The power factor of the capacitor is $\cos\phi$ and its loss angle $\delta = \alpha - \beta$.

Obviously, in addition to the value of the variable inductance L, the values of the inductance l_p and resistance r_p of the pressure coil circuit, of the resistance r_c of the current coil and C and ω must be known.

Rosa (Ref. (18)) has described several null methods of measurement of dielectric loss using wattmeters.

Electrostatic Wattmeter Method. This method has been used by many investigators. Fig. 87 (a) shows the connections for the method

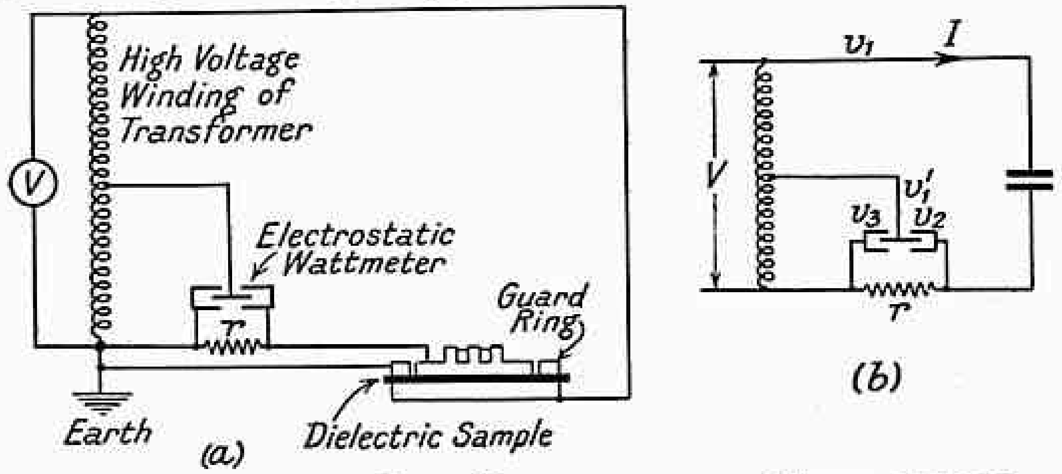


FIG. 87. DIELECTRIC LOSS MEASUREMENT BY ELECTROSTATIC WATTMETER

as used by Rayner (Ref. (20)). Fig. 87 (b) gives the equivalent diagram showing the instantaneous potentials v_1, v_2, etc., at various points; r is a non-inductive resistance.

The moving vane of the electrostatic instrument is connected to a tapping point on the high voltage winding of a transformer from which the supply is obtained.

The sample of insulating material whose dielectric loss is to be measured, is connected as shown and is provided with a guard ring which is earthed.

From the theory of the electrostatic wattmeter given in Chap. XX, it can be shown that the mean torque of the wattmeter is proportional to

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2}$$

where P = dielectric power loss

I = r.m.s. value of the current

Then, if K is the constant of the instrument and D is the deflection, we have

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2} = KD$$

from which
$$P = \frac{nKD}{r} + \frac{n-2}{2}rI^2 \quad . \quad . \quad (108)$$

If the tapping point on the transformer winding is adjusted so that $n = 2$, the second term becomes zero, and we have

$$P = \frac{2KD}{r}$$

This avoids the correction for the power loss in the resistance r.

The voltage used by Rayner in his measurements was 10,000 volts.

(b) BRIDGE METHODS. The Schering bridge method is now the most widely used of all methods of measuring dielectric loss and power factor. All bridge methods consist essentially of a

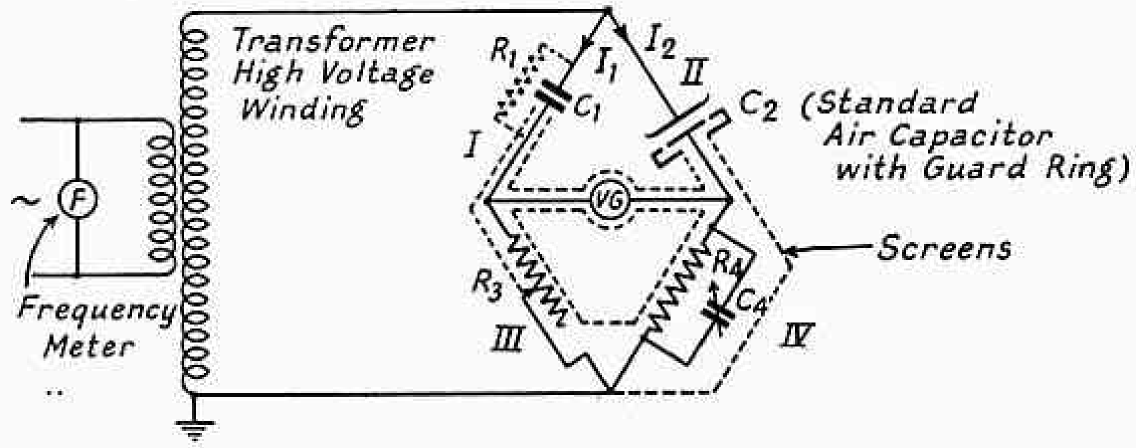


FIG. 88. CONNECTIONS OF SCHERING BRIDGE

Wheatstone bridge network, the battery supply being replaced by an a.c. supply at either power frequency or some higher frequency. The detector used depends upon the frequency, a vibration galvanometer being used for power frequency work and telephones for work at higher frequencies, the latter being often of the order of 800 to 1,000 cycles per second.

Fig. 88 gives the connections of the Schering bridge, which can be used with high or low voltages. C_1 is the capacitor whose power factor is to be measured, R_1 being an imaginary resistance representing its dielectric loss component. C_2 is a standard air capacitor. of the type described in Chapter II. R_3 and R_4 are non-inductive resistors, the former being variable. C_4 is a variable capacitor Earthed screens are provided in order to avoid errors due to intercapacitance between the high and low voltage arms of the bridge. Instead of earthing one point on the network as shown in the figure, the earth capacitance effect on the galvanometer and leads is eliminated by means of a "Wagner earth" device (Ref. (22)), which will be described in a later chapter. V.G. is a vibration galvanometer of a special design suited to the purpose. This must have a high current sensitivity, since the impedances of arms 1 and 2 of the bridge are usually very high. For the same reason, this method of measurement involves only a small power loss. Since the impedances of branches 3 and 4 are usually small compared with those of arms 1 and 2, the galvanometer and the resistances are at a potential of

CHAPTER IV

CAPACITORS, CAPACITANCE, AND DIELECTRICS

General Considerations. In Chapter I capacitance was defined with reference to a number of conductors having different charges and being at different potentials. Self- and earth-capacitances were also discussed. Before proceeding to develop formulae for the capacitances of various common arrangements of conductors encountered in practice it may be well to give these matters a little further consideration.

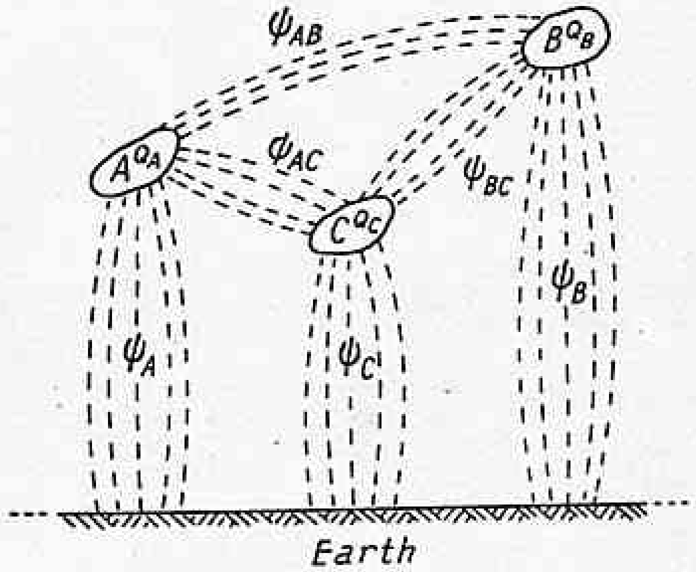


FIG. 63. SYSTEM OF CHARGED CONDUCTORS NEAR TO EARTH

Fig. 63 shows a general system of conductors in air, situated at various distances from earth and from one another. If all these conductors are at the same potential above earth, varying quantities of electric flux will pass from them to earth, these fluxes depending, in each case, upon the size and shape of the conductor, and upon its position relative to earth—i.e. upon the "earth capacitance" of each conductor. No flux will pass from one conductor to another, since they are all at the same potential above earth. The quantities of positive electricity existing upon the various conductors will be different, since their earth capacitances are different and their potentials the same.

Suppose the capacitances of the various conductors to earth are given by C_A, C_B, C_C, etc. Suppose now that the conductors are charged to different potentials V_A, V_B, V_C, etc., above earth. In this case, not only will some flux pass from each conductor to earth, but, in addition, flux will pass between any one conductor and each of the others in the system. Each of these inter-conductor fluxes will

be proportional to the difference of potential of the conductors between which it exists, and its direction will, of course, depend upon which of the two conductors concerned is at the higher potential. If conductor A is at a higher potential than any of the other conductors, fluxes will flow from it, which may be represented by ψ_{AB}, ψ_{AC}, ψ_{AD}, and so on, the second suffix letter indicating, in each case, the conductor to which the particular flux radiated from A flows. If B is at the second highest potential, the fluxes radiating from it are $-\psi_{BA}$, ψ_{BC}, ψ_{BD}, etc., and for conductor C, $-\psi_{CA}$, $-\psi_{CB}$, ψ_{CD}, etc., assuming it to be the third highest in potential. As stated above there will be, in each case, an earth flux which may be represented by ψ_A, ψ_B, ψ_C, etc.

It may be supposed that a portion of the total charge of each conductor is associated with each of the fluxes radiating from that conductor. These portions of charge will, of course, be proportional to the corresponding fluxes, and therefore will be proportional to the differences in potential between the pairs of conductors. Representing these portions of charge, in the case of A by Q_{AB}, Q_{AC}, Q_{AD}, etc., and in the case of B by Q_{BA}, Q_{BC}, Q_{BD}, etc., and so on, we have for the total charges on the various conductors

$$\begin{aligned}
Q_A &= C_A V_A + C_{AB}(V_A - V_B) + C_{AC}(V_A - V_C) + C_{AD}(V_A - V_D) + \ldots \\
Q_B &= C_B V_B + C_{AB}(V_B - V_A) + C_{BC}(V_B - V_C) + C_{BD}(V_B - V_D) + \ldots \\
Q_C &= C_C V_C + C_{AC}(V_C - V_A) + C_{BC}(V_C - V_B) + C_{CD}(V_C - V_D) + \ldots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (81)
\end{aligned}$$

Thus, if there are n capacitors, each one has n component capacitances, including its earth capacitance.

In most cases in practice we are concerned with two (or it may be three or four) conductors, which are so near together, compared with their distances from other conductors and from earth, that the capacitances due to the latter can be neglected. Thus, in the case of a capacitor having two plates, A and B, near together, it is only the capacitance C_{AB} which is considered, and this is spoken of as the capacitance of the capacitor. In the cases considered in the following pages earth capacitances and inter-capacitances with conductors other than those forming the arrangement under consideration, will be neglected unless otherwise stated. The earth capacitance, and intercapacitance with other conductors, may, however, be of considerable importance if the capacitor is of small capacitance and large dimensions. In the case of capacitors of capacitance $\frac{1}{10}$ microfarad and over, earth capacitances are usually negligible.

Capacitance of Various Systems of Conductors. 1. CAPACITANCE OF AN ISOLATED SPHERICAL CONDUCTOR. Suppose the spherical conductor to be perfectly insulated and at an infinite distance from all other conductors. Let its radius be R cm. and let the medium surrounding it have permittivity κ.

If a charge of Q units of electricity be given to the sphere, the intensity of the electric field at any point outside it is the same as it would be if the charge were concentrated at the centre of the sphere. Thus, the intensity at any point P, distant x cm. from the centre of the sphere, is, from Equation (1),

$$E = \frac{Q}{\kappa x^2}$$

and the potential of the sphere is given by

$$V = \int_R^\infty \frac{Q}{\kappa x^2}\,dx = \frac{Q}{\kappa R}$$

$$\therefore \text{The capacitance of the isolated sphere} = \frac{Q}{Q/\kappa R} = \kappa R^* \quad . \quad . \quad . \quad (82)$$

If the sphere is in air, its capacitance in electrostatic c.g.s. units is equal to its radius R, expresed in centimetres; or, in air,

$$C = \frac{R}{9 \times 10^{11}} \text{ farads}\dagger$$

2. Capacitance of a Spherical Conductor Inside a Concentric Hollow Conducting Sphere. Let the radii of the inner and outer spheres be R_1 and R_2 cm. respectively, the latter being the radius of the inner spherical surface of the outer sphere. Let κ be the permittivity of the medium between them.

If a charge of $+Q$ units be given to the inner sphere a charge of $-Q$ units will be induced on the inner surface of the outer sphere. Since, as shown in Chapter I, the intensity at any point inside a hollow charged conductor is zero, the intensity at any point between the two spheres will be that due to the inner sphere only. Taking any point P, distant x cm. from the centre of the inner sphere, and, as before, considering the charge on this sphere to be concentrated at its centre, we have, for the intensity at P,

$$E_P = \frac{Q}{\kappa x^2}$$

The potential difference between the spheres is given by

$$V = \int_{R_1}^{R_2} \frac{Q}{\kappa x^2}\,.\,dx\,. = \left[-\frac{Q}{\kappa x}\right]_{R_1}^{R_2}$$

* The potentials and capacitances here and in the rest of the chapter are expressed in e.s.c.g.s. units but can be expressed in m.k.s. units if κ_0, the permittivity of free space, is inserted (*see* Ch. II).

† This conversion to farads—by dividing by 9×10^{11}—will be omitted in the rest of the chapter.

$$\therefore V = \frac{Q}{\kappa}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Hence, the capacitance of the arrangement is

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{\kappa}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} = \frac{\kappa R_1 R_2}{R_2 - R_1} \quad . \quad . \quad . \quad (83)$$

3. Capacitance Between Two Spheres at a Relatively Great Distance Apart. In this case each sphere will have its own "self-capacitance," and also a mutual capacitance with the other sphere. Suppose that the two spheres have equal and opposite

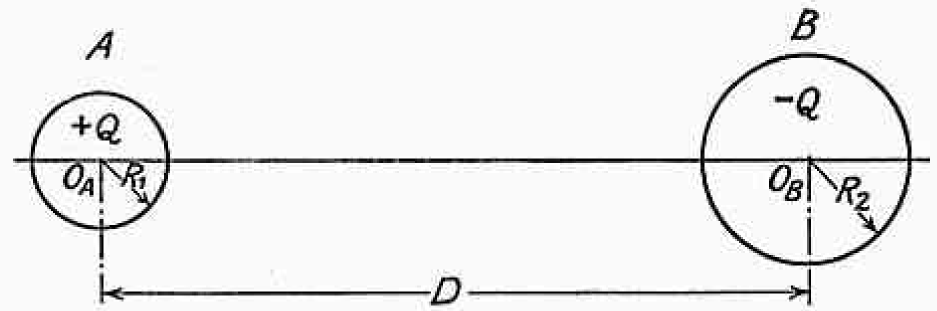


Fig. 64. Two Charged Spheres

charges, and are at a relatively great distance apart, and infinitely distant from all other bodies.*

Under these conditions, if the charges upon spheres A and B are $+Q$ and $-Q$ units, and their potentials V_1 and V_2, then the capacitance between the spheres is

$$C = \frac{Q}{V_1 - V_2}$$

Let the spheres have radii R_1 and R_2 cm. respectively, and let their distance apart be D cm. in air (see Fig. 64). Then the potential at the centre O_A of sphere A due to its own charge is $\frac{Q}{R_1}$. If the second sphere is distant from sphere A, the potential at O_A due to the charge on B is $-\frac{Q}{D}$

$$\therefore V_1 = \frac{Q}{R_1} - \frac{Q}{D}$$

By similar reasoning

$$V_2 = -\frac{Q}{R_2} + \frac{Q}{D}$$

* The capacitance of a system of two charged spheres in the general case has been fully investigated by Russell (Ref. (12)).

Thus the capacitance between the spheres

$$C = \frac{Q}{V_1 - V_2} = \frac{Q}{\frac{Q}{R_1} - \frac{Q}{D} - \left(\frac{-Q}{R_2} + \frac{Q}{D}\right)} = \frac{1}{\frac{1}{R_1} - \frac{2}{D} + \frac{1}{R_2}}$$

or

$$C = \frac{R_1 R_2 D}{D(R_1 + R_2) - 2R_1R_2}.$$

If the medium is not air but has a permittivity κ, then

$$C = \frac{\kappa . R_1R_2D}{D(R_1 + R_2) - 2R_1R_2} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

If the spheres are equal

$$C = \frac{\kappa\, RD}{2\,(D - R)}$$

where R is the common radius.

Russell (*loc. cit.*) gives the *capacitance of the two spheres in parallel*, i.e. when connected by a thin wire so that they are at the same potential as

$$C_P = \left(R_1 + R_2 - \frac{2R_1R_2}{D}\right)\frac{D^2}{D^2 - R_1R_2} \quad . \quad . \quad . \quad . \quad . \quad (85)$$

(in air), using the symbols as above. If the spheres have equal radii R, then

$$C_P = \frac{2RD}{D + R} \quad \text{or} \quad C_P = \frac{2\kappa RD}{D + R}$$

when in a medium of permittivity κ.

For *two equal spheres close together*, the capacitance between the spheres is given approximately by

$$C = \frac{R}{2}\left(1 + \frac{x}{6R}\right)\left(1{\cdot}2704 + \frac{1}{2}\log_e\frac{R}{x} + \frac{x}{18R}\right) \quad . \quad . \quad (86)$$

electrostatic c.g.s. units in air, where R is the common radius and x the nearest distance between them ($= D - 2R$).

4. Capacitance Between Two Conducting Plates. Consider two equal conducting plates, placed parallel to one another, and at a distance D cm. apart, this distance being small compared with the dimensions of the plates, so that the fringing effect at the edges of the plates can be neglected. Let the area of each plate (one side only) be A sq. cm., and let the charges on the plates be $+ Q$ and $- Q$ electrostatic units.

From Chapter I the intensity at a point between the plates is $\frac{4\pi\sigma}{\kappa}$ where σ is the density of the charge and equals $\frac{Q}{A}$. Then the potential difference between the plates is

$$V = \int_0^D \frac{4\pi Q}{\kappa A}\,.\,dx = \frac{4\pi QD}{\kappa . A}$$

Thus,
$$C = \frac{Q}{V} = \frac{Q}{4\pi QD/\kappa A} = \frac{\kappa A}{4\pi D} \quad . \quad . \quad . \quad . \quad (87)$$

Suppose that instead of there being only one dielectric in between the plates, there are several parallel layers of dielectrics of thicknesses D_1, D_2, D_3, etc., and having permittivities κ_1, κ_2, κ_3, etc., respectively, as in Fig. 65.

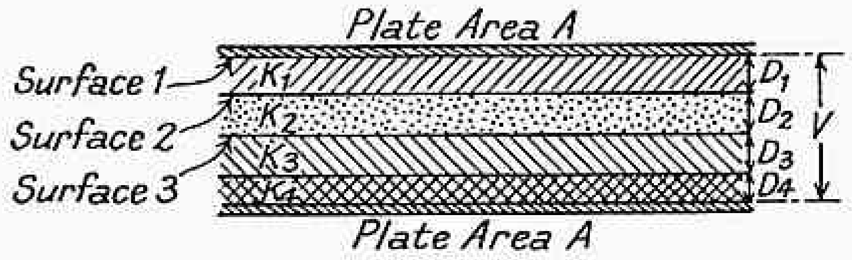


Fig. 65. Dielectrics in Series in a Plate Capacitor

Then potential difference between surfaces (1) and (2) is

$$V_{12} = \int_0^{D_1} \frac{4\pi Q}{\kappa_1 A}\,dx = \frac{4\pi Q}{\kappa_1 A}\,D_1$$

while that between surfaces (2) and (3) is

$$V_{23} = \frac{4\pi Q}{\kappa_2 A}\,.\,D_2$$

and so on. Thus, the total potential difference V between the parallel conducting plates is

$$V = V_{12} + V_{23} + V_{34} + \ldots$$
$$= \frac{4\pi Q}{A}\left(\frac{D_1}{\kappa_1} + \frac{D_2}{\kappa_2} + \frac{D_3}{\kappa_3} + \ldots\right)$$

and the capacitance between the plates is therefore

$$C = \frac{Q}{V} = \frac{A}{4\pi\left(\frac{D_1}{\kappa_1} + \frac{D_2}{\kappa_2} + \frac{D_3}{\kappa_3} + \ldots\right)} \quad . \quad . \quad (88)$$

Effect of Additional Plates. If two more similar plates are added, one of which is connected to each of the existing plates (Fig. 66), and the same dielectric placed between them, then the effective area for the whole capacitor thus formed is $3A$, and the capacitance is thus increased to $\frac{3\kappa A}{4\pi D}$.

In general, since the use of N plates creates $N-1$ spaces (each of width D cm.) the capacitance of such a capacitor with N plates is

$$C = \frac{\kappa \, . \, (N-1) A}{4\pi D} \qquad . \quad . \quad . \quad . \quad . \qquad (89)$$

By this means the capacitance of a plate capacitor can be made large whilst using plates with only a comparatively small surface area.

Although these formulae must be considered as approximations, if the plates are close together they are sufficiently accurate for most

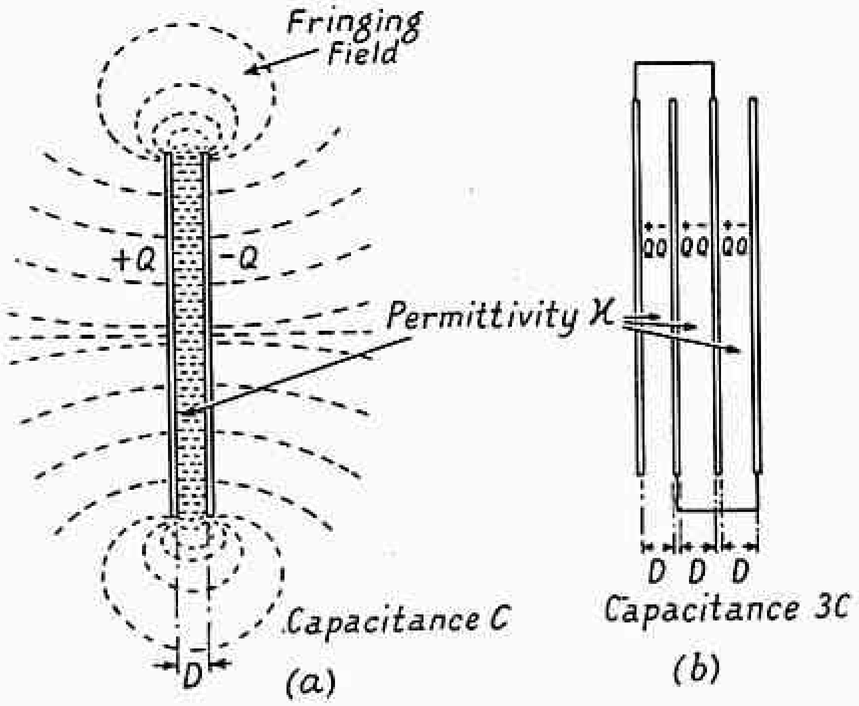


FIG. 66. PLATE CAPACITOR

practical purposes, even though the capacitor may be in the vicinity of other conductors.

5. CAPACITANCE BETWEEN TWO LONG, PARALLEL CONDUCTING CYLINDERS. This problem can be resolved into two separate cases, namely: (*a*) when the cylinders are at a distance apart which is great compared with their diameters; (*b*) when they are comparatively close together.

In the former case it is considerably easier to calculate the capacitance between them than in the latter. This case will be considered first.

Case (*a*).

Fig. 67 represents two long parallel conducting cylinders, perpendicular to the plane of the paper, each of diameter d cm. placed at a distance D cm. apart in air, D being great compared with d and the cylinders being at a great distance from all other conductors.

Let $+Q$ and $-Q$ units be the charges per centimetre axial length on A and

B respectively. In this case it may be assumed that the charges are concentrated at the axes of the cylinders.

From Equation (4), the intensity at P, distant x from cylinder A, due to this cylinder is $\frac{2Q}{x}$ which is the force (in dynes, if Q is in electrostatic units and x in centimetres) upon unit charge placed at P. This force is in the direction AB. Similarly, cylinder B would exert a force (of attraction) upon unit charge at P of $\frac{2Q}{D-x}$ dynes, also in the direction AB. Thus the total force upon unit charge at P is $2Q\left(\frac{1}{x} + \frac{1}{D-x}\right)$ dynes in direction AB. The potential difference

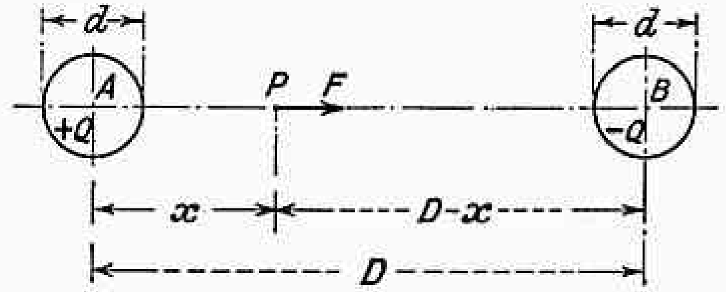


FIG. 67. PARALLEL CYLINDERS

between the cylinders—which is the work done in moving unit charge from the surface of one cylinder to the surface of the other—is

$$\int_{x=\frac{d}{2}}^{x=D-\frac{d}{2}} \left[2Q\left(\frac{1}{x} + \frac{1}{D-x}\right)\right] dx = 2Q\,[\log_e x - \log_e (D-x)]$$

$$= 4Q \log_e \frac{2D-d}{d}$$

i.e. potential difference between the cylinders

$$V = 4Q \log_e \frac{2D-d}{d} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (90)$$

$\therefore$ The capacitance between the cylinders *per centimetre axial length*

$$= \frac{Q}{V} = \frac{1}{4 \log_e \frac{2D-d}{d}}$$

If the permittivity of the medium between the cylinders is κ, then, of course,

$$C = \frac{1{\cdot}21\,\kappa}{10^{13} \log_{10}\left(\frac{2D}{d} - 1\right)} \text{ farads per cm. length}$$

or, the capacitance per mile of two such parallel cylinders in air is

$$\frac{1{\cdot}95}{10^{8} \log_{10}\left(\frac{2D}{d} - 1\right)} \text{ farads} \qquad . \quad . \quad . \quad . \qquad (91)$$

If D is great compared with d,

$$C \doteqdot \frac{1{\cdot}95}{10^8 \log_{10} \frac{2D}{d}} \text{ farads per mile}$$

Case (*b*). When the cylinders are comparatively close together the treatment of the problem differs from that of Case (*a*), owing to the fact that the charges of $+Q$ and $-Q$ cannot now be assumed to be concentrated at the axes of the cylinders. The charges must now be taken as concentrated along other axes, parallel to and in the same plane as the axes of the cylinders, but displaced so that the distance apart of the axes along which the charges are assumed to be concentrated is now less than the distance D. To derive an expression for the capacitance in this case the distribution of the electrostatic field between the cylinders must first be considered.

When the cylinders are at a great distance apart, as in Case (*a*), the lines of force of the electrostatic field radiate from the cylinders uniformly in all directions, each line cutting the surfaces of the cylinders perpendicularly. Since the potential of a point along any one line of force decreases as the distance of the point from cylinder A is increased, a number of equipotential surfaces exist which are in the form of cylinders concentric with the cylindrical conductors, the lines of force cutting all of these cylinders perpendicularly.

If the cylindrical conductors are comparatively close together these equipotential surfaces are still cylinders, but they are not concentric with the surfaces of the cylindrical conductors whose capacitance is to be determined, nor are they concentric with one another.

It can be shown* that the equations of the traces of these cylindrical equipotential surfaces in the plane of the paper are $r_1 = Mr$, where r_1 and r are the distances of any point on one of the circular traces from the traces X and Y of the axes along which the charges $+Q$ and $-Q$ may be assumed to be concentrated and from which the lines of electrostatic force radiate (these lines of force being circles, as in Fig. 68), and M is a constant which differs for different traces. By giving M different values a series of circular traces is obtained, as shown in the figure. When $M = 1$ the trace is a straight line, this being the trace of a plane the potential of all points on which is zero.

Now, since the surfaces of the cylindrical conductors are equipotential surfaces, the equations of whose traces in the plane of the paper are given by the above relationship ($r_1 = Mr$), it follows that the traces X and Y are not coincident with the axes of the conducting cylinders, but are displaced as shown in Fig. 68.

Calculation of Capacitance. To calculate the positions of the axes whose traces are X and Y, proceed as below.

* See T. F. Wall's *Electrical Engineering*, p. 46.

Let the points X and Y be displaced inwards from the centres of the two circles which are the traces of the cylindrical conductors A and B by a distance m in each case, and let their distance apart be l. Then $l = D - 2m$.

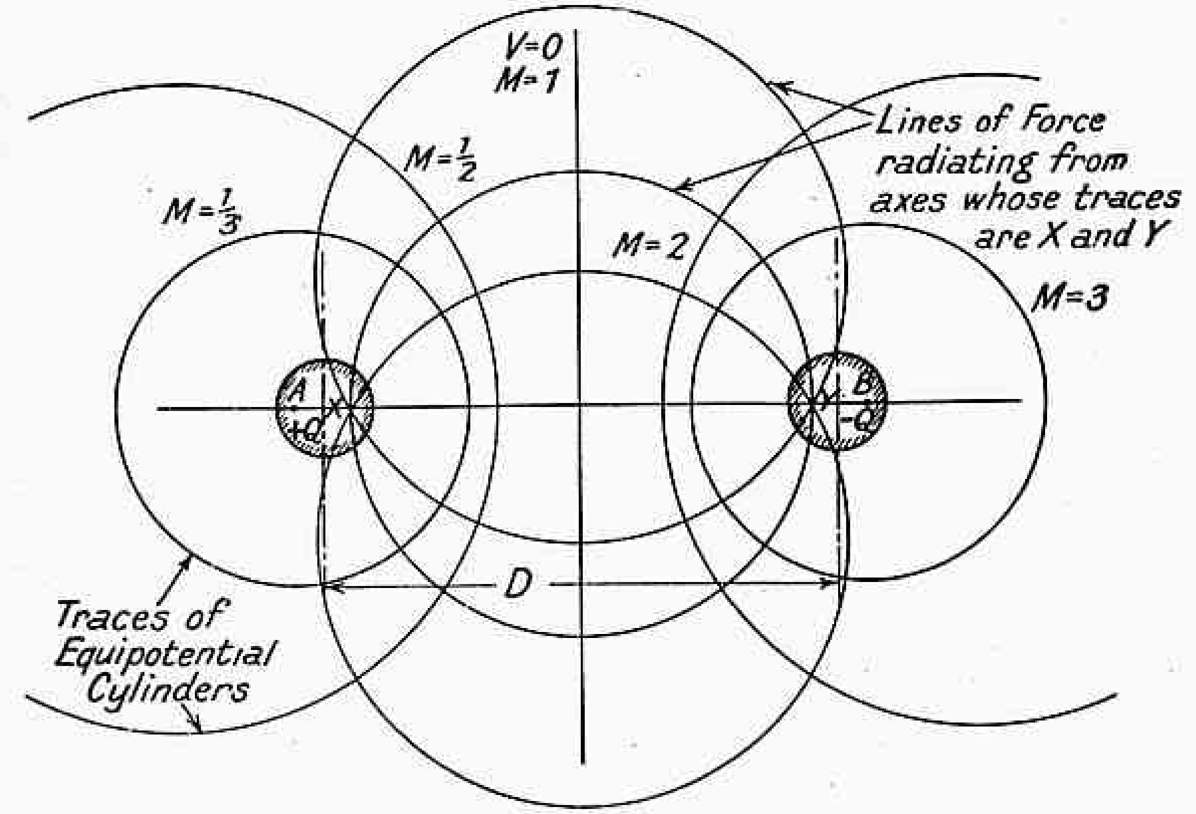


FIG. 68. ELECTROSTATIC FIELD BETWEEN CHARGED PARALLEL CYLINDERS WHICH ARE NEAR TOGETHER

Since the surfaces of the cylindrical conductors are equipotential surfaces, the equation $r_1 = Mr$ holds for their traces. Consider the point P (Fig. 69) on the trace of cylinder A on a line through X perpendicular to the line XY.

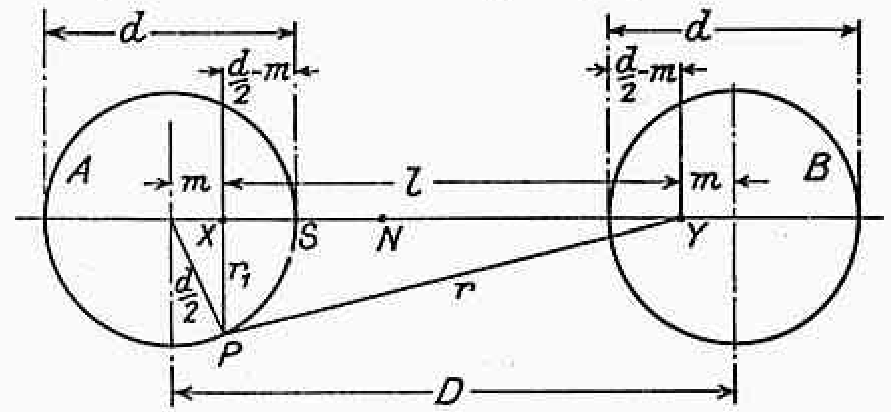


FIG. 69

Then $\left(\frac{d}{2}\right)^2 = m^2 + XP^2$, and $XP = M(PY)$, since for point P, $r_1 = XP$ and $r = PY$.

Also $l^2 + XP^2 = PY^2$

and $$m = \frac{D - l}{2}$$

For the point S,

$$r_1 = XS = \frac{d}{2} - m \text{ and } r = SY = l - \left(\frac{d}{2} - m\right)$$

Since for all points on the circular trace of A

$$r_1 = Mr$$

we have $\frac{d}{2} - m = M\left[l - \left(\frac{d}{2} - m\right)\right]$ for point S

$$\therefore M = \frac{\frac{d}{2} - m}{l - \left(\frac{d}{2} - m\right)}$$

Now $l^2 + XP^2 = PY^2 = \left(\frac{XP}{M}\right)^2 = \frac{XP^2}{\left[\frac{\frac{d}{2} - m}{l - \left(\frac{d}{2} - m\right)}\right]^2}$

and $\left(\frac{d}{2}\right)^2 = m^2 + XP^2$

$$\therefore l^2 + \left(\frac{d}{2}\right)^2 - m^2 = \frac{\left(\frac{d}{2}\right)^2 - m^2}{\left[\frac{\frac{d}{2} - m}{l - \frac{d}{2} + m}\right]^2}$$

Substituting $m = \frac{D - l}{2}$ and solving for l we have the solution

$$l = \sqrt{D^2 - d^2}$$

If d is small compared with D, we have $l = D$, as in Case (a).

Thus, to calculate the capacitance between the cylinders, the treatment is exactly the same as that of Case (a), except that the charges $+Q$ and $-Q$ per centimetre axial length are considered concentrated along parallel axes whose distance apart is now l instead of D.

We have then for the intensity of field at a point such as N (Fig. 69) distant x from X

$$E = \frac{2Q}{x} + \frac{2Q}{l - x}$$

and the potential difference between the cylinders

$$V = \int_{x = \frac{d}{2} - m}^{x = l - \left(\frac{d}{2} - m\right)} \left(\frac{2Q}{x} + \frac{2Q}{l - x}\right) dx$$

$$= 4Q \log_e \frac{l - \left(\frac{d}{2} - m\right)}{\frac{d}{2} - m}$$

or, since $l = \sqrt{D^2 - d^2}$ and $2m = D - l$

$$V = 4Q \log_e \left[\frac{\sqrt{D^2 - d^2} - (d - D)}{\sqrt{D^2 - d^2} + (d - D)}\right] \quad . \quad . \quad . \quad (92)$$

Thus capacitance per centimetre axial length is

$$C = \frac{Q}{V} = \frac{1}{4 \log_e \left[\frac{\sqrt{D^2 - d^2} - (d - D)}{\sqrt{D^2 - d^2} + (d - D)}\right]}$$

Rationalizing and simplifying, we have

$$C = \frac{1}{4 \log_e \left[\frac{D + \sqrt{D^2 - d^2}}{d}\right]}$$

If the permittivity of the medium between the cylinders is κ, we have

$$C = \frac{\kappa}{4 \log_e \frac{(D + \sqrt{D^2 - d^2}}{d}} \quad . \quad . \quad . \quad . \quad (93)$$

or $C = \dfrac{1 \cdot 95}{10^8 \log_{10}\left(\dfrac{D + \sqrt{D^2 - d^2}}{d}\right)}$ farads per mile of double conductor in air

These capacitances are given in farads *per mile*, since the arrangement of two long parallel conducting cylinders is chiefly met with in overhead transmission lines where the most useful unit of length is the mile. Formulae for the general case of two parallel cylindrical conductors have been given by Russell (Ref. (13)).

6. Capacitance between Two Coaxial Cylinders. An important case of this arrangement in practice is, of course, a concentric cable.

Consider two long conducting concentric cylinders, the diameter of the inner one being d cm. and the inner diameter of the outer one being D cm. Let $+Q$ and $-Q$ units be their charges per centimetre axial length. The lines of force of the electrostatic field will be radial, and the equipotential surfaces will be cylindrical and coaxial with the two conducting cylinders. The intensity of the field at some point at a radial distance of x cm. from the common axis of the cylinders will be $\frac{2Q}{x}$ if the dielectric separating the cylinders is air.

Thus the potential difference between the cylinders is

$$\int_{\frac{d}{2}}^{\frac{D}{2}} \frac{2Q}{x} dx = 2Q\left[\log_e \frac{D}{2} - \log_e \frac{d}{2}\right]$$

$$V = 2Q \log_e \frac{D}{d}$$

The capacitance per centimetre length is

$$C = \frac{Q}{V} = \frac{1}{2 \log_e \frac{D}{d}}$$

The general expression for a length l cm., the dielectric having a permittivity κ is

$$C = \frac{\kappa l}{2 \log_e \frac{D}{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

or $$C = \frac{3{\cdot}89\kappa}{10^8 \log_{10} \frac{D}{d}} \text{ farads per mile}$$

7. Capacitance of a Single Straight Conductor Parallel to Earth. *Method of Electric Images.* This method is based upon the imagination of an "image" of a conductor placed above the earth's surface, this image being of the same size and shape as the conductor considered and lying as far beneath the surface of the earth as the conductor considered is above the surface. The earth's surface is thus in the plane of zero potential for these two conductors—considering the image as being in actual fact a conductor placed at a distance $2H$ from the original one, H being the height of this original conductor above the earth.

Since the earth's surface is at zero potential, the electrostatic field from the charged conductor above the earth, to the surface of the earth, has the same distribution as the field which would exist between the conductor and the zero potential plane, in the case of two conductors placed at a distance of $2H$ apart.

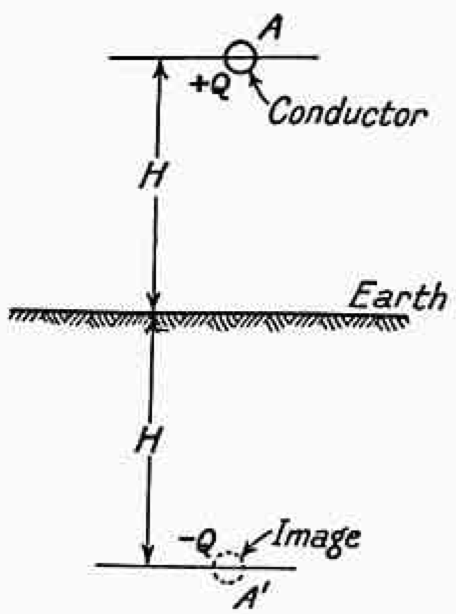


Fig. 70. Cylindrical Conductor Parallel to Earth

Fig. 70 shows the trace of a cylindrical conductor A lying parallel to the earth's surface, and at a height H cm. above the earth; A' is its image. If conductor A has a charge of $+ Q$ units per cm. axial length, then the potential difference between it and conductor A', which is supposed to have $- Q$ units per cm. axial length, is from Equation (90)

$$2V = 4Q \log_e \frac{4H - d}{d}$$

where d is the diameter of the conductors and is assumed small compared with H. V is the potential of A above that of the earth, and is also the potential of A' below earth potential.

Thus $$V = 2Q \log_e \frac{4H - d}{d}$$

and the capacitance per centimetre length of one conductor to earth is

$$C = \frac{\kappa}{2 \log_e \frac{4H - d}{d}} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

where the dielectric has permittivity κ.

The capacitance per mile of one conductor to earth in air is therefore

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \frac{4H - d}{d}} \text{ farads per mile}$$

If d is small compared with H (as is usually the case when an overhead line is considered) then

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \frac{4H}{d}} = \frac{3{\cdot}89}{10^8 \log_{10} \frac{2H}{r}} \text{ farads per mile}$$

where r is the radius of the conductor in centimetres.

If d is not small compared with H, the calculation of capacitance must be based upon Equation (92) instead of Equation (90) as above.

8. Capacitance Between Two Long, Straight Conductors, Parallel to the Earth and to One Another. Consider two long cylindrical conductors M and N parallel to earth and to one another, their diameters being d cm. and their distance apart being D cm. Let H be their height above earth and let M' and N' be their images (Fig. 71). Suppose d small compared with H.

Let M and N have charges of $+ Q$ and $- Q$ per centimetre axial length respectively and M' and N' charges of $- Q$ and $+ Q$ units per cm. length respectively.

Consider a point P on the horizontal line joining the centres of M and N and distance x cm. from M. The intensity at P is due to all four conductors M, N, M', and N'. Thus intensity at P *in the direction MN* is—

Due to M . . $\left(\frac{2Q}{x}\right)$

Due to N . . $\left(\frac{2Q}{D - x}\right)$

Due to M'. . $$\left(-\frac{2Q}{PM'} \cos a\right) = \frac{-2Q}{\sqrt{4H^2 + x^2}} \cdot \frac{x}{\sqrt{4H^2 + x^2}} = \frac{-2Qx}{4H^2 + x^2}$$

Due to N' . . $\left(\dfrac{-2Q}{\sqrt{(D-x)^2+4H^2}}\cos\beta\right)$

$$= \frac{-2Q(D-x)}{(D-x)^2+4H^2}$$

Resultant intensity at P is

$$\frac{2Q}{x}+\frac{2Q}{D-x}-\frac{2Qx}{4H^2+x^2}-\frac{2Q(D-x)}{(D-x)^2+4H^2}$$

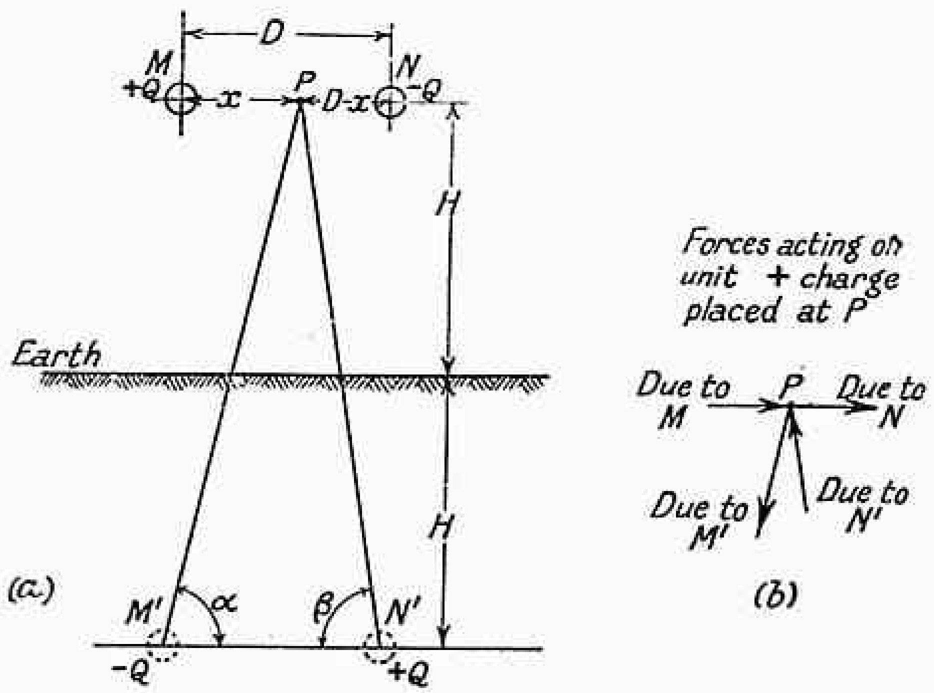


FIG. 71. TWO CHARGED PARALLEL CONDUCTORS NEAR TO EARTH

and the potential difference between M and N is

$$V=\int_r^{D-r}\left(\frac{2Q}{x}+\frac{2Q}{D-x}-\frac{2Qx}{4H^2+x^2}-\frac{2Q(D-x)}{(D-x)^2+4H^2}\right)dx$$

where r is the radius of the conductors.

Integrating, we have

$$V=2Q\left[2\log_e\frac{D-r}{r}+\log_e\frac{4H^2+r^2}{4H^2+(D-r)^2}\right] \quad . \quad . \quad (96)$$

If D is great compared with r,

$$V=4Q\log_e\frac{D}{r}+2Q\log_e\frac{4H^2}{4H^2+D^2}$$

The capacitance between the conductors is

$$C=\frac{Q}{V}=\frac{1}{4\log_e\dfrac{D}{r}+2\log_e\dfrac{4H^2}{4H^2+D^2}} \quad . \quad . \quad . \quad (97)$$

$$=\frac{1}{4\log_e\dfrac{D}{r}\left(\dfrac{2H}{\sqrt{4H^2+D^2}}\right)} \text{ per cm. length in air}$$

or $$C=\frac{1\cdot95}{10^8\log_{10}\dfrac{D}{r}\left(\dfrac{2H}{\sqrt{4H^2+D^2}}\right)} \text{ farads per mile}$$

The capacitance of two parallel cylinders which are at a great distance from earth was previously found to be

$$\frac{1\cdot95}{10^8\log_{10}\dfrac{2D}{d}}=\frac{1\cdot95}{10^8\log_{10}\dfrac{D}{r}} \text{ farads per mile}$$

D being great compared with r.

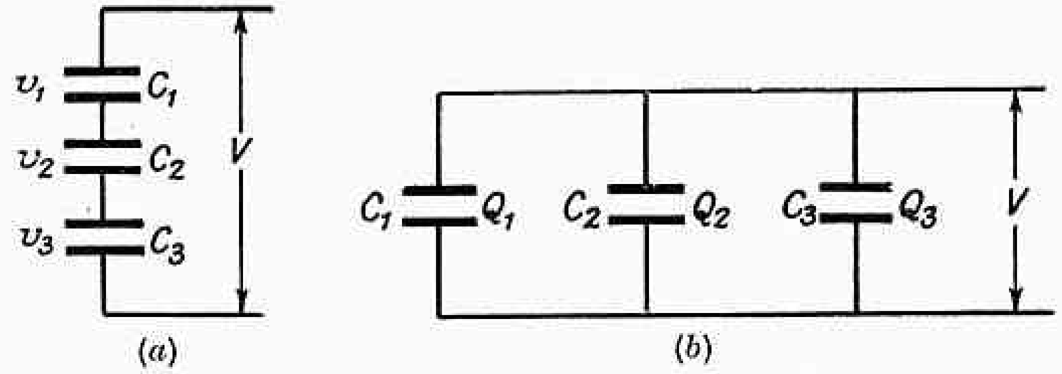


FIG. 72. CAPACITORS IN SERIES AND IN PARALLEL

Thus the proximity of the earth introduces the term $\dfrac{2H}{\sqrt{4H^2+D^2}}$ in the denominator, as shown above.

The capacitance of a system of three or more conductors, parallel and near to the earth, can be found by similar methods (Refs. (1), (5), (8)).

Capacitors in Series and Parallel. (*a*) SERIES. If a number of capacitors are connected in series, as in Fig. 72 (*a*), a potential difference of V being applied between the outer terminals, there will be potential differences v_1, v_2, v_3, etc., between the different pairs of plates.

Let the capacitances of the capacitors (neglecting earth capacitances) be C_1, C_2, C_3, etc. If a quantity of electricity Q units is given to the system of capacitors by means of a current which flows for a short time through them until they are charged to the total potential difference V, then

$$v_1=\frac{Q}{C_1},\ v_2=\frac{Q}{C_2}\ v_3=\frac{Q}{C_3}$$

and so on.

If C is the capacitance of the whole system, the potential difference for which is V, then

$$C = \frac{Q}{V} \text{ or } V = \frac{Q}{C}$$

Thus, since $\quad V = v_1 + v_2 + v_3$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \ldots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \qquad (98)$$

(*b*) Parallel. If a potential difference V is applied to a number of capacitors connected in parallel (Fig. 72 (*b*)), then the potential difference across the plates of such capacitors is, in each case, V, but the quantities of electricity given to the capacitors are now different for the different capacitors. If these quantities are Q_1, Q_2, Q_3, etc., then

$$v_1 = \frac{Q_1}{C_1} \text{ or } Q_1 = v_1 C_1$$

$$v_2 = \frac{Q_2}{C_2} \text{ or } Q_2 = v_2 C_2$$

and so on.

But $\quad v_1 = v_2 = v_3 = \ldots = V$

and the total quantity of electricity

$$Q = Q_1 + Q_2 + Q_3 + \ldots = CV$$

where C is the total capacitance.

Thus

$$CV = C_1 v_1 + C_2 v_2 + C_3 v_3 + \ldots$$
$$= V(C_1 + C_2 + C_3 + \ldots)$$
$$\therefore C = C_1 + C_2 + C_3 + \ldots \qquad (99)$$

Two-core Cable. In the case of multi-core cables generally, the earth capacitances of the cores cannot be neglected. A two-core cable consists essentially of two long parallel conductors embedded in some insulating material, the whole being enclosed by an earthed, conducting cylinder, as in Fig. 73(*a*).

This arrangement is equivalent to the system of capacitors shown in Fig. 73 (*b*). If the cores are represented by A and B, then C_{AB} is the capacitance between cores and C_A and C_B the earth capacitances of the two conductors. We thus have C_A and C_B in series with one another this series circuit being in parallel with C_{AB}, the equivalent arrangement being represented in Fig. 73 (*c*). The capacitance of C_A and C_B in series is $\dfrac{C_A\,C_B}{C_A + C_B}$, and when this is connected in parallel with C_{AB} the total, or working, capacitance is $C_{AB} + \dfrac{C_A\,C_B}{C_A + C_B}$.

Three-core Cable. The capacitances which exist in the case of a three-core cable are shown in Fig. 74 (*a*), in which C_1 is the intercore capacitance, and C_0 the earth capacitance. Diagram (*b*) shows the

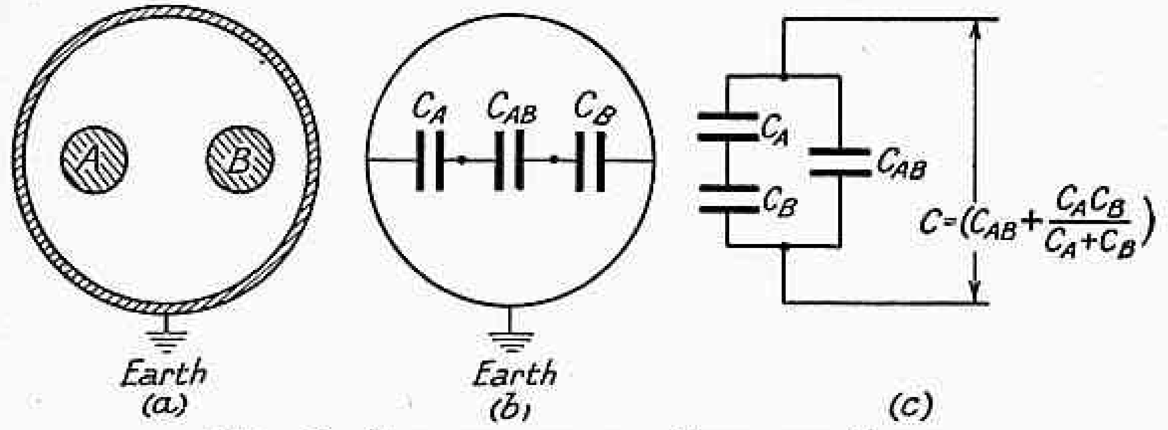


Fig. 73. Capacitance of a Two-core Cable

equivalent circuit of such a cable when used on a three-phase system of line voltage E.

To facilitate calculations of the charging current per line it is

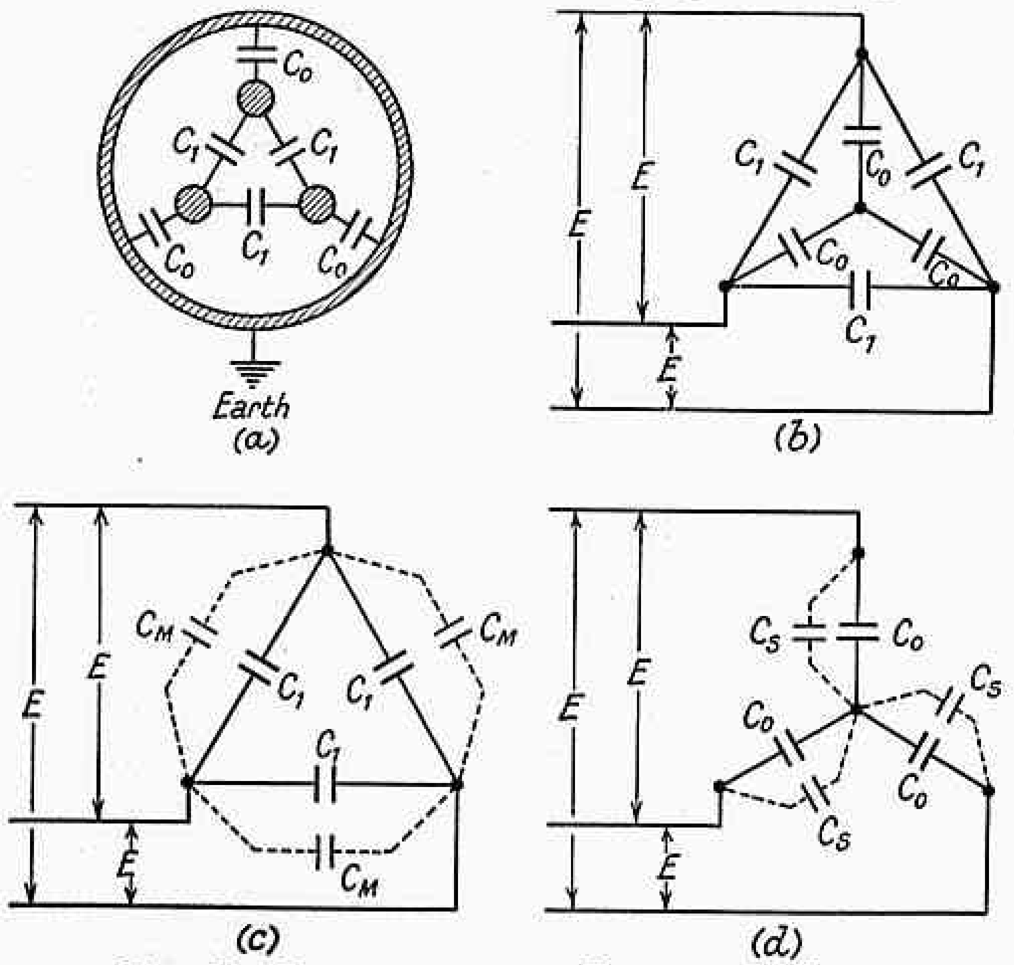


Fig. 74. Capacitance of a Three-core Cable

usual to resolve the system shown in diagram (*b*) into either an *equivalent mesh* system, as in diagram (*c*), or an *equivalent star* system as in diagram (*d*). In the first case, the three capacitances C_0 are replaced by three imaginary capacitances C_m, connected in mesh, in parallel with the inter-core capacitances C_1, and having such values that the charging current per line is the same as that for the actual cable. The magnitude of C_m is thus determined as follows—

The voltage to neutral (i.e. the voltage across each capacitor C_0) is $E/\sqrt{3}$ and the charging current taken by each C_0 is $(E/\sqrt{3}) \cdot \omega\, C_0$.

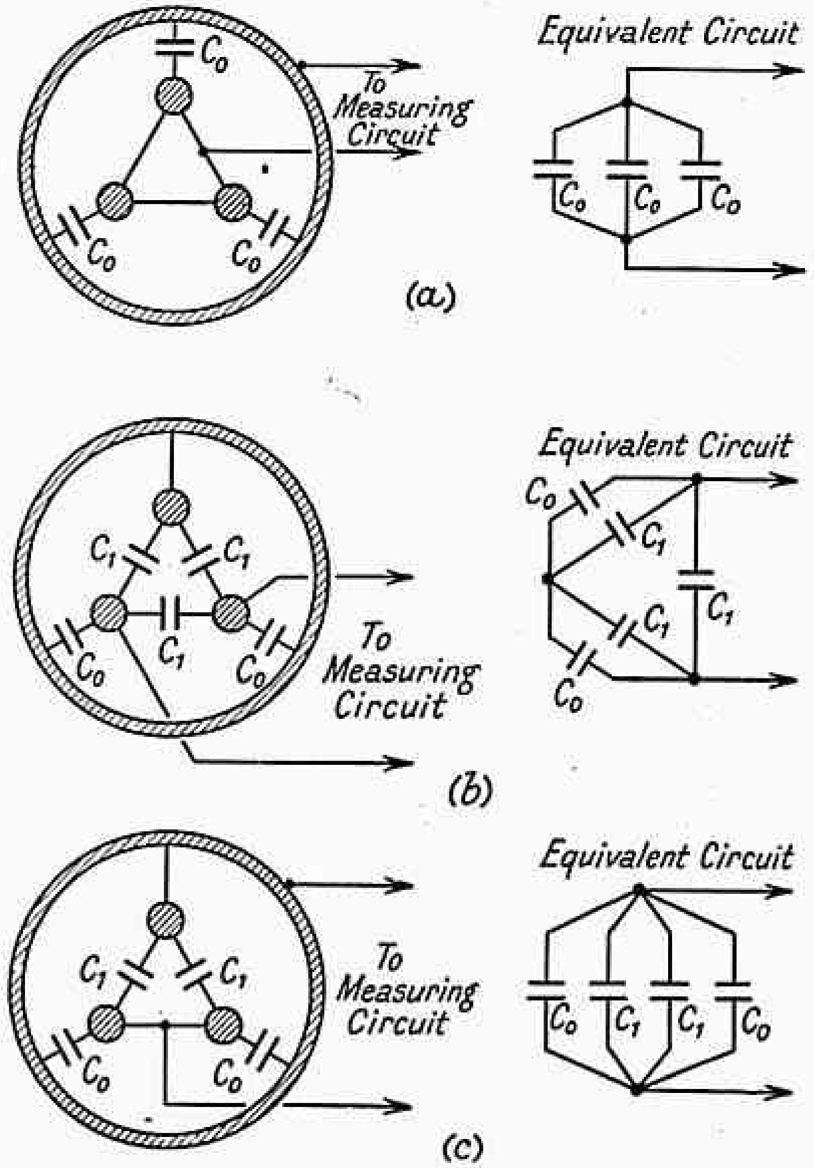


FIG. 75. CABLE CAPICITANCE MEASUREMENTS

In diagram (c) the current taken by each capacitor C_m is $E \cdot \omega\, C_m$ and the *line* current on this account is thus $\sqrt{3}$. $E \cdot \omega\, C_m$.

For equivalence this line current must be equal to $(E/\sqrt{3}) \cdot \omega\, C_0$.

Thus,
$$\sqrt{3} \cdot E \cdot \omega C_m = \frac{E}{\sqrt{3}} \cdot \omega C_0$$

or
$$C_m = \frac{C_0}{3}$$

Hence the total equivalent mesh system consists of three groups of C_1 each in parallel with C_m, i.e. three capacitances $C_1 + \frac{C_0}{3}$ connected in mesh.

Diagram (*d*) shows the equivalent star system, in which the capacitors C_1 are replaced by three capacitors C_3, each in parallel with C_0 and of such values that the line currents are the same as for the actual cable.

To determine the value of C_s—

Current taken by each capacitance $C_s = \frac{E}{\sqrt{3}} \cdot \omega C_s$

Now, current taken by each capacitance C_1 (diagram (*b*)) is $E\omega C_1$, and the line current on this account $= \sqrt{3} \cdot E\omega C_1$.

For equivalence

$$\frac{E}{\sqrt{3}} \cdot \omega C_s = \sqrt{3}\, E\omega C_1 \text{ or } C_s = 3C_1$$

So that the total equivalent star system consists of three groups of capacitors in star, each consisting of C_0 and C_3 in parallel, i.e. three capacitances of $C_0 + 3C_1$.

Measurements of Three-core Cable Capacitances. The values of the capacitances C_0 and C_1 for a given length of cable may be determined by means of two tests. First, the three cores are connected together and the capacitance between them and the sheath measured (see Fig. 75 (*a*)). The measured capacitance is obviously $3C_0$.

The second test may be of the capacitance between two cores, the third being connected to the sheath (Fig. 75 (*b*)) or between two cores connected together, and the sheath and third core connected together (Fig. 75 (*c*)).

In the former case the capacitance obtained by the measurement is

$$\frac{C_0 + C_1}{2} + C_1 = \frac{3}{2}\, C_1 + \frac{C_0}{2}$$

In the latter case the measured value is $2C_0 + 2C_1$.

The first test obviously enables C_0 to be determined and this value, substituted in either of the expressions obtained above for the two alternative methods of carrying out the second test, renders C_1 calculable.

Distributed Capacitance. In the foregoing paragraphs it has been assumed in all cases that the surfaces of the conductors considered have been assumed to be equipotential surfaces.

There are many important cases in practice when this is not so, and in these cases the calculation of capacitance cannot be carried out by the simple methods used above. In wire-wound solenoids we have capacitance between adjacent turns, and layers, and all the conductors in one layer are obviously not at the same potential. The earth capacitances of the turns in the coil also are not all the same. In such coils we have what is referred to as "distributed capacitance."

The effect of such distributed capacitance is, in many cases, small for low-frequency work, and an equivalent circuit, which represents such a coil sufficiently accurately for most purposes, can then be obtained by assuming the coil itself to be free from capacitance but as having a simple capacitor connected in parallel with it, and also having simple capacitors connected between parts of the coil and earth. The latter represent the distributed earth capacitance, while the former represents the distributed inter-turn capacitance.

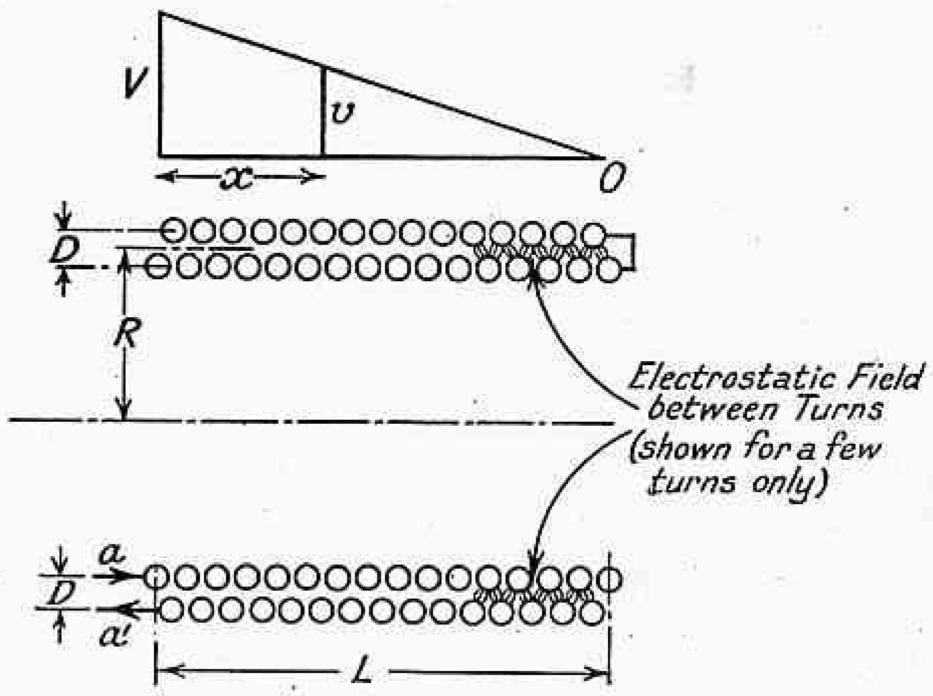


FIG. 76. CAPACITANCE OF A TWO-LAYER SOLENOID

If such a coil is to be used for very high frequency work, e.g. radio frequency work, such approximate methods of representation are not justifiable, since the distributed capacitance of the coil may, at such frequencies, become of more importance than its inductance.

Capacitance of a Two-layer Solenoid. Fig. 76 represents a solenoid of circular section, having two layers of insulated wire wound continuously so that, in effect, the layers are connected together at one end as shown. If a steady potential difference V is applied to the terminals (aa') of the coil, then the potential difference between layers will vary from V at the left-hand end of the coil to zero at the right-hand end, and the electrostatic field between adjacent turns will thus decrease from a maximum to zero, moving from left to right. Morecroft (*Principles of Radio Communication*, Chap. II) calculates the internal capacitance of such a coil by treating it as, essentially, two coaxial conducting cylinders, whose capacitance, if the layers of wire are close together compared with the diameter of the coil, is given by the formula for flat plates, assuming at first that the cylinders are equipotential surfaces.

TABLE VI

Capacitance of Various Systems of Conductors

(κ = permittivity of the medium. Dimensions in cm. Capacitance in electrostatic c.g.s. units)

No.	System of Conductors	Capacitance	Explanation of Symbols	Assumptions, etc.	Source
1	Isolated sphere . . .	κR	R = radius of sphere		
2	Concentric spheres . .	$\frac{\kappa R_1 R_2}{R_2 - R_1}$	R_1 and R_2 = radii of spheres	Spheres isolated from other conductors	
3	Two spheres distant from one another . . .	$\frac{\kappa R_1 R_2 D}{D(R_1 + R_2) - 2R_1 R_2}$	R_1 and R_2 = radii of spheres D = distance between centres	Spheres isolated from other conductors	
4	Two spheres at same potential	$\left(R_1 + R_2 - \frac{2R_1 R_2}{D}\right)\frac{D^2 \kappa}{D^2 - R_1 R_2}$	As in (3)	Approx. formula (see Ref.)	Russell (Ref. (12))
5	Two equal spheres close together	$\frac{R\kappa}{2}\left(1 + \frac{x}{6R}\right)\left(1{\cdot}2704 + \frac{1}{2}\log_e \frac{R}{x} + \frac{1}{18R}\right)$	R = radius of spheres x = shortest distance between their surfaces	Approx. formula (see Ref.)	Russell (Ref. (12))
6	Two parallel conducting plates	$\frac{\kappa \cdot A}{4\pi D}$	A = area of one plate D = distance between plates	D small. Edge effect negligible	
7	N parallel plates . . .	$\frac{\kappa(N-1)A}{4\pi D}$	As in (6)	As in (6)	
8	Two parallel plates with several dielectrics between them	$\frac{\kappa A}{4\pi\left(\frac{D_1}{\kappa_1} + \frac{D_2}{\kappa_2} + \frac{D_3}{\kappa_3} + \ldots\right)}$	A = area of one plate D_1, D_2, D_3 = thicknesses of dielectrics κ_1, etc. = permittivity of dielectrics	As in (6)	
9	Two similar long parallel conducting cylinders .	$\frac{\kappa l}{4\log_e \frac{2D - d}{d}}$	l = axial length of cylinders d = diam. of cylinders D = distance between centres	d small compared with D Cylinders isolated	
10	Two similar long parallel conducting cylinders .	$\frac{\kappa l}{4\log_e\left[\frac{D + \sqrt{D^2 - d^2}}{d}\right]}$	As in (9)	d *not* small compared with D	
11	Two coaxial conducting cylinders	$\frac{\kappa l}{2\log_e \frac{D}{d}}$	l = axial length d and D = diameters of cylinders		
12	Long straight conductor parallel to earth .	$\frac{\kappa l}{2\log_e \frac{4H - d}{d}}$	l = axial length d = diam. of conductor H = height above earth		
13	Two long similar parallel conductors parallel to earth	$\frac{\kappa l}{4\log_e \frac{D}{r} + 2\log_e \frac{4H^2}{4H^2 + D^2}}$	l = axial length r = radius of conductors D = distance apart H = height above ground		
14	Thin circular disc . .	$\frac{2\kappa r}{\pi}$	r = radius of disc	Thickness small compared with r. Disc isolated	
15	Two similar coaxial parallel discs	$\kappa\left[\frac{r^2}{4D} + \frac{r}{4\pi}\left(\log_e \frac{16\pi(D + t)r}{D^2} + \frac{t}{D}\log_e \frac{D + t}{t} + 1\right)\right]$	t = thickness of discs r = radius of discs D = distance apart	t and D small compared with r	Kirchhoff (Ref. (32))
16	Single vertical wire .	$\frac{\kappa l}{2\log_e \frac{l}{r}}$	l = length of wire r = radius of wire	Distant from earth and other conductors	
17	Distributed capacitance of two-layer solenoid .	$\frac{\kappa Rl}{6D}$	R = radius of section of solenoid D = distance between layers l = axial length of solenoid	D small compared with R	Morecroft (Ref. (4))
18	Distributed capacitance of solenoid of N layers .	$C_0 \times \frac{4}{3}\left(\frac{N-1}{N}\right)^2$	C_0 = capacitance between innermost and outermost layers		Morecroft (Ref. (4))
19	Distributed capacitance of short single-layer solenoid	$0{\cdot}07l\kappa$	l = length of one turn of wire on solenoid	Approx. formula	Breit (Ref. (33))

Thus, $C = \frac{\kappa A}{4\pi D}$ where C is the capacitance when the potential difference is the same throughout the axial length of the cylinders, A being the area of each cylindrical surface, κ the permittivity of the medium, and D the distance between the layers.

If R is the radius of the section of the solenoid (assumed the same for both layers, since their distance apart is small) and L is their axial length, then $A = 2\pi RL$ and $C = \frac{\kappa RL}{2D}$ or capacitance per centimetre axial length is $\frac{\kappa R}{2D}$.

Actually the potential difference between layers varies along the axial length from V to zero. Assuming this variation to be according to a straight line law, we have

Energy stored in axial length dx

$$dW = \frac{c \cdot v^2}{2} = \frac{\kappa R}{2D} \cdot \frac{v^2}{2} \cdot dx$$

where v is the potential difference between layers at any point of axial distance x from the left-hand end (Fig. 76). Since $\frac{V}{L} = \frac{v}{L-x}$ we have $v = \left(1 - \frac{x}{L}\right) V$ and

$$dW = \frac{\kappa \cdot R}{2D} \frac{V^2}{2} \left(1 - \frac{x}{L}\right)^2 dx$$

$\therefore$ Total energy stored is

$$W = \int_0^L \frac{\kappa R V^2}{4D} \left(1 - \frac{x}{L}\right)^2 dx$$

$$W = \frac{\kappa R V^2}{4D} \left[-\frac{L}{3} \left(1 - \frac{x}{L}\right)^3 \right]_0^L = \frac{\kappa R V^2 L}{12D}$$

Thus, if C' is the distributed capacitance (in e.s.u.) to be calculated

$$W = C' \frac{V^2}{2}$$

$$\therefore C' \frac{V^2}{2} = \frac{\kappa R V^2 L}{12D}$$

or

$$C' = \frac{\kappa RL}{6D} \qquad (100)$$

Morecroft (*loc. cit.*) gives the distributed capacitance for a solenoid of N layers as

$$C' = C_0 \times \frac{4}{3} \left(\frac{N-1}{N}\right)^2 \qquad (101)$$

where C_0 is the capacitance between the outermost and innermost layers.

Breit (*Physical Review*, XVIII, p. 133 (1921)) gives the capacitance for a short single-layer solenoid in air as approximately $0{\cdot}07l$ e.s.u., where l is the length of one turn of wire on the solenoid.

Shielding and Guard Rings. In making measurements involving the use of capacitors it is often desirable—and in some cases absolutely necessary—to shield pieces of apparatus from the effect of electrostatic fields which are external to the apparatus itself. This is done by surrounding the apparatus by an earthed metal screen which may be of thin aluminium or copper sheet, or in the form of a wire mesh. Charges which may be induced in this screen pass to earth and have no effect upon the apparatus inside.

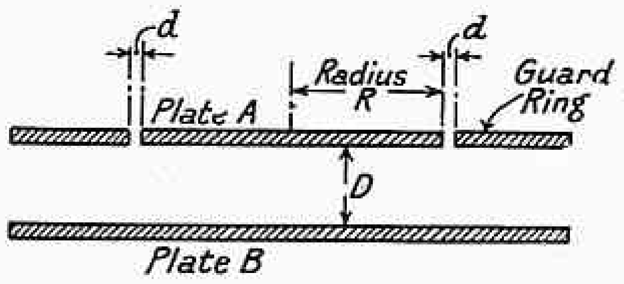


FIG. 77. GUARD RING

Guard rings are used in order to overcome the difficulty of calculating accurately the capacitance of a capacitor which has a fringing electrostatic field at its edges. The distribution of such fringing fields is somewhat uncertain and this renders exact calculations of capacitance difficult.

In calculating the capacitance of a parallel plate capacitor in a previous paragraph it was assumed that the effect of the field at the edge of the plate could be neglected. The simple formula obtained is rendered much more accurate by the use of a guard ring as shown in Fig. 77. The guard ring consists of a metal plate of the same thickness as the plate A which it surrounds, and from which it is separated by a narrow and uniform air gap. This ring is usually of the same outside dimensions as the opposing plate B of the capacitor, and is, in use, at the same potential as the plate A which it surrounds. Under these conditions the electrostatic field between the plates is perpendicular to the plates even up to the extreme edge of plate A, the fringing field being now transferred to the edges of the guard ring. The effective area of the plates to be used in the capacitance formula is now taken, of course, as the area of plate A.

A formula which corrects for the width of the air gap between plate A and the guard ring (which gap should be of zero length if no correction is to be used) has been given by Maxwell and is

$$C \doteqdot \frac{R^2}{4D} + \frac{1}{4}\cdot\frac{Rd}{D+0{\cdot}22d}\left(1+\frac{d}{2R}\right) \text{ e.s. units} \quad . \quad . \quad (102)$$

where the plate A (assumed circular) has a radius R cm., D being the distance between the plates in centimetres, and d being the width of the air gap, the dielectric being air.

When no guard ring is used, the edge effect can be taken into account in the calculation of capacitance by a formula due to Kirchhoff. This formula is

$$C = \frac{R^2}{4D} + \frac{R}{4\pi D}\left[D\left\{\log_e \frac{16\pi R(D+t)}{D^2} - 1\right\} + t\log_e\left(1+\frac{D}{t}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where R is the radius of the circular plates of the capacitor, t being the thickness of the plates and D the distance between them, the dielectric being air.

In cylindrical capacitors the guard ring takes the form of two cylinders, of the same diameter as the cylindrical electrode to which they are adjacent, and placed one at each end of, and coaxial with, this electrode. They are connected together and are, in use, charged to the same potential as the electrode between them. Their use was described in Chapter II in connection with high voltage air capacitors.

Dielectrics. The broadest definition of a dielectric is, simply, "an insulator." More precisely, a dielectric is some medium in which a constant electrostatic field can be maintained without involving the supply of any appreciable amount of energy from outside sources. The term "dielectric" is applied when an insulating material is used to separate two neighbouring conductors such as the plates of a capacitor. As will be seen later, dielectrics increase the capacitance of a system of conductors as compared with the capacitance of the same system of conductors existing in vacuo. No dielectrics are at present known which, when placed between two conductors, decrease the capacitance between them.

Three very important quantities in connection with any dielectric are—

(*a*) Its "dielectric strength."
(*b*) Its "permittivity" or "dielectric constant."
(*c*) Its "dielectric loss angle" or power factor.

(*a*) DIELECTRIC STRENGTH. This may be defined as the ability of a dielectric to withstand breakdown when a voltage is applied to it. All insulating materials should, of course, have a very high resistivity, so that only an extremely small current flows through them when a voltage is applied. This is, however, an entirely different property from dielectric strength. If a gradually increasing voltage is applied between, say, the opposite faces of a slab of an insulating material, the material becomes electrically strained, the electrostatic field in it increasing in intensity with increasing voltage. Eventually a value of the field intensity is reached at which the material "breaks down," i.e. the material is punctured and is rendered useless for insulation purposes. This effect is observed in the case of all insulating materials, although the magnitude of the field intensity, or "potential gradient," for which it occurs differs for different materials. In liquid or gaseous dielectrics the breakdown is only temporary.

The dielectric strength is expressed in volts per millimetre or per centimetre, or in kilovolts per centimetre, etc.

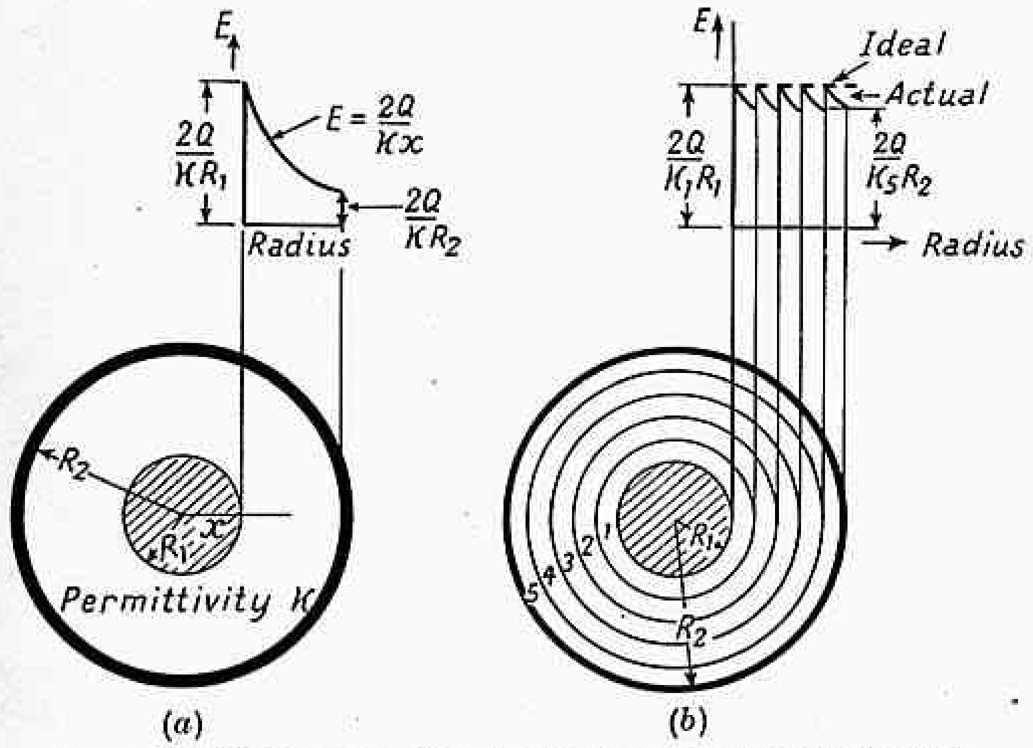
The true or intrinsic dielectric strength of solid materials can be measured only if all discharges in the ambient medium are eliminated and if the heating effect of the applied field is negligible. Such intrinsic strengths are difficult to measure, but have been obtained for a few good dielectrics and lie in the region of 5×10^6 V/cm. When the dielectric strength is measured in the conventional manner between disc or sphere electrodes the breakdown is due to intense local concentration of stress at the end of ionic discharges outside the material, and values from 5 to 50 times lower than the intrinsic value are obtained. It is these lower values which are quoted in Table VII. The dielectric strength so measured is dependent on the geometry of the electrodes, upon the nature of the ambient medium (air or oil) and upon the thickness of the specimen, but no exact laws can be quoted. If the time of a test is prolonged to days or weeks in order to represent the useful life of the material, still lower values of breakdown strength are obtained which depend either on the erosion of microscopic holes through the material by ionic bombardment or on electro-chemical changes in the structure of the insulation. In low grade materials failure may be due to thermal instability, resulting from the heat liberated by dielectric losses.

When the applied voltage is alternating, the frequency of the supply affects the dielectric strength and, also, since the maximum value of the voltage is responsible for the breakdown, the wave-form of the voltage, as well as its r.m.s. value, is important. The shape of the electrodes by means of which the voltage is applied is important, since the distribution of the electrostatic field depends upon this shape, which therefore affects the dielectric strength. The true dielectric strength is the strength at breakdown when the electrostatic field is uniform.

Potential Gradient. In practice the potential gradient is an important matter. Consider the case of a single-core cable with a conducting outer sheath. From page 6 we have for the field intensity at a point, between two coaxial cylinders, and at a distance x from their common axis $E = \dfrac{2Q}{\kappa x}$ where Q is the charge on the inner conductor per centimetre axial length. Since the potential between two points is given by $\int E dx$, E is the potential gradient at any point. If R_1 is the radius of the core, and R_2 the internal radius of the sheath, the potential gradient at the surface of the core is $\dfrac{2Q}{\kappa R_1}$, and at the internal surface of the sheath $\dfrac{2Q}{\kappa R_2}$; the gradient

in between these points varying as shown in Fig. 78 (*a*). Now, if the dielectric between the core and sheath consists of only one material, of permittivity κ, which is capable of withstanding, without breakdown, the maximum stress $\dfrac{2Q}{\kappa R_1}$ at the core surface, then the outer layers of dielectric, approaching the sheath, will not be economically used.

Graded Cables. To effect a more economical utilization of the dielectric between the core and sheath, several different dielectrics,



FIGS. 78. POTENTIAL GRADIENT IN SINGLE-CORE CABLES

of permittivities κ_1, κ_2, κ_3, etc., are used, these being arranged so that their permittivities are in descending order as the radius increases. Cables insulated in this way are referred to as "graded" cables. Obviously, if the dielectric used could be varied continuously so that κ varied inversely as the radius x, an absolutely uniform potential gradient could be obtained, between core and sheath, as shown in the dotted line in Fig. 78 (*b*). Actually the potential gradient varies in the manner shown in the full-line curve.

In the previous work the potential difference between two coaxial cylinders of radii R_1 and R_2 was found to be

$$V = \frac{2Q}{\kappa} \log_e \frac{R_2}{R_1}$$

from which
$$Q = \frac{V\kappa}{2 \log_e \dfrac{R_2}{R_1}} \quad \ldots \quad (104)$$

Substituting this value for Q, we have for the potential gradient at any radius x

$$E = \frac{2}{\kappa x} \cdot \frac{V\kappa}{2 \log_e \frac{R_2}{R_1}} = \frac{V}{x \log_e \frac{R_2}{R_1}} \qquad (105)$$

when only one dielectric, of permittivity κ, is used.

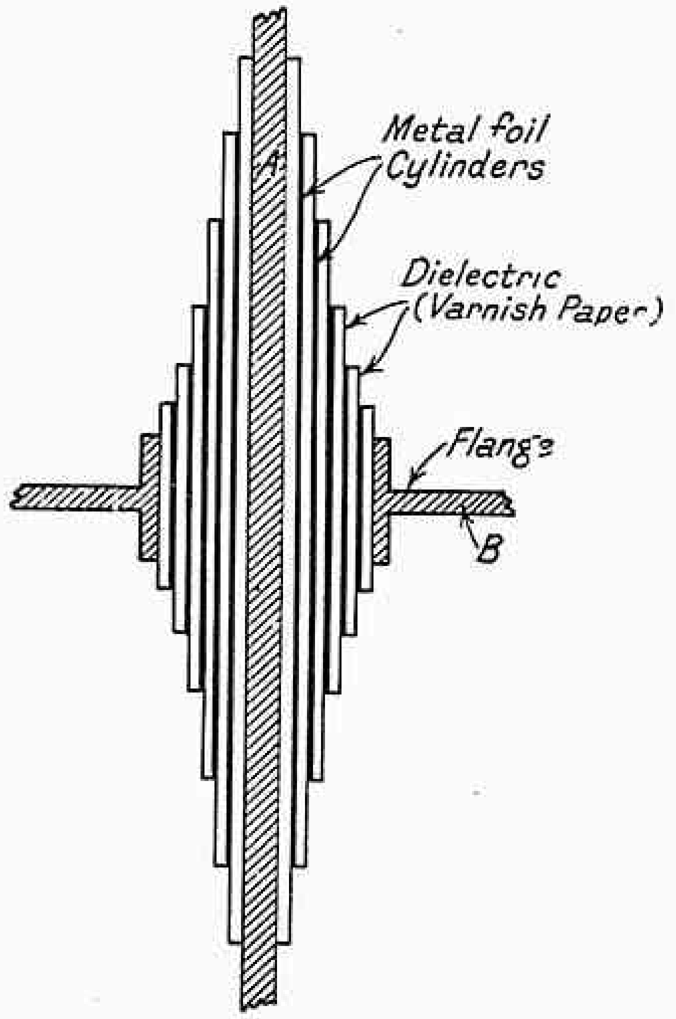


FIG. 79. CONDENSER BUSHING

Another method of obtaining a uniform potential gradient between two coaxial cylinders is by the interposition of metal intersheaths (consisting of cylindrical sheets of metal foil coaxial with the two conductors) in the dielectric, between the charged conductors. As an example of the use of such intersheaths, a "Condenser Bushing" will be considered.

Condenser Bushing. This is a type of bushing which is commonly used for the terminals of high voltage transformers and switchgear. Fig. 79 shows a conductor A which is charged to some high voltage V. This conductor is insulated from the flange B (at earth potential, say), by a condenser bushing consisting of some dielectric material with metal-foil cylindrical sheaths of different lengths and radii

embedded in it, thus splitting up what is essentially a capacitor, having the high tension conductor and flange as its plates, into a number of capacitors in series. The capacitances of the capacitors formed by the metal-foil cylinders are given by the equation

$$C = \frac{\kappa l}{2 \log_e \frac{R_2}{R_1}}$$

l being the axial length of the capacitor and R_1 and R_2 the radii of its cylindrical plates (assumed to be of negligible thickness in the

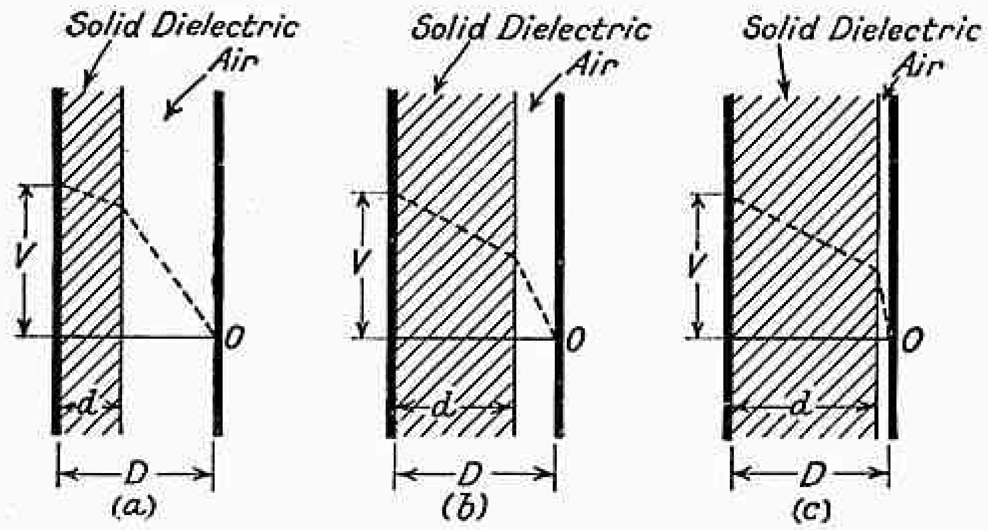


FIG. 80. EFFECT OF DIELECTRIC THICKNESS UPON POTENTIAL GRADIENT IN A PLATE CAPACITOR

case of the metal foil). If these capacitors all have the same capacitance, since Q is the same for all (being the charge per centimetre axial length of the high tension conductor), the potential differences between their plates will be equal. They can be made to have the same capacitance by suitably choosing the axial lengths of successive sheets of foil together with the ratios of their radii $\frac{R_2}{R_1}$. If the radial spaces between successive sheets of foil are made equal and the lengths adjusted to make the capacitances equal, the potential gradient in the dielectric is uniform, but the edges of foil sheets lie on a curve, thus giving unequal surfaces of dielectric between the edges of successive sheets. This is undesirable from the point of view of flashover by "creeping" along the surface. If the differences between the lengths of successive sheets are made equal, the radial potential gradient is not uniform. A compromise between the two conditions is usually adopted.

Effect of Varying Thicknesses of Solid Dielectric upon the Potential Gradient Between Parallel Plates. Fig. 80 shows the effect upon the potential gradient of varying the thickness of a slab of solid dielectric which is situated between the plates of a parallel plate capacitor, one plate being charged to a potential V volts and the other being at earth potential. The remaining space is air.

If σ is the surface density of charge on the plates and κ the permittivity of the solid dielectric, we have—

$$\text{Intensity in solid dielectric} = \frac{4\pi\sigma}{\kappa} = E_D$$

$$\text{Intensity in air space} = 4\pi\sigma = E_A$$

$$\text{Thus} \quad \kappa E_D = E_A$$

Also, if d is the thickness of solid dielectric

$$E_D \, . \, d + E_A(D - d) = V$$

Substituting for E_D we have

$$\frac{E_A}{\kappa} d + E_A(D - d) = V$$

or

$$E_A = \frac{V}{D - d\left(1 - \frac{1}{\kappa}\right)} \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Thus, increase of d increases the potential gradient in the air space, as is shown in Fig. 80. Also, if κ is much greater than 1, the potential gradient in the air space approaches the value $\frac{V}{D - d}$, which means that, in this case, the whole of the potential drop is across the air space.

The high potential gradient so produced is very likely to cause breakdown of the air in the case of a thin film of air included between a solid dielectric and a conducting plate. The air then becomes "ionized," and the insulation will ultimately fail due to damage by ionic bombardment.

The dielectric strengths of the most important insulating materials are given in Table VII under the conditions of the conventional one-minute dielectric strength test. The electrodes used in carrying out tests of dielectric strength are usually flat plates with rounded edges or smooth spheres of large diameter. In either case a fairly uniform electrostatic field is obtained.

(*b*) Permittivity. This quantity is defined as the ratio

$$\frac{\text{The capacitance of a capacitor having the material considered as its dielectric}}{\text{The capacitance of the same capacitor with air as the dielectric}} = \kappa$$

Strictly, the capacitance in the denominator should be that obtained when a vacuum exists between the plates, since the permittivity of a vacuum is unity, while that of air is about

1·0006. Most gaseous dielectrics have permittivity of the same order as that of air, while solid and liquid dielectrics have values of κ varying from about 2 upwards, as shown in Table VII.

TABLE VII

Properties of Dielectrics

Dielectric	Approx. Dielectric Strength Volts/mm.	Permittivity	Power Factor (f = 50 c/s. except where noted)
Bakelite	20,000–25,000	5–6	
Bitumen (Vulcanized) . .	14,000	4·5	
Cotton Cloth (varnished) .	3,000–4,000	4·5–5·5	0·2
Ebonite	10,000–40,000	2·8	0·01
Empire cloth . . .	10,000–20,000	2	
Fibre	5,000	4–6	
Glass (plate) . . .	5,000–12,000	6–7	0·006 (f = 800–1,000)
Guttapercha . . .	10,000–20,000	3–5	
Hard rubber (loaded) . .	10,000–25,000	3·5–4·5	0·016
Marble	6,000	8	
Mica (Muscovite) . . .	40,000–150,000	4·5–7	0·0003
Mycalex		6–7	0·002–0·005
Paper (dry)	4,000–10,000	1·9–2·9	0·005
Paraffin wax . . .	8,000	2·2	0·0003 (f = 800–1,000)
Polystyrene		2·5–2·7	0·0002
Polythene		2·3	0·0001
Porcelain	9,000–20,000	5·5–6·5	0·005–0·01
Shellac	5,000–20,000	2·3–3·8	0·008
Silica (fused transparent) .		3·8	0·0001–0·0003
Slate	3,000	6–7·5	
Steatite		4·1–6·5	0·002
Mineral insulating oil . .	25,000–30,000	2–2·5	0·0002
Water	—	40–90 (decreases with increase of temperature)	

Note. Owing to the different qualities of the various materials and to the variations in results according to the conditions of the test (e.g. frequency, and temperature) the above figures must be regarded as approximations only. The properties of dielectrics, including many of the recently introduced plastic materials are given in Refs. (39) to (44).

(*c*) Dielectric Loss and Power Factor. If a steady voltage V is applied to the plates of a perfect capacitor a "charging current" flows from the supply for a short time and gives to the capacitor a certain quantity Q of electricity, which is sufficient to produce a potential difference between the capacitor plates of V volts. When this potential difference has been attained, the current ceases to flow, the quantity of electricity Q, which has been supplied, being given by $Q = CV$ where C is the capacitance and is, of course, dependent upon the permittivity of the dielectric. In a perfect capacitor, therefore, the dielectric has only one electrical property, namely that of permittivity. It is found that with all practical dielectrics the current does not cease after a short time as above, but dies away gradually over a long period of time as shown in Fig. 81. This means that dielectrics have other properties beyond that of permittivity.

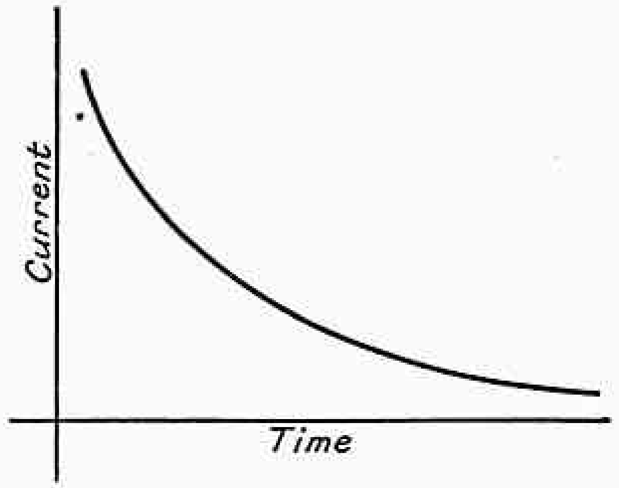


FIG. 81. CHARGING CURRENT IN AN IMPERFECT CAPACITOR

A very small "conduction" current will, of course, flow through the dielectric because of the fact that the resistance of the dielectric, though very high, is not infinite. This does not explain, however, the phenomena observed in most dielectrics, since the current is at first larger than that due to plain conduction and also it is not a constant current, but dies away gradually.

This second phenomenon is referred to as "absorption" and dielectrics in which it occurs are said to be "absorptive." All dielectrics are absorptive to some degree. If an absorptive capacitor after being charged, is discharged, the discharging connection being removed after a short time, it is found that the potential difference between the plates gradually rises again, i.e. the capacitor charges itself. This is known as the "residual" effect. Absorption is explained by assuming that there is a viscous movement of the molecules or ions of a dielectric when the plates between which it is situated are charged. In charging such a capacitor there are rapid electronic and molecular movements which correspond to the initial charging current. Thereafter there are slower molecular and ionic movements which correspond to the absorption current. Finally, there is a steady flow of ions which corresponds to the true conduction current.

The capacitance of a capacitor may thus be divided into two components, viz. the "geometric capacitance" and the "absorptive capacitance." In measuring the capacitance of a capacitor on direct current, the time of charging is thus very important. The shorter the charging time (provided this is long enough to charge the capacitor to the potential difference applied), the nearer the measured capacitance approaches the "geometric" capacitance. Fig. 82 shows the variation of the quantity of charge with time in an absorptive capacitor. The measurement of resistance of dielectrics must also be carried out, having regard to the time of application of the p.d., since

the current for a given applied voltage varies with time as shown above.

Dunsheath (Ref. (10)) represents an absorptive capacitor symbolically, as in Fig. 83. The capacitor C_1 represents the geometric

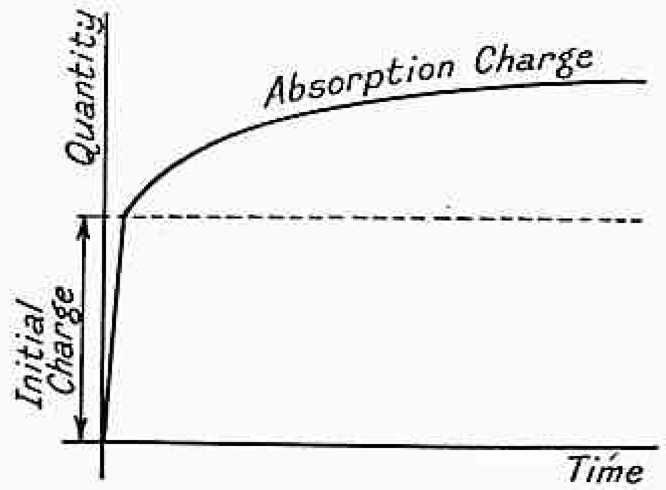
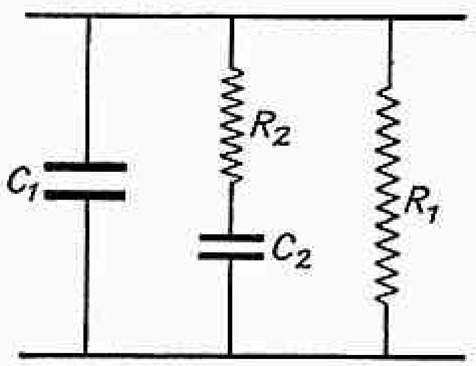


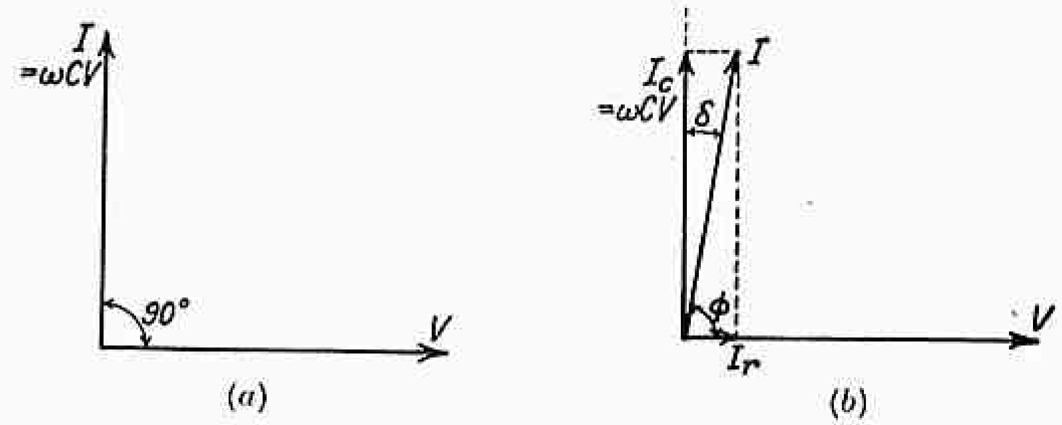
FIG. 82. ABSORPTION IN AN ABSORPTIVE CAPACITOR



(From "*High Voltage Cables*," Dunsheath.)

FIG. 83. SYMBOLIC COMBINATION TO IMITATE PRACTICAL DIELECTRIC

capacitance, the resistance R_1 represents the pure conduction effect, and C_2 and R_2 in series represents the absorption effect. In real materials the behaviour can rarely be represented by a single circuit R_2C_2. There is instead a whole spectrum of similar RC circuits in parallel, with different values of RC.



FIGS. 84. CAPACITOR VECTOR DIAGRAMS

With alternating currents the absorption of the dielectric is intimately connected with the loss of power in the dielectric. In the case of air and most other gases, the losses are very small, and such dielectrics may be regarded as almost perfect.

If a sinusoidal voltage is applied to a perfect capacitor, the current which flows into the capacitor leads the voltage in phase by 90°, as shown in the vector diagram in Fig. 84 (a). If the voltage is

$$v = V_{max} \sin \omega t$$

the current in a perfect capacitor of capacitance C farads is

$$i = \omega C \,.\, V_{max} \cos \omega t$$

Its r.m.s. value is $\omega C.V$ amps. where V is the r.m.s. value of the applied voltage. Owing to the dielectric loss, the current in capacitors used in practice leads the voltage by some angle which is slightly less than 90°, as in Fig. 84 (*b*). The angle ϕ is the "phase angle" of the capacitor, the power factor being cos ϕ. The angle δ, which equals $90 - \phi$, is called the "loss angle." Obviously the power factor may also be expressed as sin δ.

In a perfect capacitor $\phi = 90°$, and therefore $\delta = 0$. The dielectric loss in an imperfect capacitor is given by $IV \cos\phi$ or $IV \sin\delta$ where I and V are r.m.s. values of current and voltage. Thus the loss in a perfect capacitor is

$$IV \sin\delta = 0, \text{ since } \delta = 0$$

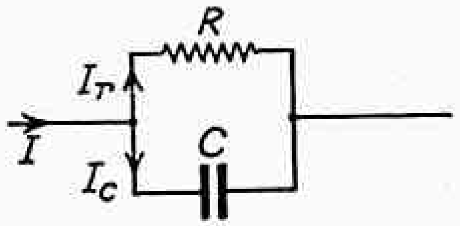


FIG. 85. SYMBOLIC REPRESENTATION OF AN IMPERFECT CAPACITOR

A capacitor having dielectric loss can be represented, at any single frequency, by a perfect capacitor in parallel with a resistance as in Fig. 85, but the value of the equivalent resistance in general varies with frequency. The current I in the capacitor can be split up into a current I_r in the resistance branch, in phase with the voltage, and a current I_c in the capacitor branch, leading the voltage by 90°. These components are shown in Fig. 84 (*b*). Then

$$I_c = \omega CV = I \cos\delta$$

where C is the effective capacitance of the capacitor,

$$\therefore C = \frac{I}{\omega V} \cos\delta$$

The dielectric loss $P = IV \sin\delta$

$$= V \sin\delta \times \frac{V\omega C}{\cos\delta}$$

$$= V^2\omega C \tan\delta \quad \text{watts} \quad . \quad . \quad . \quad (107)$$

if C is in farads and V in volts.

The works referred to at the end of the chapter should be consulted by those who wish to carry the study of dielectric loss further. Refs. (15), (16), and (40) give the effect of frequency and of temperature upon dielectric loss. W. H. F. Griffiths* has investigated the question of losses in variable air capacitors.

* *Experimental Wireless and The Wireless Engineer,* Vol. VIII, No. 90, March, 1931.

Measurement of Dielectric Loss and Power Factor. The two groups of methods of measuring dielectric losses which have been used are—

(*a*) Wattmeter methods,
(*b*) Bridge methods.

The cathode-ray oscillograph has also been applied to such measurements and is still used to investigate the dielectric properties of non-linear materials which would give no balance in a bridge circuit. One example of such a material is barium titanate.

(*a*) WATTMETER METHODS. These are now very seldom used and will be described here only briefly.

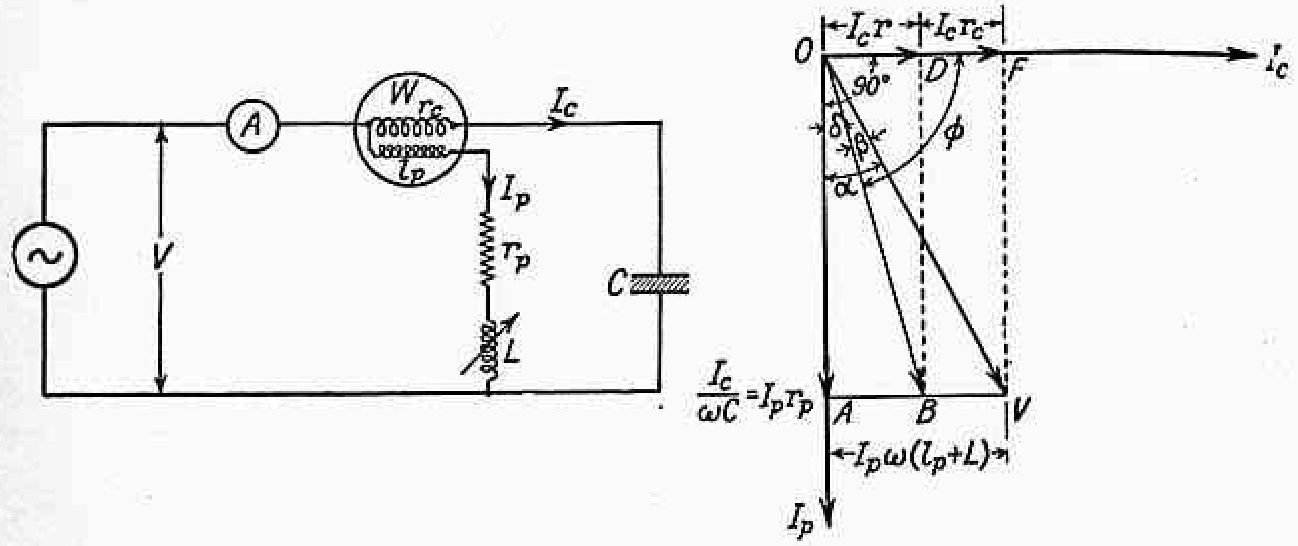


FIG. 86. WATTMETER METHOD OF MEASURING DIELECTRIC LOSS AND POWER FACTOR

Fig. 86 shows the connection diagram for a dynamometer wattmeter when used for this purpose. Owing to the very small power loss and low power factor (usually less than 0·01) the wattmeter must be very sensitive. A "null" method of use is preferable, the wattmeter reading being made zero by adjustment of the variable inductance L in the pressure coil circuit; this brings about a 90° phase difference between I_c and I_p.

Since the loss angle δ, of the capacitor C under test, is very small, as is also the angle β, we may write

$$\tan\beta = \frac{BV}{OB} = \frac{BV}{OA} \text{ approx.}$$

$$= \frac{I_c r_c}{I_c/\omega C} = \omega C r_c \text{ approx.}$$

Thus $\quad \beta = \tan^{-1}\omega C r_c$

Again $\quad \alpha = \tan^{-1}\dfrac{\omega(l_p + L)}{r_p}$, so that

$$\phi = 90 - \alpha + \beta$$

$$= 90 - \tan^{-1}\frac{\omega(l_p + L)}{r_p} + \tan^{-1}\omega C r_c$$

$$= \frac{4\pi}{10}\frac{N_1I}{l}\frac{\frac{l}{2}}{\sqrt{R^2+\frac{l^2}{4}}} = \frac{2\pi N_1I}{10\sqrt{R^2+\frac{l^2}{4}}}$$

This also gives the value of the flux density at the centre, since the core is air ($\mu = 1$). Thus the flux threading the small solenoid is

$$\frac{2\pi N_1I}{10\sqrt{R^2+\frac{l^2}{4}}} \times \pi r^2 = \phi$$

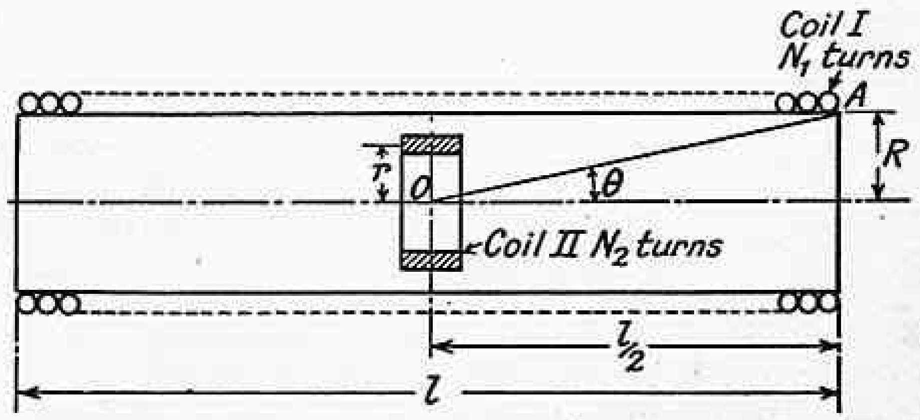


FIG. 100. MUTUAL-INDUCTANCE BETWEEN CONCENTRIC COILS

The mutual-inductance is thus $\frac{\phi N_2}{10^8 I}$ henries

or
$$M = \frac{2\pi^2 N_1N_2r^2}{10^9\sqrt{R^2+\frac{l^2}{4}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

This is, however, only an approximate expression for the mutual inductance, since the strength of field H only refers to the centre point O of the solenoid and its intensity varies both axially and radially.

Corrections. If the internal coil had negligible axial length, the mutual inductance, corrected for radial variation of field strength, would be obtained by multiplying M (above) by the expression (Ref. (12))

$$1 + \frac{3}{8}\frac{R^2r^2}{\left(R^2+\frac{l^2}{4}\right)^2} + \frac{5}{64}\frac{R^4r^2}{\left(R^2+\frac{l^2}{4}\right)^4}\left(3-\frac{l^2}{R^2}\right)$$

Correction for the axial length of the internal coil is obtained by subtracting a quantity, given by the following expression, from the mutual-inductance (Ref. (12))

$$-\frac{5}{9}\sqrt{\frac{3}{5}}\frac{S}{l}N_1N_2\left\{M_{\left(\frac{l}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\frac{S}{2}\right)} - M_{\left(\frac{l}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\frac{S}{2}\right)}\right\}$$

where S = length of internal short coil

and $M_{\left(\frac{l}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right)}$ and $M_{\left(\frac{l}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right)}$ indicate the mutual-inductances between two circles, of radii R and r, at distances apart of

$$\left(\frac{l}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right) \text{ and } \left(\frac{l}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right) \text{ respectively.}$$

As the above two corrections (for axial and radial variation of field) tend to neutralize one another, the difference between the final value of the mutual-inductance and the approximate value originally obtained is usually quite small—of the order of 1 in 1,000 for usual dimensions of a mutual-inductance for ballistic galvanometer calibration.

If both $\frac{l}{R}$ and $\frac{S}{r}$ are equal to $\sqrt{3}$, and if the ratio is small, the corrections become unnecessary (Ref. (17)).

The mutual-inductance when the short coil, instead of being situated *within* the long coil at its centre, is situated at the centre but *outside*, is obtained by the same method as above, but R and r are interchanged.

Example. Calculate the mutual-inductance between two coaxial circular coils, given that—

Length of long coil	=	80 cm.
Radius of long coil	=	4 cm.
No. of turns of long coil	=	500
Length of short coil	=	6 cm.
Radius of short coil	=	3 cm.
No. of turns of short coil	=	150

Small coil placed inside, and at the centre of, the larger coil.

Then
$$M = \frac{2\pi^2 \times 500 \times 150 \times 3^2}{10^9\sqrt{4^2+\frac{80^2}{4}}}$$

$$= \frac{332}{10^6} \text{ henries, or 332 microhenries (very nearly)}$$

6. **Self-inductance of Circular Coils of Rectangular Cross-section of Winding.** Consider a single-layer coil of axial length l cm. and radius of cross-section r cm., having N turns, with a current of I amp. flowing in it. If $\frac{l}{r}$ is great, the magnetic intensity within the coil is $\frac{4\pi}{10}\cdot\frac{NI}{l}$. If no magnetic material is present, this is also the flux density within the coil. Thus the flux inside the solenoid is $\frac{4\pi}{10}\cdot\frac{NI}{l} \times \pi r^2$, and the inductance is

$$L = \frac{4\pi^2N^2r^2}{10^9 l} \text{ henries (approx.)} \quad . \quad . \quad . \quad (128)$$

only a few volts above earth even when a high voltage supply (of the order of 100 kilovolts) is used, except in the case of breakdown of one of the capacitor arms I and II.

In use, the bridge is balanced by successive variation of R_3 and C_4 until the vibration galvanometer indicates zero deflection. Then, at balance,

$$C_1 = C_2 \cdot \frac{R_4}{R_3} \cos^2 \delta = C_2 \cdot \frac{R_4}{R_3} \text{ approx.} \quad . \quad (109)$$

since δ is small, and

$$\tan \delta = R_4 \omega \cdot C_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

where $\omega = 2\pi \times$ frequency

$\delta =$ the "loss angle" of the capacitor, sin δ giving the power factor

$C_1 =$ the effective parallel capacitance of the test capacitor

$C_2 =$ the capacitance of the standard capacitor

Theory. Consider first the impedances of the four arms of the bridge numbered I, II, III, and IV in Fig. 88.

Arm I. Consider this arm as consisting of the effective parallel capacitance of the capacitor whose power factor is to be obtained, in parallel with a resistance R_1, as shown, the latter representing its loss component.

$$\text{Total admittance of arm I} = \frac{1}{R_1} + \frac{1}{\frac{-j}{\omega C_1}}$$

$$= \frac{1}{R_1} + j\omega C_1$$

$$\therefore \text{Impedance of arm I} \quad = \frac{1}{\frac{1}{R_1} + j\omega C} = \frac{R_1}{1 + j\omega C_1 R_1} = z_1$$

Arm II. Impedance $= \frac{-j}{\omega C_2} = z_2$

Arm III. Impedance $= R_3 = z_3$

Arm IV. Impedance $= \frac{R_4}{1 + j\omega C_4 R_4} = z_4$

Under balance conditions

$$\frac{z_1}{z_3} = \frac{z_2}{z_4}$$

i.e.

$$\frac{R_1}{R_3(1 + j\omega C_1 R_1)} = \frac{\frac{-j}{\omega C_2}}{\frac{R_4}{1 + j\omega C_4 R_4}} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Rationalizing, we have

$$\frac{R_1(1 - j\omega C_1 R_1)}{R_3(1 + \omega^2 C_1^2 R_1^2)} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Equating real terms

$$\frac{R_1}{1 + \omega^2 C_1^2 R_1^2} = \frac{C_4 R_3}{C_2}$$

Now, from Fig. 89, which shows the vector diagram for the capacitor (C_1 and R_1 in parallel) when a voltage E is applied to it,

$$\cos \delta = \frac{E\omega C_1}{E\sqrt{\frac{1}{R_1^2} + \omega^2 C_1^2}} = \frac{\omega C_1 R_1}{\sqrt{1 + \omega^2 C_1^2 R_1^2}}$$

or

$$\cos^2 \delta = \frac{\omega^2 C_1^2 R_1^2}{1 + \omega^2 C_1^2 R_1^2}$$

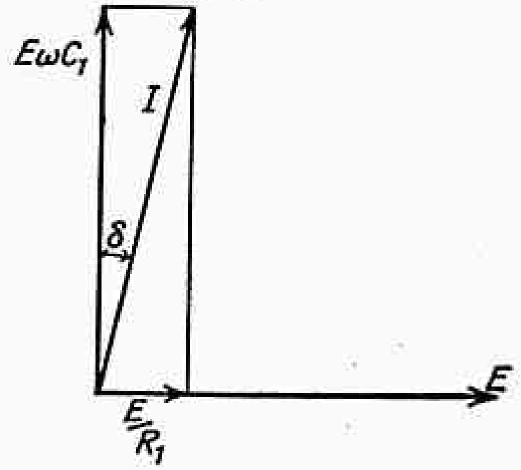


FIG. 89. VECTOR DIAGRAM FOR C_1 AND R_1 IN PARALLEL

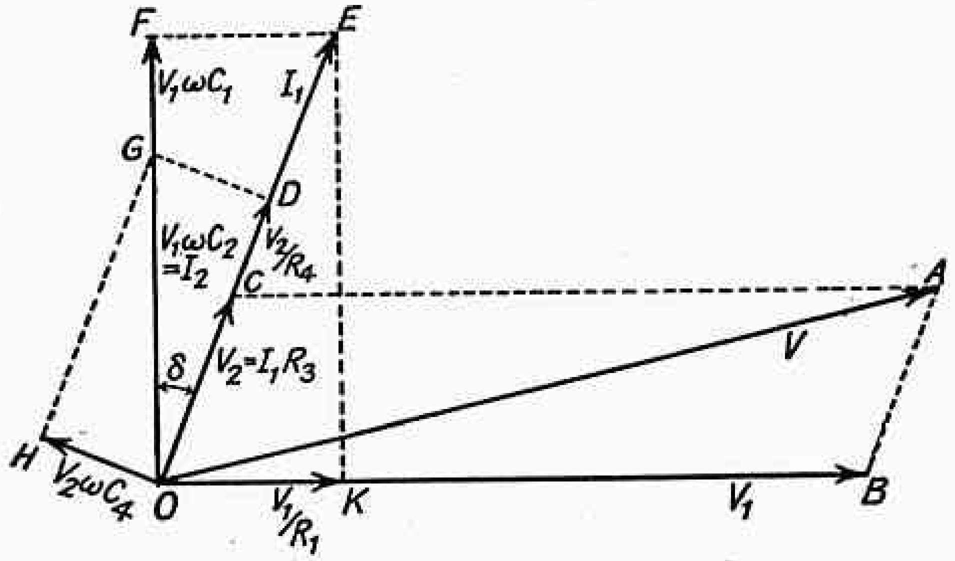


FIG. 90. VECTOR DIAGRAM FOR SCHERING BRIDGE UNDER BALANCE CONDITIONS

Substituting $\cos^2 \delta$ in the equation of real terms obtained above, we have

$$\frac{\cos^2 \delta}{\omega^2 C_1^2 R_1} = \frac{C_4 R_3}{C_2}$$

$$\therefore C_1 = \frac{C_2 \cos^2 \delta}{\omega^2 C_1 C_4 R_1 R_3}$$

From Fig. 90, showing the complete vector diagram for the bridge network under balance conditions,

$$\tan \delta = \frac{\omega C_4}{\frac{1}{R_4}} = \omega C_4 R_4$$

(which is the expression previously stated), and also

$$\tan\delta = \frac{\frac{1}{R_1}}{\omega C_1} = \frac{1}{\omega C_1 R_1}$$

$$\therefore \omega C_4 R_4 = \frac{1}{\omega C_1 R_1}$$

or

$$R_4 = \frac{1}{\omega^2 C_1 C_4 R_1}$$

Substituting R_4 in the expression for C_1 gives

$$C_1 = \frac{C_2 R_4}{R_3} \cos^2\delta$$

as previously stated.

Imperfect Capacitor as a Series Circuit. An alternative method to that of representing an imperfect capacitor diagrammatically as a perfect capacitor C_1, in parallel with a resistance R_1, is to represent it as a perfect capacitor C_s in series with a resistance R_s.

The impedances in the two cases are

$$\frac{R_1}{1 + j\omega C_1 R_1} = \frac{R_1(1 - j\omega C_1 R_1)}{1 + \omega^2 C_1{}^2 R_1{}^2} \text{ in the parallel representation}$$

and

$$R_s - \frac{j}{\omega C_s} \text{ in the series representation.}$$

By equating the real and imaginary terms in the two impedances we obtain the relationships

$$R_s = \frac{R_1}{1 + \omega^2 C_1{}^2 R_1{}^2}$$

and

$$C_s = \frac{1 + \omega^2 C_1{}^2 R_1{}^2}{\omega^2 C_1 R_1{}^2}$$

The vector diagram of Fig. 90 needs, perhaps, some explanation. Vector OA represents the voltage applied to the bridge from the supply transformer. OB is the volt drop V_1 across arm II which, when no current flows in the vibration galvanometer branch (i.e. under balance conditions), is equal in magnitude and phase to the volt drop across arm I. Vector OC is the volt drop V_2 across arm III, which is equal in magnitude and phase to that across arm IV. The vector sum of OB and OC obviously gives the total bridge voltage OA. The current I_1 flowing in arms I and III is represented by vector OE, while OG represents the current I_2 flowing in branches II and IV. OF and OK represent the component parts of current I_1 when split up between the capacitance C_1 and resistance R_1. In the same way OD and OH represent the components of the current I_2 when similarly split up between R_4 and C_4.

The magnitudes of some of the vectors, e.g. OC, are exaggerated for the sake of clearness. V_2 will, in reality, be very small compared with V_1 and V.

A direct-reading Schering bridge for the measurement of permittivity and power factor of solid dielectrics at 1,600 cycles per sec. and voltages of 100–200 is manufactured by Messrs. H. W. Sullivan, Ltd. This covers a range of capacitance up to 1,000 $\mu\mu$F.

The Cambridge Instrument Co. manufacture both low- and high-tension Schering bridges.

Muirhead and Co., Ltd. make a Schering Bridge, with a Wagner earth attachment (see p. 231) which is intended for the measurement of power factor and permittivity of insulating materials in accordance with the recommendations of *British Standard Specification No.* 234.

A portable high-voltage Schering bridge made by H. Tinsley and Co. has with it a screened, loss-free air capacitor of 100 $\mu\mu$F (within $\pm\frac{1}{2}$ per cent) and is for use at 11 kV. It may be used up to 150 kV using a compressed-air capacitor of nominal capacitance 100 $\mu\mu$F having a power factor $\not> 0{\cdot}0001$ at 50 cycles per sec. This requires an air pressure of 250 to 300 lb. per sq. in.

L. Hartshorn (Ref. (45)) adopted the Schering bridge to the measurement of very small capacitances (below 1 $\mu\mu$F) and the Hartshorn form of the bridge is the best method of measuring the permittivity and dielectric loss of sheet materials. B.S. 234 and B.S. 903 give detailed specifications for its use for this purpose.

A very full discussion of the Schering bridge, in its various forms, is given in Hague's *Alternating Current Bridge Methods*.

Dielectric Loss Measurement by Cathode-ray Oscillograph. The construction of the cathode-ray oscillograph is dealt with in Chap. XV. For the present purpose it is sufficient to know that it consists of a vacuum tube having, at one end, a filament which gives off a stream of electrons in a thin beam, or pencil, when the tube is in use. This beam passes two pairs of parallel plates, set at right angles to one another, and is deflected by potential differences applied to these pairs. A continuous path will be traced out by the beam on the fluorescent screen of the tube if the p.d.'s are alternating. This path will be a straight line if the p.d.'s are sinusoidal and are in phase but will be an ellipse if they are not in phase. The area of this ellipse is maximum—for any given maximum values of the two potential differences—when they are 90° out of phase with one another. Under these conditions, the semi-axes of the ellipse give the maximum values of the two potential differences to scale. The electron beam, having negligible inertia, can immediately take up a deflected position which is proportional, at any given time, to the deflecting force.

When used for dielectric loss measurements, a potential difference proportional to the applied voltage is applied to one pair of plates and one proportional to the integral of current through the dielectric to the other pair. This is obtained in the form of the p.d. across a relatively larger capacitor in series with the sample.

It will be shown below that the area of the ellipse traced out by the electron beam is then proportional to the power loss in the dielectric. If there is no power loss—as in the case of an air capacitor—the p.d.'s applied to the plates are in phase with one another and the path traced out is a straight line.

A record of the ellipse traced out in power loss measurements can be obtained photographically.

J. P. Minton (Ref. (23)) used a cathode-ray oscillograph for dielectric loss and power factor measurements. The full circuit arrangements are given by Hartshorn (Ref. (16)).

Fig. 91 shows a simpler arrangement than that of Minton which,

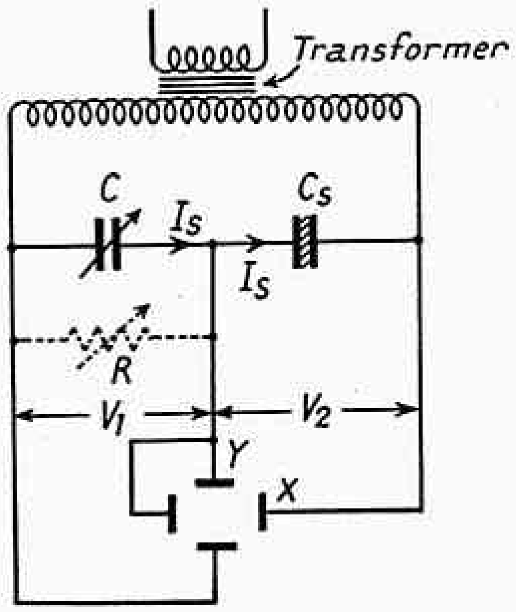


Fig. 91

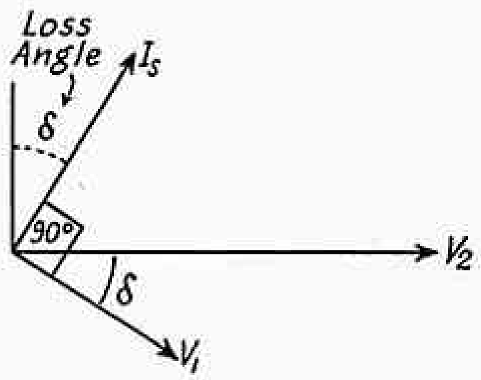


Fig. 92

Measurement of Dielectric Loss by C.R. Oscillograph

nevertheless, will serve to illustrate this method of dielectric loss measurement.

C_s is the dielectric sample and C a loss-free capacitor of much greater capacitance than C_s. The resistor R, shown dotted, may be used, if desired, for compensating the loss angle of C_s. The C.R. oscillograph plates X and Y are connected as shown. (If the voltage on C_s is low, an amplifier will be needed between C and the Y plates since the method is inaccurate unless the voltage on C is much less than that on C_s.)

In the theory of the method which is given below it is assumed that the resistor R is omitted. The vector diagram may then be drawn as in Fig. 92 which shows the voltage V_2 across C_s, the current I_s through both C_s and C and the voltage V_1 across the latter.

The power loss in C_s is $V_2 I_s \sin\delta$. From the vector diagram, if $v_2 = V_{2\,max}\sin\omega t$ then $v_1 = V_{1max}\sin(\omega t - \delta)$.

The deflection produced by the Y plates is proportional to v_1 and we may write

$$y \text{ deflection} = \alpha \,.\, V_{1\,max}\sin(\omega t - \delta)$$

$$= \alpha \,.\, \frac{I_{s\,max}}{\omega c}\sin(\omega t - \delta)$$

Also, the x deflection $= \beta \,.\, V_{2\,max}\sin\omega t$

where α and β are proportionality constants.

Now the area of the ellipse traced out on the oscillograph screen is

$$A = \int y \,.\, dx$$

$$= \int_0^T \alpha \,.\, \frac{I_{s\,max}}{\omega C} \,.\, \sin(\omega t - \delta) \,.\, \beta \,.\, V_{2\,max}\omega\cos\omega t \,.\, dt$$

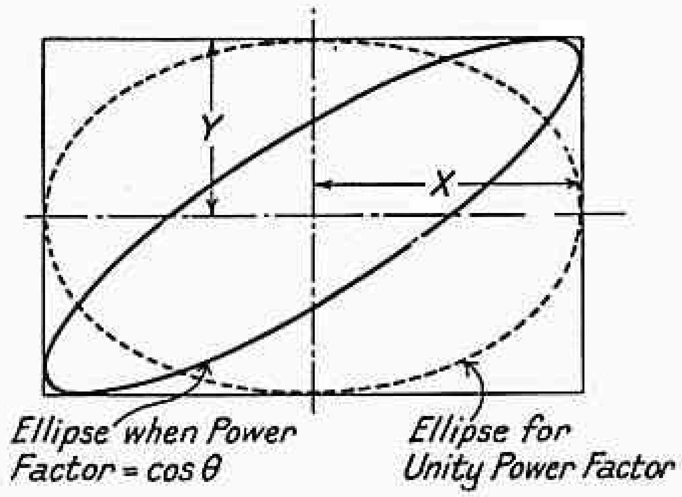


Fig. 93. Forms of Oscillograms Obtained with Cathode Ray Oscillograph

(where T is the periodic time)

$$= \frac{\alpha\beta \,.\, I_{s\,max}V_{2\,max}}{C}\int_0^T \sin(\omega t - \delta)\cos\omega t \,.\, dt$$

$$= \frac{\alpha\beta}{C} \,.\, \frac{I_{s\,max}V_{2\,max}}{2}\int_0^T [\sin(2\omega t - \delta) + \sin\delta]\, dt$$

i.e. $$A = \frac{\alpha\beta}{C} \,.\, I_s V_2 \,.\, \sin\delta \,.\, \frac{2\pi}{\omega}$$

since $$T = \frac{2\pi}{\omega}$$

Thus the area of the ellipse is proportional to $V_2 I_s \sin\delta$ which is the dielectric loss.

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CHAPTER V

INDUCTANCE

Self-inductance. Whenever the number of lines of magnetic force linking with a circuit changes, an e.m.f. is induced in the circuit; this e.m.f. is given by

$$e = -N\frac{d\phi}{dt}$$

where e = induced e.m.f.

N = number of turns with which the flux links

$\frac{d\phi}{dt}$ = rate of change of the interlinking flux in lines per second

The negative sign indicates that the direction of the e.m.f. is such as to oppose the change in the flux.

If, now, the change in the flux is due to a change in the current flowing in the circuit itself (by which current the inter-linking magnetic flux is produced) and if, also, the reluctance of the path of the magnetic flux is constant, then

$$\phi = ki$$

where i is the current in the circuit.

Thus, $$e = -Nk\frac{di}{dt}$$

or $$e = -Nk\frac{di}{dt} \times 10^{-8} \text{ volts}$$

Since $k = \frac{\phi}{i}$, the above expression can be written

$$e = -\left(N\frac{\phi}{i} \times 10^{-8}\right)\frac{di}{dt} \text{ volts}$$

or $$e = -L\frac{di}{dt} \text{ volts}$$

where L is the "coefficient of self-induction" or, simply, the "inductance" of the circuit. Obviously L is constant for any given circuit, only if k is constant—i.e. when no magnetic material is present. If i is expressed in amperes, $\frac{di}{dt}$ is in amperes per second, and k is the flux produced by 1 amp. flowing in the circuit. Then L is in henries.

It follows from the above that the inductance of a circuit, in henries, can be expressed in words as

$$\frac{\text{Number of turns} \times \text{flux produced per ampere}}{10^8}$$

(When, as in an a.c. circuit, the flux per ampere is not constant, a better definition for the inductance is given by the induced voltage divided by the rate of change of current.)

Mutual-inductance. If two coils are close together and unit current flows in one of them, then the number of "linkages" with the other coil, of the magnetic flux due to this current, is called "the coefficient of mutual-induction," or simply the "mutual-inductance" between the coils. By "linkages" is meant the product of lines of force and the number of turns on the coil.

If the current i_1 in coil 1 varies, its rate of change being $\frac{di_1}{dt}$, then the e.m.f., e_2, induced in the second coil is given by

$$e_2 = -M\frac{di_1}{dt}$$

where M is the mutual-inductance.

If i_1 is in amperes, and M is the number of linkages with coil 2, per ampere in coil 1, divided by 10^8, then

$$e_2 = -M\frac{di_1}{dt} \text{ volts} \quad . \quad . \quad . \quad (111)$$

If the current i_2 flows in coil 2 instead of coil 1, then the e.m.f. induced in coil 1 when the rate of change of current in coil 2 is $\frac{di_2}{dt}$ is given by

$$e_1 = -M\frac{di_2}{dt} \text{ volts}$$

it being assumed that M is the same in each case.

To determine the *direction of the induced e.m.f.* in coil 1 consider the current in coil 2 to be increasing; then a self-induced e.m.f. will be produced in coil 2, the direction of which is in opposition to the direction of the current. Since the same flux which induces this self-induced e.m.f. is also inducing the e.m.f. in coil 1, this latter e.m.f. will also be in a direction opposing that of the current in coil 2. If the circuit of coil 1 is closed, a current will flow, due to the induced e.m.f. and in the same direction. This current reduces the interlinking flux and thus reduces the self-inductance of coil 2. Hence there is a mutual action between the coils.

Mutual-inductance is measured, like self-inductance, in henries.

A mutual-inductance of 1 *henry exists between two circuits when* *a rate of change of current of 1 amp. per second in one circuit induces an e.m.f. of 1 volt in the other circuit.*

Relations Between Self- and Mutual-inductance. Suppose that two coils, having respectively N_1 and N_2 turns, are so close together that the whole of the flux produced by a current in one coil links with the other. Let this flux be ϕ when the current in coil 1 is i_1. Then the self-inductance of coil 1 is $L_1 = N_1 \frac{\phi}{i_1}$ and the mutual inductance is $M = N_2 \frac{\phi}{i_1} = \frac{N_2}{N_1} L_1$.

Similarly, if i_2 flows in coil 2, its self-inductance $L_2 = \frac{N_2\phi'}{i_2}$ and $M = N \frac{\phi'}{i_2} = \frac{N_1}{N_2} L_2$.

$$\therefore \frac{N_2}{N_1} L_1 = \frac{N_1}{N_2} L_2 = M$$

or

$$M^2 = L_1 L_2$$

$$M = \sqrt{L_1 L_2} \quad . \quad . \quad . \quad . \quad . \quad (112)$$

As stated above, this relationship is true only when the whole of the flux from one coil links with the other. In practice this condition is not fulfilled, although if the coils are very close together it is very nearly so. The ratio $\frac{M}{\sqrt{L_1 L_2}}$ is called the *coefficient of coupling*, and is of importance, especially in radio work. If this ratio is nearly unity, the circuits are said to be "close coupled," while if it is considerably less than unity they are said to be "loosely coupled."

Self-inductances in Series. If two coils, of self-inductances L_1 and L_2 henries, are connected in series and the mutual-inductance between the coils is M, then if the flux, produced by coil 2, linking with coil 1, is in the same direction, at any instant, as the self-produced flux of coil 1, then the effective self-inductance of coil 1 is $L_1 + M$. In the same way the effective self-inductance of coil 2 is $L_2 + M$, provided the self and mutual fluxes are in the same direction at any instant. Thus the total self-inductance of the circuit is

$$L = L_1 + L_2 + 2M$$

Expressing this generally, to include the case when the mutual and self-produced fluxes are in opposition at any instant, we have

$$L = L_1 + L_2 \pm 2M \quad . \quad . \quad . \quad (113)$$

Figs. 94 (a) and (b) show two coils connected in series, and with the directions of current in them such that their magnetic effects are (a) cumulative, (b) in opposition.

In the first case, $L = L_1 + L_2 + 2M$, and in the second case $L = L_1 + L_2 - 2M$.

1. **Inductance of Two Long, Parallel Cylinders.** Consider two long, parallel cylinders X and Y, each of radius of cross-section R cm. and carrying currents of I amp., in opposite directions as shown in Fig. 95 (a). Let the distance between the axes of the cylinders be D cm., and the surrounding medium be air. Suppose, also, that the material of which the cylinders are made is non-magnetic.

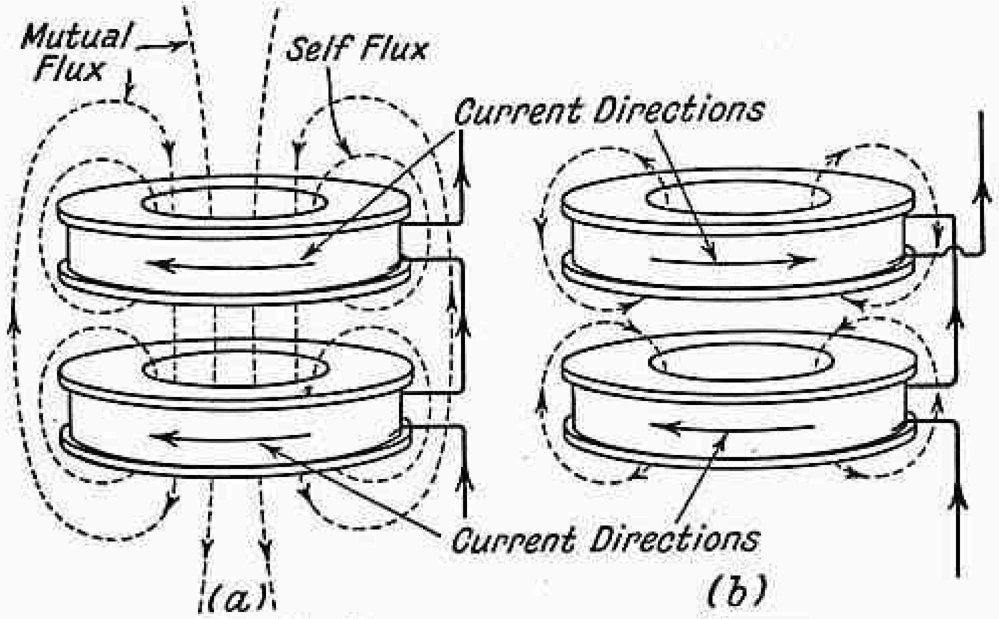

FIG. 94 SELF-INDUCTANCES IN SERIES

The flux, produced by the currents, and to which the inductance is due, is composed of two parts which must be treated separately.

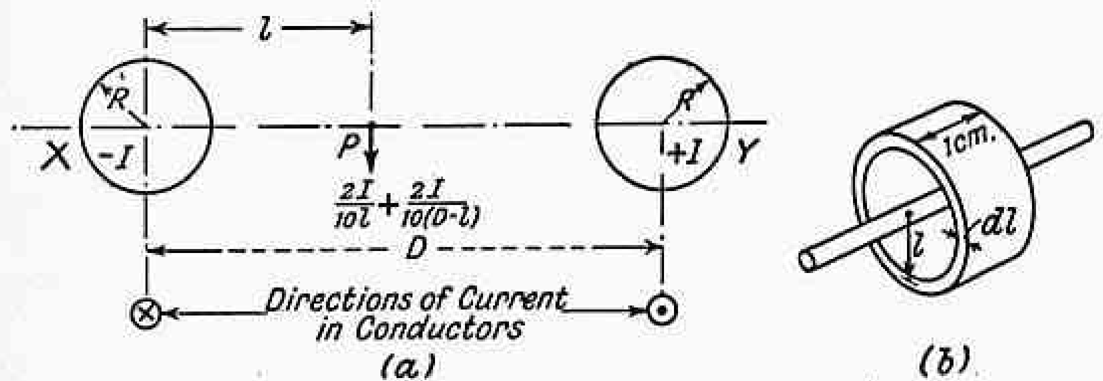

FIG. 95. INDUCTANCE OF TWO LONG PARALLEL CYLINDERS

These are (a) the flux surrounding the two conductors, and (b) the flux which exists inside the conductors themselves. These will be considered in order.

(a) The magnetic intensity at a distance r cm. from a conductor carrying I amperes is $\frac{2I}{10r}$.

Thus, the total intensity of field at a point P distant l cm. from cylinder X and $(D - l)$ cm. from Y is $\frac{2I}{10l} + \frac{2I}{10(D-l)}$, the addition of the intensities due to the two cylinders separately being because the currents in them are in opposite directions. The resultant intensity is downwards, as shown in the figure. Since the medium between the cylinders is air, the flux density B at P is also equal to $\frac{2I}{10l} + \frac{2I}{10(D-l)}$.

Thus, the flux in a ring of very small radial width dl and axial length 1 cm., the ring being of mean radius l cm. and concentric with cylinder X (see Fig. 95 (*b*)) is

$$B \times dl \times 1 = \frac{2I}{10}\left[\frac{1}{l} + \frac{1}{D-l}\right] dl$$

The total flux between the cylinders per centimetre axial length is

$$\int_R^{D-R} Bdl = \frac{2I}{10} \int_R^{D-R} \left[\frac{1}{l} + \frac{1}{D-l}\right] dl$$
$$= \frac{2I}{10}\Big[\log_e l - \log_e (D-l)\Big]_R^D$$
$$= \frac{4I}{10}\log_e \frac{D-R}{R}$$

or $\frac{4I}{10}\log_e \frac{D}{R}$ if R is small compared with D.

The flux surrounding each wire is one-half the total flux, i.e.

$$\frac{2I}{10}\log_e \frac{D}{R} \text{ lines per cm. axial length}$$

Since inductance in henries $= \dfrac{\text{No. of turns} \times \text{flux per amp.}}{10^8}$ the inductance of one conductor per centimetre axial length due to its external flux alone is

$$\frac{2}{10^9}\log_e \frac{D}{R} \text{ henries}$$

(*b*) In considering the flux existing inside each conductor, assume that the current is distributed uniformly over the cross-section of the conductor. This assumption is justified if the supply frequency is low. At high frequencies, the current flows almost entirely in the outside "skin" of the conductor and in this case the flux inside the conductor is negligibly small. The expression for inductance derived below, together with most of the succeeding expressions, gives therefore the "low-frequency" inductance. Slight modifications,

due to the negligible internal flux, are required to convert them to expressions for "high-frequency" inductance.

It is convenient to consider the conductor as being made up of a very large number of filaments, all parallel to the axis and each carrying a small fraction of the total conductor current I (see Fig. 96).

Consider an elemental ring of radius r and radial width dr as shown.

Then the current enclosed within this ring

$$I_r = \frac{\pi r^2}{\pi R^2} \times I = \frac{r^2}{R^2} I$$

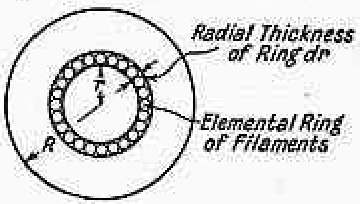

FIG. 96. CURRENT DISTRIBUTION IN A CYLINDRICAL CONDUCTOR

and the intensity of the magnetic field at radius r within the conductor is

$$B_r = \frac{2I_r}{10r}$$

the permeability of the conductor material being unity.

The flux in the elemental ring, of axial length 1 cm., radius r, and radial width dr, is $B_r dr \times 1$

$$= \frac{2I_r}{10r}\, dr \text{ lines}$$

This flux does not surround the whole conductor current, but only the current I_r. Referring it to the whole conductor, we have

$$d\phi = \frac{2I_r}{10r}\, dr \times \frac{r^2}{R^2}$$

by multiplying by $\frac{r^2}{R^2}$—the inverse ratio of the numbers of filaments.

Thus $$d\phi = \frac{2Ir^2}{10rR^2} \times \frac{r^2}{R^2}\, dr = \frac{2Ir^3}{10R^4}\, dr$$

The total flux of the inside of the conductor, referred to the whole conductor,

$$= \int_0^R \frac{2Ir^3}{10R^4}\, dr = \frac{2IR^4}{4 \times 10R^4} = \frac{I}{20} \text{ lines}$$

For *one conductor* the inductance L is $\dfrac{\text{flux}}{\text{amp.} \times 10^8}$

$$= \frac{\frac{2I}{10}\log_e \frac{D}{R} + \frac{I}{20}}{I \times 10^8} = \frac{\frac{2}{10}\log_e \frac{D}{R} + \frac{1}{20}}{10^8}$$

due to the negligible internal flux, are required to convert them to expressions for "high-frequency" inductance.

It is convenient to consider the conductor as being made up of a very large number of filaments, all parallel to the axis and each carrying a small fraction of the total conductor current I (see Fig. 96).

Consider an elemental ring of radius r and radial width dr as shown.

Then the current enclosed within this ring

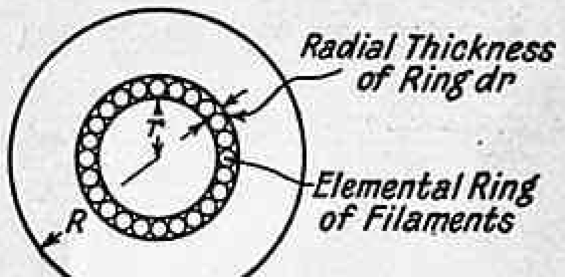

Fig. 96. Current Distribution in a Cylindrical Conductor

$$I_r = \frac{\pi r^2}{\pi R^2} \times I = \frac{r^2}{R^2} I$$

and the intensity of the magnetic field at radius r within the conductor is

$$B_r = \frac{2I_r}{10r}$$

the permeability of the conductor material being unity.

The flux in the elemental ring, of axial length 1 cm., radius r, and radial width dr, is $B_r dr \times 1$

$$= \frac{2I_r}{10r} dr \text{ lines}$$

This flux does not surround the whole conductor current, but only the current I_r. Referring it to the whole conductor, we have

$$d\phi = \frac{2I_r}{10r} dr \times \frac{r^2}{R^2}$$

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The total flux of the inside of the conductor, referred to the whole conductor,

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For *one conductor* the inductance L is $\frac{\text{flux}}{\text{amp.} \times 10^8}$

$$= \frac{\frac{2I}{10}\log_e \frac{D}{R} + \frac{I}{20}}{I \times 10^8} = \frac{\frac{2}{10}\log_e \frac{D}{R} + \frac{1}{20}}{10^8}$$

It is assumed in this formula that the circle is complete, i.e. there is no gap in it, and that the wire is of non-magnetic material.

If a gap of length g cm. is left in the circle, then

$$L = \left(1 - \frac{g}{\pi D}\right) 2\pi D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad . \quad (117)$$

If, instead of one circular turn, we have a circular coil of circular cross-section and N turns, the self-inductance of the coil is

$$L = 2\pi N^2 D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad . \quad . \quad . \quad (118)$$

where d is now the diameter of the section of the coil. The formula for a single turn is obviously a special case of the coil when $N = 1$.

For high frequencies the formula, given by Grover (Ref (2)) for a single turn, is

$$L = 2\pi D\left[\left(1 - \frac{d^2}{4D^2}\right)\log_e \frac{8D}{d} - 2\right] \times 10^{-9} \text{ henries} \quad . \quad . \quad (119)$$

4. Mutual-inductance Between Two Concentric Circles. The mutual-inductance between two concentric circles can be calculated by integration, using the equation

$$H = \frac{idl}{r^2}\sin\theta$$

Fig. 97 (*a*) shows two concentric circular wires of radii r_1 and r_2, the outer of which carries a current of i e.m. units. If the flux threading the inner circle, due to the current in an element dl of the outer circle, is calculated, then the total flux threading the inner circle, when i units of current flow in the outer, can be found by integrating over the whole circumference of the latter. The mutual inductance is then given by $\frac{\phi}{i}$ where ϕ is the total flux linking with the inner circle.

It can be shown by integration* that the flux linking with the inner circle when a current of i e.m. units flows in the outer is

$$4\pi r_1 i\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\}$$

if $r_1 - r_2$ is small and assuming the medium to have unit permeability. Thus, the mutual inductance between the circles is given by

$$M = \frac{4\pi r_1}{10^9}\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\} \text{ henries}$$

the radii r_1 and r_2 being expressed in centimetres.

If, instead of being circles of one turn only, the coils had a number

* See Drysdale and Jolley, *Electrical Measuring Instruments*, Vol. I.

$$= \left(2 \log_e \frac{D}{R} + \frac{1}{2}\right) \times 10^{-9} \text{ henries per cm. axial length}$$

The inductance per mile of one conductor is thus

$$L = 0{\cdot}0804 + 0{\cdot}740 \log_{10} \frac{D}{R} \text{ millihenries per mile} \quad (114)$$

2. Single Straight Wire Parallel to Earth

Let the axial length of the wire $= l$ cm.
,, ,, radius of wire $= R$ cm.
,, ,, height of wire above earth $= H$ cm.

Assume the wire to be of non-magnetic material and that the radius of the wire is small compared with its length. Then, using the method of images, imagine that an exactly similar conductor, running parallel to the overhead one, is embedded in the earth at a depth H cm. immediately below the latter. If the embedded conductor carries the same current as the overhead one, but in the opposite direction, then the distribution of the magnetic field will be the same as that of the single overhead conductor existing alone.

The distance between the overhead and imaginary embedded conductor is $2H$. We may, therefore, from the results of the previous paragraph, state the inductance of the overhead conductor as

$$L = 0{\cdot}0804 + 0{\cdot}740 \log_{10} \frac{2H}{R} \text{ millihenries per mile}$$

replacing D by $2H$.

The inductance of a single straight cylindrical conductor, distant from earth and other conductors, is given by

$$L = 2l\left(\log_e \frac{2l}{R} - 0{\cdot}75\right) \text{ millihenries}$$

where $l =$ length of wire in centimetres
$R =$ radius of wire in centimetres

it being assumed that the material of the wire is non-magnetic and that the surrounding medium is also non-magnetic.

If the wire is of magnetic material the inductance is given by

$$L = 2l\left(\log_e \frac{2l}{R} - 1 + \frac{\mu}{4}\right) \text{ millihenries} \quad . \quad . \quad (115)$$

where $\mu =$ permeability of the material of the wire.

3. A Single Circular Turn of Round Wire. The inductance for continuous current and low frequencies is given by Rayleigh and Niven's Formula (Ref. (1)), i.e.—

$$L = 2\pi D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad (116)$$

where $D =$ mean diameter of the turn in centimetres
$d =$ diameter of cross-section of the wire in centimetres

It is assumed in this formula that the circle is complete, i.e. there is no gap in it, and that the wire is of non-magnetic material.

If a gap of length g cm. is left in the circle, then

$$L = \left(1 - \frac{g}{\pi D}\right) 2\pi D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad . \quad (117)$$

If, instead of one circular turn, we have a circular coil of circular cross-section and N turns, the self-inductance of the coil is

$$L = 2\pi N^2 D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad . \quad . \quad . \quad (118)$$

where d is now the diameter of the section of the coil. The formula for a single turn is obviously a special case of the coil when $N = 1$.

For high frequencies the formula, given by Grover (Ref (2)) for a single turn, is

$$L = 2\pi D\left[\left(1 - \frac{d^2}{4D^2}\right)\log_e \frac{8D}{d} - 2\right] \times 10^{-9} \text{ henries} \quad . \quad . \quad (119)$$

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Fig. 97 (a) shows two concentric circular wires of radii r_1 and r_2, the outer of which carries a current of i e.m. units. If the flux threading the inner circle, due to the current in an element dl of the outer circle, is calculated, then the total flux threading the inner circle, when i units of current flow in the outer, can be found by integrating over the whole circumference of the latter. The mutual inductance is then given by $\frac{\phi}{i}$ where ϕ is the total flux linking with the inner circle.

It can be shown by integration* that the flux linking with the inner circle when a current of i e.m. units flows in the outer is

$$4\pi r_1 i\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\}$$

if $r_1 - r_2$ is small and assuming the medium to have unit permeability. Thus, the mutual inductance between the circles is given by

$$M = \frac{4\pi r_1}{10^9}\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\} \text{ henries}$$

the radii r_1 and r_2 being expressed in centimetres.

If, instead of being circles of one turn only, the coils had a number

* See Drysdale and Jolley, *Electrical Measuring Instruments*, Vol. I.

of turns, these turns being assumed to be coincident in space, then

$$M = \frac{4\pi r_1}{10^9} N_1 N_2 \left\{ \log_e \frac{8r_1}{r_1 - r_2} - 2 \right\} \text{ henries} \quad . \quad (120)$$

where N_1 = No. of turns on outer coil

N_2 = „ „ inner coil

The length $r_1 - r_2$ may be expressed as the distance between the circumferences of the coils. Since the assumption that the turns on the coils are coincident in space is not usually justified even as an approximation, a length R called the "geometrical mean distance,"

(a) (b)

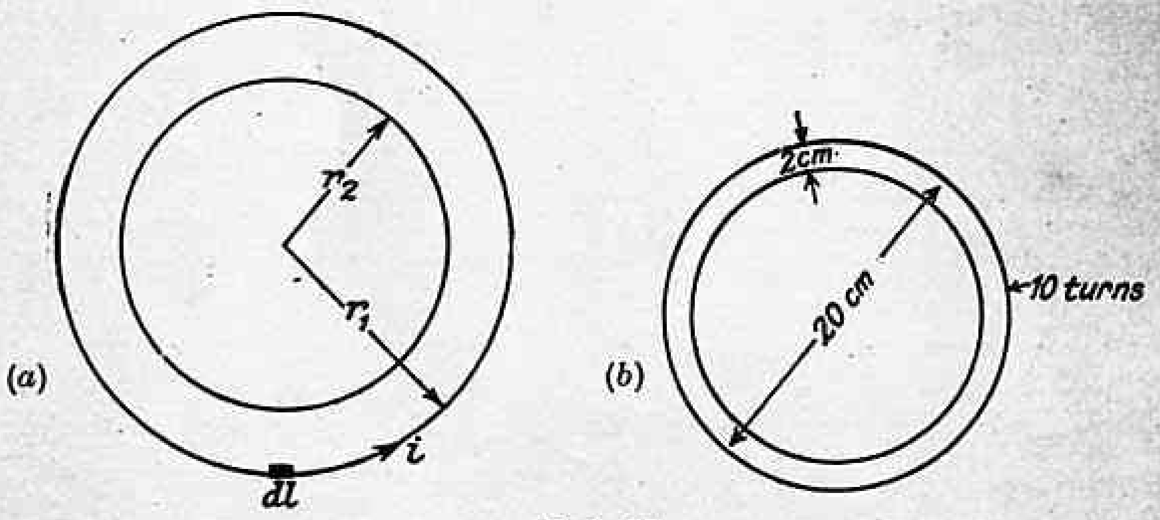

Fig. 97

first introduced by Maxwell, is used instead of $r_1 - r_2$. The mutual inductance is then given by

$$M = \frac{4\pi N_1 N_2 r_1}{10^9} \left\{ \log_e \frac{8r_1}{R} - 2 \right\} \text{ henries} \quad . \quad . \quad (121)$$

"*Geometrical Mean Distance*" may be defined as follows. Consider a point P external to a circuit. Let d_1, d_2, d_3, etc., be distances from P to various points on the circuit. Then, if an infinite number of these distances be taken, the "geometrical mean distance" R is given by

$$R = \sqrt[n]{d_1 d_2 d_3} \text{ etc., where } n_{\rightarrow\infty}$$

or

$$\log_e R = \text{limit}_{n\rightarrow\infty} \frac{1}{n} \Sigma \log_e d$$

The factor R is used in many of the formulae for the calculation of both mutual- and self-inductance. In the case of self-inductance, R is the G.M.D. of the circuit from itself, or, in the case of a multi-turn coil, of the turns from each other.

To find the self-inductance of the coil of N_1 turns, using Equation (121), with the correct value of R we have

$$L = \frac{4\pi N_1^2}{10^9} r_1 \left\{ \log_e \frac{8r_1}{0{\cdot}7788\rho} - 2 \right\} \text{ henries} \quad . \quad (122)$$

The G.M.D. of a circular area from itself is 0·7788 ρ where, in this case, ρ is the radius of the cross-section of the coil (assumed circular).

This expression, if compared with the expression given previously (Equation (118)) for the self-inductance of a similarly-shaped coil, will be found to give the same result in any particular case, provided ρ is small compared with r.

Example. Calculate the self-inductance of a coil of mean diameter 20 cm., having 10 turns, whose cross-section *is circular* and of radius 1 cm. (Fig. 97(*b*)). Then, using Equation (122), we have

(i) $$L = \frac{4\pi \times 100 \times 10}{10^9} \left[\log_e \frac{80}{0{\cdot}7788 \times 1} - 2 \right]$$

$$= 0{\cdot}000033080 \text{ henries}$$

(ii) Using Equation (118),

$$L = \frac{2\pi \times 100 \times 20}{10^9} \left[\left(1 + \frac{4}{8 \times 400}\right) \log_e \frac{8 \times 20}{2} + \frac{4}{24 \times 400} - 1{\cdot}75 \right] \text{ henries}$$

$$= \frac{4000\pi}{10^9} \left[\frac{801}{800} \times 4{\cdot}3828 + \frac{1}{2400} - \frac{7}{4} \right]$$

$$= 0{\cdot}000033150 \text{ henries}$$

Table V gives some of the more important geometrical mean distances.

Some exact expressions for the geometrical mean distance in several cases are given by Butterworth (*Dictionary of Applied Physics*, Vol. II, p. 391). For the calculation of geometrical mean distances, see Refs. (3), (4), and (12) at the end of the chapter.

If two circles are coaxial, but not concentric, and if the difference $r_1 - r_2$ between their radii is not small compared with their radii then the formula (Ref. (11)) is

$$M = \frac{4\pi\sqrt{r_1 r_2}}{10^9} \left\{ \log_e \frac{8\sqrt{r_1 r_2}}{D_1} \left[1 + \frac{3}{16}\alpha - \frac{15}{1024}\alpha^2 + \frac{35}{128^2}\alpha^3 \ldots \right] - \left[2 + \frac{1}{16}\alpha - \frac{31}{2048}\alpha^2 + \frac{247}{6(128)^2}\alpha^3 \ldots \right] \right\} \text{ henries} \quad . \quad (123)$$

where $\alpha = D_1 \sqrt{\frac{1}{r_1 r_2}}$

This equation may be written

$$M = M_0 \sqrt{r_1 r_2}$$

where M_0 equals $\frac{4\pi}{10^9}$ multiplied by the expression in brackets.

Nottage (Ref. (5)) gives a table of values of M_0 for different values of $\frac{D_1}{D_2}$ where D_1 and D_2 are the least and greatest distances between the circles (see Fig. 98). M_0 varies from zero, when $\frac{D_1}{D_2} = 1$, to 50·16, when $\frac{D_1}{D_2} = 0{\cdot}01$.

If the circles have approximately equal radii, and the distance between them is small compared with their radius

$$M = \frac{4\pi r_2}{10^9} \left(\log_e \frac{8r_2}{d} - 2 \right) \text{ henries} \quad . \quad . \quad . \quad . \quad (124)$$

where d is the distance (in centimetres) between the circles.

5. Mutual Inductance Between Two Coaxial Circular Coils of Rectangular Cross-section of Winding. From the previous paragraph an approximate formula can be derived for the mutual inductance between two coaxial circular coils of rectangular cross-section, i.e.

$$M = \frac{N_1 N_2 M_o}{10^9} \text{ henries} \quad . \quad . \quad . \quad (125)$$

where M_o is the mutual inductance between the two central turns of the two coils and can be obtained from Equation (123). N_1 and N_2 are the numbers of turns on the two coils.

TABLE VIII

GEOMETRICAL MEAN DISTANCES

Shape of Circuit	Geometrical Mean Distance (R)	Interpretation of Symbols Used
Line from itself .	$R = 0{\cdot}22313l$	l = length of line
Rectangular area from itself . .	$R = 0{\cdot}2235\,(a + b)$ (approx. expression)	a and b = sides of rectangle
Circular area from itself . . .	$R = 0{\cdot}7788r$	r = radius of circle
Annular ring from itself . . .	$\log_e R = \log_e r_1 - \log_e \frac{m}{(m^2-1)^2} + \frac{(3-m^2)}{4(m^2-1)}$	r_1 = external radius r_2 = internal radius $m = \frac{r_1}{r_2}$
Ellipse from itself .	$\log_e R = \log_e \frac{a+b}{2} - 0{\cdot}25$	a and b = semi-axes of ellipse
Two parallel straight lines . . .	$\log_e R = \frac{D^2}{l^2} \log_e D + \frac{1}{2}\left(1 - \frac{D^2}{l^2}\right) \log_e (D^2 + l^2) + 2\frac{D}{l}\tan^{-1}\frac{l}{D} - \frac{3}{2}$	l = length of lines D = distance between lines

The accuracy of this formula is of the order of 1 per cent in most practical cases.

Rayleigh's Formula. This is a more exact formula than the above, since it takes into account the dimensions of the cross-sections of the coil windings to a greater degree.

Referring to Fig. 99, let the mutual-inductance between a circle of radius r_1 with centre X, passing through point a_1, and a circle of radius r_2, centre Y, passing through o_2, be given by Mo_2a_1. There will be, in all, eight such mutual-inductances—four referred to coil 2 and four referred to coil 1. a_1, b_1, c_1, and d_1 are the mid-points of the sides of the section of coil 1 of which section o_1 is the centre point. The points a_2, b_2, c_2, d_2, and o_2 are similarly situated on the section of coil 2.

Then, by Rayleigh's formula, the mutual-inductance between the coils is given by

$$M = \frac{1}{6}\Big(Mo_1a_2 + Mo_1b_2 + Mo_1c_2 + Mo_1d_2 + Mo_2a_1 + Mo_2b_1 + Mo_2c_1 + Mo_2d_1 - 2M_o\Big) \quad . \quad . \quad . \quad (126)$$

where M_o is the mutual-inductance between the two central circles

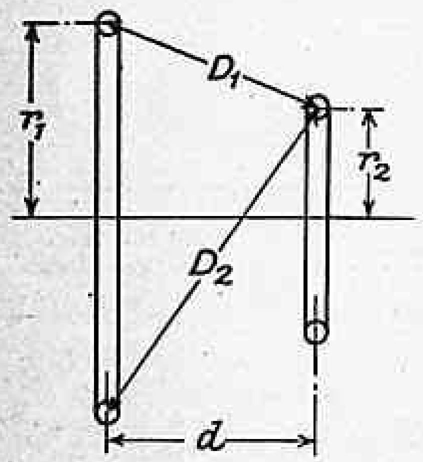

FIG. 98

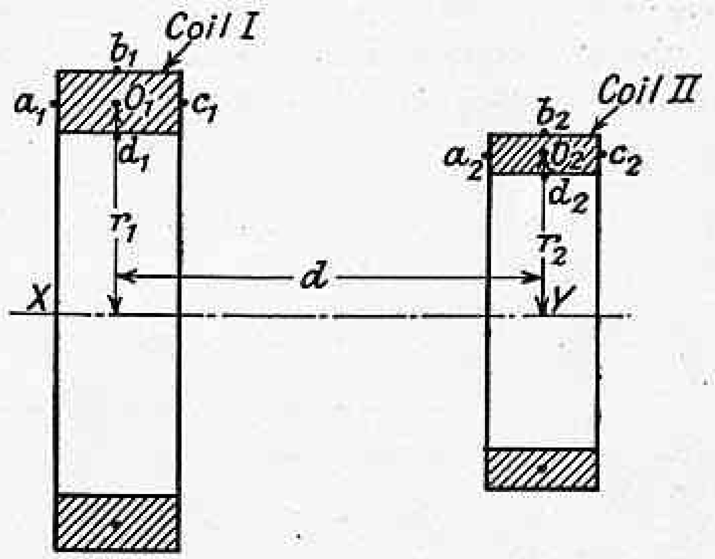

FIG. 99. MUTUAL-INDUCTANCE BETWEEN COAXIAL COILS

of the coils (through points o_1 and o_2). The mutual-inductances Mo_1a_2, etc., can be calculated as indicated in the previous paragraph.*

If instead of one of the coils being, as above, external to the other and displaced axially from it, one of the coils is inside the other at its centre, the coils being still coaxial, the mutual-inductance can be calculated as below. This case refers particularly to the mutual-inductance used in ballistic galvanometer work for calibration purposes, where a small coil is fixed inside a long circular solenoid as in Fig. 100.

Let l = length of long solenoid in centimetres
R = radius of long solenoid in centimetres
r = radius of internal short solenoid
N_1 = No. of turns on outer solenoid
N_2 = No. of turns on inner solenoid

If a current of I amp. flows in the outer solenoid, the magnetic field strength at its centre is given by

$$H = \frac{4\pi}{10}\,\frac{N_1 I}{l} \cos\theta$$

* Other methods of calculation of the mutual-inductance between two such coils, due to Lyle, and Nottage are given by the latter (Ref. (5)).