In modern terminology, he found the very simple rule, that a falling body increases its speed by equal increments in equal intervals of time, that is to say, its *acceleration* is constant.

1.2 Constant acceleration formulæ

Kinematics is the study of movement. The word 'kinematics', like the word 'cinema', comes from the Greek word *kinesis*, meaning motion. The most basic formula in kinematics is

$$\text{distance} = \text{speed} \times \text{time} \ , \tag{1.1}$$

which applies in the case of constant speed. From his investigations, Galileo developed further formulæ which apply in the case of constant acceleration. His concern was primarily with the constant acceleration produced by gravity, but the formulæ are valid in any situation, such as a rocket accelerating upwards or a car accelerating on a horizontal road, where acceleration is constant, at least for a time.

The first of these is

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The first of these is

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which gives the velocity v attained after accelerating from velocity u at a constant rate a over an interval of time t. While it is referred to as a formula, it is more than just a recipe for calculation. When a is the subject of the equation, we have

$$a = \frac{v - u}{t} \,, \tag{1.3}$$

which says acceleration is "change in velocity divided by the change in time". This is the very meaning of acceleration. No-one who understood this should have much difficulty in remembering *Formula (1.3)*.

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Galileo's second formula tells us the total displacement which has resulted by the end of the time interval. If this displacement is denoted by s, then

$$s = \frac{u + v}{2} t \;. \tag{1.4}$$

This also is a formula which has a natural meaning. On the right hand side $(u+v)/2$ is the average of the initial velocity u and the final velocity v. So *Formula (1.4)* is saying "displacement = average velocity × time". Again this is not difficult to remember. The term 'displacement' is used, rather than 'distance', to emphasize that the direction of the movement is significant. For example, when a stone is thrown 20 metres into the air and falls to earth again, the total *distance* travelled is 40 metres. But as far as displacement is concerned, the journey up cancels the journey down, so the *displacement* s is zero.

The two formulæ, $v = u + at$, and $s = \frac{1}{2}(u + v)t$, are between them the source of all that needs to be known about constant acceleration. There are other formulæ, and special ways to use them, but there is nothing actually new. Everything else is derived from these two. For example, there is a third formula which results from substituting v from *Formula (1.3)* into the right hand side of *Formula (1.4)*. Then

$$\begin{aligned} s &= \frac{u+v}{2} \times t \\ &= \frac{u+u+at}{2} \times t\,. \end{aligned}$$

Simplifying, we find

$$s = ut + \tfrac{1}{2}at^2\,. \tag{1.5}$$

Formula (1.5) re-expresses the content of *Formulæ (1.3)* and *(1.4)* in what for some circumstances is a more convenient form.

1.4 Velocity-time graphs

As its name implies, a velocity-time graph plots the velocity of a moving object against time. The graph gives us another way to look at the formulæ we have been using.

In the case of an object moving at constant speed, the graph is simply a horizontal straight line:

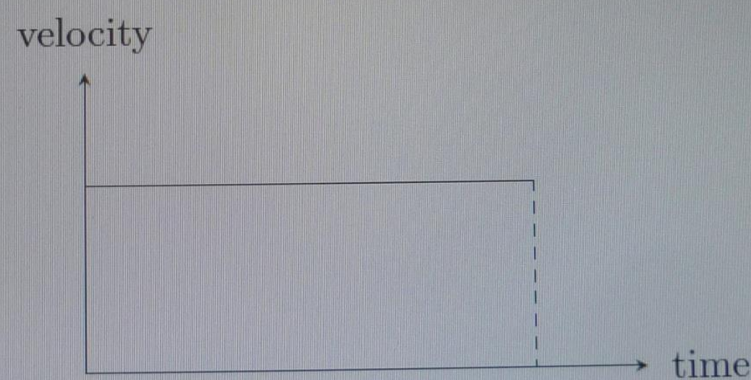

Figure 1.1: *A velocity-time graph for an object moving at constant speed.*

This is not a very exciting graph but it does illustrate a new idea. The area under the graph forms a rectangle whose height is the constant value of the speed and whose base is the time-interval under consideration. The area of the rectangle is

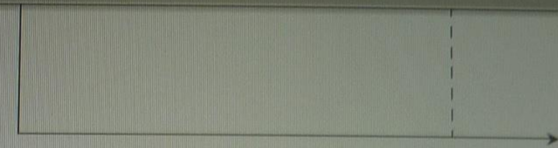

Figure 1.1: *A velocity-time graph for an object moving at constant speed.*

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$$\begin{aligned} \text{area} &= \text{height} \times \text{base} \\ &= \text{speed} \times \text{time} \\ &= \text{distance}\ . \end{aligned}$$

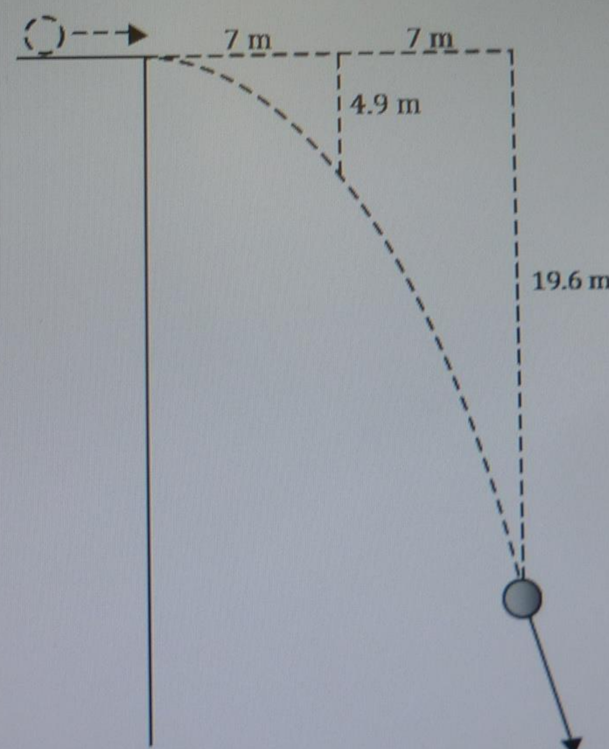
The area under the graph thus represents the distance travelled.

In a graph representing constant acceleration, velocity is represented by a plot of constant slope equal to the acceleration a, while the initial speed is u and the final speed is v:

the horizontal motion, nor is the horizontal motion altered by gravity acting vertically.

Example 2.1

A ball, travelling horizontally at 7 m/s, rolls over the edge of a cliff. Where will it be 2 seconds later?



motion, it will after one second be 7 metres clear of the cliff face. Considering the vertical motion, the distance fallen under gravity will be the same as if the ball had been dropped from rest, so the usual constant acceleration formula applies

$$s = ut + \tfrac{1}{2}at^2 ,$$

with u – the initial speed in the *downwards* direction – being zero. With $t = 1$ and, if we take the downwards direction as positive, $a = 9.8\,\mathrm{m/s}^2$, the downwards displacement is $s = 4.9\,\mathrm{m}$.

Repeating the calculation at $t = 2$, the ball is $14\,\mathrm{m}$ beyond the cliff face and has fallen through a vertical distance of $19.6\,\mathrm{m}$. These two separate pieces of information combine to fix the position of the ball.

formulated by Isaac Newton, born in 1642, the year Galileo died. He was still an unknown student in his early twenties when he made crucial discoveries in calculus, optics, and the theory of gravity. Much of this work was done in a Lincolnshire farmhouse, still there today just off the A1 trunk road, where he retreated from the plague which swept through towns and cities in 1665 – 1666.

Newton's laws of motion state the relationship between the forces and the movements which they produce.

1. The first law is:

   Every body continues in its state of rest or of uniform motion in a straight line unless compelled to change that state by forces impressed upon it.

   To illustrate this, Newton gives some examples. Projectiles would continue their motion, with the same speed in a straight line, if they were not retarded by the resistance of the air or impelled downwards by the force of gravity. Planets and comets meet with less resistance and so continue their motions for much longer times.

2. The second law is:

   The change in motion is proportional to the motive force, and in the same direction as the motive force.

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2. The second law is:

   The change in motion is proportional to the motive force, and in the same direction as the motive force.

   This law is usually expressed in the form of an equation,

$$F = ma \,, \tag{3.1}$$

where F is the motive force, m is the mass of the body to which the force is applied, and a is what Newton called the change in motion, or as we would now say, its acceleration. Because the law is about change in motion, rather than the motion itself, the effect of the motive force is superimposed on any pre-existing motion of the body.

3.3 Equilibrium

A body which is at rest, or travelling at constant speed in a straight line, is said to be in equilibrium. The word implies balance. Before Newton and Galileo, we might have presumed that Mr A sits at rest in his armchair because this happy state of quiet repose is simply the natural condition for a man in an armchair. But, with Newton, we now say that this state of rest is only possible because the upward reaction forces on Mr A from the floor and from the seat of the chair are together exactly equal to his weight. When Mr A sits, he compresses the cushion until the upwards reaction force is just right to maintain the balance.

Mr A in his armchair is an example of static equilibrium, where there is no movement. But the first and second laws show us that a perfect balance of forces is also consistent with motion at a constant speed.

Example 3.3

Mr B, mass 80 kg, stands in the corridor of a train to Glasgow travelling at 100 km/hr. What are the forces on Mr B?

— but not always — this is a useful simplification.

Example 3.4

Mr C, mass 60 kg, goes up in a lift of mass 340 kg, the tension in the cable being 4200 newtons. What is the upward reaction on Mr C's feet from the floor of the lift?

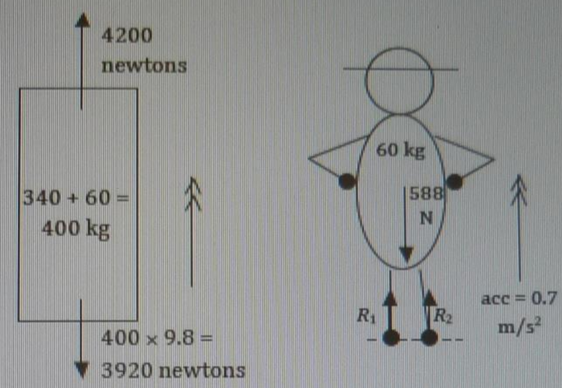


Figure 3.1: *The forces on the lift (left) and the forces on Mr C (right).*

First, apply $F = ma$ to Mr C plus the lift, considered as a single composite body of mass $340 + 60 = 400$ kg and weight $400g = 3920$ newtons.

$$\begin{aligned} F = \text{tension} - \text{weight} &= ma \\ \Rightarrow \quad 4200 - 3920 &= 400a\ , \end{aligned}$$

giving $a = 0.7\,\mathrm{m/s^2}$. Now apply $F = ma$ to Mr C, considered as a body on his own, subjected to his own weight of 588 N and the reaction forces R_1 and R_2 on his feet from the floor of the lift. Since Mr C's acceleration must be the $0.7\,\mathrm{m/s^2}$ just calculated,

$$R_1 + R_2 - 588 = 60 \times 0.7 = 42\ .$$

The total reaction force $R_1 + R_2 = R$ is therefore about 630 newtons. Since Mr C is considered as particle, we do not distinguish between the separate forces R_1 and R_2.

4.2 Terminal velocity

If an object is dropped from a sufficient height and gains speed as it falls, there comes a point when the drag force equals its weight. The ball is now in equilibrium, as defined in *Section 3.3*, and no further acceleration will occur. The speed v_t at which this happens is called the terminal velocity, and can be calculated from the equation

$$\text{Drag force} = D = \tfrac{1}{2}C_d A \rho v_t^2 = mg = \text{weight} . \qquad (4.2)$$

Example 4.2

Calculate the terminal velocity for the tennis ball of *Example 4.1*.

Substituting in the values,

$$0.5 \times 0.3 \times \pi(0.032)^2 \times 1.2 \times v_t^2 = 0.058 \times 9.8 ,$$

gives $v_t = 31\,\text{m/s}$.

Table 4.1 shows values of the terminal velocity for some other bodies of different sizes and weights.

Body	Radius, m	Mass, kg	Terminal velocity, m/s
Raindrop	0.001	4.2×10^{-6}	6.6
Table tennis ball	0.02	0.0027	8.4
Apple	0.03	0.10	34
Football	0.11	0.43	25
Iron cannonball	0.075	13.9	160
Skydiver		70	55

Table 4.1: *Terminal velocities for free fall through air.*

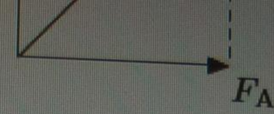


Figure 5.3: *The special case of perpendicular forces.*

5.2 Components of a force

The geometrical argument for calculating the resultant can be used in the reverse direction, to express an oblique force F as the sum of two perpendicular forces. These forces, in the x- and y-directions say, are called the x- and y-components of F.

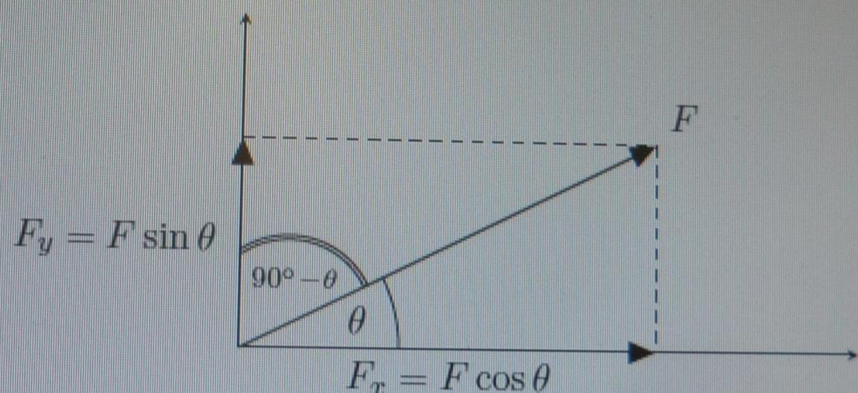
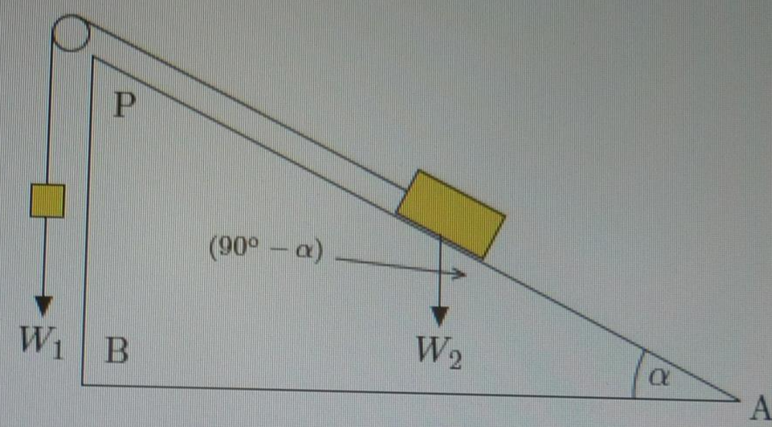


Figure 5.4: *The components of a force, F.*

From the diagram, F_x, the component of F along the x-axis, is $F_x = F\cos\theta$, where θ is the angle between F and the x-axis. Similarly, $F_y = F\sin\theta$. These properties of the force components follow exactly the same idea that we have already used for velocity components in our analysis of projectiles.

Example 5.3

Two weights W_1 and W_2 are connected by a string which passes over a smooth pulley. W_1 is suspended vertically below the pulley while W_2 rests on a smooth slope which is inclined at an angle α to the horizontal, where $\tan\alpha = 3/4$. How should W_1 and W_2 be related if the system is to rest in equilibrium?



Since the weights are at rest, this problem would be classed as an exercise in statics – as compared with *Examples 5.1* and *5.2* which are applications of dynamics. The only difference is that here the value of the acceleration happens to be zero. The arguments we use are exactly the same.

Suppose the tension in the string is T. Then, considering the equilibrium of W_1,

$$T = W_1 \ ,$$

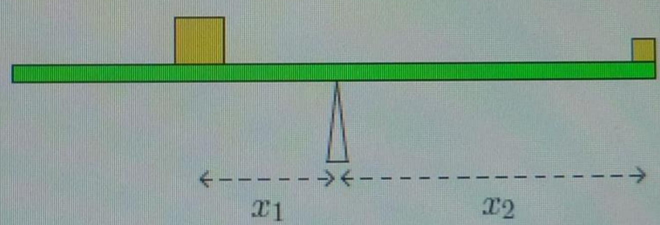
while resolving down the slope gives the condition for the equilibrium of W_2:

$$T = W_2 \cos(90° - \alpha) = W_2 \sin\alpha \ .$$

6.2 The lever

The law of the lever was formulated by Archimedes, generally reckoned, along with Newton and Gauss, as one of the three greatest mathematicians in history. Born in 287 BC, he lived in the town of Syracuse in Sicily, and was famous both as a mathematician and as an inventor of mechanical devices and engines of war. Of his work it was said, it is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. . . no amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you could have discovered it.

The law of the lever says that if two masses rest in equilibrium on a beam, or 'rigid rod', the product of the masses with their respective distances from the fulcrum are equal. In the diagram below, $m_1 \times x_1 = m_2 \times x_2$.



This Archimedes deduced from simple arguments of symmetry. Suppose, he said, that the masses are measured in some common unit. If the unit is m, suppose for the sake of illustration, $m_1 = 3m$ and $m_2 = 2m$. Then the total mass is $5m$, and the beam will be balanced if its length is divided into 5 equal parts with a unit mass placed at the centre of each.

Example 6.4

A ladder of length 6.5 m and weight 200 N rests against a wall with its base on rough ground 2.5 m away from the bottom of the wall. The wall is perfectly "smooth" so the reaction R from the wall on the ladder is exactly perpendicular to the wall. What is the minimum value of μ, the coefficient of friction between the ground and the ladder, for the ladder to remain in equilibrium in this position?

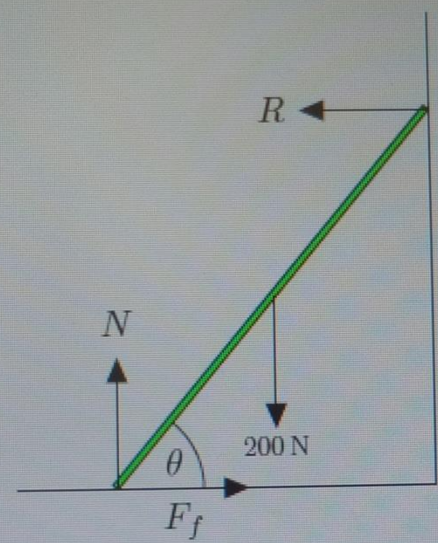


Figure 6.1: *The ladder of Example 6.4 resting against a wall.*

Let θ be the angle the ladder makes with the horizontal. From the dimensions given, $\cos\theta = 2.5/6.5 = 5/13$, and θ is an angle in a $5, 12, 13$ Pythagorean triangle with $\sin\theta = 12/13$, $\tan\theta = 12/5$. The simplest moment balance comes from taking moments about the base of the ladder, which will eliminate both N and F_f from the resulting equation. Assuming the centre of gravity of the ladder to be at its mid-point, and balancing the moment about B of the ladder's weight with the moment of the reaction at the wall, R