A fluid particle will respond to a force in the same way that a solid particle will. If a force is applied to a particle, acceleration will result as governed by Newton's second law of motion, which states that the rate of change of momentum of a body is proportional to the unbalanced force acting on it and takes place in the direction of the force. It is useful to consider the forces that a fluid particle can experience. These include:

body forces such as gravity and electromagnetism;
forces due to pressure;
forces due to viscous action;
forces due to rotation.

Assuming that the shear rate in a fluid is linearly related to shear stress, and that the fluid flow is laminar, Navier (1823) derived the equations of motion for a viscous fluid from molecular considerations. Stokes (1845) also derived the equations of motion for a viscous fluid in a slightly different form and the basic equations that govern fluid flow are now generally known as the Navier-Stokes equations of motion. The Navier-Stokes equations can also be used for turbulent flow, with

We are not going to derive the Navier-Stokes and continuity equations here as this can be found in most standard text books. However, we will state the equations here and briefly give the physical interpretation of the terms as this will help us understand the numerical schemes used to solve those equations. It will also allow us to introduce the various levels of approximations used to simplify the equations to reduce the numerical solution costs.

2.1.1 Compressible flow

The flow governing equations are the continuity equation, momentum equation (Navier-Stokes) and energy equation:

The continuity equation:

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \qquad (2.5)$$

The Navier Stokes Equation:

$$\underbrace{\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)}_{Inertial\ \ terms}=-\underbrace{\frac{\partial p}{\partial x}}_{\text{Pressure}\ \ gradient}+\underbrace{\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}_{Viscous\ \ terms}+\underbrace{F_x}_{Body\ \ force\ \ terms} \tag{2.6}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+F_y \tag{2.7}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+F_z \tag{2.8}$$

The energy equation

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+F_z \qquad (2.8)$$

The energy equation

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\Phi+\frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right]+\frac{\partial}{\partial z}\left[k\frac{\partial T}{\partial z}\right] \qquad (2.9)$$
$$+\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)$$

where Φ is the dissipation function given by:

$$\Phi=2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+0.5\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+0.5\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+0.5\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]$$
$$-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2$$

2.1.2 Incompressible flow

The above system of equation can be simplified if the density is constant. If the temperature is also is assumed constant, the system reduces to (for simplicity, body forces are also neglected, but they can be retained if needed):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.10}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{2.11}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2.12}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{2.13}$$

For incompressible flows with temperature variation, the energy equation need to be solved to obtain

2.2 Turbulent flow

In the 1880's, Osborne Reynolds carried out his historic visualisation studies of flow in a pipe. He observed that well-ordered laminar flow degenerated into a chaotic motion when the velocity in the pipe reached a certain value. This was linked to a non-dimensional quantity called the flow Reynolds number $\mathrm{Re} = \dfrac{\rho U D}{\mu}$ where U is the average velocity in the pipe and D is the diameter.

The Reynolds number represents the ratio between inertia forces and viscous forces. If this number is low, the flow is orderly with parallel streamlines. If it is increased, at some point, this structure of laminar flow loses its identity, giving rise to a flow structure characterised by large-scale eddies.

Generally speaking, viscous effects, and consequently turbulence, prevail in a region close to solid boundaries called the boundary layer. In pipe flows, the boundary layer grows as flow from, say, a plenum until it covers the whole pipe.

For external flows, such as flow over a wing or a car, the boundary layer is confined to a narrow region close to the wall. Away from the wall viscous effects are negligible and the flow is termed inviscid.

To understand how the boundary layer forms, imagine flow with free stream velocity U_∞ approaching a flat plate as shown in Fig 2.4. The flow will have no slip or zero velocity at the wall due to friction. At a distance far from the wall, the flow has a velocity U_∞. As the flow approaches the wall, a boundary layer forms where the flow varies from zero at the wall to U_∞ at the far stream.

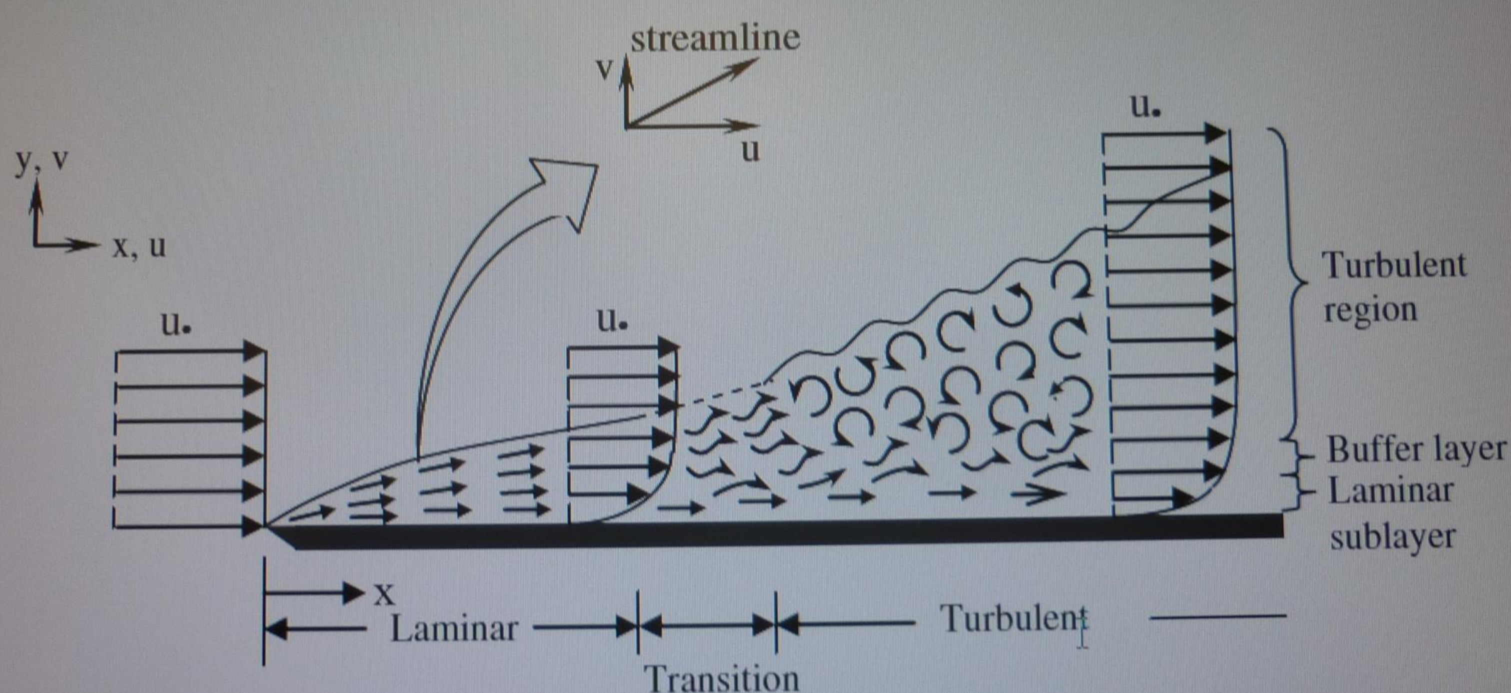

Fig 2.4 Boundary layer over a flat plate

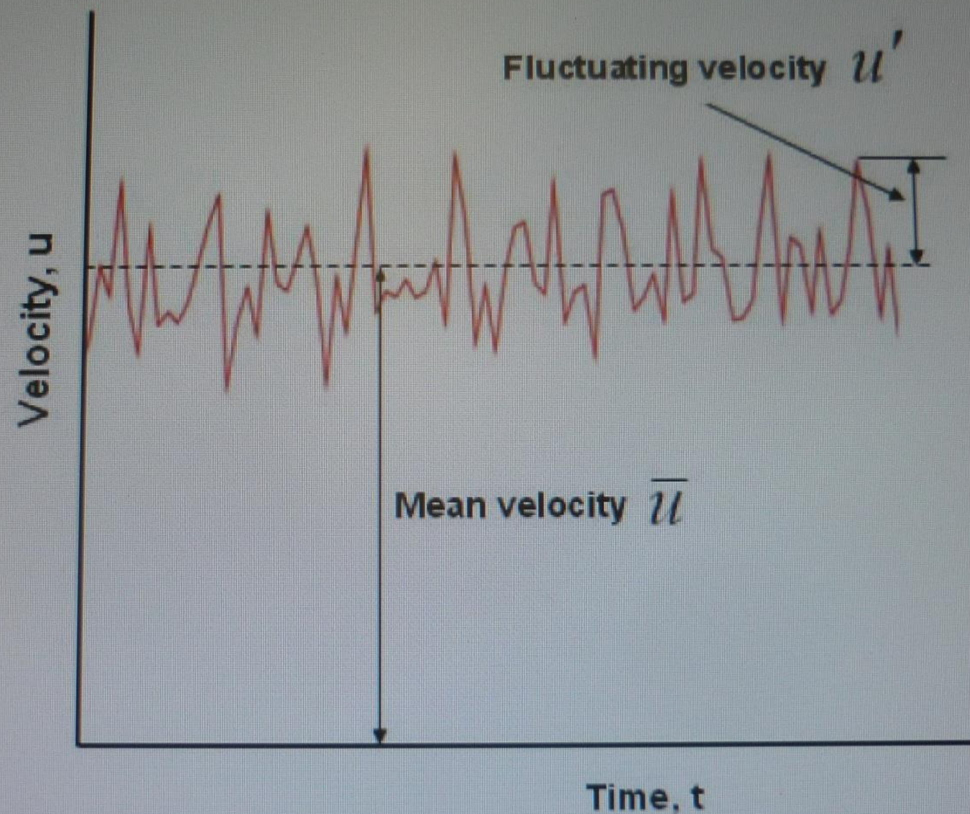

Figure 2.5 Instantaneous and average velocity in turbulent flow

2.3 Inviscid flow

In a number of engineering flows, particularly high speed flows, the boundary layer is confined to a very thin region near the wall. Outside that region viscous effects are negligible. This can be also restated as the viscous terms are very small compared to the time derivative and the advection terms in the momentum equations. In this case, it is possible to get a good description of the flow field by removing the viscous terms from the equations. This results in the inviscid flow model.

For example, for incompressible flow, the momentum equations 2.11-2.13 take the form:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x} \tag{2.25}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y} \tag{2.26}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z} \tag{2.27}$$

You should be able to notice here that turbulence does not play a role and there is no need to perform

2.4 Boundary layer approximation

It is possible to simplify the flow equations in the boundary layer by making and order of magnitude analysis of the different terms in the momentum equations and neglecting the terms that are very small relative to other terms. The resulting terms apply only in the boundary layer and provide a simpler form of the equations which can be solved easily even sometimes using an analytical approach.

The basic boundary layer assumption, usually used, is that the thickness of the boundary layer is much smaller than the extent that the flow travels in the stream-wise direction. For a flat plate boundary layer where the flow is along the x-axis and the boundary layer thickness is in the y-direction, for example, the two dimensional boundary layer equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.28)$$

$$\rho\left\{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right\}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+F_x \qquad (2.29)$$