THE GOVERNMENT OF THE REPUBLIC OF THE UNION OF MYANMAR

MINISTRY OF EDUCATION

MATHEMATICS

GRADE 9

BASIC EDUCATION CURRICULUM, SYLLABUS AND

TEXTBOOK COMMITTEE

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Preface

This text is the first of a two-volume set of High School Mathematics written in accordance with the new High School Mathematics Curriculum.

The general aims of teaching mathematics at high school level are specified as follows:

At the end of Basic Education High School Level, students should be able to

(1) gain basic mathematical knowledge and understanding,

(2) acquire necessary mathematical skills,

(3) apply mathematical knowledge and skills in real-life situations, and

(4) have an interest in, and appreciation of mathematics, together with the development of maths-related values.

With these aims in mind, the Mathematics Curriculum and Textbook Committee has selected a proper curriculum content and developed two textbooks for high school students. This textbook is intended for use in the ninth standard only. Areas covered in this text are the following:

Chapter 1 is concerned with sets and functions. The study of sets being a basic and indispensable element in the courses of modern mathematics, an attempt is made here to deepen the students' understanding of sets and enable them to deal with problems through sets wherever necessary.

The study of functions aim to develop an understanding of functions as mappings and to recognize functions as relations between sets.

Chapter 2 introduce basic ideas in co-ordinate geometry.

Chapters 3 and 4 emphasize the basic properties of exponents, radicals and logarithms. Ideas mainly developed in these two chapters are : (i) the algebra of rational exponent, and (ii) the algebra of logarithm of a positive number.

Chapter 5 deals with the methods for solving a quadratic equation are discussed. Geometrical motivations are used in finding the solution set of the system of equations.

Chapter 6 extends the ideas of ratio, proportion and variation introduced in the middle school level ; different, forms of variation are thoroughly discussed with special reference to the applicability of mathematics in real-life situations.

Chapter 7 is intended to develop three measures of central tendency the mode, the median, and the arithmetic mean.

Chapters 8 and 9 attempt to stress the formal structure of geometry and integrate geometry with arithmetic and algebra. Emphasis is laid down, upon the use of precise language in the statements of definitions, postulates, and theorems.

Following the general trend, we have presented geometry for the high school as a concrete deductive system, that is, geometry is viewed as a mathematical model of the physical world studied by deductive methods.

Chapter 10 which is the last chapter of this text, attempts to broaden the students' understanding of geometric properties, and interrelations between sides and angles of plane figures. In this regard, instruction is centered on triangles, especially right triangles at the beginning.

Besides the above-mentioned course content in each chapter of this text. the salient features as to the presentation style of the text are as follows:

1, Each of the chapter is preceded by an introduction which acts as a link between what the students have already learnt and what they are going to learn.
2. As an attempt to motivate the students' interest and to help them in the understanding of the subject, illustrative examples are first provided and then analyzed leading finally to the synthesis of a whole picture.
3. On the basis of analysis and synthesis, next comes Abstraction Phase which is essentially an abstract picture of necessary definitions and theorems for a particular topic.
4. Applicability of the mathematical ideas for a certain topic is stressed in such a manner that suitable examples, situational motivation and numerous exercise have been provided in order that the wide areas of application of mathematical concepts are delineated.
5. Each chapter is backed up a summary which is Psychologically termed Consolidation Phase, the result of which is that new ideas introduced in the chapter are well consolidated and rooted in students' minds.

Finally, one cautionary, as well as, suggestive note on how to learn mathematics is appropriate here.

"Learn mathematics by doing only".

Do not just read your textbook. You should always have a pencil and paper with you and " work through " the text. It is suggested to make summaries of your own for each unit in each chapter and frequently review them.

CONTENTS

CHAPTER 1

Sets and Functions

1.1 Sets

Since the earliest times man has formed his ideas of number, measure and shape. The ideas developed into Arithmetic, Algebra and Geometry which form the basis of Mathematics.

Modern Mathematics attempts to introduce Mathematics as a fascinating subject which deals with problem solving based on reason, cause and effect. In it, the concept of a set is a basis.

The set notation is a handy tool to describe the concepts in various branches of Mathematics and to link them into an integrated whole.

The foundation of Set Theory was first laid down by George Cantor (1845-1918) and developed over the years by many famous Mathematicians including John Venn (1834-1923) who introduced a graphic notation for representing sets.

Let's begin our study of the language of Mathematics with the notion of set. There is nothing complicated or mysterious about sets. A set is a collection of some kind. The only restriction placed on this collection is that there must be a definite way to decide whether a particular object belongs to the collection. The objects that belong to a set are called the elements, or members of the set.

Sets are often symbolized by capital letters A, B, C, and so on and members of sets by small letters a, b, c, and so on. Another symbolism for sets uses curly brackets, { }, to enclose either a list of elements of the set or some phrase that describes the elements. Thus, both {1, 2, 3} and {first three natural numbers} represent the set whose elements are the numbers 1, 2 and 3.

If you can count the elements of a set one by one until you reach a final elements that set is called finite. When you can't count the elements of a set you are dealing with an infinite set. If you start listing the even numbers 2, 4, 6, 8,... you have started on an infinite set; you can never reach a "last" element. A set that contains no members is called the empty set or null set. It is denoted by the symbol ϕ and is considered a finite set.

You will be expected to be familiar with the following sets of numbers.

1. The set N of natural numbers, consisting of the counting numbers 1, 2, 3,... and so on.

2. The set Z of integers, which contains the natural numbers, their negatives and zero.

3. The set Q of rational numbers. A typical element of Q can be expressed in the form $\frac{p}{q}$, with p and q integers and q not zero. Some examples of rational numbers are $\frac{1}{2}$, $\frac{7}{8}$, $-\frac{2}{3}$ and 8, which is rational since it can be expressed in the form $\frac{8}{1}$.

4. The set of irrational numbers. The members of this set are those real numbers that are not rational. Examples are such numbers as $\sqrt{2}$, $-\sqrt{3}$ and π

5. The set R of real numbers consist of rational and irrational numbers.

Before you can hope to do very much with these sets, however, you will need some additional terminology. Two sets are called equal if they have the same members. Thus {1, 2, 3} = {first three natural numbers} since both sets comprise the numbers 1,2, and 3.

The set A is called a **subset** of the set B if every element of A is also an element of B. The symbol for a subset is $\subset$, and $A \subset B$ is read "A is a subset of B" or" A is contained in B" thus $\{1,2,4\} \subset \{1,2,3,4\}$ and $\{1,2,4\} \subset \{1,2,4\}$ are each valid assertions. The empty set is taken to be a subset of every set.

The fact that the object x is an element of the set A is denoted by writing $x \in A$. To indicate that x is not an element of A, one writes $x \notin A$. The slant bar symbol, /, can be used consistently to negate symbols. Thus $\neq$ means "is not equal to" and $\not\subset$ means "is not a subset of ".

1.1.1 Set Builder Form

We have seen that a set may be described either in words or by listing its elements. In this section we will introduce yet another way of describing a set.

Set builder form

Let G be the set of positive integers that are less than 8.

Take a general element, say x, of G.

Then we note that

(i) x is a positive integer and

(ii) $x < 8$

Therefore we can write the set G as

$$G = \{x \mid x \text{ is a positive integer and } x < 8\}$$

It is read as "G is the set of all x such that x is a positive integer and x is less than 8".

This description of a set is known as the "**Set builder form**". In the above example, the general element x shall take the values 1, 2, 3, 4, 5, 6 and 7. Thus, the set G can be written as

$$G = \{1, 2, 3, 4, 5, 6, 7\}$$

Thus a set can be described in three ways, namely

(i) in words ;

(ii) in set builder form;

(iii) by listing its elements.

Example 1. Let A be the set of solutions of the equation $x^2 + 2x - 3 = 0$. Describe the set A (i) in set builder form (ii) by listing its elements.

Solution

(i) $A = \{x \mid x^2 + 2x - 3 = 0\}$

(ii) Since $x^2 + 2x - 3 = 0$, $(x + 3)(x - 1) = 0$

$$x = -3 \text{ or } x = 1.$$

(iii) we have

$$A = \{x \mid x^2 + 2x - 3 = 0\} = \{x \mid x = -3 \text{ or } x = 1\} = \{-3, 1\}$$

We have seen, in the above example, how a set, given in the set builder form, can be described by listing its elements. Conversely, a set given by listing the elements may be described in set builder form. But, this can be done in more than one way as we shall see in the example below.

Example 2. Describe the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ in set builder form.

Solution

The elements of A are positive integers and all are less than 8.

Therefore, we may write $A = \{x \mid x \text{ is a positive integer and } x < 8\}$

We may also see this way: the elements of A are the first seven members of the set of natural numbers. So we may write

$A = \{x \mid x \text{ is one of the first seven natural numbers}\}$

Thus the description of a set in set builder form needs not to be unique.

Example 3. Let $A = \{-1, 0, 1, 2, 3\}$ and $B = \{x \mid x \text{ is an integer and } -2 < x < 4\}$

Is $A = B$?

Solution

We will describe the set B by listing its elements. If x is an integer and $-2 < x < 4$, then $x = -1$ or $x = 0$ or $x = 1$ or $x = 2$ or $x = 3$. Therefore

$$\begin{aligned} B &= \{x \mid x \text{ is an integer and } -2 < x < 4\} \\ &= \{x \mid x = -1 \text{ or } x = 0 \text{ or } x = 1 \text{ or } x = 2 \text{ or } x = 3\} \\ &= \{-1, 0, 1, 2, 3\} \end{aligned}$$

Thus, $A = B$

Example 4. Let x, y be variables which can take values of positive integers, (i.e. positive integral values). Write down the set L whose elements are of the form (x, y) such that $x + y = 6$.

Solution

In set builder notation, we get $L = \{(x, y) \mid x + y = 6 \text{ where } x, y \text{ are positive integers}\}$

Pairs of positive integers x and y whose sum is 6 are

$x = 1$	and	$y = 5$;
$x = 2$	and	$y = 4$;
$x = 3$	and	$y = 3$;
$x = 4$	and	$y = 2$;
$x = 5$	and	$y = 1$

Thus elements of L are (1, 5), (2, 4), (3, 3) (4, 2) and (5, 1).

That is, $L = \{(x, y) \mid x + y = 6.\ x, y \text{ a positive integer}\}$

$$= \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

Exercise 1.1

Write each of the following sets (i) in set builder form (ii) by listing its elements.

1. The set N of natural numbers.
2. The set J of all positive integers.
3. The set P of all prime numbers.
4. The set A of all positive integers that lie between 1 and 13.
5. The set B of real numbers which satisfy the equation $3x^2 + 5x - 2 = 0$.

Choose a suitable description (a) or (b) or (c) in set builder form for each of the following sets.

6. $E = \{2, 4, 6, 8\}$
   (a) $E = \{x \mid x \text{ is an even integer less than } 10\}$
   (b) $E = \{x \mid x \text{ is an even positive integer less than } 10\}$
   (c) $E = \{x \mid x \text{ is a positive integer, } x < 10 \text{ and } x \text{ is a multiple of } 2\}$
7 $F = \{3, 6, 9, 12, 15, \ldots\}$
   (a) $F = \{x \mid x \text{ is a positive integer that is divisible by } 3\}$
   (b) $F = \{x \mid x \text{ is a multiple of } 3\}$
   (c) $F = \{x \mid x \text{ is a natural number that is divisible by } 3\}$
8. $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{-3, 2\}$. Is $A = B$?
9. $A = \{x \mid x \text{ is a prime number which is less than } 10\}$ and
   $B = \{x \mid x^2 - 8x + 15 = 0\}$
   (a) Is $A = B$?
   (b) Is $B \subset A$?
10. $P = \{x \mid x \text{ is an integer and } -1 < x < \frac{3}{5}\}$, $Q = \{x \mid x^3 - 3x^2 + 2x = 0\}$
    Is $P = Q$?
11. $L = \{(x, y) \mid x, y \text{ are positive integers and } x + y = 7\}$
    Write L by listing its elements.

1.1.2 Intervals

Subsets of R (the set of real number) such as

$\{x \mid -2 \leq x \leq 3\}$, $\{x \mid -2 < x < 3\}$, $\{x \mid x > 2\}$ are called **intervals**.

The set of points on a real number line associated with number of the interval is called the graph of the interval. It is a line segment of the real number line and it is usually shown as a thick black line.

If the beginning (or the end) of the segment is included a black dot is marked at the beginning (or the end) of the segment.

If both ends are included, the interval is called a **closed interval.**

$\{x \mid -2 \leq x \leq 3\}$ is a **closed interval** and is shown in Fig 1.1.

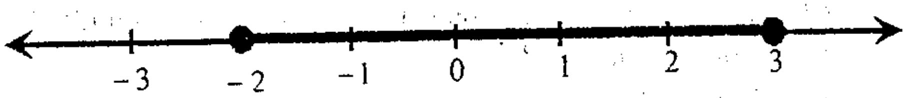


Fig. 1.1

We may also write this closed interval as $[-2, 3]$. If the beginning (or the end) of the segment is not included, an open circle is marked at the beginning (or the end) of the segment.

If both ends are not included, the interval is called an **open interval.**

$\{x \mid -2 < x < 3\}$ is an **open interval** and is shown in Fig. 1.2. We may write this open interval as $(-2, 3)$.

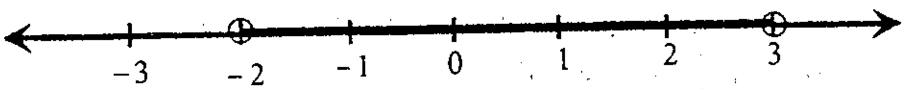


Fig. 1.2

The interval $\{x \mid x > 2\}$ consists of all real numbers greater than 2. Its graph is a set of all points on the real number line which lie to the right of the point associated with 2, and it therefore continues indefinitely towards the right. We can draw the graph as below.

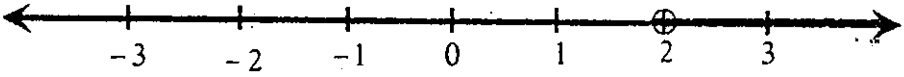


Fig. 1.3

We write the interval as $\{x \mid 2 < x < \infty\}$, where the symbol ∞ stands for "infinity". We usually write this open interval as $(2, \infty)$. We can also have the following types of intervals.

$\{x \mid -2 \leq x < 3\}$ denoted by $[-2, 3)$

$\{x \mid -2 < x \leq 3\}$ denoted by $(-2, 3]$

We can illustrate them using graph as below.

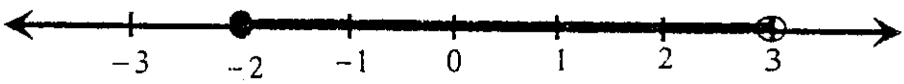


Fig. 1.4 Graph for $[-2, 3)$

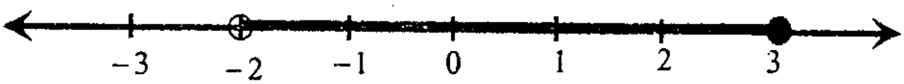


Fig. 1.5 Graph for $(-2, 3]$

Example 1. Draw the graph of the interval $\{x \mid x \leq -1\}$.

Rewrite it using the open / closed notation.

Solution

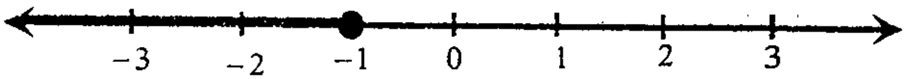


Fig. 1.6

The graph starts from -1 and extends indefinitely to the left. We write this interval as $(-\infty, -1]$.

Since the intervals are sets, we can find the union and intersection of two intervals.

Example 2. Draw the graph of the following on the real number line.

(a) $P = \{x \mid x \geq 1,\ x \in R\}$ (b) $Q = \{x \mid x < 4,\ x \in R\}$

(c) $P \cup Q$ (d) $P \cap Q$

Express $P \cap Q$ in set builder form.

Solution

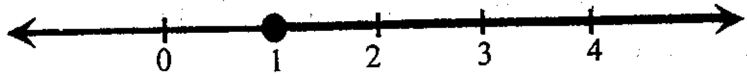


Fig.1.7 Graph for P

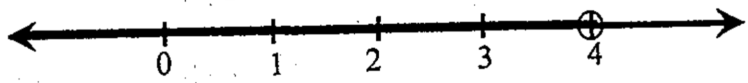


Fig.1.8 Graph for Q

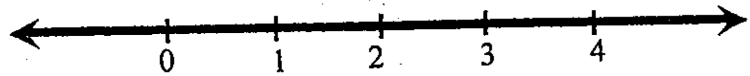


Fig. 1.9 Graph for $P \cup Q$

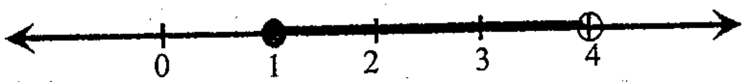


Fig. 1.10 Graph for $P \cap Q$

The first segment starts from 1 and extends indefinitely to the right. The second segment extends indefinitely to the left of 4. The third segment is the union of first and second segments. The fourth segment is the intersection of the first and second segments.

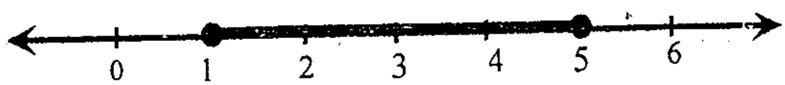
It is common to both segments.

We can express it as $\{x \mid 1 \leq x < 4\}$.

Exercise 1.2

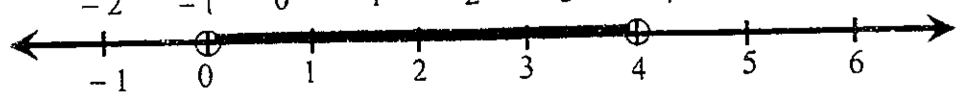
1. Using the set-builder notation, write down the intervals whose graphs are shown below.

(a)



(b)

(c)



(d)

(e)

(f)

2. Draw the graphs of the following intervals.

(a) $\{x \mid x > 2\}$ (b) $\{x \mid x \geq 3\}$

(c) $\{x \mid x \leq -1\}$ (d) $\{x \mid x > -1\}$

(e) $\{x \mid -2 \leq x \leq 2\}$ (f) $\{x \mid 0 \leq x \leq 5\}$

(g) $\{x \mid x \leq 0 \text{ or } x > 2\}$

3. Draw a graph to show the solution set of each of the following.

(a) $x - 1 < 4$ (b) $x - 1 \leq 0$ (c) $2x \leq 5$

(d) $2x - 1 > 7$ (e) $5 - x \geq 1$ (f) $\frac{1}{3}(1 - x) < 1$

4. Draw the graphs of the following on number lines below one another.

(a) $P = \{x \mid x \geq 3, x \in R\}$ (b) $Q = \{x \mid x \leq -2, x \in R\}$

(c) $P \cap Q$ (d) $P \cup Q$

5. On separate number lines draw the graphs

$$S = \{x \mid x > -4\}$$

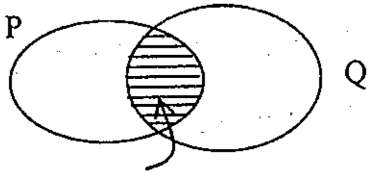
$$T = \{x \mid x < 3\}$$

and $S \cap T$, where $x \in R$.

Give a set- builder description of $S \cap T$.

1.1.3 Intersection of Sets

Definition : The intersection of two sets P and Q is the set of elements which belong to both P and Q. It is denoted by $P \cap Q$ and is read **P intersection Q**.



Overlapping region = $P \cap Q$

Fig. 1. 11

Example 1. Let A = the set of even integers between 1 and 9 and

$B = \{x \mid x^3 - 8x^2 + 12x = 0\}$. Find $A \cap B$.

Solution

A = the set of even integers between 1 and 9 = {2, 4, 6, 8}

$B = \{x \mid x^3 - 8x^2 + 12x = 0\}$

$= \{x \mid x(x^2 - 8x + 12) = 0\}$

$= \{x \mid x(x-2)(x-6) = 0\}$

$= \{x \mid x = 0 \text{ or } x = 2 \text{ or } x = 6\} = \{0, 2, 6\}$

$\therefore A \cap B = \{2, 6\}$

Example 2. Let A = the set of positive even integers and

$B = \{x \mid x$ is a positive integer and x is divisible by 3}. Find $A \cap B$.

Solution

Here, A = the set of positive even integers

= {2, 4, 6, 8, 10, 12,...}

$B = \{x \mid x$ is a positive integer and x is divisible by 3}

= {3, 6, 9, 12, 15, 18,...}

$A \cap B = \{6, 12, 18, ...\}$

Note : The elements of A are multiples of 2, the elements of B are multiples of 3 and the elements of $A \cap B$ are multiples of 6.

The least (i.e., the smallest) element of $A \cap B$ is 6 and this element is called the least common multiple (L.C.M.) of 2 and 3.

1.1.4 Union of Sets

Definition : The union of two sets A and B is the set of elements which belong either to A or to B, or both A and B.

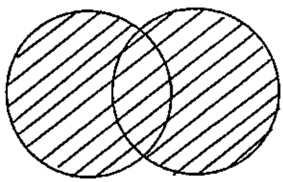


Fig. 1.12
$A \cup B$ is shaded

In listing the elements of $A \cup B$, all you need is to write down elements of A first and then the elements of B. Note that an element in a set cannot be repeated in the list.

Example 1. $A = \{ x \mid x$ is an even positive integer less than 13$\}$ and

$B = \{ x \mid x$ is a positive integer less than 20 and x is divisible by 3$\}$

Find $A \cup B$.

Solution

$$A = \{2, 4, 6, 8, 10, 12\}$$

$$B = \{3, 6, 9, 12, 15, 18\}$$

$$A \cup B = \{2, 4, 6, 8, 10, 12, 3, 9, 15, 18\}$$

Exercise 1.3

1. $M = \{ x \mid x$ is an integer, and $-3 < x < 6\}$,
   N = the set of positive integers that are less than 8. Find $M \cap N$.

2. $A = \{ x \mid x$ is a positive integer that is divisible by 3$\}$

$B = \{ x \mid x$ is a positive integer that is divisible by 5$\}$

Find (a) $A \cap B$ (b) L.C.M of 3 and 5.

3. $J = \{1, 2, 3, 4, \ldots\}$ = the set of positive integers and

$P = \{ x \mid x$ is a prime number$\}$. Find $J \cap P$.

4. $A = \{ x \mid x$ is a positive even integer$\}$

$B = \{ x \mid x$ is a prime number$\}$

$C = \{ x \mid x$ is a positive integer that is divisible by 3$\}$

Find (a) $A \cap (B \cap C)$ and

(b) $(A \cap B) \cap C$

Show that $A \cap (B \cap C) = (A \cap B) \cap C$.

5. Let $A = \{ x \mid x$ is a positive integer that is divisible by 2.$\}$

$B = \{ x \mid x$ is a positive integer that is divisible by 3.$\}$

$C = \{ x \mid x$ is a positive integer that is divisible by 5.$\}$

List the elements of the sets A, B and C.

Find (a) $A \cap (B \cap C)$ and (b) $(A \cap B) \cap C$.

Show that $A \cap (B \cap C) = (A \cap B) \cap C$.

(The resulting set will simply be called $A \cap B \cap C$. The smallest element of $A \cap B \cap C$ is called the L.C.M of 2, 3 and 5.)

Find the L.C.M.

6. Let $A = \{ x \mid x$ is a positive integer less than 7$\}$

and $B = \{ x \mid x$ is an integer and $-3 \leq x \leq 4\}$

List the members of A and B, and then write down $A \cup B$.

7. Let $A = \{x \mid x \text{ is an integer, } 0 < x < 6\}$

and $B = \{x \mid x \text{ is a positive integer less than 13 and } x \text{ is a multiple of } 3\}$

List the members of A and B, and then find $A \cup B$.

8. If $P = \{x \mid x^2 + 2x - 3 = 0\}$ and $Q = \{x \mid x \text{ is an integer, } -1 \leq x \leq 4\}$,

find $P \cap Q$ and $P \cup Q$.

1.1.5 Difference of Two Sets

Definition : Difference of two sets A and B, $A \setminus B$, is the set whose elements are those of A, but are not in B.

Note : To evaluate $A \setminus B$, we work out as follows.

Step 1 : List the elements of A.

Step 2 : Strike out the elements that appear in B.

Step 3 : Form the set with the remaining elements to get $A \setminus B$.

Example 1. Let $A = \{x \mid x \text{ is an integer, } 0 < x < 8\}$

and $B = \{x \mid x \text{ is a positive integer less than 17 and } x \text{ is a multiple of } 4\}$

List the members of A and B, and then find $A \setminus B$ and $B \setminus A$. Is $A \setminus B = B \setminus A$?

Solution

$A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{4, 8, 12, 16\}$

$A \setminus B = \{1, 2, 3, 5, 6, 7\}$

$B \setminus A = \{8, 12, 16\}$

$\therefore A \setminus B \neq B \setminus A$

1.1.6 The Universal Set "S"

Definition: The set containing the totality of elements for any particular discussion or situation is called the **universal set** and is denoted by "S".

All the sets we considered in the context of any partioular situation will be subsets of "S".

For example, if we are discussing the set

$$A = \{0, 2, 4, 6\},$$

then a universal set can be

(i) the set of all non-negative even numbers $= \{0, 2, 4, 6, 8, \ldots\}$

or (ii) the set of non-negative even numbers less than $8 = \{0, 2, 4, 6\}$

or (iii) the set of whole number $= \{0, 1, 2, 3, 4, \ldots\}$

1.1.7 Complement of a Set

Consider the universal set $S = \{1, 2, 3, 4, 5\}$; $A = \{2, 4, 5\}$ is then a subset of S. Collecting all the members of S that are **NOT** in A, we get the subset $\{1, 3\}$. This subset $\{1, 3\}$ is called the **complement** of the set A with respect to S, and is denoted by A' (This is read A prime). See the Venn diagram.

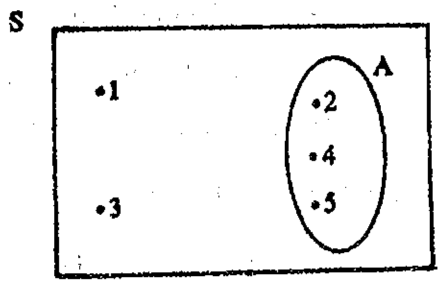


Fig. 1.13

Thus, if $S = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4, 5\}$

then $A' = \{1, 3\}$

Note that $A' = S \setminus A$

Example 1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$.

Find (a) A' (b) $A' \cap A$ (c) $A' \cup A$ (d) $(A')'$ (e) ϕ' (f) S'

Solution

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7, 9\}$

$\therefore$ (a) $A' = \{2, 4, 6, 8\}$

(b) $A' \cap A = \phi$

(c) $A \cup A = \{2, 4, 6, 8, 1, 3, 5, 7, 9\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = S$

(d) Since A' $= \{2, 4, 6, 8\}$ and $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we get

$$(A')' = \{1, 3, 5, 7, 9\} = A$$

(e) $\phi' = S \setminus \phi = S$

(f) $S' = S \setminus S = \phi$

Remark : This example illustrates the following properties which are true in general.

(a) $A \cap A' = \phi$

(b) $A \cup A' = S$

(c) $(A')' = A$

Exercise 1.4

1. Find the sets $A \setminus B$ and $B \setminus A$ in the following.

   (a) $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 5\}$

   (b) $A = \{p, q, r, s, t\}$, $B = \{x, y, z\}$

   (c) $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$

2. S = the set of natural numbers, and T = the set of even numbers. Describe in words the complement of T.

   Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$.

   Find (a) A' (b) B' (c) $A' \cap B'$ (d) $A \cup B$ and (e) $(A \cup B)'$ What can you say about $A' \cap B'$ and $(A \cup B)'$?

4. $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 5, 7\}$

   $B = \{1, 3, 5, 7\}$, $C = \{2, 4, 6, 8\}$.

   (i) List the sets A', B' and C'.

   (ii) State whether each of the following is true or false:

   (a) $A \cap A' = \phi$ (b) $A \cup A' = \phi$

   (c) $A \subset B$ (d) $B \subset A$

   (e) $A' \subset B'$ (f) $B' \subset A'$

   (g) $B' = C$ (h) $B \subset C'$

1.1.8 Number of elements in a set

For any set A, let n (A) (read "n of A") be the number of elements in the set A.

In general, the relationship between the number of elements in each of two sets, their union and their intersection is given by the following theorem.

Theorem

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example 1. If A = {a, b, c, d, e, f}, B = {a, e, i, o, u, w, y} then verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Solution

$A \cup B = \{a, b, c, d, e, f, i, o, u, w, y\}$, $A \cap B = \{a, e\}$

$n(A \cup B) = 11$, $n(A \cap B) = 2$, $n(A) = 6$, $n(B) = 7$,

$n(A) + n(B) - n(A \cap B) = 6 + 7 - 2 = 11 = n(A \cup B)$

Example 2. One hundred students were asked whether they were taking Mathematics (M) or Physics (P).The responses showed that 61 were taking Mathematics.

18 were taking both. 12 were taking neither.

(a) How many were taking Physics?

(b) How many were taking Mathematics but not Physics?

(c) How many were not taking Physics?

(d) Find $n((M \cap P)')$

Solution

(a) $n(M) = 61$

$n(M \cap P) = 18$

12 were taking neither.

∴ the rest, 88 were taking at least one of the courses.

i.e, $n(M \cup P) = 88$

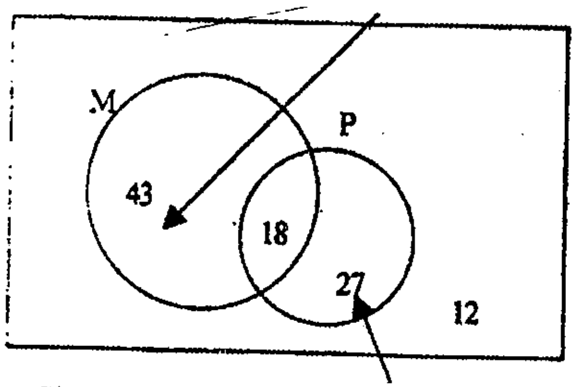
But $n(M \cup P) = n(M) + n(P) - n(M \cap P)$

$\therefore 88 = 61 + n(P) - 18$

$\therefore n(P) = 45$

Since n (M) = 61, 61 – 18 = 43 go here.

Fig. 1.14



Since n (P) = 45, 45 – 18 = 27 go here.

(b) No: of students taking Mathematics but not Physics = n (M \ P) = 43

(c) No: of students not taking Physics = n (P') = 43 + 12 = 55

(d) $n((M \cap P)') = 43 + 27 + 12 = 82$

Example 3. If the universal set contains 500 elements, n (A) = 240

$n(A \cup B) = 460$, $n(A \cap B) = 55$, find n (B').

Solution

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$460 = 240 + n(B) - 55$$

$$\therefore n(B) = 275$$

$$n(B') = n(S \setminus B) = 500 - 275 = 225$$

Example 4.

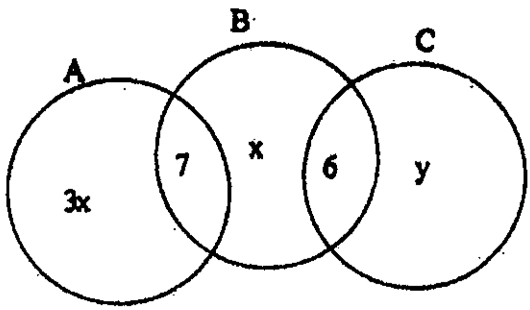


Fig. 1.15

A, B and C are sets such that the universal set $S = A \cup B \cup C$. The number of elements in each subset is shown in the Venn diagram.

(a) Find x, if n (A) = n (B)

(b) Find y, if n ((A ∪ B)') = n (A ∩ B)

(c) Find n (S).

Solution

(a) $n(A) = n(B)$

$3x + 7 = 7 + x + 6$

$2x = 6$

$x = 3$

(b) $n((A \cup B)') = n(A \cap B)$

$\therefore y = 7$

(c) $n(S) = 3x + 7 + x + 6 + y = 4x + 13 + y = 12 + 13 + 7 = 32$

1.1.9 Some Laws on sets

$$\left.\begin{array}{ll}(1) & A \cap (B \cap C) = (A \cap B) \cap C \\ (2) & A \cup (B \cup C) = (A \cup B) \cup C\end{array}\right\} \text{Associative Laws}$$

$$\left.\begin{array}{ll}(3) & A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ (4) & A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\end{array}\right\} \text{Distributive Laws}$$

$$\left.\begin{array}{ll}(5) & (A \cap B)' = A' \cup B' \\ (6) & (A \cup B)' = A' \cap B'\end{array}\right\} \text{De Morgan's Laws}$$

(7) $(A')' = A$

(8) $A \backslash B = A \cap B'$

The above laws can be proved for any sets A, B and C. But in this course we will only verify those laws by using suitable sets.

Example 1. Let the universal set S be the set of natural numbers less than 9. A = {1, 2, 5, 6, 8} and B is the set of even numbers. Verify the followings.

(a) $A \backslash B = A \cap B'$ (b) $(A')' = A$

Solution

$$S = \{1, 2, 3, 4, 5, 6, 7, 8\},\ A = \{1, 2, 5, 6, 8\},\ B = \{2, 4, 6, 8\}$$

(a) $A \setminus B = \{1, 5\}$

$B' = S \setminus B = \{1, 3, 5, 7\}$

$A \cap B' = \{1, 5\}$

$\therefore\ A \setminus B = A \cap B'$

(b) $A' = S \setminus A = \{3, 4, 7\}$

$(A')' = S \setminus A' = \{1, 2, 5, 6, 8\}$

$\therefore\ (A')' = A$

Example 2. Let the universal set S be the set of positive even integers less than 12. $A = \{x \in S \mid 2 \leq x < 8\}$. B is the set of prime numbers. C = {2, 6, 10}. Verify the following.

(a) $A \cap (B \cap C) = (A \cap B) \cap C$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution

$$S = \{2, 4, 6, 8, 10\},\ A = \{2, 4, 6\},\ B = \{2\}, C = \{2, 6, 10\}$$

(a) $B \cap C = \{2\}$

$A \cap (B \cap C) = \{2\}$

$A \cap B = \{2\}$

$(A \cap B) \cap C = \{2\}$

$\therefore\ A \cap (B \cap C) = (A \cap B) \cap C$

(b) $B \cup C = \{2, 6, 10\}$

$A \cap (B \cup C) = \{2, 6\}$

$A \cap B = \{2\}$

$A \cap C = \{2, 6\}$

$\therefore\ (A \cap B) \cup (A \cap C) = \{2, 6\}$

$\therefore\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 3. The universal set S = {2, 3, 4, 5, 6, 7, 8, 9, 10},

A = {prime numbers}, B = {even numbers}, C = {square numbers}.

Verify the followings.

(a) $A \cap B \subset C$ (b) $B \setminus C = B \cap C'$

(c) $(A \cup B)' = A' \cap B'$ (d) $(A \cap B)' = A' \cup B'$

(e) $A \setminus B \subset B'$

Solution

S = {2,3,4,5,6,7,8,9,10}, A = {2,3,5,7}, B = {2,4,6,8,10}, C = {4, 9}

(a) $A \cap B = \{2\}$

$A \cap B \subset A$

(b) $B \setminus C = \{2, 6, 8, 10\}$

$C' = \{2, 3, 5, 6, 7, 8, 10\}$

$B \cap C' = \{2, 6, 8, 10\}$

$\therefore B \setminus C = B \cap C'$

(c) $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$

$(A \cup B)' = S \setminus (A \cup B) = \{9\}$

$A' = S \setminus A = \{4, 6, 8, 9, 10\}$

$B' = \{3, 5, 7, 9\}$

$A' \cap B' = \{9\}$

$\therefore (A \cup B)' = A' \cap B'$

(d) $A \cap B = \{2\}$

$(A \cap B)' = S \setminus (A \cap B) = \{3, 4, 5, 6, 7, 8, 9, 10\}$

$A' \cup B' = \{3, 4, 5, 6, 7, 8, 9, 10\}$

$\therefore (A \cap B)' = A' \cup B'$

(e) $A \setminus B = \{3, 5, 7\}$

$B' = \{3, 5, 7, 9\}$

$\therefore A \setminus B \subset B'$

1.1.10 Product Sets

If a and b are any two elements, then (a, b) denotes an ordered pair, a is called the first element and b the second element. Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$, $b = d$.

Definition: The product of two sets A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. It is denoted by $A \times B$, and is read "A cross B". In symbol,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Note : In general,

$$A \times B \neq B \times A$$

Example 1. If $A = \{2, 4\}$, $B = \{1, 5, 6\}$, then

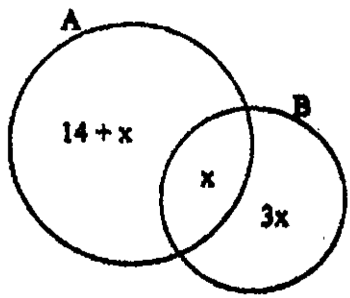
$$A \times B = \{(2, 1), (2, 5), (2, 6), (4, 1), (4, 5), (4, 6)\}$$

$$B \times A = \{(1, 2), (1, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$$

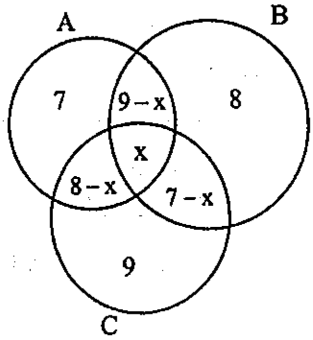
$$A \times A = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$$

Exercise 1.5

1. A and B are two sets and the numbers of elements are as shown in the Venn diagram. Given that $n(A) = n(B)$, calculate (a) x (b) $n(A \cup B)$.



2. A, B and C are three sets and the numbers of elements are as shown in the Venn diagram. Given that the universal set $S = A \cup B \cup C$ and $n(S) = 34$, find (a) the value of x (b) $n(A \cap B \cap C')$.

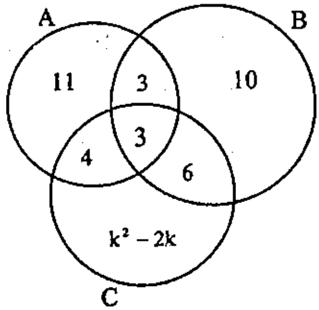


3. A, B and C are three sets and the numbers of elements are as shown in the Venn diagram. The universal set $S == A \cup B \cup C$.

(a) State the value of $n((B \cup C)')$

(b) If $n(C) = n(A)$,

find the two possible values of k.



4. $S = \{ x \mid x \text{ is an integer}, 1 \leq x \leq 10\}$, $A = \{ x \mid x - 1 \geq 3\}$

$B = \{ x \mid 8 < 4x < 30\}$

List the elements of the sets S, A, B.

Verify that (a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$

5. $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $A = \{x \mid x \text{ is a factor of } 18\}$,

$B = \{x \mid 3x - 1 > 20\}$.

Verify that (a) De Morgan's Laws (b) $A \setminus B = A \cap B'$

6. Given that $S = \{x \mid x \text{ is an integer}, 30 \leq x \leq 50\}$.

$A = \{x \mid x \text{ is a prime number}, x > 30\}$

$B = \{x \mid x \text{ is a multiple of } 5\}$, $C = \{x \mid x \text{ is an odd number}\}$

(a) Find $A \cap B$, $A \cup C$

(b) Verify that $n(B \cup C) = n(B) + n(C) - n(B \cap C)$

(c) $A' \setminus C = A' \cap C'$

7. Let $S = \{1, 2, 3, 4, 5, 7\}$, $A = \{x \mid x \text{ is a prime number}\}$

$B = \{x \mid 1 < x < 5\}$. Find $A \times B$, $B \times A$.

8. If $A = \{1, 2\}$, $B = \{2, 3, 4\}$, find $(B \setminus A) \times (A \cup B)$.

1.2 Relations, Functions and Graphs

Let us consider a simple example.

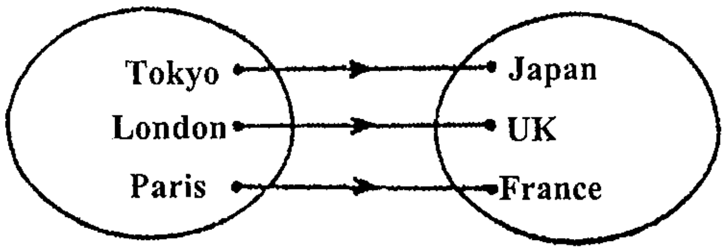


Fig. 1.16

Is Tokyo related to Japan? The answer is "yes", if we define "related to" as "is the capital of". Since Tokyo is the capital of Japan, Tokyo is related to Japan. Similarly, London is related to UK and Paris is related to France.

Is Tokyo related to UK? The answer is "no", if we define "related to" as "is the capital of". Since Tokyo is not the capital of UK, Tokyo is not related to UK. Similarly, Tokyo is not related to France. London is not related to Japan, and so on.

We have described a relation as a mapping diagram (or an arrow diagram) where the arrow represents the relation from one set to another set.

Are there other ways of representing the same information?

We have illustrated one method using word statements.

Tokyo is the capital of Japan.

London is the capital of UK.

Paris is the capital of France.

We may leave out the words "is the capital of" and simply write down pairs of names such as

Tokyo	,	Japan
London	,	UK
Paris	,	France

Anyone who has had an elementary course in Geography can observe the relation even if it is not stated explicitly.

We may also represent a relation as a set as follows:

{(Tokyo, Japan), (London, UK), (Paris, France)}

Each element is an ordered pair.

We may also represent a relation as a table

Table 1.1

Capital	Tokyo	London	Paris
Country	Japan	UK	France

Let us consider another example consisting of a set of countries and a set of official languages.

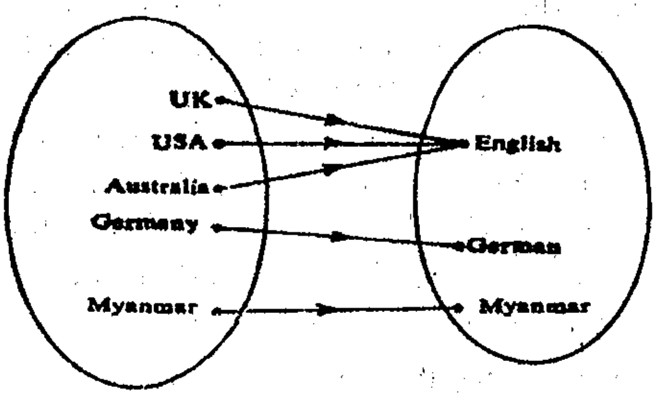


Fig. 1.17

What is the official language of UK? The answer is "English". However, if we ask which countries use English as the official language, the answer will include UK, USA and Australia.

We may also describe the relation by a graph as shown in Fig. 1.18.

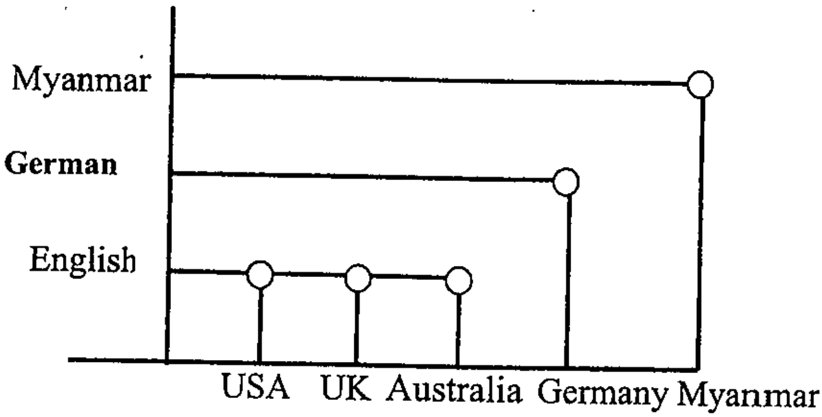


Fig. 1.18

We shall discuss the concepts of relations, functions and graphs in the following sections.

1.2.1 Relation between Two Sets

Let us consider two Sets A and B.

We can describe a relation as a rule involving an element of the set A and an element of the set B.

Example 1. Let $A = \{1, 2\}$ and $B = \{10, 20\}$. Let the relation from A to B be "is one-tenth of". Draw an arrow diagram.

Solution

We will consider each element of the set A and compare with each element of the set B. The relation is "one-tenth of".

1	is related to	10	(because 1 is one –tenth of 10).
1	is not related to	20	(because 1 is not one–tenth of 20).
2	is not related to	10	(because 2 is not one–tenth of 10).
2	is related to	20	(because 2 is one–tenth of 20).

Hence, we can draw an arrow diagram as in Fig. 1.19.

The arrow stands for "is one-tenth of".

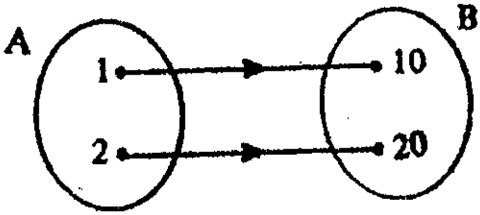


Fig. 1.19

For the two given sets mentioned above, we may define several relations such as

(a) "is not one-tenth of"

(b) "is one-tenth of"

(c) "is not greater than one-fifth of"

(d) "is less than"

Example 2. Let A = {1, 2} and B = {10, 20}. Let the relation from A to B be "is not one-tenth of". Draw an arrow diagram.

Solution

1 is not related to	10	(because 1is one-tenth of 10).
1 is related to	20	(because 1 is not one-tenth of 20).
2 is related to	10	(because 2 is not one-tenth of 10).
2 is not related to	20	(because 2 is one-tenth of 20).

Hence, we can draw an arrow diagram as in Fig. 1.20.

The arrow stands for "is not one-tenth of".

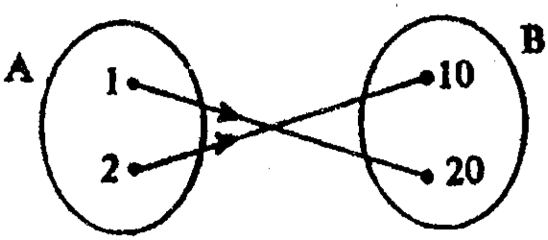


Fig. 1.20

Example 3. Let A = {1, 2} and B = {10, 20}. Let the relation form A to B be "is one-fifth of". Draw an arrow diagram.

Solution

1 is not related to	10	(because 1 is not one-fifth of 10).
1 is not related to	20	(because 1 is not one-fifth of 20).
2 is related to	10	(because 2 is one-fifth of 10).
2 is not related to	20	(because 2 is not one-fifth of 20).

We can draw an arrow diagram as in **Fig.1.21**.The arrow stands for "is one-fifth of". Note that there is no arrow coming out of 1 and no arrow going into 20. This is because 1 is not related to any element of the set B and no element of A is related to 20.

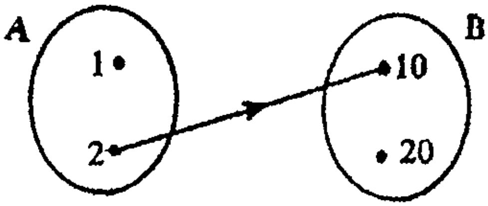


Fig. 1.21

Example 4. Let A = {1, 2} and B = {10, 20}. Let the relation from A to B be "is not greater than one-fifth of". Draw an arrow diagram.

Solution

Let x be an element of A. i.e., $x \in A$

Let y be an element of B. i.e., $y \in B$

The given relation is "is not greater than one-fifth of". We can say that x is related to y if x is not greater than one-fifth of y.

i.e., $x \not> \frac{y}{5}$

1 is related to 10 (because $1 \not> \frac{10}{5}$)

1 is related to 20 (because $1 \not> \frac{20}{5}$)

2 is related to 10 (because $2 \not> \frac{10}{5}$)

2 is related to 20 (because $2 \not> \frac{20}{5}$)

We can draw the arrow diagram as in Fig. 1.22. The arrow stands for "is not greater than one-fifth of".

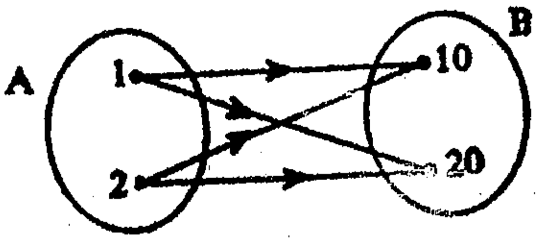


Fig. 1.22

We can see that every element of A is related to every element of B.

Example 5. Let A = {1, 2} and B = {10, 20}. Let the relation from A to B be described by "is less than". Draw an arrow diagram.

Solution

1 is related to 10 (because 1 is less than 10).

1 is related to 20 (because 1 is less than 20).

2 is related to 10 (because 2 is less than 10).

2 is related to 20 (because 2 is less than 20).

The arrow diagram is the same as in Fig. 1.22.

The arrow stands for "is less than".

Examples 4 and 5 show that for given sets A and B, an arrow diagram may represent more than one form of verbal statements.

Example 6. Let C = {1, 5} and D = {5, 10}. Let the relation from C to D be described by "is not greater than one-fifth of". Draw an arrow diagram.

Solution

1 is related to 5 (because $1 \ngtr \frac{5}{5}$)

1 is related to 10 (because $1 \ngtr \frac{10}{5}$)

5 is not related to 5 (because $5 > \frac{5}{5}$)

5 is not related to 10 (because $5 > \frac{10}{5}$)

The arrow diagram is shown in Fig. 1.23.

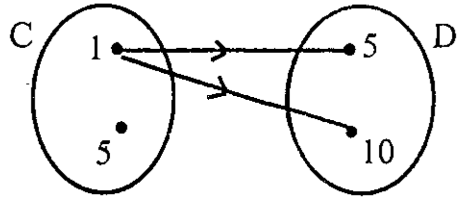


Fig. 1.23

Example 7. Let C = {1, 5} and D = {5, 10}. Let a relation from C to D be described by "is less than". Draw an arrow diagram.

Solution

1 is related to	5	(because	$1 < 5$)
1 is related to	10	(because	$1 < 10$)
5 is not related to	5	(because	$5 \nless 5$)
5 is related to	10	(because	$5 < 10$)

The arrow diagram is shown in Fig. 1.24.

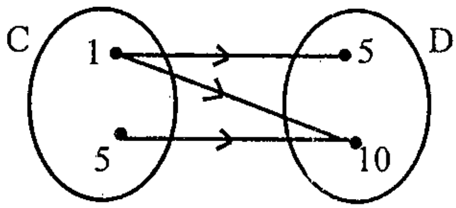


Fig. 1.24

Examples 4, 5, 6 and 7 show that sets are important in talking about relations. We need to know the elements of the two sets involved in a relation before we can draw an arrow diagram.

We may notice that the elements in a set need not be integers and also that the words which describe a relation need not have arithmetical meaning.

Exercise 1.6

1. Relation can be described in words. Some examples are

"is twice of"

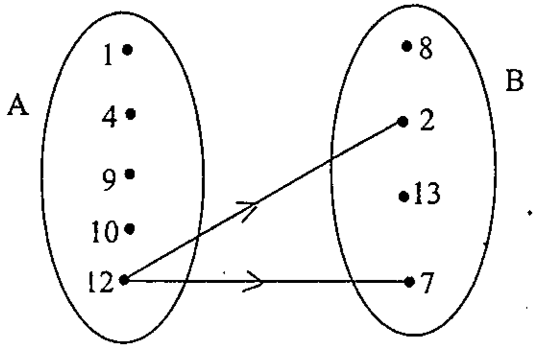
"is more than 3 times of"

"is the daughter of"

Write down five more examples.

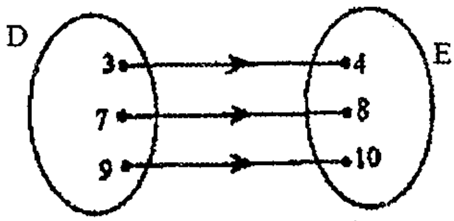
For each of the three examples given above, find a pair of sets A and B so that the words describe a relation from A to B.

2. Take the sets A and B as shown in the following figure. Copy and complete the arrow diagram to describe the relation "is greater than "from A to B.

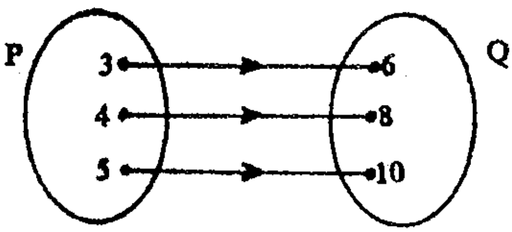


3. Using the same sets as in question 2, draw an arrow diagram to describe the relation "is less than "from A to B.

4. A relation from D to E is described in the following figure. Describe it in words.



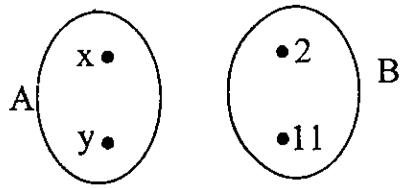
5. A relation from P to Q is described in the following figure. Describe it in words.



6. S = {0, 1, 2, 5} and T = {2, 3, 4, 6, 7}. The given relation from S to T "is one less thar ". Draw the corresponding arrow diagram.

7. Draw the arrow diagram to describe the relation "is a factor of "from the set A = {2, 3, 5, 7, 11} to the set B = {1, 6, 12, 17, 30, 35}.

8. Draw ten different arrow diagrams for the sets A and B shown in the following figure.



1.2.2 Set of Ordered Pairs and Graph

{1, 10} is a set of two elements and {10, 1} denotes the same set. The order in which the elements are written is not important.

Let A and B be any two sets, and let $x \in A$, $y \in B$ be any two elements. A pair of such elements x and y, written as (x, y), is called an ordered pair. In an ordered pair, we consider the ordering to be important.

For example, take A = {g, h} and B = {2, 3}. Possible ordered pairs are

(g, 2), (g, 3), (h, 2), (h, 3);

(2, g), (3, g), (2, h), (3, h);

(g, g), (g, h), (h, g), (h, h);

(2, 2), (2, 3), (3, 2), (3, 3);

We should remember that (2, 3) and (3, 2) are not the same. We consider these ordered pairs to be distinct.

We can use ordered pairs to describe a relation.

Consider an ordered pair (x, y) with $x \in A$ and $y \in B$. If x is related to y, we will take the ordered pair (x, y). If x is not related to y, we will not take the ordered pair (x, y).

All the ordered pairs (x, y) chosen in this way, form a set. This set, denoted by R, can be used to find out whether an element x in A is related to an element y in B or not.

If $(x, y) \in R$, then an arrow from x to y exists in the arrow diagram of the given relation. Conversely, each arrow in the diagram indicates an ordered pair in R.

Example 1. Let A = {5, 10} and B = {1, 5}.

Describe the relation shown in the following diagram using ordered pairs.

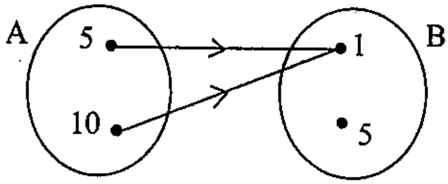


Fig. 1.25

Solution

R = {(5, 1), (10, 1)} is the set of ordered pairs which describes the relation.

Example 2. Let A = {1, 2, 3, 4} and B = {3, 5, 2, 7}

Let R = {(3, 5), (2, 2), (2, 3), (3, 7), (1, 3)}

describe a relation from A to B. Draw the corresponding arrow diagram.

Solution

The arrow diagram is shown in Fig. 1.26.

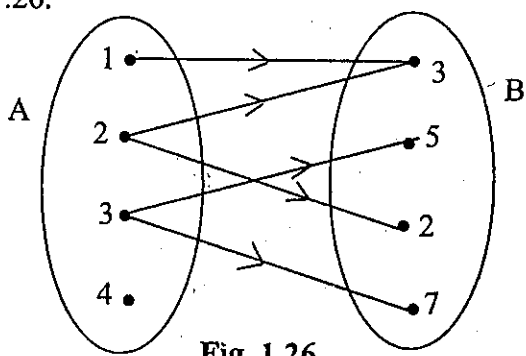


Fig. 1.26

Graph of a relation

A graph is a set of ordered pairs which describe a relation. Let A = $\{x_1, x_2\}$ and B = $\{y_1, y_2, y_3\}$.

Let $x_1 \in A$ be related to $y_1 \in B$. Place the elements of A on a horizontal line and the elements of B on a vertical line. Draw vertical lines through the elements of A and horizontal lines through the elements of B.

Since the ordered pair (x_1, y_1) belongs to the relation, a dot is placed at the intersection of the vertical line passing through x_1 and the horizontal line passing through y_1 as shown. In this way each ordered pair corresponds to a dot.

The elements of the graph of the relation $\{(x_1, y_1), (x_2, y_2)\}$ is represented by dots in Fig. 1.27.

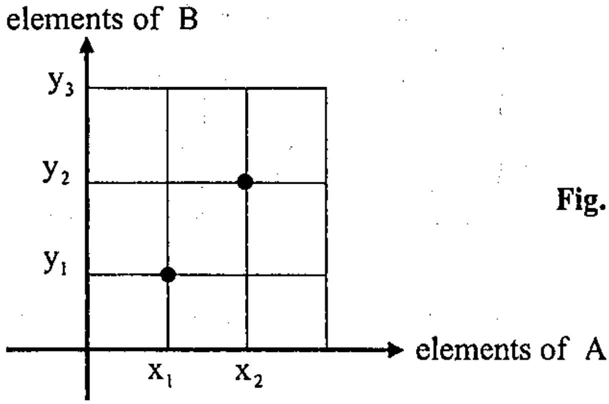


Fig. 1.27

Example 3. $A = \{5, 10\}$ and $B = \{1, 2\}$. Draw the graph of the relation $R = \{(5, 1), (10, 1)\}$ from the set A to the set B.

Solution

We draw vertical lines passing through the elements of A. We draw horizontal lines passing through the elements of B. We place dots on the points (5, 1) and (10, 1) belonging to the relation R. The graph consists of only two points (5, 1) and (10, 1).

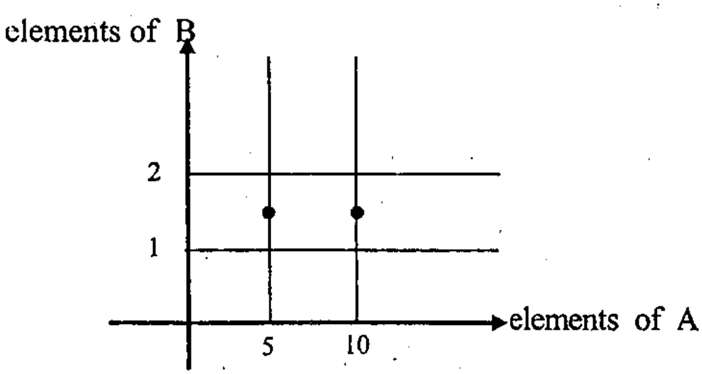


Fig. 1.28

Example 4. Let $A = \{x, y, z, w\}$ and $B = \{3, 5, 7, 9\}$. Draw the graph of the relation $\{(x, 3), (y, 3), (y, 5), (y, 7), (z, 5), (z, 9)\}$.

Solution

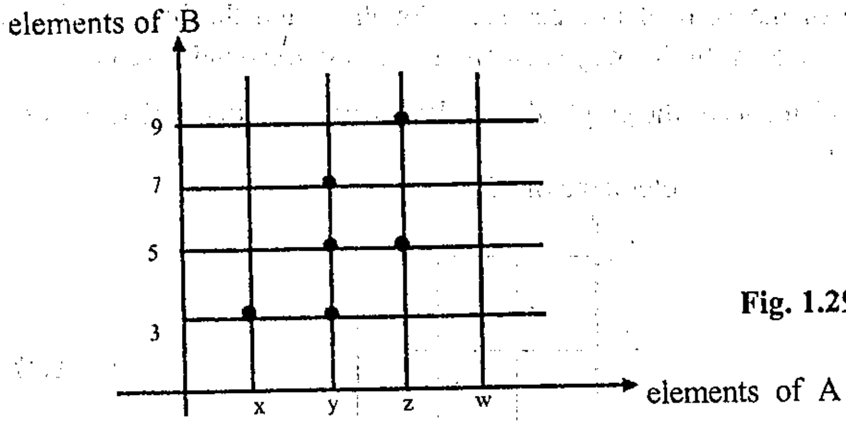


Fig. 1.29

The graph consists of six points (x, 3), (y, 3), (y, 5), (y, 7), (z, 5) and (z, 9).

Exercise 1.7

1. $A = \{2, 3, 5, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Take the relation "is a factor of" from A to B. Draw an arrow diagram. Write down the set of ordered pairs which describes this relation.
2. $X = \{2, 6, 8\}$ and $Y = \{1, 3, 5\}$. Let the given relation from X to Y be "x is two times y" where $x \in X$ and $y \in Y$. Write down the set of ordered pairs which describes this relation. Draw the graph of this relation.
3. $A = B = \{1, 2, 3, 6\}$. Take the relation "is a factor of" from A to B. Draw an arrow diagram.
4. Write down the set of ordered pairs which describes the relation "is less than" from A to B, where $A = B = \{1, 2, 3, 4\}$. Draw the graph of this relation.
5. Repeat the above for the relation "is greater than".

1.2.3 Functions

A function is a special relation. A function from a set A to a set B is a relation such that each element of A is related to exactly one element of B.

This means that in the arrow diagram, one and only one arrow must point out from each element of A.

Example 1. Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$. Let the relation from A to B be described by the following arrow diagram. Is it a function?

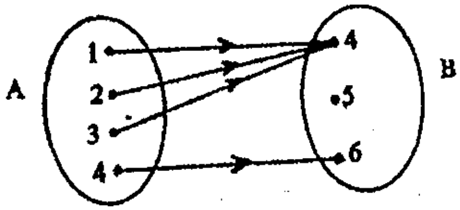


Fig. 1.30

Solution

In the diagram, only one arrow points out from each element of A. The given relation is a function since each element of A is related to exactly one element of B.

Example 2. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Let the relation from A to B be $R = \{(1, 4), (2, 4), (3, 4), (1, 6)\}$.

Is this a function?

Solution

The given relation R is not a function since 1 is related to both 4 and 6.

Definition : If a relation from A to B is a function and if $a \in A$ is related to $b \in B$, b is called the image of a.

We write

$$a \longmapsto b$$

and we say :"a maps to b"

The first set A is called the domain of the function.

The range of a function is the set of images of all $a \in A$ where A is the domain. The range is a subset of B.

Example 3. Let $A = \{1, 2, 3\}$, and $B = \{4, 5, 6\}$. Let the function $R = \{(1, 4), (2, 4), (3, 4)\}$. Find the domain and range of the function.

Solution

The function R maps the set A to the set B.

The domain is the first set, i.e., A.

Domain = {1, 2, 3}

To find the range, we must first find the images.

1 maps to 4 $\quad 1 \longmapsto 4$

2 maps to 4 $\quad 2 \longmapsto 4$

3 maps to 4 $\quad 3 \longmapsto 4$

Since all elements of the set A map to the element 4 of B, the range is a set with one element, 4.

Range = {4}

Example 4. Let A = {1, 3, 5, 7, 9} and B = {1, 2, 3, ..., 12}.

Let a function from A to B be defined as $n \longmapsto n + 2$ whenever $n \in A$. Find the set of ordered pairs describing this function. Find the range.

Solution

$1 \longmapsto 3 \qquad (1 \longmapsto 1 + 2, \text{ i.e., } 3)$

$3 \longmapsto 5$

$5 \longmapsto 7$

$7 \longmapsto 9$

$9 \longmapsto 11$

The required set of ordered pairs is {(1, 3), (3, 5), (5, 7), (7, 9), (9, 11)}.

The range of this function is {3, 5, 7, 9, 11}.

Example 5. Let A = {1, 3, 5, 7, 9} and

B = {2, 4, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19, 20}.

Let the function from A to B be $n \longmapsto 2n$ whenever $n \in A$.

Describe the function using a set of ordered pairs. Find the range.

Solution

Let us consider the elements of A.

For each $n \in A$, the image of n is 2n.

$$1 \longmapsto 2$$
$$3 \longmapsto 6$$
$$5 \longmapsto 10$$
$$7 \longmapsto 14$$
$$9 \longmapsto 18$$

The function is the set of ordered pairs

$\{(1, 2), (3, 6), (5, 10), (7, 14), (9, 18)\}$.

The range of this function is $\{2, 6, 10, 14, 18\}$.

We observe that the range is a subset of B.

Example 6. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{-1, 1\}$.

A function from A to B is such that $n \longmapsto (-1)^n$ whenever $n \in A$. Find the set of ordered pairs describing this function. Find the range.

Solution

$$1 \longmapsto (-1)^1 = -1$$
$$2 \longmapsto (-1)^2 = 1$$
$$3 \longmapsto (-1)^3 = -1$$
$$4 \longmapsto (-1)^4 = 1$$
$$5 \longmapsto (-1)^5 = -1$$
$$6 \longmapsto (-1)^6 = 1$$

The set of ordered pairs describing the function is

$\{(1, -1), (2, 1), (3, -1), (4, 1), (5, -1), (6, 1)\}$

The range of this function is $\{-1, 1\}$.

We observe that the range is B itself.

The arrow diagram is shown in Fig. 1.31.

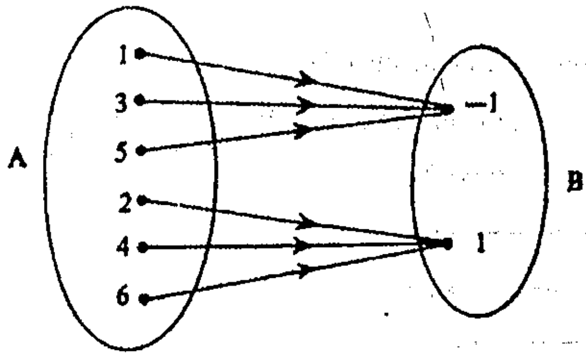


Fig. 1.31

So far, we have considered functions from a set of numbers to another set of numbers. The following example involves a triangle.

Example 7. Assume that we have an equilateral triangle ABC made of cardboard. Let us "turn it over" as in Fig. 1.32.

Can we describe the process as a function?

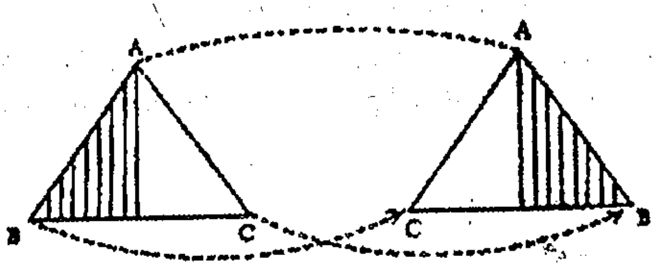


Fig. 1.32

Solution

Δ ABC is an equilateral triangle.

Let S = {A, B, C} be the set of vertices.

The process of "turning the triangle over" can be described by the arrow diagram in Fig. 1.33.

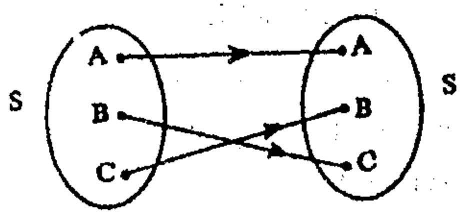


Fig. 1.33

This function is also described by the set of ordered pairs

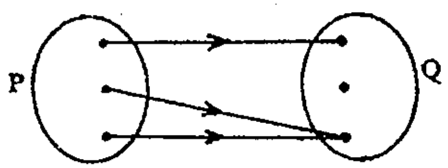
$\{(A, A), (B, C), (C, B)\}$.

The range of the function is $\{A, B, C\}$ which is the set S itself.

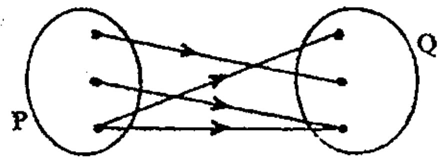
Exercise 1.8

1. Each of the arrow diagram shown below describes a relation from a set P to Q. Which of these relations are functions?

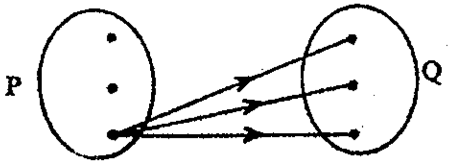
(a)



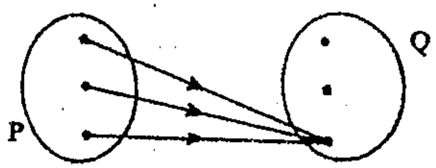
(b)



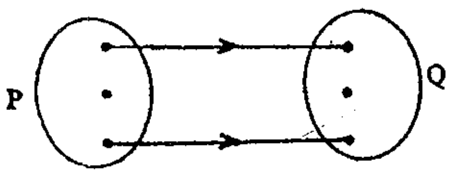
(c)



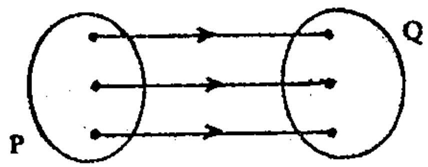
(d)



(e)



(f)



2. $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. A mapping is given by

$a \longmapsto 1$.

$b \longmapsto 2$

$c \longmapsto 2$

Draw the arrow diagram for this function. Write down the set of ordered pairs describing this function.

3. $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. A relation from X to Y is given by the set of ordered pairs $\{(a, 3), (b, 2), (b, 4), (c, 1)\}$. Draw the arrow diagram. Is it a function? Why?

4. $S = \{1, 2, 3, 4\}$ and $T = \{8, 9\}$. If the image of each odd number in S is 9 and the image of each even number is 8, describe the function by an arrow diagram. Write down the set of ordered pairs for this function.

5. Let $S = \{1, 2, 3, \ldots, 20\}$. A relation from S to S is described by the set of ordered pairs (x, y) where $y \in S$, $x \in S$ and $y = 3x$. Write down all the ordered pairs for this relation. Is this relation a function? Why?

6. $K = \{1, 2, 3, 4, 5, 6\}$. A function from K to K is such that for $n \in K$

$$n \longmapsto 1 \text{ if } n \text{ is odd, and}$$

$$n \longmapsto \frac{n}{2} \text{ if } n \text{ is even.}$$

Describe this function by a set of ordered pairs.

7. $E = \{3, 4, 5, \ldots, 21\}$ and $F = \{2, 4, 6, 8, 10\}$.

A function from F to E is such that $n \longmapsto n + 3$ for each $n \in F$. Find the set of ordered pairs for the function. Write down the range.

8. Describe all the possible functions from $A = \{a, b, c\}$ to $B = \{p, q\}$ by drawing arrow diagrams.

1.2.4 One – to – one Function

A function from a set A to a set B is called a one – to–one function if no two different elements of A map to the same element of B.

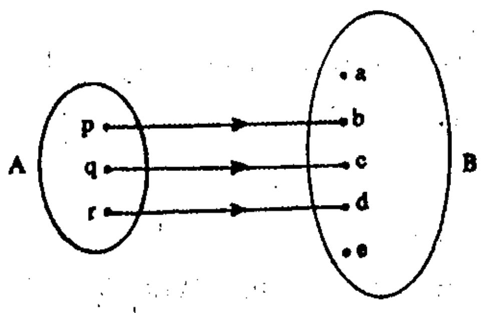


Fig. 1.34

Figure 1.34 describes a one-to-one function. There is exactly one arrow coming out from each element of A and there is at most one arrow going into each element of B. Thus, the function $\{(p, b), (q, c), (r, d)\}$ is one-to-one.

Some elements of B do not have any arrow going into it.

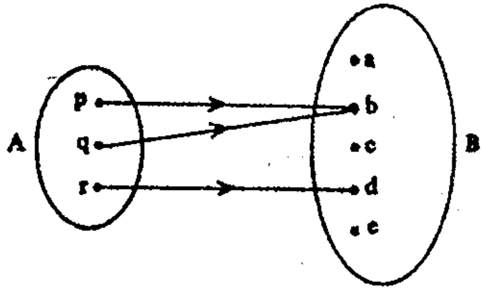


Fig. 1.35

Next consider Fig. 1.35.

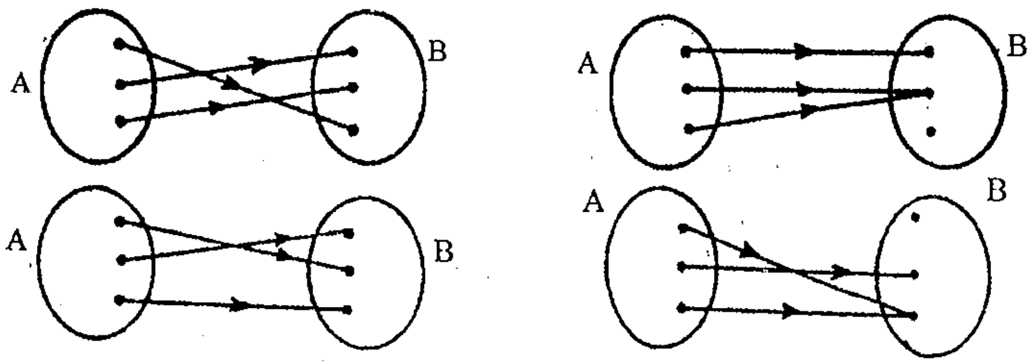
The function {(p, b), (q, b), (r, d)} is not one-to-one because two different elements p and q map to the same element b. The element b has two arrows going into it.

In general, if the arrow diagram of a function contains a pair of arrows of the form shown in Fig. 1.36, then the function is not one-to-one.

Fig. 1.36

Exercise 1.9

1. Which of the arrow diagram given below describe a one-to-one function?



Draw an arrow diagram to describe a one-to-one function from the set A = {m, n} to the set B = {p, q}.

3. Find three different arrow diagrams to describe a one-to-one function from the set A = {m, n} to the set B = {p, q, r}.

4. By drawing arrow diagrams, show that there are six different one-to-one functions from the set A = {l, m} to the set B = {p, q, r}.

5. Draw arrow diagrams to describe at least seven different one-to-one functions from the set A = {2, 4, 6, 8} to the set B = {1, 3, 5, 7}.

1.2.5 Graphs

We have seen in Section 1.2.2 that a graph is a set of ordered pairs which describe a relation. We have also seen some simple graphs.

Example 1. Let A = {0, 1, 2, 3, 4, 5} and Z be the set of integers. Let the given function from A to Z be $x \longmapsto 2x,\ x \in A$

(a) Show the function in table form.

(b) Draw the corresponding graph.

Solution

Consider the elements in A.

For each $x \in A$, the image of x is 2x.

We have $0 \longmapsto 0$

$1 \longmapsto 2$

$2 \longmapsto 4$

$3 \longmapsto 6$

$4 \longmapsto 8$

$5 \longmapsto 10$

(a) We can construct the following table.

Table 1.2

x	0	1	2	3	4	5
2x	0	2	4	6	8	10

(b) The corresponding graph is shown in Fig. 1.37

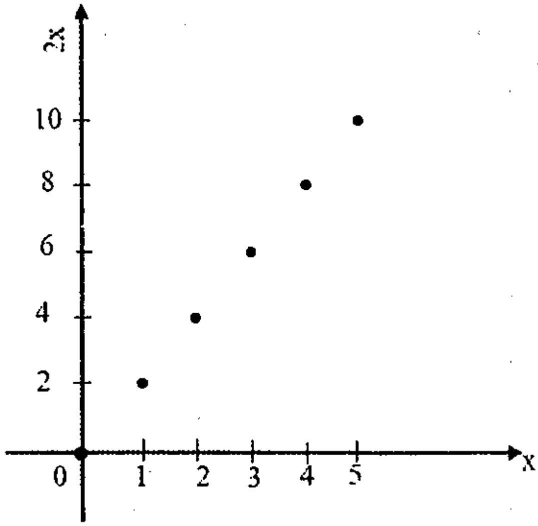


Fig. 1.37

Example 2. Let A = {0, 1, 2, 3, 4, 5} and Z be the set of integers. Let the function from A to Z be

$$x \longmapsto 2x + 1, \; x \in A$$

(a) Show the function in table form. (b) Draw the corresponding graph.

Solution

Consider the elements in A.

For each $x \in A$, the image of x is 2x + 1.

We have
$$0 \longmapsto 1$$
$$1 \longmapsto 3$$
$$2 \longmapsto 5$$
$$3 \longmapsto 7$$
$$4 \longmapsto 9$$
$$5 \longmapsto 11$$

(a) We can construct the following table.

Table 1.3

x	0	1	2	3	4	5
2x + 1	1	3	5	7	9	11

(b) The corresponding graph is shown in Fig. 1.38.

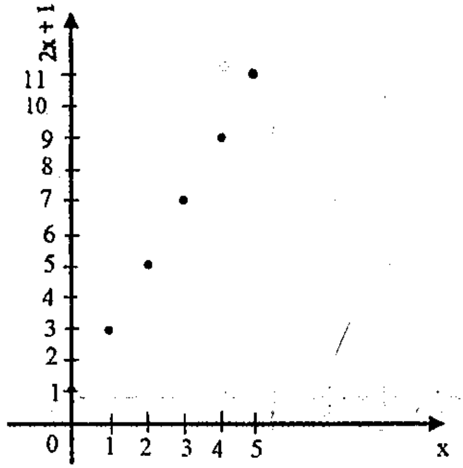


Fig. 1.38

Example 3. Let B = {– 2, – 1, 0, 1, 2} and Z be the set of integers. Let the function from B to Z be $x \longmapsto x^2$.

(a) Show the function in the table form.

(b) Draw the corresponding graph.

Solution

Consider the elements in B. For each $x \in B$, the image of x is x^2.

(a) We can construct the following table.

Table 1.4

x	– 2	– 1	0	1	2
x^2	4	1	0	1	4

(a) The corresponding graph is shown in Fig. 1.39.

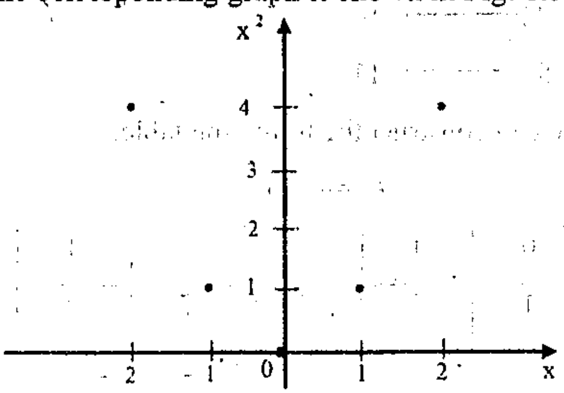


Fig 1.39

Exercise 1.10

1. Make a table for the function $x \longmapsto x + 1$ from the set $\{0, 1, 2, 3, 4\}$ to the set Z. Draw the graph.

2. Make tables for the functions $x \longmapsto 2x$, $x \longmapsto 3x$, $x \longmapsto 4x$ from the set $\{0,1,4\}$ to the set Z. Draw the graphs of these functions. What do you observe?

3. Make a table for the function $x \longmapsto x$ from the set $\{0, 1, 2, 3\}$ to the set Z, and draw a graph. Draw a smooth curve through the points.

4. Make tables for the functions.

   (a) $x \longmapsto x$, (b) $x \longmapsto -x$, (c) $x \longmapsto x + 2$,

   (d) $x \longmapsto -x + 2$, (e) $x \longmapsto 2x - 1$.

   from the set $A = \{-5, -4, -3, -1, 0, 1, 2, 3, 4\}$ to the set Z. Draw the graphs. Can you describe a property which is common to these graphs?

5. Draw the graph of the relation
   $R = \{(-4, -3), (-3, -2), (-2, -1), (-1, 0), (0, 1), (1, 2), (2, 3), (3, 4)\}$.

SUMMARY

1. A set is a collection of clearly defined objects.
2. Each object can be checked whether it belongs to the set or not.
3. Equal sets have exactly the same elements.
4. The empty set is the set with no elements.
5. P is a subset of Q if every element of P is also an element of Q.
6. The intersection of two sets P and Q is the set of elements which are common to both P and Q.
7. The union of two sets A and B is the set of elements which belong either to A, or to B, or both A and B.
8. The difference of two sets A and B, A \ B, is the set whose elements are those of A, but are not in B.

9. The open interval (a, b) is the subset $\{x \mid a < x < b\}$ of R and the closed interval [a, b] is the subset $\{ x \mid a \leq x \leq b\}$ of R.

10. A function is a special relation. A function from a set A to a set B is a relation such that each element of A is related to exactly one element of B.

11. If a relation from A to B is a function and if $a \in A$ is related to $b \in B$, b is called the image of a. The first set A is called the domain of the function. The range of the function is the set of images of all $a \in A$ where A is the domain. The range is a subset of B.

12. A function from a set A to a set B is called a one – to –one function if no two different elements of A map to the same element of B.

CHAPTER 2

Introduction to Coordinate Geometry

Rene Descartes' greatest contribution to mathematics was the discovery of coordinate systems and their applications to problems of geometry. Coordinate system used in this book is referred to as Cartesian coordinate system.

We have seen how coordinate systems work on a line.

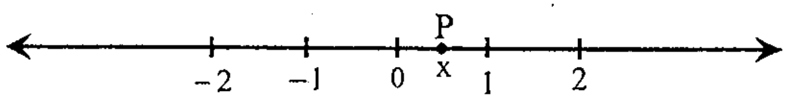


Fig. 2.1

Once we have set up a coordinate system on a line, every number corresponds to a point on the line and every point on the line corresponds to a number.

We shall now extend this idea to the points in a plane. In a plane, a point will correspond not to a single number, but to an ordered pair of numbers. The scheme works like this. First we take a line X'OX in the plane. This line will be called the X-axis. An arrowhead in an axis indicates the positive direction of that axis.

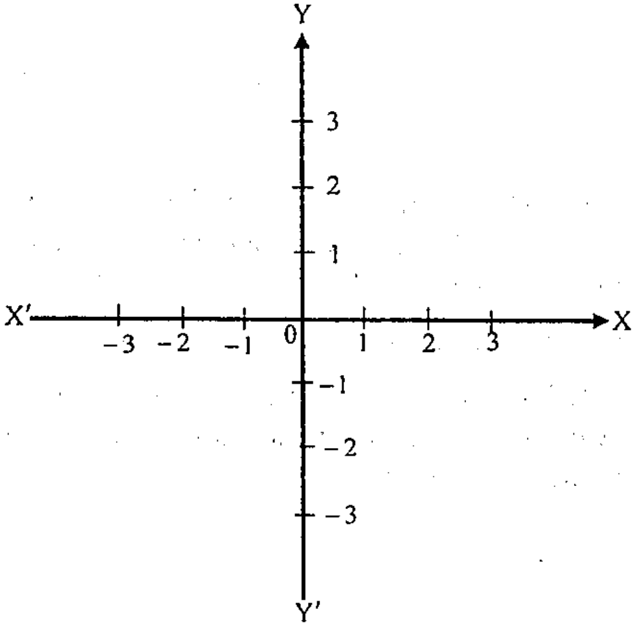


Fig. 2.2

Now we draw Y'OY perpendicular to the X-axis through the point with coordinate 0. On Y'OY we have set up a coordinate system in such a way that the zero point on Y is the zero point on X. The line Y'OY will be called the Y-axis. As before, we indicate the positive direction with an arrowhead. The point O where the line X'OX intersects the line Y'OY is called the origin.

We can now describe any point P of the plane by an ordered pair of numbers, as follows. Draw PM, a perpendicular to the X-axis. Let x be the coordinate of M on the line X'OX. The number x is called the x-coordinate of P.

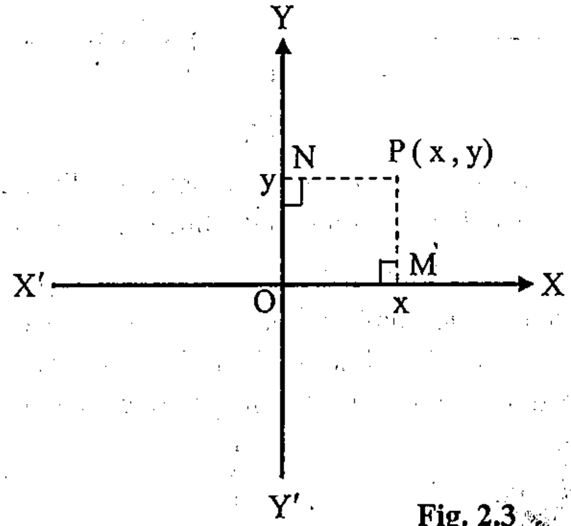


Fig. 2.3

We then draw PN, a perpendicular to the Y-axis. Let the foot of the perpendicular be the point N. Let y be the coordinate of N on the line Y'OY. The number y is called the y-coordinate of P.

In short, we indicate that P has these coordinates by writing P (x, y).

Just as a single line separates the plane into two pieces (each of which is a half-plane), so the two axes separate the plane into four parts, called quadrants. The four quadrants are identified by numbers.

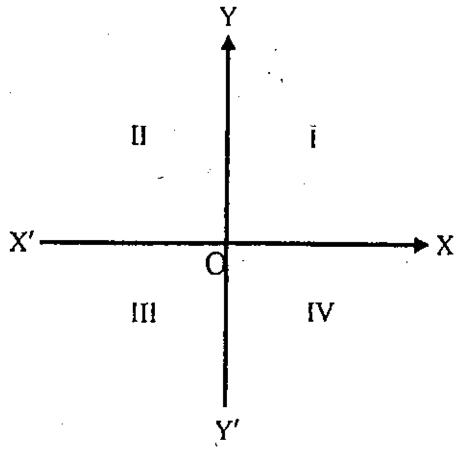


Fig. 2.4

We have shown that under the scheme that we have set up, every point P determines an ordered pair of real numbers. Does it work in reverse? That is, does every ordered pair (a, b) of real numbers determine a point? It is easy to see that the answer is "Yes".

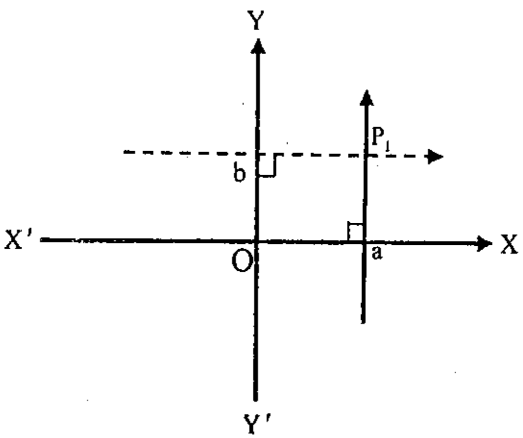


Fig. 2. 5

At the point of the X-axis with coordinate x = a, we draw a perpendicular. Next we draw another perpendicular at the point with coordinate y = b. The point where these perpendiculars intersect is the point with coordinates (a, b).

Thus we have a one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. Such a correspondence is called a rectangular coordinate system. To describe such a coordinate system, we need to choose

(1) a line X'OX to be the X-axis,

(2) a line Y'OY perpendicular to X'OX to be the Y-axis, and

(3) a positive direction on each of the axes.

Once we have made these choices, the coordinate systems on both axes are determined, and they in turn determine the coordinates of all points of the plane. With reference to a coordinate system, every point P determines an ordered pair (a, b) and every ordered pair (a, b) determines a point. It will therefore do on harm to ignore the difference between points and number-pairs. This will enable us to use such convenient phrases as "the point (2, 3)" and "P = (2, 3)".

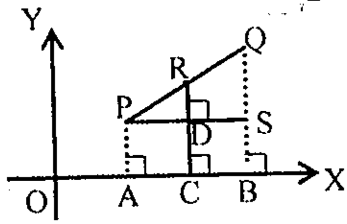
2.1 Mid-point Formula

Suppose P = (a, b) and Q = (c, d). If R is the mid- point of PQ, then

$$R = \left(\frac{a+c}{2}, \frac{b+d}{2}\right).$$

Let R = (x, y)

$$x = OC = OA + AC = a + \frac{1}{2}$$



$$(AB) = a + \frac{1}{2}(c - a) = \frac{a+c}{2}$$

$$y = RC = RD + DC = \frac{1}{2}(QS) + PA = \frac{1}{2}(d - b) + b = \frac{b+d}{2}$$

$$\therefore R = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

Example 1. M is the midpoint of the line joining the points A and B. Find

(a) the coordinates of M if A and B have coordinates (–1, 2) and (3, –4) respectively.

(b) the coordinates of A if M and B have coordinates (2, –1) and (3, 2) respectively.

Solution

(a) Since M is the midpoint of AB, $M = \left[\frac{(-1)+3}{2}, \frac{2+(-4)}{2}\right]$
$= (1, -1)$.

(b) Suppose the coordinates of A are (a, b). Since M (2, –1) is the midpoint of AB.

$$(2, -1) = \left[\frac{a+3}{2}, \frac{b+2}{2}\right]$$

i.e. $\quad 2 = \dfrac{a+3}{2} \quad$ and $\quad -1 = \dfrac{b+2}{2}$

$a = 1 \quad$ and $\quad b = -4$

The coordinates of A are $(1, -4)$.

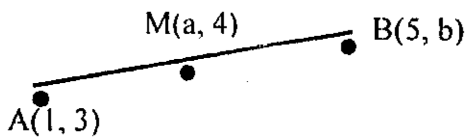
Example 2. The point M (a, 4) is the midpoint of the line segment with endpoints at A (1,3) and B (5, b). Find the values of a and b.

Solution Since M is the midpoint of AB,

$a = \dfrac{1+5}{2}$, $\quad 4 = \dfrac{3+b}{2}$

$= 3$, $\quad 8 = 3 + b$

$b = 5$



2.2 The Slope of a Non vertical Line

The X-axis and all lines parallel to it are called horizontal lines.

The Y-axis and all lines parallel to it are called vertical lines.

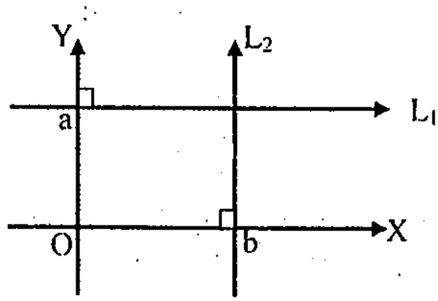


Fig. 2.6

In Fig. 2.6, it is easy to see that all points of the horizontal line L_1 have the same y -coordinate a, because the point (0, a) is the common foot of all perpendiculars to the Y-axis from points of L_1. Similarly, all points of the vertical line L_2 have the same x-coordinate b.

Of course, a segment is called horizontal if the line containing it is horizontal, and a segment is called vertical if the line containing it is vertical.

Two of the important topics discussed in geometry concerned themselves with parallelism and perpendicularity of two lines. We would like to take another look at these topics, but from an analytic standpoint. This can be done, though, only after an understanding of the term slope has been established.

We have often heard people speak of a hill as having a "steep" slope or a "gradual" slope; yet precisely what did they mean? Are the slopes of the hills in Fig. 2.7 (a) and 2.7 (b) "steeper" than that of Fig. 2.7 (c)?

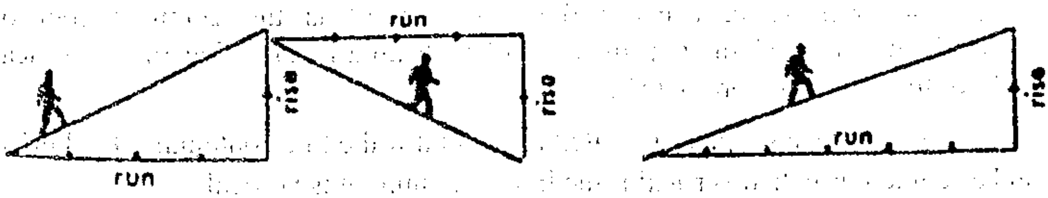
(a) (b) (c)

Fig. 2.7

Is it possible to distinguish between the slopes of the hills in Fig. 2.7(a) and 2.7 (b)? that is, in Fig. 2.7 (a) we seem to be going uphill, while in Fig. 2.7 (b) our direction is downhill. To answer these questions, the mathematician has defined the slope of a hill in terms of a ratio.

$$\text{slope of a hill} = \frac{\text{rise}}{\text{run}}$$

Let us examine our three hills to see exactly what this means. The "rise" of a hill is the change in our vertical position as we move from the bottom to the top of the hill or from the top to the bottom The "run" is the distance we moved horizontally in travelling from the bottom to the top of the hill or from the top to the bottom. By measuring the "rise" and "run" in Fig. 2.8 (a) we discover the "rise", or vertical change, is only two units, while the



(a) (b) (c)

Fig. 2.8

"run", or horizontal change, is four units. Hence, we say that the slope of the hill in Fig. 2.8 (a) is

$$\text{slope}_1 = \frac{\text{rise}}{\text{run}} = \frac{2}{4} = \frac{1}{2}$$

In the same way, by measuring the "rise" and "run" in Fig. 2.8 (c) we note that they are 2 units and 6 units respectively, and hence, the slope of that hill is

$$\text{slope}_3 = \frac{\text{rise}}{\text{run}} = \frac{2}{6} = \frac{1}{3}$$

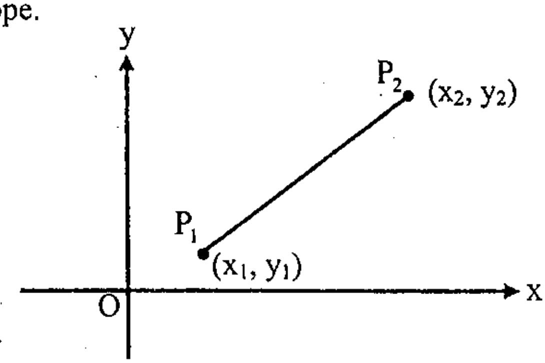
In terms of these numbers, we can conclude that the slope of the first hill is greater than that of the third, for the fraction $\frac{1}{2}$ is greater than $\frac{1}{3}$.

The hill in Fig. 2.8 (b) presents a bit of a problem, for here the "rise" is actually negative. In going from the top of the hill to the bottom, the vertical position of the person has decreased two units, while his horizontal position has increased four units. Thus, the slope becomes

$$\text{slope}_2 = \frac{\text{rise}}{\text{run}} = \frac{-2}{4} = \frac{-1}{2}$$

Hence, it appears that the "rise" may be either positive or negative depending on whether the vertical change has been positive or negative. This, in turn, will determine the sign of the slope.

Fig. 2. 9



Now, how does all this tie in with our coordinate system? Rather than considering hills and their slopes, we now investigate line segments and their slopes. Hence, we must ask ourselves to question:

(1) How great is the "rise" between P_1 and P_2; that is, what is the extent of the vertical change?

(2) How great is the "run" between P_1 and P_2; that is, what is the extent of the horizontal change?

To answer both of these questions it is necessary to find the coordinates of the point where the line through P_2 parallel to the Y-axis intersects the line through P_1 that is parallel to the X-axis.

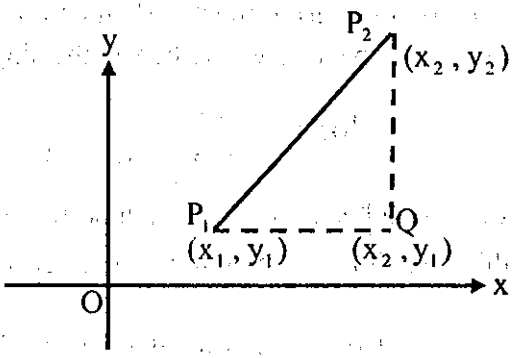


Fig. 2.10

Examining this diagram, we realize that the coordinates of Q will have to be (x_2, y_1). From this it follows that the "run" must be $(x_2 - x_1)$, while the "rise" is $(y_2 - y_1)$. This brings us to the point where we can safely formulate the definition of the slope of a line segment.

Definition: The slope of a line segment whose endpoints are (x_1, y_1) and (x_2, y_2) is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Some facts about slopes are obvious from the definition.

1. If the points P_1 and P_2 are interchanged, the slope is the same as before, because

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}.$$

In other words, the slope of a segment does not depend on the order in which its endpoints are named.

On the other hand, it is important to name the coordinates in the same order in the numerator and denominator. The formula $\frac{y_1 - y_2}{x_2 - x_1}$ is not a correct formula for the slope.

3. For non-vertical segments, the slope formula always gives as a number, because the denominator $x_2 - x_1$ cannot be 0.

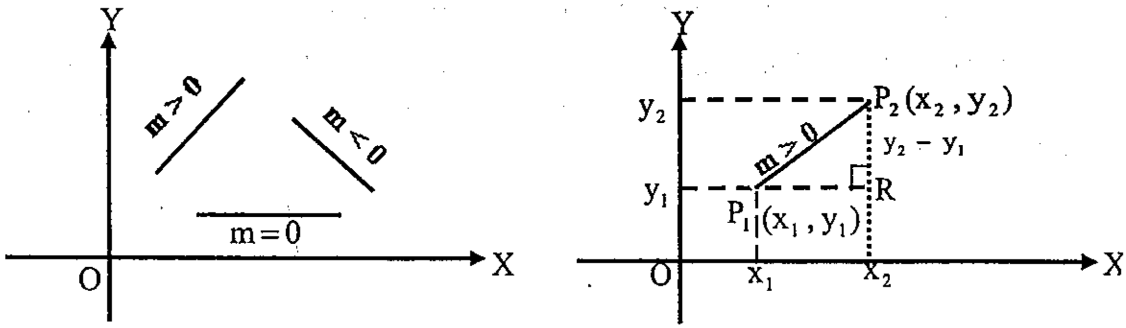
4. For vertical segments, the slope formula never gives us a number, because in this case the denominator $x_2 - x_1$ is equal to 0. In fact, there is no such thing as the slope of a vertical segment.

5. If a segment is horizontal, its slope is 0. (The numerator $y_2 - y_1$ is 0 and the denominator $x_2 - x_1$ is not 0.)

6. If a segment is not horizontal (or vertical), then its slope is not 0.

7. If a segment rises from left to right, its slope is positive.

   If the segment descends from left to right, slope is negative. (See Fig. 2.11)



(i) **Fig.2.11** **(ii)**

If a segment has positive slope, then its slope is the ratio of two distances as in Fig. 2.11 (ii). Since $x_1 < x_2$ and $y_1 < y_2$, we have $P_1R = x_2 - x_1$ and $RP_2 = y_2 - y_1$.

Therefore, $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{RP_2}{P_1R}$$

If a segment has negative slope, then its slope is negative of the ratio of two distances.

Here, with $x_1 < x_2$ and $y_2 < y_1$, we have $P_1R = x_2 - x_1$

as before, but

$$RP_2 = y_1 - y_2 = -(y_2 - y_1)$$

Therefore $$m = \frac{-(y_2 - y_1)}{x_2 - x_1} = \frac{RP_2}{P_1R}$$

These ideas relate slopes to our geometry, and make it easy to see why the following theorem is true.

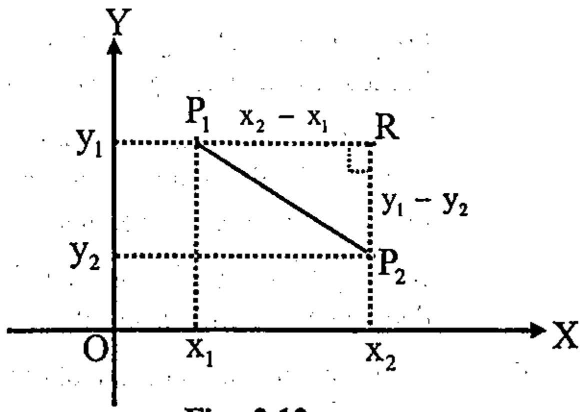


Fig. 2.12

Theorem 1

On a non-vertical line, all segments have the same slope.

Proof : If the line is horizontal, this statement is obvious, because all segments on the line must have slope equal to 0. The interesting cases are indicated by the following figures.

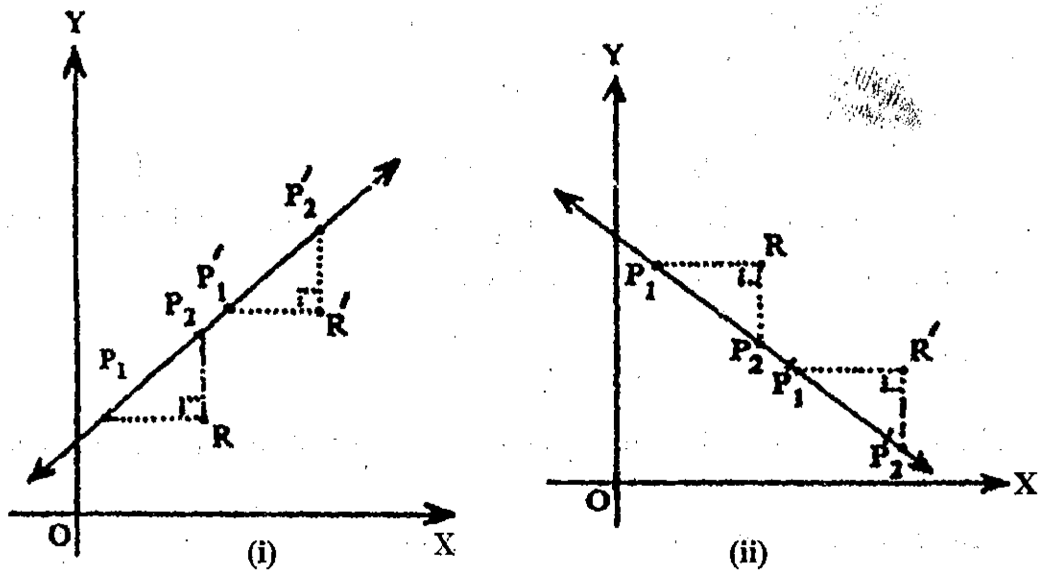


Fig 2. 13

In Fig. 2.13 (i), we have

$$\Delta P_1RP_2 \sim \Delta P'_1R'P'_2$$

so that $\dfrac{RP_2}{R'P'_2} = \dfrac{P_1R}{P'_1R'}$ (or) $\dfrac{RP_2}{P_1R} = \dfrac{R'P'_2}{P'_1R'}$

Therefore $P_1 P_2$ and $P'_1 P'_2$ have the same slope.

In Fig. 2.13 (ii), we also have

$$\Delta P_1RP_2 \sim DP'_1RP'_2$$

This gives, as before, $\dfrac{RP_2}{P_1R} = \dfrac{R'P'_2}{P'_1R'}$

Hence the slopes of $P_1 P_2$ and $P'_1 P'_2$ are equal.

This leads to the following definition.

Definition : The slope of a non-vertical line is the number which is the slope of any segment of the line.

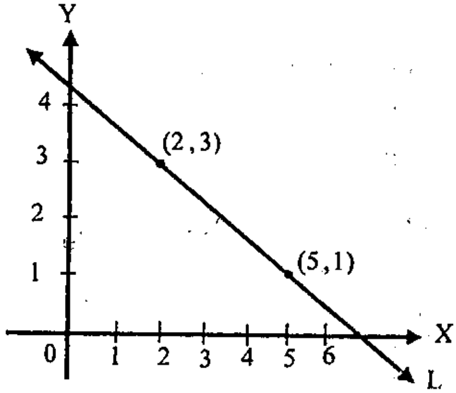


Fig. 2.14

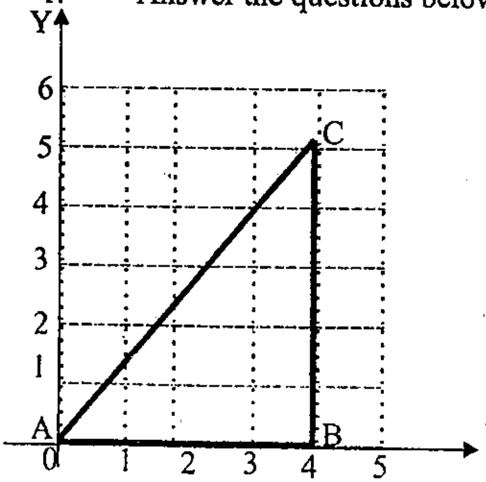
In Fig. 2.14 the slope of L is

$$\frac{1-3}{5-2} = -\frac{2}{3} \text{ (By Defn.)}$$

Any other segment of the same line would give the same value.

Exercise 2.1

1. Answer the questions below for each figure.



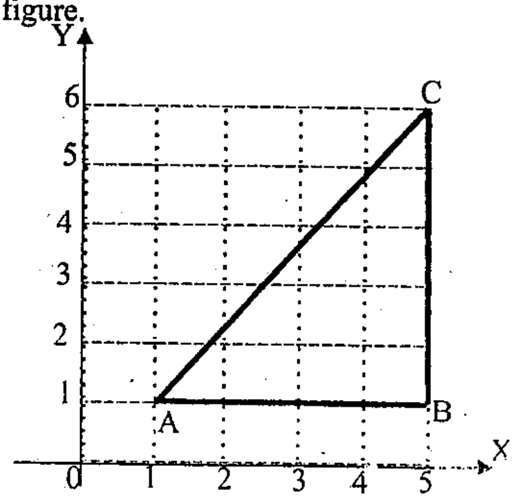


Fig. 2.15

(a) What are the coordinates of A, B and C?

(b) What is BC?

(c) What is the slope of AC?

2. Draw a set of coordinate axes. Locate four points, A, B, C, D that have an x-coordinate of 3. Locate four points, P, Q, , S, that have a y-coordinate of – 2. Lable each point with its coordinates.

3. Give the slope of each segment shown in Fig. 2.16.

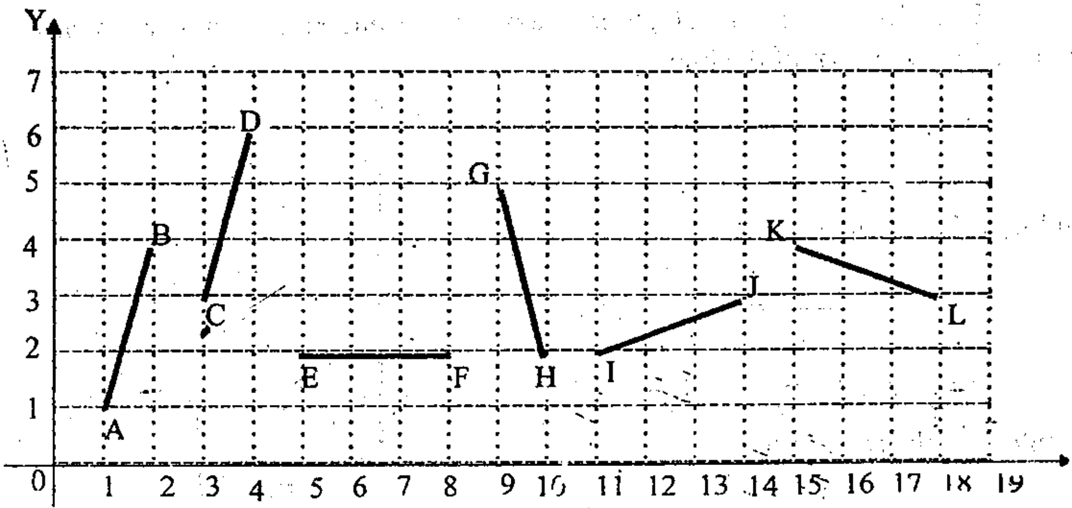


Fig. 2.16

4. Which pairs of points given below will determine horizontal lines? Which ones vertical lines?

(a) (5, 7) and (– 3, 7)

(b) (5, 2) and (– 3, 5)

(c) (3, 3) and (– 3, 3)

(d) (0, 0) and (0, 5)

(e) (a, b) and (a, c)

5 Find the slope of each line which contains each pair of points listed below.

(a) (0, 0) and (8, 4) (b) (10, 5) and (6, 8)

(c) (2, – 2) and (4, 2) (d) (0, 3) and (– 2, 3)

(e) (– 2, 0) and (0, 6) (f) (15, 6) and (– 2, 23)

6. Find the slope of each line which contains each of the pairs of points listed below.

(a) $(-5, 7)$ and $(3, 8)$ (b) $(63, 49)$ and $(-7, 9)$

(c) $(\frac{5}{2}, \frac{4}{3})$ and $(-\frac{13}{2}, \frac{16}{3})$ (d) $(2a, 3b)$ and $(-a, b)$

(e) $(5\sqrt{2}, 6\sqrt{3})$ and $(\sqrt{8}, \sqrt{12})$ (f) $(0, n)$ and $(n, 0)$

7. The vertices of a triangle are the points A $(-2, 3)$, B $(5, -4)$ and C $(1, 8)$. Find the slope of each side and also the slope of each median.

8. The vertices of a parallelogram are the points R(1, 4), S(3, 2), T(4, 6), V(2, 8). Find the slope of each side.

9. Determine the slope of each side of the quadrilateral whose vertices are A(5, 6), B (13, 6), C (11, 2), D (1, 2). Can you tell what kind of a quadrilateral it is?

10. A quadrilateral has as vertices the points M (a, b), N (c, d), Q (c + d, e),P (a + d, e). Find the slope of each side.

11. C is the mid-point of AB, A is the point $(-3, -2)$, and B is the point (2, 8).

What is the slope of BC?

12. Given the points D $(-4, 6)$, E (1, 1), F (4, 6), find the slopes of DE and EF.

Are D, E and F collinear? Why?

13. Draw a coordinate system and plot the point (2, 0). Now plot three other points whose x-coordinates are greater than 0 and less than 8 and which lie on a line with slope equal to 2, containing (2, 0).

14. A line having a slope of -1 contains the point $(-2, 5)$. What is the y-coordinate of the point on the line whose x-coordinate is 8?

15. Draw a coordinate system and plot the point $(-3, 1)$. Now plot three other points whose x-coordinates are greater than 0 and less than 10, and which lie on a line with slope equal to $-\frac{1}{3}$ and containing $(-3, 1)$.

2.3 Parallel and Perpendicular Lines

Using slopes, we can rather easily tell whether two non-vertical lines are parallel.

1. If two non-vertical lines are parallel, then they have the same slope.

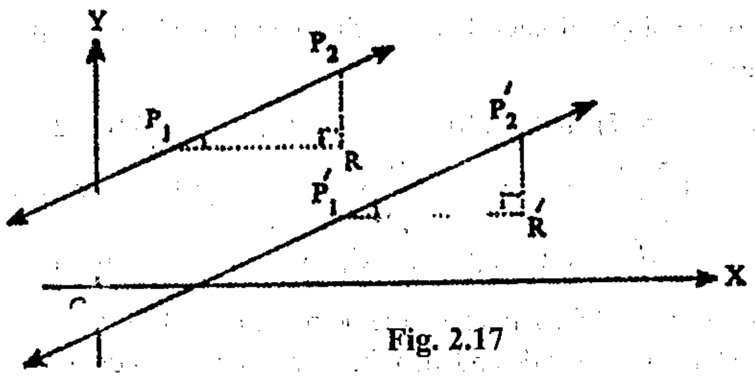


Fig. 2.17

This follows from the fact that

$$\Delta P_1RP_2 \sim \Delta P'_1R'P'_2$$

2. If two different non-vertical lines intersect, then their slopes are different.

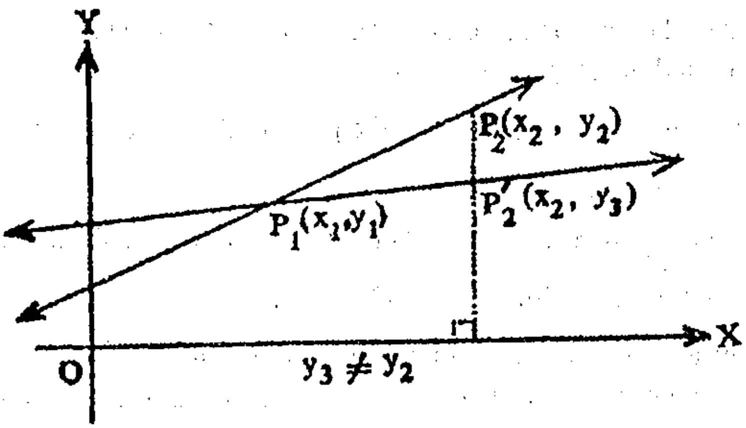


Fig. 2.18

If the two lines intereset at P_1, as in Fig. 2.18 then the slopes of the lines are

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m' = \frac{y_3 - y_1}{x_2 - x_1}$$

Since $y_2 \neq y_3$, $m \neq m'$

Thus, we have proved the following theorem.

Theorem 2.

Two non-vertical lines are parallel if and only if they have the same slope.

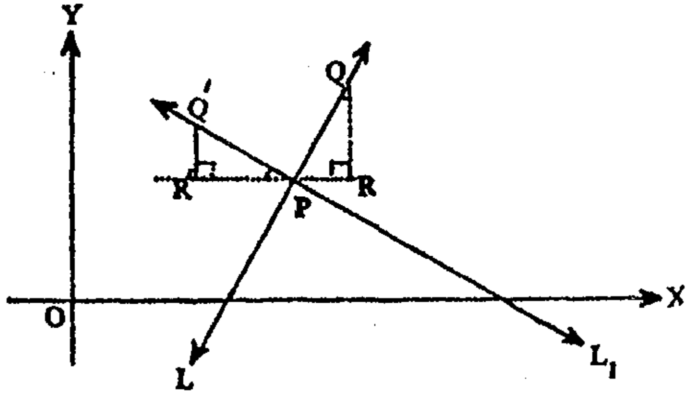


Fig. 2.19

Suppose now that we have two perpendicular lines, intersecting at P. Suppose that neither of our lines is vertical.

We take a point Q, on one of the lines, above and to the right of P, and complete the right triangle PRQ. Take a point Q' on the other line, above and to the left of P, making PQ' = PQ.

Complete the right triangle Q'R'P.

From Fig. 2.19

$$\Delta\ PQR \cong \Delta\ Q'PR'$$

Therefore

$$\frac{RQ}{PR} = \frac{R'P}{Q'R'}$$

But the slope of L is

$$m = \frac{RQ}{PR}$$

and the slope of L_1 is

$$m' = -\frac{Q'R'}{R'P} = -\frac{PR}{RQ}$$

Therefore

$$m' = -\frac{1}{m}$$

That is, the slopes of perpendicular lines are negative reciprocals of each other.

The same scheme works in reverse.

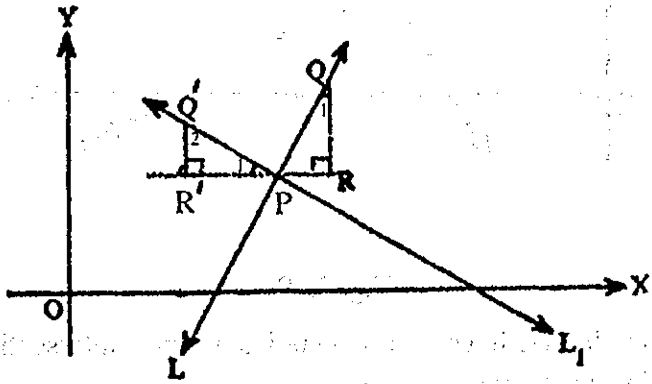


Fig. 2.20

Given than $m' = \dfrac{-1}{m}$, we construct $\Delta\,PRQ$ as before. Take R' so that R'P = RQ, and complete the right triangle $\Delta\,Q'R'P$, with Q' on L'. We then have

$$\Delta\,PQR \;\cong\; \Delta\,Q'PR' \qquad\qquad (SAS)$$

Therefore $\angle 1$ and $\angle 2$ are complementary, and $L \perp L_1$. We sum up this discussion in the following theorem.

Theorem 3.

Two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of each other.

The last two theorems cannot be applied to the case where one of the two given lines is vertical. But for vertical lines, the facts are plain. If L is vertical then the lines parallel to L are simply the other vertical lines. And the lines perpendicular to a vertical line are simply the horizontal lines.

Exercise 2.2

1. Lines L_1, L_2, L_3 and L_4 have slopes $\frac{2}{3}$, -4, $-1\frac{1}{2}$, $\frac{1}{4}$ respectively. Which pairs of lines are perpendicular?

2. Consider the points A (– 1, 5), B (5, 1), C (6, – 2), D (0, 2). Find the slopes of AB, BC, CD and AD. Is quadrilateral ABCD a parallelogram?

3. Without plotting the points, determine which of the quadrilaterals whose vertices are given here are parallelograms.

   (a) A (– 2, – 2), B (4, 2), C (9, 1), D (3, – 3).

   (b) K (– 5, – 2), L (– 4, 2), M (4, 6), N (3, 1).

   (c) P (5, 6), Q (7, – 3), R (– 2, – 12), S (– 4, – 3).

4. The vertices of a triangle are A (16, 0), B (9, 2) and C (0, 0).

   (a) What are the slopes of its sides?

   (b) What are the slopes of its altitudes?

5. Given the points E (– 4, 0), G (3, 5) and K (8, – 2). Show that the product of the slope of EG and the slope of GK is – 1.

6. Prove that the quadrilateral with vertices A (– 2, 2), B (2, – 2), C (4, 2) and D (2,4) is a trapezoid with perpendicular diagonals.

7. Consider the points W (0, 3), X (6, 4), Y (12, – 3), Z (– 2, – 12).

   Which two lines determined by these points are perpendicular? Prove your answer.

8. Four points taken in pairs determine six segments. For each set of four points given below, find out which segments are parallel.

   (a) A (– 2, 7), B (3, 6), C (4, 2), D (9, 1)

   (b) A (5, 3), B (–5, –2), C (6, –2), D (4, 5)

9. Prove that the triangle whose vertices are H (– 12, 1), K (9,.3) and M (11, – 18) is a right triangle.

10. Show that the line through (3n, 0) and (0,7n) is parallel to the line through (0, 21n) and (9n, 0).

11. If the line containing points $(-8, m)$ and $(2, 1)$ is parallel to the line containing points $(11, -1)$ and $(7, m + 1)$, what must be the value of m?

12. What values of k will make the line containing points $(k, 3)$ and $(-2, 1)$ parallel to the line through $(5, k)$ and $(1, 0)$?

13. In problem 12, what values of k will make the lines perpendicular?

14. Given the points $P(1, 2)$, $Q(5, -6)$ and $R(b, b)$, determine the value of b so that $\angle PQR$ is a right angle.

15. Find the slopes of the six lines determined by the points $A(-5, 4)$, $B(3, 5)$, $C(7, -2)$, $D(-1, -3)$. Prove that ABCD is a rhombus.

2.4 The Distance Formula

If we know the coordinates of two points P_1 and P_2, then the points are determined. Therefore the distance between them is determined. We shall now find a way to calculate this distance P_1P_2 in terms of the coordinates (x_1, y_1) and (x_2, y_2).

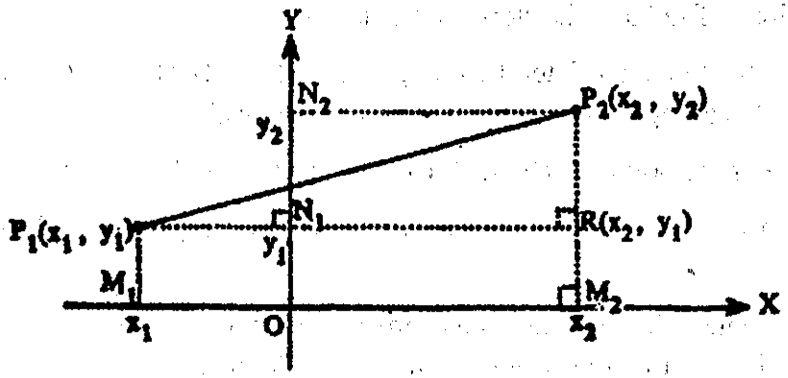


Fig. 2.21

Let the feet of the perpendiculars from P_1 and P_2 be M_1, N_1, M_2, N_2 as indicated in

Fig. 2.21. Let R be the point where the horizontal line through P_1 intersects the vertical line through P_2. Then by the Pythagoras Theorem

$$(P_1 P_2)^2 = (P_1 R)^2 + (RP_2)^2.$$

$P_1R = M_1 M_2$, because opposite sides of a rectangle are congruent.

$RP_2 = N_1 N_2$, for the same reason.

Therefore, by substitution,

$$(P_1 P_2)^2 = (M_1 M_2)^2 + (N_1 N_2)^2$$

But we know, by the property of a number line,

$$M_1 M_2 = | x_2 - x_1 |$$

$$\text{and} \qquad N_1 N_2 = | y_2 - y_1 |,$$

$$\text{Therefore} \qquad (P_1 P_2)^2 = | x_2 - x_1 |^2 + | y_2 - y_1 |^2$$

Since the square of a number is the same as the square of its absolute value, this expression can be written in the form $(P_1 P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Since $P_1 P_2 \geq 0$, we get, $\quad P_1 P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

In deriving this result, we have proved the following theorem.

Theorem 4. (The Distance Formula)

The distance between two points (x_1, y_1) and (x_2, y_2) is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

For example, if P_1 (3, 4) and P_2 (– 2, 1), the formula tells us that

$$P_1 P_2 = \sqrt{(-2-3)^2 + (1-4)^2} = \sqrt{(-5)^2 + (-3)^2} = \sqrt{25+9} = \sqrt{34}$$

Exercise 2.3

1. Use the distance formula to find the distance between:

(a)	(0, 0) and (3, 4)	(b)	(8, 11) and (15, 35)
(c)	(3, 8) and (– 5, – 7)	(d)	(– 2, 3) and (– 1, 4)
(e)	(5, – 1) and (– 3, – 5)	(f)	(– 6, 3) and (4, – 2)

2. Find the perimeter of a triangle whose vertices are A(5, 7), B (1, 10) and C (–3,- 8).

3. Δ PQR has vertices P (8, 0), Q (– 3, 2) and R (10, 2).

   (a) Find the length of each side. (b) Find the area of Δ PQR .

4. Δ KLM has vertices K (– 5, 18), L (10, – 2) and M (– 5, – 10).

   (a) Find the perimeter of Δ KLM. (b) Find the area of Δ KLM.

5. The vertices of a quadrilateral are D (4, – 3), E (7, 10), F (– 8, 2), G (– 1, – 5). Find the length of each diagonal.

6. Prove that the triangle whose vertices are A(2,3), B (– 1, – 1), C (3, – 4) is isosceles.

7. A triangle has vertices G (0, 7), H (5, – 5) and K (10, 7). Find the length of the altitude to the shortest side.

2.5 How to describe a Line by an Equation

Vertical line

Consider a vertical line passing through the point at (3, 0) as shown in Fig. 2.22.

It is observed that all the points of the line have 3 as their x-coordinates. This may be expressed as x = 3. This equation is satisfied by all points on the straight line.

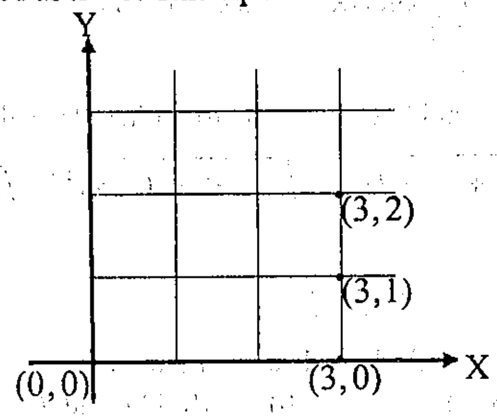


Fig. 2.22

Thus $\{(x, y) \mid x = 3\}$ describes the vertical straight line through (3, 0).

If the line intersects the X-axis at (a, 0), then $\{(x, y) \mid x = a\}$ represents the line through the point (a, 0).

We say that, the equation of the vertical line through the point (a, 0), is x = a.

Horizontal line

The horizontal line L intersects the Y-axis at (0, c), then

$\{(x, y) \mid y = c\}$ represents the line.

The equation of this horizontal line is y = c.

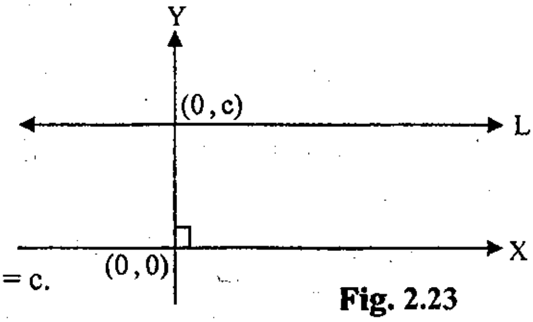


Fig. 2.23

Slanting line

For the nonvertical and nonhorizontal lines, we need to use the slope. Suppose the line L passes through the point (0, c) and has slope m. If P (x, y) is any other point of L, then

$$\frac{y - c}{x - 0} = m$$

because all segment of L have slope m.

i.e., $y = mx + c$

Thus we have $\{(x, y) \mid y = mx + c\}$ describing the line L.

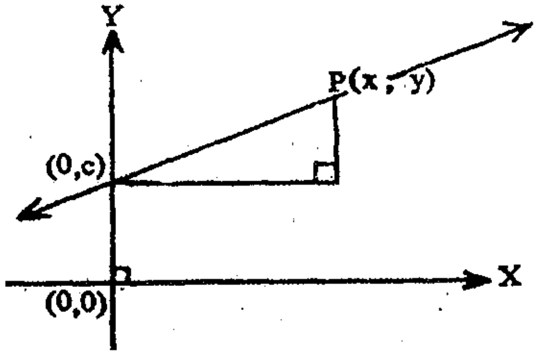


Fig 2. 24

We say that the equation of L is y = mx + c.

c is called the intercept on the Y-axis and m is the slope of the line. y = mx + c is called the equation of the straight line in **slope-intercept form.**

Next, consider a line passing through the point (x_1, y_1) and having a slope m.

If P (x, y) is any other point on the line, then $m = \dfrac{y - y_1}{x - x_1}$.

$$y - y_1 = m(x - x_1).$$

Thus we have $\{(x, y) \mid y - y_1 = m(x - x_1)\}$ describing the line passing through the point (x_1, y_1) and having a slope m.

We say that the equation of the line passing through the point (x_1, y_1) and having a slope m, is $y - y_1 = m(x - x_1)$. This is known as the **point – slope form.**

Consider a line passing through two points (x_1, y_1) and (x_2, y_2). Then the slope of this line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Then the equation of the line through the point (x_1, y_1) and having slope
$y - y_1 = m(x - x_1)$.

Hence, the line through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is known as the two – point form.

Exercise 2.4

1. Write an equation of the line through point P and having slope m given that

   (a) $P(0, 1), m = 3$ (b) $P(0, -4), m = -2$

   (c) $P(0, 2), m = \frac{3}{4}$ (d) $P(0, 6), m = \frac{5}{4}$

   (e) $P(0, 5), m = 0$ (f) $P(0, 6), m = \frac{-2}{3}$

2. For each pair of points, first find the slope of the line which contains them, and then write an equation of the line.

   (a) $(-6, 0)$ and $(0, 4)$ (b) $(0, 6)$ and $(2, -2)$

3. For each of the equations below, written in slope-intercept form, determine the slope and the y-intercept, and sketch the graph.

   (a) $y = 2x + 6$ (b) $y = -2x + 6$ (c) $y = \frac{2}{3}x$

   (d) $y = 2x - 6$ (e) $y = \frac{2}{3}x - 6.$

4. Find an equation of the line with slope equal to -5 and containing the point $(0, 4)$.

5. Prove that the line through two points (4, 3) and (2, 5) cuts off equal intercepts on the x - axis and y axis.

6. Find the equation of the line through the point (2, - 1) and perpendicular to $4x - 3y = 5$.

7 Find the equation of the line through the point (– 4, 6) and concurrent with the lines

$3x - 2y + 3 = 0$ and $5x + 6y - 2 = 0$. Show that this line passes through the origin.

SUMMARY

1. A point P having coordinates x and y is written as P (x, y).

2. Suppose P = (a, b) and Q = (c, d). If R is the mid- point of PQ, then

$$R = \left(\frac{a+c}{2}, \frac{b+d}{2}\right).$$

3. For nonvertical segment P_1P_2 joining $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$, the slope is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

4. Two nonvertical lines are parallel if and only if they have the same slope.

5. Two nonvertical lines are perpendicular if and only if the product of their slopes is – 1.

6. The distance between $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$ is

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

7 Equation of the line passing through the point (x_1, y_1) and having a slope m is

$y - y_1 = m (x - x_1)$.

8. Equation of the line through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

CHAPTER 3

Exponents and Radicals

Much of the power of Mathematics comes from its symbolism. A good symbol can be used to communicate a great deal of information clearly and precisely in a small amount of space. A good illustration is the use of positive integers as exponents. For example , it is obvious that writing $b^4 \cdot b^3 = b^7$ has more advantage than writing $(b \cdot b \cdot b \cdot b)\ (b \cdot b \cdot b) = b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b$.

The concept of exponents is an old one. In fact, the first law of exponents was first used by Archimedes. However, the complete theory of exponents was only presented in 1655 by John Wallis in his book "Arithmetic of Infinities".

In this chapter, we will first discuss positive integral exponents and then proceed with the development of the zero exponent , negative integral exponents and rational exponents. We then discuss radicals and exponential equations.

3.1 Positive Integral Exponents

Factors often occur more than once in a given product. For example , a is a double factor in $a \times a$, and 2 occurs four times in the product $2 \times 2 \times 2 \times 2$. In writing a product, a raised dot is often used instead of $\times$ to denote multiplication. Thus,

$$9 \cdot 9 = 9 \times 9,$$
$$7 \cdot 7 = 7 \times 7,$$
and $$b \cdot b = b \times b.$$

Such products of repeated factors are called **powers**.

Powers can be represented by special symbols to abbreviate the operation. Thus, we write

$$3 \cdot 3 = 3^2 ,$$
$$3 \cdot 3 \cdot 3 = 3^3 ,$$
$$b \cdot b = b^2 ,$$
$$b \cdot b \cdot b = b^3 ,$$
$$b \cdot b \cdot b \cdot b = b^4 ,$$ and so on.

In general, if b is a real number and n is a positive integer, then we may write

$$\underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{\textbf{n factors}} = b^n$$

and call b^n the n^{th} power of b .

The small raised symbol n is called the exponent and b is called **the base of the power** . Thus

exponent

$$\mathbf{b^n} \quad = \textbf{the } \mathbf{n^{th}} \textbf{ power of b}$$

base

The exponent tells the number of times the base b occurs as a factor in a product.

In $b \cdot b \cdot b = b^3$, b^3 is called the **exponential form,** $b \cdot b \cdot b$ is called the **factored** form or **expanded form.**

Let us define the power b^n formally as follows.

Definition 1. If b is any real number and n is a positive integer, the n^{th} power of b is

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{\textbf{n factors}}$$

where the base is b and the exponent is n .

The following rules of exponents are immediate consequences of the above definition. We assume that m and n are positive integers and a and b are any real numbers.

Rule 1. (Rule of Multiplication)

$$b^m \cdot b^n = b^{m+n}$$

Illustrations : $3^2 \cdot 3^4 = 3^{2+4} = 3^6$

$x^2 \cdot x^4 = x^{2+4} = x^6$

Rule 2. (Rule of Division)

$$\frac{b^m}{b^n} = \begin{cases} b^{m-n} & ,\text{if } m > n \\ 1 & ,\text{if } m = n \\ \dfrac{1}{b^{n-m}} & ,\text{if } m < n,\ b \neq 0 \end{cases}$$

Illustrations : $(m > n)$: $\frac{2^5}{2^3} = 2^{5-3} = 2^2$

$\frac{x^5}{x^3} = x^{5-3} = x^2$

$(m = n)$: $\frac{2^5}{2^5} = 1$

$\frac{x^5}{x^5} = 1, (x \neq 0)$

$(m < n)$: $\frac{2^3}{2^5} = \frac{1}{2^{5-3}} = \frac{1}{2^2}$

$\frac{x^3}{x^5} = \frac{1}{x^{5-3}} = \frac{1}{x^2}$, $(x \neq 0)$

Rule 3. (Rule of a Power of a Power)

$$(b^m)^n = b^{mn}$$

Illustrations : $(2^3)^2 = 2^{3 \cdot 2} = 2^6$

$(x^3)^2 = x^{3 \cdot 2} = x^6$

When exponents have exponents , "work down".

For example , $2^{3^2} = 2^{(3^2)} = 2^9$

$(2^{3^2} \neq (2^3)^2 = 8^2.)$

Rule 4. (Rule of a Power of a Product)

$$(ab)^n = a^n \cdot b^n$$

Illustrations : $(2 \cdot 3)^4 = 2^4 \cdot 3^4$

$$(xy)^4 = x^4 \cdot y^4$$

Rule 5. (Rule of a Power of a Quotient)

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \qquad b \neq 0$$

Illustrations : $\left(\frac{3}{2}\right)^5 = \frac{3^5}{2^5}$

$$\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$$

The proper use of these rules can simplify computations as in the following examples.

Example 1. Simplify and name the rules used.

(i) $2^2 \cdot 3^2$ (ii) $(xy)^4$ (iii) $x^6 \cdot x^9$

(iv) $(a^3)^5$

Solution

(i) $2^2 \cdot 3^2 = (2 \cdot 3)^2$ (Rule of a Power of a Product)

$= 6^2 = 36$

(ii) $(xy)^4 = x^4y^4$ (Rule of a Power of a Product)

(iii) $x^6 \cdot x^9 = x^{6+9} = x^{15}$ (Rule of Multiplication)

(iv) $(a^3)^5 = a^{3 \times 5} = a^{15}$ (Rule of a Power of a Power)

Example 2. Simplify and name the rules used.

$$\left(\frac{-8a^6b^3}{2a^4b}\right)^3$$

Solution 1.

$$\left(\frac{-8a^6b^3}{2a^4b}\right)^3 = (-4\,a^{6-4}b^{3-1})^3 \qquad \text{(Rule of Division)}$$

$$= (-4)^3\,(a^2)^3\,(b^2)^3 \qquad \text{(Rule of a Power of a Product)}$$

$$= -\ 64a^6b^6 \qquad \text{(Rule of a Power of a Power)}$$

Solution 2. $$\left(\frac{-8a^6b^3}{2a^4b}\right)^3 = \frac{(-8)^3\,(a^6)^3\,(b^3)^3}{(2)^3\,(a^4)^3\,(b)^3} \qquad \text{(Rule of a Power of a Quotient)}$$

$$= \frac{-512\,a^{18}b^9}{8\,a^{12}b^3} \qquad \text{(Rule of a Power of a Power)}$$

$$= -64\,a^{18-12}b^{9-3} \qquad \text{(Rule of Division)}$$

$$= -64\,a^6b^6.$$

In solution 2 removing the parentheses before dividing leads to a longer solution. The ability to apply the rules in an order that yields an economical simplification comes through experience. In the next example, division is not done until the final step.

The exponent notation together with factoring is useful in arithmetic simplification too, as shown in the following example.

Example 3. Evaluate $\left(\frac{8\cdot 9}{5}\right)^3 \left(\frac{5^2}{12}\right)^2$

Solution

$$\left(\frac{8\cdot 9}{5}\right)^3\left(\frac{5^2}{12}\right)^2 = \left(\frac{2^3\cdot 3^2}{5}\right)^3\left(\frac{5^2}{2^2\cdot 3}\right)^2 = \frac{2^9\cdot 3^6}{5^3}\cdot\frac{5^4}{2^4\cdot 3^2}$$

$$= \frac{2^9\cdot 3^6\cdot 5^4}{2^4\cdot 3^2\cdot 5^3} = 2^5\cdot 3^4\cdot 5 = 32\cdot 81\cdot 5 = 12960$$

Exercise 3.1

1. Simplify by using the rules of exponents.

(a) $\dfrac{32\,a^2b^2}{16\,a^3b^3}$ (b) $\dfrac{16\,a^5b^2}{(2ab)^3}$ (c) $\dfrac{-(4xy)^2}{(-2xy)^3}$

(d) $\dfrac{(2)^{2^3}}{(2^2)^3}$

2. Simplify in a step-by-step fashion and name the rule used at each step.

(a) $\left(\dfrac{a^3}{b^7}\right)^4$ (b) $\left(\dfrac{-210\,x^{11}y^{14}z^{17}}{315\,x^9y^{11}z^{16}}\right)^6$

(c) $\left(\dfrac{x^5}{y^6}\right)^4\left(\dfrac{y^3}{x^2}\right)^7$ (d) $\left(\dfrac{4a^3b^7}{-5\,c^4d^3}\right)^3\left(\dfrac{-5c^3d^3}{2a^2b^5}\right)^4$

3. Simplify

(a) $(c^m\,d^m)^{2m}$ (b) $\left(\dfrac{3^n}{5^m}\right)^2\left(\dfrac{3^n}{5^m}\right)^3$

(c) $\dfrac{(x^{a-b}\,x^{b-c})^a\left(\dfrac{x^a}{x^c}\right)^c}{(x^a\cdot x^c)^a\div(x^{a+c})^c}$ (d) $\dfrac{(x^2-y^2)^2\,(x+y)}{(x-y)^2}$

4. Evaluate

(a) $\dfrac{5^2\times 12^4\times 32^2\,(3^2\times 4^2\times 5)^2}{(3\times 15\times 2^3)^{10}}$ (b) $\left(\dfrac{49}{36}\right)^3\left(\dfrac{180}{35}\right)^4\left(\dfrac{22}{154}\right)^2$

3.2 Zero and Negative Integral Exponents

Definition (1) applies only if n is a positive integer. According to this definition, the symbols such as 5^0, 4^{-2} and $3^{\frac{1}{2}}$ have no meaning yet. Is it possible to extend the rules of exponents to include these symbols? Can we choose definitions for these new exponents such that they will obey the five rules already established for positive integral exponents?

For the multiplication rule $b^m \cdot b^n = b^{m+n}$ to hold for a zero exponent, we must define b^0 so that

$$\boxed{b^0} \cdot b^n = b^{0+n} = b^n$$

Since $\boxed{1} \cdot b^n = b^n$, it is reasonable to make the following definition for b^0.

Definition 2. For any real number b, if $b \neq 0$, then $b^0 = 1$.

For example, $12^0 = 1$, $(-15)^0 = 1$, $\left(\frac{2}{3}\right)^0 = 1$, $(\sqrt{3})^0 = 1$, $(3.1752)^0 = 1$

and so on.

Note that no meaning is assigned to the symbol 0^0, since it cannot be given a useful meaning.

Definition (2) has extended the rules of exponents to include zero exponent.

If the multiplication rule is to hold for negative integral exponents, we must define b^{-n} so that

$$b^n \quad \boxed{b^{-n}} = b^{n+(-n)} = b^0 = 1$$

Since $b^n \cdot \boxed{\frac{1}{b^n}} = 1$, it is reasonable to define b^{-n} to the reciprocal of b^n.

Definition 3 : For any real number b, if $b \neq 0$, then

$$b^{-n} = \frac{1}{b^n}$$

For example $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

$$\left(\frac{1}{2}\right)^{-3} = \frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{8}} = 8$$

$$(-6)^{-2} = \frac{1}{(-6)^2} = \frac{1}{36}.$$

An immediate consequence of definition (3) is that

$$\frac{1}{b^{-n}} = b^n$$

We can check that definitions (2) and (3) obey the fundamental rules of exponents.

Example 1. Evaluate (a) $\frac{2^{-5}}{5^{-2}}$ (b) $(3x)^0 - 3x^0 - 3^{-1} - 3^0, x \neq 0.$

Solution

(a) $$\frac{2^{-5}}{5^{-2}} = \frac{\frac{1}{2^5}}{\frac{1}{5^2}} = \frac{5^2}{2^5} = \frac{25}{32}.$$

(b) $$(3x)^0 - 3x^0 - 3^{-1} - 3^0 = 1 - 3(1) - \frac{1}{3} - 1 = -3 - \frac{1}{3} = -3\frac{1}{3}.$$

Example 2. Simplify and express the answer with positive exponents $(xy^{-3})^{-1}$.

Solution 1

$$(xy^{-3})^{-1}$$

$$= \frac{1}{xy^{-3}}$$

$$= \frac{y^3}{x}.$$

Solution 2

$$(xy^{-3})^{-1}$$

$$= \left(\frac{x}{y^3}\right)^{-1}$$

$$= \frac{1}{\frac{x}{y^3}} = \frac{y^3}{x}.$$

Solution 3

$$(xy^{-3})^{-1}$$

$$= x^{-1}y^3$$

$$= \frac{y^3}{x}.$$

Example 3. Simplify and state any number which x cannot represent.

(a) $(x+1)^0$ (b) $(3x)^{-2} + 3x^{-2}$

Solution

(a) $(x+1)^0 = 1, \; x \neq -1$

(b) $$(3x)^{-2} + 3x^{-2} = \frac{1}{(3x)^2} + 3\left(\frac{1}{x^2}\right), \; (x \neq 0)$$

$$= \frac{1}{9x^2} + \frac{3}{x^2} = \frac{1+27}{9x^2} = \frac{28}{9x^2}$$

When the base is a fraction and the exponent is a negative integer, use the rule

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}, \; a \neq 0 \text{ and } b \neq 0.$$

We can verify this statement as follows.

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^{n}} \qquad \text{Definition (3)}$$

$$= \frac{1}{\frac{a^n}{b^n}} \qquad \text{(Rule 5)}$$

$$= \frac{b^n}{a^n} \qquad \text{(Division of fractions)}$$

$$= \left(\frac{b}{a}\right)^{n} \qquad \text{(Rule 5)}$$

Example 4. Evaluate (a) $\left(\frac{8}{7}\right)^{-1}\left(\frac{1}{2}\right)^{-5}$ (b) $\left(\frac{-3}{2}\right)^{-2} + \left(\frac{-3}{2}\right)^{-3}$

Solution

(a) $\left(\frac{8}{7}\right)^{-1}\left(\frac{1}{2}\right)^{-5} = \left(\frac{7}{8}\right)\left(\frac{2}{1}\right)^{5} = \frac{7}{8}\cdot\frac{2^5}{1} = \frac{7}{8}\cdot 32 = 28$

(b) $\left(\frac{-3}{2}\right)^{-2} + \left(\frac{-3}{2}\right)^{-3} = \left(\frac{-2}{3}\right)^{2} + \left(\frac{-2}{3}\right)^{3} = \frac{4}{9} - \frac{8}{27} = \frac{12-8}{27} = \frac{4}{27}.$

Exercise 3.2

1. Simplify and express answers with positive exponents.

(a) $(-3x^{4})(4x^{-7})$

(b) $\left(\dfrac{5^{0}x^{-3}y^{-4}}{2^{-2}w^{-2}z^{5}}\right)^{3}$

(c) $\left(\dfrac{m^{-5}n^{-3}}{p^{4}q^{-6}}\right)^{4}$

(d) $\left(\dfrac{a^{m-2n}\cdot a^{3(m+n)}}{a^{2m-n}}\right)^{-3}$

(e) $\left(\dfrac{2x^{-3}y^{2}}{3^{-1}y^{3}}\right)^{-2}\left(\dfrac{4x^{-2}y^{3}}{3x^{5}}\right)^{3}\div\left(\dfrac{81x^{-2}}{y^{-3}}\right)^{-2}$

2. Evaluate the following.

(a) -3^{-2}

(b) $6^{0}-2^{0}$

(c) $(-3)^{-1}+(-1)^{-3}$

(d) $-2(-1)^{-2}$

(e) $\dfrac{11^{-6}}{2^{-8}}\div\left(\dfrac{11}{2}\right)^{-6}$

(f) $(-5)^{0}-(-5)^{-1}-(-5)^{-2}-(-5)^{-3}$

(g) $(-1)^{(-1)^{(-1)}}$

(h) $\dfrac{(2\cdot 20^{3})^{-2}\ (6\cdot 10^{-3})^{2}}{(4\cdot 10^{-1})^{2}\ (2\cdot 10^{2})^{-3}}$

3. Simplify the following.

(a) $(2^{n}3^{m})^{2}\cdot(2^{m})^{-1}$

(b) $\dfrac{5^{x+1}}{(5^{x})^{x-1}}\div\dfrac{25^{x+1}}{(5^{x-1})^{x+1}}$

(c) $\dfrac{(2^{-3}-3^{-2})^{-1}}{(2^{-3}+3^{-2})^{-1}}$

(d)

$$\left[54\cdot 3^{n-6}+15\cdot 9^{\frac{1}{2}n-3}\right]-\frac{6^{n+1}}{2^{n}\cdot 27}$$

4. Simplify and state any numbers which x cannot represent.

(a) $(x-3)^{0}$

(b) $6x^{0}-(6x)^{0}$

(c) $4x^{-2}-(4x)^{-2}$

(d) $\dfrac{2x+y}{x^{-1}+2y^{-1}}$

5. Simplify and express answers with positive exponents.

(a) $(a^{-1} + b^{-1})^{-2}$ (b) $\dfrac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$

(c) $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$ (d) $\dfrac{(x + y^{-1})^2}{1 + x^{-1} y^{-1}}$

3.3 Rational Exponents

In the last section, b^n was defined only for the case where n is an integer. We now wish to define powers like $5^{\frac{1}{2}}$ and $(-8)^{\frac{2}{3}}$ so that the five familiar rules of exponents still hold. Before we can proceed to extend the concept of an exponent to rational numbers, we need to discuss the roots of a number.

Definition 4 : If n is a positive integer and a and b are real numbers, such that $a^n = b$, then **a is called the n^{th} root of b.**

Example 1. 2 is a cube root of 8, since $2^3 = 8$

Example 2. -3 is a fifth root of -243, since $(-3)^5 = -243$

In general, if n is odd, then there is only one real n^{th} root of b, no matter whether b is negative, zero or positive. In that case, the real n^{th} root of b is denoted by $\sqrt[n]{b}$ and is called the principal n^{th} root of b.

Example 3. 3 and -3 are two real square roots of 9, since $(3)^2 = 9$ and $(-3)^2 = 9$.

In general, **if n is even and b is positive, then there are two real n^{th} roots of b, one positive and the other negative.** In that case, the positive n^{th} root of b is denoted by $\sqrt[n]{b}$ and is called the principal n^{th} root of b.

Example 4. $\sqrt{4} = 2$, because 2 is the positive square root of 4.

Example 5. $\sqrt[3]{-8} = -2$, because -2 is the only real cube root of -8.

There is no real number x such that $x = \sqrt{-4}$; we can never have $x^2 = -4$, since the square of any nonzero real number is positive. It is important to understand the difference between the symbols $\sqrt[n]{-b}$ and $-\sqrt[n]{b}$. Note that $\sqrt[n]{-b}$ is the principal n^{th} root of negative of b, whereas $-\sqrt[n]{b}$ is negative of the principal n^{th} root of b.

Example 6. (a) $\sqrt{(-3)^2} = \sqrt{9} = 3$, (not -3)

(b) $\sqrt{(3)^2} = 3$

Note that :

$$\sqrt{x^2} = x, \text{ if } x \text{ is positive or zero}$$
$$= -x, \text{ if } x \text{ is negative}$$

We are now ready to extend the exponential concept to include fractional exponents. Once again our guideline will be to preserve the earlier rules for integral exponents. The first step is to consider exponents of the form $\frac{1}{n}$, where n is a positive integer. If our rules for exponents are to work,then $\left(b^{\frac{1}{n}}\right)^n = b^{(\frac{1}{n})(n)} = b$.

So $b^{\frac{1}{n}}$ is an n^{th} root of b (provided such a root exists). This suggests the following definition.

Definition 5. For a real number b and an integer n $(n \geq 2)$

$b^{\frac{1}{n}} = \sqrt[n]{b}$, when n is even, b must be positive.

Example 7. Simplify : (a) $16^{\frac{1}{2}}$ (b) $(-8)^{\frac{1}{3}}$ (c) $(-1)^{\frac{1}{2}}$

Solution

(a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

(b) $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$

(c) $(-1)^{\frac{1}{2}} = \sqrt{-1}$: this is not a real number and therefore $(-1)^{\frac{1}{2}}$ has no meaning in terms of real numbers.

We now consider the more general fractional exponent expression $a^{\frac{m}{n}}$. If m and n are positive integers, to preserve the rules for exponents we want to write

$$\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}$$

This observation leads to the following definition.

Definition 6: If m and n are positive integers and $\frac{m}{n}$ is a rational number **in lowest terms**, then for any real number b.

$$b^{\frac{m}{n}} = (\sqrt[n]{b})^m, \text{ when } n \text{ is even, b must be positive.}$$

Example 8. Simplify $(-64)^{\frac{2}{3}}$.

Solution

$$(-64)^{\frac{2}{3}} = (\sqrt[3]{-64})^2 = (-4)^2 = 16$$

We may simplify by using exponent rule as follows :

$$(-64)^{\frac{2}{3}} = \left[(-64)^2\right]^{\frac{1}{3}} = \sqrt[3]{4096} = 16.$$

Obviously the first approach needs much less work. With some experience we will get used to the simpler method.

Example 9. Find what is wrong with the following "proof" that $2 = -2$.

$$-2 = \sqrt[3]{-8} = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2.$$

Solution

Since $\frac{2}{6}$ is not in lowest terms, we are not permitted to write

$$(-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2}.$$

Example 10. Simplify $(-8a^3)^{\frac{2}{3}}$

Solution

$$(-8a^3)^{\frac{2}{3}} = \left(\sqrt[3]{-8a^3}\right)^2 = (-2a)^2 = 4a^2$$

Definition 7. If m and n are positive integers, then for any real number $b \neq 0$,

$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$$

Example 11. Evaluate $16^{-\frac{3}{2}}$

Solution

$$16^{-\frac{3}{2}} = \frac{1}{16^{\frac{3}{2}}} = \frac{1}{(\sqrt{16}\,)^3} = \frac{1}{4^3} = \frac{1}{64}.$$

Example 12. Simplify and express the answer with positive exponents.

$$\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m}$$

Solution

$$\left(\frac{2^m\sqrt{2^{-m}}}{\sqrt{2^{m+1}}}\right)^{-2m} = \left(\frac{2^m\,2^{-\frac{m}{2}}}{2^{\frac{m+1}{2}}}\right)^{-2m} = \left(2^{m-\frac{m}{2}-\frac{m+1}{2}}\right)^{-2m}$$

$$= \left(2^{-\frac{1}{2}}\right)^{-2m} = 2^m.$$

Using fractional exponents many possibilities are now opened up for factoring. Consider, for example,

$$x - y = \left(x^{\frac{1}{2}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2 \quad = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)$$

$$x + y = \left(x^{\frac{1}{3}}\right)^3 + \left(y^{\frac{1}{3}}\right)^3$$

$$= \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) \cdot \left(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}\right)$$

Also observe that for any fractional number m,

$$x^m - y^m = \left(x^{\frac{m}{2}}\right)^2 - \left(y^{\frac{m}{2}}\right)^2$$

$$= \left(x^{\frac{m}{2}} + y^{\frac{m}{2}}\right)\left(x^{\frac{m}{2}} - y^{\frac{m}{2}}\right)$$

Thus using this process repeatedly, we get

$$x - y = \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)$$

$$= \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)$$

$$= \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{8}} + y^{\frac{1}{8}}\right)\left(x^{\frac{1}{8}} - y^{\frac{1}{8}}\right)$$

and continuing we could arrive at any desired number of factors.

Example 13. Simplify $\dfrac{x - y^{-1}}{\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)}$

Solution

$$\frac{x - y^{-1}}{\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)} = \frac{\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)}{\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)} = \frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{\frac{1}{4}} - y^{-\frac{1}{4}}}$$

$$= \frac{\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)\left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)}{\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)} = x^{\frac{1}{4}} + y^{-\frac{1}{4}}$$

$$= x^{\frac{1}{4}} + \frac{1}{y^{\frac{1}{4}}} = \frac{x^{\frac{1}{4}} y^{\frac{1}{4}} + 1}{y^{\frac{1}{4}}}$$

It can be shown that, the existing rules of exponents also hold for fractional exponents, and hence the Fundamental Rules for exponents hold for all rational exponents (i.e.. positive integers, negative integers, zero and fractions.).

Rules for Exponents

If p and q are any rational numbers, then

E 1. $a^p \cdot a^q = a^{p+q}$

E 2. $a^p \div a^q = a^{p-q}$, $a \neq 0$

E 3. $(a^p)^q = a^{pq}$

E 4. $(ab)^p = a^p \cdot b^p$

E 5. $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$, $b \neq 0$

Exercise 3.3

1. Evaluate the following.

(a) $16^{\frac{1}{2}}$ (b) $27^{-\frac{5}{3}}$ (c) $(-27)^{-\frac{2}{3}}$

(d) $9^{-\frac{3}{2}}$ (e) $\left(\frac{4}{9}\right)^{\frac{3}{2}}$ (f) $\left(\frac{-125}{8} \div \frac{1}{64}\right)^{\frac{1}{3}}$

2. Simplify and express answers with positive exponents.

(a) $\left(\frac{-136\, x^{-3}\, y^{-\frac{5}{6}}}{-34\, x^{-5}\, y^{-\frac{17}{18}}}\right)^{-\frac{3}{2}}$ (b) $\left(\frac{210\, m^5\, n^0}{-63\, m^{-3}\, n^{-\frac{1}{3}}}\right)^2$

(c) $\left(\dfrac{a^{-2}\, y}{a^{\frac{3}{2}}\; y^{\frac{-5}{2}}} \right) \div \left(\dfrac{a\; y^{-\frac{1}{2}}}{y^{-1}\; a^{\frac{1}{2}}} \right)^{-2}$

(d) $\left[\dfrac{(x^{-2}\sqrt{y}\,)^{-2}}{\sqrt[3]{x}\; y^{-3}} \div \dfrac{x^{-\frac{5}{6}}\,\sqrt{y^3}}{x^{-\frac{11}{12}}\; y^{-\frac{1}{2}}} \right]^{-4}$

3. Evaluate the following.

(a) $\sqrt[3]{4^2} \cdot 4^{\frac{2}{3}} \cdot \left(\dfrac{1}{4}\right)^{-\frac{2}{3}}$

(b) $\sqrt{\dfrac{2^8 \times 27^{-3} \times 81 \times 3^8}{3^2}}$

(c) $\left[\left(\dfrac{3}{4}\right)^{-4} \right]^{-0.5} \cdot \sqrt{\left(\dfrac{4}{3}\right)^{-1}} \div 16^{-0.5}$

(d) $(27)^{\frac{2}{3}} + (16)^{\frac{3}{4}} - \dfrac{2}{(8)^{-\frac{2}{3}}} + \dfrac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}$

4. Simplify the following.

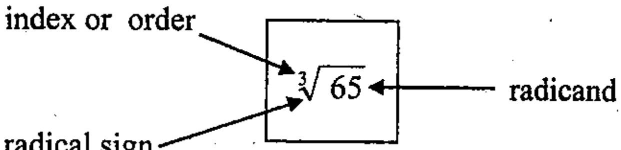
(a) $\dfrac{x - 7x^{\frac{1}{2}}}{x - 5\sqrt{x} - 14} \div \left(1 + \dfrac{2}{\sqrt{x}}\right)^{-1}$

(b) $\sqrt[a]{\left(\dfrac{\sqrt[b]{x}}{\sqrt[c]{x}}\right)} \cdot \sqrt[b]{\left(\dfrac{\sqrt[c]{x}}{\sqrt[a]{x}}\right)} \cdot \sqrt[c]{\left(\dfrac{\sqrt[a]{x}}{\sqrt[b]{x}}\right)}$

3.4 Radicals

In the last section, the principal n^{th} root of b is denoted by $\sqrt[n]{b}$ and by definition(5) $\sqrt[n]{b} = b^{\frac{1}{n}}$. There is another name for roots of a number.

Definition 8 : The symbol $\sqrt[n]{b}$ is called a radical, $\sqrt[n]{}$ is the **radical sign**, n is the **order or index**, and b is called the **radicand**.



A radical

There are times when the radical notation is more natural than the exponential notation. Any changes we make in the form of a radical must be made in accordance with our rules of exponents. In practice, we make use of the following common changes in form.

Rules For Radical

If m, n and k are positive integers,

RD 1. $(\sqrt[n]{b})^n = \sqrt[n]{b^n} = b$

RD 2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

RD 3. $\sqrt[m]{\sqrt[n]{b}} = \sqrt[mn]{b}$

RD 4. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $(b \neq 0)$

RD 5(a). $\sqrt[n]{b^m} = \sqrt[kn]{b^{km}}$

(b). $\sqrt[n]{b^m} = \sqrt[\frac{n}{k}]{b^{\frac{m}{k}}}$

Remember that when the index of a radical is even, all radicands are non-negative.

The most common ways in which one might wish to change a radical are :

1. To remove factors from the radicand.

Example 1. $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$.

2. To reduce to a radical of lower order or index.

Example 2. $\sqrt[4]{3^2} = \sqrt{3}$ by RD 5 (b)

3. To reduce a radical with fractional radicand to an equivalent radical with an integral radicand.

Example 3. $\sqrt{\frac{3}{2}} = \sqrt{\frac{3\times 2}{2\times 2}} = \sqrt{\frac{6}{2^2}} = \frac{\sqrt{6}}{\sqrt{2^2}} = \frac{1}{2}\sqrt{6}$

This process is called **rationalizing the denominator**.

4. To introduce coefficients under the radical sign.

Example 4. $5a\sqrt[3]{3} = \sqrt[3]{(5a)^3} \cdot \sqrt[3]{3} = \sqrt[3]{(5a)^3 \cdot 3} = \sqrt[3]{375a^3}$

When all or some of the first three operations have been done, we say that we have simplified the radical, and the result is said to be in **simplified form or simplest form**.

Example 5. $\sqrt{\sqrt[3]{128}}$ Simplify and write the result in simplified form.

Solution $\sqrt{\sqrt[3]{128}} = \sqrt[6]{2^7}$ (RD 3)

$= \sqrt[6]{2^6 \cdot 2} = 2\sqrt[6]{2}.$

Exercise 3.4

1. Write in radical form.

(a) $5^{\frac{1}{2}}$ (b) $(-9)^{\frac{1}{3}}$ (c) $2^{-\frac{1}{2}}$ (d) $(-\frac{3}{4})^{-\frac{1}{7}}$

2. Write with a fractional exponent.

(a) $\sqrt[6]{c^5}$ (b) $(\sqrt[3]{-7})^2$ (c) $(\sqrt[3]{-\frac{1}{5}})^3$ (d) $\sqrt[5]{a^4\sqrt[3]{b^5}}$

3. Change to an expression in which all factors are under the same radical.

(a) $6\sqrt{2}$ (b) $3a\sqrt[3]{x}$ (c) $2\sqrt[5]{2}$ (d) $2\sqrt[3]{\frac{1}{2}}$

4. Simplify.

(a) $\sqrt{32}$ (b) $\sqrt[5]{-32}$ (c) $\sqrt{\sqrt{625}}$ (d) $\sqrt[3]{\dfrac{81x^2}{4y}}$

(e) $\dfrac{9^{\frac{1}{2}}}{\sqrt[3]{27}}$ (f) $\sqrt{\dfrac{2}{3}} \cdot \sqrt{\dfrac{75}{98}}$ (g) $\sqrt[3]{\dfrac{-216}{8 \times 10^3}}$ (h) $\sqrt[n]{\dfrac{32}{2^{5+n}}}$

5. Rationalize the denominator.

(a) $\dfrac{4\sqrt{35}}{3\sqrt{7}}$ (b) $\dfrac{20}{\sqrt{5}}$ (c) $\dfrac{18}{\sqrt[3]{2}}$ (d) $\dfrac{\sqrt[3]{32}}{\sqrt[4]{27}}$

(e) $\dfrac{\sqrt[3]{36a^2}}{\sqrt[3]{9a}}$ (f) $\dfrac{\sqrt[3]{2}}{\sqrt[6]{12}}$ (g) $\dfrac{1}{\sqrt[3]{xy^2}}$ (h) $\sqrt[m]{\dfrac{2x^2y^{3m}}{9x^5y^{4m-1}}}$

6. Reduce the orders as far as possible.

(a) $\sqrt[4]{25}$ (b) $\sqrt[6]{4}$ (c) $\sqrt[6]{8}$ (d) $\sqrt[9]{8y^3}$

(e) $\sqrt[6]{27^3}$ (f) $\sqrt[8]{a^2b^4}$ (g) $\sqrt[12]{64a^2b^6}$

7. Find the simplified forms.

(a) $\sqrt{\dfrac{9}{50}}$ (b) $\sqrt[3]{-\dfrac{192}{49}}$ (c) $\sqrt[4]{16}$ (d) $2\sqrt[3]{56}$

3.5 Operations with Radicals

The product of two radicals of the same order is found directly from the rule

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}. \qquad \text{(RD 2)}$$

The product of two radicals of different orders can be found using this rule after they have been reduced to equivalent radicals of the same order.

Example 1. Multiply $\sqrt{6}$ by $\sqrt{15}$

Solution

$$\sqrt{6} \cdot \sqrt{15} = \sqrt{6 \cdot 15} = \sqrt{2 \cdot 3 \cdot 3 \cdot 5} = 3\sqrt{10}$$

Example 2. Multiply $\sqrt[3]{12}$ by $\sqrt{10}$

Solution

Since the radicals are of different orders, we have to convert them to the same order by using rule RD 5(a).

$$\sqrt[3]{12} \cdot \sqrt{10} = \sqrt[6]{12^2} \cdot \sqrt[6]{10^3} = \sqrt[6]{12^2 \cdot 10^3}$$
$$= \sqrt[6]{2^4 \cdot 3^2 \cdot 2^3 \cdot 5^3} = \sqrt[6]{2^6 \cdot 2 \cdot 3^2 \cdot 5^3}$$
$$= 2 \cdot \sqrt[6]{2 \cdot 3^2 \cdot 5^3} = 2 \cdot \sqrt[6]{2250}\,.$$

Example 3. Simplify $\sqrt{3}\ (\sqrt{3} + \sqrt{8}\,)$

Solution

$$\sqrt{3}\ (\sqrt{3} + \sqrt{8}\,) = (\sqrt{3}\,)^2 + \sqrt{3} \cdot \sqrt{8} = 3 + \sqrt{24} = 3 + 2\sqrt{6}\,.$$

Addition and subtraction of radicals are possible under certain conditions, and are achieved through the use of the distributive property. Consider, for example, the sum $3\sqrt{2} + 7\sqrt{2}$

$$3\sqrt{2} + 7\sqrt{2} = (3+7)\sqrt{2} = 10\sqrt{2}\,.$$

Thus radicals having the same index and same radicand can be added or subtracted. They are called **similar or like radicals.**

Example 4. Simplify $\sqrt{200} + \sqrt{50} - \sqrt{18}$

Solution

Although each radical has the same index, they do not have the same radicand. However, each can be simplified :

$$\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$$
$$\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$$
$$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

Thus $\sqrt{200} + \sqrt{50} - \sqrt{18} = 10\sqrt{2} + 5\sqrt{2} - 3\sqrt{2}$

$$= (10 + 5 - 3)\sqrt{2} = 12\sqrt{2}.$$

This cannot be simplified further. The algebraic sum of unlike radicals can only be indicated.

Example 5. Simplify $\sqrt{24} - \sqrt{\frac{2}{3}} - \sqrt[3]{\frac{2}{9}}$

Solution

$$\sqrt{24} - \sqrt{\frac{2}{3}} - \sqrt[3]{\frac{2}{9}} = \sqrt{4.6} - \frac{\sqrt{2}}{\sqrt{3}} - \frac{\sqrt[3]{2}}{\sqrt[3]{9}}$$

$$= 2\sqrt{6} - \frac{\sqrt{6}}{3} - \frac{\sqrt[3]{2}}{\sqrt[3]{3^2}} \times \frac{\sqrt[3]{3}}{\sqrt[3]{3}}$$

$$= 2\sqrt{6} - \frac{1}{3}\sqrt{6} - \frac{1}{3}\sqrt[3]{6}$$

$$= (2 - \frac{1}{3})\sqrt{6} - \frac{1}{3}\sqrt[3]{6} = \frac{5}{3}\sqrt{6} - \frac{1}{3}\sqrt[3]{6}$$

Example 6. Simplify the expression $\dfrac{\sqrt{2} - 3\sqrt{3}}{2\sqrt{2} - \sqrt{3}}$

Solution

The technique required here is to rationalize the denominator.

The denominator can be rationalized as follows.

$$\frac{\sqrt{2} - 3\sqrt{3}}{2\sqrt{2} - \sqrt{3}} = \frac{\sqrt{2} - 3\sqrt{3}}{2\sqrt{2} - \sqrt{3}} \cdot \frac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}} = \frac{2(2) - 6\sqrt{6} + \sqrt{6} - 3(3)}{(2\sqrt{2})^2 - (\sqrt{3})^2}$$

$$= \frac{-5 - 5\sqrt{6}}{4(2) - 3} = \frac{-5 - 5\sqrt{6}}{5} = -(1 + \sqrt{6}).$$

In the preceding example, we say that the denominator was rationalized by multiplying by its conjugate. This process is based on the fact that

$$(a + b)(a - b) = a^2 - b^2$$

Each of the two factors is called the conjugate of the other. Thus

(1) $a+\sqrt{b}$ and $a-\sqrt{b}$

(2) $a + b\sqrt{c}$ and $a - b\sqrt{c}$

(3) $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$

(4) $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$

are conjugate radicals.

Exercise 3.5

1. Perform the indicated operation and express the results in simplified forms.

(a) $3 \cdot 3\sqrt{3} \cdot 3\sqrt{9}$

(b) $\sqrt[6]{\dfrac{4x^4}{27}} \cdot \dfrac{1}{5}\sqrt[3]{9x}$

(c) $(\sqrt{7}-\sqrt{6})(\sqrt{7}+\sqrt{6})(\sqrt{x}+1)(\sqrt{x}-1)$

(d) $(a - \sqrt{b})(-\sqrt{b}-a)(a^2+b)$

(e) $\sqrt{6}(3\sqrt{2}-\sqrt{43}-5\sqrt{6})$

(f) $7\sqrt{7} + 4\sqrt{45} - 3\sqrt{125}$

(g) $\sqrt{75} - \dfrac{3}{4}\sqrt{48} - 5\sqrt{12}$

(h) $\sqrt{2x^2} + 5\sqrt{32x^2} - 2\sqrt{98x^2}$

(i) $\sqrt{x^2}y + \sqrt{8x^2}y + \sqrt{200x^2}y$

(j) $\sqrt{20a^3} + a\sqrt{5a} + \sqrt{80a^3}$

2. Rationalize the denominator and simplify.

(a) $\dfrac{12}{3\sqrt{3}}$ (b) $\dfrac{12}{\sqrt{5}-\sqrt{3}}$

(c) $\dfrac{20}{3-\sqrt{2}}$ (d) $\dfrac{4\sqrt{2}-3\sqrt{3}}{5\sqrt{3}-3\sqrt{2}}$

(e) $\dfrac{5\sqrt{3}-4\sqrt{5}}{2\sqrt{5}+3\sqrt{3}}$ (f) $\dfrac{1}{3+\sqrt{3}}+\dfrac{1}{\sqrt{3}-3}+\dfrac{1}{\sqrt{3}}$

3. Simplify.

(a) $\sqrt{\dfrac{x+1}{x-1}}+\sqrt{\dfrac{x-1}{x+1}}-\sqrt{\dfrac{1}{x^2-1}}$

(b) $\dfrac{7+\sqrt{5}}{7-\sqrt{5}}+\dfrac{\sqrt{11}-3}{\sqrt{11}+3}$

(c) $\dfrac{3+2\sqrt{3}}{(\sqrt{3}-1)^3}$

(d) $\sqrt{\dfrac{\sqrt[5]{32}\cdot\sqrt{4}}{2^{-2}-2^{-3}}}$

3.6 Exponential Equations

When a variable appears as an exponent in an equation such as $3^x = 81$, the equation is called an **exponential equation.**

Some exponential equations may be solved using the fact that, **if $b^x = b^y$, then** $x = y$. for $b \neq 0$ and $b \neq 1$.

Example 1. Find the solution of $4^{3x-1} = (\frac{1}{2})^{x-1}$

Solution

$$4^{3x-1} = (\tfrac{1}{2})^{x-1}$$
$$(2^2)^{3x-1} = (2^{-1})^{x-1}$$
$$2^{6x-2} = 2^{1-x}$$
$$6x-2 = 1-x$$
$$7x = 3$$
$$x = \frac{3}{7}.$$

Example 2. Solve $2^{2x} - 5 \cdot 2^x + 4 = 0$

Solution Let $2^x = a$.

Then
$$a^2 - 5a + 4 = 0$$
$$(a-4)(a-1) = 0$$
$$a = 4 \quad \text{or} \quad a = 1$$
$$2^x = 4 \quad \text{or} \quad 2^x = 1$$
$$2^x = 2^2 \quad \text{or} \quad 2^x = 2^0$$
$$x = 2 \quad \text{or} \quad x = 0$$

Exercise 3.6

Solve the following equations

(1) $2^x \cdot 2^3 = 2^{12}$ (2) $3^{x-1} = 81$

(3) $8^{-x} = \dfrac{1}{64}$ (4) $(-2)^{-5x} = -32$

(5) $10^{-x} = 0.0001$ (6) $3^{2x-1} \cdot 3^{4x+8} = (\dfrac{1}{27})^{2x-5}$

(7) $2^{3x-1} = 2^4 \cdot 2^{x+3}$ (8) $2^x \cdot 3^x = 216$

(9) $4^x + 4^{x+1} = 20$ (10) $(0.2)^{-x} = 125$

SUMMARY

Important Definitions and Rules

For any real numbers a , b and positive integers m , n.

1. $b^n = b.\ b.\ b \ldots b$ (n factors)
2. $b^0 = 1,$ $(b \neq 0)$
3. $b^{-n} = \dfrac{1}{b^n}$, $\dfrac{1}{b^{-n}} = b^n,$ $(b \neq 0)$
4. $x = \sqrt[n]{b}$ means $b = x^n$

5. $b^{\frac{1}{n}} = \sqrt[n]{b}$, when n is even, b must be positive.

6. $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$, when n is even, b must be positive.

7. $b^{-\frac{m}{n}} = \sqrt[n]{b^{-m}} = (\sqrt[n]{b})^{-m}$, when n is even, b must be positive.

8. The symbols $\sqrt{b}$, $\sqrt[3]{b}$, $\sqrt[n]{b}$ are also called radicals of order 2, 3 and n.

Rules for Exponents.

For any rational numbers m and n,

E 1. $b^m \cdot b^n = b^{m+n}$

E 2. $b^m \div b^n = b^{m-n}$

E 3. $(b^m)^n = b^{mn}$

E 4. $(ab)^n = a^n \; b^n$

E 5. $(\frac{a}{b})^n = \frac{a^n}{b^n}$, $(b \neq 0)$

Rules for Radicals.

For any positive integers m , n and k,

RD 1. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

RD 2. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

RD 3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

RD 4. $\sqrt[n]{a^m} = \sqrt[kn]{a^{km}}$

RD 5. $\sqrt[n]{a^m} = \sqrt[\frac{n}{k}]{a^{\frac{m}{k}}}$, $(k \neq 0)$

CHAPTER 4

Logarithms

The algebra of exponents presented in chapter 3 offers some easy arithmetic computations. Consider the following problem.

Simplify : $\dfrac{729 \times 27}{81}$

By using exponents

$$\frac{729 \times 27}{81} = \frac{3^6 \times 3^3}{3^4} = 3^5 = 243.$$

This example displays the elegance of evaluation using exponents, but it may not be readily apparent like this for all cases. Consider again the following problem.

Evaluate : $\dfrac{(200)(98{\cdot}7)\ (85{\cdot}3)}{(8{\cdot}5)^3}$

We can see that it requires rather tedious computation. Hence, a method that will allow us to find a very good approximation, but with less work, will be developed in this chapter. This computational short cut will be accomplished by making use of logarithms. In this chapter, we will learn about logarithms of numbers and their use to shorten tedious computations.

4.1 Scientific Notations

Scientists deal with very large and very small numbers. For example, the speed of light is 29,979,000,000 cm per second and the mass of the proton is 0.000,000,000,000,000,000,000,001,65 gram. They are awkward to write and print and errors are often made in copying them. To write such very large and very small numbers, scientists make use of a compact form called **scientific notation.** It is extensively used in all branches of modern science.

Definition 1: A positive number is in scientific notation when it is written in the form $a \times 10^n$ where n is an integer and a is a real number satisfying the inequality $1 \leq a < 10$.

For example , scientists would normally write as follows.

Speed of light : 2.9979×10^{10} cm per second

Mass of proton : 1.65×10^{-24} gram

To illustrate the idea, let us consider the following numbers:

$$523{,}000{,}000 = 5.23 \times 10^8$$

$$52{,}300{,}000 = 5.23 \times 10^7$$

$$523{,}000 = 5.23 \times 10^5$$

$$0.000{,}523 = 5.23 \times 10^{-4}$$

$$0.000{,}005{,}23 = 5.23 \times 10^{-6}$$

It is easy to verify that these are all correct. For example :

$$5.23 \times 10^5 = 5.23 \times 100{,}000 = 523{,}000$$

$$5.23 \times 10^{-4} = 5.23 \times \frac{1}{10^4} = \frac{5.23}{10000} = 0.000{,}523$$

Look again at these illustrations. We will see that there is a simple rule for finding the scientific notation form for any given number.

The rule is :

> **Place the decimal point after the first nonzero digit. (This produces the number between 1 and 10.) Then determine the power of 10 by counting the number of places we have moved the original decimal point to the marked decimal point. If we moved the point to the left, then power is positive; and if we moved it to the right, it is negative.**

Example 1. Write the following numbers in scientific notation .

(a) 51,000 (b) 0.00075

Solution

(a) $51000 = 51000.0$

$$5 \odot 1000 \ .0 = 5.1 \times 10^4.$$

decimal point moved

$\underbrace{\text{four place to the left}}$

(b) $0\,.\,0007\odot 5 = 7.5 \times 10^{-4}$.

decimal point moved

four places to the right

The scientific notation is helpful when we are faced with a computation that involves large or small numbers.

Example 2. Evaluate $\dfrac{(2{,}750{,}000)(0.015)}{750}$

Solution

$$\frac{(2{,}750{,}000)(0.015)}{750} = \frac{(2.75\times10^{6})(1.5\times10^{-2})}{7.5\times10^{2}}$$

$$= \frac{(2.75)(1.5)(10)^{6}(10)^{-2}}{7.5\times10^{2}}$$

$$= \frac{(2.75)(1.5)}{7.5} \times \frac{(10)^{6}(10)^{-2}}{10^{2}}$$

$$= 0.55 \times 10^{2} = 55.$$

We will now present some basic facts about approximate numbers and rules for estimating the accuracy of a computation.

If an engineer writes 6.2 inches as the length of a rod, he means that the length lies in the range 6.15 to 6.25 inches inclusive. The length is **accurate to two figures** or **accurate to one decimal place.**

Similarly, a length of 768.3 feet would mean a length in the range 768.25 feet to 768.35 feet. The length would be **accurate to four figures** or **accurate to one decimal place.**

An approximate number written in scientific notation $a \times 10^{n}$ indicates an accuracy to the number of digits in a. The figures or digits present in a are called **significant figures.**

For example 2.600×10^6 indicates four-figure accuracy and the digits 2, 6, 0, 0 are significant.

How many significant figures are there in the number 0.0305? We will rewrite it in scientific notation.

$$0.0305 = \overbrace{\underbrace{3.05}_{\text{significant digits : 3, 0, 5}}}^{\text{three figures accuracy}} \times 10^{-2}$$

Here $a = 3.05$ and $n = -2$. There are three significant figures and the significant digits are 3, 0, 5.

It is considered wise to estimate the answer to each problem before solving it. The estimate can then be used as a rough check on the working.

There are several ways to perform approximate arithmetic. We should choose the one that makes arithmetic easy.

For example, what is the approximate value of the expression $\dfrac{129 \times 3240}{\sqrt{87.6}}$.

We write

$$\frac{129 \times 3240}{\sqrt{87.6}} \simeq \frac{130 \times 3000}{9} = \frac{390000}{9} \simeq 43333$$
$$\simeq 43{,}000.$$

We have rounded 129 up to 130, 3240 down to 3000 and $\sqrt{87.6}$ down to 9. Therefore 43,000 seems to be a reasonable estimate.

The following example illustrates the use of scientific notation to estimate the value of an expression.

Example 3. Find an estimate of x to one significant figure,

$$\text{if } x = \frac{7499 \times 49.34 \times 276}{647}.$$

Solution Rounding each number to one significant figure, we get

$$x \approx \frac{7000 \times 50 \times 300}{600}$$

Rewriting the approximations in scientific notation

$$x \approx \frac{7\times10^{3}\times5\times10\times3\times10^{2}}{6\times10^{2}}$$

$$\approx \frac{7\times5\times3}{6}\cdot10^{(3+1+2-2)} = 1.75\times10^{5}$$

To one significant figure

$$x \approx 2\times10^{5} = 200{,}000.$$

Exercise 4.1

1. Write in scientific notation.

   (a) 24.86 (b) 2.486 (c) 0.2486

   (d) 0.002486 (e) 0.073 (f) 0.0086

   (g) 0.934 (h) 7 (i) 0.00056857

   (j) 6.843,250

2. Write each number in ordinary decimal form.

   (a) 7.84×10^{4} (b) 7.89×10^{-4}

   (c) 2.25×10^{5} (d) 4.01×10^{-3}

3. Given that $0.00073 = 7.3\times10^{p}$ and $892000 = 8.92\times10^{q}$, find the values of p and q.

4. Simplify and give the answers in scientific notation.

   (a) $2.3\times10^{2} + 1.7\times10^{2}$ (b) $4.6\times10^{-3} - 2.5\times10^{-3}$

   (c) $1.5\times10^{3} - 5\times10^{2}$ (d) $\dfrac{4.5\times10^{6}}{1.5\times10^{-2}}$

   (e) $\dfrac{1.4\times10^{3}}{2.8\times10^{5}}$ (f) $\dfrac{7.6\times10^{5}}{1.9\times10^{-2}}$

5. Compute using scientific notation.

(a) $\dfrac{2.5\times10^2}{0.25\times0.002}$ (b) $\dfrac{33{,}000{,}000\times0.4}{1.1\times30}$

6. How many significant figures are there in each of the following numbers.

(a) 21.75 (b) 2.175 (c) 0.2175

(d) .0075 (e) 89400 (f) 0.00046

7. Give a one-significant figure estimate of each of the following.

(a) $\dfrac{343.7\times72.484}{(0.62)^2}$ (b) $\dfrac{(50.78)\ (0.07)(0.345)}{231}$

(c) $\dfrac{(-11011)\ (0.953)}{(1.72)(0.3418)}$ (d) $\dfrac{(-767.5)\ (3.14)}{(0.079)^3}$

4.2 Definition of the Logarithm

In Chapter 3, we discussed exponents and exponential equations.

What is the solution of the exponential equation $2^x = 8$?

By inspection, we can say that $x = 3$ is the solution.

What is the solution of the equation $2^x = 7$? In this case it is not easy to see what the solution is.

A basic property of real numbers states that

"Given a positive number N, and a positive number b other than 1, the equation $N = b^x$ has exactly one solution for x. "

We will call this solution to be the logarithm of N to the base b and write

$$x = \log_b N$$

In other words, $\log_b N$ is simply that exponent to which we should raise the base b in order to obtain the positive number N.

Definition 2 : If N and b are positive real numbers, b $\neq$ 1, then the equations $N = b^x$ and $x = \log_b N$ are equivalent.

That is, if $b^x = N$, then $x = \log_b N$ and if $\log_b N = x$, then $b^x = N$.

For example,

exponent

$\log_2 8 = 3$ because $8 = 2^3$

Number

Base

Example 1. Express the following in logarithmic form.

(a) $3^4 = 81$ (b) $4^{\frac{1}{2}} = 2$ (c) $5^0 = 1$

Solution

(a) $\log_3 81 = 4$ (b) $\log_4 2 = \frac{1}{2}$ (c) $\log_5 1 = 0$

Example 2. Write each statement in exponential form.

(a) $\log_2 64 = 6$ (b) $\log_4 2 = \frac{1}{2}$ (c) $\log_{10} 2 = 0.3010$

Solution

(a) $2^6 = 64$ (b) $4^{\frac{1}{2}} = 2$ (c) $10^{0.3010} = 2$

As an immediate consequence of this definition, we obtain the following properties.

L 1. $N = b^{\log_b N}$

L 2. $x = \log_b b^x$

These two properties will be used frequently. Definition (2) says that logarithms are exponents. The following points are worth noting.

*(a) Logarithms are real numbers.

*(b) Logarithms of negative numbers and zero are not defined since b^x is always positive.

*(c) There are no logarithms for a base which is negative or 1.

Example 3. Find the value of each logarithm.

(a) $\log_5 25$ (b) $\log_6 \sqrt{6}$ (c) $\log_{10} 0.01$

Solution

(a) There are two methods.

Solution 1	**Solution 2**
Let $\log_5 25 = x$	$\log_5 25 = \log_5 5^2$
then $5^x = 25$	$= 2$ (using L2)
$5^x = 5^2$	
$x = 2$	

(b) $\log_6 \sqrt{6} = \log_6 6^{\frac{1}{2}} = \dfrac{1}{2}$ (using L2)

(c) $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$ (using L2)

Example 4. Solve for x. (a) $\log_b x = 0$ (b) $\log_5 5^x = 3$

Solution

(a) If $\log_b x = 0$, then $x = 1$ $(b > 0,\ b \neq 1)$

(b) Since $\log_5 5^x = x$ (using L 2)

$x = 3$

It should not be difficult for us to use the definition of a logarithm to show that the following properties are true for any positive base b other than 1.

L 3. $\log_b b = 1$

L 4. $\log_b 1 = 0$

Exercise 4.2

1. Write the following equations in logarithmic form.

(a) $3^4 = 81$ (b) $9^{\frac{3}{2}} = 27$ (c) $10^{-3} = 0.001$

(d) $3^{-1} = \frac{1}{3}$ (e) $2^5 = 32$ (f) $16^{\frac{3}{4}} = 8$

2. Write each statement in exponential form.

(a) $\log_2 4 = 2$ (b) $\log_9 3 = \frac{1}{2}$

(c) $\log_{10} 100 = 2$ (d) $\log_{32} 4 = \frac{2}{5}$

(e) $\log_{10} 3 = 0.4771$ (f) $\log_4 64 = 3$

3. Find the value of b, x, or y in each of the following problems.

(a) $\log_7 49 = y$ (b) $\log_b 10 = 1$

(c) $\log_9 3 = y$ (d) $\log_5 x = 3$

(e) $\log_5 x = -3$

4. Solve the following equations.

(a) $\log_b x = 0$ (b) $\log_2 \frac{1}{8} = x$

(c) $\log_{\sqrt{3}} x = 2$ (d) $\log_b \frac{16}{81} = 4$

(e) $\log_{0.2} 5 = y$ (f) $\log_{100} 10 = x$

(g) $\log_{\frac{1}{3}} 27 = y$ (h) $\log_b \frac{1}{128} = -7$

(i) $2^{\log_2 x} = 5$ (j) $5^{\log_x 5} = 5$

(k) $x^{\log_x x} = 5$

5. Evaluate: (a) $10^{1+\log_{10} 3}$ (b) $10^{2+\log_{10} 5}$

6. If $x = 2^4$ and $10^{0.3010} = 2$, find without the use of tables, the logarithm of x to the base 10.

7. If $a^{0.6} = 4$, find $\log_4 a$.

8. If $\log_b x = 2$, what are $\log_{\frac{1}{b}} x$ and $\log_b \frac{1}{x}$? Is it true that

$\log_{\frac{1}{b}} x = \log_b \frac{1}{x}$?

4.3 Properties of Logarithms

The three basic properties of exponents concern products, quotients and powers.

$$\text{Product} : b^p \cdot b^q = b^{p+q}$$

$$\text{Quotient} : b^p \div b^q = b^{p-q}$$

$$\text{Power} : (b^p)^q = b^{pq}$$

Since logarithms are exponents, the properties of exponents hold for logarithms. We state them in the following theorem.

Theorem 1:

If M, N, b are positive numbers, $b \neq 1$ and p is any real number, then

$$\textbf{L 5}: \log_b (MN) = \log_b M + \log_b N$$

$$\textbf{L 6}: \log_b N^p = p \log_b N$$

$$\textbf{L 7}: \log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Proof :

1. According to L 1, we have

$$M = b^{\log_b M} \quad \text{and} \quad N = b^{\log_b N}$$

$$M.N = b^{\log_b M} . b^{\log_b N} = b^{\log_b M + \log_b N}$$

which is of the form

$$MN = b^x \quad , \text{where } x = \log_b M + \log_b N$$

Hence by definition

$$\log_b (MN) = \log_b M + \log_b N$$

2. Since $N = b^{\log_b N}$, we have

$$N^p = \left(b^{\log_b N}\right)^p = b^{p \log_b N}$$

Hence by definition

$$\log_b N^p = p \log_b N$$

3. From L 5 and L 6, we have

$$\log_b \left(\frac{M}{N}\right) = \log_b (MN^{-1})$$

$$= \log_b M + \log_b N^{-1} = \log_b M - \log_b N$$

A tabular summary of the basic properties of exponents and logarithms.

Properties of Exponents		Properties of Logarithms
Product	$b^p \cdot b^q = b^{p+q}$	$\log_b (MN) = \log_b M + \log_b N$
Quotient	$b^p \div b^q = b^{p-q}$	$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$
Power	$(b^p)^q = b^{pq}$	$\log_b N^p = p \log_b N$

Example 1. Write each expression as a single logarithm

(a) $\log_b 20 + \log_b 67 - \log_b 201$

(b) $\log_2 3 + \log_2 15 - \log_2 5$

Solution

(a) $\log_b 20 + \log_b 67 - \log_b 201$

$= \log_b (20 \times 67) - \log_b 201$

$= \log_b \left(\frac{20 \times 67}{201}\right)$

$= \log_b \frac{20}{3}$

(b) $\log_2 3 + \log_2 15 - \log_2 5$

$= \log_2 \frac{3 \times 15}{5}$

$= \log_2 3^2 = \log_2 9$

Example 2. Evaluate

(a) $\log_2 \sqrt{128}$

(b) $\log_{10} 50 + \log_{10} 2$

(a) $\log_2 \sqrt{128}$

$= \log_2 (2^7)^{\frac{1}{2}}$

$= \log_2 2^{\frac{7}{2}} = \frac{7}{2}$ (using L 2)

(b) $\log_{10} 50 + \log_{10} 2$

$= \log_{10} (50 \times 2)$

$= \log_{10} 100$

$= \log_{10} 10^2 = 2$

Example 3. If $\log_{10} 2 = .3010$ and $\log_{10} 3 = .4771$ find (a) $\log_{10} 6$ and (b) $\log_{10} 4.5$

Solution

(a) $\log_{10} 6 = \log_{10}(2 \times 3)$

$= \log_{10} 2 + \log_{10} 3$

$= .3010 + .4771$

$= .7781$

(b) $\log_{10} 4.5$

$= \log_{10} \frac{45}{10} = \log_{10} \frac{9}{2}$

$= \log_{10} 9 - \log_{10} 2$

$= \log_{10} 3^2 - \log_{10} 2$

$= 2 \log_{10} 3 - \log_{10} 2$

$= 2(.4771) - .3010$

$= .9542 - .3010$

$= .6532$

Exercise 4.3

1. Replace $\square$ with the appropriate number.

(a) $\log_3 24 = \log_3 6 + \log_3 \square$ (b) $\log_5 30 = \log_5 60 - \log_5 \square$

(c) $\log_2 \square = 3\log_2 3$ (d) $\log_{10} 9 = \square \log_{10} 3$

(e) $\log_8 5 = \log_8 \square - \log_8 11$

2. Write each expression as a single logarithm.

(a) $\log_b 20 + \log_b 57 - \log_b 241$

(b) $3\log_b 8 - \dfrac{1}{2}\log_b 12$

(c) $\log_b x + 2\log_b y - \log_b a$

(d) $\log_2 3 + \log_4 15$

(e) $\log_3 2 + \log_9 81$

3. Write each expression in terms of log 2, log 3 and log 5 to the same base.

(a) $\log 8$ (b) $\log 15$ (c) $\log 270$

(d) $\log \dfrac{27\sqrt[3]{5}}{16}$ (e) $\log \dfrac{216}{\sqrt[3]{32}}$ (f) $\log(648\sqrt{125})$

4. Evaluate each expression.

(a) $\log_2 128$ (b) $\log_3 81^4$ (c) $\log_{\frac{1}{2}} 8$

(d) $\log_8 2$ (e) $\log_3 \dfrac{\sqrt{3}}{81}$ (f) $\dfrac{\log_3 \sqrt{3}}{\log_3 81}$

(g) $\dfrac{\log 25}{\log 5}$ (h) $\log_4 8$

5. Use $\log_{10} 2 = .3010$ and $\log_{10} 3 = .4771$ to evaluate each expression.

(a) $\log_{10} 6$ (b) $\log_{10} 1.5$ (c) $\log_{10} \sqrt{3}$

(d) $\log_{10} 4$ (e) $\log_{10} 4.5$ (f) $\log_{10} 8$

(g) $\log_{10} 18$ (h) $\log_{10} 5$ (i) $\log_{10} \sqrt{20}$

6. Solve the following equations for x.

(a) $\log_a \frac{18}{5} + \log_a \frac{10}{3} - \log_a \frac{6}{7} = \log_a x$

(b) $\log_b x = 2 - a + \log_b \left(\frac{a^2 b^a}{b^2}\right)$

(c) $\log x^3 - \log x^2 = \log 5x - \log 4x$

(d) $\log_{10} x + \log_{10} 3 = \log_{10} 6$

(e) $8 \log x = \log a^{\frac{3}{2}} + \log 2 - \frac{1}{2} \log a^3 - \log \frac{2}{a^4}$

7. If $\log_{10} 5 = .6990$ and $\log_{10} x = .2330$ then what is the value of x?

8. If $\log_e I = -(\frac{R}{L})t + \log_e I_0$, show that $I = I_0 e^{\frac{-Rt}{L}}$.

9. If $\log_b y = \frac{1}{2}\log_b x + c$, show that $y = b^c\sqrt{x}$.

10. Show that

(a) $\frac{1}{4}\log_{10} 8 + \frac{1}{4}\log_{10} 2 = \log_{10} 2$

(b) $4 \log_{10} 3 - 2 \log_{10} 3 + 1 = \log_{10} 90$

11. Show that, (a) $a^{2 \log_a 3} + b^{3 \log_b 2} = 17$

(b) $3 \log_6 1296 = 2 \log_4 4096$

12. Given that $\log_{10} 12 = 1.0792$ and, $\log_{10} 24 = 1.3802$, deduce the values of $\log_{10} 2$ and $\log_{10} 6$.

13. If $\log_x a = 5$ and $\log_x 3a = 9$, find a and x.

14. (a) If $\log_{10} 2 = a$, find in terms of a, $\log_{10} 8 + \log_{10} 25$.

(b) If $a = 10^x$ and $b = 10^y$, express $\log_{10}(a^4 b^3)$ in terms of x and y.

15. (a) If $\log_2(4x-4)=2$, find the value of $\log_4 x$.

(b) If $\frac{1}{2}\log_3 M + 3\log_3 N = 1$, then prove that $MN^6 = 9$.

4.4 Change of Base

Logarithms to any base are easily obtained by using logarithms computed for some given base. It is possible to determine $\log_a N$ from $\log_b N$ and $\log_b a$ by means of a property that we shall now derive.

Theorem 2

If a and b are two bases and if N is any positive number, then $\log_a N = \dfrac{\log_b N}{\log_b a}$

Proof : We may write $N = a^{\log_a N}$

Hence $\log_b N = \log_b a^{(\log_a N)} = (\log_a N)\log_b a$

$$\therefore \log_a N = \frac{\log_b N}{\log_b a}$$

If we let $N = b$, the above equality becomes

$$\log_a b = \frac{\log_b b}{\log_b a}$$

But $\log_b b = 1$, so we have the following corollary.

$$\log_a b = \frac{1}{\log_b a}$$

Example 1. If $\log_2 8 = 3$, find the value of $\log_8 2$.

Solution

By using change of base, we have $\log_8 2 = \dfrac{1}{\log_2 8} = \dfrac{1}{\log_2 2^3} = \dfrac{1}{3}$

Example 2. Show that $\log_{a^2} b = \dfrac{1}{2}\log_a b$

Solution

$$\text{L.H.S} = \log_{a^2} b = \frac{\log_a b}{\log_a a^2}$$

$$= \frac{\log_a b}{2} = \frac{1}{2}\log_a b = \text{R.H.S}$$

In general, we can prove the following property.

$$\text{L 8.} \quad \log_{b^n} a = \frac{1}{n}\log_b a$$

Example 3. If a, b and k are positive real numbers other than 1, then prove the following property.

$$\text{L 9.} \quad (a)^{\log_k b} = (b)^{\log_k a}$$

Solution

Using L 1, $b = a^{\log_a b}$

so that $a^{\log_k b} = a^{\log_k (a^{\log_a b})} = a^{(\log_a b)(\log_k a)}$

$$= \left(a^{\log_a b}\right)^{\log_k a} = (b)^{\log_k a} = b^{\log_k a}$$

Exercise 4.4

1. Find the value of each :

   (a) $2^{\frac{\log 3}{\log 2}}$ (b) $5^{\frac{1}{\log_7 5}}$ (c) $\log_3 5 \times \log_{25} 27$

2. If $\log_4 16 = 2$, find $\log_{16} 4$.

3. If $\log_a b + \log_b a^2 = 3$, find b in terms of a.

4. Show that

   (a) $\log_4 x = 2\log_{16} x$ (b) $\log_b x = 3\log_{b^3} x$

   (c) $\log_2 x = (1 + \log_2 3)\log_6 x$

5. If $a = \log_b c, b = \log_c a, c = \log_a b$, prove that $abc = 1$.

6. Show that

   (a) $(\log_{10} 4 - \log_{10} 2)\log_2 10 = 1$

   (b) $(2\log_2 3)(\log_9 2 + \log_9 4) = 3$

7. Compute : (a) $3^{\log_2 5} - 5^{\log_2 3}$ (b) $4^{\log_2 3}$

   (c) $2^{\log_{2\sqrt{2}} 27}$

4.5 Common Logarithms

John Napier invented logarithms to simplify arithmetic calculations. We have discussed many possible logarithm bases, but only two are commonly used. The first type uses the constant e (after Euler) as a base in all advanced mathematics courses. These are called natural logarithms and, we shall see the importance of this logarithm when we study calculus. An approximate value of e is

$$e \triangleq 2.71828$$

and like π, e is an irrational number. The other type of logarithm uses as base of 10. These are called common logarithms and fit in nicely with our decimal (base 10) system of numeration. For simplicity, we write log x for $\log_{10}$ x (omitting the base).

Thus $y = \log x$ is equivalent to $10^y = x$. In particular,

$$
\begin{array}{llllll}
\log 100 & = & \log 10^2 & = 2 & \text{because } 10^2 & = 100 \\
\log 1000 & = & \log 10^3 & = 3 & \text{because } 10^3 & = 1000 \\
\log 10 & = & \log 10^1 & = 1 & \text{because } 10^1 & = 10 \\
\log 1 & = & \log 10^0 & = 0 & \text{because } 10^0 & = 1 \\
\log 0.1 & = & \log 10^{-1} & = -1 & \text{because } 10^{-1} & = 0.1 \\
\log 0.001 & = & \log 10^{-3} & = -3 & \text{because } 10^{-3} & = 0.001
\end{array}
$$

Indeed, for any integer n, $\log 10^n = n$. It is for this reason that common logarithms are used in problems involving arithmetic calculations. But what about all the other numbers? For numbers between 1 and 10, we use a four-figure table of common logarithms originally developed by an English mathematician Henry Briggs (published in 1624). The table gives approximate values of log z for $1 \leq z \leq 9.9$. In Fig. 4.1, a part of the table is shown.

											Difference column								
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0																			
1.1																			
1.2																			
.																			
.																			
.																			
3.0																			
3.1	.4914	.4928	.4942	.4955	$\boxed{.4969}$	.4983	.4997	.5011	.5024	.5038	1	3	4	6	7	8	10	11	12
3.2																			
3.3																			

Fig. 4.1

Thus to find log 3.142 we have to look at the intersection of the row labelled 3.1 and the columns labelled 4 and 2 as shown in Fig. 4.1. Then adding the values, we will get $\log\ 3.142 \mathrel{\hat{=}} 0.4969 + 0.0003 \mathrel{\hat{=}} 0.4972$.

Similarly we can see that

$$\log 3 = \log 3.000 \mathrel{\hat{=}} 0.4771$$

$$\log 3.7 = \log 3.700 \mathrel{\hat{=}} 0.5682.$$

$$\log 3.75 = \log 3.750 \mathrel{\hat{=}} 0.5740$$

$$\log 3.758 \mathrel{\hat{=}} 0.5740 + 0.0009 \mathrel{\hat{=}} 0.5749$$

A glance at the table of logarithms shows that it gives values of log z for $1 \le z \le 9.9$. To obtain the value of log z for z outside this range, we simply rewrite z in scientific notation as:

$$z = a\ .\ 10^n \quad (1 \le a < 10)$$

Since $\quad z = a\ .\ 10^n = 10^n\ .\ a$

$$\log z = \log 10^n + \log a$$

$$\log z = n + \log a,\ 1 \le a < 10,\ n \text{ integer} \qquad \ldots (1)$$

Since $1 \le a < 10$, we can use the table of logarithms to find log a. It will be noticed that

$$0 \le \log a < 1 \qquad \ldots (2)$$

Combining (1) and (2) we conclude that

The logarithm of any positive real number z can be written as a sum of an integer (namely, n) and a nonnegative real number less than 1 (namely, log a).

The integer n is called the **characteristic** of log z while the nonnegative real number log a is called the **mantissa** of log z. Observe that, the mantissa is always positive and is strictly less than 1. Thus (1) can be rewritten as follows:

$$\log z = n + \log a \mathrel{\hat{=}} \text{characteristic} + \text{mantissa}$$

In actual practice, we write

$$\log z = n + \log a = \text{characteristics} + \text{mantissa}$$

Example 1. Find log 23450.

Solution $23450 = 2.345 \times 10^4$

$$\log 23450 = 4 + \log 2.345 = 4 + (0.3692 + 0.0009)$$

$$= \underbrace{\underset{\text{Characteristic}}{4} \, . \, \underbrace{3701}_{\text{Mantissa}}}_{\text{Logarithm}}$$

Example 2. Find log 51.7, log 5.17, log 0.517 and log 0.0517.

Solution

$$
\begin{aligned}
51.7 &= 5.17 \times 10^1 \\
5.17 &= 5.17 \times 10^0 \\
0.517 &= 5.17 \times 10^{-1} \\
0.0517 &= 5.17 \times 10^{-2} \\
\log 51.7 &= 1 + \log 5.170 \\
&= 1 + 0.7135 = 1.7135 \\
\log 5.17 &= \log 5.170 = 0 + 0.7135 = 0.7135 \\
\log 0.517 &= \log 0.5170 = -1 + 0.7135 = \bar{1}.7135 \\
\log 0.0517 &= \log 0.0517 = -2 + 0.7135 = \bar{2}.7135
\end{aligned}
$$

The number $\bar{2}.7135$ denotes the sum $0.7135 + (-2)$

That is $\bar{2}.7135 = 0.7135 + (-2)$

We do not write $0.7135 + (-2)$ as $-(1.2865)$ because if we write like this we would not know at a glance what the mantissa and characteristic of the logarithm are ; $\bar{2}$ is read as " bar 2 ".

Now we have the following rule :

To find log z, for any positive number z :

1. **Write z in scientific notation**

   $\mathbf{z = a \times 10^n}$

2. **Look up log a in the table.**

3. **Write log z = n + log a**

4. **If $z = a.\ 10^{-n}$, $\log z = \bar{n} + \log a$**

A table of common logarithms can be used in a second way. Given a logarithm, we can find the corresponding number N, called the antilogarithm, or simply antilog.

Example 3. Find antilog 3.6075.

It means to find N if $\log N = 3.6075$.

Solution

We reverse the process used to find logarithms.

Here 3 is the characteristic and 0.6075 is the mantissa.

$$3.6075 = 3 + 0.6075$$

Here we find 0.6075 in the body of the table and see that it is at the intersection of the 4.0 row and 5 column.

Thus $\log 4.05 = 0.6075$.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0																			
.																			
.																			
.																			
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10
4.1																			
4.2																			

Fig. 4.2

$$\log N = 3 + 0.6075 = \log 10^3 + \log 4.05 = \log (10^3 \times 4.05)$$

$$N = 10^3 \times 4.05$$

Hence antilog $3.6075 = N = 4050$

Note that the three equations

$\log N = 3.6075$

$N = \text{Antilog } 3.6075$ and $10^{3.6075} = N$ are equivalent

Example 4. Find N = antilog 0.6644.

Solution

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0																			
.																			
.																			
.																			
4.6	.6628	$\boxed{.6637}$	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1	2	3	4	5	6	7	7	8

Fig 4.3

In the body of the table, we can find 0.6637 and 0.0007 as shown in Fig. 4.3.

Since $0.6644 = 0.6637 + 0.0007$, $0.6644 = \log 4.617$

Hence, $N = \text{antilog } 0.6644 = 4.617$

Exercise 4.5

1. Find the characteristic, mantissa and logarithm of each number.

(a) 98 (b) 462 (c) 0.00462

(d) 60000 (e) 2.74 (f) 0.00002

(g) 0.6666 (h) 2222 (i) 68.360

2. Find the logarithms.

(a) $\log 273$ (b) $\log 2.73$ (c) $\log 0.000273$

(d) $\log 1$ (e) $\log 10$ (f) $\log \frac{3}{4}$

(g) $\log 0.7806$ (h) $\log 42.030$ (i) $\log (0.015)^2$

(j) $\log \frac{1}{20}$

3. Find the antilogarithms.

(a) antilog 2.8248 (b) antilog 0.8248

(c) antilog $\bar{2}.8248$ (d) antilog 1.0043

4. Find N if log N is

(a) 0.9032 (b) 0.5613 (c) 0.8600

5. Solve for the indicated variable.

(a) $\log N = 0.0808$ (b) $\log 5.277 = x$ (c) $\log N^2 = 0.9500$

4.6 Application of Common Logarithm

For 300 years, scientists depended on logarithms to reduce the drudgery associated with long computations. Properties of logarithm replace multiplications by additions, powers by multiplications, divisions by subtractions;

$$\log MN = \log M + \log N$$

$$\log M^p = p \log M$$

$$\log \frac{M}{N} = \log M - \log N$$

In addition to these laws, let us make three observations:

1. If $M = N$ $(M, N > 0)$, then $\log M = \log N$.
2. If $\log_b M = \log_b N$, then $M = N$.
3. If $M = N$, then $b^M = b^N$.

Today, the use of the slide rule and the advent of high-speed computing devices have almost removed the need to perform routine numerical computations by logarithms. Nevertheless, we introduce the techniques for computation by means of logarithm tables.

Example 1. Use logarithms to evaluate $\dfrac{(200)(98.7)(85.3)}{(8.5)^3}$

Solution Let $Q = \dfrac{(200)(98.7)(85.3)}{(8.5)^3}$

The first step is to write the appropriate logarithmic equation.

$$\log Q = \log (200)(98.7)(85.3) - \log (8.5)^3$$
$$= \log 200 + \log 98.7 + \log 85.3 - 3 \log 8.5$$

Now using this equation as a guide, we can perform the necessary work with an appropriate scheme, as shown:

$$\left.\begin{array}{lcl} \log 200 & = & 2.3010 \\ \log 98.7 & = & 1.9943 \\ \log 85.3 & = & 1.9309 \end{array}\right\} \text{add}$$

$$\left.\begin{array}{lcl} & & 6.2262 \\ 3 \log 8.5 = 3(0.9294) & = & 2.7882 \end{array}\right\} \text{subtract}$$

$$\begin{aligned} \log Q &= 3.4380 \\ Q &= \text{antilog } 3.4380 \\ &= 10^3 \times 2.741 \\ Q &= 2741 \end{aligned}$$

Example 2. Use logarithms to evaluate $\dfrac{\sqrt[3]{-753.9}}{(38.07)^2(6.44)}$

Solution

In spite of the fact that there are no logarithms of negative numbers, we are still to do computations involving negative quantities. Since

$\sqrt[3]{-753.9} = -\sqrt[3]{753.9}$, the given expression can be written as

$$\frac{\sqrt[3]{-753.9}}{(38.07)^2(6.44)} = \frac{-\sqrt[3]{753.9}}{(38.07)^2(6.44)}$$

$$\text{Let } Q = \frac{\sqrt[3]{753.9}}{(38.07)^2(6.44)}$$

Then, $\log Q = \log (753.9)^{\frac{1}{3}} - \log [(38.07)^2 (6.44)]$

$$= \frac{1}{3}\log 753.9 - [2 \log 38.07 + \log 6.44]$$

$$\left.\begin{array}{l} \log 753.9 = 2.8773 \longrightarrow \frac{1}{3}\log 753.9 = 0.9591 \longrightarrow 10.9591 - 10 \\ \log 38.07 = 1.5806 \longrightarrow \left.\begin{array}{r} 2\log 38.07 = 3.1612 \\ \log 6.44 = 0.8089 \end{array}\right\} \text{add} \\ \hline \qquad\qquad 3.9701 \longrightarrow 3.9701 \end{array}\right\} \text{subtract}$$

$$\begin{aligned} \log Q &= 6.9890 - 10 \\ &= -4 + 0.9890 \\ Q &= 10^{-4} \times 9.75 = 0.000975 \end{aligned}$$

Thus $\dfrac{\sqrt[3]{-753.9}}{(38.07)^2(6.44)} = -0.000975$

We can also use logarithms to solve difficult exponential equations.

Example 3. Solve $4^x = 15$

Solution $4^x = 15$

$$\begin{aligned} \log 4^x &= \log 15 \\ x \log 4 &= \log 15 \\ x &= \frac{\log 15}{\log 4} = \frac{1.1761}{0.6021} \\ \log x &= \log 1.1761 - \log 0.6021 = 0.0705 - \bar{1}.7797 \\ &= 1.0705 - 0.7797 = 0.2908 \\ x &= \text{antilog } 0.2908 \\ &= 1.954 \end{aligned}$$

Exercise 4.6

Compute the following, using logarithms.

1. $\dfrac{(-200)(98.7-85.3)}{(8.5)^3}$

2. $\dfrac{(274)^{\frac{1}{3}}}{(927)(818)}$

3. $\dfrac{(28.3)\sqrt{.621}}{597}$

4. $\left(\dfrac{-28.3}{(597)(621)}\right)^2$

5. $\dfrac{\sqrt[3]{-22.380}}{(0.00625)^2(7992)^{\frac{1}{2}}}$

6. $\dfrac{\sqrt[3]{(186)^2}}{(600)^{\frac{1}{4}}}$

Solve.

7. $6^x = 0.2$

8. $5^x = 0.50$

9. $x^{2.5} = 25$

10. $5^x 6^x = 100$

11. $3^{2x+1} = 8$

12. $12^{1-x} = 40$

SUMMARY

Important Definitions and Rules

1. A positive number is in scientific notation when it is written in the form $a \cdot 10^n$, $1 \le a < 10$, n is an integer.

2. Logarithms are exponents.
   $\log_b N = n$ means $b^n = N$ where N and b are positive real numbers, $b \neq 1$.

3. (a) $N = b^{\log_b N}$

   (b) $\log_b b^x = x$

   (c) $\log_b 1 = 0$

   (d) $\log_b b = 1$

   (e) $\log_b (MN) = \log_b M + \log_b N$

   (f) $\log_b M^k = k \log_b M$

(g) $\log_b \frac{M}{N} = \log_b M - \log_b N$

(h) $\log_b N = \frac{\log_a N}{\log_a b}$

(i) $\log_{b^n} a = \frac{1}{n} \log_b a$

(j) $a^{\log_k b} = b^{\log_k a}$

CHAPTER 5

Equations

By 2000 B.C. Babylonian Arithmetic had evolved into a well-developed algebra. Not only were quadratic equations solved, both by substituting in a general formula and by completing the square, but some cubic (third degree) and biquadratic (fourth degree) equations were also discussed. In this chapter we shall study two methods of solving a quadratic equation, and we shall make an assumption that the variables are real numbers.

5.1 Quadratic Equation

Let us consider the open sentence $x^2 + 4x - 5 = 0$ ... (1)

It contains an x^2 term and no higher power of x. It is an example of a quadratic equation. To solve equation (1), we have to find all the replacements for x which make equation (1) a true sentence. These replacements are the solutions of the quadratic equations. They are also called the roots of the quadratic equation. The only replacements for the variable x which make the given quadratic equation a true sentence are -5 and 1. Hence the solution set of the given quadratic equation is $\{-5, 1\}$.

The equation $ax^2 + bx + c = 0$ ... (2)

where a is non-zero, is called a **quadratic equation** or **an equation of the second degree in x.**

What are the solutions of equation (2)? How can we find them?

We will describe the methods for solving quadratic equations in the following sections.

5.2 Solving Quadratic Equations by Completing the Square

16, x^2, $9x^2$, $(x-2)^2$, $(x+p)^2$ are examples of perfect squares.

If the quadratic equation can be expressed in the form $(x+p)^2 = q$, $(q > 0)$

then, by taking square root of both sides, we get

$$x + p = \pm\sqrt{q}, \qquad (q > 0)$$

$$x + p - p = \pm\sqrt{q} - p$$

$$x = -p \pm \sqrt{q}$$

(Note : To avoid ambiguity in the meaning of $\sqrt{}$, $\sqrt{q}$ is defined to be the positive square root of q ; the negative square root of q is denoted by $-\sqrt{q}$. For example, $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.)

Consider $x^2 + 2px$, which is not a perfect square. By adding p^2 we obtain $x^2 + 2px + p^2$ which is now a perfect square $(x + p)^2$. Here we may observe that p is half of coefficient of x. The process of adding a constant term to a quadratic expression to make it a perfect square is called **completing the square.** In applying the method, we must first ensure that the coefficient of x^2 is unity.

Example 1. Solve $x^2 - 8x + 8 = 0$ by completing the square, rounding off the roots to 2 decimal places.

Solution

$$x^2 - 8x + 8 = 0$$

$$x^2 - 8x + 8 - 8 = 0 - 8$$

$$x^2 + 2x(-4) = -8$$

$$x^2 + 2x(-4) + (-4)^2 = -8 + (-4)^2$$

$$(x - 4)^2 = 8$$

$$x - 4 = \pm\sqrt{8}$$

$$x - 4 + 4 = \pm\sqrt{8} + 4$$

$$x = 4 \pm \sqrt{8}$$

therefore, $x \triangleq 4 + 2.83$ or $x \triangleq 4 - 2.83$

$$x \triangleq 6.83 \quad \text{or} \quad x \triangleq 1.17$$

Note : (i) we use the symbol "$\simeq$" to mean approximately equal.

(ii) The solution set of the given equation is $\{4 + \sqrt{8}\ ,\ 4 - \sqrt{8}\ \}$

(iii) 6.83 and 1.17 are the approximate solutions of the given equation and they do not exactly satisfy the given equation.

Example 2. Solve $2x^2 + 6x - 1 = 0$ and get the approximate solutions to 2 decimal places.

Solution

$$2x^2 + 6x - 1 = 0$$

$$x^2 + \frac{6}{2}x - \frac{1}{2} = 0$$

$$x^2 + \frac{6}{2}x - \frac{1}{2} + \frac{1}{2} = 0 + \frac{1}{2}$$

$$x^2 + 3x = \frac{1}{2}$$

$$x^2 + 2x \cdot \frac{3}{2} = \frac{1}{2}$$

$$x^2 + 2x \cdot \frac{3}{2} + (\frac{3}{2})^2 = \frac{1}{2} + (\frac{3}{2})^2$$

$$(x + \frac{3}{2})^2 = \frac{1}{2} + \frac{9}{4}$$

$$(x + 1.5)^2 = 2.75$$

Therefore $\quad x + 1.5 \simeq \pm 1.66$

$$x \simeq -1.5 \pm 1.66$$

$$x \simeq -3.16 \text{ or } x \simeq 0.16$$

Example 3. Find the solution set of $x^2 + 1 = 0$

Solution

$$x^2 + 1 = 0$$

$$x^2 + 1 - 1 = 0 - 1$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

Since $\sqrt{-1}$ is not a real number, i.e., $\sqrt{-1} \notin R$, $+\sqrt{-1}$ and $-\sqrt{-1}$ do not belong to the replacement set of the given equation. So the solution set is ϕ .

Exercise 5.1

1. Find the number which must be added to each to make a perfect square.

(a)	$x^2 + 6x$	(b)	$y^2 + 8y$	(c)	$z^2 + 10z$
(d)	$t^2 - 6t$	(e)	$m^2 - 2m$	(f)	$n^2 - 12n$
(g)	$x^2 + x$	(h)	$p^2 + 3p$	(i)	$x^2 + \frac{2x}{3}$

2. Complete the square in each of the following.

(a) $x^2 + 6x + \ldots = (x + \ldots)^2$
(b) $y^2 - 6y + \ldots = (y - \ldots)^2$
(c) $z^2 + 12z + \ldots = (z + \ldots)^2$
(d) $t^2 - 14t + \ldots = (t - \ldots)^2$
(e) $m^2 + m + \ldots = (m + \ldots)^2$
(f) $n^2 - n + \ldots = (n - \ldots)^2$
(g) $x^2 - 0.6x + \ldots = (x - \ldots)^2$
(h) $y^2 + 5y + \ldots = (y + \ldots)^2$

3. Solve each of the following quadratic equations by completing the square.

(a)	$x^2 - 14x + 40 = 0$	(b)	$x^2 + 10x + 25 = 0$
(c)	$x^2 + x - 2 = 0$	(d)	$x^2 + 6x + 9 = 0$
(e)	$x^2 + 5x + 6 = 0$	(f)	$x^2 + 3x - 40 = 0$
(g)	$2x^2 + x - 6 = 0$	(h)	$2x^2 + 3x - 14 = 0$

4. Solve the following by completing the square, rounding off the roots of one decimal place.

(a)	$x^2 + 4x + 1 = 0$	(b)	$x^2 - 6x + 3 = 0$
(b)	$x^2 + 6x + 4 = 0$	(d)	$x^2 - 2x - 1 = 0$
(e)	$x^2 + 8x - 1 = 0$	(f)	$x^2 + 3x + 1 = 0$
(g)	$x^2 - 3x - 2 = 0$	(h)	$2x^2 + 2x - 1 = 0$

5.3 Solving Quadratic Equations by Formula

Every quadratic equation can be expressed in the standard form

$$ax^2 + bx + c = 0, \quad a \neq 0$$

We shall now get the formula which can be used to find the solution set of any quadratic equation.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} - \frac{c}{a} = 0 - \frac{c}{a}$$

$$x^2 + 2x.\frac{b}{2a} = -\frac{c}{a}$$

$$x^2 + 2x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} - \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

The result is known as the **quadratic formula.** To use this formula, we must first express the quadratic equation in standard form.

Example 1. Find the solution set of $x^2 - 3x + 2 = 0$

Solution

Compare $x^2 - 3x + 2 = 0$ with the standard form

$ax^2 + bx + c = 0.$

$a = 1, b = -3, c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm \sqrt{1}}{2}$$

$$x = 2 \text{ or } x = 1$$

The solution set is $\{ 1, 2 \}$

Example 2. Solve $2x^2 - 3x + 4 = 0$.

Solution

Compare $2x^2 - 3x + 4 = 0$ with the standard form

$ax^2 + bx + c = 0$.

$a = 2,\ b = -3,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(4)}}{2 \times 2} = \frac{3 \pm \sqrt{-23}}{4}$$

Since $\sqrt{-23}$ is not a real number (i.e. $\sqrt{-23} \notin R$) and since we are considering only the real roots, the solution set is empty.

Example 3. Solve $2x^2 + 3x = 4$. Round off the roots to one decimal place.

Solution

$2x^2 + 3x = 4$

$2x^2 + 3x - 4 = 0$. Compare it with the standard form $ax^2 + bx + c = 0$.

$a = 2, b = 3, c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-4)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{41}}{4}$$

$$x \approx \frac{-3 \pm 6.40}{4}$$

$$x \approx 0.85 \quad \text{or} \quad x \approx -2.35$$

$$x \approx 0.9 \quad \text{or} \quad x \approx -2.4 \quad \text{to one decimal place.}$$

Exercise 5.2

1. Find the solution sets of the following equations by using quadratic formula:

(a) $4t^2 - 8t - 5 = 0$ (b) $4u^2 - 20u + 25 = 0$

(c) $3v^2 + 13v = 10$ (d) $3x^2 + 20x = 7$

(e) $4y^2 = 5(3y + 5)$

2. Solve the following quadratic equations by the formula, rounding off the roots to one decimal place.

(a) $x^2 + 2x - 1 = 0$ (b) $y^2 + 1 = 0$

(c) $z^2 + 6z + 4 = 0$ (d) $a^2 - 6a + 3 = 0$

(e) $t^2 - 4t - 8 = 0$ (f) $b^2 = 12b + 5$

(g) $2p^2 + p - 4 = 0$ (h) $2c^2 - 3c - 4 = 0$

(i) $2y^2 + 12y + 9 = 0$ (j) $3x^2 - 2x - 4 = 0$

5.4 System of Equation in Two Variables, One Linear and One Quadratic

When one of the given equations is linear, and the other is quadratic, we may, from the linear equation, express one of the variables in terms of the other, and substitute in the quadratic equation. Then the resulting equation will be quadratic in one variable.

Consider the system of equations

$$\left.\begin{aligned} y &= x^2 \\ y &= x + 2 \end{aligned}\right\} \ldots (a)$$

The solution set of the open sentence $y = x^2$ is an infinite set of points

$$\{ (x, y) \mid y = x^2 \}.$$

which represents a parabola ; the solution set of the open sentence $y = x + 2$ is

$$\{ (x, y) \mid y = x + 2 \}$$

which represents a straight line. The graphs of the given system of equations in (a) are shown in Fig.5.1.

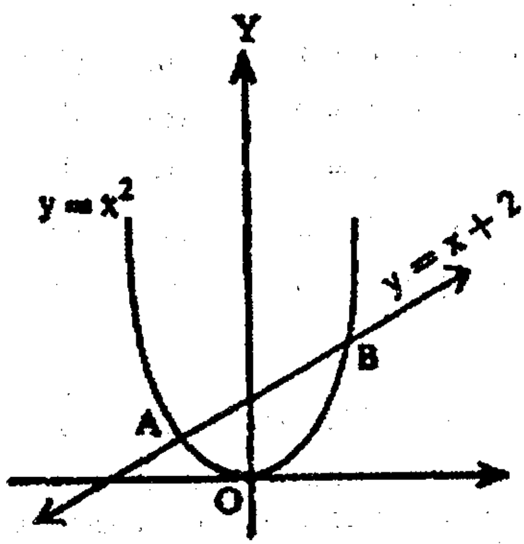


Fig. 5 . 1

The points of intersection A and B correspond to the elements of the solution set of the given system of equations; the solution set is

$$\{ (x, y) \mid y = x^2 \} \cap \{ (x, y) \mid y = x + 2 \}$$

or $\{ (x, y) \mid y = x^2 \text{ and } y = x + 2 \}$

To find this solution set, we use the substitution method.

The working may be set down conveniently as follows:

$$y = x^2 \qquad \ldots (1)$$

$$y = x + 2 \qquad \ldots (2)$$

Substituting $y = x + 2$ in equation, (1) we get

$$x + 2 = x^2$$

$$x^2 - x - 2 = x + 2 - x - 2$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \qquad \text{or} \qquad x = 2$$

Substituting $x = -1$ in equation (2),

$$y = -1 + 2 = 1$$

Substituting $x = 2$ in equation (2),

$$y = 2 + 2 = 4$$

Therefore the solution set is $\{ (-1, 1), (2, 4) \}$

Next, let us consider another system of equations.

$$\left.\begin{aligned} v &= x^2 - x - 2 \\ y &= 0 \end{aligned}\right\} \ldots (b)$$

The solution set of the open sentence $y = x^2 - x - 2$ is an infinite set of points

$$\{ (x, y) \mid y = x^2 - x - 2 \},$$

which represents a parabola. To draw the graph of this parabola, we find the sample points from Table 5.1.

Table 5.1

$y = x^2 - x - 2$

x	-3	-2	-1	0	1	2	3
x^2 $-x$ -2	9 3 -2	4 2 -2	1 1 -2	0 0 -2	1 -1 -2	4 -2 -2	9 -3 -2
x^2-x-2	10	4	0	-2	-2	0	4

The sample points are $(-3,10),(-2,4),(-1,0),(0,-2),(1,-2),(2,0)$ and $(3,4)$. Plot these points and draw a smooth curve through these points.(Fig.5.2)

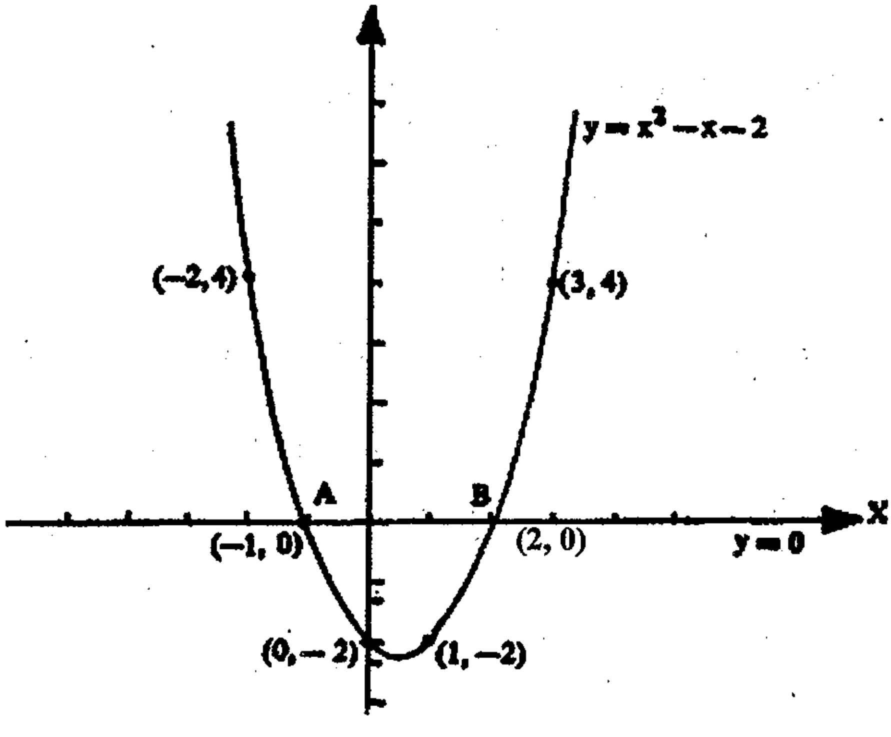


Fig. 5.2

The solution set of the open sentence $y = 0$ is $\{(x, y) \mid y = 0\}$ which represents the X-axis. (Fig. 5.2)

The points of intersection A and B correspond to the elements of the solution set of the given system of equations; the solution set is

$$\{ (x, y) \mid y = x^2 - x - 2 \} \cap \{ (x, y) \mid y = 0 \}$$

or $$\{ (x, y) \mid y = x^2 - x - 2 \text{ and } y = 0 \}$$

The working may be set down conveniently as follows:

$$y = x^2 - x - 2 \qquad \dots (1)$$

$$y = 0 \qquad \dots (2)$$

Substituting $y = 0$ in equation (1), we get

$$0 = x^2 - x - 2$$

$$x^2 - x - 2 = 0 \qquad \dots *$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \quad \text{or} \quad x = 2$$

For $x = -1$ or $x = 2$, y is always zero.

Therefore the solution set is $\{ (-1, 0), (2, 0) \}$.

The system (a) has the solution set $\{(-1, 1), (2, 4)\}$ and the system (b) has the solution set $\{(-1, 0), (2, 0)\}$. In both systems, we observe that the x-coordinates of the solutions are respectively the same and they are the roots of the same quadratic equation $x^2 - x - 2 = 0$ marked by (*) in the workings. Thus the system (a) suggests a method for finding the roots of a quadratic equation $ax^2 + bx + c = 0$ by using the graphs (i) $y = x^2$ (parabola) and (ii) $y = mx + c$ (straight line); and the system (b) suggests another method for finding the roots by using the graphs

(i) $y = ax^2 + bx + c$ (parabola) and (ii) $y = 0$ (X-axis).

Example 1. Find the solution set of the system of equations :

$$x^2 + y^2 = 25$$

$$y = x - 1$$

Solution

$$x^2 + y^2 = 25 \quad \ldots (1)$$
$$y = x - 1 \quad \ldots (2)$$

Substituting $y = x - 1$ in equation (1), we get

$$x^2 + (x-1)^2 = 25$$
$$x^2 + x^2 - 2x + 1 - 25 = 25 - 25$$
$$2x^2 - 2x - 24 = 0$$
$$x^2 - x - 12 = 0$$
$$(x+3)(x-4) = 0$$
$$x = -3 \quad \text{or} \quad x = 4$$

Substituting $x = -3$ in equation (2),

$$y = -3 - 1$$
$$y = -4$$

Substituting $x = 4$ in equation (2),

$$y = 4 - 1$$
$$y = 3$$

Therefore the solution set is $\{ (-3, -4), (4, 3) \}$

Example 2. Find the solution set of

$$x^2 - xy + 2y^2 = 8$$
$$3x - 2y = 2$$

Solution

$$x^2 - xy + 2y^2 = 8 \quad \ldots (1)$$
$$3x - 2y = 2 \quad \ldots (2)$$

Rewriting (2) in the equivalent form

$$y = \frac{3x-2}{2} \quad \ldots (3)$$

Substituting $y = \frac{3x-2}{2}$ in equation (1), we get

$$x^2 - \frac{x(3x-2)}{2} + \frac{2(3x-2)^2}{4} = 8$$

$$2x^2 - x(3x-2) + (3x-2)^2 = 16$$

$$2x^2 - 3x^2 + 2x + 9x^2 - 12x + 4 = 16$$

$$8x^2 - 10x + 4 - 16 = 16 - 16$$

$$8x^2 - 10x - 12 = 0$$

$$4x^2 - 5x - 6 = 0$$

$$(4x + 3)(x - 2) = 0$$

$$x = -\frac{3}{4} \quad \text{or} \quad x = 2$$

Substituting $x = -\frac{3}{4}$ in equation (3),

$$y = \frac{3(-\frac{3}{4}) - 2}{2} = -\frac{17}{8}$$

Substituting $x = 2$ in equation (3),

$$y = \frac{3 \cdot 2 - 2}{2} = 2$$

Hence the solution set is $\{(-\frac{3}{4}, -\frac{17}{8}), (2, 2)\}$

Example 3. Solve graphically the quadratic equation $3 - 4x - 2x^2 = 0$.

Solution

$$3 - 4x - 2x^2 = 0$$

$$2x^2 = 3 - 4x$$

$$x^2 = \frac{3}{2} - 2x$$

Let us now consider the system of the equations

$$y = x^2$$

$$y = \frac{3}{2} - 2x$$

The sample points of the parabola $y = x^2$ can be found from Table 5.2.

Table 5.2

$y = x^2$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9

The sample points are $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$.

Take 0.5 of an inch as the unit for measuring x and 0.3 of an inch as the unit for measuring y.

Plot the sample points of the parabola and draw a smooth curve through these points. Then the graph of $y = x^2$ will be as in Fig. 5.3.

Next the sample points of the straight line $y = \frac{3}{2} - 2x$ can be found from Table 5.3.

Table 5.3

$y = \frac{3}{2} - 2x$

x	0	1
$\frac{3}{2}$ $-2x$	$\frac{3}{2}$ 0	$\frac{3}{2}$ -2
$\frac{3}{2} - 2x$	$\frac{3}{2}$	$-\frac{1}{2}$

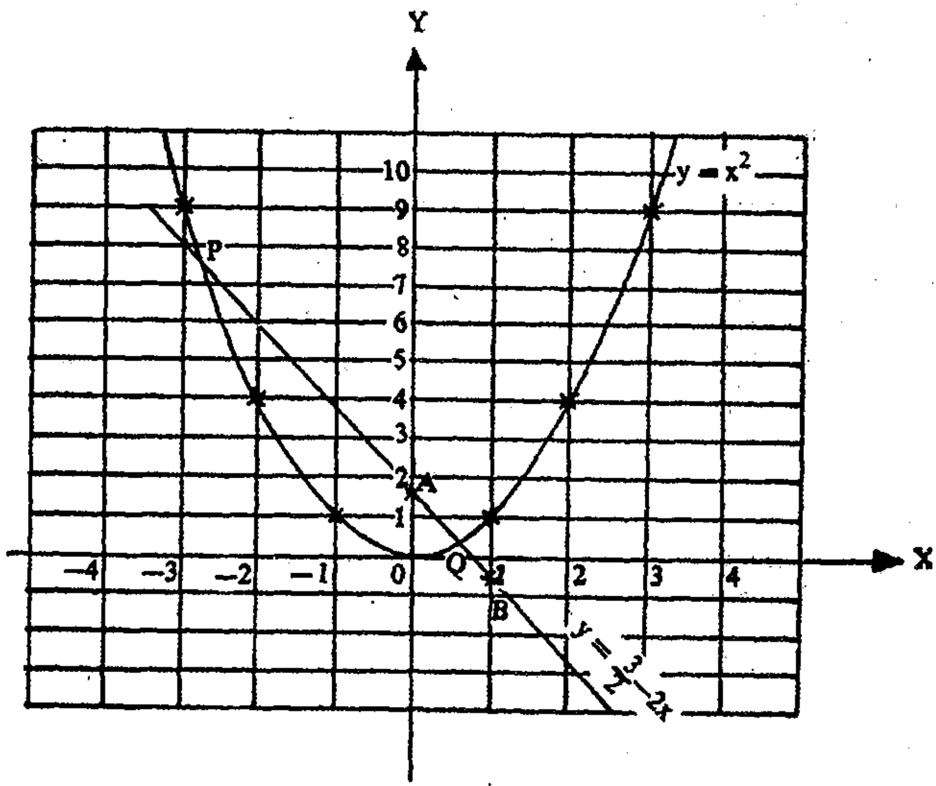


Fig. 5.3

The sample points of the straight line are A $(0, \frac{3}{2})$ and B $(1, -\frac{1}{2})$. Plot these points and draw a straight line through these points. Then the graph of $y = \frac{3}{2} - 2x$ will be as in Fig. 5.3.

Let this line cut the parabola in P and Q. The x-coordinates of P and Q are the roots of the equation $3 - 4x - 2x^2 = 0$.

These x-coordinates are respectively equal to -2.6 and 0.6 approximately.

Therefore the approximate roots are -2.6 and 0.6.

Alternative solution

Given quadratic equation is $3 - 4x - 2x^2 = 0$

Let us consider the system of equations

$$y = 3 - 4x - 2x^2$$
$$y = 0$$

The sample points of the parabola $y = 3 - 4x - 2x^2$ can be found from Table 5.4.

Table 5.4

$y = 3 - 4x - 2x^2$

x	-3	-2	-1	0	1	2	3
3	3	3	3	3	3	3	3
$-4x$	12	8	4	0	-4	-8	-12
$-2x^2$	-18	-8	-2	0	-2	-8	-18
$3-4x-2x^2$	-3	3	5	3	-3	-13	-27

The sample points are $(-3, -3)$, $(-2, 3)$, $(-1, 5)$, $(0, 3)$, $(1, -3)$, $(2, -13)$, $(3, -27)$.

Take 0.5 of an inch as the unit for measuring x, and 0.2 of an inch as the unit for measuring y.

Plot the above sample points and draw a smooth curve through these points. Then the graph of the parabola $y = 3 - 4x - 2x^2$ will be as in Fig. 5.4.

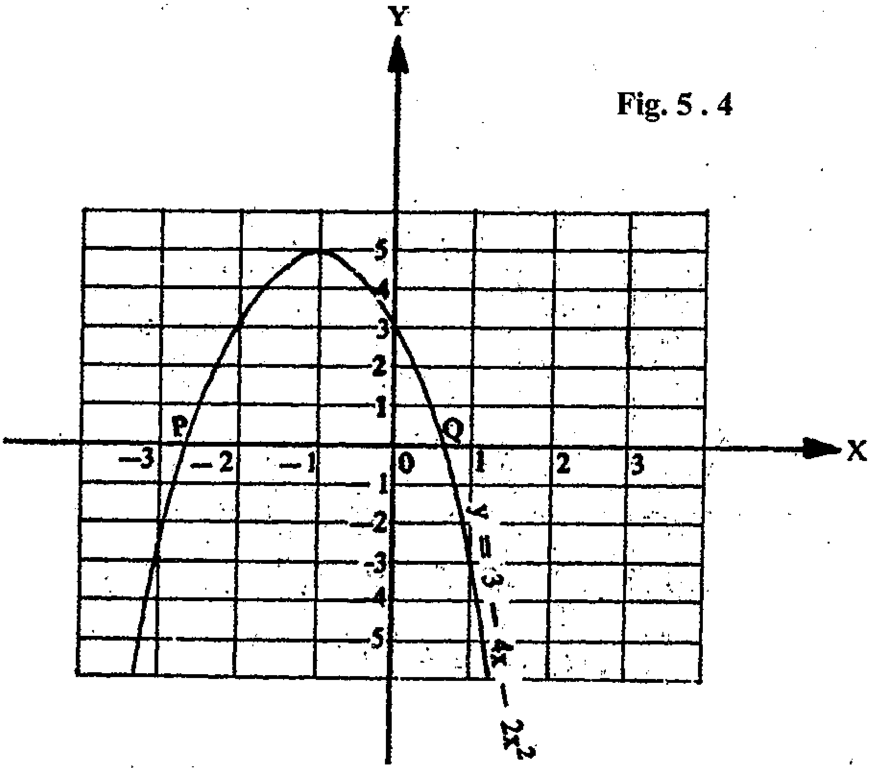


Fig. 5.4

Let this curve cuts the X-axis (the line $y = 0$) at P and Q. The x-coordinates of P and Q are the roots of the equation $3 - 4x - 2x^2 = 0$

These x-coordinates are respectively equal to -2.6 and 0.6 approximately.

Therefore the approximate roots are -2.6 and 0.6.

Example 4. A rectangular garden has an area of 720 sq.yd. If the length be reduced by 6 yd and the breadth be increased by 6 yd, then the resulting area is the same as the original area. Find the length and breadth of the garden.

Solution Suppose that the length and breadth of the garden are x yd and y yd respectively. Then the new length and breadth will be $(x - 6)$ yd and $(y + 6)$ yd. We have the following.

$$xy = 720 \qquad \ldots (1)$$

$$(x - 6)(y + 6) = 720 \qquad \ldots (2)$$

Rewriting (2) in the equivalent form

$$xy + 6x - 6y - 36 = 720$$

$$xy + 6x - 6y - 36 + 36 = 720 + 36$$

$$xy + 6x - 6y = 756 \qquad \ldots (3)$$

Substituting $xy = 720$ in equation (3),

$$720 + 6x - 6y = 756$$

$$720 + 6y - 6y - 720 = 756 - 720$$

$$6x - 6y = 36$$

$$x - y = 6$$

$$x - y + y = 6 + y$$

$$x = 6 + y \qquad \ldots (4)$$

Substituting $x = 6 + y$ in equation (1),

$$(y + 6)\ y = 720$$

$$y^2 + 6y - 720 = 720 - 720$$

$$(y + 30)(y - 24) = 0$$

Since y must be positive, $y + 30$ cannot be 0.

So $y - 24 = 0$ i.e., $y = 24$

Substituting $y = 24$ in equation (4),

$$x = 24 + 6 = 30$$

Exercise 5.3

1. Find the solution set of each of the systems of equations ;

(a) $x^2 = 4y$
$y = 9$

(b) $x^2 - y^2 = 9$
$x = 5$

(c) $y = x^2$
$y = x + 6$

(d) $y = 3 - 2x$
$y = x^2$

(e) $x^2 - 8x + y = 0$
$y = 2x$

(f) $x - y = 2$
$y = x^2 - 6x + 5$

(g) $x^2 + y^2 = 1$
$x - y = 0$

(h) $x^2 + y^2 = 20$
$2y - x = 0$

(i) $y = x^2 - 2x + 5$
$y = 4x$

(j) $y = \frac{8}{x}$
$y = 7 + x$

(k) $xy = 4$
$y = 8 - 2x$

(l) $x^2 + y^2 = 25$
$y = x + 1$

(m) $y^2 = 4x$
$2x + y = 4$

(n) $4y^2 - 3x^2 = 1$
$x - 2y = 1$

(o) $x^2 + xy + y^2 = 7$
$2x + y = 1$

(p) $x^2 + 5x + y = 4$
$x + y = 8$

(q) $x^2 - xy - y^2 = 1$
$2x + y = 1$

2. Solve graphically the quadratic equations:

(a) $x^2 - 2x - 3 = 0$ (b) $x^2 + x - 6 = 0$

(c) $x^2 - 2x = 4$ (d) $\frac{x^2}{4} + x - 2 = 0$

(e) $4x^2 - 16x + 9 = 0$

3. The sum of squares of two numbers is 58. If the first number and twice the second add up to 13, find the numbers.

4. The sum of the reciprocals of two positive numbers is $\frac{7}{36}$ and the product of the numbers is 108. Find the numbers.

5. A rectangle has an area of 65 square meters. If the length and breadth are increased by 5 meters, the area of the resulting rectangle is 15 square meters less than three times the area of the original rectangle. Find the dimensions of the original rectangle.

6. The product of two numbers is 108, and the sum of their reciprocals is $\frac{2}{9}$. Find the numbers.

7. The sum of the lengths of the two legs of a right triangle is 28 square inches and the area is 96 square inches. Find the length of the hypotenuse of the right triangle.

SUMMARY

1. The equation $ax^2 + bx + c = 0$, where $a \neq 0$, is a quadratic equation in standard form. This is also called an equation of the second degree in x. A root of an equation is any member of its solution set.

2. Methods of solving a quadratic equation;

   (a) By completing the square: Reduce to the form $(x + p)^2 = q$

   (b) By the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

3. System of equations in two variables, one linear and one quadratic, can be solved by the method of substitution.

CHAPTER 6

Ratio, Proportion and Variation

6.1 Ratio

A ratio is one way of comparing two similar quantities. It expresses the relative magnitudes of two quantities of the same kind without stating what the magnitudes actually are. In fact it shows how many times the one is contained in the other. The comparison in this case, is of the nature of division. The ratio of a to b is equal to the quotient, a divided by b and is denoted symbolically as a : b. The first number a is called the antecedent and the second number b the consequent.

Example 1. Simplify the ratios

(a) 24 ft to 84 ft (b) 4 gallons : 6 quarts

Solution

(a) $\dfrac{24\text{ ft}}{84\text{ ft}} = \dfrac{2}{7}$

(b) Since 1 gallon contains 4 quarts,

4 gallons contains 16 quarts and

$$4\text{ gals} : 6\text{ qts} = 16\text{qts} : 6\text{ qts} = \frac{16\text{ qts}}{6\text{ qts}} = \frac{8}{3}$$

The cancelling of common units of measure in Example (1) indicates that the resulting ratio is a valid size comparison. Note that this could not be done in 12 ounces/ 16 feet because no common unit of measure exists in this case.

Example 2. Express each ratio as a percent.

(a) $\dfrac{2}{5}$ (b) K 39 : K 52 (K = Kyats)

Solution

(a) $\dfrac{2}{5} = \dfrac{40}{100} = 40\ \%$

(b) $\dfrac{\text{K }39}{\text{K }52} = \dfrac{3}{4} = \dfrac{75}{100} = 75\ \%$

Example 2 suggests that we may compare 2 to 5 by saying 2 is 40 % of 5. Similarly, we say that K 39 is 75 % of K 52.

Example 3. A 21-foot board is to be cut so that the lengths of the two pieces are in the ratio 2 : 5. Find the lengths of the two pieces.

Solution

Since the lengths of the two pieces are to be in the ratio 2 : 5,

let them be 2x ft and 5x ft.

Since their sum must be the total length,

$$2x + 5x = 21$$

$$7x = 21$$

$$x = 3$$

The two required lengths are 2x = 6 ft and 5x = 15 ft.

A comparison involving three numbers a, b, c may be indicated by a "ratio" such as 3 : 5 : 7. This notation means that

$$\frac{a}{b} = \frac{3}{5},\ \frac{b}{c} = \frac{5}{7} \text{ and } \frac{a}{c} = \frac{3}{7}$$

$$\text{or } \frac{a}{3} = \frac{b}{5} = \frac{c}{7}$$

Example 4. Determine the angles of a triangle so that their degree measures are in the ratio 3 : 5: 7.

Solution

Since the degree measure of the angles are to be in the ratio 3 : 5 : 7, let them be $3x^\circ$, $5x^\circ$ and $7x^\circ$.

Since their sum must be 180°,

$$3x + 5x + 7x = 180^\circ$$

$$15x = 180^\circ$$

$$x = 12^\circ$$

The angles are $3x = 36^\circ$, $5x = 60^\circ$ and $7x = 84^\circ$

In the following example we use the property that a ratio is not altered in value, if both its terms are multiplied or divided by the same quantity.

The property is obvious, for the ratio a : b has the same value as

$$ma : mb \qquad \text{or} \qquad \frac{a}{m} : \frac{b}{m}$$

Example 5. If $a : b = 5 : 3$, find the value of $3a + 4b : 4a + 3b$

Solution

$$\frac{a}{b} = \frac{5}{3}$$

$$\frac{3a+4b}{4a+3b} = \frac{3(\frac{a}{b})+4}{4(\frac{a}{b})+3} = \frac{3(\frac{5}{3})+4}{4(\frac{5}{3})+3} = \frac{27}{29}$$

Example 6. A map is drawn using a scale, 4 inches to a mile.

Express this scale by a ratio of the form 1 : n.

Solution

In order to compare two quantities, they must be expressed in the same unit.

$$1 \text{ mile} = 1760 \text{ yards} = 5280 \text{ feet}$$

$$= 5280 \times 12 \text{ inches}$$

The required scale is $4 : 5280 \times 12$

$1 : 5280 \times 3$

$1 : 15840$

Exercise 6.1

1. Simplify each of the following ratios.

(a) 30 inches to 75 inches

(b) 20 gallons to 5 quarts

(c) 2 years to 10 months

(d) 6600 feet to 2 miles

(e) 6 ounces to 3 pounds

2. Express each of the following ratios as a percentage.

(a) $\frac{3}{4}$ (b) $\frac{7}{2}$ (c) $\frac{1}{25}$ (d) $\frac{33}{55}$ (e) $72:32$

3. An 18-foot board is to be cut so that the lengths of the two pieces are in the ratio $7:3$. Find the lengths.

4. A rope that is 20 feet long is cut into two pieces. The ratio of the smaller piece to the larger piece is $3:5$. Find the length of the shorter piece.

5. Determine the angles of a triangle so that their degree measures are in the ratio $6:11:13$.

6. The perimeter of a triangle is to be 51 inches and the lengths of its sides are to be in the ratio $2:3:4$. Find the lengths of the sides.

7. Find two numbers whose ratio is $3:5$, such that the difference of their squares is 64.

8. The ratio of the perpendicular sides of a right triangle is $3:4$. Determine the ratio of each of the sides to the hypotenuse.

9. If $x:y=3:4$, find the value of $7x-4y:3x+y$.

10. Find two numbers which are in the ratio $5:3$, and whose difference is 34.

11. PQRS is a straight line in which R divides PS in the ratio $7:3$, Q divides PS in the ratio $3:5$. Find the ratios $PQ:RQ$ and $PQ:RS$.

12. Two numbers are in the ratio $5:12$ and the second number exceeds the first number by 42. Find the numbers.

13. What is the ratio of x to y if $10x+3y:5x+2y=9:5$?

6.2 Proportion

A proportion is the statement of the equality of two ratios.

A proportion may be written in any of the forms.

$$\frac{a}{b}=\frac{c}{d}, \qquad a \div b = c \div d, \qquad a:b=c:d, \quad a:b::c:d.$$

a, b, c and d are called proportionals. We call a and d the extremes and b and c the means of the proportion.

The properties of proportionals are given below.

Property I. (The Mean – extreme Product Property).

In a proportion, the product of the means is equal to the product of the extremes.

If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$

Proof :

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b}.bd = \frac{c}{d}.bd \quad \Rightarrow ad = bc$$

Property II. (The Upside – downable Property).

In a proportion, the ratios may be inverted.

$$\text{If } \frac{a}{b} = \frac{c}{d} \quad \text{then } \frac{b}{a} = \frac{d}{c}$$

Proof :

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{1}{(\frac{a}{b})} = \frac{1}{(\frac{c}{d})} \quad \Rightarrow \frac{b}{a} = \frac{d}{c}$$

This property is also known as "Invertendo".

Property III. (The Interchangeable Property).

In a proportion, the means (or extremes) may be interchanged.

$$\text{If } \frac{a}{b} = \frac{c}{d} \quad \text{then } \frac{a}{c} = \frac{b}{d}$$

$$\text{or } \frac{d}{b} = \frac{c}{a}$$

Proof :

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad ad = bc \Rightarrow \frac{ad}{cd} = \frac{bc}{cd} \Rightarrow \frac{a}{c} = \frac{b}{d}$$

$$\frac{ad}{ab} = \frac{bc}{ab} \Rightarrow \frac{d}{b} = \frac{c}{a}$$

This property is also known as "Alternando".

Property IV. (The Denominator Addition Property).

$$\text{If } \frac{a}{b} = \frac{c}{d} \quad \text{then } \frac{a+b}{b} = \frac{c+d}{d}$$

Proof : $$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} + 1 = \frac{c}{d} + 1 \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$$

This property is also known as "Componendo"

Property V. (The Denominator Subtraction Property).

$$\text{If } \frac{a}{b} = \frac{c}{d} \quad \text{then } \frac{a-b}{b} = \frac{c-d}{d}$$

Proof : $$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} - 1 = \frac{c}{d} - 1 \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$$

This property is also known as "Dividendo"

Property VI. (The Equal Ratios Theorem).

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \ldots, \text{ then } \frac{a+c+e+\ldots}{b+d+f+\ldots} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Proof :

Since the equal ratios have a common value, say k,

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = k$$

$$\frac{a}{b} = k \Rightarrow a = bk$$

Similarly $c = dk$, $e = fk$

$$\therefore \frac{a+c+e+\ldots}{b+d+f+\ldots} = \frac{bk+dk+fk+\ldots}{b+d+f+\ldots} = k$$

Thus we have $$\frac{a+c+e+\ldots}{b+d+f+\ldots} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Note. Using Properties IV and V, we can prove the following.

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

This property is also known as "Componendo and Dividendo".

Example 1. Solve for x in the proportion $15 : 2 = 35 : x$.

Solution

$$\frac{15}{2} = \frac{35}{x}$$

$$15x = 2 \times 35$$

$$x = \frac{2 \times 35}{15} = \frac{14}{3}$$

Example 2. Two gallons of a solution of acid and water contains $1\frac{1}{2}$ quarts of acid. How many quarts of acid are there in nine gallons of a solution of the same concentration?

Solution

Let x = number of quarts of acid in the new solution. The ratio of acid to solution in the original sample is

$$\frac{1\frac{1}{2} \text{ qts}}{2 \text{ gals}} \text{ or } \frac{\frac{3}{2} \text{ qts}}{2 \times 4 \text{ qts}} = \frac{3}{2} \div 8 = \frac{3}{16}$$

The same ratio for the new solution is

$$\frac{x}{9 \times 4} = \frac{x}{36}$$

Hence $$\frac{x}{36} = \frac{3}{16}$$

$$16x = 3 \times 36$$

$$x = \frac{3 \times 36}{16} = \frac{27}{4} = 6\frac{3}{4} \text{ quarts}$$

Example 3. The formula for the surface area S of a sphere of radius r is $S = 4\pi r^2$. If the radius of one sphere is twice that of another, compare their surface areas.

Solution

The ratio of the radii is 2 : 1

The ratio of the surface areas is

$$\frac{S_1}{S_2} = \frac{4\pi r_1^2}{4\pi r_2^2} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

$$\frac{S_1}{S_2} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$$

or $\quad S_1 = 4\,S_2$

Thus doubling the radius increase the surface area fourfold.

Example 4. If a : b = c : d = e :f prove that

$a^3 + c^3 + e^3 : b^3 + d^3 + f^3 = ace : bdf.$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then a = bk, c = dk, e = fk

$$\frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{b^3k^3 + d^3k^3 + f^3k^3}{b^3 + d^3 + f^3} = \frac{k^3(b^3 + d^3 + f^3)}{b^3 + d^3 + f^3} = k^3$$

$$\frac{ace}{bdf} = \frac{bk \cdot dk \cdot fk}{bdf} = k^3$$

Thus $\frac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \frac{ace}{bdf}$

That is $a^3 + c^3 + e^3 : b^3 + d^3 + f^3 = ace : bdf$

Example 5. If $\dfrac{a}{b+c-a} = \dfrac{b}{c+a-b} = \dfrac{c}{a+b-c}$,

prove that each ratio = 1 provided $a + b + c \neq 0$.

Solution

Using Property VI (equal ratio theorem)

We have, each ratio $= \dfrac{a+b+c}{(b+c-a)+(c+a-b)+(a+b-c)}$

$$= \frac{a+b+c}{a+b+c} = 1 \quad (\because a+b+c \neq 0)$$

Example 6. If $\dfrac{ay-bx}{c} = \dfrac{cx-az}{b} = \dfrac{bz-cy}{a}$, show that $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Solution Let each of the given ratios = k.

Then we have $ay - bx = kc$

$cx - az = kb$

$bz - cy = ka$

Thus $(ay - bx)\,c = kc^2$

$(cx - az)\,b = kb^2$

$(bz - cy)\,a = ka^2$

Hence by addition, $k\,(a^2 + b^2 + c^2) = 0$

So, $k = 0$, $\quad (\because a^2 + b^2 + c^2 > 0)$

Hence $ay - bx = 0$

$ay = bx$

$$\frac{x}{a} = \frac{y}{b} \qquad \dots (1)$$

Also, $cx - az = 0$

$cx = az$

$$\frac{x}{a} = \frac{z}{c} \qquad \dots (2)$$

Hence, from equations (1) and (2), $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

If $\frac{a}{b} = \frac{b}{c}$, then b is called the **mean proportional** between a and c.

Example 7. Find the mean proportional between 8 and 12.

Solution

Let x = the mean proportional.

$$\frac{8}{x} = \frac{x}{12}$$

$$x \cdot x = 8 \cdot 12$$

$$x^2 = 96$$

$$x = \pm\sqrt{96} = \pm 4\sqrt{6}$$

Either $4\sqrt{6}$ or $-4\sqrt{6}$ is a mean proportional between 8 and 12.

Example 8. Working alone, Maung Hla can complete a job in 3 hours. Maung Aye can complete the same job in 2 hours. How long will it take them working together?

Solution

Let x = time (in hour) to do the job together

Then $\frac{1}{x}$ = portion of job done in 1 hour working together

Also $\frac{1}{3}$ = portion of job done by Maung Hla in 1 hour.

$\frac{1}{2}$ = portion of job done by Maung Aye in 1 hour.

The required equation is $\frac{1}{3} + \frac{1}{2} = \frac{1}{x}$

multiplying by 6x, we have

$$2x + 3x = 6$$

$$5x = 6$$

$$x = 1\frac{1}{5}$$

Together they need $1\frac{1}{5}$ hours, or 1 hour 12 minutes.

Exercise 6. 2

1. Two rectangles have the same width but one is three times as long as the other. Find the ratio of their areas.
2. The lengths of the sides of two squares are in the ratio of 3 : 2. What is the ratio of their areas?
3. If the radius of one circle is $\frac{1}{4}$ as long as that of another, compare their areas.
4. The denominator of a fraction is 3 more than the numerator. If 5 is added to the numerator and 4 is subtracted from the denominator, the value of the new fraction is 2. Find the original fraction.
5. Two numbers are in the ratio 2 : 3. If 9 be added to each of them, the numbers are in the ratio 3 : 4. Find the numbers.
6. The sides of a triangle are 8, 10 and 12 ft. If the longest side of a similar triangle is 30 ft, find the other sides of that triangle.
7. The areas of similar triangles are proportional to the squares of corresponding sides. Find the lengths of the sides of a triangle similar to a triangle with sides 8, 12 and 12 ft long but with twice the area.
8. Find the mean proportional between each pair of numbers given below.

   (a) 4 and 9 (b) 8 and 27.
9. If $\frac{a}{b} = \frac{b}{c}$, prove that ab + bc is a mean proportional between $a^2 + b^2$ and $b^2 + c^2$. (Hint: Let $\frac{a}{b} = \frac{b}{c} = k$. Then $a = bk$, $b = ck$, $a = ck^2$).

10. Find two mean proportionals between 2 and 54. (Hint: Let b and c be the required two mean proportionals. Then $\frac{2}{b} = \frac{b}{c} = \frac{c}{54}$)

11. Working together Ma Mya and Ma Hla can paint their room in 3 hours. If it takes Ma Mya 5 hours to do the job alone, how long would it take Ma Hla to paint the room working by herself?

12. A takes twice as long as B to complete a certain job. Working together they can complete the job in 6 hours. How long will it take A to complete the job by herself?

13. If $(a + 3b + 2c + 6d)(a - 3b - 2c + 6d) = (a - 3b + 2c - 6d)(a + 3b - 2c - 6d)$, prove that $a : b = c : d$ [Hint: Use Componendo - Dividendo

i.e., Properties IV and V combined to the proportion

$$\frac{(a+3b)+(2c+6d)}{(a+3b)-(2c+6d)} = \frac{(a-3b)+(2c-6d)}{(a-3b)-(2c-6d)}$$]

14. If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$, show that each of these ratios is equal to $\frac{1}{2}$ or -1.

15. If $(7x - 5y) : (7x + 5y) = 11 : 31$, find the value of $(5x^2 - 4y^2) : (5x^2 + 4y^2)$.

16. If $x : a = y : b = z : c$, show that $\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} = \frac{(x + y + z)^3}{(a + b + c)^2}$

17. If $\frac{\ell}{x-y} = \frac{m}{y-z} = \frac{n}{z-x}$, prove that $\ell + m + n = 0$.

6.3 Variation

We have seen that a formula is an equation relating two or more variables in a specific context. Formulas are usually written in a form showing how one variable is determined by the remaining ones. Thus the formula $C = 2\pi r$ for the circumference of a circle of radius r shows that C is determined by multiplying r by 2π. C is called the dependent variable and r is the independent variable. Similarly in $d = rt$, d is dependent while r and t are independent. However if we write $t = \frac{d}{r}$, then t is dependent and d and r are the independent variables. Since formulas have widespread applications, it is profitable to have a vocabulary to describe certain types of dependence that most frequently occur.

Direct Variation, by Formula

In the formula for the circumference of a circle $C = 2\pi r$, r can be any positive real number. To study the way in which C (the circumference) depends on r, we construct a table of ordered pairs (r, C) and the resulting graph is as shown in Figure 6.1.

r	C
1	2π
2	4π
3	6π
4	8π
5	10π

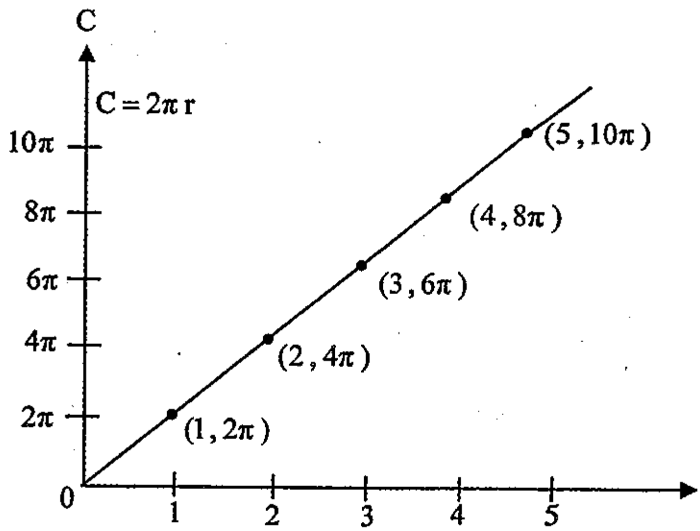


Fig. 6.1

It is apparent that to each value of r there corresponds a unique value of C, and as r increases C also increases. We can see from the table that for each ordered pair (C, r) the quotient $\frac{C}{r}$ is always equal to the constant 2π. One can recognize that the values of r and the corresponding values of C are in direct proportion. That is the dependence of C on r is of the nature of direct proportion.

We may now extend the ideas of direct proportion and for this reason it will be convenient to extend our mathematical language. Thus, the direct proportion illustration above can be described in the language of variation as follows.

(i) C varies directly as r, simply C varies as r.

(ii) $C \propto r$ (read as in (i)).

(iii) $C = 2\pi r$, where 2π is a constant, called the constant of variation, which can be determined from the data.

(iv) Circumference of a circle varies as its radius.

These four statements mean exactly the same thing.

The study of these facts lead to the following definition.

Definition

If two variables x and y are so related that the quotient $\frac{y}{x}$ is a nonzero constant k for all permissible values of x and y, then y is said to vary directly as x according to the variation formula $y = kx$, k is called the proportionality constant or the constant of variation.

This relationship between y and x may also be stated as "y is directly proportional to x" or in short as "y is proportional to x".

Variation is expressed symbolically by $\propto$. Thus, $y \propto x$, read "y varies as x", is equivalent to $y = kx$, where k is a constant. For most cases the use of the equation is preferred. Numerous examples of direct variation can be found in geometry. Here are some illustrations.

$A = \pi r^2$; A varies directly as the square of r, and π is the constant of variation.

Area of an equilateral triangle of side s: $A = \frac{\sqrt{3}}{4} s^2$

A varies directly as s^2 and $\frac{\sqrt{3}}{4}$ is the constant of variation.

Volume of a cube of side c : $V = c^3$

V varies directly as c^3 and 1 is the constant of variation.

Thus

We have now established a formula for direct variation.

$$y \propto x \quad \Leftrightarrow \quad y = kx$$

Next we shall consider two useful methods of description of direct variation.

(i) The rate method

If (x_1, y_1), (x_2, y_2), (x_3, y_3), ... are ordered pairs which satisfy the equation $y = kx$, we can equally say that

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots = k$$

This description of direct variation by the rate method is useful in some application.

Example 1. In a French test, the possible mark is 65. Convert a mark of 39 out of 65 to a percentage mark.

Old mark	New mark
65	100
39	x

Using rate method $\dfrac{100}{65} = \dfrac{x}{39}$

$$x = 60$$

(ii) The ratio method

There is another way of looking at direct variation. If (x_1, y_1), (x_2, y_2) are ordered pairs which satisfy the equation $y = kx$, then

$$y_1 = kx_1$$

and $$y_2 = kx_2$$

Dividing, $\frac{y_2}{y_1} = \frac{kx_2}{kx_1}$ i.e., $\frac{y_2}{y_1} = \frac{x_2}{x_1}$

which is the ratio method for direct proportion.

Example 2. The cost of petrol for a car journey varies directly as the number of kilometres. If petrol for a journey of 380 km costs 32.96 kyats, estimate the cost of petrol for 640 km.

Let c kyats be the cost of petrol for a journey of x km. Then c = kx. The data may be tabulated as follows:

Distance x km	Cost c kyats
380	32.96
640	c

Using the ratio method,

$$\frac{640}{380} = \frac{c}{32.96}$$

$$c = 55.51$$

Check this result by the rate method.

Other forms of direct variation

If a relation is of the type "y varies as x", its graph is a straight line through the origin, and conversely, if the graph of y against x is a straight line through the origin, y varies as x. Furthermore, the graph can be used to find the constant of variation since this gives the gradient of the line.

Example 3. A ball-bearing, starting from rest, is allowed to roll down a straight channel which is inclined at an angle to the horizontal. The distance, d metres, travelled in t seconds is given below:

t	0	0.5	1.0	1.5	2.0	2.5	3.0
d	0	0.5	2.0	4.5	8.0	12.5	18.0

It is evident that the ratio d : t is not constant, so that d does not vary as t, this is confirmed by the graph in Figure 6.2.

Notice that the graph resembles the graph of the mapping $x \longmapsto x^2$

To test this, we construct a table of ordered pairs (t^2, d) and draw a graph of d against t^2.

t^2	0	0.25	1.00	2.25	4.00	6.25	9.00
d	0	0.5	2.0	4.5	8.0	12.5	18.0

Since the graph in Figure 6.3 is a straight line, we deduce that d varies as t^2, and so $d = kt^2$.

From the table (or graph), $k = 2$, and hence the formula is $d = 2t^2$.

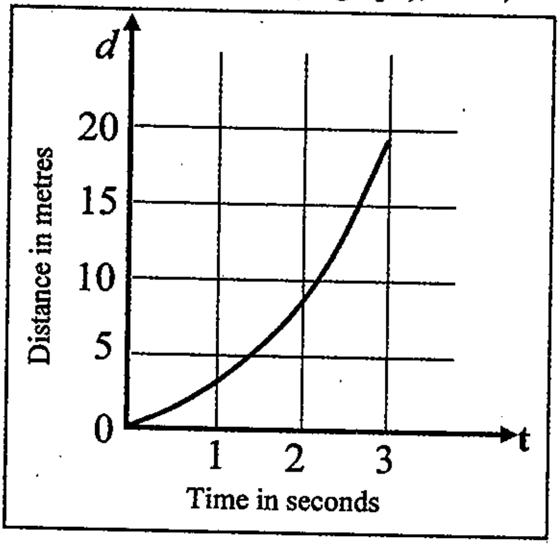


Fig. 6.2

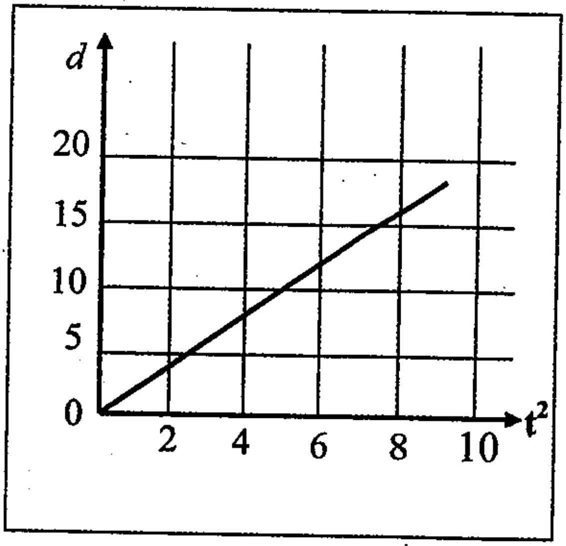


Fig. 6.3

In general, y is said to vary directly as the n^{th} power of x, written $y \propto x^n$, if $\frac{y}{x^n}$ is a constant, i.e. if $y = kx^n$, where k is a constant.

Example 4. The distance of the visible horizon varies as the square root of the height of the observation point above sea-level. At a height of 4 metres it is known that the distance is 5.2 kilometres. Find the distance at a height of 9 metres.

Let the distance be d km at a height of h m. Then the law of variation is

$$d \propto \sqrt{h} \qquad \text{i.e.,} \qquad d = k\sqrt{h}$$

(i) Method of direct formula application

$$d = k\sqrt{h}$$
$$5.2 = k\sqrt{4}$$
$$k = 2.6$$
$$d = 2.6\sqrt{h}$$

When $h = 9$, $d = 2.6\sqrt{9}$

$$= 7.8$$

The distance is 7.8 kilometres.

(ii) Ratio method

$d_1 = k\sqrt{h_1}$ and $d_2 = k\sqrt{h_2}$

$$\frac{d_2}{d_1} = \frac{\sqrt{h_2}}{\sqrt{h_1}}$$

Hence $\dfrac{d_2}{5.2} = \dfrac{\sqrt{9}}{\sqrt{4}} = \dfrac{3}{2}$

Data

$$2d_2 = 3 \times 5.2$$
$$d_2 = 7.8$$

The distance is 7.8 kilometres.

Figure 6.4 and 6.5 show the graphs of d against h and d against $\sqrt{h}$.

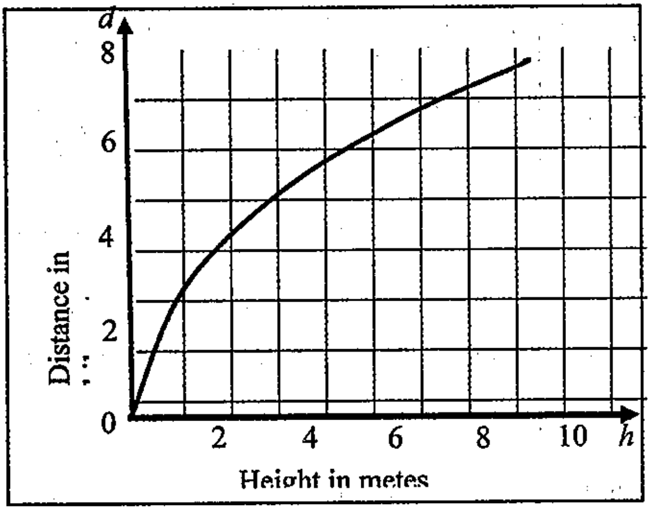


Fig. 6.4

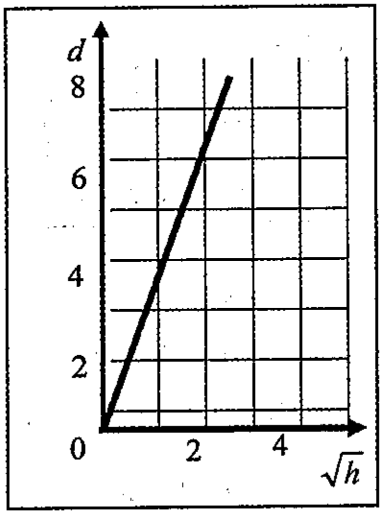


Fig. 6.5

Inverse variation

Example 5. The table shows some ordered pairs from the one-to-one correspondence between the number of men and the number of days required to complete a piece of work.

number of men $= x$	1	2	3	5	6	10	15
number of days $= y$	30	15	10	6	5	3	2

It is clear that x does not vary directly as y.

The graph in fig. 6.6 illustrates the relationship between x and y.

This graph is a rectangular hyperbola.

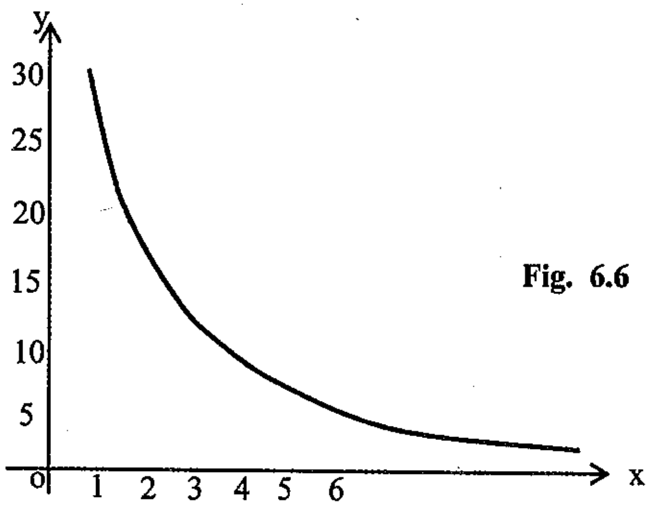


Fig. 6.6

However, in this example, we see that $xy = 30$ for each pair of values of x and y. Although the values of x and y change, this product is always a constant which is 30 in this case.

We may consider the same situation by rewriting the equation in the form

$y = \dfrac{30}{x}$,

or $y = 30 \cdot \dfrac{1}{x}$

The table shows the corresponding values of the ordered pairs $(\frac{1}{x}, y)$

$\frac{1}{x}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{3}$	1
y	2	3	5	6	10	30

The graph of y against $\frac{1}{x}$ is given in figure 6.7. This graph is a straight line through the origin.

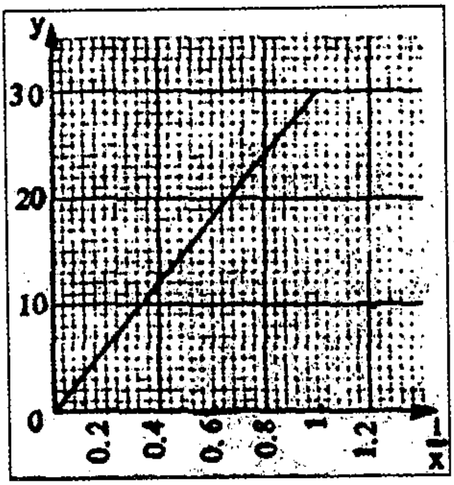


Fig. 6.7

Thus, we deduce that y varies as $\frac{1}{x}$. In the language of variation we say that y varies inversely as x in such a situation. This discussion leads to the following definition.

Definition

If two variables v and t are so related that the product vt is a nonzero constant k for all permissible values of v and t, then v is said to vary inversely as t according to the variation formula $v = \frac{k}{t}$. You will recognise this as inverse proportion. Methods for solving problems correspond to the methods used for solving problems in inverse proportion.

Example 6. v varies inversely as t, and v = 30 when t = 4. Find v when t = 9.

Method 1.

$$v = \frac{k}{t}$$

$$30 = \frac{k}{4}$$

$$k = 120$$

$$v = \frac{120}{t}$$

When $t = 9$,

$$v = \frac{120}{9}$$

$$= 13\frac{1}{3}$$

Method 2.

If (t_1, v_1) and (t_2, v_2) are ordered pairs which satisfy the equation

$vt = k$ then $v_1 t_1 = k$ and $v_2 t_2 = k$

$$v_1 t_1 = v_2 t_2$$

$$\frac{v_2}{v_1} = \frac{t_1}{t_2}$$

Hence

$$\frac{v_2}{30} = \frac{4}{9}$$

$$9v_2 = 120$$

$$v_2 = \frac{120}{9}$$

$$= 13\frac{1}{3}$$

Data

v	t
30	4
v_2	9

Other forms of inverse variation.

In the same way that the language of variation extends the idea of direct variation, $y \propto x$, to other laws of variation such as $y \propto x^2$, $y \propto \sqrt{x}$, $y \propto x^3$, it also extends the idea of inverse variation, $y \propto \frac{1}{x}$, to $y \propto \frac{1}{\sqrt{x}}$, $y \propto \frac{1}{x^2}$, etc. The last mentioned is a particularly important one, called the inverse square law, which is of importance in science.

In general, y is said to vary inversely as the n^{th} power of x, written $y \propto \frac{1}{x^n}$, if $yx^n = k$, or $y = \frac{k}{x^n}$, where k is a constant.

Example 7. Two electrical charges attract one another with a force, P units, which varies inversely as the square of the distance, x units, between them. P = 5.4 when x = 6, find P when x = 9.

Method 1.

$$P = \frac{k}{x^2}$$

$$5.4 = \frac{k}{6^2}$$

$$k = 194.4$$

$$P = \frac{194.4}{x^2}$$

When $x = 9$, $P = \dfrac{194.4}{81}$

$$= 2.4$$

Method 2.

$P_1 = \dfrac{k}{x_1^{\,2}}$ and $P_2 = \dfrac{k}{x_2^{\,2}}$

$$\frac{P_2}{P_1} = \frac{x_1^{\,2}}{x_2^{\,2}}$$

Data

P	x
5.4	6
P_2	9

Hence $\dfrac{P_2}{5.4} = \dfrac{6^2}{9^2} = \dfrac{4}{9}$

$$9P_2 = 4 \times 5.4$$

$$P_2 = 2.4$$

Exercise 6.3

1. Assuming that y varies as x, complete this table:

x	16	17	18	19	20	21	22	...	...
y	40	...	...	...	...	...	...	62.5	74

2. Convert the following marks, out of a possible 80, to percentage marks.

Mark out of 80	27	43	44	50	52	54	57	63	76
Percentage mark	...	...	...	...	...	...	...	...	...

3. The tension T newtons in a spring varies directly as the extension x millimetres. Use this information to complete the following table:

x	0.2	0.4	0.5	...
T	50	100	...	175

4. Express each of the following as an algebraic equation, denoting the constant of variation in each case by k :

(a) $p \propto q^2$ (b) $r \propto \sqrt{A}$

(c) $W \propto d^3$ (d) $D \propto \tan x$

5. $y \propto x$. Given that $y = 3.8$ when $x = 2.2$, calculate:

   (a) y when $x = 4.5$ (b) x when $y = 10.7$.

6. y varies as the cube root of x, and $y = 1$ when $x = 4$. Find :

   (a) y when $x = 729$ (b) x when $y = 5$.

7. y varies as the square of x, and $y= 6561$ when $x = 243$. Find y when $x = 729$.

8. If y varies as $(x + 1)^3$, and $y = 3$ when $x = 5$. Find y when $x = 8$.

9. If $x \propto y^2$, and $x = a^2$ when $y = 2a$, show that $y^2 = 4x$.

10. y varies inversely as x, and $y = 3$ when $x = 4$. Find an equation connecting y and x. Hence find y when $x = 6$.

11. Given that $y \propto \frac{1}{x}$, and $y = 6$ when $x = 14$, obtain an equation in x and y. Find y when $x = 28$.

12. The pressure P pascals (newtons per square metre) of a given mass of gas at constant temperature varies inversely as the volume V m^3 ; when $P = 600$, $V = 2$. Find :

    (a) P when $V = 3$ (b) V when $P = 800$

13. Express each of the following as an algebraic equation :

    (a) $p \propto \frac{1}{v}$ (b) $y \propto \frac{1}{\sqrt{x}}$ (c) $y \propto \frac{1}{x^3}$

14. y varies inversely as x^2, and $y = 9$ when $x = 2$. Find :

    (a) y when $x = 10$ (b) x when $y = 4$

15. $y \propto \frac{1}{\sqrt{x}}$ and $y = 5$ when $x = 9$. Find y when $x = 25$.

16. Given that y varies inversely as the square of x , and $y = 48$ when $x = 3$, calculate y when $x = 4$.

17. y is inversely proportional to $\sqrt{x}$. If y = 5 when x = 16, find:

(a) y when x = 100 (b) x when y = 60.

18. $y \propto \frac{1}{x^3}$, and y = 10 when x = 2. Find y when x = 10.

Variation involving several variables

The volume V of a cylinder with radius r and height h is given by the formula $V = \pi r^2 h$, and depends on both r and h, which can be changed independently of the other.

(i) If r is multiplied by 3, V is multiplied by 3^2, i.e., 9, and so if h is constant, $V \propto r^2$ (see Figure 6.8 (i)).

(ii) If h is multiplied by 2, V is multiplied by 2 and so if r is constant, $V \propto h$. (see Figure 6.8 (ii)).

(iii) If r is multiplied by 3, and h is multiplied by 2, V is multiplied by $3^2 \times 2$, i.e, 18, and we say that V varies jointly as r^2 and h i.e, $V \propto r^2h$ and $V = kr^2h$ (see Figure 6.8 (iii)). This is known as joint variation.

Note: The word ' jointly ' is frequently omitted from the statement.

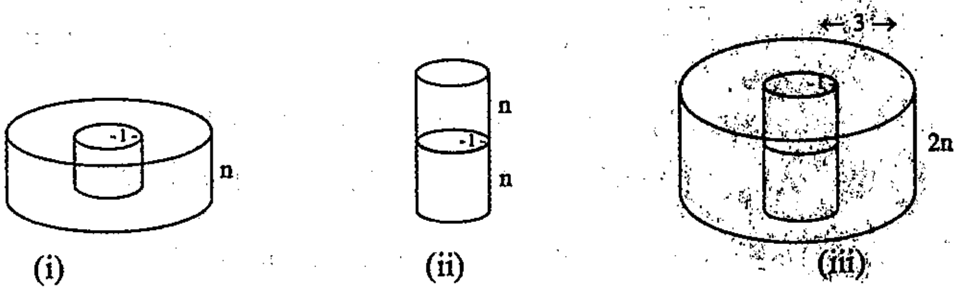


Fig. 6.8

In general, y is said to vary jointly as x and z, written $y \propto xz$, if $y = kxz$, where k is the constant of variation. In other words, we can define joint variation as follows.

Definition

A variable y is said to vary jointly as x and z if y varies as the product of x and z.

That is $y = k\,xz$.

The ideas of joint variation can be extended as in the following example.

Example. y varies as x^2 and inversely as z. $y = 12$ when $x = 6$ and $z = 9$.

Find y when $x = 9$ and $z = 12$.

Method 1.

$$y = \frac{kx^2}{z}$$

$$12 = \frac{k \times 6^2}{9}$$

$$k = \frac{9 \times 12}{36} = 3$$

$$y = \frac{3x^2}{z}$$

When $x = 9$, $z = 12$,

$$y = \frac{3 \times 9^2}{12}$$

$$y = 20.25$$

Method 2.

$$y_1 = \frac{kx_1^{\,2}}{z_1} \text{ and } y_2 = \frac{kx_2^{\,2}}{z_2}$$

$$\frac{y_2}{y_1} = \frac{x_2^{\,2}}{x_1^{\,2}} \; \frac{z_1}{z_2}$$

Data

[redacted]

$$\text{Hence } \frac{y_2}{12} = \frac{9^2}{6^2} \times \frac{9}{12} = \frac{27}{16}$$

$$16y_2 = 12 \times 27$$

$$y_2 = \frac{12 \times 27}{16}$$

$$= 20.25$$

Note. So far we have been using symbols for measures of quantity, i.e., for numbers. But often in the language of variation the symbols are used also for the quantities themselves. This practice is followed in the next Exercise; remember that the constant of variation, if used, depends on the units.

Exercise 6.4

1. Express each of the following algebraically as an equation:

   (a) A varies jointly as r and h.

   (b) P varies as t and inversely as v.

   (c) Q varies jointly as x and the square of y.

   (d) F varies as m_1 and m_2 and inversely as the square of d.

2. x varies jointly as y and z. $x = 24$ when $y = 3$ and $z = 2$.

   (a) Find an equation connecting x, y and z.

   (b) Find x when $y = 5$ and $z = 4$.

3. y varies as x and z and $y = 12$ when $x = 2$ and $z = 3$.

   (a) Obtain an equation connecting x, y and z.

   (b) Find y when $x = 4$ and $z = 5$.

   (c) Find x when $y = 18$ and $z = 9$.

4. $p \propto qr$, and $p = 18$ when $q = 4$ and $r = 9$. Find p when $q = 6$ and $r = 2.5$.

5. F varies as y and inversely as z, and $F = 8$ when $y = 4$ and $z = 3$,

   (a) Find a formula for F.

   (b) Calculate F when $y = 5$ and $z = 2$.

6. Q varies as x^2 and inversely as y. If $Q = 1$ when $x = 3$ and $y = 18$, find Q when $x = 10$ and $y = 32$.

7. T varies directly as x and inversely as the square root of y. $T = 12$ when $x = 6$ and $y = 4$. Find T when $x = 15$ and $y = 9$.

8. y varies directly as x and z and inversely as the square of t. $y = 100$ when $x = 25$, $z = 2$ and $t = 1$. Find

   (a) y when $x = 12$, $z = 5$ and $t = 4$.

   (b) x when $y = 36$, $z = 3$ and $t = 2$.

Example. V varies directly as r^2 and h so that $V \propto r^2 h$.

(a) Find the effect on V if r is tripled and h is halved.

(b) What is the percentage change in V if r is increased by 20 % and h is decreased by 25 %?

Solution

(a) If r is multiplied by 3 and h is multiplied by $\frac{1}{2}$, then V is multiplied by

$3^2 \times \frac{1}{2}$, i.e. 4.5.

(b) It r is increased by 20 % and h is decreased by 25 %, then the percentage change, in V is

$$\left(\frac{120}{100}\right)^2 \times \frac{75}{100}, \text{ i.e., } \frac{36}{25} \times \frac{3}{4} \text{ or } \frac{108}{100}$$

Hence V is increased by 8 %.

Exercise 6.5

1. If w varies directly as z and w = 11 when z = 49.5, find the following.
   (a) The proportionality constant
   (b) The value of w when z = 84.

2. y varies jointly as x and z. y = 42 when x = 5 and z = 2.4.
   (a) Find a formula for y.
   (b) Calculate y for x = 2 and z = 9.

3. x varies as the square of y and inversely as z. x = 20 when y = 4 and z = 12.
   Find x when y = 1.2 and z = 3.

4. x varies as the square root of y and inversely as z^2. x = 18 when y = 81 and z = 3.

   Find x when y = 1.44 and $z = \sqrt{3}$.

5. y varies directly as x^2 and inversely as $\sqrt{z}$. Find y in terms of x and z given that

   y = 8 when x = 2 and z = 25. Calculate y when x = 3 and z = 1.

6. Q varies as p and q and inversely as t. Given that $Q = 1$ when $p = 3$, $q = 4$ and $t = 6$, calculate Q for $p = 4$, $q = 9$ and $t = 12$.

7. If $a \propto x^2$, $b \propto x$, c is constant, and $y = a + b + c$, and $y = 5, 12, 25$ when $x = 1, 2, 3$ respectively, find the relation between x and y.

8. If the area of a circle varies as the square of its radius, find the radius of a circle whose area equals the sum of the areas of the circles whose radii are 3 cm, 4 cm, and 5 cm respectively.

9. If $x^3 + 3xy^2 = 3x^2y + y^3$, show that x varies directly y.

10. If $\frac{y}{x} \propto (x + y)$, and $\frac{x}{y} \propto (x^2 - xy + y^2)$, show that $x^3 + y^3$ is constant.

11. If $a \propto bc$ and $b \propto ac$, show that c is constant.

12. If the area of a circle varies as the square of the radius, show that the area of a circle whose radius is 17 inches is equal to the sum of the areas of two circles whose radii are 15 inches and 8 inches.

13. A trapezium has two parallel sides x cm, y cm and d cm is the distance between two parallel sides.
    (a) Write down the formula for the area.
    (b) If x, y and d are doubled, what is the effect on the area?

14. The surface area of a sphere varies directly as the square of its radius
    (a) Write the variation formula.
    (b) If the surface area is doubled, what is the effect on the radius?

15. The volume of a sphere varies directly as the cube of its radius
    (a) Write the variation formula.
    (b) If the volume is tripled, what is the effect on the radius?

SUMMARY

1. Ratio

A ratio is one way of comparing two similar quantities. It expresses the relative magnitudes of two quantities of the same kind without stating what the magnitudes actually are.

2. Proportion

A proportion is the statement of the equality of two ratios.

3. Variation

If two variables x and y are so related that the quotient $\frac{y}{x}$ is a nonzero constant k for all permissible values of x and y, then y is said to vary directly as x according to the variation formula $y = kx$.

If the product xy is a nonzero constant k for all permissible values of x and y, then y is said to vary inversely as x and the corresponding variation formula is $y = \frac{k}{x}$.

CHAPTER 7

Introduction to Statistics and Probability

Statistics is concerned with collecting, organizing, summarizing, presenting and analysing data relation to human affairs or to natural phenomena.

Analysis of data may be done in several ways. Two common methods are:

(a) By representing data using diagrams such as pictographs, pie charts, bar charts, line graphs, histograms and frequency polygons.

(b) By calculating the measure of average and spread of given data.

In this chapter we will discuss the common measures of average such as mean, median and mode.

7.1 Concept of Measures of Central Tendency

Observations of data show that there is a tendency to cluster at some particular location on a scale of measurement. There are several possible ways to measure the central tendency of a given set of data. They are

(a) arithmetic mean

(b) median and

(c) mode.

They are also known as measures of location.

The Mode

Whenever data is processed in the form of a frequency distribution, the following questions arise,

(1) What is the most frequent value?

The answer is that value having the highest frequency.

The most frequent value is called the mode of the sample.

It is also known as the modal value.

(2) What is the most frequent class interval?

The answer is that class interval having the highest frequency.

The most frequent class interval is called modal class.

The mode is the most popular or most fashionable and frequently occurring measure in a sample being considered.

The Median

To find the median of a collection of data it is first necessary to arrange the data in ascending order. The middle number of the array is called the median.

Let the numbers be $x_1, x_2, x_3, \ldots x_n$.

If n is odd, the median is the value of x_k where $k = \frac{(n+1)}{2}$

For example, the median of x_1, x_2, x_3, x_4, x_5 is x_3 because x_3 is the middle measure.

Note $\frac{(5+1)}{2} = 3$

If n is even, there is no "true" middle measure. We define the median to be average of x_k and x_{k+1} where $k = \frac{n}{2}$.

For example, the median of $x_1, x_2, x_3, x_4, x_5, x_6$ is the average of x_3 and x_4.

Note $\frac{6}{2} = 3$.

The Mean

The arithmetical average of a sample is called the mean. It is usually denoted by the symbol μ (pronounced as "mu").

To find μ, the mean of a sample of numbers, such as examination marks, measures of weights, heights and prices etc:

(1) find T, the total of the measures of the items of the sample.

(2) divide T by the number of measures n. This may be represented as

$$\mu = \frac{T}{n}$$

If we have a collection of data $x_1, x_2, \ldots, x_n$, then we can write the mean μ as

$$\mu = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

where the symbol Σ (pronounced "sigma") means sum of.

Example 1. Calculate the mean, median and mode of 3,4, 5, 6, 8, 8, 8,10, 11.

Solution

$n = 9$

$$\text{mean} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{9} x_i}{9} = \frac{63}{9} = 7$$

Since n is odd, median = x_k where $k = \frac{(n+1)}{2} = \frac{(9+1)}{2} = 5$.

Thus, median = $x_5 = 8$.

Since 8 occurs most, mode = 8

Example 2. Calculate the mean, median and mode of 8, 9, 9, 9, 10,11,12, 12.

Solution $n = 8$

$$\text{mean} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{8} x_i}{8} = \frac{80}{8} = 10$$

Since n is even, median = $\frac{x_4 + x_5}{2} = \frac{9 + 10}{2} = 9.5$

Since 9 occurs most, mode = 9.

Example 3. What is the modal class in the following table?

Class interval	Frequency
56-57	5
58-59	12
60-61	11
62-63	4

Table 7.1

Solution 58-59 is the class with the highest frequency, 12. Thus 58-59 is the modal class.

Example 4. Find the median of the following collection of data:
2, 3, 4, 7, 8, 9, 3, 4, 3, 6, 8, 10, 9.

Solution Arrange the numbers in ascending order:
2, 3, 3, 3, 4, 4, 6, 7, 8, 8, 9, 9, 10.

Here the median is 6 because 6 is the middle term in the above arrangement of measures.

Exercise 7.1

1. Find the mean, median, and mode of each of the following (to one decimal place where necessary).

   (a) 2, 3, 4, 4, 7 (b) 2, 3, 4, 5, 6, 7, 8 (c) 6, 9, 7, 8, 5, 9, 7, 9, 10, 9

   (d) 1, 5, 4, 2, 1, 1, 3, 5, 4, 3, 2, 4, 4 (e) 12, 13, 13, 13, 14, 15, 15, 16, 17

2. What is the mode or modal value of the following samples?

   (a) 3, 4, 5, 4, 5, 4, 5, 3, 6, 3, 5.

   (b) 12, 1.2, 2.1, 12, 1.2, 1.2, 12.1.

   (c) 0.5, 5, 0.45, 0.5, 45.

3. Find the median of the following samples.

   (a) 12, 18, 14, 17, 25, 18, 15.

   (b) 101, 97, 215, 356, 245, 150.

   (c) 33, 18, 29, 37, 33, 26, 31, 32, 42, 25, 19.

4. Find the mean (average) of 2.2, 5.7, 12.8,6.7,2.1.

5. A class of 36 pupils has an average attendance of 35 for a period of 4 weeks. What is the total number of attendance recorded for the period?

7.2 Mean of a distribution

The examples discussed in Section 7.1 assume that the given data are already sorted (i.e.,arranged in an ascending order). However, in real life, the given data is rarely sorted. Here is a list of marks obtained by 40 students. The marks are not sorted.

42	73	54	52	85	58	48	54	60	54
58	48	70	52	53	53	53	60	75	55
60	55	50	53	25	58	53	45	65	68
58	57	82	30	55	49	57	63	72	28

Since no number is less than 25 and no number is greater than 90, we might decide to group the marks in class intervals of 5, starting at 25. This means that the first group would include 25, 26, 27, 28, 29, and the second group 30, 31, 32, 33, 34, and so on.

Consider the first four numbers in the list.

42 lies in the range 40 – 44.

73 lies in the range 70 – 74.

54 lies in the range 50 – 54.

52 also lies in the range 50 – 54.

We can form a table for tallying numbers. Using the four numbers above, we get the following table.

Marks	Tally
25 – 29	
30 – 34	
35 – 39	
40 – 44	|
45 – 49	
50 – 54	||
55 – 59	
60 – 64	
65 – 69	
70 – 74	|
75 – 79	
80 – 84	
85 – 89	

Table 7.2

So far we have treated only four entries. There are two entries in the range 50 – 54.

We assume that the frequencies are spread evenly over each interval. This is the same as assuming that they are all concentrated at the midpoint of the interval. Choosing a class interval of 5, we can construct the following table.

Table 7.3

Marks	Midpoint (x)	Tally	Frequency (f)	f . x
25 – 29	27	II	2	54
30 – 34	32	I	1	32
35 – 39	37		0	0
40 – 44	42	I	1	42
45 – 49	47	IIII	4	188
50 – 54	52	~~IIII~~ ~~IIII~~ I	11	572
55 – 59	57	~~IIII~~ IIII	9	513
60 – 64	62	IIII	4	248
65 – 69	67	II	2	134
70 – 74	72	III	3	216
75 – 79	77	I	1	77
80 – 84	82	I	1	82
85 – 89	87	I	1	87
			$\sum f = 40$	$\sum fx = 2245$

$$\text{Mean marks} = \frac{\sum fx}{\sum f} = \frac{2245}{40} = 56.1 \text{ marks}$$

Since the data, or information, have been classed together we can find the modal class. For grouped data, we define the modal class to be the class interval in which the greatest frequency of marks occurs. In the above example the modal class would be 50– 54 marks. 50 marks and 54 marks are called **the class limits of the interval.**

Before rounding off, some marks may lie outside the class limits. For example, 54.2 marks lies outside the class limits 50 – 54 and 55 – 59. In such case, we define class boundaries 49.5 marks and 54. 5 marks for the modal class above.

For the class interval 50 – 54, a lower class limit is 50 and an upper limit is 54 marks while a lower class boundary is 49.5 marks and an upper class boundary is 54.5 marks. Then, in the above example, every measure x would be contained within class boundaries as follows :

$$24.5 \le x < 29.5$$

$$29.5 \le x < 34.5$$

$$34.5 \le x < 39.5$$

Exercise 7.2

1. Paper clips are sold in boxes which normally contain 100 clips. The number of clips in a random sample of 150 boxes are shown in the following table.

Paper clips	95	96	97	98	99	100	101	102	103	104
Frequency	1	5	10	22	30	37	20	15	10	0

Construct a frequency table with the headings "number of clips (x)", "frequency (f)" and "total number of clips (f.x)". Hence calculate the mean number of clips in a box.

2. The scores in the final round of Masters Golf Tournament in 1971 were as follows :

66	70	69	68	70	72	69	71	72
66	63	69	69	73	73	69	73	74
69	66	67	67	70	71	69	74	80

(a) Construct a frequency table.

(b) Illustrate these data in a histogram.

(c) Find the median and mode.

3. The following table gives the data concerning heights of 100 male pupils of a certain school.

Height of 100 male pupils at a school

Height(inches)	Number of pupils
30 – 32	5
33 – 35	18
36 – 38	42
39 – 41	27
42 – 44	8
	Total 100

(a) Construct a frequency table with the headings "height in inches", "midpoints (x)", "frequency (f)" and "f. x". Hence calculate the mean height.

(b) What is the modal class?

(c) Find the class limits for each class interval.

(d) Write down the class boundaries for each class interval.

4. Calculate the mean of the following numbers :

(a) without grouping them

(b) grouping them in class intervals

0 – 2, 3 – 5, 6 – 8, …, 18 – 20

8	12	15	10	11	11	16	9	8	11	13
2	12	20	10	9	15	5	7	6	14	8
18	8	13	16	19	5	7	20	8	9	12

5. The following table gives the weights of 40 male students to the nearest pound.

(a) Calculate the mean without grouping them.

(b) Construct a frequency table by using suitable class intervals.

(c) Calculate the mean from the frequency table.

38	64	50	32	44	25	49	57
46	58	40	47	36	48	52	44
68	26	38	36	63	19	54	65
46	35	42	47	33	53	40	35
61	45	35	42	56	56	45	28

7.3 The Mean of a Frequency Distribution

We can calculate the mean of a frequency distribution as follows:

Let $x_1, x_2, \ldots, x_N$ be a collection of data with corresponding frequencies $f_1, f_2, \ldots, f_N$. That is, x_1 occurs f_1 times, x_2 occurs f_2 times and so on. Then to find the mean of the frequency distribution:

(a) find the total of products $f_1 . x_1, f_2 . x_2, \ldots, f_N . x_N$.

(b) divide the total by the sum of the frequencies.

This may be written as $$\mu = \frac{f_1.x_1 + f_2.x_2 + \ldots + f_N.x_N}{f_1 + f_2 + \ldots + f_N}$$

$$= \frac{\sum_{i=1}^{N} f_i x_i}{\sum_{i=1}^{N} f_i} = \frac{\sum f.x}{\sum f}$$

These calculation can be summarized by constructing a frequency table as follows:

Marks (x)	Frequency(f)	Products(f.x)
x_1	f_1	$f_1.x_1$
x_2	f_2	$f_2.x_2$
.	.	.
.	.	.
.	.	.
x_N	f_N	$f_N.x_N$
	$\sum f$	$\sum f.x$

Table 7.4

$$\mu = \frac{\sum f.x}{\sum f}$$

Example 1. Find the mean of a frequency distribution given in table 7.3.

x	1	2	3	4	5
f	2	3	5	10	2

Table 7.5

Solution

To find the mean, we construct the following frequency table.

Marks (x)	Frequency(f)	Products(f.x)
1	2	2
2	3	6
3	5	15
4	10	40
5	2	10
	$\sum f = 22$	$\sum f.x = 73$

Table 7.6

$$\mu = \frac{\sum f.x}{\sum f} = \frac{73}{22} = 3.31$$

Calculation of the mean from the assumed mean

In the following, we will present how the mean of a collection of data can be found without using the formula $\mu = \frac{\sum x}{n}$.

To find the mean:

(a) first write down an **assumed mean** (say) e. Then,

(b) calculate the deviations from the **assumed mean.** That is, if we have a collection of data $x_1, x_2, \ldots, x_n$, the deviations of these numbers from the assumed mean e are $d_1 = x_1 - e$, $d_2 = x_2 - e$, …, $d_n = x_n - e$,

(c) calculate the total deviation from the assumed mean,

(d) calculate the average deviation from the assumed mean,

(e) calculate the value, (assumed mean + average deviation) which is the required mean.

This can be written as

$$\text{the mean, } \mu = e + \frac{\sum_{i=1}^{n} d_i}{n} = e + \frac{\sum d}{n}$$

Example 2. Find the mean of numbers 5, 8, 11, 9, 12, 6, 14 and 10, taking the assumed mean e as (a) 9 (b) 20.

Solution

For the assumed mean e = 9, we can construct the following table.

	Number x_i	Deviations $d_i = x_i - e$
	5	−4
	8	−1
	11	2
e →	9	0
	12	3
	6	−3
	14	5
	10	1
	Total =	3

Table 7.7

The mean of deviations = $\frac{\sum d}{n} = \frac{3}{8} = .375$

Thus, the mean of given numbers = e + $\frac{\sum d}{n} = 9.375$

(b) The deviations of the given numbers from 20 are, –15,–12,–9,–11,–8, –14, –6, –10 and $\sum d = -85$

Thus, the mean of given numbers = e + $\frac{\sum d}{n}$

$$= 20 + \frac{-85}{8} = 20 - 10.625 = 9.375$$

Example 3. Find the mean of the following frequency distribution of heights.

Heights (inches)	Midpoint (x)	Deviation $d = x - e$	Frequency f	f.d
60-62	61	–6	5	–30
63-65	64	–3	18	–54
66-68	67 ← e	0	42	0
69-71	70	3	27	81
72-74	73	6	8	48
			n = f = 100	f.d = 45

Table 7.6

Sum of deviations = $\sum f.d = 45$

The mean deviations = $\frac{45}{100} = .45$

Mean height = 67 +.45 = 67.45 inches

In this case, mean height 67.45 lies in the class interval 66-68.

Exercise 7.3

1. Find in each case
   (a) the mean of the sample,
   (b) the deviations from the mean, and
   (c) the sum of the deviations from the mean
   (1) 20, 12, 17, 13, 8, 14.
   (2) 15, 21, 32, 46, 54, 71, 76.
   (3) 2.4, 2.8, 3.6, 7.2.
   (4) 1.9, 1.4, 1.1, 0.97, 0.18.
2. Find the mean of the numbers 8, 9, 7, 5, 6, 10 and 11 by taking the assumed mean e as (a) 7, (b) 18.
3. Find the mean of the following frequency distribution of marks by taking as assumed mean as (a) 67, (b) 70.

Score	Frequency
50-54	4
55-59	6
60-64	8
65-69	16
70-74	10
75-79	3
80-84	2
85-89	1
	Total = 50

4. Calculate the mean of the frequency distribution of the life times of 40 radio tubes tested at an electrical department.

life time (hours)	Number of tubes
300-399	2
400-499	4
500-599	5
600-699	7
700-799	6
800-899	8
900-999	4
1000-1099	2
1100-1199	2
	Total = 40

5. Find the mean weight of the following frequency distribution of weights.

Weight (lb)	Number of student
118-126	6
127-135	10
136-144	18
145-153	24
154-162	10
163-171	8
172-180	4
	Total = 80

7.4 Introduction to Probability

We encounter chance phenomena daily. We are familiar with expressions such as "the chances are", "the odds are against it", "that is most improbable", and "this event is very likely". Our experience is rich with what we will formally call "chance experiments". In this chapter, we will carry out a number of **experiments**, and study their **outcomes**. These will be random experiments or chance experiments in which the exact outcomes of each trial cannot be predicted.

For example,

(i)	Experiment	: Tossing a coin
	Outcomes	: Head or Tail
(ii)	Experiment	: Rolling a die
	Outcomes	: 1, 2, 3, 4, 5, or 6 turns up
(iii)	Experiment	: Choosing a number from a set of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
	Outcomes	: 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10.

These experiments have three points in common :

(1) In none of them it is possible to predict before hand how the experiment will turn out.

(2) It is possible, however, to make a list of all the possible outcomes of each experiment.

(3) One has "feelings" or "beliefs" based on experience and intuition about the "likelihoods" or "probabilities" of the various outcomes.

Probability

Consider a chance experiment with a finite number of distinct possible outcomes. Let us say that we consider an experiment which has r distinct possible outcomes, labeled $w_1, w_2, \ldots, w_r$. If the experiment is repeated N times, then some of the N results of the experiment may be of type w_1, some of type w_2, and so forth. Let N_1 denote the number of results of type w_1, N_2 the number of results of type w_2, and so on. Then we have,

$$N_1 + N_2 + \ldots + N_r = N$$

Then, the proportion of results of type w_1 is $\frac{N_1}{N}$. This proportion is also known as **relative frequency** of type w_1. Thus, we have,

$$\text{relative frequency of type } w_2 = \frac{N_2}{N} = \frac{\text{number of trials giving results of type } w_2}{\text{total number of trials}}$$

$$\text{relative frequency of type } w_3 = \frac{N_3}{N} = \frac{\text{number of trials giving results of type } w_3}{\text{total number of trials}}$$

$$\text{relative frequency of type } w_r = \frac{N_r}{N} = \frac{\text{number of trials giving results of type } w_r}{\text{total number of trials}}$$

It can easily be seen that

$$\frac{N_1}{N} + \frac{N_2}{N} + \ldots + \frac{N_r}{N} = 1$$

The frequency theory of probability asserts that the following limits exist an N goes to infinity :

$$\lim_{N \to \infty} \frac{N_i}{N} = P_i$$

The limiting values $P_1, P_2, \ldots, P_r$ are called the theoretical probabilities of the outcomes $w_1, w_2, \ldots, w_r$ respectively. These assertions are really not about mathematical limits but rather about empirical phenomena. To verify these limits, one would have to be able to repeat the experiment an infinite number of times, which is impossible. However it is useful to interpret the frequency probability theory in a purely intuitive way: Associated with outcomes $w_1, w_2, \ldots, w_r$ are theoretical probabilities $P_1, P_2, \ldots, P_r$. When the experiment is repeated a "large" number of times, the observed proportions $\frac{N_1}{N}, \frac{N_2}{N}, \ldots, \frac{N_r}{N}$ are "close" to these theoretical probabilities.

Now, let us talk about outcomes of an experiment. When we see no reason why one outcome of an experiment should occur more frequently than another, we say that the outcomes are equally likely.

Thus, a "Head" and a "Tail" are equally likely to turn upon the toss of an unbiased coin, and each of the six numbers on a die is equally likely to turn up when the die is rolled.

In such cases, we say that where an experiment has several equally likely outcomes;

$$\begin{matrix}\text{The probability of} \\ \text{"favourable" outcome}\end{matrix} = \frac{\text{the number of favourable outcomes}}{\text{the number of possible outcomes}}$$

We must be sure that the outcomes are equally likely to occur. If they are, then the values of the probability given by the limit of the relative frequency in a large number of trials, and that given by the above definition, will be equal.

Example 1. When a coin is tossed, what is the probability that the outcome is:

(a) Head turns up (b) Tail turns up?

Solution

In this experiment, the set of possible outcomes is { H, T }, where H denotes head and T denotes tail. This set contains two members.

In (a) the set of favourable outcome is { H } with one member. So the probability that "Head turns up" is

$$\frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \frac{1}{2}$$

In (b) the set of favourable outcome is {T} with one member. So the probability that "Tail turns up" is $\frac{1}{2}$.

Example 2. When a die is rolled, what is the probability that the outcomes is :

(a) "4 turns up"

(b) "An odd number turns up"?

Solution In this example, the set of possible outcomes is {1,2,3,4,5,6} containing 6 members.

In (a) the set of favourable outcomes is {4} with one member. So the probability that "4 turns up" is

$$\frac{\text{the number of favourable outcomes}}{\text{the number of possible outcomes}} = \frac{1}{6}$$

In (b) the set of favourable outcomes is {1, 3, 5} with three members. So the probability that "An odd number turns up" is $\frac{3}{6}$ or $\frac{1}{2}$.

Example 3. A bag contains 10 red balls and 30 black balls.

(a) When a ball is drawn, what is the probability of "a red ball"?

(b) If the ball drawn is red and **not replaced**, and another ball is then drawn, what is the probability that it will be red also?

Solution

(a) Total number of balls in the bag = 10 + 30 = 40.

Thus the number of possible outcomes = 40.

Since there are 10 red balls in the bag,

the number of favourable outcomes = 10.

So, P (a red ball) = $\frac{10}{40} = \frac{1}{4}$.

(b) If the first ball drawn is red and not replaced, the number of balls left in the bag for the next draw is 40 – 1 = 39

$\therefore$ number of possible outcomes = 39, and

Number of favourable outcomes = number of red balls in the bag = 10–1= 9.

Thus, P (a red ball in the second draw) = $\frac{9}{39} = \frac{3}{13}$.

Example 4. A letter is chosen at random from the letters of the word ORANGE: What is the probability that it is:

(a) N (b) a vowel.

Solution In both cases, the set of possible outcomes is

{O, R, A, N, G, E} containing 6 members.

In (a), the set of favourable outcomes is {N}, with one member. So the probability that "letter drawn is N" is:

$$\frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \frac{1}{6}$$

In (b), the set of favourable outcomes is {O, A, E} with 3 members. So the probability of a vowel is $\frac{3}{6}$ or $\frac{1}{2}$. Briefly, writing P for probability, we have :

(a) P (N) = $\frac{1}{6}$ (b) P (a vowel) = $\frac{3}{6} = \frac{1}{2}$

Exercise 7. 4

1. A box contains 10 white balls and 20 black balls. If a ball is drawn, what is the probability that it is : (a) white (b) black ?

2. A letter is chosen at random from the letters of the word ' PENCIL '. What is the probability that it is : (a) N (b) a vowel ?

3. What is the probability that on rolling a die the number that turns up will be

   (a) odd (b) less than 3 (c) 6 (d) prime ?

4. The names of the 25 boys and 10 girls in a class are written on similar pieces of paper, and placed in a cup. If one is drawn out at random, what is the probability that it is the name of : (a) a boy (b) a girl ?

5. If a whole number from 1 to 20 both inclusive is selected, and if each number has an equal chance of being selected, what is the probability that the number will be

   (a) even (b) greater than 1 (c) prime ?

6. There are 50 pens of the same maker in a show-case. 18 are blue, 15 white, 10 green, 7 red. If it is equally likely that any one of them will be the first to be sold, what is the probability that it will be:

   (a) white (b) green (c) red or blue (d) neither red nor blue ?

7. A bag contains 5 white marbles, 3 black and 2 red.

   (a) What is the probability that if one marble is chosen at random it will be white?

   (b) If in fact a white marble is chosen, and not replaced, what is the probability that a black marble will be chosen next?

8. If a letter is chosen at random from the letters of the word MISSISSIPPI, what is the probability that it is: (a) S (b) I (c) neither I nor S ?

9. In a class of 30 pupils, 18 like popular songs only, 7 like classical songs, the rest like neither. If a pupil in the class is chosen at random, what is the probability that a pupil likes popular songs?

10. A symmetrical triangular pyramid (tetrahedron) has the number 1,2,3,4 painted on its faces. Find the probability that when it is tossed it will land so that:

(a) the 2 face is downwards,

(b) the sum of the numbers on the three faces showing is even. ?

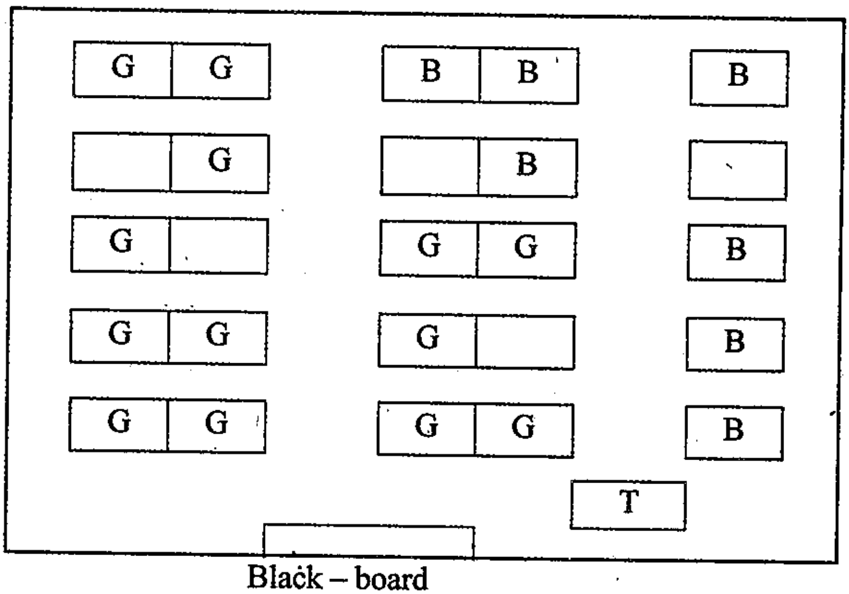
11. A box contains three white and two black counters. What is the probability that when a counter is drawn at random it will be white?If the counter is white, and is not replaced, what is the probability that the next one will be white also?

12. A carton of twelve pens contains three faulty pens. What is the probability that a pen chosen at random will be faulty?
If it is not faulty and is not replaced, what is the probability that the next pen chosen will not be faulty?

13. Figure 7.1, shows the distributions of boys (B) and girls (G) at the desks of a classroom. If the name of each pupil is written on a piece of paper and placed in a box, state the probability that when a paper is drawn at random from the box it will contain the name of:

(a) a girl, (b) a boy, (c) a pupil sitting on his or her own,
(d) a pupil sitting next to another pupil, (e) a pupil sitting next to a wall,
(f) a pupil sitting at the back of the room, (g) a pupil sitting directly behind a girl,
(h) a pupil sitting directly in front of a boy. ?

Fig. 7.1



SUMMARY

Statistics

There are three measures of central tendency, viz, mean, median, and mode.

1. Mean is the arithmetical average and its formula

$$\text{mean} = \frac{\text{sum of all measures}}{\text{number of all measures}}$$

2. Mode is the most frequent value.

3. Median is the middle number in the arrangement of the data in ascending order of magnitude.

4. Mean μ = assumed mean + $\frac{\sum d}{n}$

Probability The probability of

$$\text{"favourable" outcome} = \frac{\text{the number of favourable outcomes}}{\text{the number of possible outcomes}}$$

CHAPTER 8

Similarity

In everyday life, we learn to recognize objects more on the basis of their shapes than on the basis of their actual sizes. This chapter will be concerned with a study of the properties of geometric figures having "the same shape". Later on we will explain precisely what we mean by "the same shape". Many of these properties discussed in this chapter are used extensively in industries and in engineering where models and scale drawings play key roles. Conclusions about a proposed product can be reached at a considerable saving in both time and money on the basis of experimentation with small models.

In this chapter, we will prove some theorems about similar triangles. We will then prove Pythagoras Theorem using the above theorems. We may observe that the theorems on similar triangles depend very much on the properties of parallel lines, ratios and proportions. Before we read this chapter, chapter 6 about ratio and proportion, should be reviewed.

8.1 The Idea of Similarities and Similar Triangles

We have studied congruence of two triangles. We notice that if two triangles are congruent they have the "same shape" and the "same size". We now learn some properties of geometric figures that are of the "same shape" but not necessarily of the "same size". Such figures are said to be "similar".

For example, any two circles are similar; any two squares are similar; any two equilateral triangles are similar.

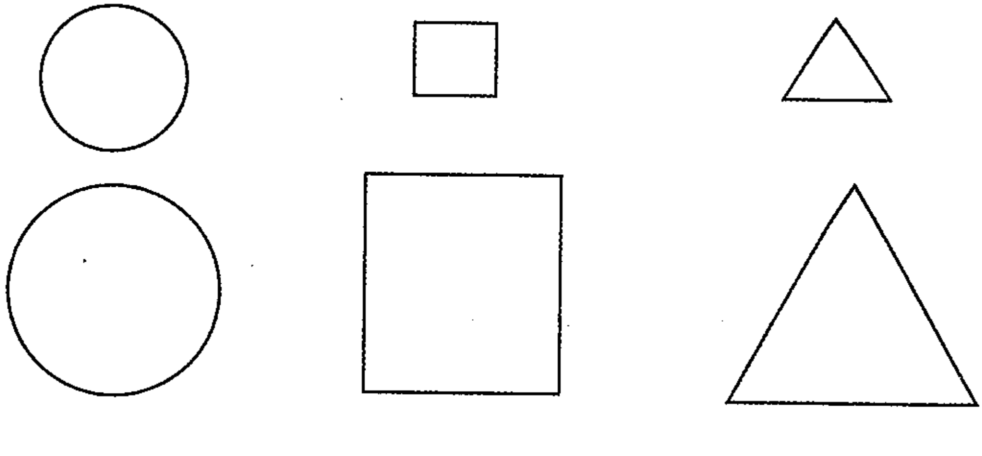


Fig. 8.1

Another way of expressing this is to say that two figures are similar if one of them is an exact scale model of the other.

We need to make the concept "same shape" more precise. In each pair shown below, the two figures do not have the same shape.

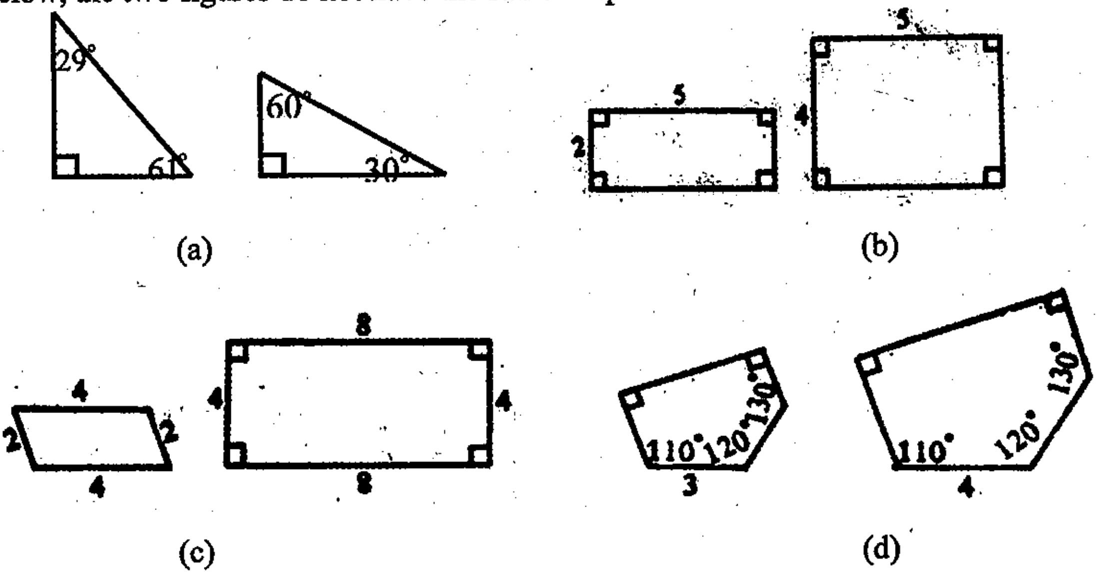
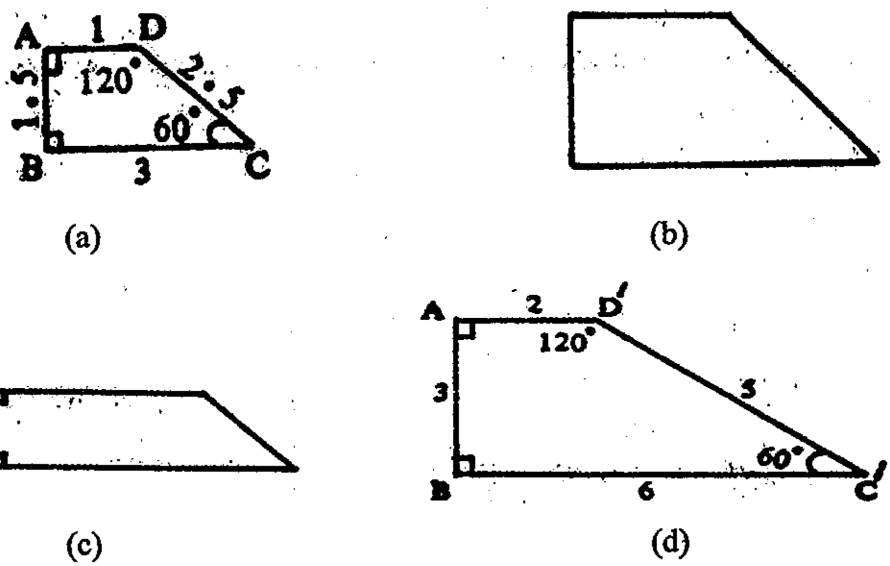


Fig. 8.2

These examples suggest that a definition of similarity of two figures might involve both distances (lengths) and angles.



(e) **Fig 8 . 3** (f)

Which of the quadrilaterals in Fig. 8. 3 look alike? While they all have certain things in common, only (a) and (d) have the same shape but are of different sizes. The others, as compared to (a) appear to have angles of different sizes, or to have some sides proportionally too long or too short.

Consider the correspondence ABCD $\longleftrightarrow$ A' B' C' D'. We notice that the corresponding angles are equal, that is, $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and $\angle D = \angle D'$. The corresponding sides are not equal and so if we compare them by forming the ratios of their lengths, we find

$$\frac{AB}{A'B'} = \frac{1.5}{3} = \frac{1}{2}$$

$$\frac{BC}{B'C'} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{CD}{C'D'} = \frac{2.5}{5} = \frac{1}{2}$$

$$\frac{DA}{D'A'} = \frac{1}{2}$$

Each ratio is equal to $\frac{1}{2}$.

In general, if all the ratios of the lengths of the corresponding sides are the same, we say the **corresponding sides are proportional.**

Thus the definition of a similarity for any polygon requires two conditions.

(i) corresponding angles must be equal, and
(ii) corresponding sides must be proportional.

We are now ready to define similarity of triangles.

Definition 1 : Two triangles whose corresponding angles are equal and whose corresponding sides are proportional are said to be similar.

The symbol " ~ " will be used to denote similarity in the same way that " ≅ " denotes congruence; thus " Δ ABC ~ Δ PQR" means the triangle ABC is similar to the triangle PQR.

As a convention, similar triangles and polygons will be named so that the order of the letters indicates the correspondence between the two figures. Thus the statement Δ ABC ~ Δ PQR will always indicate that which angles are corresponding angles and which sides are corresponding sides.

(i) $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$

and

(ii) $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CA}{RP}$

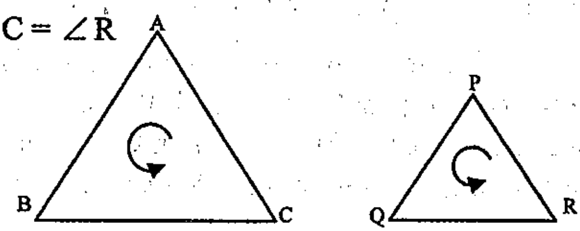
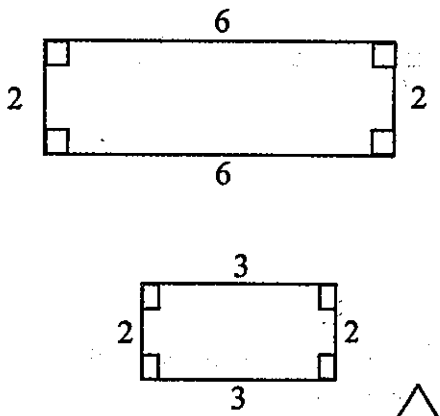


Fig. 8. 4

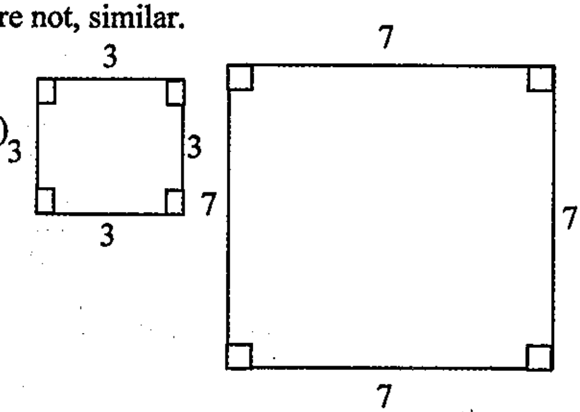
Exercise 8.1

1. State why the two polygons are, or are not, similar.

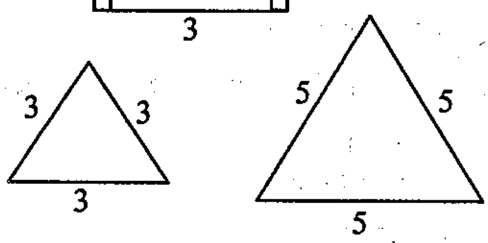
(a)



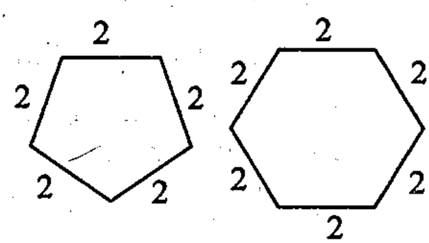
(b)



(c)



(d)



2. Complete the proportions.

(a) If $\Delta\ ABC \sim \Delta\ DEF$ then $\frac{AB}{?} = \frac{BC}{?} = \frac{?}{DF}$

(b) If $\Delta\ GHI \sim \Delta\ KLM$ then $\frac{?}{HI} = \frac{?}{GH} = \frac{?}{GI}$

3. State whether the proportions are correct for the indicated similar triangles.

(a) $\Delta\ ABC \sim \Delta\ XYZ$

$$\frac{AB}{XY} = \frac{BC}{YZ}$$

(b) $\Delta\ DEF \sim \Delta\ HIJ$

$$\frac{DE}{HI} = \frac{EF}{IJ}$$

(c) $\Delta\ RST \sim \Delta\ LMK$

$$\frac{RT}{LM} = \frac{ST}{MK}$$

(d) $\Delta\ XYZ \sim \Delta\ UVW$

$$\frac{XY}{UV} = \frac{XZ}{VW}$$

4. If $\Delta\ ABC \cong \Delta\ A'B'C'$, does it follow that $\Delta\ ABC \sim \Delta\ A'B'C'$? Why?

5. Which of the following statements are true?

(a) Any two isosceles triangles are similar.

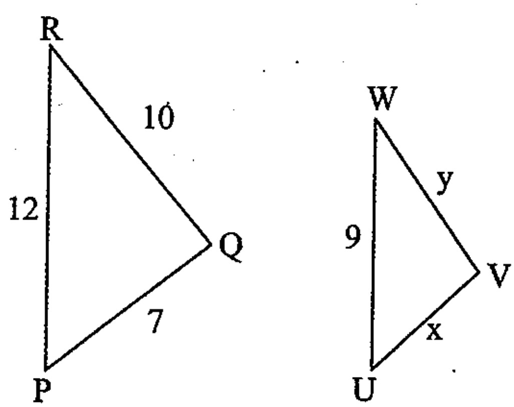
(b) Any two parallelograms are similar.

(c) Any two rectangles are similar.

(d) If two polygons are similar, they have the same number of sides.

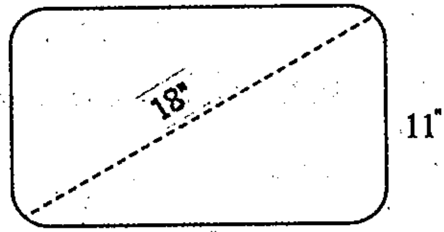
6. Given : $\Delta\ PQR \sim \Delta\ UVW$ and lengths of sides are as marked.

Find : x and y.



7. The measures of two angles of Δ XYZ are 82° and 16°. Find the measures of the angles of a triangle similar to Δ XYZ.

8. Most TV screens have similar shape. The measure of a diagonal is used to denote the screen size. Thus the TV screen as shown is an 18" screen.



Find the approximate dimensions of

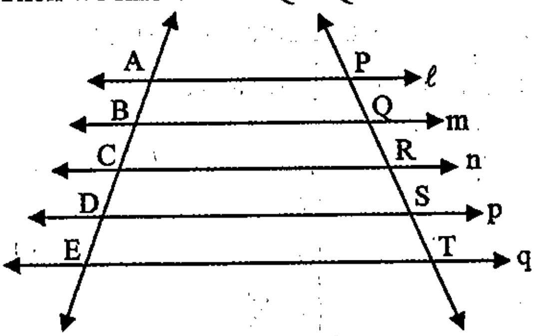
(a) a 9" screen (b) a 21" screen.

8.2 The Basic Proportionality Theorem

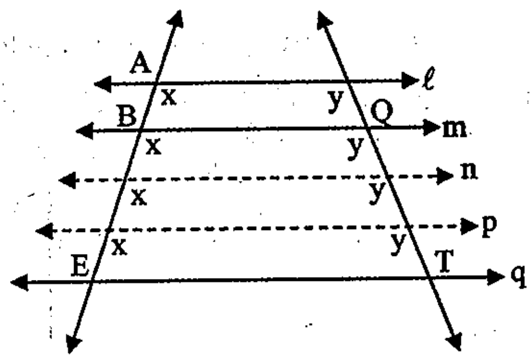
Before we prove theorems on similar triangles, we prove some useful properties relating to transversals to parallel lines and proportional division in triangles. These proofs rest essentially upon some properties of the real number system. It is desirable to state a postulate about parallel lines and proportional division which will shorten proofs and provide us with a more direct method.

To help understand the postulate, we will use the following particular example.

In Fig. 8.5 (a), lines ℓ, m, n, p and q are parallel and AB = BC = CD = DE. Then we know that PQ = QR = RS = ST.



(a)



(b)

Fig. 8 . 5

Let AB $= x$, PQ $= y$ so that

BE $= 3x$, QT $= 3y$

Considering only the lines ℓ, m and q,

$$\frac{AB}{BE} = \frac{x}{3x} = \frac{1}{3}$$

$$\frac{PQ}{QT} = \frac{Y}{3Y} = \frac{1}{3}$$

and hence $\frac{AB}{BE} = \frac{PQ}{QT}$

This shows three parallel lines ℓ, m and q divide the transversals proportionally. We will state it as a postulate.

Postulate 1: If three parallel lines intersect two transversals, then the lines divide the transversals proportionally.

According to the above postulate, if ℓ // m // n, then $\frac{a}{b} = \frac{c}{d}$

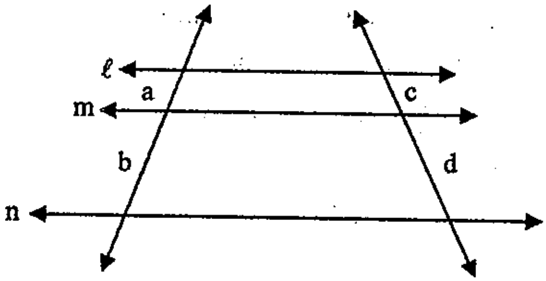


Fig. 8.6

We now prove the Basic Proportionality Theorem.

Theorem 1 (The Basic Proportionality Theorem-BPT)

If a line intersecting the interior of a triangle is parallel to one side, then the line divides the other two sides proportionally.

Given : In Δ ABC, DE // BC,

To Prove : $\frac{AD}{DB} = \frac{AE}{EC}$

Proof : Through A, draw AF // BC

Since DE // BC (given)

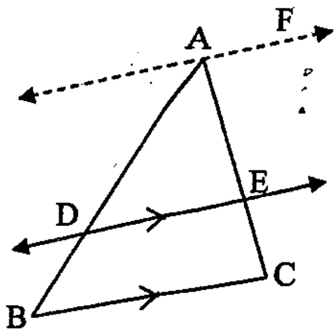


Fig. 8.7

AF // DE // BC

AB and AC are transversals.

According to the postulate 1, $\frac{AD}{DB} = \frac{AE}{EC}$

Corollary 1.1

Using properties of proportions, it can be shown that the following three proportions are equivalent.

(1) $\frac{AD}{DB} = \frac{AE}{EC}$

(2) $\frac{AD}{AB} = \frac{AE}{AC}$

(3) $\frac{AB}{DB} = \frac{AC}{EC}$

The following corollary is the converse of the BPT.

Corollary 1.2 (CBPT)

If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.

Given : $\frac{PQ}{QR} = \frac{PY}{YZ}$

To prove : QY // RZ

Proof : Assume QY $\nparallel$ RZ.

There exists a ray QK such that QK // RZ.

By BPT, $\frac{PQ}{QR} = \frac{PK}{KZ}$

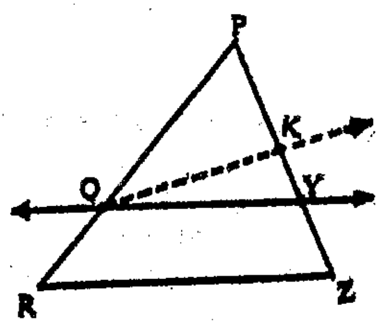


Fig. 8.8

$$\text{or} \qquad \frac{PY}{YZ} = \frac{PK}{KZ} \quad \left(\frac{PQ}{QR} = \frac{PY}{YZ}\right)$$

$$\frac{PY + YZ}{YZ} = \frac{PK + KZ}{KZ}$$

$$\frac{PZ}{YZ} = \frac{PZ}{KZ}$$

$$YZ = KZ$$

Y coincides with K.

This means $QK = QY$

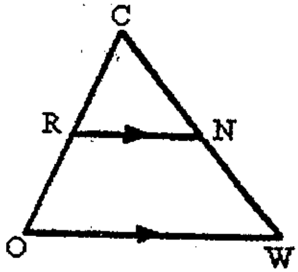
Thus the assumption that $QY \nparallel RZ$ is not true.

Hence $QY \parallel RZ$

Exercise 8.2

1. Use the Basic Proportionality Theorem (BPT) and its corollary to complete the proportions for the adjoining figures.

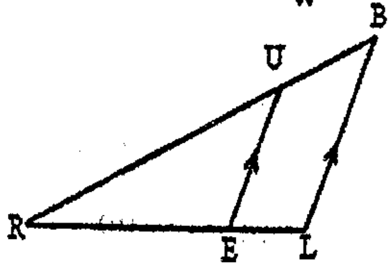
In Δ COW, RN // OW.



(a) $\frac{OR}{RC} = ?$

(b) $\frac{NW}{CW} = ?$

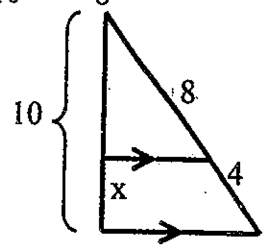
In Δ RBL, EU // LB.



(c) $\frac{EL}{RE} = ?$

(d) $\frac{RU}{RB} = ?$

2. To find x in the figure, a student wrote the proportion $\frac{x}{10} = \frac{4}{8}$



(a) Is this correct?

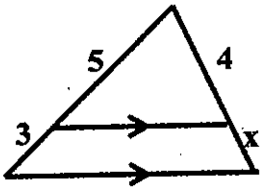
(b) Another student wrote the proportion

$\dfrac{x}{x-4} = \dfrac{4}{8}$. Is he correct?

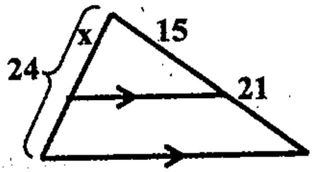
(c) Write a simpler proportion that will give the correct answer.

3. Find x in each of the figures below. (They are not drawn to scale.)

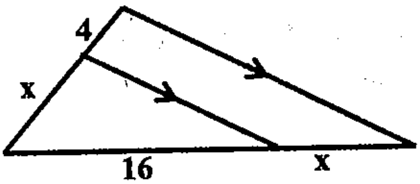
(a)



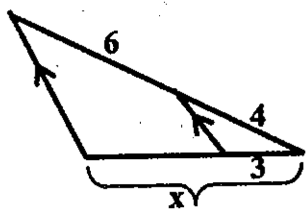
(b)



(c)

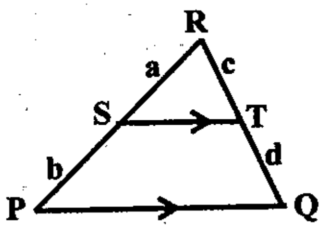


(d)



4. Given ΔPQR with ST // PQ and lengths of segments are marked.

Which of the following proportions are correct?

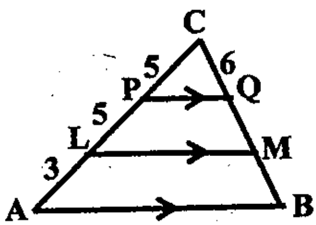


(a) $\dfrac{b}{a} = \dfrac{d}{c}$ (b) $\dfrac{a+b}{a} = \dfrac{c+d}{d}$

(c) $\dfrac{c}{d+c} = \dfrac{a}{b+a}$ (d) $\dfrac{a}{c} = \dfrac{b}{d}$

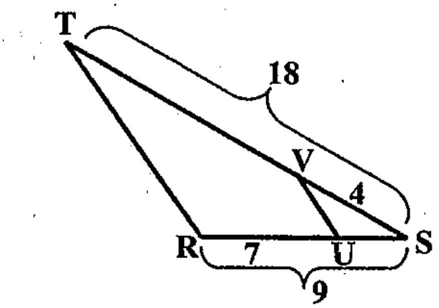
(e) $\dfrac{a}{b} = \dfrac{c}{d}$ (f) $\dfrac{a-b}{b} = \dfrac{c-d}{c}$

5. If PQ // LM // AB, and the lengths are as shown, how long are the segments MQ and BM?



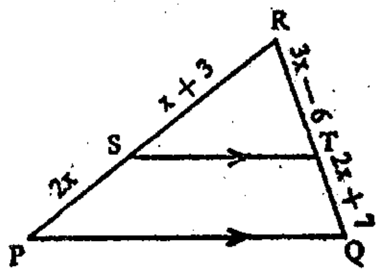
6. If the segments in the figure have the lengths indicated, is UV // RT ?

Justify your answer.



7. Given the figure as marked with ST // PQ.

Find the values of the segments PS, SR, RT and TQ.



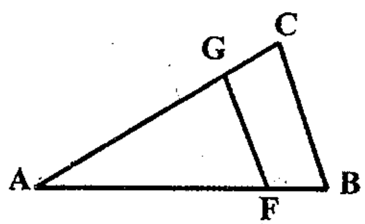
8. Which of the following sets of lengths will make FG // BC?

(a) AB = 14, AF = 6, AC = 7, AG = 3

(b) AB = 12, FB = 3, AC = 8, AG = 6

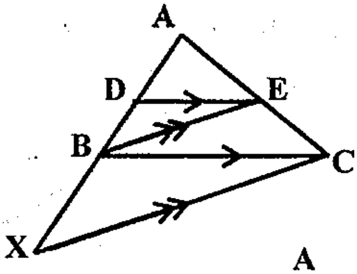
(c) AF = 6, FB = 5, AG = 9, GC = 8

(d) AC = 21, GC = 9, AB = 14, AF = 5



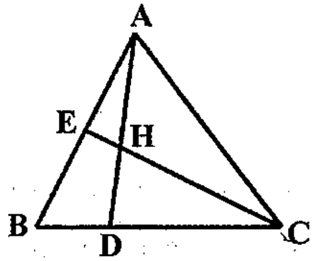
9. Given : $\frac{AD}{DB} = \frac{2}{1}$

Prove : $\frac{AX}{XB} = \frac{3}{1}$



10. Given : $AE = EB$, $\frac{BD}{DC} = \frac{2}{3}$

Find : The ratio $\frac{CH}{CE}$



8.3 Basic Theorems on Similar Triangles

At present we have no method of proving that two triangles are similar other than to show that two triangles meet all the requirements named in the definition of similar triangles.

For triangles, it turns out that if one of the two condition holds, then so does the other. That is, if corresponding angles are equal, the corresponding sides are proportional, and conversely, if corresponding sides are proportional, the corresponding angles are equal. These facts are given in the AAA Similarity Theorem and the SSS Similarity Theorem.

Theorem 2 (The AAA Similarity Theorem)

If the angles of one triangle are equal, respectively, to the angles of another triangle, then the two triangles are similar.

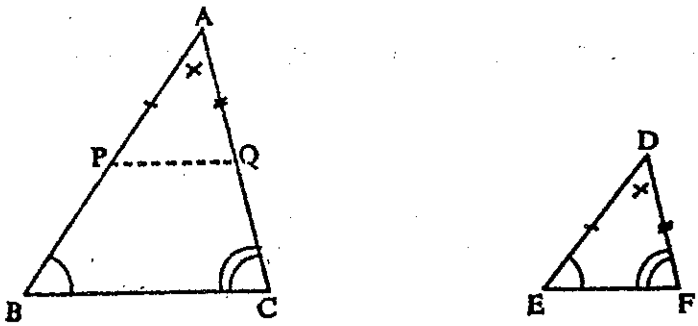


Fig. 8.9

Given : In Δ ABC and Δ DEF,

$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

To Prove : $\Delta ABC \sim \Delta DEF$

Proof : Let P and Q be points on AB and AC such that $AP = DE$ and $AQ = DF$.

By SAS congruence theorem,

We have $\Delta APQ \cong \Delta DEF$, so that

$$\angle APQ = \angle E$$

Since $\angle E = \angle B$, it follows that

$$\angle APQ = \angle B$$

Hence $PQ \parallel BC$

By BPT, $$\frac{AB}{AP} = \frac{AC}{AQ}$$

i.e., $$\frac{AB}{DE} = \frac{AC}{DF} \text{ (since } AP = DE,\ AQ = DF)$$

Instead of PQ, if we draw a comparable segment parallel to CA or AB, it can be shown that

$$\frac{BC}{EF} = \frac{AB}{DE} \text{ or } \frac{BC}{EF} = \frac{AC}{DF}$$

and hence

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

Thus, $\Delta ABC \sim \Delta DEF$.

Theorem 2 suggests the following corollary,

Corollary 2.1 (The AA Corollary)

If two angles of one triangle are equal to two angles of a second triangle, then the triangles are similar,

A line parallel to one side of a triangle may form another triangle with the given triangle. In the following figure, line XY is parallel to BC and $\Delta ABC \sim \Delta AXY$, since indicated angles are equal.

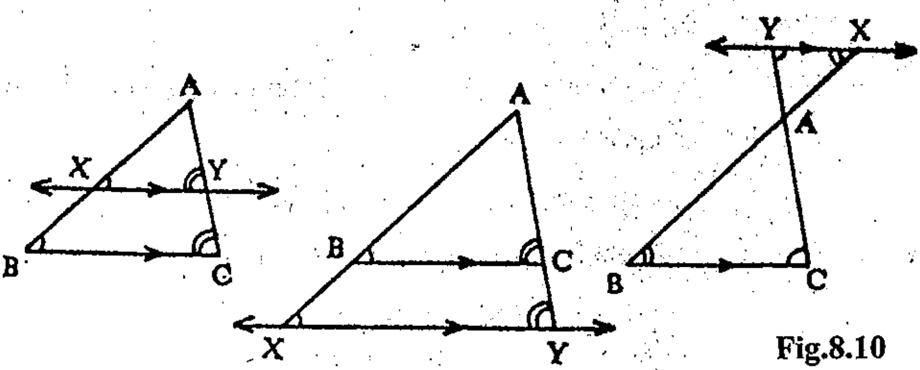


Fig.8.10

Thus we have the following corollary.

Corollary 2.2

If a line parallel to one side of a triangle determines a second triangle, then the second triangle will be similar to the original triangle.

Theorem 3 (The SAS Similarity Theorem)

If an angle of one triangle is equal to an angle of another, and the sides including these angles are proportional, then the triangles are similar.

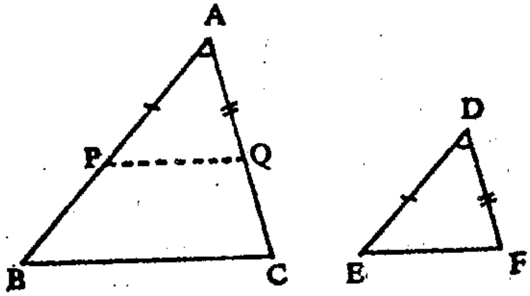


Fig. 8. 11

Given : Δ ABC and Δ DEF with $\angle A = \angle D$ and $\dfrac{AB}{DE}=\dfrac{AC}{DF}$

To Prove : $\Delta ABC \sim \Delta DEF$

Proof : Assume AB > DE, then AC > DF.

Let P and Q be points on AB and AC such that

AP= DE , AQ= DF

By SAS congruent theorem, we have

$$\Delta\ APQ \cong \Delta\ DEF, \text{ and}$$

$$\frac{AB}{DE} = \frac{AC}{DF} \text{ which becomes}$$

$$\frac{AB}{AP} = \frac{AC}{AQ}$$

By CBPT we have

$$PQ \parallel BC, \text{ and hence}$$

$$\angle B = \angle APQ$$

$$\text{Since } \angle A = \angle A$$

It follows by the AA Corollary that

$$\Delta\ ABC \sim \Delta\ APQ$$

$$\text{But } \Delta\ APQ \cong \Delta\ DEF, \text{ and so } \Delta\ APQ \sim \Delta\ DEF$$

$$\Delta\ ABC \sim \Delta\ DEF$$

The next theorem is the last of the three basic theorems on similar triangles.

Theorem 4 (The SSS Similarity Theorem)

If the corresponding sides of two triangles are proportional, then the triangles are similar.

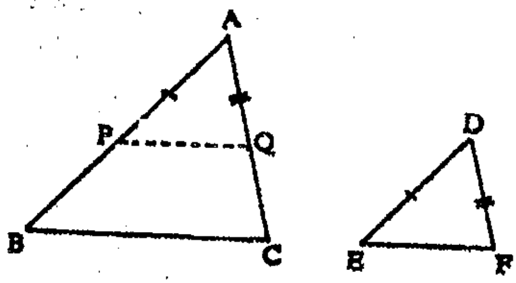


Fig 8.12

Given : ΔABC and ΔDEF with $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

To Prove : $\Delta ABC \sim \Delta DEF$

Proof : Assume AB > DE, then AC > DF

Let P and Q be points on AB and AC such that

AP = DE, AQ = DF,

$\frac{AB}{DE} = \frac{AC}{DF}$ becomes $\frac{AB}{AP} = \frac{AC}{AQ}$

and $\angle A = \angle A$

By SAS similarity theorem,we have

$$\Delta ABC \sim \Delta APQ$$

$$\frac{PQ}{BC} = \frac{AP}{AB} = \frac{DE}{AB} \text{ (AP = DE) and so}$$

$$PQ = BC \cdot \frac{DE}{AB} \quad \text{... (i)}$$

Since $\frac{AB}{DE} = \frac{BC}{EF}$, we have

$$EF = BC \cdot \frac{DE}{AB} \quad \text{... (ii)}$$

From (i) and (ii), we get PQ = EF

Thus $\Delta APQ \cong \Delta DEF$ (SSS Congruence Theorem)

Hence $\Delta ABC \sim \Delta DEF$.

We have now three answers to the question "When are two triangles similar?"

The answers are given in the proofs of theorems 2, 3 and 4. We notice that there is a correspondence between the triangle congruence and similarity theorems.

AAA Similarity corresponds to ASA Congruence.

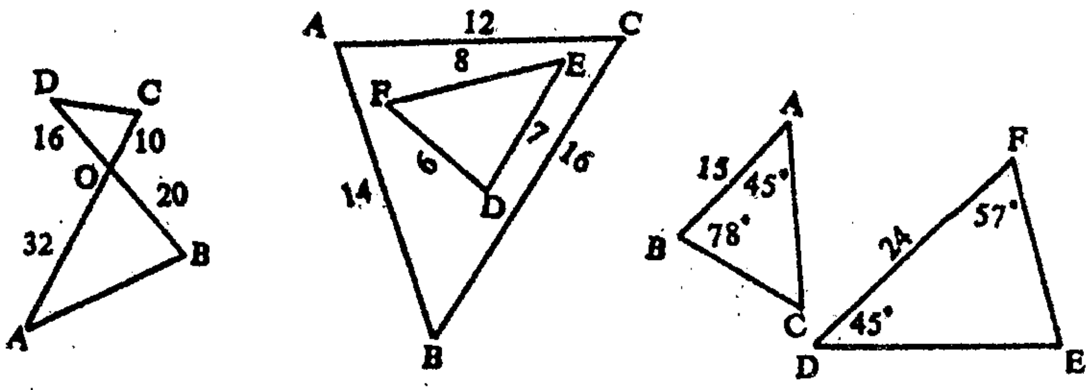
SAS Similarity corresponds to SAS Congruence.

SSS Similarity corresponds to SSS Congruence.

With the application of these three theorems, some of the most famous theorems in geometry can be proved.

Example 1. For each pair of triangles, indicate whether the two triangles similar or not. If they are similar, state why.

(a) (b) (c)



Solution

(a) Yes. By SAS Similarity Theorem

(For $\angle AOB = \angle DOC$, $\frac{AO}{OB} = \frac{8}{5} = \frac{OD}{OC}$)

(b) Yes. By SSS Similarity Theorem

(For $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA} = \frac{1}{2}$)

(c) Yes. By AAA Similarity Theorem

(For $\angle C = 180^\circ - (78^\circ + 45^\circ) = 57^\circ$

$\angle E = 180^\circ - (57^\circ + 45^\circ) = 78^\circ$

so that $\angle A = \angle D$, $\angle B = \angle ,E$ $\angle C = \angle F$)

Example 2. In the given figure, A, P, B and C, P, D are collinear.

(a) Explain why the triangles are similar and state the fact in the proper form.

(b) Find the value of x.

Solution

(a) $\Delta\ APC \sim \Delta\ DPB$ by SAS similarity theorem.

For $\angle APC = \angle\ DPB$ .... (1)

$$\frac{PA}{PD} = \frac{6}{9} = \frac{2}{3}$$

$$\frac{PC}{PB} = \frac{5}{7.5} = \frac{10}{15} = \frac{2}{3}$$

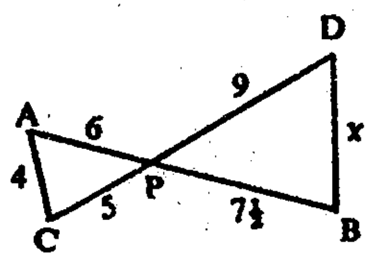
$$\frac{PA}{PD} = \frac{PC}{PB} \quad \text{... (2)}$$

(b) Since $\Delta APC \sim \Delta\ DPB$

$$\frac{BD}{CA} = \frac{PD}{PA}$$

$$\frac{BD}{4} = \frac{9}{6} = \frac{3}{2}$$

$$BD = 4(\frac{3}{2}) = 6 \text{ units}$$



When we were called upon to prove two segments congruent, we often were able to do so by showing that they were corresponding sides of congruent triangles. Similarly, to prove four segments are in proportion we can often do so by showing that they are corresponding sides of similar triangles.

We must exercise care in writing proportions based on similar triangles. Corresponding sides of similar triangles lie opposite the corresponding equal angles. We find it helpful to mark the equal angles before we attempt to write any proportions involving corresponding sides.

If $\Delta ABC \sim \Delta RST$, we can state

$$\frac{AB}{RS} = \frac{BC}{ST} = \frac{AC}{RT}$$

We can write this extended proportion without referring to a figure. The order of the letters in the notation "$\Delta ABC \sim \Delta\ RST$" tells us that:

AB corresponds to RS,

BC corresponds to ST,

AC corresponds to RT,

Study the Fig.8.13 and note how we identify the pairs of corresponding sides for the two triangles.

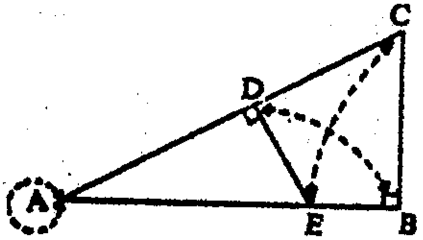


Fig . 8.13

Since $\angle A = \angle A$, we pair vertex A with vertex A. $\angle ADE$ and $\angle ABC$ are right angles, and hence $\angle ADE = \angle ABC$. We now pair vertex D with vertex B. The remaining vertices, E and C, are then paired.

$$A \longleftrightarrow A$$
$$D \longleftrightarrow B$$
$$E \longleftrightarrow C$$

We now state, with vertices in proper order, that

$$\Delta ADE \sim \Delta ABC$$

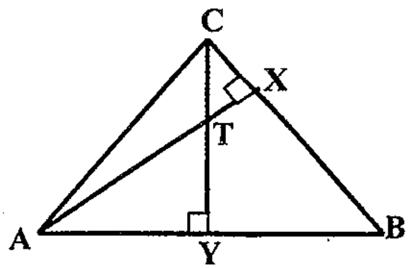
With this notation it is easy to write the correct extended proportion about the corresponding sides.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}$$

We can prove that the lengths of line segments are proportional by proving that they are corresponding sides of similar triangles. Try to find two triangles which have these segments as sides. A careful drawing will help you to find the correct triangles. Then prove that the triangles are similar.

Example 3.

Given : $AX \perp BC$; $CY \perp AB$



To Prove : $\frac{AY}{TY} = \frac{CX}{TX}$

Planning the proof : Try to find two triangles with sides AY, TY and CX, TX, and then prove these two triangles similar. In this problem we can find Δ AYT and ΔCXT.

Proof : In Δ AYT and Δ CXT

$\angle$ AYT = 90° = $\angle$CXT

$\angle$ATY = $\angle$CTX (vertically opposite angles)

By AA corollary, Δ AYT ~ Δ CXT.

Hence $\frac{AY}{CX} = \frac{TY}{TX}$

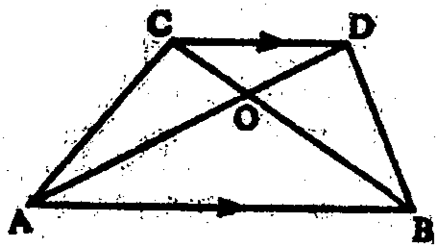
i.e. $\frac{AY}{TY} = \frac{CX}{TX}$.

On occasion we may be called upon to prove that the product of the lengths of two segments is equal to the product of the lengths of two other segments. To do this, we can often prove that two triangles are similar, write a proportion, and then apply a property of proportion. Example 4 illustrates this type of proof.

Example 4.

Given : CD // AB

To Prove : OB . OD = OA . OC



Planning the proof: Proving OB . OD = OA . OC is equivalent to proving

$$\frac{OB}{OA} = \frac{OC}{OD} \quad \text{or} \quad \frac{OB}{OC} = \frac{OA}{OD}$$

Now see whether there are two triangles, one of which has OB, OC or OB, OA as sides and the other has OA, OD or OC, OD as sides. In this case, we can find Δ AOB and Δ COD. Try to prove these two triangles similar.

Proof:

Since CD // AB, by the corollary 2.2, we have

Δ AOB $\sim$ Δ DOC

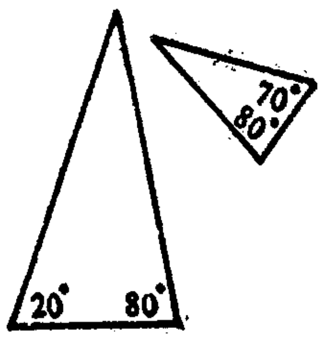
$$\frac{OB}{OC} = \frac{OA}{OD}$$

or $OB \cdot OD = OA \cdot OC$.

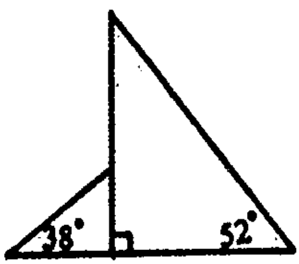
Exercise 8.3

1. Use the given information to tell whether each pair of triangles is similar. Give a reason for each answer.

(a)



(b)



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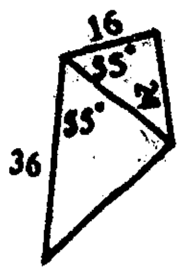
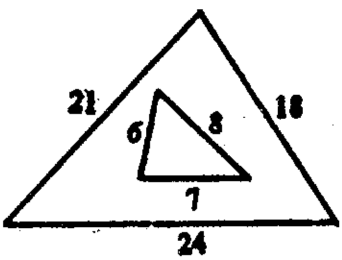
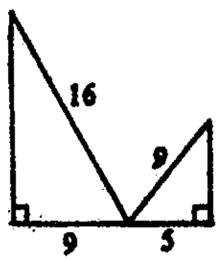


(d)



(e)



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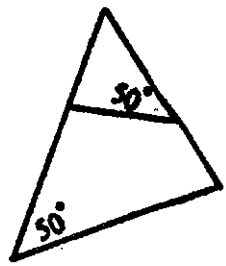
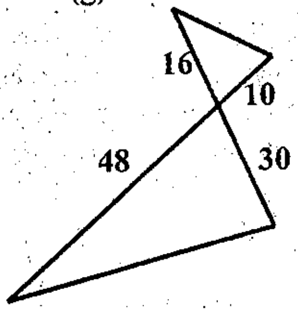
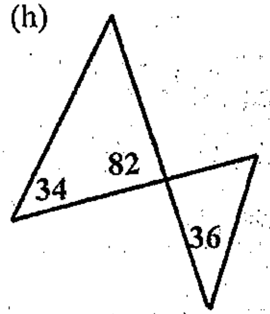


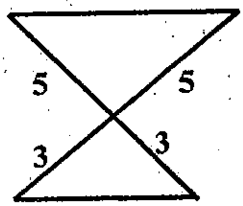
(g)



(h)

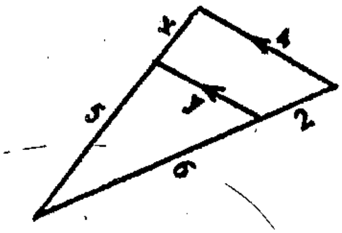


(i)

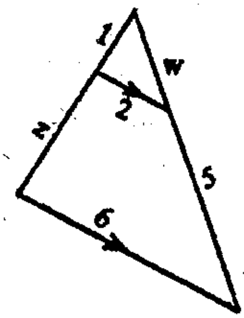


2. In each of the following triangles, the lengths of certain segments are marked. Find the values of x, y, z, w and v.

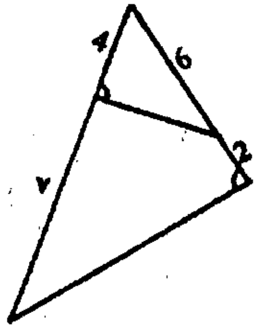
(a)



(b)



(c)

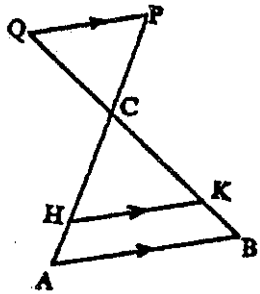


3. Name two ratios equal to the given ratios

(a) $\dfrac{HK}{AB}$

(b) $\dfrac{CH}{CK}$

(c) $\dfrac{CP}{PQ}$

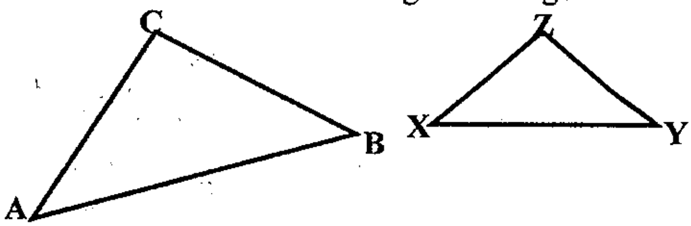


4. Suppose $\Delta\ ABC \sim \Delta\ PQR$. Then

(a) if $\dfrac{AB}{PQ} = \dfrac{BC}{x}$, what is x? (b) If $\dfrac{AB}{AC} = \dfrac{y}{PR}$, what is y?

5. How many pairs of corresponding angles are needed to prove two triangles similar if nothing is known about the sides?

6. Which conditions are sufficient to make the two given triangles similar?



(a) $\angle A = \angle X, \angle B = \angle Y$

(b) $\Delta ABC \cong \Delta XYZ$

(c) $\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$

(d) $AB = 20$, $BC = 16$, $AC = 12$,
$XY = 24$, $YZ = 20$, $XZ = 16$.

(e) $\angle A = \angle X$, $AC = 5$, $XZ = 6$, $AB = 10$, $XY = 12$.

(f) $XZ = \dfrac{5}{4} AC$, $YZ = \dfrac{5}{4} BC$, $\angle C = \angle Z$.

(g) $AC = XZ$, $BC = YZ$, $\angle A = \angle X$.

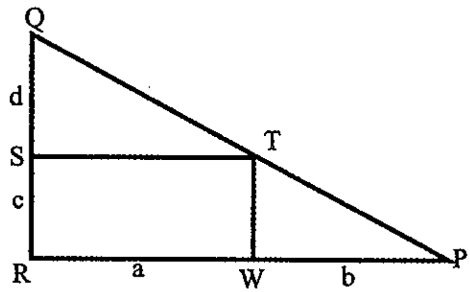
(h) Both the triangles are similar to the same triangle.

7. In the figure, RWTS is a rectangle, marked lengths are in cm.

(a) Find a relation connecting a, b, c, d.

(b) If $a = 8$, $b = 12$, $c = 4.8$, find d.

(c) If $c = 7$, $d = 3$, $a + b = 16$, find a.

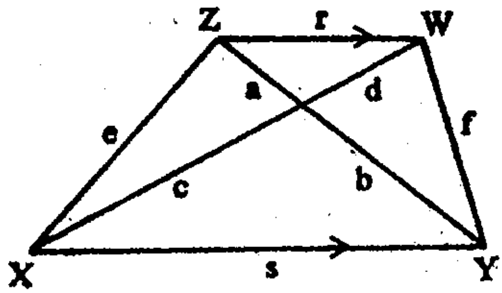


8. In the figure, XYWZ is a trapezium; marked lengths are in cm.

(a) Find the relations connecting a, b, c, d, r, s.

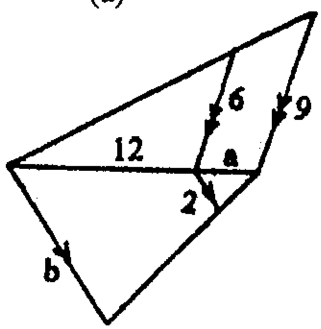
(b) If a = 7, d = 5, r = 10, s = 16, find all the other lengths you can.

(c) If YZ = 14, c = 12, d = 4, s = 15, find all the other lengths you can.

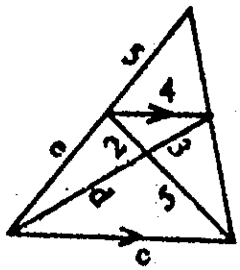


9. Find the marked lengths in each of the figures.

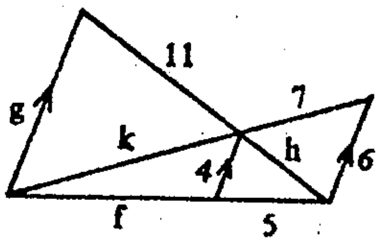
(a)



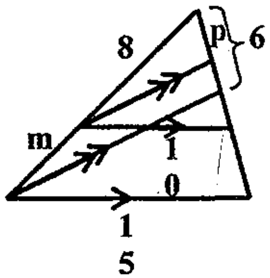
(b)



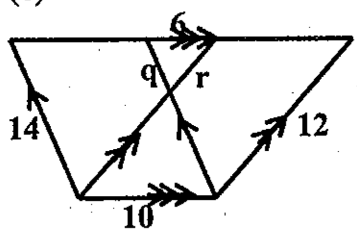
(c)



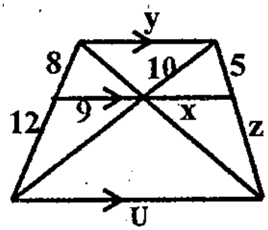
(d)



(e)



(f)

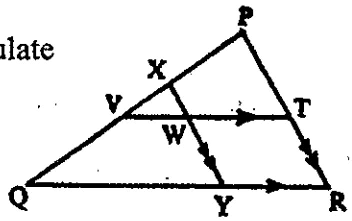


10. In the figure, XY // PR and VT // QR.

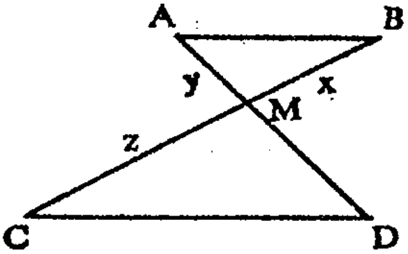
If $\frac{PT}{TR} = \frac{3}{2}$, $\frac{QY}{YR} = \frac{2}{1}$ and PQ = 15 cm, calculate

(a) the lengths of PV, PX and XV.

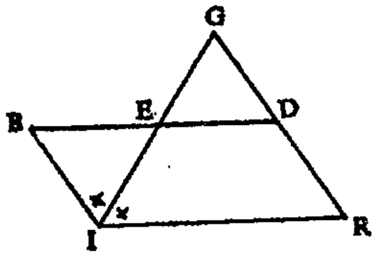
(b) the numerical values of $\frac{YW}{WX}$ and $\frac{VW}{QY}$.



1. In the figure, x, y and z are the lengths of MB, MA and MC. There are two possible lengths of MD for which the triangles are to be similar. Explain and compute these lengths.



12.

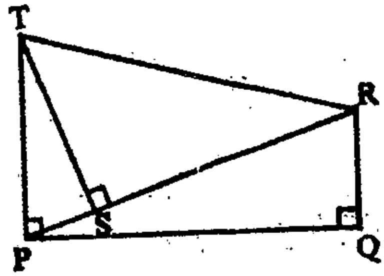


Given : Parallelogram BIRD ;

IG bisects $\angle$BIR.

Prove : $\dfrac{BE}{EI} = \dfrac{RG}{GI}$

13.

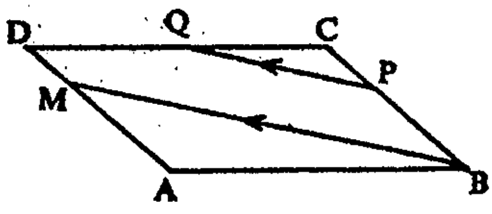


Given : RQ $\perp$ PQ, PQ $\perp$PT,

ST$\perp$PR.

Prove : ST $\cdot$ RQ = PS $\cdot$ PQ

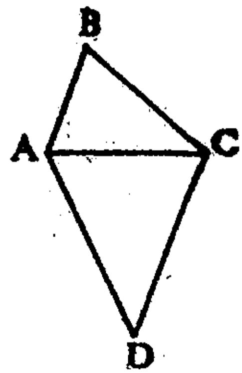
14.



Given : Parallelogram ABCD ;

PQ// MB.

15.



Prove : $\Delta ABM \sim \Delta CQP$.

ΔABC and Δ CAD are drawn on opposite sides of AC such that AB : BC : CA = CA : AD : DC. Prove that DC // AB.

16. Prove the following properties of similar triangles.

(a) The corresponding altitudes of similar triangles are proportional to any pair of corresponding sides.

(b) The corresponding medians of similar triangles are proportional to any pair of corresponding sides.

(c) The perimeters of two similar triangles are proportional to any pair of corresponding sides.

(d) The bisectors of corresponding angles of two similar triangles are proportional to any pair of corresponding sides.

(e) If $\Delta ABC \sim \Delta DEF$ and $\Delta DEF \sim \Delta PQR$, then $\Delta ABC \sim \Delta PQR$.

8.4 The Angle Bisector Theorem

Lines and angles commonly associated with triangles have many properties. Some of these are rather obvious while others are not. In this section we prove one of the many properties of triangles which is by no means obvious.

An idea to be used in this section is the notion of dividing a segment internally or externally in a given ratio.

Definition 2 : Let B be a point on the line containing segment AC. Then $\frac{AB}{BC}$ is called the ratio in which B divides AC.

Examples

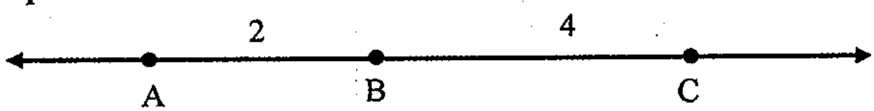
(1) 

Fig. 8. 14

Since $\frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$, B divides AC in the ratio $\frac{1}{2}$.

In this case, B is said to divide $\overline{AC}$ internally in the ratio 1 : 2.

(2)

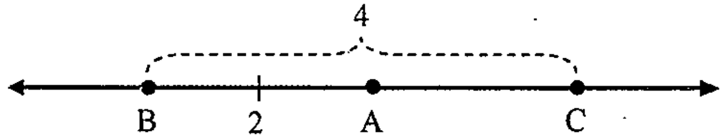


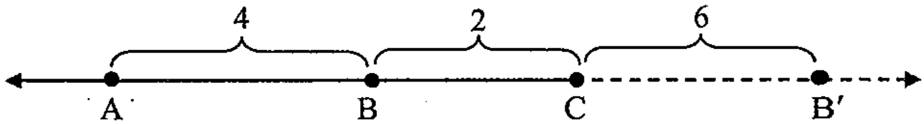
Fig.8.15

$$AB = 2,\ BC = 4,\ \frac{AB}{BC} = \frac{2}{4} = \frac{1}{2}$$

Thus B divides AC in the ratio 1:2.

In this case, B is said to divide AC externally in the ratio 1:2.

For a given ratio and segment there are usually two points which divide the segment in the given ratio. One is an internal point, the other is an external point. This is shown in Fig.8.16 for the ratio 2:1. B divides AC internally and B' divides AC externally.



$$\frac{AB}{BC} = \frac{AB'}{B'C} = \frac{2}{1}$$

Fig. 8.16

Many of us think that an angle bisector in a triangle also bisects the opposite side. This is true only for isosceles triangles. In general the opposite side is divided internally in a specific ratio.

Theorem 5.A (The Angle Bisector Theorem-ABT)

The bisector of an interior angle of a triangle divides the opposite side internally into a ratio equal to the ratio of the other two sides of the triangle.

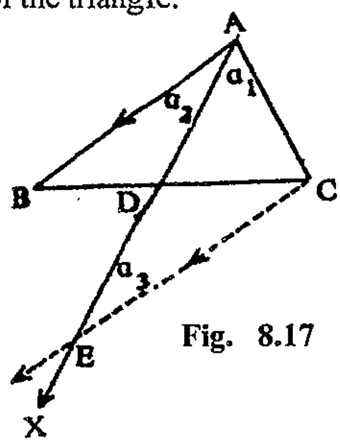


Fig. 8.17

Given : AX bisects $\angle BAC$.

To Prove : $\dfrac{AB}{AC} = \dfrac{BD}{DC}$

Proof : Draw CE // AB, intersecting AX at E.

Since CE // AB,

$a_3 = a_2'$, and then

$a_3 = a_1$, $(a_2 = a_1)$

Thus AC = CE

and also Δ ABD ~ Δ ECD (why?)

Hence $\dfrac{AB}{EC} = \dfrac{BD}{CD}$ or $\dfrac{AB}{AC} = \dfrac{BD}{DC}$

Theorem 5.B

The bisector of an exterior angle of a triangle divides the opposite side externally into a ratio equal to the ratio of the other two sides of the triangle.

(Hint : To prove $\dfrac{BD'}{D'C} = \dfrac{AB}{AC}$, draw CE // D'A,

and first prove AE = AC, then show that $\dfrac{BD'}{D'C} = \dfrac{BA}{AE}$)

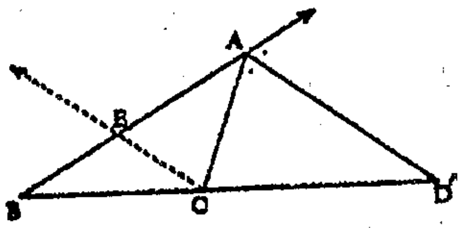


Fig. 8.18

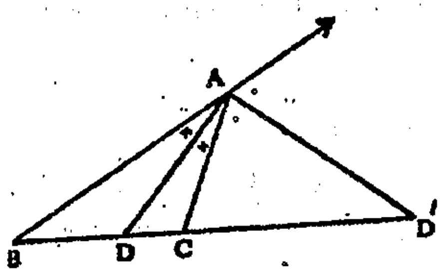


Fig. 8.19

From the Angle Bisector Theorem and its corollary, we notice that

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{BD'}{D'C}$$

or $$\frac{BD}{DC} = \frac{BD'}{D'C}$$

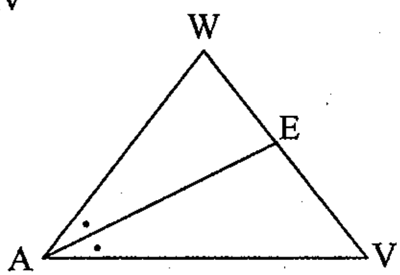
i.e, the segment BC is divided internally and externally in the same ratio at D and D'. The segment BC is said to be divided **harmonically** at D and D'.

Exercise 8.4

1. Which of the following proportions follow from the fact that AE bisects $\angle$ WAV in Δ WAV?

(a) $\dfrac{WE}{EV} = \dfrac{WA}{AV}$ (b) $\dfrac{WE}{EV} = \dfrac{VA}{AW}$

(c) $\dfrac{WE}{WA} = \dfrac{EV}{AV}$ (d) $\dfrac{AV}{AW} = \dfrac{VE}{EW}$

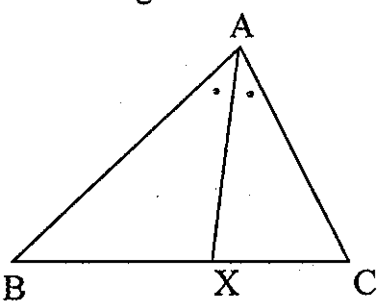


2. AX bisects $\angle$ CAB. Complete the following statements:

(a) AC : AB =...

(b) AB : AC = ...

(c) XC : XB = ...

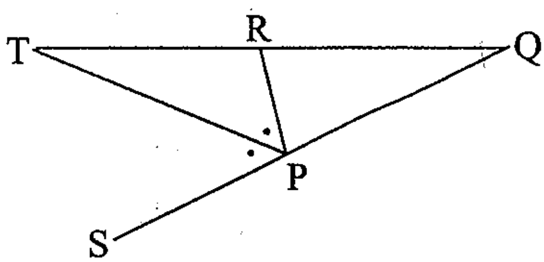


3. PT bisects $\angle$ RPS. Complete the following statements:

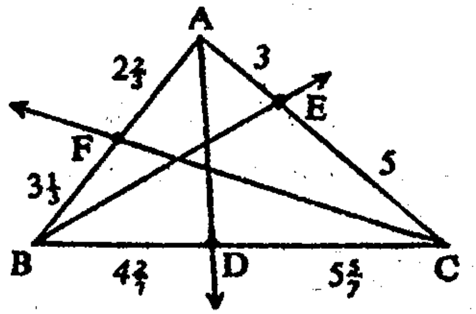
(a) PQ : PR = ...

(b) TR : PR = ...

(c) QR : TR = ...



4. What can you say about the rays AD, BE and CF?



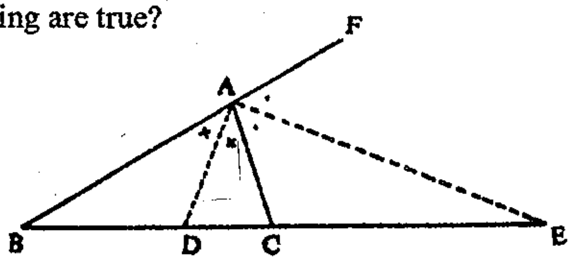
5. If AD and AE are bisectors of the interior and exterior angles at A of Δ ABC, then which of the following are true?

(a) $\angle$ DAE = 90°

(b) BD : DC = BC : CE

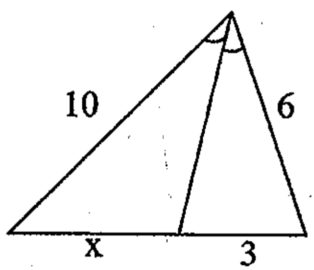
(c) BD : DC = BE : CE

(d) AD : AE = DC : CE

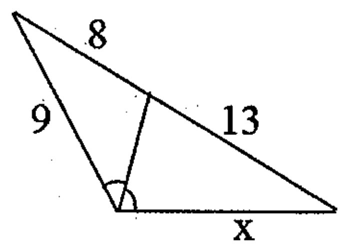


6. Find x in each of the following figures.

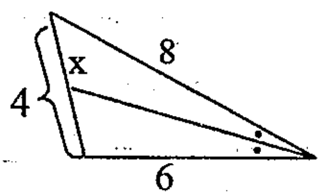
(a)



(b)



(c)



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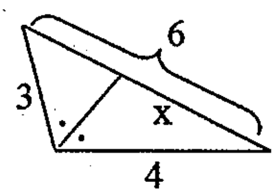
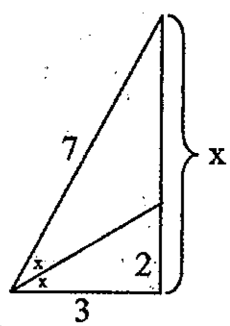
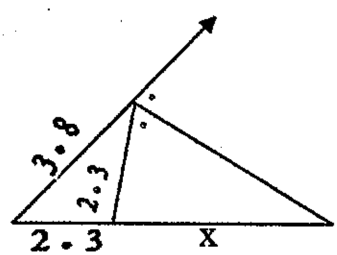


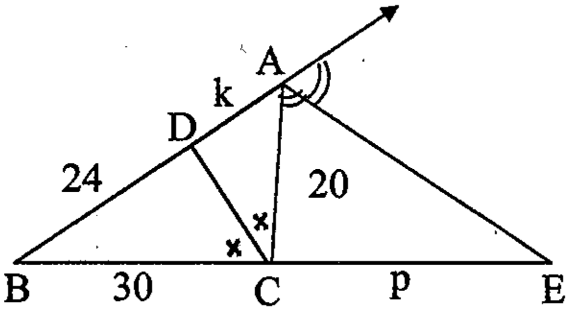
(e)



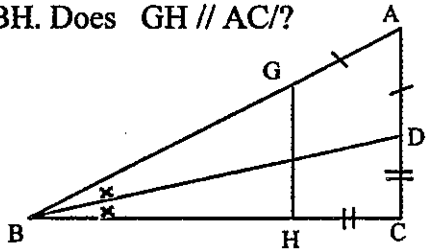
(f)



7. Find the unknown marked lengths in the figure.



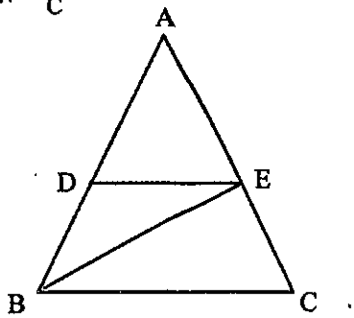
8. AB = 12 cm, BC = 9 cm, CA = 7 cm. BD bisects $\angle$B and AG = AD, CH=CD. Calculate BG, BH. Does GH // AC/?



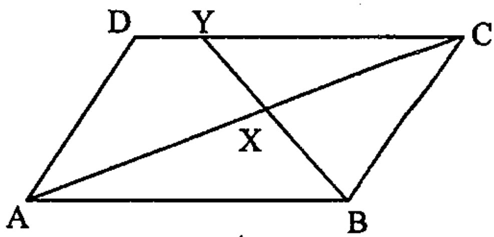
9. In Δ ABC, DE // BC, AD = 2.7 cm.

DB = 1.8 cm and BC = 3 cm.

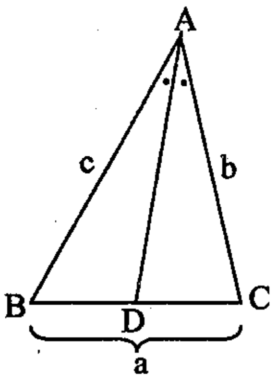
Prove that $\overline{BE}$ bisects $\angle$ABC.



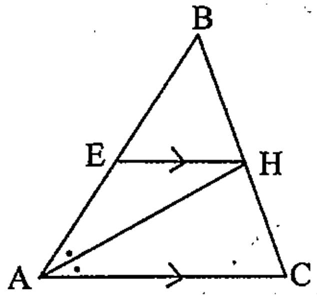
10. In parallelogram ABCD, AB = 3.6 cm, BC =2.7 cm. AX = 3.2 cm, XC = 2.4 cm. Prove that Δ BCY is isosceles.



11. Calculate BD and DC in terms of a, b, c.



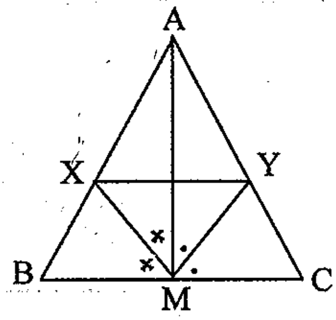
12.



Given : AH bisects $\angle BAC$ in Δ ABC. EH//AC

Prove : $\dfrac{BE}{EA} = \dfrac{BA}{AC}$

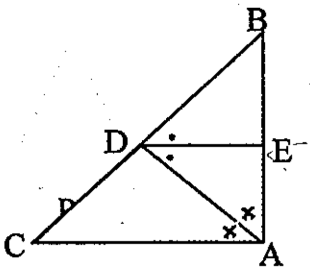
13.



Given :In Δ ABC, BM = MC; MX bisects $\angle AMB$, MY bisects $\angle AMC$,

Prove : XY //BC

14.

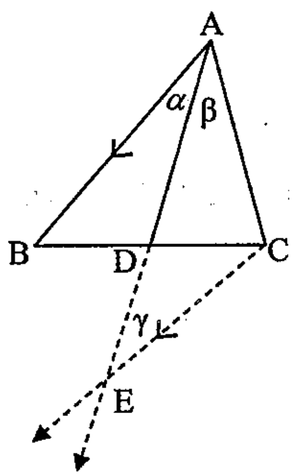


Given : In ΔABC, $\angle A = 2\angle C$, AD bisects $\angle BAC$ and DE bisects $\angle ADB$.

: $\dfrac{BE}{EA} = \dfrac{BA}{AC}$

15. Supply the reasons for the following proof of the Converse of the Angle Bisector Theorem (CABT)

If a ray from the vertex of one angle of a triangle divides the opposite side into segments that have the same ratio as the other two sides, then it bisects the angle.



Given : Δ ABC with AD such that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

To Prove : AD bisects $\angle BAC$.

Proof : Draw CE // AB.

Let AD and CE intersect at E.

$\Delta ABD \sim \Delta ECD$ (why?)

$\dfrac{AB}{EC} = \dfrac{BD}{CD}$ (why?)

$$\text{But} \quad \frac{AB}{AC} = \frac{BD}{DC} \quad \text{(why?)}$$

$$\frac{AB}{AC} = \frac{AB}{CE} \quad \text{(why?)}$$

$$AC = CE$$

$$\gamma = \beta \quad \text{(why?)}$$

$$\text{but} \quad \alpha = \gamma \quad \text{(why?)}$$

$$\alpha = \beta.$$

8.5. The Pythagoras Theorem

With our knowledge of similar triangles, we can prove the most famous theorem in Geometry, which is attributed to the Greek mathematician Pythagoras, who lived in the 6 th century B. C. This theorem gives a relationship between the three sides of a right triangle. The first proof of this theorem is usually attributed to the Pythagoreans (a sect founded by Pythagoras).

This theorem is an indispensable tool for practical calculations ; that is why we may have been using it for a long time without formally proving it. First, we state and prove a theorem that is useful for the proof of the Pythagoras Theorem.

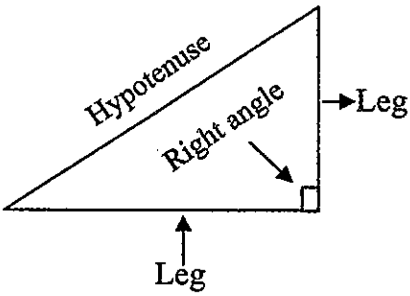


Fig. 8.20

Recall the definition that a triangle with a right angle is called a right triangle and the sides which determine the right angle are called legs of the right triangle, and the side opposite the right angle is called the hypotenuse.

Theorem 6

The altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original triangle.

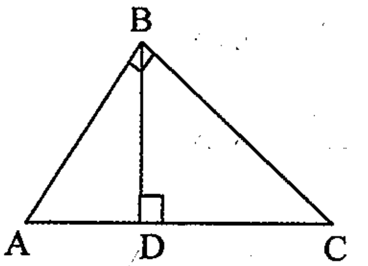


Fig. 8.21

Given : Right triangle ABC with altitude BD to hypotenuse AC.

To Prove : $\Delta ADB \sim \Delta BDC \sim \Delta ABC$

Proof : Since Δ ADB and Δ ABC are right triangles with a common acute angle at A, $\Delta ADB \sim \Delta ABC$. (AA corollary)

Similarly Δ BDC and ΔABC have a common acute angle at C.

$\Delta BDC \sim \Delta ABC$

Hence $\Delta ADB \sim \Delta BDC \sim \Delta ABC$.

From the similarity of Δ ADB and Δ BDC, we can write $\frac{AD}{BD} = \frac{BD}{CD}$, which suggests the following corollary.

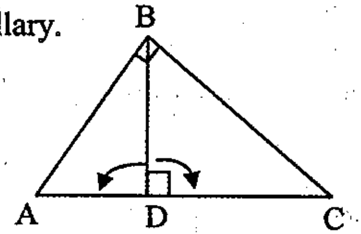


Fig. 8.22

Corollary 6.1

The altitude to the hypotenuse of a right triangle is the **geometric mean** of the segments into which it separates the hypotenuse.

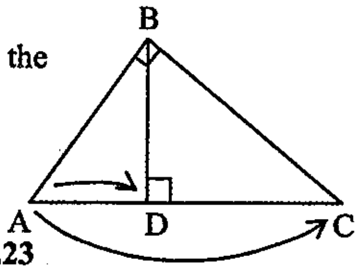


Fig.8.23

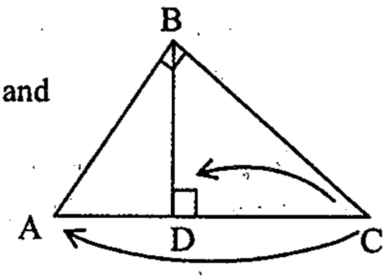


Fig.8.24

Considering the similarities of Δ ADB, Δ ABC and Δ BDC, Δ ABC we get the following proportions :

$$\frac{AD}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{DC}{BC} = \frac{BC}{AC}$$

or

$AB^2 = AD.AC$, $BC^2 = DC.AC = CD.CA$

Corollary 6.2

Each leg of a right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Now we will state and prove the most famous theorem in geometry.

Theorem 7. (Pythagoras Theorem)

In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides

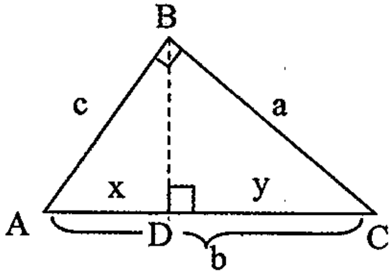


Fig. 8.25

Given : Δ ABC is a right triangle with $\angle B = 90^\circ$

To Prove : $b^2 = c^2 + a^2$

Proof : Draw $BD \perp AC$,

Then $\Delta ADB \sim \Delta BDC \sim \Delta ABC$ (Theorem 6)

$$c^2 = xb$$

and $a^2 = yb$

By addition,

$$xb + yb = c^2 + a^2$$
$$b(x+y) = c^2 + a^2$$
$$b(b) = c^2 + a^2$$
$$b^2 = c^2 + a^2.$$

The converse of the Pythagoras Theorem provides a way of showing whether or not a triangle is a right triangle. It is stated here without proof.

Converse of Pythagoras Theorem

If a triangle has sides with lengths a, b and c and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

8.6 Special Right Triangles

There are two special types of right triangles that are of particular interest. One is the isosceles right triangle ; such a triangle is formed by two sides and a diagonal of a square (Fig. 8.26 (a)). The other type is the right triangle with acute angles of measures 30° and 60° ; an altitude of an equilateral triangle determines two such triangles. (Fig. 8.26(b)).

(a) **Fig. 8.26** (b)

The following theorems are based on Pythagoras Theorem, and therefore their proofs are left as exercises. There are frequent opportunities in geometry to apply these two theorems.

Theorem 8.

In a 45° – 45° right triangle, the length of hypotenuse is equal to the length of each leg times $\sqrt{2}$. (Fig. 8.27)

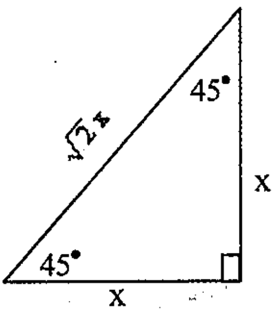


Fig. 8.27

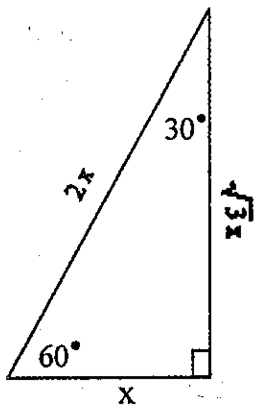
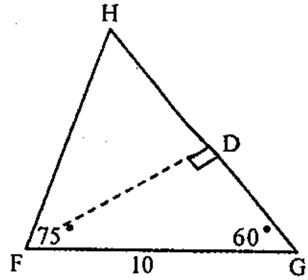


Fig. 8.28

Theorem 9.

In a $30^\circ - 60^\circ$ right triangle, the leg opposite the 30° angle is one-half the length of the hypotenuse, and the other leg is equal to the length of the hypotenuse times $\frac{\sqrt{3}}{2}$. (Fig. 8.28)



Example 1.

Given :$FG = 10$; $\angle F = 75^\circ$; $\angle G = 60^\circ$

To find : GH and FH

Solution : Draw $FD \perp HG$

Then ΔFGD is a $30^\circ - 60^\circ$ right triangle.

Hence $DG = \frac{1}{2}(10) = 5$

$FD = \frac{\sqrt{3}}{2}(10) = 5\sqrt{3}$

and $\angle HFD = 75^\circ - 30^\circ = 45^\circ$

$\angle FDH = 90^\circ$

Δ FHD is a $45^\circ - 45^\circ$ right triangle.

Hence $HD = FD = 5\sqrt{3}$

$FH = \sqrt{2}\ FD = \sqrt{2}\ (5\sqrt{3}) = 5\sqrt{6}$

and $GH = DG + HD = 5 + 5\sqrt{3}$.

Exercise 8.5

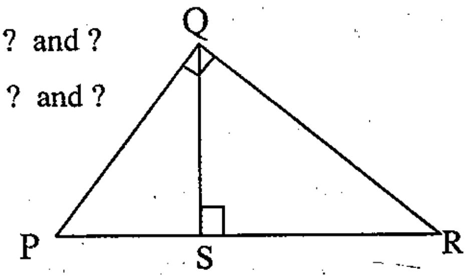
1. In the figure $\angle PQR = 90^\circ$, $QS \perp PR$. Complete each of the following true statements.

   (a) $\Delta PQR \sim \Delta ? \sim \Delta ?$

   (b) QS is the geometric mean between ? and ?

   (c) QR is the geometric mean between ? and ?

   (d) $\dfrac{?}{PQ} = \dfrac{PQ}{?}$



2. Use the right triangle PQR for problem 1. State the definition, theorem, or corollary that supports each conclusion.

   (a) $\Delta PQR \sim \Delta PSQ$

   (b) $\dfrac{QS}{PS} = \dfrac{SR}{QS}$

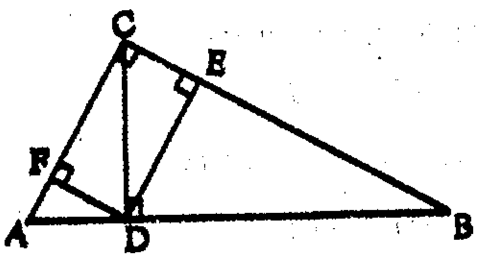
   (c) $\dfrac{PS}{PQ} = \dfrac{PQ}{PR}$

   (d) $\dfrac{RQ}{PR} = \dfrac{QS}{PQ}$

   (e) $\Delta PRQ \sim \Delta QRS$

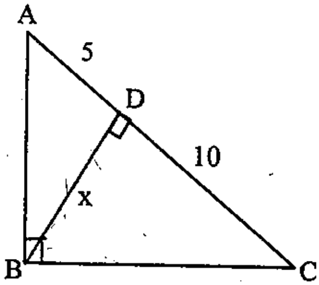
   (f) $\Delta PQS \sim \Delta QRS$

3. In the figure $CD \perp AB$ and $\angle C = 90^\circ$. If $DE \perp BC$, $DF \perp CA$, write out all the triangles that are similar to ΔABC.

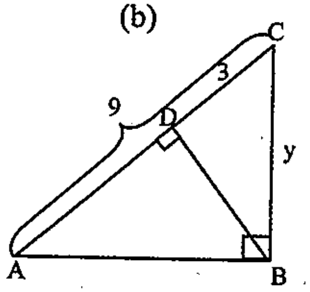


4. Find the length of each marked segment.

(a)



(b)



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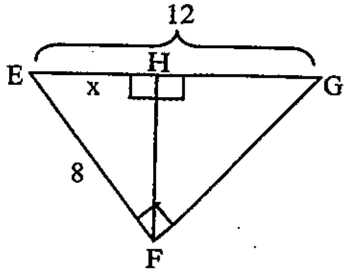

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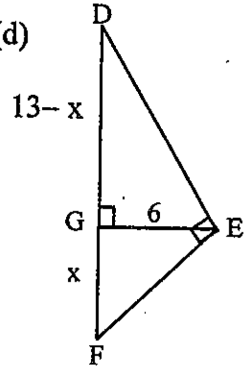

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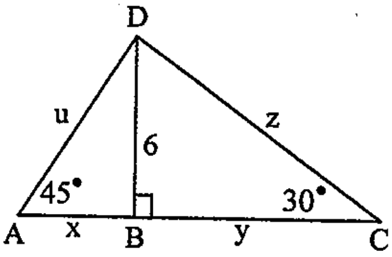

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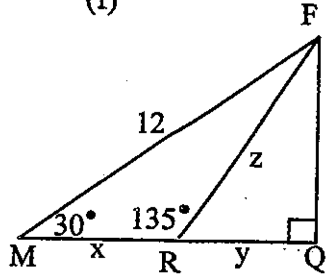


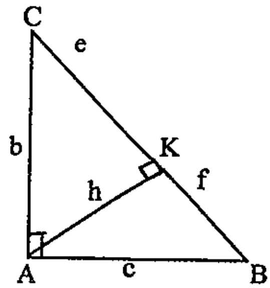
5. In the figure AK is the altitude to the hypotenuse of ΔABC.

(a) If $e = 5$, and $h = 15$, find f, b and c.

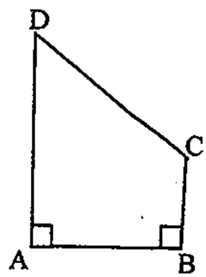
(b) If $b = 4\sqrt{3}$ and $e = 4$, find f, h and c.

(c) If $b = 3\sqrt{10}$ and $f = 13$, find e, h and c.

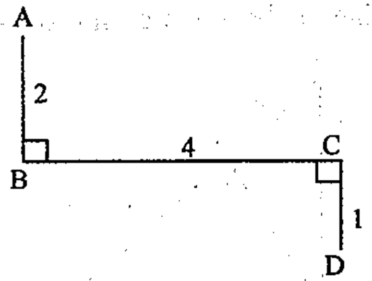
(d) If $b = f = 8$, find e, h and c.



6. In the figure, if AD = 10 cm, AB = 8 cm, BC = 4 cm, find the length of CD.

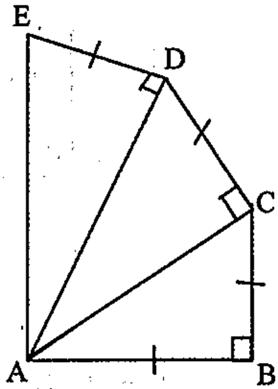


7. In the figure, find the distance of D from A.

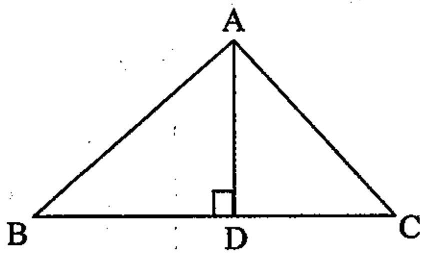


8. A parallelogram with sides 8 cm and 15 cm has a diagonal of 17 cm. Is it a rectangle?

9. In the figure, Δ ABC, Δ ACD, Δ ADE are right triangles and AB = BC = CD = DE. Show that AE = 2AB.



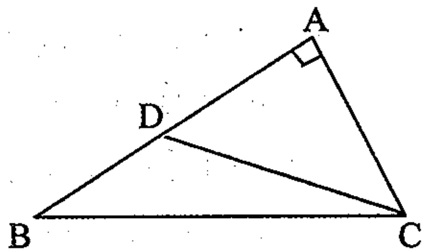
10.



Given : $AD \perp BC$

Prove : $AB^2 - AC^2 = BD^2 - DC^2$

11.

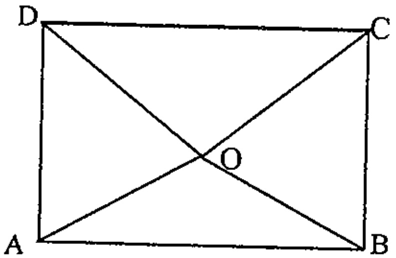


Given : $\angle BAC = 90^\circ$

D is any point on AB.

Prove : $BC^2 + AD^2 = AB^2 + CD^2$

12.

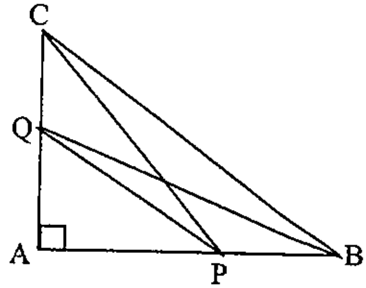


Given : O is any point within a rectangle ABCD.

Prove : $OA^2 + OC^2 = OB^2 + OD^2$

(Hint : use problem 11.)

13.



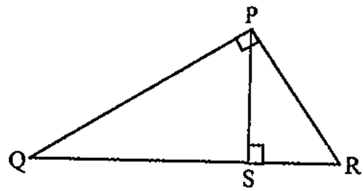
Given : $\angle A = 90^\circ$; P and Q are any two points on AB and AC

Prove : $BQ^2 + CP^2 = BC^2 + PQ^2$

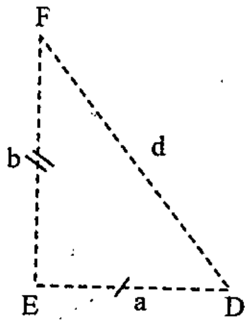
(Hint : use problem 11.)

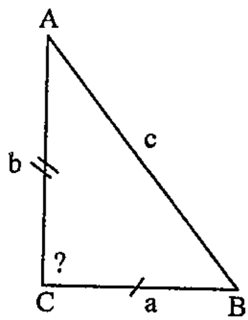
14. Given : $\angle QPR = 90^\circ$, $PS \perp QR$.

Prove : $\dfrac{1}{PS^2} = \dfrac{1}{PQ^2} + \dfrac{1}{PR^2}$



15. Complete the following proof of the Converse of Pythagoras Theorem.





Given : In ΔABC, $c^2 = a^2 + b^2$

To Prove : $\angle C = 90^\circ$

Proof : Construct the auxiliary triangle DEF such that

$\angle E = 90^\circ$, DE = ..., and EF = ...

(a) Why is $d^2 = a^2 + b^2$?

(b) Why is $c^2 = a^2 + b^2$?

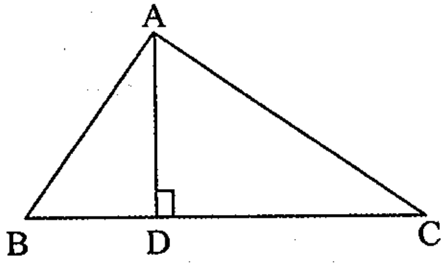
(c) How does it follow that $d^2 = c^2$?

(d) Why is $d = c$?

(e) Why is $\Delta ACB \cong \Delta FED$?

(f) What can you conclude about $\angle C$, and hence ΔACB as a result?

16.

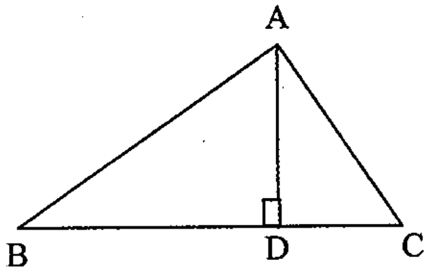


Given : In ΔABC, $AD \perp BC$,

$BC^2 - 2AD^2 = BD^2 + DC^2$

Prove : $\angle BAC = 90^\circ$

17.



Given : In ΔABC, $AD \perp BC$

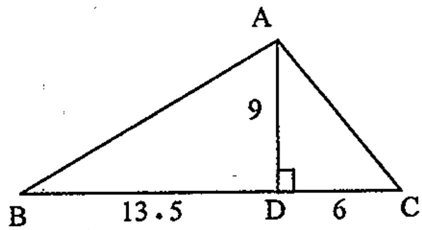
$AD^2 = BD \cdot DC$

Prove : $\angle BAC = 90^\circ$

18. Given : $AD \perp BC$, $AD = 9$ cm,

$BD = 13.5$ cm, $DC = 6$ cm.

Prove : $\angle BAC = 90^\circ$.



SUMMARY

1. In any triangle

(a) a line parallel to one side and intersecting the other two sides divides them proportionally. (BPT)

(b) a ray that bisects an angle divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides. (ABT)

2. In two similar triangles

   (a) corresponding angles have equal measures.

   (b) the lengths of corresponding sides are in proportion.

   (c) the ratio of the lengths of two corresponding altitudes is equal to the ratio of the lengths of two corresponding sides.

3. Two triangles are similar

   (a) if two angles of one triangle are congruent to two angles of the other triangle. (AA)

   (b) if the lengths of two sides of a triangle are proportional to the lengths of two sides of another triangle and the included angles are congruent. (SAS)

   (c) if the lengths of their corresponding sides are proportional. (SSS)

4. In any right triangle

   (a) the altitude to its hypotenuse forms two right triangles that are similar to each other and to the given triangle.

   (b) the altitude drawn to the hypotenuse is geometric mean of, or (the mean proportional between) the segments of the hypotenuse.

   (c) each leg is the mean proportional between the hypotenuse and the segment of the hypotenuse, adjacent to the leg.

   (d) the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. (Pythagoras Theorem).

5. (a) In a $30^\circ - 60^\circ$ right triangle, the shorter leg is $\frac{1}{2}$ times as long as the hypotenuse.

   (b) In an isosceles right triangle, the hypotenuse is $\sqrt{2}$ times as long as each of the legs.

CHAPTER 9

Circles, Chords and Tangents

The circle and the lines associated with it will be studied in this chapter.

The circle has many properties which no other plane figure possesses. Of all the plane figures, the circle is the only one which can be rotated about a point without changing its position. It is perhaps the most appealing of all simple geometric figures. Plato called it the "perfect shape".

The symmetry of the circle makes it a beautiful shape, and most people find it appealing to the eyes. Have you ever considered how useful the circle is? Without the circle there would be, for example, no wheels, bicycles, automobiles. But the circle has been a source of difficulty to mathematicians and other thinkers. For example, many generations wondered exactly how many times a circle contained its diameter. In other words, they wanted to know more about the number we call pi which is written as π.

Not until the notation of irrational numbers was admitted for consideration, did the identity of pi as a nonterminating, nonrepeating decimal come to light. In the mean time, mathematicians had spent many fruitless hours trying to resolve impossible situations, including one of the most famous of classic geometry problems: how to construct with geometric tools a square with area identical to a given circle. This problem is popularly known as **"squaring the circle"**.

Though pi has been properly identified, mathematicians are still bothered by some by things concerning the circle. For example, it is easy to see that if the number of sides of a regular polygon is increased without bound, it becomes indistinguishable from a circle. The question then is: "Does it indeed become a circle? if so, exactly when does it cease, to be a polygon and become a circle?"

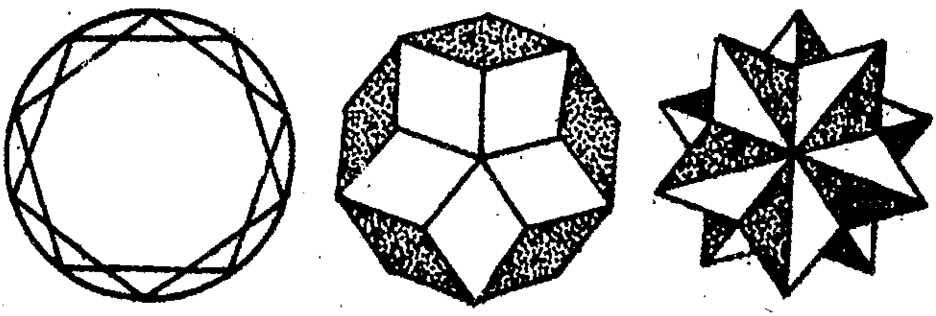


Fig. 9.1

The designs shown in Fig. 9.1 on the last page are based on a circle. See if you can draw these designs.

9.1 Circles, Radii and Chords

Roughly speaking, a circle is the boundary of a round region in a plane. Here are some questions and answers to help you recall some basic ideas about circles.

What is the basic classification for a circle? A simple closed path.

Are all the points of the circle in one plane? Yes.

Are all points of the circle equidistant from a point inside? Yes.

What is this point inside called? The centre.

Is the centre point on the circle? No.

What is the distance from the centre to a point of the circle called? The radius.

These questions and answers suggest the following definition.

Definition 1 : A circle is the set of all points in a plane that are at a given distance from a given point. The given point is called the **centre** of the circle and the given distance is called the **radius** of the circle.

We use the symbol $\odot$ and the centre point to name a circle. Thus Fig 9.2 shows $\odot$ P, read as "circle P".

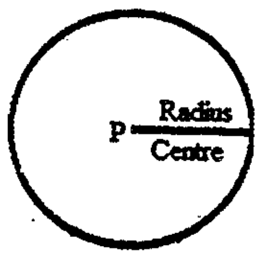
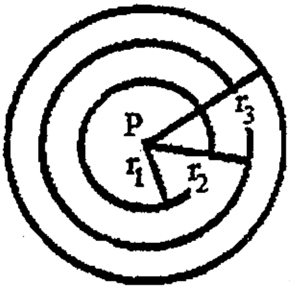


Fig. 9.2

Circles that have the same radius are called congruent circles. Circles with the same centre but different radii are called concentric circles.



In Fig. 9.3 P is the common centre of the three concentric circles.

Fig. 9.3

Definition 2. A chord of a circle is a segment whose end points lie on the circle. A **diameter** of a circle is a chord containing the centre of the circle. A **secant** is a line that intersects a circle in two points.

Thus every chord determines a secant, and every secant contains a chord. Each of the words "diameter", "radius" and "chord" is used in two ways. It refers to both a line segment and the number that represents its length.

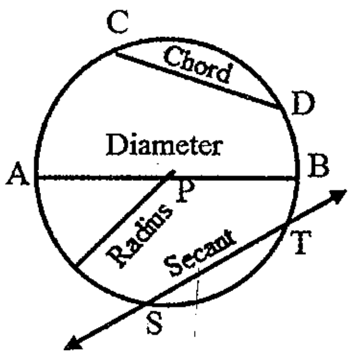


Fig. 9.4

9.2 Properties of Chords

Many of the basic properties of chords in a circle are illustrated in Fig. 9.5. Δ APB is isosceles. If the radius PD is perpendicular to the chord AB, we can easily prove that it bisects the chord. Conversely, if the radius bisects the chord, we can show that it must be perpendicular to the chord.

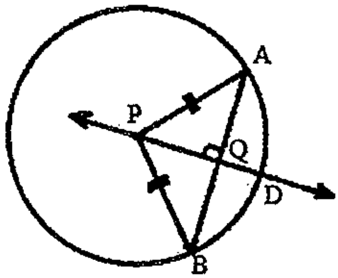


Fig. 9.5

Symbolically

if PD $\perp$ AB, then AQ = QB and

if AQ = QB, then PD $\perp$ AB

We will state these relationships, along with a third one, as theorems.

Theorem 1

If a line through the centre of a circle is perpendicular to a chord, it also bisects the chord.

Theorem 2

If a line through the centre of a circle bisects a chord, it is also perpendicular to the chord.

Theorem 3

The perpendicular bisector of a chord of a circle passes through the centre.

Theorem 3 provides a means for constructing the centre of a circle. Since each perpendicular bisector of a chord contains the centre, two such perpendicular bisectors will intersect at the centre. The steps in the construction are shown in Fig. 9.6.

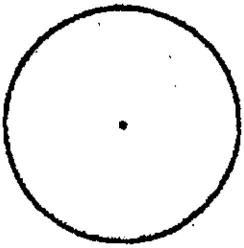
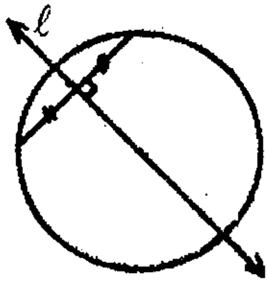
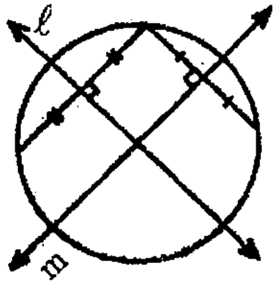
 Give : A circle	 **Step 1** Construct a chord and its perpendicular bisector ℓ	 **Step 2** Construct a second chord and its perpendicular bisector m. The intersection of ℓ and m is at the centre of the circle.

Fig. 9.6

Corollary 3.1

No circle contains three collinear points.

If three points P, Q, R of the circle were collinear, bisectors of the chords PQ and QR, would be parallel. This is impossible, because both lines pass through the centre.

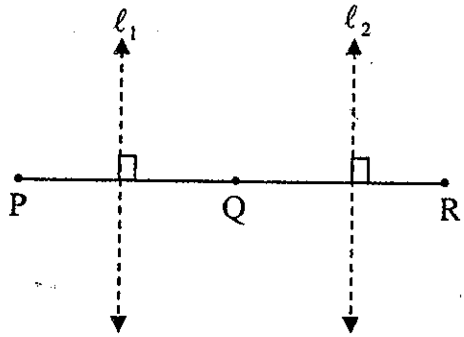


Fig 9.7

By using theorems on congruent triangles, the following theorems can be proved.

Theorem 4.

In the same circle or in congruent circles, chords equidistant from the centre are equal.

Theorem 5.

In the same circle or in congruent circles, any two equal chords are equidistant from the centre.

We can combine theorems 4 and 5 as a single theorem using "if and only if" as follows:

In a circle or congruent circles, chords are equal if and only if they are equidistant from the centre of the circle.

Now let us consider how chords of different lengths in congruent circles are related.

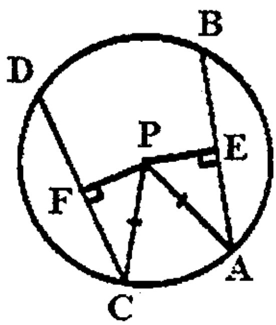


Fig. 9.8

Theorem 6

In a circle or congruent circles, if one of two chords is the larger chord, then it is nearer the centre of the circle.

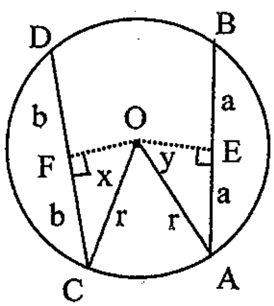


Fig. 9.9

Proof : Consider the lengths of the chords as 2a and 2b where 2b > 2a. Call their distance from the centre of the circle x and y, respectively.

Applying Pythagoras theorem to the right triangles AEO and CFO,

$r^2 = y^2 + a^2$ and

$r^2 = x^2 + b^2$

$x^2 + b^2 = y^2 + a^2$

$x^2 - y^2 = a^2 - b^2$

Since $0 < a < b$, $a^2 < b^2$

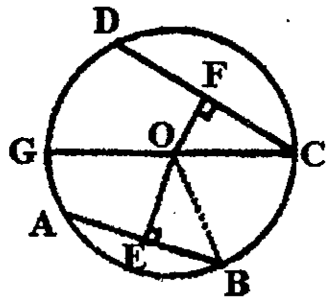
and $a^2 - b^2$ is negative.

Hence $x^2 - y^2$ is negative.

$x^2 < y^2$

Hence $x < y$.

Example 1. In the given figure, CG is the diameter of the given circle. AB = 6, CD = 8 and OE = 4



(a) Find OF.

(b) Find the length of the diameter .

(c) Supply the correct inequality sign between OC and OE.

Solution

(a) Join OB.

Since $OE \perp AB$, $AE = EB = \frac{1}{2}AB = \frac{1}{2}(6) = 3$

From the right triangle OEB

$OB^2 = OE^2 + EB^2 = 4^2 + 3^2 = 25$

Therefore $OB = \sqrt{25} = 5$, and $OC = OB = 5$

Since $OF \perp CD$, $CF = FD = \frac{1}{2}CD = \frac{1}{2}(8) = 4$

Now, from the right triangle OFC

$OF^2 = OC^2 - CF^2 = 5^2 - 4^2 = 9$

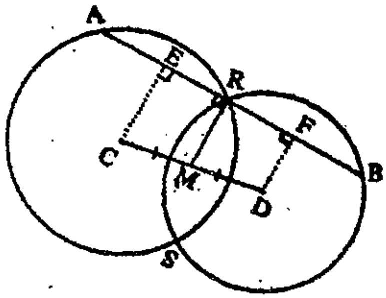
Therefore OF = 3.

(b) $CG = 2\,(OC) = 2\,(5) = 10$.

(c) $OB = 5$ and $OE = 4$, $OB > OE$, $OC > OE$ ($OB = OC$).

Example 2. Two circles of unequal radii, intersect at points R and S. M is the midpoint of CD, where C and D are the centres of the circles. A line through R perpendicular to MR intersects the circles again at A and B. Prove that AR = BR.

Solution



Given : $\odot$ C and $\odot$ D intersect at R and S ; CM = MD and ARB $\perp$ MR.

To Prove : AR = BR

Proof : Introduce CE and DF both perpendicular to AB.

Since CE $\perp$ AB, DF $\perp$ AB and MR $\perp$ AB, CE // MR// DF.

and CM = MD (given)

Hence ER = RF (by the Equal Intercept Theorem)

But $ER = \frac{1}{2} AR$

$RF = \frac{1}{2} BR$

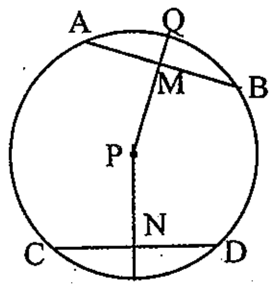
$\frac{1}{2} AR = \frac{1}{2}\ BR$

or AR = BR.

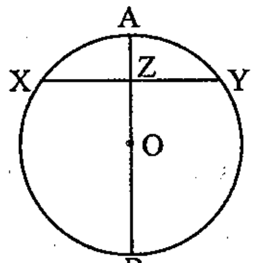
We notice that in solving problems about chords of a circle, it will often be found useful to draw perpendicular from the centre of the circle to the chords.

Exercise 9.1

1. State the theorem or corollary which justifies each conclusion below. P is the centre of the circle.
   (a) If $PN \perp CD$, then $CN = ND$.
   (b) Points A, Q and B are noncollinear.
   (c) If $PM = PN$, $PM \perp AB$ and $PN \perp CD$, then $AB = CD$.
   (d) If $AB = CD$, $PM \perp AB$ and $PN \perp CD$, then $PM = PN$.
   (e) If $CD > AB$, then $PN < PM$.

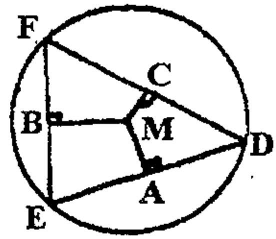


2. ⊙O has chords AB and XY as shown :
   (a) $AB \perp XY$ and $XZ = ZY$, what other facts can you conclude about the figure? Explain.
   (b) AB is a diameter and $AB \perp XY$. What can you conclude? Explain.
   (c) AB is a diameter and $XZ = ZY$. What can you conclude? Explain.



3. True or false?
   (a) A diameter of a circle is also a chord of the circle.
   (b) The longest chord of a circle is a diameter.
   (c) In a circle of radius 10 cm, no chord has length 25 cm.
   (d) In a circle of radius 10 cm, no chord has length.01 cm.

4. An **inscribed polygon** is a polygon with all its vertices as points of the circle and its sides as chords of the circle and the circle is said to be **circumscribed** about the polygon. ΔDEF is inscribed in $\odot M$. $MA \perp ED$, $MB \perp EF$ and $MC \perp DF$. $MA = 3$, $MB = 4$ and $MC = 2$ unit.



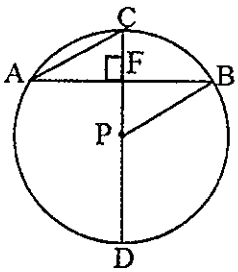
(a) Name the longest sides of ΔDEF.

(b) Name the shortest side of ΔDEF.

(c) Name the smallest angle of ΔDEF.

5. Which theorem or corollary tells us that a line cannot intersect a circle in three or more points?

6. For each part of the problem, answer in the following way. Write "extra" if more information is given than is needed to get a numerical answer. Write "not enough" if not enough information is given. Write "OK" if just enough information is given to allow a numerical solution. Write "contradictory" if the given information is contradictory.(Note :You do not need to solve ; just decide whether or not you can.) In the figure, P is the centre of the circle and $AB \perp CD$.



(a) $AF = 5$, $AB = ?$

(b) $PB = 7$, $CD = ?$

(c) $CF = 3$, $FP = 2$, $PD = 6$, $CD = ?$

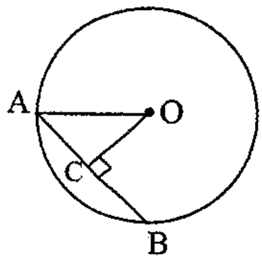
(d) $PB = 13$, $PF = 5$, $AB = ?$

(e) $AB = 16$, $CD = 20$, $CF = 4$, $PB = ?$

(f) $CF = 7$, $PB = 17$, $FB = 10$, $CD = ?$

(g) $CD = 30$, $AB = 24$, $AC = ?$

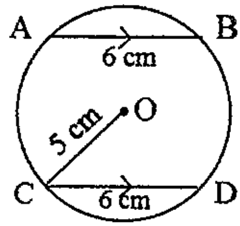
7. In $\odot O$, $OC \perp AB$ as shown in the figure :



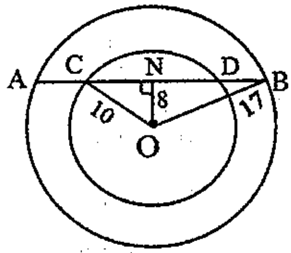
(a) if $OA = 17"$, $OC = 8"$ find AB.

(b) if $OA = 5"$, $AB = 8"$ find OC.

8. Find the distance between the parallel chords AB and CD.



9. In the figure, circles are concentric, with centre O. Find AC.



10. A diameter and a chord of a circle have a common endpoint. If the length of the diameter is 40 cm and the length of the chord is 24 cm, how far is the chord from the centre of the circle?

11. P is a point inside a circle of radius 6 cm whose distance from the centre is 3.6 cm. Find the lengths of

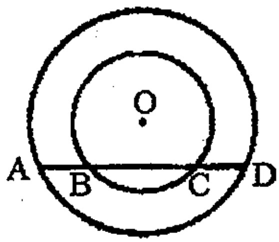
(a) the longest chord,

(b) the shortest chord of the circle, which can be drawn through P.

12. Through a given point P within a circle, the longest chord that can be drawn is 10 cm long and the shortest chord is 6 cm long. What is the radius of the circle and how far is P from the centre?

13. In the figure, O is the centre of the concentric circle.

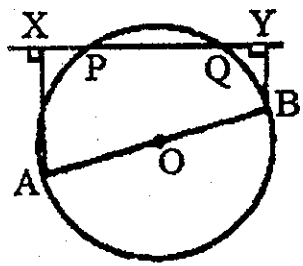
Prove that AB = CD.



14. In the figure, PQ is a chord and $\overline{AB}$ is a diameter.

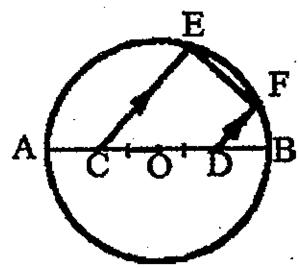
AX, BY are the perpendiculars to PQ.

Prove that PX = QY. Is the result still true if P and Q lie on the opposite sides of AB?



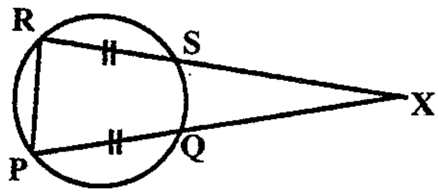
15. From two points C, D in the diameter AB of $\odot$ O, equidistant from the centre O, parallel lines are drawn to meet the circumference at E, F.

Show that $\angle CEF = \angle DFE = 90^\circ$.



16. Congruent chords PQ and RS are produced to meet at X.

Prove that ΔXPR is isosceles.



9.3 Tangent Lines and Tangent Circles

A circle separates the plane into three sets of points ; those on the circle, those interior to the circle and, those exterior to the circle. As shown in Fig. 9.10, A is a point in the interior of $\odot$ P, and PB is the radius of $\odot$ P.

Since PA + AB = PB, PA < PB.

Also, C is a point in the exterior of $\odot$ P.

Hence PB + BC = PC and PB < PC.

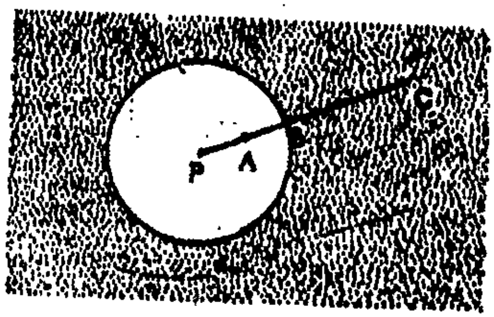


Fig. 9.10

All these possible situations may be summarized as follows :

Let r be the radius of a circle with centre P.

(a) $PA < r$ if and only if A is in the interior of the circle.

(b) $PA = r$ if and only if A is on the circle.

(c) $PA > r$ if and only if A is in the exterior of the circle.

For a circle and a line in the same plane, there are three possibilities for intersection. A line can intersect the circle in no point, one point, or two points as shown in Fig. 9.11.

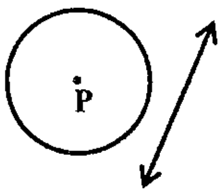
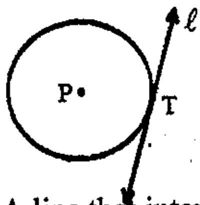
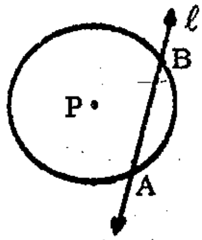
No point	One point	Two points
 The line and the circle do not intersect.	 A line that intersects the circle in exactly one point. (This line is a tangent.)	 A line that intersects the circle in two points. (This line is a secant).

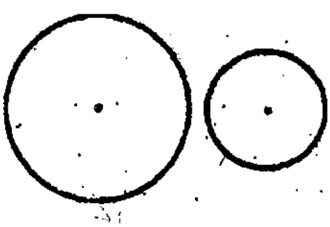
Fig 9.11

Definition 3 :

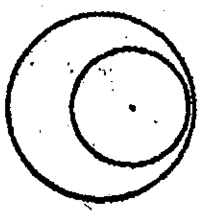
A tangent to a circle is a line in the plane of the circle that intersects the circle in exactly one point. This point is called the point of tangency or point contact.

We have described the three possible situations for the intersection of a line and a circle. If we consider two different circles, there are six possibilities for their intersections. These situations are shown in Fig. 9.12.

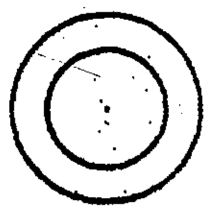
Disjoint Circles **Concentric circles**



Externally disjoint circles



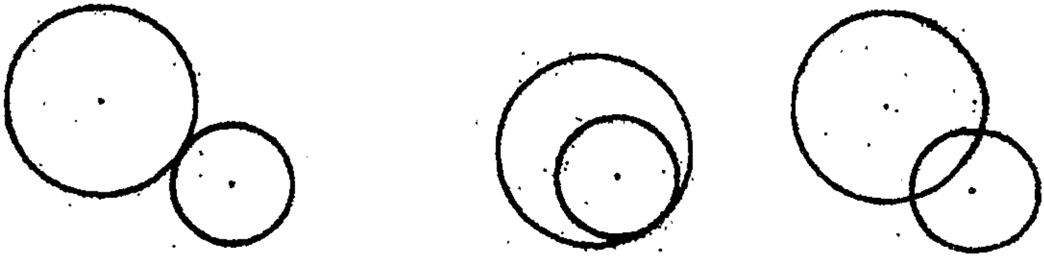
Internally disjoint circles



Intersecting Circles

Circles with one point of intersection

Circles with two points of intersection



Externally Tangent Circles

Internally Tangent Circles

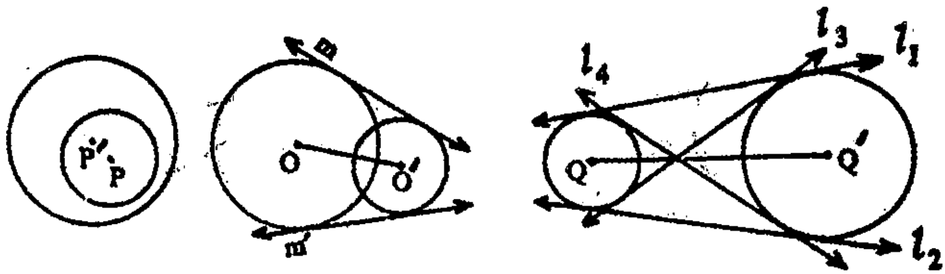
Fig. 9.12

Definition 4 :

Coplanar circles (circles in the same plane) that intersect in exactly one point are called tangent circles.

Tangent circles are said to be tangent internally if one circle, except for the point of tangency, is in the interior of the other circle. They are said to be tangent externally if each circle, except for the point of tangency, is in the exterior of the other.

A line tangent to each of two circles is called a **common tangent**. It is a **common internal tangent** if it intersects the segment that has the centres as end points. It is a **common external tangent** if it does not intersect the segment having the centres as end points.



(a) (b) (c)

Fig. 9.13

In Fig. 9.13 (a), the circles have no common tangents. In Fig. 9.13 (b), lines m and m' are common external tangents. In Fig. 9.13(c) l_1 and l_2 are common external tangents and lines l_3 and l_4 are common internal tangents. Note that l_1 and l_2 does not intersect QQ' but l_3 and l_4 do intersect QQ'.

Now we shall explore several properties of a tangent.

Theorem 7.

The line perpendicular to a radius at its outer endpoint (the endpoint on the circle) is tangent to the circle.

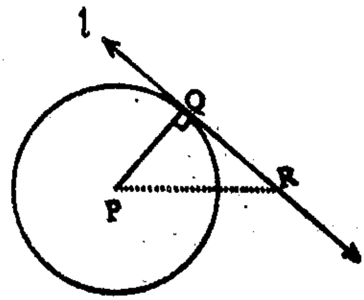


Fig. 9.14

Given : $\odot$ P, with radius PQ and line $l \perp PQ$.

To prove : l is tangent to $\odot$ P.

Proof : Let R be any point on l, distinct from Q. Since $l \perp PQ$, ΔPQR is a right triangle and so $PR > PQ$.

Thus R is not on the circle.

l intersects $\odot$ P in exactly one point. So, l is a tangent.

This theorem tells us that certain lines perpendicular to radii at their endpoints are tangents. From the next theorem, we can deduce that they are the only tangents.

Theorem 8.

If a line is tangent to a circle, then it is perpendicular to a radius of the circle at the outer endpoint of the radius.

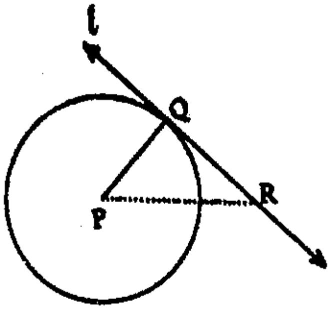


Fig. 9.15

Given : $\odot$ P, l is tangent to $\odot$ P at Q.

To Prove : $PQ \perp l$

Proof : Let R be any point on l distinct from Q. l is a tangent to $\odot$ P at Q and R is in the exterior of $\odot$ P.

Therefore $PR > PQ$

Hence PQ is the shortest segment from P to l. But the shortest segment from a given point of a given line is perpendicular to the line.

Thus $PQ \perp l$

Corollary 8.1

If a line is perpendicular to a tangent to a circle at the point of tangency, it contains the centre of the circle.

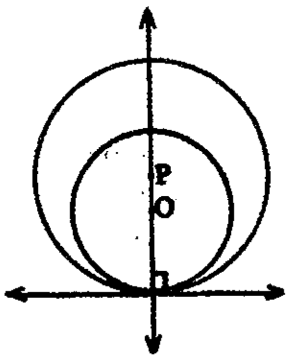


Fig. 9.16

Corollary 8.2

If two circles are tangent internally or externally, the line containing their centres also contains the point of tangency.

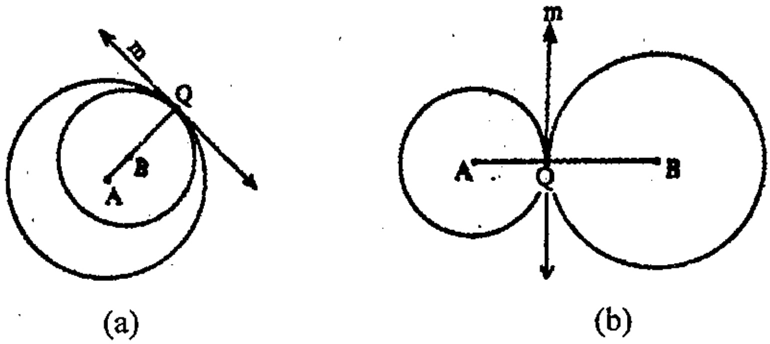


Fig. 9.17

From theorems (7) and (8), we can see the following:

There is exactly one tangent to a circle at a given point on the circle. This tangent is perpendicular to the radius at its outer endpoint.

Definition 5 : A tangent segment of a circle is a segment of a tangent whose endpoints are the point of tangency and an exterior point of the circle.

AT and BT are examples of tangent segments in Fig. 9.18.

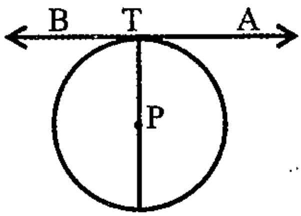


Fig. 9.18

Theorem 9.

For any circle, the two tangent segments from the same exterior point are congruent and form congruent angles with the line containing the exterior point and the centre of the circle.

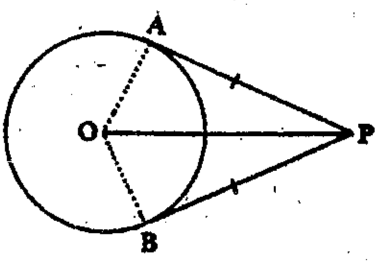


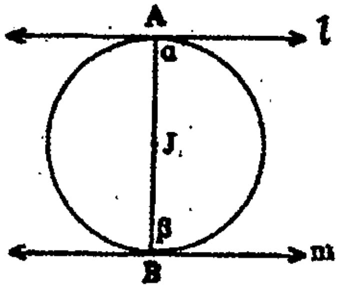
Fig. 9.19

Given : PA and PB are tangents to ⊙ O at A and B.

To Prove : PA = PB and OP bisects $\angle APB$.

The proof is left as an exercise.

Example 1.



Given : AB is a diameter of ⊙ J. l and m are tangents.

Prove : $l \parallel m$

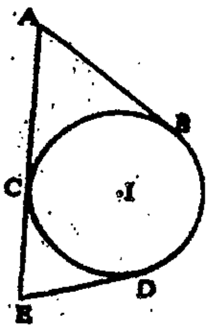
Proof : Since l is tangent to ⊙ J at A, $JA \perp l$.

Similarly $JB \perp m$.

Hence $\alpha + \beta = 90^\circ + 90^\circ = 180^\circ$

$l \parallel m$

Example 2.



Given : AB, AE and ED are tangents to ⊙ I.

Prove : AE = AB + DE

Proof : Since AB and AC are tangent segments to ⊙ I at B and C,

AB = AC

Similarly EC = ED

Hence AE = AC + CE = AB + DE

Exercise 9.2

1. Draw a circle with centre P and radius PQ = 1.25 cm. Locate a point A such that PA = 2 cm, and a point B such that PB = 1 cm. Now complete the following statements.

(a) A lies in the .... of the circle because ....

(b) B lies in the .... of the circle because ....

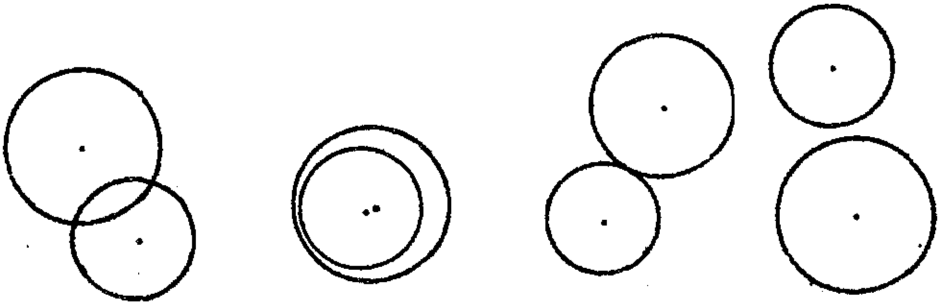
2. How many tangents do you think can be drawn to a circle.

   (a) at a point on the circle?

   (b) from a point outside it?

   (c) through a point inside?

3. Give the number of common tangents to the two circles.



(a) (b) (c) (d)

4. In the following figure one arrangement of three circles having different radii so that each circle is tangent to the other two circles is shown.

   (a) Name two internally tangent circles.

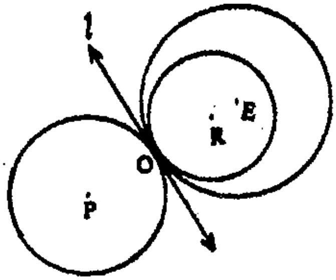
   (b) Name two externally tangent circles.

   (c) Of what two circles is l a common internal tangent?

   (d) Of what two circles is l a common external tangent?

   (e) Make sketches showing at least three other arrangements.

   (f) Name four collinear points.



5. Can a pair of circles be drawn so that they have

   (a) four common tangents?

   (b) exactly two common tangents?

   (c) only one common tangent?

   (d) no common tangent?

   (e) more than four common tangents?

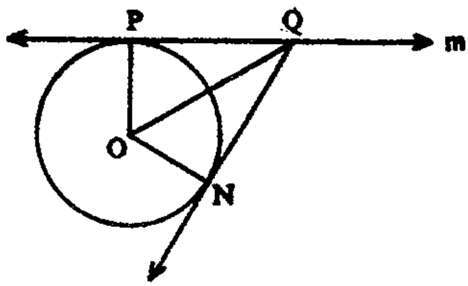
6. m is tangent to $\odot$ O at P. QN is tangent to $\odot$ O at N. Complete the statements.

(a) $\angle$ OPQ = ...

(b) If PQ = 10 cm, then QN = ...

(c) $\angle$ PON + $\angle$ PQN = ...

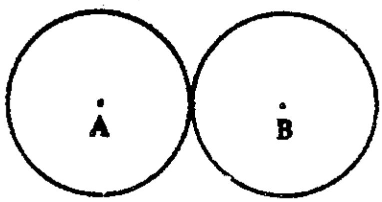
(d) If $\angle$ PQN = 60°, then $\angle$OQP = ...



7. Two circles have radii 3 and 5. Tell whether the circles intersect in 0, 1 or 2 points, if the length of the segment with their centres as endpoint is :

(a) 10 (b) 8 (c) 6 (d) 5 (e) 2 (f) 1.

8. Circles A and B are two congruent externally tangent circles. If $\odot$C is tangent to both of these circles, where must C lie?



9. Write the letter that best completes each statement.

(i) If a line is tangent to a circle, it is .... the radius drawn to the point of tangency.

(a) parallel to (b) congruent to

(c) perpendicular to (d) the bisector of

(ii) If PA and PB are tangent segments to $\odot$ O at A and B respectively, then ...

(a) $\angle$ P = $\angle$ AOB (b) $\angle$ P < $\angle$ AOB

(c) $\angle$ P + $\angle$AOB = 180° (d) $\angle$P + $\angle$AOB = 90°

(iii) If two circles intersect in two points, the line containing their centres is .... the common chord.

(a) perpendicular to (b) only bisector of

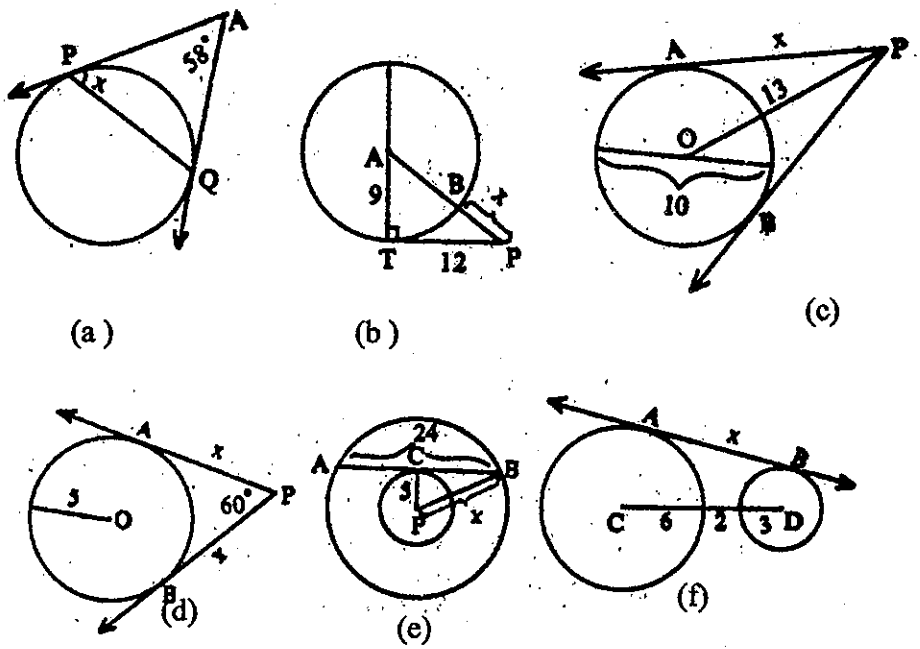
(c) the perpendicular bisector of (d) parallel to

(iv) Two equal circles in the same plane cannot have ...... common tangents.

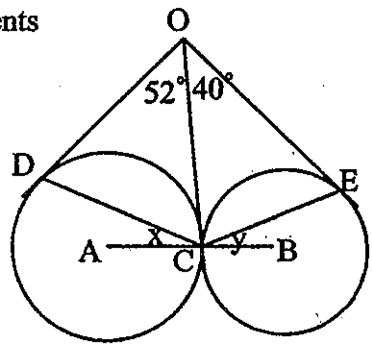
(a) only one (b) two (c) three

(d) four

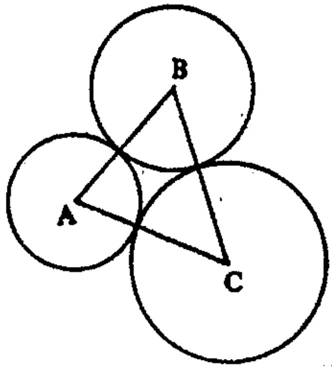
10. Find x in each of the given figures.



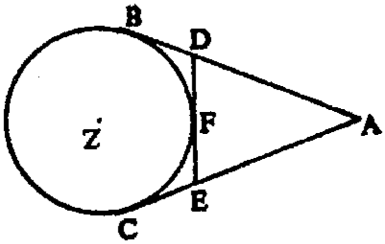
11. $\odot$A and $\odot$B are tangent externally at C.
OC, OD and OE are common tangent segments as shown in the figure.
Prove that OD = OC = OE and $x - y = 6°$.



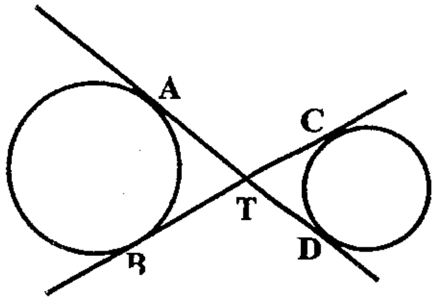
12. In the figure, $\odot A$, $\odot B$ and $\odot C$ are tangent to each other. If AB = 10, AC = 14 and BC = 18, find the radius of each circle.



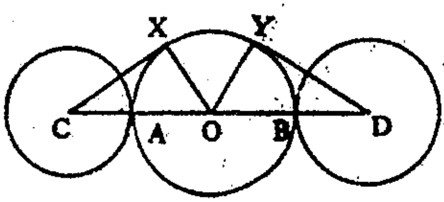
13. Tangent segments AB and AC are drawn to $\odot Z$ from A. DE is tangent to the circle at F. Prove that ΔADE has a perimeter equal to AB + AC.



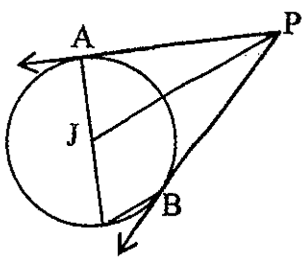
14. In the figure, AD and BC are common internal tangents intersecting at T.
    (a) Prove that AD = BC.
    (b) Prove that TA. TC = TB. TD
    (c) True or false? AB // CD.



15. Three circles with centres C,O and D are tangent to one another externally as shown in the figure. CX and DY are tangent. Prove that $\angle XOY = \angle ACX + \angle BDY$.

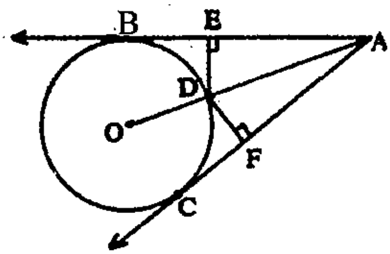


16. Given : $\odot J$, tangent PA and PB, diameter AC and line segment JP.
    Prove : CB // JP



17. Given : Tangents AB and AC to $\odot$O and AO intersects the circle in D.

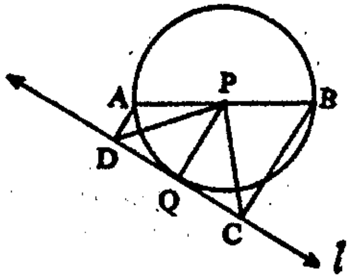
Prove : D is equidistant from AB andAC. (i.e. DE = DF)



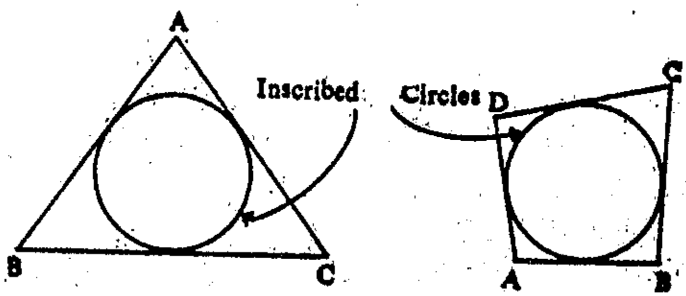
18. Given : In the figure, AB is a diameter of $\odot$P. l is tangent to $\odot$P at Q.

AD and BC are each perpendicular to l.

Prove : PD = PC.



19. A polygon is **circumscribed** about a circle, if each side of it is tangent to the circle, and the circle is said to be **inscribed** in the polygon.



Circumscribed triangle Circumscribed quadrilateral

(a) ABCD is a circumscribed quadrilateral.

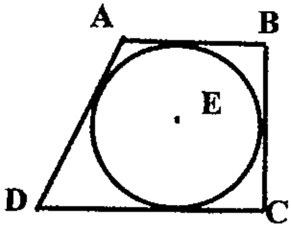
Prove that AB + CD = AD + BC.

(b) ABCDEF is a circumscribed hexagon. Prove that

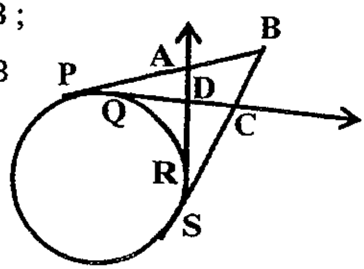
AB + CD + EF = BC + DE + FA.

(c) Generalize this result.

20. The sides of quadrilateral ABCD are tangents to ⊙E. AB = 14, DC = 20, and AD = 16. Find BC.



21. In the given figure, the tangents at P and S meet at B ; the tangents at Q and R meet at D ; RD intersects PB in A and QD intersects SB in C.
Prove that AB + AD = CB + CD.



SUMMARY

1. **Important words and symbols**

Circle , ⊙

Circle with centre P, ⊙ P

Concentric circles

Tangent circles

Tangent segment

Common tangent

Inscribed circle

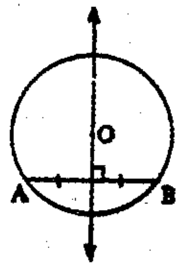
Inscribed triangle

Circumscribed circle

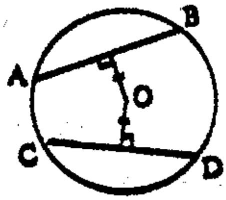
Circumscribed triangle

2. Important Theorems

1. In a circle, a line through the centre bisects a chord if and only if it is perpendicular to that chord.



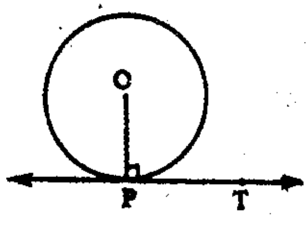
2. In a circle, two chords are equal if and only if they are equidistant from the centre.



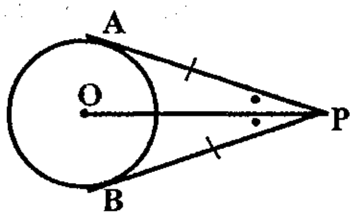
3. In a circle, one chord is longer than a second chord if and only if the first chord is closer to the centre.

4. Any three noncollinear points lie on a circle.

5. A line is perpendicular to a radius of a circle at its outer endpoint if and only if the line is tangent to the circle.



6. Tangent segments to a circle from an external point are equal.



CHAPTER 10

Trigonometric Ratios and their Applications

The word " trigonometry " is derived from the words "tri" (meaning three), "gon" (meaning sides) and "metry" (meaning to measure). Thus trigonometry deals with the measurement of sides and angles of a triangle.

It has been widely used in astronomy, surveying, geography, physics, navigation etc. The captains of the ships employed trigonometry to calculate the distances from far off island, sea shores, cliffs and other ships on the high sea.

To study trigonometry, the students should already be acquainted with the theorems on similar triangles. Then only they will find it easy to understand the definition of trigonometric ratios. To begin with, we shall only consider the acute angles in this book.

10.1 Angles

In trigonometry an angle is determined by rotating a ray about its endpoint from an initial position to terminal position.

Consider a line OP which is free to rotate in the XY - plane. O is taken to be the origin about with a line OP rotates.

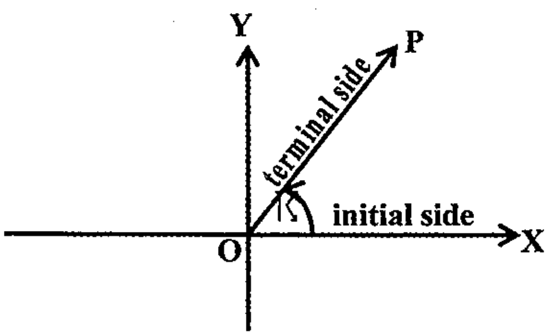


Fig. 10.1

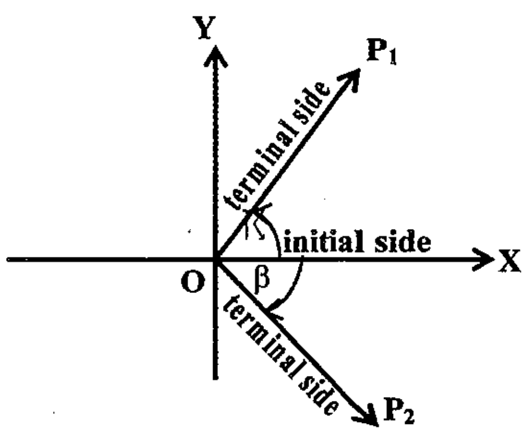


Fig.10.2

When the line OP is rotated, it is possible to vary the size of the angle θ between OP and OX (see Fig. 10.1). Angles measured from the X-axis (i.e. OX) in an anticlockwise direction are positive angles. Angles measured from the X-axis in a clockwise direction are negative angles.

Therefore in Fig. 10.2, the angle α (i.e. $\angle P_1OX$) is positive while the angle β (i.e. $\angle P_2OX$) is negative.

One complete revolution of the line OP from OX denotes an angel of 360°. The range of values of θ between 0° and 360° in each quadrant is as shown in Fig. 10.3.

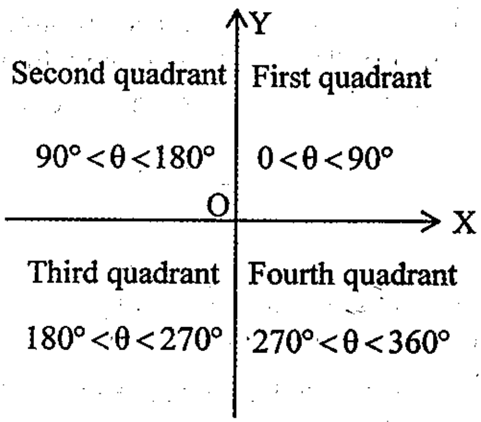


Fig. 10.3

10.2 The Relation between Degree and Radian Measure

Two kinds of units commonly used for measuring angles are radian measure and degree measure. The radian measure is employed almost exclusively in advanced mathematics and in many branches of science. In this chapter first we introduce the concept of radian and study the relation between degrees and radians.

Consider a circle with centre O and radius r units as shown in Fig. 10.4. Let arc AB be an arc on the circle of length equal to r. We define the magnitude of angle AOB which the arc AB subtends at the centre as one radian.

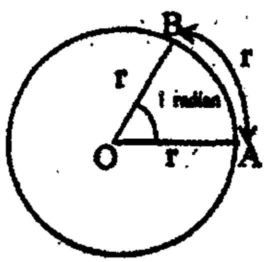


Fig. 10.4

Since the circumference of a circle is equal to $2\pi r$, it subtends a central angel of 2π radians. That is there are 2π radians in a complete rotation of $360°$.

$$2\pi \text{ radians} = 360°$$

and hence $\boxed{\pi \text{ radians} = 180°}$ which is a fundamental relation between radians and degrees.

$$\text{We have } 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57°19'$$

$$1° = \frac{\pi}{180} \text{ radians} \approx 0.01746 \text{ radian}$$

Example 1.

Express the following in radian measures.

(a) $45°$ (b) $120°$

Solution

(a) $45° = 45 \times \dfrac{\pi}{180} = \dfrac{\pi}{4}$ radians

(b) $120° = 120 \times \dfrac{\pi}{180} = \dfrac{2\pi}{3}$ radians

Example 2.

Express the following in degree measures.

(a) $\dfrac{\pi}{3}$ radians (b) $\dfrac{4\pi}{5}$ radians

Solution

(a) $\dfrac{\pi}{3}$ radians $= \dfrac{\pi}{3} \times \dfrac{180}{\pi} = 60°$

(b) $\dfrac{4\pi}{5}$ radians $= \dfrac{4\pi}{5} \times \dfrac{180}{\pi} = 144°$

Note : Usually when the units of an angle are not specified, it is understood that the angle is expressed in radians.

10.3 Arc Length and Area of Sector of a Circle

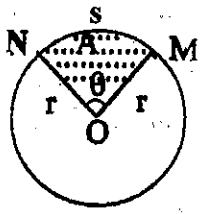


Fig. 10.5

Let the arc MN subtend an angle of magnitude θ radians at the centre of a circle of radius r as shown in Fig. 10.5. Clearly the length s of arc MN is proportional to the angle θ and we have

$$\frac{\text{length of arc MN}}{\text{length of circumference}} = \frac{\text{angle subtended by arc MN}}{\text{angle subtended by circumference}}$$

i.e. $$\frac{s}{2\pi r} = \frac{\theta}{2\pi} \qquad \text{(or)} \qquad \theta(\text{in radians}) = \frac{s}{r}$$

Furthermore, as the area of the sector MON (the shaded region shown in Fig. 10.5) is also, proportional to the angle θ, we have

$$\frac{\text{area of sector MON}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

i.e $$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \qquad \text{(or)} \qquad \boxed{A = \frac{1}{2} r^2 \theta}$$

where θ is given in radian measures.

Example 3. An arc BC subtends an angle of 144° at the centre O of a circle of radius 10 cm. Find (i) the length of arc BC
(ii) the area of the sector BOC

Solution

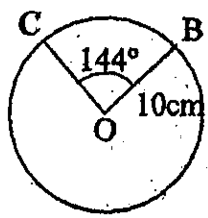


Fig. 10.6

(i) As $\theta = 144^\circ = 144 \times \frac{\pi}{180} = \frac{4\pi}{5}$ radians and r = 10 cm, we have

the length of arc BC $= r\theta = 10 \times \frac{4\pi}{5} = 8\pi$ cm

(ii) The area of the sector BOC $= \frac{1}{2} r^2\theta = \frac{1}{2} \times 10^2 \times \frac{4\pi}{5} = 40\pi \text{ cm}^2$

Exercise 10.1

1. Convert each of the following to radians.

   (a) 120° (b) 75° (c) 108° (d) 225°

   (e) 150° (f) 135° (g) 160°

2. Convert each of the following to degrees.

   (a) $\frac{\pi}{2}$ (b) $\frac{4\pi}{3}$ (c) π (d) $\frac{5\pi}{6}$

   (e) $\frac{3\pi}{2}$ (f) $\frac{2\pi}{3}$ (g) $\frac{\pi}{3}$ (h) $\frac{3\pi}{4}$

3. A central angle θ subtends an arc of $\frac{11\pi}{2}$ cm on a circle of radius 6 cm.

   (i) Find the measure of θ in radians.

   (ii) Find the area of the sector of the circle which has θ is its central angle.

4. The area of a sector of a circle is 143 cm^2 and the length of the arc of the sector is 11 cm. Find the radius of the circle.

5. The diagram shows a sector OAB of radius 8 cm and a perimeter of 23 cm. Find (i) $\angle AOB$ in radian, (ii) area of sector AOB

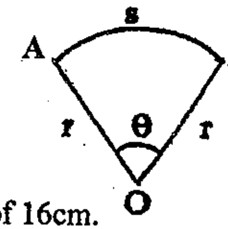
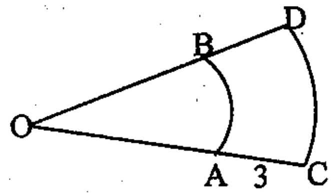


Fig.10.7

6. A sector cut from a circle of radius 3cm has a perimeter of 16cm. Find that area of this sector.

7. A piece of wire of fixed length L cm, is bent to form the boundary a sector of a circle. The circle has radius r cm and the angle of the sector is $\theta = \left(\frac{32}{r} - 2\right)$ radians. Find the length L and show that the area of the sector, A cm^2 is given by $A = 16r - r^2$.

8. The figure shows two sectors in which the arcs AB and CD are arcs of concentric circles, center O. $\angle AOB = \frac{2}{3}$ radians, AC = 3m and the area of sector AOB is $12cm^2$, calculate



(i) OA (ii) the area and the perimeter of ABDC.

10.4 Six Trigonometric Ratios

Consider an acute angle A, that is an angle whose measure in degree is less than 90°. On one arm of the angle A, select a point C, and construct a perpendicular at C to form a right triangle ABC. (See Fig. 10.8)

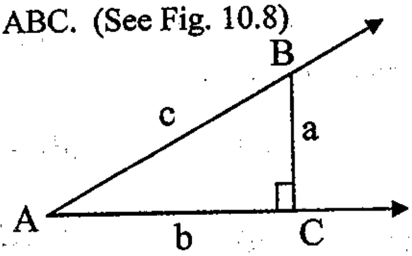


Fig. 10.8

Denote the lengths of the segments BC , AC , AB by the letters a , b , c respectively. We say that

BC is the side which is opposite to angle A.

AC is the side which is adjacent to angle A.

AB is the hypotenuse.

Consider the ratios $\frac{a}{c}$, $\frac{b}{c}$ and $\frac{a}{b}$. The ratio $\frac{a}{c}$ is called the sine of angle A , the ratio $\frac{b}{c}$ is called the cosine of angle A , and the ratio $\frac{a}{b}$ is called the tangent of angle A. Thus

$$\text{sine of } \angle A = \frac{a}{c} = \frac{\text{length of opposite side of angle A}}{\text{length of hypotenuse}}$$

(sine of $\angle A$ is abbreviated as sin A)

$$\text{cosine of } \angle A = \frac{b}{c} = \frac{\text{length of adjacent side of angle A}}{\text{length of hypotenuse}}$$

(cosine of $\angle A$ is abbreviated as cos A)

$$\text{tangent of } \angle A = \frac{a}{b} = \frac{\text{length of opposite side of angle A}}{\text{length of adjacent side of angle A}}$$

(tangent of $\angle A$ is abbreviated as tan A)

From now on , we shall measure the angles in degrees. Thus Ð A = α means that the measure of angle A is α degrees . Thus if Ð A = α then we may write $\sin A = \sin \alpha$, $\cos A = \cos \alpha$ and $\tan A = \tan \alpha$.

We will show that each of these ratios depends only on the size of angle A and not on the location of points B and C on its arms.

Take another point C' , construct a perpendicular C' B' to form a right triangle AB'C' . Since ΔABC and $\Delta AB'C'$ are equiangular, they are similar.

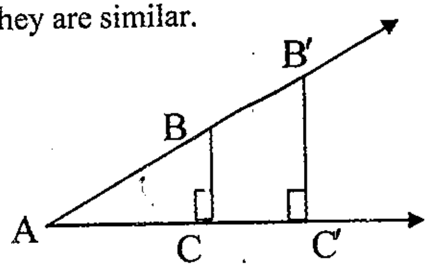


Fig. 10 . 9

Therefore $\dfrac{BC}{B'C'} = \dfrac{AB}{AB'} = \dfrac{AC}{AC'}$

$$\frac{BC}{AB} = \frac{B'C'}{AB'} \qquad \text{... (i)}$$

$$\frac{AC}{AB} = \frac{AC'}{AB'} \qquad \text{... (ii)}$$

$$\frac{BC}{AC} = \frac{B'C'}{AC'} \qquad \text{... (iii)}$$

From equation (i) we see that the ratio $\dfrac{\text{length of opposite side of angle A}}{\text{length of hypotenuse}}$ is constant and does not depend on the size of triangle .

Thus sin A , cos A and tan A do not depend on the size of the triangle.

Example 1. (a) Using the right triangle ABC, find sin A, cos A, tan A and sin B, cos B , tan B .

(b) Using the right triangle AB'C' , find sin A , cos A,tan A and sin B' , cos B' , tan B'.

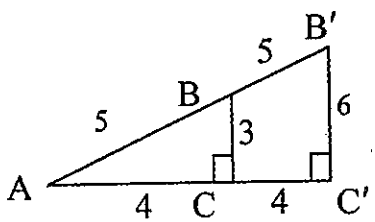


Fig. 10.10

Solution

(a) From the right triangle ABC,

$$\sin A = \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{3}{5}$$

$$\cos A = \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{4}{5}$$

$$\tan A = \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{BC}{AC} = \frac{3}{4}$$

$$\sin B = \frac{\text{opposite side of } \angle B}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{4}{5}$$

$$\cos B = \frac{\text{adjacent side of } \angle B}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{3}{5}$$

$$\tan B = \frac{\text{opposite side of } \angle B}{\text{adjacent side of } \angle B} = \frac{AC}{BC} = \frac{4}{3}$$

(b) From the right triangle AB'C'

$$\sin A = \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{B'C'}{AB'} = \frac{6}{10} = \frac{3}{5}$$

$$\cos A = \frac{\text{adjacent side of } \angle A}{\text{hypotenuse}} = \frac{AC'}{AB'} = \frac{8}{10} = \frac{4}{5}$$

$$\tan A = \frac{\text{opposite side of } \angle A}{\text{adjacent side of } \angle A} = \frac{B'C'}{AC'} = \frac{6}{8} = \frac{3}{4}$$

$$\sin B' = \frac{\text{opposite side of } \angle B'}{\text{hypotenuse}} = \frac{AC'}{AB'} = \frac{8}{10} = \frac{4}{5}$$

$$\cos B' = \frac{\text{adjacent side of } \angle B'}{\text{hypotenuse}} = \frac{B'C'}{AB'} = \frac{6}{10} = \frac{3}{5}$$

$$\tan B' = \frac{\text{opposite side of } \angle B'}{\text{adjacent side of } \angle B'} = \frac{AC'}{B'C'} = \frac{8}{6} = \frac{4}{3}$$

In addition to the above three ratios , we now define three other ratios, namely , cotangent of angle A , written as cot A , secant of angle A , written as sec A and cosecant of angle A , written as cosec A . Thus

$$\cot A = \frac{b}{a} = \frac{\text{length of adjacent side of angle } A}{\text{length of opposite side of angle } A}$$

$$\sec A = \frac{c}{b} = \frac{\text{length of hypotenuse}}{\text{length of adjacent side of angle } A} \quad \text{and}$$

$$\text{cosec } A = \frac{c}{a} = \frac{\text{length of hypotenuse}}{\text{length of opposite side of angle } A}$$

Example 2. Find the six trigonometric ratios for angle B of the given right triangle.

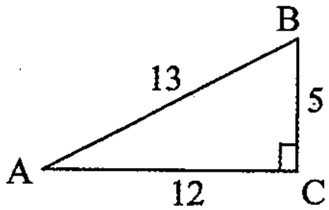


Fig. 10.11

Solution

Opposite side of angle B $= AC = 12$

adjacent side of angle B $= BC = 5$

hypotenuse $= AB = 13$

$$\sin B = \frac{12}{13}, \qquad \text{cosec } B = \frac{13}{12}$$

$$\cos B = \frac{5}{13}, \qquad \sec B = \frac{13}{5}$$

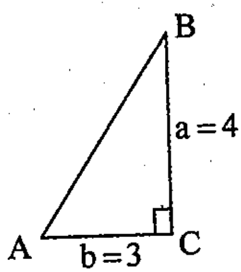
$$\tan B = \frac{12}{5}, \qquad \cot B = \frac{5}{12}$$

Example 3. Triangle ABC is a right triangle with $a = 4$, $b = 3$ and $\angle C = 90°$. Find sin A and sin B .

Solution

By the Pythagoras Theorem

$$c^2 = a^2 + b^2$$
$$= 16 + 9 = 25$$
$$c = 5$$
$$\sin A = \frac{BC}{AB} = \frac{4}{5}$$
$$\sin B = \frac{AC}{AB} = \frac{3}{5}$$



Example 4. In the given right triangle, find AC in terms of x. Also find the six ratios of $\angle A$ in terms of x.

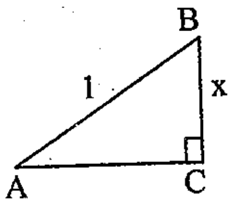


Fig. 10.12

Solution

By the Pythagoras Theorem,

$$AB^2 = AC^2 + BC^2$$
$$AC^2 = AB^2 - BC^2 = 1 - x^2$$
$$AC = \sqrt{1-x^2}$$

$$\sin A = \frac{x}{1} = x$$
$$\cos A = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$
$$\tan A = \frac{x}{\sqrt{1-x^2}}$$
$$\operatorname{cosec} A = \frac{1}{x}$$
$$\sec A = \frac{1}{\sqrt{1-x^2}}$$
$$\cot A = \frac{\sqrt{1-x^2}}{x}$$

Example 5. Given right triangle ABC with $\angle C = 90^\circ$ and $\tan A = \frac{5}{12}$.

Find sin A and cos A .

Solution

Since $\tan A = \frac{5}{12}$, the opposite side and the adjacent side of $\angle A$ are in ratio 5 : 12.

Therefore , let BC = 5k and AC = 12 k , where k is the constant of proportionality.

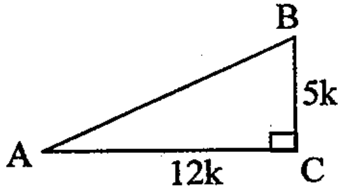


Fig. 10.13

By the Pythagoras Theorem ,

$$AB^2 = AC^2 + BC^2$$

$$= (12k)^2 + (5k)^2$$

$$AB = 13k$$

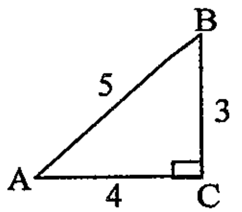
$$\text{Hence } \sin A = \frac{5k}{13k} = \frac{5}{13} \quad \text{and} \quad \cos A = \frac{12k}{13k} = \frac{12}{13}$$

Exercise 10.2

1. Given right triangle with lengths of sides as indicated. Find the following trigonometric ratios.

   sin A , cos A , tan A , cot A , sec A , cosec A .

   sin B , cos B , tan B , cot B , sec B , cosec B .



2. Given $\cos A = \frac{5}{13}$, determine the values of tan A and sin A .

3. Use Pythagoras Theorem to find the third side of each of the following right triangle. In each case consider a $\Delta\, ABC$ with right angle at C and find the six trigonometric ratios for angle A and angle B .

(i) $a = 3,\ b = 5$ (ii) $a = 1,\ b = 2$

(iii) $a = 4,\ b = 5$ (iv) $a = 3,\ c = 4$

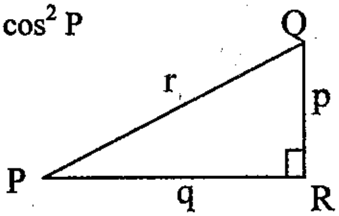
(v) $a = 12, c = 13$ (vi) $b = 2,\ c = 5$

4. Given a right triangle PQR, calculate each of the following :

(i) $(\cos P)(\sec P)$ (ii) $(\tan Q)(\cot Q)$

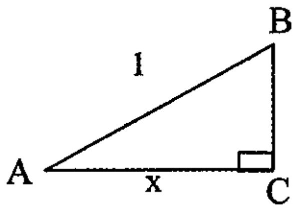
(iii) $(\tan P)\left(\dfrac{1}{\cot P}\right)$ (iv) $(\sin Q)\left(\dfrac{1}{\operatorname{cosec} Q}\right)$

(v) $\sin^2 Q + \cos^2 P$ (vi) $\sec^2 Q + \tan^2 Q$



5. For the given right triangle ABC , show that

$(\sin A)(\cos A) = x\sqrt{1 - x^2}$.



10.5 Value of the Trigonometric Ratios for some Special Angles

1. Trigonometric ratios for an angle of 45°.

In a 45°–45° right triangle by considering one with the two equal sides of length 1 unit each , the length of the hypotenuse is $\sqrt{2}$.

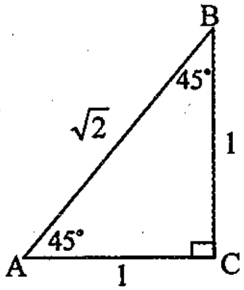


Fig. 10.14

This may be referred to as a $1, 1, \sqrt{2}$ triangle. We can now write the six trigonometric ratios for an angle of 45°.

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \cot 45^\circ = \frac{1}{1} = 1$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \sec 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\tan 45^\circ = \frac{1}{1} = 1, \qquad \text{cosec } 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Notice that we also get the same results with any similar isosceles right triangle with sides $x, x, x\sqrt{2}$ for any $x > 0$.

Example 1. Using an isosceles right triangle with sides $3, 3, 3\sqrt{2}$, find $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$.

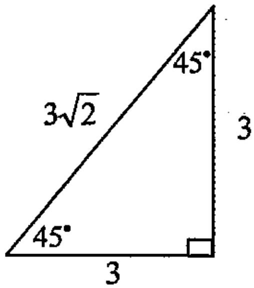


Fig. 10.15

Solution

$$\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{3}{3} = 1$$

2. Trigonometric ratios for angles of 30° and 60°.

In a $30^\circ - 60^\circ$ right triangle with the shorter leg of 1 unit in length, the length of the hypotenuse is 2 and the length of the other leg is $\sqrt{3}$. This may be referred to as $1, \sqrt{3}, 2$ triangle.

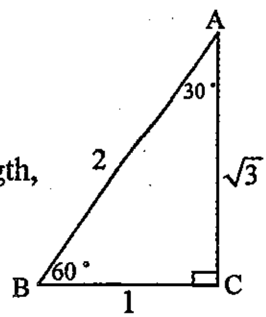


Fig. 10.16

We can now write the six trigonometric ratios for angles of 30° and 60°.

$$\sin 30^\circ = \frac{1}{2}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\operatorname{cosec} 30^\circ = \frac{2}{1} = 2, \qquad \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \qquad \sec 60^\circ = \frac{2}{1} = 2$$

$$\cot 30^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}, \qquad \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Notice that we also get the same result by using any similar right triangle with sides x, $x\sqrt{3}$, $2x$ for any $x > 0$.

Example 2. Using a right triangle with sides 2, $2\sqrt{3}$, 4, as shown in Fig. 10.17 find $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$ and $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$

Solution

$$\sin 30^\circ = \frac{2}{4} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

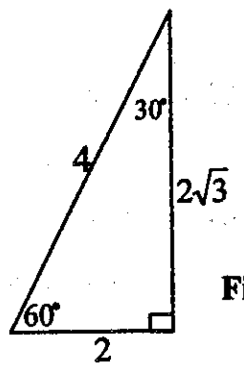


Fig. 10.17

$$\sin 60^\circ = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{2}{4} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

Table summarizing the trigonometric ratios for special angles .

Tables 10.1

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Example 3. In the following triangle determine the length of the side marked x .

Solution

To find x , we must choose a ratio that includes x and 10 cm.

Thus $\cos B = \frac{BC}{AB} = \frac{x}{10}$

But $\angle B = 60^\circ$ and we have found that

$$\cos 60^\circ = \frac{1}{2}$$

Thus $\frac{x}{10} = \frac{1}{2}$

$$x = \frac{10}{2} = 5$$

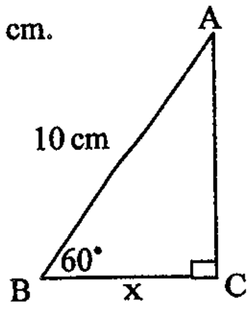


Fig. 10.18

Example 4. A ladder is placed along a wall such that its upper end is touching the top of the wall. The foot of the ladder is 5 ft away from the wall and the ladder is making an angle of 60° with the level of the ground. Find the height of the wall.

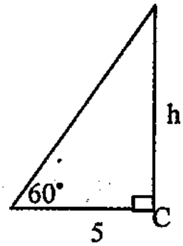


Fig. 10.19

Solution

Let h = height of the wall in feet.

Then $\tan 60^\circ = \frac{h}{5}$

$$h = 5 \tan 60^\circ = 5\sqrt{3} \text{ ft.}$$

Example 5. Find the side marked x from the given triangle.

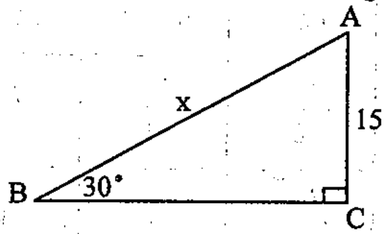


Fig. 10.20

Solution

To find x, we must choose a ratio that includes AB and AC.

$$\operatorname{cosec} B = \frac{AB}{AC} = \frac{x}{15}$$

$$\operatorname{cosec} 30^\circ = \frac{x}{15}$$

$$x = 15 \operatorname{cosec} 30^\circ = 15 \times 2 = 30$$

Exercise 10.3

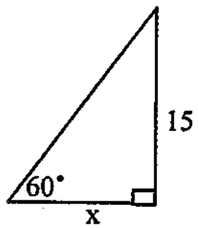
Draw a right triangle and find $\angle A$.

1. $\sin A = \frac{1}{2}$ 2. $\cos A = \frac{\sqrt{2}}{2}$ 3. $\sec A = \sqrt{2}$

4. $\tan A = \sqrt{3}$ 5. $\cot A = \sqrt{3}$ 6. $\operatorname{cosec} A = 2$

Solve each triangle for the side marked x .

7.



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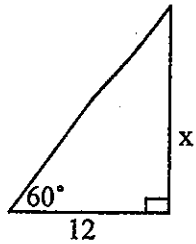

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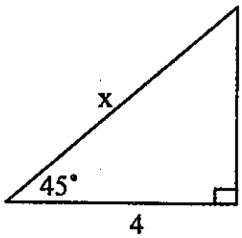

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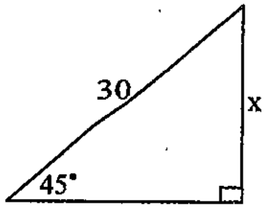

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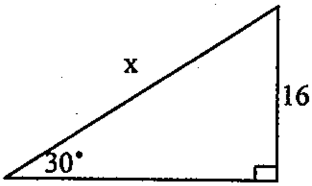

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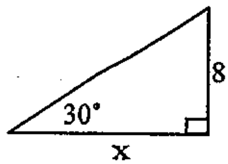
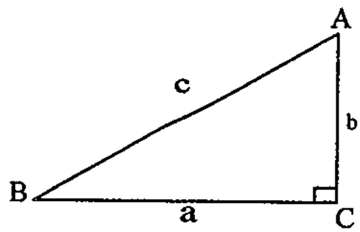


From the right triangle ABC , find the indicated sides .



13. $\angle A = 30^\circ$, $c = 30$, find a .

14. $\angle A = 60^\circ$, $b = 15$, find c .

15. $\angle A = 45^\circ$, $c = 16$, find b .

16. $\angle B = 30^\circ$, $a = 10$, find c .

17. $\angle A = 45^\circ$, $b = 8$, find c .

18. $\angle A = 60^\circ$, $a = 12$, find b .

10.6 Basic Identities

Using a right triangle ABC , we may find the relationships that exist among the trigonometric ratios. As stated earlier , the angles are measured in degrees in the following.

Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma = 90^\circ$

$$\sin\alpha = \frac{a}{c} = \cos\beta = \cos(90^\circ - \alpha)$$

$$\cos\alpha = \frac{b}{c} = \sin\beta = \sin(90^\circ - \alpha)$$

$$\tan\alpha = \frac{a}{b} = \cot\beta = \cot(90^\circ - \alpha)$$

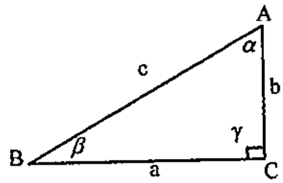


Fig. 10.21

$$\cot\alpha = \frac{b}{a} = \tan\beta = \tan(90^\circ - \alpha)$$

$$\sec\alpha = \frac{c}{b} = \text{cosec}\,\beta = \text{cosecs}\,(90^\circ - \alpha)$$

$$\text{cosec}\,\alpha = \frac{c}{a} = \sec\beta = \sec(90^\circ - \alpha)$$

Two angles are said to be complementary if the sum of their angle measures is 90° . The trigonometric ratio of an acute angle is equal to the co-named ratio of its complement. (sine and cosine , tangent and cotangent , secant and cosecant).

For example

$$\sin 30^\circ = \cos 60^\circ$$

$$\cos 70^\circ = \sin 20^\circ$$

$$\tan 40^\circ = \cot 50^\circ$$

$$\cot 10^\circ = \tan 80^\circ$$

$$\sec 25^\circ = \text{cosec}\ 65^\circ$$

$$\text{cosec}\ 35^\circ = \sec 55^\circ$$

The followings are some useful basic identities . We will use the Greek letter θ (theta) to represent the measure of an arbitrary angle.

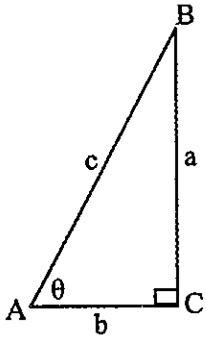


Fig. 10.22

$$\left.\begin{aligned} \sin\theta &= \frac{a}{c} \\ \operatorname{cosec}\theta &= \frac{c}{a} \end{aligned}\right\} \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\left.\begin{aligned} \cos\theta &= \frac{b}{c} \\ \sec\theta &= \frac{c}{b} \end{aligned}\right\} \sec\theta = \frac{1}{\cos\theta}$$

$$\left.\begin{aligned} \tan\theta &= \frac{a}{b} \\ \cot\theta &= \frac{b}{a} \end{aligned}\right\} \cot\theta = \frac{1}{\tan\theta}$$

$$\left.\begin{aligned} \frac{\sin\theta}{\cos\theta} &= \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} \\ \tan\theta &= \frac{a}{b} \end{aligned}\right\} \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\left.\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \cot\theta &= \frac{1}{\tan\theta} \end{aligned}\right\} \cot\theta = \frac{\cos\theta}{\sin\theta}$$

From the right triangle ABC , using the Pythagoras Theorem , we get

$a^2 + b^2 = c^2$ ... (i)

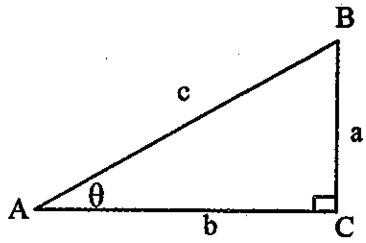


Fig. 10.23

Divide equation (i) by c^2,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Thus

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{... (ii)}$$

Divide equation (ii) by $\cos^2\theta$, we get

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

Thus $\tan^2\theta + 1 = \sec^2\theta$

Divide equation (ii) by $\sin^2\theta$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Thus $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Note : We denote $(\sin\theta)^2$, $(\cos\theta)^2$ and $(\tan\theta)^2$ by $\sin^2\theta$, $\cos^2\theta$ and $\tan^2\theta$ respectively.

Example 1. Prove that $\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \tan\theta$.

Solution L.H.S $= \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\sin\theta}{\sqrt{\cos^2\theta}} \quad (\because \sin^2\theta + \cos^2\theta = 1)$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{R.H.S}$$

Example 2. Verify the identity $(1 - \sin\theta)(1 + \sin\theta) = \frac{1}{1+\tan^2\theta}$

Solution L.H.S $= (1 - \sin\theta)(1 + \sin\theta) = 1 - \sin^2\theta$

$$= \cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \text{R.H.S}$$

Exercise 10.4

1. Given that $x = 2\sin\theta - 3$ and $y = 2\cos\theta + 1$, show that $(x + 3)^2 + (y - 1)^2 = 4$.

2. Given that $x = 3\tan\theta$ and $y = 2\cos^2\theta$, show that $x^2y + 9y = 18$.

3. If $a = x\sin\theta - y\cos\theta$, $b = x\cos\theta + y\sin\theta$, show that $a^2 + b^2 = x^2 + y^2$.

4. Prove the identity $1 - \frac{\sin^2 x}{1+\cos x} = \cos x$.

5. Prove the identity $\frac{1}{1+\sin^2\alpha} + \frac{1}{1+\operatorname{cosec}^2\alpha} = 1$.

6. Prove the identity $\frac{\tan\alpha}{\sec\alpha - 1} + \frac{\tan\alpha}{\sec\alpha + 1} = 2\operatorname{cosec}\alpha$.

7. Show that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

8. Prove that $\cos^2 x + \cot^2 x \cos^2 x = \cot^2 x$.

9. Prove that $\frac{\tan^2\theta + 1}{\tan\theta \operatorname{cosec}^2\theta} = \tan\theta$.

10. Given that $p = \cos\theta + \sin\theta$ and $q = \cos\theta - \sin\theta$,
    (i) show that $p^2 - q^2 = 4\sin\theta\cos\theta$
    (ii) express $\frac{p}{q}$ in terms of $\tan\theta$.

10.7 Degrees , Minutes and Seconds

A degree is subdivided into 60 equal parts called minutes. Again a minute is subdivided into 60 equal parts called seconds. Thus we have ,

60 seconds = 1 minute

60 minutes = 1 degree

10.8 Use of Trigonometric Tables

We have seen , with the help of geometry how to calculate the trigonometric ratios of certain angles like 30° and 60°. But how about sin 17°or cos 32° 48' etc. Using techniques beyond the scope of this book, a table has been prepared containing the values of trigonometric ratios for all angles from 0° to 90° at intervals of 6 minutes.

We can use these tables (i) to find the value of a trigonometric ratio of a given angle, and (ii) to find an angle corresponding to a given value of a trigonometric ratio of that angle.

Let us familiarize ourselves with the tables. In the first column, we find the angles from 0° to 90°. Then the next ten columns are arranged at intervals of 6 minutes starting from 0'. For the values of certain angles whose miniutes parts are not multiples of 6, the last 5 columns, known as difference columns will be useful.

In table 10.2 a part of sine table is shown.

Table 10.2

	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
10	.1736	.1754	-	-	-	-	-	-	-	-	3	6	9	11	14
11	.1908	.1925	-	-	-	-	-	-	-	-	3	6	9	11	14
12	.2079	.2096	-	-	-	-	-	-	-	-	3	6	9	11	14
13	.2250	$\boxed{.2267}$	-	-	-	-	-	-	-	-	3	6	8	11	14
14	.2419	.2436	-	-	-	-	-	-	-	-	3	6	8	11	14

Thus to find sin 13° 6' we have to look the intersection of the row labelled 13 and the column labelled 6'. We will get sin 13° 6' = .2267

Example 1. With the use of trigonometric tables determine the values of

(a) sin 31° 22′ (b) cos 15° 44′

	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
30	.5000	5015	5030	5045	5060	-	-	-	-	-	3	5	8	10	13
31	.5150	5165	5180	$\boxed{5195}$	5210	-	-	-	-	-	2	5	7	$\boxed{10}$	12
32	.5299	5314	5329	5344	5358	-	-	-	-	-	2	5	7	10	12

(a) From the above sine tables, sin 31° 18′ = 0.5195

Mean difference for 4′ = $\frac{10}{0.5205}$ (to be added)

Thus sin 31° 22′ = 0.5205

	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
15	9659	9655	9650	9646	9641	9636	9632	$\boxed{9627}$	9622	9617	1	$\boxed{2}$	2	3	4
16	-	-	-	-	-	9588	9583	9578	-	-	1	2	2	3	4
17	-	-	-	-	-	9537	9532	9527	-	-	-	-	-	-	-

(b) From the above cosine tables, cos 15° 42′ = 0.9627

Mean difference for 2′ = $\frac{2}{0.9625}$ (to be subtracted)

Thus cos 15° 44′ = 0.9625

Example 2. Determine the angle θ , given that $\tan\theta = 0.4688$.

	0'	6'	[illegible]	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
25	0.4663	$\boxed{4684}$	4706	4727	-	-	-	-	-	-	$\boxed{4}$	7	11	14	18
26	0.4877	4899	4921	4942	-	-	-	-	-	-	4	7	11	15	18
27	0.5095	5117	-	-	-	-	-	-	-	-	-	-	-	-	-

$$\text{Given value} = 0.4688$$

$$\text{from the table } \tan 25°\,6' = \frac{0.4684}{4}$$

From difference columns 4 corresponds to an increase of 1' in the angle , and hence the angle is 25° 7'.

Example 3. There is a lake and a surveyor wants to measure the distance between two points on the opposite banks. He cannot measure across the lake directly. But he can measure AC , where C is a point on the straight line perpendicular to AB , and the angle ACB.

Solution

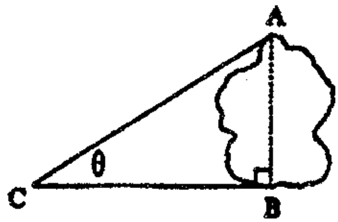


Fig. 10.24

Suppose AC = 305 ft and $\theta = 32°$

$$\text{Since } \sin\theta = \frac{AB}{AC}$$

$$AB = AC \sin\theta$$

The surveyor looks for sin 32° in his tables and finds that

$$\sin 32° = 0.5299$$

Hence $AB = 305 \times 0.5299 = 161.62$ ft

Thus the tables help the surveyors to solve such kinds of problems.

The tables can also be used in solving problems such as finding the heights of towers or trees without actually climbing and measuring.

The following is an example .

Example 4. To find the height of a flag post without climbing it, measure a certain distance , say , AC , from the foot of the post. Then measure the angle made by the top of the post with AC .

Solution

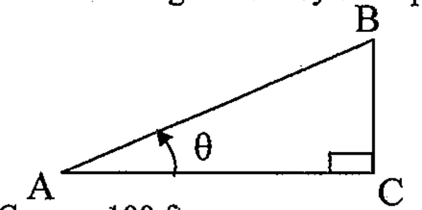


Fig. 10.25

Suppose $AC = 100$ ft

and $\theta = 22°$

Since $\tan\theta = \dfrac{BC}{AC}$

$$BC = AC \tan\theta = 100 \tan 22° = 100 \times 0.4040 = 40.4 \text{ ft}$$

10.9 Angle of Elevation and Angle of Depression

Suppose we are viewing an object. The line of sight or line of vision is a straight line from our eye to the object we are viewing.

If the object is above the horizontal from the eye, we have to lift up our head to view the object. In the process our eye moves through an angle. This angle is called the angle of elevation. (See Fig. 10.26)

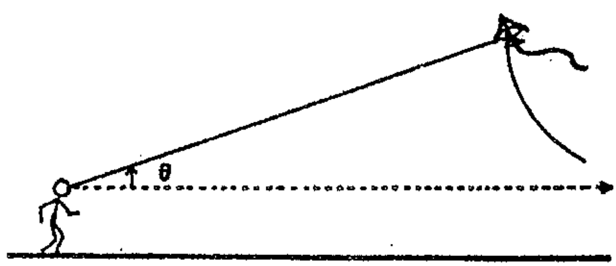


Fig. 10.26

If the object is below the horizontal from the eye, we have to move our head downwards to view the object. In the process , our eye moves through an angle. This angle is called the angle of depression. (See Fig. 10.27)

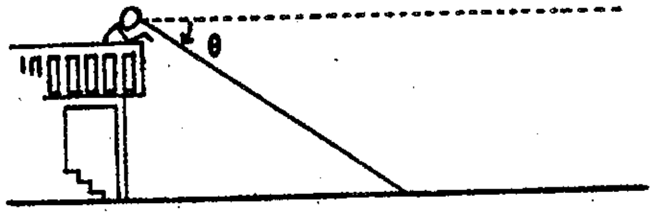


Fig. 10.27

Example 1. A kite is at the end of a 45 metres string that is taut. It is 30 metres above the ground. What is the angle of elevation of the kite ?

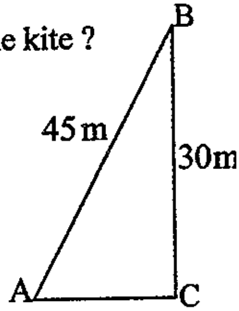


Fig. 10.28

Solution

We wish to find $\angle A$ in Fig. 10.28 .

$$\sin A = \frac{30}{45} = .6667 = \sin 41° 49'$$

$\angle A = 41° 49'$, which is the angle of elevation.

Example 2. From the top of a house the angle of depression of a point on the ground is 25°. The point is 45 metres from the base of the building . How high is the building ?

Solution

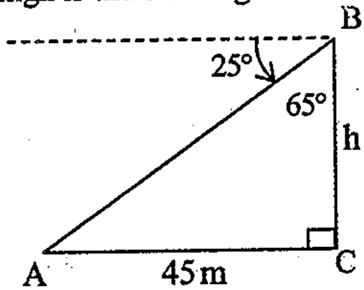


Fig. 10.29

The angle of depression is not within the triangle .However , the complement of the angle, 65°, is the measure of $\angle B$ inside the triangle.

Thus

$$\cot\ 65^\circ = \frac{h}{45}$$

$$h = 45 \cot 65^\circ = 45 \times 0.4663 = 20.98 \text{ metres}$$

Exercise 10.5

1. Find the following values by using the table of trigonometric ratios.

   (a) $\sin 15^\circ$ (b) $\cos 40^\circ$ (c) $\tan 12^\circ$ (d) $\sin 3^\circ$

   (e) $\cos 52^\circ 40'$ (f) $\tan 35^\circ$ (g) $\sin 18^\circ 47'$ (h) $\cos 26^\circ$

2. Find angle t, given that

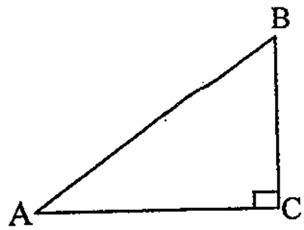
   (a) $\sin\ t = 0.3090$ (b) $\tan\ t = 0.3057$

   (c) $\tan\ t = 2.9042$ (d) $\cos t = 0.2079$

   (e) $\sin\ t = 0.9613$ (f) $\tan\ t = 0.5543$

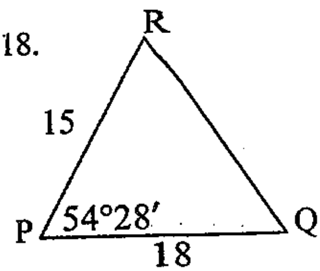
   (g) $\cos t = 0.4540$ (h) $\sin\ t = 0.5150$

3. In the given right triangle ABC, AB = 20 ft, $\angle A = 38^\circ$. Find BC and AC.



4. In a right triangle ABC with right angle at C, $\angle A = 42^\circ$ and AC = 7. Find BC.

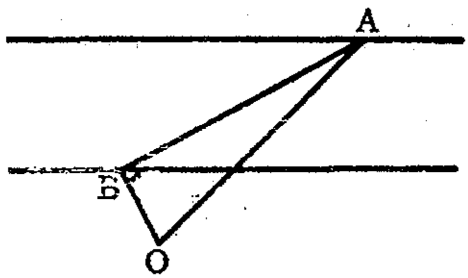
5. In triangle PQR, $\angle P = 54^\circ 28'$, PR = 15 and PQ = 18. Find the length of the perpendicular drawn from R to PQ.



6. In ΔABC, $\angle A = 70°$, $AC = 12$ and $AB = 20$. Find the length of perpendicular drawn from C to AB. Also find the area of the triangle ABC.

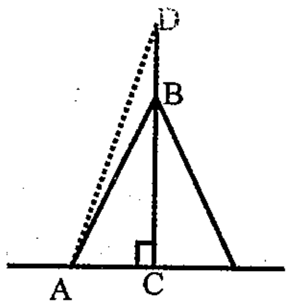
7. Find the area of the triangle ABC, given $AB = 30$, $BC = 16$ and $\angle B = 27°$.

8. In surveying for new highway, an engineer drove two large intakes A and B, on opposite banks of a river, to make the sites of bridge abutments. Then, taking point O, 100 ft from B and such that $OB \perp AB$, he measured the angle AOB. If measure of angle AOB $= 73°$, what is the distance across the river from A to B ?



9. A ladder on a fire truck can be extended to a maximum length of 68 ft, when elevated to its maximum angle of 70°. The base of the ladder is mounted on the truck 7 ft above the ground. How high above the ground will the ladder reach ?

10. A radio tower is anchored by long cables, such as AB in the figure. If A is 250 ft from the base of the tower and if measure of angle BAC = 59°, how long is the cable ? How high is the tower DC, if $m\angle DAC = 71°$?



Miscellaneous Exercises

1. Solve the triangles : $\angle B = 55° 30'$, $\angle A = 34° 30'$, $c = 10$.

2. Solve the isosceles triangle: (Draw the perpendicular from the vertex to the base).

(a) $b = c = 12$, $\angle B = 44^\circ$

(b) $a = b = 9$, $\angle A = 62^\circ$

(c) $b = c = 11$, $\angle A = 80^\circ\, 36'$

3. A mountain railway rises for 400 yards at a uniform slope of $16^\circ\, 24'$ with the horizontal. What is the horizontal distance between its two ends ?

4. A circus artist is climbing a rope stretched from the top of a pole and fixed at the ground. The height of the pole is 15 yards and the angle made by the rope with ground level is $35^\circ\, 18'$. Calculate the distance covered by the artist in climbing to the top of the pole.

5. A vertical mast is secured from its top by straight cables 600 ft long fixed into the ground. The cables make angles of 66° with the ground. What is the height of the mast ?

6. A tree is kept upright by a pole 40 ft long which leans against it at a height of 25 ft from the ground. If the foot of the pole is on level ground , what angle does the pole make with the ground ?

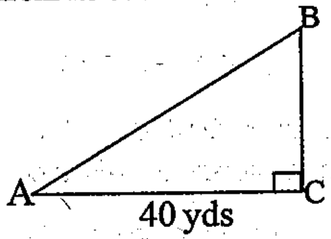
7. A kite flying at a height of 65 yards is attached to a string inclined at $45^\circ$to the horizontal. What is the length of the string ? Assume there is no slack in the string.

8. An observer , $5\frac{1}{2}$ ft tall , is 20 yards away from a tower 30 yards high. Determine the angle of elevation from his eye to the top of the tower.

9. What is the length of the shadow of a stick 10ft high when the sun's altitude is 50° ?

10. Find the altitude of the sun if the shadow cast by a tower 60 ft high is 50 ft long.

11. A surveyor wants to determine the height of a lighthouse (see figure). He measures the angle at A and finds that $\tan A = \frac{3}{4}$. What is the height of the lighthouse if A is 40 yards from its base ?



12. The length of a string between a kite and a point on the ground is 90 metres. If the string makes an angle θ with the level ground such that $\tan \theta = 5$, how high is the kite ? Assume there is no slack in the string.

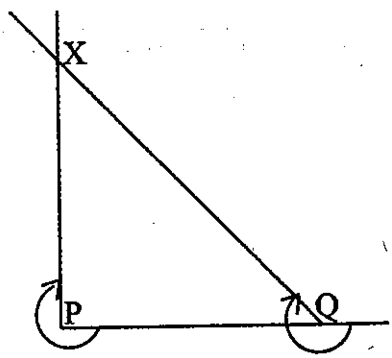
13. Two men are on the opposite sides of a tower. They measure the angles of elevation to the top of the tower as 20° and 24° respectively. If the height of the tower is 40 yards , find the distance between the men.

14. A, B and C are three points on horizontal ground, AB = 14 m, AC = 10 m. A vertical mast AT stands at the point A and the angle of elevation of the top of this mast from the points B is 20°, calculate the height of the mast and find the angle of depression from the top of the mast to the point C on the ground.

15. A, B and C are three points on level ground B is 17 m due east of C and BC=BA The point P on AB is such that BP = 8 m and CP = 15 m, find the $\angle$ ACP and AC. If a vertical mast stand at the point P and the angle of elevation of the top of the mast from A is 30°, calculate the height of the mast.

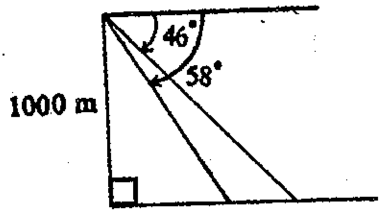
16. Two points X and Y, on level ground, are such that XY = 240 m. A vertical mast YT of height 30 m stands at Y. Calculate the angle of elevation of T from X.

17. Two lighthouses, P and Q, are 10 km apart with Q due east of P. Each lighthouse emits a narrow beam of light. Initially both beams are pointing due east and then at the same instant they start to rotate in a clockwise direction. The beam from P



takes 8 seconds to make a complete revolution, while the beam from Q takes 9 seconds. Six seconds after the beams start to rotate, they cross at a point X. Calculate $\angle PXQ$ and the distance PX.

18. A captain of an aeroplane flying at an altitude of 1000 meters sights two ships as shown in the figure. If the angles of depression are 58° and 46° respectively , determine the distance between the ships.



SUMMARY

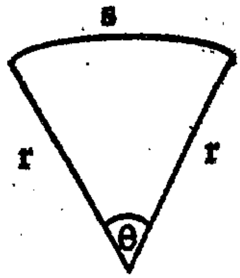
1. π radian = 180°
2. For a sector of a circle enclosed by two radii that subtend an angle of θ radians at the centre, the are length s is given by

$$s = r\theta$$

and the area of the sector A is given by

$$A = \frac{1}{2} r^2\theta$$

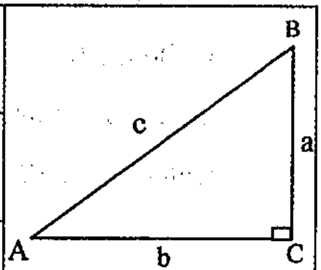
where r is the radius of the circle.



3. Six Trigonometric Ratios

Ratio	Abbreviation	Definition	For ΔABC	
sine of $\angle A$	sin A	$\frac{\text{side opposite}}{\text{hypotenuse}}$	$\frac{a}{c}$	

cosine of $\angle A$	cos A	$\frac{\text{side adjacent}}{\text{hypotenuse}}$	$\frac{b}{c}$
tangent of $\angle A$	tan A	$\frac{\text{side opposite}}{\text{side adjacent}}$	$\frac{a}{b}$
cotangent of $\angle A$	cot A	$\frac{\text{side adjacent}}{\text{side opposite}}$	$\frac{b}{a}$
secant of $\angle A$	sec A	$\frac{\text{hypotenuse}}{\text{side adjacent}}$	$\frac{c}{b}$
cosecant of $\angle A$	cosec A	$\frac{\text{hypotenuse}}{\text{side opposite}}$	$\frac{c}{a}$



LIST OF SYMBOLS

Following are short explanations of the symbols used in this book .

Symbols	Meaning
$=$	Equals ; is equal to ; is the same as
$\neq$	Not equal to ; is different from
$<$	Is less than (apply only to real numbers)
$\leq$	Is less than or equal to (...)
$>$	Is greater than (...)
$\geq$	Is greater than or equal to (...)
$\equiv$	Is identically equal to
$\cong$	Is congruent to : for geometric figures
$\sim$	Is similar to : for geometric figures
//	Is parallel to : for lines
$\perp$	Is perpendicular to : for lines
AB	The positive number which is the distance between the two points A and B .

$\angle ABC$	The angle having B as vertex and BA and BC as sides.
ΔABC	The triangle having A , B , and C as vertices
$\{a, b, c\}$	The set whose elements are a , b , and c
$A \subset B$	Set A is a subset of set B
$x \in A$	x is an element of set A
$A \cap B$	The intersection of sets A and B
$A \cup B$	The union of sets A and B
$\{x \mid x < 3\}$	The set of all numbers less than 3
$ABC \leftrightarrow DEF$	The correspondence which matches A with D , B with E , and C with F
$\odot P$	The circle with centre P
$\sqrt{a}$	The positive square root of a
$\lvert a \rvert$	The absolute value of a