13

Sinusoidal Alternating Waveforms

13.1 INTRODUCTION

The analysis thus far has been limited to dc networks, networks in which the currents or voltages are fixed in magnitude except for transient effects. We will now turn our attention to the analysis of networks in which the magnitude of the source varies in a set manner. Of particular interest is the time-varying voltage that is commercially available in large quantities and is commonly called the *ac voltage*. (The letters *ac* are an abbreviation for *alternating current*.) To be absolutely rigorous, the terminology *ac voltage* or *ac current* is not sufficient to describe the type of signal we will be analyzing. Each waveform of Fig. 13.1 is an alternating waveform available from commercial supplies. The term *alternating* indicates only that the waveform alternates between two prescribed levels in a set time sequence (Fig. 13.1). To be absolutely correct, the term *sinusoidal, square wave,* or *triangular* must also be applied. The pattern of particular interest is the *sinusoidal* ac voltage of Fig. 13.1. Since this type of signal is encountered in the vast majority of instances, the abbreviated phrases *ac voltage* and *ac current* are commonly applied without confusion. For the other patterns of Fig. 13.1, the descriptive term is always present, but frequently the *ac* abbreviation is dropped, resulting in the designation *square-wave* or *triangular* waveforms.

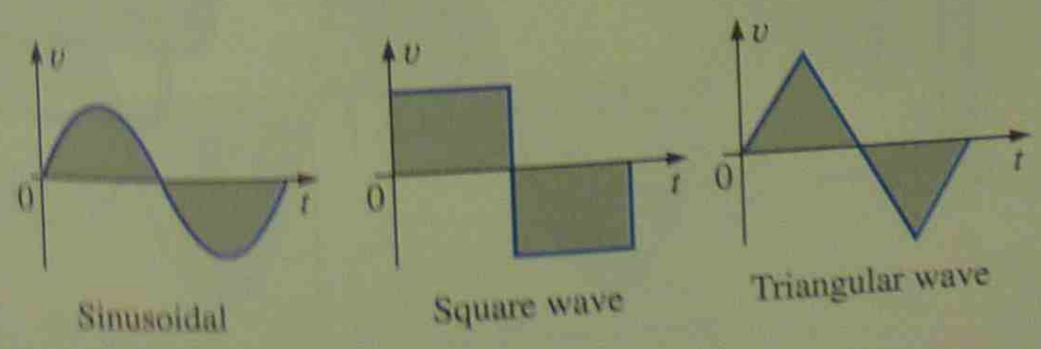


FIG. 13.1
Alternating waveforms.

One of the important reasons for concentrating on the sinusoidal ac voltage is that it is the voltage generated by utilities throughout the world. Other reasons include its application throughout electrical, electronic, communication, and industrial systems. In addition, the chapters to follow will reveal that the waveform itself has a number of characteristics that will result in a unique response when it is applied to the basic electrical elements. The wide range of theorems and methods introduced for dc networks will also be applied to sinusoidal ac systems. Although the application of sinusoidal signals will raise the required math level, once the notation given in Chapter 14 is understood, most of the concepts introduced in the dc chapters can be applied to ac networks with a minimum of added difficulty.

The increasing number of computer systems used in the industrial community requires, at the very least, a brief introduction to the terminology employed with pulse waveforms and the response of some fundamental configurations to the application of such signals. Chapter 22 will serve such a purpose.

13.2 SINUSOIDAL ac VOLTAGE CHARACTERISTICS AND DEFINITIONS

Generation

Sinusoidal ac voltages are available from a variety of sources. The most common source is the typical home outlet, which provides an ac voltage that originates at a power plant; such a power plant is most commonly fueled by water power, oil, gas, or nuclear fusion. In each case an *ac generator* (also called *alternator*), as in Fig. 13.2(a), is the primary component in the energy-conversion process. The power to the shaft developed by one of the energy sources listed will turn a *rotor* (constructed of alternating magnetic poles) inside a set of windings housed

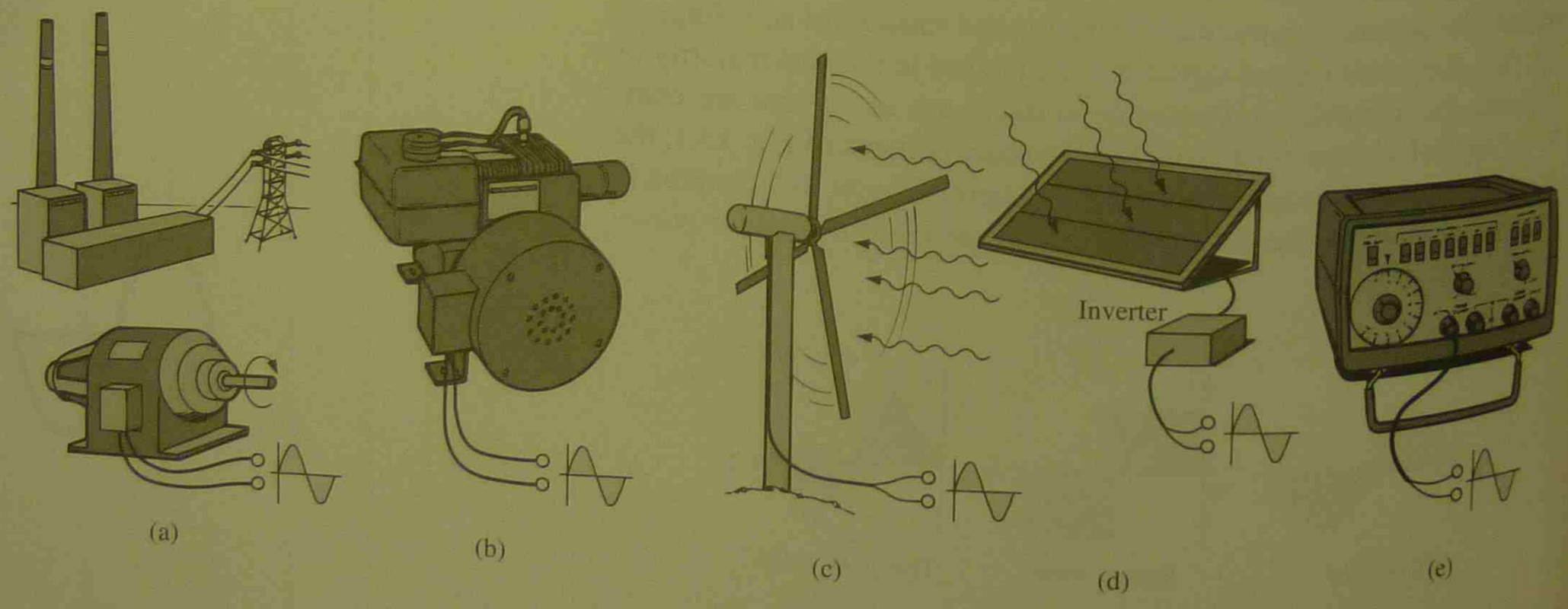


FIG. 13.2
Various sources of ac power. (a) Generating plant; (b) portable ac generator; (c) wind-power station; (d) solar panel; (e) function generator.

in the *stator* (the stationary part of the dynamo) and induce a voltage across the windings of the stator, as defined by Faraday's law,

$$e = N\frac{d\phi}{dt}$$

Through proper design of the generator, a sinusoidal ac voltage is developed that can be transformed to higher levels for distribution through the power lines to the consumer. For isolated locations where power lines have not been installed portable ac generators [Fig. 13.2(b)] are available that run on gasoline. As in the larger power plants, however, an ac generator is an integral part of the design.

In an effort to conserve our natural resources, wind power and solar energy are receiving increasing interest from various districts of the world that have such energy sources available in level and duration that make the conversion process viable. The turning propellers of the wind-power station [Fig. 13.2(c)] are connected directly to the shaft of an ac generator to provide the ac voltage described above. Through light energy absorbed in the form of *photons*, solar cells [Fig. 13.2(d)] can generate dc voltages. Through an electronic package called an *inverter*, the dc voltage can be converted to one of a sinusoidal nature. Boats, RVs, etc., make frequent use of the inversion process in isolated areas.

Sinusoidal ac voltages with characteristics that can be controlled by the user are available from *function generators*, such as the one in Fig. 13.2(e). By setting the various switches and controlling the position of the knobs on the face of the instrument, sinusoidal voltages of different peak values and different repetition rates can be made available. The function generator plays an integral role in the investigation of the variety of theorems, methods of analysis, and topics to be introduced in the chapters to follow.

Definitions

The sinusoidal waveform of Fig. 13.3 with its additional notation will now be used as a model in defining a few basic terms. These terms can,

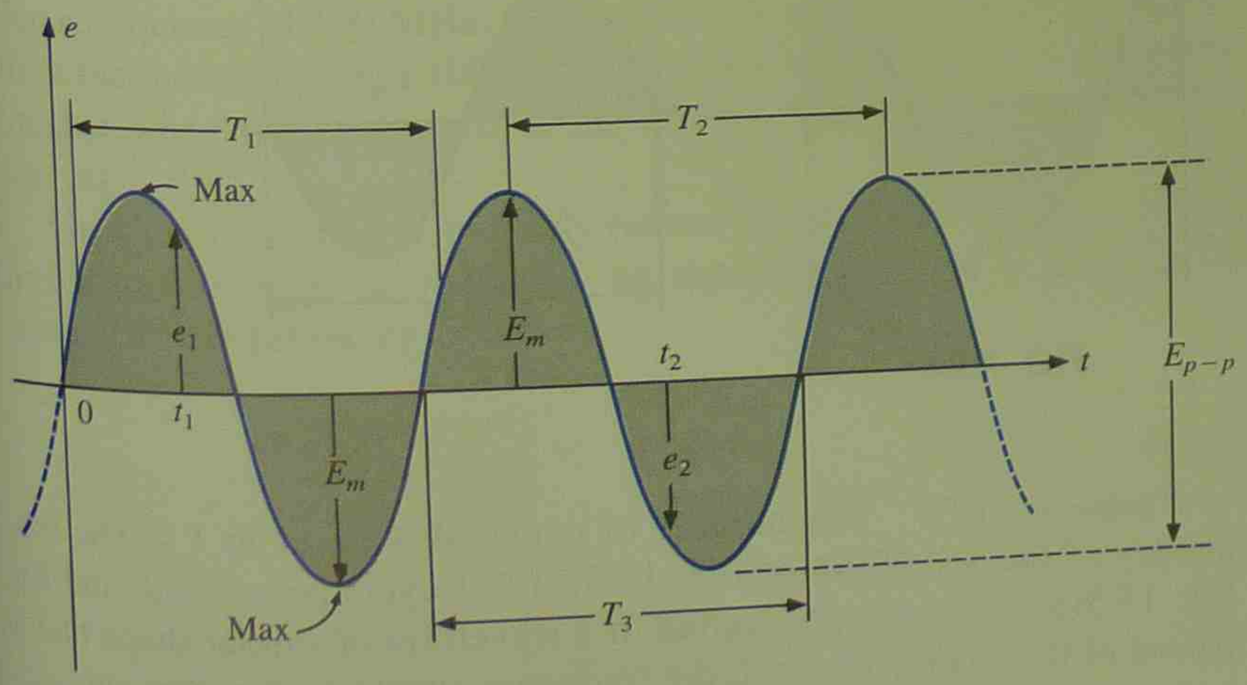


FIG. 13.3
Sinusoidal voltage.

however, be applied to any alternating waveform. It is important to remember as you proceed through the various definitions that the vertical scaling is in volts or ampere and the horizontal scaling is *always* in units of time.

Waveform: The path traced by a quantity, such as the voltage in Fig. 13.3, plotted as a function of some variable such as time (as above), position, degrees, radians, temperature, and so on.

Instantaneous value: The magnitude of a waveform at any instant of time; denoted by lowercase letters (e_1, e_2).

Peak amplitude: The maximum value of a waveform as measured from its *average*, or *mean*, value, denoted by uppercase letters (such as E_m for sources of voltage and V_m for the voltage drop across a load). For the waveform of Fig. 13.3 the average value is zero volts and E_m is as defined by the figure.

Peak value: The maximum instantaneous value of a function as measured from the zero-volt level. For the waveform of Fig. 13.3 the peak amplitude and peak value are the same, since the average value of the function is zero volts.

Peak-to-peak value: Denoted by $E_{p\text{-}p}$ or $V_{p\text{-}p}$, the full voltage between positive and negative peaks of the waveform, that is, the sum of the magnitude of the positive and negative peaks.

Periodic waveform: A waveform that continually repeats itself after the same time interval. The waveform of Fig. 13.3 is a periodic waveform.

Period (T): The time interval between successive repetitions of a periodic waveform; the period $T_1 = T_2 = T_3$ in Fig. 13.3, so long as successive *similar points* of the periodic waveform are used in determining T.

Cycle: The portion of a waveform contained in *one period* of time. The cycles within T_1, T_2, and T_3 of Fig. 13.3 may appear different in Fig. 13.4, but they are all bounded by one period of time and therefore satisfy the definition of a cycle.

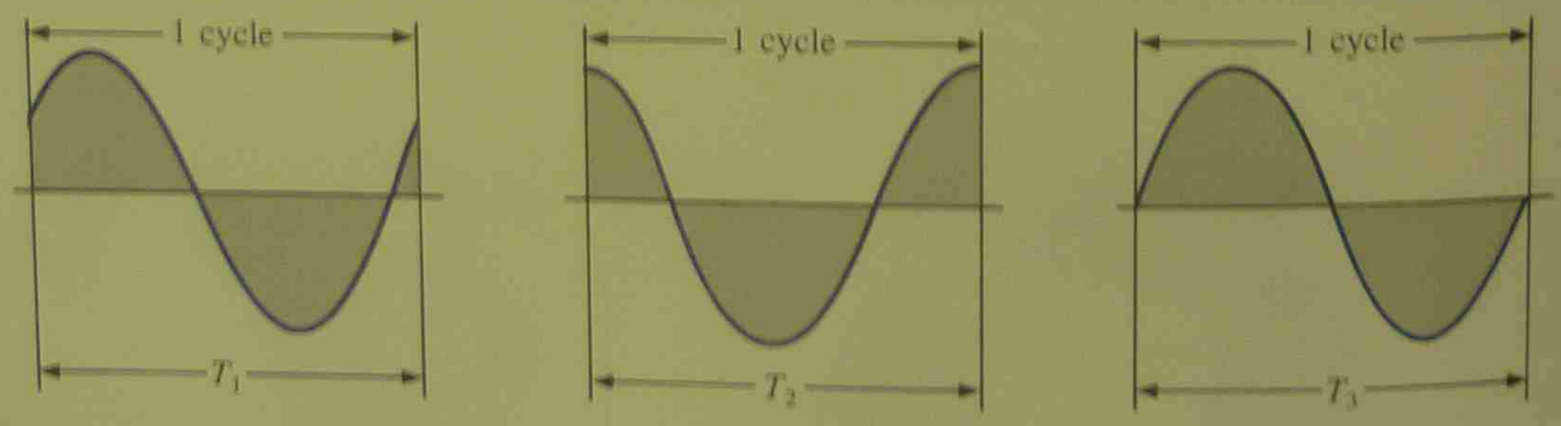
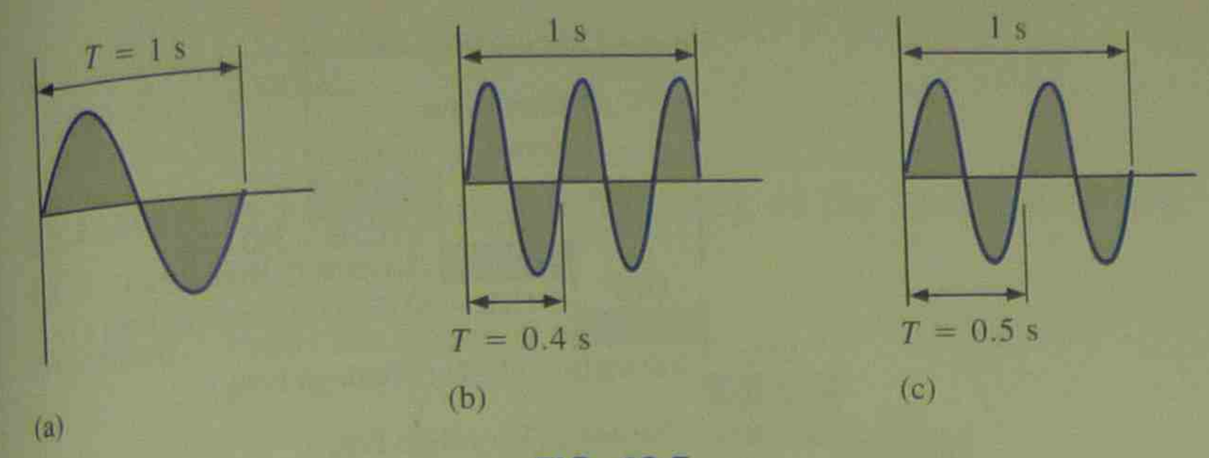


FIG. 13.4

Frequency (f): The number of cycles that occur in 1 s. The frequency of the waveform of Fig. 13.5(a) is 1 cycle per second, and for Fig. 13.5(b), $2\frac{1}{2}$ cycles per second. If a waveform of similar shape had a period of 0.5 s [Fig. 13.5(c)], the frequency would be 2 cycles per second.



(a) (b) (c)

FIG. 13.5

The unit of measure for frequency is the *hertz* (Hz), where

$$\boxed{1 \text{ hertz (Hz)} = 1 \text{ cycle per second (c/s)}} \qquad \textbf{(13.1)}$$

The unit hertz is derived from the surname of Heinrich Rudolph Hertz (Fig. 13.6), who did original research in the area of alternating currents and voltages and their effect on the basic R, L, and C elements. The frequency standard for North America is 60 Hz, whereas for Europe it is predominantly 50 Hz.

Using a log scale (described in detail in Chapter 21) a frequency spectrum from 1 to 1000 GHz can be scaled off on the same axis as shown in Fig. 13.7. A number of terms in the various spectrums are probably familiar to the reader from everyday experiences. Note that the audio range (human ear) extends from only 15 Hz to 20 kHz, but the transmission of radio signals can occur between 3 kHz and 300 GHz. The uniform process of defining the intervals of the radio frequency spectrum from VLF to EHF is quite evident from the length of the bars in the figure (although keep in mind that it is a log scale so the frequencies encompassed within each segment are quite different). Other frequencies of particular interest (TV, CB, microwave, etc.) are also included for reference purposes. Although it is numerically easy to talk about frequencies in the megahertz and gigahertz range, keep in mind that a frequency of 100 MHz, for instance, represents a sinusoidal waveform that passes through 100,000,000 cycles in only 1 s—an incredible number when we compare it to the 60 Hz of our conventional power sources.

Since the frequency is inversely related to the period—that is, as one increases, the other decreases by an equal amount—the two can be related by the following equation:

$$\boxed{f = \frac{1}{T}} \qquad \begin{aligned} f &= \text{Hz} \\ T &= \text{seconds (s)} \end{aligned} \qquad \textbf{(13.2)}$$

or

$$\boxed{T = \frac{1}{f}} \qquad \textbf{(13.3)}$$

German (Hamburg, Berlin, Karlsruhe)
(1857–1894)
Physicist
Professor of Physics,
Karlsruhe Polytechnic and University of Bonn



Courtesy of the Smithsonian Institution Photo No. 66,606

Spurred on by the earlier predictions of the English physicist James Clerk Maxwell, Heinrich Hertz produced *electromagnetic waves* in his laboratory at the Karlsruhe Polytechnic while in his early 30s. The rudimentary *transmitter* and *receiver* were in essence the first to broadcast and receive radio waves. He was able to measure the *wavelength* of the electromagnetic waves and confirmed that the *velocity of propagation* is in the same order of magnitude as light. In addition, he demonstrated that the *reflective* and *refractive* properties of electromagnetic waves are the same as those for heat and light waves. It was indeed unfortunate that such an ingenious, industrious individual should pass away at the very early age of 37 due to a bone disease.

FIG. 13.6 HEINRICH RUDOLPH HERTZ

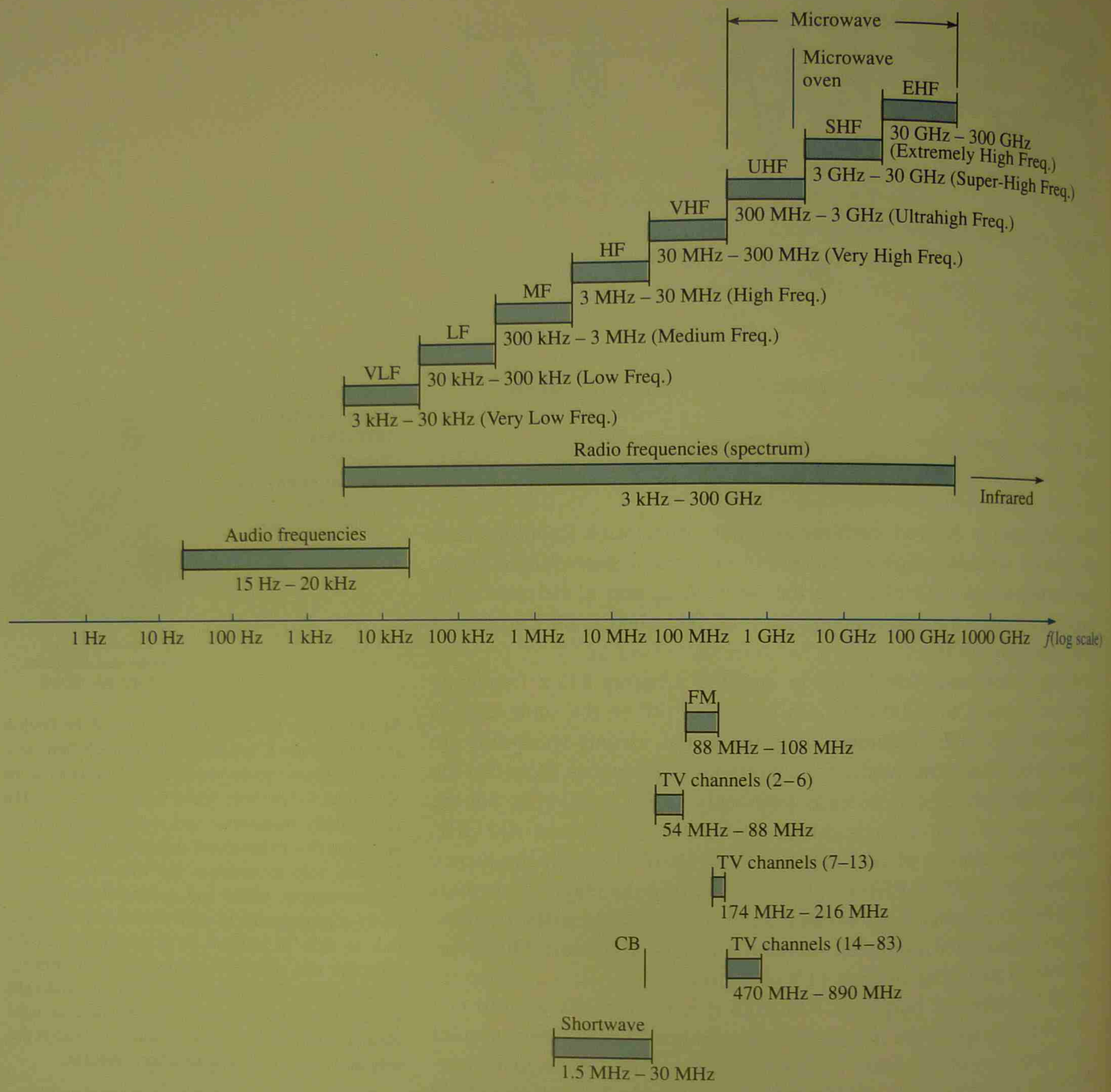


FIG. 13.7

EXAMPLE 13.1 Find the period of a periodic waveform with a frequency of

a. 60 Hz

b. 1000 Hz

Solutions:

a. $T = \frac{1}{f} = \frac{1}{60\ \text{Hz}} \cong 0.01667$ s, or **16.67 ms**

(a recurring value since 60 Hz is so prevalent)

b. $T = \frac{1}{f} = \frac{1}{1000\ \text{Hz}} = 10^{-3}\ \text{s} = \mathbf{1\ ms}$

EXAMPLE 13.2 Determine the frequency of the waveform of Fig. 13.8.

Solution: From the figure, $T = (25\ \text{ms} - 5\ \text{ms}) = 20\ \text{ms}$ and

$$f = \frac{1}{T} = \frac{1}{20 \times 10^{-3}\ \text{s}} = \mathbf{50\ Hz}$$

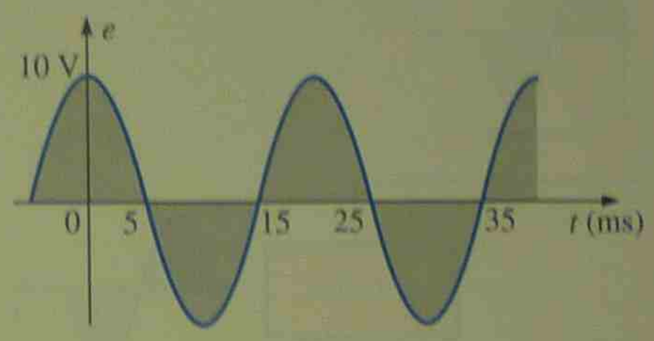
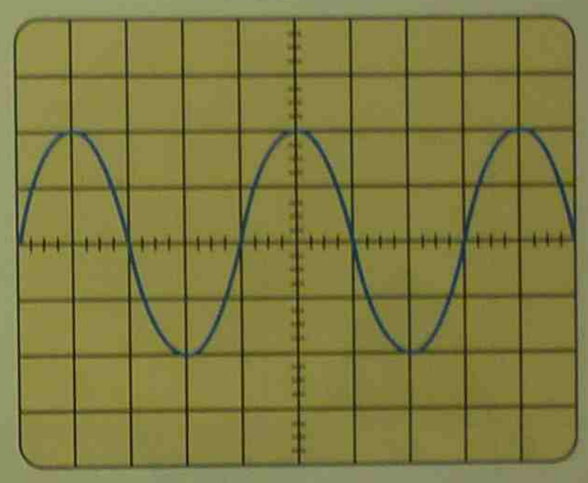


FIG. 13.8

EXAMPLE 13.3 The oscilloscope is an instrument that will display alternating waveforms such as those described above. A sinusoidal pattern appears on the oscilloscope of Fig. 13.9 with the indicated vertical and horizontal sensitivities. The vertical sensitivity defines the voltage associated with each vertical centimeter (cm) of the display. Virtually all oscilloscope screens are cut into a crosshatch pattern of lines separated by 1 cm in the vertical and horizontal directions. The horizontal sensitivity defines the time period associated with each horizontal centimeter of the display.

For the pattern of Fig. 13.9 and the indicated sensitivities, determine the period, frequency, and peak value of the waveform.



Vertical sensitivity = 0.1 V/cm
Horizontal sensitivity = 50 μs/cm

FIG. 13.9

Solution: One cycle spans 4 cm. The period is

$$T = 4(50\ \mu\text{s}) = \mathbf{200\ \mu s}$$

and the frequency is

$$f = \frac{1}{T} = \frac{1}{200 \times 10^{-6}\ \text{s}} = \mathbf{5\ kHz}$$

The vertical height above the horizontal axis encompasses 2 cm. Therefore,

$$V_m = (2\ \text{cm})(0.1\ \text{V/cm}) = \mathbf{0.2\ V}$$

Defined Polarities and Direction

In the following analysis, we will find it necessary to establish a set of polarities for the sinusoidal ac voltage and a direction for the sinusoidal ac current. In each case, the polarity and current direction will be for an instant of time in the positive portion of the sinusoidal waveform. This is shown in Fig. 13.10 with the symbols for the sinusoidal ac voltage and current. A lowercase letter is employed for each to indicate that the quantity is time dependent; that is, its magnitude will change with time. The need for defining polarities and current direction will become quite obvious when we consider multisource ac networks. Note in the last sentence the absence of the term *sinusoidal* before the phrase *ac networks*. This will occur to an increasing degree as we progress; *sinusoidal* is to be understood unless otherwise indicated.

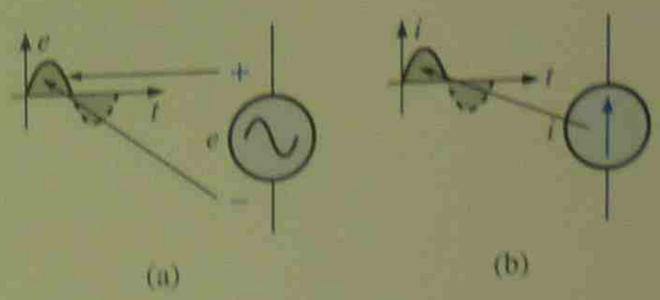
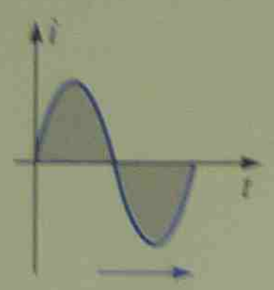


FIG. 13.10
(a) Sinusoidal ac voltage sources; (b) sinusoidal current sources.



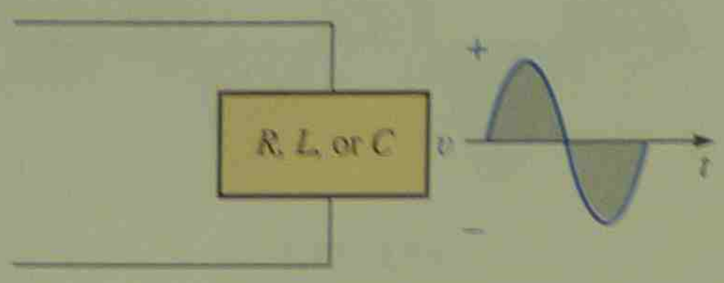
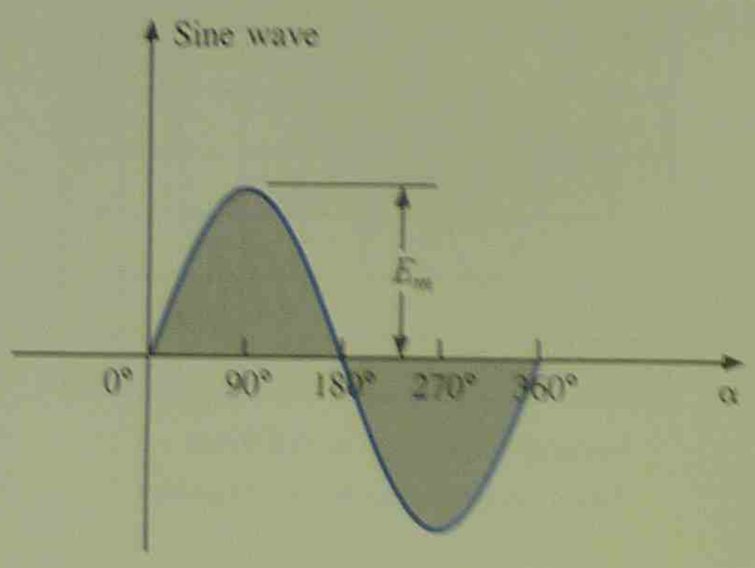


FIG. 13.11



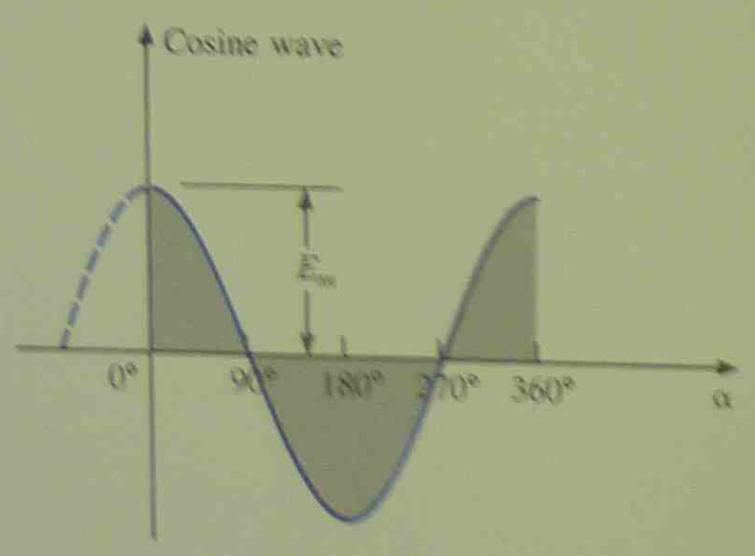


FIG. 13.12

13.3 THE SINE WAVE

The terms defined in the previous section can be applied to any type of periodic waveform, whether smooth or discontinuous. The sinusoidal waveform is of particular importance, however, since it lends itself readily to the mathematics and the physical phenomena associated with electric circuits. Consider the power of the following statement:

The sine wave is the only alternating waveform whose shape is unaffected by the response characteristics of R, L, and C elements.

In other words, if the voltage across (or current through) a resistor, coil, or capacitor is sinusoidal in nature, the resulting current (or voltage, respectively) for each will also have sinusoidal characteristics as shown in Fig. 13.11. If a square wave or a triangular wave were applied, such would not be the case. It must be pointed out that the above statement is also applicable to the cosine wave, since the waves differ only by a 90° shift on the horizontal axis, as shown in Fig. 13.12.

The unit of measurement for the horizontal axis of Fig. 13.12 is the *degree*. A second unit of measurement frequently used is the *radian* (rad). It is defined by a quadrant of a circle such as in Fig. 13.13 where the distance subtended on the circumference equals the radius of the circle.

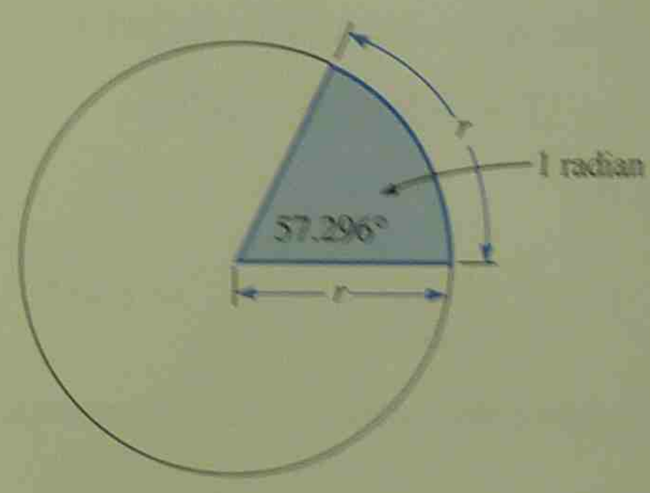


FIG. 13.13
Defining the radian.

If we define x as the number of intervals of r (the radius) around the circumference of the circle, then

$$C = 2\pi r = x \cdot r$$

and we find

$$x = 2\pi$$

Therefore, there are 2π rad around a 360° circle, as shown in Fig. 13.14, and

$$2\pi \text{ rad} = 360° \tag{13.4}$$

with

$$1 \text{ rad} \cong 57.3° \tag{13.5}$$

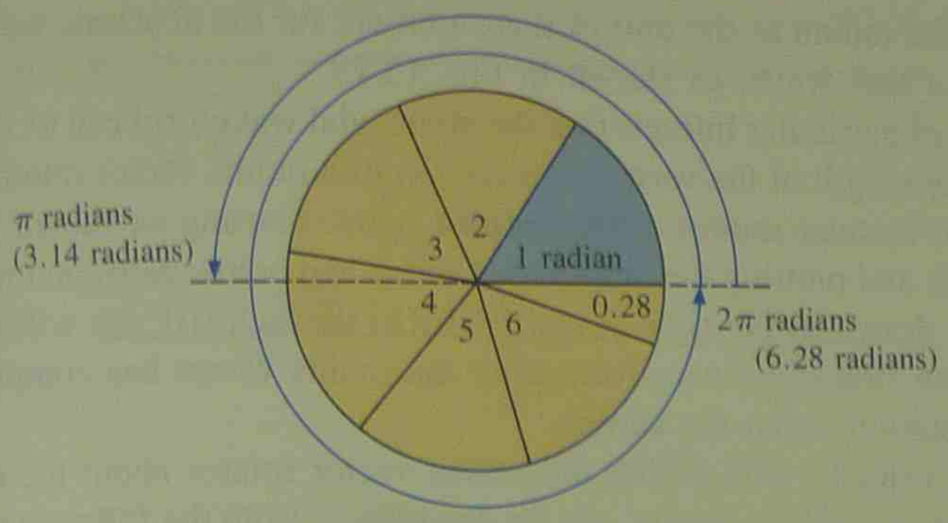


FIG. 13.14

A number of electrical formulas contain a multiplier of π. This is one reason it is sometimes preferable to measure angles in radians rather than in degrees.

The quantity π is the ratio of the circumference of a circle to its diameter.

π has been determined to an extended number of places primarily in an attempt to see if a repetitive sequence of numbers appears. It does not. A sampling of the effort appears below:

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ \ldots$$

For our purposes, the following approximation will often be applied:

$$\boxed{\pi = 3.14} \tag{13.6}$$

For 180° and 360°, the two units of measurement are related as shown in Fig. 13.14. The conversion equations between the two are the following:

$$\boxed{\text{Radians} = \left(\frac{\pi}{180^\circ}\right) \times (\text{degrees})} \tag{13.7}$$

$$\boxed{\text{Degrees} = \left(\frac{180^\circ}{\pi}\right) \times (\text{radians})} \tag{13.8}$$

Applying these equations, we find

$$\textbf{90°}: \quad \text{Radians} = \frac{\pi}{180^\circ}(90^\circ) = \frac{\pi}{2}\ \textbf{rad}$$

$$\textbf{30°}: \quad \text{Radians} = \frac{\pi}{180^\circ}(30^\circ) = \frac{\pi}{6}\ \textbf{rad}$$

$$\frac{\pi}{3}\text{rad}: \quad \text{Degrees} = \frac{180^\circ}{\pi}\left(\frac{\pi}{3}\right) = \mathbf{60^\circ}$$

$$\frac{3\pi}{2}\text{rad}: \quad \text{Degrees} = \frac{180^\circ}{\pi}\left(\frac{3\pi}{2}\right) = \mathbf{270^\circ}$$

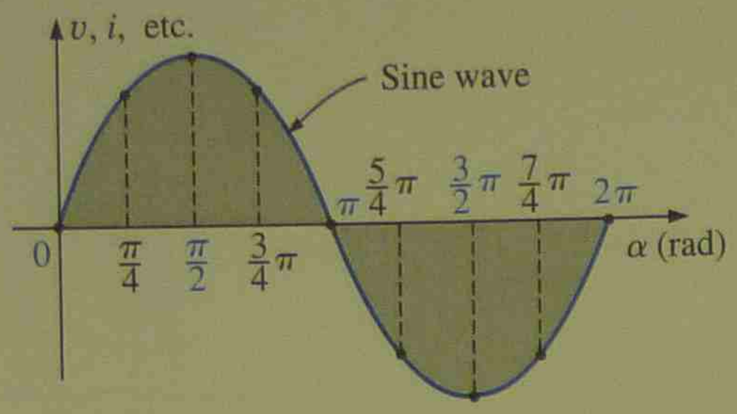


FIG. 13.15

Using the radian as the unit of measurement for the abscissa, we would obtain a sine wave, as shown in Fig. 13.15.

It is of particular interest that the sinusoidal waveform can be derived from the length of the *vertical projection* of a radius vector rotating in a uniform circular motion about a fixed point. Starting as shown in Fig. 13.16(a) and plotting the amplitude (above and below zero) on the coordinates drawn to the right [Figs. 13.16(b) through (i)], we will trace a complete sinusoidal waveform after the radius vector has completed a 360° rotation about the center.

The velocity with which the radius vector rotates about the center, called the *angular velocity,* can be determined from the following equation:

$$\text{Angular velocity} = \frac{\text{distance (degrees or radians)}}{\text{time (seconds)}} \tag{13.9}$$

Substituting into Eq. (13.9) and assigning the Greek letter omega (ω) to the angular velocity, we have

$$\omega = \frac{\alpha}{t} \tag{13.10}$$

and

$$\alpha = \omega t \tag{13.11}$$

Since ω is typically provided in radians per second, the angle α obtained using Eq. (13.11) is usually in radians. If α is required in degrees, Eq. (13.8) must be applied. The importance of remembering the above will become obvious in the examples to follow.

In Fig. 13.16, the time required to complete one revolution is equal to the period (T) of the sinusoidal waveform of Fig. 13.16(i). The radians subtended in this time interval are 2π. Substituting, we have

$$\omega = \frac{2\pi}{T} \quad \text{(rad/s)} \tag{13.12}$$

In words, this equation states that the smaller the period of the sinusoidal waveform of Fig. 13.16(i), or the smaller the time interval before one complete cycle is generated, the greater must be the angular velocity of the rotating radius vector. Certainly this statement agrees with what we have learned thus far. We can now go one step further and apply the fact that the frequency of the generated waveform is inversely related to the period of the waveform; that is, $f = 1/T$. Thus,

$$\omega = 2\pi f \quad \text{(rad/s)} \tag{13.13}$$

This equation states that the higher the frequency of the generated sinusoidal waveform, the higher must be the angular velocity. Equations

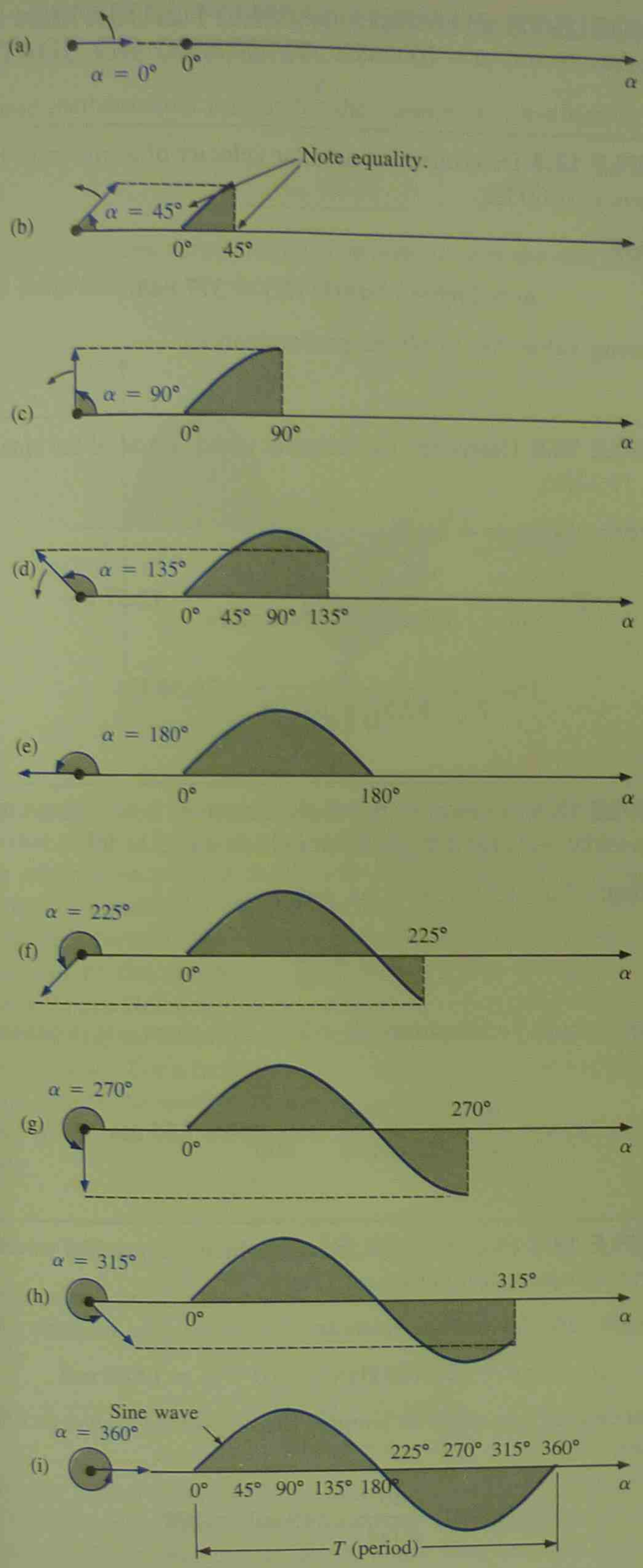

(a)
α = 0°
0°
α
(b)
Note equality.
α = 45°
0°
45°
(c)
α = 90°
0°
90°
α
(d)
α = 135°
0° 45° 90° 135°
α
(e)
α = 180°
0°
180°
α
(f)
α = 225°
225°
0°
α
(g)
α = 270°
270°
0°
α
(h)
α = 315°
315°
0°
α
(i)
Sine wave
α = 360°
225° 270° 315° 360°
0° 45° 90° 135° 180°
α
T (period)

FIG. 13.16

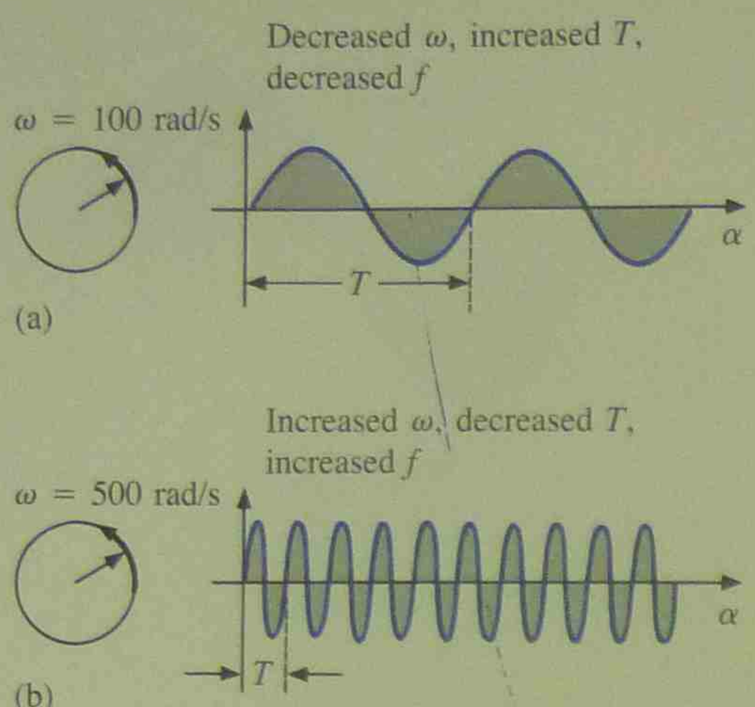


FIG. 13.17

(13.12) and (13.13) are verified somewhat by Fig. 13.17, where for the same radius vector, $\omega = 100$ rad/s and 500 rad/s.

EXAMPLE 13.4 Determine the angular velocity of a sine wave having a frequency of 60 Hz.

Solution:

$$\omega = 2\pi f = (2\pi)(60 \text{ Hz}) \cong \mathbf{377 \text{ rad/s}}$$

(a recurring value due to 60-Hz predominance).

EXAMPLE 13.5 Determine the frequency and period of the sine wave of Fig. 13.17(b).

Solution: Since $\omega = 2\pi/T$,

$$T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{500 \text{ rad/s}} = \frac{2\pi \text{ rad}}{500 \text{ rad/s}} = \mathbf{12.57 \text{ ms}}$$

and

$$f = \frac{1}{T} = \frac{1}{12.57 \times 10^{-3} \text{ s}} = \mathbf{79.58 \text{ Hz}}$$

EXAMPLE 13.6 Given $\omega = 200$ rad/s, determine how long it will take the sinusoidal waveform to pass through an angle of 90°.

Solution: Eq. (13.11): $\alpha = \omega t$, and

$$t = \frac{\alpha}{\omega}$$

However, α must be substituted as $\pi/2$ (= 90°) since ω is in radians per second:

$$t = \frac{\alpha}{\omega} = \frac{\pi/2 \text{ rad}}{200 \text{ rad/s}} = \frac{\pi}{400} \text{ s} = \mathbf{7.85 \text{ ms}}$$

EXAMPLE 13.7 Find the angle through which a sinusoidal waveform of 60 Hz will pass in a period of 5 ms.

Solution: Eq. (13.11): $\alpha = \omega t$, or

$$\alpha = 2\pi f t = (2\pi)(60 \text{ Hz})(5 \times 10^{-3} \text{ s}) = \mathbf{1.885 \text{ rad}}$$

If not careful, one might be tempted to interpret the answer as 1.885°. However,

$$\alpha\,(^\circ) = \frac{180^\circ}{\pi \text{ rad}}(1.885 \text{ rad}) = \mathbf{108^\circ}$$

13.4 GENERAL FORMAT FOR THE SINUSOIDAL VOLTAGE OR CURRENT

The basic mathematical format for the sinusoidal waveform is

$$\boxed{A_m \sin \alpha} \qquad \textbf{(13.14)}$$

where A_m is the peak value of the waveform and α is the unit of measure for the horizontal axis as shown in Fig. 13.18.

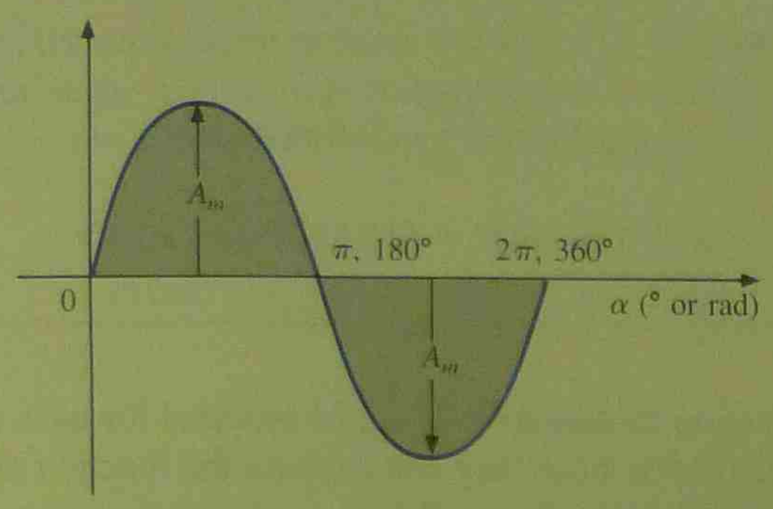


FIG. 13.18

The equation $\alpha = \omega t$ states that the angle α through which the rotating vector of Fig. 13.16 will pass is determined by the angular velocity of the rotating vector and the length of time the vector rotates. For example, for a particular angular velocity (fixed ω), the longer the radius vector is permitted to rotate (that is, the greater the value of t), the greater will be the number of degrees or radians through which the vector will pass. Relating this statement to the sinusoidal waveform, for a particular angular velocity, the longer the time, the greater the number of cycles shown. For a fixed time interval, the greater the angular velocity, the greater the number of cycles generated.

Due to Eq. (13.11), the general format of a sine wave can also be written

$$\boxed{A_m \sin \omega t} \qquad \textbf{(13.15)}$$

with ωt as the horizontal unit of measure.

For electrical quantities such as current and voltage, the general format is

$$i = I_m \sin \omega t = I_m \sin \alpha$$
$$e = E_m \sin \omega t = E_m \sin \alpha$$

where the capital letters with the subscript m represent the amplitude and the lowercase letters i and e represent the instantaneous value of current or voltage, respectively, at any time t. This format is particularly important, since it presents the sinusoidal voltage or current as a func-

tion of time, which is the horizontal scale for the oscilloscope. Recall that the horizontal sensitivity of a scope is in time per centimeter and not degrees per centimeter.

EXAMPLE 13.8 Given $e = 5 \sin \alpha$, determine e at $\alpha = 40°$ and $\alpha = 0.8\pi$.

Solution: For $\alpha = 40°$,

$$e = 5 \sin 40° = 5(0.6428) = \mathbf{3.214\ V}$$

For $\alpha = 0.8\pi$,

$$\alpha\,(°) = \frac{180°}{\pi}(0.8\pi) = 144°$$

and

$$e = 5 \sin 144° = 5(0.5878) = \mathbf{2.939\ V}$$

The conversion to degrees will not be required for most modern-day scientific calculators, since they can perform the function directly. First be sure the calculator is in the RAD mode and then simply enter the radian measure and use the appropriate trigonometric key (sin, cos, tan, etc.).

The angle at which a particular voltage level is attained can be determined by rearranging the equation

$$e = E_m \sin \alpha$$

in the following manner:

$$\sin \alpha = \frac{e}{E_m}$$

which can be written

$$\boxed{\alpha = \sin^{-1} \frac{e}{E_m}} \qquad (13.16)$$

Similarly, for a particular current level,

$$\boxed{\alpha = \sin^{-1} \frac{i}{I_m}} \qquad (13.17)$$

The function $\sin^{-1}$ is available on all scientific calculators.

EXAMPLE 13.9

a. Determine the angle at which the magnitude of the sinusoidal function $v = 10 \sin 377t$ is 4 V.
b. Determine the time at which the magnitude is attained.

Solutions:

a. Eq. (13.16):

$$\alpha_1 = \sin^{-1}\frac{v}{E_m} = \sin^{-1}\frac{4\text{ V}}{10\text{ V}} = \sin^{-1} 0.4 = \mathbf{23.578^\circ}$$

However, Fig. 13.19 reveals that the magnitude of 4 V (positive) will be attained at two points between 0° and 180°. The second intersection is determined by

$$\alpha_2 = 180^\circ - 23.578^\circ = \mathbf{156.422^\circ}$$

In general, therefore, keep in mind that Eqs. (13.16) and (13.17) will provide an angle with a magnitude between 0° and 90°.

b. Eq. (13.11): $\alpha = \omega t$, and so $t = \alpha/\omega$. However, α must be in radians. Thus,

$$\alpha(\text{rad}) = \frac{\pi}{180^\circ}(23.578^\circ) = 0.411\text{ rad}$$

and

$$t_1 = \frac{\alpha}{\omega} = \frac{0.411\text{ rad}}{377\text{ rad/s}} = \mathbf{1.09\text{ ms}}$$

For the second intersection,

$$\alpha(\text{rad}) = \frac{\pi}{180^\circ}(156.422^\circ) = 2.73\text{ rad}$$

$$t_2 = \frac{\alpha}{\omega} = \frac{2.73\text{ rad}}{377\text{ rad/s}} = \mathbf{7.24\text{ ms}}$$

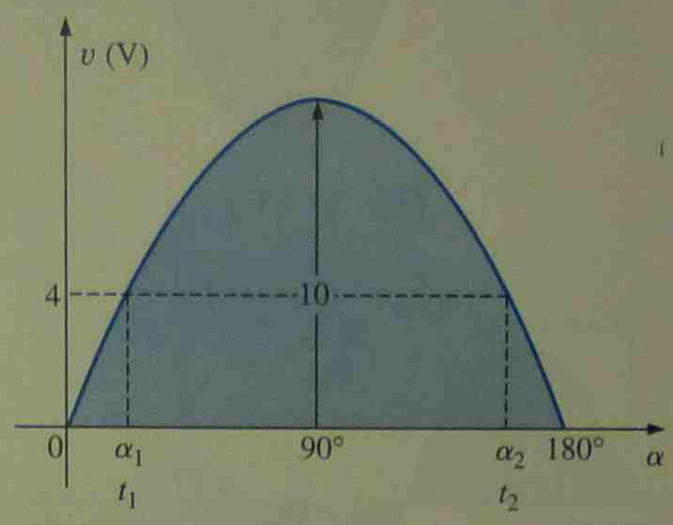


FIG. 13.19

The sine wave can also be plotted against *time* on the horizontal axis. The time period for each interval can be determined from $t = \alpha/\omega$, but the most direct route is simply to find the period T from $T = 1/f$ and break it up into the required intervals. This latter technique will be demonstrated in Example 13.10.

Before reviewing the example, take special note of the relative simplicity of the mathematical equation that can represent a sinusoidal waveform. Any alternating waveform whose characteristics differ from those of the sine wave cannot be represented by a single term, but may require two, four, six, or perhaps an infinite number of terms to be represented accurately. A further description of nonsinusoidal waveforms can be found in Chapter 24.

EXAMPLE 13.10 Sketch $e = 10 \sin 314t$ with the abscissa

a. angle (α) in degrees.
b. angle (α) in radians.
c. time (t) in seconds.

Solutions:

a. See Fig. 13.20. (Note that no calculations are required.)

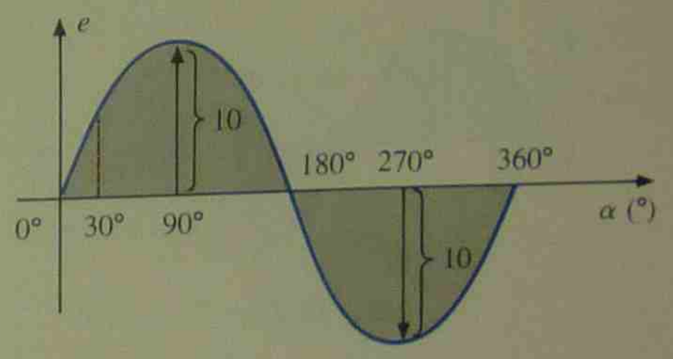


FIG. 13.20

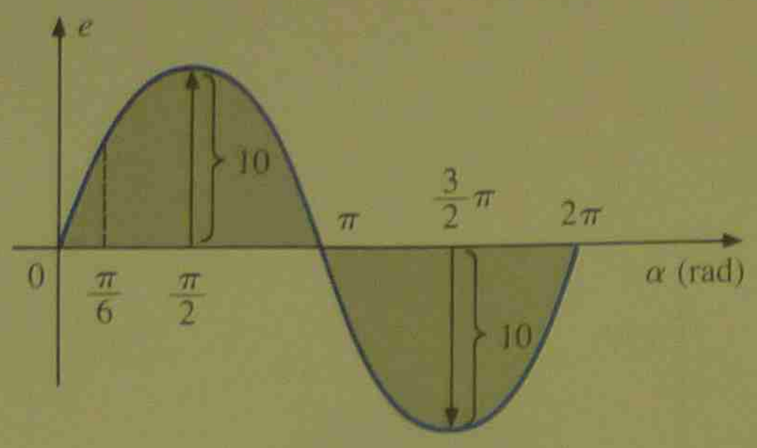


FIG. 13.21

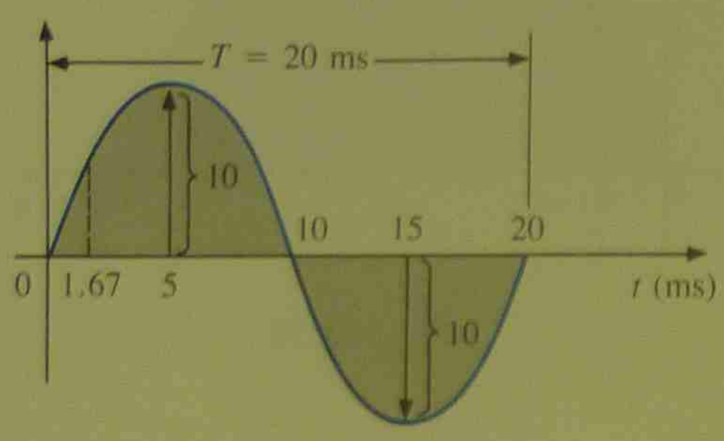


FIG. 13.22

b. See Fig. 13.21. (Once the relationship between degrees and radians is understood, there is again no need for calculations.)

c. 360°: $T = \dfrac{2\pi}{\omega} = \dfrac{6.28}{314} = 20 \text{ ms}$

180°: $\dfrac{T}{2} = \dfrac{20}{2} \times 10^{-3} = 10 \text{ ms}$

90°: $\dfrac{T}{4} = \dfrac{20}{4} \times 10^{-3} = 5 \text{ ms}$

30°: $\dfrac{T}{12} = \dfrac{20}{12} \times 10^{-3} = 1.67 \text{ ms}$

See Fig. 13.22.

EXAMPLE 13.11 Given $i = 6 \times 10^{-3} \sin 1000t$, determine i at $t = 2$ ms.

Solution:

$$\alpha = \omega t = 1000t = (1000 \text{ rad/s})(2 \times 10^{-3} \text{ s}) = 2 \text{ rad}$$

$$\alpha\,(^\circ) = \frac{180^\circ}{\pi \text{ rad}}(2 \text{ rad}) = 114.59^\circ)$$

$$i = (6 \times 10^{-3})(\sin 114.59^\circ)$$
$$= (6 \text{ mA})(0.9093) = \mathbf{5.46 \text{ mA}}$$

13.5 PHASE RELATIONS

Thus far, we have considered only sine waves that have maxima at $\pi/2$ and $3\pi/2$, with a zero value at 0, π, and 2π, as shown in Fig. 13.21. If the waveform is shifted to the right or left of 0°, the expression becomes

$$\boxed{A_m \sin(\omega t \pm \theta)} \tag{13.18}$$

where θ is the angle in degrees or radians that the waveform has been shifted.

If the waveform passes through the horizontal axis with a *positive-going* (increasing with time) slope *before* 0°, as shown in Fig. 13.23, the expression is

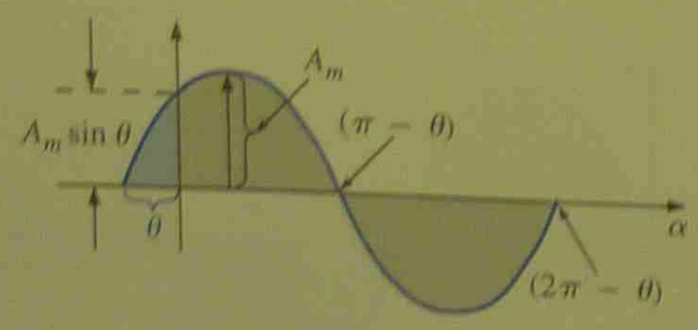


FIG. 13.23

$$\boxed{A_m \sin(\omega t + \theta)} \tag{13.19}$$

At $\omega t = \alpha = 0°$, the magnitude is determined by $A_m \sin \theta$. If the waveform passes through the horizontal axis with a positive-going slope *after* 0°, as shown in Fig. 13.24, the expression is

$$\boxed{A_m \sin(\omega t - \theta)} \tag{13.20}$$

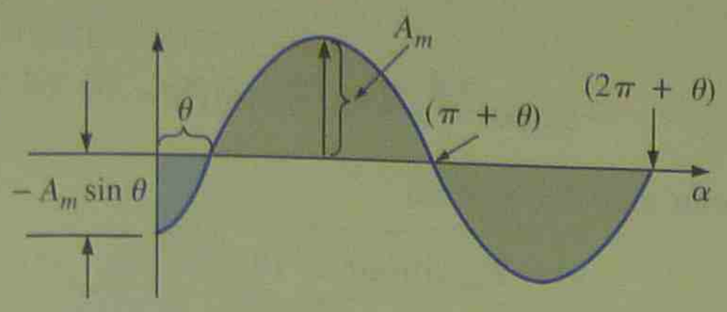


FIG. 13.24

And at $\omega t = \alpha = 0°$, the magnitude is $A_m \sin(-\theta)$, which by a trigonometric identity is $-A_m \sin \theta$.

If the waveform crosses the horizontal axis with a positive-going slope 90° ($\pi/2$) sooner, as shown in Fig. 13.25, it is called a *cosine wave*. That is,

$$\sin(\omega t + 90°) = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos \omega t \qquad \textbf{(13.21)}$$

or

$$\sin \omega t = \cos(\omega t - 90°) = \cos\left(\omega t - \frac{\pi}{2}\right) \qquad \textbf{(13.22)}$$

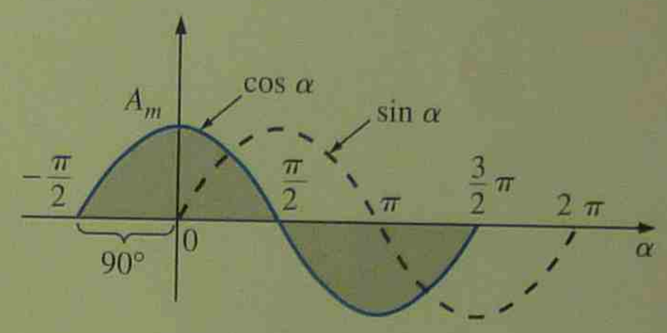


FIG. 13.25

The terms *lead* and *lag* are used to indicate the relationship between two sinusoidal waveforms of the *same frequency* plotted on the same set of axes. In Fig. 13.25, the cosine curve is said to *lead* the sine curve by 90°, and the sine curve is said to *lag* the cosine curve by 90°. The 90° is referred to as the phase angle between the two waveforms. In language commonly applied, the waveforms are *out of phase* by 90°. Note that the phase angle between the two waveforms is measured between those two points on the horizontal axis through which each passes with the *same slope*. If both waveforms cross the axis at the same point with the same slope, they are *in phase*.

A few additional geometric relations that may prove useful in applications involving sines or cosines in phase relationships are the following:

$$\begin{aligned} \sin(-\alpha) &= -\sin \alpha \\ \cos(-\alpha) &= \cos \alpha \\ -\sin(\alpha) &= \sin(\alpha \pm 180°) \\ -\cos(\alpha) &= \cos(\alpha \pm 180°) \end{aligned} \qquad \textbf{(13.23)}$$

If a sinusoidal expression should appear as

$$e = -E_m \sin \omega t$$

the negative sign is associated with the sine portion of the expression, not the peak value E_m. In other words, the expression, if not for convenience, would be written

$$e = E_m(-\sin \omega t)$$

Since

$$-\sin \omega t = \sin(\omega t \pm 180°)$$

the expression can also be written

$$e = E_m \sin(\omega t \pm 180°)$$

revealing that a negative sign can be replaced by a 180° change in phase angle (+ or −). That is,

$$\begin{aligned} e &= -E_m \sin \omega t = E_m \sin(\omega t + 180°) \\ &= E_m \sin(\omega t - 180°) \end{aligned}$$

A plot of each will clearly show their equivalence. There are, therefore, two correct mathematical representations for the functions.

The *phase relationship* between two waveforms indicates which one leads or lags, and by how many degrees or radians.

EXAMPLE 13.12 What is the phase relationship between the sinusoidal waveforms of each of the following sets?

a. $v = 10 \sin(\omega t + 30°)$
   $i = 5 \sin(\omega t + 70°)$
b. $i = 15 \sin(\omega t + 60°)$
   $v = 10 \sin(\omega t - 20°)$
c. $i = 2 \cos(\omega t + 10°)$
   $v = 3 \sin(\omega t - 10°)$
d. $i = -\sin(\omega t + 30°)$
   $v = 2 \sin(\omega t + 10°)$
e. $i = -2 \cos(\omega t - 60°)$
   $v = 3 \sin(\omega t - 150°)$

Solutions:

a. See Fig. 13.26.
   i leads v by 40°, or v lags i by 40°.

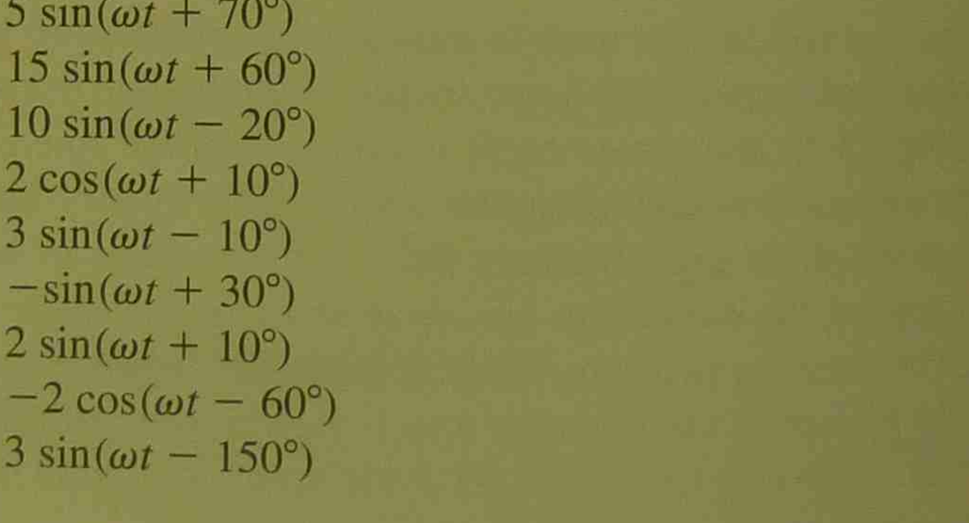


FIG. 13.26

b. See Fig. 13.27.

***i* leads *v* by 80°, or *v* lags *i* by 80°.**

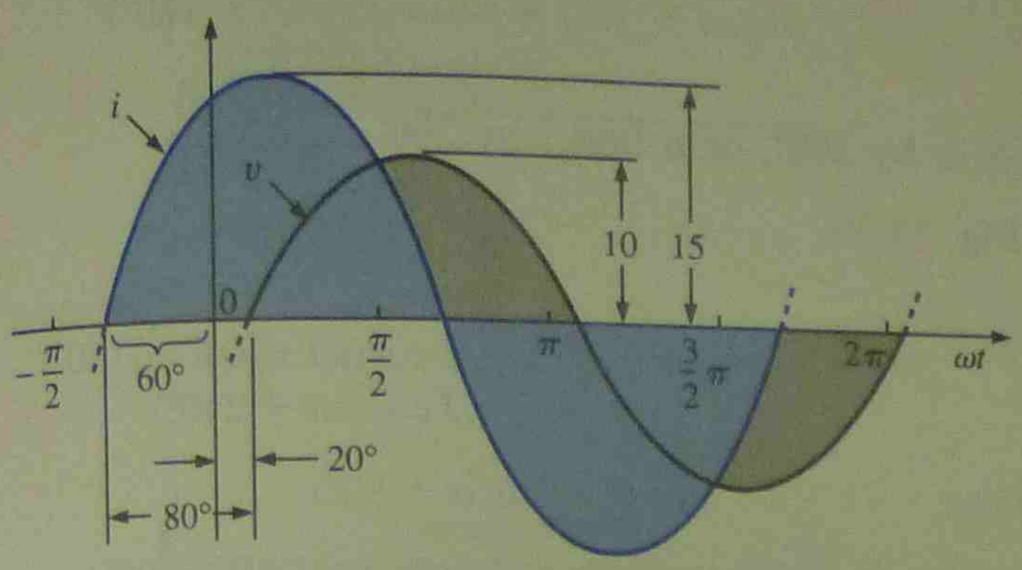


FIG. 13.27

c. See Fig. 13.28.

$$i = 2\cos(\omega t + 10°) = 2\sin(\omega t + 10° + 90°)$$
$$= 2\sin(\omega t + 100°)$$

***i* leads *v* by 110°, or *v* lags *i* by 110°.**

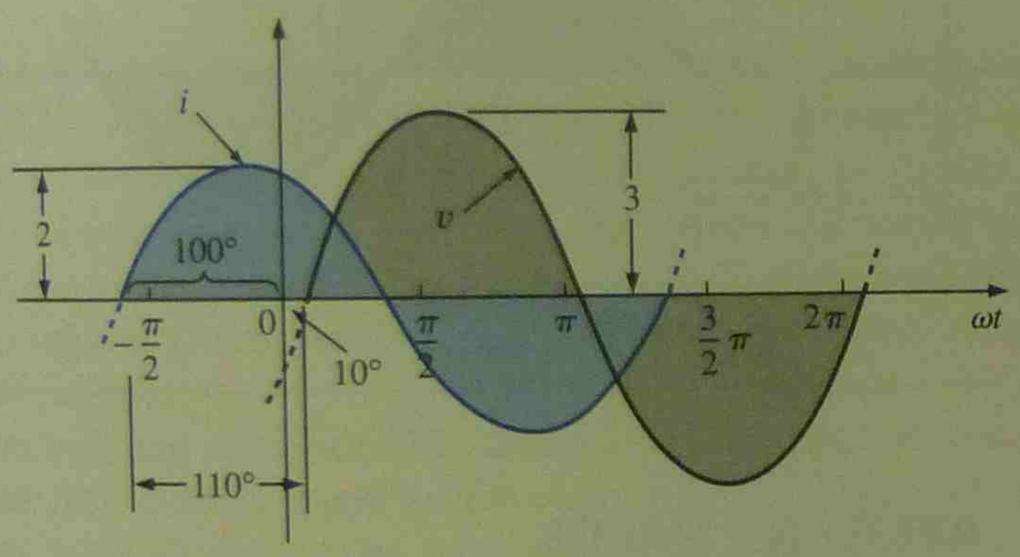


FIG. 13.28

d. See Fig. 13.29.

Note: $-\sin(\omega t + 30°) = \sin(\omega t + 30° - 180°)$

$$= \sin(\omega t - 150°)$$

***v* leads *i* by 160°, or *i* lags *v* by 160°.**

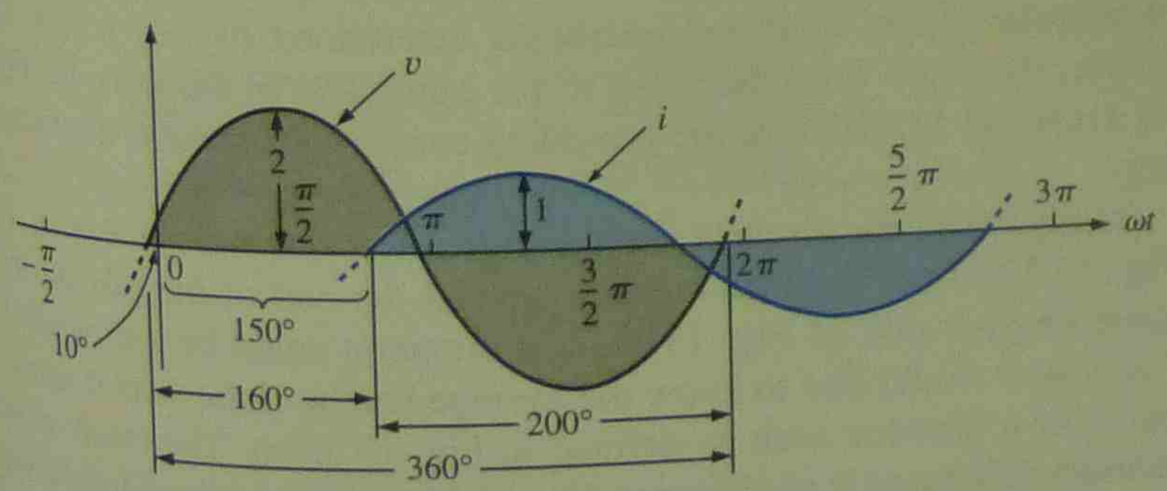


FIG. 13.29

Or using

$$-\sin(\omega t + 30°) = \sin(\omega t + 30° + 180°) \quad \text{(Note)}$$
$$= \sin(\omega t + 210°)$$

***i* leads *v* by 200°, or *v* lags *i* by 200°.**

e. See Fig. 13.30.

$$i = -2\cos(\omega t - 60°) = 2\cos(\omega t - 60° - 180°) \quad \text{(By choice)}$$
$$= 2\cos(\omega t - 240°)$$

However, $\cos\alpha = \sin(\alpha + 90°)$

so that $2\cos(\omega t - 240°) = 2\sin(\omega t - 240° + 90°)$
$= 2\sin(\omega t - 150°)$

***v* and *i* are in phase.**

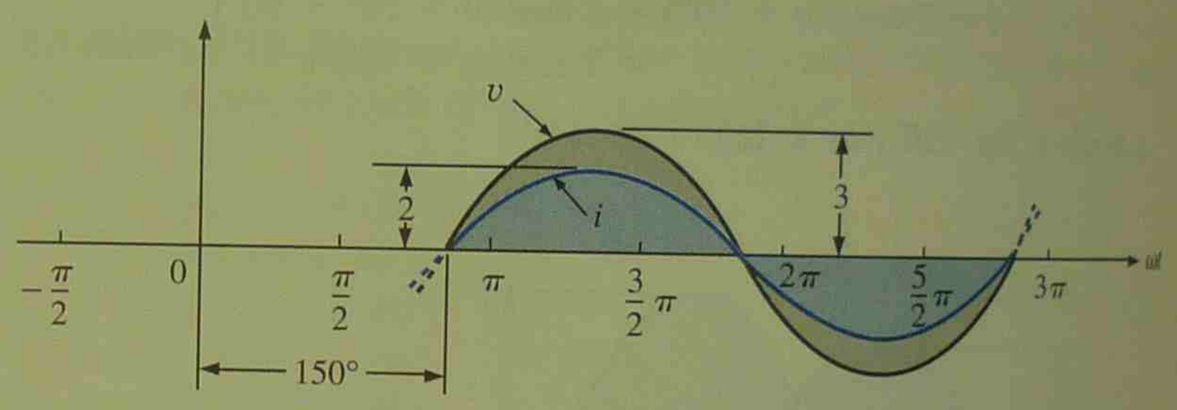


FIG. 13.30

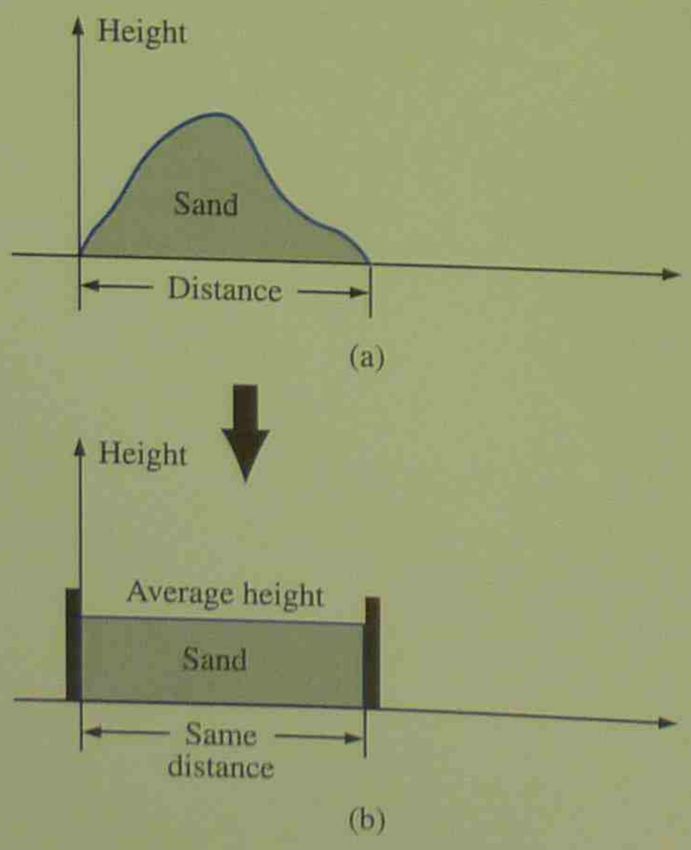


FIG. 13.31
Defining average value.

13.6 AVERAGE VALUE

Even though the concept of the *average value* is an important one in most technical fields, its true meaning is often misunderstood. In Fig. 13.31(a), for example, the average height of the sand may be required to determine the volume of sand available. The average height of the sand is that height obtained if the distance from one end to the other is maintained while the sand is leveled off, as shown in Fig. 13.31(b). The area under the mound of Fig. 13.31(a) will then equal the area under the rectangular shape of Fig. 13.31(b) as determined by $A = b \times h$. Of course, the depth (into the page) of the sand must be the same for Fig. 13.31(a) and 13.31(b) for the preceding conclusions to have any meaning.

In Fig. 13.31 the distance was measured from one end to the other. In Fig. 13.32(a) the distance extends beyond the end of the pile for the same original pile of Fig. 13.31. The situation could be one where a landscaper would like to know the average height of the sand if spread out over a distance such as defined in Fig. 13.32(a). The result of an increased distance is as shown in Fig. 13.32(b). The average height has decreased compared to Fig. 13.31. Quite obviously, therefore, the longer the distance, the lower is the average value.

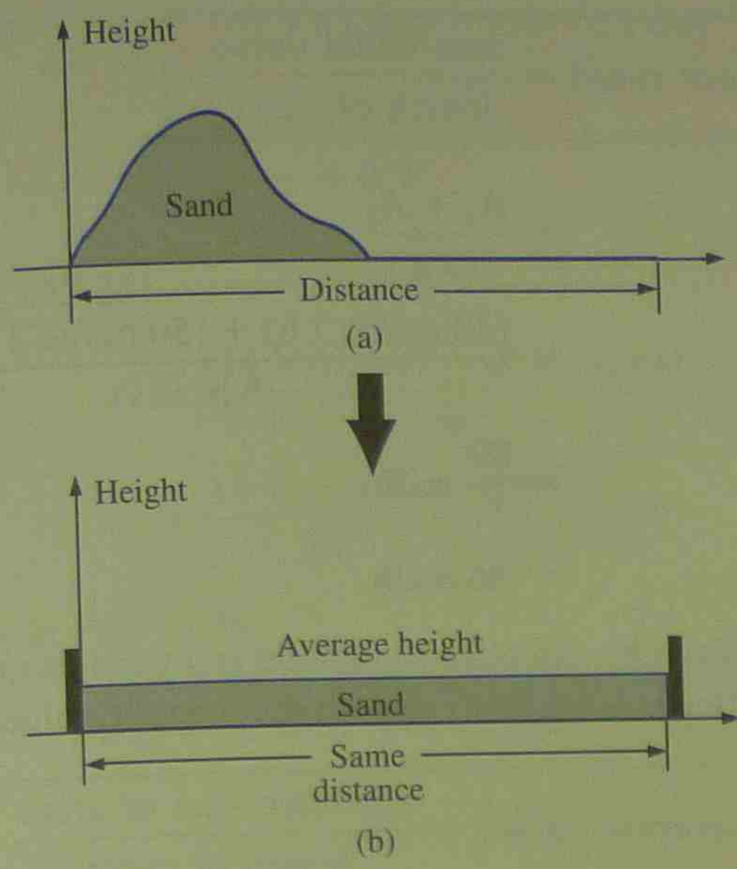


FIG. 13.32
Effect of distance on average value.

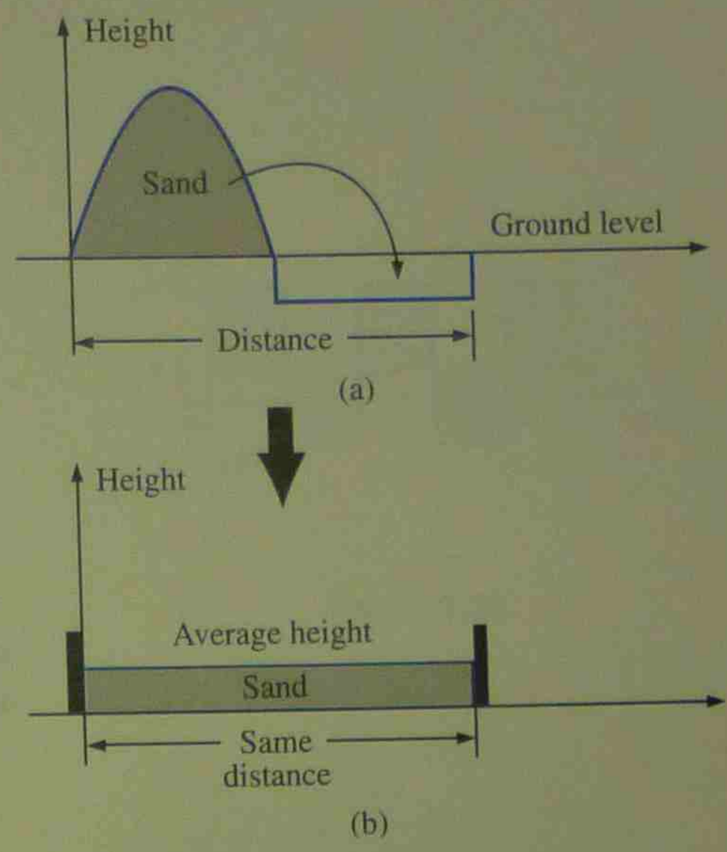


FIG. 13.33
Effect of depressions (negative excursions) on average value.

If the distance parameter includes a depression, as shown in Fig. 13.33(a), some of the sand will be used to fill the depression, resulting in an even lower average value for the landscaper, as shown in Fig. 13.33(b). For a sinusoidal waveform, the depression would have the same shape as the mound of sand (over one full cycle), resulting in an average value at ground level (or zero volts for a sinusoidal voltage over one full period).

After traveling a considerable distance by car, some drivers like to calculate their average speed for the entire trip. This is usually done by dividing the miles traveled by the hours required to drive that distance. For example, if a person traveled 180 mi in 5 h, the average speed was 180 mi/5 h, or 36 mi/h. This same distance may have been traveled at various speeds for various intervals of time, as shown in Fig. 13.34.

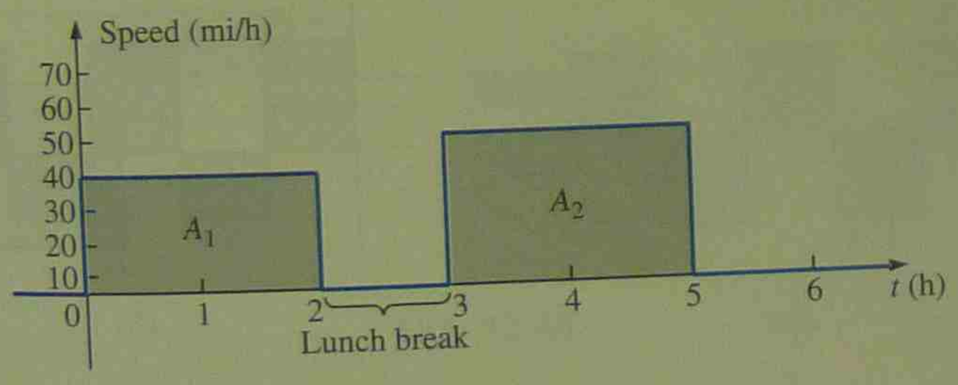


FIG. 13.34

By finding the total area under the curve for the 5 h and then dividing the area by 5 h (the total time for the trip), we obtain the same result of 36 mi/h; that is,

$$\text{Average speed} = \frac{\text{area under curve}}{\text{length of curve}} \tag{13.24}$$

$$= \frac{A_1 + A_2}{5\text{ h}}$$

$$= \frac{(40\text{ mi/h})(2\text{ h}) + (50\text{ mi/h})(2\text{ h})}{5\text{ h}}$$

$$= \frac{80}{5}\text{ mi/h}$$

$$= \mathbf{36\text{ mi/h}}$$

Equation (13.24) can be extended to include any variable quantity, such as current or voltage, if we let G denote the average value, as follows:

$$G\text{ (average value)} = \frac{\text{algebraic sum of areas}}{\text{length of curve}} \tag{13.25}$$

The algebraic sum of the areas must be determined, since some area contributions will be from below the horizontal axis. Areas above the axis will be assigned a positive sign, and those below, a negative sign. A positive average value will then be above the axis, and a negative value, below.

The average value of *any* current or voltage is the value indicated on a dc meter. In other words, over a complete cycle, the average value is the equivalent dc value. In the analysis of electronic circuits to be considered in a later course, both dc and ac sources of voltage will be applied to the same network. It will then be necessary to know or determine the dc (or average value) and ac components of the voltage or current in various parts of the system.

EXAMPLE 13.13 Determine the average value of the waveforms of Fig. 13.35.

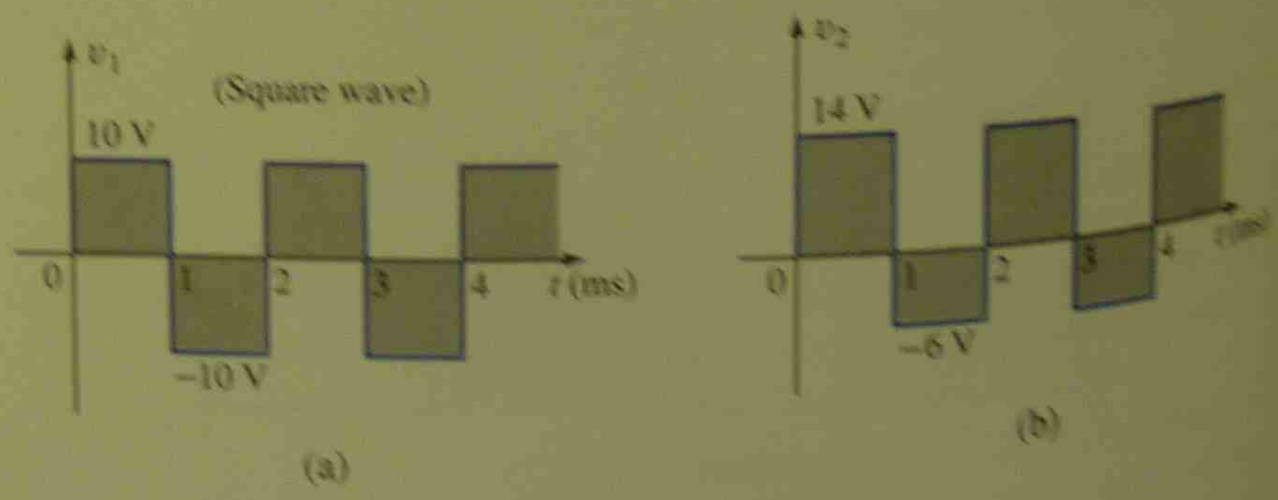


FIG. 13.35

Solutions:

a. By inspection, the area above the axis equals the area below over one cycle, resulting in an average value of zero volts. Using Eq. (13.25)

$$G = \frac{(10\text{ V})(1\text{ ms}) - (10\text{ V})(1\text{ ms})}{2\text{ ms}}$$

$$= \frac{0}{2\text{ ms}} = \mathbf{0\text{ V}}$$

b. Using Eq. (13.25):

$$G = \frac{(14\text{ V})(1\text{ ms}) - (6\text{ V})(1\text{ ms})}{2\text{ ms}}$$

$$= \frac{14\text{ V} - 6\text{ V}}{2} = \frac{8\text{ V}}{2} = \mathbf{4\text{ V}}$$

as shown in Fig. 13.36.

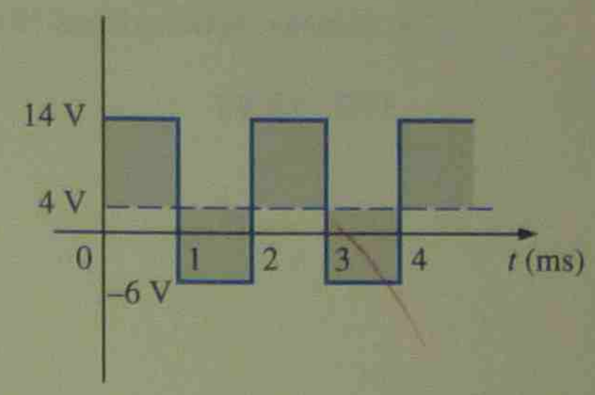


FIG. 13.36

In reality, the waveform of Fig. 13.35(b) is simply the square wave of Fig. 13.35(a) with a dc shift of 4 V. That is,

$$v_2 = v_1 + 4\text{ V}$$

EXAMPLE 13.14 Find the average values of the following waveforms over one full cycle:

a. Fig. 13.37.

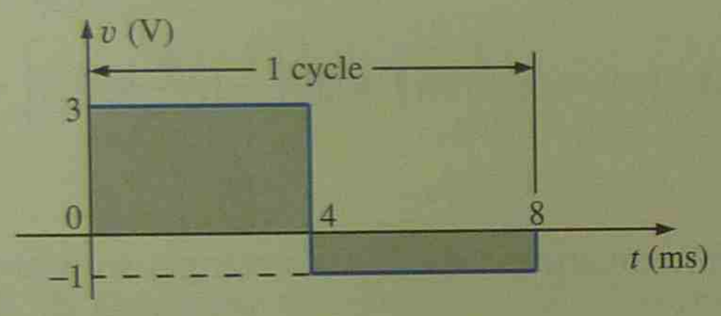


FIG. 13.37

b. Fig. 13.38.

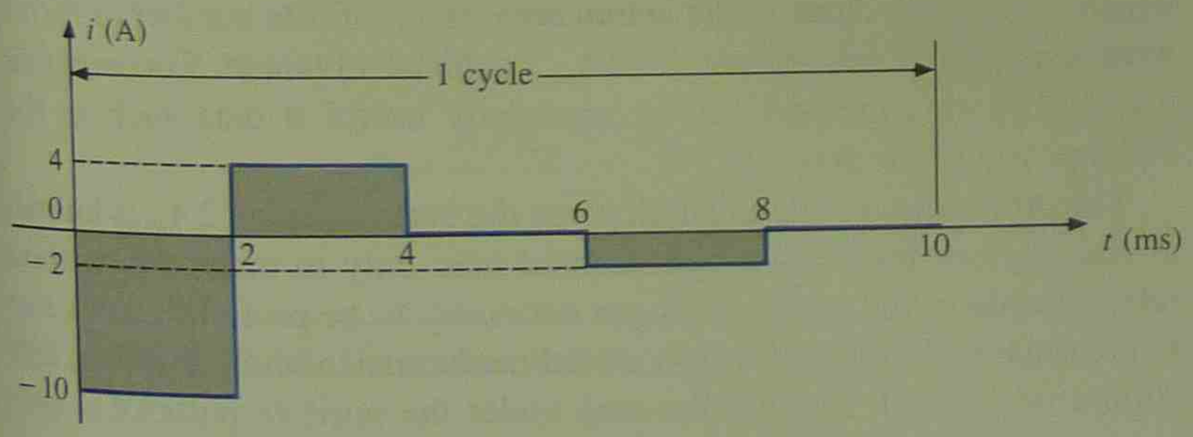
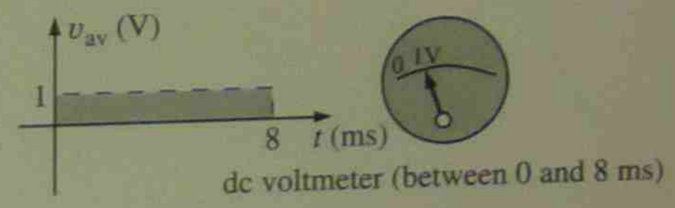


FIG. 13.38

Solutions:

a. $G = \dfrac{+(3\text{ V})(4\text{ ms}) - (1\text{ V})(4\text{ ms})}{8\text{ ms}} = \dfrac{12\text{ V} - 4\text{ V}}{8} = \mathbf{1\text{ V}}$

Note Fig. 13.39.



dc voltmeter (between 0 and 8 ms)

FIG. 13.39

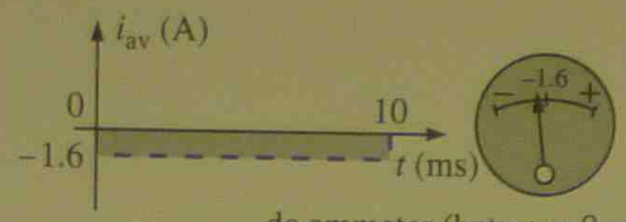


FIG. 13.40

b. $$G = \frac{-(10\text{ V})(2\text{ ms}) + (4\text{ V})(2\text{ ms}) - (2\text{ V})(2\text{ ms})}{10\text{ ms}}$$

$$= \frac{-20\text{ V} + 8\text{ V} - 4\text{ V}}{10} = -\frac{16\text{ V}}{10} = \mathbf{-1.6\text{ V}}$$

Note Fig. 13.40.

We found the areas under the curves in the preceding example by using a simple geometric formula. If we should encounter a sine wave or any other unusual shape, however, we must find the area by some other means. We can obtain a good approximation of the area by attempting to reproduce the original wave shape using a number of small rectangles or other familiar shapes, the area of which we already know through simple geometric formulas. For example,

the area of the positive (or negative) pulse of a sine wave is $2A_m$.

Approximating this waveform by two triangles (Fig. 13.41), we obtain (using *area* = 1/2 *base* × *height* for the area of a triangle) a rough idea of the actual area:

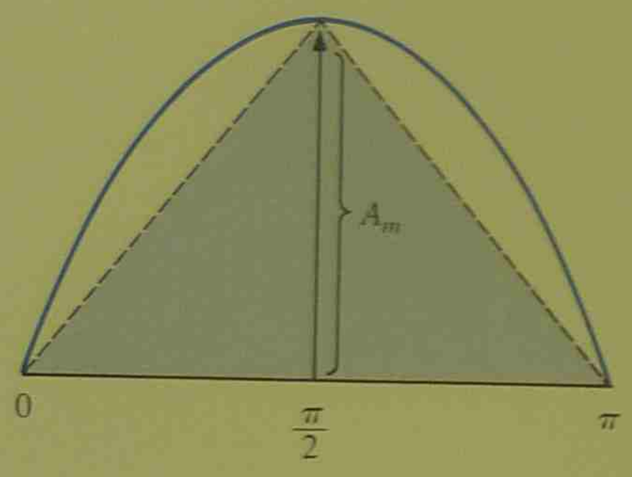


FIG. 13.41

$$\text{Area (shaded)} = 2\left(\frac{1}{2}bh\right) = 2\left[\overbrace{\left(\frac{1}{2}\right)}^{b}\overbrace{\left(\frac{\pi}{2}\right)}^{h}(A_m)\right] = \frac{\pi}{2}A_m$$

$$\cong 1.58A_m$$

A closer approximation might be a rectangle with two similar triangles (Fig. 13.42):

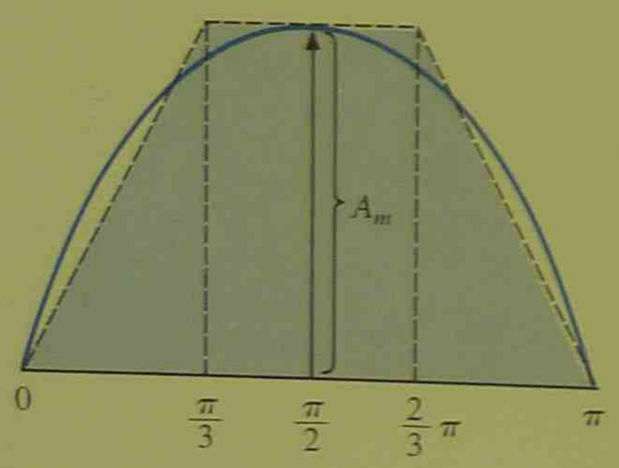


FIG. 13.42

$$\text{Area} = A_m\frac{\pi}{3} + 2\left(\frac{1}{2}bh\right) = A_m\frac{\pi}{3} + \frac{\pi}{3}A_m = \frac{2}{3}\pi A_m$$

$$= 2.094A_m$$

which is certainly close to the actual area. If an infinite number of forms were used, an exact answer of $2A_m$ could be obtained. For irregular waveforms, this method can be especially useful if data such as the average value are desired.

The procedure of calculus that gives the exact solution $2A_m$ is known as *integration*. Integration is presented here only to make the method recognizable to the reader; it is not necessary to be proficient in its use to continue with this text. It is a useful mathematical tool, however, and should be learned. Finding the area under the positive pulse of a sine wave using integration, we have

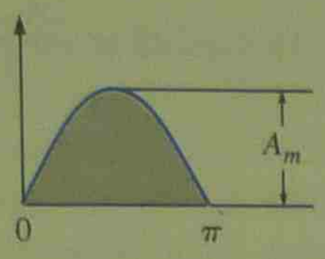
$$\text{Area} = \int_0^{\pi} A_m \sin\alpha\, d\alpha$$

where $\int$ is the sign of integration, 0 and π are the limits of integration, $A_m \sin\alpha$ is the function to be integrated, and $d\alpha$ indicates that we are integrating with respect to α.

Integrating, we obtain

$$\begin{aligned}\text{Area} &= A_m[-\cos\alpha]_0^{\pi} \\ &= -A_m(\cos\pi - \cos 0^\circ) \\ &= -A_m[-1-(+1)] = -A_m(-2)\end{aligned}$$

$$\boxed{\text{Area} = 2A_m} \tag{13.26}$$

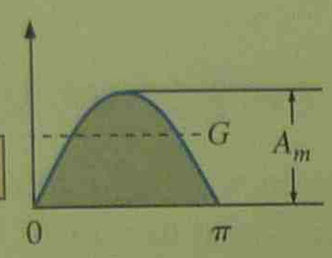


Since we know the area under the positive (or negative) pulse, we can easily determine the average value of the positive (or negative) region of a sine wave pulse by applying Eq. (13.25):

$$G = \frac{2A_m}{\pi}$$

and

$$\boxed{G = 0.637A_m} \tag{13.27}$$



For the waveform of Fig. 13.43,

$$G = \frac{(2A_m/2)}{\pi/2} = \frac{2A_m}{\pi}$$

(average the same as for a full pulse)

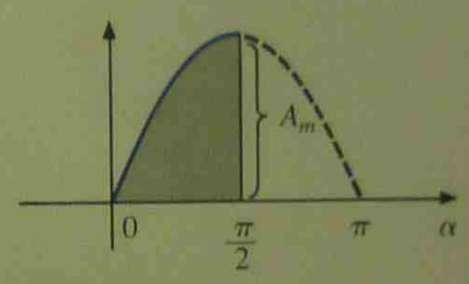


FIG. 13.43

EXAMPLE 13.15 Determine the average value of the sinusoidal waveform of Fig. 13.44.

Solution: By inspection it is fairly obvious that

the average value of a pure sinusoidal waveform over one full cycle is zero.

Equation (13.25):

$$G = \frac{+2A_m - 2A_m}{2\pi} = \mathbf{0\ V}$$

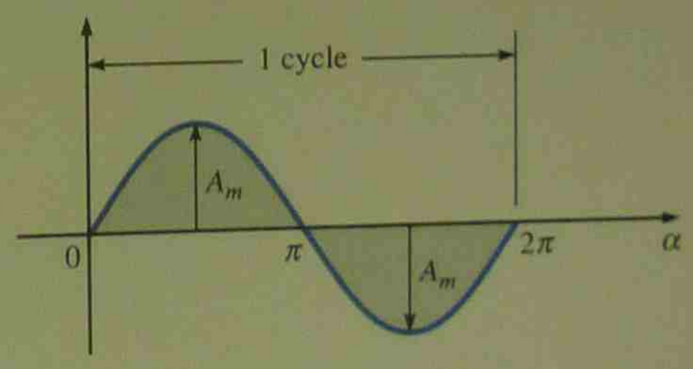


FIG. 13.44

EXAMPLE 13.16 Determine the average value of the waveform of Fig. 13.45.

Solution: The peak-to-peak value of the sinusoidal function is 16 mV + 2 mV = 18 mV. The peak amplitude of the sinusoidal waveform is, therefore, 18 mV/2 = 9 mV. Counting down 9 mV from 2 mV (or 9 mV up from −16 mV) results in an average or dc level of −7 mV, as noted by the dashed line of Fig. 13.45.

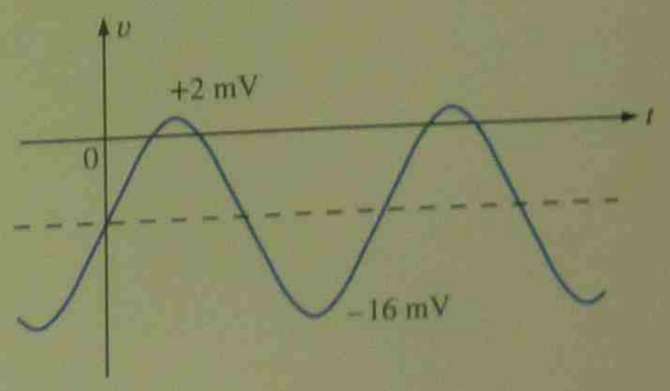


FIG. 13.45

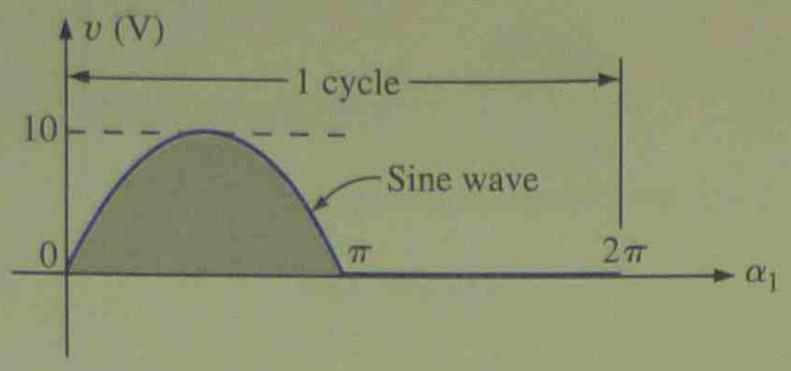


FIG. 13.46

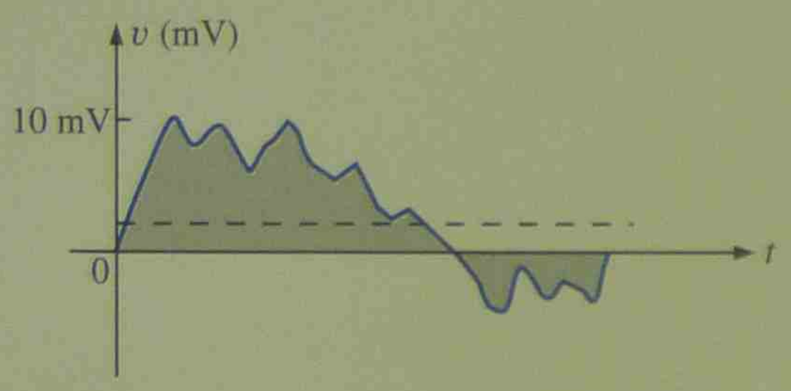


FIG. 13.47

EXAMPLE 13.17 Determine the average value of the waveform of Fig. 13.46.

Solution:

$$G = \frac{2A_m + 0}{2\pi} = \frac{2(10\text{ V})}{2\pi} \cong \mathbf{3.18\text{ V}}$$

EXAMPLE 13.18 For the waveform of Fig. 13.47 determine whether the average value is positive or negative and its approximate value.

Solution: From the appearance of the waveform, the average value is positive and in the vicinity of 2 mV. Occasionally, judgments of this type will have to be made.

13.7 EFFECTIVE VALUES

This section will begin to relate dc and ac quantities with respect to the power delivered to a load. It will help us determine the amplitude of a sinusoidal ac current required to deliver the same power as a particular dc current. The question frequently arises, How is it possible for a sinusoidal ac quantity to deliver a net power if, over a full cycle, the net current in any one direction is zero (average value = 0)? It would almost appear that the power delivered during the positive portion of the sinusoidal waveform is withdrawn during the negative portion, and since the two are equal in magnitude, the net power delivered is zero. However, understand that *irrespective of direction,* current of any magnitude through a resistor will deliver power *to that resistor.* In other words, during the positive or negative portions of a sinusoidal ac current, power is being delivered at *each instant of time* to the resistor. The power delivered at each instant will, of course, vary with the magnitude of the sinusoidal ac current, but there will be a net flow during either the positive or negative pulses with a net flow over the full cycle. The net power flow will equal twice that delivered by either the positive or negative regions of the sinusoidal quantity.

A fixed relationship between ac and dc voltages and currents can be derived from the experimental setup shown in Fig. 13.48. A resistor in a

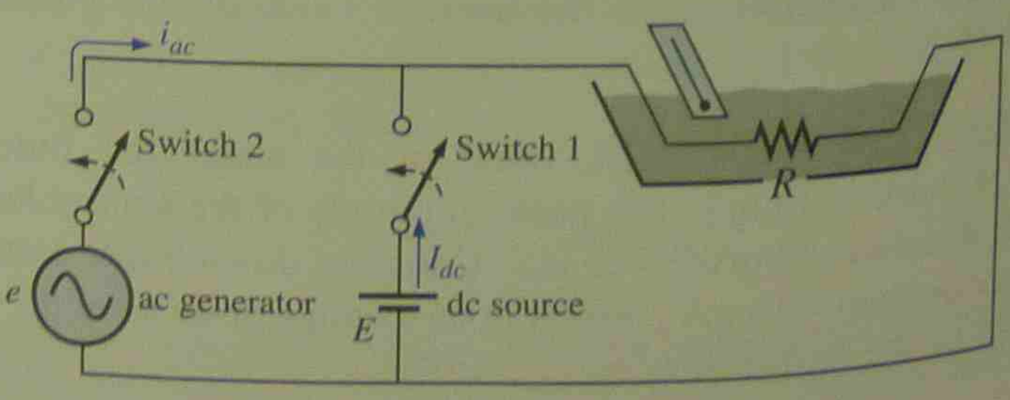


FIG. 13.48

water bath is connected by switches to a dc and an ac supply. If switch 1 is closed, a dc current I, determined by the resistance R and battery voltage E, will be established through the resistor R. The temperature reached by the water is determined by the dc power dissipated in the form of heat by the resistor.

If switch 2 is closed and switch 1 left open, the ac current through the resistor will have a peak value of I_m. The temperature reached by the water is now determined by the ac power dissipated in the form of heat by the resistor. The ac input is varied until the temperature is the same as that reached with the dc input. When this is accomplished, the average electrical power delivered to the resistor R by the ac source is the same as that delivered by the dc source.

The power delivered by the ac supply at any instant of time is

$$P_{ac} = (i_{ac})^2 R = (I_m \sin \omega t)^2 R = (I_m^2 \sin^2 \omega t) R$$

but

$$\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t) \qquad \text{(trigonometric identity)}$$

Therefore,

$$P_{ac} = I_m^2 \left[\frac{1}{2}(1 - \cos 2\omega t)\right] R$$

and

$$\boxed{P_{ac} = \frac{I_m^2 R}{2} - \frac{I_m^2 R}{2} \cos 2\omega t} \qquad \textbf{(13.28)}$$

The *average power* delivered by the ac source is just the first term, since the average value of a cosine wave is zero even though the wave may have twice the frequency of the original input current waveform. Equating the average power delivered by the ac generator to that delivered by the dc source,

$$P_{av(ac)} = P_{dc}$$

$$\frac{I_m^2 R}{2} = I_{dc}^2 R \quad \text{and} \quad I_m = \sqrt{2} I_{dc}$$

or

$$I_{dc} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

which, in words, states that

the equivalent dc value of a sinusoidal current or voltage is $1/\sqrt{2}$ or 0.707 of its maximum value.

The equivalent dc value is called the effective value of the sinusoidal quantity.

In summary,

$$\boxed{I_{eq\ dc} = I_{eff} = 0.707 I_m} \qquad \textbf{(13.29)}$$

or
$$I_m = \sqrt{2}I_{\text{eff}} = 1.414I_{\text{eff}} \tag{13.30}$$

and
$$E_{\text{eff}} = 0.707E_m \tag{13.31}$$

or
$$E_m = \sqrt{2}E_{\text{eff}} = 1.414E_{\text{eff}} \tag{13.32}$$

As a simple numerical example, it would require an ac current with a peak value of $\sqrt{2}(10) = 14.14$ A to deliver the same power to the resistor in Fig. 13.48 as a dc current of 10 A. The effective value of any quantity plotted as a function of time can be found by using the following equation derived from the experiment just described:

$$I_{\text{eff}} = \sqrt{\frac{\int_0^T i^2(t)\,dt}{T}} \tag{13.33}$$

or
$$I_{\text{eff}} = \sqrt{\frac{\text{area } (i^2(t))}{T}} \tag{13.34}$$

which, in words, states that to find the effective value, the function $i(t)$ must first be squared. After $i(t)$ is squared, the area under the curve is found by integration. It is then divided by T, the length of the cycle or period of the waveform, to obtain the average or *mean* value of the squared waveform. The final step is to take the *square root* of the mean value. This procedure gives us another designation for the effective value, the *root-mean-square* (rms) value.

EXAMPLE 13.19 Find the effective values of the sinusoidal waveforms in each part of Fig. 13.49.

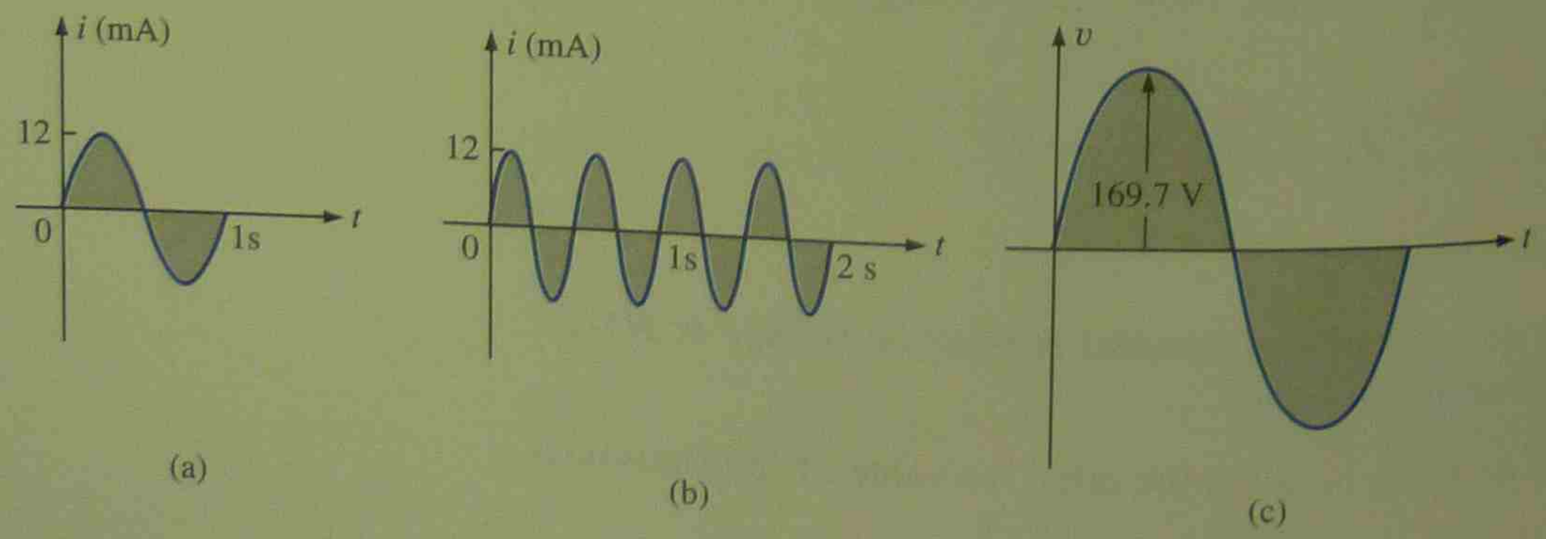


FIG. 13.49

Solution: For part (a), $I_{\text{eff}} = 0.707(12 \times 10^{-3}\text{ A}) = \mathbf{8.484\ mA}$. For part (b), again $I_{\text{eff}} = \mathbf{8.484\ mA}$. Note that frequency did not change the

effective value in (b) above as compared to (a). For part (c), $V_{\text{eff}} = 0.707(169.73\text{ V}) \cong \mathbf{120\text{ V}}$, as available from a home outlet.

EXAMPLE 13.20 The 120-V dc source of Fig. 13.50(a) delivers 3.6 W to the load. Determine the peak value of the applied voltage (E_m) and the current (I_m) if the ac source (Fig. 13.50(b)) is to deliver the same power to the load.

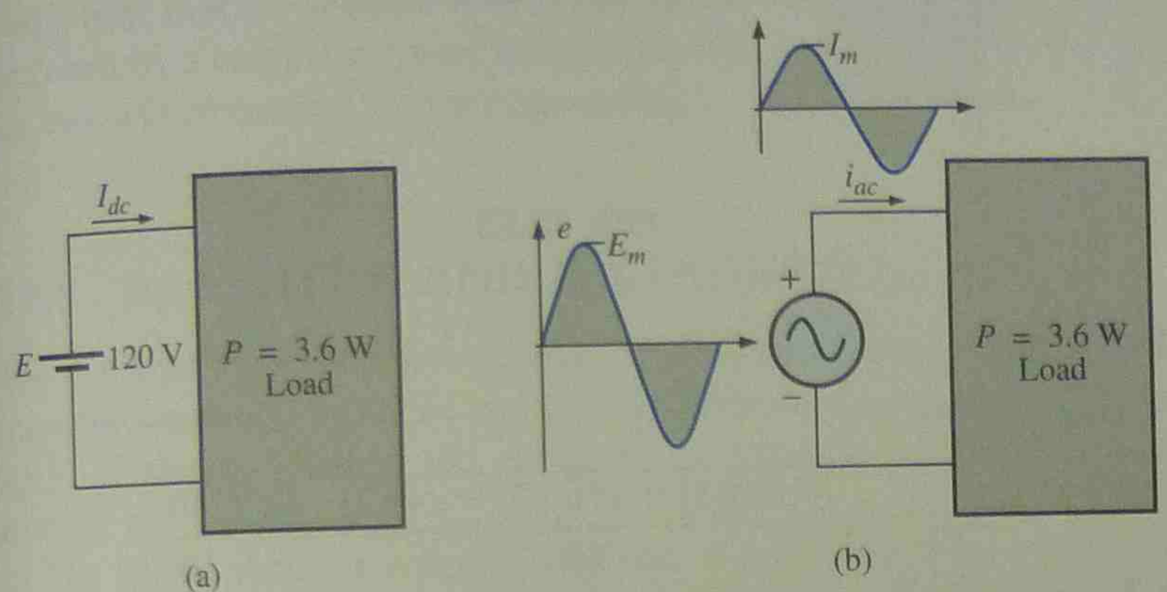


FIG. 13.50

Solution:

$$P_{\text{dc}} = V_{\text{dc}}I_{\text{dc}}$$

and
$$I_{\text{dc}} = \frac{P_{\text{dc}}}{V_{\text{dc}}} = \frac{3.6\text{ W}}{120\text{ V}} = 30\text{ mA}$$

$$I_m = \sqrt{2}I_{\text{dc}} = (1.414)(30\text{ mA}) = \mathbf{42.42\text{ mA}}$$

$$E_m = \sqrt{2}E_{\text{dc}} = (1.414)(120\text{ V}) = \mathbf{169.68\text{ V}}$$

EXAMPLE 13.21 Find the effective or rms value of the waveform of Fig. 13.51.

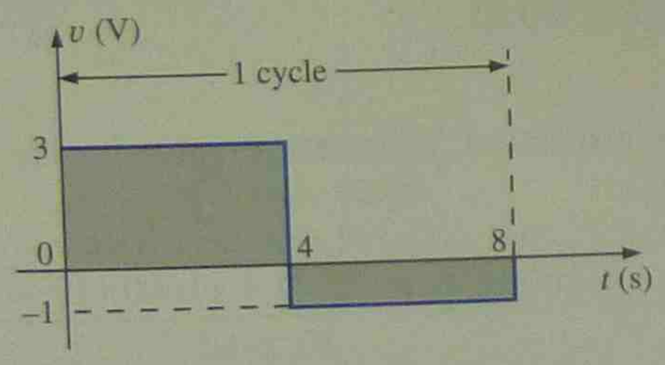


FIG. 13.51

Solution:

v^2 (Fig. 13.52):

$$V_{\text{eff}} = \sqrt{\frac{(9)(4) + (1)(4)}{8}} = \sqrt{\frac{40}{8}} = \mathbf{2.236\text{ V}}$$

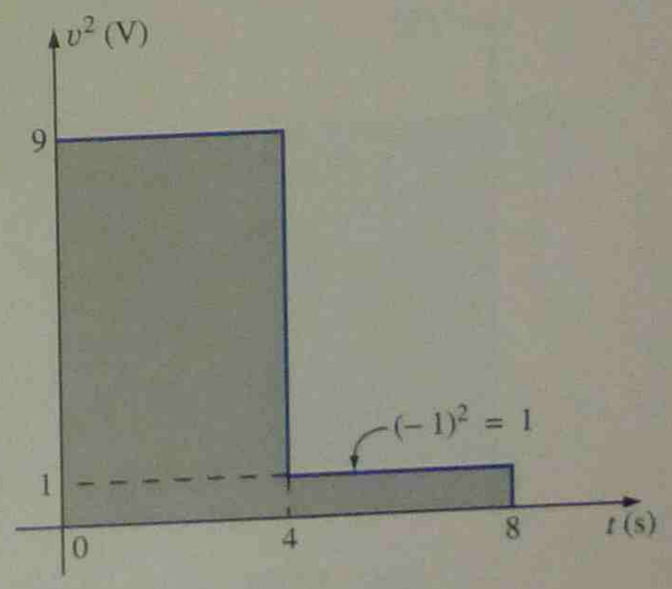


FIG. 13.52

EXAMPLE 13.22 Calculate the effective value of the voltage of Fig. 13.53.

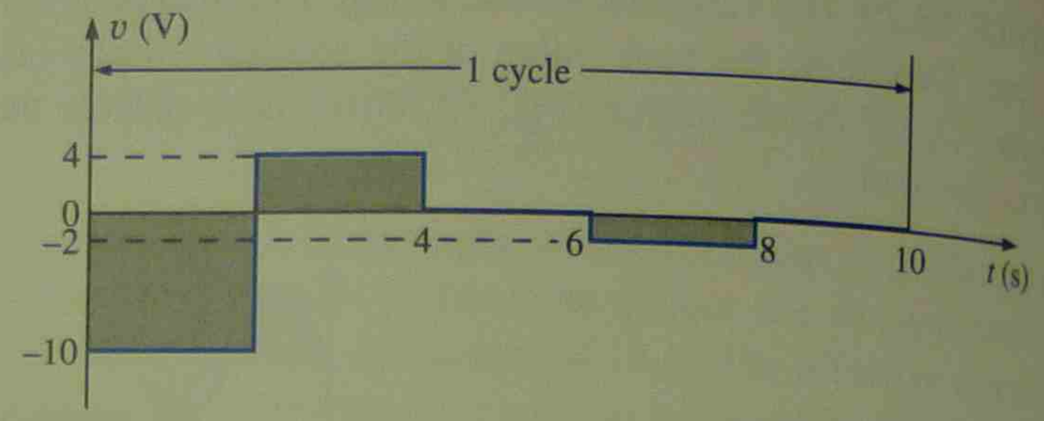


FIG. 13.53

Solution:

v^2 (Fig. 13.54):

$$V_{\text{eff}} = \sqrt{\frac{(100)(2) + (16)(2) + (4)(2)}{10}} = \sqrt{\frac{240}{10}}$$

$$= \mathbf{4.899\ V}$$

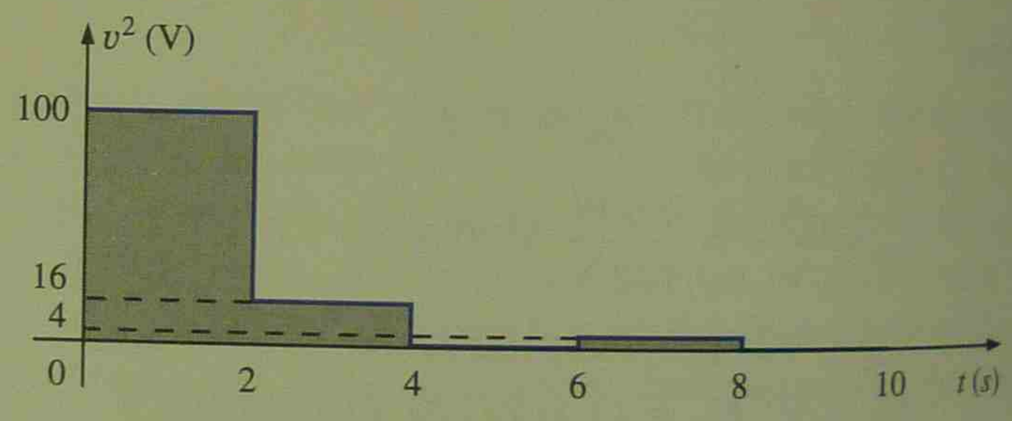


FIG. 13.54

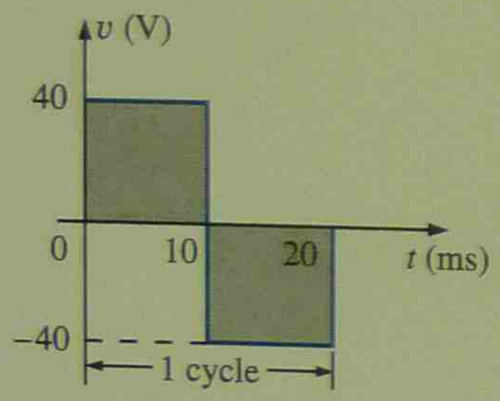


FIG. 13.55

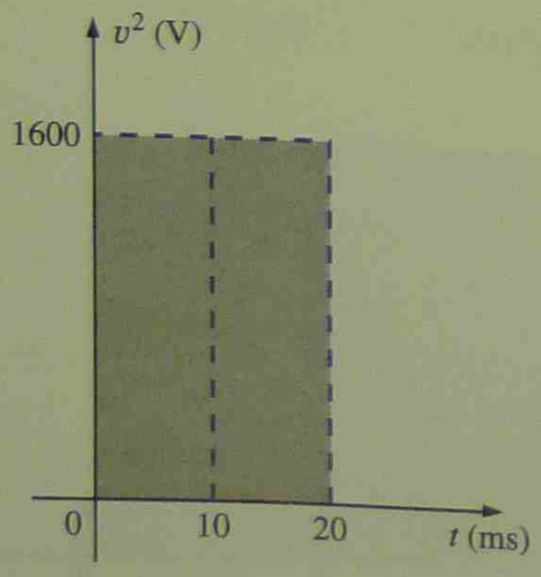


FIG. 13.56

EXAMPLE 13.23 Determine the average and effective values of the square wave of Fig. 13.55.

Solution: By inspection, the average value is zero.

v^2 (Fig. 13.56):

$$V_{\text{eff}} = \sqrt{\frac{(1600)(10 \times 10^{-3}) + (1600)(10 \times 10^{-3})}{20 \times 10^{-3}}}$$

$$= \sqrt{\frac{32{,}000 \times 10^{-3}}{20 \times 10^{-3}}} = \sqrt{1600}$$

$$V_{\text{eff}} = \mathbf{40\ V}$$

(the maximum value of the waveform of Fig. 13.56)

The waveforms appearing in these examples are the same as those used in the examples on the average value. It might prove interesting to compare the effective and average values of these waveforms.

The effective values of sinusoidal quantities such as voltage or current will be represented by E and I. These symbols are the same as those used for dc voltages and currents. To avoid confusion, the peak value of a waveform will always have a subscript m associated with it: $I_m \sin \omega t$. *Caution:* When finding the effective value of the positive pulse of a sine wave, note that the squared area is *not* simply $(2A_m)^2 = 4A_m^2$; it must be found by a completely new integration. This will always be the case for any waveform that is not rectangular.

13.8 ac METERS AND INSTRUMENTS

The d'Arsonval movement employed in dc meters can also be used to measure sinusoidal voltages and currents if the *bridge rectifier* of Fig. 13.57 is placed between the signal to be measured and the average reading movement.

The bridge rectifier composed of four diodes (electronic switches) will convert the input signal of zero average value to one having an average value sensitive to the peak value of the input signal. The conversion process is well described in most basic electronics texts. Fundamentally, conduction is permitted through the diodes in such a manner as to convert the sinusoidal input of Fig. 13.58(a) to one having the appearance of Fig. 13.58(b). The negative portion of the input has been effectively "flipped over" by the bridge configuration. The resulting waveform of Fig. 13.58(b) is called a *full-wave rectified waveform.*

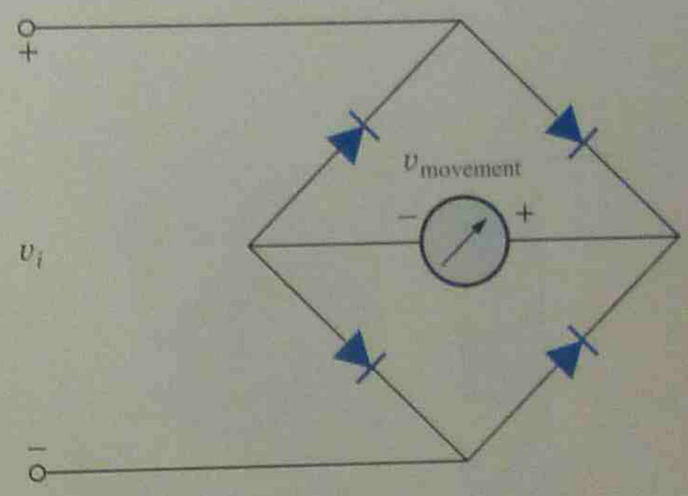


FIG. 13.57
Full-wave bridge rectifier.

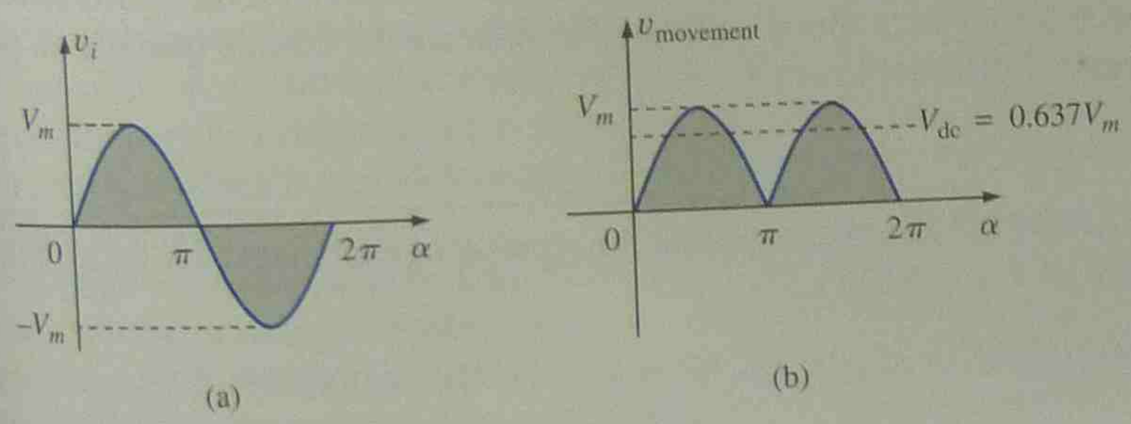


FIG. 13.58
(a) Sinusoidal input; (b) full-wave rectified signal.

The zero average value of Fig. 13.58(a) has been replaced by a pattern having an average value determined by

$$G = \frac{2V_m + 2V_m}{2\pi} = \frac{4V_m}{2\pi} = \frac{2V_m}{\pi} = 0.637V_m$$

The movement of the pointer will therefore be directly related to the peak value of the signal by the factor 0.637.

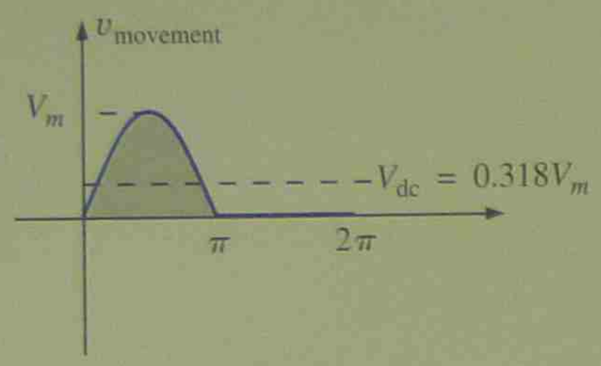


FIG. 13.59
Half-wave rectified signal.

Forming the ratio between the rms and dc levels will result in

$$\frac{V_{rms}}{V_{dc}} = \frac{0.707V_m}{0.637V_m} \cong 1.11$$

revealing that the scale indication is 1.11 times the dc level measured by the movement. That is,

$$\boxed{\text{Meter indication} = 1.11(\text{dc or average value})} \quad \textit{full-wave} \quad (13.35)$$

Some ac meters use a half-wave rectifier arrangement that results in the waveform of Fig. 13.59, which has half the average value of Fig. 13.58(b) over one full cycle. The result is

$$\boxed{\text{Meter indication} = 2.22(\text{dc or average value})} \quad \textit{half-wave} \quad (13.36)$$

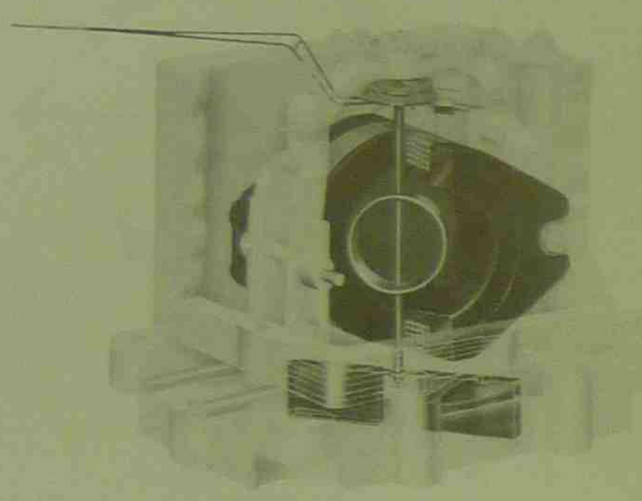


FIG. 13.60
Electrodynamometer movement. (Courtesy of Weston Instruments, Inc.)

A second movement, called the electrodynamometer movement (Fig. 13.60), can measure both ac and dc quantities without a change in internal circuitry. The movement can, in fact, read the effective value of any periodic or nonperiodic waveform, because a reversal in current direction reverses the fields of both the stationary and the movable coils, so the deflection of the pointer is always up-scale.

The VOM, introduced in Chapter 2, can be used to measure both dc and ac voltages using a d'Arsonval movement and the proper switching networks. That is, when the meter is used for dc measurements, the dial setting will establish the proper series resistance for the chosen scale and permit the appropriate dc level to pass directly to the movement. For ac measurements, the dial setting will introduce a network that employs a full- or half-wave rectifier to establish a dc level. As discussed above, each setting is properly calibrated to indicate the desired quantity on the face of the instrument.

EXAMPLE 13.24 Determine the reading of each meter for each situation of Fig. 13.61.

Solution: For part (a), situation (1): By Eq. (13.35),

$$\text{Meter indication} = 1.11(20\text{ V}) = \mathbf{22.2\text{ V}}$$

For part (a), situation (2):

$$V_{rms} = 0.707V_m = 0.707(20\text{ V}) = \mathbf{14.14\text{ V}}$$

For part (b), situation (1):

$$V_{rms} = V_{dc} = \mathbf{25\text{ V}}$$

For part (b), situation (2):

$$V_{rms} = 0.707V_m = 0.707(15\text{ V}) \cong \mathbf{10.6\text{ V}}$$

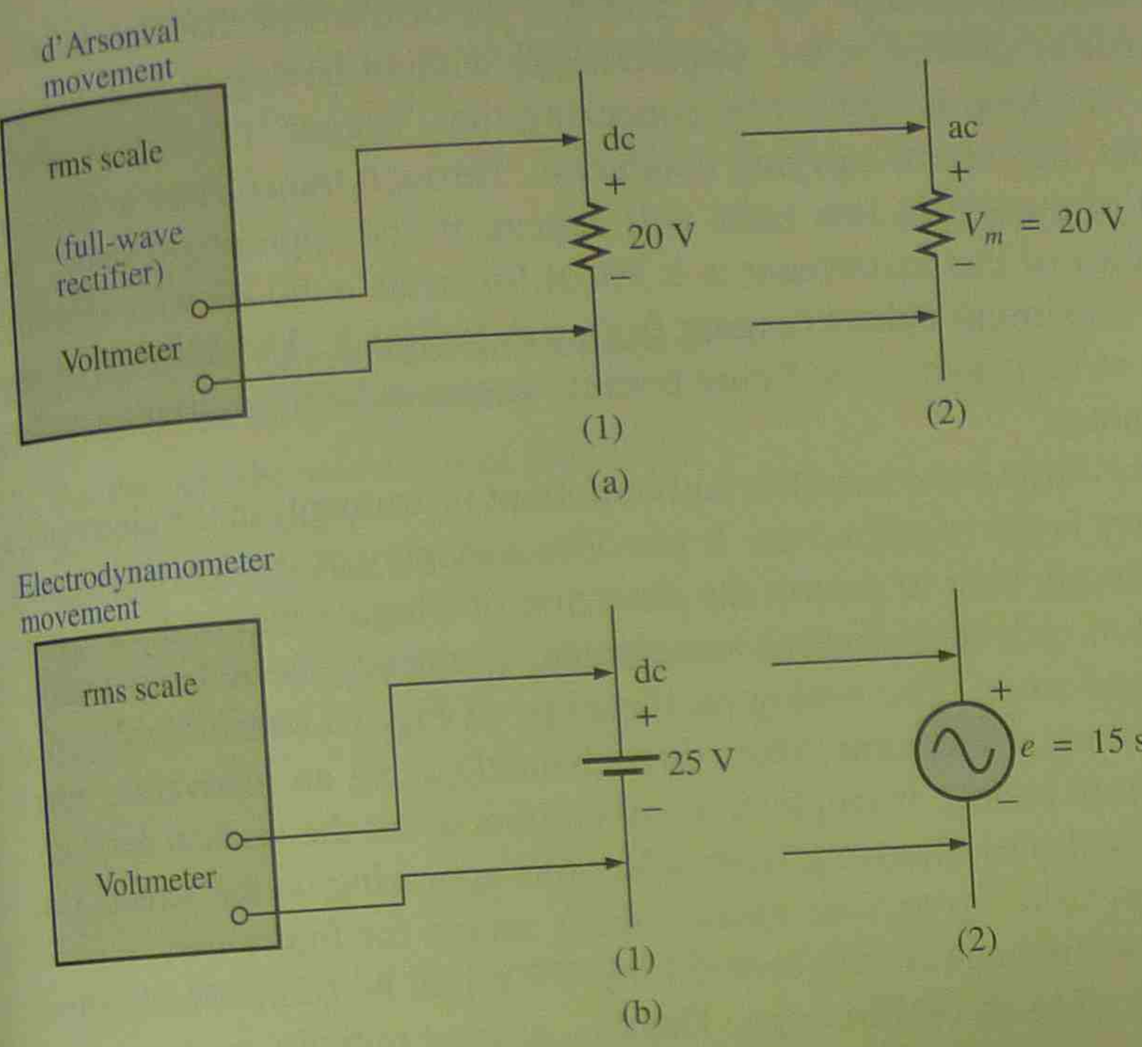


FIG. 13.61

Most DMMs employ a full-wave rectification system to convert the input ac signal to one with an average value. In fact, for the DMM of Fig. 2.27, the same scale factor of Eq. (13.35) is employed. That is, the average value is scaled up by a factor of 1.11 to obtain the rms value. In the digital meters, however, there are no moving parts such as in the d'Arsonval or electrodynamometer movements to display the signal level. Rather, the average value is sensed by a multiprocessor integrated circuit (IC), which in turn determines which digits should appear on the digital display.

Digital meters can also be used to measure nonsinusoidal signals, but the scale factor of each input waveform must first be known (normally provided by the manufacturer in the operator's manual). For instance, the scale factor for an average responding DMM on the ac rms scale will produce an indication for a square-wave input that is 1.11 times the peak value. For a triangular input, the response is 0.555 times the peak value. Obviously, for a sine wave input the response is 0.707 times the peak value.

For any instrument, it is always good practice to read (if only briefly) the operator's manual if it appears you will use the instrument on a regular basis.

For frequency measurements, the frequency counter of Fig. 13.62 provides a digital readout of sine, square, and triangular waves from 5 Hz to 100 MHz at input levels from 30 mV to 42 V. Note the relative simplicity of the panel and the high degree of accuracy available.



FIG. 13.62

Frequency counter. (Courtesy of Tektronix, Inc.)



FIG. 13.63
Amp-Clamp. (Courtesy of Simpson Instruments, Inc.)

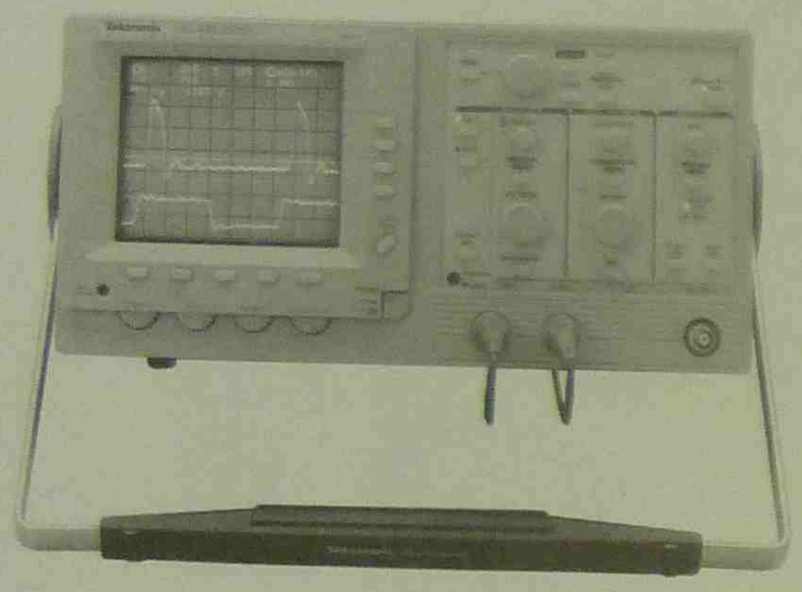


FIG. 13.64
Dual-channel oscilloscope. (Courtesy of Tektronix, Inc.)

The Amp-Clamp® of Fig. 13.63 is an instrument that can measure alternating current in the ampere range without having to open the circuit. The loop is opened by squeezing the "trigger"; then it is placed around the current-carrying conductor. Through transformer action, the level of current in rms units will appear on the appropriate scale. The accuracy of this instrument is ±3% of full scale at 60 Hz, and its scales have maximum values ranging from 6 A to 300 A. The addition of two leads as indicated in the figure permits its use as both a voltmeter and an ohmmeter.

One of the most versatile and important instruments in the electronics industry is the oscilloscope. It provides a display of the waveform on a cathode-ray tube to permit the detection of irregularities and the determination quantities such as magnitude, frequency, period, dc component, and so on. The analog oscilloscope of Fig. 13.64 can display two waveforms at the same time (dual-channel) using an innovative interface (front panel). It employs menu buttons to set the vertical and horizontal scales by choosing from selections appearing on the screen. One can also store up to four measurement setups for future use.

A student accustomed to watching TV might be confused when first introduced to an oscilloscope. There is, at least initially, an assumption that the oscilloscope is generating the waveform on the screen—much like a TV broadcast. However, it is important to clearly understand that

an oscilloscope displays only those signals generated elsewhere and connected to the input terminals of the oscilloscope. The absence of an external signal will simply result in a horizontal line on the screen of the scope.

On most modern-day oscilloscopes, there is a switch or knob with the choice DC/GND/AC as shown in Fig. 13.65(a) that is often ignored or treated too lightly in the early stages of scope utilization. The effect of each position is fundamentally as shown in Fig. 13.65(b). In the DC mode the dc and ac components of the input signal can pass directly to the display. In the AC position the dc input is blocked by the capacitor, but the ac portion of the signal can pass on through to the screen. In the GND position the input signal is prevented from reaching the scope display by a direct ground connection, which reduces the scope display to a single horizontal line. The DC/AC option is an excellent way to determine the dc content of the input signal. All that is required is to note the vertical swing of the input signal when the switch is moved between the AC and DC positions. The lack of any change in the waveform indicates a lack of dc in the input signal, but an upward swing

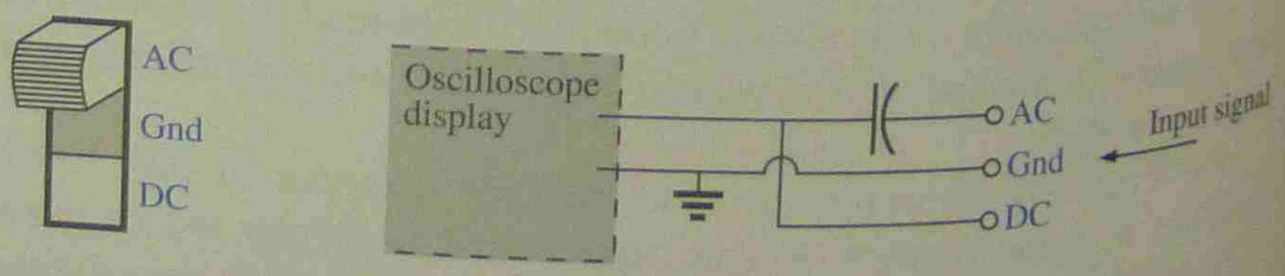


FIG. 13.65

indicates a positive dc level and a downward shift a negative dc level. Using the vertical sensitivity (V/cm) permits a direct measurement of the dc content.

PROBLEMS

SECTION 13.2 Sinusoidal ac Voltage Characteristics and Definitions

1. For the periodic waveform of Fig. 13.66:
   a. Find the period T.
   b. How many cycles are shown?
   c. What is the frequency?
   *d. Determine the positive amplitude and peak-to-peak value (think!).

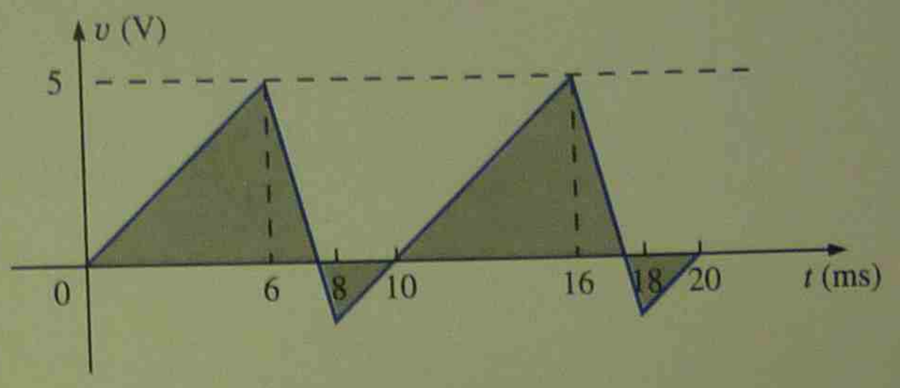


FIG. 13.66

2. Repeat Problem 1 for the periodic waveform of Fig. 13.67.

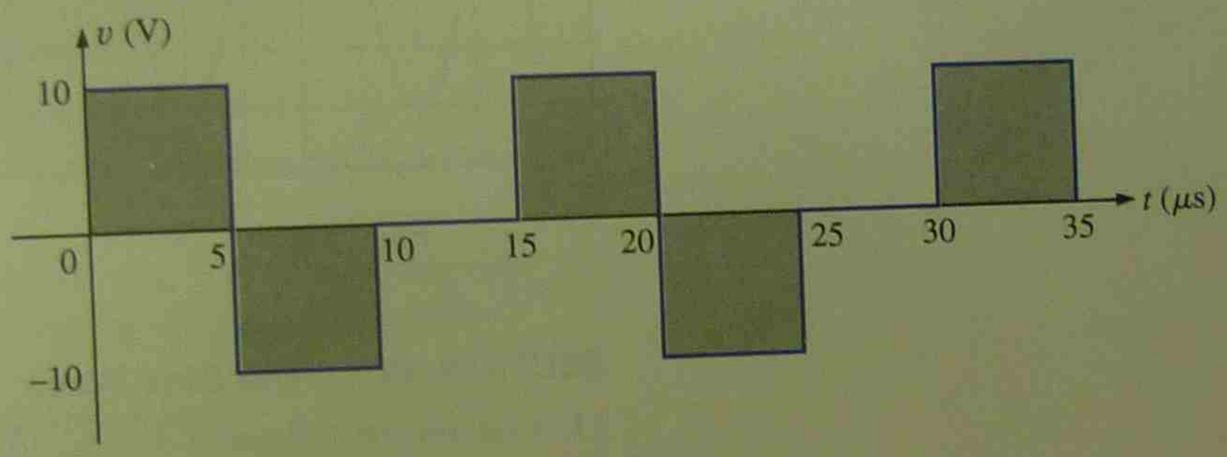


FIG. 13.67

3. Determine the period and frequency of the sawtooth waveform of Fig. 13.68.

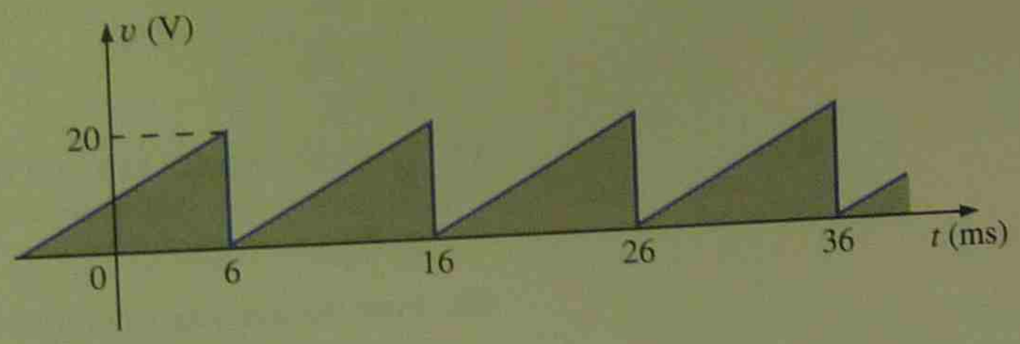


FIG. 13.68

4. Find the period of a periodic waveform whose frequency is
   a. 25 Hz
   b. 35 MHz
   c. 55 kHz
   d. 1 Hz

5. Find the frequency of a repeating waveform whose period is
   a. 1/60 s
   b. 0.01 s
   c. 34 ms
   d. 25 μs

6. Find the period of a sinusoidal waveform that completes 80 cycles in 24 ms.

7. If a periodic waveform has a frequency of 20 Hz, how long (in seconds) will it take to complete 5 cycles?
8. What is the frequency of a periodic waveform that completes 42 cycles in 6 s?
9. Sketch a periodic square wave like that appearing in Fig. 13.67 with a frequency of 20,000 Hz and a peak value of 10 mV.
10. For the oscilloscope pattern of Fig. 13.69,
    a. Determine the peak amplitude.
    b. Find the period.
    c. Calculate the frequency.
    Redraw the oscilloscope pattern if a +25-mV dc level were added to the input waveform.

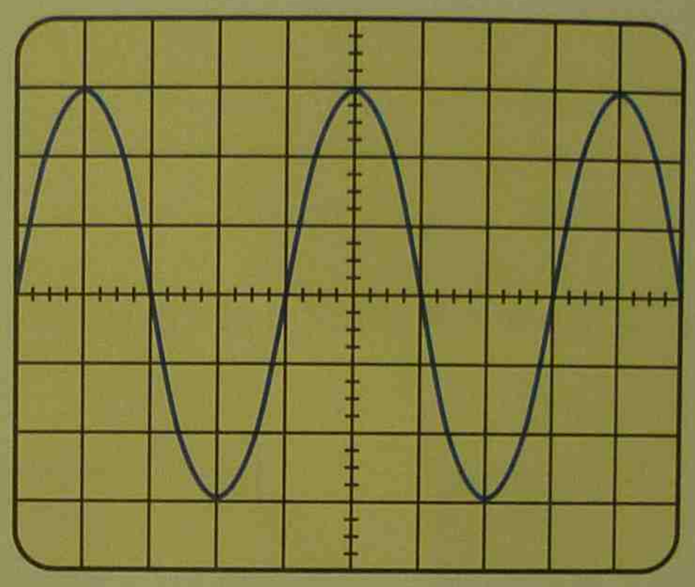


FIG. 13.69

SECTION 13.3 The Sine Wave

11. Convert the following degrees to radians:
    a. 45° **b.** 60°
    c. 120° **d.** 270°
    e. 178° **f.** 221°
12. Convert the following radians to degrees:
    a. $\pi/4$ **b.** $\pi/6$
    c. $\frac{1}{10}\pi$ **d.** $\frac{7}{6}\pi$
    e. 3π **f.** 0.55π
13. Find the angular velocity of a waveform with a period of
    a. 2 s **b.** 0.3 ms
    c. 4 μs **d.** $\frac{1}{25}$ s
14. Find the angular velocity of a waveform with a frequency of
    a. 50 Hz **b.** 600 Hz
    c. 2 kHz **d.** 0.004 MHz
15. Find the frequency and period of sine waves having an angular velocity of
    a. 754 rad/s **b.** 8.4 rad/s
    c. 6000 rad/s **d.** $\frac{1}{16}$ rad/s
16. Given $f = 60$ Hz, determine how long it will take the sinusoidal waveform to pass through an angle of 45°.
17. If a sinusoidal waveform passes through an angle of 30° in 5 ms, determine the angular velocity of the waveform.

SECTION 13.4 General Format for the Sinusoidal Voltage or Current

18. Find the amplitude and frequency of the following waves:
    - **a.** $20 \sin 377t$
    - **b.** $5 \sin 754t$
    - **c.** $10^6 \sin 10{,}000t$
    - **d.** $0.001 \sin 942t$
    - **e.** $-7.6 \sin 43.6t$
    - **f.** $(\frac{1}{42}) \sin 6.28t$
19. Sketch $5 \sin 754t$ with the abscissa
    - **a.** angle in degrees.
    - **b.** angle in radians.
    - **c.** time in seconds.
20. Sketch $10^6 \sin 10{,}000t$ with the abscissa
    - **a.** angle in degrees.
    - **b.** angle in radians.
    - **c.** time in seconds.
21. Sketch $-7.6 \sin 43.6t$ with the abscissa
    - **a.** angle in degrees.
    - **b.** angle in radians.
    - **c.** time in seconds.
22. If $e = 300 \sin 157t$, how long (in seconds) does it take this waveform to complete 1/2 cycle?
23. Given $i = 0.5 \sin \alpha$, determine i at $\alpha = 72°$.
24. Given $v = 20 \sin \alpha$, determine v at $\alpha = 1.2\pi$.
*25. Given $v = 30 \times 10^{-3} \sin \alpha$, determine the angles at which v will be 6 mV.
*26. If $v = 40$ V at $\alpha = 30°$ and $t = 1$ ms, determine the mathematical expression for the sinusoidal voltage.

SECTION 13.5 Phase Relations

27. Sketch $\sin(377t + 60°)$ with the abscissa
    - **a.** angle in degrees.
    - **b.** angle in radians.
    - **c.** time in seconds.
28. Sketch the following waveforms:
    - **a.** $50 \sin(\omega t + 0°)$
    - **b.** $-20 \sin(\omega t + 2°)$
    - **c.** $5 \sin(\omega t + 60°)$
    - **d.** $4 \cos \omega t$
    - **e.** $2 \cos(\omega t + 10°)$
    - **f.** $-5 \cos(\omega t + 20°)$
29. Find the phase relationship between the waveforms of each set:
    - **a.** $v = 4 \sin(\omega t + 50°)$
      $i = 6 \sin(\omega t + 40°)$
    - **b.** $v = 25 \sin(\omega t - 80°)$
      $i = 5 \times 10^{-3} \sin(\omega t - 10°)$
    - **c.** $v = 0.2 \sin(\omega t - 60°)$
      $i = 0.1 \sin(\omega t + 20°)$
    - **d.** $v = 200 \sin(\omega t - 210°)$
      $i = 25 \sin(\omega t - 60°)$
*30. Repeat Problem 29 for the following sets:
    - **a.** $v = 2 \cos(\omega t - 30°)$
      $i = 5 \sin(\omega t + 60°)$
    - **b.** $v = -1 \sin(\omega t + 20°)$
      $i = 10 \sin(\omega t - 70°)$
    - **c.** $v = -4 \cos(\omega t + 90°)$
      $i = -2 \sin(\omega t + 10°)$

14 The Basic Elements and Phasors

14.1 INTRODUCTION

The response of the basic R, L, and C elements to a sinusoidal voltage and current will be examined in this chapter with special note of how frequency will affect the ''opposing'' characteristic of each element. Phasor notation will then be introduced to establish a method of analysis that permits a direct correspondence with a number of the methods, theorems, and concepts introduced in the dc chapters.

14.2 THE DERIVATIVE

It is fundamental to the understanding of the response of the basic R, L, and C elements to a sinusoidal signal that the concept of the *derivative* be examined in some detail. It will not be necessary that you become proficient in the mathematical technique but simply that you understand the impact of a relationship defined by a derivative.

Recall from Section 10.11 that the derivative dx/dt is defined as the rate of change of x with respect to time. If x fails to change at a particular instant, $dx = 0$, and the derivative is zero. For the sinusoidal waveform, dx/dt is zero only at the positive and negative peaks ($\omega t = \pi/2$ and $\frac{3}{2}\pi$ in Fig. 14.1), since x fails to change at these instants of time. The derivative dx/dt is actually the slope of the graph at any instant of time.

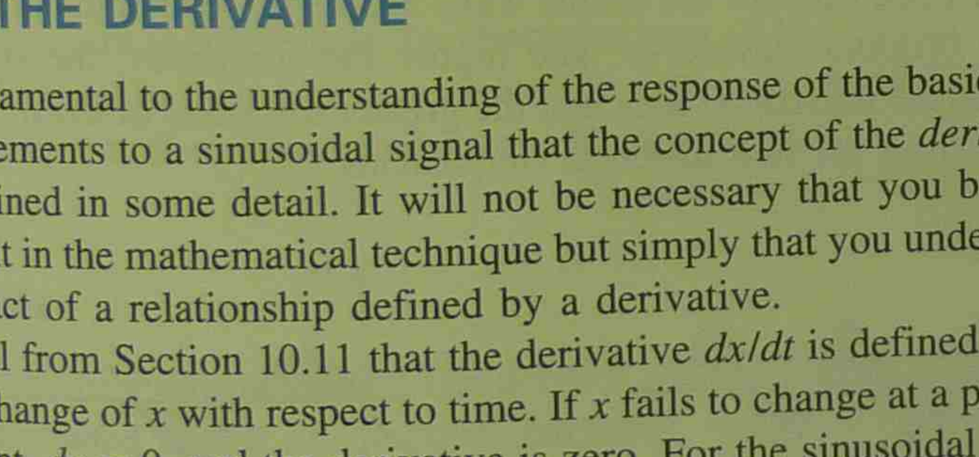


FIG. 14.1

A close examination of the sinusoidal waveform will also indicate that the greatest change in x will occur at the instants $\omega t = 0$, π, and 2π. The derivative is therefore a maximum at these points. At 0 and 2π, x increases at its greatest rate, and the derivative is given a positive sign since x increases with time. At π, dx/dt decreases at the same rate as it increases at 0 and 2π, but the derivative is given a negative sign since x decreases with time. Since the rate of change at 0, π, and 2π is the same, the magnitude of the derivative at these points is the same also. For various values of ωt between these maxima and minima, the derivative will exist and have values from the minimum to the maximum inclusive. A plot of the derivative in Fig. 14.2 shows that

the derivative of a sine wave is a cosine wave.

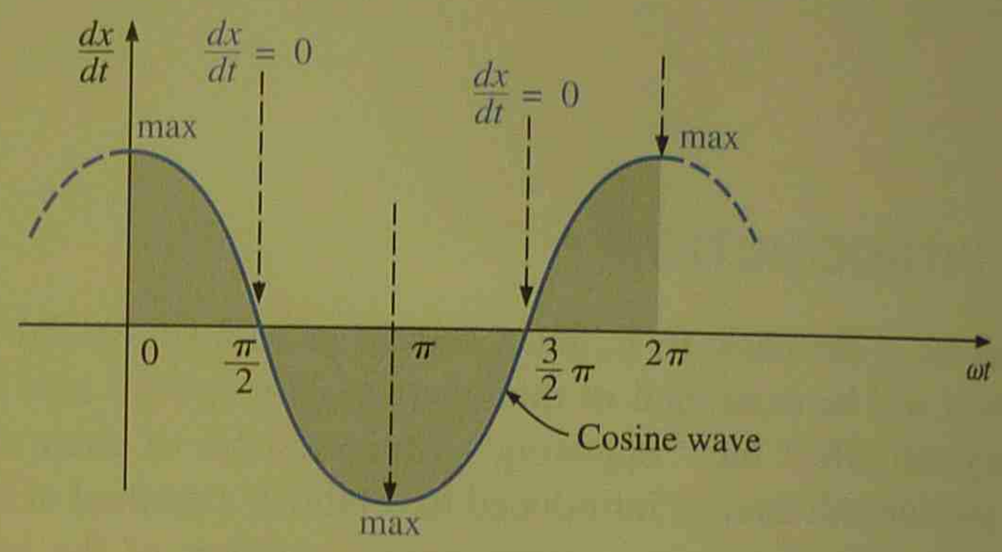


FIG. 14.2
Derivative of the sine wave of Fig. 14.1.

The peak value of the cosine wave is directly related to the frequency of the original waveform. The higher the frequency, the steeper the slope at the horizontal axis and the greater the value of dx/dt, as shown in Fig. 14.3 for two different frequencies.

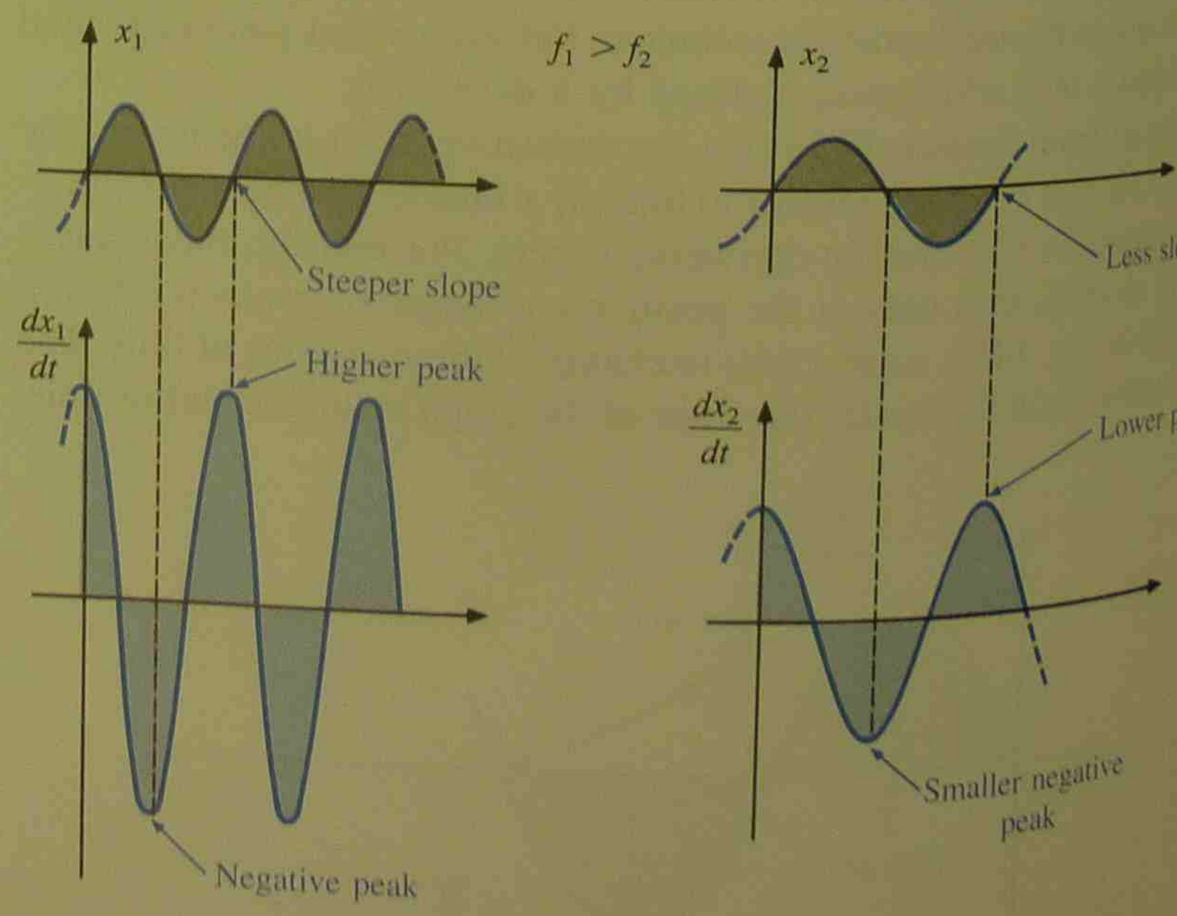


FIG. 14.3
Effect of frequency on the peak value of the derivative.

Note in Fig. 14.3 that even though both waveforms (x_1 and x_2) have the same peak value, the sinusoidal function with the higher frequency produces the larger peak value for the derivative. In addition, note that

the derivative of a sine wave has the same period and frequency as the original sinusoidal waveform.

For the sinusoidal voltage

$$e(t) = E_m \sin(\omega t \pm \theta)$$

the derivative can be found directly by differentiation (calculus) to produce the following:

$$\frac{d}{dt}e(t) = \omega E_m \cos(\omega t \pm \theta) = 2\pi f E_m \cos(\omega t \pm \theta) \qquad \textbf{(14.1)}$$

The mechanics of the differentiation process will not be discussed or investigated here nor will they be required to continue with the text. Note, however, that the peak value of the derivative, $2\pi f E_m$, is a function of the frequency of $e(t)$ and the derivative of a sine wave is a cosine wave.

14.3 RESPONSE OF BASIC *R*, *L*, AND *C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

Now that we are familiar with the characteristics of the derivative of a sinusoidal function, we can investigate the response of the basic elements R, L, and C to a sinusoidal voltage or current.

Resistor

For power-line frequencies and frequencies up to a few hundred kilohertz, resistance is, for all practical purposes, unaffected by the frequency of the applied sinusoidal voltage or current. For this frequency region, the resistor R of Fig. 14.4 can be treated as a constant, and Ohm's law can be applied as follows. For $v = V_m \sin \omega t$,

$$i = \frac{v}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t$$

where

$$I_m = \frac{V_m}{R} \qquad \textbf{(14.2)}$$

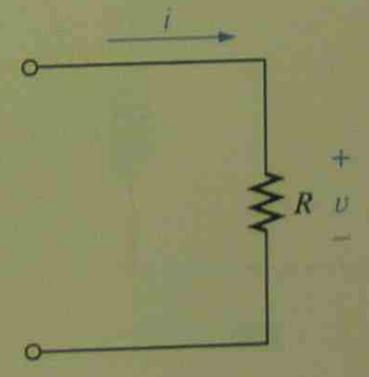


FIG. 14.4

In addition, for a given i,

$$v = iR = (I_m \sin \omega t)R = I_m R \sin \omega t = V_m \sin \omega t$$

where

$$V_m = I_m R \qquad \textbf{(14.3)}$$

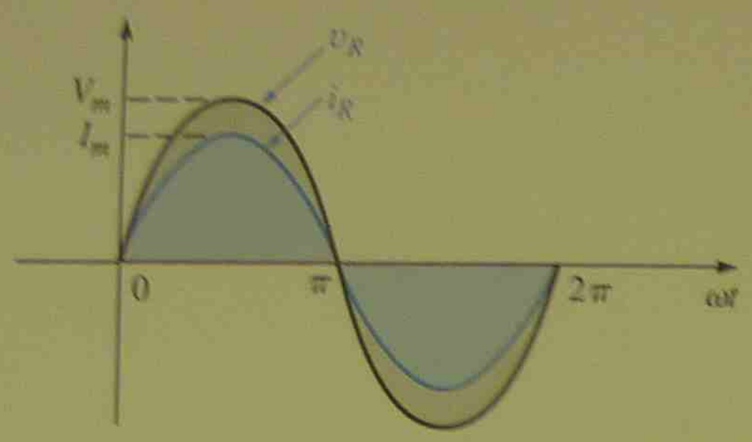


FIG. 14.5

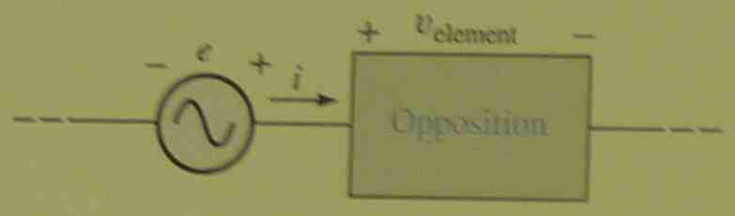


FIG. 14.6

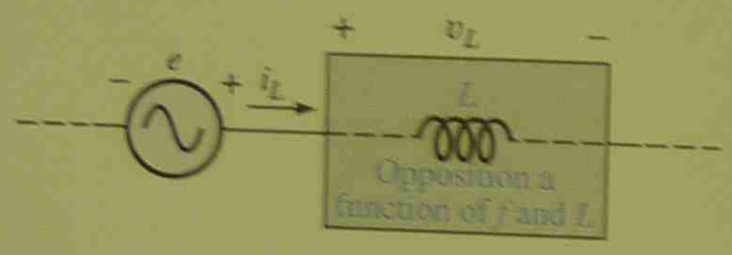


FIG. 14.7

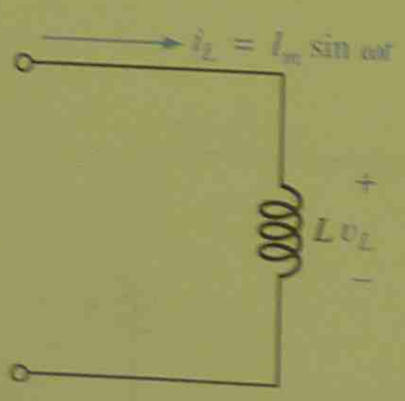


FIG. 14.8

A plot of v and i in Fig. 14.5 reveals that

for a purely resistive element, the voltage across and the current through the element are in phase with their peak values related by Ohm's law.

Inductor

For the series configuration of Fig. 14.6, the voltage $v_{element}$ of the boxed-in element opposes the source e and thereby reduces the magnitude of the current i. The magnitude of the voltage across the element is determined by the opposition of the element to the flow of charge, or current i. For a resistive element, we have found that the opposition is its resistance and that $v_{element}$ and i are determined by $v_{element} = iR$.

For the inductor, we found in Chapter 12 that the voltage across an inductor is directly related to the rate of change of current through the coil. Consequently, the higher the frequency, the greater will be the rate of change of current through the coil, and the greater the magnitude of the voltage. In addition, we found in the same chapter that the inductance of a coil will determine the rate of change of flux linking a coil for a particular change in current through the coil. The higher the inductance, the greater the rate of change of the flux linkages, and the greater the resulting voltage across the coil.

The inductive voltage, therefore, is directly related to the frequency (or, more specifically, the angular velocity of the sinusoidal ac current through the coil) and the inductance of the coil. For increasing values of f and L in Fig. 14.7, the magnitude of v_L will increase as described above.

Utilizing the similarities between Figs. 14.6 and 14.7, we find that increasing levels of v_L are directly related to increasing levels of opposition in Fig. 14.6. Since v_L will increase with both ω $(= 2\pi f)$ and L, the opposition of an inductive element is as defined in Fig. 14.7.

We will now verify some of the preceding conclusions using a more mathematical approach and then define a few important quantities to be employed in the sections and chapters to follow.

For the inductor of Fig. 14.8, we recall from Chapter 12 that

$$v_L = L\frac{di_L}{dt}$$

and, applying differentiation,

$$\frac{di_L}{dt} = \frac{d}{dt}(I_m \sin \omega t) = \omega I_m \cos \omega t$$

Therefore, $$v_L = L\frac{di_L}{dt} = L(\omega I_m \cos \omega t) = \omega L I_m \cos \omega t$$

or

$$v_L = V_m \sin(\omega t + 90°)$$

where

$$V_m = \omega L I_m$$

Note that the peak value of v_L is directly related to ω ($= 2\pi f$) and L as predicted in the discussion above.

A plot of v_L and i_L in Fig. 14.9 reveals that

for an inductor v_L leads i_L by 90°, or i_L lags v_L by 90°.

If a phase angle is included in the sinusoidal expression for i_L, such as

$$i_L = I_m \sin(\omega t \pm \theta)$$

then

$$v_L = \omega L I_m \sin(\omega t \pm \theta + 90°)$$

The opposition established by an inductor in a sinusoidal ac network can now be found by applying Eq. (4.1):

$$\text{Effect} = \frac{\text{cause}}{\text{opposition}}$$

which for our purposes can be written

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

Substituting values, we have

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{\omega L I_m}{I_m} = \omega L$$

revealing that the opposition established by an inductor in an ac sinusoidal network is directly related to the product of the angular velocity ($\omega = 2\pi f$) and the inductance, verifying our earlier conclusions.

The quantity ωL, called the *reactance* (from the word *reaction*) of an inductor, is symbolically represented by X_L and is measured in ohms; that is,

$$X_L = \omega L \qquad \text{(ohms, } \Omega\text{)} \qquad \textbf{(14.4)}$$

In an Ohm's law format, its magnitude can be determined from

$$X_L = \frac{V_m}{I_m} \qquad \text{(ohms, } \Omega\text{)} \qquad \textbf{(14.5)}$$

Inductive reactance is the opposition to the flow of current, which results in the continual interchange of energy between the source and the magnetic field of the inductor. In other words, inductive reactance, unlike resistance (which dissipates energy in the form of heat), does not dissipate electrical energy (ignoring the effects of the internal resistance of the inductor).

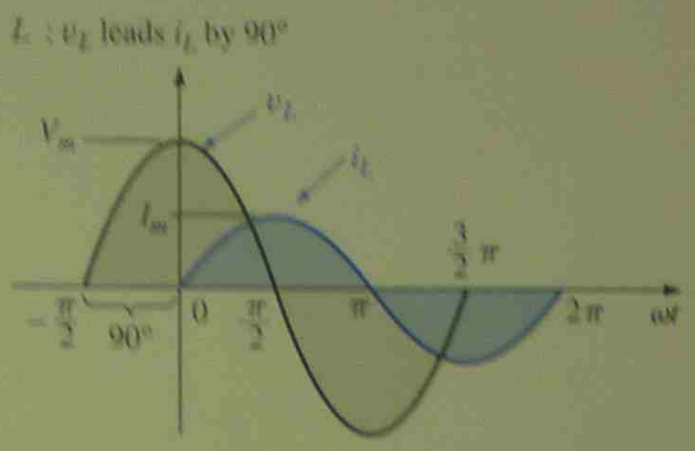


FIG. 14.9

Capacitor

Let us now return to the series configuration of Fig. 14.6 and insert the capacitor as the element of interest. For the capacitor, however, we will

determine i for a particular voltage across the element. When this approach reaches its conclusion, the relationship between the voltage and current will be known, and the opposing voltage ($v_{element}$) can be determined for any sinusoidal current i.

Our investigation of the inductor revealed that the inductive voltage across a coil opposes the instantaneous change in current through the coil. For capacitive networks, the voltage across the capacitor is limited by the rate at which charge can be deposited on, or released by, the plates of the capacitor during the charging and discharging phases, respectively. In other words, an instantaneous change in voltage across a capacitor is opposed by the fact that there is an element of time required to deposit charge on (or release charge from) the plates of a capacitor, and $V = Q/C$.

Since capacitance is a measure of the rate at which a capacitor will store charge on its plates

for a particular change in voltage across the capacitor, the greater the value of capacitance, the greater will be the resulting capacitive current.

In addition, the fundamental equation relating the voltage across a capacitor to the current of a capacitor [$i = C\,(dv/dt)$] indicates that

for a particular capacitance, the greater the rate of change of voltage across the capacitor, the greater the capacitive current.

Certainly, an increase in frequency corresponds to an increase in the rate of change of voltage across the capacitor and to an increase in the current of the capacitor.

The current of a capacitor is therefore directly related to the frequency (or, again more specifically, the angular velocity) and the capacitance of the capacitor. An increase in either quantity will result in an increase in the current of the capacitor. For the basic configuration of Fig. 14.10, however, we are interested in determining the opposition of the capacitor as related to the resistance of a resistor and ωL for the inductor. Since an increase in current corresponds to a decrease in opposition, and i_C is proportional to ω and C, the opposition of a capacitor is inversely related to $\omega\ (= 2\pi f)$ and C.

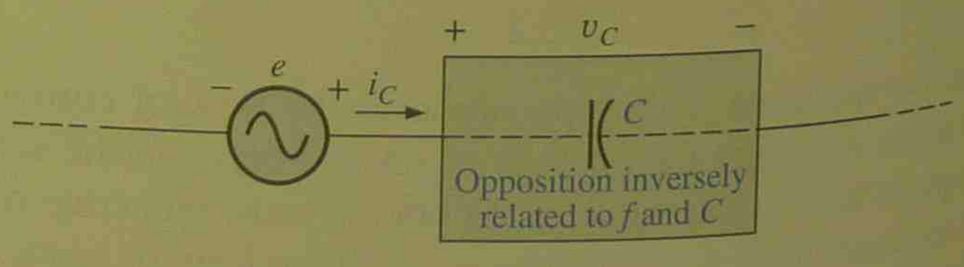


FIG. 14.10

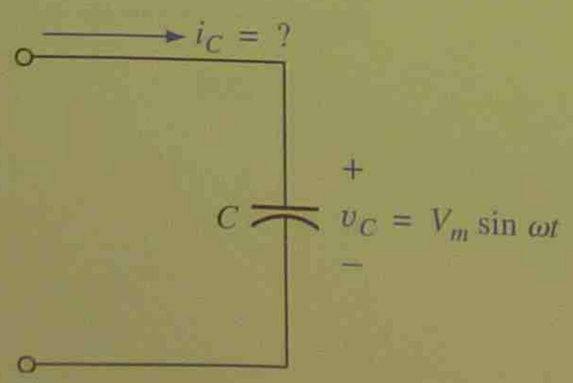


FIG. 14.11

We will now verify, as we did for the inductor, some of the above conclusions using a more mathematical approach.

For the capacitor of Fig. 14.11, we recall from Chapter 10 that

$$i_C = C\frac{dv_C}{dt}$$

determine i for a particular voltage across the element. When this approach reaches its conclusion, the relationship between the voltage and current will be known, and the opposing voltage ($v_{element}$) can be determined for any sinusoidal current i.

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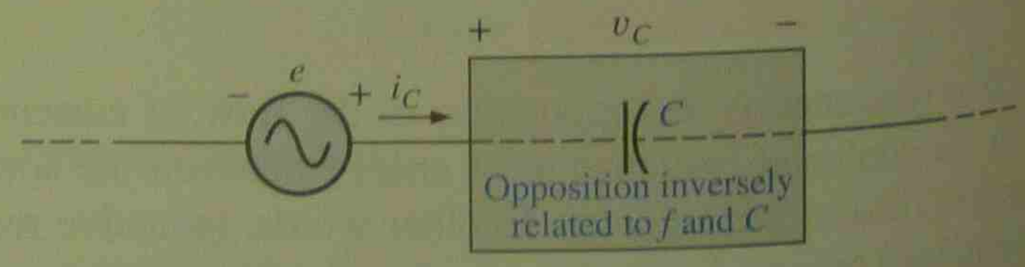


FIG. 14.10

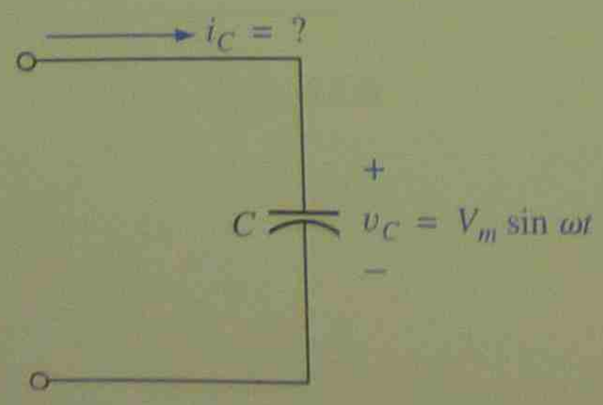


FIG. 14.11

We will now verify, as we did for the inductor, some of the above conclusions using a more mathematical approach.

For the capacitor of Fig. 14.11, we recall from Chapter 10 that

$$i_C = C\frac{dv_C}{dt}$$

and, applying differentiation,

$$\frac{dv_C}{dt} = \frac{d}{dt}(V_m \sin \omega t) = \omega V_m \cos \omega t$$

Therefore,

$$i_C = C\frac{dv_C}{dt} = C(\omega V_m \cos \omega t) = \omega C V_m \cos \omega t$$

or

$$i_C = I_m \sin(\omega t + 90°)$$

where

$$I_m = \omega C V_m$$

Note that the peak value of i_C is directly related to $\omega\,(= 2\pi f)$ and C, as predicted in the discussion above.

A plot of v_C and i_C in Fig. 14.12 reveals that

for a capacitor, i_C leads v_C by 90°, or v_C lags i_C by 90°.*

If a phase angle is included in the sinusoidal expression for v_C, such as

$$v_C = V_m \sin(\omega t \pm \theta)$$

then

$$i_C = \omega C V_m \sin(\omega t \pm \theta + 90°)$$

Applying

$$\text{Opposition} = \frac{\text{cause}}{\text{effect}}$$

and substituting values, we obtain

$$\text{Opposition} = \frac{V_m}{I_m} = \frac{V_m}{\omega C V_m} = \frac{1}{\omega C}$$

which agrees with the results obtained above.

The quantity $1/\omega C$, called the *reactance* of a capacitor, is symbolically represented by X_C and is measured in ohms; that is,

$$X_C = \frac{1}{\omega C} \quad \text{(ohms, } \Omega\text{)} \tag{14.6}$$

In an Ohm's law format, its magnitude can be determined from

$$X_C = \frac{V_m}{I_m} \quad \text{(ohms, } \Omega\text{)} \tag{14.7}$$

Capacitive reactance is the opposition to the flow of charge, which results in the continual interchange of energy between the source and

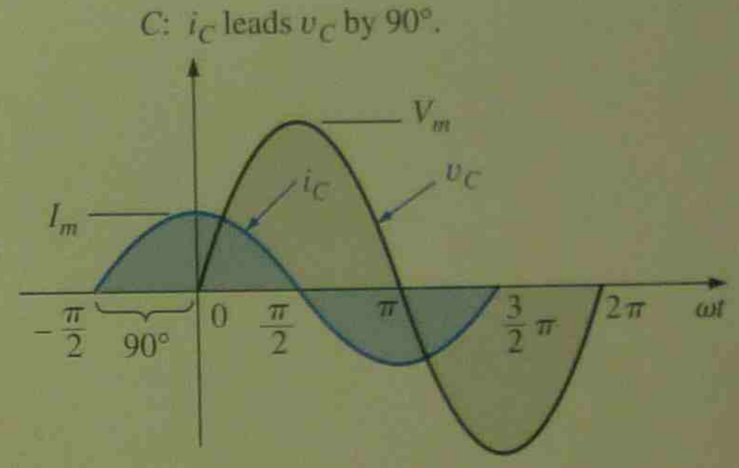


FIG. 14.12

*A mnemonic phrase sometimes used to remember the phase relationship between the voltage and current of a coil and capacitor is "*ELI* the *ICE* man." Note that the *L* (inductor) has the *E* before the *I* (*e* leads *i* by 90°), and the *C* (capacitor) has the *I* before the *E* (*i* leads *e* by 90°).

the electric field of the capacitor. Like the inductor, the capacitor does *not* dissipate energy in any form (ignoring the effects of the leakage resistance).

In the circuits just considered, the current was given in the inductive circuit, and the voltage in the capacitive circuit. This was done to avoid the use of integration in finding the unknown quantities. In the inductive circuit,

$$v_L = L\frac{di_L}{dt}$$

but

$$\boxed{i_L = \frac{1}{L}\int v_L\, dt} \tag{14.8}$$

In the capacitive circuit,

$$i_C = C\frac{dv_C}{dt}$$

but

$$\boxed{v_C = \frac{1}{C}\int i_C\, dt} \tag{14.9}$$

Shortly, we shall consider a method of analyzing ac circuits that will permit us to solve for an unknown quantity with sinusoidal input without having to use direct integration or differentiation.

It is possible to determine whether a network with one or more elements is predominantly capacitive or inductive by noting the phase relationship between the input voltage and current.

If the current leads the voltage, the network is predominantly capacitive, and if the voltage leads the current, it is predominantly inductive.

Since we now have an equation for the reactance of an inductor or capacitor, we do not need to use derivatives or integration in the examples to be considered. Simply applying Ohm's law, $I_m = E_m/X_L$ (or X_C), and keeping in mind the phase relationship between the voltage and current for each element, will be sufficient to complete the examples.

EXAMPLE 14.1 The voltage across a resistor is indicated. Find the sinusoidal expression for the current if the resistor is 10 Ω. Sketch the curves for v and i.

a. $v = 100 \sin 377t$

b. $v = 25 \sin(377t + 60°)$

Solutions:

a. Eq. 14.2: $I_m = \dfrac{V_m}{R} = \dfrac{100\text{ V}}{10\ \Omega} = 10\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{10 \sin 377t}$$

The curves are sketched in Fig. 14.13.



FIG. 14.13

b. Eq. 14.2: $I_m = \dfrac{V_m}{R} = \dfrac{25\text{ V}}{10\ \Omega} = 2.5\text{ A}$

(v and i are in phase), resulting in

$$i = \mathbf{2.5\sin(377t + 60°)}$$

The curves are sketched in Fig. 14.14.

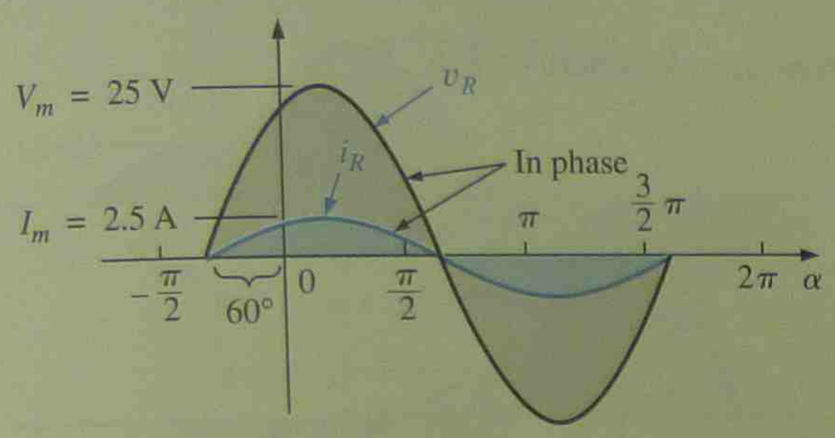


FIG. 14.14

EXAMPLE 14.2 The current through a 5-Ω resistor is given. Find the sinusoidal expression for the voltage across the resistor for $i = 40\sin(377t + 30°)$.

Solution: Eq. 14.3: $V_m = I_m R = (40\text{ A})(5\ \Omega) = 200\text{ V}$

(v and i are in phase), resulting in

$$v = \mathbf{200\sin(377t + 30°)}$$

EXAMPLE 14.3 The current through a 0.1-H coil is provided. Find the sinusoidal expression for the voltage across the coil. Sketch the v and i curves.

a. $i = 10\sin 377t$

b. $i = 7\sin(377t - 70°)$

Solutions:

a. Eq. 14.4: $X_L = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$

Eq. 14.5: $V_m = I_m X_L = (10\text{ A})(37.7\ \Omega) = 377\text{ V}$

and we know for a coil that v leads i by 90°. Therefore,

$$v = \mathbf{377\sin(377t + 90°)}$$

The curves are sketched in Fig. 14.15.

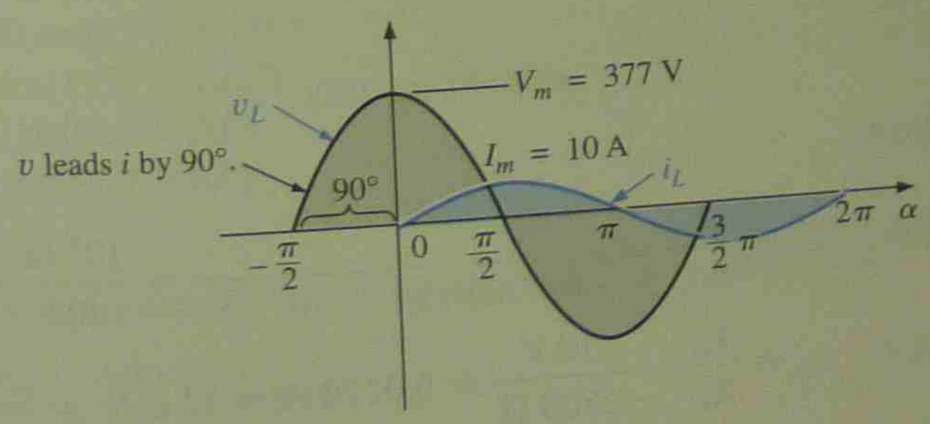


FIG. 14.15

b. X_L remains at 37.7 Ω.

$V_m = I_m X_L = (7\text{ A})(37.7\ \Omega) = 263.9\text{ V}$

and we know for a coil that v leads i by 90°. Therefore,

$$v = 263.9 \sin(377t - 70° + 90°)$$

and

$$\boldsymbol{v = 263.9 \sin(377t + 20°)}$$

The curves are sketched in Fig. 14.16.

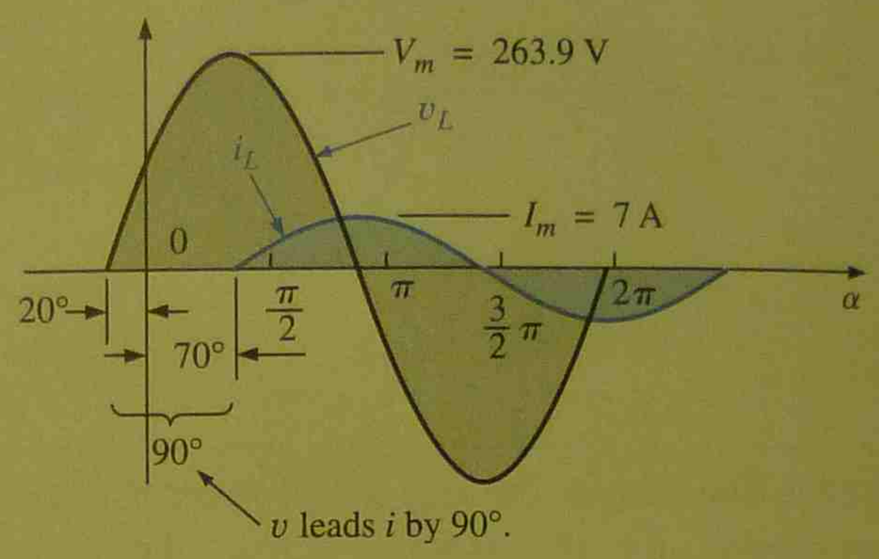


FIG. 14.16

EXAMPLE 14.4 The voltage across a 0.5-H coil is provided below. What is the sinusoidal expression for the current?

$$v = 100 \sin 20t$$

Solution:

$$X_L = \omega L = (20\text{ rad/s})(0.5\text{ H}) = 10\ \Omega$$

$$I_m = \frac{V_m}{X_L} = \frac{100\text{ V}}{10\ \Omega} = 10\text{ A}$$

and we know that i lags v by 90°. Therefore,

$$\boldsymbol{i = 10 \sin(20t - 90°)}$$

EXAMPLE 14.5 The voltage across a 1-μF capacitor is provided below. What is the sinusoidal expression for the current? Sketch the v and i curves.

$$v = 30 \sin 400t$$

Solution:

Eq. 14.6: $$X_C = \frac{1}{\omega C} = \frac{1}{(400\text{ rad/s})(1 \times 10^{-6}\text{ F})} = \frac{10^6\ \Omega}{400} = 2500\ \Omega$$

Eq. 14.7: $$I_m = \frac{V_m}{X_C} = \frac{30\text{ V}}{2500\ \Omega} = 0.0120\text{ A} = 12\text{ mA}$$

and we know for a capacitor that i leads v by 90°. Therefore,

$$i = \mathbf{12 \times 10^{-3} \sin(400t + 90°)}$$

The curves are sketched in Fig. 14.17.

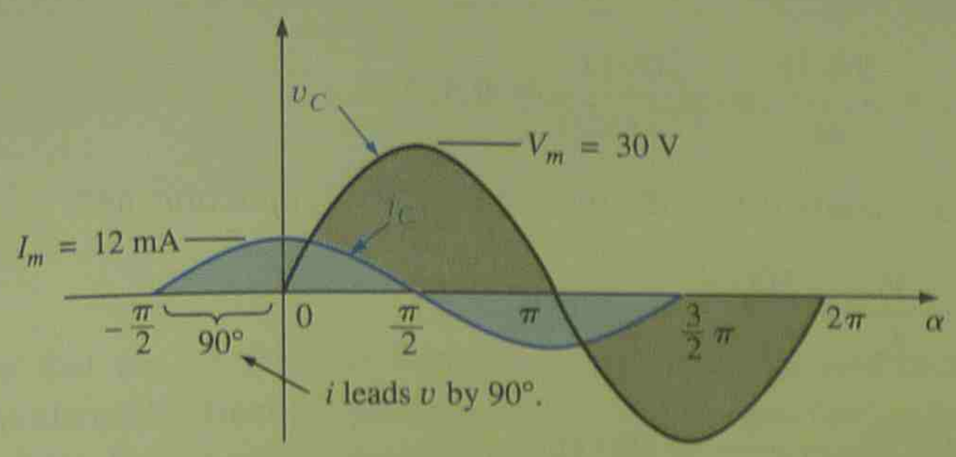


FIG. 14.17

EXAMPLE 14.6 The current through a 100-μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor.

$$i = 40 \sin(500t + 60°)$$

Solution:

$$X_C = \frac{1}{\omega C} = \frac{1}{(500 \text{ rad/s})(100 \times 10^{-6} \text{ F})} = \frac{10^6\ \Omega}{5 \times 10^4} = \frac{10^2\ \Omega}{5} = 20\ \Omega$$

$$V_m = I_m X_C = (40 \text{ A})(20\ \Omega) = 800 \text{ V}$$

and we know for a capacitor that v lags i by 90°. Therefore,

$$v = 800 \sin(500t + 60° - 90°)$$

and

$$v = \mathbf{800 \sin(500t - 30°)}$$

EXAMPLE 14.7 For the following pairs of voltages and currents, determine whether the element involved is a capacitor, inductor, or resistor, and determine the value of C, L, or R, if sufficient data are provided (Fig. 14.18):

a. $v = 100 \sin(\omega t + 40°)$
   $i = 20 \sin(\omega t + 40°)$
b. $v = 1000 \sin(377t + 10°)$
   $i = 5 \sin(377t - 80°)$
c. $v = 500 \sin(157t + 30°)$
   $i = 1 \sin(157t + 120°)$
d. $v = 50 \cos(\omega t + 20°)$
   $i = 5 \sin(\omega t + 110°)$

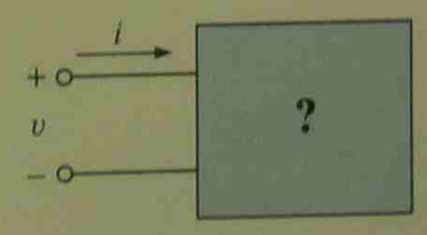


FIG. 14.18

Solutions:

a. Since v and i are *in phase*, the element is a *resistor*, and

$$R = \frac{V_m}{I_m} = \frac{100 \text{ V}}{20 \text{ A}} = \mathbf{5\ \Omega}$$

b. Since v *leads* i by 90°, the element is an *inductor*, and

$$X_L = \frac{V_m}{I_m} = \frac{1000\text{ V}}{5\text{ A}} = 200\,\Omega$$

so that $X_L = \omega L = 200\,\Omega$

$$\text{or } L = \frac{200\,\Omega}{\omega} = \frac{200\,\Omega}{377\text{ rad/s}} = \mathbf{0.531\text{ H}}$$

c. Since i *leads* v by 90°, the element is a *capacitor*, and

$$X_C = \frac{V_m}{I_m} = \frac{500\text{ V}}{1\text{ A}} = 500\,\Omega$$

$$\text{so that } X_C = \frac{1}{\omega C} = 500\,\Omega$$

$$\text{or } C = \frac{1}{\omega 500\,\Omega} = \frac{1}{(157\text{ rad/s})(500\,\Omega)} = \mathbf{12.74\,\mu F}$$

d. $v = 50\cos(\omega t + 20°) = 50\sin(\omega t + 20° + 90°)$
$= 50\sin(\omega t + 110°)$

Since v and i are *in phase*, the element is a *resistor*, and

$$R = \frac{V_m}{I_m} = \frac{50\text{ V}}{5\text{ A}} = \mathbf{10\,\Omega}$$

For dc circuits, the frequency is zero, and the reactance of a coil is

$$X_L = 2\pi fL = 2\pi(0)L = 0\,\Omega$$

The use of the short-circuit equivalence for the inductor in dc circuits (Chapter 12) is now validated. At very high frequencies, $X_L\uparrow = 2\pi f\uparrow L$ is very large, and for some practical applications the inductor can be replaced by an open circuit.

The capacitor can be replaced by an open circuit equivalence in dc circuits since $f = 0$, and

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(0)C} = \infty\,\Omega$$

once again substantiating our previous action (Chapter 10). At very high frequencies, for finite capacitances,

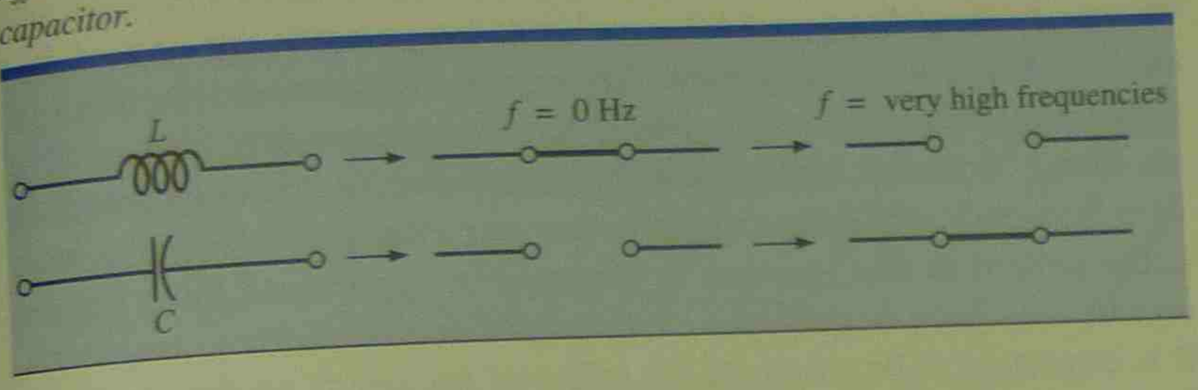
$$X_C\downarrow = \frac{1}{2\pi f\uparrow C}$$

is very small, and for some practical applications the capacitor can be replaced by a short circuit.

Table 14.1 reviews the preceding conclusions.

TABLE 14.1

Effect of high and low frequencies on the circuit model of an inductor and capacitor.



Now that we are familiar with phase relationships and understand how the elements affect the phase relationship between the applied voltage and resulting current, the use of the oscilloscope to measure the phase angle can be introduced. Recall from past discussions that the oscilloscope can be used only to display voltage levels versus time. However, now that we realize that the voltage across a resistor is in phase with the current through a resistor, we can consider the phase angle associated with the voltage across any resistor actually to be the phase angle of the current. For example, suppose we want to find the phase angle introduced by the unknown system of Fig. 14.19(a). In Fig. 14.19(b) a resistor was added to the input leads, and the two channels of a dual trace (most modern-day oscilloscopes can display two signals at the same time) were connected as shown. One channel will display the input voltage v_i, whereas the other will display v_R, as shown in Fig. 14.19(c). However, as noted before, since v_R and i_R are in phase, the phase angle appearing in Fig. 14.19(c) is also the phase angle between v_i and i_i. The addition of a "sensing" resistor (a resistor of a magnitude that will not adversely affect the input characteristics of the system), therefore, can be used to determine the phase angle introduced by the system and can be used to determine the magnitude of the resulting current. The details of the connections that must be made and how the actual phase angle is determined will be left for the laboratory experience.

FIG. 14.19

Using an oscilloscope to determine the phase angle of a system.

14.4 FREQUENCY RESPONSE OF THE BASIC ELEMENTS

The analysis of Section 14.3 was limited to a particular applied frequency. What is the effect of varying the frequency on the level of opposition offered by a resistive, inductive, or capacitive element? We are aware from the last section that the inductive reactance increases with frequency while the capacitive reactance decreases. However, what is the pattern to this increase or decrease in opposition? Since applied signals may have frequencies extending from a few hertz to megahertz, it is important to be aware of the effect of frequency on the opposition level.

R

Thus far we have assumed that the resistance of a resistor is independent of the applied frequency. However, in the real world each resistive element has stray capacitance levels and lead inductance that are sensitive to the applied frequency. However, the capacitive and inductive levels involved are usually so small that their real effect is not noticed until the megahertz range. The resistance-versus-frequency curves for a number of carbon composition resistors are provided in Fig. 14.20. Note that the lower resistance levels seem to be less affected by the frequency level. The 100-Ω resistor is essentially stable up to about 300 MHz, whereas the 100-kΩ resistor starts its radical decline at about 15 MHz.

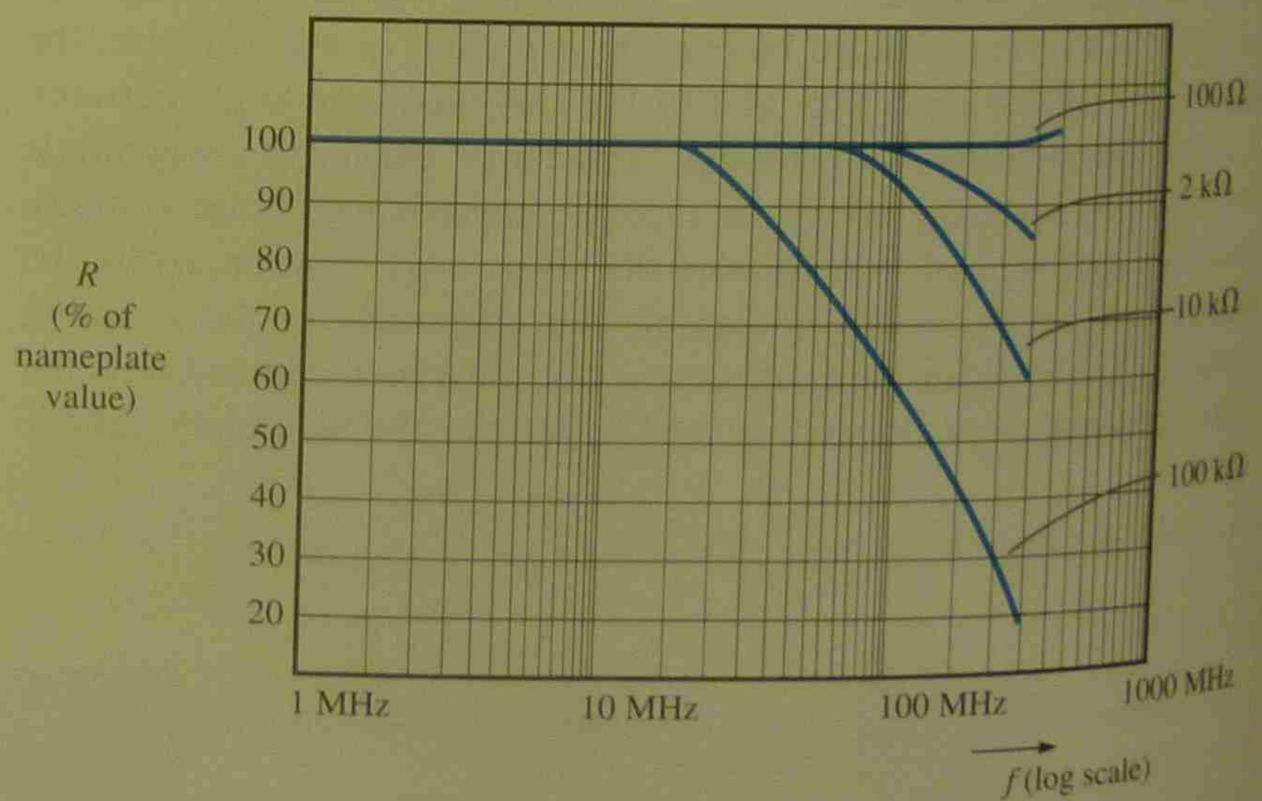


FIG. 14.20
Typical resistance versus frequency curves for carbon compound resistors.

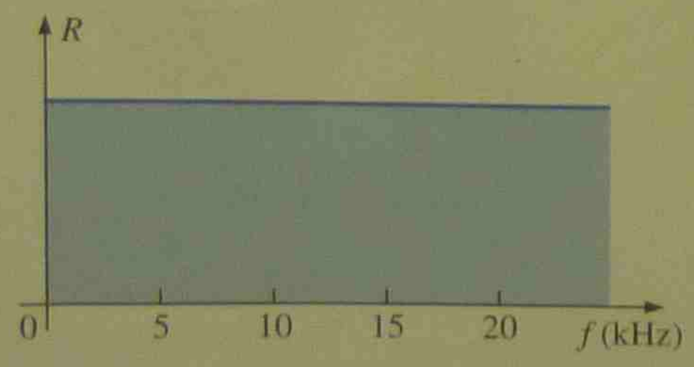


FIG. 14.21
R vs. f for the range of interest.

Frequency, therefore, does have impact on the resistance of an element, but for our current frequency range of interest, we will assume the resistance-versus-frequency plot of Fig. 14.21 (like Fig. 14.20 up to 15 MHz), which essentially specifies that the resistance level of a resistor is independent of frequency.

L

The nameplate value of an inductor or capacitor will also change at very high frequencies for many of the same reasons already introduced for the resistor. However, for our current frequency range of interest it will be assumed that the nameplate values of inductors (5 mH, 10 μH, etc.) and capacitors (0.1 μF, 1000 pF, etc.) will remain fixed.

However, the equation

$$X_L = \omega L = 2\pi f L = 2\pi L f$$

is directly related to the straight-line equation

$$y = mx + b = (2\pi L)f + 0$$

with a slope (m) of $2\pi L$ and a y-intercept (b) of zero. X_L is the y variable and f is the x variable, as shown in Fig. 14.22.

The larger the inductance, the greater the slope ($m = 2\pi L$) for the same frequency range, as shown in Fig. 14.22. Keep in mind, as reemphasized by Fig. 14.22, that the opposition of an inductor at very low frequencies approaches that of a short circuit, while at high frequencies the reactance approaches that of an open circuit.

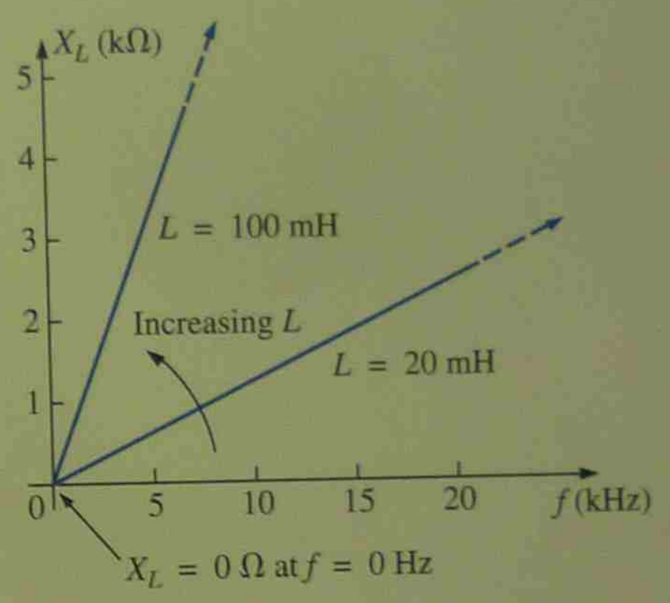


FIG. 14.22
X_L vs. frequency.

For the capacitor, the reactance equation

$$X_C = \frac{1}{2\pi f C}$$

can be written

$$X_C f = \frac{1}{2\pi C}$$

which matches the basic format of a hyperbola,

$$yx = k$$

with $y = X_C$, $x = f$, and the constant $k = 1/(2\pi C)$.

At $f = 0$ Hz, the reactance of the capacitor is so large, as shown in Fig. 14.23, that it can be replaced by an open-circuit equivalent. As the frequency increases, the reactance decreases, until eventually a short-circuit equivalent would be appropriate. Note how an increase in capacitance causes the reactance to drop off more rapidly with frequency.

In summary, therefore, as the applied frequency increases, the resistance of a resistor remains constant, the reactance of an inductor increases linearly, and the reactance of a capacitor decreases nonlinearly.

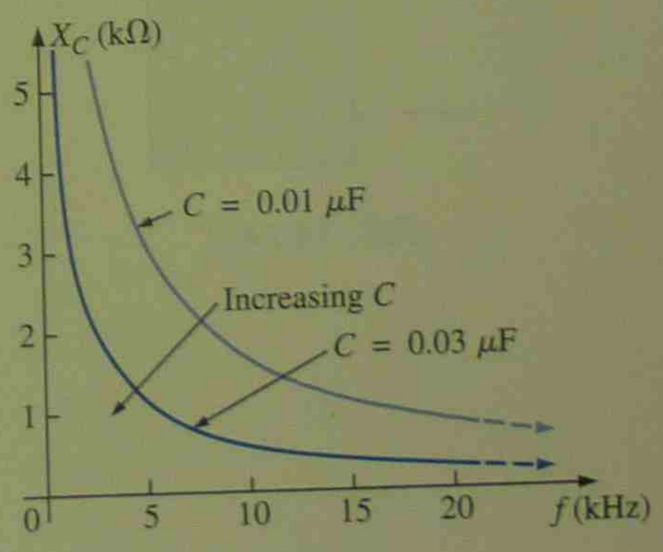


FIG. 14.23
X_C vs. frequency.

EXAMPLE 14.8 At what frequency will the reactance of a 200-mH inductor match the resistance level of a 5-kΩ resistor?

Solution: The resistance remains constant at 5 kΩ for the frequency range of the inductor. Therefore,

$$R = 5000\ \Omega = X_L = 2\pi f L = 2\pi L f$$
$$= (6.28\ \text{rad})(200 \times 10^{-3}\ \text{H})f = 1.256f$$

and $$f = \frac{5000 \text{ Hz}}{1.256} \cong \mathbf{3.98\ kHz}$$

EXAMPLE 14.9 At what frequency will an inductor of 5 mH have the same reactance as a capacitor of 0.1 μF?

Solution:

$$X_L = X_C$$

$$2\pi fL = \frac{1}{2\pi fC}$$

$$f^2 = \frac{1}{4\pi^2 LC}$$

and $$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{6.28 \text{ rad}\sqrt{(5 \times 10^{-3}\text{ H})(0.1 \times 10^{-6}\text{ F})}}$$

$$= \frac{1}{6.28\sqrt{5 \times 10^{-10}}} = \frac{1}{(6.28)(2.236 \times 10^{-5})}$$

$$f = \frac{10^5 \text{ Hz}}{14.04} \cong \mathbf{7.12\ kHz}$$

14.5 AVERAGE POWER AND POWER FACTOR

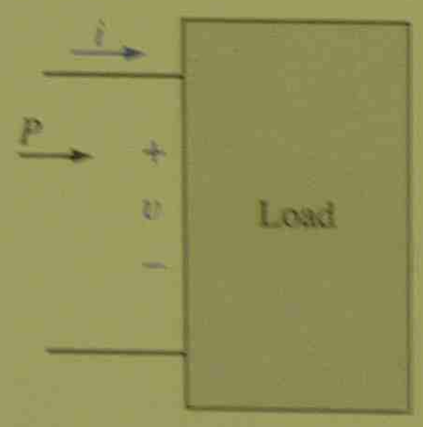


FIG. 14.24

The instantaneous power to the load of Fig. 14.24 is

$$p = vi$$

If we consider the general case where

$$v = V_m \sin(\omega t + \theta_v) \quad \text{and} \quad i = I_m \sin(\omega t + \theta_i)$$

then $$p = vi = V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i)$$
$$= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$

Using the trigonometric identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

the function $\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$ becomes

$$\sin(\omega t + \theta_v) \sin(\omega t + \theta_i)$$
$$= \frac{\cos[(\omega t + \theta_v) - (\omega t + \theta_i)] - \cos[(\omega t + \theta_v) + (\omega t + \theta_i)]}{2}$$
$$= \frac{\cos(\theta_v - \theta_i) - \cos(2\omega t + \theta_v + \theta_i)}{2}$$

so that

$$p = \overbrace{\left[\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\right]}^{\text{Fixed value}} - \overbrace{\left[\frac{V_m I_m}{2}\cos(2\omega t + \theta_v + \theta_i)\right]}^{\text{Time-varying junction}}$$

A plot of v, i, and p on the same set of axes is shown in Fig. 14.25.

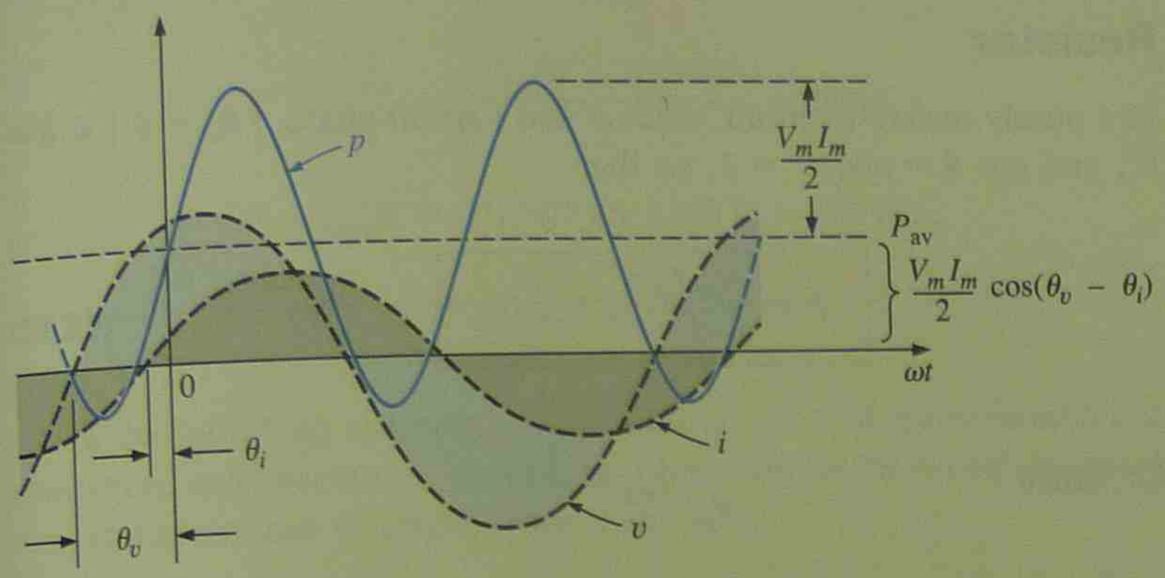


FIG. 14.25

Note that the second factor in the preceding equation is a cosine wave with an amplitude of $V_m I_m/2$, and a frequency twice that of the voltage or current. The average value of this term is zero, producing no net transfer of energy in any one direction.

The first term in the preceding equation, however, has a constant magnitude (no time dependence) and therefore provides some net transfer of energy. This term is referred to as the *average power*, the reason for which is obvious from Fig. 14.25. The average power, or *real* power as it is sometimes called, is the power delivered to and dissipated by the load. It corresponds to the power calculations performed for dc networks. The angle $(\theta_v - \theta_i)$ is the phase angle between v and i. Since $\cos(-\alpha) = \cos\alpha$

the magnitude of average power delivered is independent of whether v leads i or i leads v.

Defining θ as equal to $|\theta_v - \theta_i|$, where $|\quad|$ indicates that only the magnitude is important and the sign is immaterial, we have

$$\boxed{P = \frac{V_m I_m}{2}\cos\theta} \quad \text{(watts, W)} \qquad \textbf{(14.10)}$$

where P is the average power in watts. This equation can also be written

$$P = \left(\frac{V_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta$$

Or, since $V_{\text{eff}} = \dfrac{V_m}{\sqrt{2}}$ and $I_{\text{eff}} = \dfrac{I_m}{\sqrt{2}}$

Eq. (14.10) becomes

$$\boxed{P = V_{\text{eff}} I_{\text{eff}} \cos\theta} \tag{14.11}$$

Let us now apply Eqs. (14.10) and (14.11) to the basic R, L, and C elements.

Resistor

In a purely resistive circuit, since v and i are in phase, $|\theta_v - \theta_i| = \theta = 0°$, and $\cos\theta = \cos 0° = 1$, so that

$$\boxed{P = \frac{V_m I_m}{2} = V_{\text{eff}} I_{\text{eff}}} \quad \text{(W)} \tag{14.12}$$

Or, since $$I_{\text{eff}} = \frac{V_{\text{eff}}}{R}$$

then $$\boxed{P = \frac{V_{\text{eff}}^2}{R} = I_{\text{eff}}^2 R} \quad \text{(W)} \tag{14.13}$$

Inductor

In a purely inductive circuit, since v leads i by 90°, $|\theta_v - \theta_i| = \theta = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos 90° = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal inductor (no associated resistance) is zero watts.

Capacitor

In a purely capacitive circuit, since i leads v by 90°, $|\theta_v - \theta_i| = \theta = |-90°| = 90°$. Therefore,

$$P = \frac{V_m I_m}{2} \cos(90°) = \frac{V_m I_m}{2}(0) = \mathbf{0\ W}$$

The average power or power dissipated by the ideal capacitor (no associated resistance) is zero watts.

EXAMPLE 14.10 Find the average power dissipated in a network whose input current and voltage are the following:

$$i = 5 \sin(\omega t + 40°)$$
$$v = 10 \sin(\omega t + 40°)$$

Solution: Since v and i are in phase, the circuit appears to be purely resistive at the input terminals. Therefore,

$$P = \frac{V_m I_m}{2} = \frac{(10\text{ V})(5\text{ A})}{2} = \mathbf{25\text{ W}}$$

or

$$R = \frac{V_m}{I_m} = \frac{10\text{ V}}{5\text{ A}} = 2\ \Omega$$

and

$$P = \frac{V_{\text{eff}}^2}{R} = \frac{[(0.707)(10\text{ V})]^2}{2} = \mathbf{25\text{ W}}$$

or

$$P = I_{\text{eff}}^2 R = [(0.707)(5\text{ A})]^2(2) = \mathbf{25\text{ W}}$$

For the following examples, the circuit consists of a combination of resistances and reactances producing phase angles between the input current and voltage different from 0° or 90°.

EXAMPLE 14.11 Determine the average power delivered to networks having the following input voltage and current:

a. $v = 100\sin(\omega t + 40°)$
   $i = 20\sin(\omega t + 70°)$

b. $v = 150\sin(\omega t - 70°)$
   $i = 3\sin(\omega t - 50°)$

Solutions:

a. $V_m = 100,\ \theta_v = 40°$
   $I_m = 20,\ \theta_i = 70°$
   $\theta = |\theta_v - \theta_i| = |40° - 70°| = |-30°| = 30°$
   and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(100\text{ V})(20\text{ A})}{2}\cos(30°) = (1000\text{ W})(0.866)$$
$$= \mathbf{866\text{ W}}$$

b. $V_m = 150\text{ V},\ \theta_v = -70°$
   $I_m = 3\text{ A},\ \theta_i = -50°$
   $\theta = |\theta_v - \theta_i| = |-70° - (-50°)|$
   $= |-70° + 50°| = |-20°| = 20°$

and

$$P = \frac{V_m I_m}{2}\cos\theta = \frac{(150\text{ V})(3\text{ A})}{2}\cos(20°) = (225\text{ W})(0.9397)$$
$$= \mathbf{211.431\text{ W}}$$

Power Factor

The frequency and the elements of the parallel network of Fig. 14.26(a) were chosen such that

$$i_1 = 50 \sin \omega t$$
$$i_2 = 50 \sin(\omega t - 90°)$$
$$i_3 = 50 \sin(\omega t + 90°)$$

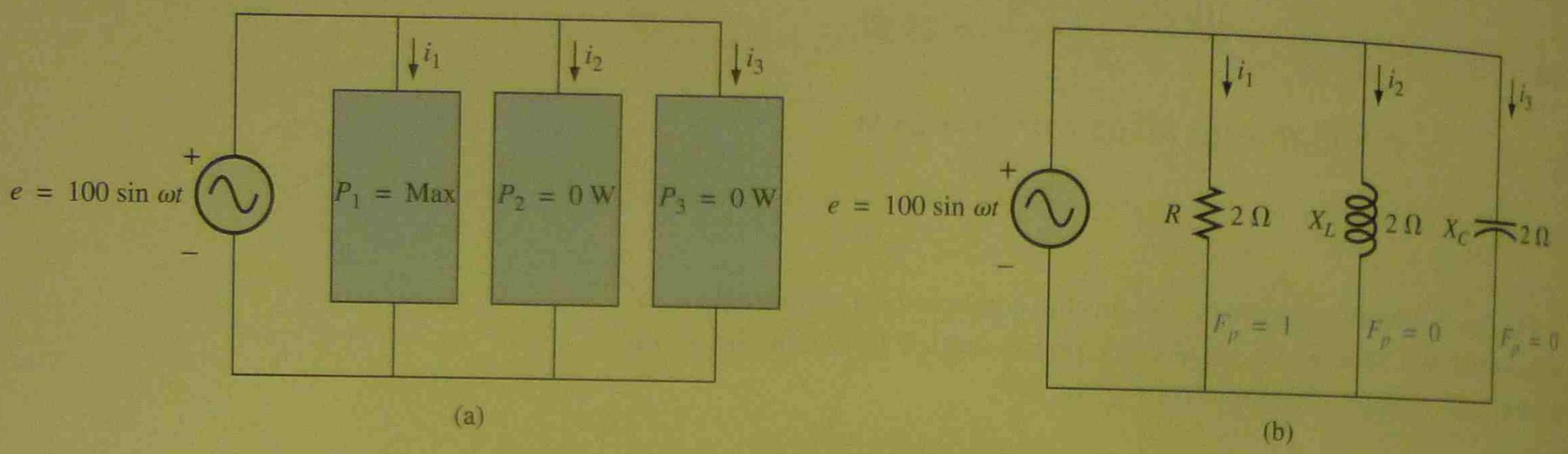


FIG. 14.26

Therefore, for each parallel branch, the peak values (or effective values) of the voltage and current are the same. Yet, as indicated in Fig. 14.26(a), the power to two of the branches is zero, and a maximum to the third. In our power equation, the only factor that accounts for this variation is cos θ, since the product VI is the same for each load. This factor, cos θ, is called the *power factor* and is symbolically represented by F_p; that is,

$$\text{Power factor} = F_p = \cos \theta \tag{14.14}$$

The more reactive a load, the lower the power factor, and the smaller the average power delivered. The more resistive the load, the higher the power factor, and the greater the real power delivered. The elements within each load of Fig. 14.26(a) appear in Fig. 14.26(b). Note the power factor for each element in the figure.

In terms of the average power and the terminal voltage and current,

$$F_p = \cos \theta = \frac{P}{V_{\text{eff}} I_{\text{eff}}} \tag{14.15}$$

The terms *leading* and *lagging* are often written in conjunction with the power factor. *They are defined by the current through the load.* If the current leads the voltage across a load, the load has a leading power factor. If the current lags the voltage across the load, the load has a lagging power factor. In other words,

capacitive networks have leading power factors, and inductive networks have lagging power factors.

The importance of the power factor on power distribution systems is examined in Chapter 19. In fact, one section is devoted to power factor correction.

EXAMPLE 14.12 Determine the power factors of the following loads, and indicate whether they are leading or lagging:

a. Fig. 14.27
b. Fig. 14.28
c. Fig. 14.29

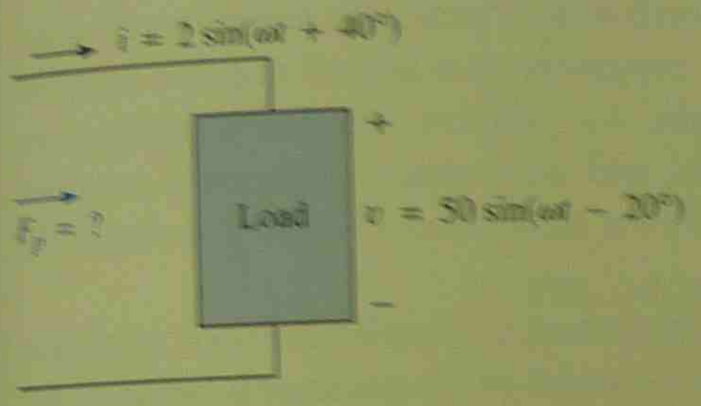


FIG. 14.27

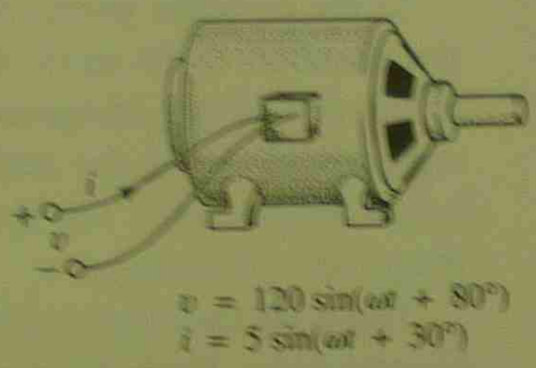


FIG. 14.28

Solutions:

a. $F_p = \cos\theta = \cos|40° - (-20°)| = \cos 60° = \mathbf{0.5\ leading}$

b. $F_p = \cos\theta = \cos|80° - 30°| = \cos 50° = \mathbf{0.6428\ lagging}$

c. $F_p = \cos\theta = \dfrac{P}{V_{eff}I_{eff}} = \dfrac{100\text{ W}}{(20\text{ V})(5\text{ A})} = \dfrac{100\text{ W}}{100\text{ W}} = 1$

The load is resistive, and F_p is neither leading nor lagging.

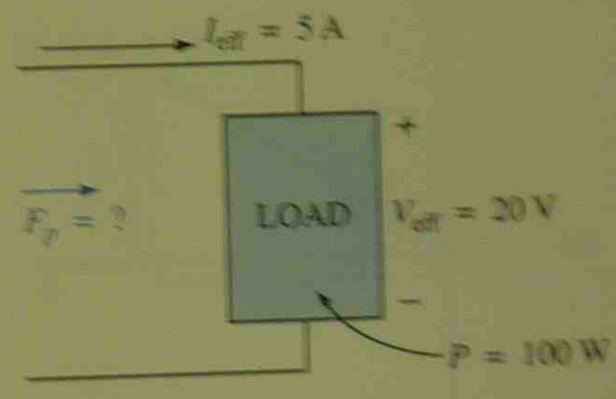


FIG. 14.29

14.6 COMPLEX NUMBERS

In our analysis of dc networks, we found it necessary to determine the algebraic sum of voltages and currents. Since the same will also be true for ac networks, the question arises, How do we determine the algebraic sum of two or more voltages (or currents) that are varying sinusoidally? Although one solution would be to find the algebraic sum on a point-to-point basis (as shown in Section 14.12), this would be a long and tedious process in which accuracy would be directly related to the scale employed.

It is the purpose of this chapter to introduce a system of *complex numbers*, which, when related to the sinusoidal ac waveform, will result in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate. In the following chapters, the technique will be extended to permit the analysis of sinusoidal ac networks in a manner very similar to that applied to dc networks. The methods and theorems as described for dc networks can then be applied to sinusoidal ac networks with little difficulty.

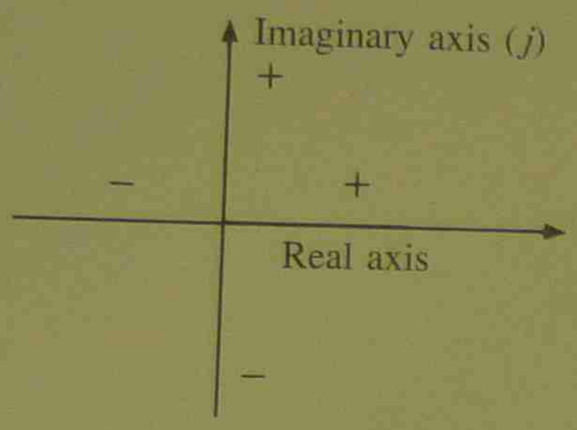


FIG. 14.30

A *complex number* represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.30. For reasons that will be obvious later, the real axis is sometimes called the *resistance* axis, and the imaginary axis, the *reactance* axis. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that any number not on the real axis would not exist—hence the term *imaginary* for the vertical axis.

In the complex plane, the horizontal or real axis represents all positive numbers to the right of the imaginary axis and all negative numbers to the left of the imaginary axis. All positive imaginary numbers are represented above the real axis, and all negative imaginary numbers, below the real axis. The symbol j (or sometimes i) is used to denote the imaginary component.

There are two forms used to represent a complex number: *rectangular* and *polar*. Each can represent a point in the plane or a radius vector drawn from the origin to that point.

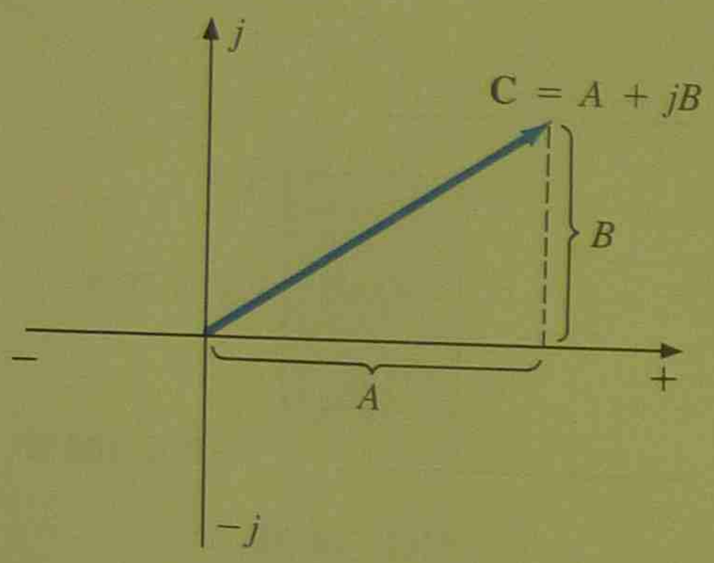


FIG. 14.31

14.7 RECTANGULAR FORM

The format for the rectangular form is

$$\mathbf{C} = A + jB \tag{14.16}$$

as shown in Fig. 14.31.

EXAMPLE 14.13 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$

b. $\mathbf{C} = 0 - j6$

c. $\mathbf{C} = -10 - j20$

Solutions:

a. See Fig. 14.32.

b. See Fig. 14.33.

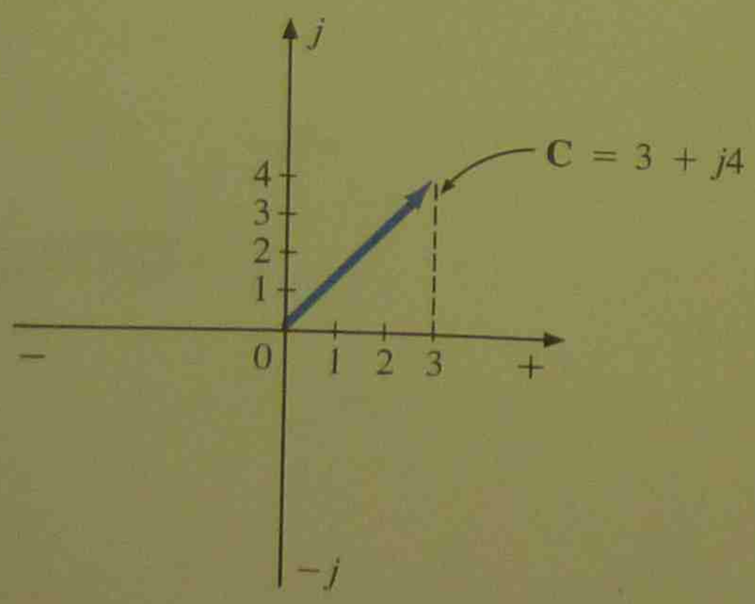


FIG. 14.32

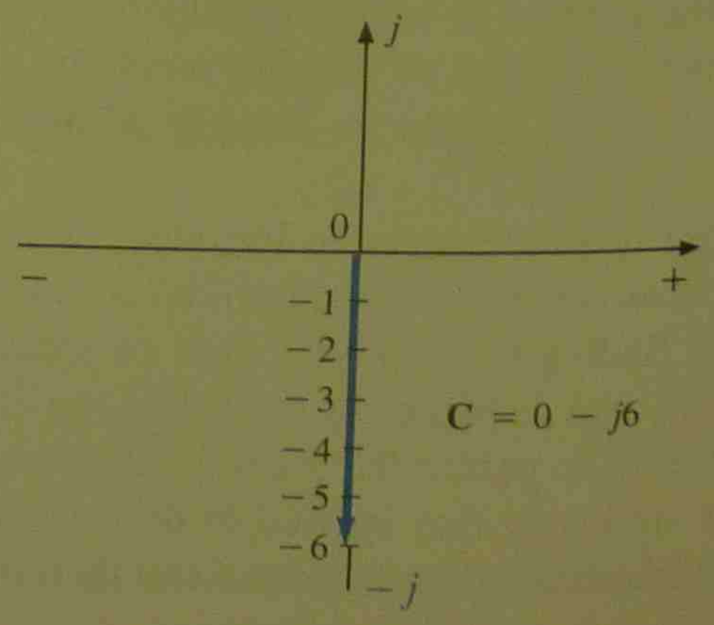


FIG. 14.33

c. See Fig. 14.34.

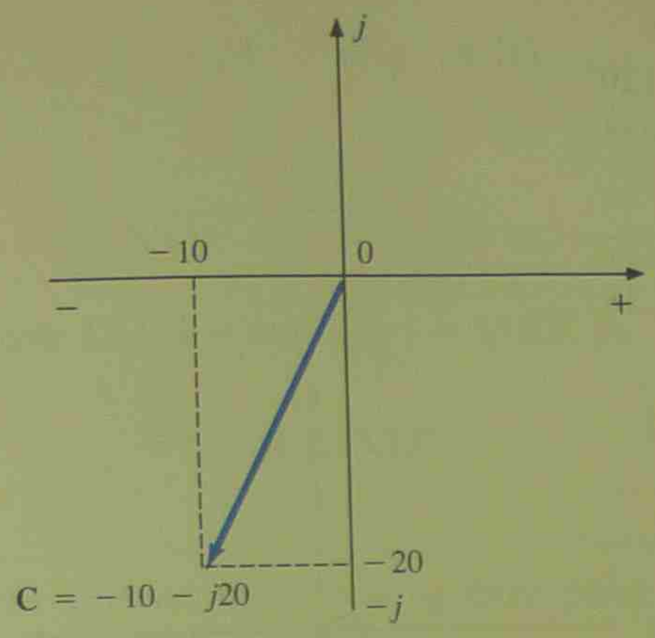


FIG. 14.34

14.8 POLAR FORM

The format for the polar form is

$$\mathbf{C} = C\angle\theta \tag{14.17}$$

where C indicates magnitude only and θ is always measured counterclockwise (CCW) from the *positive real axis*, as shown in Fig. 14.35. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

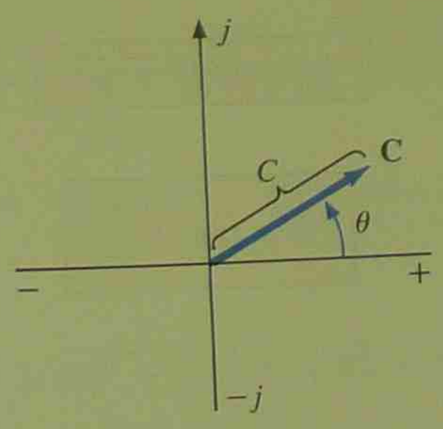


FIG. 14.35

A negative sign in front of the polar form has the effect shown in Fig. 14.36. Note that it results in a complex number directly opposite the complex number with a positive sign.

$$-\mathbf{C} = -C\angle\theta = C\angle\theta \pm \pi \tag{14.18}$$

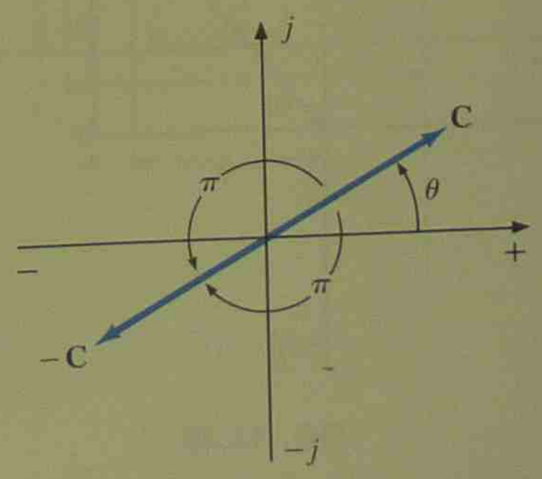


FIG. 14.36

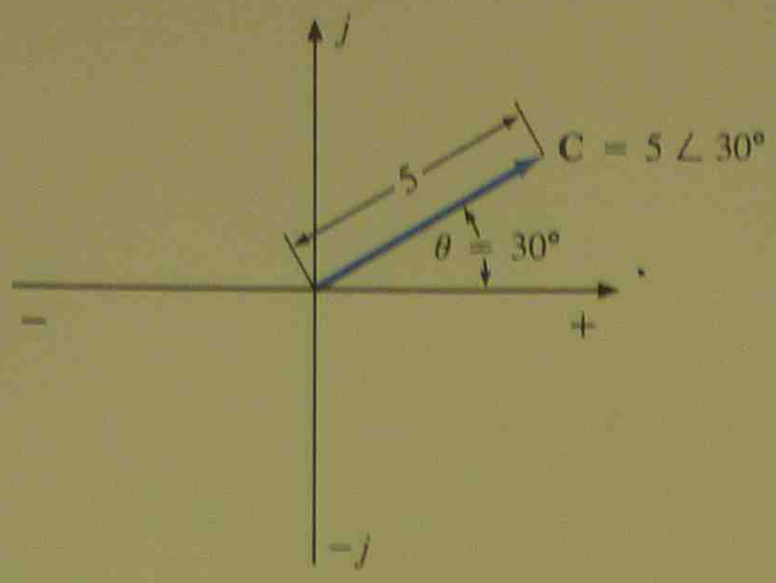


FIG. 14.37

EXAMPLE 14.14 Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5\angle 30°$

b. $\mathbf{C} = 7\angle -120°$

c. $\mathbf{C} = -4.2\angle 60°$

Solutions:

a. See Fig. 14.37.

b. See Fig. 14.38.

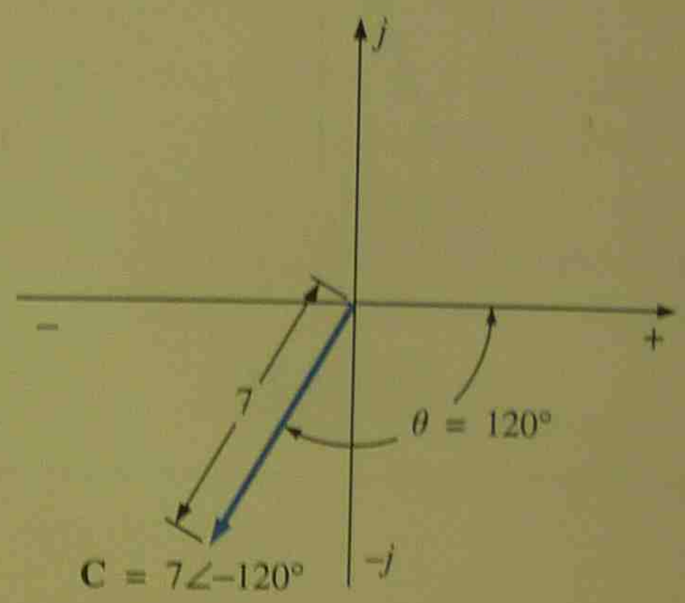


FIG. 14.38

$\mathbf{C} = -4.2\angle 60° = 4.2\angle 60° + 180°$
$= 4.2\angle +240°$

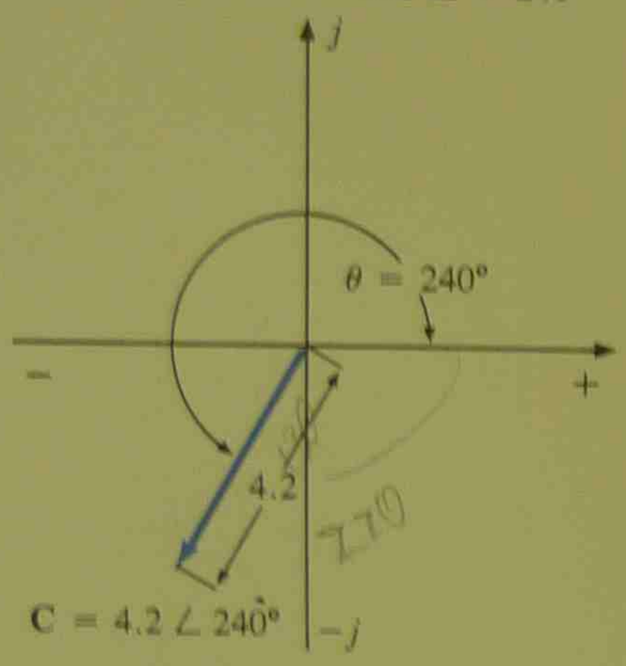


FIG. 14.39

c. See Fig. 14.39.

14.9 CONVERSION BETWEEN FORMS

The two forms are related by the following equations.

Rectangular to Polar

$$C = \sqrt{A^2 + B^2} \tag{14.19}$$

$$\theta = \tan^{-1}\frac{B}{A} \tag{14.20}$$

Note Fig. 14.40.

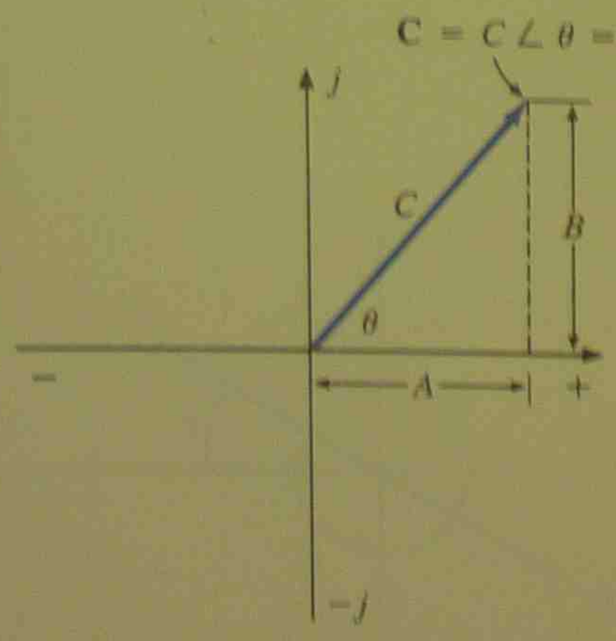


FIG. 14.40

Polar to Rectangular

$$A = C\cos\theta \tag{14.21}$$

$$B = C\sin\theta \tag{14.22}$$

EXAMPLE 14.15 Convert the following from rectangular to polar form:

$$\mathbf{C} = 3 + j4 \qquad \text{(Fig. 14.41)}$$

Solution:

$$C = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Example 14.18$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

and

$$\mathbf{C} = \mathbf{5 \angle 53.13^\circ}$$

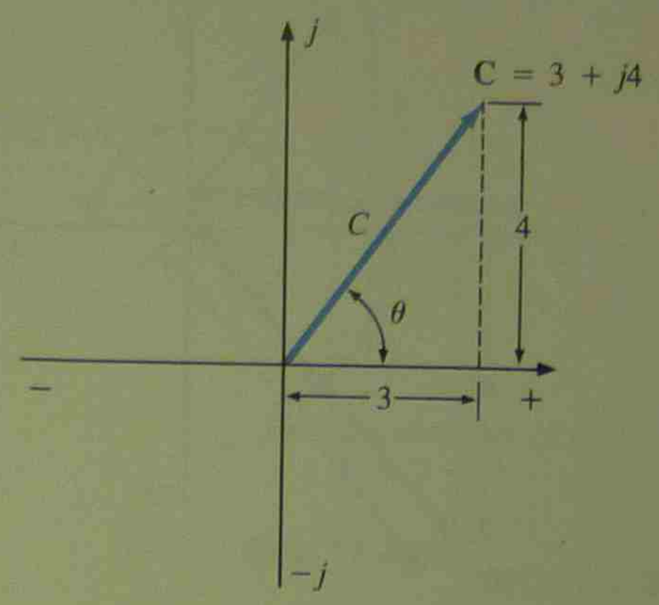


FIG. 14.41

EXAMPLE 14.16 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10 \angle 45^\circ \qquad \text{(Fig. 14.42)}$$

Solution:

$$A = 10 \cos 45^\circ = (10)(0.707) = 7.07$$
$$B = 10 \sin 45^\circ = (10)(0.707) = 7.07$$

and

$$\mathbf{C = 7.07 + j7.07}$$

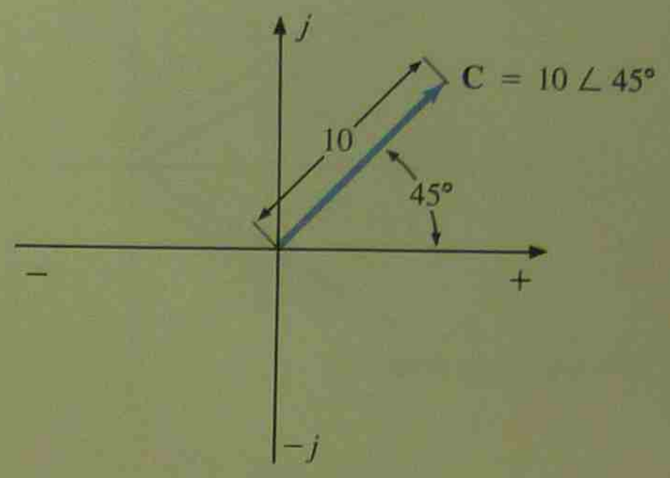


FIG. 14.42

If the complex number should appear in the second, third, or fourth quadrant, simply convert it in that quadrant, and carefully determine the proper angle to be associated with the magnitude of the vector.

EXAMPLE 14.17 Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j3 \qquad \text{(Fig. 14.43)}$$

Solution:

$$C = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57^\circ$$

$$\theta = 180 - 26.57^\circ = 153.43^\circ$$

and

$$\mathbf{C = 6.71 \angle 153.43^\circ}$$

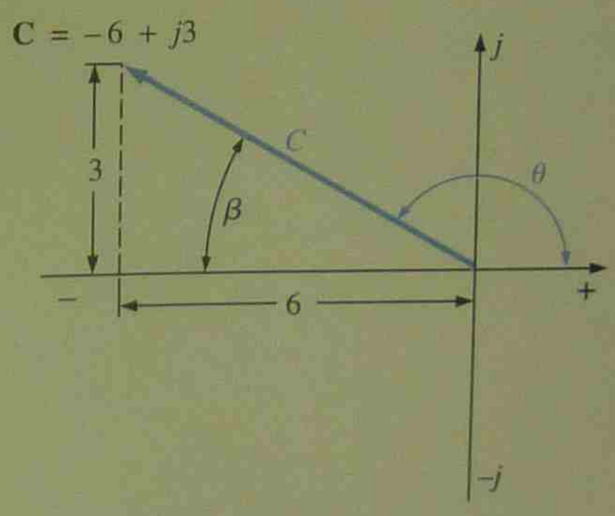


FIG. 14.43

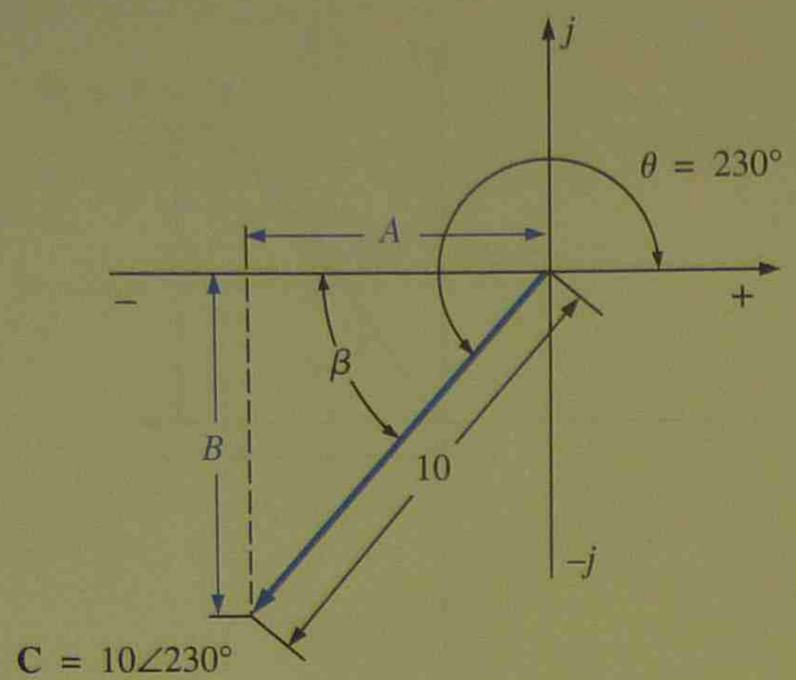


FIG. 14.44

EXAMPLE 14.18 Convert the following from polar to rectangular form:

$$\mathbf{C} = 10\angle 230° \qquad \text{(Fig. 14.44)}$$

Solution:

$$A = C\cos\beta = 10\cos(230° - 180°) = 10\cos 50°$$
$$= (10)(0.6428) = 6.428$$

$$B = C\sin\beta = 10\sin 50° = (10)(0.7660) = 7.660$$

and

$$\mathbf{C} = \mathbf{-6.428 - j7.660}$$

14.10 MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of addition, subtraction, multiplication, and division. A few basic rules and definitions must be understood before considering these operations.

Let us first examine the symbol j associated with imaginary numbers. By definition,

$$\boxed{j = \sqrt{-1}} \tag{14.23}$$

Thus,

$$\boxed{j^2 = -1} \tag{14.24}$$

and

$$j^3 = j^2 j = -1j = -j$$

with

$$j^4 = j^2 j^2 = (-1)(-1) = +1$$
$$j^5 = j$$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and

$$\boxed{\frac{1}{j} = -j} \tag{14.25}$$

Complex Conjugate

The *conjugate* or *complex conjugate* of a complex number can be found by simply changing the sign of the imaginary part in the rectangular

form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is

$$2 - j3$$

as shown in Fig. 14.45. The conjugate of

$$\mathbf{C} = 2\angle 30^\circ$$

is

$$2\angle -30^\circ$$

as shown in Fig. 14.46.

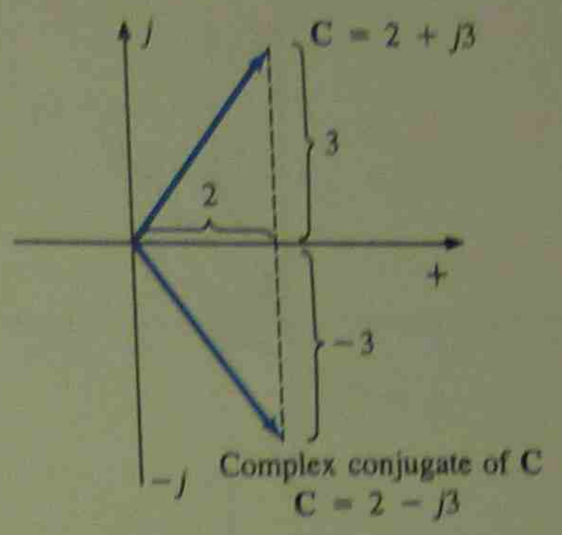


FIG. 14.45

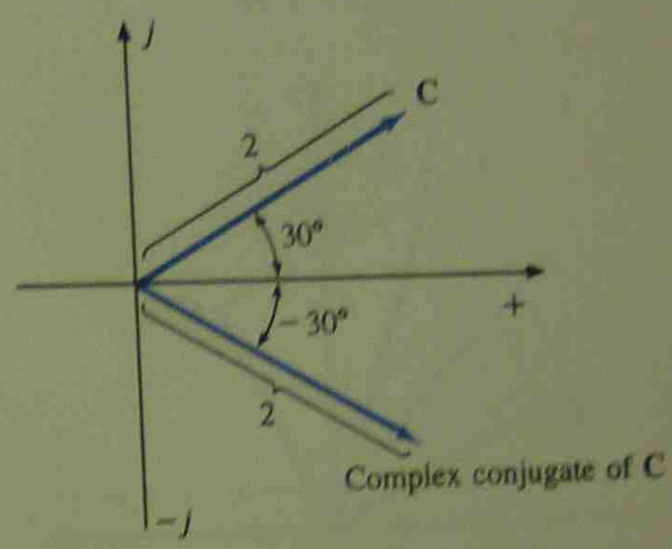


FIG. 14.46

Reciprocal

The *reciprocal* of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = A + jB$$

is

$$\frac{1}{A + jB}$$

and of $C\angle\theta$,

$$\frac{1}{C\angle\theta}$$

We are now prepared to consider the four basic operations of *addition, subtraction, multiplication,* and *division* with complex numbers.

Addition

To add two or more complex numbers, simply add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm A_1 \pm jB_1 \quad \text{and} \quad \mathbf{C}_2 = \pm A_2 \pm jB_2$$

then

$$\boxed{\mathbf{C}_1 + \mathbf{C}_2 = (\pm A_1 \pm A_2) + j(\pm B_1 \pm B_2)} \tag{14.26}$$

There is really no need to memorize the equation. Simply set one above the other and consider the real and imaginary parts separately, as shown in Example 14.19.

EXAMPLE 14.19

a. Add $\mathbf{C}_1 = 2 + j4$ and $\mathbf{C}_2 = 3 + j1$.
b. Add $\mathbf{C}_1 = 3 + j6$ and $\mathbf{C}_2 = -6 + j3$.

Solutions:

a. By Eq. (14.26),

$$\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = 5 + j5$$

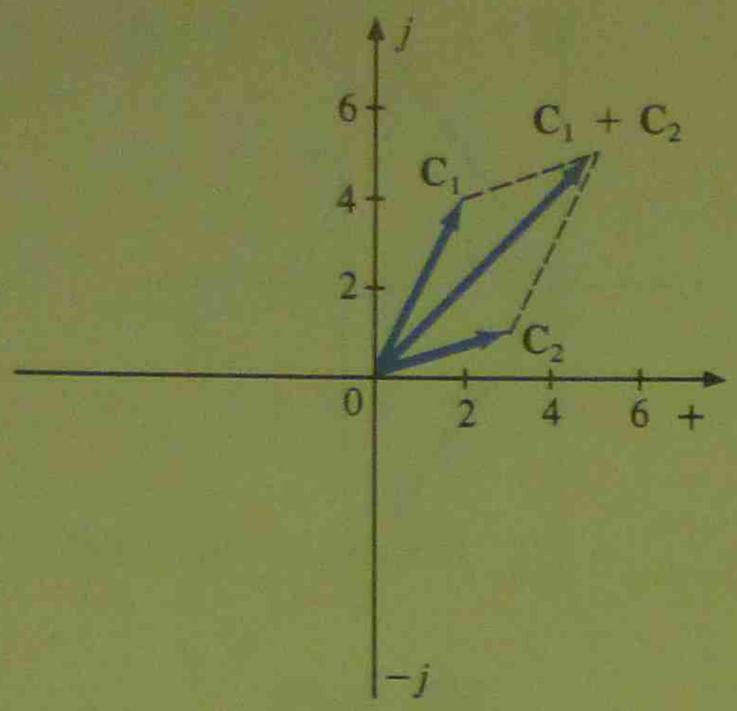


FIG. 14.47

Note Fig. 14.47. An alternative method is

$$\begin{array}{r} 2 + j4 \\ 3 + j1 \\ \hline \downarrow \quad \downarrow \\ \mathbf{5 + j5} \end{array}$$

b. By Eq. (14.26),

$$\mathbf{C}_1 + \mathbf{C}_2 = (3 - 6) + j(6 + 3) = \mathbf{-3 + j9}$$

Note Fig. 14.48. An alternative method is

$$\begin{array}{r} 3 + j6 \\ -6 + j3 \\ \hline \downarrow \quad \downarrow \\ \mathbf{-3 + j9} \end{array}$$

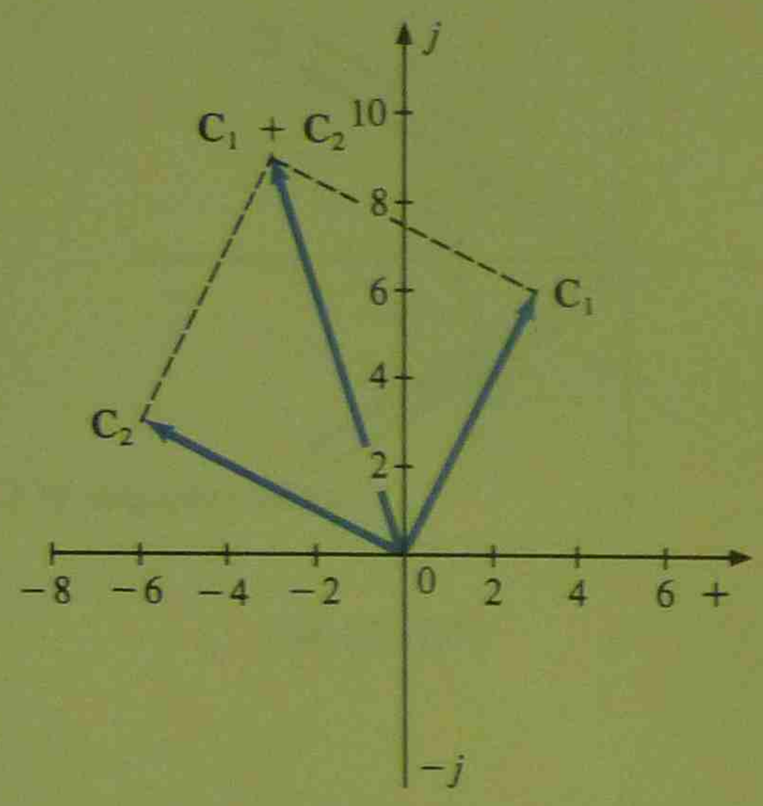


FIG. 14.48

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if

$$\mathbf{C}_1 = \pm A_1 \pm jB_1 \quad \text{and} \quad \mathbf{C}_2 = \pm A_2 \pm jB_2$$

then

$$\boxed{\mathbf{C}_1 - \mathbf{C}_2 = [\pm A_2 - (\pm A_2)] + j[\pm B_1 - (\pm B_2)]} \tag{14.27}$$

Again, there is no need to memorize the equation if the alternative method of Example 14.20 is employed.

EXAMPLE 14.20

a. Subtract $\mathbf{C}_2 = 1 + j4$ from $\mathbf{C}_1 = 4 + j6$.
b. Subtract $\mathbf{C}_2 = -2 + j5$ from $\mathbf{C}_1 = +3 + j3$.

Solutions:

a. By Eq. (14.27),

$$\mathbf{C}_1 - \mathbf{C}_2 = (4 - 1) + j(6 - 4) = \mathbf{3 + j2}$$

Note Fig. 14.49. An alternative method is

$$\begin{array}{r} 4 + j6 \\ -(1 + j4) \\ \hline \downarrow \quad \downarrow \\ \mathbf{3 + j2} \end{array}$$

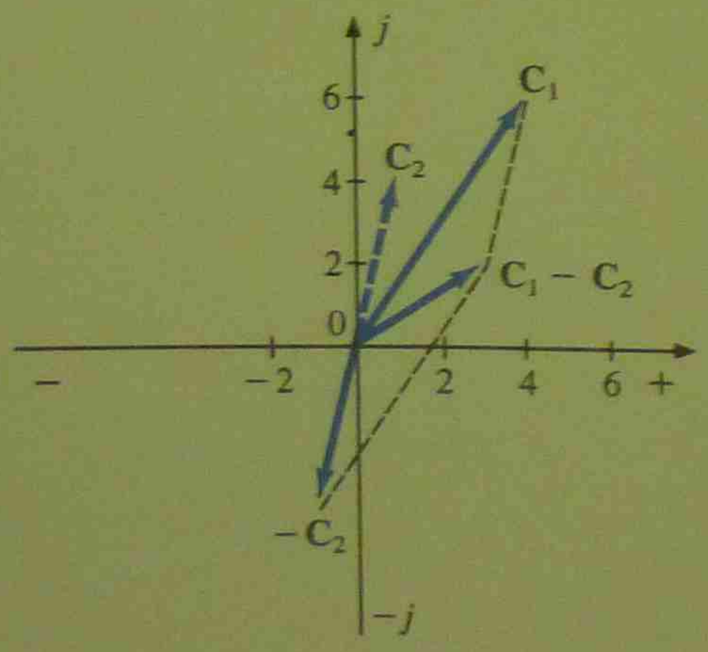


FIG. 14.49

b. By Eq. (14.27),

$$\mathbf{C}_1 - \mathbf{C}_2 = [3 - (-2)] + j(3 - 5) = \mathbf{5 - j2}$$

Note Fig. 14.50. An alternative method is

$$\begin{array}{r} 3 + j\,3 \\ -(-2 + j\,5) \\ \hline \downarrow \quad \downarrow \\ \mathbf{5 - j\,2} \end{array}$$

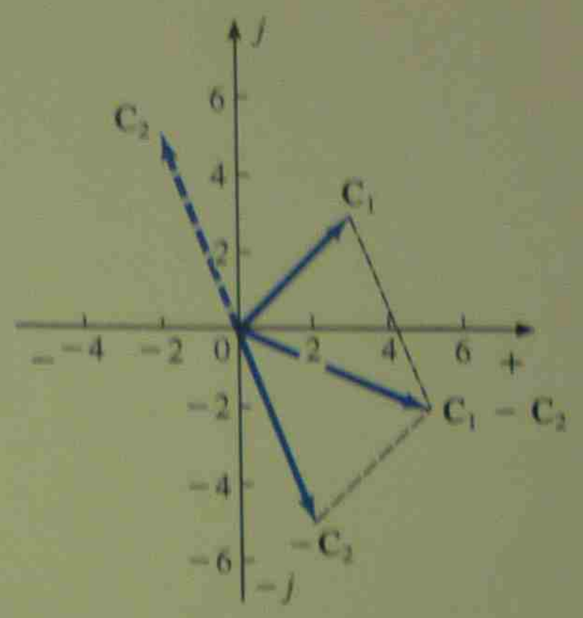


FIG. 14.50

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or differ only by multiples of 180°.

EXAMPLE 14.21

$$2\angle 45^\circ + 3\angle 45^\circ = \mathbf{5\angle 45^\circ}$$

Note Fig. 14.51. Or

$$2\angle 0^\circ - 4\angle 180^\circ = \mathbf{6\angle 0^\circ}$$

Note Fig. 14.52.

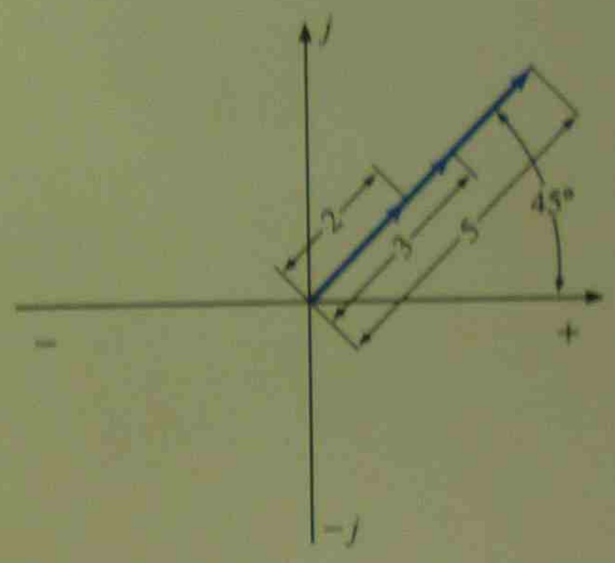


FIG. 14.51

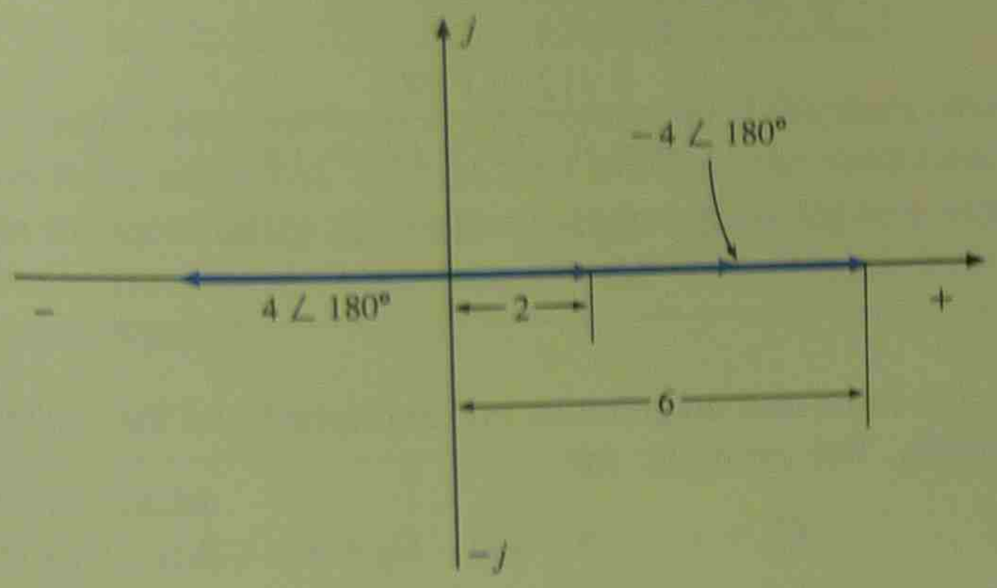


FIG. 14.52

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

$$\mathbf{C}_1 = A_1 + jB_1 \quad \text{and} \quad \mathbf{C}_2 = A_2 + jB_2$$

then

$$\mathbf{C}_1 \cdot \mathbf{C}_2: \quad \begin{array}{l} A_1 + jB_1 \\ A_2 + jB_2 \\ \hline A_1A_2 + jB_1A_2 \\ \qquad + jA_1B_2 + j^2B_1B_2 \\ \hline A_1A_2 + j(B_1A_2 + A_1B_2) + B_1B_2(-1) \end{array}$$

and
$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = (A_1A_2 - B_1B_2) + j(B_1A_2 + A_1B_2)} \quad (14.28)$$

In Example 14.22(b), we obtain a solution without resorting to memorizing Eq. (14.28). Simply carry along the j factor when multiplying each part of one vector with the real and imaginary parts of the other.

EXAMPLE 14.22

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2 + j3 \quad \text{and} \quad \mathbf{C}_2 = 5 + j10$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = -2 - j3 \quad \text{and} \quad \mathbf{C}_2 = +4 - j6$$

Solutions:

a. Using the format above, we have

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = [(2)(5) - (3)(10)] + j[(3)(5) + (2)(10)]$$
$$= \mathbf{-20 + j35}$$

b. Without using the format, we obtain

$$\begin{array}{r} -2 - j3 \\ +4 - j6 \\ \hline -8 - j12 \\ \quad + j12 + j^2 18 \\ \hline -8 + j(-12 + 12) - 18 \end{array}$$

and
$$\mathbf{C}_1 \cdot \mathbf{C}_2 = \mathbf{-26 = 26\angle 180^\circ}$$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = C_1\angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = C_2\angle\theta_2$$

we write

$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = C_1C_2\,\underline{/\theta_1 + \theta_2}} \quad (14.29)$$

EXAMPLE 14.23

a. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 5\angle 20^\circ \quad \text{and} \quad \mathbf{C}_2 = 10\angle 30^\circ$$

b. Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if

$$\mathbf{C}_1 = 2\angle -40^\circ \quad \text{and} \quad \mathbf{C}_2 = 7\angle +120^\circ$$

Solutions:

a. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (5\angle 20^\circ)(10\angle 30^\circ) = (5)(10)\,\underline{/20^\circ + 30^\circ} = \mathbf{50\angle 50^\circ}$

b. $\mathbf{C}_1 \cdot \mathbf{C}_2 = (2\angle -40^\circ)(7\angle +120^\circ) = (2)(7)\,\underline{/-40^\circ + 120^\circ} = \mathbf{14\angle +80^\circ}$

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

$$(10)(2 + j3) = 20 + j30$$

and

$$50\angle 0°(0 + j6) = j300 = 300\angle 90°$$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$$\mathbf{C}_1 = A_1 + jB_1 \quad \text{and} \quad \mathbf{C}_2 = A_2 + jB_2$$

then

$$\frac{C_1}{C_2} = \frac{(A_1 + jB_1)(A_2 - jB_2)}{(A_2 + jB_2)(A_2 - jB_2)} = \frac{(A_1A_2 + B_1B_2) + j(A_2B_1 - A_1B_2)}{A_2^2 + B_2^2}$$

and

$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{A_1A_2 + B_1B_2}{A_2^2 + B_2^2} + j\frac{A_2B_1 - A_1B_2}{A_2^2 + B_2^2}} \qquad \textbf{(14.30)}$$

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator squared.

EXAMPLE 14.24

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 1 + j4$ and $\mathbf{C}_2 = 4 + j5$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = -4 - j8$ and $\mathbf{C}_2 = +6 - j1$.

Solutions:

a. By Eq. (14.30),

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(1)(4) + (4)(5)}{4^2 + 5^2} + j\frac{(4)(4) - (1)(5)}{4^2 + 5^2}$$

$$= \frac{24}{41} + \frac{j11}{41} \cong \mathbf{0.585 + j0.268}$$

b. Using an alternative method, we obtain

$$\begin{array}{r} -4 - j8 \\ +6 + j1 \\ \hline -24 - j48 \\ \quad -j4 - j^2 8 \\ \hline -24 - j52 + 8 = -16 - j52 \end{array}$$

$$\begin{array}{r} +6 - j1 \\ +6 + j1 \\ \hline 36 + j6 \\ \quad - j6 - j^2 1 \\ \hline 36 + \ 0 + \ \ 1 = 37 \end{array}$$

and
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{-16}{37} - \frac{j52}{37} = \mathbf{-0.432 - j1.405}$$

To divide a complex number in rectangular form by a real number, both the real part and the imaginary part must be divided by the real number. For example,

$$\frac{8 + j10}{2} = 4 + j5$$

and
$$\frac{6.8 - j0}{2} = 3.4 - j0 = 3.4 \angle 0°$$

In *polar* form, division is accomplished by simply dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = C_1 \angle \theta_1 \quad \text{and} \quad \mathbf{C}_2 = C_2 \angle \theta_2$$

we write

$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{C_1}{C_2} \underline{/\theta_1 - \theta_2}} \tag{14.31}$$

EXAMPLE 14.25

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 15 \angle 10°$ and $\mathbf{C}_2 = 2 \angle 7°$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 8 \angle 120°$ and $\mathbf{C}_2 = 16 \angle -50°$.

Solutions:

a. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{15\angle 10°}{2\angle 7°} = \frac{15}{2} \underline{/10° - 7°} = \mathbf{7.5 \angle 3°}$$

b. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{8\angle 120°}{16\angle -50°} = \frac{8}{16} \underline{/120° - (-50°)} = \mathbf{0.5 \angle 170°}$$

We obtain the *reciprocal* in the rectangular form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{A + jB} = \left(\frac{1}{A + jB}\right)\left(\frac{A - jB}{A - jB}\right) = \frac{A - jB}{A^2 + B^2}$$

$$\frac{1}{A+jB} = \frac{A}{A^2+B^2} - j\frac{B}{A^2+B^2} \tag{14.32}$$

and

In the polar form, the reciprocal is

$$\frac{1}{C\angle\theta} = \frac{1}{C}\angle -\theta \tag{14.33}$$

Some concluding examples using the four basic operations follow.

EXAMPLE 14.26 Perform the following operations, leaving the answer in polar or rectangular form:

a. $$\frac{(2+j3)+(4+j6)}{(7+j7)-(3-j3)} = \frac{(2+4)+j(3+6)}{(7-3)+j(7+3)}$$
$$= \frac{(6+j9)(4-j10)}{(4+j10)(4-j10)}$$
$$= \frac{[(6)(4)+(9)(10)]+j[(4)(9)-(6)(10)]}{4^2+10^2}$$
$$= \frac{114-j24}{116} = \mathbf{0.983 - j0.207}$$

b. $$\frac{(50\angle 30°)(5+j5)}{10\angle -20°} = \frac{(50\angle 30°)(7.07\angle 45°)}{10\angle -20°} = \frac{353.5\angle 75°}{10\angle -20°}$$
$$= 35.35\angle 75° - (-20°) = \mathbf{35.35\angle 95°}$$

c. $$\frac{(2\angle 20°)^2(3+j4)}{8-j6} = \frac{(2\angle 20°)(2\angle 20°)(5\angle 53.13°)}{10\angle -36.87°}$$
$$= \frac{(4\angle 40°)(5\angle 53.13°)}{10\angle -36.87°} = \frac{20\angle 93.13°}{10\angle -36.87°}$$
$$= 2\angle 93.13° - (-36.87°) = \mathbf{2.0\angle 130°}$$

d. $$3\angle 27° - 6\angle -40° = (2.673 + j1.362) - (4.596 - j3.857)$$
$$= (2.673 - 4.596) + j(1.362 + 3.857)$$
$$= \mathbf{-1.923 + j5.219}$$

14.11 TECHNIQUES OF CONVERSION

For many years the technologist was dependent on tables, charts, and the slide rule to perform conversions between complex numbers. Today, calculators such as that shown in Fig. 14.53 are available that can perform the conversion to eight-place accuracy. In fact, this particular calculator is programmed to perform this particular operation by your pressing just a few buttons. The →R and →P refer to the rectangular and polar forms, respectively.

You will appreciate the speed and accuracy of calculator conversion after using other techniques of conversion. There are inexpensive calculators that have the necessary functions to perform the conversions

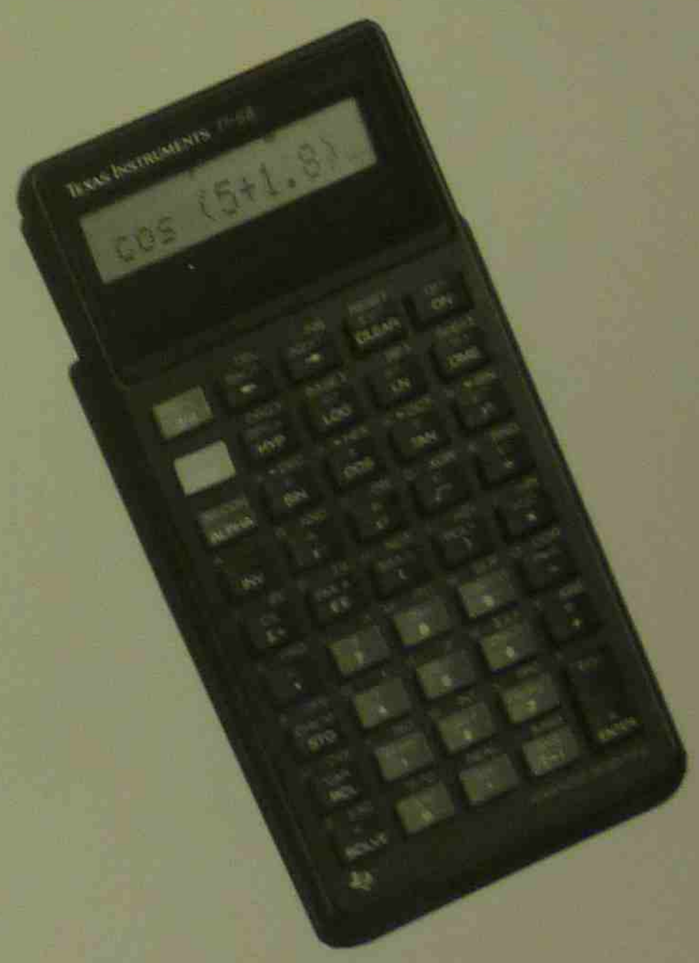


FIG. 14.53
Scientific calculator. (Courtesy of Texas Instruments, Inc.)

using Eqs. (14.19) through (14.22). For those who seriously expect to stay in this field, a calculator with the proper functions would be a wise investment.

14.12 PHASORS

As noted earlier in this chapter, the addition of sinusoidal voltages and currents will frequently be required in the analysis of ac circuits. One lengthy but valid method of performing this operation is to place both sinusoidal waveforms on the same set of axes and add algebraically the magnitudes of each at every point along the abscissa, as shown for $c = a + b$ in Fig. 14.54. This, however, can be a long and tedious process with limited accuracy. A shorter method uses the rotating radius

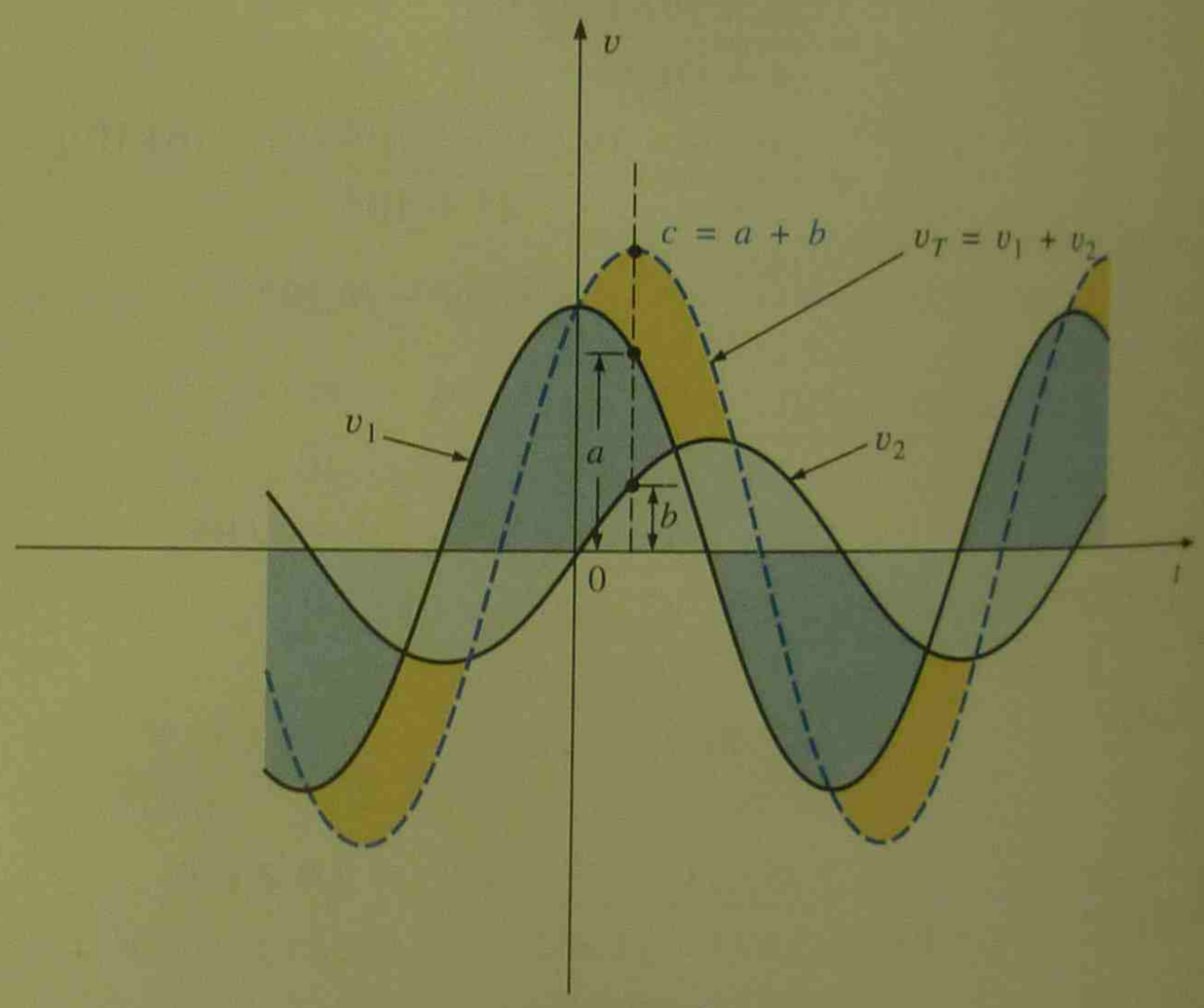


FIG. 14.54

vector first appearing in Fig. 13.16. This *radius vector,* having a *constant magnitude* (length) with *one end fixed at the origin,* is called a *phasor* when applied to electric circuits. During its rotational development of the sine wave, the phasor will, at the instant $t = 0$, have the positions shown in Fig. 14.55(a) for each waveform in Fig. 14.55(b).

Note in Fig. 14.55(b) that v_2 passes through the horizontal axis at $t = 0$ s, requiring that the radius vector in Fig. 14.55(a) be on the horizontal axis to insure a vertical projection of zero volts at $t = 0$ s. Its length in Fig. 14.55(a) is equal to the peak value of the sinusoid as required by the radius vector of Fig. 13.16. The other sinusoid has passed through 90° of its rotation by the time $t = 0$ s is reached and therefore has its maximum vertical projection as shown in Fig. 14.55(a). Since the vertical projection is a maximum, the peak value of the sinu-

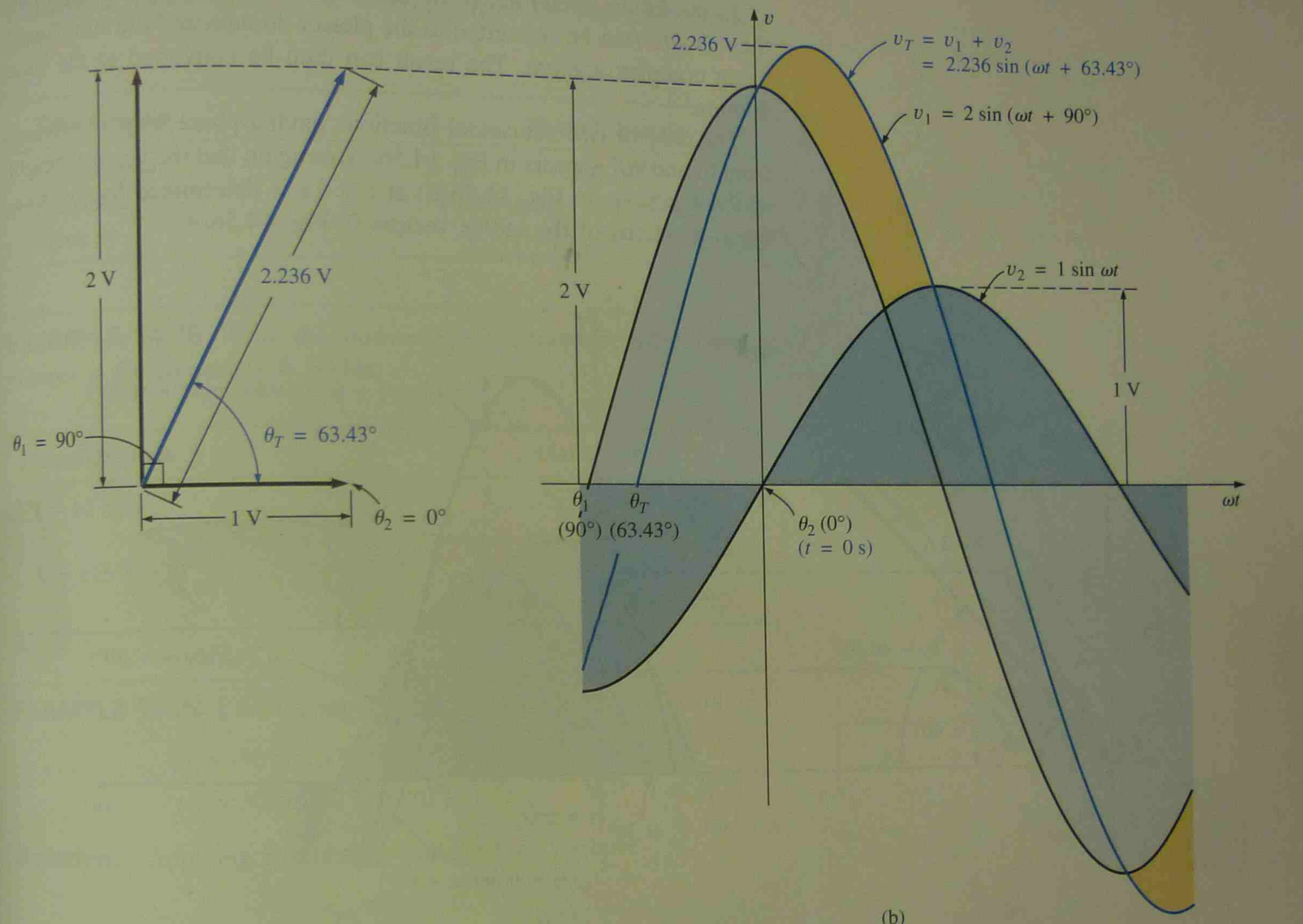


FIG. 14.55

(a) The phasor representation of the sinusoidal waveforms of Fig. 14.55(b); (b) finding the sum of two sinusoidal waveforms of v_1 and v_2.

soid that it will generate is also attained at $t = 0$ s, as shown in Fig. 14.55(b). Note also that $v_T = v_1$ at $t = 0$ s since $v_2 = 0$ V at this instant.

It can be shown [see Fig. 14.55(a)] using the vector algebra described in Section 14.10 that

$$1 \text{ V} \angle 0° + 2 \text{ V} \angle 90° = 2.236 \text{ V} \angle 63.43°$$

In other words, if we convert v_1 and v_2 to the phasor form using

$$v = V_m \sin(\omega t \pm \theta) \Rightarrow V_m \angle \pm \theta$$

and add them using vector algebra, we can find the phasor form for v_T with very little difficulty. It can then be converted to the time domain and plotted on the same set of axes as shown in Fig. 14.55(b). Figure 14.55(a), showing the magnitudes and relative positions of the various phasors, is called a *phasor diagram*. It is actually a "snapshot" of the rotating radius vectors at $t = 0$ s.

In the future, therefore, if the addition of two sinusoids is required, they should first be converted to the phasor domain and the sum found using complex algebra. The result can then be converted to the time domain.

The case of two sinusoidal functions having phase angles different from 0° and 90° appears in Fig. 14.56. Note again that the vertical height of the functions in Fig. 14.56(b) at $t = 0$ s is determined by the rotational positions of the radius vectors in Fig. 14.56(a).

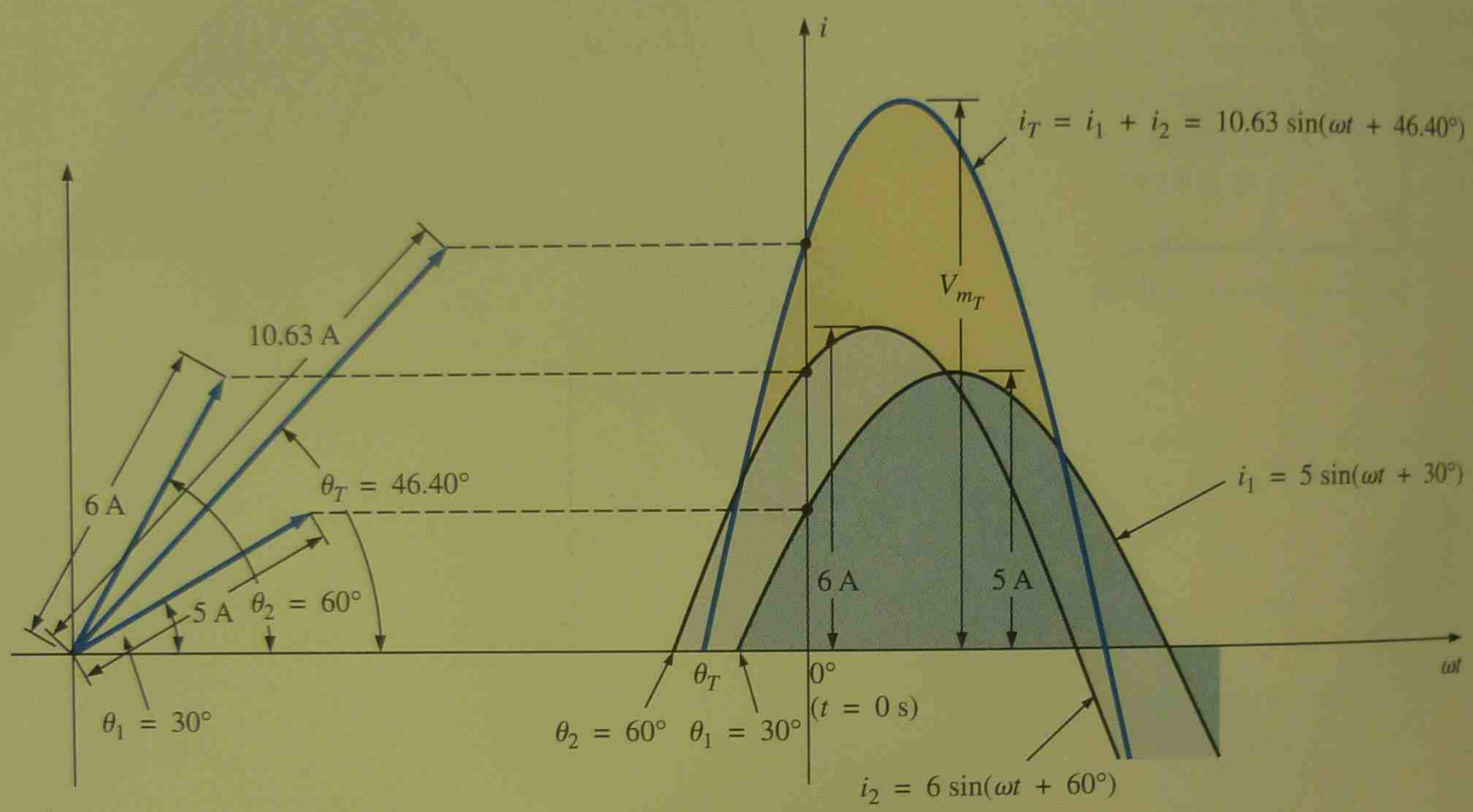


FIG. 14.56

Since the effective, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *effective value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V \angle \theta \quad \text{and} \quad \mathbf{I} = I \angle \theta$$

where V and I are effective values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

EXAMPLE 14.27 Convert the following from the time to the phasor domain:

Time Domain	Phasor Domain
a. $\sqrt{2}(50)\sin \omega t$	$\mathbf{50\ \angle 0^\circ}$
b. $69.6\sin(\omega t + 72^\circ)$	$(0.707)(69.6)\ \angle 72^\circ = \mathbf{49.21\ \angle 72^\circ}$
c. $45\cos \omega t$	$(0.707)(45)\ \angle 90^\circ = \mathbf{31.82\ \angle 90^\circ}$

EXAMPLE 14.28 Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10\ \angle 30^\circ$	$i = \sqrt{2}(10)\sin(2\pi\, 60t + 30^\circ)$ and $\boldsymbol{i = 14.14\sin(377t + 30^\circ)}$
b. $\mathbf{V} = 115\ \angle -70^\circ$	$v = \sqrt{2}(115)\sin(377t - 70^\circ)$ and $\boldsymbol{v = 162.6\sin(377t - 70^\circ)}$

EXAMPLE 14.29 Find the input voltage of the circuit of Fig. 14.57 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30^\circ) \\ v_b &= 30\sin(377t + 60^\circ) \end{aligned}\right\} f = 60\ \text{Hz}$$

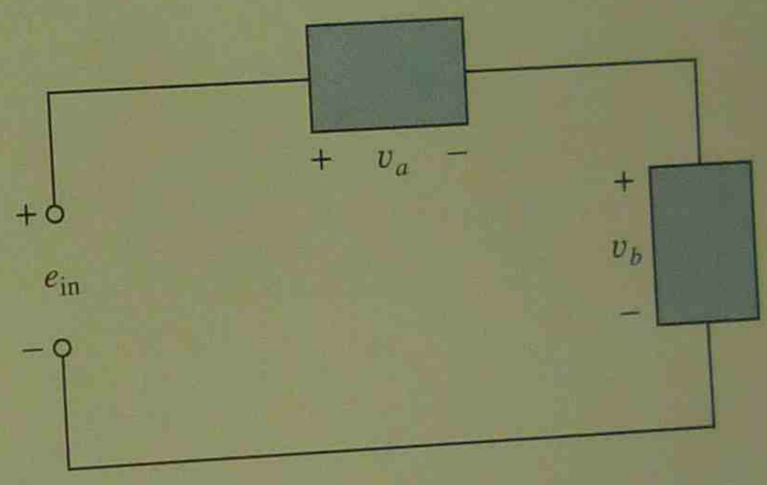


FIG. 14.57

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30^\circ) \Rightarrow \mathbf{V}_a = 35.35\ \text{V}\ \angle 30^\circ$$
$$v_b = 30\sin(377t + 60^\circ) \Rightarrow \mathbf{V}_b = 21.21\ \text{V}\ \angle 60^\circ$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\ \text{V}\ \angle 30^\circ = 30.61 + j17.68$$
$$\mathbf{V}_b = 21.21\ \text{V}\ \angle 60^\circ = 10.61 + j18.37$$

Then

$$\mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b = (30.61 + j17.68) + (10.61 + j18.37)$$
$$= 41.22 + j36.05$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22 + j36.05 = 54.76\ \text{V}\ \angle 41.17^\circ$$

Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\ \text{V}\ \angle 41.17^\circ \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17^\circ)$$

and

$$e_{in} = \boldsymbol{77.43\sin(377t + 41.17^\circ)}$$

A plot of the three waveforms is shown in Fig. 14.58. Note that at each instant of time, the sum of the two waveforms does in fact add up to e_{in}. At $t = 0$ ($\omega t = 0$), e_{in} is the sum of the two positive values, while at a value of ωt almost midway between $\pi/2$ and π, the sum of the positive value of v_a and the negative value of v_b results in $e_{in} = 0$.

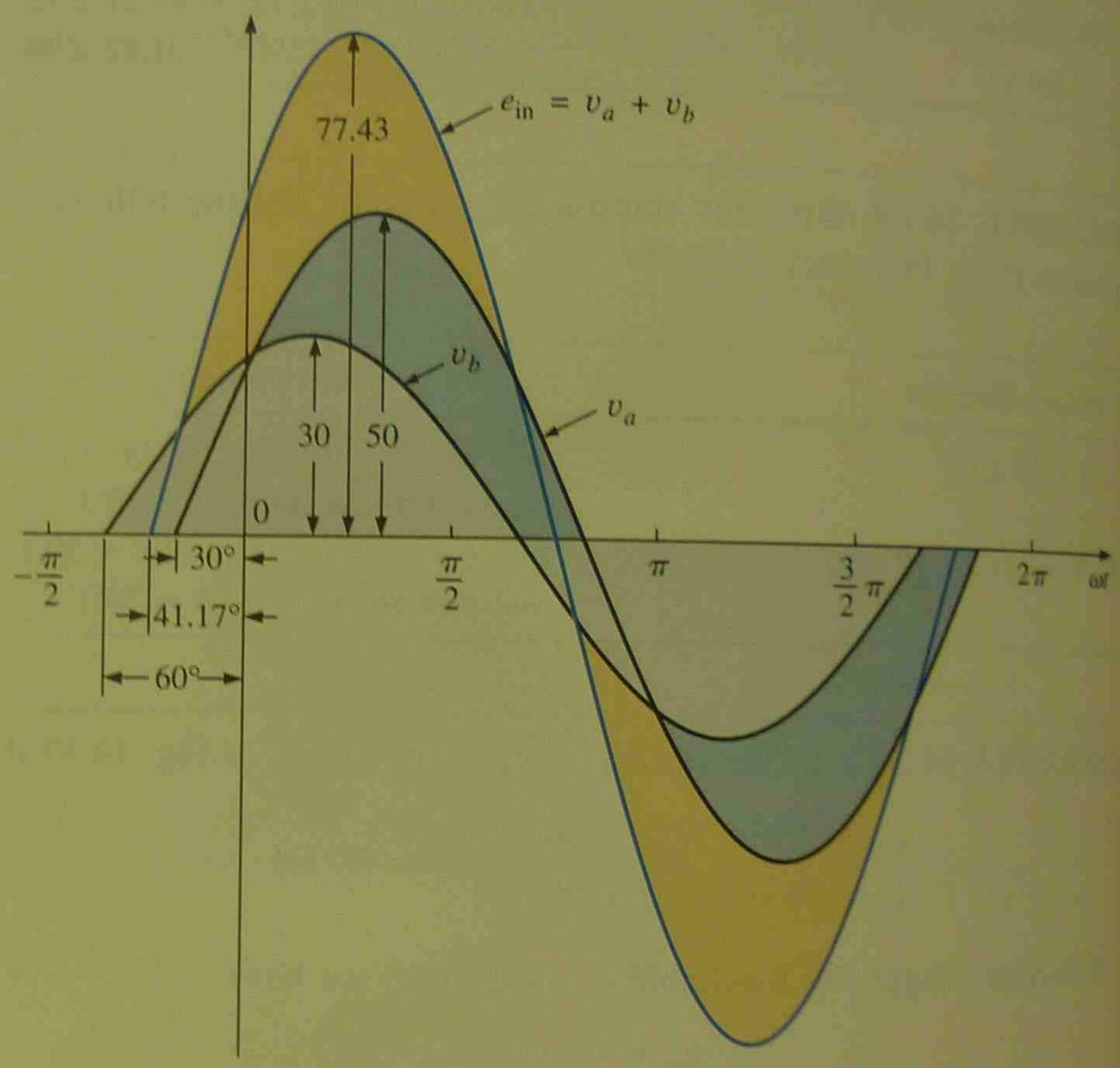


FIG. 14.58

EXAMPLE 14.30 Determine the current i_2 for the network of Fig. 14.59.

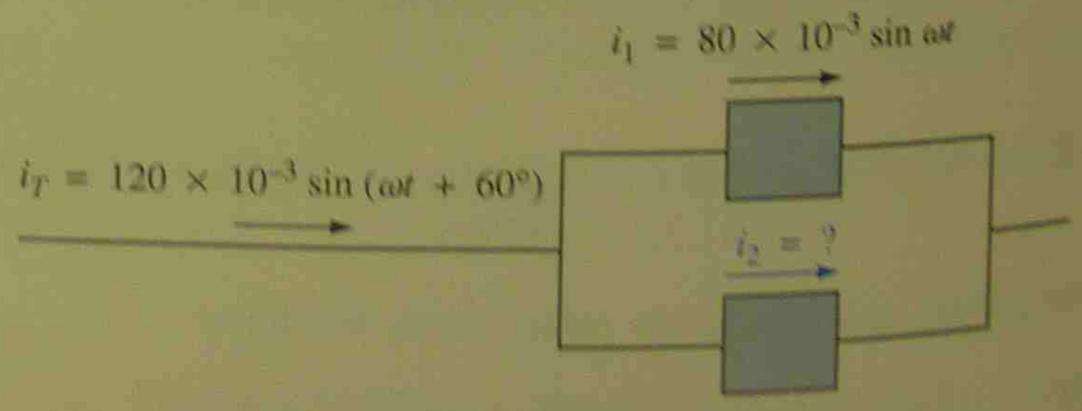


FIG. 14.59

Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA} \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA} \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA} \angle 60° = 42.42 \times 10^{-3} + j73.47 \times 10^{-3}$$
$$\mathbf{I}_1 = 56.56 \text{ mA} \angle 0° = 56.56 \times 10^{-3} + j0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \times 10^{-3} + j73.47 \times 10^{-3}) - (56.56 \times 10^{-3} + j0)$$

and

$$\mathbf{I}_2 = -14.14 \times 10^{-3} + j73.47 \times 10^{-3}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3}) \sin(\omega t + 100.89°)$$

and

$$i_2 = \mathbf{105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$$

A plot of the three waveforms appears in Fig. 14.60. The waveforms clearly indicate that $i_T = i_1 + i_2$.

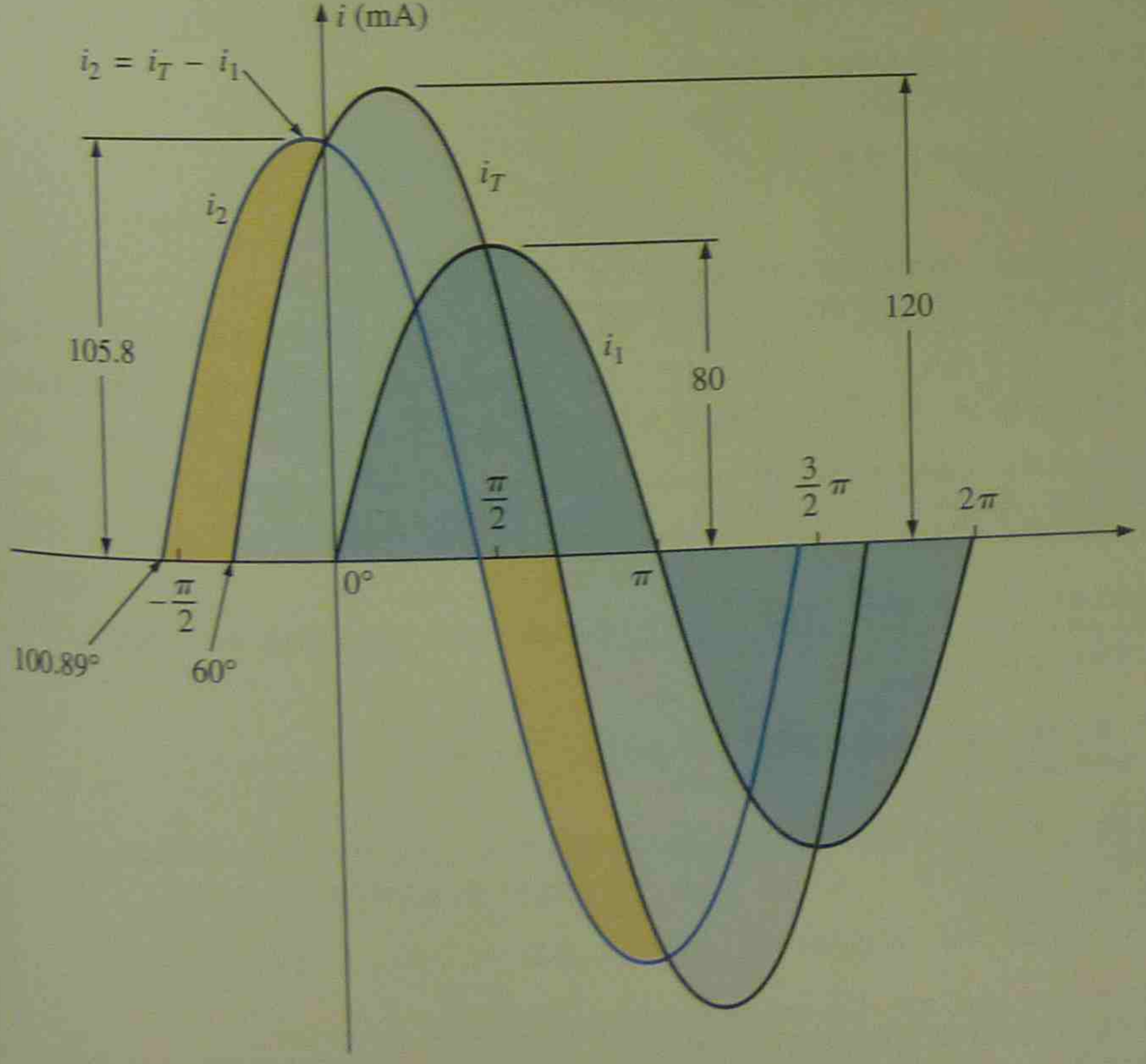


FIG. 14.60

15

Series and Parallel ac Circuits

15.1 INTRODUCTION

In this chapter, phasor algebra will be used to develop a quick, direct method for solving both the series and the parallel ac circuits. The close relationship that exists between this method for solving for unknown quantities and the approach used for dc circuits will become apparent after a few simple examples are considered. Once this association is established, many of the rules (current divider rule, voltage divider rule, and so on) for dc circuits can be readily applied to ac circuits.

SERIES ac CIRCUITS

15.2 IMPEDANCE AND THE PHASOR DIAGRAM

In Chapter 14, we found, for the purely resistive circuit of Fig. 15.1, that v and i were in phase, and the magnitude

$$I_m = \frac{V_m}{R} \qquad \text{or} \qquad V_m = I_m R$$

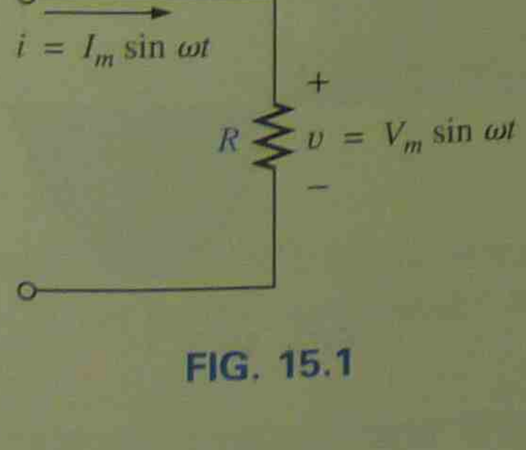


FIG. 15.1

In phasor form,

$$v = V_m \sin \omega t \Rightarrow \mathbf{V} = V \angle 0^\circ$$

where $V = 0.707V_m$.

Applying Ohm's law and using phasor algebra, we have

$$\mathbf{I} = \frac{V\angle 0^\circ}{R \angle \theta_R} = \frac{V}{R} \underline{/0^\circ - \theta_R}$$

Since i and v are in phase, the angle associated with i also must be 0°. To satisfy this condition, θ_R must equal 0°. Substituting $\theta_R = 0^\circ$, we find

$$\mathbf{I} = \frac{V \angle 0^\circ}{R \angle 0^\circ} = \frac{V}{R} \underline{/0^\circ - 0^\circ} = \frac{V}{R} \angle 0^\circ$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{R}\right) \sin \omega t$$

The fact that $\theta_R = 0^\circ$ will now be employed in a polar format for resistance to insure the proper phase relationship between the voltage and current of a resistor. It will be a very useful "tool" when the networks become more complex and phase relationships become less obvious. It is important, however, to realize that even though the format $R \angle 0^\circ$ looks very similar to the phasor notation for sinusoidal voltages and currents, the terminology *phasor* is reserved solely for voltages and currents. The resistance R is fixed at one level (there is no effective, or peak-to-peak, value) and the angle is always 0° to represent the phase angle between the voltage and current of a resistor (not the intersection on the horizontal axis of a sinusoidal plot).

EXAMPLE 15.1 Using complex algebra, find the current i for the circuit of Fig. 15.2. Sketch the waveforms of v and i.

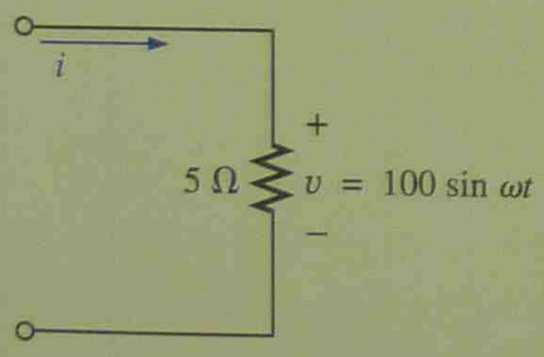


FIG. 15.2

Solution: Note Fig. 15.3:

$$v = 100 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 70.71 \text{ V} \angle 0^\circ$$

$$\mathbf{I} = \frac{V \angle \theta}{R \angle 0^\circ} = \frac{70.71 \text{ V} \angle 0^\circ}{5\ \Omega \angle 0^\circ} = 14.14\ A \angle 0^\circ$$

and

$$i = \sqrt{2}(14.14) \sin \omega t = \mathbf{20 \sin \omega t}$$

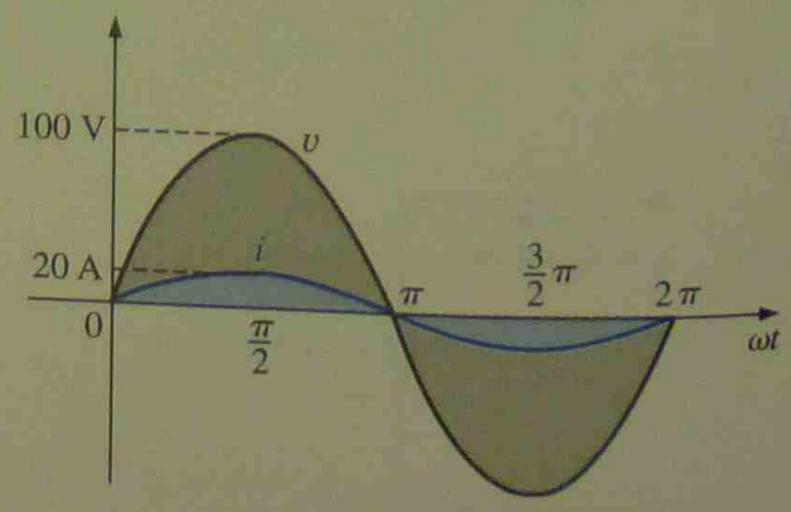


FIG. 15.3

EXAMPLE 15.2 Using complex algebra, find the voltage v for the circuit of Fig. 15.4. Sketch the waveforms of v and i.

Solution: Note Fig. 15.5:

$i = 4\sin(\omega t + 30°) \Rightarrow$ phasor form $\mathbf{I} = 2.828\text{ A}\angle 30°$

$\mathbf{V} = (I\angle\theta)(R\angle 0°) = (2.828\text{ A}\angle 30°)(2\,\Omega\angle 0°) = 5.656\text{ V}\angle 30°$

and $v = \sqrt{2}(5.656)\sin(\omega t + 30°) = \mathbf{8.0\sin(\omega t + 30°)}$

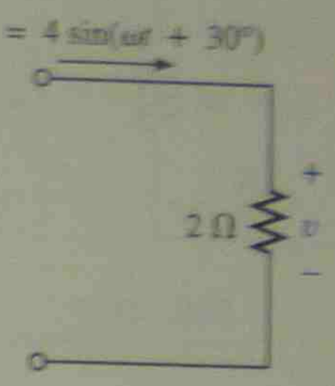


FIG. 15.4

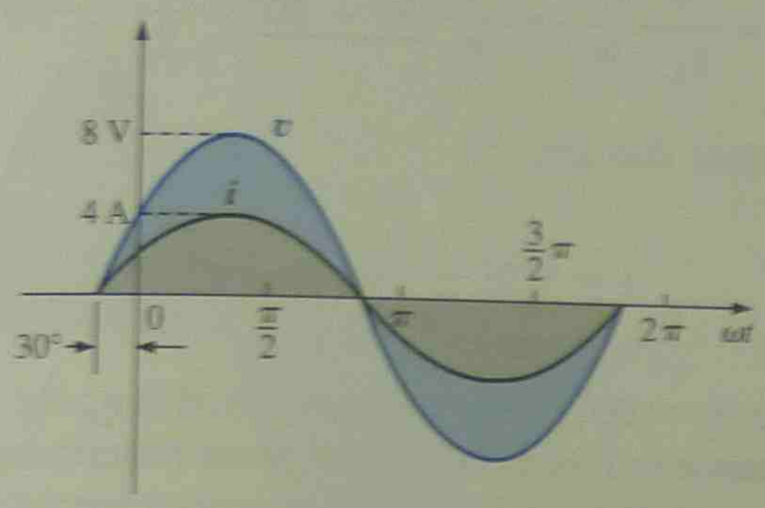


FIG. 15.5

It is often helpful in the analysis of networks to have a *phasor diagram*, which shows at a glance the *magnitudes* and *phase relations* between the various quantities within the network. For example, the phasor diagrams of the circuits considered in the preceding examples would be as shown in Fig. 15.6. In both cases, it is immediately obvious that v and i are in phase, since they both have the same phase angle.

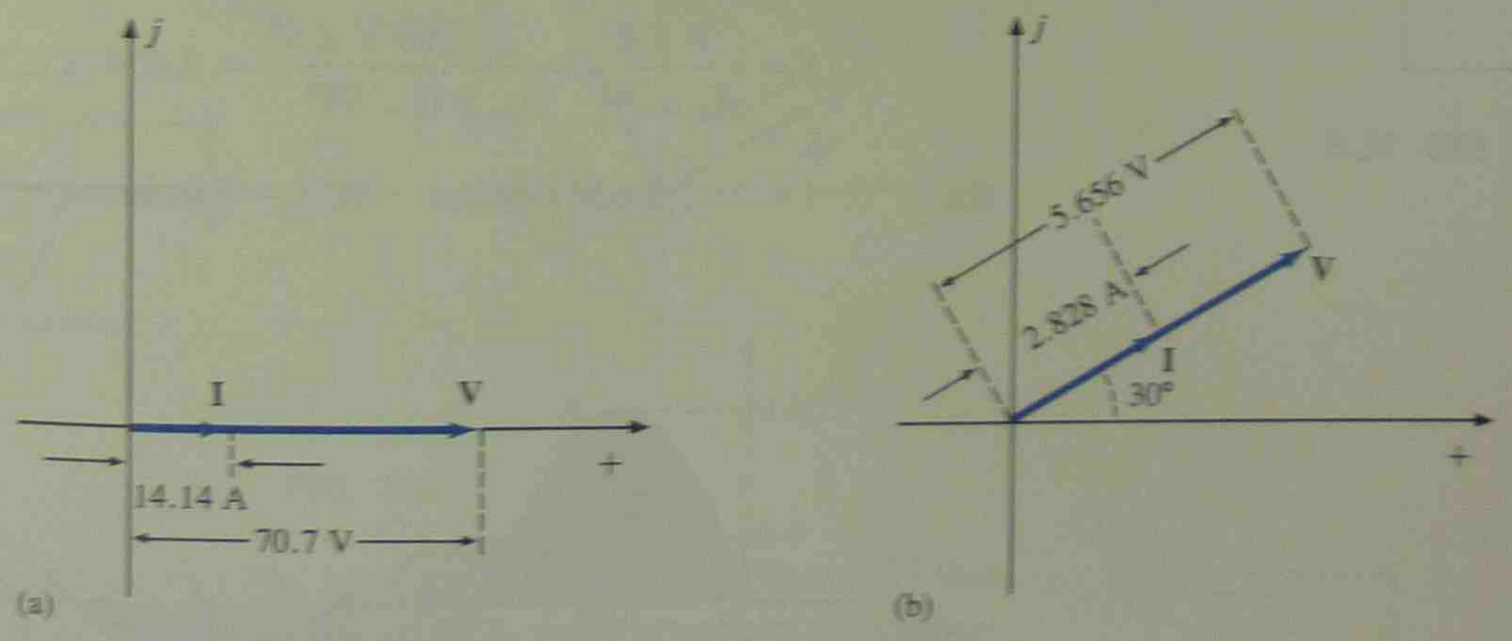


FIG. 15.6

For the pure inductor of Fig. 15.7, it was learned in Chapter 13 that the voltage leads the current by 90°, and that the reactance of the coil X_L is determined by ωL.

$v = V_m \sin \omega t \Rightarrow$ phasor form $\mathbf{V} = V\angle 0°$

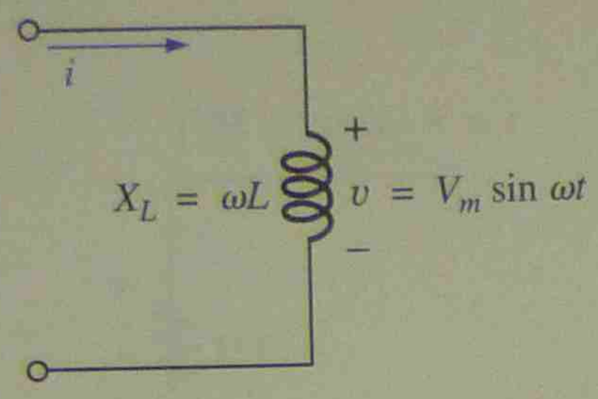


FIG. 15.7

By Ohm's law,

$$\mathbf{I} = \frac{V \angle 0^\circ}{X_L \angle \theta_L} = \frac{V}{X_L} \angle 0^\circ - \theta_L$$

Since v leads i by 90°, i must have an angle of −90° associated with it. To satisfy this condition, θ_L must equal +90°. Substituting $\theta_L = 90^\circ$, we obtain

$$\mathbf{I} = \frac{V \angle 0^\circ}{X_L \angle 90^\circ} = \frac{V}{X_L} \angle 0^\circ - 90^\circ = \frac{V}{X_L} \angle -90^\circ$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_L}\right)\sin(\omega t - 90^\circ)$$

The fact that $\theta_L = 90^\circ$ will now be employed in a polar format for the inductive reactance to insure the proper phase relationship between the voltage and current of an inductor. As described for the resistor, however, it is important to realize that the format is not a phasor notation but one that includes the phase shift introduced by inductive elements; the magnitude is fixed at X_L and the associated angle is always 90°.

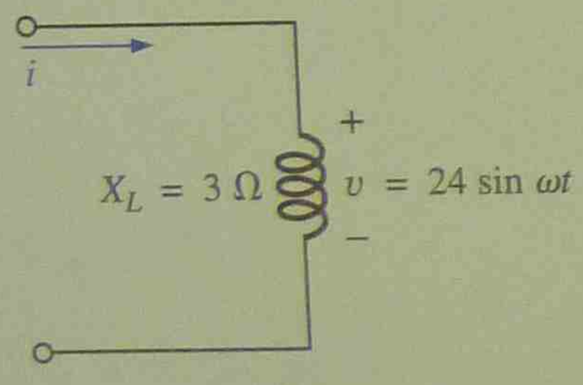


FIG. 15.8

EXAMPLE 15.3 Using complex algebra, find the current i for the circuit of Fig. 15.8. Sketch the v and i curves.

Solution: Note Fig. 15.9:

$$v = 24 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 16.968\ \text{V} \angle 0^\circ$$

$$\mathbf{I} = \frac{V \angle \theta}{X_L \angle 90^\circ} = \frac{16.968\ \text{V} \angle 0^\circ}{3\ \Omega \angle 90^\circ} = 5.656\ \text{A} \angle -90^\circ$$

and $$i = \sqrt{2}(5.656)\sin(\omega t - 90^\circ) = \mathbf{8.0 \sin(\omega t - 90^\circ)}$$

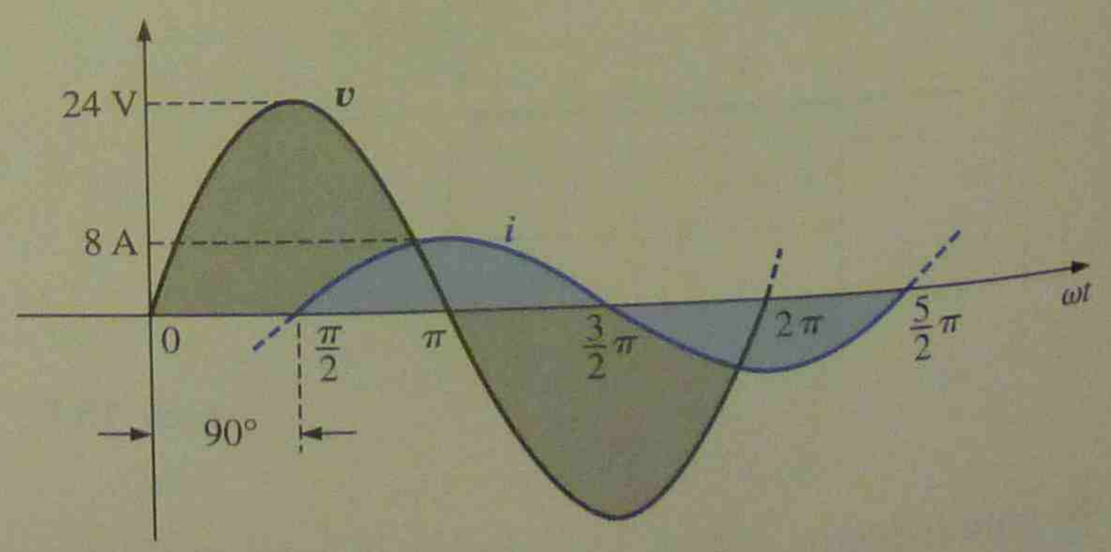


FIG. 15.9

EXAMPLE 15.4 Using complex algebra, find the voltage v for the circuit of Fig. 15.10. Sketch the v and i curves.

Solution: Note Fig. 15.11:

$$i = 5\sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 3.535 \text{ A} \angle 30°$$

$$\mathbf{V} = (I \angle \theta)(X_L \angle 90°) = (3.535 \text{ A} \angle 30°)(4\ \Omega \angle +90°)$$
$$= 14.140 \text{ V} \angle 120°$$

and $$v = \sqrt{2}(14.140)\sin(\omega t + 120°) = \mathbf{20\sin(\omega t + 120°)}$$

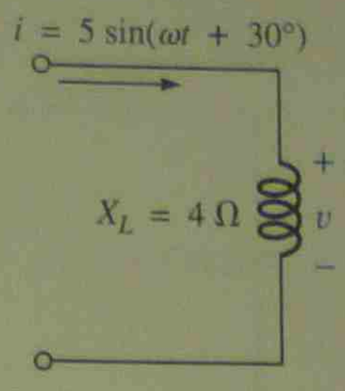


FIG. 15.10

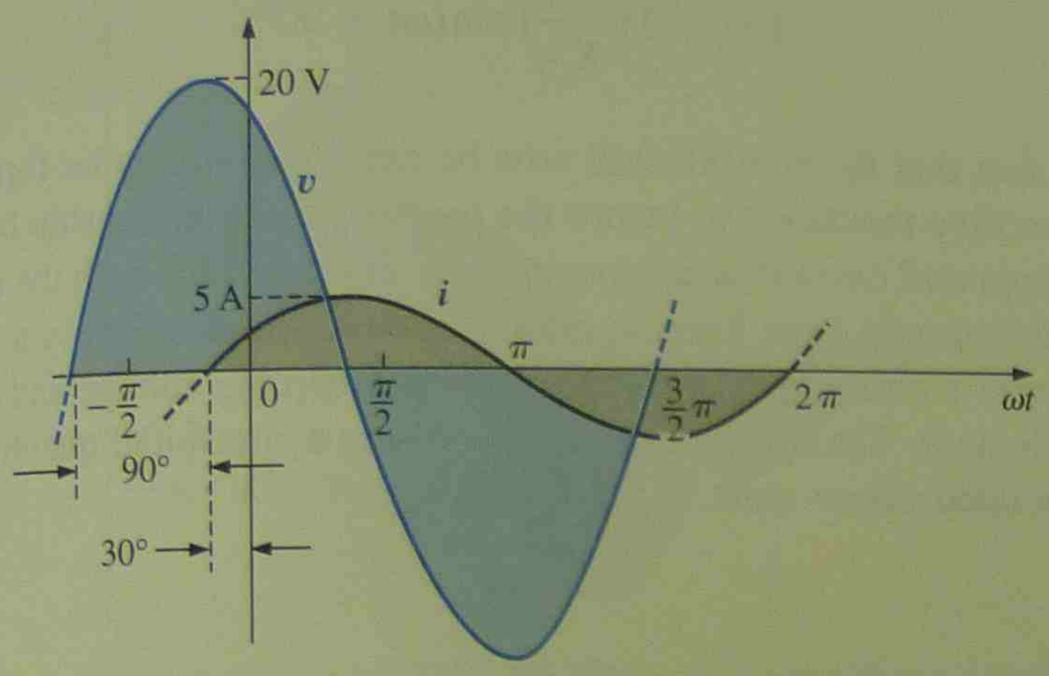


FIG. 15.11

The phasor diagrams for the two circuits of the preceding examples are shown in Fig. 15.12. Both indicate quite clearly that the voltage leads the current by 90°.

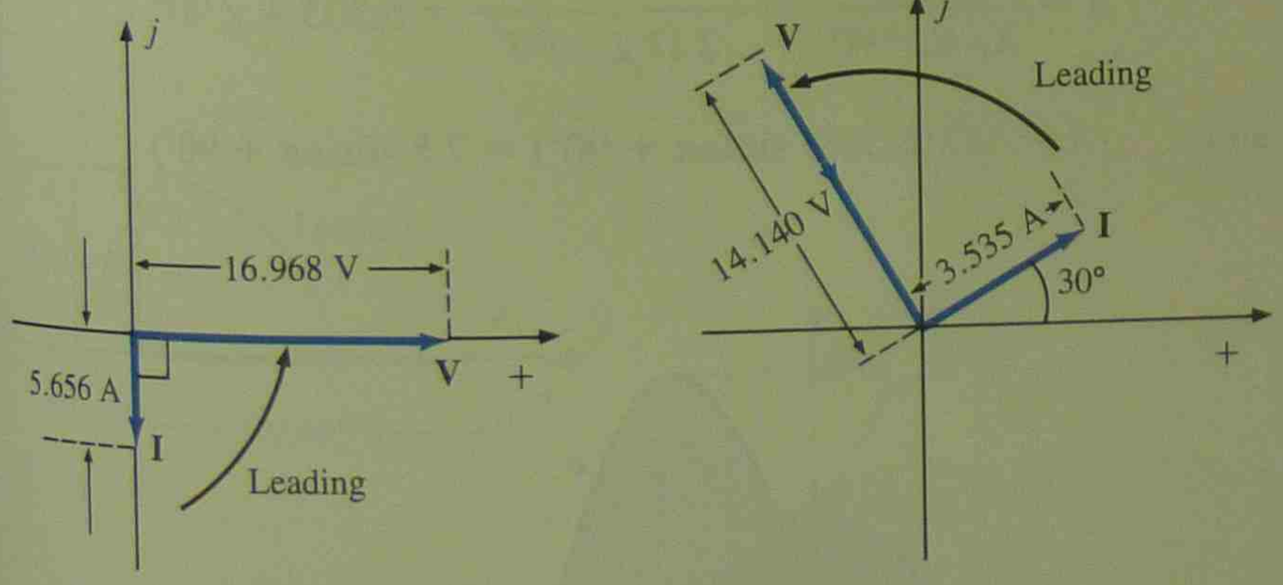


FIG. 15.12

For the pure capacitor of Fig. 15.13, it was learned in Chapter 13 that the current leads the voltage by 90°, and that the reactance of the capacitor X_C is determined by $1/\omega C$.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$

Applying Ohm's law and using phasor algebra, we find

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle \theta_C} = \frac{V}{X_C} \angle 0° - \theta_C$$

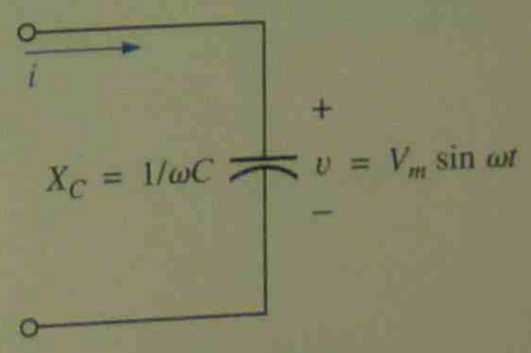


FIG. 15.13

Since we know i leads v by 90°, i must have an angle of +90° associated with it. To satisfy this condition, θ_C must equal −90°. Substituting $\theta_C = -90°$ yields

$$\mathbf{I} = \frac{V\angle 0°}{X_C\angle -90°} = \frac{V}{X_C}\angle 0° - (-90°) = \frac{V}{X_C}\angle 90°$$

so, in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_C}\right)\sin(\omega t + 90°)$$

The fact that $\theta_C = -90°$ will now be employed in a polar format for the capacitive reactance to insure the proper phase relationship between the voltage and current of a capacitor. As described for both the resistor and inductor, one must keep in mind that the format is simply a "tool" that insures a correct phase relationship between the voltage and current of the element. The format does not represent a sinusoidal quantity with its associated phase shift.

EXAMPLE 15.5 Using complex algebra, find the current i for the circuit of Fig. 15.14. Sketch the v and i curves.

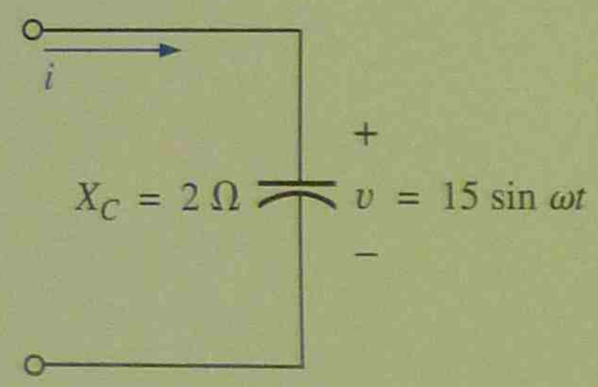


FIG. 15.14

Solution: Note Fig. 15.15:

$$v = 15 \sin \omega t \Rightarrow \text{phasor notation } \mathbf{V} = 10.605\ \text{V}\angle 0°$$

$$\mathbf{I} = \frac{V\angle\theta}{X_C\angle -90°} = \frac{10.605\ \text{V}\angle 0°}{2\ \Omega\angle -90°} = 5.303\ \text{A}\angle 90°$$

and $$i = \sqrt{2}(5.303)\sin(\omega t + 90°) = \mathbf{7.5\ sin(\omega t + 90°)}$$

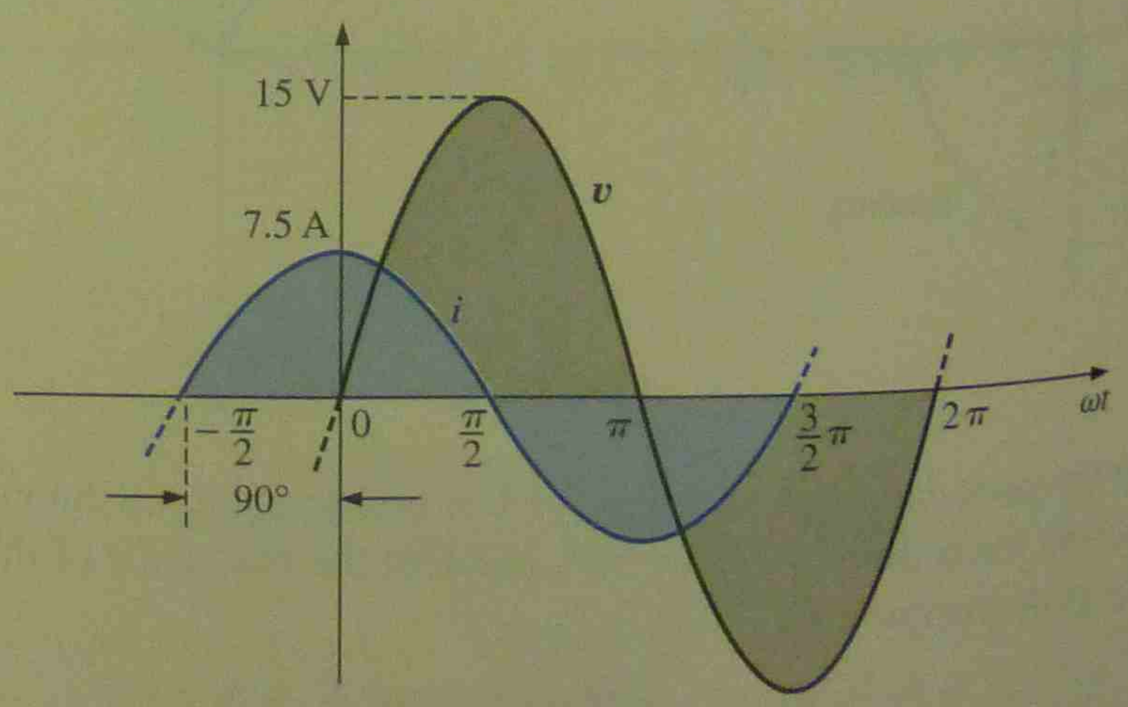


FIG. 15.15

EXAMPLE 15.6 Using complex algebra, find the voltage v for the circuit of Fig. 15.16. Sketch the v and i curves.

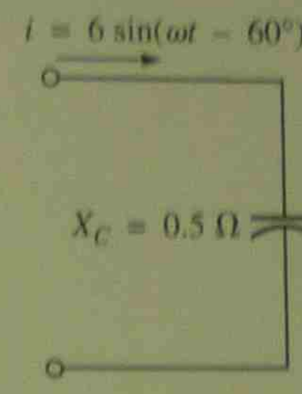


FIG. 15.16

Solution: Note Fig. 15.17:

$$i = 6\sin(\omega t - 60^\circ) \Rightarrow \text{phasor notation } \mathbf{I} = 4.242\text{ A} \angle -60^\circ$$

$$\mathbf{V} = (I \angle \theta)(X_C \angle -90^\circ) = (4.242\text{ A} \angle -60^\circ)(0.5\ \Omega \angle -90^\circ)$$
$$= 2.121\text{ V} \angle -150^\circ$$

and $$v = \sqrt{2}(2.121)\sin(\omega t - 150^\circ) = \mathbf{3.0\sin(\omega t - 150^\circ)}$$

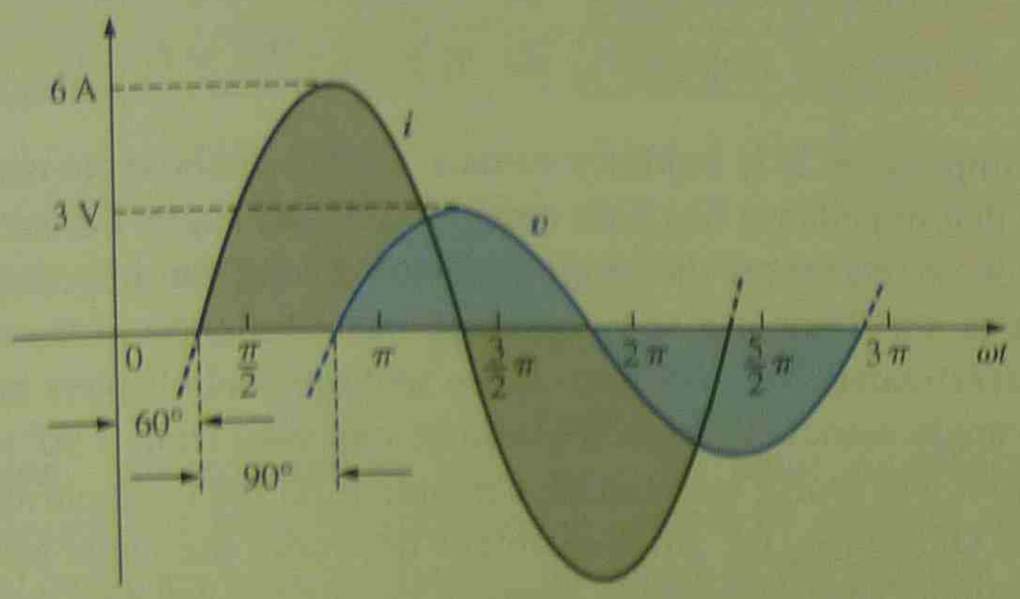


FIG. 15.17

The phasor diagrams for the two circuits of the preceding examples are shown in Fig. 15.18. Both indicate quite clearly that the current i leads the voltage v by 90°.

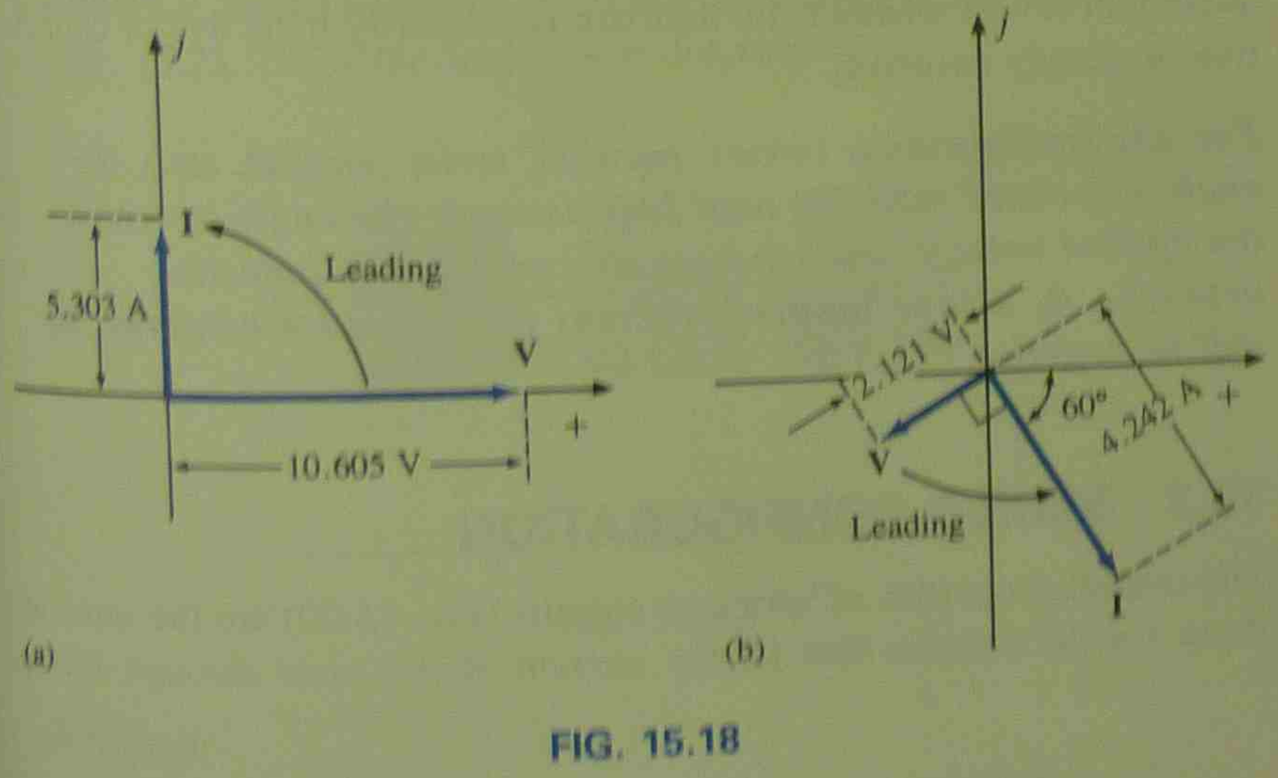


FIG. 15.18

Now that an angle is associated with resistance, inductive reactance, and capacitive reactance, each can be placed on a complex plane diagram, as shown in Fig. 15.19. For any network, the resistance will *always* appear on the positive real axis, the inductive reactance on the positive imaginary axis, and the capacitive reactance on the negative imaginary axis.

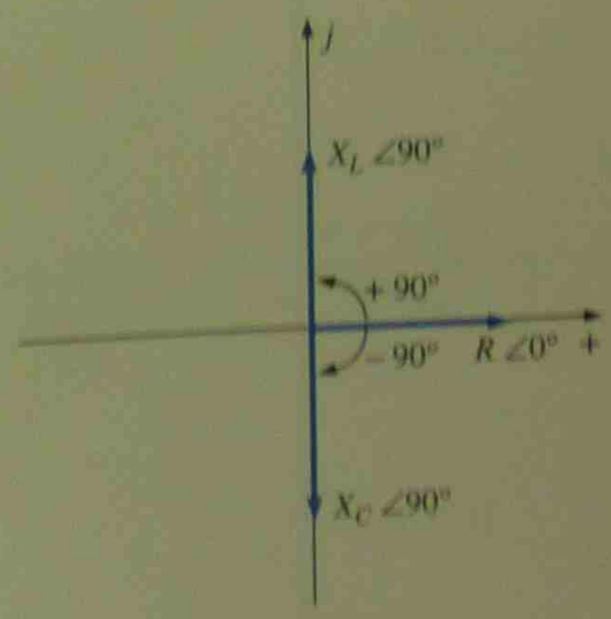


FIG. 15.19

Any *one or combination* of these elements in an ac circuit defines the *impedance* of the circuit. It is a measure of how much the circuit will *impede*, or hinder, the flow of current through it. The diagram of Fig. 15.19 is referred to as an *impedance diagram*. The symbol for impedance is Z and the unit of measurement is the ohm.

For the individual elements,

$$\text{Resistance:} \quad \mathbf{Z}_R = R\angle 0^\circ = R + j0 \tag{15.1}$$

$$\text{Inductive reactance:} \quad \mathbf{Z}_L = X_L\angle 90^\circ = 0 + jX_L \tag{15.2}$$

$$\text{Capacitive reactance:} \quad \mathbf{Z}_C = X_C\angle -90^\circ = 0 - jX_C \tag{15.3}$$

The impedance $\mathbf{Z}$ is boldface roman, as shown above, to emphasize the fact that impedance has both a magnitude and angle. Further, unlike resistance or reactance, the angle associated with the impedance of a network is not fixed at a specific value but a function of the network being investigated. We will find in the sections and chapters to follow that the angle associated with impedance can vary from -90° to $+90^\circ$ (in the first and fourth quadrants), depending on the components of the network. Of course, for single-element networks the angle associated with the impedance will be the same as that of the resistive or reactive element as revealed by Eqs. (15.1) to (15.3). It is important to maintain an awareness that impedance, like resistance or reactance, is not a phasor quantity representing a time-varying function with a particular phase shift. It is simply an operating "tool" that is extremely useful in determining the magnitude and angle of quantities in a sinusoidal ac network.

Once the total impedance of a network is determined, its magnitude will define the resulting current level (through Ohm's law), whereas its angle will reveal whether the network is primarily inductive or capacitive or simply resistive.

For any configuration (series, parallel, series-parallel, etc.), the angle associated with the total impedance is the angle by which the applied voltage leads the source current. For inductive networks, θ_T will be positive, whereas for capacitive networks, θ_T will be negative.

15.3 SERIES CONFIGURATION

The overall properties of series ac circuits (Fig. 15.20) are the same as those for dc circuits; that is, *the current is the same through series*

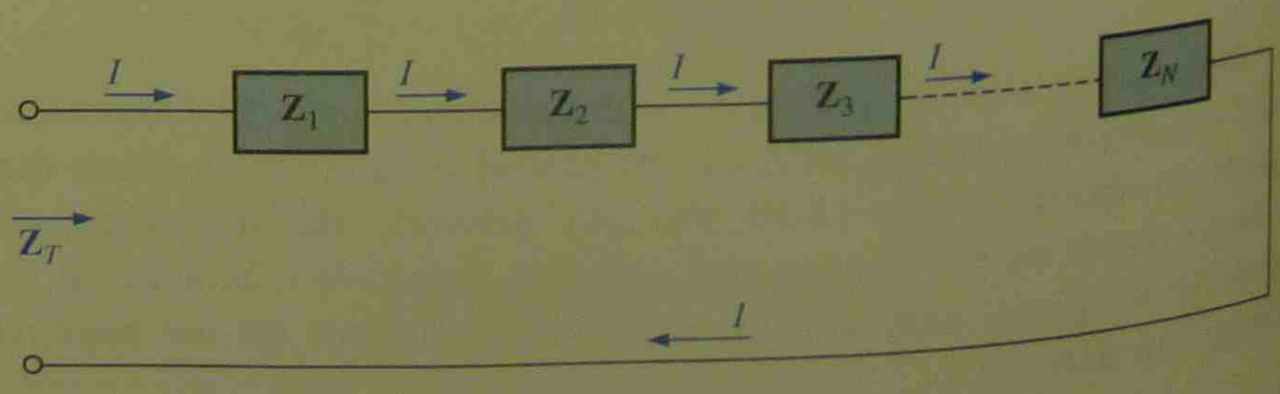


FIG. 15.20

elements, and the total impedance of a system is the sum of the individual impedances:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_N \tag{15.4}$$

EXAMPLE 15.7 Draw the impedance diagram for the circuit of Fig. 15.21 and find the total impedance.

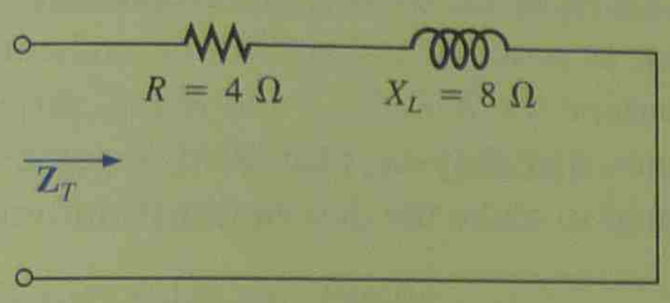


FIG. 15.21

Solution: As indicated by Fig. 15.22, the input impedance can be found graphically from the impedance diagram by properly scaling the real and imaginary axes and finding the length of the resultant vector Z_T and angle θ_T. Or, by using vector algebra, we obtain

$$\begin{aligned}\mathbf{Z}_T &= \mathbf{Z}_1 + \mathbf{Z}_2 \\ &= R\angle 0^\circ + X_L\angle 90^\circ \\ &= R + jX_L = 4\,\Omega + j8\,\Omega \\ \mathbf{Z}_T &= \mathbf{8.944\,\Omega\angle 63.43^\circ}\end{aligned}$$

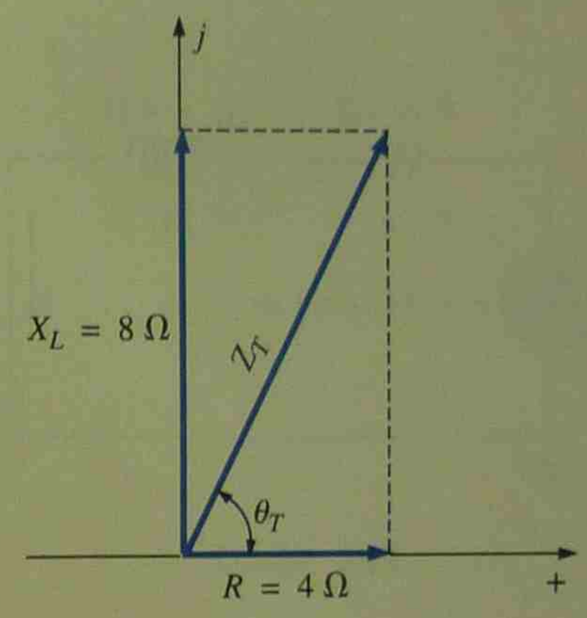


FIG. 15.22

EXAMPLE 15.8 Determine the input impedance to the series network of Fig. 15.23. Draw the impedance diagram.

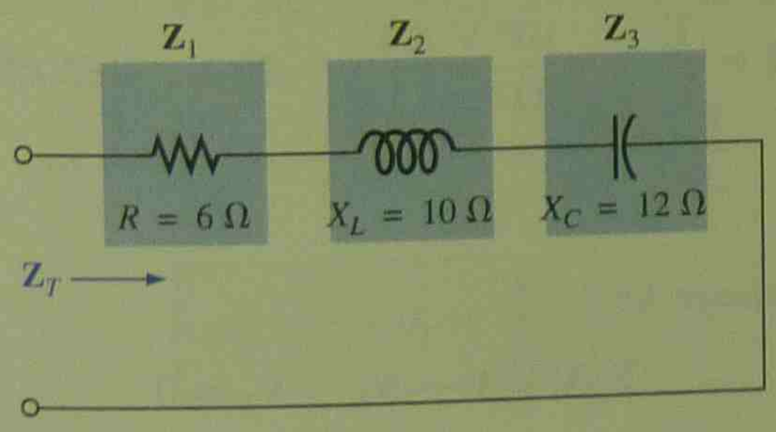


FIG. 15.23

Solution:

$$\begin{aligned}\mathbf{Z}_T &= \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 \\ &= R\angle 0^\circ + X_L\angle 90^\circ + X_C\angle -90^\circ \\ &= R + jX_L - jX_C \\ &= R + j(X_L - X_C) = 6\,\Omega + j(10\,\Omega - 12\,\Omega) = 6\,\Omega - j2\,\Omega \\ \mathbf{Z}_T &= \mathbf{6.325\,\Omega\angle -18.43^\circ}\end{aligned}$$

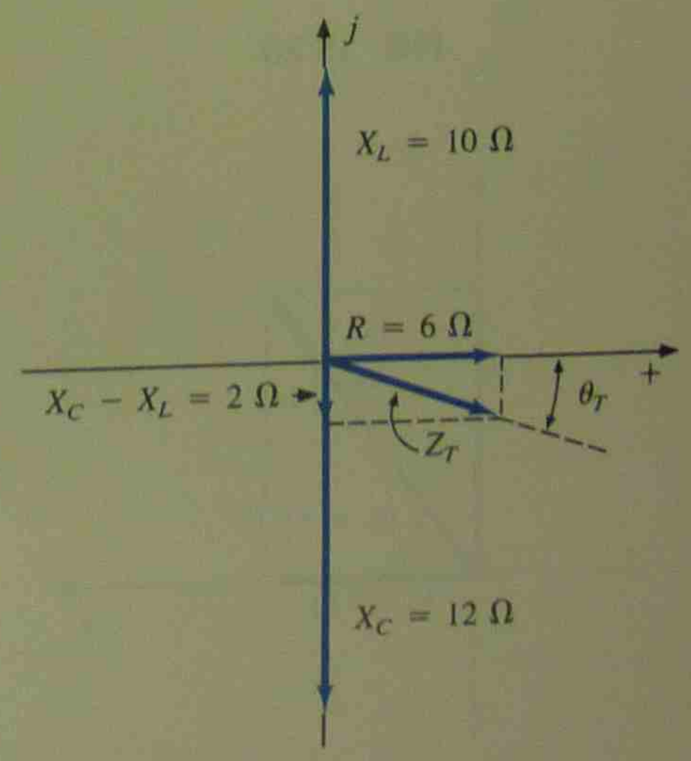


FIG. 15.24

The impedance diagram appears in Fig. 15.24. Note that in this example, series inductive and capacitive reactances are in direct opposi-

tion. For the circuit of Fig. 15.23, if the inductive reactance were equal to the capacitive reactance, the input impedance would be purely resistive. We will have more to say about this particular condition in a later chapter.

In many of the circuits to be considered, $3 + j4 = 5\angle 53.13^\circ$ and $4 + j3 = 5\angle 36.87^\circ$ will be used quite frequently to insure that the approach is as clear as possible and not lost in mathematical complexity.

Let us now examine the *R-L*, *R-C*, and *R-L-C* series networks. Their basic nature dictates that they be examined in some detail. Numerical values were assigned to make the description as informative as possible.

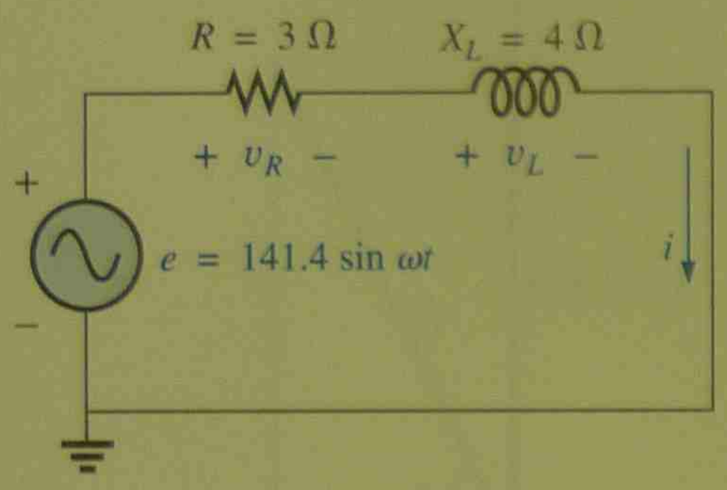


FIG. 15.25

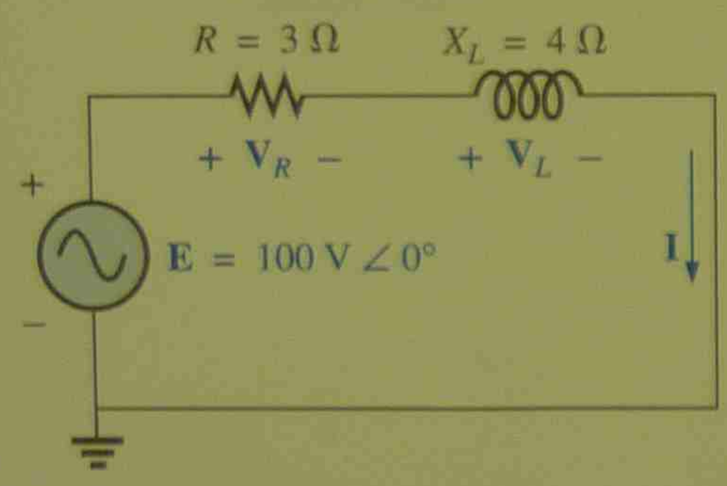


FIG. 15.26

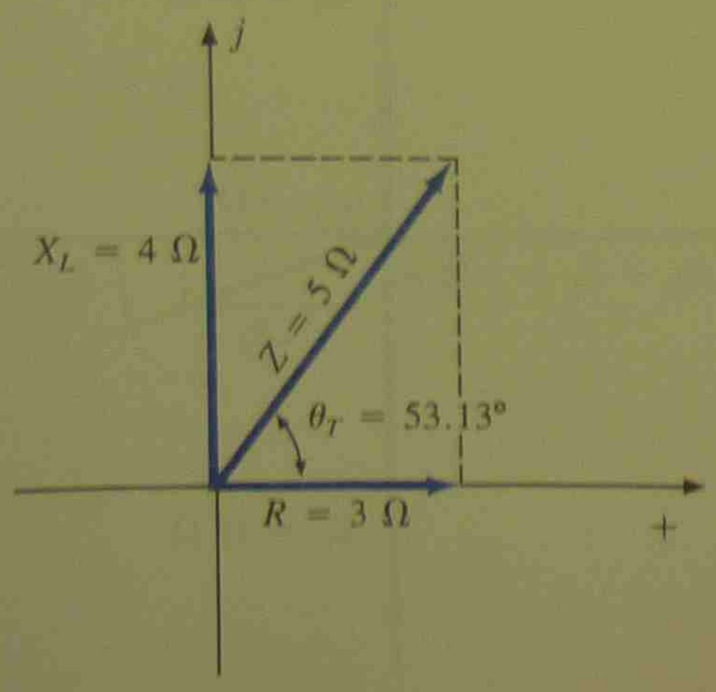


FIG. 15.27

R-L (Fig. 15.25)

Phasor notation:

$$e = 141.4 \sin \omega t \Rightarrow \mathbf{E} = 100\ \text{V}\angle 0^\circ$$

Note Fig. 15.26.

$\mathbf{Z}_T$:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 = 3\ \Omega\angle 0^\circ + 4\ \Omega\angle 90^\circ = 3\ \Omega + j4\ \Omega$$

and $$\mathbf{Z}_T = \mathbf{5\ \Omega\angle 53.13^\circ}$$

Impedance diagram: As shown in Fig. 15.27.

I:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{100\ \text{V}\angle 0^\circ}{5\ \Omega\angle 53.13^\circ} = \mathbf{20\ A\angle -53.13^\circ}$$

$\mathbf{V}_R$, $\mathbf{V}_L$:

Kirchhoff's voltage law:

$$\Sigma_{\circlearrowright}\mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_L = 0$$

or $$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L$$

In rectangular form,

$$\mathbf{V}_R = 60\ \text{V}\angle -53.13^\circ = 36\ \text{V} - j48\ \text{V}$$
$$\mathbf{V}_L = 80\ \text{V}\angle +36.87^\circ = 64\ \text{V} + j48\ \text{V}$$

and

$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L = (36\ \text{V} - j48\ \text{V}) + (64\ \text{V} + j48\ \text{V}) = 100\ \text{V} + j0$$
$$= 100\ \text{V}\angle 0^\circ$$

as applied.

Phasor diagram: Note that for the phasor diagram of Fig. 15.28, **I** is in phase with the voltage across the resistor and lags the voltage across the inductor by 90°.

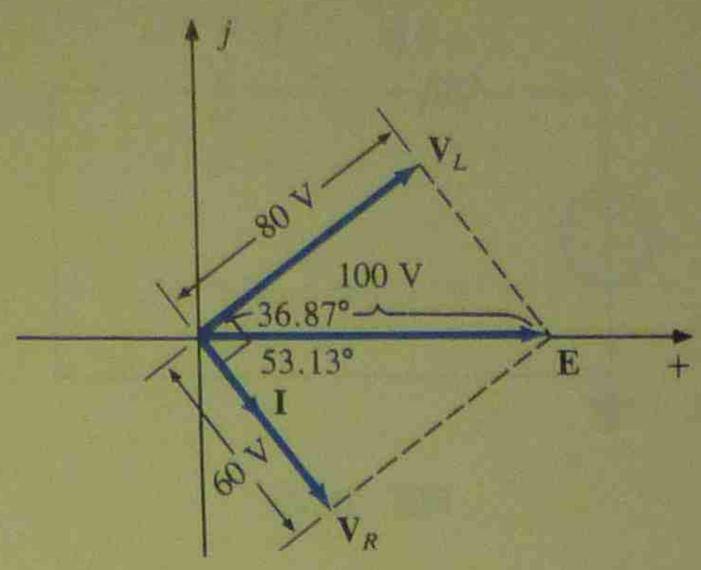


FIG. 15.28

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta_T$$
$$= (100 \text{ V})(20 \text{ A}) \cos 53.13° = (2000 \text{ W})(0.6)$$
$$= \mathbf{1200\ W}$$

where E and I are effective values and θ_T is the phase angle between E and I, or

$$P_T = I^2R$$
$$= (20 \text{ A})^2(3\ \Omega) = (400)(3)$$
$$= \mathbf{1200\ W}$$

where I is the effective value, or, finally,

$$P_T = P_R + P_L = V_R I \cos \theta_R + V_L I \cos \theta_L$$
$$= (60 \text{ V})(20 \text{ A}) \cos 0° + (80 \text{ V})(20 \text{ A}) \cos 90°$$
$$= 1200 \text{ W} + 0$$
$$= \mathbf{1200\ W}$$

where θ_R is the phase angle between V_R and I, and θ_L is the phase angle between V_L and I.

Power factor: The power factor F_p of the circuit is $\cos 53.13° = \mathbf{0.6}$ **lagging,** where 53.13° is the phase angle between **E** and **I.**

If we write the basic power equation $P = EI \cos \theta$ as follows:

$$\cos \theta = \frac{P}{EI}$$

where E and I are the input quantities and P is the power delivered to the network, and then perform the following substitutions from the basic series ac circuit:

$$\cos \theta = \frac{P}{EI} = \frac{I^2R}{EI} = \frac{IR}{E} = \frac{R}{E/I} = \frac{R}{Z_T}$$

we find

$$\boxed{F_p = \cos \theta_T = \frac{R}{Z_T}} \qquad \textbf{(15.5)}$$

Reference to Fig. 15.27 also indicates that θ is the impedance angle θ_T as written in Eq. (15.5), further supporting the fact that the impedance angle θ_T is also the phase angle between the input voltage and current for a series ac circuit. To determine the power factor, it is necessary only to form the ratio of the total resistance to the magnitude of the input impedance.

For the case at hand,

$$F_p = \cos \theta = \frac{R}{Z_T} = \frac{3\ \Omega}{5\ \Omega} = \mathbf{0.6\ lagging}$$

as found above.

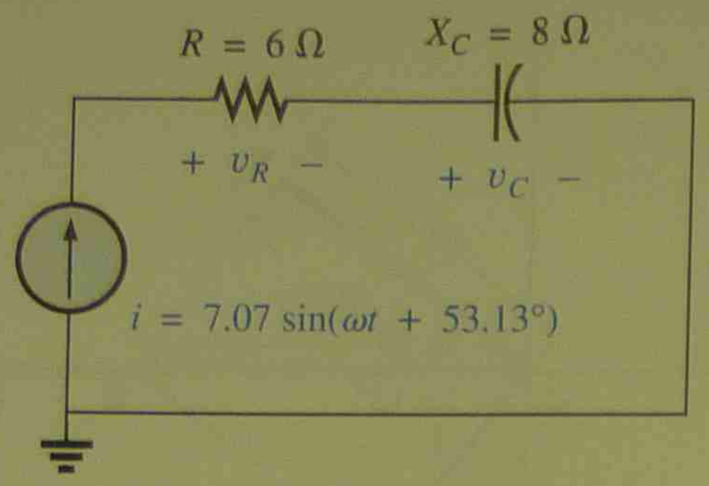


FIG. 15.29

R-C (Fig. 15.29)

Phasor notation:

$$i = 7.07 \sin(\omega t + 53.13^\circ) \Rightarrow \mathbf{I} = 5 \text{ A} \angle 53.13^\circ$$

Note Fig. 15.30.

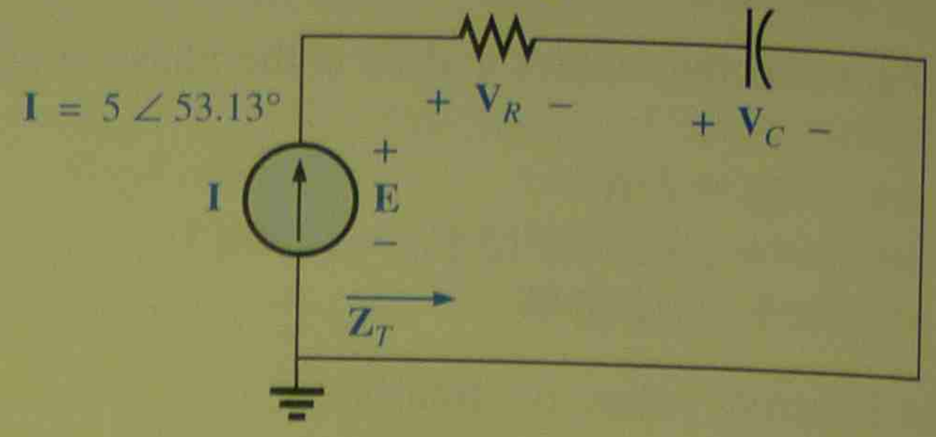


FIG. 15.30

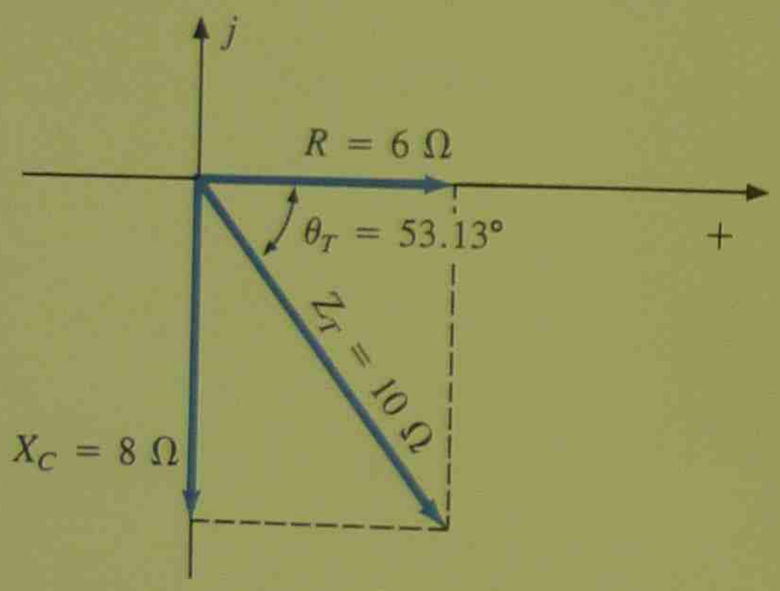


FIG. 15.31

$\mathbf{Z}_T$:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 = 6\ \Omega \angle 0^\circ + 8\ \Omega \angle -90^\circ = 6\ \Omega - j8\ \Omega$$

and $$\mathbf{Z}_T = \mathbf{10\ \Omega \angle -53.13^\circ}$$

Impedance diagram: As shown in Fig. 15.31.

E:

$$\mathbf{E} = \mathbf{I}\mathbf{Z}_T = (5 \text{ A} \angle 53.13^\circ)(10\ \Omega \angle -53.13^\circ) = \mathbf{50\ V \angle 0^\circ}$$

$\mathbf{V}_R$, $\mathbf{V}_C$:

$$\mathbf{V}_R = (I \angle \theta)(R \angle 0^\circ) = (5 \text{ A} \angle 53.13^\circ)(6\ \Omega \angle 0^\circ) = \mathbf{30\ V \angle 53.13^\circ}$$

$$\mathbf{V}_C = (I \angle \theta)(X_C \angle -90^\circ) = (5 \text{ A} \angle 53.13^\circ)(8\ \Omega \angle -90^\circ) = \mathbf{40\ V \angle -36.87^\circ}$$

Kirchhoff's voltage law:

$$\Sigma_{\circlearrowright} \mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_C = 0$$

or $$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_C$$

which can be verified by vector algebra as demonstrated for the *R-L* circuit.

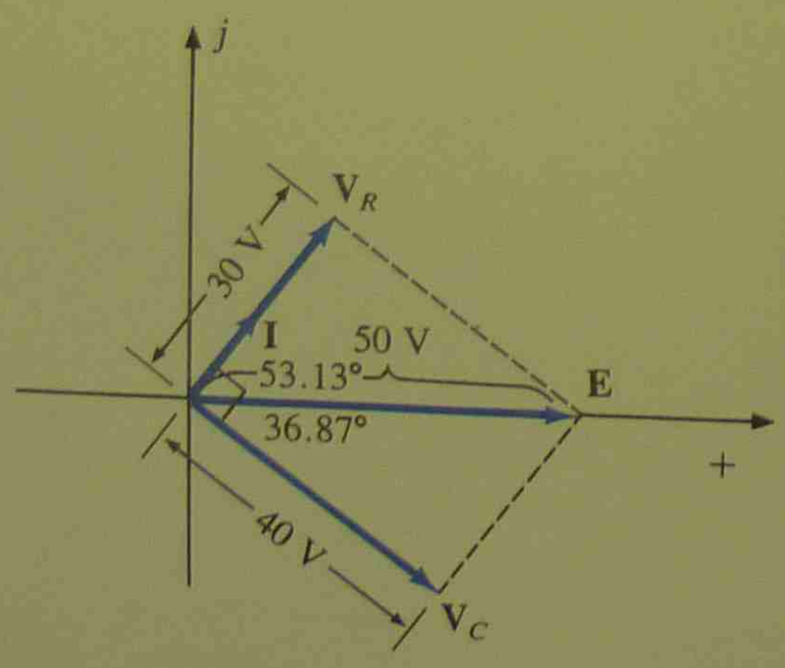


FIG. 15.32

Phasor diagram: Note on the phasor diagram of Fig. 15.32 that the current **I** is in phase with the voltage across the resistor and leads the voltage across the capacitor by 90°.

Time domain: In the time domain,

$$e = \sqrt{2}(50) \sin \omega t = \mathbf{70.70 \sin \omega t}$$

$$v_R = \sqrt{2}(30) \sin(\omega t + 53.13^\circ) = \mathbf{42.42 \sin(\omega t + 53.13^\circ)}$$

$$v_C = \sqrt{2}(40) \sin(\omega t - 36.87^\circ) = \mathbf{56.56 \sin(\omega t - 36.87^\circ)}$$

A plot of all of the voltages and the current of the circuit appears in Fig. 15.33. Note again that i and v_R are in phase and that v_C lags i by 90°.

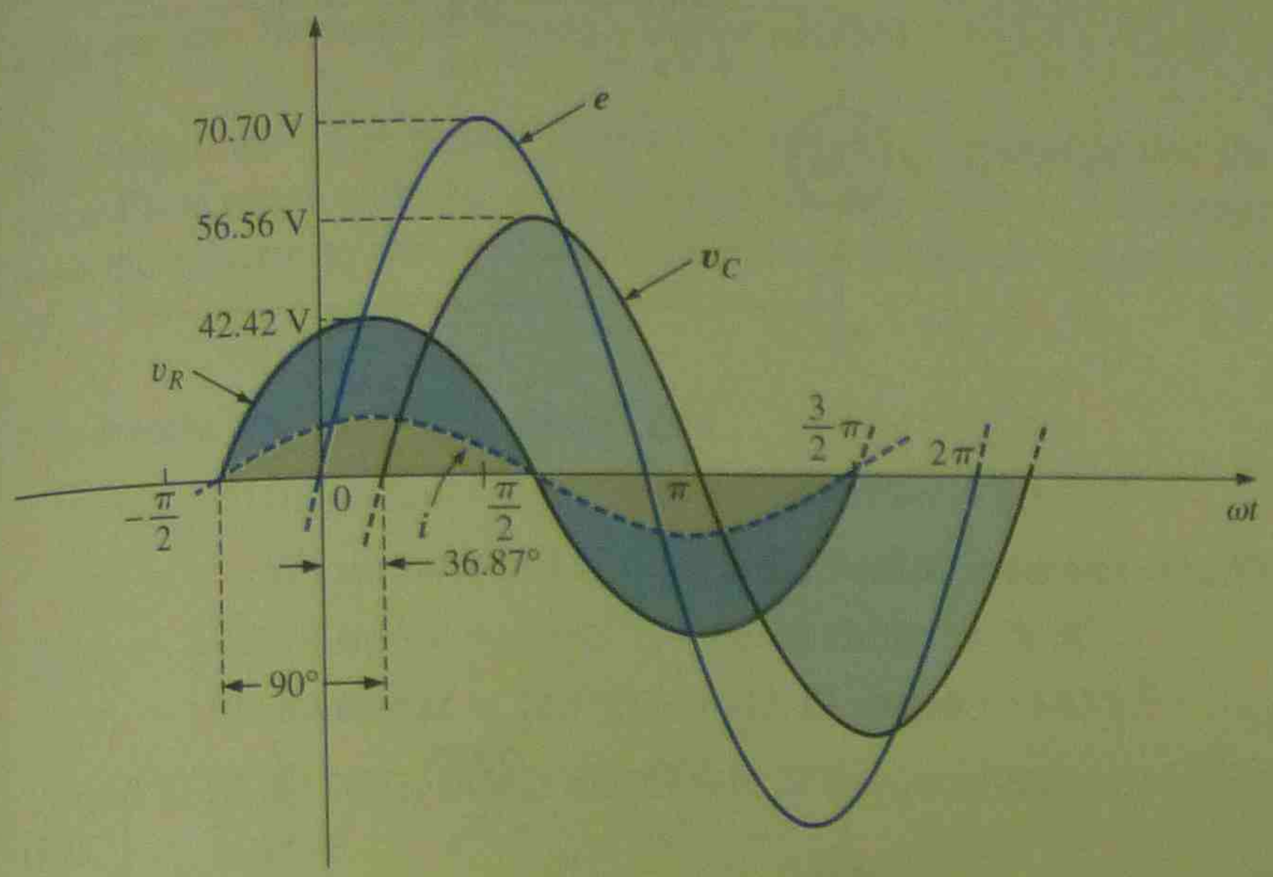


FIG. 15.33

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta_T = (50 \text{ V})(5 \text{ A}) \cos 53.13^\circ$$
$$= (250)(0.6) = \mathbf{150 \text{ W}}$$

or

$$P_T = I^2R = (5 \text{ A})^2(6\ \Omega) = (25)(6)$$
$$= \mathbf{150 \text{ W}}$$

or, finally,

$$P_T = P_R + P_C = V_R I \cos \theta_R + V_C I \cos \theta_C$$
$$= (30 \text{ V})(5 \text{ A}) \cos 0^\circ + (40 \text{ V})(5 \text{ A}) \cos 90^\circ$$
$$= 150 \text{ W} + 0$$
$$= \mathbf{150 \text{ W}}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta = \cos 53.13^\circ = \mathbf{0.6 \text{ leading}}$$

Using Eq. (15.5), we obtain

$$F_p = \cos \theta = \frac{R}{Z_T} = \frac{6\ \Omega}{10\ \Omega}$$
$$= \mathbf{0.6 \text{ leading}}$$

as determined above.

R-L-C (Fig. 15.34)

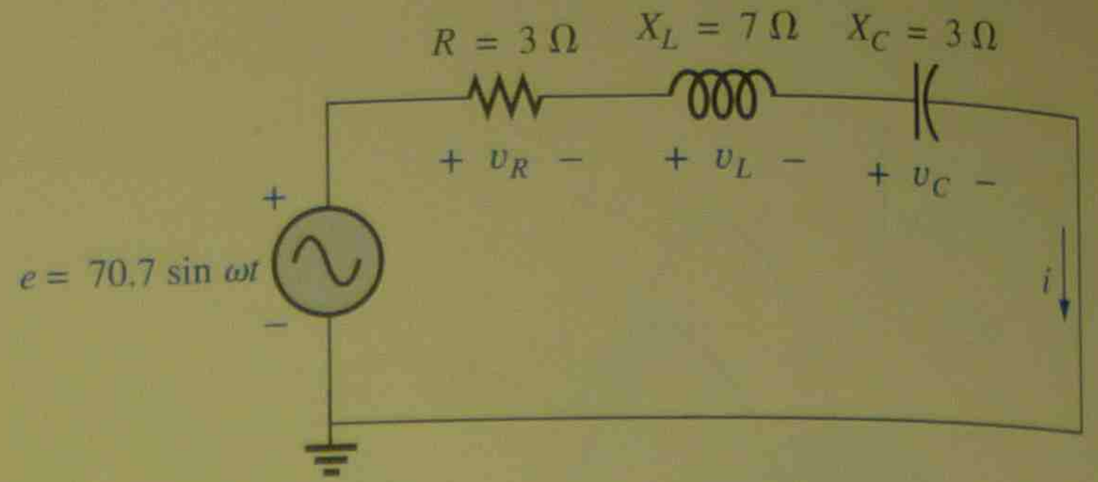


FIG. 15.34

Phasor notation: As shown in Fig. 15.35.

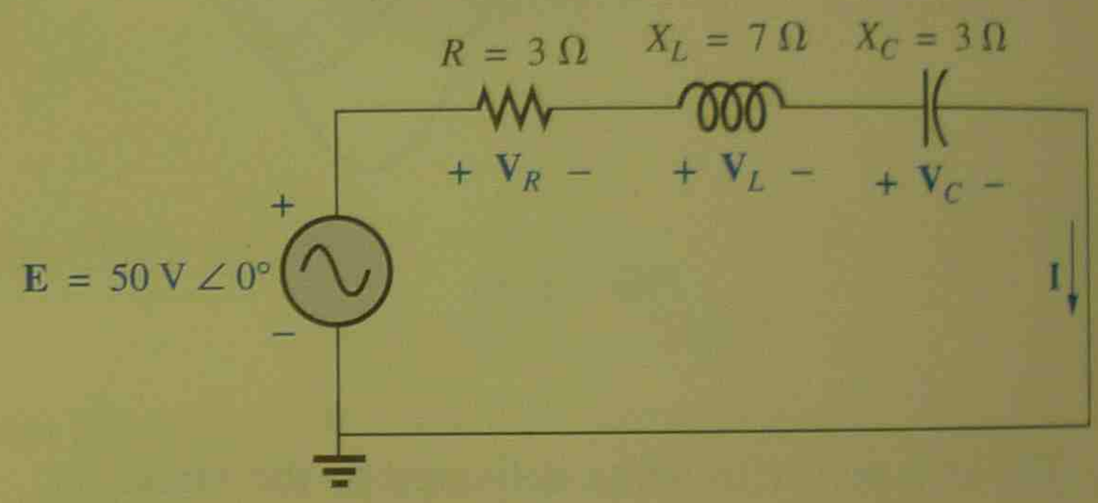


FIG. 15.35

$\mathbf{Z}_T$:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = R\angle 0^\circ + X_L\angle 90^\circ + X_C\angle -90^\circ$$
$$= 3\ \Omega + j7\ \Omega - j3\ \Omega = 3\ \Omega + j4\ \Omega$$

and $$\mathbf{Z}_T = \mathbf{5\ \Omega\ \angle 53.13^\circ}$$

Impedance diagram: As shown in Fig. 15.36.

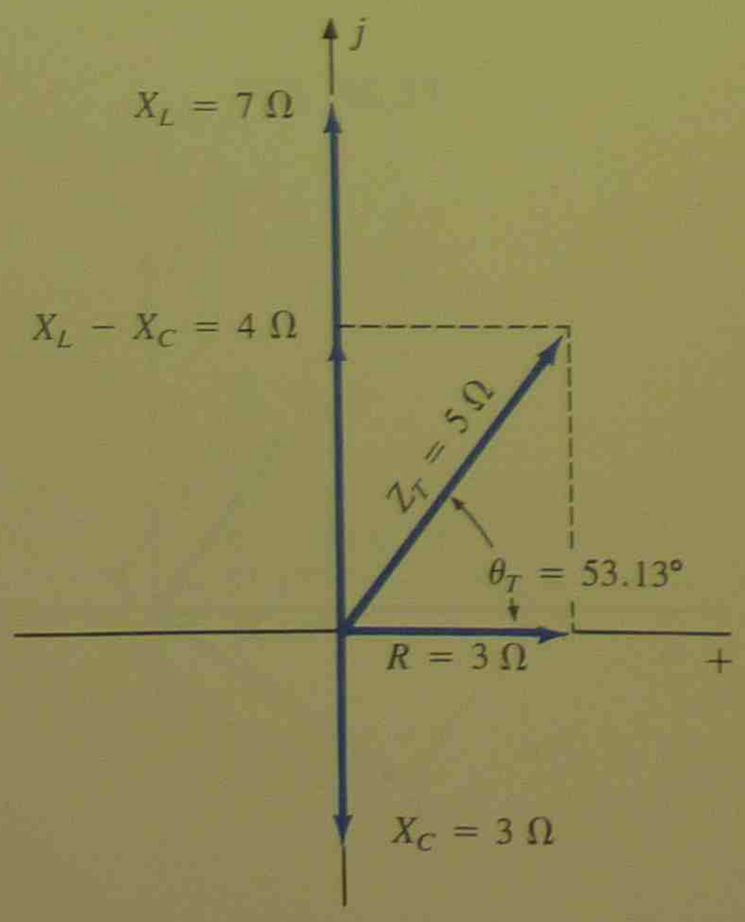


FIG. 15.36

$\mathbf{I}$:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{50\ \text{V}\ \angle 0^\circ}{5\ \Omega\ \angle 53.13^\circ} = \mathbf{10\ A\ \angle -53.13^\circ}$$

$\mathbf{V}_R$, $\mathbf{V}_L$, $\mathbf{V}_C$:

$$\mathbf{V}_R = (I\angle\theta)(R\angle 0^\circ) = (10\ \text{A}\ \angle -53.13^\circ)(3\ \Omega\ \angle 0^\circ)$$
$$= \mathbf{30\ V\ \angle -53.13^\circ}$$
$$\mathbf{V}_L = (I\angle\theta)(X_L\angle 90^\circ) = (10\ \text{A}\ \angle -53.13^\circ)(7\ \Omega\ \angle 90^\circ)$$
$$= \mathbf{70\ V\ \angle 36.87^\circ}$$
$$\mathbf{V}_C = (I\angle\theta)(X_C\angle -90^\circ) = (10\ \text{A}\ \angle -53.13^\circ)(3\ \Omega\ \angle -90^\circ)$$
$$= \mathbf{30\ V\ \angle -143.13^\circ}$$

Kirchhoff's voltage law:

$$\Sigma_{\circlearrowright}\mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_L - \mathbf{V}_C = 0$$

or

$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

which can also be verified through vector algebra.

Phasor diagram: The phasor diagram of Fig. 15.37 indicates that the current **I** is in phase with the voltage across the resistor, lags the voltage across the inductor by 90°, and leads the voltage across the capacitor by 90°.

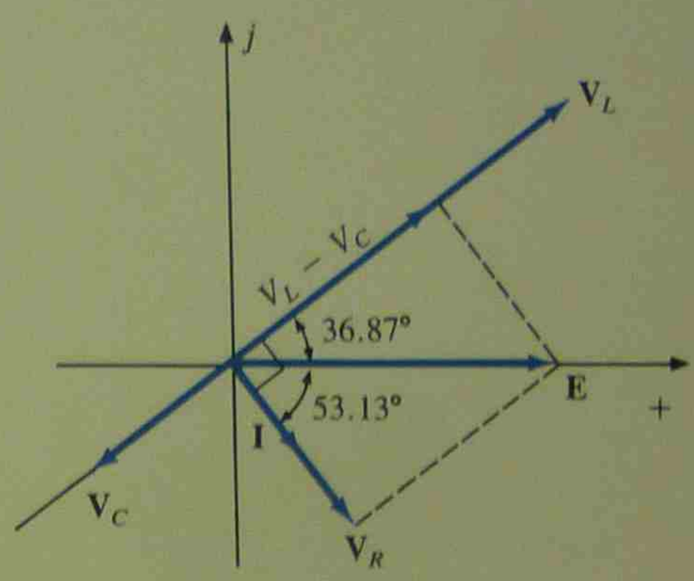


FIG. 15.37

Time domain:

$$i = \sqrt{2}(10)\sin(\omega t - 53.13°) = \mathbf{14.14\sin(\omega t - 53.13°)}$$

$$v_R = \sqrt{2}(30)\sin(\omega t - 53.13°) = \mathbf{42.42\sin(\omega t - 53.13°)}$$

$$v_L = \sqrt{2}(70)\sin(\omega t + 36.87°) = \mathbf{98.98\sin(\omega t + 36.87°)}$$

$$v_C = \sqrt{2}(30)\sin(\omega t - 143.13°) = \mathbf{42.42\sin(\omega t - 143.13°)}$$

A plot of all the voltages and the current of the circuit appears in Fig. 15.38.

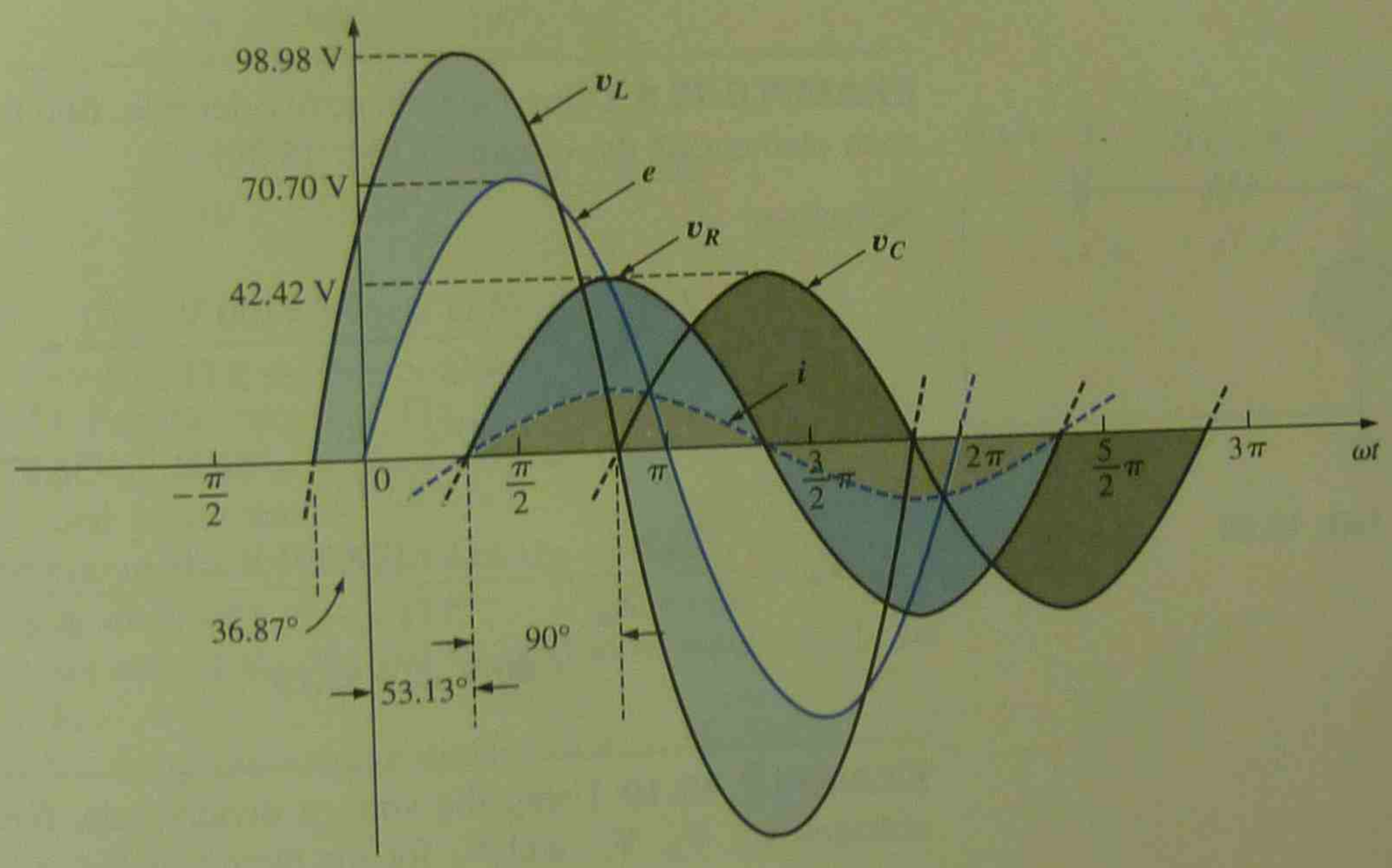


FIG. 15.38

Power: The total power in watts delivered to the circuit is

$$P_T = EI\cos\theta_T = (50\text{ V})(10\text{ A})\cos 53.13° = (500)(0.6) = \mathbf{300\text{ W}}$$

or

$$P_T = I^2R = (10\text{ A})^2(3\ \Omega) = (100)(3) = \mathbf{300\text{ W}}$$

or

$$P_T = P_R + P_L + P_C$$
$$= V_R I\cos\theta_R + V_L I\cos\theta_L + V_C I\cos\theta_C$$
$$= (30\text{ V})(10\text{ A})\cos 0° + (70\text{ V})(10\text{ A})\cos 90° + (30\text{ V})(10\text{ A})\cos 90°$$
$$= (30\text{ V})(10\text{ A}) + 0 + 0 = \mathbf{300\text{ W}}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta_T = \cos 53.13^\circ = \mathbf{0.6\ lagging}$$

Using Eq. (15.5), we obtain

$$F_p = \cos \theta = \frac{R}{Z_T} = \frac{3\ \Omega}{5\ \Omega} = \mathbf{0.6\ lagging}$$

15.4 VOLTAGE DIVIDER RULE

The basic format for the voltage divider rule in ac circuits is exactly the same as that for dc circuits:

$$\boxed{\mathbf{V}_x = \frac{\mathbf{Z}_x\mathbf{E}}{\mathbf{Z}_T}} \tag{15.6}$$

where $\mathbf{V}_x$ is the voltage across one or more elements in series that have total impedance $\mathbf{Z}_x$, $\mathbf{E}$ is the total voltage appearing across the series circuit, and $\mathbf{Z}_T$ is the total impedance of the series circuit.

EXAMPLE 15.9 Using the voltage divider rule, find the voltage across each element of the circuit of Fig. 15.39.

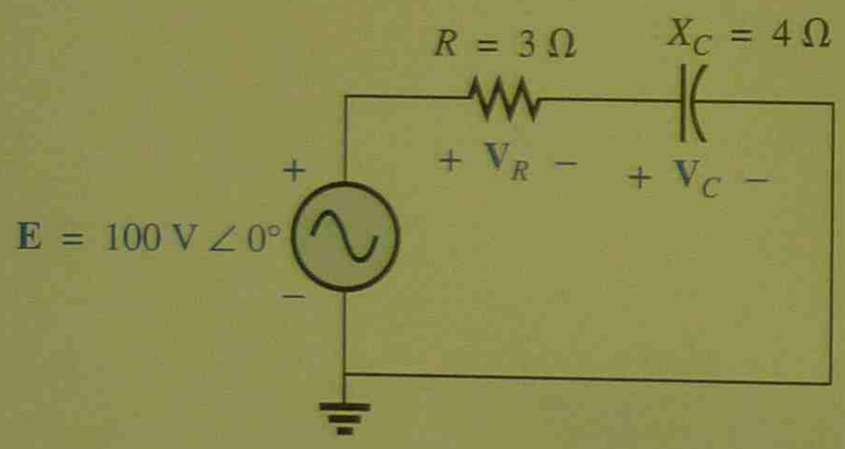


FIG. 15.39

Solution:

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(4\ \Omega \angle -90^\circ)(100\ \text{V} \angle 0^\circ)}{4\ \Omega \angle -90^\circ + 3\ \Omega \angle 0^\circ} = \frac{400 \angle -90^\circ}{3 - j4}$$

$$= \frac{400 \angle -90^\circ}{5 \angle -53.13^\circ} = \mathbf{80\ V \angle -36.87^\circ}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(3\ \Omega \angle 0^\circ)(100\ \text{V} \angle 0^\circ)}{5\ \Omega \angle -53.13^\circ} = \frac{300 \angle 0^\circ}{5 \angle -53.13^\circ}$$

$$= \mathbf{60\ V \angle +53.13^\circ}$$

EXAMPLE 15.10 Using the voltage divider rule, find the unknown voltages $\mathbf{V}_R$, $\mathbf{V}_L$, $\mathbf{V}_C$, and $\mathbf{V}_1$ for the circuit of Fig. 15.40.

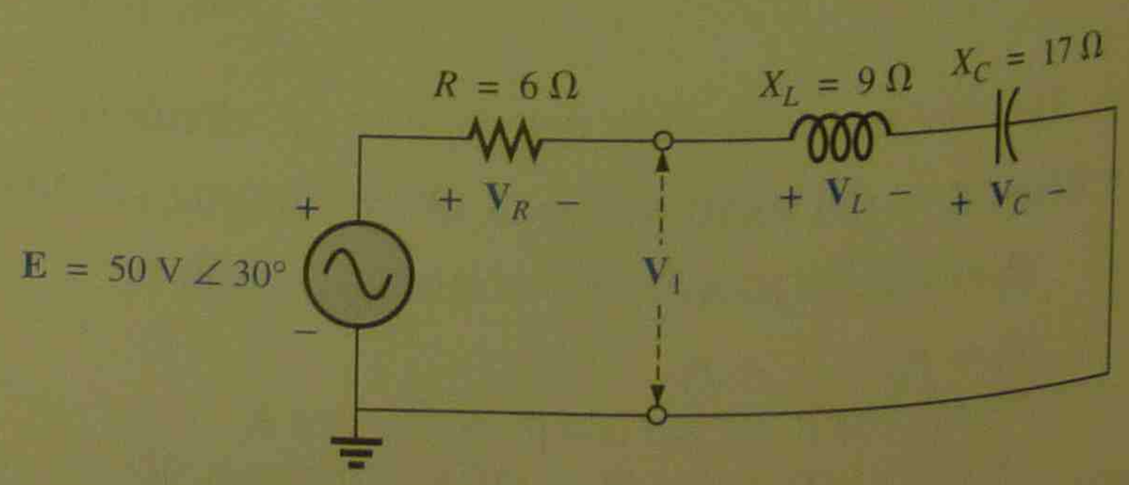


FIG. 15.40

Solution:

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = \frac{(6\ \Omega\ \angle 0°)(50\ \text{V}\ \angle 30°)}{6\ \Omega\ \angle 0° + 9\ \Omega\ \angle 90° + 17\ \Omega\ \angle -90°}$$

$$= \frac{300\ \angle 30°}{6 + j9 - j17} = \frac{300\ \angle 30°}{6 - j8}$$

$$= \frac{300\ \angle 30°}{10\ \angle -53.13°} = \mathbf{30\ V\ \angle 83.13°}$$

$$\mathbf{V}_L = \frac{\mathbf{Z}_L\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega\ \angle 90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°} = \frac{450\ \angle 120°}{10\ \angle -53.13°}$$

$$= \mathbf{45\ V\ \angle 173.13°}$$

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_T} = \frac{(17\ \Omega\ \angle -90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°} = \frac{850\ \angle -60°}{10\ \angle -53°}$$

$$= \mathbf{85\ V\ \angle -6.87°}$$

$$\mathbf{V}_1 = \frac{(\mathbf{Z}_L + \mathbf{Z}_C)\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega\ \angle 90° + 17\ \Omega\ \angle -90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°}$$

$$= \frac{(8\ \angle -90°)(50\ \angle 30°)}{10\ \angle -53.13°}$$

$$= \frac{400\ \angle -60°}{10\ \angle -53.13°} = \mathbf{40\ V\ \angle -6.87°}$$

EXAMPLE 15.11 For the circuit of Fig. 15.41:

a. Calculate $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ in phasor form.
b. Calculate the total power factor.
c. Calculate the average power delivered to the circuit.
d. Draw the phasor diagram.
e. Obtain the phasor sum of $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$, and show that it equals the input voltage $\mathbf{E}$.
f. Find $\mathbf{V}_R$ and $\mathbf{V}_C$ using the voltage divider rule.

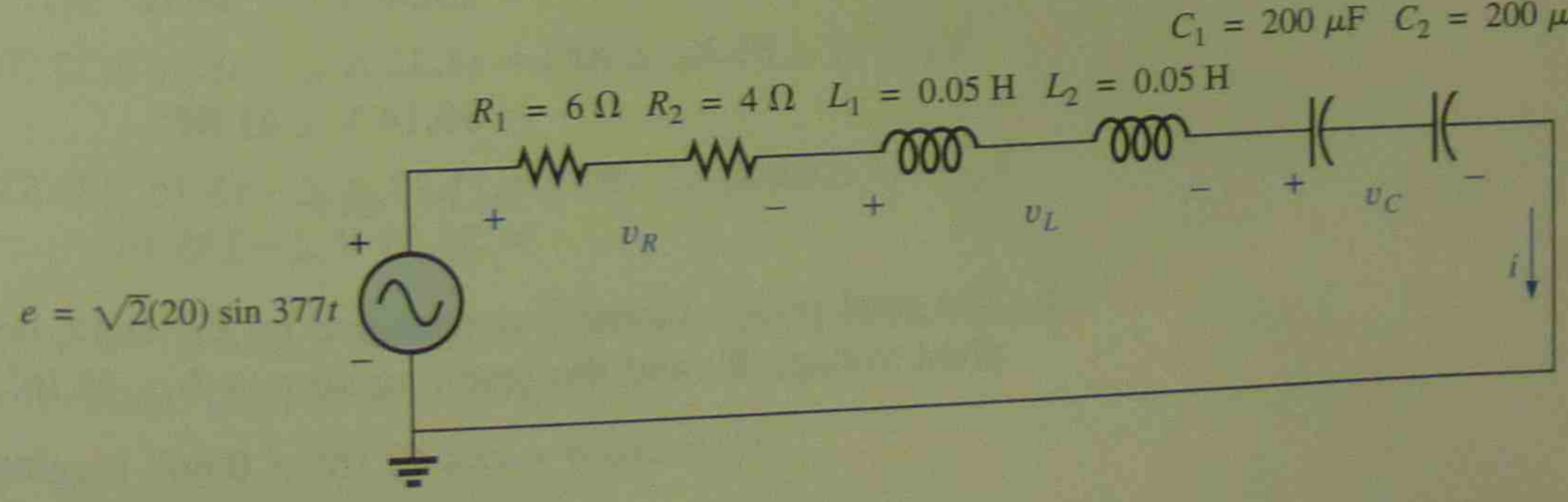


FIG. 15.41

Solutions:

a. Combining common elements and finding the reactance of the inductor and capacitor, we obtain

$$R_T = 6\,\Omega + 4\,\Omega = 10\,\Omega$$

$$L_T = 0.05\text{ H} + 0.05\text{ H} = 0.1\text{ H}$$

$$C_T = \frac{200\,\mu\text{F}}{2} = 100\,\mu\text{F}$$

$$X_L = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.70\,\Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(377\text{ rad/s})(100 \times 10^{-6}\text{ F})} = \frac{10^6\,\Omega}{37{,}700} = 26.53\,\Omega$$

Redrawing the circuit using phasor notation results in Fig. 15.42.

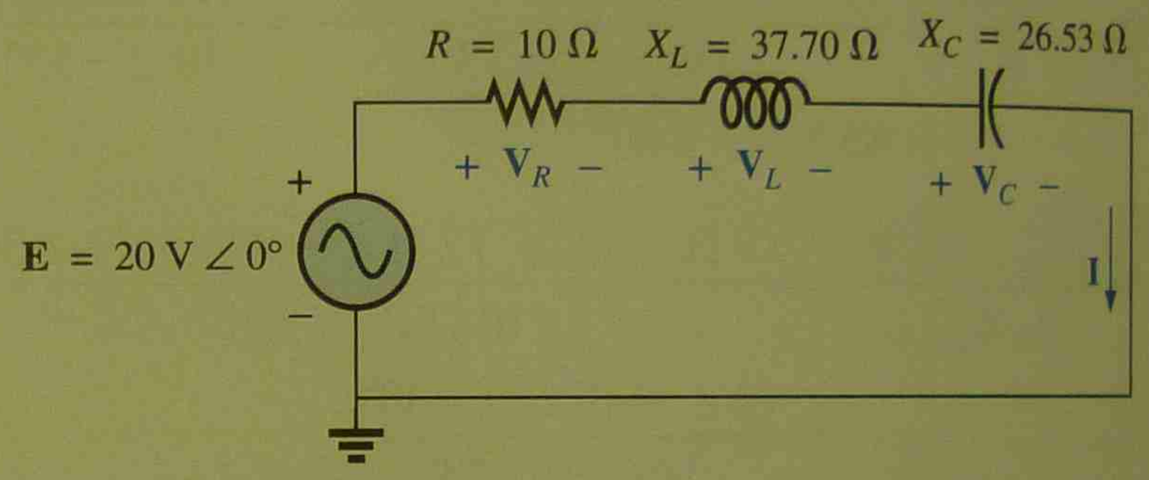


FIG. 15.42

For the circuit of Fig. 15.42,

$$\mathbf{Z}_T = R\angle 0^\circ + X_L\angle 90^\circ + X_C\angle -90^\circ$$
$$= 10\,\Omega + j\,37.70\,\Omega - j\,26.53\,\Omega$$
$$= 10\,\Omega + j\,11.17\,\Omega = \mathbf{15\,\Omega\angle 48.16^\circ}$$

The current **I** is

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{20\text{ V}\angle 0^\circ}{15\,\Omega\angle 48.16^\circ} = \mathbf{1.33\text{ A}\angle -48.16^\circ}$$

The voltage across the resistor, inductor, and capacitor can be found using Ohm's law:

$$\mathbf{V}_R = (I\angle\theta)(R\angle 0^\circ) = (1.33\text{ A}\angle -48.16^\circ)(10\,\Omega\angle 0^\circ)$$
$$= \mathbf{13.30\text{ V}\angle -48.16^\circ}$$

$$\mathbf{V}_L = (I\angle\theta)(X_L\angle 90^\circ) = (1.33\text{ A}\angle -48.16^\circ)(37.70\,\Omega\angle 90^\circ)$$
$$= \mathbf{50.14\text{ V}\angle 41.84^\circ}$$

$$\mathbf{V}_C = (I\angle\theta)(X_C\angle -90^\circ) = (1.33\text{ A}\angle -48.16^\circ)(26.53\,\Omega\angle -90^\circ)$$
$$= \mathbf{35.28\text{ V}\angle -138.16^\circ}$$

b. The total power factor is determined by the angle between the applied voltage **E**, and the resulting current **I** is 48.16°:

$$F_p = \cos\theta = \cos 48.16^\circ = \mathbf{0.667\text{ lagging}}$$

or

$$F_p = \cos\theta = \frac{R}{Z_T} = \frac{10\,\Omega}{15\,\Omega} = \mathbf{0.667\text{ lagging}}$$

c. The total power in watts delivered to the circuit is

$$P_T = EI\cos\theta = (20\text{ V})(1.33\text{ A})(0.667) = \mathbf{17.74\text{ W}}$$

d. The phasor diagram appears in Fig. 15.43.

e. The phasor sum of $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ is

$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$
$$= 13.30\text{ V}\angle -48.16^\circ + 50.14\text{ V}\angle 41.84^\circ + 35.28\text{ V}\angle -138.16^\circ$$
$$\mathbf{E} = 13.30\text{ V}\angle -48.16^\circ + 14.86\text{ V}\angle 41.84^\circ$$

Therefore,

$$E = \sqrt{(13.30\text{ V})^2 + (14.86\text{ V})^2} = \mathbf{20\text{ V}}$$

and $\theta_E = \mathbf{0^\circ}$ (from phasor diagram)

and $\mathbf{E} = 20\angle 0^\circ$

f. $$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_T} = \frac{(10\ \Omega\angle 0^\circ)(20\text{ V}\angle 0^\circ)}{15\ \Omega\angle 48.16^\circ} = \frac{200\angle 0^\circ}{15\angle 48.16^\circ}$$
$$= \mathbf{13.3\text{ V}\angle -48.16^\circ}$$

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_T} = \frac{(26.5\ \Omega\angle -90^\circ)(20\text{ V}\angle 0^\circ)}{15\ \Omega\angle 48.16^\circ} = \frac{530.6\angle -90^\circ}{15\angle 48.16^\circ}$$
$$= \mathbf{35.37\text{ V}\angle -138.16^\circ}$$

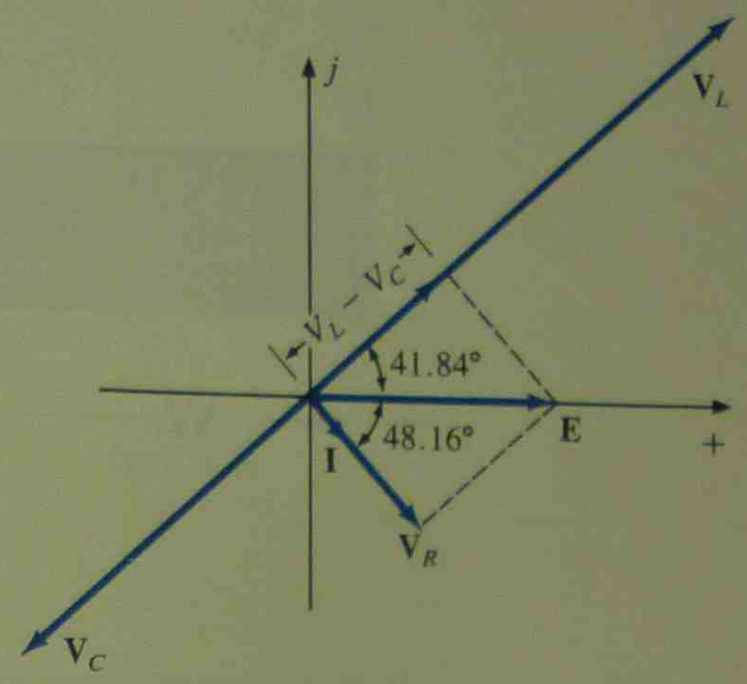


FIG. 15.43

15.5 FREQUENCY RESPONSE OF THE *R-C* CIRCUIT

Thus far, the analysis of series circuits has been limited to a particular frequency. We will now examine the effect of frequency on the response of an *R-C* series configuration such as that in Fig. 15.44.

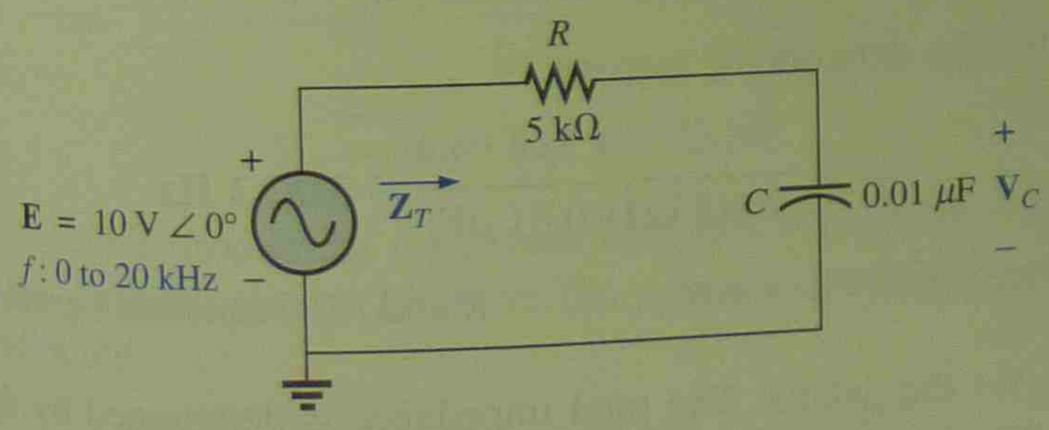


FIG. 15.44

The magnitude of the source is fixed at 10 V but the frequency range of analysis will extend from zero to 20 kHz.

$\mathbf{Z}_T$:

Let us first determine how the impedance of the circuit $\mathbf{Z}_T$ will vary with frequency for the specified frequency range of interest. Before getting

into specifics, however, let us first develop a sense for what we should expect by noting the impedance versus frequency curve of each element as drawn in Fig. 15.45.

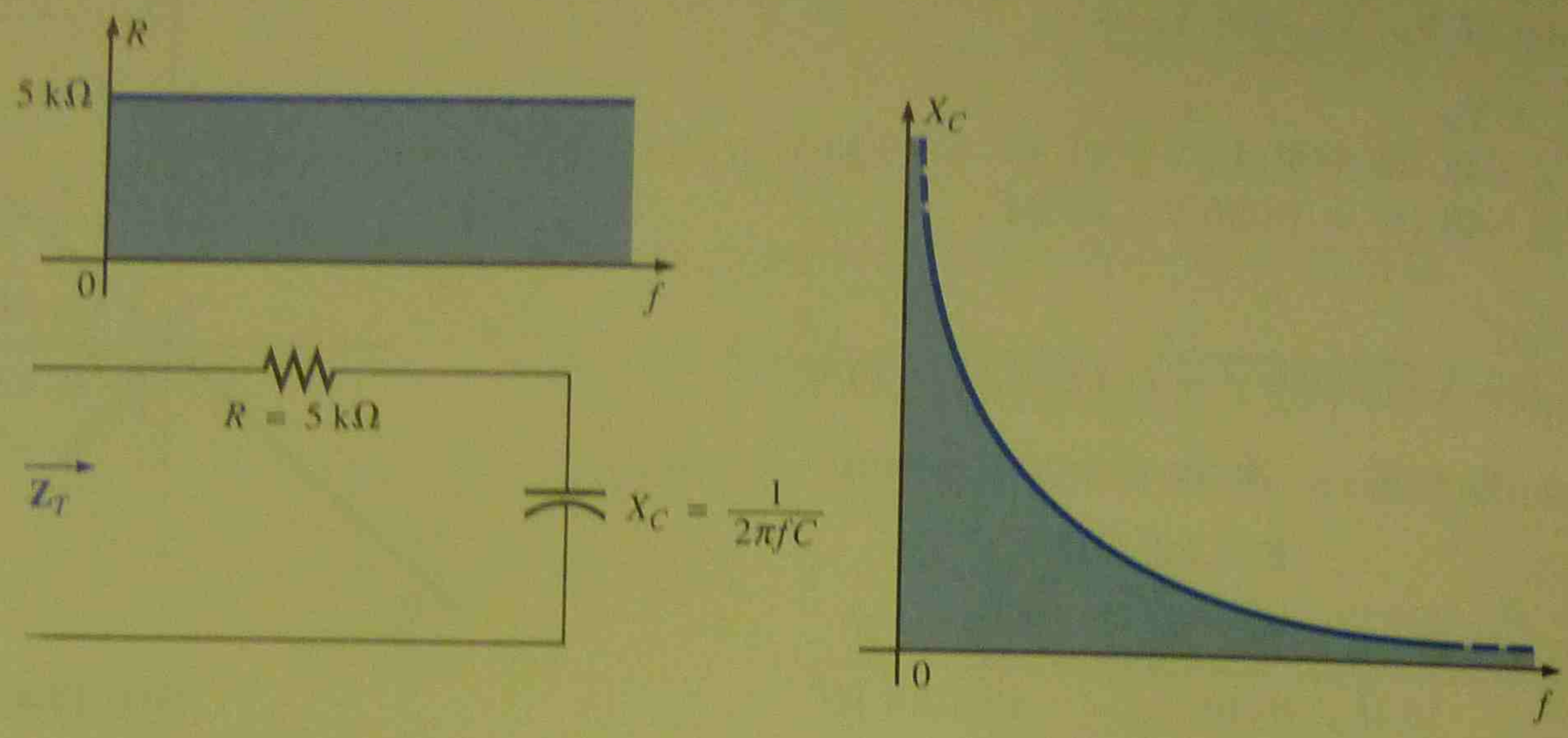


FIG. 15.45

At low frequencies the reactance of the capacitor will be quite high and considerably more than the level of the resistance R, suggesting that the total impedance will be primarily capacitive in nature. At high frequencies the reactance X_C will drop below the $R = 5\text{-k}\Omega$ level and the network will start to shift toward one of a purely resistive nature (at 5 kΩ). The frequency at which $X_C = R$ can be determined in the following manner:

$$X_C = \frac{1}{2\pi f_1 C} = R$$

and

$$\boxed{f_1 = \frac{1}{2\pi RC}} \qquad X_C = R \tag{15.7}$$

which for the network of interest is

$$f_1 = \frac{1}{2\pi(5\text{ k}\Omega)(0.01\ \mu\text{F})} \cong \mathbf{3183.1\ Hz}$$

For frequencies less than f_1, $X_C > R$ and for frequencies greater than f_1, $R > X_C$.

Now for the details. The total impedance is determined by the following equation:

$$\mathbf{Z}_T = R - jX_C$$

and

$$\boxed{\mathbf{Z}_T = Z_T \angle \theta_T = \sqrt{R^2 + X_C^2} \angle -\tan^{-1}\frac{X_C}{R}} \tag{15.8}$$

The magnitude and angle of the total impedance can now be found at any frequency of interest by simply substituting into Eq. (15.8). The

presence of the capacitor suggests that we start from a low frequency (100 Hz) and then open the spacing until we reach the upper limit of interest (20 kHz).

$f = 100$ Hz:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(100\text{ Hz})(0.01\ \mu\text{F})} = 159.16\text{ k}\Omega$$

and

$$Z_T = \sqrt{R^2 + X_C^2} = \sqrt{(5\text{ k}\Omega)^2 + (159.16\text{ k}\Omega)^2} = 159.24\text{ k}\Omega$$

with

$$\theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{159.16\text{ k}\Omega}{5\text{ k}\Omega} = -\tan^{-1}31.83 = -88.2°$$

and $$\mathbf{Z_T = 159.24\text{ k}\Omega\ \angle -88.2°}$$

which compares very closely with $Z_C = 159.16\text{ k}\Omega\ \angle -90°$ if the circuit were purely capacitive ($R = 0\ \Omega$). Our assumption that the circuit is primarily capacitive at low frequencies is therefore confirmed.

$f = 1$ kHz:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(1\text{ kHz})(0.01\ \mu\text{F})} = 15.92\text{ k}\Omega$$

and

$$Z_T = \sqrt{R^2 + X_C^2} = \sqrt{(5\text{ k}\Omega)^2 + (15.92\text{ k}\Omega)^2} = 16.69\text{ k}\Omega$$

with

$$\theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{15.92\text{ k}\Omega}{5\text{ k}\Omega} = -\tan^{-1}3.18 = -72.54°$$

and $$\mathbf{Z_T = 16.69\text{ k}\Omega\ \angle -72.54°}$$

A noticeable drop in the magnitude has occurred and the impedance angle has dropped almost 17° from the purely capacitive level.

Continuing:

$f = 5$ kHz: $\mathbf{Z_T = 5.93\text{ k}\Omega\ \angle -32.48°}$

$f = 10$ kHz: $\mathbf{Z_T = 5.25\text{ k}\Omega\ \angle -17.66°}$

$f = 15$ kHz: $\mathbf{Z_T = 5.11\text{ k}\Omega\ \angle -11.98°}$

$f = 20$ kHz: $\mathbf{Z_T = 5.06\text{ k}\Omega\ \angle -9.04°}$

Note how close the magnitude of Z_T at $f = 20$ kHz is to the resistance level of 5 kΩ. In addition, note how the phase angle is approaching that associated with a pure resistive network (0°).

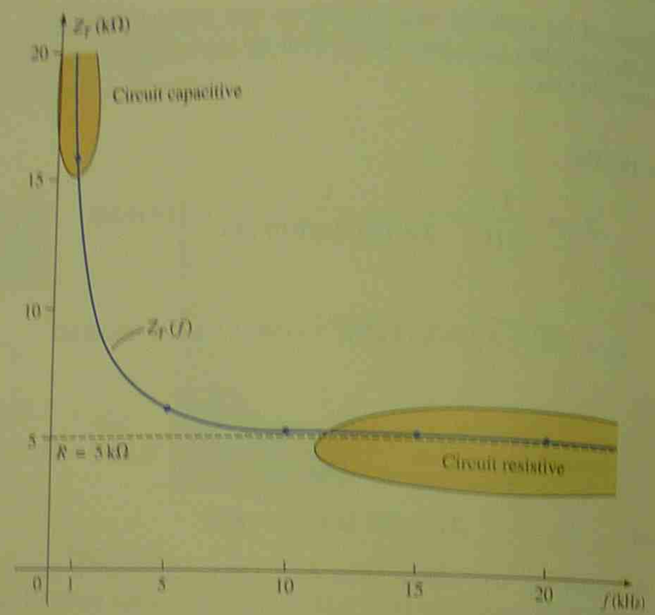


FIG. 15.46

A plot of Z_T versus frequency in Fig. 15.46 completely supports our assumption based on the curves of Fig. 15.45. The plot of θ_T versus frequency in Fig. 15.47 further suggests the fact that the total impedance made a transition from one of a capacitive nature ($\theta_T = -90°$) to one with resistive characteristics ($\theta_T = 0°$).

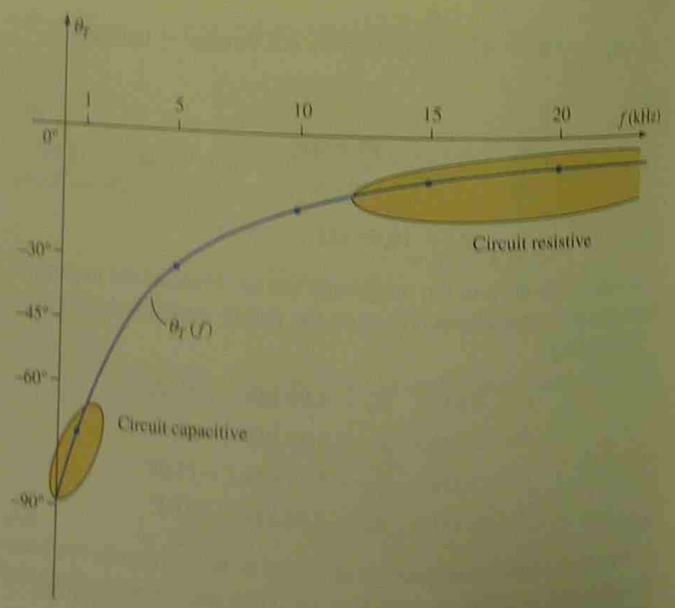


FIG. 15.47

Applying the voltage divider rule to determine the voltage across the capacitor in phasor form yields

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{(X_C \angle -90°)(E \angle 0°)}{R - jX_C} = \frac{X_C E \angle -90°}{R - jX_C} = \frac{X_C E \angle -90°}{\sqrt{R^2 + X_C^2}\angle -\tan^{-1} X_C/R}$$

or $$\mathbf{V}_C = V_C \angle \theta_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} \angle -90° + \tan^{-1}(X_C/R)$$

The magnitude of $\mathbf{V}_C$ is therefore determined by

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} \tag{15.9}$$

and the phase angle θ_C by which $\mathbf{V}_C$ leads $\mathbf{E}$ is given by

$$\theta_C = -90° + \tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{R}{X_C} \tag{15.10}$$

To determine the frequency response, X_C must be calculated for each frequency of interest and inserted into Eqs. (15.9) and (15.10).

To begin our analysis, it makes good sense to consider the case of $f = 0$ Hz (dc conditions).

$f = 0$ Hz:

$$X_C = \frac{1}{2\pi(0)C} = \frac{1}{0} \Rightarrow \text{very large value}$$

Applying the open-circuit equivalent for the capacitor based on the above calculation will result in the following:

$$\mathbf{V}_C = \mathbf{E} = 10\ \text{V} \angle 0°$$

If we apply Eq. (15.9), we find

$$X_C^2 \gg R^2$$

and

$$\sqrt{R^2 + X_C^2} \cong \sqrt{X_C^2} = X_C$$

and

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{X_C E}{X_C} = E$$

with

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1} 0 = 0°$$

verifying the above conclusions.

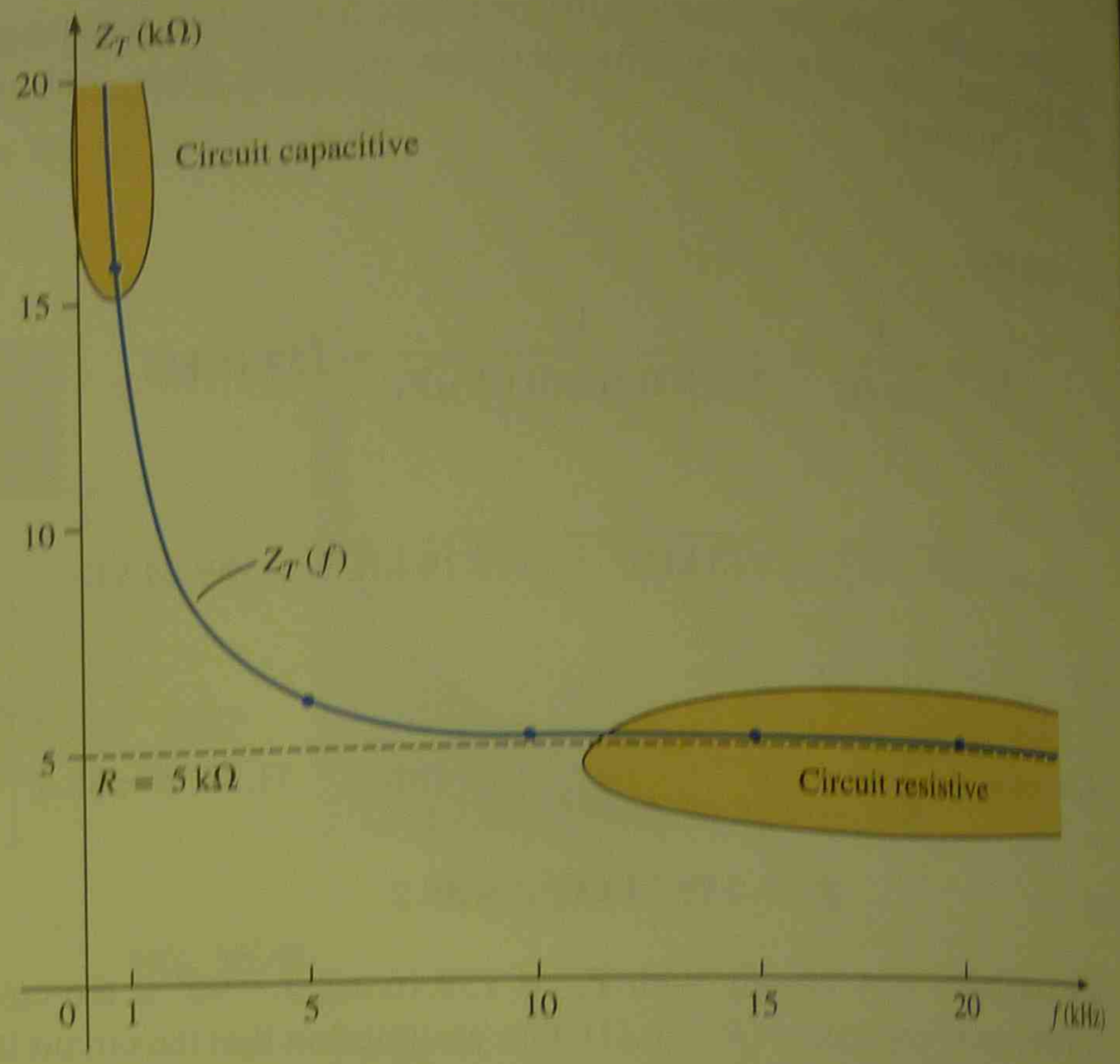


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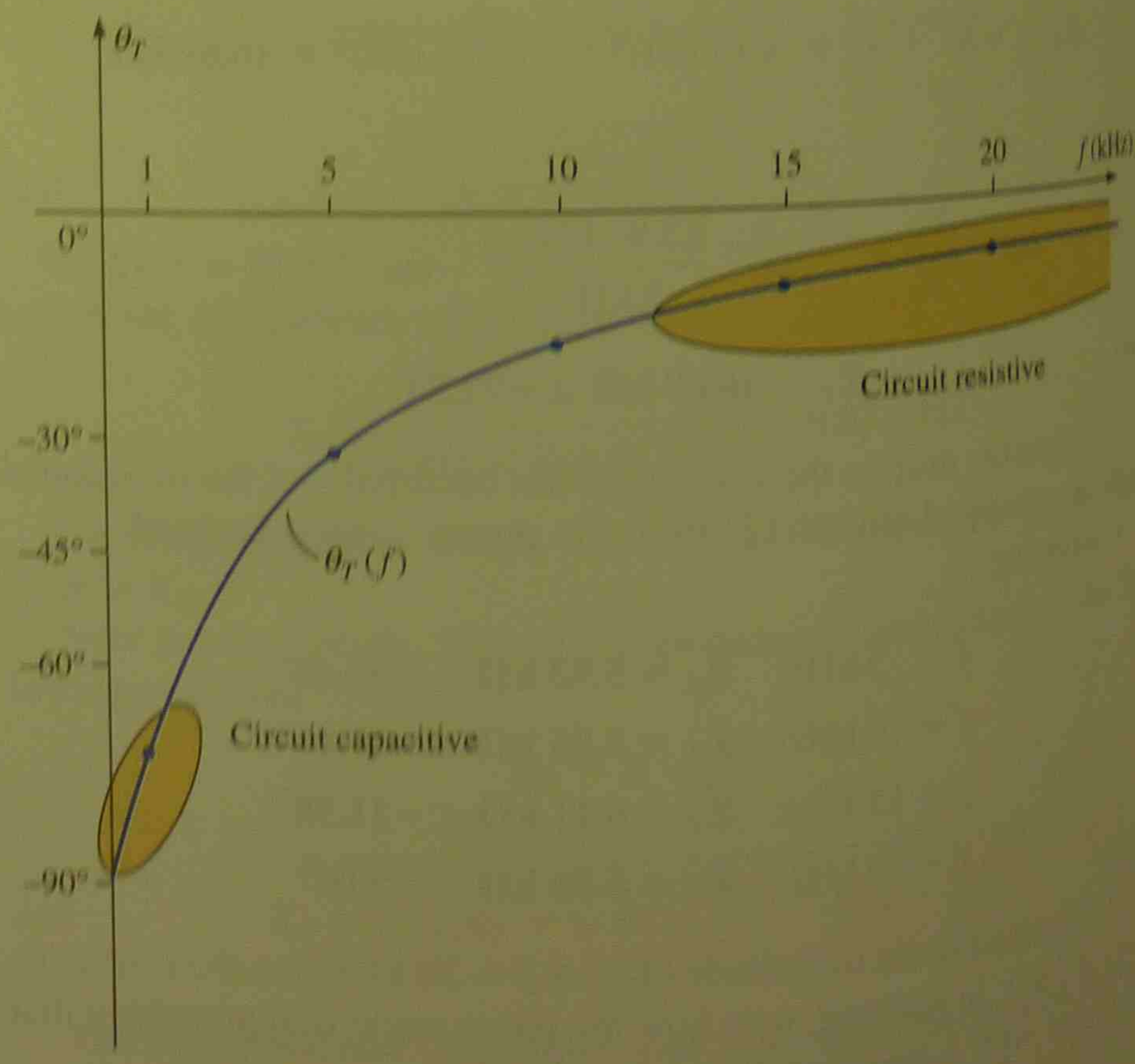


FIG. 15.47

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$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_C}$$

$$= \frac{(X_C \angle -90°)(E \angle 0°)}{R - jX_C} = \frac{X_C E \angle -90°}{R - jX_C}$$

$$= \frac{X_C E \angle -90°}{\sqrt{R^2 + X_C^2}\,\underline{/-\tan^{-1} X_C/R}}$$

$$= \frac{X_C E}{\sqrt{R^2 + X_C^2}}\,\underline{/-90° + \tan^{-1}(X_C/R)}$$

or $$\mathbf{V}_C = V_C \angle \theta_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}}\,\underline{/-90° + \tan^{-1}(X_C/R)}$$

The magnitude of $\mathbf{V}_C$ is therefore determined by

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} \tag{15.9}$$

and the phase angle θ_C by which $\mathbf{V}_C$ leads $\mathbf{E}$ is given by

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To determine the frequency response, X_C must be calculated for each frequency of interest and inserted into Eqs. (15.9) and (15.10).

To begin our analysis, it makes good sense to consider the case of $f = 0$ Hz (dc conditions).

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Applying the open-circuit equivalent for the capacitor based on the above calculation will result in the following:

$$\mathbf{V}_C = \mathbf{E} = 10\ \mathrm{V} \angle 0°$$

If we apply Eq. (15.9), we find

$$X_C^2 \gg R^2$$

and $$\sqrt{R^2 + X_C^2} \cong \sqrt{X_C^2} = X_C$$

and $$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{X_C E}{X_C} = E$$

with $$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1} 0$$

$$= 0°$$

verifying the above conclusions.

$f = 1$ kHz:

Applying Eq. (15.9):

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(6.28)(1 \times 10^3\ \text{Hz})(0.01 \times 10^{-6}\ \text{F})} \cong \mathbf{15.92\ k\Omega}$$

$$\sqrt{R^2 + X_C^2} = \sqrt{(5\ \text{k}\Omega)^2 + (15.92\ \text{k}\Omega)^2} \cong 16.69\ \text{k}\Omega$$

and

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{(15.92\ \text{k}\Omega)(10)}{16.69\ \text{k}\Omega} = \mathbf{9.54\ V}$$

Applying Eq. (15.10):

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1}\frac{5\ \text{k}\Omega}{15.9\ \text{k}\Omega}$$

$$= -\tan^{-1} 0.314$$

$$= \mathbf{-17.46^\circ}$$

and

$$\mathbf{V}_C = \mathbf{9.53\ V} \angle \mathbf{-17.46^\circ}$$

As expected, the high reactance of the capacitor at low frequencies has resulted in the major part of the applied voltage appearing across the capacitor.

If we plot the phasor diagrams for $f = 0$ Hz and $f = 1$ kHz, as shown in Fig. 15.48, we find that $\mathbf{V}_C$ is beginning a clockwise rotation with increase in frequency that will increase the angle θ_C and decrease the phase angle between **I** and **E.** Recall that for a purely capacitive network, **I** leads **E** by 90°. As the frequency increases, therefore, the capacitive reactance is decreasing, and eventually $R \gg X_C$ with $\theta_C = -90°$, and the angle between **I** and **E** will approach 0°. Keep in mind as we proceed through the other frequencies that θ_C is the phase angle between $\mathbf{V}_C$ and **E** and that the magnitude of the angle by which **I** leads **E** is determined by

$$\boxed{|\theta_I| = 90^\circ - |\theta_C|} \qquad (15.11)$$

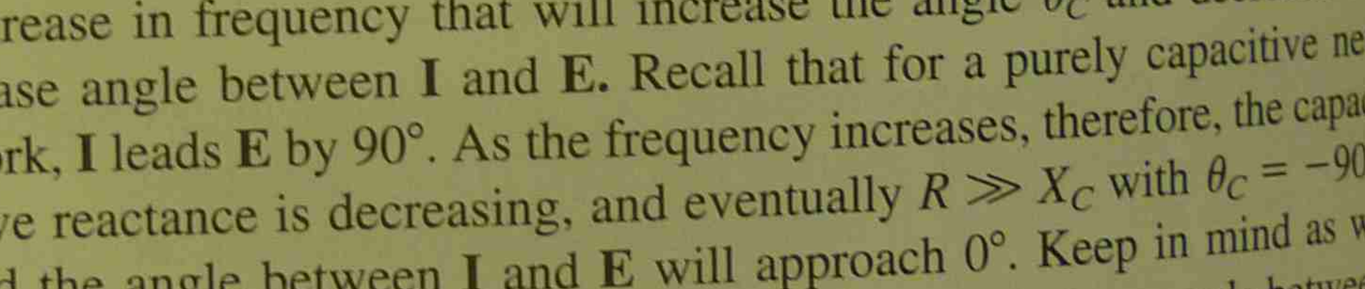


FIG. 15.48

$f = 5$ kHz:

Applying Eq. (15.9):

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(6.28)(5 \times 10^3\ \text{Hz})(0.01 \times 10^{-6}\ \text{F})} \cong \mathbf{3.18\ k\Omega}$$

Note the dramatic drop in X_C from 1 kHz to 5 kHz. In fact, X_C is now less than the resistance R of the network, and the phase angle determined by $\tan^{-1}(X_C/R)$ must be less than 45°. Here,

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{(3.18\,\text{k}\Omega)(10\,\text{V})}{\sqrt{(5\,\text{k}\Omega)^2 + (3.18\,\text{k}\Omega)^2}} = \mathbf{5.37\ V}$$

with

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1}\frac{5\,\text{k}\Omega}{3.2\,\text{k}\Omega}$$
$$= -\tan^{-1} 1.56$$
$$= \mathbf{-57.38^\circ}$$

f = 10 kHz:

$$X_C \cong \mathbf{1.59\ k\Omega},\ V_C = \mathbf{3.03\ V},\ \theta_C = \mathbf{-72.34^\circ}$$

f = 15 kHz:

$$X_C \cong \mathbf{1.06\ k\Omega},\ V_C = \mathbf{2.07\ V},\ \theta_C = \mathbf{-78.02^\circ}$$

f = 20 kHz:

$$X_C \cong \mathbf{795.78\ \Omega},\ V_C = \mathbf{1.57\ V},\ \theta_C = \mathbf{-80.96^\circ}$$

The phasor diagrams for $f = 5$ kHz and $f = 20$ kHz appear in Fig. 15.49 to show the continuing rotation of the $\mathbf{V}_C$ vector.

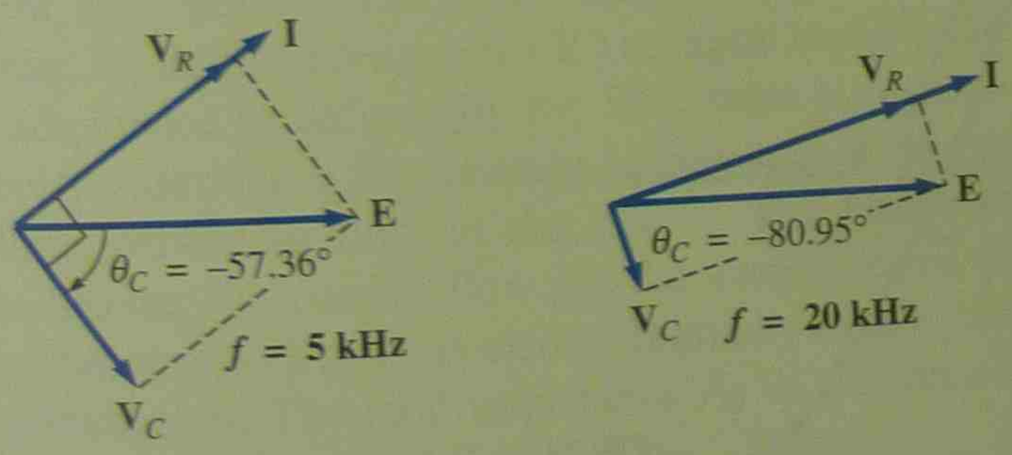


FIG. 15.49

Note also from Figs. 15.48 and 15.49 that the vector $\mathbf{V}_R$ and the current $\mathbf{I}$ have grown in magnitude with reduction in the capacitive reactance. Eventually, at very high frequencies X_C will approach zero ohms and the short-circuit equivalent can be applied, resulting in $V_C \cong 0$ V and $\theta_C \cong -90^\circ$, producing the phasor diagram of Fig. 15.50. The network is then resistive and the phase angle between $\mathbf{I}$ and $\mathbf{E}$ is essentially zero degrees and V_R and I are their maximum values.



FIG. 15.50

A plot of V_C versus frequency appears in Fig. 15.51. At low frequencies $X_C \gg R$ and V_C is very close to E in magnitude. As the applied frequency increases, X_C decreases in magnitude along with V_C as V_R captures more of the applied voltage. A plot of θ_C versus frequency is provided in Fig. 15.52. At low frequencies the phase angle between $\mathbf{V}_C$ and $\mathbf{E}$ is very small since $\mathbf{V}_C \cong \mathbf{E}$. Recall that if two phasors are equal, they must have the same angle. As the applied frequency increases, the

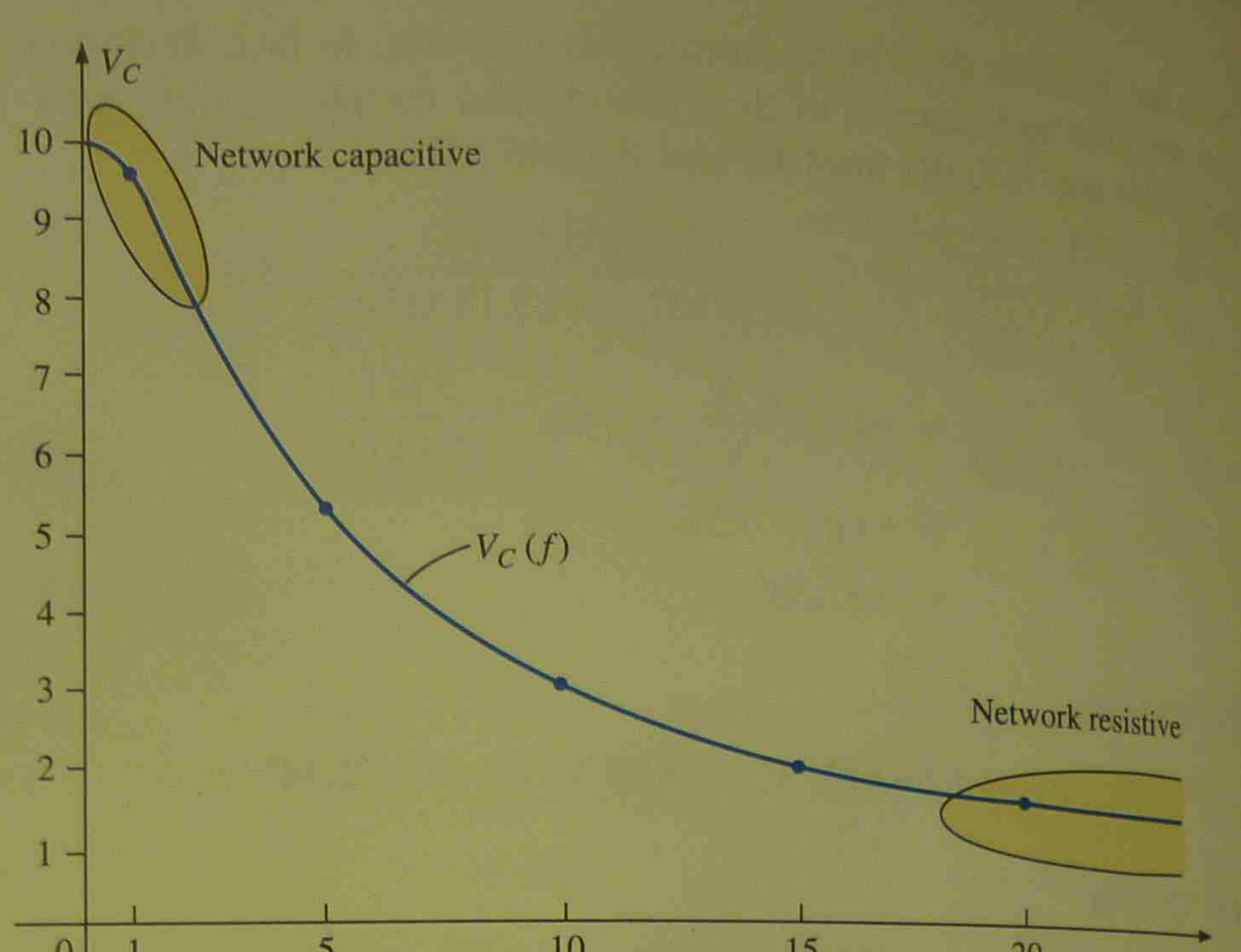


FIG. 15.51

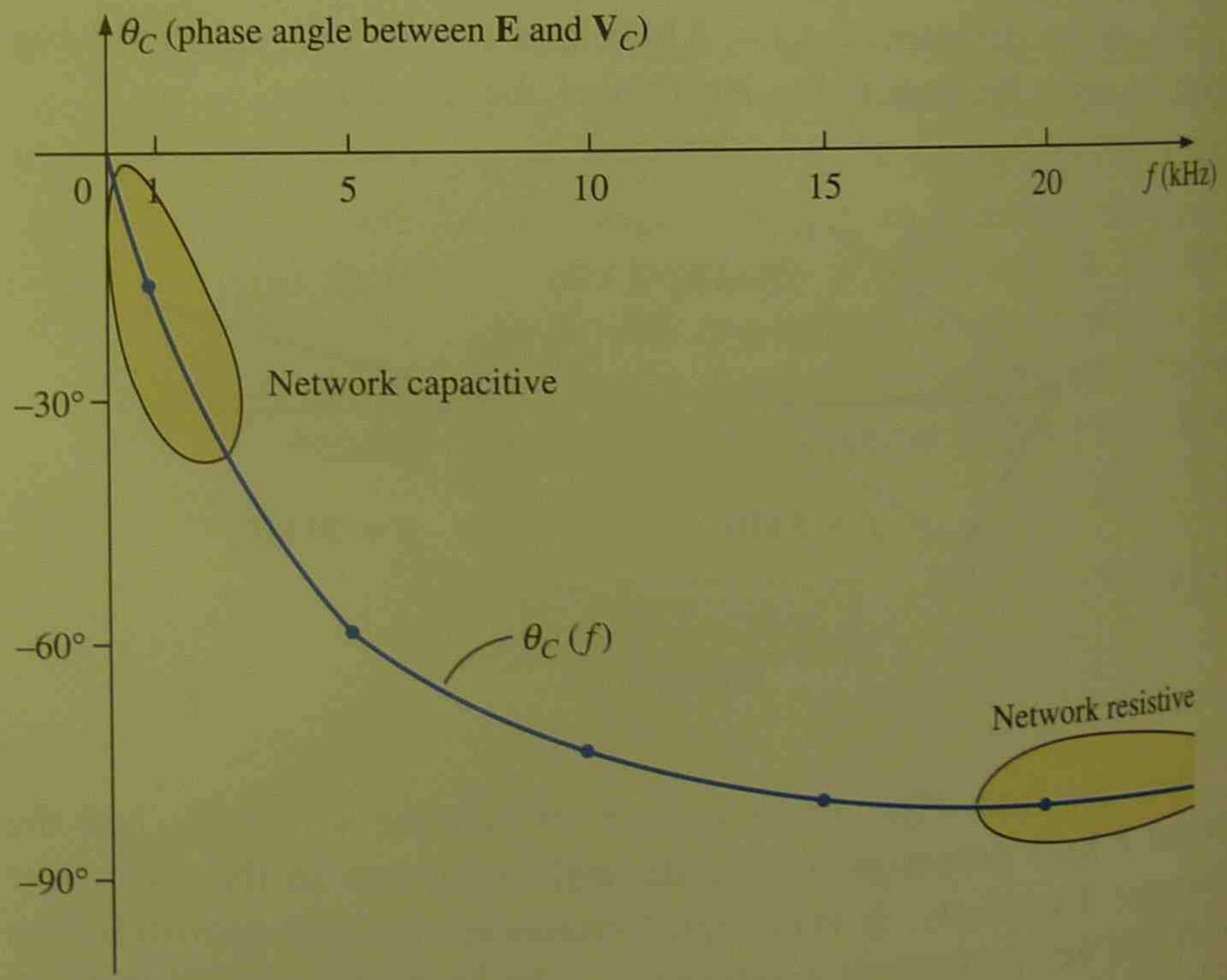


FIG. 15.52

network becomes more resistive and the phase angle between $\mathbf{V}_C$ and $\mathbf{E}$ approaches 90°. At high frequencies keep in mind that $\mathbf{I}$ and $\mathbf{E}$ are approaching an in-phase situation and the angle between $\mathbf{V}_C$ and $\mathbf{E}$ will approach that between $\mathbf{V}_C$ and $\mathbf{I}$, which we know must be 90° ($\mathbf{I}_C$ leading $\mathbf{V}_C$).

A plot of V_R versus frequency would approach E volts from zero volts with increase in frequency, but remember $V_R \neq E - V_C$ due to

vector relationship. The phase angle between **I** and **E** could be plotted directly from Fig. 15.52 using Eq. (15.11).

In Chapter 21, the analysis of this section will be extended to a much wider frequency range using a log axis for frequency. It will be demonstrated that an *R-C* circuit such as that in Fig. 15.44 can be used as a filter to determine which frequencies will have the greatest impact on the stage to follow. From our current analysis, it is obvious that any network connected across the capacitor will receive the greatest potential level at low frequencies and be effectively "shorted out" at very high frequencies.

The analysis of a series *R-L* circuit would proceed in much the same manner, except that X_L and V_L would increase with frequency and the angle between **I** and **E** would approach 90° (voltage leading the current) rather than 0°. If $\mathbf{V}_L$ were plotted versus frequency, $\mathbf{V}_L$ would approach E, and X_L would eventually attain a level at which the open-circuit equivalent would be appropriate.

PARALLEL ac CIRCUITS

15.6 ADMITTANCE AND SUSCEPTANCE

The discussion for parallel ac circuits will be very similar to that for dc circuits. In dc circuits, *conductance* (G) was defined as equal to $1/R$. The total conductance of a parallel circuit was then found by adding the conductance of each branch. The total resistance R_T is simply $1/G_T$.

In ac circuits, we define *admittance* (**Y**) as equal to $1/\mathbf{Z}$. The unit of measure for admittance as defined by the SI system is *siemens*, which has the symbol S. Admittance is a measure of how well an ac circuit will *admit*, or allow, current to flow in the circuit. The larger its value, therefore, the heavier the current flow for the same applied potential. The total admittance of a circuit can also be found by finding the sum of the parallel admittances. The total impedance $\mathbf{Z}_T$ of the circuit is then $1/\mathbf{Y}_T$; that is, for the network of Fig. 15.53.

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots + \mathbf{Y}_N \tag{15.12}$$

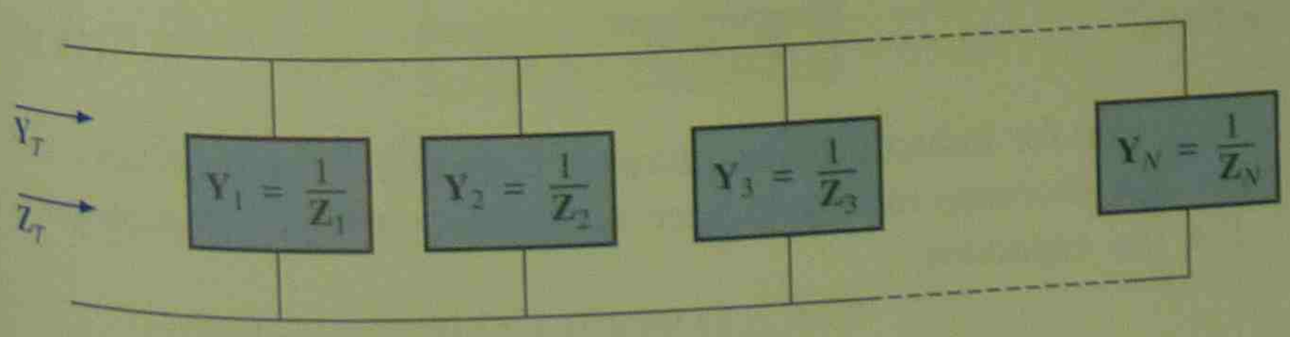


FIG. 15.53

or, since $\mathbf{Z} = 1/\mathbf{Y}$,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_N} \tag{15.13}$$

For two impedances in parallel,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

If the manipulations used in Chapter 6 to find the total resistance of two parallel resistors are now applied, the following similar equation will result:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{15.14}$$

For three parallel impedances,

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3} \tag{15.15}$$

As pointed out in the introduction to this section, conductance is the reciprocal of resistance, and

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{R\angle 0^\circ} = G\angle 0^\circ \tag{15.16}$$

The reciprocal of reactance ($1/X$) is called *susceptance* and is a measure of how *susceptible* an element is to the passage of current through it. Susceptance is also measured in *siemens*, and is represented by the capital letter B.

For the inductor,

$$\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{1}{X_L\angle 90^\circ} = \frac{1}{X_L}\angle -90^\circ \tag{15.17}$$

Defining

$$B_L = \frac{1}{X_L} \quad \text{(siemens, S)} \tag{15.18}$$

we have

$$\mathbf{Y}_L = B_L\angle -90^\circ \tag{15.19}$$

Note that for inductance, an increase in frequency or inductance will result in a decrease in susceptance or, correspondingly, in admittance.

For the capacitor,

$$\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = \frac{1}{X_C\angle -90^\circ} = \frac{1}{X_C}\angle 90^\circ \tag{15.20}$$

$$B_C = \frac{1}{X_C} \quad \text{(siemens, S)} \tag{15.21}$$

we have

$$\mathbf{Y}_C = B_C \angle 90^\circ \tag{15.22}$$

For the capacitor, therefore, an increase in frequency or capacitance will result in an increase in its susceptibility.

For parallel ac circuits, the *admittance diagram* is used with the three admittances represented as shown in Fig. 15.54.

Note in Fig. 15.54 that the conductance (like resistance) is on the positive real axis, whereas inductive and capacitive susceptances are in direct opposition on the imaginary axis.

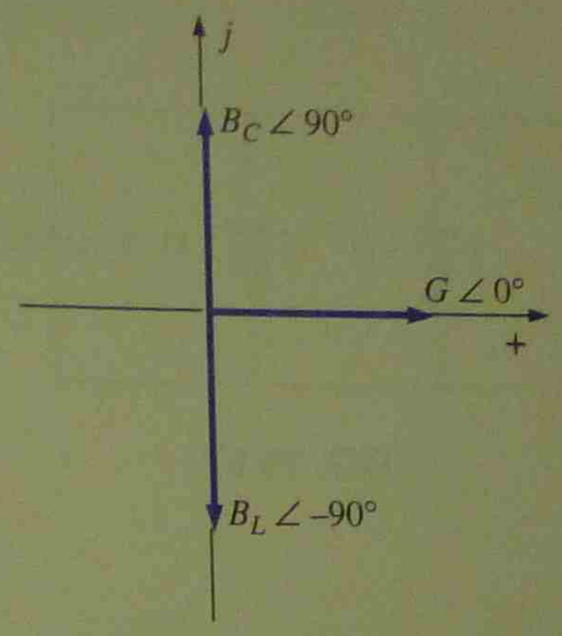


FIG. 15.54

For any configuration (series, parallel, series-parallel, etc.), the angle associated with the total admittance is the angle by which the source current leads the applied voltage. For inductive networks, θ_T is negative, whereas for capacitive networks, θ_T is positive.

EXAMPLE 15.12 For the network of Fig. 15.55:

a. Find the admittance of each parallel branch.
b. Determine the input admittance.
c. Calculate the input impedance.
d. Draw the admittance diagram.

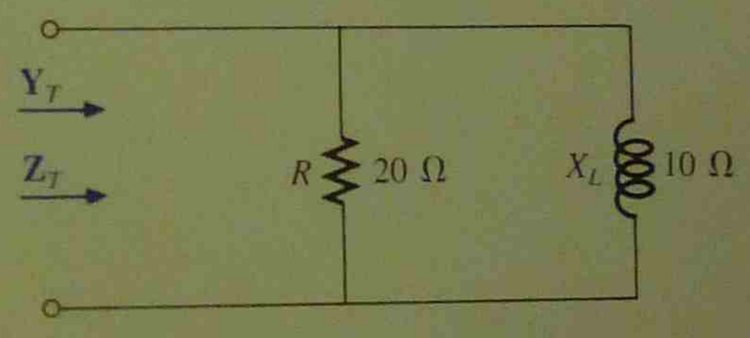


FIG. 15.55

Solutions:

a. $\mathbf{Y}_R = G \angle 0^\circ = \dfrac{1}{R} \angle 0^\circ = \dfrac{1}{20\ \Omega} \angle 0^\circ$

$= \mathbf{0.05\ S \angle 0^\circ = 0.05\ S + j0}$

$\mathbf{Y}_L = B_L \angle -90^\circ = \dfrac{1}{X_L} \angle -90^\circ = \dfrac{1}{10\ \Omega} \angle -90^\circ$

$= \mathbf{0.1\ S \angle -90^\circ = 0 - j0.1\ S}$

b. $\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L = (0.05\ \text{S} + j0) + (0 - j0.1\ \text{S})$

$= \mathbf{0.05\ S - j0.1\ S} = G - jB_L$

c. $\mathbf{Z}_T = \dfrac{1}{\mathbf{Y}_T} = \dfrac{1}{0.05\ \text{S} - j0.1\ \text{S}} = \dfrac{1}{0.112\ \text{S} \angle -63.43^\circ}$

$= \mathbf{8.93\ \Omega \angle 63.43^\circ}$

or Eq. (15.17):

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R \mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(20\ \Omega \angle 0^\circ)(10\ \Omega \angle 90^\circ)}{20\ \Omega + j10\ \Omega}$$

$$= \frac{200\ \Omega \angle 90^\circ}{22.36 \angle 26.57^\circ} = \mathbf{8.93\ \Omega \angle 63.43^\circ}$$

d. The admittance diagram appears in Fig. 15.56.

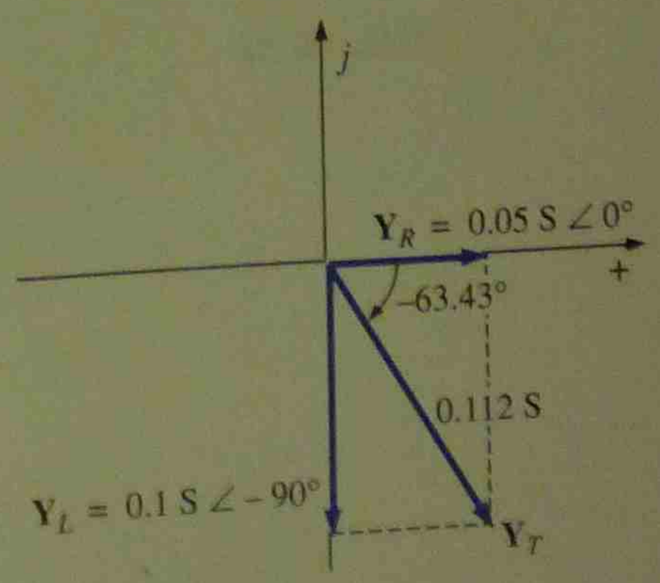


FIG. 15.56

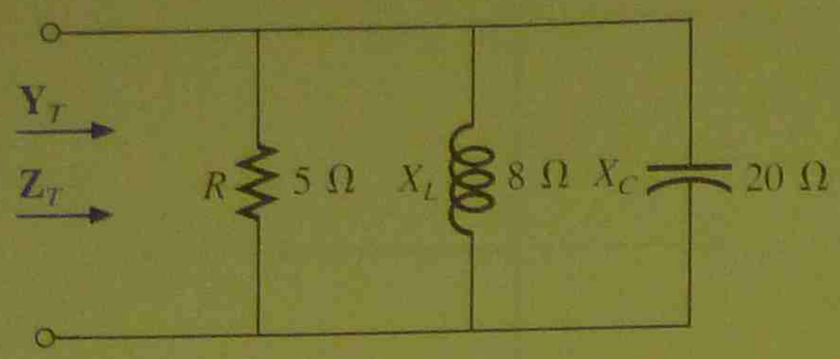


FIG. 15.57

EXAMPLE 15.13 Repeat Example 15.12 for the parallel network of Fig. 15.57.

Solutions:

a. $\mathbf{Y}_R = G\angle 0^\circ = \dfrac{1}{R}\angle 0^\circ = \dfrac{1}{5\ \Omega}\angle 0^\circ$

$= \mathbf{0.2\ S\angle 0^\circ = 0.2\ S + j0}$

$\mathbf{Y}_L = B_L\angle -90^\circ = \dfrac{1}{X_L}\angle -90^\circ = \dfrac{1}{8\ \Omega}\angle -90^\circ$

$= \mathbf{0.125\ S\angle -90^\circ = 0 - j0.125\ S}$

$\mathbf{Y}_C = B_C\angle 90^\circ = \dfrac{1}{X_C}\angle 90^\circ = \dfrac{1}{20\ \Omega}\angle 90^\circ$

$= \mathbf{0.050\ S\angle +90^\circ = 0 + j0.050\ S}$

b. $\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}$

$= (0.2\ \text{S} + j0) + (0 - j0.125\ \text{S}) + (0 + j0.050\ \text{S})$

$= 0.2\ \text{S} - j0.075\ \text{S} = \mathbf{0.2136\ S\angle -20.56^\circ}$

c. $\mathbf{Z}_T = \dfrac{1}{0.2136\ \text{S}\angle -20.56^\circ} = \mathbf{4.68\ \Omega\angle 20.56^\circ}$

or

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_L\mathbf{Z}_C}{\mathbf{Z}_R\mathbf{Z}_L + \mathbf{Z}_L\mathbf{Z}_C + \mathbf{Z}_R\mathbf{Z}_C}$$

$$= \frac{(5\ \Omega\angle 0^\circ)(8\ \Omega\angle 90^\circ)(20\ \Omega\angle -90^\circ)}{(5\ \Omega\angle 0^\circ)(8\ \Omega\angle 90^\circ) + (8\ \Omega\angle 90^\circ)(20\ \Omega\angle -90^\circ) + (5\ \Omega\angle 0^\circ)(20\ \Omega\angle -90^\circ)}$$

$$= \frac{800\angle 0^\circ}{40\angle 90^\circ + 160\angle 0^\circ + 100\angle -90^\circ}$$

$$= \frac{800}{160 + j40 - j100} = \frac{800}{160 - j60}$$

$$= \frac{800}{170.88\angle -20.56^\circ}$$

$$= \mathbf{4.68\ \Omega\angle 20.56^\circ}$$

d. The admittance diagram appears in Fig. 15.58.

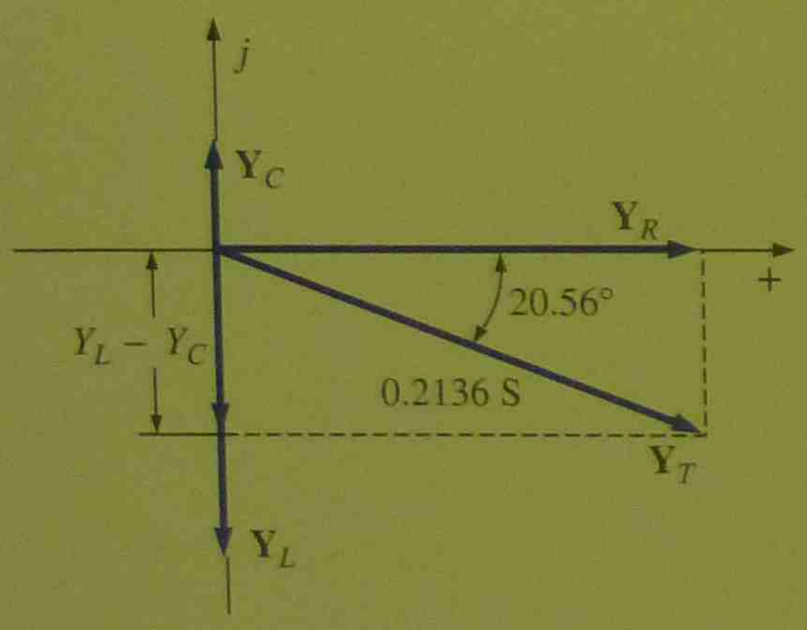


FIG. 15.58

On many occasions, the inverse relationship $\mathbf{Y}_T = 1/\mathbf{Z}_T$ or $\mathbf{Z}_T = 1/\mathbf{Y}_T$ will require that we divide the number 1 by a complex number having a real and an imaginary part. This division, if not performed in the polar form, requires that we multiply the numerator and denominator by the conjugate of the denominator, as follows:

$$\mathbf{Y}_T = \frac{1}{\mathbf{Z}_T} = \frac{1}{4\ \Omega + j6\ \Omega} = \frac{1}{4\ \Omega + j6\ \Omega}\frac{(4\ \Omega - j6\ \Omega)}{(4\ \Omega - j6\ \Omega)} = \frac{4 - j6}{4^2 + 6^2}$$

and

$$\mathbf{Y}_T = \frac{4}{52}\ \text{S} - j\frac{6}{52}\ \text{S}$$

To avoid this laborious task each time we want to find the reciprocal of a complex number in rectangular form, a format can be developed using the following complex number, which is symbolic of any impedance or admittance in the first or fourth quadrant:

$$\frac{1}{a_1 \pm jb_1} = \left(\frac{1}{a_1 \pm jb_1}\right)\left(\frac{a_1 \mp jb_1}{a_1 \mp jb_1}\right) = \frac{a_1 \mp jb_1}{a_1^2 + b_1^2}$$

or

$$\boxed{\frac{1}{a_1 \pm jb_1} = \frac{a_1}{a_1^2 + b_1^2} \mp j\frac{b_1}{a_1^2 + b_1^2}} \qquad \textbf{(15.23)}$$

Note that the denominator is simply the sum of the squares of each term. The sign is inverted between the real and imaginary parts. A few examples will develop some familiarity with the use of this equation.

EXAMPLE 15.14 Find the admittance of each set of series elements in Fig. 15.59.

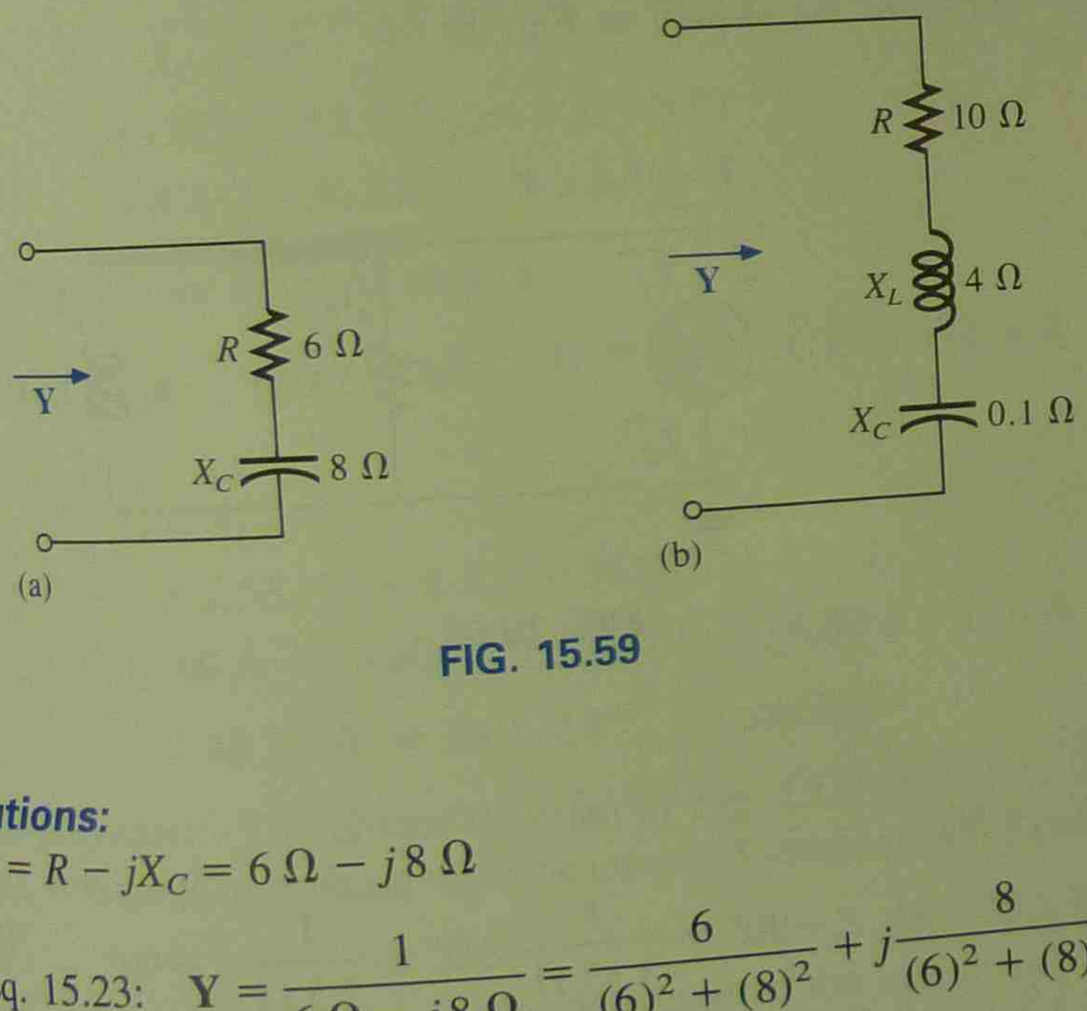


FIG. 15.59

Solutions:

a. $\mathbf{Z} = R - jX_C = 6\,\Omega - j\,8\,\Omega$

Eq. 15.23: $$\mathbf{Y} = \frac{1}{6\,\Omega - j\,8\,\Omega} = \frac{6}{(6)^2 + (8)^2} + j\frac{8}{(6)^2 + (8)^2}$$

$$= \frac{\mathbf{6}}{\mathbf{100}}\,\mathbf{S} + j\frac{\mathbf{8}}{\mathbf{100}}\,\mathbf{S}$$

b. $\mathbf{Z} = 10\,\Omega + j\,4\,\Omega + (-j\,0.1\,\Omega) = 10\,\Omega + j\,3.9\,\Omega$

Eq. 15.23:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{10\,\Omega + j\,3.9\,\Omega} = \frac{10}{(10)^2 + (3.9)^2} - j\frac{3.9}{(10)^2 + (3.9)^2}$$

$$= \frac{10}{115.21} - j\frac{3.9}{115.21} = \mathbf{0.087\ S - j0.034\ S}$$

15.7 *R-L*, *R-C*, AND *R-L-C* PARALLEL ac NETWORKS

Let us now examine a few of the basic parallel ac networks and determine a variety of important levels and characteristics. The format will be very similar to that employed for series ac circuits.

R-L (Fig. 15.60)

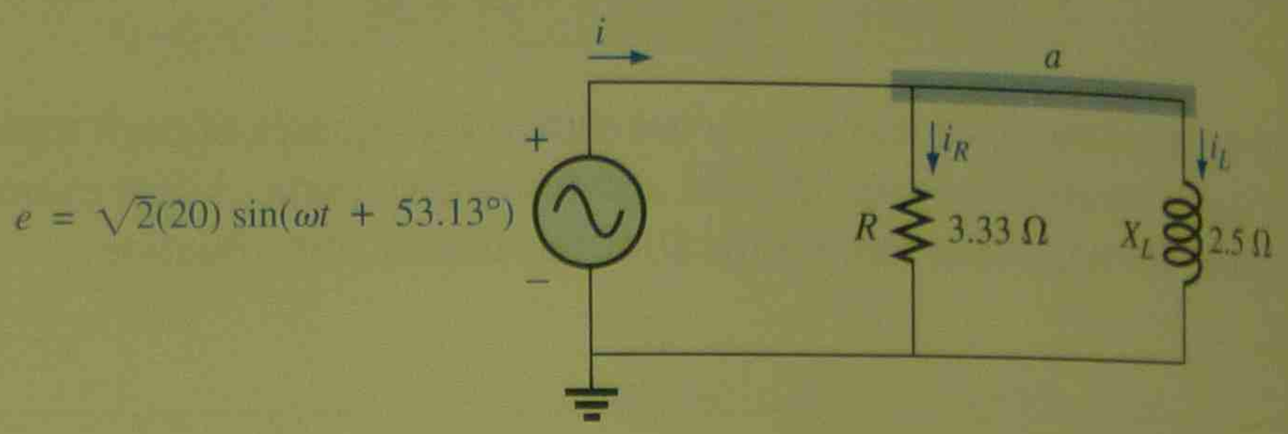


FIG. 15.60

Phasor notation: As shown in Fig. 15.61.

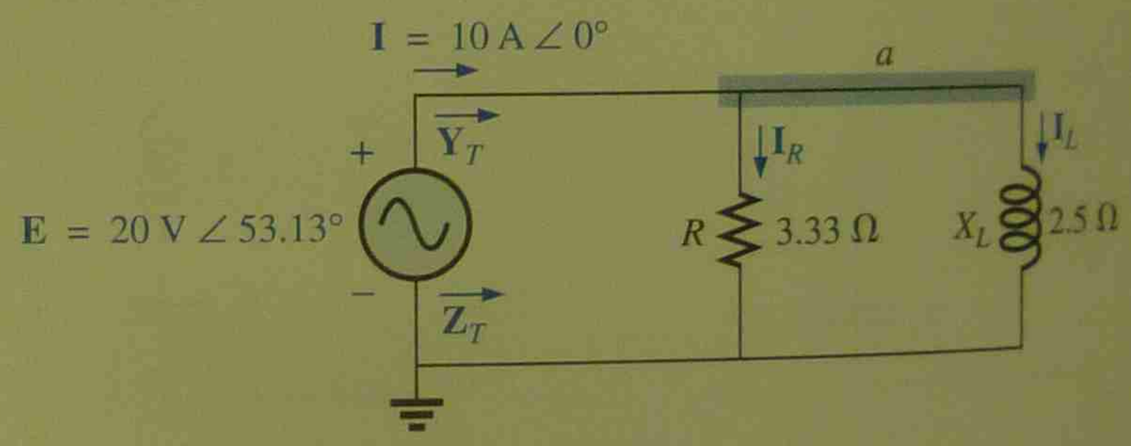


FIG. 15.61

$\mathbf{Y}_T$, $\mathbf{Z}_T$:

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L$$

$$= G\angle 0° + B_L\angle -90° = \frac{1}{3.33\ \Omega}\angle 0° + \frac{1}{2.5\ \Omega}\angle -90°$$

$$= 0.3\text{ S}\angle 0° + 0.4\text{ S}\angle -90° = 0.3\text{ S} - j\,0.4\text{ S}$$

$$= \mathbf{0.5\text{ S}\angle -53.13°}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{0.5\text{ S}\angle -53.13°} = \mathbf{2\ \Omega\angle 53.13°}$$

Admittance diagram: As shown in Fig. 15.62.

I:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \mathbf{EY}_T = (20\text{ V}\angle 53.13°)(0.5\text{ S}\angle -53.13°) = \mathbf{10\text{ A}\angle 0°}$$

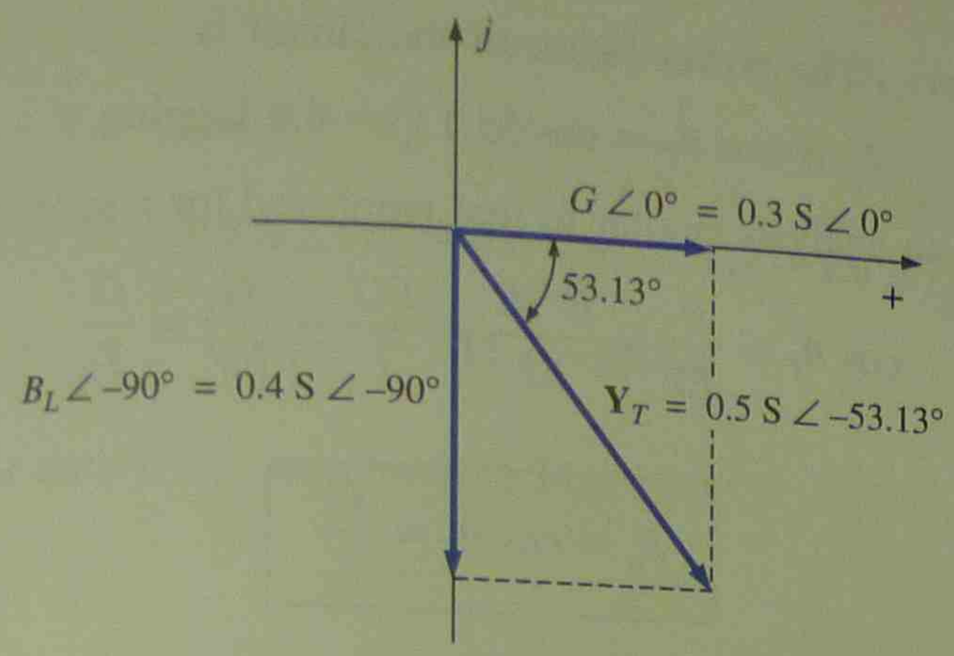


FIG. 15.62

I_R, I_L:

$$\mathbf{I}_R = \frac{E\angle\theta}{R\angle 0^\circ} = (E\angle\theta)(G\angle 0^\circ)$$

$$= (20\text{ V}\angle 53.13^\circ)(0.3\text{ S}\angle 0^\circ) = \mathbf{6\text{ A}\angle 53.13^\circ}$$

$$\mathbf{I}_L = \frac{E\angle\theta}{X_L\angle 90^\circ} = (E\angle\theta)(B_L\angle -90^\circ)$$

$$= (20\text{ V}\angle 53.13^\circ)(0.4\text{ S}\angle -90^\circ)$$

$$= \mathbf{8\text{ A}\angle -36.87^\circ}$$

Kirchhoff's current law: At node *a*,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_L = 0$$

or

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L$$

$$10\text{ A}\angle 0^\circ = 6\text{ A}\angle 53.13^\circ + 8\text{ A}\angle -36.87^\circ$$

$$10\text{ A}\angle 0^\circ = (3.60\text{ A} + j4.80\text{ A}) + (6.40\text{ A} - j4.80\text{ A}) = 10\text{ A} + j0$$

and $$\mathbf{10\text{ A}\angle 0^\circ = 10\text{ A}\angle 0^\circ} \quad \text{(checks)}$$

Phasor diagram: The phasor diagram of Fig. 15.63 indicates that the applied voltage **E** is in phase with the current $\mathbf{I}_R$ and leads the current $\mathbf{I}_L$ by 90°.

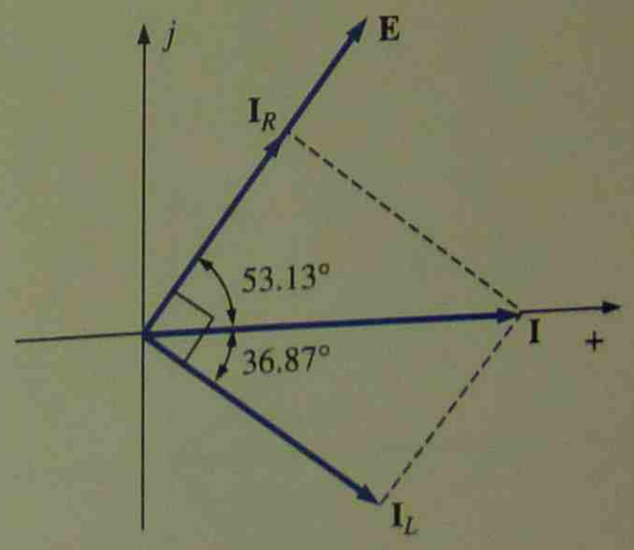


FIG. 15.63

Power: The total power in watts delivered to the circuit is

$$P_T = EI\cos\theta_T$$
$$= (20\text{ V})(10\text{ A})\cos 53.13^\circ = (200\text{ W})(0.6)$$
$$= \mathbf{120\text{ W}}$$

or $$P_T = I^2R = \frac{V_R^2}{R} = V_R^2G = (20\text{ V})^2(0.3\text{ S}) = \mathbf{120\text{ W}}$$

or, finally,

$$P_T = P_R + P_L = EI_R\cos\theta_R + EI_L\cos\theta_L$$
$$= (20\text{ V})(6\text{ A})\cos 0^\circ + (20\text{ V})(8\text{ A})\cos 90^\circ = 120\text{ W} + 0$$
$$= \mathbf{120\text{ W}}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta_T = \cos 53.13^\circ = \mathbf{0.6\ lagging}$$

or, through an analysis similar to that employed for a series ac circuit,

$$\cos \theta_T = \frac{P}{EI} = \frac{E^2/R}{EI} = \frac{EG}{I} = \frac{G}{I/V} = \frac{G}{Y_T}$$

and

$$\boxed{F_p = \cos \theta_T = \frac{G}{Y_T}} \tag{15.24}$$

where G and Y_T are the magnitudes of the total conductance and admittance of the parallel network. For this case,

$$F_p = \cos \theta_T = \frac{0.3\ \text{S}}{0.5\ \text{S}} = \mathbf{0.6\ lagging}$$

Impedance approach: The current **I** can also be found by first finding the total impedance of the network:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(3.33\ \Omega\ \angle 0^\circ)(2.5\ \Omega\ \angle 90^\circ)}{3.33\ \Omega\ \angle 0^\circ + 2.5\ \Omega\ \angle 90^\circ}$$

$$= \frac{8.325\ \angle 90^\circ}{4.164\ \angle 36.87^\circ} = \mathbf{2\ \Omega\ \angle 53.13^\circ}$$

And then, using Ohm's law, we obtain

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{20\ \text{V}\ \angle 53.13^\circ}{2\ \Omega\ \angle 53.13^\circ} = \mathbf{10\ A\ \angle 0^\circ}$$

R-C (Fig. 15.64)

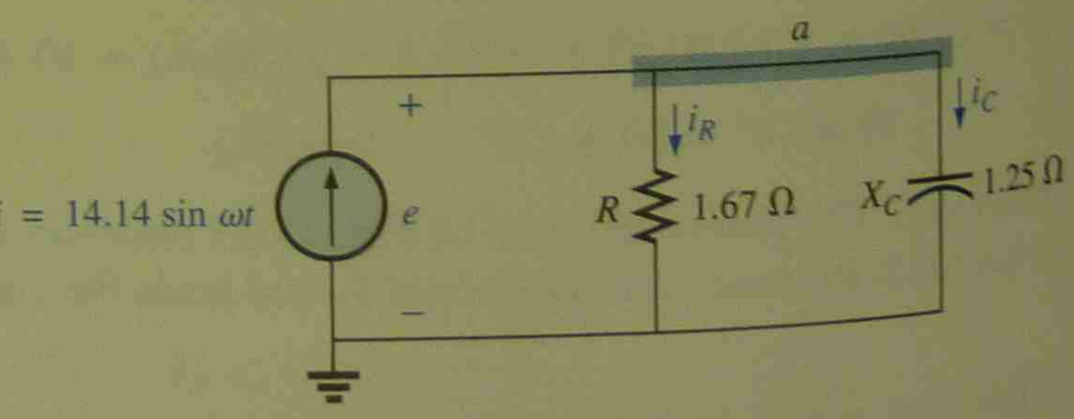


FIG. 15.64

Phasor notation: As shown in Fig. 15.65.

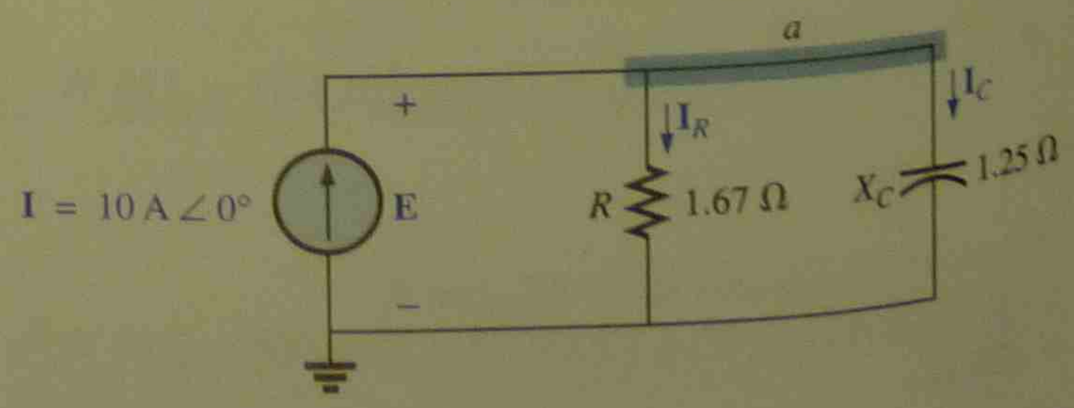


FIG. 15.65

Y_T, Z_T:

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_C = G\angle 0^\circ + B_C\angle 90^\circ = \frac{1}{1.67\ \Omega}\angle 0^\circ + \frac{1}{1.25\ \Omega}\angle 90^\circ$$

$$= 0.6\ \text{S}\angle 0^\circ + 0.8\ \text{S}\angle 90^\circ = 0.6\ \text{S} + j0.8\ \text{S} = \mathbf{1.0\ S\angle 53.13^\circ}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{1.0\ \text{S}\angle 53.13^\circ} = \mathbf{1\ \Omega\angle -53.13^\circ}$$

Admittance diagram: As shown in Fig. 15.66.

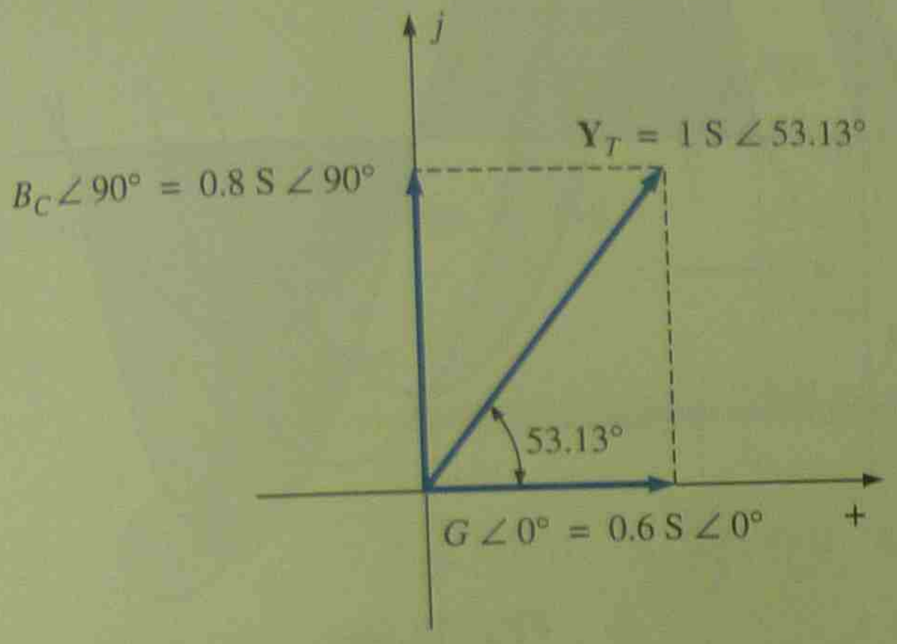


FIG. 15.66

E:

$$\mathbf{E} = \mathbf{I}\mathbf{Z}_T = \frac{\mathbf{I}}{\mathbf{Y}_T} = \frac{10\ \text{A}\angle 0^\circ}{1\ \text{S}\angle 53.13^\circ} = \mathbf{10\ V\angle -53.13^\circ}$$

I_R, I_C:

$$\mathbf{I}_R = (E\angle\theta)(G\angle 0^\circ)$$
$$= (10\ \text{V}\angle -53.13^\circ)(0.6\ \text{S}\angle 0^\circ) = \mathbf{6\ A\angle -53.13^\circ}$$

$$\mathbf{I}_C = (E\angle\theta)(B_C\angle 90^\circ)$$
$$= (10\ \text{V}\angle -53.13^\circ)(0.8\ \text{S}\angle 90^\circ) = \mathbf{8\ A\angle 36.87^\circ}$$

Kirchhoff's current law: At node *a*,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_C = 0$$

or

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_C$$

which can also be verified (as for the *R-L* network) through vector algebra.

Phasor diagram: The phasor diagram of Fig. 15.67 indicates that **E** is in phase with the current through the resistor $\mathbf{I}_R$ and lags the capacitive current $\mathbf{I}_C$ by 90°.

Time domain:

$$e = \sqrt{2}(10)\sin(\omega t - 53.13^\circ) = \mathbf{14.14\sin(\omega t - 53.13^\circ)}$$
$$i_R = \sqrt{2}(6)\sin(\omega t - 53.13^\circ) = \mathbf{8.48\sin(\omega t - 53.13^\circ)}$$
$$i_C = \sqrt{2}(8)\sin(\omega t + 36.87^\circ) = \mathbf{11.31\sin(\omega t + 36.87^\circ)}$$

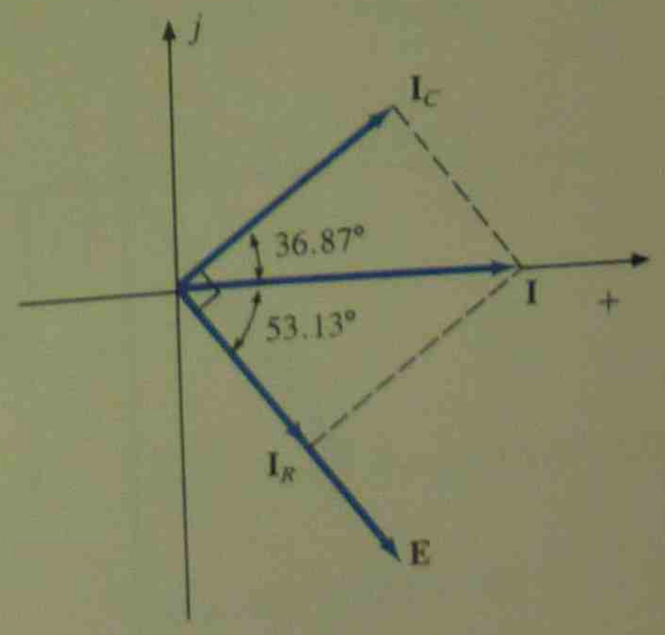


FIG. 15.67

A plot of all of the currents and the voltage appears in Fig. 15.68. Note that e and i_R are in phase and e lags i_C by 90°.

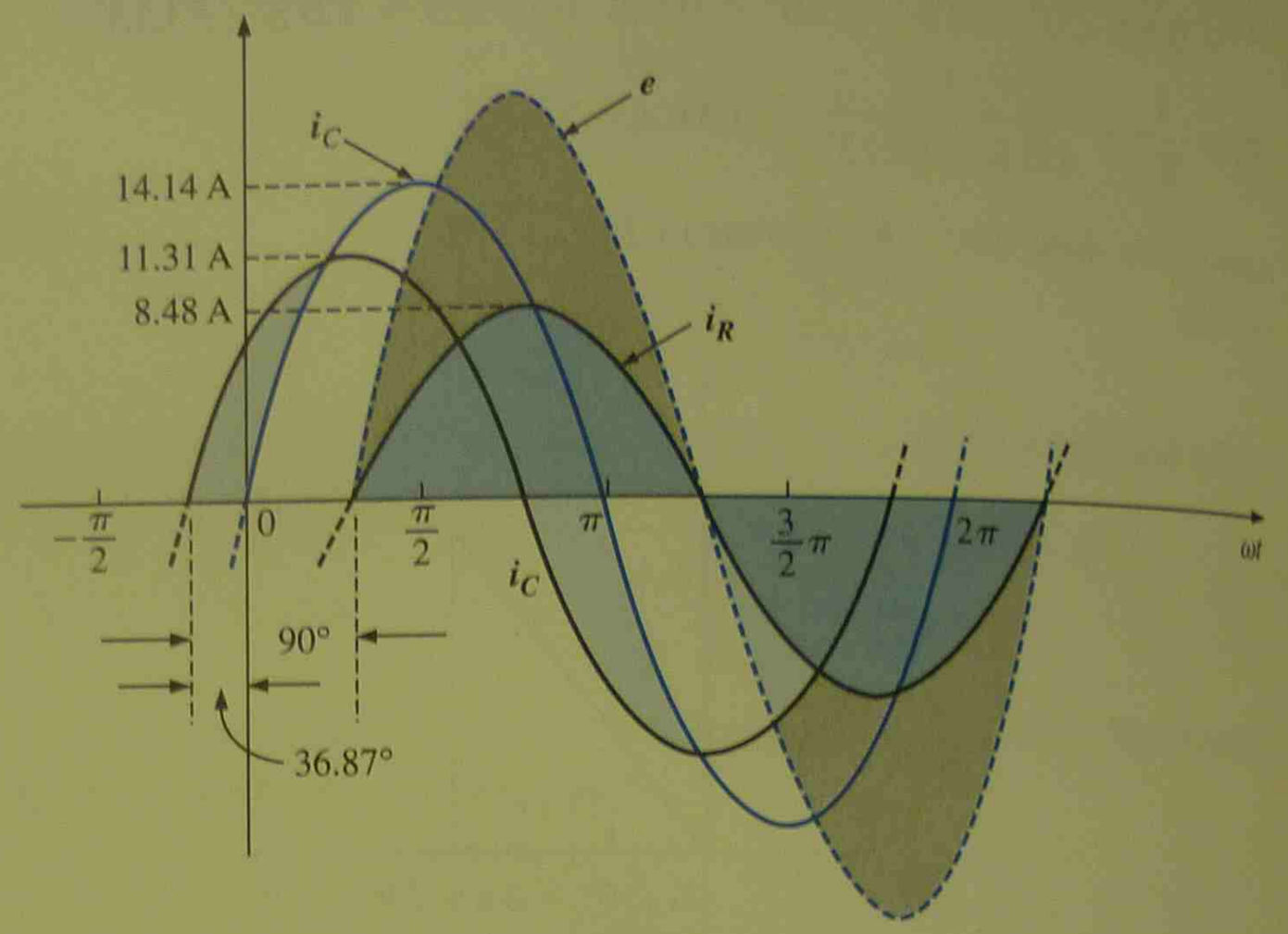


FIG. 15.68

Power:

$$P_T = EI\cos\theta = (10\text{ V})(10\text{ A})\cos 53.13^\circ = (10)^2(0.6)$$
$$= \mathbf{60\text{ W}}$$

or
$$P_T = E^2G = (10\text{ V})^2(0.6\text{ S}) = \mathbf{60\text{ W}}$$

or, finally,

$$P_T = P_R + P_C = EI_R\cos\theta_R + EI_C\cos\theta_C$$
$$= (10\text{ V})(6\text{ A})\cos 0^\circ + (10\text{ V})(8\text{ A})\cos 90^\circ$$
$$= \mathbf{60\text{ W}}$$

Power factor: The power factor of the circuit is

$$F_p = \cos 53.13^\circ = \mathbf{0.6\text{ leading}}$$

Using Eq. (15.24), we have

$$F_p = \cos\theta_T = \frac{G}{Y_T} = \frac{0.6\text{ S}}{1.0\text{ S}} = \mathbf{0.6\text{ leading}}$$

Impedance approach: The voltage **E** can also be found by first finding the total impedance of the circuit:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{(1.67\ \Omega\ \angle 0^\circ)(1.25\ \Omega\ \angle -90^\circ)}{1.67\ \Omega\ \angle 0^\circ + 1.25\ \Omega\ \angle -90^\circ}$$
$$= \frac{2.09\ \angle -90^\circ}{2.09\ \angle -36.81^\circ} = \mathbf{1\ \Omega\ \angle -53.19^\circ}$$

and then, using Ohm's law, we find

$$\mathbf{E} = \mathbf{IZ}_T = (10\text{ A}\ \angle 0^\circ)(1\ \Omega\ \angle -53.19^\circ) = \mathbf{10\text{ V}\ \angle -53.19^\circ}$$

R-L-C (Fig. 15.69)

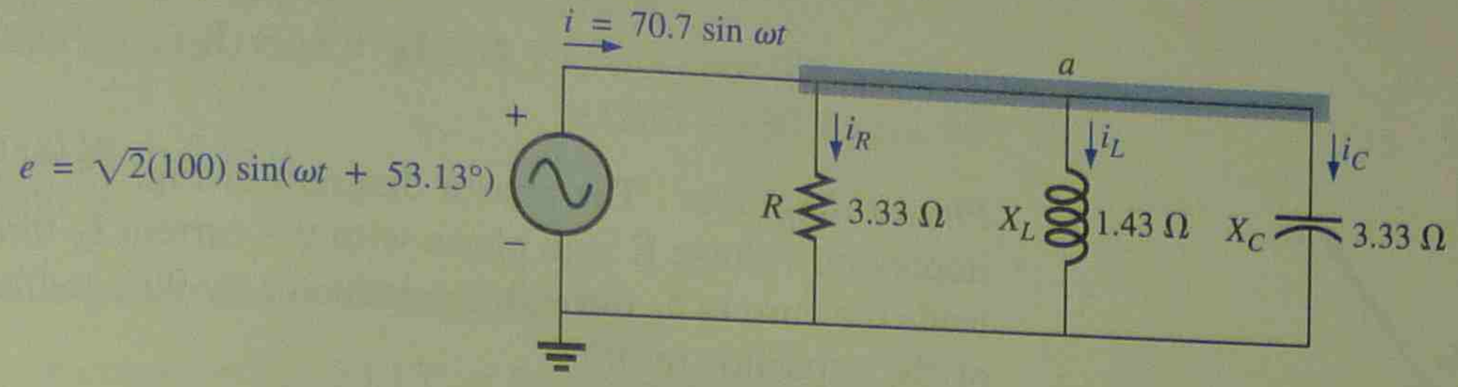


FIG. 15.69

Phasor notation: As shown in Fig. 15.70.

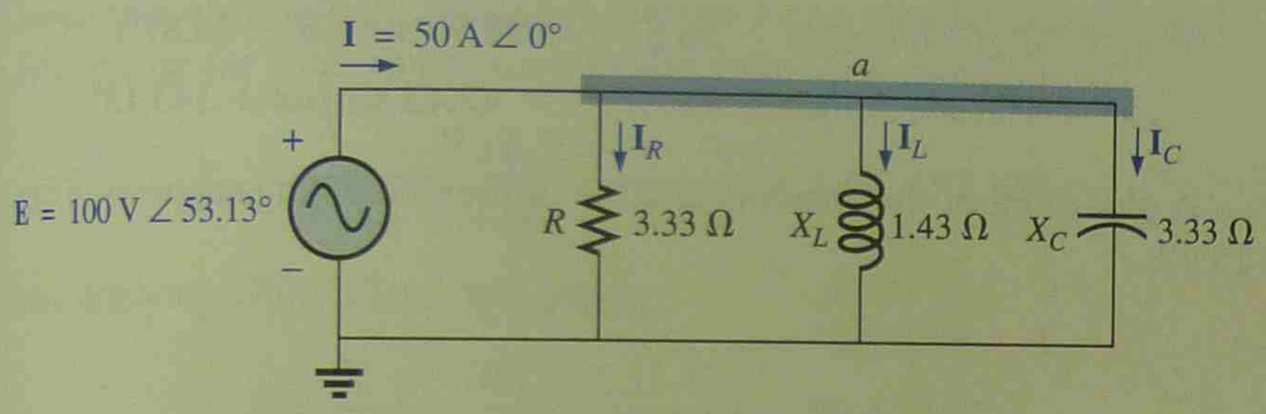


FIG. 15.70

Y_T, Z_T:

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C = G\angle 0° + B_L\angle -90° + B_C\angle 90°$$

$$= \frac{1}{3.33\ \Omega}\angle 0° + \frac{1}{1.43\ \Omega}\angle -90° + \frac{1}{3.33\ \Omega}\angle 90°$$

$$= 0.3\ \text{S}\angle 0° + 0.7\ \text{S}\angle -90° + 0.3\ \text{S}\angle 90°$$

$$= 0.3\ \text{S} - j0.7\ \text{S} + j0.3\ \text{S}$$

$$= 0.3\ \text{S} - j0.4\ \text{S} = \mathbf{0.5\ S\angle -53.13°}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{0.5\ \text{S}\angle -53.13°} = \mathbf{2\ \Omega\angle 53.13°}$$

Admittance diagram: As shown in Fig. 15.71.

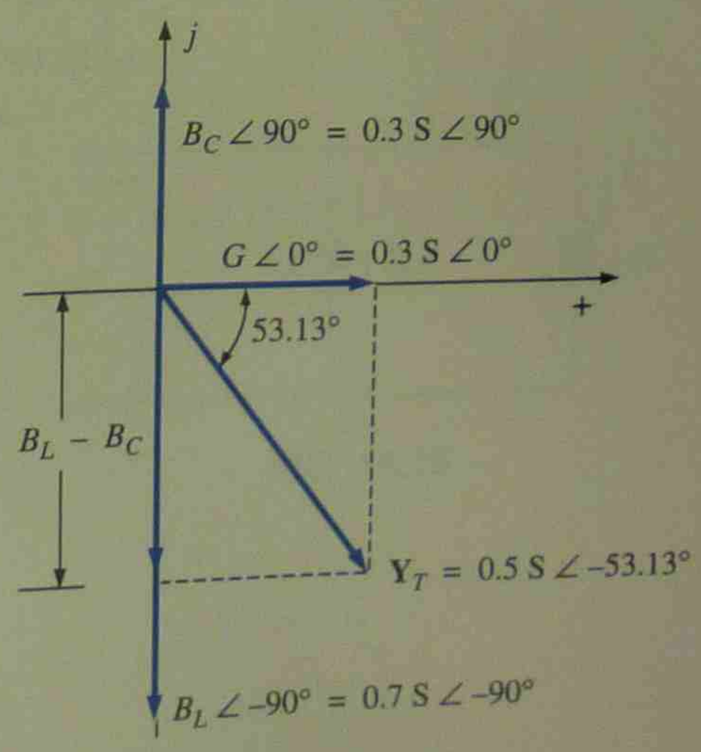


FIG. 15.71

I:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \mathbf{E}\mathbf{Y}_T = (100\ \text{V}\angle 53.13°)(0.5\ \text{S}\angle -53.13°) = \mathbf{50\ A\angle 0°}$$

I_R, I_L, I_C:

$$\mathbf{I}_R = (E\angle\theta)(G\angle 0°)$$
$$= (100\ \text{V}\angle 53.13°)(0.3\ \text{S}\angle 0°) = \mathbf{30\ A\angle 53.13°}$$

$$\mathbf{I}_L = (E\angle\theta)(B_L\angle -90°)$$
$$= (100\ \text{V}\angle 53.13°)(0.7\ \text{S}\angle -90°) = \mathbf{70\ A\angle -36.87°}$$

$$\mathbf{I}_C = (E\angle\theta)(B_C\angle 90°)$$
$$= (100\ \text{V}\angle 53.13°)(0.3\ \text{S}\angle +90°) = \mathbf{30\ A\angle 143.13°}$$

Kirchhoff's current law: At node *a*,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_L - \mathbf{I}_C = 0$$

or

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L + \mathbf{I}_C$$

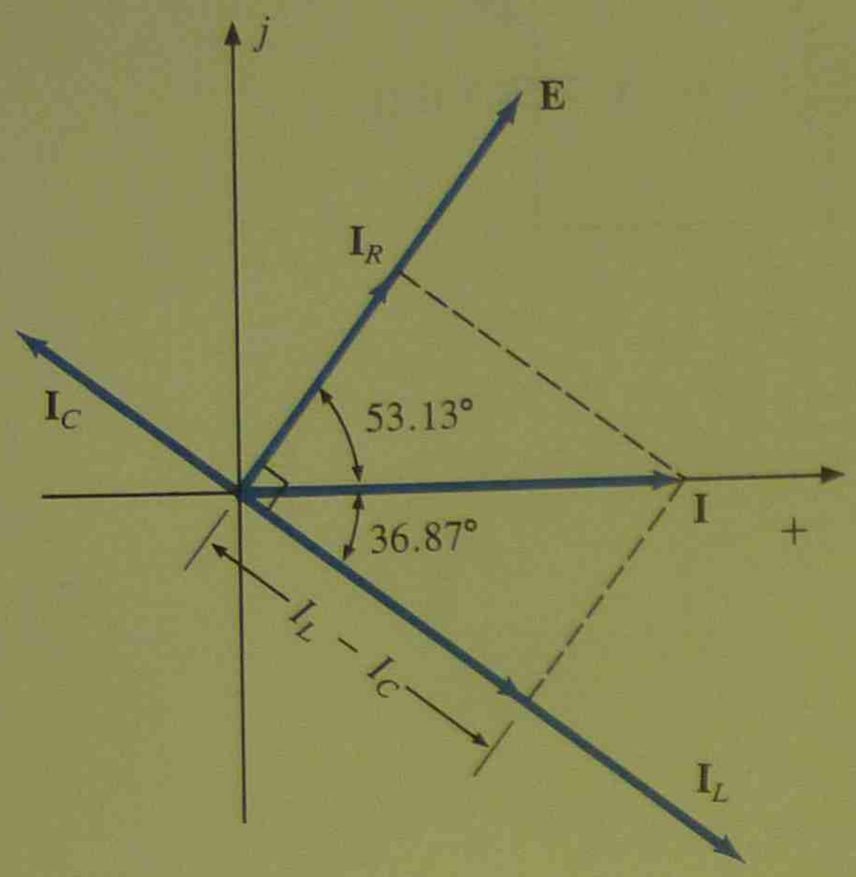


FIG. 15.72

Phasor diagram: The phasor diagram of Fig. 15.72 indicates that the impressed voltage **E** is in phase with the current $\mathbf{I}_R$ through the resistor, leads the current $\mathbf{I}_L$ through the inductor by 90°, and lags the current $\mathbf{I}_C$ of the capacitor by 90°.

Time domain:

$$i = \sqrt{2}(50)\sin \omega t = \mathbf{70.70 \sin \omega t}$$

$$i_R = \sqrt{2}(30)\sin(\omega t + 53.13°) = \mathbf{42.42 \sin(\omega t + 53.13°)}$$

$$i_L = \sqrt{2}(70)\sin(\omega t - 36.87°) = \mathbf{98.98 \sin(\omega t - 36.87°)}$$

$$i_C = \sqrt{2}(30)\sin(\omega t + 143.13°) = \mathbf{42.42 \sin(\omega t + 143.13°)}$$

A plot of all of the currents and the impressed voltage appears in Fig. 15.73.

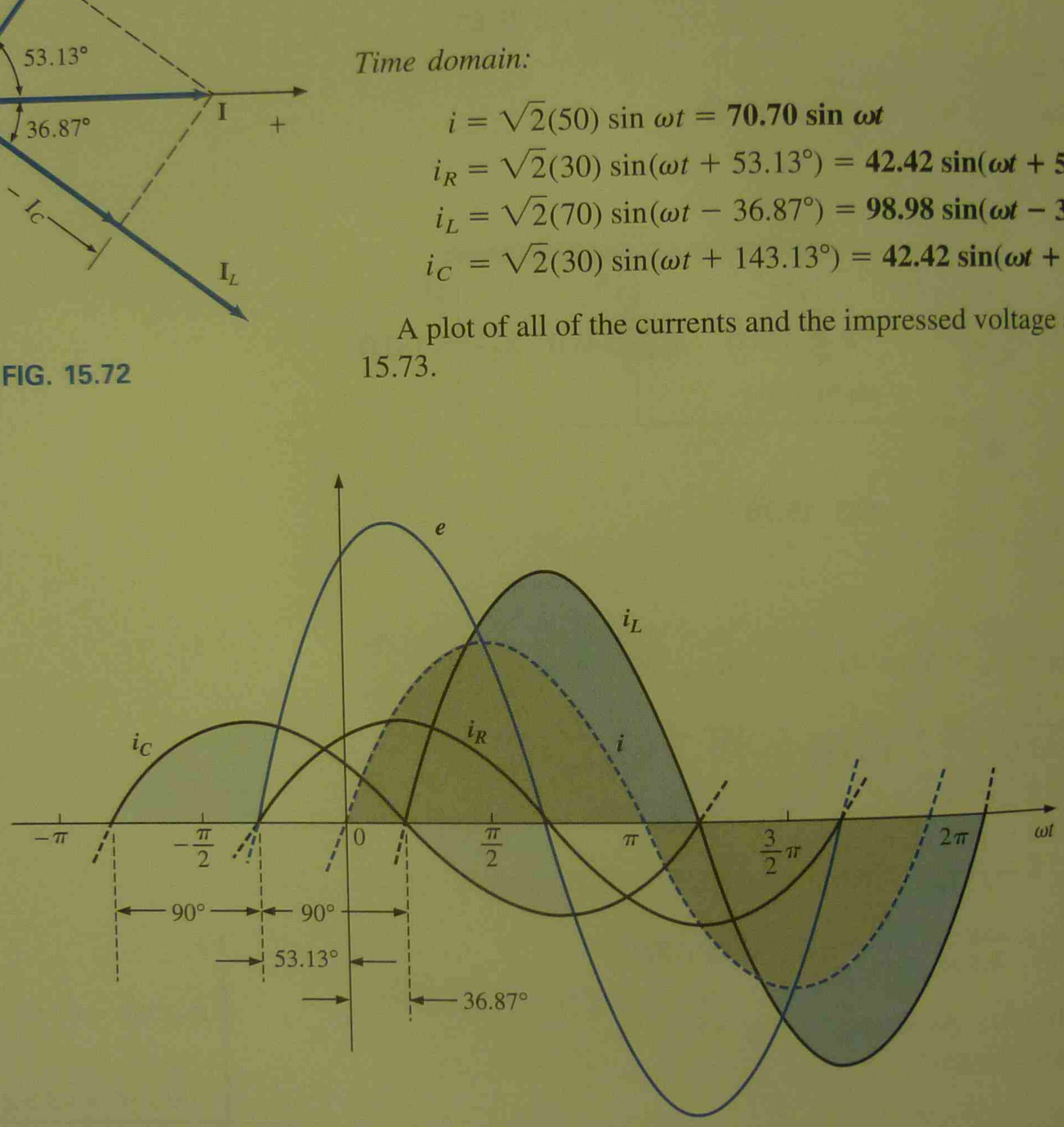


FIG. 15.73

Power: The total power in watts delivered to the circuit is

$$P_T = EI\cos\theta = (100\text{ V})(50\text{ A})\cos 53.13° = (5000)(0.6)$$
$$= \mathbf{3000\text{ W}}$$

or

$$P_T = E^2G = (100\text{ V})^2(0.3\text{ S}) = \mathbf{3000\text{ W}}$$

or, finally,

$$\begin{aligned} P_T &= P_R + P_L + P_C \\ &= EI_R \cos\theta_R + EI_L \cos\theta_L + EI_C \cos\theta_C \\ &= (100\text{ V})(30\text{ A})\cos 0^\circ + (100\text{ V})(70\text{ A})\cos 90^\circ \\ &\qquad + (100\text{ V})(30\text{ A})\cos 90^\circ \\ &= 3000\text{ W} + 0 + 0 \\ &= \mathbf{3000\text{ W}} \end{aligned}$$

Power factor: The power factor of the circuit is

$$F_p = \cos\theta_T = \cos 53.13^\circ = \mathbf{0.6\ lagging}$$

Using Eq. (15.24), we obtain

$$F_p = \cos\theta_T = \frac{G}{Y_T} = \frac{0.3\text{ S}}{0.5\text{ S}} = \mathbf{0.6\ lagging}$$

Impedance approach: The input current **I** can also be determined by first finding the total impedance in the following manner:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_L\mathbf{Z}_C}{\mathbf{Z}_R\mathbf{Z}_L + \mathbf{Z}_L\mathbf{Z}_C + \mathbf{Z}_R\mathbf{Z}_C} = \mathbf{2\ \Omega\ \angle 53.13^\circ}$$

and, applying Ohm's law, we obtain

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{100\text{ V}\ \angle 53.13^\circ}{2\ \Omega\ \angle 53.13^\circ} = \mathbf{50\text{ A}\ \angle 0^\circ}$$

15.8 CURRENT DIVIDER RULE

The basic format for the current divider rule in ac circuits is exactly the same as that for dc circuits; that is, for two parallel branches with impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as shown in Fig. 15.74,

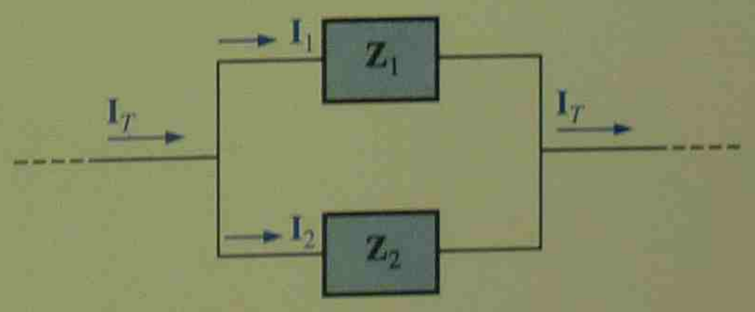


FIG. 15.74

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad \text{or} \quad \mathbf{I}_2 = \frac{\mathbf{Z}_1\mathbf{I}_T}{\mathbf{Z}_1 + \mathbf{Z}_2}} \qquad (15.25)$$

EXAMPLE 15.15 Using the current divider rule, find the current through each impedance of Fig. 15.75.

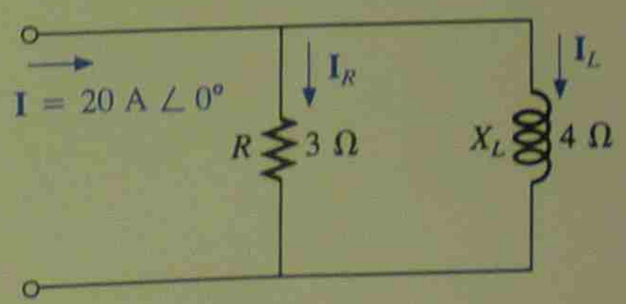


FIG. 15.75

Solution:

$$\begin{aligned} \mathbf{I}_R &= \frac{\mathbf{Z}_L\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(4\ \Omega\ \angle 90^\circ)(20\text{ A}\ \angle 0^\circ)}{3\ \Omega\ \angle 0^\circ + 4\ \Omega\ \angle 90^\circ} = \frac{80\text{ A}\angle 90^\circ}{5\ \angle 53.13^\circ} \\ &= \mathbf{16\text{ A}\ \angle 36.87^\circ} \end{aligned}$$

$$\begin{aligned} \mathbf{I}_L &= \frac{\mathbf{Z}_R\mathbf{I}_T}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(3\ \Omega\ \angle 0^\circ)(20\text{ A}\ \angle 0^\circ)}{5\ \Omega\ \angle 53.13^\circ} = \frac{60\text{ A}\angle 0^\circ}{5\ \angle 53.13^\circ} \\ &= \mathbf{12\text{ A}\ \angle -53.13^\circ} \end{aligned}$$

EXAMPLE 15.16 Using the current divider rule, find the current through each parallel branch of Fig. 15.76.

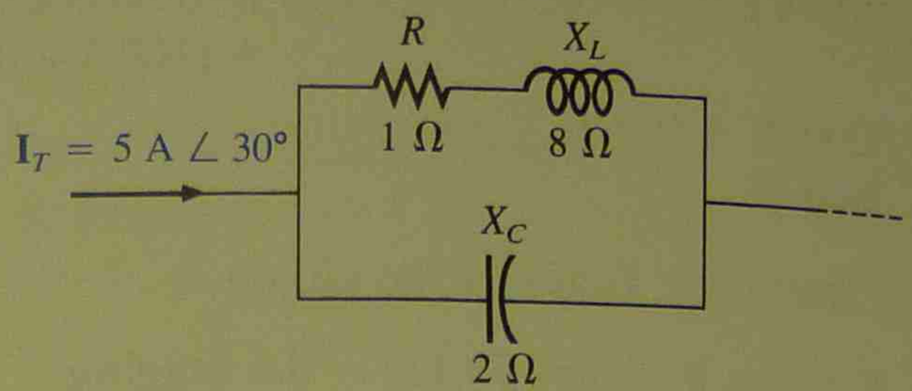


FIG. 15.76

Solution:

$$\mathbf{I}_{R\text{-}L} = \frac{\mathbf{Z}_C\mathbf{I}_T}{\mathbf{Z}_C + \mathbf{Z}_{R\text{-}L}} = \frac{(2\ \Omega \angle -90°)(5\text{ A} \angle 30°)}{-j2\ \Omega + 1\ \Omega + j8\ \Omega} = \frac{10\text{ A} \angle -60°}{1 + j6}$$

$$= \frac{10\text{ A} \angle -60°}{6.083 \angle 80.54°} \cong \mathbf{1.644\text{ A} \angle -140.54°}$$

$$\mathbf{I}_C = \frac{\mathbf{Z}_{R\text{-}L}\mathbf{I}_T}{\mathbf{Z}_{R\text{-}L} + \mathbf{Z}_C} = \frac{(1\ \Omega + j8\ \Omega)(5\text{ A} \angle 30°)}{6.08\ \Omega \angle 80.54°}$$

$$= \frac{(8.06 \angle 82.87°)(5\text{ A} \angle 30°)}{6.08 \angle 80.54°} = \frac{40.30\text{ A} \angle 112.87°}{6.083 \angle 80.54°}$$

$$= \mathbf{6.625\text{ A} \angle 32.33°}$$

15.9 FREQUENCY RESPONSE OF THE PARALLEL *R-L* NETWORK

In Section 15.5 the frequency response of a series *R-C* circuit was analyzed. Let us now note the impact of frequency on the total impedance and inductive current for the parallel *R-L* network of Fig. 15.77 for a frequency range of 0 through 40 kHz.

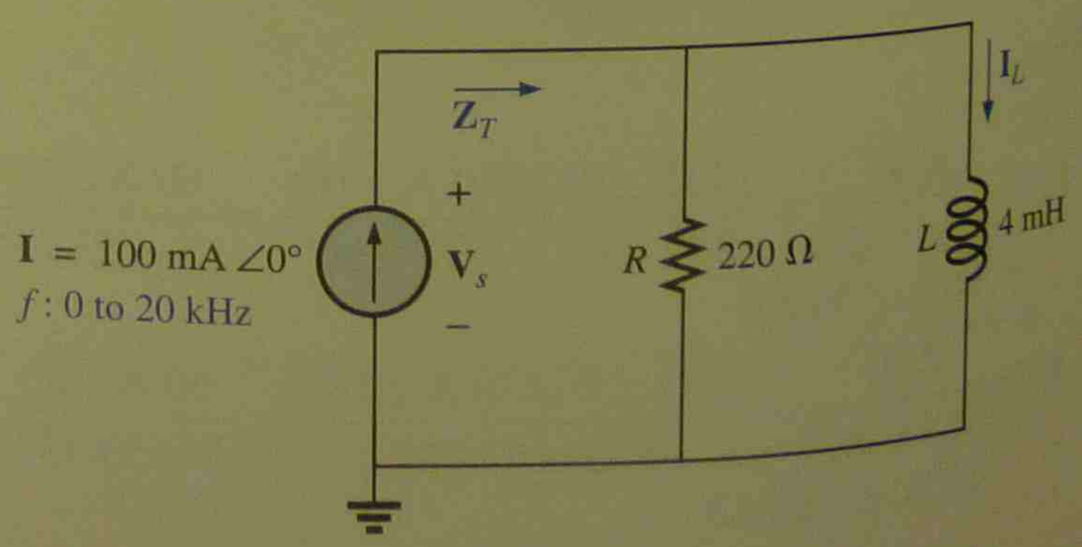


FIG. 15.77

Z_T:

Before getting into specifics, let us first develop a "sense" for the impact of frequency on the network of Fig. 15.77 by noting the impedance versus frequency curves of the individual elements, as shown in Fig. 15.78. The fact that the elements are now in parallel requires that

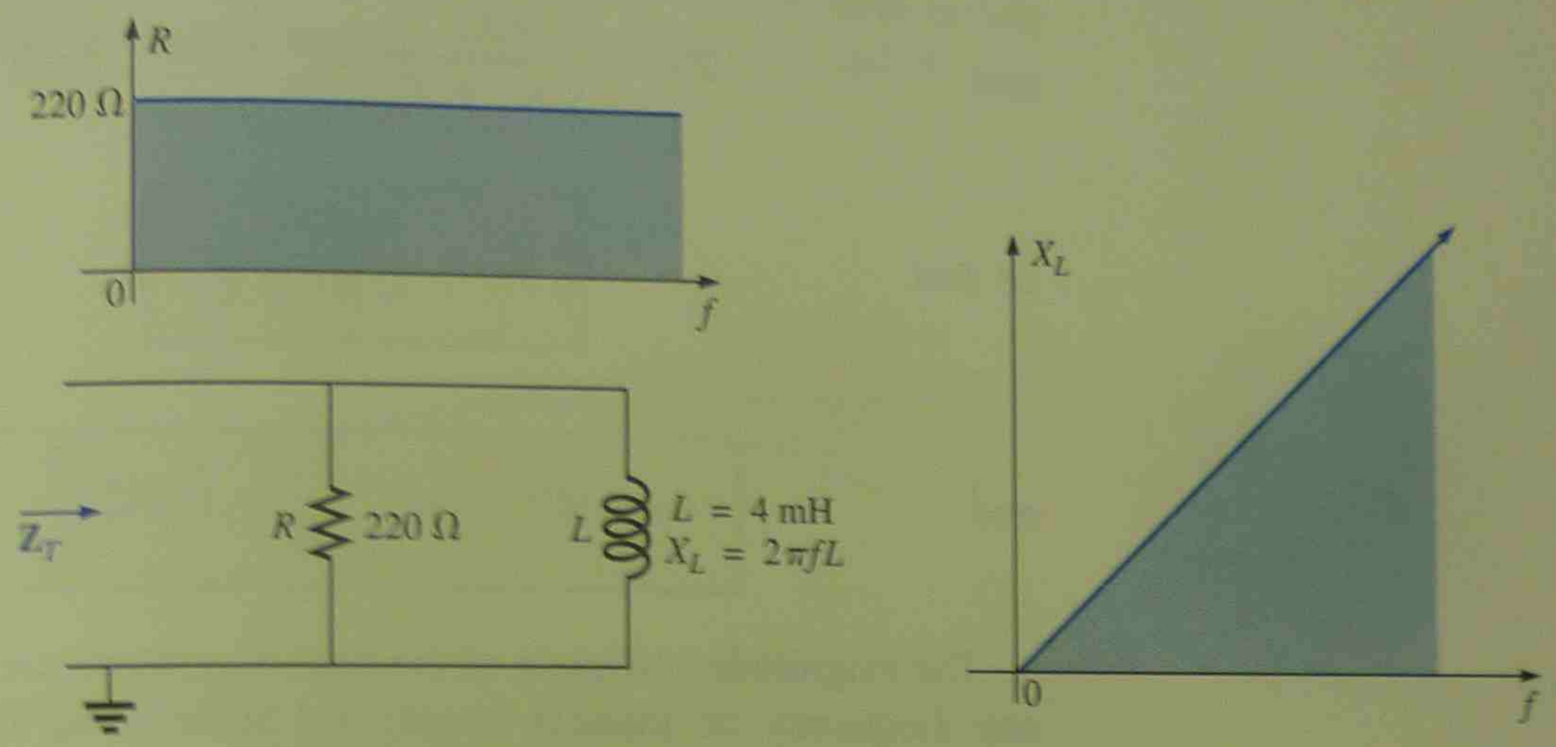


FIG. 15.78

we consider their characteristics in a different manner than occurred for the series *R-C* circuit of Section 15.5. Recall that for parallel elements, the element with the smallest impedance will have the greatest impact on the total impedance at that frequency. In Fig. 15.78, for example, X_L is very small at low frequencies compared to R, establishing X_L as the predominant factor in this frequency range. In other words, at low frequencies the network will be primarily inductive and the angle associated with the total impedance will be close to 90°, as with a pure inductor. As the frequency increases, X_L will increase until it equals the impedance of the resistor (220 Ω). The frequency at which this situation occurs can be determined in the following manner:

$$X_L = 2\pi f_2 L = R$$

and

$$\boxed{f_2 = \frac{R}{2\pi L}} \tag{15.26}$$

which for the network of Fig. 15.77 is

$$f_2 = \frac{R}{2\pi L} = \frac{220\ \Omega}{2\pi(4 \times 10^{-3}\ \text{H})}$$

$$\cong \mathbf{8.75\ kHz}$$

falling within the frequency range of interest.

For frequencies less than f_2, $X_L < R$, and for frequencies greater than f_2, $X_L > R$. A general equation for the total impedance in vector form

can be developed in the following manner:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L}$$

$$= \frac{(R\angle 0°)(X_L\angle 90°)}{R + jX_L} = \frac{RX_L\angle 90°}{\sqrt{R^2 + X_L^2}\angle \tan^{-1} X_L/R}$$

and

$$\mathbf{Z}_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}}\underline{/90° - \tan^{-1} X_L/R}$$

so that

$$\boxed{\mathbf{Z}_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}}} \tag{15.27}$$

and

$$\boxed{\theta_T = 90° - \tan^{-1}\frac{X_L}{R} = \tan^{-1}\frac{R}{X_L}} \tag{15.28}$$

The magnitude and angle of the total impedance can now be found at any frequency of interest simply by substituting Eqs. (15.27) and (15.28).

f = 1 kHz:

$$X_L = 2\pi fL = 2\pi(1\text{ kHz})(4 \times 10^{-3}\text{ H}) = 25.12\,\Omega$$

and

$$Z_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}} = \frac{(220\,\Omega)(25.12\,\Omega)}{\sqrt{(220\,\Omega)^2 + (25.12\,\Omega)^2}} = \mathbf{24.96\,\Omega}$$

with

$$\theta_T = \tan^{-1}\frac{R}{X_L} = \tan^{-1}\frac{220\,\Omega}{25.12\,\Omega}$$

$$= \tan^{-1} 8.76 = 83.49°$$

and

$$\mathbf{Z}_T = \mathbf{24.96\,\Omega\angle 83.49°}$$

This value compares very closely with $X_L = 25.12\,\Omega\angle 90°$, which it would be if the network were purely inductive ($R = \infty\,\Omega$). Our assumption that the network is primarily inductive at low frequencies is therefore confirmed.

Continuing:

$f =$ 5 kHz: $\mathbf{Z}_T = \mathbf{109.1\,\Omega\angle 60.23°}$

$f =$ 10 kHz: $\mathbf{Z}_T = \mathbf{165.5\,\Omega\angle 41.21°}$

$f =$ 15 kHz: $\mathbf{Z}_T = \mathbf{189.99\,\Omega\angle 30.28°}$

$f =$ 20 kHz: $\mathbf{Z}_T = \mathbf{201.53\,\Omega\angle 23.65°}$

$f =$ 30 kHz: $\mathbf{Z}_T = \mathbf{211.19\,\Omega\angle 16.27°}$

$f =$ 40 kHz: $\mathbf{Z}_T = \mathbf{214.91\,\Omega\angle 12.35°}$

At $f = 40$ kHz, note how closely the magnitude of Z_T has approached the resistance level of 220 Ω and how the associated angle with the total

impedance is approaching zero degrees. The result is a network with terminal characteristics that are becoming more and more resistive as the frequency increases, which further confirms the earlier conclusions developed by the curves of Fig. 15.78.

Plots of Z_T versus frequency in Fig. 15.79 and θ_T in Fig. 15.80 clearly reveal the transition from an inductive network to one that has resistive characteristics. Note that the transition frequency of 8.75 kHz occurs right in the middle of the knee of the curves for both Z_T and θ_T.

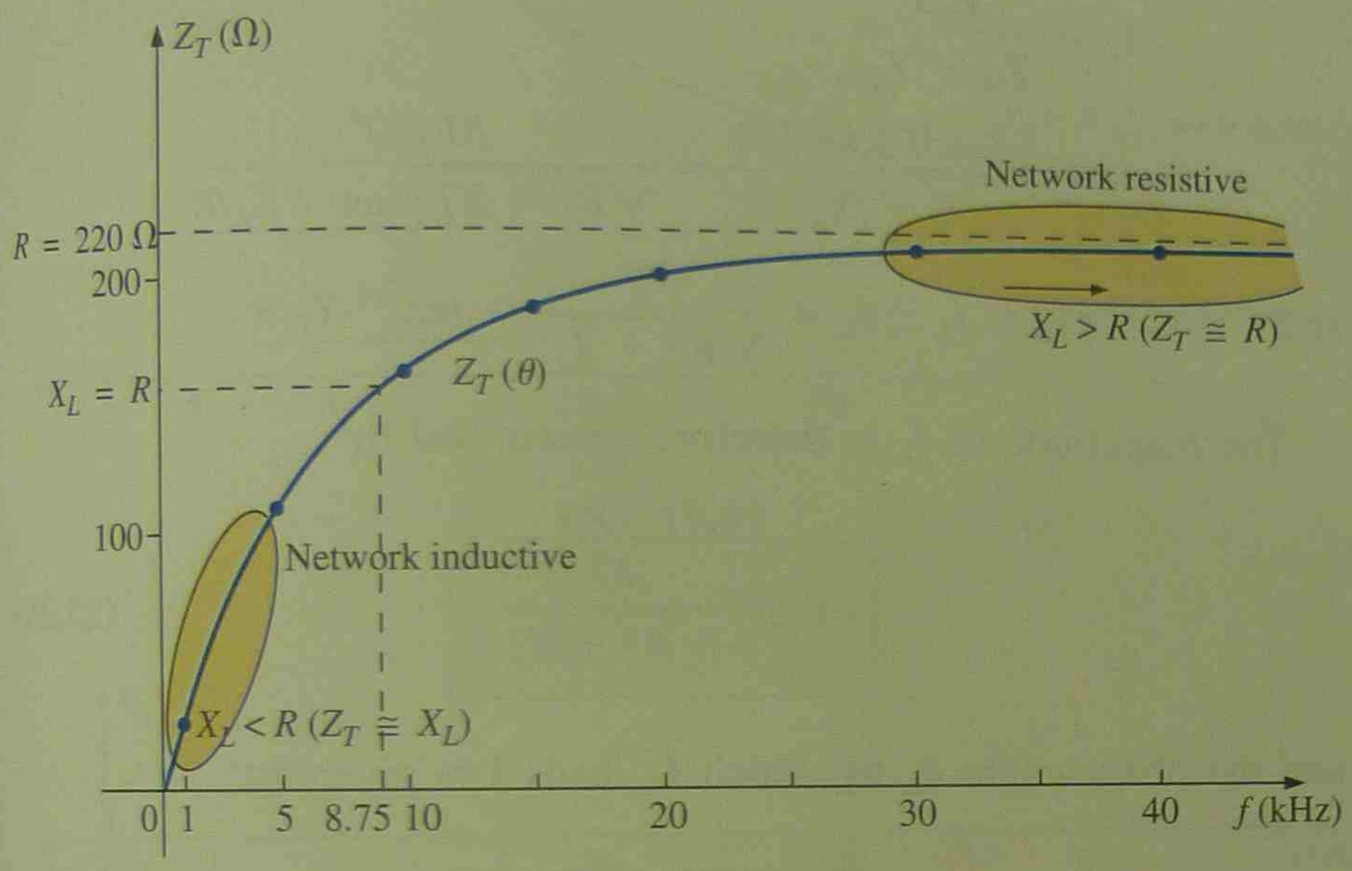


FIG. 15.79

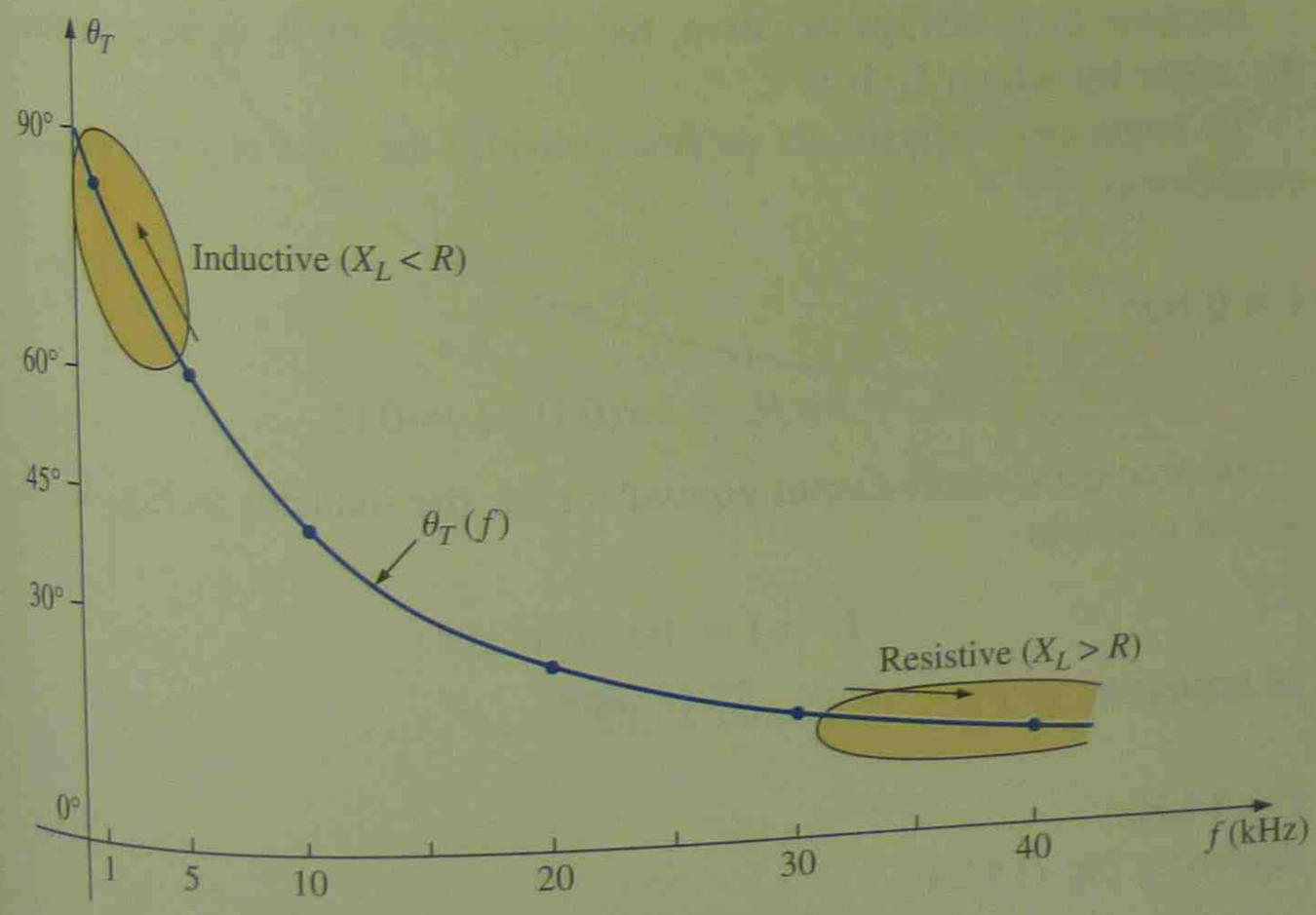


FIG. 15.80

A review of Figs. 15.46 and 15.79 will reveal that a series *R-C* and a parallel *R-L* network will have an impedance level that approaches the resistance of the network at high frequencies. The capacitive circuit

approaches the level from above, whereas the inductive network does the same from below. For the series R-L circuit and the parallel R-C network, the total impedance will begin at the resistance level and then display the characteristics of the reactive elements at high frequencies.

$\mathbf{I}_L$:

Applying the current divider rule to the network of Fig. 15.77 will result in the following:

$$\mathbf{I}_L = \frac{\mathbf{Z}_R\mathbf{I}}{\mathbf{Z}_R + \mathbf{Z}_L}$$

$$= \frac{(R\angle 0^\circ)(I\angle 0^\circ)}{R + jX_L} = \frac{RI\angle 0^\circ}{\sqrt{R^2 + X_L^2}\,\underline{/\tan^{-1} X_L/R}}$$

and

$$\mathbf{I}_L = I_L\angle\theta_L = \frac{RI}{\sqrt{R^2 + X_L^2}}\,\underline{/-\tan^{-1} X_L/R}$$

The magnitude of I_L is therefore determined by

$$\boxed{I_L = \frac{RI}{\sqrt{R^2 + X_L^2}}} \tag{15.29}$$

and the phase angle θ_L by which $\mathbf{I}_L$ leads $\mathbf{I}$ is given by

$$\boxed{\theta_L = -\tan^{-1}\frac{X_L}{R}} \tag{15.30}$$

Because θ_L is always negative, the magnitude of θ_L is, in actuality, the angle by which $\mathbf{I}_L$ lags $\mathbf{I}$.

To begin our analysis, let us first consider the case of $f = 0$ Hz (dc conditions).

$f = 0$ Hz:

$$X_L = 2\pi fL = 2\pi(0\text{ Hz})L = 0\,\Omega$$

Applying the short-circuit equivalent for the inductor in Fig. 15.77 would result in

$$\mathbf{I}_L = \mathbf{I} = 100\text{ mA}\angle 0^\circ$$

as appearing in Figs. 15.81 and 15.82.

$f = 1$ kHz:

Applying Eq. (15.28):

$$X_L = 2\pi fL = 2\pi(1\text{ kHz})(4\text{ mH}) = 25.12\,\Omega$$

and

$$\sqrt{R^2 + X_L^2} = \sqrt{(220\,\Omega)^2 + (25.12\,\Omega)^2} = 221.43\,\Omega$$

and

$$I_L = \frac{RI}{\sqrt{R^2 + X_L^2}} = \frac{(220\,\Omega)(100\text{ mA})}{221.43\,\Omega} = \mathbf{99.35\text{ mA}}$$

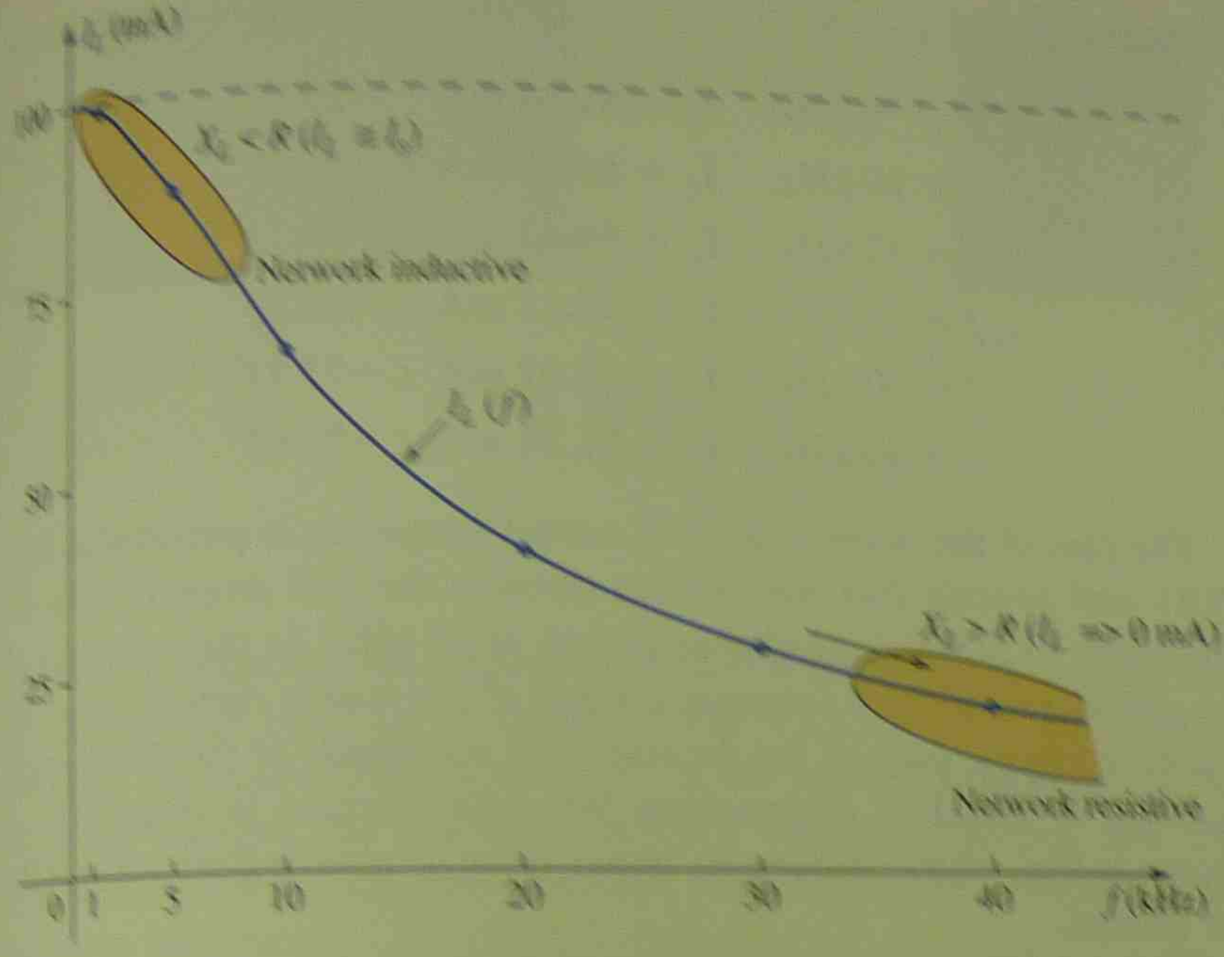


FIG. 15.81

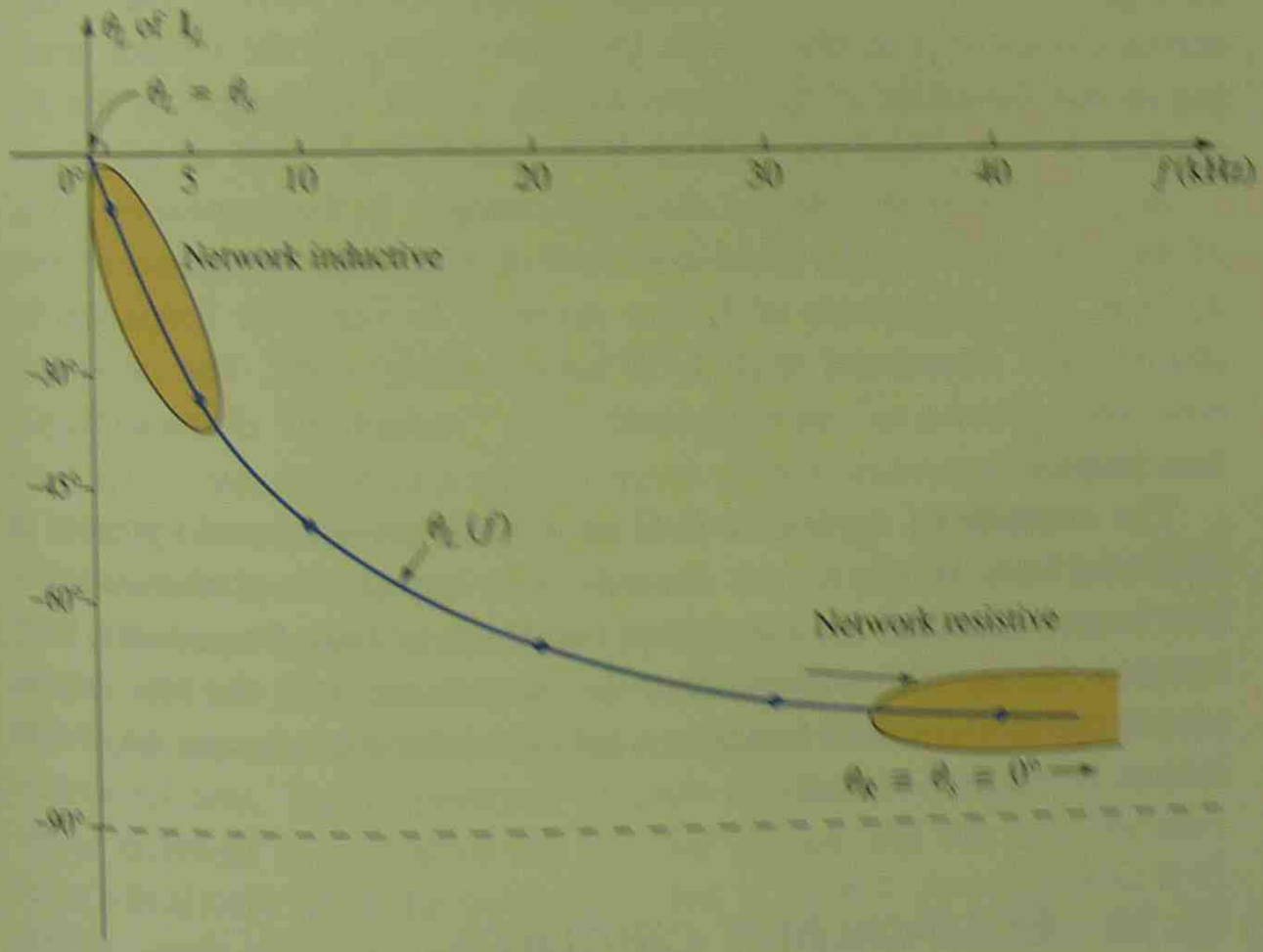


FIG. 15.82

with

$$\theta_L = \tan^{-1}\frac{X_L}{R} = -\tan^{-1}\frac{25.12\ \Omega}{220\ \Omega} = -\tan^{-1} 0.114 = -6.51°$$

and

$$\mathbf{I}_L = 99.35\ \text{mA} \angle -6.51°$$

The result is a current $\mathbf{I}_L$ that is still very close to the source current $\mathbf{I}$ in both magnitude and phase.

Continuing:

$f = 5$ kHz: $\mathbf{I}_L = \mathbf{86.84\ mA} \angle \mathbf{-29.72^\circ}$
$f = 10$ kHz: $\mathbf{I}_L = \mathbf{65.88\ mA} \angle \mathbf{-48.79^\circ}$
$f = 15$ kHz: $\mathbf{I}_L = \mathbf{50.43\ mA} \angle \mathbf{-59.72^\circ}$
$f = 20$ kHz: $\mathbf{I}_L = \mathbf{40.11\ mA} \angle \mathbf{-66.35^\circ}$
$f = 30$ kHz: $\mathbf{I}_L = \mathbf{28.02\ mA} \angle \mathbf{-73.73^\circ}$
$f = 40$ kHz: $\mathbf{I}_L = \mathbf{21.38\ mA} \angle \mathbf{-77.65^\circ}$

The plot of the magnitude of I_L versus frequency is provided in Fig. 15.81 and reveals that the current through the coil dropped from its maximum of 100 mA to almost 20 mA at 40 kHz. As the reactance of the coil increased with frequency, more of the source current chose the lower-resistance path of the resistor. The magnitude of the phase angle between $\mathbf{I}_L$ and $\mathbf{I}$ is approaching 90° with increase in frequency, as shown in Fig. 15.82, leaving its initial value of zero degrees at $f = 0$ Hz far behind.

At $f = 1$ kHz the phasor diagram of the network appears as shown in Fig. 15.83. First note that the magnitude and phase angle of $\mathbf{I}_L$ is very close to that of $\mathbf{I}$. Since the voltage across a coil must lead the current through a coil by 90°, the voltage $\mathbf{V}_s$ appears as shown. The voltage across a resistor is in phase with the current through the resistor, resulting in the direction of $\mathbf{I}_R$ shown in Fig. 15.83. Of course, at this frequency $R > X_L$, and the current I_R is relatively small in magnitude.

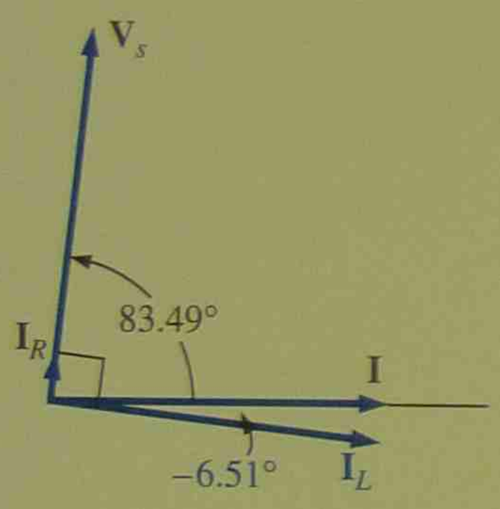


FIG. 15.83
f = 1 kHz.

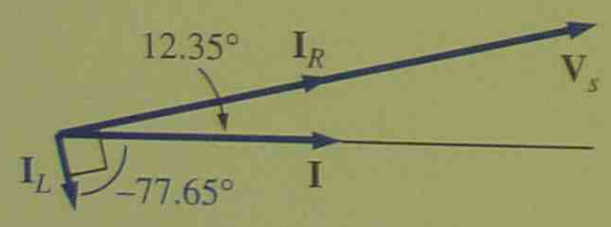


FIG. 15.84
f = 40 kHz.

At $f = 40$ kHz the phasor diagram changes to that appearing in Fig. 15.84. Note that now $\mathbf{I}_R$ and $\mathbf{I}$ are close in magnitude and phase because $X_L > R$. The magnitude of $\mathbf{I}_L$ has dropped to very low levels and the phase angle associated with $\mathbf{I}_L$ is approaching −90°. The network is now more ''resistive'' as compared to its ''inductive'' characteristics at low frequencies.

The analysis of a parallel *R-C* or *R-L-C* network would proceed in much the same manner, with the inductive impedance predominating at low frequencies and the capacitive reactance at high frequencies. Keep in mind through the analysis that the impedance with the least impedance is the predominant factor in a parallel network, whereas the reverse is true in a series circuit.

15.10 EQUIVALENT CIRCUITS

In a series ac circuit, the total impedance of two or more elements in series is often equivalent to an impedance that can be achieved with fewer elements of different values, the elements and their values being determined by the frequency applied. This is also true for parallel circuits. For the circuit of Fig. 15.85(a),

$$\mathbf{Z}_T = \frac{\mathbf{Z}_C\mathbf{Z}_L}{\mathbf{Z}_C + \mathbf{Z}_L} = \frac{(5\ \Omega \angle -90^\circ)(10\ \Omega \angle 90^\circ)}{5\ \Omega \angle -90^\circ + 10\ \Omega \angle 90^\circ} = \frac{50 \angle 0^\circ}{5 \angle 90^\circ}$$
$$= 10\ \Omega \angle -90^\circ$$

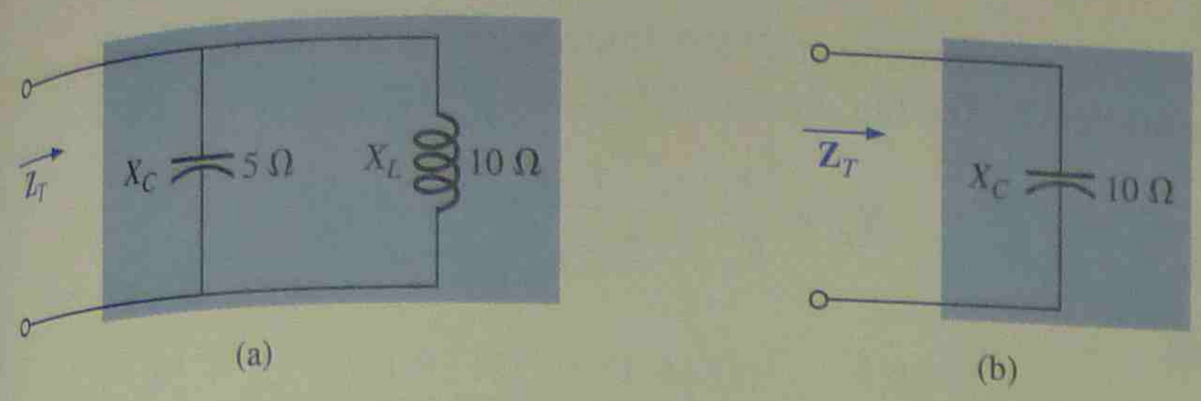


FIG. 15.85

The total impedance at the frequency applied is equivalent to a capacitor with a reactance of 10 Ω, as shown in Fig. 15.85(b). Always keep in mind that this equivalence is true only at the applied frequency. If the frequency changes, the reactance of each element changes, and the equivalent circuit will change—perhaps from capacitive to inductive in the above example.

Another interesting development appears if the impedance of a parallel circuit, such as the one of Fig. 15.86(a), is found in rectangular form. In this case,

$$\mathbf{Z}_T = \frac{\mathbf{Z}_L\mathbf{Z}_R}{\mathbf{Z}_L + \mathbf{Z}_R} = \frac{(4\ \Omega\ \angle 90^\circ)(3\ \Omega\ \angle 0^\circ)}{4\ \Omega\ \angle 90^\circ + 3\ \Omega\ \angle 0^\circ}$$

$$= \frac{12\ \angle 90^\circ}{5\ \angle 53.13^\circ} = 2.40\ \Omega\ \angle 36.87^\circ$$

$$= 1.920\ \Omega + j\,1.440\ \Omega$$

which is the impedance of a series circuit with a resistor of 1.92 Ω and an inductive reactance of 1.44 Ω, as shown in Fig. 15.86(b).

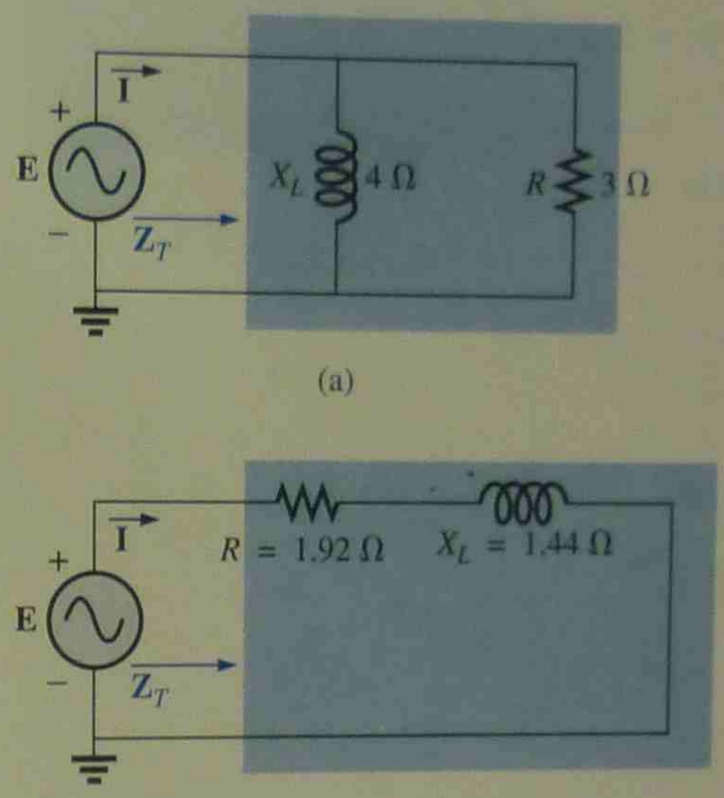
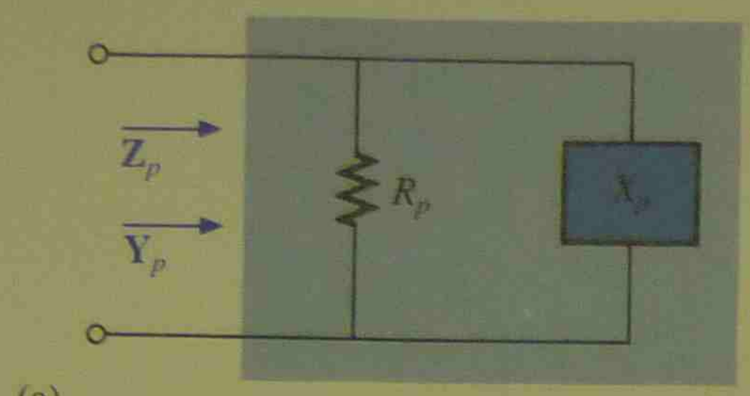


FIG. 15.86

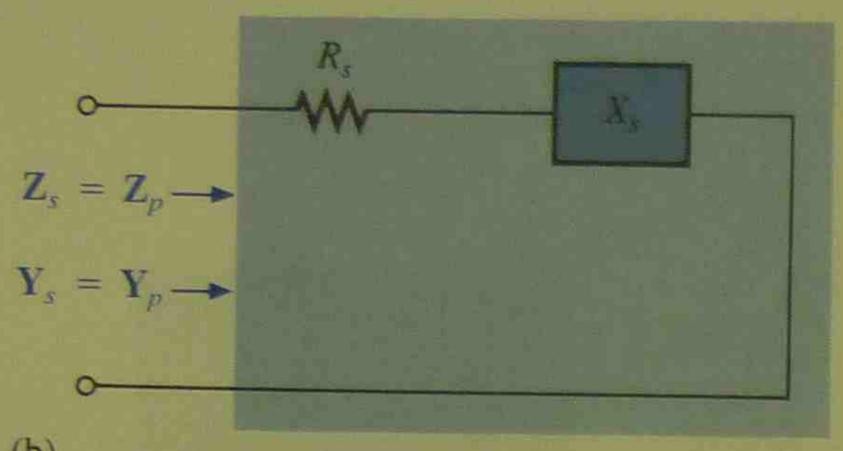
The current **I** will be the same in each circuit of Fig. 15.85 or Fig. 15.86 if the same input voltage **E** is applied. For a parallel circuit of one resistive element and one reactive element, the series circuit with the same input impedance will always be composed of one resistive and one reactive element. The impedance of each element of the series circuit will be different from that of the parallel circuit, but the reactive elements will always be of the same type; that is, an *R-L* circuit and an *R-C* parallel circuit will have an equivalent *R-L* and *R-C* series circuit, respectively. The same is true when converting from a series to a parallel circuit. In the discussion to follow, keep in mind that

the term equivalent refers only to the fact that for the same applied potential, the same impedance and input current will result.

To formulate the equivalence between the series and parallel circuits, the equivalent series circuit for a resistor and reactance in parallel can be



(a)



(b)

FIG. 15.87

found by determining the total impedance of the circuit in rectangular form; that is, for the circuit of Fig. 15.87(a),

$$\mathbf{Y}_p = \frac{1}{R_p} + \frac{1}{\pm jX_p} = \frac{1}{R_P} \mp j\frac{1}{X_p}$$

and

$$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \frac{1}{(1/R_p) \mp j(1/X_p)}$$
$$= \frac{1/R_p}{(1/R_p)^2 + (1/X_p)^2} \pm j\frac{1/X_p}{(1/R_p)^2 + (1/X_p)^2}$$

Multiplying the numerator and denominator of each term by $R_p^2 X_p^2$ results in

$$\mathbf{Z}_p = \frac{R_p X_p^2}{X_p^2 + R_p^2} \pm j\frac{R_p^2 X_p}{X_p^2 + R_p^2}$$
$$= R_s \pm jX_s \qquad \text{[Fig. 15.87(b)]}$$

and

$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} \tag{15.31}$$

with

$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} \tag{15.32}$$

For the network of Fig. 15.86,

$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} = \frac{(3\ \Omega)(4\ \Omega)^2}{(4\ \Omega)^2 + (3\ \Omega)^2} = \frac{48\ \Omega}{25} = \mathbf{1.920\ \Omega}$$

and

$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} = \frac{(3\ \Omega)^2(4\ \Omega)}{(4\ \Omega)^2 + (3\ \Omega)^2} = \frac{36\ \Omega}{25} = \mathbf{1.440\ \Omega}$$

which agrees with the previous result.

The equivalent parallel circuit for a circuit with a resistor and reactance in series can be found by simply finding the total admittance of the system in rectangular form; that is, for the circuit of Fig. 15.87(b),

$$\mathbf{Z}_s = R_s \pm jX_s$$

$$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = \frac{1}{R_s \pm jX_s} = \frac{R_s}{R_s^2 + X_s^2} \mp j\frac{X_s}{R_s^2 + X_s^2}$$
$$= G_p \mp jB_p = \frac{1}{R_p} \mp j\frac{1}{X_p} \qquad \text{[Fig. 15.87(a)]}$$

or

$$R_p = \frac{R_s^2 + X_s^2}{R_s} \tag{15.33}$$

with
$$X_p = \frac{R_s^2 + X_s^2}{X_s} \tag{15.34}$$

For the above example,

$$R_p = \frac{R_s^2 + X_s^2}{R_s} = \frac{(1.92\ \Omega)^2 + (1.44\ \Omega)^2}{1.92\ \Omega} = \frac{5.76\ \Omega}{1.92} = \mathbf{3.0\ \Omega}$$

and
$$X_p = \frac{R_s^2 + X_s^2}{X_s} = \frac{5.76\ \Omega}{1.44} = \mathbf{4.0\ \Omega}$$

as shown in Fig. 15.86(a).

EXAMPLE 15.17 Determine the series equivalent circuit for the network of Fig. 15.88.

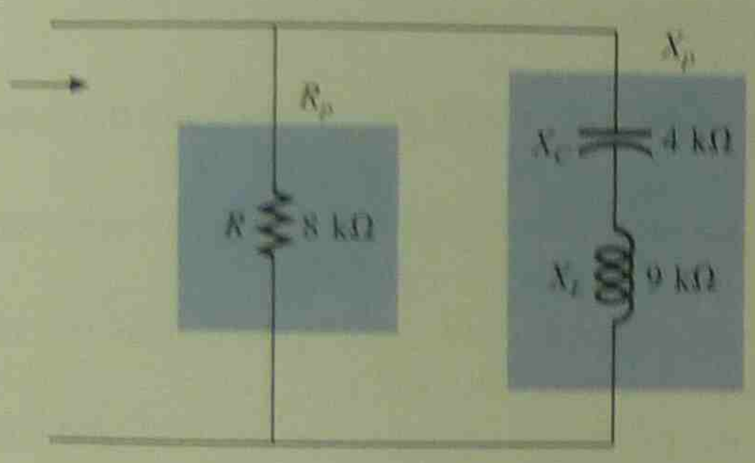


FIG. 15.88

Solution:

$$R_p = 8\ \text{k}\Omega$$
$$X_p\ (\text{resultant}) = |X_L - X_C| = |9\ \text{k}\Omega - 4\ \text{k}\Omega|$$
$$= 5\ \text{k}\Omega$$

and
$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} = \frac{(8\ \text{k}\Omega)(5\ \text{k}\Omega)^2}{(5\ \text{k}\Omega)^2 + (8\ \text{k}\Omega)^2} = \frac{200\ \text{k}\Omega}{89} = \mathbf{2.247\ k\Omega}$$

with
$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} = \frac{(8\ \text{k}\Omega)^2(5\ \text{k}\Omega)}{(5\ \text{k}\Omega)^2 + (8\ \text{k}\Omega)^2} = \frac{320\ \text{k}\Omega}{89}$$
$$= \mathbf{3.596\ k\Omega} \quad \textbf{(inductive)}$$

The equivalent series circuit appears in Fig. 15.89.

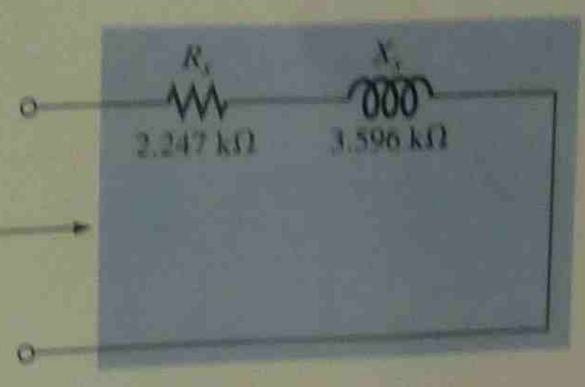


FIG. 15.89

EXAMPLE 15.18 For the network of Fig. 15.90:

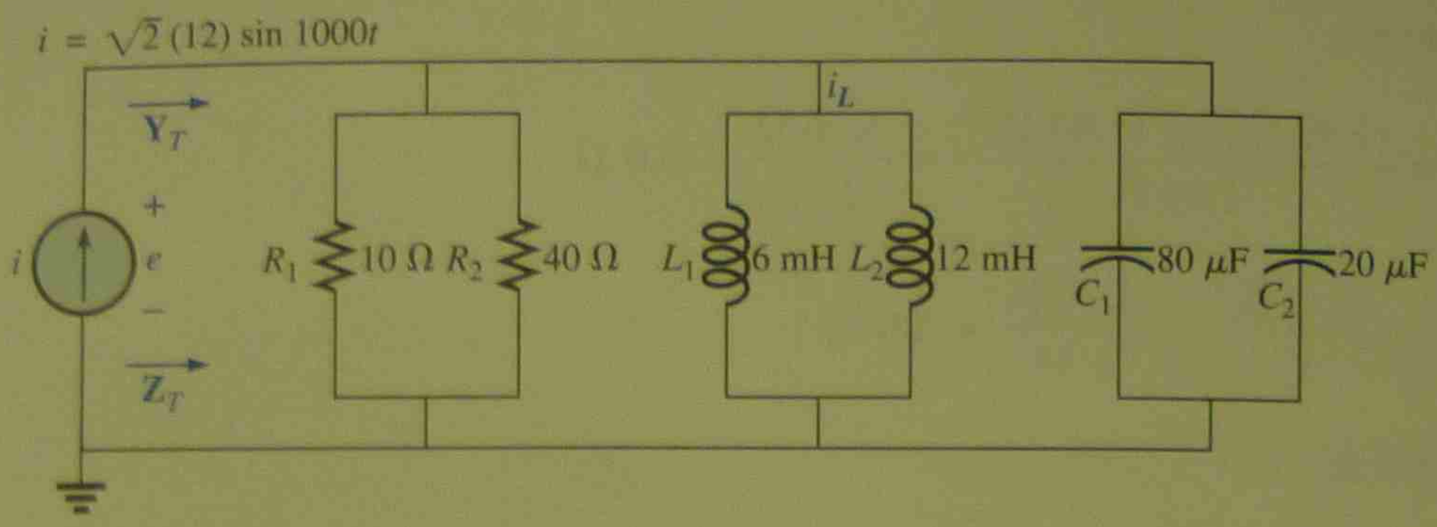


FIG. 15.90

a. Determine $\mathbf{Y}_T$.
b. Sketch the admittance diagram.
c. Find $\mathbf{E}$ and $\mathbf{I}_L$.
d. Compute the power factor of the network and the power delivered to the network.
e. Determine the equivalent series circuit as far as the terminal characteristics of the network are concerned.
f. Using the equivalent circuit developed in part (e), calculate $\mathbf{E}$ and compare with the result of part (c).
g. Determine the power delivered to the network and compare with the solution of part (d).
h. Determine the equivalent parallel network from the equivalent series circuit and calculate the total admittance $\mathbf{Y}_T$. Compare the result with the solution of part (a).

Solutions:

a. Combining common elements and finding the reactance of the inductor and capacitor, we obtain

$$R_T = 10\ \Omega \,\|\, 40\ \Omega = 8\ \Omega$$
$$L_T = 6\ \text{mH} \,\|\, 12\ \text{mH} = 4\ \text{mH}$$
$$C_T = 80\ \mu\text{F} + 20\ \mu\text{F} = 100\ \mu\text{F}$$
$$X_L = \omega L = (1000\ \text{rad/s})(4\ \text{mH}) = 4\ \Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(1000\ \text{rad/s})(100\ \mu\text{F})} = 10\ \Omega$$

The network is redrawn in Fig. 15.91 with phasor notation. The total admittance is

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C$$
$$= G\angle 0° + B_L\angle -90° + B_C\angle +90°$$
$$= \frac{1}{8\ \Omega}\angle 0° + \frac{1}{4\ \Omega}\angle -90° + \frac{1}{10\ \Omega}\angle +90°$$

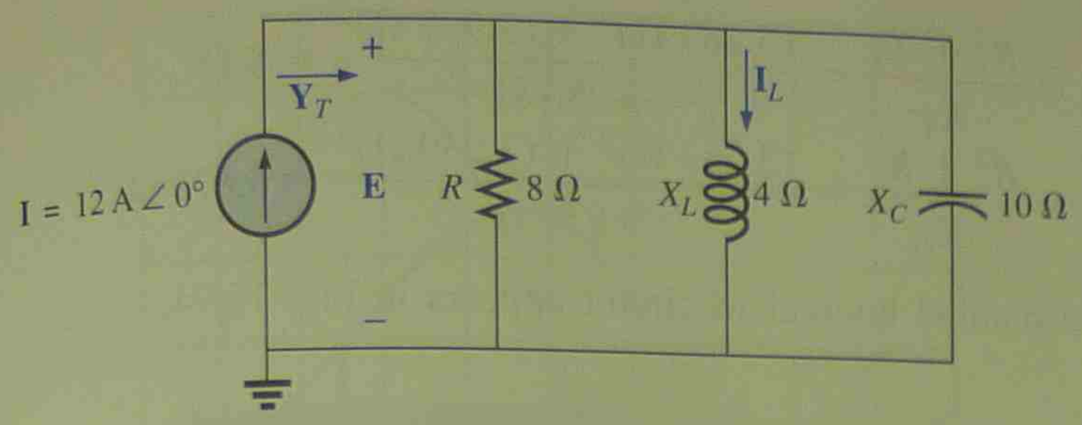


FIG. 15.91

$$= 0.125\text{ S}\angle 0° + 0.25\text{ S}\angle -90° + 0.1\text{ S}\angle +90°$$
$$= 0.125\text{ S} - j0.25\text{ S} + j0.1\text{ S}$$
$$= 0.125\text{ S} - j0.15\text{ S} = \mathbf{0.195\text{ S}\angle -50.194°}$$

b. See Fig. 15.92.

c. $$\mathbf{E} = \mathbf{I}\mathbf{Z}_T = \frac{\mathbf{I}}{\mathbf{Y}_T} = \frac{12\text{ A}\angle 0°}{0.195\text{ S}\angle -50.194°} = \mathbf{61.538\text{ V}\angle 50.194°}$$

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{\mathbf{E}}{\mathbf{Z}_L} = \frac{61.538\text{ V}\angle 50.194°}{4\,\Omega\angle 90°} = \mathbf{15.385\text{ A}\angle -39.81°}$$

d. $$F_P = \cos\theta = \frac{G}{Y_T} = \frac{0.125\text{ S}}{0.195\text{ S}} = \mathbf{0.641}\text{ lagging (E leads I)}$$

$$P = EI\cos\theta = (61.538\text{ V})(12\text{ A})\cos 50.194°$$
$$= \mathbf{472.75\text{ W}}$$

e. $$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{0.195\text{ S}\angle -50.194°} = 5.128\,\Omega\angle +50.194°$$
$$= 3.283\,\Omega + j3.939\,\Omega$$
$$= R + jX_L$$

$$X_L = 3.939\,\Omega = \omega L$$

and $$L = \frac{3.939\,\Omega}{\omega} = \frac{3.939\,\Omega}{1000\text{ rad/s}} = \mathbf{3.939\text{ mH}}$$

The series equivalent circuit appears in Fig. 15.93.

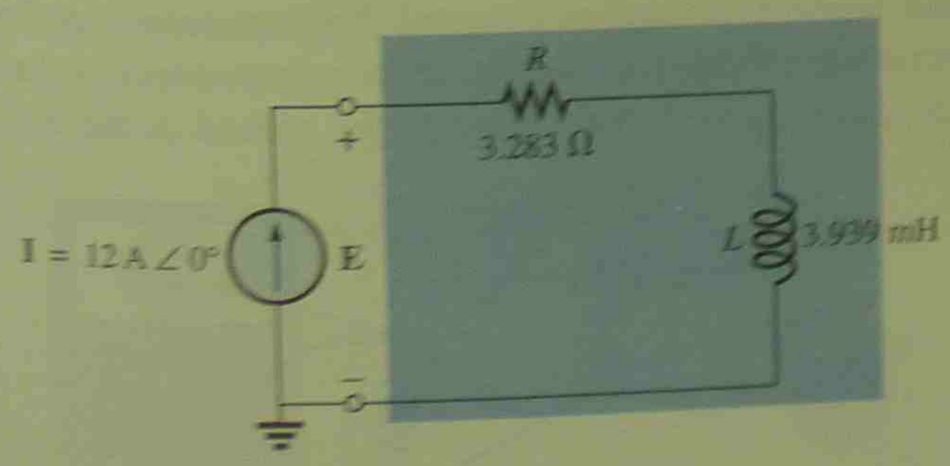


FIG. 15.93

f. $$\mathbf{E} = \mathbf{I}\mathbf{Z}_T = (12\text{ A}\angle 0°)(5.128\,\Omega\angle 50.194°)$$
$$= \mathbf{61.536\text{ V}\angle 50.194°}\text{, as above}$$

g. $$P = I^2R = (12\text{ A})^2 3.283\,\Omega = \mathbf{472.75\text{ W}}$$ (as above)

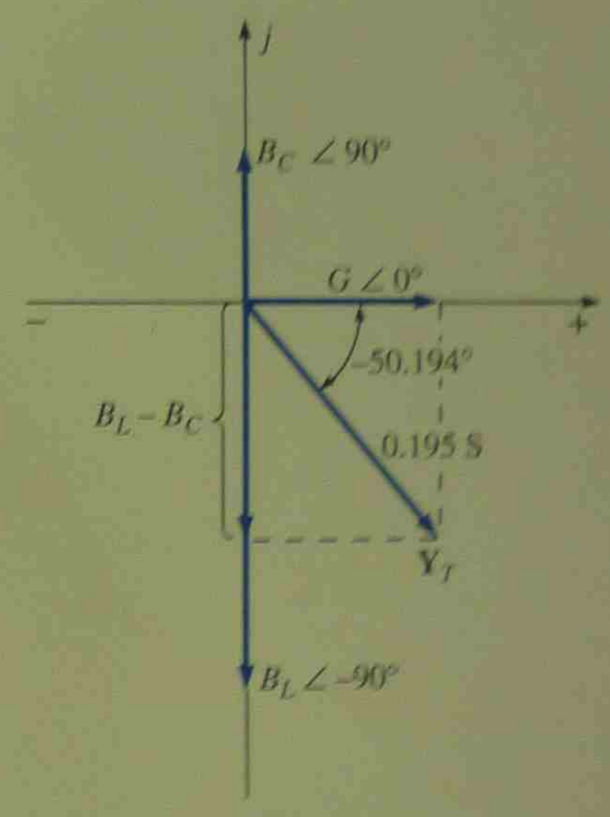


FIG. 15.92

h. $$R_p = \frac{R_s^2 + X_s^2}{R_s} = \frac{(3.283\ \Omega)^2 + (3.939\ \Omega)^2}{3.283\ \Omega} = \mathbf{8\ \Omega}$$

$$X_p = \frac{R_s^2 + X_s^2}{X_s} = \frac{(3.283\ \Omega)^2 + (3.939\ \Omega)^2}{3.939\ \Omega} = \mathbf{6.675\ \Omega}$$

The parallel equivalent circuit appears in Fig. 15.94.

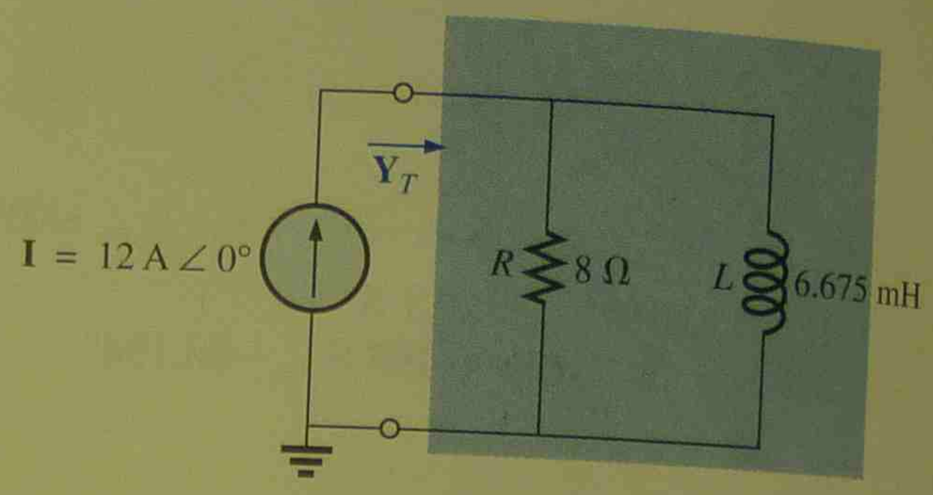


FIG. 15.94

$$\mathbf{Y}_T = G\angle 0° + B_L\angle -90° = \frac{1}{8\ \Omega}\angle 0° + \frac{1}{6.675\ \Omega}\angle -90°$$
$$= 0.125\ \text{S}\angle 0° + 0.15\ \text{S}\angle -90°$$
$$= 0.125\ \text{S} - j0.15\ \text{S} = \mathbf{0.195S}\angle \mathbf{-50.194°} \quad \text{(as above)}$$

15.11 PHASE MEASUREMENTS (DUAL-TRACE OSCILLOSCOPE)

The phase shift between the voltages of a network or between the voltages and currents of a network can be found using a dual-trace (two signals displayed at the same time) oscilloscope. Phase-shift measurements can also be performed using a single-trace oscilloscope by properly interpreting the resulting Lissajous patterns obtained on the screen. This latter approach, however, will be left for the laboratory experience.

In Fig. 15.95 channel 1 of the dual-trace oscilloscope is hooked up to display the applied voltage e. Channel 2 is connected to display the voltage across the inductor v_L. Of particular importance is the fact that the ground of the scope is connected to the ground of the oscilloscope

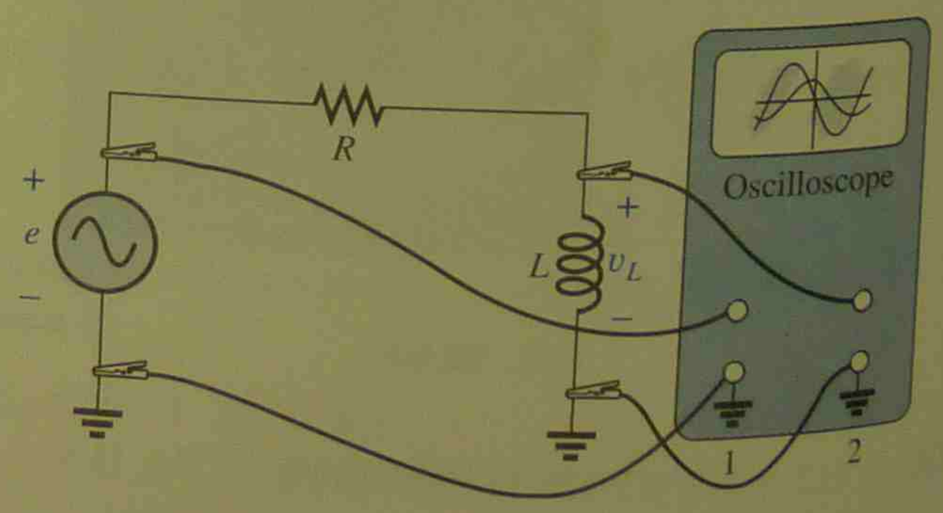


FIG. 15.95
Determining the phase relationship between e and v_L.

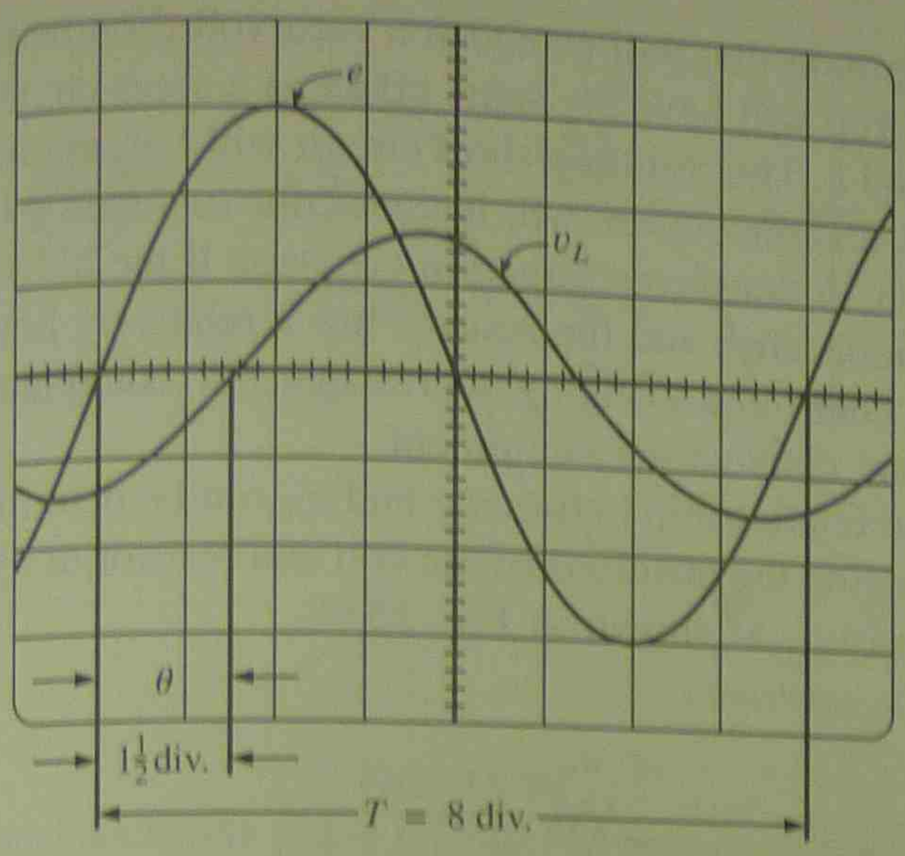


FIG. 15.96
Determining the phase angle between e and v_L

for both channels. In other words, there is only one common ground for the circuit and oscilloscope. The resulting waveforms may appear as shown in Fig. 15.96.

For the chosen horizontal sensitivity, each waveform of Fig. 15.96 has a period T defined by eight horizontal divisions and the phase angle between the two waveforms is defined by $1\frac{1}{2}$ divisions. Using the fact that each period of a sinusoidal waveform encompasses 360°, the following ratios can be set up to determine the phase angle θ.

$$\frac{8 \text{ divisions}}{360^\circ} = \frac{1.5 \text{ divisions}}{\theta}$$

and

$$\theta = \left(\frac{1.5}{8}\right)360^\circ = \mathbf{67.5^\circ}$$

In general,

$$\theta = \frac{(Div.\ for\ \theta)}{(Div.\ for\ T)} \times 360^\circ \qquad \textbf{(15.35)}$$

If the phase relationship between e and v_R is required, the oscilloscope *must not* be hooked up as shown in Fig. 15.97. Points a and b have

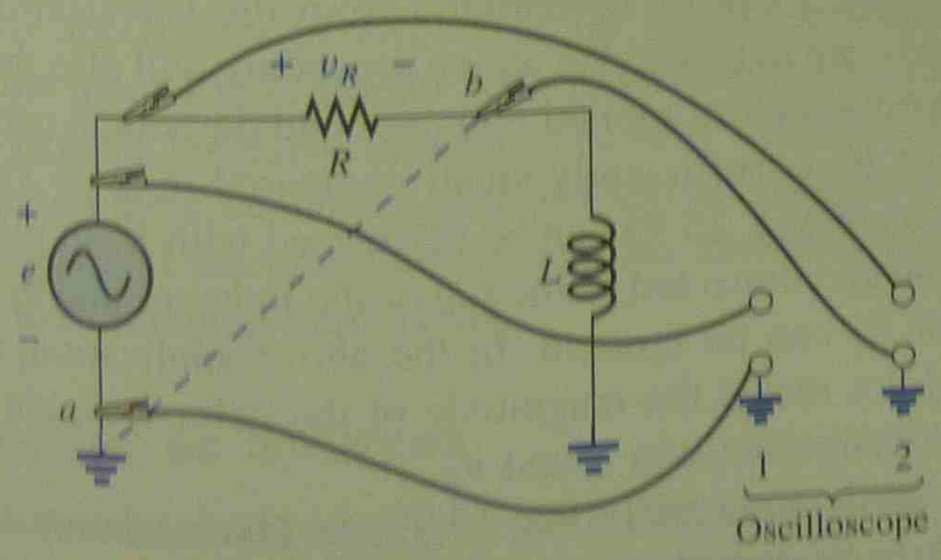


FIG. 15.97
An improper phase-measurement connection.

a common ground that will establish a zero-volt drop between the two points; this drop will have the same effect as a short-circuit connection between a and b. The resulting short circuit will "short out" the inductive element, and the current will increase due to the drop in impedance for the circuit. A dangerous situation can arise if the inductive element has a high impedance and the resistor has a relatively low impedance. The current, controlled solely by the resistance R, could jump to dangerous levels and damage the equipment.

The phase relationship between e and v_R can be determined by simply interchanging the positions of the coil and resistor or by introducing a sensing resistor, as shown in Fig. 15.98.

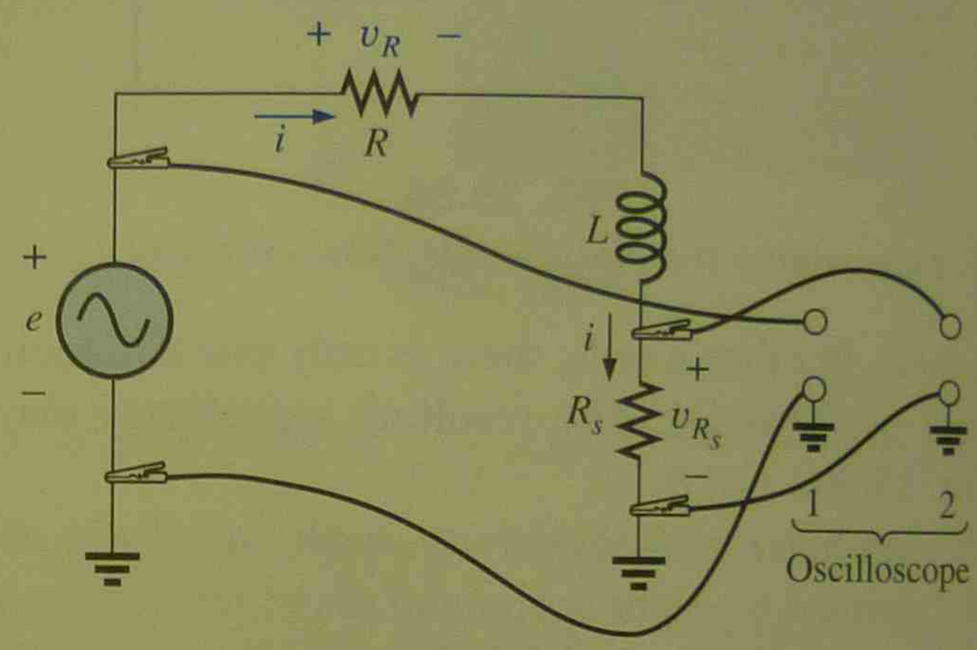


FIG. 15.98

Determining the phase relationship between e and v_R or e and i using a sensing resistor.

A sensing resistor is exactly that, introduced to "sense" a quantity without adversely affecting the behavior of the network. In other words, the sensing resistor must be small enough compared to the other impedances of the network not to cause a significant change in the voltage and current levels or phase relationships. Note that the sensing resistor is introduced in a way that will result in one end being connected to the common ground of the network. In Fig. 15.98, channel 2 will display the voltage v_{R_s}, which is in phase with the current i. However, the current i is also in phase with the voltage v_R across the resistor R. The net result is that the voltages v_{R_s} and v_R are in phase and the phase relationship between e and v_R can be determined from the waveforms e and v_R. Since v_{R_s} and i are in phase, the above procedure will also determine the phase angle between the applied voltage e and the source current i. If the magnitude of R_s is sufficiently small compared to R or X_L, the phase measurements of Fig. 15.95 can be performed with R_s in place. That is, channel 2 can be connected to the top of the inductor and to ground and the effect of R_s can be ignored. In the above application, the sensing resistor will not reveal the magnitude of the voltage v_R but simply the phase relationship between e and v_R.

For the parallel network of Fig. 15.99, the phase relationship between two of the branch currents, i_R and i_L, can be determined using a sensing resistor, as shown in the figure. Channel 1 will display the voltage v_R

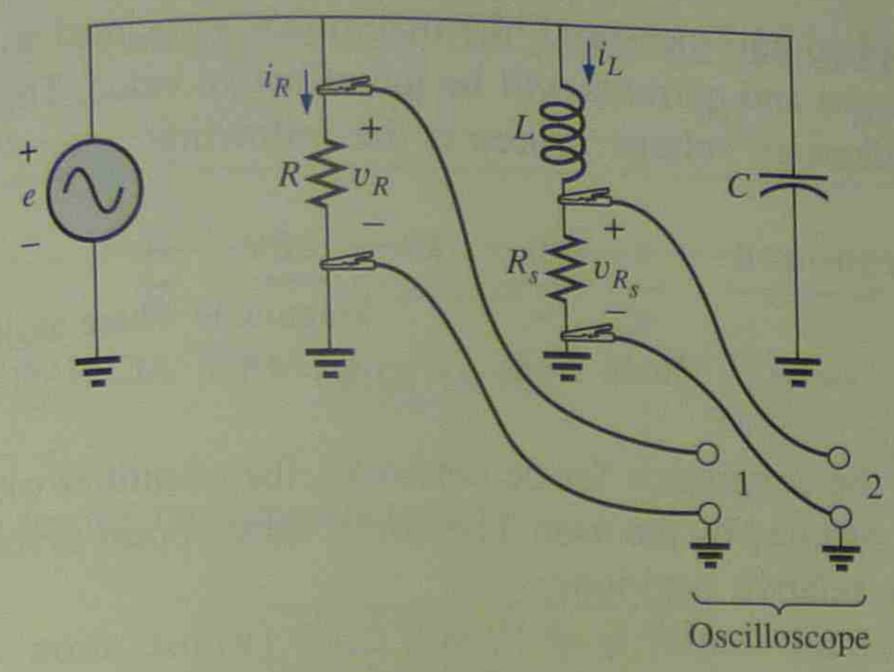


FIG. 15.99
Determining the phase relationship between i_R and i_L.

and channel 2 will display the voltage v_{R_s}. Since v_R is in phase with i_R and v_{R_s} is in phase with the current i_L, the phase relationship between v_R and v_{R_s} will be the same as that between i_R and i_L. In this case, the magnitudes of the current levels can be determined using Ohm's law and the resistance levels R and R_s, respectively.

If the phase relationship between e and i_s of Fig. 15.99 is required, a sensing resistor can be employed, as shown in Fig. 15.100.

In general, therefore, for dual-trace measurements of phase relationships, be particularly careful of the grounding arrangement and fully utilize the in-phase relationship between the voltage and current of a resistor.

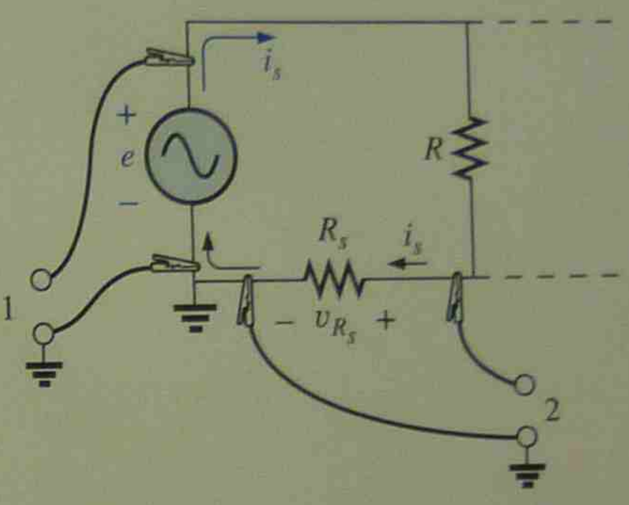


FIG. 15.100
Determining the phase relationship between e and i_s.

15.12 COMPUTER ANALYSIS

We have now covered sufficient material to make effective use of the PSpice software package. Although all the examples to appear in this section can be solved using BASIC, this section will be limited to PSpice and the commands required to apply the package to ac systems.

PSpice

As with the analysis just described, there is a great deal of similarity between the application of PSpice to dc and ac systems. The basic format of the input file is similar with passive elements (R, L, and C) entered in exactly the same manner. The two elements that have the most impact on the command statements are the frequency or frequencies of interest and the phase angles applied and obtained, as demonstrated by the following commands.

Independent ac Sources

The term *independent* (as described more than once in the text) simply means that the magnitude of the source is independent of any other parameters of the network. Since effective values are the most fre-