cm and the other measurements are as shown.

The proportion of the square which is black is closest to

(A) 36%
(B) 14%
(C) 40%
(D) 28%



End of Section 1.

Section II

Total marks (78)
Attempt Questions 23 – 28
Allow about 2 hours for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 23 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) A couple arrange a \$75 000 housing loan at 14.5% p.a. interest. The loan is to be repaid over a period of 25 years. Their bank provides the following table.

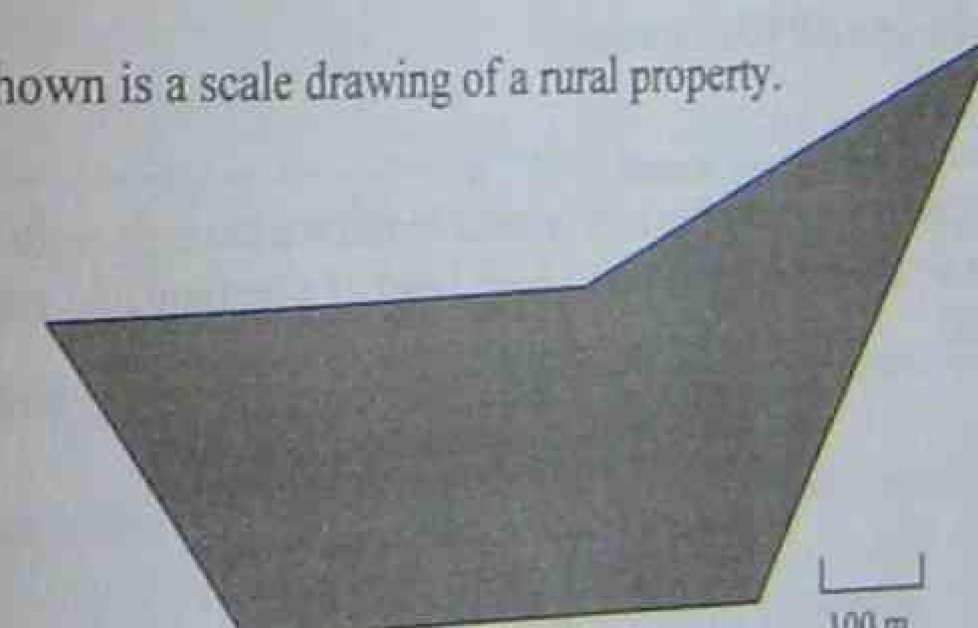
Interest Rate p.a.	Monthly Loan Repayment per \$1000
14.00%	\$12.04
14.25%	\$12.23
14.50%	\$12.42
14.75%	\$12.61
15.00%	\$12.81
15.25%	\$13.00

(i) Use the table above to calculate their monthly repayments. 1

(ii) Calculate the total amount to be repaid by the end of the 25 year period. 1

(iii) What is the equivalent simple interest rate on a \$75 000 loan that would generate the same amount of interest? 2

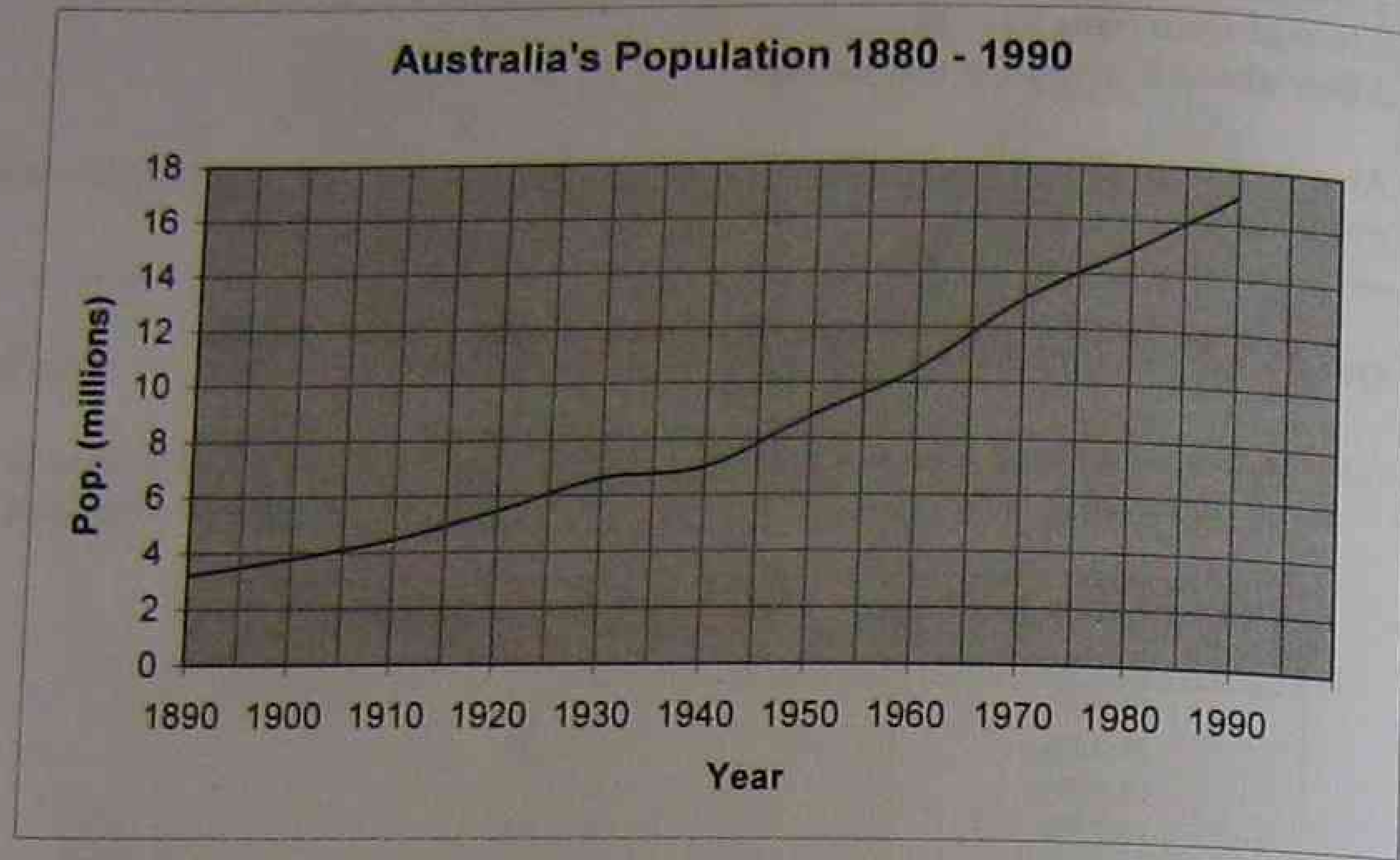
(b) The plan shown is a scale drawing of a rural property.



(i) Calculate the length of the perimeter of the property. 1

(ii) Calculate the area of the property in square metres. (Hint: Divide the plan into triangles and take the necessary measurements.) 2

(c) This line graph shows how the population of Australia increased over a period of 100 years.



(i) What was Australia's population in 1920? **1**

(ii) In which decade did Australia's population increase at the highest rate? **1**

(iii) What was the percentage increase in Australia's population between 1890 and 1990? **1**

(iv) Use the graph to estimate Australia's population in 2000. Give reasons for your answer. **1**

(d) Use appropriate drawing instruments to draw a triangle with sides 8.4 cm, 7.1 cm and 5.2 cm. Note that a high level of accuracy and neatness is required. **2**

End of Question 23.

Question 24 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) The number of matches in 10 packets was found to be

48 46 50 49 49 45 51 53 50 49

(i) Calculate the range of the sample. **1**

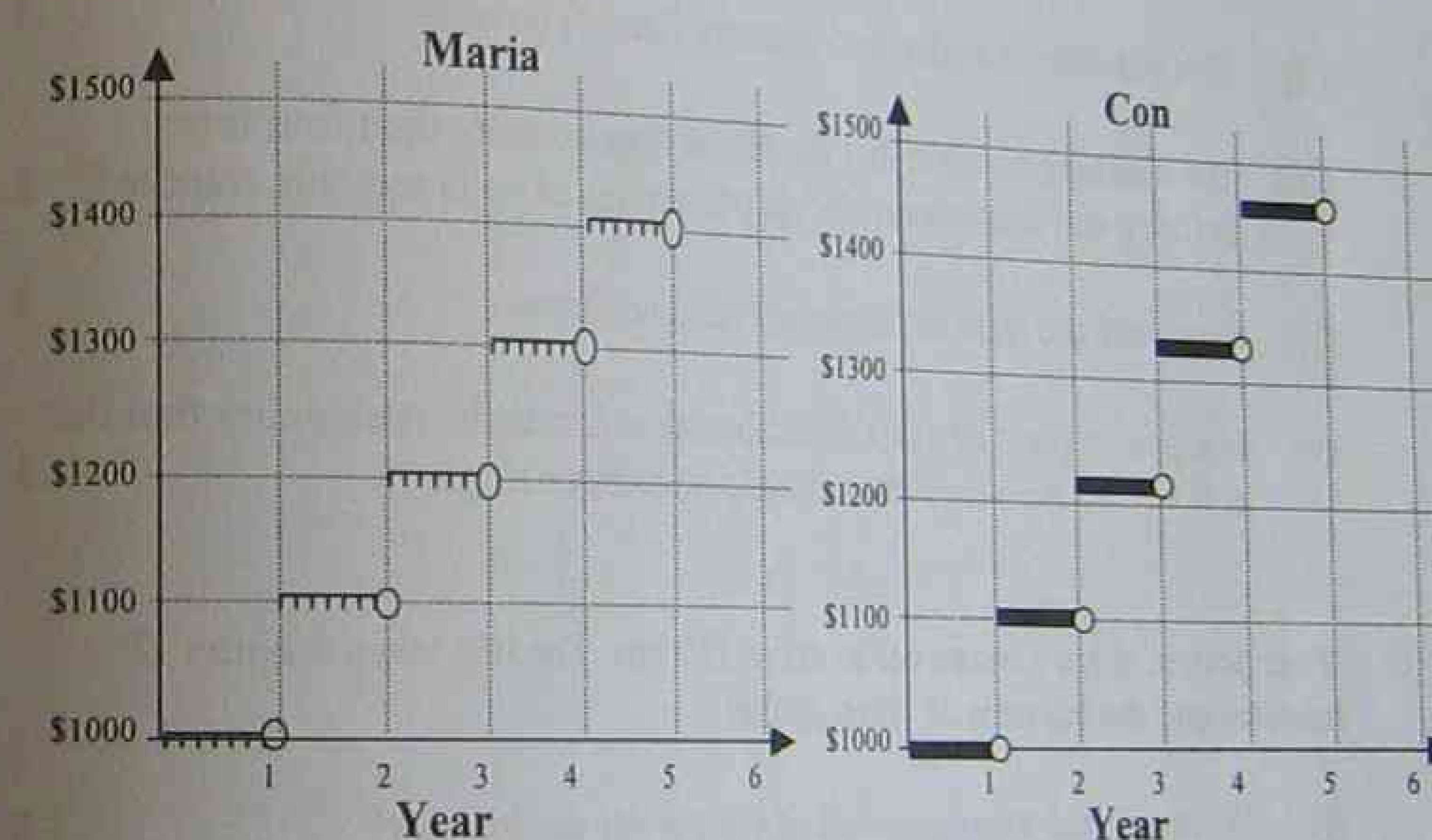
(ii) Calculate the mode of the sample. **1**

(iii) What is the median of the sample? **1**

(iv) Calculate the mean of the sample. **1**

(v) Which of the four statistics above gives the best indication of the number of matches in a box? Give reasons for your answer. **2**

(b) Con and Maria each open an investment account with $1000. The amount in the account is shown in the graph below.



(i) Which investment is worth more at the end of one year? **1**

(ii) What is the value of Maria's investment after five years? **1**

(iii) Describe the type and rate of interest earned by Maria. **2**

(iv) Describe the type and rate of interest earned by Con. **2**

(v) What will Con's investment by worth after ten years at this rate? **1**

End of Question 24.

Question 25 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) An office manager installed a device to record the number of rings on a switchboard telephone before it was answered.

No. rings	Frequency	Cum. f	Rel.f
1	3		
2	5		
3	13		
4	10		
5	7		
6	7		
7	2		
8	1		
9	1		
10	1		
Total			

(i) How many calls were monitored? **1**

(ii) Complete the cumulative frequency (cum.f) column. **1**

(iii) The manager required that the phone ring no more than four times before a call is answered. What percentage of calls met this standard? **1**

(iv) Complete the relative frequency (rel.f) column. **1**

(v) If a person rings into this switchboard, estimate the probability that the call will be answered immediately after three rings. **1**

(b) Four people share a motor car to drive 320 km. The trip takes 5 hours 20 minutes and the car uses 27 litres of fuel.

(i) What was the average speed of the car for the journey? **1**

(ii) What was the average fuel consumption in litres per 100 km for the journey? **1**

(iii) Calculate each person's share of the fuel cost if petrol was 97.9 cents per litre. **2**

Question 25 continues on the next page.

(c) A nursery follows this schedule of charges.

Plant type	Cost per plant ($)
Banksia	9
Gum tree	12
* Delivery charge is $2 per plant	

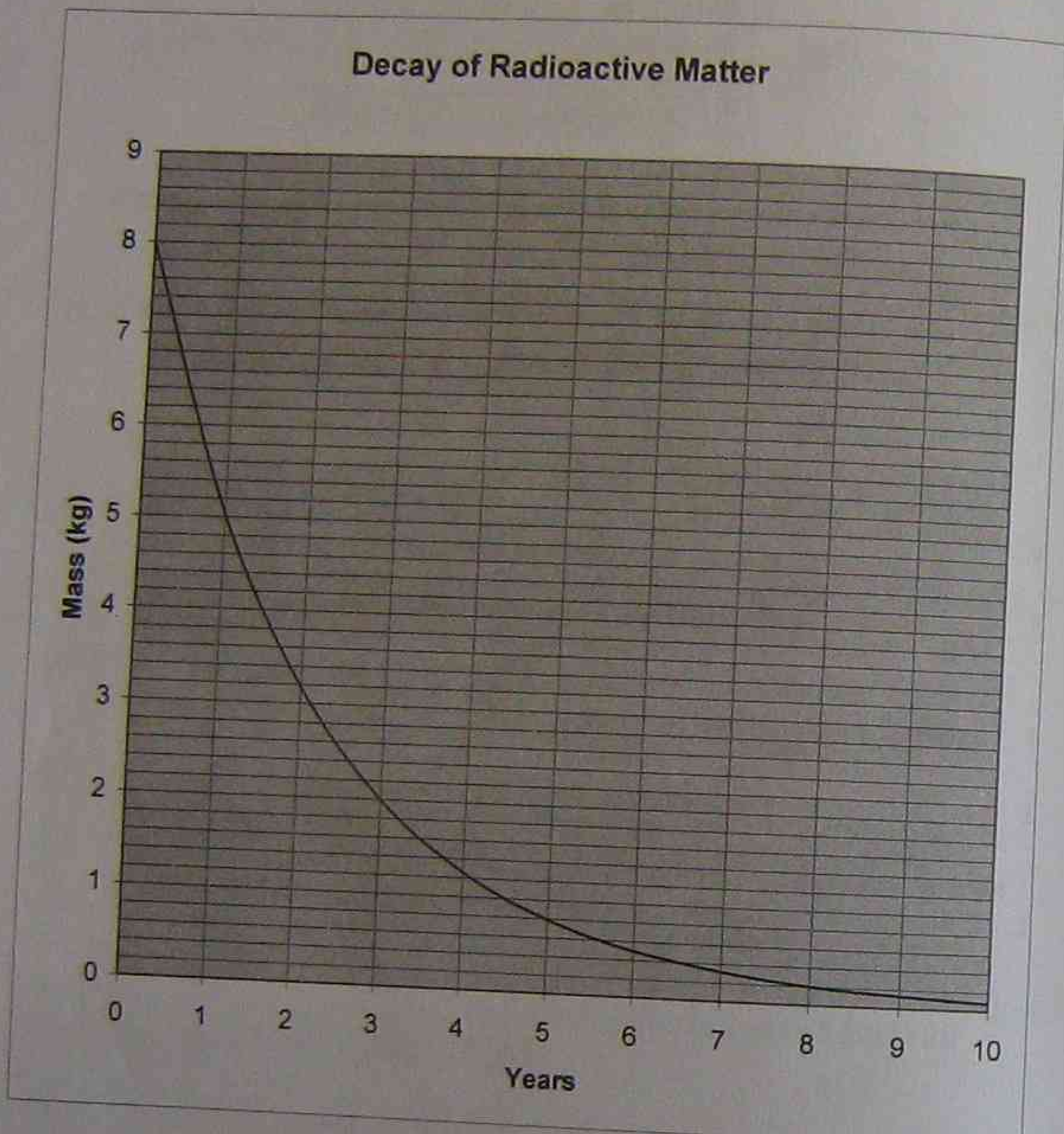
(i) Write an expression for the total cost, including delivery, of 6 banksias and m gum trees. **2**

(ii) If the total charge for an order of gum trees was $126, how many plants were purchased? **2**

End of Question 25.

Question 26 (13 marks) Use a SEPARATE writing booklet. **Marks**

(a) The mass of radioactive material was measured annually and results graphed.



(i) What was the initial mass of material? **1**

(ii) The half life is the time for the mass to reduce by 50%. What is the half life of this substance? **1**

(iii) By what factor is the mass multiplied each year? **1**

(iv) Would the mass ever be zero? Explain. **1**

Question 26 continues on the next page.

(b) A soccer squad of 12 players decides to choose the reserve and the captain for a match by drawing the names randomly from a hat, without replacing the first name before drawing the second.

(i) What is the probability of a particular player being neither the captain nor the reserve? **1**

(ii) There are two brothers in the squad. What is the probability that one will be captain and the other the reserve? **2**

(iii) The name of the reserve is drawn first but the result is not told to the players. What is the probability that a particular player will now become captain? **1**

(c) Turn to the diagram on the next page (tear-off page). The two shapes are similar pentagons.

(i) By measuring corresponding sides, determine the ratio of the side lengths of the small pentagon to those of the larger pentagon. Write your answer in the form 1 : x **2**

(ii) Draw lines of perspective to find the centre of enlargement or vanishing point. Mark the point O on the diagram. **1**

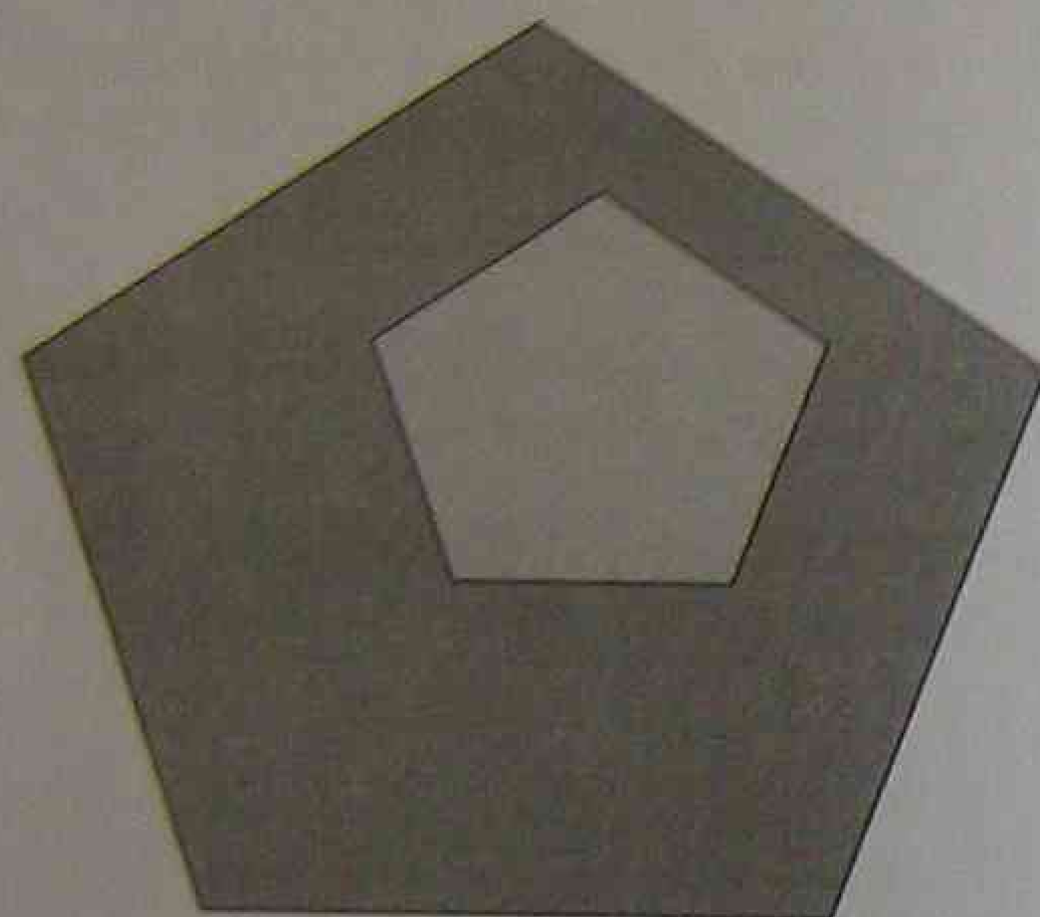
(iii) How many times greater than the area of the small pentagon is the area of the large pentagon? **1**

(iv) Draw a pentagon similar to these which has side lengths three times those of the smaller pentagon. Use O as the centre of enlargement. **1**

End of Question 26.

Diagram for Question 26 (c).

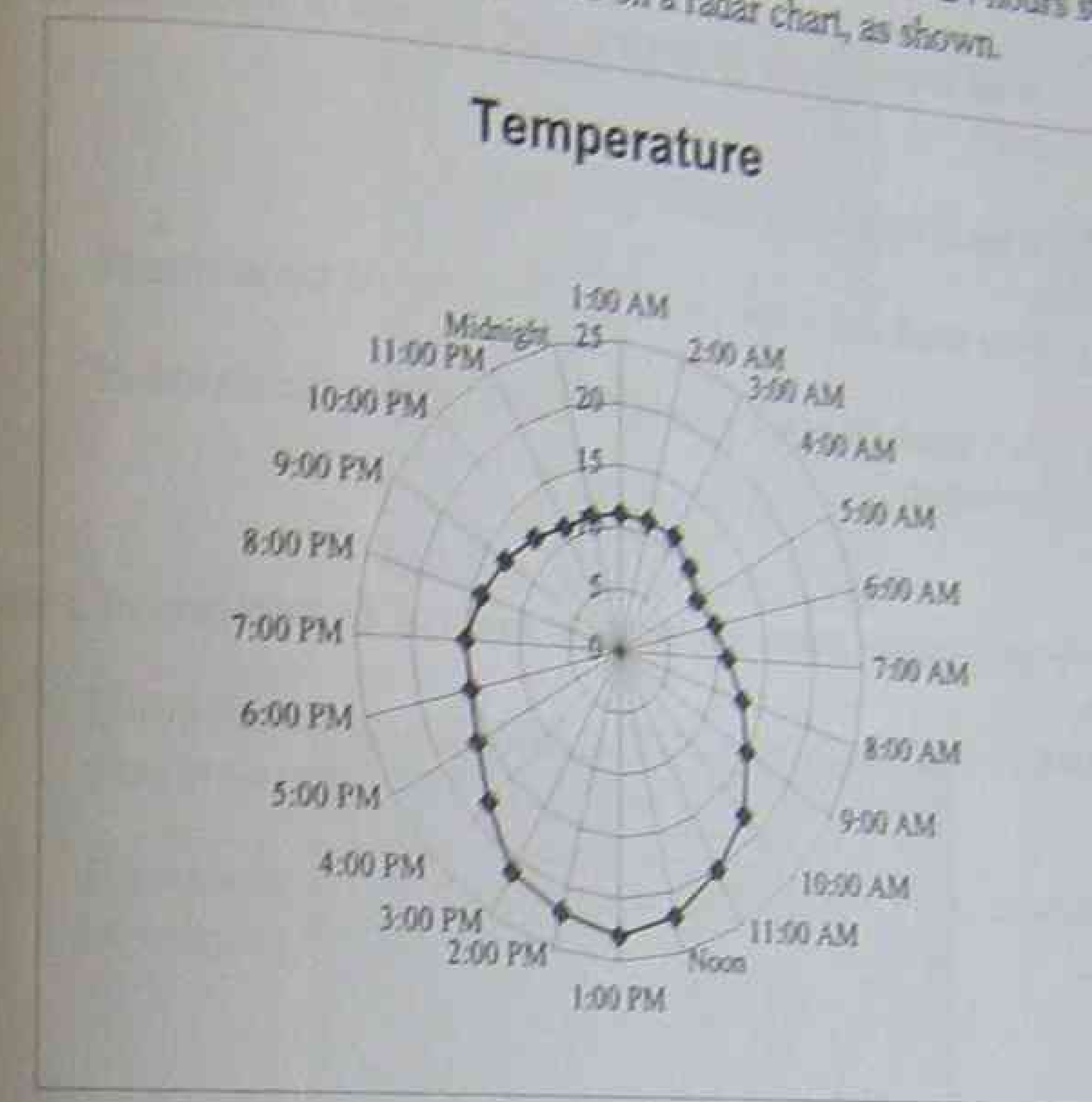
Tear along this line



Question 27 (13 marks) Use a SEPARATE writing booklet.

Marks

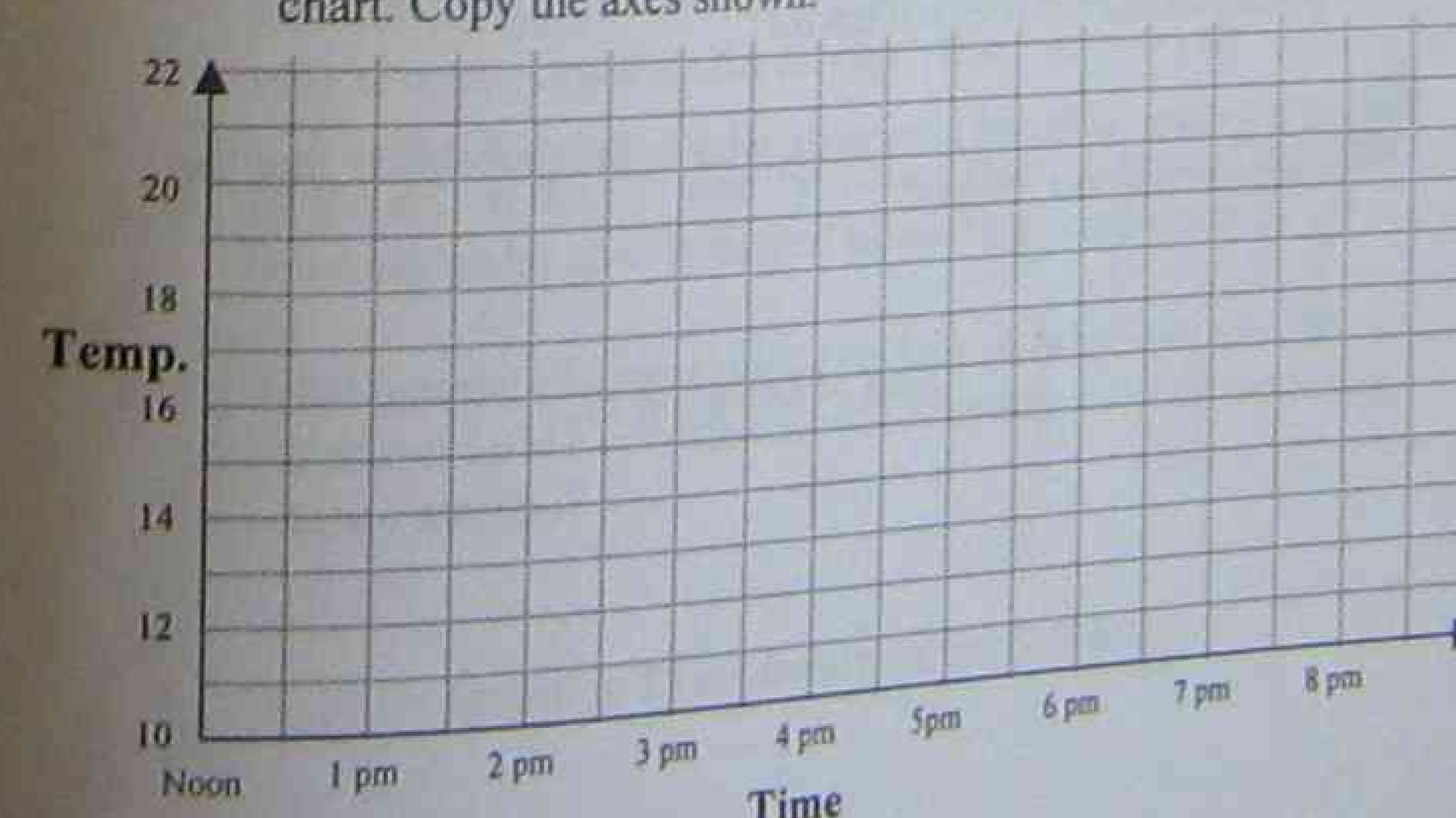
(a) The room temperature in °C was recorded every hour for 24 hours starting at midnight. The results were plotted on a radar chart, as shown.



(i) Complete this table of values from the radar chart. 2

Time	1am	2 am	3 am	4 am	5 am	6 am	7 am	8 am	9 am	10 am
Temp.										

(ii) Complete a line graph to show the same information as the radar chart. Copy the axes shown. 2



(b) A person saving for their retirement invests \$5000 in an account every 3 months. The interest is compounded quarterly with an interest rate of 8% p.a..

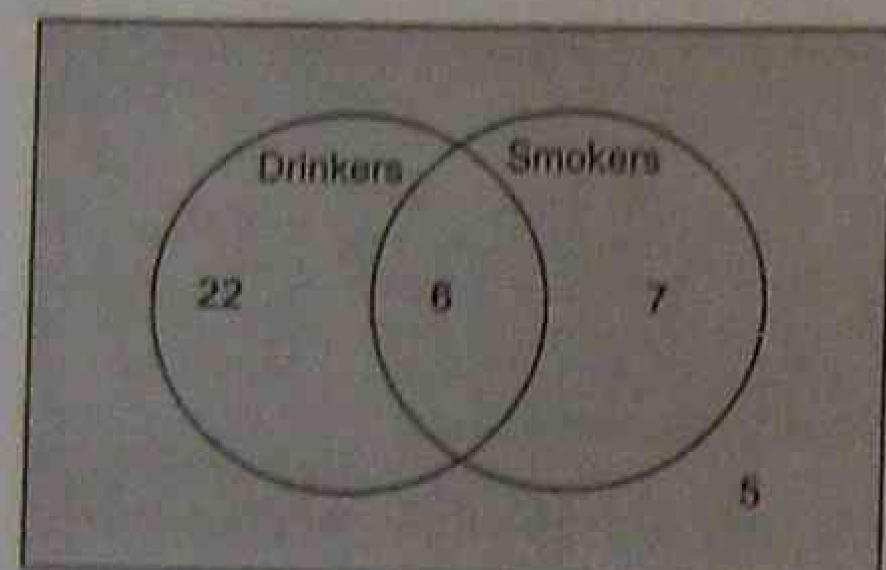
(i) What is the value of the annuity after 20 years of investment at this rate? **2**

(ii) What is the present value of the annuity? **1**

(iii) How much money was actually deposited into the account? **1**

(iv) How much interest was actually earned during the period of investment? **1**

(c) The Venn Diagram below illustrates the habits of the members of a social club.



(i) How many of the members smoke? **1**

(ii) How many of the members neither drink nor smoke? **1**

(iii) If a member was chosen at random, what is the probability that the person drinks but does not smoke? **1**

(iv) If a smoker is chosen at random, what is the probability that the member is also a drinker? **1**

End of Question 27.

Question 28 (13 marks) Use a SEPARATE writing booklet.

Marks

(a) Tickets are being sold in a raffle, as shown in this advertisement.

(i) If 400 tickets are sold, what is the financial expectation of someone who buys one ticket? **1**

(ii) How many tickets would a person have to buy in order to be certain of winning at least one prize? **1**

(b) A cotton grower buys tax deductible equipment to the value of \$1.3 million. After one year, the grower has a choice of how to claim the depreciation that has occurred. The equipment can be considered obsolete after 12 years.

(i) Using a flat rate (straight line) method of calculating depreciation, determine how much could be claimed each year. **1**

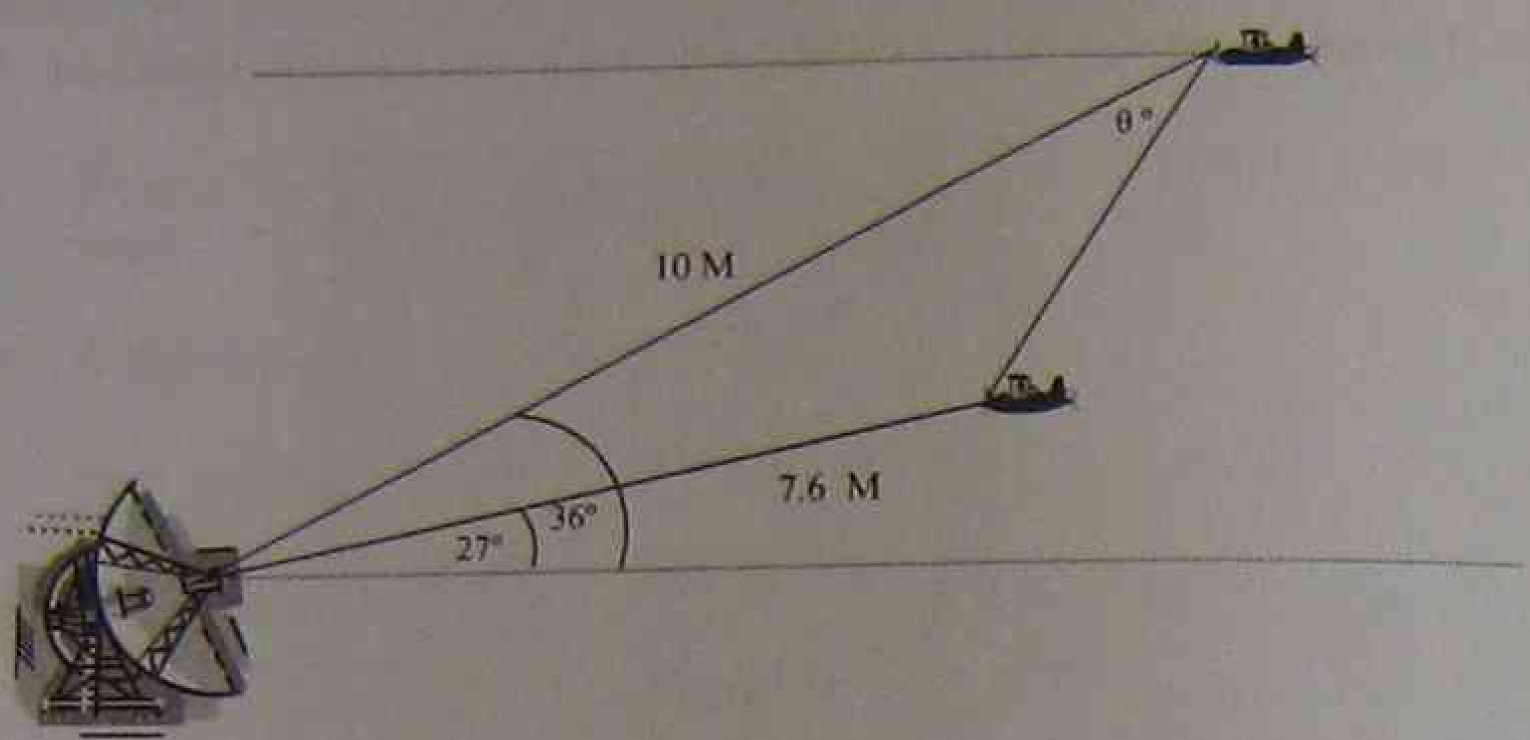
(ii) The taxation Office allows the grower to use the declining balance method using a 22% deduction each year based on the previous year's value. Calculate the salvage value after one year. **1**

(iii) At the end of the 12^{th} year, how much is the equipment considered to be worth under each system of calculating depreciation? **2**

(iv) What amount of depreciation could the grower claim as a tax deduction for the seventh year of ownership under the declining balance method? **2**

Question 28 continues on the next page.

(c) Two aeroplanes are approaching an airport, as shown.



At the moment illustrated, the airport radar identifies one 'plane 7.6 M away at an angle of elevation of 27°. Another plane on the same approach path is 10 M away at an angle of 36° from horizontal.

(i) What is the angle of depression of the radar disc as seen by the more distant of the two aircraft? **1**

(ii) Calculate the direct distance between the two aircraft? **2**

(iii) What is the angle of depression of the lower aircraft from the higher aircraft? (Hint: find the angle marked θ in the sketch first.) **2**

End of paper.

Formulae Sheet

Area of an annulus

$A = \pi(R^2 - r^2)$

where R is the radius of then outer circle,
r is the radius of the inner circle.

Area of an ellipse

$A = \pi ab$

where a = length of the semi-major axis
and b is the length of the semi-minor axis.

Area of a sector

$$A = \frac{\theta}{360}\pi r^2$$

where θ is the number of degrees of the central angle.

Arc length of a circle

$$l = \frac{\theta}{360}2\pi r$$

where θ is the number of degrees of the central angle.

Surface area of a sphere

$A = 4\pi r^2$

Simpson's Rule

$$A = \frac{h}{3}(d_1 + 4d_2 + d_3)$$

where h = distance between successive measurements
d_1 is the first measurement
d_2 is the middle measurement
d_3 is the last measurement.

Volume

Cone $V = \frac{1}{3}\pi r^2 h$

Cylinder $V = \pi r^2 h$

Pyramid $V = \frac{1}{3}Ah$

Sphere $V = \frac{4}{3}\pi r^3$

where A is the area of the base
h is the perpendicular height.

Mean of a set of scores

$$\bar{x} = \frac{1}{n}\sum x$$

$$\bar{x} = \frac{\sum fx}{\sum f}$$

where x is the score value
$\bar{x}$ is the mean

z-scores

$$z = \frac{x - \bar{x}}{s}$$

where s is the standard deviation.

Probability of an event

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

Simple interest

$I = Prn$

where P is the principal or initial amount
r is the percentage interest rate expressed as a decimal
n is the number of periods

Compound interest

$A = P(1 + r)^n$

where A is the final balance
P is the principal or initial amount
n is the number of periods
r is the percentage interest rate expressed as a decimal.

Future value (A) of an annuity

$$A = M\frac{\{(1+r)^n - 1\}}{r}$$

where M is the contribution per period, paid at the end of the period.

Present value (N) of an annuity

$$N = M\frac{\{(1+r)^n - 1\}}{r(1+r)^n}$$

or

$$N = \frac{A}{(1+r)^n}$$

Straight-line formula for depreciation

$s = V_0 - Dn$

where S is the salvage value of the asset after n periods
V_0 is the purchase price of the asset
D is the amount of depreciation apportioned per period
n is the number of periods.

Declining balance formula for depreciation

$S = V_0(1 - r)^n$

where S is the salvage value of the asset after n periods
r is the percentage interest rate per period, expressed as a decimal.

Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Area of a triangle

$A = \frac{1}{2} ab \sin C$

Cosine Rule

$c^2 = a^2 + b^2 - 2ab \cos C$

or

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Gradient of a straight line

$$m = \frac{\text{vertical change in position}}{\text{horizontal change in position}}$$

Gradient-intercept form of a line

$y = mx + b$

where m is the gradient
b is the Y intercept.

ANSWERS AND CONSIDERATIONS

Financial Mathematics

Multiple choice

1. D $47\frac{1}{2} \times \$4.50$
2. C $(985.70 - 300) per week, then take away bills.
3. C $485 ÷ 39h
4. A $615 + $2.20 x 2 x 89
5. A
6. B $2.05 + $1.25 x 19
7. A $32 652 ÷ $52^1/_7$ x 1.175
8. B Taxable income = $29 650 x 0.9
   Tax = $3060 + $2034.90
   $26 685 - $5094.90 ÷ $52^1/_7$ x 2
9. C ($169.10 - $92.60) ÷ 0.9
10. A $8000 x 0.095
11. D $(1 + 0.085)^7 \times 50\,000 - 50\,000$
12. B $^5/_{100}$ ÷ 12
13. C
14. A Balances, for X: $(1.06)^{10} = 1.7908$
    For Y: $(1.014375)^{40} = 1.7699$
    For Z: $(0.062 \times 10 + 1 = 1.62$
15. C 650 = 3800 x r x 7
16. C Balance = $5000(1.0175)^{80}$
17. B Have to test B, C and D. Clearly not flat.
18. D $(1.075)^{18} \times 2000 = 7351$, so $5351 interest
19. B $(1 + ^r/_2)^{20} = 2$. Check each r given
20. C 452 ÷ 38.5 = 11.74 /h.
    Take-home pay = 11.74 x 1.5 x 0.615
21. B 20 cents x 100
22. A $9261.50 + 0.44125 x extra
    = $10 000 so extra = $1673.65
    Income = $38 000 + $1673.65
23. D Tax on $37 072 = $8491.50 + $412.72
24. B $23 560 x 0.0125
25. C $^{10}/_{11}$ x $890
26. C 0.2 x $2600 = $520, then $^{52000}/_{80\,00}$
27. D $5 goes to $6, i.e.20% extra
28. C $20 000+ bracket
    $2380 + 0.3 x extra = $5950
    Extra = $11 900
29. A Old: 0.17 x $9500 = $1615
    New: 0.2 x $10 100 = $2020
30. A 28 x 0.0005753 x $900
31. B Say 156 repayments.
    Total repayments = 156 x $45 on $6000 borrowed. Interest = $1020
    so $^{1020}/_{6000 \times 3}$ x 100
32. D Rate = 0.23 ÷ 365.25
    = 0.0006297056
    Balance = $220(1.0006297056)^{30}$
    = $224.19
33. A Flat: $3600 x $^6/_{100}$ x 4 = $864
    Reducible: $87.88 x 48 = $4218.24, giving $618.24 interest.
34. A

N	A ($)	A + I ($)	A + I − R ($)
1	10 000.00	10 075.00	9867.42
2	9867.42	9941.43	9733.85
3	9733.85	9806.85	9599.27
4	9599.27	9671.27	9463.68
5	9463.68	9534.67	9327.09
6	9327.09	9397.04	9189.46

35. C Solve $9000 = \frac{M\{(1 + ^{15}/_{1200})^{72} - 1\}}{^{15}/_{1200}(1 + ^{15}/_{1200})^{72}}$
36. B 10 yrs @ 6%: 100000 = M x 7.36009
    M = 13 586.79
    Repayments = $M x 10
    = $135 867.90
    6 yrs @ 10%: 100000 = M x 4.35526
    M = 22 960.74
    Repayments = $M x 6
    = $137 764.45
    Difference = $1896.55
37. C $A = \frac{1\{(1 + 0.00024640657)^{10958} - 1\}}{0.00024640657}$
    = 56 388.89
38. D $A = \frac{200\{(1.00833333)^{240} - 1\}}{0.0083333(1.00833333)^{240}}$
39. A $40\,000 = \frac{M\{(1.01)^{72} - 1\}}{1.01\,(1.01)^{72}}$
40. C $2400(0.75)^3$
41. D St. line: $10 000 deprec. each year.
    Dec. bal.:$12 000, $10 560, $9292.80, $8177.66, ...
42. B $100 = 2500(1 - r)^7$
    $(1 - r)^7 = ^1/_{25}$
    r = 0.37
43. B 100 ÷ 7 = 14.28 ...
44. D $600 in 15 yrs is $40/yr
    $^{40}/_{600}$ = 6.67%
45. B 20% depreciation reverses to 25% appreciation.
    $400(1.25)^{10}$ = $3725
46. A $20 000 - $4000 = $16 000 in 8 years
    $2000 per year is 10%
47. B Not straight line. $^{9500}/_{12000}$ = 79%, so 21% depreciation.

Free response

1. (a) \$2600
   (b) \$2600 ÷ (90 x 6) = about \$4.81
   (c) \$5 if sell-out guaranteed, \$6 or more to make profit.
2. (a) 3.4 x \$2.80 + \$3.50 = \$13.02
   (b) Separately, 2 x \$13.02 = \$26.04
   Together, 6.8 x \$2.80 + \$3.50 = \$22.54, thus saving \$3.50
   (c) \$20 - \$3.50 = \$16.50
   \$16.50 ÷ \$2.80 = 5.89 kg
   so heaviest parcel would be 5 kg
3. (a) \$1240.44 ÷ 14 x 365 = \$32 340
   (b) \$1009.92
   (c) \$1240.44 ÷ 70 = \$17.72
   (d) \$1009.92 - \$662 = \$347.92
4. (a) \$39 650
   (b) \$32.03
   (c) \$45.75
   (d) \$30.50 x 200 x 7 = \$42 700
5. \$630 x 1.175 = \$740.25
6. (a) \$0.995 x 15670 = \$15 591.65
   (b) \$9.95 x 0.65 x 0.15 x 15 670 = \$15 201.86
7. (a) (i) \$5000 + \$3000 + \$3750 = \$11 750
   (ii) \$11 750 + 2% of \$180 000 = \$15 350
   (b) 2% of extra = \$830
   Extra = \$41 500
   Selling price = \$391 500
8. (a) $64000(1+0.07)^{10}$ = \$125 897 say \$126 000
   (b) About \$19 000
   (c) 0.475 x 19 102 = \$9073.45
   (d) $145000(1+0.02)^{7}$ = \$166 559
9. 13 unit of ratio = \$28 951
   1 unit of ratio = \$2227
   8 units of ratio = \$17 816
10. $23000(1+0.05)^{19}$ = \$58 120
11. (a) $3000(1+0.01)^{24}$ = \$3809.20
    (b) \$809.20
12. (a) I = 15T
    (b) 3% p.a.
13. (a) Old: \$3060 + 35.5% of \$9300 = 6361.50
    New: \$2380 + 30% of \$10 000 = \$5380
    (b) Old: \$14 556.50 + 47% of \$50 000 = \$38 056.50
    New: \$15 580 + 47% of \$40 000 = \$34 380
    Pays \$3676.50 less with new system
    (c) Old: \$3060 + 35.5% of \$14 300 = \$8136.50
    New: \$2380 + 30% of \$15 000 = \$6880
    Difference is \$1256.50. GST adds \$2080 so \$823.50 worse off per year, \$31.67 worse off per fortnight.
14. (a) From 13 (a), \$981.50
    (b) 52 x \$40 = \$2080 so about \$1100 worse off.
15. (a) £10.75
    (b) £96.75
16. (a) 47 cents
    (b) \$402.37
    (c) 14.95%
17. (a) \$9660
    (b) \$4636.80
    (c) \$14 296.80 ÷ 48 = \$297.85
18. (a) \$1000 + 36 x \$300 = 11 800
    (b) \$9000
    (c) \$1800
    (d) $\frac{1800}{9000} \times \frac{100}{3}$ = 6.67%
19. (a) $A = \dfrac{12000\{(1+0.035)^{11}-1\}}{0.035}$ = \$157 703.90
    (b) $\frac{175\,000}{157\,703.90}$ x 12 000 = \$13 316
    (c) $A = \dfrac{12000\{(1+0.035)^{4}-1\}}{0.035}$ = \$50 579.31
    (d) $\$50578.31(1.025)^{6}$ = \$58 656.50
    (e) $A = \dfrac{12000\{(1+0.025)^{6}-1\}}{0.025}$ = \$76 652.84
    Gemma's total = \$135 309.34
    Loss = \$157 703.90 - \$135 309.34 = \$22 394.56
20. $300\,000 = \dfrac{M\{(1+\frac{0.0725}{12})^{60}-1\}}{\frac{0.0725}{12}}$
    M = \$4163.31
21. (a) $A = \dfrac{5\{(1+0.005)^{252}-1\}}{0.005}$ = \$2514.37
    (b) \$5 x \$252 = \$1260
    (c) (i) $\frac{10000}{\$2514.37}$ x 5 = \$19.88 (say \$20)
    (ii) $N = 10\,000 \div (1+0.005)^{252}$ = \$2845.46
22. (a) (i) $40\,000 = \dfrac{M\{(1+\frac{15}{1200})^{12}-1\}}{\frac{15}{1200}(1+\frac{15}{1200})^{12}}$
    M = 3610.33
    (ii) $40\,000 = \dfrac{M\{(1+\frac{15}{1200})^{36}-1\}}{\frac{15}{1200}(1+\frac{15}{1200})^{36}}$
    M = 1386.61
    (iii) $40\,000 = \dfrac{M\{(1+\frac{15}{1200})^{120}-1\}}{\frac{15}{1200}(1+\frac{15}{1200})^{120}}$
    M = 645.34
    (iv) $40\,000 = \dfrac{M\{(1+\frac{15}{1200})^{240}-1\}}{\frac{15}{1200}(1+\frac{15}{1200})^{240}}$
    M = 526.71
    (b) (i) \$43 323.96
    (ii) \$49 917.96
    (iii) \$77 440.80
    (iv) \$126 410.40
    (c) Under 10 years, nearly 9 years.
23. (a) \$86 000
    (b) (i) $86\,000 = \dfrac{M\{(1+\frac{7.3}{1200})^{300}-1\}}{\frac{7.3}{1200}(1+\frac{7.3}{1200})^{300}}$
    M = 624.39
    (ii) \$624.39 x 12 ÷ 365 x 7 = \$143.69 at least
    (iii) \$624.39 x 12 = \$7492.68
    (iv) \$187 317
    (v) \$101 317
    (c) (i) $80\,000 = \dfrac{M\{(1+\frac{7.3}{1200})^{300}-1\}}{\frac{7.3}{1200}(1+\frac{7.3}{1200})^{300}}$
    M = 580.83
    (ii) \$174 249
    (iii) \$94 249
    (iv) \$7068
24. Use K button and count presses. 7 years
25. (a) \$2564.54
    (b) \$1875
26. (a) \$65 000
    (b) \$1 000 000 ÷ \$65 000 = 15.4 years
    (c) $1\,000\,000(1-r)^{10} = 350\,000$
    r = 0.1, i.e. 10%
27. (a)

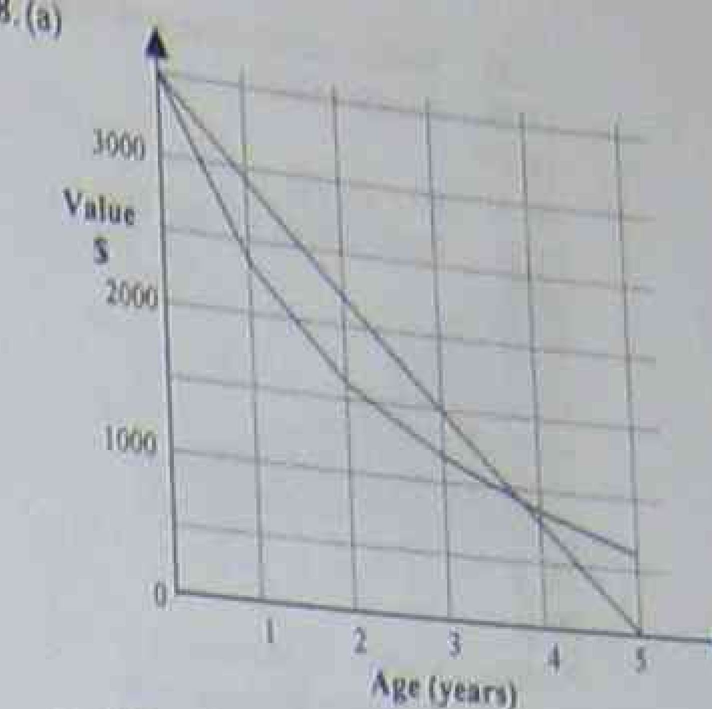
Age (yrs)	Value (\$)	Dep (\$)
0	3200	768
1	2432	583.68
2	1848.32	443.60
3	1404.72	337.13
4	1067.59	256.22
5	811.37	194.73

(b)

Age (yrs)	Value (\$)	Dep (\$)
0	3200	640
1	2560	640
2	1920	640
3	1280	640
4	640	640
5	0	

(c) Printer can be "written off" under straight line depreciation. Cost is recovered totally in 5 years. More recouped earlier under declining balance method.

28. (a)



    (b) $\frac{1050}{3500}$ x 100 = 30%
    (c) See graph above.
    (d) Just under 4 years
    (e) $590(0.7)^{2}$ = \$289.10, say \$290
29. $S = 55000(0.65)^{6}$ = \$4148
30. (a) $5000 = 22000(1-r)^{10}$
    r = 0.14 so rate is 14%
    (b) (i) \$2000(dep) + 120 x \$287 = \$36 440
    (ii) \$16 440 ÷ 10 = \$1664
    (iii) $\frac{1644}{20000}$ x 100 = 8.22%
31. Four times after 6 years, eight times after 9 years
32. (a) Tax on first ring = \$25
    so $\frac{25}{250}$ x 100 = 10%
    (b) \$250, \$500
33. (a) $30(1+0.005)^{1970}$ = \$554 961.58
    (b) After two years, the account would be overdrawn. The bank might then charge interest on the debt!

Data Analysis

Multiple choice

1. D $\frac{130}{360}$ x 280 kg
2. B
3. C
4. A
5. D Total land area not given
6. D
7. C
8. D
9. C
10. B
11. B
12. B
13. C
14. D
15. C
16. C Four scores greater than or equal to 8

17. D Middle scores when ranked are 6, 7
18. B Length of box
19. C
20. A $n\sigma \div (n-1)$
21. D Need total of 150 in 5 lectures
22. C
23. D
24. D
25. C
26. D
27. C
28. A
29. B
30. A
31. C $\sigma_1 = 1.789$, $\bar{x}_1 = 17$,
$\sigma_2 = 2.21$, $\bar{x}_2 = 17.67$
32. C
33. D $\sigma_g = 9.58$, $\sigma_b = 9.93$
34. C $< 1\sigma = 16\%$
35. A
36. D $\pm 2\sigma$
37. C
38. B A is truncated, C is skewed (Pascal distribution), D is skewed and truncated.
39. A Not more than 2σ above mean
40. C
41. B
42. B
43. B
44. D

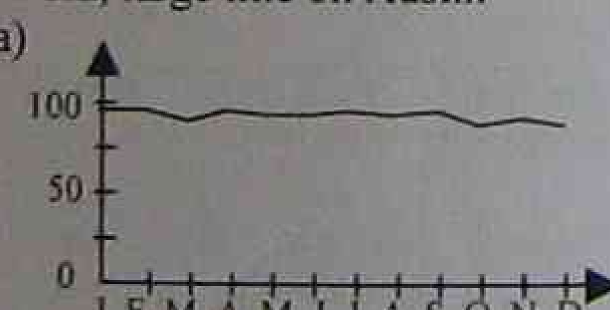
Free response

1. (a) 130L
   (b) Steady increase for wine. Beer increased until mid '70s then declined.
   (c) Compare in 'beer equivalent' terms:
   1972: 130 + 31 = 161
   1980: 128 + 90 = 218
   1985: 117 + 104 = 221 so 1985 was highest (just).
2. e.g. * Researcher biased in choice of person.
   * People shy about answering.
   * Sample limited to suburban people.
   * Sample limited to those who shop at that time, maybe more married women.
   * People deliberately giving false answers.
   * People in a hurry will not answer.
   * Age bias – many people at work.
3. (a) Column graph, (line graph maybe)
   (b) Sector, divided bar (column graph maybe).
4. (a) 2 cereals: "Weetbix and Corn Flakes"
   Unusual: "Sunflower seeds"
   Misunderstood: "Days of Our Lives"
   Not constant: "Ricies on Mondays, Cornies on Tuesdays, ..."
   (b) "Here is a list of breakfast cereals. Pick the one that you ate most of today."
5. 18 – 30: 167, 31 – 60: 287, over 60: 46
6. (a) Quantitative Discrete
   (b) Quantitative continuous
   (c) Categorical
   (d) Quantitative continuous
   (e) Categorical
   (f) Quantitative continuous
7. (a) Samples will be different, e.g.

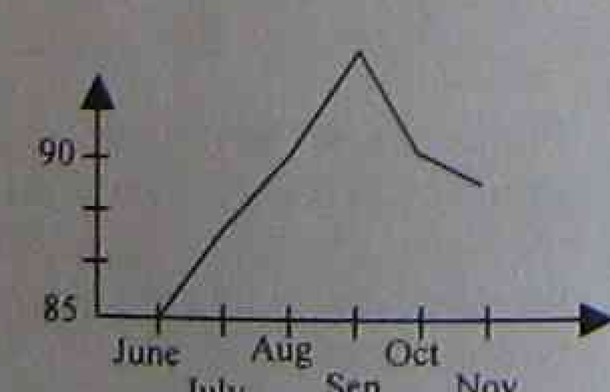
Number	Freq.
0	6
1	3
2	8
3	5
4	3
5	2
6	6
7	6
8	4
9	7

   (b) The expected frequency for each number is 5.
8. (a) Privacy issues
   (b) Security
   (c) Tax, privacy
   (d) Busy, privacy
9. * 3D where only one dimension counts
   * Misplaced zero on vertical axis
   * Bold, large title on Austin

10. (a)



(b)

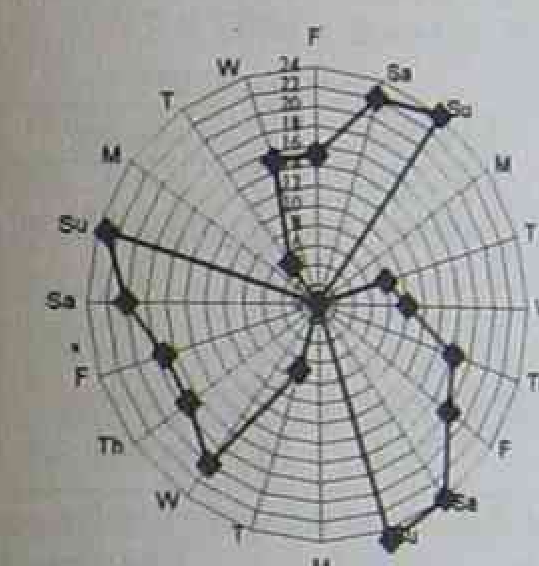


(c) $\bar{x} = 88.3$, $\sigma = 2.59$
(d) Scores of 86 to 91: 9 out of 12, 75%

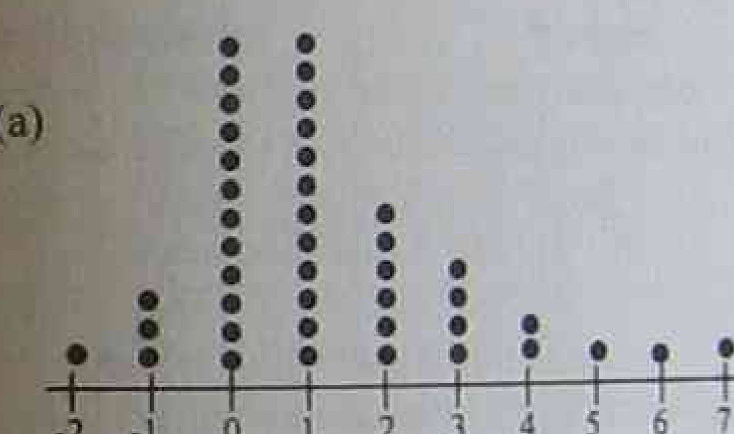
11. (a) (32 + 24 + 20) ÷ 25⅓
   (b)

Class	Centre	Freq.	f x centre
1 – 5	3	1	3
6 – 10	8	3	24
11 – 15	13	2	26
16 – 20	18	7	126
21 - 25	23	5	115
Total		18	294

   (c) 16.78 (raw), 16.33 (grouped)
   (d) Column graph, radar chart, line graph (maybe).
   (e)



12. (a)

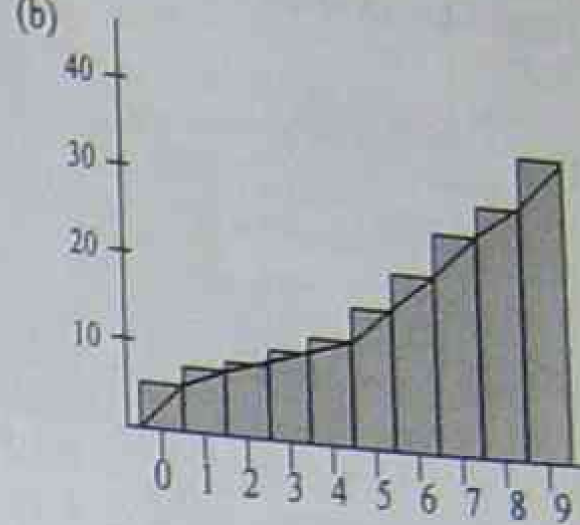


   (b) Bus was early.
   (c) (i) 22nd highest is 1
   (ii) 0 or 1
   (iii) 1.33
   (d) All much the same. Mode is most likely, median less affected by one bad day.
   (e) 16 out of 43 is 37%

13. (a)

No.	freq	cum. f
0	4	4
1	2	6
2	2	8
3	2	10
4	2	12
5	5	17
6	4	21
7	5	26
8	4	30
9	6	36

   (b)



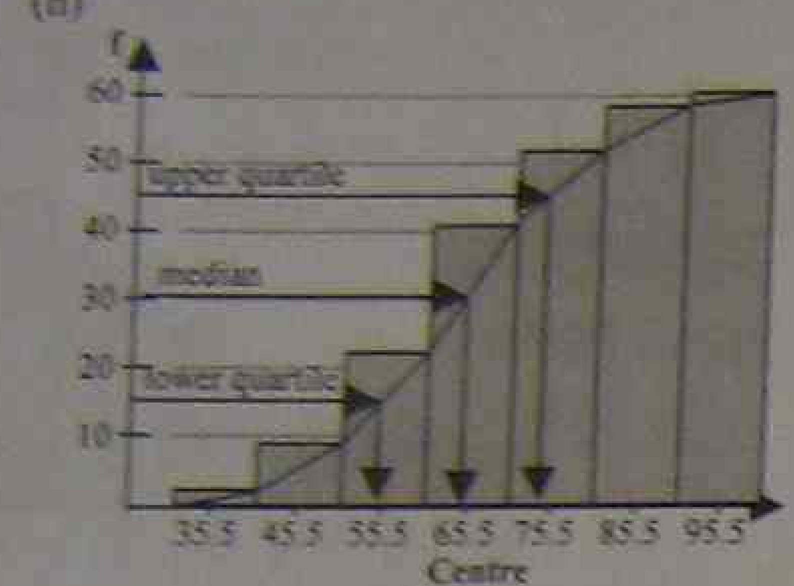
   (c) Lowest score: 0, highest score: 9, median: 6, upper quartile: 8, lower quartile: 3
   (d)

14. (a) 125.5, 135.5, 145.5, ..., 175.5, 185.5
   (b) (i) 155.23 cm
   (ii) 14.2 cm
   (c) Between 141 and 169.43 lie about 1170 scores out of 1704, 68.7%
15. 45 ÷ 9 = 5
16. 0
17. {3, 9, 9}, {4, 7, 10}, {5, 5, 11}
18. (a) Mode
   (b) Mean
   (c) Median
19. (a) Mean = 64.77, st.dev. = 12.42
   (b) 8
   (c) $^{52}/_{60} = {}^{13}/_{15} = 0.87$
   (d) 65.5, using a frequency table and finding average of 30th and 31st scores.
   (e) No. scores between 52.35 and 77.19 = 40 which is 67% of scores, compared with 68% for normal dist.
   (f)

Class	Class centre	f	Cum. f
30 – 39	35.5	1	1
40 – 49	45.5	7	8
50 – 59	55.5	14	22
60 – 69	65.5	19	41
70 – 79	75.5	11	52
80 – 89	85.5	7	59
90 – 99	95.5	1	60
Total	BLANK	60	BLANK

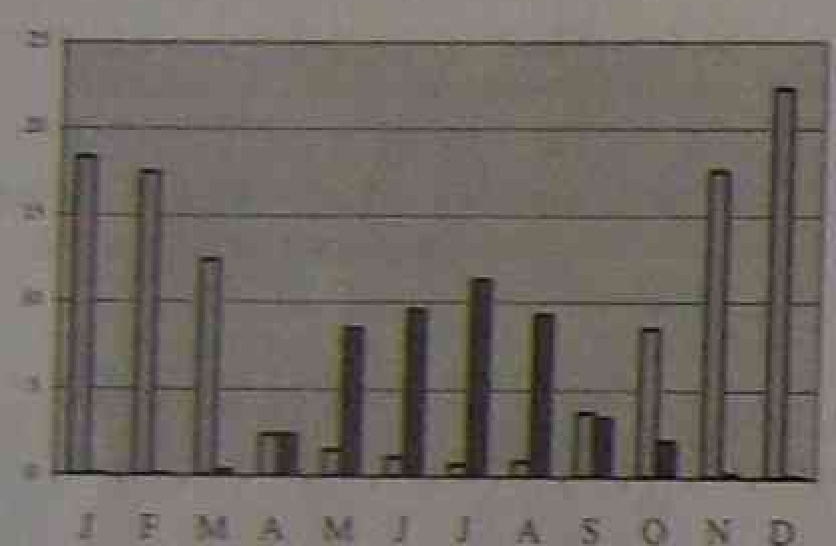
   (g) 64 which is close to correct.

(b)



Median is about 65, close to true value. Interquartile range is about 19.

20. (a) Series 2 (winter months)
(b) Sea
(c) December
(d)



21. (a)

Statistic	Set 1	Set 2
Highest score	43	55
Lowest score	15	10
Upper quartile	40	38
Lower quartile	20	22.5
Median	34	29

(b) Set 2

22. (a) About 100 million
(b) (i) Europe increased about 60 million in 25 years = 2.4 million / yr
(ii) Africa increased by about 278 million = 11 million / yr

23. (a) (i) 49%
(ii) 38%
(b) 19%
(c) 59%

24. (a) 12
(b) (i) 543 – 516 = 27 cm
(ii) 522 – 501 = 21 cm
(c) 543 – 522 = 21 cm
(d) 6, 7, 8 or 9
(e) (i) 527.75 cm
(ii) 521 cm
(iii) $(529 + 526) \div 2 = 527.5$ cm
(f) 527.75 – 509.91 = 17.83 cm

25. (a) Uniform distribution, e.g. die is rolled.
(b) Bimodal, continuous, e.g. time taken from home to city at different times of the day.
(c) Negatively skewed, continuous, e.g. HSC marks in a subject
(d) Symmetrical, discrete, e.g. The number of heads when five coins are tossed.

26. (a) 2σ above the mean = 78
(b) 0.015% of 1000 = 6
(c) Mark below 59

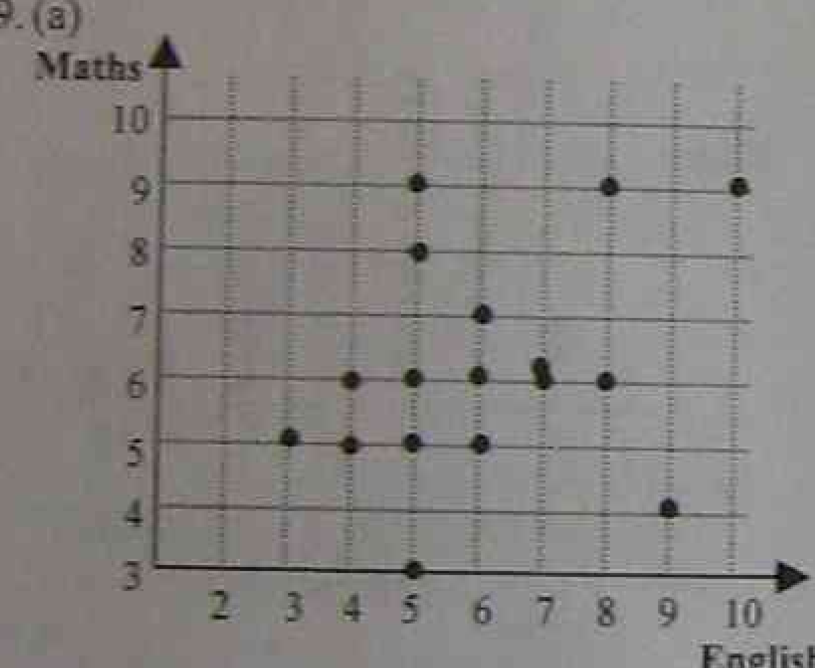
27. (a) $\pm 2\sigma$ gives 74.998 and 75.002 mm
(b) $0.003 \div 75 = 0.004\%$
(c) 75 mm
(d) $5\% \div 2 \times 1000 = 25$

28.

Subject	Albert	Bernie
English	1.0	2.0
Maths	2.0	-0.5
Science	0.5	0.25
History	2.0	2.5
Geography	1.0	0.5
French	-1.0	0
TOTAL	5.5	4.75

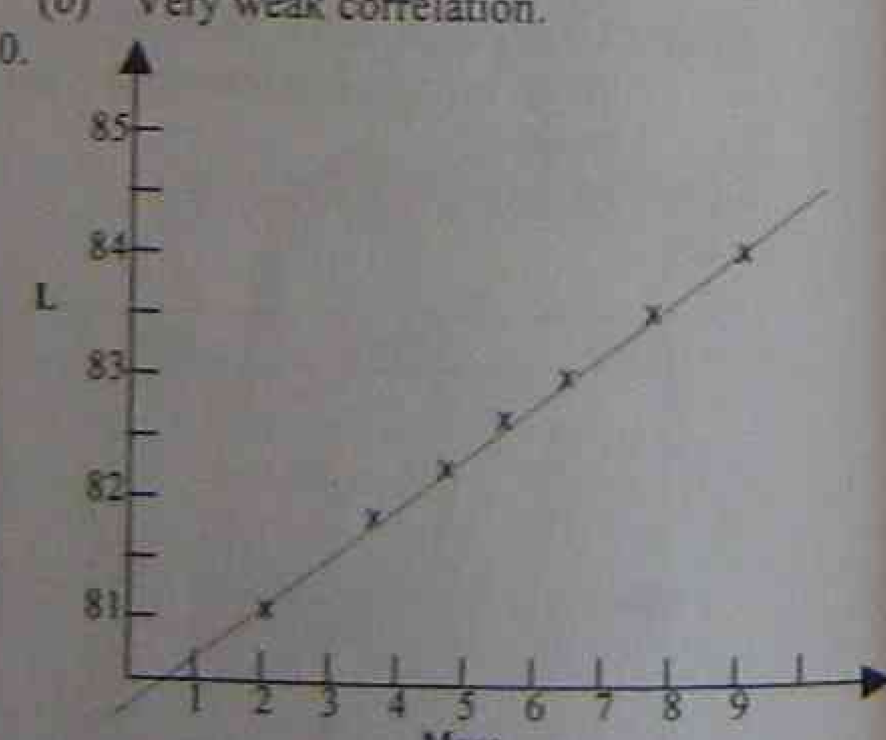
so Albert has a higher total z-score.

29. (a)



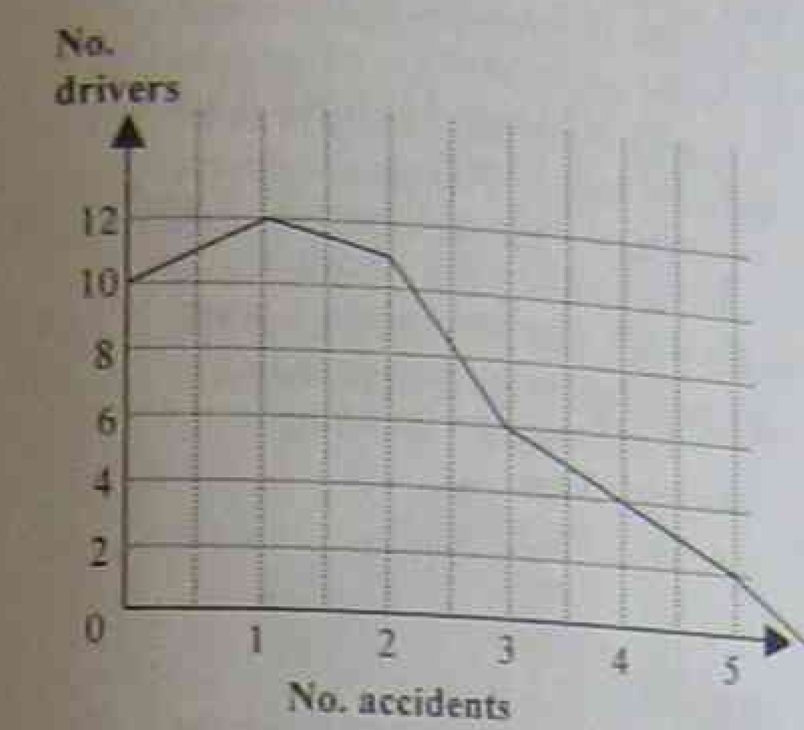
(b) Very weak correlation.

30.



(b) $^{3}/_{7}$
(c) About 80.6 mm
(d) About 12 kg

31. (a) 45
(b)

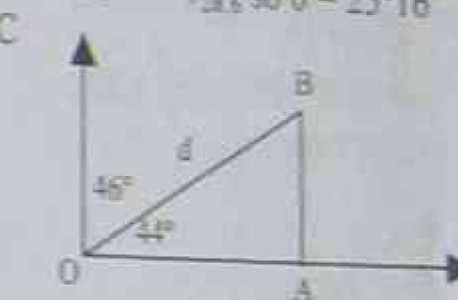


(c) 33
(d) 23rd highest is 2
(e) (i) 1.7
(ii) 1.4
(f) Truncated at 0, very much skewed so, not normal.

Measurement

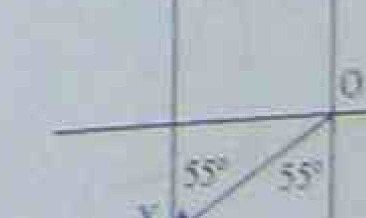
Multiple choice

1. B 57 km < 61 km, 50 min < 20 min
2. D 600 m / min, 36 000 m /h = 36 km/h
3. A Area = $\frac{1}{2} \pi \times 3^2 = 14.14$ m², therefore 1414 g needed.
4. C $380 \div 32 = 11.8$ km/L or $32 \div 3.8 = 8.42$ L / 100km
5. B Volume of tank = 12 000 cm³ so tank holds 12 L. 3 mins to fill.
6. B Error is ± 0.05 m or 50 cm.
7. C Accuracy limited to 2 figures by data in the question.
8. C 1200 : 1500 = 4 : 5
9. B $^{17}/_{25}$ of $17 450 = $11 866
10. B $^{1}/_{11}$ is concentrate = 1.818 L
11. D 6 faces, each 45 x 50.
12. A Area base x height
13. A Double any one dimension
14. A 2(4 x 7) + 2(10 x 7) + 10 x 4
15. B 22 x 220 tiles
16. D One wrap around the box takes 2(6 + 3) cm, two bands required so 4 x 9 = 36. Top / bottom each take three strips 3 cm long = 18 cm. Total 54 cm.
17. B Think tent shape
18. A Uniform cross section would be pentagonal.
19. C Capacity is 1.01088 m³, each m³ holds 1000L.
20. B $^{10}/_{3}(45 + 4 \times 53 + 37)$
21. A $^{15}/_{3}(0 + 4 \times 16 + 20) + {}^{15}/_{3}(20 + 4 \times 18 + 0)$
22. D Area curved surface = 93 x2π x 8
Area hemisphere = ½ x 4π x 8²
23. B $^{1}/_{5} \times \pi \times 15^2 = 141.37$
24. D Volume multiplied by 8
25. D 6.3 x 200 m = 1.26 km
26. A 50 x 50 mm = 2.5 m
27. A 5500 mm x 8000 mm = 44 m²
28. D $^{y}/_{5} = {}^{8}/_{3}$ so $y = {}^{40}/_{3}$
29. A $^{m}/_{3} = {}^{20}/_{17}$ so $m = 3 \times {}^{20}/_{17}$
30. B $^{r}/_{5} = {}^{12}/_{13}$ so $r = 5 \times {}^{12}/_{13}$
31. D If frame is to be filled, height must be multiplied by 4 so length becomes 72 cm, 18 cm too much, so sides must be trimmed.
32. D
33. A $^{d}/_{3.8} = \sin 52° 27'$ so $d = 3.8 \sin 52° 27'$
34. C $^{7.2}/_{h} = \sin 61° 5'$ so $h = 7.2 \div \sin 61° 5'$
35. B $\tan\theta = {}^{13.5}/_{28.6}$ so $\theta = 25°16'$
36. C



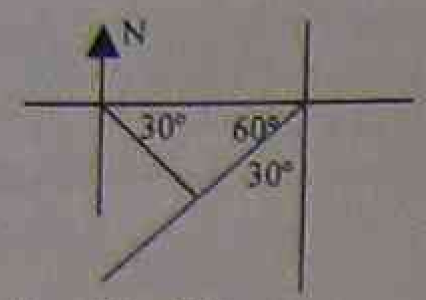
$^{40}/_{d} = \cos 44°$ so $d = 40 \div \cos 44°$

37. B Ignore ΔADC, $^{h}/_{5} = \tan 40°$, $h = 5 \tan 40°$
38. B Height is $\sqrt{6^2 - 5^2} = \sqrt{11}$
$2^2 + 11 = y^2$ so $y = \sqrt{15}$
39. D $^{a}/_{b}$ but $c = \sqrt{a^2 + b^2}$
40. A Radius = 4 cm so $\tan\theta = {}^{10}/_{4} = 2.5$, $\theta = 68° 12'$
41. B sin(90 – A) = cos A, A = 58° 26'
42. C AC = 5
43. D Side opposite B is $\sqrt{y^2 - x^2}$
44. D $\tan\theta° = {}^{3}/_{4}$, $\theta = 36° 52'$
45. A Half AB ÷ 8 = sin 30° so AB = 2 x 4
Also note symmetry, hence equilateral triangle.
46. B Side opposite A = $\sqrt{3^2 + 5^2} = \sqrt{34}$
AB² = 34 + 25, AB = 7.68
47. A



48. D ½ xy sin 50° or ½ yz sin 65° or, after finding missing angle, ½ xy sin 65°

49. B



50. A ½ x 19 x 6.2

51. A sine rule

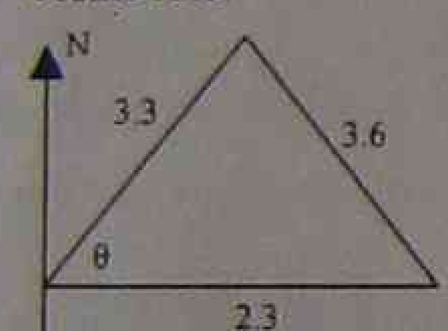
52. B $d^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \cos 35°$
d = 2.869 ......

53. B $86^2 = 47^2 + 61^2 - 2 \times 47 \times 61 \cos \theta$
θ = 104° 49'

54. A sine rule: $c = \dfrac{25.8 \sin 127°}{\sin 18°}$
= 66.68

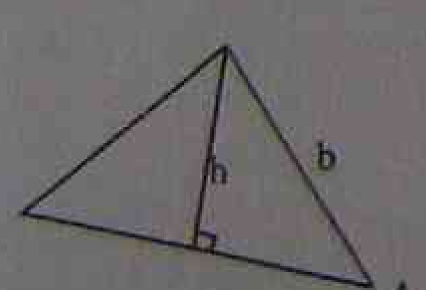
55. B sine rule: need to use angle opposite 6.8 (call it A), then subtract from 180°
$\dfrac{\sin A}{6.8} = \dfrac{\sin 98°}{17.2}$
$\sin A = \dfrac{6.8 \sin 98°}{17.2}$
A = 23° 3'
B = 180° - 98° - 23° = 59°

56. C cosine rule:



$3.6^2 = 3.3^2 + 2.3^2 - 2 \times 3.3 \times 2.3 \cos \theta$
θ = 77° 45' so bearing is 180° + 12° 15'

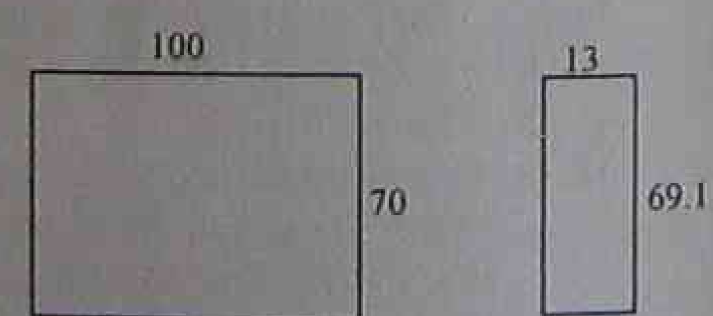
57. A
58. C
59. C
60. B
61. B
62. D 5h is 75°, X is west of Y.
63. C 1.2 of 10 000 km is 5000 km
64. B $^1/_{12}$ of circumference, $^1/_{12} \times 2\pi \times 100$
= 52.359 ....
65. A 59° is $^{59}/_{360}$ of circumference of earth
$^{59}/_{360} \times 2\pi \times 6400 = 6590$
66. B Only B is more northerly than M.
67. A No right angle, only one angle.
68. B
69. A



70. D

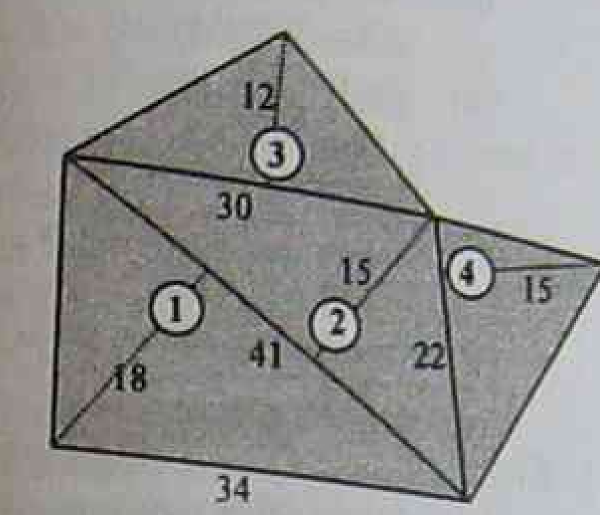
Free response

1. (a) 350 x $0.48 = $168
   (b) $20.60 ÷ $1.03 = 20 kg
2. (a) 90 ÷ 8.6 x 100 = 1046.5 km
   (b) 4.2 x 8.6 = 36.12 L
3. (a) 7 ½ hours
   (b) 15 km in 5 ½ h is 2.7 km/h
4. (a) 2h 17 min
   (b) $126.74 \div 2^{17}/_{60} = 55.5$ km/h
5. 630 ÷ 280 = 2 h 15 min
6. 1 mL of water weighs 1 gram
   20 L of water weighs 20 kg
   Total weight = 20.45 kg
7. (a) 5 km
   (b) 100.0
   (c) 0 cm
   (d) 12 000
8. (a) $^{0.5}/_{50} \times 100 = 1\%$
   (b) $^{0.5}/_{2000} \times 100 = 0.0025\%$
   (c) $^{0.5}/_{2} \times 100 = 25\%$
   (d) $^{0.000005}/_{0.0005} \times 100 = 10\%$
9. Parallax error in reading scale
   Incorrect alignment of zero end
   Misreading of scale. (Probably others.)
10. (a) 2×10^5
    (b) $\frac{1}{4} \times 10^{-5} = 2.5 \times 10^{-6}$
    (c) 7.6×10^{-7}
11. (a) 340 000
    (b) 0.00861
    (c) 597 800 000
12. (a) 6 x 2L = 12L
    (b) $^1/_6$ would be concentrate = 250 mL
    so 1250 mL of water
13. 1 share earns $2379.33
    8 shares earn $19 034.67
    7 shares earn $16 655.33
14. 1 onion ring costs 16.67 cents
    10 onion rings cost $1.67
15. $3 \times 10^5 \times 60 \times 60 \times 24 \times 365\frac{1}{4}$
    $= 9.47 \times 10^{12}$ km
16. (a) Each flat end has area $\pi \times 11^2 = 121\pi$
    Area of curved surface = $\pi \times 22 \times 13$
    $= 286\pi$
    Total area = 528π or 1659 cm²
    (b)



100 ÷ 13 = 7 and a bit

17. Find the difference in areas of two ellipses.
    Outer: semi-major axis measures about 87 ÷ 2 = 43.5 mm, semi-minor: 38 ÷ 2 = 19 mm
    Area is about 2597 mm² (using πab)
    Inner: measures about 74 ÷2 = 37 mm by 29 ÷ 2 = 14.5 mm.
    Area is about 1686 mm². Area of annulus is about 911 mm², so 9 cm²
18. Area of cross section x width =
    ½ x 4.1 x 2.3 x 3.8 = 17.9 m³
19. 4.2 cm x 2 cm : 4.2 m x 2 m = 8.4 m²
20. Several ways of dividing shape, e.g.



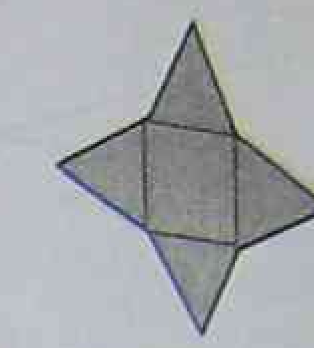
Area 1: ½ x 41 x 18
Area 2: ½ x 41 x 15
Area 3: ½ x 30 x 12
Area 4: ½ x 22 x 15
Total: 1021.5 mm²
(x 150 000² / 1000²)
=22 983 750 m²
= 2298 ha, say 2300 ha

21.

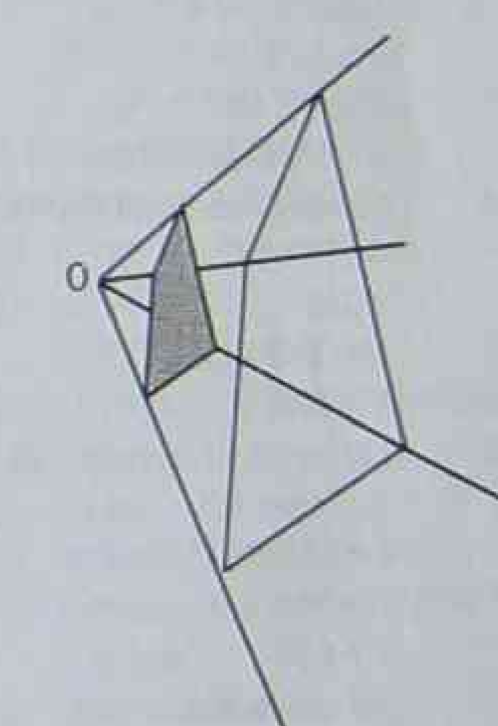
22.

23. (a) 45° gives ⅛ of area of circle
    Area = ⅛ x π(14² - 12²)
    $= {}^{52\pi}/_8$
    $= {}^{13\pi}/_2$ (m²)
    (b) Note units.
    Volume = $10 \times {}^{13\pi}/_{200}$ m³
    = 2.04 m³

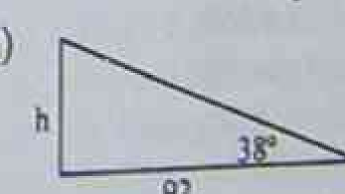
24.



25. Min. diameter 24.5 mm,
    min. length 237.5 mm,
    min. volume π x (24.5)² x 237.5
    = 447 863 mm³
    Max. diam 25.5 mm
    Max. length 238.5 mm,
    Max. volume π x (25.5)² x 238.5
    = 487 213 mm³
26. (a) 9.4 m
    (b) (i) 5.4 m x 3.6 m
    (ii) 8.3 m x 4.2 m
    (iii) 3.5 m x 4.4 m
    (c) (i) 4.2 m
    (ii) 4.2 x $245 = $1029
    (d) 2000 x 1500 ÷ 150² = 133.33 so 134
    (but in reality, 140 tiles needed.)
27. $\dfrac{h}{40} = \dfrac{2}{1.6}$
    $h = {}^{80}/_{1.6} = 50$ m
28.

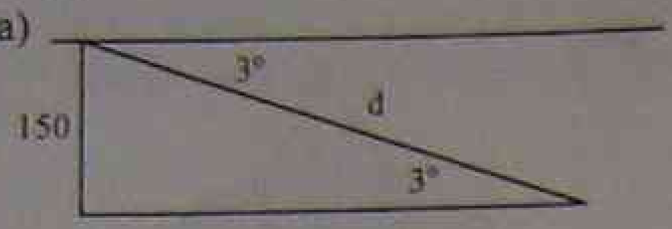


29. (a)



(b) $\dfrac{h}{92} = \tan 38°$
h = 92 tan 38°
= 72 m

30. (a)

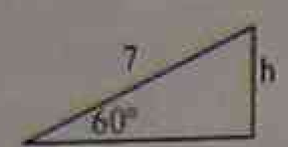


(b) $\sin 3° = {}^{150}/_{d}$
$d = {}^{150}/_{\sin 3°}$
$= 2866$ m

31. ${}^{BC}/_{13.6} = \cos 38°23'$
$BC = 13.6 \cos 38°23'$
$= 10.66$ (m)

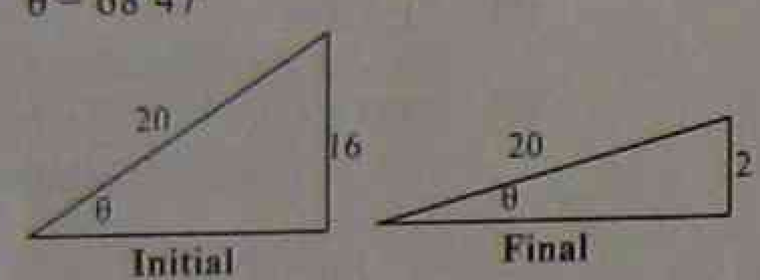
32. $18.8^2 = PQ^2 + 12.3^2$
$PQ = 14.2$

33. (a) ${}^{h}/_{7} = \sin 60°$
$h = 7 \sin 60°$
$= 6.06$ (m)



(b) ${}^{d}/_{7} = \cos 75°$
$d = 7 \cos 75°$
$= 1.81$
$\cos \theta = {}^{1.81}/_{5}$
$\theta = 68°47'$

34.



Initially, $\sin \theta = {}^{16}/_{20}$
$\theta = 53°8'$
Finally, $\sin \theta = {}^{2}/_{20}$
$\theta = 5°44'$ so difference is $47°24'$

35. Square of common hypotenuse
$= 12^2 + 14^2$
$= 340$
$x^2 = 340 - 11^2$
$x = 14.8$

36. Let length of common side be c.
${}^{c}/_{35} = \tan 38°$
$c = 35 \tan 38°$
$c^2 = y^2 + 27^2$
$y = 4.33$

37. Let common side be c.
${}^{c}/_{4.7} = \sin 28°$
$c = 4.7 \sin 28°$
$\cos \theta = {}^{1.7}/_{4.7 \sin 28°}$
$\theta = 39°37'$

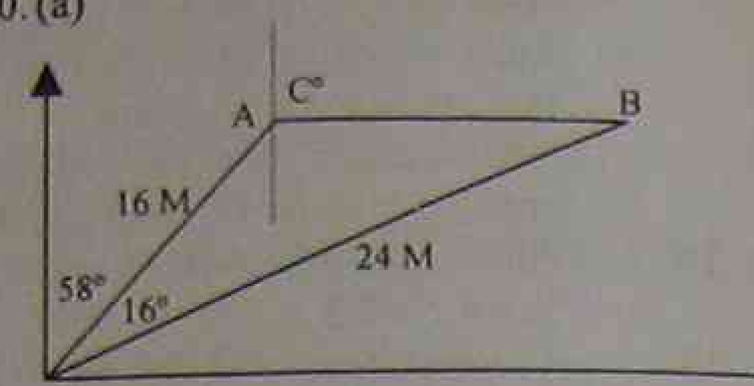
38. (a) $26.1^2 = 29.8^2 + 32.7^2$
$- 2 \times 29.8 \times 32.7 \cos Y$
$y = 49°6'$

(b) Area $= ½ \times 32.7 \times 29.8 \sin 49°6'$
$= 368.3$ cm²

39. (a) $CH^2 = 230 + 453^2$
$- 2 \times 230 \times 453 \cos 13°$
$CH = 235$ m

(b) Same as angle CHA
$\dfrac{\sin CHA}{230} = \dfrac{\sin 13°}{235}$
$\sin CHA = (230 \sin 13°) \div 235$
CHA measures $12° 43'$

40. (a)



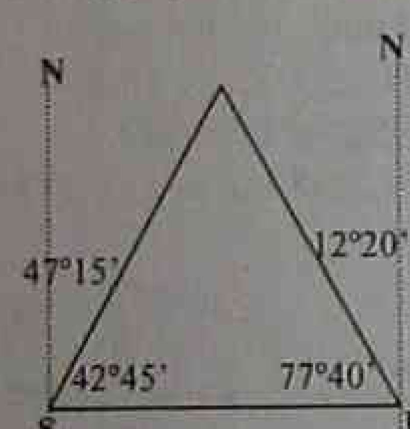
(b) $AB^2 = 16^2 + 24^2 - 2 \times 16 \times 24 \cos 16°$
$AB = 9.7$ (M)

(c) $\dfrac{\sin B}{16} = \dfrac{\sin 16°}{9.68}$
$B = 27°6'$
so angle $TAB = 180° - 16° - 27°$
$= 127°$

(d) Bearing is $360° - 127° - 122°$
$= 111°$

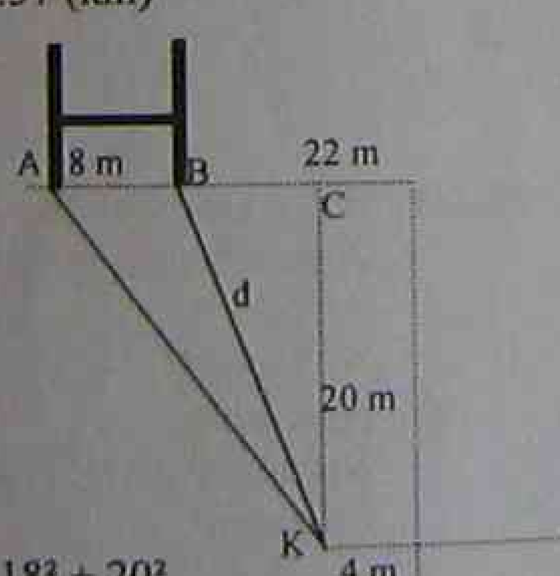
41. Angle QBP measures $168°26'$
Angle QPB measures $3°20'$
$\dfrac{QP}{\sin 168°26'} = \dfrac{5.2}{\sin 3°20'}$
$QP = (5.2 \sin 168°26') \div \sin 3°20'$
$= 17.9$ (km)

42. (a)



(b) $\dfrac{b}{\sin 77°40'} = \dfrac{2}{\sin 59°35'} = \dfrac{s}{\sin 42°45'}$
$b = 2.27$ (km)
$s = 1.57$ (km)

43. (a)



$d^2 = 18^2 + 20^2$
$d = 27$ (m)

(b) By subtraction, AKC – BKC
$\tan AKC = {}^{26}/_{20}$
$AKC = 52°25'$
$\tan BKC = {}^{18}/_{20}$
$BKC = 41°59'$
so angle subtended is $10°26'$

(c) Angles in (b) change to $\tan^{-1}({}^{22}/_{20})$ and $\tan^{-1}({}^{14}/_{20})$. Difference is $12°44'$

44. (a) $735^2 = 695^2 + 476^2$
$- 2 \times 695 \times 476 \cos A$
$A = 75°10'$ or
$695^2 = 735^2 + 476^2$
$- 2 \times 735 \times 476 \cos B$
$B = 66° 4'$ or
$476^2 = 695^2 + 735^2$
$- 2 \times 695 \times 735 \cos C$
$C = 38°46'$

(b) Use area $= ½ ab \sin C$ with one of the angles
$½ \times 695 \times 735 \sin 38°46'$
$= 159\ 898$ m²
= about 16 ha

45. (a) Irregular hexagon

(b) $25 \times 55 + 26 \times 45 + {}^{5}/_{2} \times (15 + 26) + {}^{5}/_{2}(3 + 15$
$= 2692 ½$ m²

(c) About 3.7

(d) CD = 13 m, DE = 12 m
Boundary = 178 m

(e) $\tan C = {}^{12}/_{5}$
$C = 67°23'$
so angle $BCD = 157°23'$

46. (a) 82°

(b) $BC^2 = 17^2 + 28^2 - 2 \times 17 \times 28 \cos 82°$
$BC = 30.7$

(c) (i) $48^2 = 34^2 + 21^2$
$- 2 \times 34 \times 21 \cos AOD$
Angle AOD = 120°
(ii) 263°

(d) $½ \times 28 \times 21 \sin 79° = 371$ (km²)

47. ${}^{70°}/_{360°} \times 2\pi \times 6400 = 7819$ km

48. (a) 2000 M

(b) $\dfrac{2000}{2\pi \times 6400} \times 360 = 17°54'$

(c) Subtract one hour

49. (a) The meridian of longitude 140°E

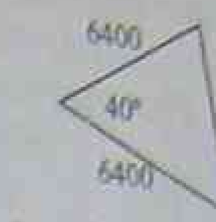
(b) The parallel of latitude 58°N

50. (a) 0°127°E

(b) (i) 40° (ii) 90° (iii) 127°

(c) (i) ${}^{40}/_{360} \times 2\pi \times 6400 = 4468$ km
(ii) ${}^{50}/_{360} \times 2\pi \times 6400 = 5585$ km

(d)



$l^2 = 6400^2 + 6400^2$
$- 2 \times 6400 \times 6400 \cos 40°$
$l = 4378$ km

(e) $127 \div 15 = 8$h 28 min

51. (a) Trapezium: $½ \times 46.5(53.2 + 84.1)$
$= 3192$ m²

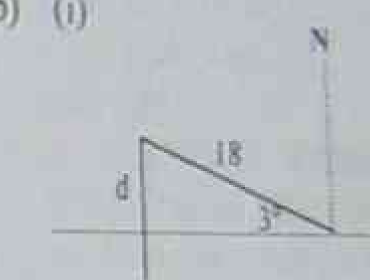
(b)

$d^2 = 46.5^2 + 30.9^2$
$d = 55.3$ (m)
Sum of sides = 239.6 m

(c) $46.5 \times \$15.86 = \$737.49$

52. (a) $3 \times 6 = 18$ M

(b) (i)



(ii) ${}^{d}/_{18} = \sin 3°$
$d = 10 \sin 3°$
$= 0.94$ (M)

53. (a) 200 min = 3h 20 min

(b) 0.4 m³

(c) $\pi r^2 = 0.2$
$r = 0.25$ (m)

(d) $(1.1)^3 = 1.331$ so 33.1% increase

54. (a) 45 km

(b) 65 km

(c) 40 km in 2 h = 20 km/h

(d) Between 11 and noon, between 1pm and 3pm

(e) About 10:20 a.m.

(f) 10:30 a.m., 1pm, 3pm

(g) 85 km (because of the backtracking)

55. (a) $A = ½ \times 48.6(x + 38.72)$

(b) $24.3(x + 38.72) = 10\ 100$
$x + 38.72 = 415.6$
$x = 376.9$

56. (a) 120 cm³ $\div 40 = 3$ cm³ per wash

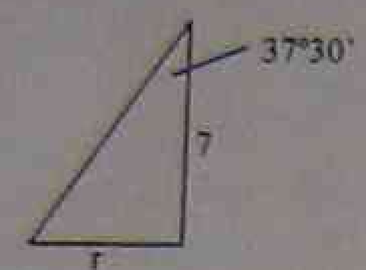
(b) i.e. $3 \times 1 \times 5 = 15$,
105 cm³ used in 35 washes.

57. (a) $V = \pi \times 1.4^2 \times 8.3$
$= 51.1$ m³

(b) Each crate holds 1 m³
so 52 crates required.

(c) 102.2 min = 1h 42min

58. (a)



$^r/_7$ = tan 37°30'
r = 7 tan 37°30'
= 5.37 m
So diameter in 10.7 m

(b) Area = π x 5.37²
= 90.6 m²

59. V = $^4/_3$ x π x 5³
= 523.6 cm³
Therefore mass = 4.08 kg

Probability

Multiple choice

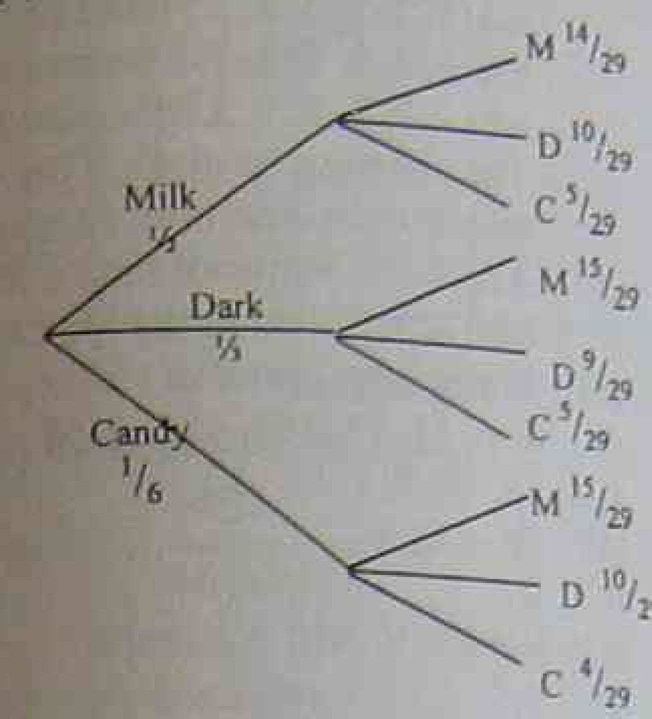
1. B
2. C
3. D
4. C
5. B Still 0.5
6. A $^{15}/_{55} = ^3/_{11}$
7. D
8. D All other are between 0 and 1
9. D
10. A Either 2, 3 or 5
11. C 90 000 possibilities
12. C
13. D
14. C (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)
15. B
16. D
17. A ⅓ x ½ x 2
18. C
19. D 0.79 x 0.75 = 0.5925
20. C ½ x ½ x ½
21. A
22. C 5 x 4 x 3 x 2 x 1
23. D All six boys are equally likely
24. D
25. C $^{31}/_{100}$ x 3.1 million
26. A Expect $^4/_7$ x 1000
27. D
28. B (1 – 0.176) x ½
29. A $\frac{1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}$
30. D
31. B
32. C 60 x $^1/_6$
33. C 20 ÷ 43 = 0.465
34. D Can't throw 1. P(3, 6 or 11) = $^2/_{36}$ x $^5/_{36}$ x $^2/_{36}$ = $^9/_{36}$
35. C Expected return for all others < \$5
36. D All money is returned to the players

Program	No. Viewers	Rel. f
Nature	96	0.08
Gala Perf.	393	0.328
First Love	241	0.201
Midwk Movie	145	0.121
Not watching	325	0.271
TOTAL	1200	1.000

Free response

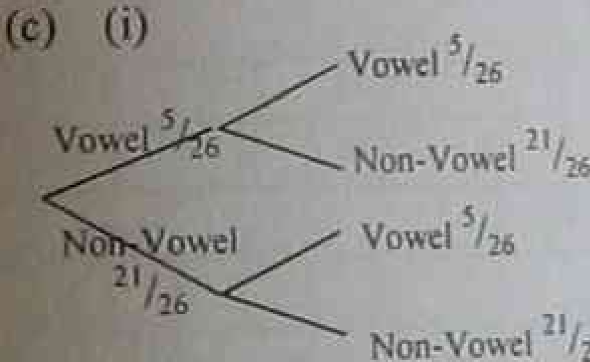
1. From 1000 to 9999, 9000 numbers
2. (a) M, T, W, T, F, S, S
   (b) {M, W, F}, {T, S}
3. (a) Improbable
   (b) Impossible
   (c) Unlikely
   (d) Unlikely
   (e) Improbable
4. There are 7 bars. Each can be 'on' or 'off'. 2^7 = 128 (but some might be difficult to distinguish from others.
5. This is called the post hoc fallacy. Once an experiment or trial has taken place, the probability of an event is either 0 or 1, i.e. it either happened or it didn't happen. Probability only applies before the experiment takes place. Before the shot, hitting the ball through the only open window might have been a thousand to one but this could only be determined experimentally and you would have to break a lot of windows to do that.
6. (a) $^2/_8$ = ¼
   (b) B will give CAB, D will give CAD, E will give ACE so $^3/_8$
7. (a) $^1/_{52}$
   (b) $^4/_{52} = ^1/_{13}$
   (c) $^2/_{52} = ^1/_{26}$
   (d) ¼
   (e) $^8/_{52} = ^2/_{13}$
   (f) $^{12}/_{52} = ^3/_{13}$
8. (a)

   (b) 0.3275 x 2.4 million = about 786 000
9. (a) $^2/_{48} = ^1/_{24}$
   (b) $^8/_{48} = ^1/_6$
   (c) $^9/_{48} = ^3/_{16}$
   (d) $^{12}/_{48} = ^1/_4$
10. (a) $^1/_{18}$
    (b) $^2/_{18} = ^1/_9$
    (c) $^1/_{18}$
    (d) $^5/_{18}$
    (e) $^9/_{18}$ = ½
    (f) $^9/_{18}$ = ½
    (g) $^3/_{18} = ^1/_6$
11. (a) $^1/_{52}$
    (b) $^1/_{52}$ x $^{51}/_{51}$ + $^{51}/_{52}$ x $^1/_{51}$ = $^1/_{26}$
12. (a)



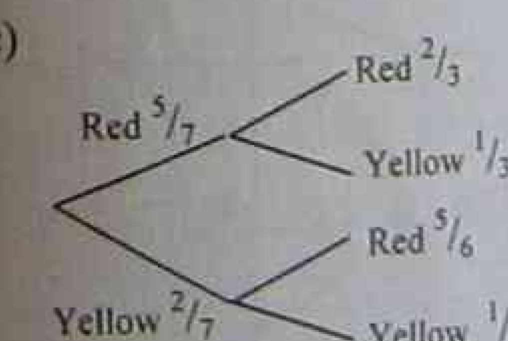
(b) (i) $^1/_2$ x $^{14}/_{29}$ = $^7/_{29}$
(ii) $^1/_2$ x $^{14}/_{29}$ = $^7/_{29}$
(iii) $^1/_6$ x $^4/_{29}$ = $^2/_{87}$
(iv) 1- P(no dark) = 1 – (⅔ x $^{19}/_{29}$)
= $^{49}/_{87}$

13. $^1/_7$ Matching second sock is one out of seven left
14. ¼ ½ x ½ x ½ + ½ x ½ x ½
15. (a) $^5/_{26}$
    (b) $^5/_{26}$ x $^4/_{25}$ = $^2/_{65}$
    (c) (i)



(ii) $^5/_{26}$ x $^{21}/_{26}$ + $^{21}/_{26}$ x $^5/_{26}$ = $^{105}/_{338}$

16. (a) (i) $^5/_7$
    (ii) $^2/_7$
    (b) $^1/_6$
    (c)



(d) (i) $^1/_7$
(ii) $^2/_7$
(iii) $^2/_7$
(iv) $^4/_7$

17. 3 x 9 x 20 = 540
18. (a) 280 (8 x 7 x 6 x 5 x 4 ways of filling the positions but 5 x 4 x 3 x 2 of these are the same people)
    (b) ⅝
19. (a) 10 combination, 6 arrangements of each so 60 arrangements.
    (b) 10, one for each combination
20. (a) $^{26}/_{60} = ^{13}/_{30}$
    (b) $^{24}/_{60} = ^2/_5$
21. (a) 0.487³ = 0.1155
    (b) 0.365
22. (a) No. Wet days are not uniformly spread across the year. April is a wetter month than most others.
    (b) No. The days are not independent; it is more likely that, if one day is wet, the next day will also be wet.
23. (a) $^1/_{20}$ or 0.05
    (b) 95%
    (c) 95%
24. (a) (i) P(10, J, Q, K) = $^{16}/_{51}$
    (ii) i.e. an Ace. $^3/_{50}$
    (iii) Sit
    (b) (i) Total value of remaining cards = 326 points on 50 cards. Average 6.52
    (ii) Yes, on average, will not bust, or, 27 of remaining 50 cards will not cause bust.
    (iii) No. Only 22 of remaining 49 cards will not cause bust.
    (c) If one card is an Ace, a total of 16 leaves only 18 out of 50 not causing a bust. Conservatives sit on 15. If neither card is an Ace, still only 20 out of 50 chance of not busting.
25. (a) $^1/_2$
    (b) $^1/_{16}$ Son, son, son, daughter
    (c) Can never be certain
    (d) There would be more boys than girls because couple would never have more than two girls.
26. (a) 153
    (b) $^{79}/_{153}$
    (c) $^3/_{79}$
    (d) $^{17}/_{93}$
27. (a)

	Carriers	Non-car.	TOTAL
Pos.	2	20 000	20 002
Neg.	0	979 998	979 998
TOTAL	2	999 998	1000000

(b) 20 000
(c) 4 will be missed, so 196 correct
(d) $^2/_{20002} = ^1/_{10001}$
(e) (i) 0.02
(ii) 0.02 x 20 000 = 400

Algebraic Modelling

Multiple choice

1. A $7 \times 150 \div 19$ mg
2. B
3. A
4. D
5. D
6. B
7. C
8. C
9. D
10. A
11. C
12. D
13. C
14. D
15. A Each swing takes 0.6 seconds.
16. C
17. C
18. B
19. D
20. A
21. A
22. D $1.7^2 + 2 \times 1.7 \times 6400 = 21\ 763$
23. A Cuts X axis at 5 and 2, head down.
24. D
25. B When line is most above curve.
26. D Moves up 2 units, cuts Y axis at (0, 1)
27. D

Free response

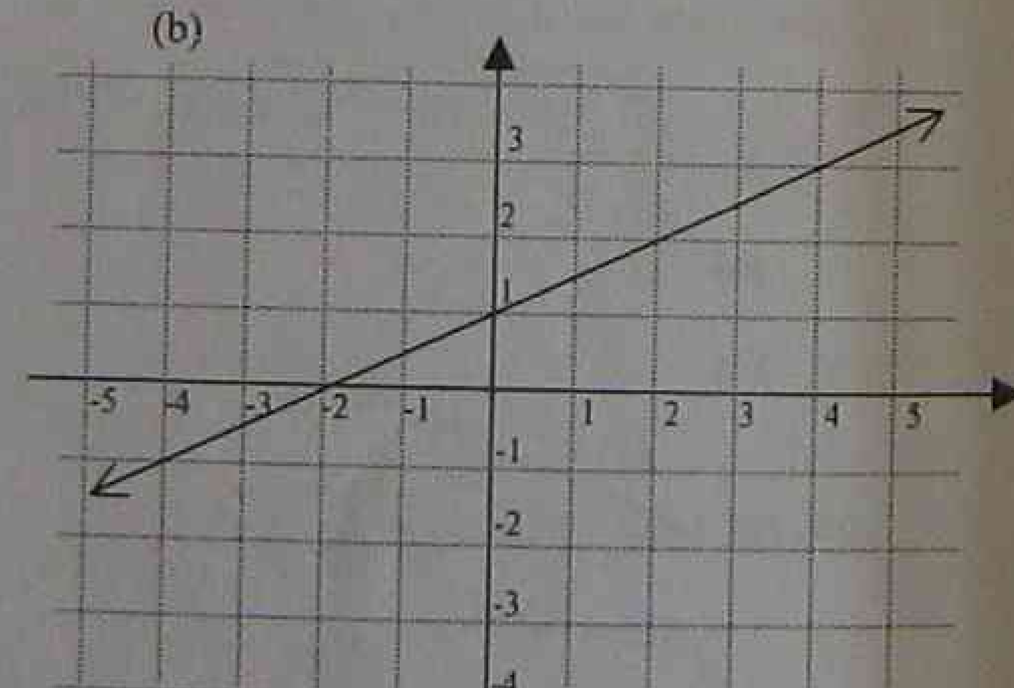
1. (a) $a = -12$
   (b) $b = 1\frac{1}{3}$
   (c) $c = -13\frac{1}{3}$
   (d) $d = -52$
   (e) $e = 2$
   (f) $f = -\frac{4}{13}$
   (g) $7g = 46$
   $g = 6\frac{4}{7}$
   (h) $2h = 86$
   $h = 43$
   (i) $5b - 2b = 40$
   $b = 13\frac{1}{3}$
   (j) $k = 0$
2. $2n - 2 = 3n + 3$
   $n = -5$
3. $5r + 8 = 5r + 40$
   $-5r \quad -5r$
   $8 = 40$. No values of r make this true
4. $2k - 1 = -3 + 2k + 2$
   $2k - 1 = 2k - 1$ true for all values of k
5. (a) $S = \frac{23}{2}(14 + 66) = 920$
   (b) $D = 4$
   (c) $B = 70 \div (\frac{7}{4})^2 = 22\frac{6}{7}$
   (d) $A = 9614.64$

(e) $k = 2.2 \times 10^{-9}$

6. $2\pi \times 62 = 389.6\ (m^2)$
7. 0.000036215 , 0.00004 corrected
8. (a) Any number will do!
   e.g. 1, 2, 3, 4, 5, .. counting numbers
   e.g. 1, 2, 3, 5, 8, .. Fibonacci
   e.g. 1, 2, 3, 1, 2, 3, .. waltz steps
   e.g. 1, 2, 3, 2, 1, 2, 3, .. oscillating
   The rule must be given if the next term is to be determined uniquely.
   (b) Arithmetic sequence (add 4 to get next term). Next terms 19, 23, 27.
   (c) Multiplying previous term by 3:
   2, 6, <u>18</u>, 54, 162, 486, <u>1458</u>
9. 1, 5, 10, 10, 5, 1
10. (a) $5p - 3$
    (b) 0
    (c) $y + 1$
    (d) $4z^5$
    (e) $4a^6$
    (f) $\frac{1}{3xy}$
11. (a) $26^2 = 10^2 + b^2$
    $b = 24$
    (b) $120 = \frac{5}{9}(F - 32)$
    $F = 248$
    (c) $160\ 000 = 4\pi r^2$
    $r = 112.8$
12. (a)

X	-2	0	1	4
Y	0	1	1½	3

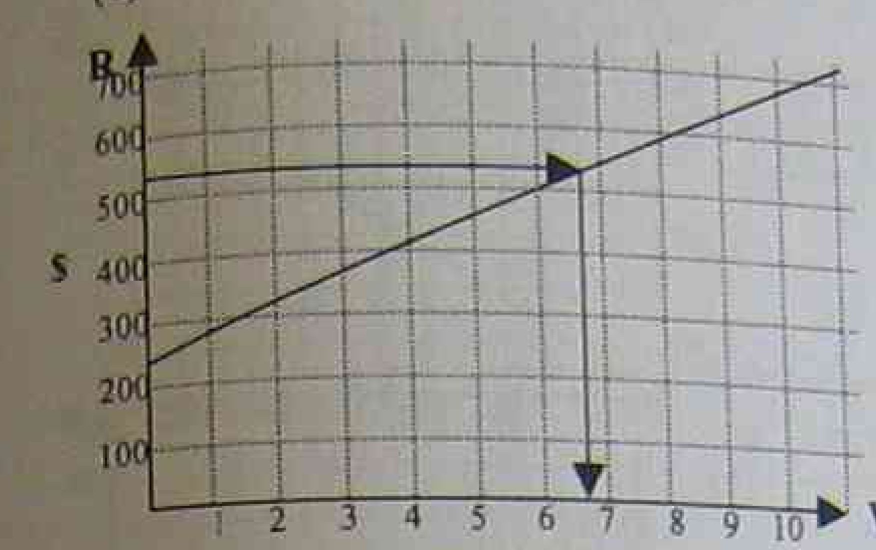
(b)



(c) Gradient is ½ , Y-intercept is 1.

13. (a) $412

(b)

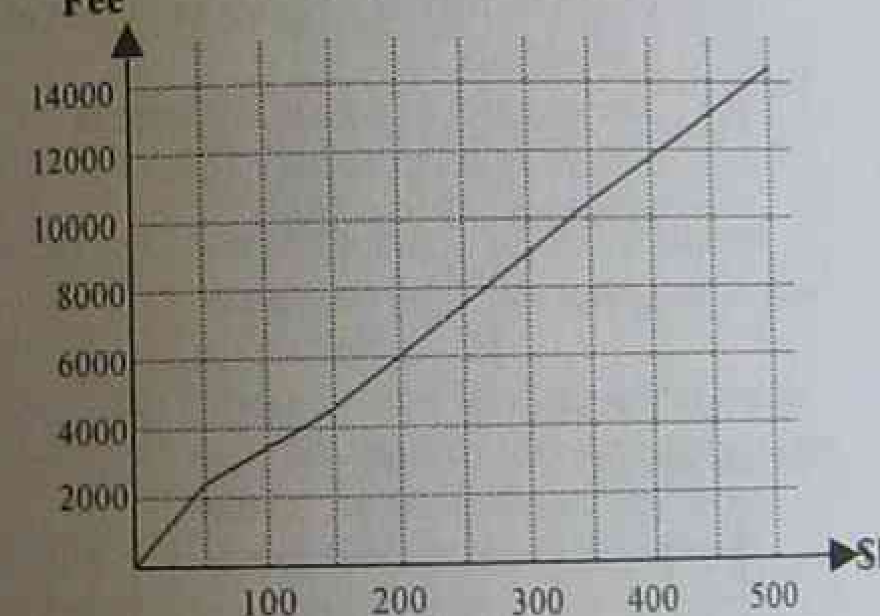


(c) About $680 000

(d) $R = \dfrac{43V}{10000} + 240$

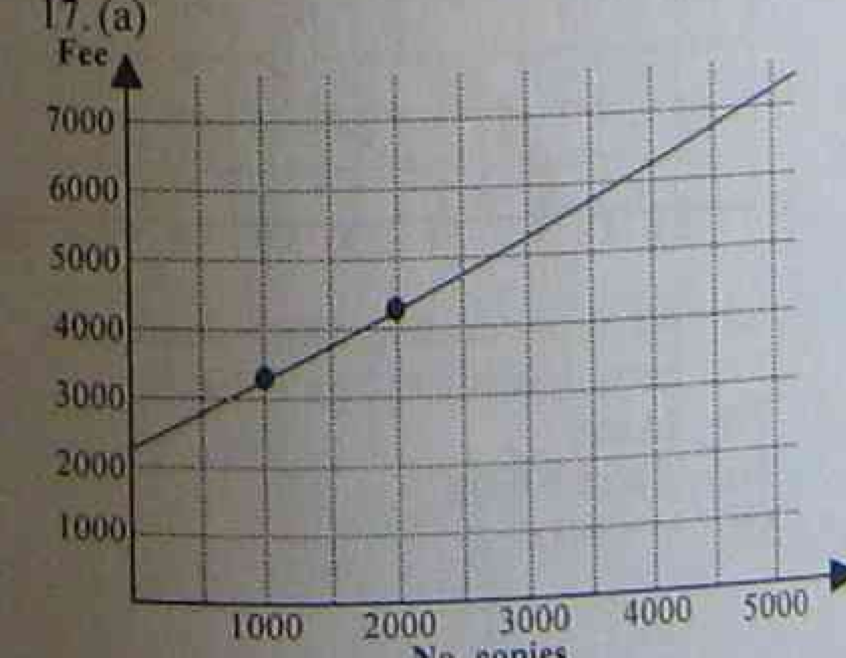
When R = 530, V = 674 419

14. (a) 60 m/minute
    (b) 20 m/minute
    (c) 420 m
15. (a) $300 \div 0.6235 = \$481.15$
    (b) $\$455 \times 0.6235 = \$283.69$
16. (a)



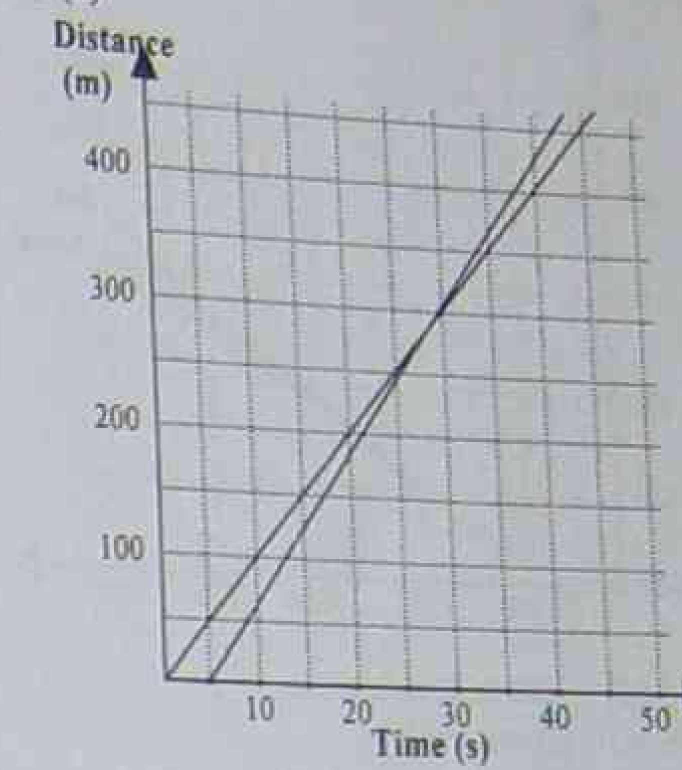
(b) and (c) $10500 fee for $350k
$4500 extra is 2% of amount over $350k
Extra = 4500 x 50
= $225 000
Sale was $575 000

17. (a)



(b) $2200
(c) $1
(d) $P = n + 220$

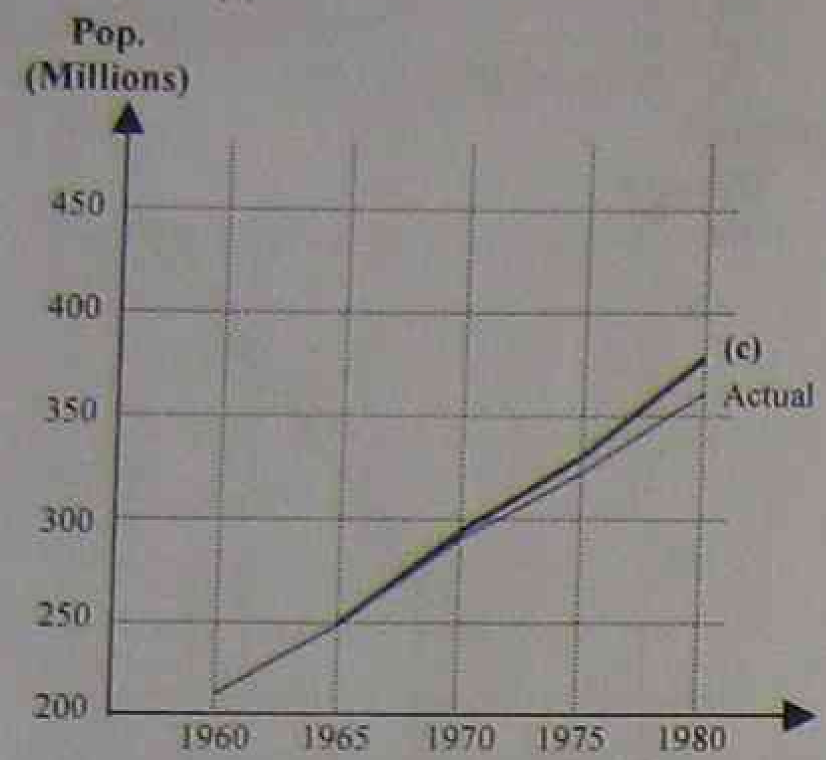
18. (a)



(b) 30 seconds after hare started
(c) 300 m
(d) Dog will run 500 m in 41⅔ seconds
Hare gets 5 s start so dog gets 46⅔ s to run 500 m, 10.7 m/s or better
(e) Hare will take 50 s to run 500 m
Dog will take 41⅔ s so starter must wait 8⅓ seconds before releasing dog.

19. $(3.1)^3 = 29.79$
    $(3.1)^4 = 92.35$
    $(3.1)^{3.3} = 52$
    $(3.1)^{3.25} = 39.52$ so x is closer to 3.3
20. $t^2 = 1600$ so $t = 40$ or -40
21. (a) $\frac{36y^3}{35x}$
    (b) $\frac{24}{25c^2}$
    (c) $-3x^3 + 4x^2 + 5x$
    (d) $2x^4$
22. (a) $r = \sqrt[3]{\frac{3V}{4\pi}}$
    (b) $V = 21\ 688.370$
    (c) $r = 0.620$ (m)
23. If x = 1, dividing by (x – 1) is dividing by zero which is illegal.
24. 4×10^2
25. 2.469×10^{-9}
26. $10 \times 10^6 \times 10^3$ joules $= m \times 9 \times 10^{16}$
    $m = 1.1 \times 10^{-7}$ kg or 0.00011 grams (not much matter for so much energy!)
27. $T^2 = \frac{k}{l}$
    $(0.6)^2 = \frac{k}{0.089}$
    $l = 0.032$
28. (a) $\frac{249}{217} = 1.147$

(b) & (c)



(d) Linear is adequate.

29. (a) About 1:05

(b) (i) 6°

(ii) 6x°

(c) (i) ½°

(ii) $^x/_2$°

(d) Initially 30° but reducing

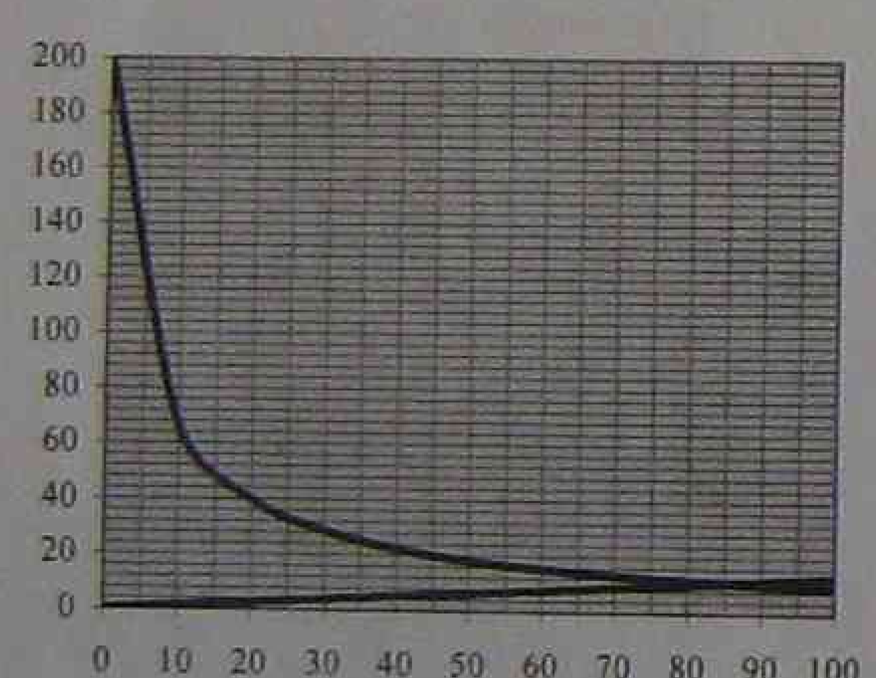
Angle = $30° + (^x/_2 - 6x)°$

= 0 when superimposed.

$x = 5\,^5/_{11}$

so time is about 1:05:27

30.



Break even when about 85 items are made.

31. (a)

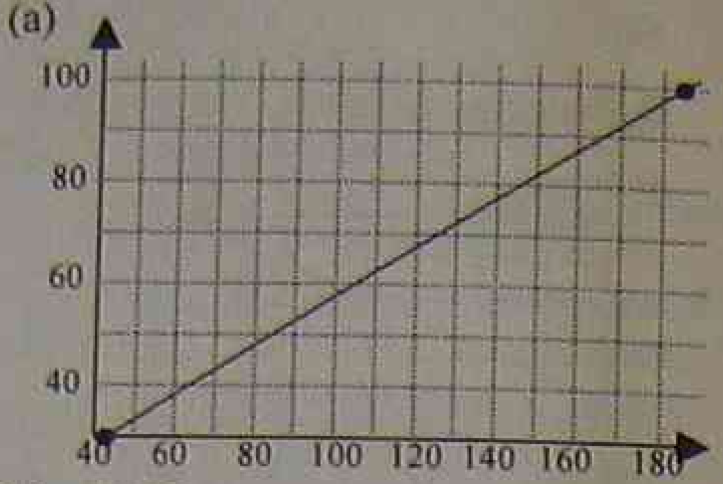
T	0	1	2	3	4	5	6	7
P x 1000	4.5	8	13	22	38	64	108	184

(b) 1 min 20 s

(c) Factor is about 1.7

184 000 x $(1.7)^2$ = 532 000

32. (a)

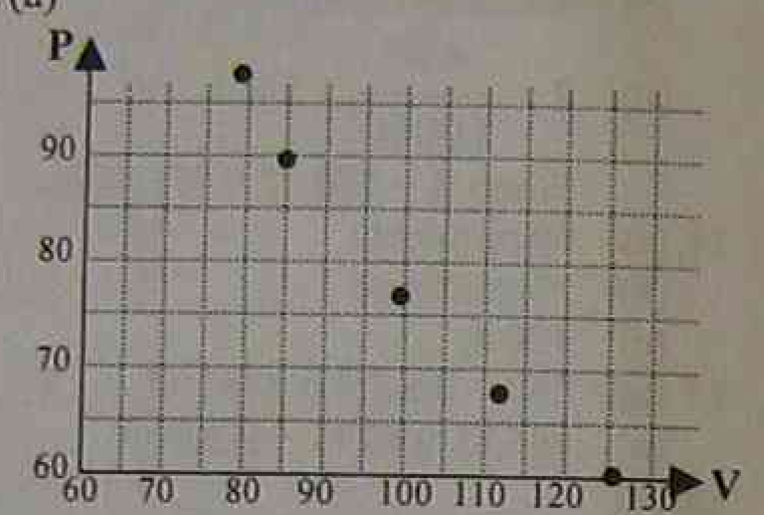


(b) (i) 39

(ii) 58

(iii) 83

33. (a)



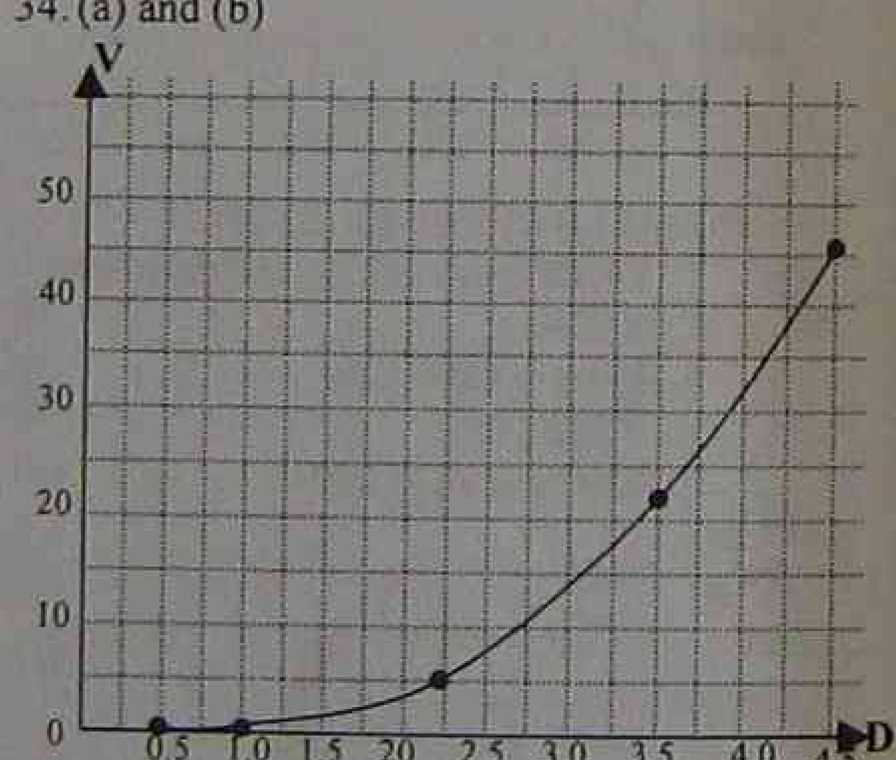
Could be part of a parabola or part of a hyperbola.

(b) The product of the coordinates is always equal to about 7560. Equation is PV = 7560

or $P = \dfrac{7560}{V}$

(c) 126 kP

34. (a) and (b)



(c) As we are dealing with volumes, the curve is likely to be a cubic.

(d) 14 mL

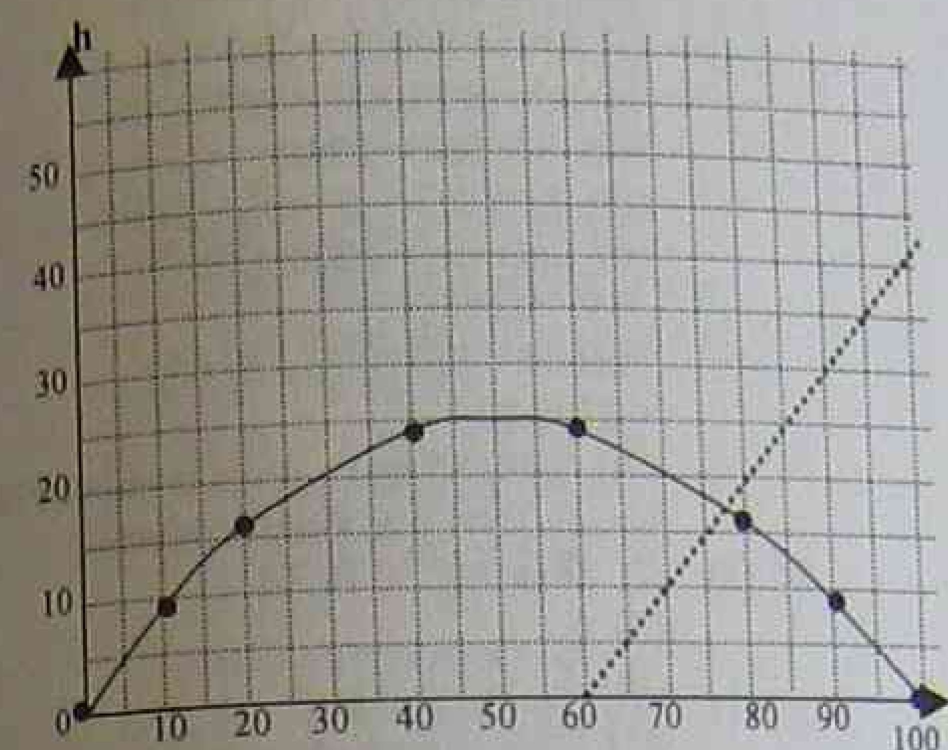
(e) About 3.9 cm

(f) $V = ½ D^3$

35. (a)

x	0	10	20	40	60	80
h	0	9	16	24	24	16

(b)



(c) By symmetry, at x = 50, h = 25

(d) x = 100

(e) See broken line on graph.

(f) At about (77, 17) so 77 horizontally and at 17 m of altitude. (More exactly: 77.45 and 17.45)

36. (a) Can only buy a whole number of stamps so graph between points has no meaning.

(b) There is little likelihood that shares will continue to follow a straight line model for more than a few days, certainly not for as long as two months.

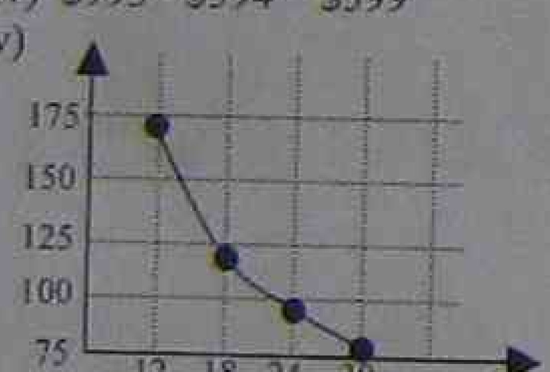
Sample Exam Paper 1

Section 1

1. D 12km in 16 mins, 3km in 4 mins, 45 km in 60 mins.
2. C Solve $5 = -2(3 - x)$
3. A B and C are not exactly true.
4. A 90° difference is 6 hours and Vitebsk is ahead of Resistencia.
5. A $5400(1.06)^7$
6. B $(140 \times 1218 + 1500) \div 141$
7. C $240\ 000 \div 330 = 727$, closest to 36×20
8. B Use $x^{1/3}$ on calculator
9. C Hyperbola
10. A Circumference is proportional to radius so $60 \div 3 \times 10 = 200$
11. B 100/15.9 km on 1 L, then x 80
12. C Two standard deviations below the mean to be rejected. 95% lie within 2 s.d. so 2.5% each end.
13. D Doubling the radius will quadruple the area of the base.
14. B $2000 \text{ cm} \div 7\pi \text{ cm}$
15. D Cone is one-third of prism
16. B Both have range 25, median of Y is greater than median of Y
17. D $230 x 1.15 = $264.50 which is $72 + 52 x $3.50
18. A $FV = 400\ 000(1 + 0.03)^{10}$
19. C $65 780 x 0.0125
20. D 3 out of 12 is ¼.
21. A
22. C Expected return is $2.50, half of $5

Section 2

23. (a) (i) $118
(ii) 18 months
(iii) $594
(iv) $993 - $594 = $399
(v)



Accurate graph gives about 14 mths.

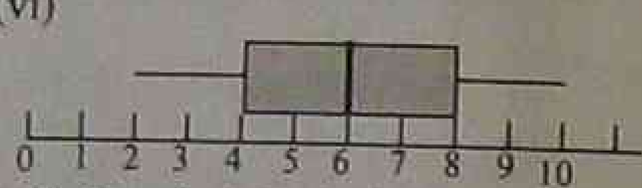
(b) (i) Rhombus (Diagonals bisect at right angles.)
(ii) ½ product of diagonals
$= \frac{1}{2} \times 280 \times 160 \text{ m}^2$
= 2.24 ha
(iii) Using Pythagoras' Thm, each side is x where $x^2 = 140^2 + 80^2$
so $x \approx 161.25$ m
Perimeter ≈ 645 m

(c) (i) x = 2 or −2 (must have both)
(ii) $\pi \times 2 \times 3 = 6\pi$
or about 18.8 sq. units

24. (a) (i) About 78 kW
(ii) About 2400 RPM
(iii) 1600 RPM and 4000 RPM
(iv) About 106 kW
(v) 500 RPM

(b) (i) 8% of $1000 = $80
(ii) 6% of $1000 = $60
(iii) 3 x $80 = $240
(iv) $\$1000(1 + 0.06)^3 = \$1190$
(v) Balance = $1000(1.06)^{10}$ = $1790.85
Interest = $1790.85 - $1000 = $790.85
(vi) $790.85 in 10 years is about $79 p.a. or 7.9% p.a. simple interest.

25. (a) (i) 28 in 7A, 30 in 7B
(ii) 6
(iii) 6
(iv) 6
(v) 10 − 2 = 8
(vi)



(vii) Standard deviation:
s.d. for 7A = 1.96, s.d. for 7B = 2.59

(b) (i) 2 x 2 x 6 = 24
(ii) 24 cubes occupy 24 x 8 cm³
Air space = 300 cm³ - 192 cm³ = 108 cm³
(iii) 1000 cm³ holds 1 litre so the box would 0.3 L.

26. (a) (i) About 58.7 cents
(ii) About 59.6 cents
(iii) Day 6
(iv) About 53.5 cents (± 0.3)

(b) (i) $\sin\theta = \dfrac{1700}{2600}$
θ = 40°50' to neatest minute
(ii) $2600^2 = x^2 + 1700^2$
x = 1967 mm to nearest mm

(c) (i) 130 (sum of frequencies)
(ii) $\dfrac{70}{130} = \dfrac{7}{13}$
(iii) $\dfrac{15}{130} \times 2000$ = about 231
(iv) Not necessarily biased as this pattern could be explained by statistical variation for this small number of spins.

27. (a) (i)

Sum of faces	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(ii) 11/36 x ½ = 11/72
(iii) 1/12 probability of win,
∴ winner should receive $12

(b) (i) 5 or (0, 5)
(ii) If x = 7, y = k = 19
(iii) 4y = - ax + 17
$y = \dfrac{-ax}{4} + 4\frac{1}{4}$
(iv) -a = 8 so a = -8

(c) (i) $28 000{ (1 + 0.08)⁵- 1}/ 0.08 = $164 265
(ii) $\$240\ 000\ (1 + 0.02)^5 = \$264\ 979$
(iii) $\$240\ 000\ (1 + 0.0116)^5$
= $254 247 which is less than $90 000 + $164 265

28. (a) (i) $82.5 \le r < 83.5$
(ii) Using $V = \dfrac{4}{3}\pi r^3$
$2352.0 \le V < 2438.6$ to 5 sig. figs.

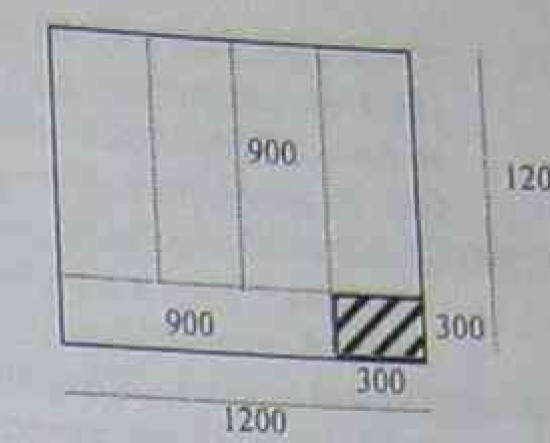
(b) (i) 80 km/h
(ii) Between ½ h and ¾ h
(iii) ¼ x 15 + ¼ x 30 + ¼ x ½ x 110 + ¼ x ½ x 120 = 40 (km)
(iv) $\dfrac{1}{12}(0 + 4 \times 30 + 30) +$
$\dfrac{1}{12}(30 + 4 \times 80 + 40) = 45$ (km)

(c) (i) Use v = k/t
50 x 20 ÷ 6 = 166.7 m/s
(ii) Halve velocity, double time: 40 s
(iii) Zero (hyperbola)

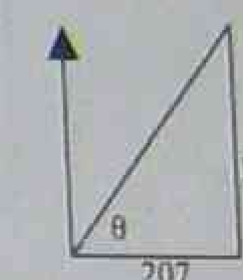
Sample Exam Paper 2

Section 1

1. C $70 \div 1.81^2 = 21.37$
2. B 26 more males than females in every 1000 so 400 000 x 26
3. B



4. D 75 x 3 = 225
5. A 1.8 x 1.8 x 1.2 x $138
6. B 10a + 15b − 24a + 42b
7. D Divide both by 1.26
8. A Solutions must be −1 and 3
9. D Two terms at 4%, $20\ 000(1 + 0.04)^2$
10. C $I = k/18^2$ so $k/6^2 = 9I$
11. B tan θ = 128/207
so θ = 32°
Bearing = 90 - θ



12. A 158° of longitude x 4 minutes
632 min = 10 h 32 min and Sydney is ahead.
13. C Both graph represent the data accurately but the graph on the left has a scale which exaggerates differences.
14. D In order, 3, 4, 7, 8, 8, so 7 is in the middle.
15. C High positive correlation indicates scatter graph with points close to an oblique straight line.
16. A All faces equally likely, uniform distribution.
17. C Try each rate in $240\ 000\ (1 + r)^{13}$
18. B 41 h x $5.80
19. B $61.75 ÷ 9 ½
20. D 2000×0.8^5
21. A One s.d. below mean: z = -1
22. C Interest paid only on $700 owed. Pays $1040 so $340 interest, 340/700 = 49%

(ii)

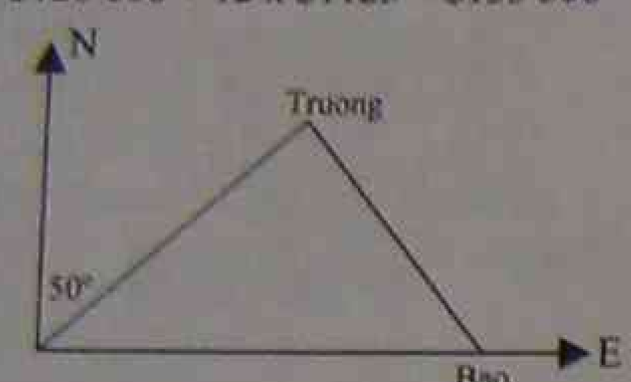
Section 2

23. (a) (i) $11.61
(ii) $900 ÷ $11.61 = $77 519
(iii) Instalment = $13.22 per thousand At 6%, term is under 8 years, i.e. more than two years faster.
(iv) $1365.30 ÷ 90 = $15.17 so it's an 8 year loan. To repay in 4 years, $25.36 x 90 = $2282.40 per month

(b) (i) Single digit: 1 possibility
Two digit number: 9 + 9

Three digit: 18 x 8 + 100 so total = 263
(ii) Only possibilities are 11, 101, 110 so probability = 0.003
(iii) 90 possibilities so p = 0.09
(c) (i) 80° is 2/9 of a circle so area = 2/9 x π x 8 x 8 = 44.7 cm², 45 cm² corrected.
(ii) Area = ½ x 8 x 8 x sin 80° = 31.5 cm², 32 cm² corrected.
(iii) Shaded area is the difference: 44.7 cm² − 31.5 cm², 13 cm² corrected.

24. (a) (i) 0.1125 x $120 000 ÷ 12 = $1125
(ii) $120 000 + 12 x $1125 = $133 500
(b) (i)



(ii) Let d be the distance between them Truong will have walked 15 km, Bao 10.5 km. Using cos rule
$d^2 = 15^2 + 10.5^2 - 2 \times 10.5 \times 15 \cos 40°$
distance = 9.7 km to one dec. pl.
(c) (i) Angle is 96° which is about 27%
(ii) 124° represents 62 people. Probability = 62/180 = about 0.34
(iii)

Earnings	f	Class Centre
$1 - $10 000	48	$5 000.50
$10 001 - $20 000	34	15 000.50
$20 001 - $30 000	36	25 000.50
$30 001 - $40 000	62	35 000.50

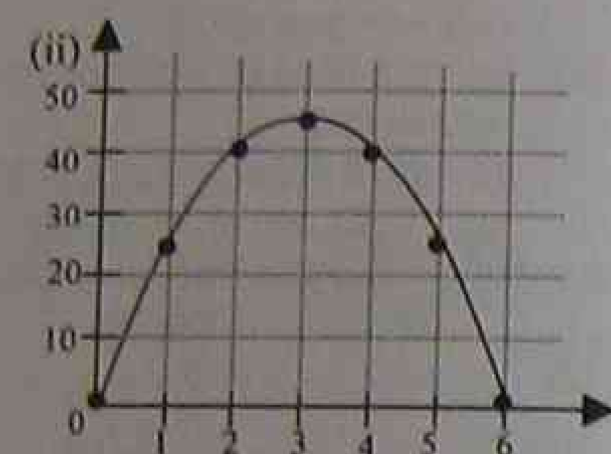
(iv) $21 222.72
(v) Not accurate because scores will not be symmetrically distributed in each class.
(vi) $12 071.80

25. (a) (i) Zero as it was empty.
(iii) Roughly 9/4
(iv) Let time be t, depth be d, then $d = \frac{9}{4}t$ or similar with gradient from part (iii).
(v) Answer will be about 45 cm
(vi) It will have fairly constantly cross section so a cylinder or any prism shape.
(b) (i) 5.6 m x 3.9 m
(ii) 5.6 m x $285 = $1596
(iii) 5.6 m, 3.9 m, 1.9 m1.9 m, 3.9 m
(iv) Area of ceiling is 21.84 m². Wall opposite door: 5.6 m x 2.9 m = 16.24 m², wall with door: 16.24 m² − 2 x 1.8 = 12.64 m². Wall opposite window: 3.9 m x 2.9 m = 11.31 m². Wall with window: 11.31 m² − 1.2 x 1.2 = 9.87 m². Total area = 71.9 m²
(c) (i) 4 x 3 x 2 x 1 = 24
(ii) 12

26. (a) (i) 5 km
(ii) 1 : 500 000
(iii) 30 km
(iv) 6.4 cm x 10.4 cm represents 32 km x 52 km = 1664 km²
(b) (i)

t	0	1	2	3	4	5	6
h	0	25	40	45	40	25	0

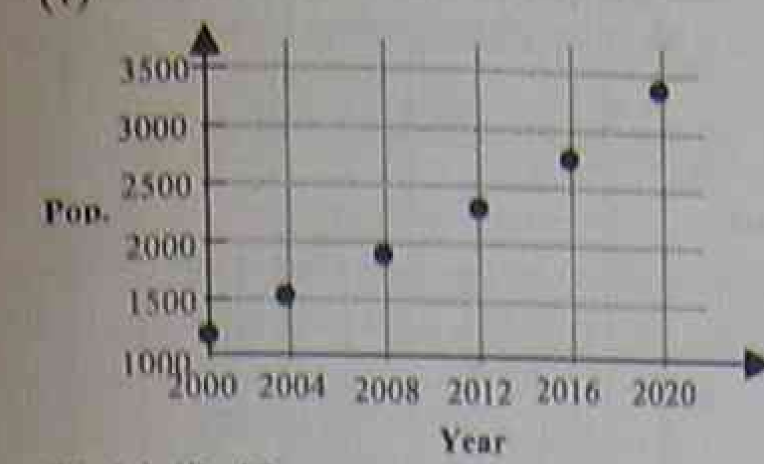
(ii)



(iii) 45 m
(iv) 6 seconds
(c) (i) m = s
(ii) There is a reasonably strong positive correlation between the two variables.

27. (a) (i) 300
(ii) 82 km/h, the upper limit of the highest class.
(iii) Speed of 150th / 151st car is in the class with centre 72. From the polygon, the median is about 71 km/h
(iv) Top 75 speeds above 74 km/h
Bottom 75 speeds below 64
Therefore interquartile range = 10 km/h
(v) 120th slowest speed = about 69 km/h
(b) (i) $A = \frac{2}{3}[0 + 4 \times 1.9 + 1.3] + \frac{2}{3}[1.3 + 4 \times 3.4 + 0]$
$= 15.87$ km², about 16 km²
(ii) 100 times your answer in (i), about 1587 ha.
(iii) 1253 ÷ 15.87 = about 79 people
(iv) $1253(1 + 0.05)^{20}$ = about 3324

(v)



28. (a) (i) 10
(ii) 20 000
(iii) 8
(iv) If 1 000 000 take the test, 8 will get a correct positive result and 200 000 will get a false positive result. Prob = 8 / 200 000 = 0.0004
(b) (i) $8942 + 43% of $7000 = $11 952
(ii) Pays 20% of $15 100 = $3020 before rise, $3060 + $34 = $3094 after so difference is $74 (in a rise of $300)
(iii) $52 140 ÷ 1920 hours = $27.16 /h
(iv) $14 102 is 28.2% of $50 000

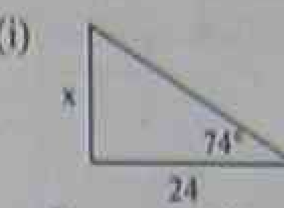
Sample Exam Paper 3

Section 1

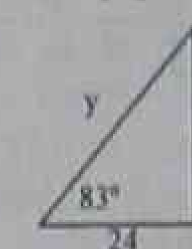
1. D $34.50 + 4 x $5.50 = $56.50
2. A $26 + $3.40 = $29.40
3. B Height ÷ 70 = tan 38°. Multiply both sides by 70.
4. A (130 x 100) ÷ (46 x 19.8) = 14.27%
5. B $40a^2 + 5a^3 = 8$
6. D 12 ÷ 3.5 = 3.43 h = 3h 26 min
7. C $3x + 3 = 8 - 4x$
$7x = 5$
$x = \frac{5}{7}$
8. D Depreciation of 20% is equal to appreciation of 25%
$11 500(1.25)^5$ = $35 095
9. B $Av_{1st\ wk} = 29.29$, $Av_{2nd\ wk} = 23.14$, difference = 6.15
10. D $h^2 = 3.8^2 - 3^2$ so h = 2.33 (m)
11. D 30° difference is 2 hours, Y is west of X
12. C 180 x 60 mm = 10.8 m
13. D Must have the 5 number summary statistics.
14. D $(0.1)^4 = 0.0001$ so zero is closest
15. B Line parallel to X axis and 5 units above it.
16. C 2 x 2 + 7 x 1 = 11
17. B 6 ¼ x $18.50 x 1.7 = $196.56
18. D Ali's median is 19, so is Barb's Ali played 13, Barb played 12 Ali's range is 47, Barb's is 43
(2 + 8 + 13 + 16 + 18 + 20 + 24 + 25 + 25 + 31 + 37 + 45) ÷ 12 = 22
19. B $60 is 6% of $1000
20. C $27y^3 = (3y)^3$
21. A Direct but negative. Must be straight line
22. A One flavour omitted for each combination so only 4 combinations.

Section 2

23. (a) (i)

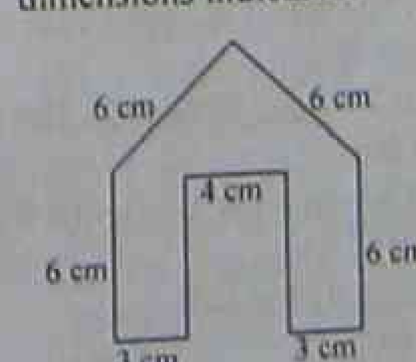


$\frac{x}{24} = \tan 74°$
$x = 24 \tan 74°$
= 83.7 (m)
(ii)



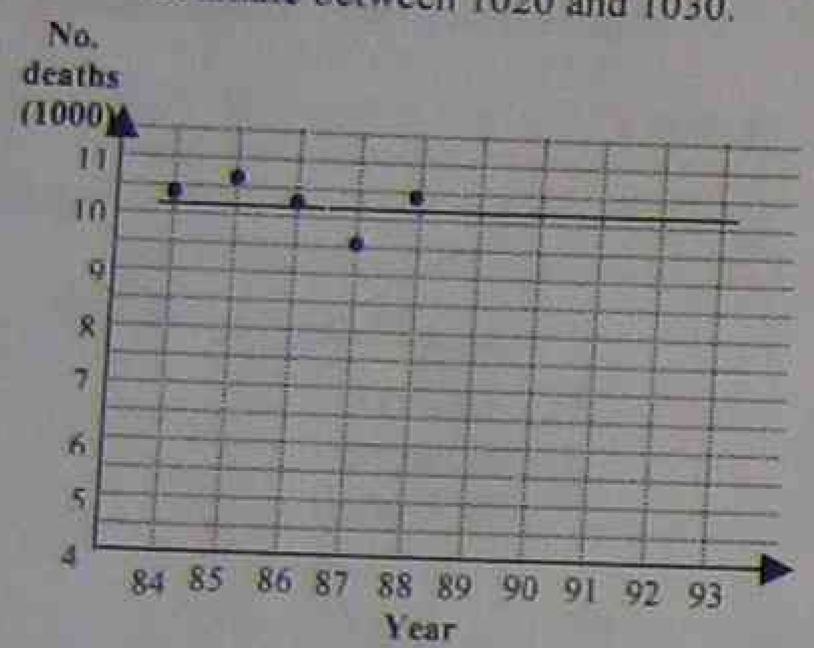
$\frac{24}{y} = \cos 83°$
$y = \frac{24}{\cos 83°}$
y = 197 (m)
(iii) d = 83.7 + 197 − 2 x 38.7 x 197 cos 7
d = 114
(b) (i) Boundary = 50π + 250 + 250 = 657 m
(ii) Circle with radius 25 m has area π x 25 x 25 = 1963 m². Rectangle 250 m x 50 m has area 12 500 m² so total is 14 463 m²
(iii) Area of annulus is $\pi(26^2 - 25^2)$ = 160 m². Area of straights = 2 x 250 x 1 = 500 m² so total is 660 m²
(c) (i) 366 days
(ii) $\frac{20}{366} = 0.055$
(iii) Months with 31 days, e.g. Jan, May, July, Aug ....

24. (a) (i) Isosceles triangles, rectangles.
(ii) Scale drawing should have the dimensions indicated.



(iii) $600^2 = x^2 + 500^2$
so x = 332
(iv) cos y = $^{500}/_{600}$
so y = 33° 33'

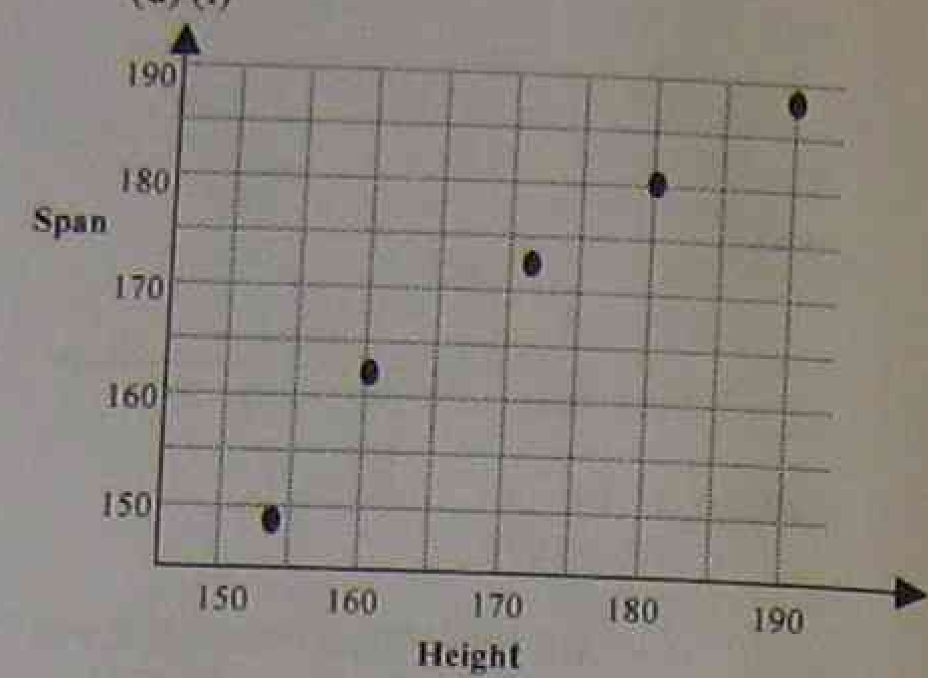
25. (a) (i) 8779 ÷ 10 = 877.9
(ii) 581 x 100 ÷ 1384 = 42%
(iii) The line is roughly horizontal. Estimate between 1020 and 1030.



(b) (i) 11.4 x 20 ÷ 19 = 12
(ii) Less than 2 s.d. below mean is 2 ½ % of distribution.
(iii) ±1 s.d. from mean: 68%
(c) (i) $^{10}/_{11}$ x $14.95 = $13.59
(ii) 10 348 x $13.59 = $140 629.32
(iii) 70% x $140 629.32
= $98 440.52
(iv) 0.12 x $13.59 x 10 348
= $16 875.52

26. (a) (i) A: 159, B: 161, C: 145 ∴ Belinda
(ii) A: 327, B: 331, C: 325
∴ Anastasia
(iii) A: 5.716, B: 5.516, C: 6.033 ∴ Constance
(iv) All three systems can be considered fair and unfair. Comparing only English/Maths ignores half of the students' work. Adding raw scores ignores mean and spread difference between subjects. z-scores assumes a normal distribution and gives unbalanced advantage for one high mark.
(b) (i) 10% of 3 million is $300 000
(ii) In first year, machine depreciates by $720 000 so its value after 1 year is $2 280 000.
In the 2nd year, 24% of $2 280 000 is $547 200.
(c) (i) (15 + 6k + 28 + 72) ÷ (k + 16)
= $^{(6k+115)}/_{(k+16)}$
(ii) 6k + 115 = 133
6k = 18
k = 3

(d) (i)



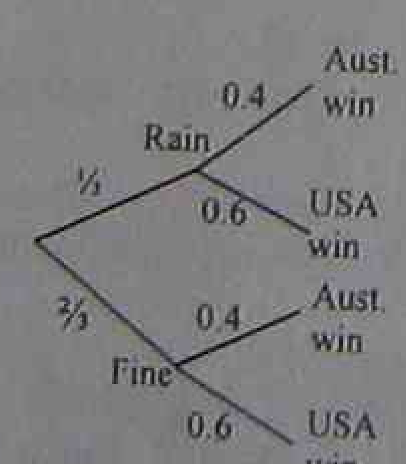
(ii) A very high degree of positive correlation is evident.

27. (a) (i) 2x = -9 so x = - 4 ½
(ii) 3y = 5y + 10
2y = -10
y = -5
(b) (i)

$10000	$91.67	$10091.67	$9874.27
$9874.27	$90.51	$9964.78	$9747.38
$9747.38	$89.35	$9836.73	$9619.33

Note that the interest rate is 0.11 ÷ 12 per month.
(ii)$217.40 x 12 x 5 = $13 044
(iii) $3 044 over 5 years is $608.80 per year = 0.0688 or about 6% p.a.
(c) $E = \frac{(1 + 0.0825)^4 - 1}{4}$
= 0.0933 or about 9.33%
(d) Semi-major axis measures 5.35 cm
Semi-minor axis measures 1.7 cm
Area = π x 5.35 x 1.7
Area is about 28.6 cm²

28. (a) (i) Sensible to change to centimetres
V = π x 7.5² x 18 cm³
= 3180 cm³ or mL
(ii) SA = ½ x 4π x 7.5²
= 353 cm²
(iii) 180 + 75 = 255 mm
(b) (i)



(ii) ⅔ x 0.6 = 0.4

(iii) 0.4 x $15 = $6 so the punter would expect a 60% return on his investment.
(c) (i) $D = \frac{2 \times 200^2 \times \sin 30° \cos 30°}{9.8}$
= 3535 m
(ii) Zero (straight up, straight down)
(iii) $D = = \frac{2 \times 200^2 \times \sin 60° \cos 60°}{9.8}$
but sin 60° = cos 30°
and cos 60° = sin 30°
so distance = 3535 m

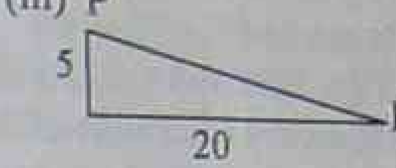
Sample Exam Paper 4

Section 1

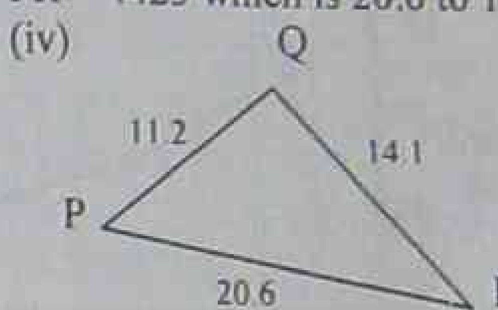
1. B 20a = 2a + 2
18a = 2
2. B (50 x 24) ÷960 = 1¼ h or 75 min
3. D 47 other cards for first draw, 46 for the second draw. P(one of them is the ace)
= 1 – P(neither is the ace)
= 1 - $^{46}/_{47}$ x $^{45}/_{46}$
= 1 - $^{45}/_{47}$
= $^{2}/_{47}$
4. C I = $8400 x 0.09 x 2
= $1512 giving total $9912
$9912 ÷ 24 = $413
5. D 2 x 1000 x 1000 m ÷ 26 500
= 75.47 mm
6. A Answer is 1301.66 ....
7. C 2x = 18.66 ..... so x = 9.33 ....
8. D None of these statements necessarily follows from the statistics (though they might be true).
9. C Total area = 4π x 6400² km² so land area ⅓ of that.
10. D Mean is 3, s.d. is 1.84 so limit of ±2 s.d. is –0.68 to 6.68. All scores lie in this range.
11. A Should follow a great circle.
12. C 3h difference so 5pm
13. A Gradient ½ ,Y intercept 1
14. A 288 ÷ 36 = 8 so the radius must have doubled.
15. D 23g + 35c + 20g + 20c
16. C The sum of the differences from the mean is zero (definition of a mean).
17. D 35 h at normal time, 6h at time-and-a-half, 5h double time making 54h at $6.65 /h
18. B 3.15% per ½ year, 20 terms.
7000(1 + 0.0315)^{20} = $13 016
19. B Average of 9th and 10th scores
20. C 1 arrangement out of 6, 16.67%
21. A 15% of $4000 + 22% of extra = $897
22% of extra = $897 - $600
Extra = $297 x 100 ÷ 22 = $1350
Price was $4000 + $1350 = $5300
22. B Number pattern is 12,16, 20.
All four expressions work for Pattern 1 and Pattern 2 but only 4(n + 2) gives 20 when n = 3.

Section 2

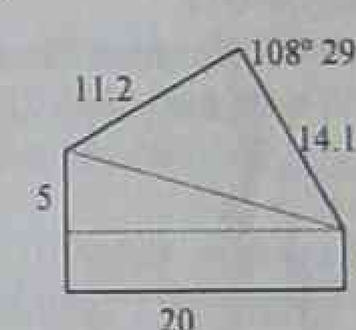
23. (a) (i) 40.3 x $35 = $1410.50
(ii) 5% of %15 000 + 3% of $45 000 + 2 ½ % of $40 000 + 2% of $245 000 = $2375
(iii) P



PR² = 5² + 20²
PR = √425 which is 20.6 to 1 dec.pl.
(iv) Q



20.6² = 11.2² + 14.1² – 2 x 11.2 x 14.1cos Q
cos Q = -0.31696 ......
Q = 108° 29'
(v)

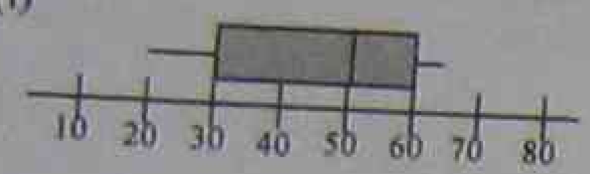


A = 100 + 50
+ ½ x 14.1 x 11.2 sin 108° 29'
= 224.887 ..... m² or about 225 m²
(b) (i) (5, 1), (4, 2), (3, 3), (2, 4), (1, 5)
(ii) P(double) = $^1/_6$
so 300 x $^1/_6$ = 50 times
(iii) 7 as there are 6 ways of achieving this total.

24. (a) (i) A = $^{40}/_3$ [0 + 4 x 36 + 44] + $^{40}/_3$ [44 + 4 x 39 + 0]
= 5173 m²
(ii) V = 400 x 5173
= 2 069 200 m³
(iii) 1 m³ = 1 kL
so volume is 2 069 200 kL
(b) (i) Yes. $^{19}/_{50}$ contained fewer than 50 matches so 62% contained 50 or more.

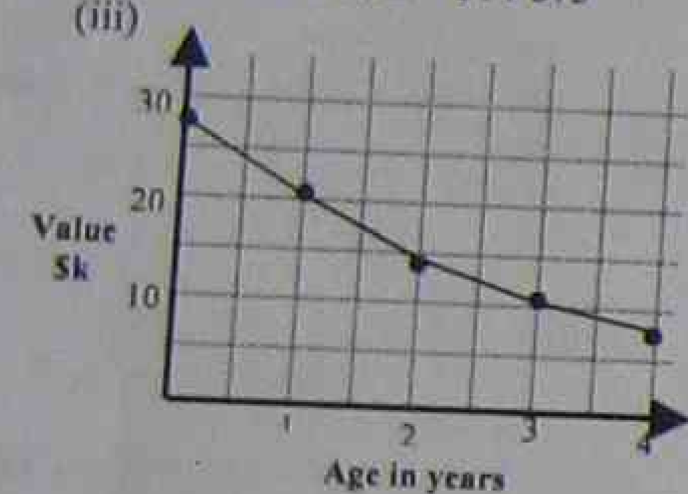
(ii) Upper quartile (37^{th} / 38^{th} scores) are both 50. Lower quartile (12^{th} / 13^{th} scores) both 49 so i.r. = 1
(iii) 25^{th} / 26^{th} highest are both 50 so median is 50
(iv) 40 – 19 = 21 out of 50 in the sample contained 50 matches. Best estimate is 0.42 (say $^2/_5$)

(c) (i)



(ii) Range is 63 – 20 = 43

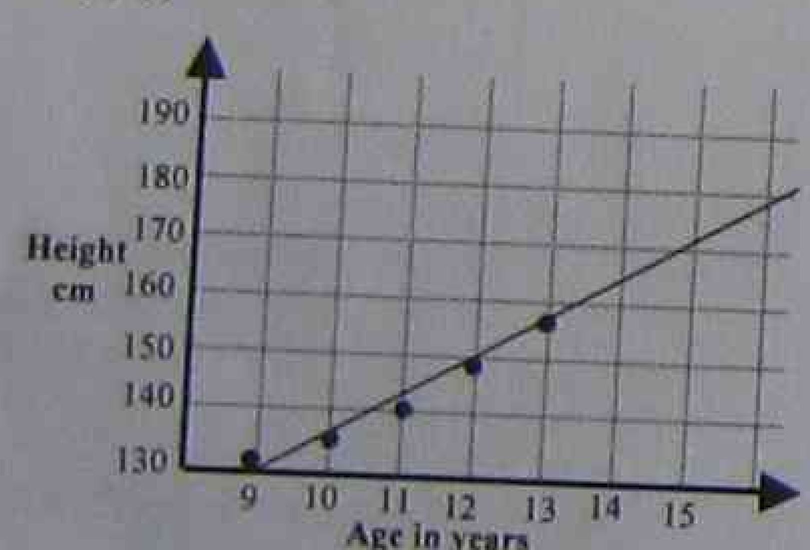
25. (a) (i) 0.72 x \$28 695 = \$20 660.40
(ii) \$20 660 x 0.72 = \$14 875
(iii)



(iv) Worth \$9565 after about 3 ½ years

(b) (i) 13 hours
(ii) 30° S 90° E
(iii) 10 000 km from Equator to Pole 30° is ⅓ of that, 3333 km

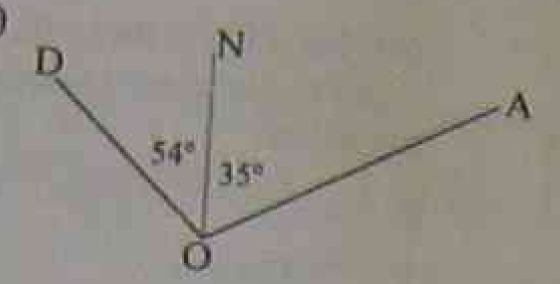
(c) (i)



(iii) Gradient is about 9 cm/year so at age 21, Xavier expects to be about 227 cm tall.
(iv) Linear modelling not appropriate as growth stops at about 17 years.

26. (a) (i) \$12 000 x 0.075 x 6 = \$5400
(ii) \$600 interest is \$200 per year. $^{200}/_{2800}$ is about 7.1%

(b) (i)



35° + 54° = 89°
(ii) Using cos rule
$DA^2 = 27^2 + 41^2 - 2 \times 27 \times 41 \cos 89°$
Length of DA is about 48.7 m
(iii) Angle is 105° so
Area = ½ x 41 x 58 sin 105°
= 1148.5 m²

(c) (i)

	Lying	Truth	Total
Correct	72	18	90
Wrong	8	2	10
Total	80	20	100

(ii) Out of 100, 18 Truthful and 8 Liars will give result indicating that they are telling the truth, so $^8/_{26}$ = $^4/_{13}$ chance they are lying.
(iii) Similarly, $^2/_{74}$ = $^1/_{37}$

27. (a) (i) Normal
(ii) 200
(iii) 96%
(iv) 16% of 3780 = about 605

(b) (i) $1^2 + 3 \times 1 = 4$ and $2^2 + 3 \times 2 = 10$ so $x^2 + 3x$ is equal to 7 somewhere between x = 1 and x = 2
(ii) $(1.5)^2 + 3 \times 1.5 = 6.75$
$(1.6)^2 + 3 \times 1.6 = 7.36$
$(1.55)^2 + 3 \times 1.55 = 7.05$ so solution lies between 1.50 and 1.55, i.e. 1.5 correct to one decimal place.

(c) (i) Growth factor 1.6 in 10 years so growth of about 4.8% p.a.
(ii) \$150 000$(1 + 0.04)^{10}$ = \$222 037
(iii) Capital gain = \$240 000 - \$222 037 = \$17 963
47 ½ % of \$17 963 = \$8532.43

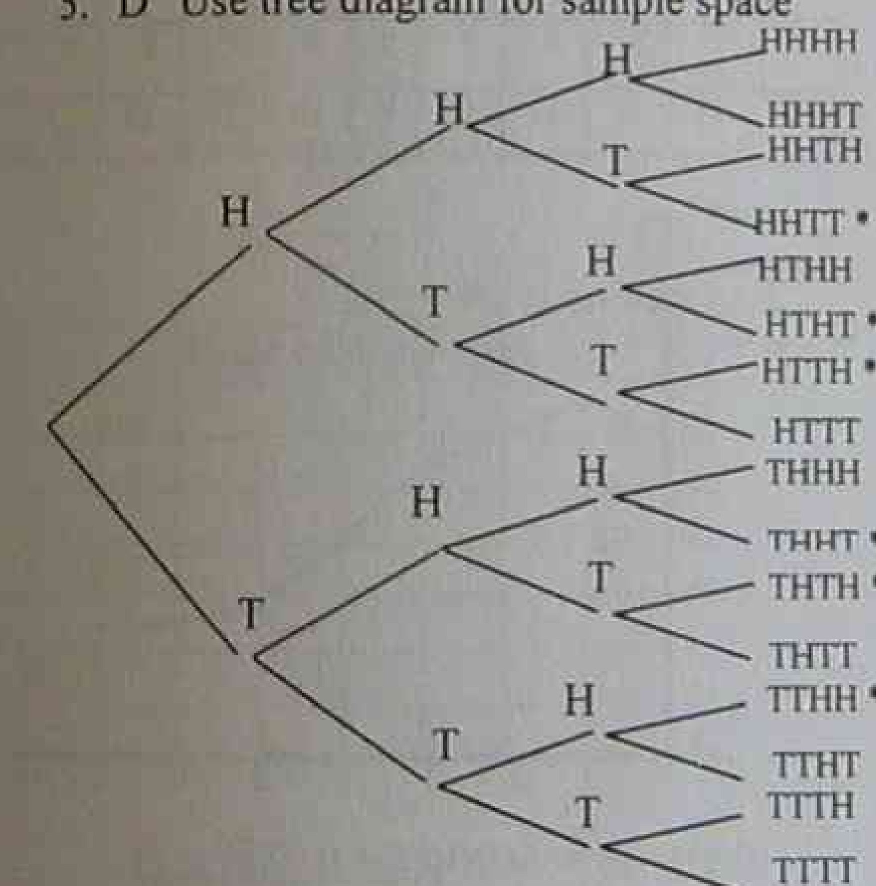
28. (a) (i) 125 people
(ii) They would be 85+ years old by 1990, i.e. only 1 of the 4 people
(iii) 59^{th} / 60^{th} oldest are both in the 51 – 55 age group
(iv) 46 – 50 age group
(v) Declining. Very few young people compared to older age groups and this has been a trend for about 40 years.

(b) (i) Use annuity formula with 78 terms and $r = ^{0.1}/_{26}$
$A = 200\dfrac{\{(1 + 0.003846)^{78} - 1\}}{0.003846}$
= \$18 152.27
(ii) \$200 for seven years gives
$= 200\dfrac{\{(1 + 0.003846)^{182} - 1\}}{0.003846}$
= \$52 574.63
Therefore, to get \$200 000, he needs to invest $^{200\,000}/_{52\,574.63}$ times as much
$^{200\,000}/_{52\,574.63}$ x \$200 = \$760.82
(iii) $A = 1500\dfrac{\{(1 + 0.04)^6 - 1\}}{0.04}$
= \$9949.46

Sample Exam Paper 5

Section 1

1. A 4.5 x 3.1 x 2+ 5.8 x 3.1 x 2 = 63.86 m²
63.86 ÷ 14 = 4.56
so two coats takes 9.12 L
2. A 360 x 0.8L difference x 40.90 = \$259
3. B $2as = v^2 - u^2$
4. B 0.150 x 120 = 18 m²
5. D Use tree diagram for sample space

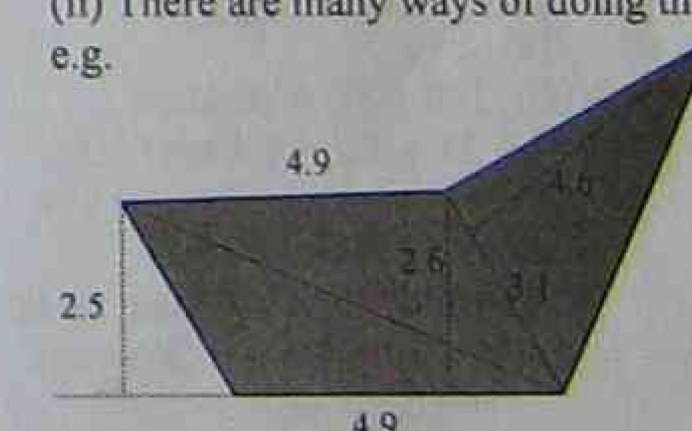


6. C 150 000$(1 + 0.07)^5$ = \$210 382
7. A 1 4949 200 x 3.629 = 542 506 468
8. A 1 km = 8 ÷ 14.816 M
2.3 km = 8 x 2.3 ÷ 14.816 M
9. B 79.7 ÷ 65 km/min
= 79.7 x 60 ÷ 65 km/h
= 73.6 km/h
10. D Can't have longitude greater than 180°
11. C Easy points are (0, 1) and (5, 3)
Gradient = (3 – 1) ÷ (5 – 0)
12. D Multiply each term by 12
13. D Use stats function on calculator
14. A Old: \$56 160 p.a. so difference is \$1840
15. D Bronze is less than half of the circle.
16. D Interquartile range is 13, mean cannot be determined from box & whisker ∴ neither is true.
17. D 12 = 5 x 1 + ½ x a x 1
½ a = 7
a = 14
18. B Arrangement doesn't matter.
52 ways of choosing 1^{st} card
51 ways of choosing 2^{nd} card
50 ways of choosing 3^{rd} card
49 ways of choosing 4^{th} card
48 ways of choosing 5^{th} card
19. C Just because they are the same distance from O doesn't mean that they are in the same place.
20. B Sine Rule: $\dfrac{y}{\sin 59°} = \dfrac{27}{\sin 46°}$
21. C Can land on "Chance" by throwing 3 or 5 only, $^2/_6$
22. A $Area_{annulus} = \pi \times 5^2 - \pi \times 4^2$
$= 25\pi - 16\pi$
$= 9\pi$ or about 28.27
$Area_{diagonal\ bar}$ = about 8 x 1 (rectangle)
so area or rebus is about 36 cm²
Total = 100 cm²
∴ about 36%

Section 2

23. (a) (i) \$12.42 x 75 = \$931.50
(ii) \$931.50 x 12 x 25 = 279 450
(iii) Interest = $\dfrac{204\,450}{25 \times 75\,000}$
= 0.10904 or about 10.9%

(b) (i) (4.9 + 3.0 + 4.9 + 4.5 + 5.1) x 100
= 2240 m
(ii) There are many ways of doing this. e.g.

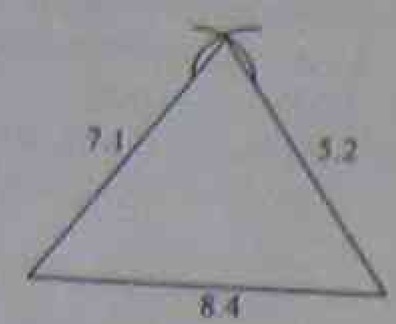


½ (490 x 250) + ½ (490 x 260)
+ ½ (310 x 460)
= 196 250 m²

(c) (i) About 5 ½ million
(ii) Steepest gradient in the 1960s.
(iii) About 3.2 million in 1890
About 17 million in 1990
(13.8 x 100) ÷ 3.2 = about 430%

(iv) Continuing the trend for 1990s, increase of 1 million, ∴ about 18 million

(d)



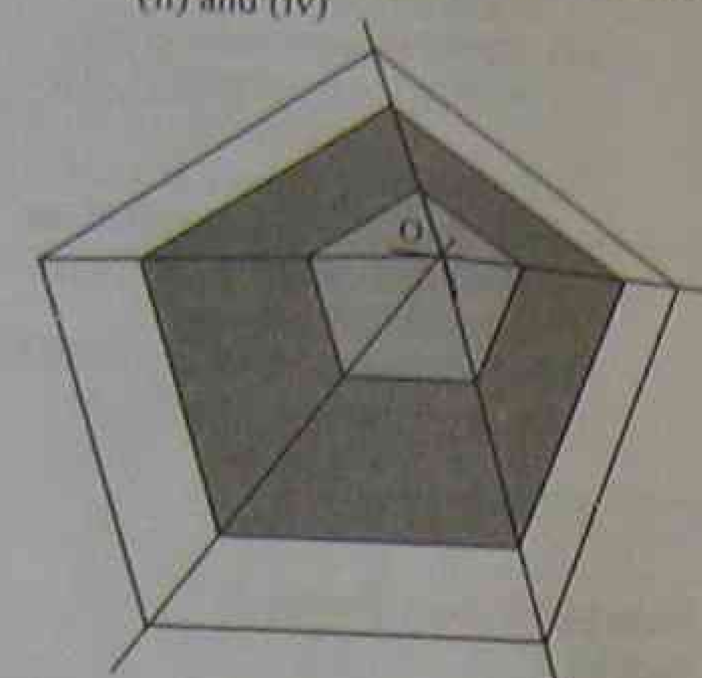
24. (a) (i) 53 – 45 = 8
(ii) 49
(iii) 5^{th} and 6^{th} scores are both 49
(iv) 49
(v) Median is less affected by outliers.
(b) (i) Both worth the same amount.
(ii) $1500
(iii) Flat / simple interest, 10% p.a.
(iv) Compound, 10% p.a. compounded annually
(v) $1000(1 + 0.1)^{12} = \$2593.74$

25. (a) (i) Total of frequency column = 50
(ii) and (iv)

No. rings	f	Cum. f	Rel.f
1	3	3	0.06
2	5	8	0.10
3	13	21	0.26
4	10	31	0.20
5	7	38	0.14
6	7	45	0.14
7	2	47	0.04
8	1	48	0.02
9	1	49	0.02
10	1	50	0.02
Total	50	[redacted]	1.00

(iii) $^{31}/_{50} = 62\%$
(v) $^{13}/_{50}$
(b) (i) 320 ÷ 5⅓ = 60 km/h
(ii) 27 ÷ 3.2 = 8.4 km / 100L
(iii) Total cost = 27 x $0.979
= $26.43
∴ each share = about $6.60
(c) (i) 6 x 9 + 12 x m + 2(6 + m)
= 14m + 66
(ii) Let no. be m
12m + 2m = 126
14m = 126
m = 9

26. (a) (i) 8 kg
(ii) 1.5 years
(iii) ⅝ left so factor is about 0.625
(iv) Never, but it will get very close.
(b) (i) $^{10}/_{12} = ^{5}/_{6}$
(ii) $^{2}/_{12} \times ^{1}/_{11} = ^{1}/_{66}$
(iii) Still $^{1}/_{12}$
(c) (i) Depending on the sides chosen, the answer is about 22 : 50 or about 2 : 2.3
(ii) and (iv)

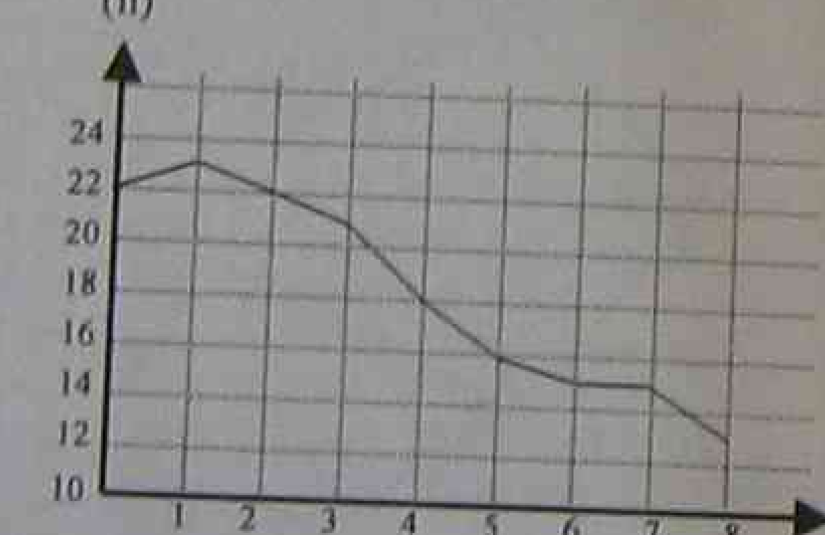


(iii) 2.3^2 or about 5.3 times
(iv) Note: treble the distance of each vertex of the small pentagon from O.

27. (a) (i)

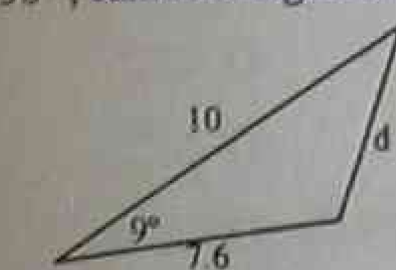
T	1	2	3	4	5	6	7	8	9	10
°C	11	11	11	10	9	10	11	13	15	18

(ii)



(b) (i) $A = 5000\dfrac{\{(1 + 0.02)^{80} - 1\}}{0.02}$
= $968 859.79
(ii) $N = A \div (1 + r)^n$
$= 968\,859.70 \div (1.02)^{80}$
= $198 722.57
(iii) 80 x $5000 = $400 000
(iv) $968 859.79 - $400 000
= $568 859.79
(c) (i) 6 + 7 = 13
(ii) 5
(iii) $^{22}/_{40} = 0.55$
(iv) $^{6}/_{13}$

28. (a) (i) $^{1}/_{400} \times \$300 + ^{1}/_{400} \times \$150 + ^{1}/_{400} \times \$50 = \$1.25$
(ii) 398
(b) (i) $1 300 000 ÷ 12 = $108 333
(ii) 78% of $1.3 million = $1 014 000
(iii) Nothing under straight line depreciation. $\$1.3(1 - 0.22)^{12}$ million = $65 929 under declining balance method.
(iv) After 6 yrs, worth $\$1.3(1 - 0.22)^{6}$ million = $292 759
After 7 yrs, worth $\$1.3(1 - 0.22)^{7}$ million = $228 352
Difference is $64 407
(c) (i) 36°, same as angle of elevation.
(ii)



cosine rule:
$d^2 = 10^2 + 7.6^2 - 2 \times 10 \times 7.6 \cos 9°$
d = 2.76 (M)
(iii) Sine rule is easier than cos rule.
$\dfrac{\sin\theta}{7.6} = \dfrac{\sin 9°}{2.76}$
$\sin\theta = \dfrac{7.6 \sin 9°}{2.76}$
$\theta = 25.52°$
= 26° to the nearest degree.
Angle of depression is 26° + 36° = 62°

Multiple Choice Answer Sheets

Sample Exam Paper 1

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭
6.	A ⬭	B ⬭	C ⬭	D ⬭
7.	A ⬭	B ⬭	C ⬭	D ⬭
8.	A ⬭	B ⬭	C ⬭	D ⬭
9.	A ⬭	B ⬭	C ⬭	D ⬭
10.	A ⬭	B ⬭	C ⬭	D ⬭
11.	A ⬭	B ⬭	C ⬭	D ⬭
12.	A ⬭	B ⬭	C ⬭	D ⬭
13.	A ⬭	B ⬭	C ⬭	D ⬭
14.	A ⬭	B ⬭	C ⬭	D ⬭
15.	A ⬭	B ⬭	C ⬭	D ⬭
16.	A ⬭	B ⬭	C ⬭	D ⬭
17.	A ⬭	B ⬭	C ⬭	D ⬭
18.	A ⬭	B ⬭	C ⬭	D ⬭
19.	A ⬭	B ⬭	C ⬭	D ⬭
20.	A ⬭	B ⬭	C ⬭	D ⬭
21.	A ⬭	B ⬭	C ⬭	D ⬭
22.	A ⬭	B ⬭	C ⬭	D ⬭

Sample Exam Paper 2

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭
6.	A ⬭	B ⬭	C ⬭	D ⬭
7.	A ⬭	B ⬭	C ⬭	D ⬭
8.	A ⬭	B ⬭	C ⬭	D ⬭
9.	A ⬭	B ⬭	C ⬭	D ⬭
10.	A ⬭	B ⬭	C ⬭	D ⬭
11.	A ⬭	B ⬭	C ⬭	D ⬭
12.	A ⬭	B ⬭	C ⬭	D ⬭
13.	A ⬭	B ⬭	C ⬭	D ⬭
14.	A ⬭	B ⬭	C ⬭	D ⬭
15.	A ⬭	B ⬭	C ⬭	D ⬭
16.	A ⬭	B ⬭	C ⬭	D ⬭
17.	A ⬭	B ⬭	C ⬭	D ⬭
18.	A ⬭	B ⬭	C ⬭	D ⬭
19.	A ⬭	B ⬭	C ⬭	D ⬭
20.	A ⬭	B ⬭	C ⬭	D ⬭
21.	A ⬭	B ⬭	C ⬭	D ⬭
22.	A ⬭	B ⬭	C ⬭	D ⬭

Sample Exam Paper 3

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D
6.	A	B	C	D
7.	A	B	C	D
8.	A	B	C	D
9.	A	B	C	D
10.	A	B	C	D
11.	A	B	C	D
12.	A	B	C	D
13.	A	B	C	D
14.	A	B	C	D
15.	A	B	C	D
16.	A	B	C	D
17.	A	B	C	D
18.	A	B	C	D
19.	A	B	C	D
20.	A	B	C	D
21.	A	B	C	D
22.	A	B	C	D

Sample Exam Paper 4

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D
6.	A	B	C	D
7.	A	B	C	D
8.	A	B	C	D
9.	A	B	C	D
10.	A	B	C	D
11.	A	B	C	D
12.	A	B	C	D
13.	A	B	C	D
14.	A	B	C	D
15.	A	B	C	D
16.	A	B	C	D
17.	A	B	C	D
18.	A	B	C	D
19.	A	B	C	D
20.	A	B	C	D
21.	A	B	C	D
22.	A	B	C	D

Sample Exam Paper 5

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D
6.	A	B	C	D
7.	A	B	C	D
8.	A	B	C	D
9.	A	B	C	D
10.	A	B	C	D
11.	A	B	C	D
12.	A	B	C	D
13.	A	B	C	D
14.	A	B	C	D
15.	A	B	C	D
16.	A	B	C	D
17.	A	B	C	D
18.	A	B	C	D
19.	A	B	C	D
20.	A	B	C	D
21.	A	B	C	D
22.	A	B	C	D

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James Moule, BSc DipEd AMusA, has been teaching Mathematics in state and private schools since 1972. As state Mathematics Consultant for the NSW Department of Education, he was a co-author of the syllabus in Mathematics K-6 and led the writing team which produced the Mathematics in Practice course. He has written a large number of curriculum support documents for the Department of Education, especially for Years 7-8 Mathematics. He is now Deputy Principal at East Hills Girls Technology High School.

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