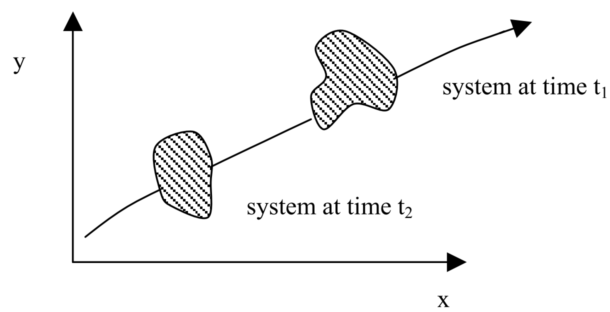
Governing Equations for a Control Volume

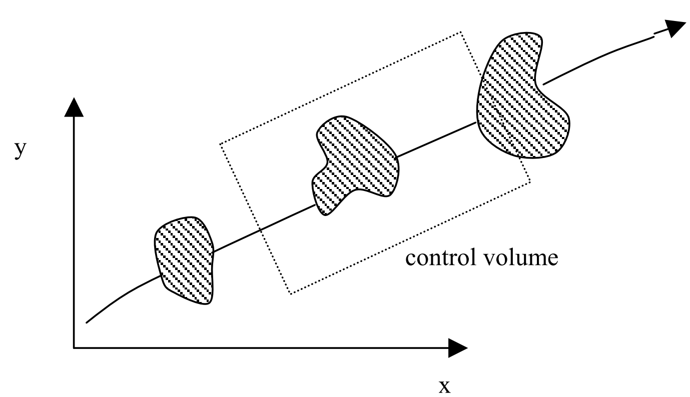
Reading Material: Van Wylen & Sonntag, 109-117

System Versus Control Volume Approach

- Consider one fluid "particle" (i.e., system) and track it with time.
- Mass does not enter or leave **system** with time.



- Not practical in many cases involving flowing fluid because of large number of "particles"
- **control volume** is an imaginary volume through which fluid may flow



- Basic conservation laws governing mass, momentum, energy are written **for system**, that is, a fixed assemblage of mass.
- Here, we will first present the basic laws for a system, then express them for a control volume.

- Later, we will show how to convert between system and control volume forms of the equations

Mathematical Description of a Continuum

- **extensive property**

  -depends on mass (S, U, E, etc.)

  -Ψ will be used as a general symbol to represent extensive properties

- **intensive property**

  -independent of mass (s, u, e, etc.)

  -ψ will be used as a general symbol to represent intensive properties

- for a material that obeys the continuum postulate, define

$$\psi = \lim_{\Delta m \to 0}\left(\frac{\Delta\Psi}{\Delta m}\right) = \frac{d\Psi}{dm} \qquad (3.2.1)$$

- it follows that,

$$\Psi = \int_{system} \psi \, dm \qquad (3.2.2)$$

- for example, if ψ = v (specific volume), then Ψ = V(volume)
- since ρ = dm/dV, then (3.2.2) can also be written as a volume integral,

$$\Psi = \int_V \psi \; \rho \; dV \qquad (3.2.3)$$

- the following table summarizes the variables of interest to us

variable	Ψ	integral definition	value of ψ
mass	m	$= \int_V (1)\rho \, dV$	1
momentum	$\vec{\mathbf{M}}$	$= \int_V \vec{\mathbf{V}}\rho \, dV$	$\vec{\mathbf{V}}$
stored energy	E	$= \int_V e\rho \, dV$	$e = u + \vec{\mathbf{V}}^2 + gz$
entropy	S	$= \int_V s\rho \, dV$	s

System - Control Volume Relationship

- For a control volume of fixed shape, it can be shown that,

$$\boxed{\frac{D\Psi}{Dt} = \int_V \frac{\partial \psi\rho}{\partial t} dV - \int_A \psi\rho\vec{\mathbf{V}} \bullet d\mathbf{A}} \qquad (3.3.1)$$

where,

d**A** = **n** dA (vector differential area)

n = unit normal in the direction of the outward normal to the control surface

Ψ = extensive quantity (m, $\vec{\mathbf{M}}$, E, S)

ψ = corresponding intensive quantity (1, $\vec{\mathbf{V}}$, e, s)

$\vec{\mathbf{V}}$ = $\mathbf{V_x i} + \mathbf{V_y j} + \mathbf{V_z k}$

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + V_x \frac{\partial()}{\partial x} + V_y \frac{\partial()}{\partial y} + V_z \frac{\partial()}{\partial z}$$

Conservation of Mass

- For system, as a "particle" is tracked, mass = constant, or

$$\frac{Dm}{Dt} = 0 \tag{3.4.1}$$

- Comparing (3.4.1) with (3.3.1), we can equate Ψ = mass, and ψ = 1, yielding

$$\boxed{\frac{Dm}{Dt} = 0 = \int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho \vec{\mathbf{V}} \bullet d\mathbf{A}} \tag{3.4.2}$$

- this is the integral form of conservation of mass (note, 1-volume integral and one surface integral)

- applying the **Divergence Theorem** (e.g., let, $\vec{\mathbf{B}}$ represent a vector quantity)

$$\int_V (\nabla \bullet \vec{\mathbf{B}}\, dV) = \int_A \vec{\mathbf{B}} \bullet dA \tag{3.4.3}$$

to the conservation of mass yields the equation in differential form

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{\mathbf{V}}) = 0} \tag{3.4.4}$$

Conservation of Momentum

- Newton's 2nd Law is expressed for a "system"

$$\vec{\mathbf{F}}_{ext} = \frac{D\vec{\mathbf{M}}}{Dt} \tag{3.5.1}$$

- Comparing (3.3.1) with (3.5.1) we can equate $\Psi = \vec{\mathbf{M}}$ and $\psi = \vec{\mathbf{V}}$, yielding

$$\vec{\mathbf{F}}_b + \vec{\mathbf{F}}_s = \vec{\mathbf{F}}_{ext} = \int_V \frac{\partial \rho \vec{\mathbf{V}}}{\partial t} dV + \int_A \vec{\mathbf{V}}(\rho \vec{\mathbf{V}} \bullet dA) \tag{3.5.2}$$

- This is the integral form of conservation of momentum, where,

  $\vec{\mathbf{F}}_b$ = body forces (e.g., gravity) = $\int_{\mathbf{V}} \vec{\mathbf{b}} \rho dV$

  $\vec{\mathbf{b}}$ = body force per unit mass

  $\vec{\mathbf{F}}_s$ = surfaces forces = $\vec{\mathbf{F}}_n + \vec{\mathbf{F}}_{shear}$

- note, for **<u>inviscid fluid</u>**,

$$\vec{F}_n = -\int_A P \, dA,$$

$$\vec{F}_{shear} = 0$$

- applying the Divergence Theorem to the conservation of momentum to convert the surface integrals into volume integrals

$$\int_A P\,\mathbf{n}_i \bullet \mathbf{dA} = \int_{\mathbf{V}} \frac{\partial \mathbf{P}}{\partial \mathbf{x_i}} dV \qquad (3.5.3)$$

$$\int_A \mathbf{V}_i(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) = \int_V \nabla \bullet (\rho \mathbf{V}_i \vec{\mathbf{V}}) dV \qquad (3.5.4)$$

- substituting these into (3.5.2) and taking a differential sized volume yields,

$$\frac{\partial(\rho \mathbf{V}_i)}{\partial t} + \nabla \bullet (\rho \mathbf{V_i} \vec{\mathbf{V}}) = \rho \mathbf{b_i} - \frac{\partial P}{\partial x_i} \qquad (3.5.5)$$

- using the continuity equation, gives Euler's equation

$$\rho \frac{D\mathbf{V}_i}{Dt} = \rho \mathbf{b}_i - \frac{\partial P}{\partial x_i} \qquad (3.5.6)$$

or, in vector form for all 3 components,

$$\rho \frac{D\vec{\mathbf{V}}}{Dt} = \rho \vec{\mathbf{b}} - \nabla \bullet P \qquad (3.5.7)$$

Conservation of Energy (1st Law of Thermodynamics)

- for a fluid particle (i.e., a "system"), conservation of energy states

$$\frac{DE}{Dt} = \delta \dot{Q} - \delta \dot{W} \qquad (3.6.1)$$

where,

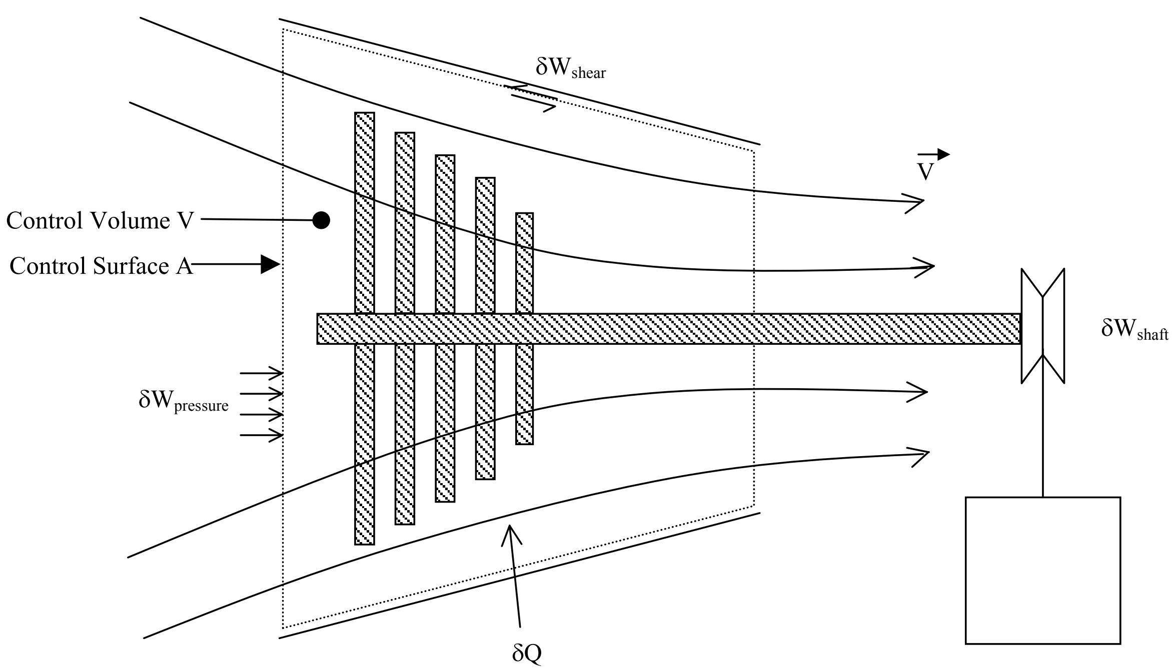
$\frac{DE}{Dt}$ = change in total energy of the particle

$\delta \dot{Q}$ = rate of heat transfer to system

$\delta \dot{W}$ = rate of work done by system

- comparing (3.3.1) with (3.6.1), we can equate Ψ = E and ψ = e, yielding for a control volume,

$$\delta\dot{Q} - \delta\dot{W} = \int_V \frac{\partial \rho e}{\partial t} dV - \int_A e(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) \tag{3.6.2}$$



- To complete the transformation, we must express the heat transfer and work terms in terms of parameters pertinent to the control volume.

- The work term can be broken into two parts:

  --W_{shaft} = shaft work (work done by a rotating shaft crossing the boundary)

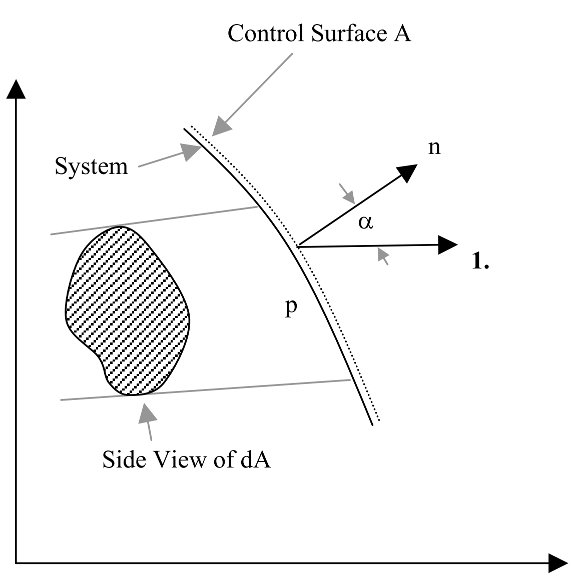
  --W_{flow} = flow work (work done by surface forces as a fluid crosses the control surface)

- The surface forces can be resolved into normal (pressure) and tangential (shear stresses) components.

- Here, we will lump the shear component into a term W_{shear}

- The work done by the pressure stresses may be either work done in pushing the mass out of the control volume (positive work), or work received when the surroundings push mass into the control volume (negative).

- The following figure shows the determination of flow work because of the normal stresses.



- The normal force acting on the infinitesimal vector area **dA** is given by,

$$d\vec{\mathbf{F}}_n = P\ \mathbf{dA} \tag{3.6.3}$$

where the differential force $\mathbf{dF}_n$ is the force exerted by the fluid inside the control volume on the surroundings, and is thus positive.

- The rate of doing work is obtained by multiplying the component of $\mathbf{dF_n}$ in the direction of $\vec{\mathbf{V}}$, by the magnitude of $\vec{\mathbf{V}}$.

$$\delta\dot{W}_n = dF_n\ \mathbf{V}\cos(\alpha) = d\vec{\mathbf{F}}_\mathbf{n} \bullet \vec{\mathbf{V}} \tag{3.6.4}$$

$$\delta\dot{W}_n = P\ \vec{\mathbf{V}} \bullet \mathbf{dA} = Pv(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) \tag{3.6.5}$$

- The total work done by the normal stresses is obtained by integrating $\delta\dot{W}_n$ over the entire control surface,

$$\dot{W}_n = \int_A d\dot{W}_n = \int_A Pv(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) \tag{3.6.6}$$

- Combining flow, shear, and shaft work yields the total work,

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{shear} + \int_A Pv(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) \qquad (3.6.7)$$

- combining (3.6.2) with (3.6.7), we can recast the energy equation for a control volume as,

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_{shear} = \int_V \frac{\partial}{\partial t}\left(\rho(u + \frac{1}{2}\mathbf{V}^2 + gz) \right) dV$$

$$+ \int_A (h + \frac{1}{2}\mathbf{V}^2 + gz)(\rho\vec{\mathbf{V}} \bullet \mathbf{dA}) \qquad (3.6.8)$$

- Note that we have combined terms from the flow work (Pv) with the energy flux term (u) to get enthalpy flux (i.e., h = u + Pv)

- As we did for the momentum equation, convert the surface integrals to volume integrals by the divergence theorem and take differential sized control volume to yield,

$$\boxed{\rho \frac{D(h + \frac{1}{2}\mathbf{V}^2 + gz)}{Dt} - \frac{\partial P}{\partial t} = \delta\dot{Q} - \delta\dot{W}_{shaft} - \delta\dot{W}_{shear}} \qquad (3.6.9)$$

2nd Law

- For fluid particle (i.e., system)

$$\frac{DS}{Dt} \geq \frac{\delta\dot{Q}}{T} \qquad (3.7.1)$$

- Comparing (3.7.1) with (3.3.1), we can equate Ψ = S and ψ = s, yielding

$$\int_V \frac{\partial(\rho s)}{\partial t} dV + \int_A s(\rho\vec{\mathrm{V}} \bullet \mathrm{dA}) \geq \frac{\dot{Q}}{T} \qquad (3.7.2)$$

- As we did for the momentum and energy equations, convert the surface integrals to volume integrals by the divergence theorem and take differential sized control volume to yield,

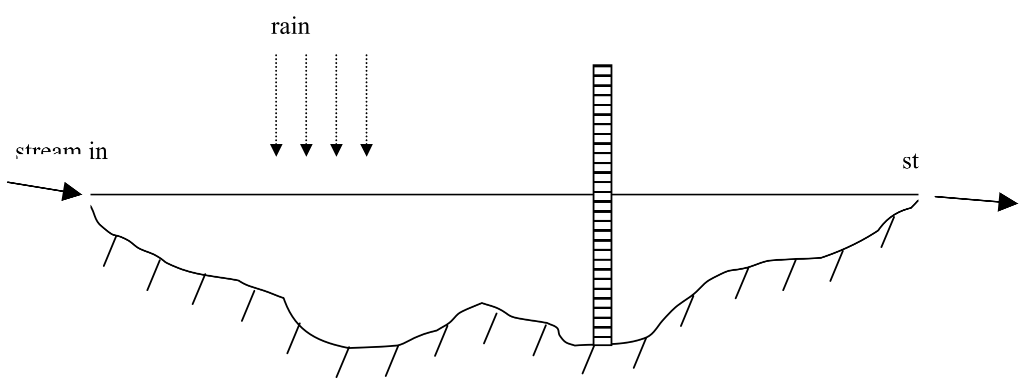
$$\boxed{\rho \frac{Ds}{Dt} \geq \frac{\delta \dot{Q}}{T}} \qquad (3.7.3)$$

the differential form of the 2nd law.

Quantitative Definition of Heat

- Given a constant composition (moles), the heat flux of a system is the difference in internal energy between initial and final states, less work done in that process.

- **Good example: Farm pond**



- water level is energy

- stream in/out is mechanical work, W

- rainfall in/out is heat, Q

- Keypoints:

  - pond level is analogous to energy

  - at any time, one can measure level, but impossible to determine how much came from rain vs. stream

  - One can measure stream flow rates in/out, but not rain in/out

  - By covering pond (adiabatic), rain is excluded

  - One can then calibrate pond level from flow rate (W)

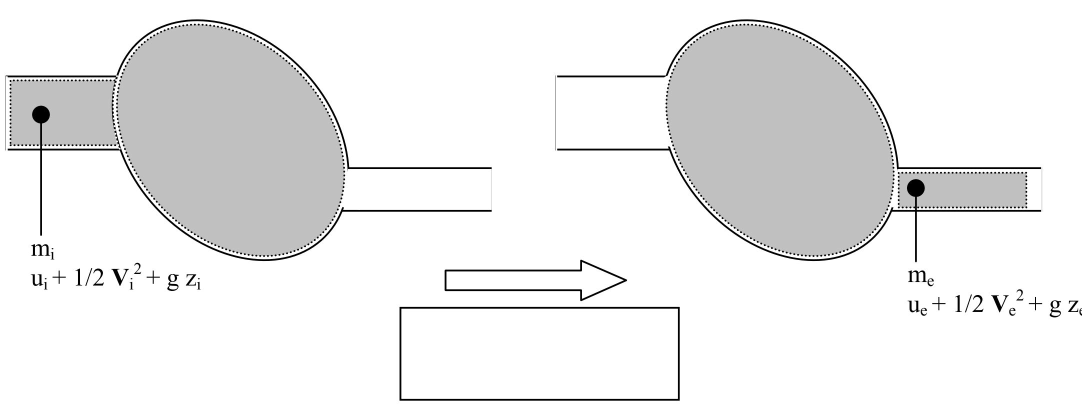
  - By removing cover, rain in/out (Q) is determined

Conservation Equations for Discrete Inlets/Outlets

- The integral forms of the conservation equations (see table on page 18) are cast for an arbitrary control volume.

- In many engineering problems, it is adequate to make certain assumptions about the features of a control volume.

- For example, consider a control volume that surrounds a mechanical device as shown below:



- In the previous figure, there are:

  -a number of discrete passages for the flow to enter and leave the device

  -heat transfer interactions with the control volume

  -work interactions with the device

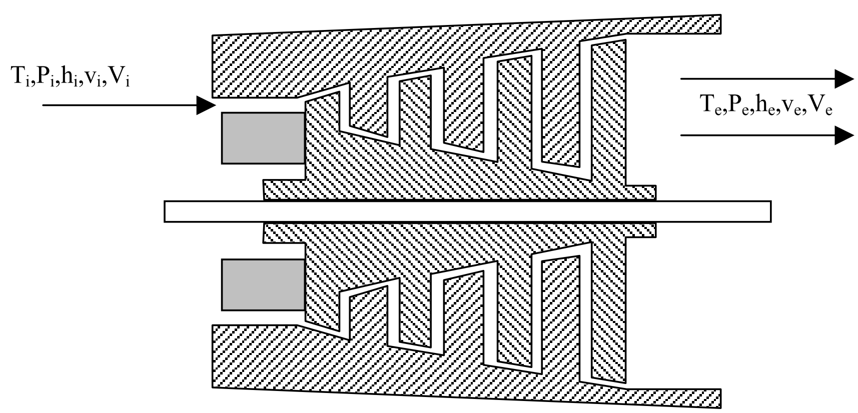
- It is reasonable in many analyses to make the following assumptions about the flow entering and leaving the device:

  -flow entering/leaving the device is in a direction normal to the opening, with uniform flow velocity and properties

  -flow entering the device is characterized by uniform properties h_i, P_i, v_i, $\mathbf{V}_i$, etc.

  -flow exiting the device is characterized by properties h_e, P_e, v_e, $\mathbf{V}_e$, etc.

<u>Example: axial flow turbine</u>



- This is sometimes called **one-dimensional** flow, and under these assumptions, the **mass flow rate** crossing an inlet/outlet boundary is

$$\dot{m}_i = \rho_i \mathbf{V}_i A_i \quad (3.10.1a)$$

$$\dot{m}_e = \rho_e \mathbf{V}_e A_e \quad (3.10.1b)$$

- Quite often, **volumetric flow rate** is given rather than **mass flow rate**

$$\dot{\mathbf{V}}_i = \mathbf{V}_i A_i = \dot{m}_i v_i \quad (3.10.2a)$$

$$\dot{\mathbf{V}}_e = \mathbf{V}_e A_e = \dot{m}_e v_e \quad (3.10.2b)$$

- Another assumption made about the figure on the previous page is that the substance inside the control volume has uniform properties defined by P_{cv}, T_{cv}, s_{cv}, h_{cv}, u_{cv}, etc.
- That is, there are no spatial variations in properties inside the control volume
- Consequently, the integral form of the conservation equations can be written in a form that describes the differential rate of change of the control volumes properties:

- Conservation of mass:

$$\frac{dm_{cv}}{dt} = \sum_i \dot{m}_i - \sum_e \dot{m}_e \quad (3.10.3)$$

Term	Description
(1)	Time rate of change of mass (m_{cv}) within control volume
(2)	Mass flux into control volume
(3)	Mass flux out of control volume

- **Conservation of energy**:

$$\frac{dE_{cv}}{dt} = \dot{Q}_{cv} - \dot{W}_{cv} + \sum_i \dot{m}_i \left(h_i + \tfrac{1}{2}\mathbf{V}_i^2 + gz_i\right)$$

$$- \sum_e \dot{m}_e \left(h_e + \tfrac{1}{2}\mathbf{V}_e^2 + gz_e\right) \qquad (3.10.4)$$

Term	Description
(1)	Time rate of change of total energy (E_{cv}) within control volume
(2)	Heat Flux across control volume
(3)	Work interactions with control volume (excluding flow work)
(4)	Energy flux into control volume
(5)	Energy flux out of control volume

- **Entropy equation:**

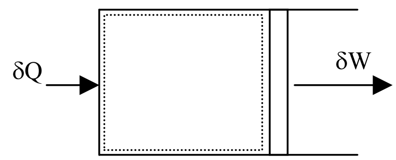
$$\frac{dS_{CV}}{dt} = \sum_j \frac{\dot{Q}_j}{T_j} + \sum_i \dot{m}_i s_i - \sum_e \dot{m}_e s_e + \dot{\sigma}_{CV} \qquad (3.10.5)$$

Term	Description
(1)	Time rate of change of entropy within control volume
(2)	Entropy transfer resulting from Q at point on surface where $T = T_j$
(3)	Entropy flux into control volume
(4)	Entropy flux out of control volume
(5)	Entropy production within control volume due to internal irreversibilities

Summary of Equations

Forms of the Governing Equations for a <u>System</u> (Fixed Mass)

Continuity	$\frac{Dm}{Dt} = 0$
Energy	$\frac{DE}{Dt} = \delta\dot{Q} - \delta\dot{W}$
Entropy	$\frac{DS}{Dt} \geq \frac{\delta\dot{Q}}{T}$



Example: piston-cylinder

Governing Equations for a <u>Control Volume with Discrete Inlets/Outlets</u>

Continuity

$$\frac{dm_{CV}}{dt} = \sum_i m_i - \sum_e m_e$$

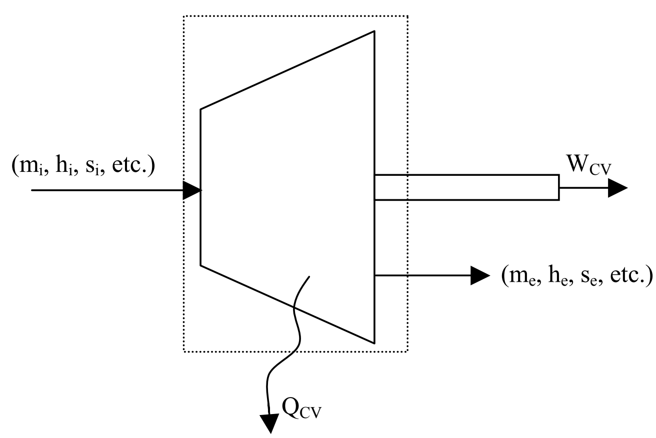
Energy

$$\frac{dE_{CV}}{dt} = \dot{Q}_{CV} - \dot{W}_{CV} + \sum_i m_i (h_i + \frac{1}{2}\mathbf{V}_i^2 + gz_i)$$

$$- \sum_e m_e (h_e + \frac{1}{2}\mathbf{V}_e^2 + gz_e)$$

Entropy

$$\frac{dS_{CV}}{dt} = \sum_j \frac{Q_j}{T_j} + \sum_i \dot{m}_i s_i - \sum_e \dot{m}_e s_e + \dot{\sigma}_{CV}$$



Example: turbine

<u>Integral Forms</u> of the Governing Equations (Arbitrary Control Volume)

Continuity	$0 = \int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho \vec{\mathbf{V}} \bullet d\mathbf{A}$
Energy	$\delta \dot{Q} - \delta \dot{W} = \int_V \frac{\partial \rho e}{\partial t} dV + \int_A e(\rho \vec{\mathbf{V}} \bullet d\mathbf{A})$
Entropy	$\int_V \frac{\partial \rho s}{\partial t} dV + \int_A s(\rho \vec{\mathbf{V}} \bullet dA) \geq \frac{\dot{Q}}{T}$

<u>Differential Forms</u> of the Governing Equations (apply at a point)

Continuity $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{\mathbf{V}}) = \mathbf{0}$

Energy

$$\rho \frac{D(h + \frac{1}{2}\mathbf{V}^2 + gz)}{Dt} - \frac{\partial P}{\partial t} = \delta \dot{Q} - \delta \dot{W}_{shaft} - \delta \dot{W}_{shear}$$

Entropy $\rho \frac{Ds}{Dt} \geq \frac{\delta \dot{Q}}{T}$