A New Approach to Digital PID Controller Design

L. H. Keel, J. I. Rego, and S. P. Bhattacharyya

***Abstract*—In this note, we present a new approach to the problem of designing a digital proportional-integral-derivative (PID) controller for a given but arbitrary linear time invariant plant. By using the Tchebyshev representation of a discrete-time transfer function and some new results on root counting with respect to the unit circle, we show how the digital PID stabilizing gains can be determined by solving sets of linear inequalities in two unknowns for a fixed value of the third parameter. By sweeping or gridding over this parameter, the entire set of stabilizing gains can be recovered. The precise admissible range of this parameter can be predetermined. This solution is attractive because it answers the question of whether there exists a stabilizing solution or not and in case stabilization is possible the entire set of gains is determined constructively. Using this characterization of the stabilizing set we present solutions to two design problems: 1) maximally deadbeat design, where we determine for the given plant, the smallest circle within the unit circle wherein the closed loop system characteristic roots may be placed by PID control and 2) maximal delay tolerance, where we determine, for the given plant the maximal-loop delay that can be tolerated under PID control. In each case, the set of controllers attaining the specifications is calculated. Illustrative examples are included.**

***Index Terms*—Deadbeat control, digital PID controller, stability, Tchebyshev representation.**

I. Introduction

There is renewed interest in proportional-integral-derivative (PID) controllers (see [1], [2]) because of two reasons. First, they are extensively used in applications in all industries (see [3], [4, Ch. 6]). Second, despite the existence of some results [5], [6] modern optimal control methods are not suitable to deal with fixed structure and fixed order controllers (see [7, p. 3]). Thus, there is much that remains to be done to modernize PID design methods.

Here, we develop some new results on discrete time PID controllers. First, the complex plane image of a real polynomial or rational function over a circle of radius ρ centered at the origin, is determined and expressed in terms of Tchebyshev polynomials [8]. In [9], Tchebyshev polynomials are used in robust control problems related to discrete time systems. They have also been used to develop a discrete time version of Foster's theorem and subsequently used to give necessary and sufficient conditions for Schur stability [10], [11]. In this note, a formula is first developed for root counting with respect to circular regions in terms of this Tchebyshev representation. This formula which differs from the root counting formulas given in[12], [13] is an extension of an initial result presented in [14], and constitutes the generalization of Hermite Bieler type results for Schur stability. Using these results, we show how the PID controller can be reparametrized so that the stabilizing set is obtained as the solution of sets of linear inequalities in two variables for a fixed value of the third variable. The admissible range of this third variable can be exactly determined. By sweeping or gridding over the third variable, the complete stabilizing set can be determined constructively. The solution shows that the stabilizing set for any plant, when it is nonempty, consists of unions of convex polygons in the space of the PID gains.

The aforementioned solution technique is extended to solve two previously unsolved design problems. The first problem is related to deadbeat control wherein one places all closed loop characteristic roots at the origin so that the transients are zeroed out in a finite number of steps. In general, deadbeat control is not possible using PIDs and a reasonable goal is to place the closed loop characteristic roots as close to the origin as possible so that the transient error decays quickly. Such designs have been advocated in the literature on sampled data control systems (see specifically [15, p. 292]). We show how the stabilization solution obtained by us can be exploited to give a constructive determination of such "maximally" deadbeat designs. The second problem involves the determination of the maximum delay in the loop that a given plant under PID control can be made to tolerate. We show how our solution can also be extended to determine this maximum delay for a given plant.

II. Tchebyshev Representation and Root Clustering

The stabilization results to be developed later in the note require us to determine the complex plane image of polynomials and rational functions on a circle of radius ρ centered at the origin.

A. Tchebyshev Representation of Real Polynomials and Rational Functions

Let us consider a polynomial $P(z) = a_n z^n + \cdots + a_0$ with real coefficients. The image of $P(z)$ evaluated on the upper half of the circle $\mathcal{C}_\rho$ of radius ρ, centered at the origin is

$$\left\{P(z) \;:\; z = \rho e^{j\theta}, \qquad 0 \le \theta \le \pi\right\}. \tag{1}$$

It is well known ([8, p. 71]) that, with $u = -\cos\theta$,

$$P(\rho e^{j\theta}) = R(u,\rho) + j\sqrt{1-u^2}\,T(u,\rho) =: P_c(u,\rho) \tag{2}$$

where

$$R(u,\rho) = a_n c_n(u,\rho) + a_{n-1} c_{n-1}(u,\rho) + \cdots + a_1 c_1(u,\rho) + a_0$$
$$T(u,\rho) = a_n s_n(u,\rho) + a_{n-1} s_{n-1}(u,\rho) + \cdots + a_1 s_1(u,\rho)$$

and

$$c_k(u,\rho) = \rho^k c_k(u),\; s_k(u,\rho) = \rho^k s_k(u), \qquad k = 0,1,2\ldots$$

and $c_k(u)$ and $s_k(u)$ are real polynomials in u satisfying the recursive relations

$$s_k(u) = -\frac{c_k'(u)}{k}, \qquad k = 1,2,\ldots \tag{3}$$

$$c_{k+1}(u) = -u c_k(u) - (1-u^2) s_k(u), \qquad k = 1,2,\ldots \tag{4}$$

and are known as the Tchebyshev polynomials of the first and second kind, respectively. The complex plane image of $P(z)$ as z traverses the upper half of the circle $\mathcal{C}_\rho$ can be obtained by evaluating $P_c(u,\rho)$ as u runs from -1 to $+1$.

Now, let $Q(z)$ be a ratio of two real polynomials $P_1(z)$ and $P_2(z)$. We compute the image of $Q(z)$ on $\mathcal{C}_\rho$ and write it as the corresponding Tchebyshev representation $Q_c(u,\rho)$ as follows. Let

$$P_i(z)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}} = R_i(u,\rho) + j\sqrt{1-u^2}\,T_i(u,\rho) \tag{5}$$

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L. H. Keel is with the Center of Excellence in Information Systems, Tennessee State University, Nashville, TN 37203 USA (e-mail: keel@gauss.tsuniv.edu).

J. I. Rego is with the Department of Electrical Engineering, Federal University of Rio Grande Do Norte, Natal 59078-370, Brazil.

S. P. Bhattacharyya is with the Department of Electrical Engineering, Texas A&M University, College Station, TX 77843 USA (e-mail: bhatt@eesun1.tamu.edu).

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for $i = 1, 2$. Then, it is easy to show that

$$Q(\rho e^{j\theta}) = \frac{\overbrace{\left(R_1(u,\rho)R_2(u,\rho)+(1-u^2)T_1(u,\rho)T_2(u,\rho)\right)}^{R(u,\rho)}}{R_2^2(u,\rho)+(1-u^2)T_2^2(u,\rho)} + j\frac{\sqrt{1-u^2}\,\overbrace{(T_1(u,\rho)R_2(u,\rho)-R_1(u,\rho)T_2(u,\rho))}^{T(u,\rho)}}{R_2^2(u,\rho)+(1-u^2)T_2^2(u,\rho)} =: Q_c(u,\rho). \tag{6}$$

This representation will be needed in a later section on the solution of the maximally deadbeat problem.

B. Interlacing Conditions for Root Clustering and Schur Stability

The formulas of the last section can be used to derive conditions for root clustering in circular regions, that is for the roots to lie strictly within a circle of radius ρ. For Schur stability we simply take $\rho = 1$. As before let $P(z)$ be a real polynomial of degree n and

$$P(\rho e^{j\theta}) = \bar{R}(\theta,\rho) + j\bar{T}(\theta,\rho) = R(u,\rho) + j\sqrt{1-u^2}T(u,\rho) \tag{7}$$

where $u = -\cos\theta$, and $R(u,\rho)$ and $T(u,\rho)$ are real polynomials of degree n and $n-1$, respectively, in u, for fixed ρ.

Theorem 1: $P(z)$ has all its zeros strictly within $\mathcal{C}_\rho$ if and only if

1) $R(u,\rho)$ has n real distinct zeros r_i, $i = 1, 2, \cdots, n$ in (−1,1);
2) $T(u,\rho)$ has $n-1$ real distinct zeros t_j, $j = 1, 2, \cdots, n-1$ in (−1,1);
3) the zeros r_i and t_j interlace

$$-1 < r_1 < t_1 < r_2 < t_2 < \cdots < t_{n-1} < r_n < +1.$$

Proof: Let $t_j = -\cos\alpha_j$, $\alpha_j \in (0,\pi)$, $j = 1, 2, \ldots, n-1$ or

$$\alpha_j = -\cos^{-1} t_j, \qquad j = 1, 2, \ldots, n-1 \quad \alpha_0 = 0, \alpha_n = \pi$$

and let $\beta_i = -\cos^{-1} r_i$, $i = 1, 2, \ldots, n$, $\beta_i \in (0,\pi)$.

Then, $(\alpha_0, \alpha_1, \cdots, \alpha_n)$ are the $n+1$ zeros of $\bar{T}(\theta,\rho) = 0$ and $(\beta_1, \beta_2, \cdots, \beta_{n-1})$ are the n zeros of $\bar{R}(\theta,\rho) = 0$. The condition (c) means that α_i and β_j satisfy:

$$0 = \alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n = \pi. \tag{8}$$

Conditions 1)–3) imply that the plot of $P(\rho e^{j\theta})$ for $\theta \in [0,\pi]$ turns counterclockwise through exactly $2n$ quadrants and this condition is equivalent to $P(z)$ having n zeros inside the circle $\mathcal{C}_\rho$. ▽ ▽ ▽

III. Root Counting Formulas

A. Phase Unwrapping and Root Distribution

Let $\phi_P(\theta) := \mathrm{ArgP}(\rho e^{j\theta})$ denote the *phase* of $P(z)$ evaluated at $z = \rho e^{j\theta}$ and let $\Delta_{\theta_1}^{\theta_2}[\phi_P(\theta)]$ denote the net change in or *unwrapped phase* of $P(\rho e^{j\theta})$ as θ increases from θ_1 to θ_2. Similarly notation applies to the rational function $Q(z)$ with Tchebyshev representation $Q_C(u,\rho)$: let $\phi_{Q_C}(u) = \mathrm{ArgQ_C}(u,\rho)$ denote the phase of $Q_C(u,\rho)$ and $\Delta_{u_1}^{u_2}[\phi_{Q_c}(u)]$ the net change in or unwrapped phase of $Q_C(u,\rho)$ as u increases from u_1 to u_2.

Lemma 1: Let $P(z)$ have i roots in the interior of the circle $\mathcal{C}_\rho$ and no roots on the circle. Then

$$\Delta_0^\pi\left[\phi_P(\theta)\right] = \pi i.$$

Proof: From geometric considerations it is easily seen that each interior root contributes 2π to $\Delta_0^{2\pi}[\phi_P(\theta)]$ and, therefore, because of the symmetry of roots about the real axis the interior roots contribute $i\pi$ to $\Delta_0^\pi[\phi_P(\theta)]$. ▽ ▽ ▽

We state the corresponding result for a rational function. The proof is similar to the previous lemma and is omitted.

Lemma 2: Let $Q(z) = (P_1(z))/(P_2(z))$ where the real polynomials $P_1(z)$ and $P_2(z)$ have i_1 and i_2 roots, respectively, in the interior of the circle $\mathcal{C}_\rho$ and no roots on the circle. Then

$$\Delta_0^\pi\left[\phi_Q(\theta)\right] = \pi(i_1 - i_2) = \Delta_{-1}^{+1}\left[\phi_{Q_C}(u)\right].$$

B. Root Counting and Tchebyshev Representation

Let us begin with a real polynomial $P(z)$ and its Tchebyshev representation $P_C(u,\rho) = R(u,\rho) + \sqrt{1-u^2}T(u,\rho)$ as developed before. Henceforth, let 31 $t_1, \ldots, t_k$ denote the real distinct zeros of $T(u,\rho)$ of odd multiplicity, for $u \in (-1,1)$, ordered as follows:

$$-1 < t_1 < t_2 < \cdots < t_k < +1.$$

Suppose also that $T(u,\rho)$ has p zeros at $u = -1$ and let $f^i(x_0)$ denote the ith derivative to $f(x)$ evaluated at $x = x_0$. Let us also define

$$\mathrm{Sgn}[x] = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

We begin with the following.

Theorem 2: Let $P(z)$ be a real polynomial with no roots on the circle $\mathcal{C}_\rho$ and suppose that $T(u,\rho)$ has p zeros at $u = -1$. Then, the number of roots i of $P(z)$ in the interior of the circle $\mathcal{C}_\rho$ is given by

$$i = \frac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]\Big(\mathrm{Sgn}\left[R(-1,\rho)\right] + 2\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}\left[R(t_j,\rho)\right] + (-1)^{k+1}\mathrm{Sgn}\left[R(+1,\rho)\right]\Big). \tag{9}$$

Proof: Recall that $P(\rho e^{j\theta}) = \bar{R}(\theta,\rho) + j\bar{T}(\theta,\rho)$ and define θ_i, $i = 1, \cdots, k$ through $t_i = -\cos\theta_i$, for $\theta_i \in [0,\pi]$. Let $\theta_0 := 0$, $t_0 := -1$ and $\theta_{k+1} := \pi$, and note that the θ_i, $i = 0, 1 \ldots, k+1$ are zeros of $\bar{T}(\theta,\rho)$. The proof depends on the following elementary and easily verified facts which are first stated. (In the following, θ_i^+ denotes the point immediately to the right of θ_i).

(a) $\Delta_0^\pi\left[\phi(\theta)\right] = \pi i$

(b) $\Delta_0^\pi\left[\phi(\theta)\right] = \Delta_0^{\theta_1}\left[\phi(\theta)\right] + \Delta_{\theta_1}^{\theta_2}\left[\phi(\theta)\right] + \cdots + \Delta_{\theta_k}^{\pi}\left[\phi(\theta)\right]$

(c) $\Delta_{\theta_i}^{\theta_{i+1}}\left[\phi(\theta)\right] = \dfrac{\pi}{2}\mathrm{Sgn}\left[\bar{T}\left(\theta_i^+,\rho\right)\right]\left(\mathrm{Sgn}\left[\bar{R}(\theta_i,\rho)\right) - \mathrm{Sgn}\left[\bar{R}(\theta_{i+1},\rho)\right]\right),\ i = 0, 1, \ldots, k$

(d) $\mathrm{Sgn}\left[\bar{T}\left(\theta_i^+,\rho\right)\right] = -\mathrm{Sgn}\left[\bar{T}\left(\theta_{i+1}^+,\rho\right)\right], \qquad i = 0, 1, \cdots, k$

(e) $\mathrm{Sgn}\left[\bar{T}(0^+,\rho)\right] = \mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]$

(f) $\mathrm{Sgn}\left[\bar{R}(\theta_i,\rho)\right] = \mathrm{Sgn}\left[R(t_i,\rho)\right], \qquad i = 0, 1, \cdots, k.$

Using (a)–(f), we have

$$\begin{aligned}
\pi i &= \Delta_0^\pi\left[\phi(\theta)\right] \\
&= \Delta_0^{\theta_1}\left[\phi(\theta)\right] + \cdots + \Delta_{\theta_k}^{\pi}\left[\phi(\theta)\right] \quad \text{by (a) and (b)} \\
&= \frac{\pi}{2}\Big\{\mathrm{Sgn}\left[\bar{T}(0^+,\rho)\right]\left(\mathrm{Sgn}\left[\bar{R}(0,\rho)\right] - \mathrm{Sgn}\left[\bar{R}(\theta_1,\rho)\right]\right) \\
&\quad + \cdots + \mathrm{Sgn}\left[\bar{T}(\theta_k^+,\rho)\right]\Big(\mathrm{Sgn}\left[\bar{R}(\theta_k,\rho)\right] \\
&\quad - \mathrm{Sgn}\left[\bar{R}(\pi,\rho)\right]\Big)\Big\} \quad \text{by (c)} \\
&= \frac{\pi}{2}\mathrm{Sgn}\left[\bar{T}\left(0^+,\rho\right)\right]\Big\{\left(\mathrm{Sgn}\left[\bar{R}(0,\rho)\right] - \mathrm{Sgn}\left[\bar{R}\left(\theta_1,\rho\right)\right]\right) \\
&\quad - \left(\mathrm{Sgn}\left[\bar{R}\left(\theta_1,\rho\right)\right] - \mathrm{Sgn}\left[\bar{R}\left(\theta_2,\rho\right)\right]\right) + \cdots + (-1)^k \\
&\quad \times \left(\mathrm{Sgn}\left[\bar{R}\left(\theta_k,\rho\right)\right] - \mathrm{Sgn}\left[\bar{R}(\pi,\rho)\right]\right)\Big\} \quad \text{by (d)} \\
&= \frac{\pi}{2}\mathrm{Sgn}[T^{(p)}(-1,\rho)]\Big\{\mathrm{Sgn}\left[\bar{R}(0,\rho)\right] - 2\mathrm{Sgn}\left[\bar{R}(\theta_1,\rho)\right] \\
&\quad + 2\mathrm{Sgn}\left[\bar{R}(\theta_2,\rho)\right] + \cdots + (-1)^k\mathrm{Sgn}\left[\bar{R}(\theta_k,\rho)\right] \\
&\quad + (-1)^{k+1}\mathrm{Sgn}\left[\bar{R}(\pi,\rho)\right]\Big\} \quad \text{by (e)}
\end{aligned}$$

$$= \frac{\pi}{2}\mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]\{\mathrm{Sgn}[R(-1,\rho)] - 2\mathrm{Sgn}[R(t_1,\rho)] + 2\mathrm{Sgn}[R(t_2,\rho)] + \cdots + (-1)^k 2\mathrm{Sgn}[R(t_k,\rho)] + (-1)^{k+1}\mathrm{Sgn}[R(+1,\rho)]\Big\} \quad \text{by (f)}$$

from which the result follows. ▽ ▽ ▽

The previously derived result can now be extended to the case of rational functions. Let $Q(z) = (P_1(z))/(P_2(z))$ where $P_i(z)$, $i = 1, 2$ are real rational functions. Let $R_i(u,\rho) + j\sqrt{1-u^2}T_i(u,\rho)$, $i = 1, 2$ denote the Tchebyshev representations of $P_i(z)$, $i = 1, 2$ and $Q_C(u,\rho)$ denote the Tchebyshev representation of $Q(z)$ on the circle $\mathcal{C}_\rho$. Let $R(u,\rho)$, $T(u,\rho)$ be defined by

$$R(u,\rho) = R_1(u,\rho)R_2(u,\rho) + (1-u^2)T_1(u,\rho)T_2(u,\rho)$$
$$T(u,\rho) = T_1(u,\rho)R_2(u,\rho) - R_1(u,\rho)T(u,\rho).$$

Suppose that $T(u,\rho)$ has p zeros at $u = -1$ and let $t_1 \cdots t_k$ denote the real distinct zeros of $T(u,\rho)$ of odd multiplicity ordered as follows:

$$-1 < t_1 < t_2 < \cdots < t_k < +1.$$

Theorem 3: Let $Q(z) = (P_1(z))/(P_2(z))$ where $P_i(z)$, $i = 1, 2$ are real polynomials with i_1 and i_2 zeros, respectively, inside the circle $\mathcal{C}_\rho$ and no zeros on it. Then

$$i_1 - i_2 = \frac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]\left(\mathrm{Sgn}[R(-1,\rho)] + 2\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}[R(t_j,\rho)] + (-1)^{k+1}\mathrm{Sgn}[R(+1,\rho)]\right). \quad (10)$$

Proof: The proof is based on the representation of $Q_C(u,\rho)$ developed in (6). Since the denominator of (6) is strictly positive for $u \in [-1, +1]$, it follows that the phase unwrapping can be computed from the numerator. The rest of the proof is similar to the proof for the polynomial case and is left to the reader. ▽ ▽ ▽

IV. Parameter Separation and Stabilization with PID Controllers

Consider the control system in unity feedback configuration wherein the plant is a represented by its discrete time transfer function $G(z) = (N(z))/(D(z))$ with $N(z)$, $D(z)$ being polynomials with real coefficients and with degree $D(z) = n$ and degree $N(z) \le n$. The closed loop system is stable iff the characteristic polynomial, denoted by $\delta(z)$, is Schur stable. The general formula of a discrete PID controller, using backward differences to preserve causality, is

$$C(z) = K_P + K_I T \cdot \frac{z}{z-1} + \frac{K_D}{T}\cdot\frac{z-1}{z} = \frac{\left(K_P + K_I T + \frac{K_D}{T}\right)z^2 + \left(-K_P - \frac{2K_D}{T}\right)z + \frac{K_D}{T}}{z(z-1)}.$$

Therefore, we can use

$$C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z-1)} \quad (11)$$

where

$$K_P = -K_1 - 2K_0 \quad K_I = \frac{K_0 + K_1 + K_2}{T}$$
$$K_D = K_0 T. \quad (12)$$

The characteristic polynomial becomes

$$\delta(z) = z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right)N(z). \quad (13)$$

Multiplying the characteristic polynomial by $z^{-1}N(z^{-1})$, we have

$$z^{-1}\delta(z)N(z^{-1}) = (z-1)D(z)N(z^{-1}) + (K_2 z + K_1 + K_0 z^{-1})N(z)N(z^{-1}).$$

Using the Tchebyshev representations, we have

$$\begin{aligned} z^{-1}\delta(z)N(z^{-1}) &= -(u+1)P_1(u) - (1-u^2)P_2(u) \\ &\quad - [(K_0+K_2)u - K_1]P_3(u) \\ &\quad + j\sqrt{1-u^2}[-(u+1)P_2(u) \\ &\quad + P_1(u) + (K_2 - K_0)P_3(u)] \\ &= R(u, K_0, K_1, K_2) \\ &\quad + j\sqrt{1-u^2}T(u, K_0, K_2). \end{aligned}$$

Now, let $K_3 := K_2 - K_0$ and rewrite $R(u, K_0, K_1, K_2)$ and $T(u, K_0, K_2)$ as follows:

$$R(u, K_0, K_1, K_2) = -(u+1)P_1(u) - (1-u^2)P_2(u) - [(2K_2 - K_3)u - K_1]P_3(u) \quad (14)$$
$$T(u, K_3) = P_1(u) - (u+1)P_2(u) + K_3 P_3(u). \quad (15)$$

We observe the parameter separation previously achieved: K_3 appears only in the imaginary part and K_1, K_2, K_3 appear linearly in the real part. Thus, by applying root counting formulas to the rational function on the left, and imposing the stability requirement yields linear inequalities in the parameters for fixed K_3. The solution is completed by sweeping over the range of K_3 for which an adequate number of real roots t_k exist. We illustrate with an example.

Example 1:

$$G(z) = \frac{1}{z^2 - 0.25}$$

Then

$$R_D(u) = 2u^2 - 1.25, \ T_D(u) = -2u, \ R_N(u) = 1, \ T_N(u) = 0$$
$$P_1(u) = 2u^2 - 1.25, \ P_2(u) = -2u, \ P_3(u) = 1.$$

Recall (14). Since $G(z)$ is of order 2 and $C(z)$, the PID controller, is of order 2, the number of roots of $\delta(z)$ inside the unit circle is required to be 4 for stability. From Theorem 2

$$i_i - i_2 = \underbrace{(i_\delta + i_{N_r})}_{i_1} - \underbrace{(l+1)}_{i_2} \quad (16)$$

where i_δ and i_{N_r} are the numbers roots of $\delta(z)$ and the reverse polynomial of $N(z)$, respectively, and l is the order of $N(z)$. Since the required i_δ is 4, $i_{N_r} = 0$, and $l = 0$, $i_1 - i_2$ is required to be 3. To illustrate the example in detail, we first fix $K_3 = 1.3$. Then, the real roots of $T(u, K_3)$ in $(-1, 1)$ are -0.4736 and -0.0264. Furthermore, $\mathrm{Sgn}[T(-1)] = 1$, and from Theorem 2, $i_1 - i_2 = 3$ requires that

$$\frac{1}{2}\mathrm{Sgn}[T(-1)](\mathrm{Sgn}[R(-1)] - 2\mathrm{Sgn}[R(-0.4736)] + 2\mathrm{Sgn}[R(-0.0264)] - \mathrm{Sgn}[R(1)]) = 3.$$

In this example, we have $\mathrm{Sign}[R(t_j)]$, $j = 0, 1, 2, 3$, and each $\mathrm{Sign}[R(\cdot)]$ may assume the value $+1$ or -1 since 0 is excluded as we are testing for stability. This leads to $2^4 = 16$ possible strings which need to be tested. In general, it is easy to devise a sorting algorithm to

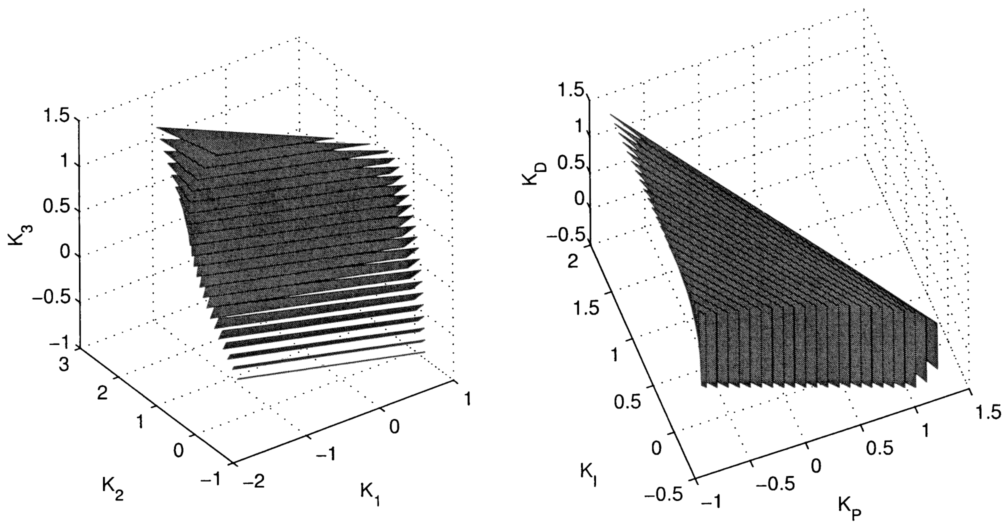

Fig. 1. Stability regions in (K_1, K_2, K_3), and (K_P, K_I, K_D) spaces.

pick out the feasible strings. Here, we have only one valid sequence satisfying the aforementioned equation, namely

Sgn $[R(-1)]$	Sgn $[R(-0.4736)]$	Sgn $[R(-0.0264)]$
1	-1	1

	Sgn $[R(1)]$	$2(i_1 - i_2)$
	-1	6.

From this valid sequence, we have the following set of linear inequalities:

$$
\begin{aligned}
-1.3+K_1+2K_2 &> 0 \quad -0.9286+K_1+0.9472 < 0 \\
1.1286+K_1+0.0528K_2 &> 0 \quad -0.2+K_1-2K_2 < 0.
\end{aligned}
$$

This set of inequalities characterize the stability region in (K_1, K_2) space for the fixed $K_3 = 1.3$. By repeating this procedure for the range of K_3, we obtain the the stability region shown in the left of Fig. 1. Consider the following relation:

$$
\begin{aligned}
\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} &= \begin{bmatrix} -2 & -1 & 0 \\ \frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\ T & 0 & 0 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -1 & 0 \\ \frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\ T & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}.
\end{aligned}
$$

Using this relation, we plot the stabilizing region in (K_P, K_I, K_D) space in the right of Fig. 1.

Remark 1: From (16) and Theorem 2, it is clear that a necessary condition for stabilization is that $T(u, K_3)$ must have two real zeros in (-1, 1). In this example, this specifies the admissible range of K_3 to be (-0.75, 1.5). In a like manner, the range of K_3 can always be predetermined from the requirement on the number of real roots of $T(u, K_3)$.

Remark 2: An alternative approach to determine the stabilizing set is via D-decomposition. In this approach, one sets $\delta(e^{j\theta}, K_P, K_I, K_D) = 0$ and determines the corresponding solution surfaces in the (K_P, K_I, K_D) space. These surfaces partition this space into disjoint open regions each with a fixed number of roots in the interior of the unit circle. The stabilizing regions will then have to be picked out by testing an arbitrary point from each region. On the other hand, our approach directly determines the stabilizing regions.

V. Maximally Deadbeat Control via PID Controllers

An important design technique in digital control is deadbeat control wherein one places all closed-loop poles at the origin. If this is used in conjunction with integral control the tracking error is zeroed out in a finite number of sampling steps. Deadbeat control requires in general that we be able to control all the poles of the system. However, such a pole placement design is in general not possible when a lower order controller is used. Thus, we are motivated to design a PID controller that places the closed-loop poles as close to the origin as possible. The transient response of such a system will decay out faster than any other design and therefore the fastest possible convergence of the error under PID control will be achieved.

The design scheme to be developed will attempt to place the closed loop poles in a circle of minimum radius ρ. Let $\mathcal{S}_\rho$ denote the set of PID controllers achieving such a closed loop root cluster. We show below how $\mathcal{S}_\rho$ can be computed for fixed ρ. The minimum value of ρ can be found by determining the value ρ^* for which $\mathcal{S}_{\rho^*} = \phi$ but $\mathcal{S}_\rho \neq \phi$, $\rho > \rho^*$.

Now, let us again consider the PID controller

$$
C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z-1)} \tag{17}
$$

and the characteristic polynomial

$$
\delta(z) = z(z-1)D(z) + (K_2 z^2 + K_1 z + K_0)N(z). \tag{18}
$$

Note that

$$
\begin{aligned}
D(z)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= R_D(u,\rho) + j\sqrt{1-u^2}\,T_D(u,\rho) \\
N(z)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= R_N(u,\rho) + j\sqrt{1-u^2}\,T_N(u,\rho)
\end{aligned}
$$

and

$$
\begin{aligned}
N(\rho^2 z^{-1})\big|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= N(z)\big|_{z=-\rho u - j\rho\sqrt{1-u^2}} \\
&= R_N(u,\rho) - j\sqrt{1-u^2}\,T_N(u,\rho).
\end{aligned}
$$

We now evaluate

$$
\rho^2 z^{-1} \delta(z) N(\rho^2 z^{-1}) = \rho^2 z^{-1}
$$

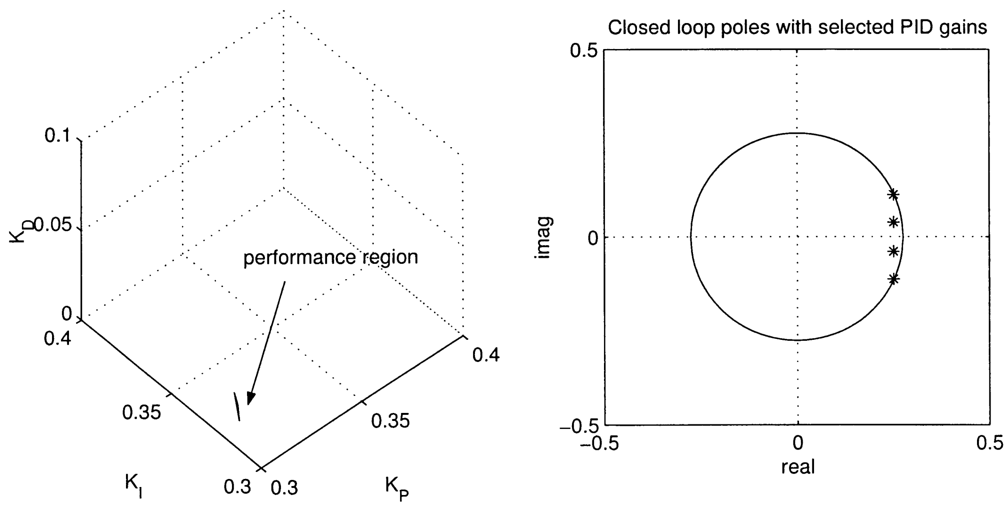

Fig. 2. Stability regions with $\rho = 0.275$ and closed-loop poles of the selected PID gains.

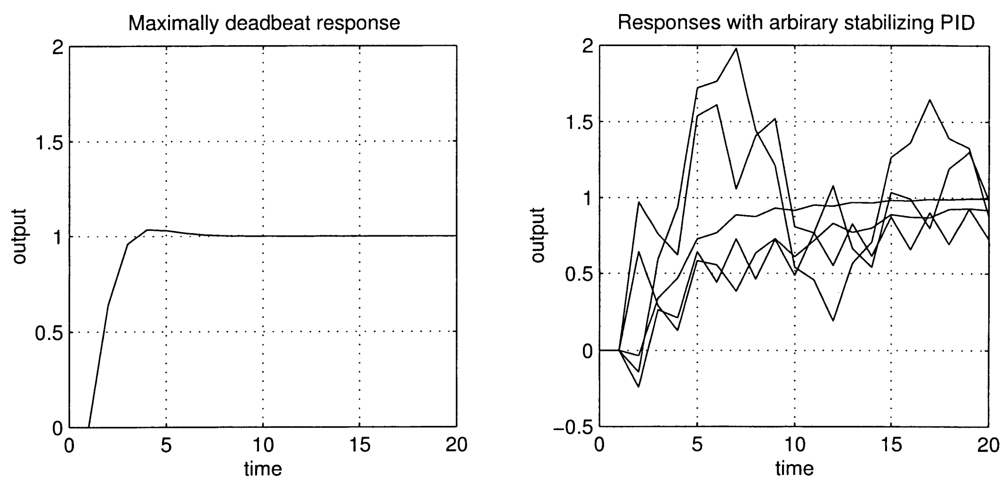

Fig. 3. Maximally deadbeat design and arbitrary stabilization.

$$\cdot \underbrace{\left[z(z-1)D(z) + (K_2 z^2 + K_1 z + K_0)N(z)\right]}_{\delta(z)} N(\rho^2 z^{-1})$$

over the circle $\mathcal{C}_\rho$

$$\begin{aligned} \rho^2 z^{-1}\delta(z)N(\rho^2 z^{-1})\Big|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= -\rho^2(\rho u + 1)P_1(u,\rho) \\ &\quad - \rho^3(1-u^2)P_2(u,\rho) - \left[(K_0 + K_2\rho^2)\right. \\ &\quad \left. \times \rho u - K_1\rho^2\right] P_3(u,\rho) \\ &\quad + j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u + 1)P_2(u,\rho)\right. \\ &\quad \left. + (K_2\rho^2 - K_0)\rho P_3(u,\rho)\right] \end{aligned}$$

where

$$\begin{aligned} P_1(u,\rho) &= R_D(u,\rho)R_N(u,\rho) + (1-u^2)T_D(u,\rho)T_N(u,\rho) \\ P_2(u,\rho) &= R_N(u,\rho)T_D(u,\rho) - T_N(u,\rho)R_D(u,\rho) \\ P_3(u,\rho) &= R_N^2(u,\rho) + (1-u^2)T_N^2(u,\rho). \end{aligned}$$

By letting

$$K_3 := K_2\rho^2 - K_0 \tag{19}$$

we have

$$\begin{aligned} \rho^2 z^{-1}\delta(z)N(\rho^2 z^{-1})\Big|_{z=-\rho u + j\rho\sqrt{1-u^2}} &= -\rho^2(\rho u + 1)P_1(u,\rho) \\ &\quad - \rho^3(1-u^2)P_2(u,\rho) - \left[(2K_2\rho^2 - K_3)\right. \\ &\quad \left. \times \rho u - K_1\rho^2\right] P_3(u,\rho) \\ &\quad + j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u + 1)P_2(u,\rho)\right. \\ &\quad \left. + K_3\rho P_3(u,\rho)\right]. \end{aligned}$$

To determine the set of controllers achieving root clustering inside a circle of radius ρ, we proceed as before: Fix K_3, use the root counting formulas of Section IV, develop linear inequalities in K_2, K_3 and sweep over the requisite range of K_3. This procedure is then performed as ρ decreases until the set of stabilizing PID parameters just disappears. The following example illustrates this scheme.

Example 2: We consider the same plant used in Example 1. Fig. 2 (left) shows the stabilizing set in the PID gain space at $\rho = 0.275$. For a smaller value of ρ, the stabilizing region in PID parameter space disappears. This means that there is no PID controller available to push all closed loop poles inside a circle of radius smaller than 0.275. From this we select a point inside the region that is: $K_0 = 0.0048$, $K_1 =$

-0.3195, $K_2 = 0.6390$, and $K_3 = 0.0435$. From the relationship in (19), we have

$$\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} = \begin{bmatrix} -1 & -2\rho^2 & 2 \\ \frac{1}{T} & \frac{\rho^2}{T}+\frac{1}{T} & -\frac{1}{T} \\ 0 & \rho^2 T & -T \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0.3099 \\ 0.3243 \\ 0.0048 \end{bmatrix}.$$

Fig. 2 (right) shows the closed-loop poles that lie inside the circle of radius $\rho = 0.275$. The roots are $0.2500 \pm j0.1118$ and $0.2500 \pm j0.0387$.

To illustrate further, we select several sets of stabilizing PID parameters from the set obtained in Example 1 (i.e., $\rho = 1$) and compare the step responses between them. Fig. 3 shows that the maximally deadbeat design produces nearly deadbeat response.

VI. Maximum Delay Tolerance Design

In some control systems an important design parameter is the delay tolerance of the loop, that is the maximum delay that can be inserted into the loop without destabilizing it. In digital control a delay of k sampling instants is represented by z^{-k}. We use this to determine the maximum delay that a control-loop under PID control can be designed to tolerate. This gives the limit of delay tolerance achievable for the given plant under PID control.

Let the plant be $G(z) = (N(z))/(D(z))$. We consider the problem of finding the maximum delay L^* such that the plant can be stabilized by a PID controller. In other words, finding the maximum values of L^* such that the stabilizing PID gain set for the plant

$$z^{-L}G(z) = \frac{N(z)}{z^L D(z)}, \qquad \text{for } L = 0, 1, \ldots, L^* \tag{20}$$

is not empty. Let $\mathcal{S}_i$ be the set of PID gains that stabilizes the plant $z^{-i}G(z)$. Then, it is clear that

$$\cap_{i=0}^{L}\mathcal{S}_i \text{ stabilizes } z^i G(z) \text{ for all } i = 0, 1, \ldots, L. \tag{21}$$

VII. Concluding Remarks

In this note, we have given a solution to the problem of stabilization of a digital control system using PID controllers. The solution is complete in the sense that a constructive yes or no answer to whether stabilization is possible, is given and in case it is possible the entire set is determined by solving sets of linear inequalies in two variables obtained by gridding over the third variable. This approach is akin to the geometric approach to synthesis and design advocated in [16]. These solution sets open up the possibility of improved and optimal design using PID controllers. The questions of loop shaping, time domain response shaping, and robust designs are important candidates for research.

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Stability Analysis of Swarms

Veysel Gazi and Kevin M. Passino

***Abstract*—In this note, we specify an "individual-based" continuous-time model for swarm aggregation in n-dimensional space and study its stability properties. We show that the individuals (autonomous agents or biological creatures) will form a cohesive swarm in a finite time. Moreover, we obtain an explicit bound on the swarm size, which depends only on the parameters of the swarm model.**

***Index Terms*—Biological systems, multiagent systems, stability analysis, swarms.**

I. Introduction

For a long time, it has been observed that certain living beings tend to perform swarming behavior. Examples of swarms include flocks of birds, schools of fish, herds of animals, and colonies of bacteria. It is known that such a cooperative behavior has certain advantages such

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V. Gazi was with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210 USA. He is now withthe Department of Electrical and Electronics Engineering, Atilim University, 06836 Ankara, Turkey.

K. M. Passino is with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210 USA (e-mail: passino@ee.eng.ohio-state.edu).

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