11 Magnetic Circuits

11.1 INTRODUCTION

Magnetism plays an integral part in almost every electrical device used today in industry, research, or the home. Generators, motors, transformers, circuit breakers, televisions, computers, tape recorders, and telephones all employ magnetic effects to perform a variety of important tasks.

The compass, used by Chinese sailors as early as the second century A.D., relies on a *permanent magnet* for indicating direction. The permanent magnet is made of a material, such as steel or iron, that will remain magnetized for long periods of time without the need for an external source of energy.

In 1820, the Danish physicist Hans Christian Oersted discovered that the needle of a compass would deflect if brought near a current-carrying conductor. For the first time it was demonstrated that electricity and magnetism were related, and in the same year the French physicist André-Marie Ampère performed experiments in this area and developed what is presently known as *Ampère's circuital law*. In subsequent years, men such as Michael Faraday, Karl Friedrich Gauss, and James Clerk Maxwell continued to experiment in this area and developed many of the basic concepts of *electromagnetism*—magnetic effects induced by the flow of charge, or current.

There is a great deal of similarity between the analyses of electric circuits and magnetic circuits. This will be demonstrated later in this chapter when we compare the basic equations and methods used to solve magnetic circuits with those used for electric circuits.

Difficulty in understanding methods used with magnetic circuits will often arise in simply learning to use the proper set of units, not because of the equations themselves. The problem exists because three different systems of units are still being used in the industry. To the extent practical, SI will be used throughout this chapter. For the CGS and English systems, a conversion table is provided in Appendix G.

11.2 MAGNETIC FIELDS

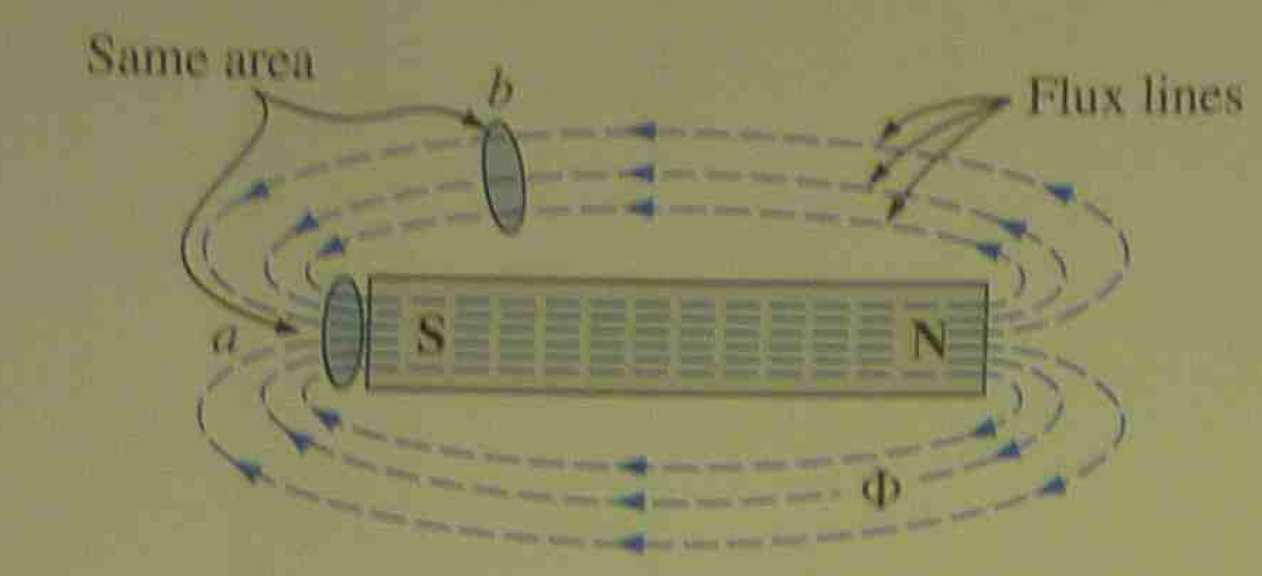


FIG. 11.1

In the region surrounding a permanent magnet there exists a magnetic field, which can be represented by magnetic flux lines similar to electric flux lines. Magnetic flux lines, however, do not have origins or terminating points like electric flux lines but exist in continuous loops, as shown in Fig. 11.1. The symbol for magnetic flux is the Greek letter Φ (phi).

The magnetic flux lines radiate from the north pole to the south pole, returning to the north pole through the metallic bar. Note the equal spacing between the flux lines within the core and the symmetric distribution outside the magnetic material. These are additional properties of magnetic flux lines in homogeneous materials (that is, materials having uniform structure or composition throughout). It is also important to realize that the continuous magnetic flux line will strive to occupy as small an area as possible. This will result in magnetic flux lines of minimum length between the like poles, as shown in Fig. 11.2. The strength of a magnetic field in a particular region is directly related to the density of flux lines in that region. In Fig. 11.1, for example, the magnetic field strength at *a* is twice that at *b* since there are twice as many magnetic flux lines associated with the perpendicular plane at *a* than at *b*. Recall from childhood experiments how the strength of permanent magnets was always stronger near the poles.

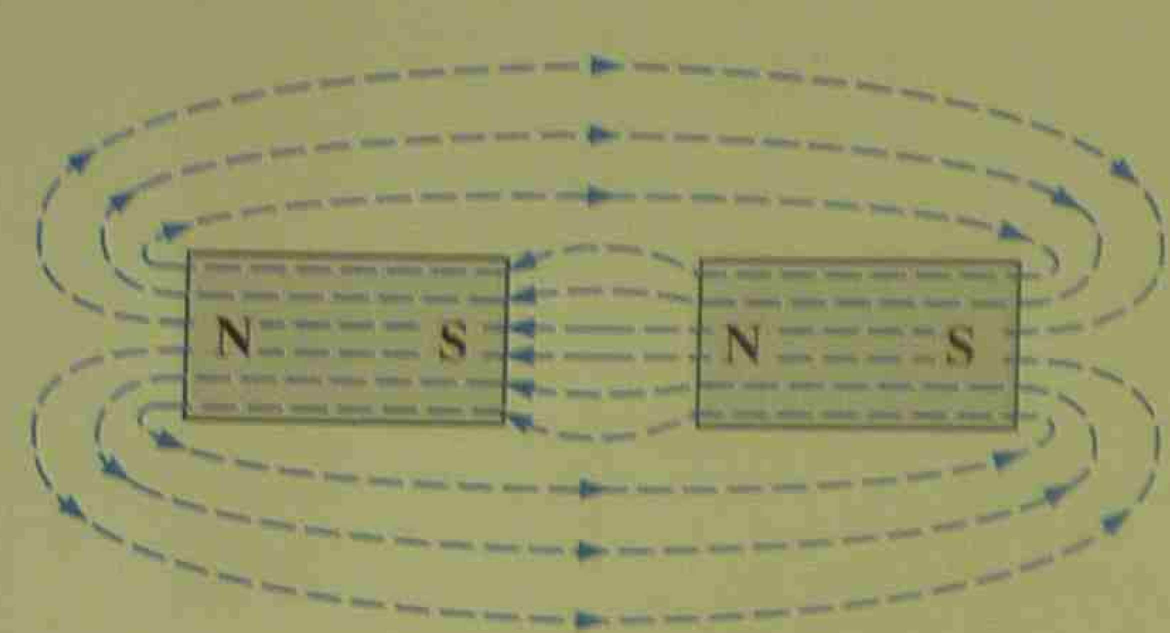


FIG. 11.2

If unlike poles of two permanent magnets are brought together, the magnets will attract, and the flux distribution will be as shown in Fig. 11.2. If like poles are brought together, the magnets will repel, and the flux distribution will be as shown in Fig. 11.3.

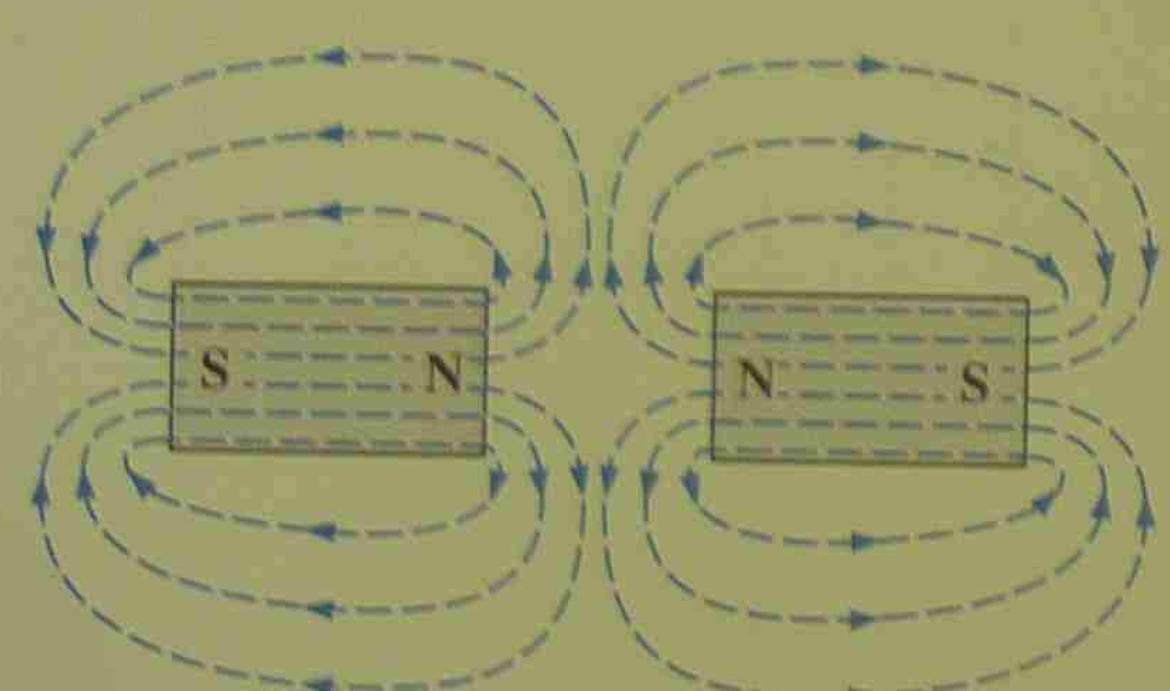


FIG. 11.3

If a nonmagnetic material, such as glass or copper, is placed in the flux paths surrounding a permanent magnet, there will be an almost unnoticeable change in the flux distribution (Fig. 11.4). However, if a magnetic material, such as soft iron, is placed in the flux path, the flux lines will pass through the soft iron rather than the surrounding air because flux lines pass with greater ease through magnetic materials than through air. This principle is put to use in the shielding of sensitive electrical elements and instruments that can be affected by stray magnetic fields (Fig. 11.5).

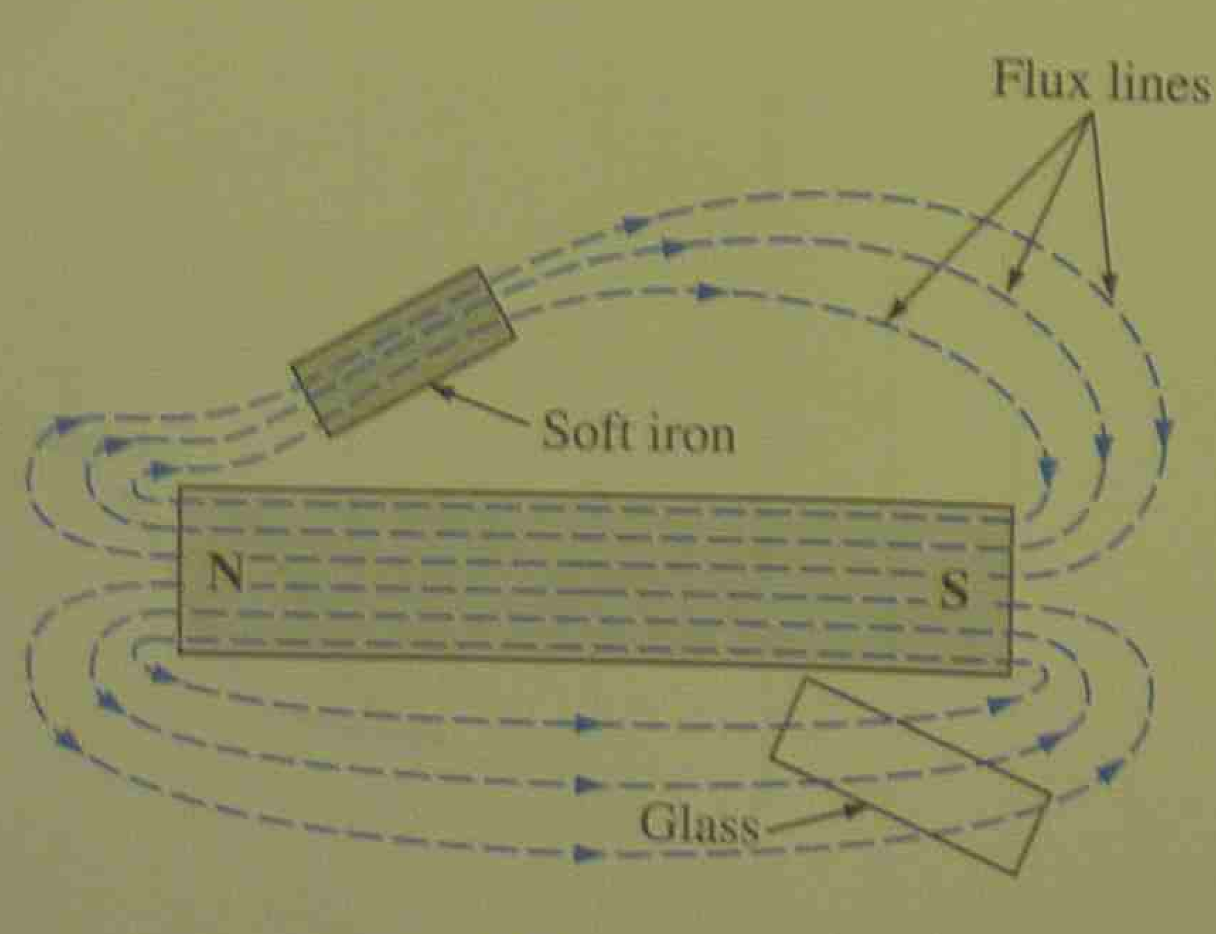


FIG. 11.4

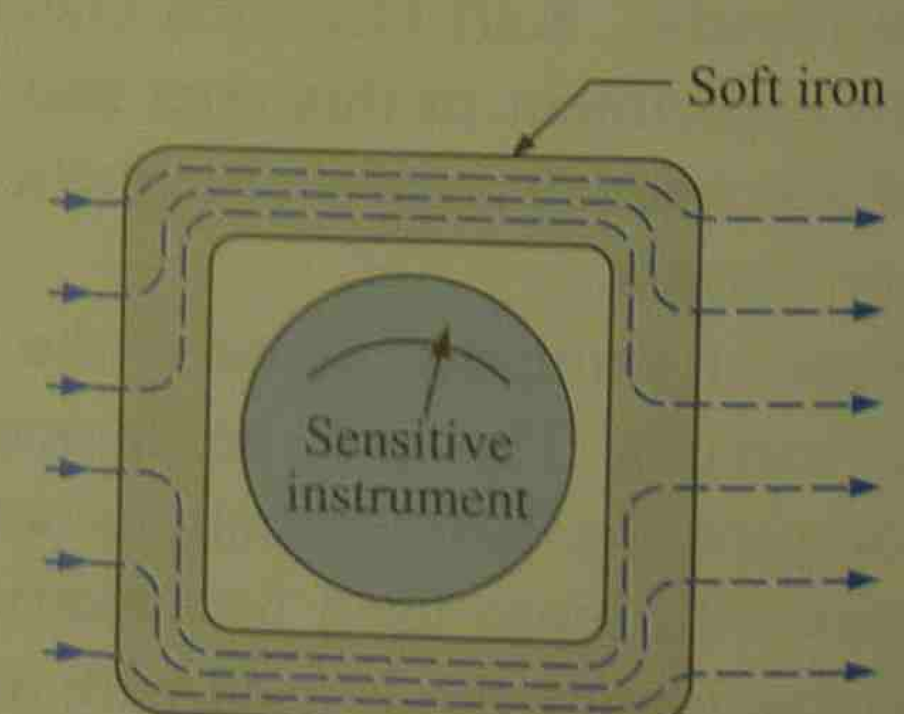


FIG. 11.5

As indicated in the introduction, a magnetic field (represented by concentric magnetic flux lines, as in Fig. 11.6) is present around every wire that carries an electric current. The direction of the magnetic flux

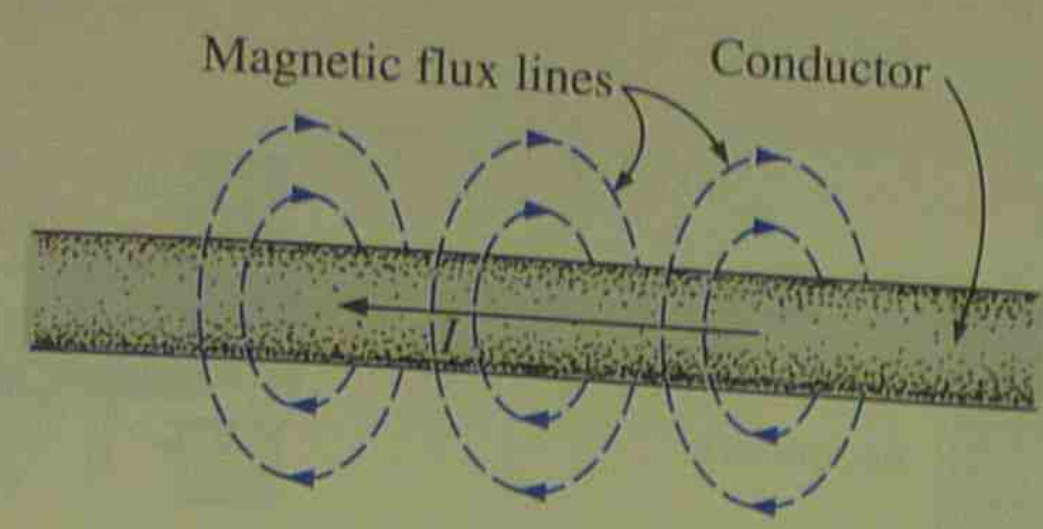


FIG. 11.6

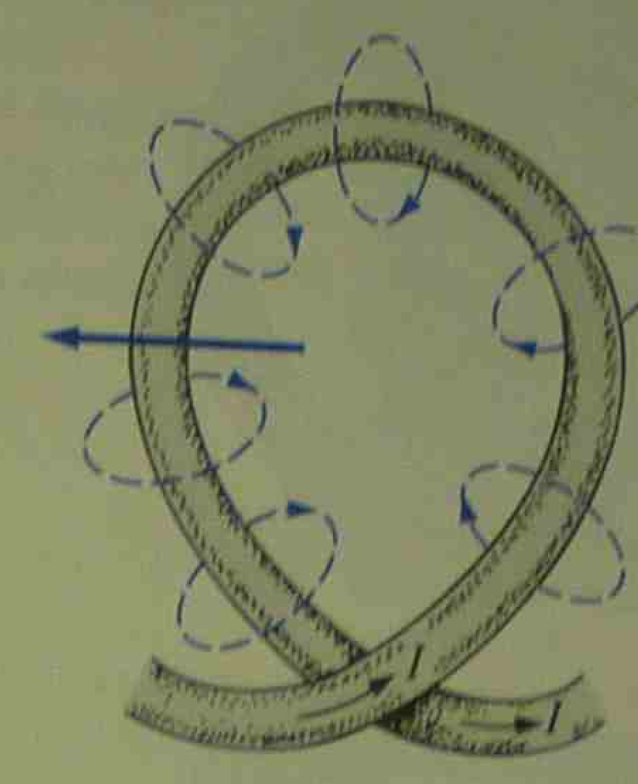


FIG. 11.7

lines can be found simply by placing the thumb of the *right* hand in the direction of *conventional* current flow and noting the direction of the fingers. (This method is commonly called the *right-hand rule.*) If the conductor is wound in a single-turn coil (Fig. 11.7), the resulting flux will flow in a common direction through the center of the coil.

A coil of more than one turn would produce a magnetic field that would exist in a continuous path through and around the coil (Fig. 11.8).

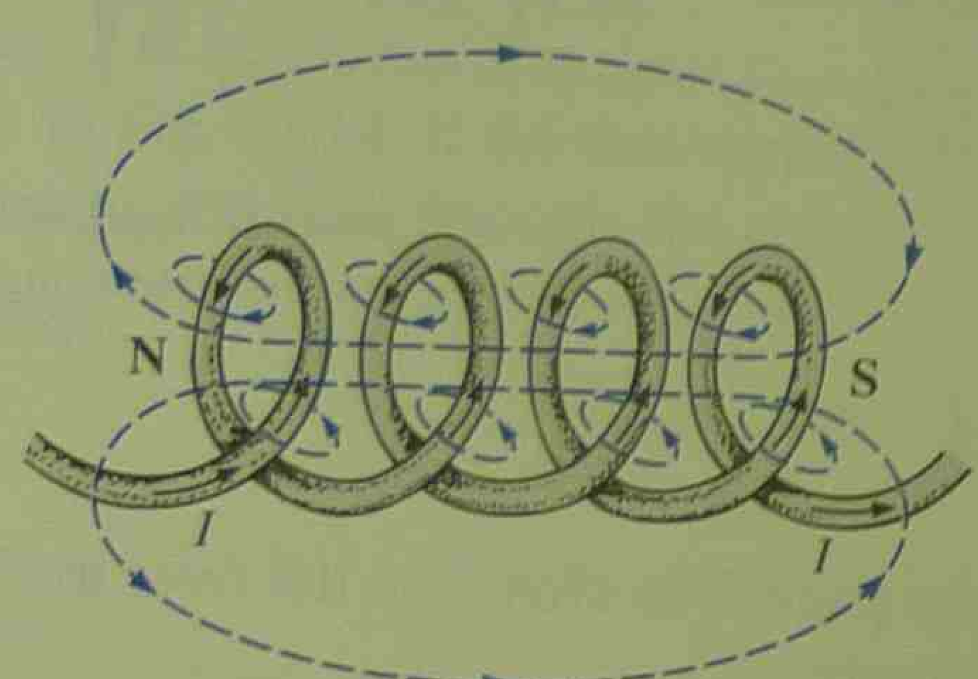


FIG. 11.8

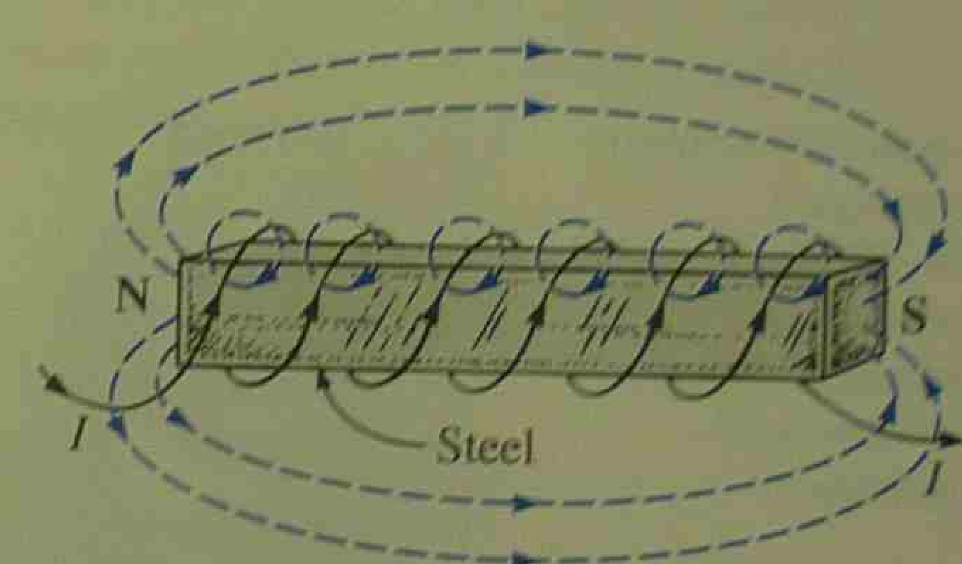


FIG. 11.9

The flux distribution of the coil is quite similar to that of the permanent magnet. The flux lines leaving the coil from the left and entering to the right simulate a north and south pole, respectively. The principal difference between the two flux distributions is that the flux lines are more concentrated for the permanent magnet than for the coil. Also, since the strength of a magnetic field is determined by the density of the flux lines, the coil has a weaker field strength. The field strength of the coil can be effectively increased by placing certain materials, such as iron, steel, or cobalt, within the coil to increase the flux density within the coil. By increasing the field strength with the addition of the core, we have devised an *electromagnet* (Fig. 11.9) which, in addition to having all the properties of a permanent magnet, also has a field strength that can be varied by changing one of the component values (current, turns, and so on). Of course, current must pass through the coil of the electromagnet in order for magnetic flux to be developed, whereas there is no need for the coil or current in the permanent magnet. The direction of flux lines can be determined for the electromagnet (or in any core with a wrapping of turns) by placing the fingers of the right hand in the direction of current flow around the core. The thumb will then point in the direction of the north pole of the induced magnetic flux. This is demonstrated in Fig. 11.10. A cross section of the same electromagnet was included in the figure to introduce the convention for directions

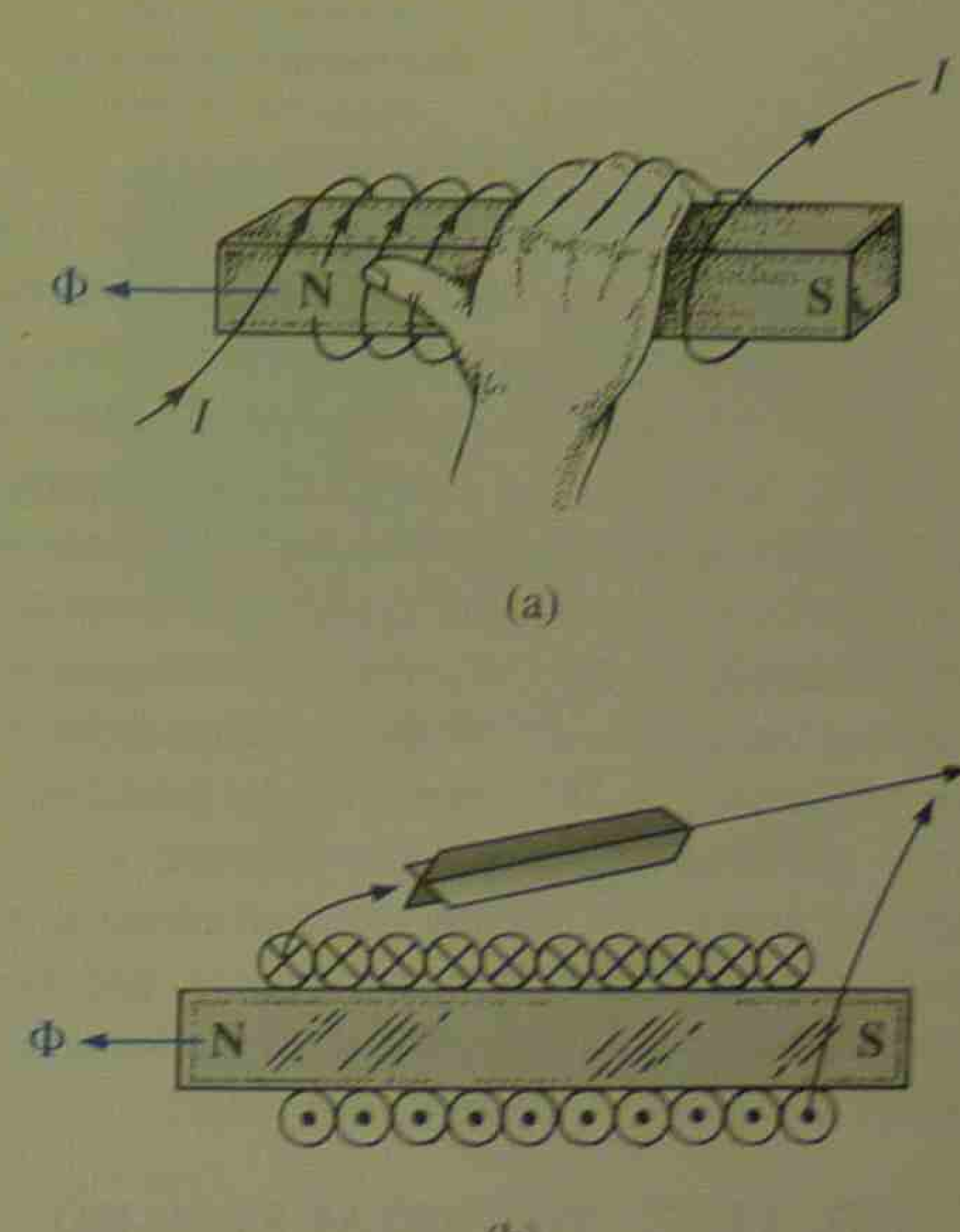


FIG. 11.10

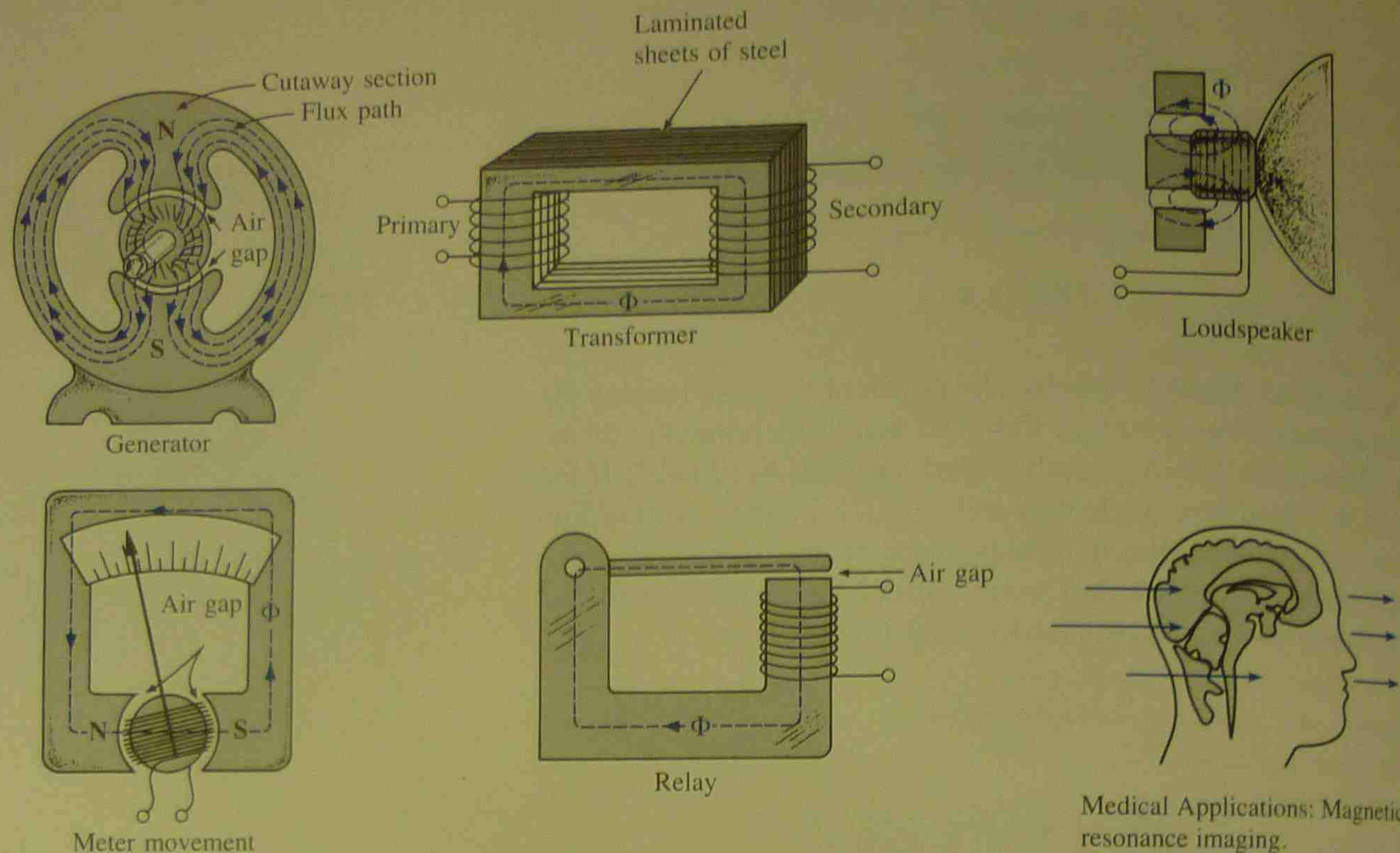


FIG. 11.11

Courtesy of the Smithsonian Institution Photo No. 52,604

German (Wittenberg, Göttingen)
(1804–1891)
Physicist
Professor of Physics, University of Göttingen

An important contributor to the establishment of a system of *absolute units* for the electrical sciences, which was beginning to become a very active area of research and development. Established a definition of electric current in an electromagnetic system based on the magnetic field produced by the current. He was politically active and, in fact, was dismissed from the faculty of the University of Göttingen for protesting the suppression of the constitution by the King of Hanover in 1837. However, he found other faculty positions and eventually returned to Göttingen as director of the astronomical observatory. Received honors from England, France, and Germany, including the Copley Medal of the Royal Society.

FIG. 11.12 WILHELM EDUARD WEBER

perpendicular to the page. The cross and dot refer to the tail and head of the arrow, respectively.

Other areas of application for electromagnetic effects are shown in Fig. 11.11. The flux path for each is indicated in each figure.

11.3 FLUX DENSITY

In the SI system of units, magnetic flux is measured in *webers* (note Fig. 11.12) and has the symbol Φ. The number of flux lines per unit area is called the *flux density*, is denoted by the capital letter *B*, and is measured in *teslas* (note Fig. 11.15). Its magnitude is determined by the following equation:

$$B = \frac{\Phi}{A} \qquad \begin{aligned} B &= \text{teslas (T)} \\ \Phi &= \text{webers (Wb)} \\ A &= \text{square meters (m}^2\text{)} \end{aligned} \qquad (11.1)$$

where Φ is the number of flux lines passing through the area *A* (Fig. 11.13). The flux density at position *a* in Fig. 11.1 is twice that at *b* because twice as many flux lines are passing through the same area.

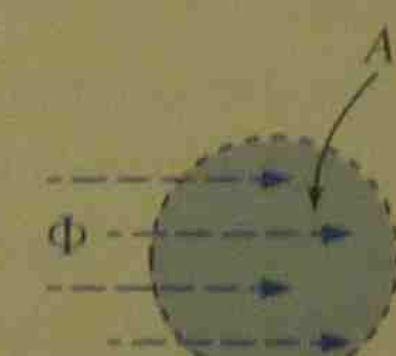


FIG. 11.13

By definition,

$$1\ \text{T} = 1\ \text{Wb/m}^2$$

EXAMPLE 11.1 For the core of Fig. 11.14, determine the flux density B in teslas.

Solution:

$$B = \frac{\Phi}{A} = \frac{6 \times 10^{-5}\ \text{Wb}}{1.2 \times 10^{-3}\ \text{m}^2} = \mathbf{5 \times 10^{-2}\ T}$$

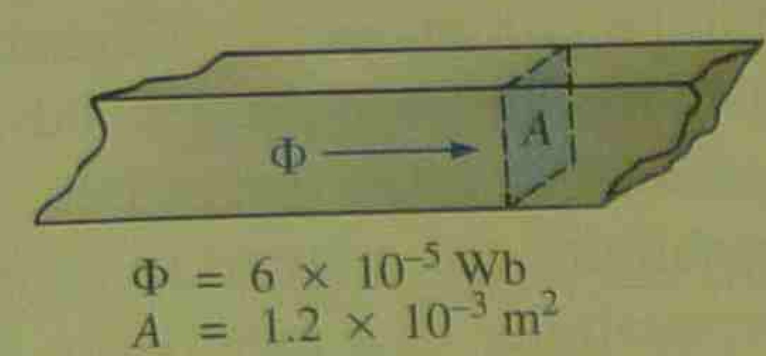


FIG. 11.14

EXAMPLE 11.2 In Fig. 11.14, if the flux density is 1.2 T and the area is 0.25 in.^{2}, determine the flux through the core.

Solution: By Eq. (11.1),

$$\Phi = BA$$

However, converting 0.25 in.^{2} to metric units,

$$A = 0.25\ \text{in.}^2\left(\frac{1\ \text{m}}{39.37\ \text{in.}}\right)\left(\frac{1\ \text{m}}{39.37\ \text{in.}}\right) = 1.613 \times 10^{-4}\ \text{m}^2$$

and

$$\Phi = (1.2\ \text{T})(1.613 \times 10^{-4}\ \text{m}^2) = \mathbf{1.936 \times 10^{-4}\ Wb}$$

An instrument designed to measure flux density in gauss (CGS system) appears in Fig. 11.16. Appendix G reveals that 1 T = 10^4 gauss. The magnitude of the reading appearing on the face of the meter in Fig. 11.16 is therefore

$$1.964\ \text{gauss}\left(\frac{1\ \text{T}}{10^4\ \text{gauss}}\right) = 1.964 \times 10^{-4}\ \text{T}$$

11.4 PERMEABILITY

If cores of different materials with the same physical dimensions are used in the electromagnet described in Section 11.2, the strength of the magnet will vary in accordance with the core used. This variation in strength is due to the greater or lesser number of flux lines passing through the core. Materials in which flux lines can readily be set up are said to be *magnetic* and to have *high permeability*. The permeability (μ) of a material, therefore, is a measure of the ease with which magnetic

Croatian-American
(Smiljan, Paris, Colorado Springs, New York City)
(1856–1943)
Electrical Engineer and Inventor
Recipient of the Edison Medal in 1917

Courtesy of the Smithsonian Institution Photo No. 52,223

Nikola Tesla is often regarded as one of the most innovative and inventive individuals in the history of the sciences. He was the first to introduce the *alternating-current machine*, removing the need for commutator bars of dc machines. After emigrating to the United States in 1884, he sold a number of his patents on *ac machines*, *transformers*, and *induction coils* (including the *Tesla coil* as we know it today) to the Westinghouse Electric Company. Some say that his most important discovery was made at his laboratory in Colorado Springs, where in 1900 he discovered *terrestrial stationary waves*. The range of his discoveries and inventions is too extensive to list here but extends from lighting systems to *polyphase power systems* to a *wireless world broadcasting system*.

FIG. 11.15 NIKOLA TESLA

FIG. 11.16
Digital display gaussmeter. (Courtesy of LDJ Electronics, Inc.)

flux lines can be established in the material. It is similar in many respects to conductivity in electric circuits. The permeability of free space μ_o (vacuum) is

$$\mu_o = 4\pi \times 10^{-7} \frac{\text{Wb}}{\text{A}\cdot\text{m}}$$

As indicated above, μ has the units of Wb/A·m. Practically speaking, the permeability of all nonmagnetic materials, such as copper, aluminum, wood, glass, and air, is the same as that for free space. Materials that have permeabilities slightly less than that of free space are said to be *diamagnetic*, and those with permeabilities slightly greater than that of free space are said to be *paramagnetic*. Magnetic materials, such as iron, nickel, steel, cobalt, and alloys of these metals, have permeabilities hundreds and even thousands of times that of free space. Materials with these very high permeabilities are referred to as *ferromagnetic*.

The ratio of the permeability of a material to that of free space is called its *relative permeability;* that is,

$$\mu_r = \frac{\mu}{\mu_o} \qquad (11.2)$$

In general, for ferromagnetic materials, $\mu_r \geq 100$, and for nonmagnetic materials, $\mu_r = 1$.

Since μ_r is a variable, dependent on other quantities of the magnetic circuit, values of μ_r are not tabulated. Methods of calculating μ_r from the data supplied by manufacturers will be considered in a later section.

11.5 RELUCTANCE

The resistance of a material to the flow of charge (current) is determined for electric circuits by the equation

$$R = \rho\frac{l}{A} \qquad \text{(ohms, } \Omega\text{)}$$

The *reluctance* of a material to the setting up of magnetic flux lines in the material is determined by the following equation:

$$\mathcal{R} = \frac{l}{\mu A} \qquad \text{(rels, or At/Wb)} \qquad (11.3)$$

where $\mathcal{R}$ is the reluctance, l is the length of the magnetic path, and A is its cross-sectional area. The *t* in the units At/Wb is the number of turns of the applied winding. More is said about ampere-turns (At) in the next section. Note that the resistance and reluctance are inversely proportional to the area, indicating that an increase in area will result in a reduction in each and an *increase* in the desired result: current and flux. For an increase in length the opposite is true, and the desired effect is reduced. The reluctance, however, is inversely proportional to the permeability, while the resistance is directly proportional to the resistivity. The larger the μ or smaller the ρ, the smaller the reluctance and resistance, respectively. Obviously, therefore, materials with high permeability, such as the ferromagnetics, have very small reluctances and will result in an increased measure of flux through the core. There is no widely accepted unit for reluctance, although the *rel* and the At/Wb are usually applied.

11.6 OHM'S LAW FOR MAGNETIC CIRCUITS

Recall the equation

$$\text{Effect} = \frac{\text{cause}}{\text{opposition}}$$

appearing in Chapter 4 to introduce Ohm's law for electric circuits. For magnetic circuits, the effect desired is the flux Φ. The cause is the *magnetomotive force* (mmf) $\mathscr{F}$, which is the external force (or "pressure") required to set up the magnetic flux lines within the magnetic material. The opposition to the setting up of the flux Φ is the reluctance $\mathscr{R}$.

Substituting, we have

$$\Phi = \frac{\mathscr{F}}{\mathscr{R}} \qquad (11.4)$$

The magnetomotive force $\mathscr{F}$ is proportional to the product of the number of turns around the core (in which the flux is to be established) and the current through the turns of wire (Fig. 11.17). In equation form,

$$\mathscr{F} = NI \qquad \text{(ampere-turns, At)} \qquad (11.5)$$

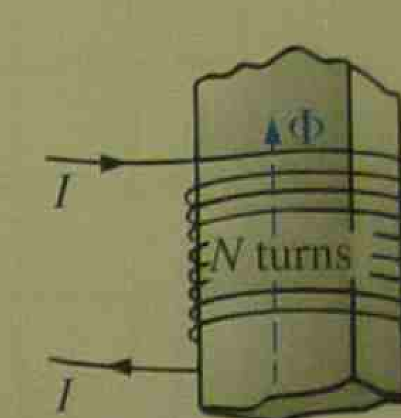


FIG. 11.17

The equation clearly indicates that an increase in the number of turns or the current through the wire will result in an increased "pressure" on the system to establish flux lines through the core.

Although there is a great deal of similarity between electric and magnetic circuits, one must continue to realize that the flux Φ is not a "flow" variable such as current in an electric circuit. Magnetic flux is established in the core through the alteration of the atomic structure of the core due to external pressure and is not a measure of the flow of some charged particles through the core.

11.7 MAGNETIZING FORCE

The magnetomotive force per unit length is called the *magnetizing force* (H). In equation form,

$$H = \frac{\mathscr{F}}{l} \qquad \text{(At/m)} \qquad (11.6)$$

Substituting for the magnetomotive force will result in

$$\boxed{H = \frac{NI}{l}} \qquad \text{(At/m)} \tag{11.7}$$

For the magnetic circuit of Fig. 11.18, if $NI = 40$ At and $l = 0.2$ m, then

$$H = \frac{NI}{l} = \frac{40\text{ At}}{0.2\text{ m}} = 200\text{ At/m}$$

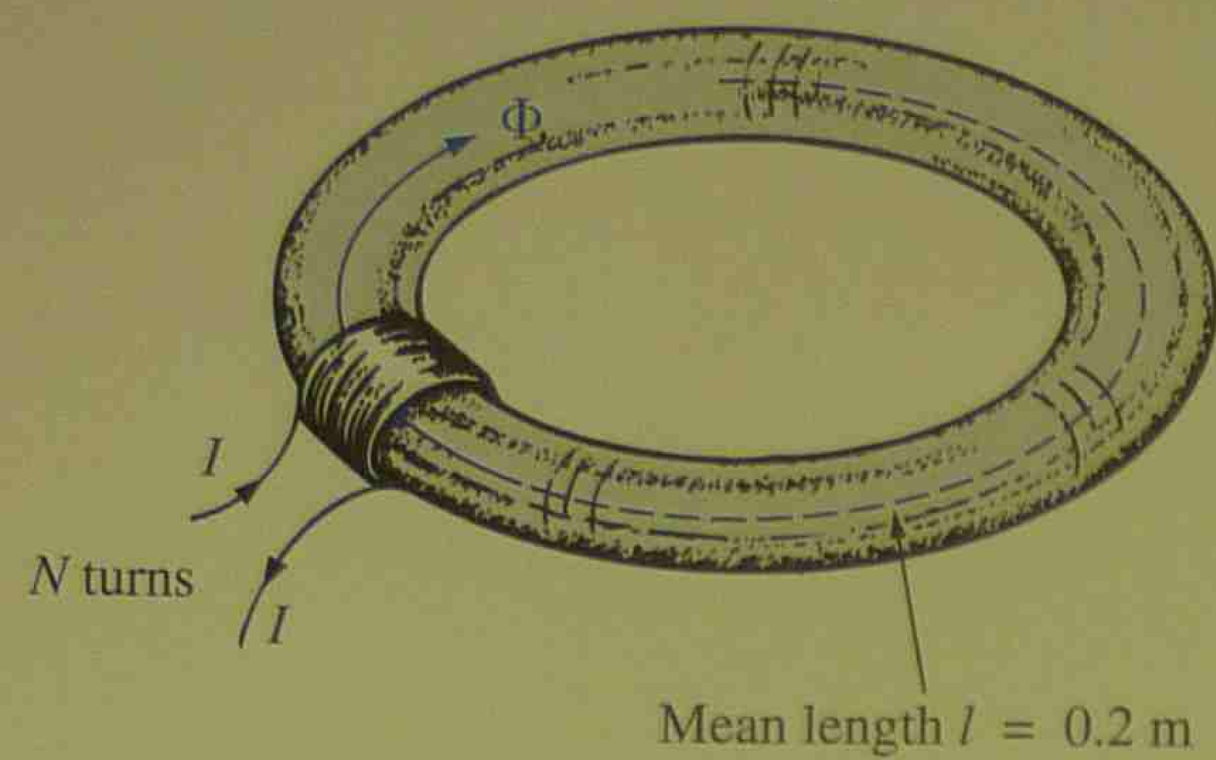


FIG. 11.18

In words, the result indicates that there are 200 At of "pressure" per meter to establish flux in the core.

Note in Fig. 11.18 that the direction of the flux Φ can be determined by placing the fingers of the right hand in the direction of current around the core and noting the direction of the thumb. It is interesting to realize that *the magnetizing force is independent of the type of core material*—it is determined solely by the number of turns, the current, and the length of the core.

The applied magnetizing force has a pronounced effect on the resulting permeability of a magnetic material. As the magnetizing force increases, the permeability rises to a maximum and then drops to a minimum, as shown in Fig. 11.19 for three commonly employed magnetic materials.

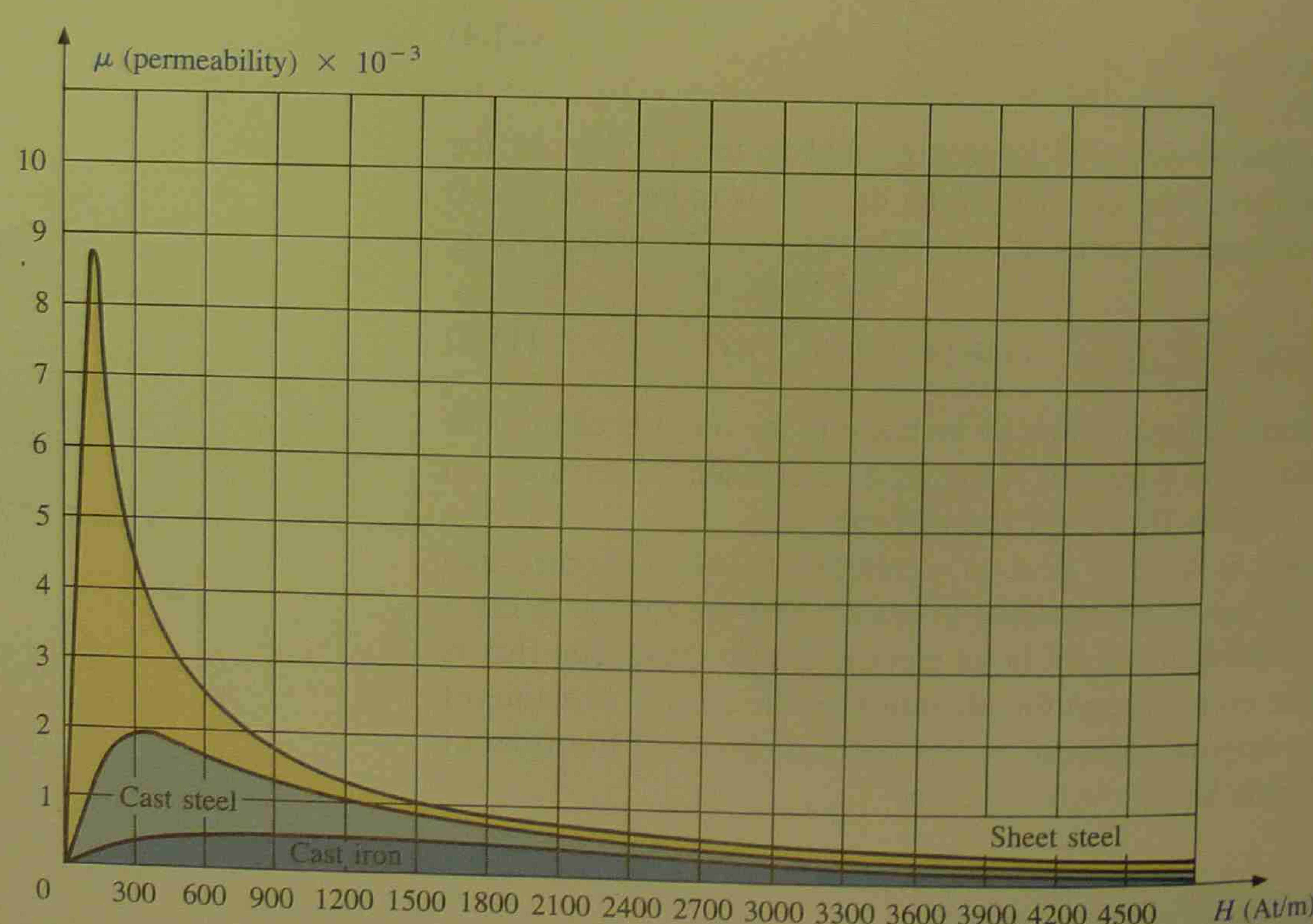


FIG. 11.19

The flux density and the magnetizing force are related by the following equation:

$$\boxed{B = \mu H} \tag{11.8}$$

This equation indicates that for a particular magnetizing force, the greater the permeability, the greater will be the induced flux density.

Now that henries (H) and the magnetizing force (H) use the same capital letter, it must be pointed out that all units of measurement in the text, such as henries, use roman letters, such as H, whereas variables such as the magnetizing force use italic letters, such as H.

11.8 HYSTERESIS

A curve of the flux density B versus the magnetizing force H of a material is of particular importance to the engineer. Curves of this type can usually be found in manuals, descriptive pamphlets, and brochures published by manufacturers of magnetic materials. A typical B-H curve for a ferromagnetic material such as steel can be derived using the setup of Fig. 11.20.

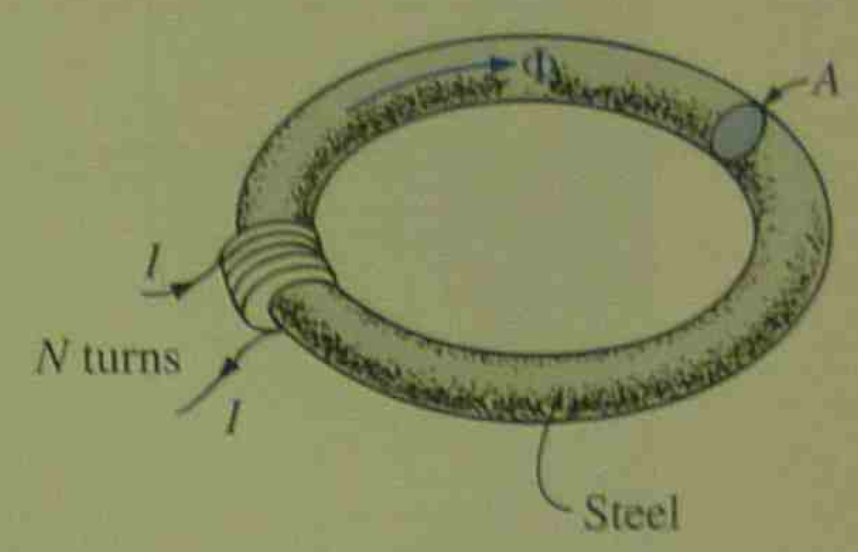


FIG. 11.20

The core is initially unmagnetized and the current $I = 0$. If the current I is increased to some value above zero, the magnetizing force H will increase to a value determined by

$$H\uparrow = \frac{NI\uparrow}{l}$$

The flux Φ and the flux density B ($B = \Phi/A$) will also increase with the current I (or H). If the material has no residual magnetism and the magnetizing force H is increased from zero to some value H_a, the B-H curve will follow the path shown in Fig. 11.21 between o and a. If the magnetizing force H is increased until saturation (H_s) occurs, the curve will continue as shown in the figure to point b. When saturation occurs, the flux density has, *for all practical purposes*, reached its maximum value. Any further increase in current through the coil increasing $H = NI/l$ will result in a very small increase in flux density B.

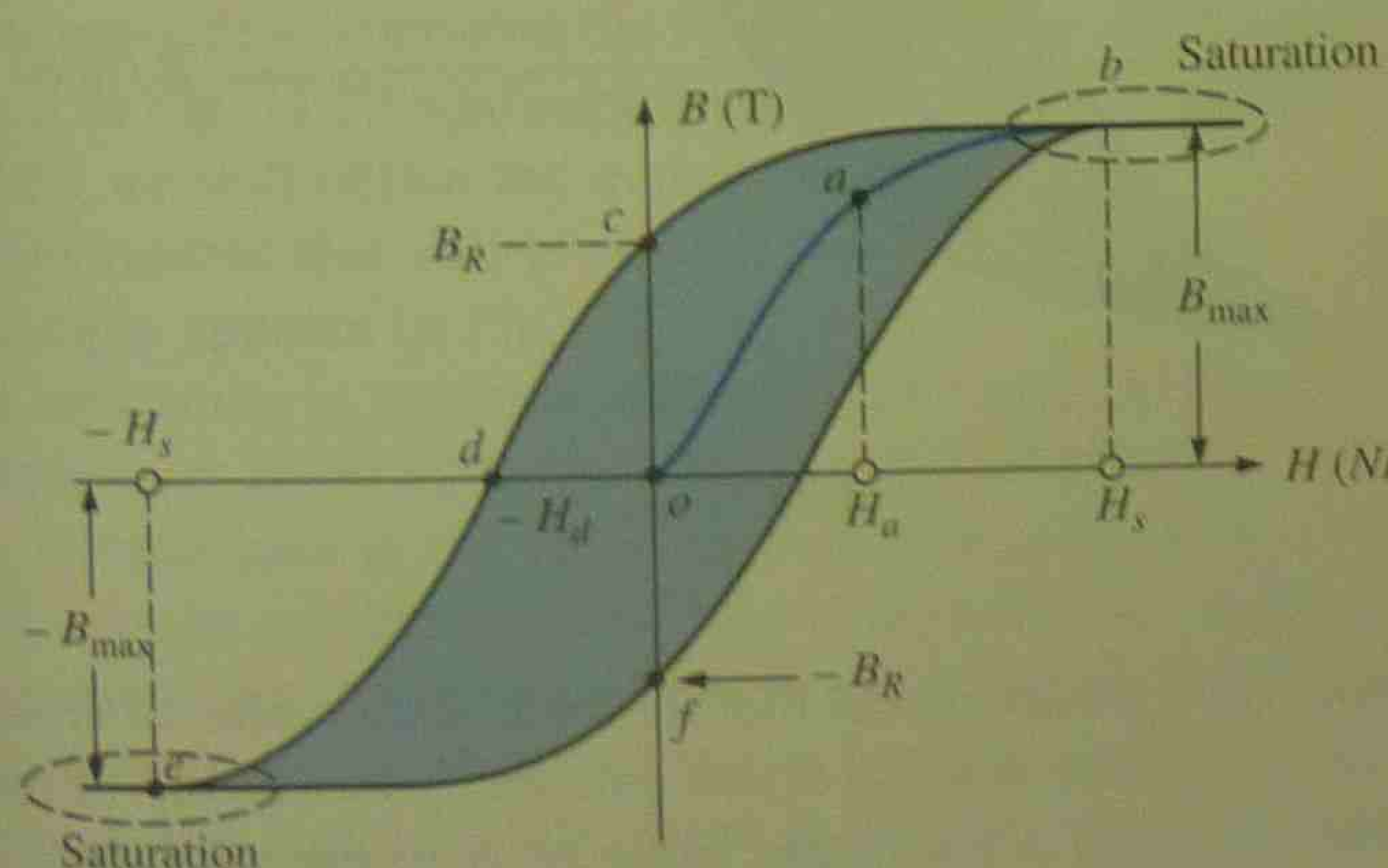


FIG. 11.21

If the magnetizing force is reduced to zero by letting I decrease to zero, the curve will follow the path of the curve between b and c. The flux density B_R, which remains when the magnetizing force is zero, is called the *residual flux density*. It is this residual flux density that makes it possible to create permanent magnets. If the coil is now removed from the core of Fig. 11.20, the core will still have the magnetic properties determined by the residual flux density, a measure of its "retentivity." If the current I is reversed, developing a magnetizing force, $-H$, the flux density B will decrease with increase in I. Eventually, the flux density will be zero when $-H_d$ (the portion of curve from c to d) is reached. The magnetizing force $-H_d$ required to "coerce" the flux density to reduce its level to zero is called the *coercive force*, a measure of the coercivity of the magnetic sample. As the force $-H$ is increased until saturation again occurs and is then reversed and brought back to zero, the path *def* will result. If the magnetizing force is increased in the positive direction ($+H$), the curve will trace the path shown from f to b. The entire curve represented by *bcdefb* is called the *hysteresis curve* for the ferromagnetic material, from the Greek *hysterein*, meaning "to lag behind." The flux density B *lagged* behind the magnetizing force H during the entire plotting of the curve. When H was zero at c, B was not zero but had only begun to decline. Long after H had passed through zero and had become equal to $-H_d$ did the flux density B finally become equal to zero.

If the entire cycle is repeated, the curve obtained for the same core will be determined by the maximum H applied. Three hysteresis loops for the same material for maximum values of H less than the saturation value are shown in Fig. 11.22. In addition, the saturation curve is repeated for comparison purposes.

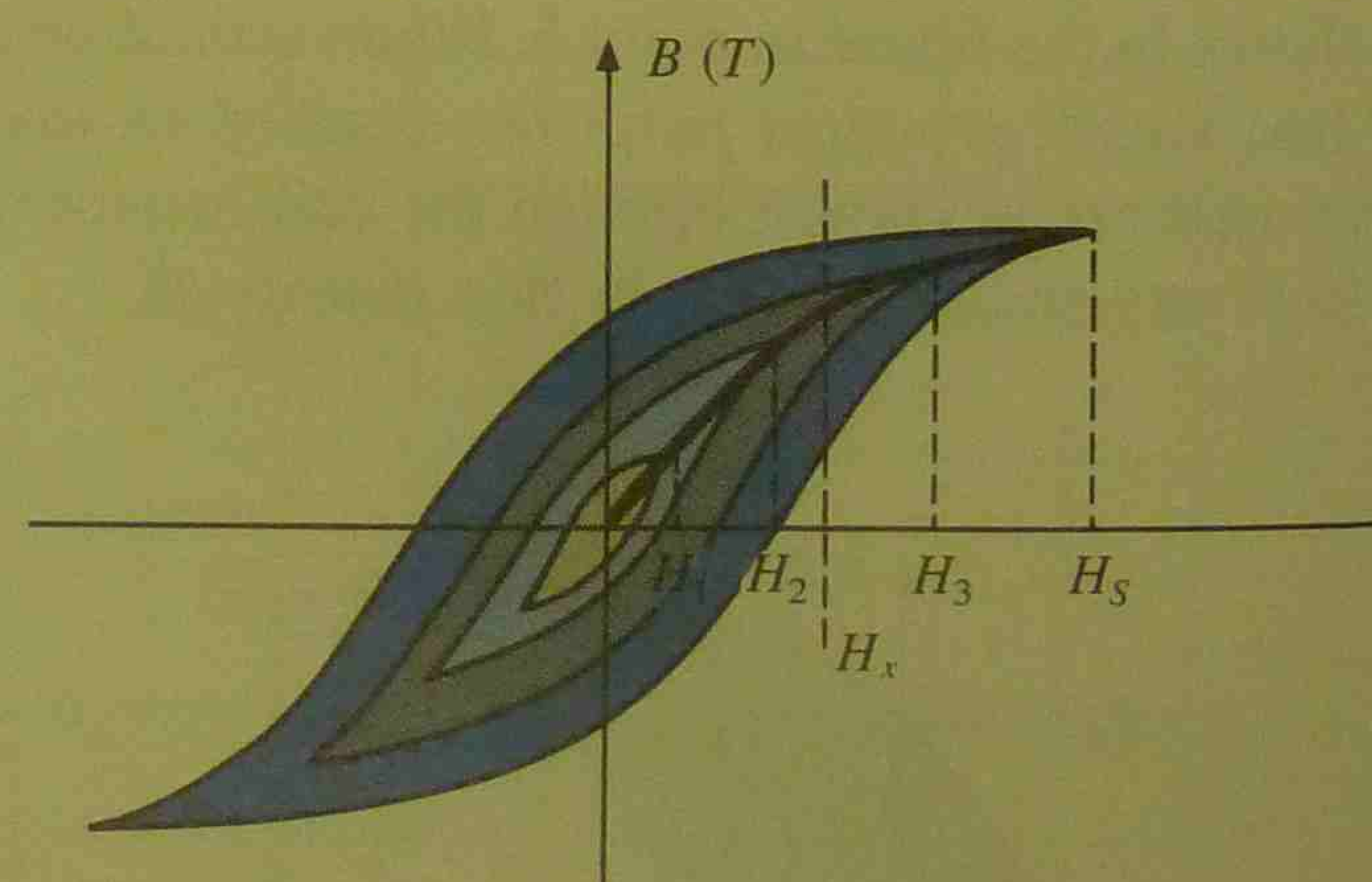


FIG. 11.22

Note from the various curves that for a particular value of H, say, H_x, the value of B can vary widely, as determined by the history of the core. In an effort to assign a particular value of B to each value of H, we compromise by connecting the tips of the hysteresis loops. The resulting curve, shown by the heavy, solid line in Fig. 11.22 and for various materials in Fig. 11.23, is called the *normal magnetization curve*. An expanded view of one region appears in Fig. 11.24.

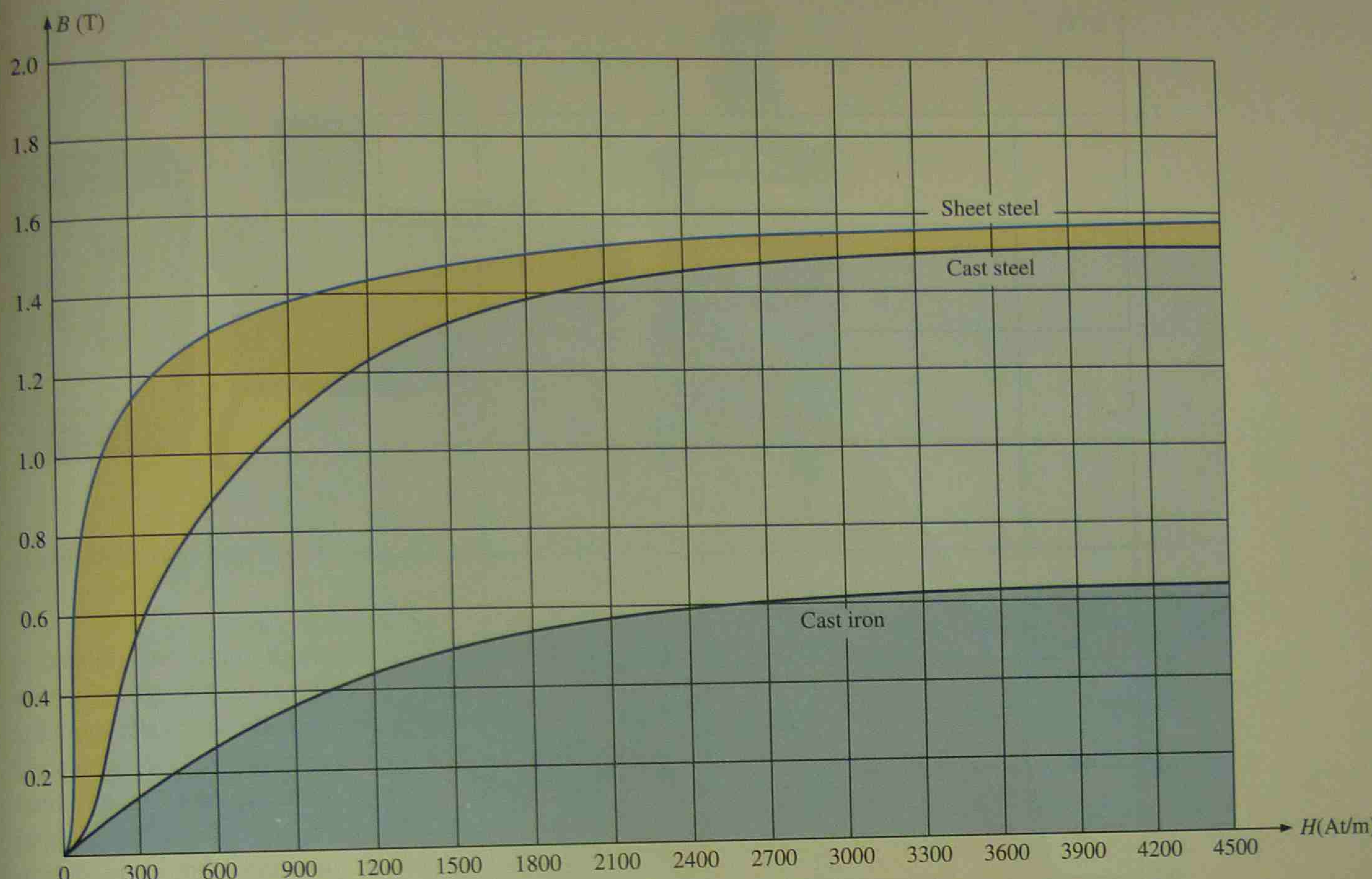
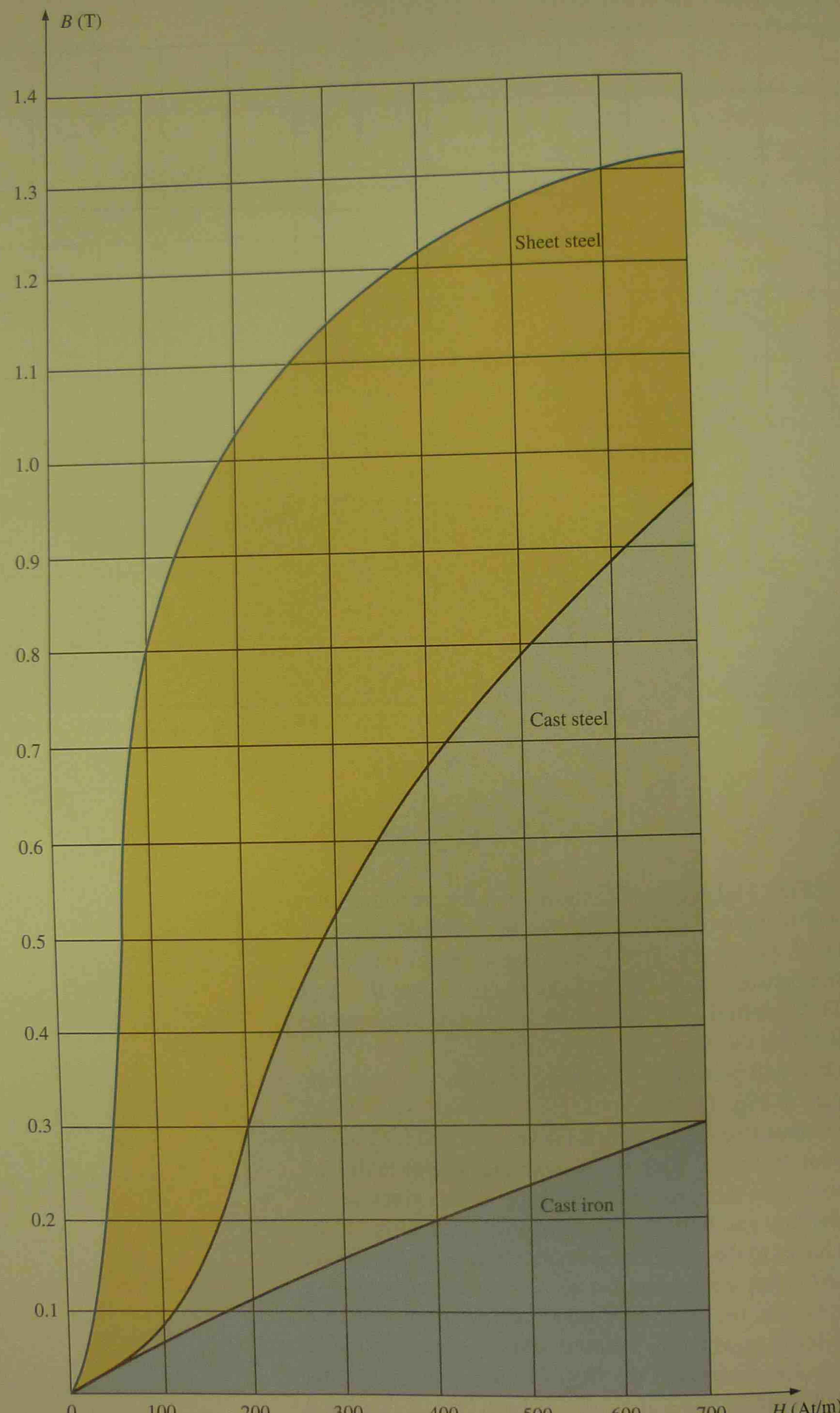


FIG. 11.23

A comparison of Figs. 11.19 and 11.23 shows that for the same value of H, the value of B is higher in Fig. 11.23 for the materials with the higher μ in Fig. 11.19. This is particularly obvious for low values of H. This correspondence between the two figures must exist, since $B = \mu H$. In fact, if in Fig. 11.23 we find μ for each value of H using the equation $\mu = B/H$, we will obtain the curves of Fig. 11.19.

An instrument that will provide a plot of the B-H curve for a magnetic sample appears in Fig. 11.25.

It is interesting to note that the hysteresis curves of Fig. 11.22 have a *point symmetry* about the origin. That is, the inverted pattern to the left of the vertical axis is the same as that appearing to the right of the vertical axis. In addition, you will find that a further application of the same magnetizing forces to the sample will result in the same plot. For a current I in $H = NI/l$ that will move between positive and negative maximums at a fixed rate, the same B-H curve will result during each cycle. Such will be the case when we examine ac (sinusoidal) networks in the later chapters. The reversal of the field (Φ) due to the changing current direction will result in a loss of energy that can best be described by first introducing the *domain theory of magnetism.*

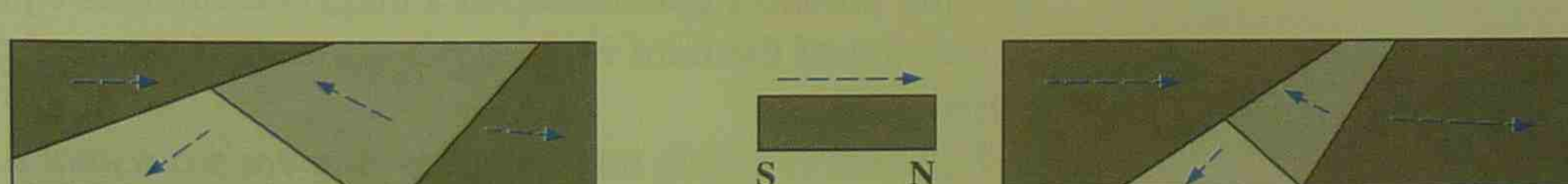


B (T)
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
200
300
400
500
600
700
H (At/m)
Sheet steel
Cast steel
Cast iron

FIG. 11.24

FIG. 11.25
Model 9600 vibrating sample magnetometer. (Courtesy of LDJ Electronics, Inc.)

Within each atom, the orbiting electrons (described in Chapter 2) are also spinning as they revolve around the nucleus. The atom, due to its spinning electrons, has a magnetic field associated with it. In nonmagnetic materials, the net magnetic field is effectively zero, since the magnetic fields due to the atoms of the material oppose each other. In magnetic materials such as iron and steel, however, the magnetic fields of groups of atoms numbering in the order of 10^{12} are aligned, forming very small bar magnets. This group of magnetically aligned atoms is called a *domain*. Each domain is a separate entity; that is, each domain is independent of the surrounding domains. For an unmagnetized sample of magnetic material, these domains appear in a random manner, such as shown in Fig. 11.26(a). The net magnetic field in any one direction is zero.



(a) (b)

FIG. 11.26

When an external magnetizing force is applied, the domains that are nearly aligned with the applied field will grow at the expense of the less favorably oriented domains, such as shown in Fig. 11.26(b). Eventually, if a sufficiently strong field is applied, all of the domains will have the orientation of the applied magnetizing force, and any further increase in external field will not increase the strength of the magnetic flux through the core—a condition referred to as *saturation*. The elasticity of the above is evidenced by the fact that when the magnetizing force is removed, the alignment will be lost to some measure and the flux density will drop to B_R. In other words, the removal of the magnetizing force will result in the return of a number of misaligned domains within the core. The continued alignment of a number of the domains, however, accounts for our ability to create permanent magnets.

At a point just before saturation, the opposing unaligned domains are reduced to small cylinders of various shapes referred to as *bubbles*. These bubbles can be moved within the magnetic sample through the application of a *controlling* magnetic field. It is these magnetic bubbles that form the basis of the recently designed bubble memory system for computers.

11.9 AMPÈRE'S CIRCUITAL LAW

It was mentioned in the introduction to this chapter that there is a broad similarity between the analyses of electric and magnetic circuits. This has already been demonstrated to some extent for the quantities in Table 11.1.

TABLE 11.1

	Electric Circuits	Magnetic Circuits
Cause	E	$\mathcal{F}$
Effect	I	Φ
Opposition	R	$\mathcal{R}$

If we apply the "cause" analogy to Kirchhoff's voltage law ($\Sigma_{\circlearrowright} V = 0$), we obtain the following:

$$\Sigma_{\circlearrowright} \mathcal{F} = 0 \quad \text{(for magnetic circuits)} \qquad (11.9)$$

which, in words, states that the algebraic sum of the rises and drops of the mmf around a closed loop of a magnetic circuit is equal to zero; that is, the sum of the mmf rises equals the sum of the mmf drops around a closed loop.

Equation (11.9) is referred to as *Ampère's circuital law*. When it is applied to magnetic circuits, sources of mmf are expressed by the equation

$$\mathcal{F} = NI \quad \text{(At)} \qquad (11.10)$$

The equation for the mmf drop across a portion of a magnetic circuit can be found by applying the relationships listed in Table 11.1. That is, for electric circuits,

$$V = IR$$

resulting in the following for magnetic circuits:

$$\mathcal{F} = \Phi\mathcal{R} \quad \text{(At)} \qquad (11.11)$$

where Φ is the flux passing through a section of the magnetic circuit and $\mathcal{R}$ is the reluctance of that section. The reluctance, however, is seldom calculated in the analysis of magnetic circuits. A more practical equation for the mmf drop is

$$\boxed{\mathscr{F} = Hl} \quad \text{(At)} \qquad \textbf{(11.12)}$$

as derived from Eq. (11.6), where H is the magnetizing force on a section of a magnetic circuit and l is the length of the section. As an example of Eq. (11.9), consider the magnetic circuit appearing in Fig. 11.27 constructed of three different ferromagnetic materials.

Applying Ampère's circuital law, we have

$$\Sigma_{\circlearrowleft} \mathscr{F} = 0$$

$$\underbrace{+NI}_{\text{Rise}} - \underbrace{H_{ab}l_{ab}}_{\text{Drop}} - \underbrace{H_{bc}l_{bc}}_{\text{Drop}} - \underbrace{H_{ca}l_{ca}}_{\text{Drop}} = 0$$

or

$$\underbrace{NI}_{\substack{\text{Impressed}\\ \text{mmf}}} = \underbrace{H_{ab}l_{ab} + H_{bc}l_{bc} + H_{ca}l_{ca}}_{\text{mmf drops}}$$

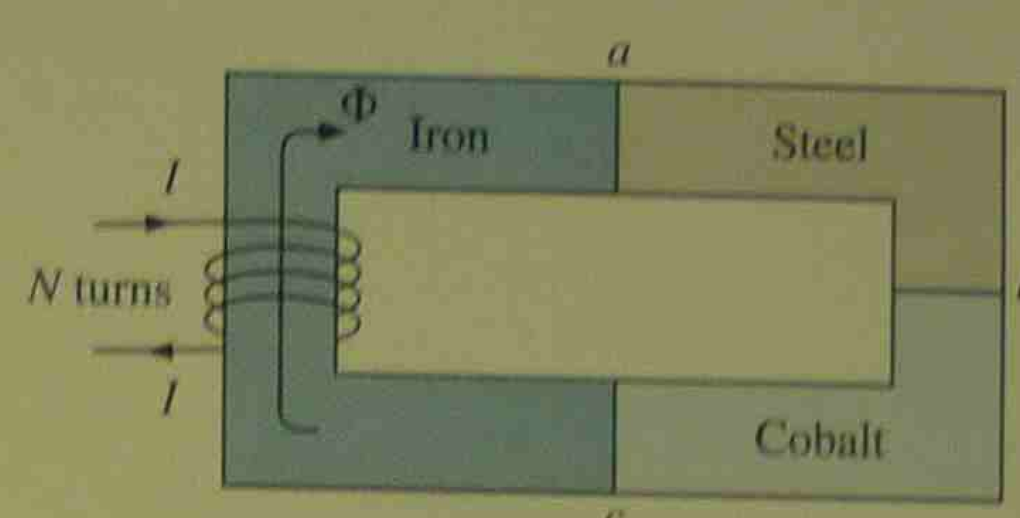


FIG. 11.27

All the terms of the equation are known except the magnetizing force for each portion of the magnetic circuit, which can be found by using the *B-H* curve if the flux density B is known.

11.10 THE FLUX Φ

If we continue to apply the relationships described in the previous section to Kirchhoff's current law, we will find that the sum of the fluxes entering a junction is equal to the sum of the fluxes leaving a junction; that is, for the circuit of Fig. 11.28,

$$\Phi_a = \Phi_b + \Phi_c \qquad \text{(at junction } a)$$

or

$$\Phi_b + \Phi_c = \Phi_a \qquad \text{(at junction } b)$$

both of which are equivalent.

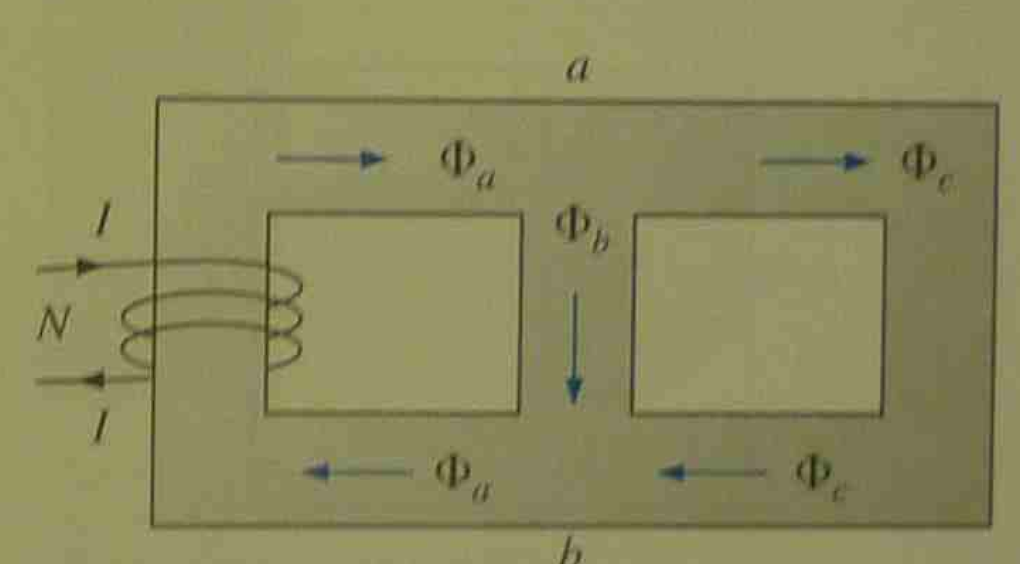


FIG. 11.28

11.11 SERIES MAGNETIC CIRCUITS: DETERMINING *NI*

We are now in a position to solve a few magnetic circuit problems, which are basically of two types. In one type, Φ is given, and the impressed mmf *NI* must be computed. This is the type of problem encountered in the design of motors, generators, and transformers. In the other type, *NI* is given, and the flux Φ of the magnetic circuit must be found. This type of problem is encountered primarily in the design of magnetic amplifiers and is more difficult since the approach is "hit or miss."

As indicated in earlier discussions, the value of μ will vary from point to point along the magnetization curve. This eliminates the possibility of finding the reluctance of each "branch" or the "total reluctance" of a network as was done for electric circuits where ρ had a fixed value for any applied current or voltage. If the total reluctance could be determined, Φ could then be determined using the Ohm's law analogy for magnetic circuits.

For magnetic circuits, the level of B or H is determined from the other using the B-H curve, and μ is seldom calculated unless asked for.

An approach frequently employed in the analysis of magnetic circuits is the *table* method. Before a problem is analyzed in detail, a table is prepared listing in the extreme left-hand column the various sections of the magnetic circuit. The columns on the right are reserved for the quantities to be found for each section. In this way, the individual doing the problem can keep track of what is required to complete the problem and also of what the next step should be. After a few examples, the usefulness of this method should become clear.

This section will consider only *series* magnetic circuits in which the flux Φ is the same throughout. In each example, the magnitude of the magnetomotive force is to be determined.

EXAMPLE 11.3. For the series magnetic circuit of Fig. 11.29:

a. Find the value of I required to develop a magnetic flux of $\Phi = 4 \times 10^{-4}$ Wb.

b. Determine μ and μ_r for the material under these conditions.

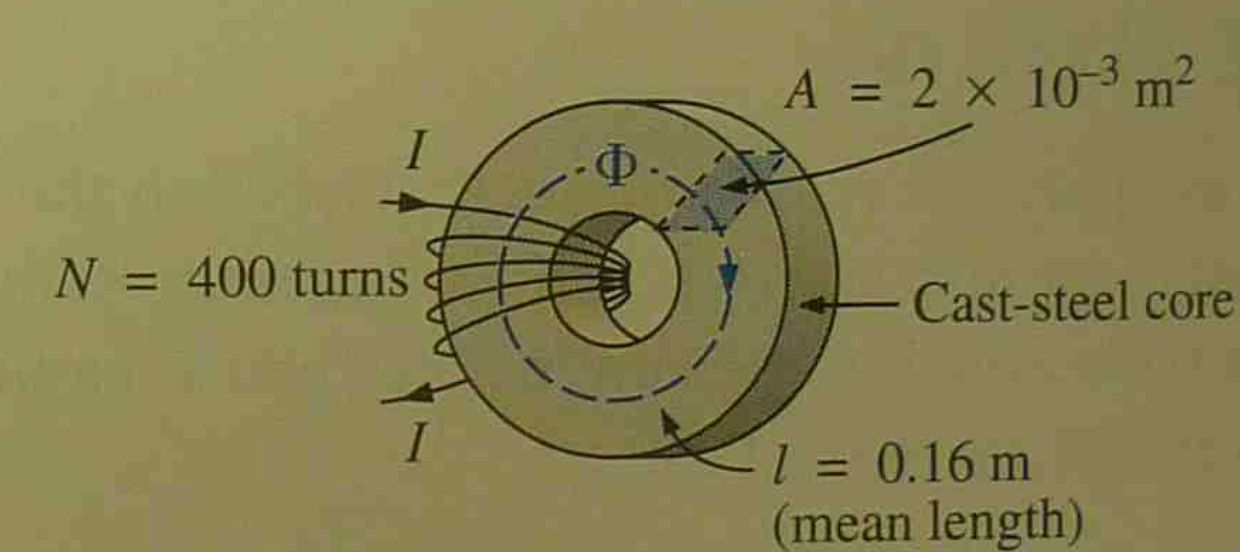


FIG. 11.29

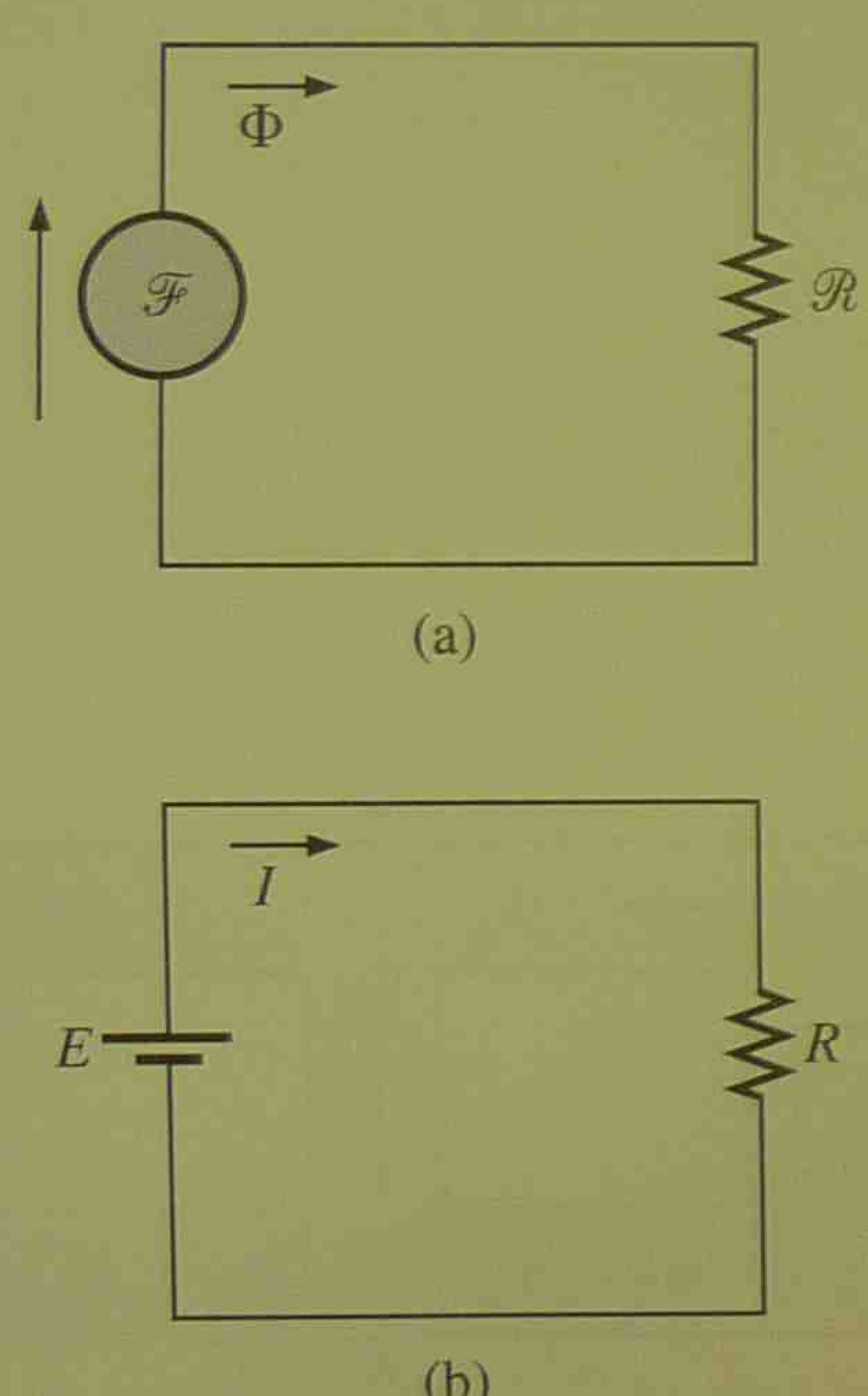


FIG. 11.30
(a) Magnetic circuit equivalent and (b) electric circuit analogy.

Solutions: The magnetic circuit can be represented by the system shown in Fig. 11.30(a). The electric circuit analogy is shown in Fig. 11.30(b). Analogies of this type can be very helpful in the solution of magnetic circuits. Table 11.2 is for part (a) of this problem. The table is fairly trivial for this example but it does define the quantities to be found.

TABLE 11.2

Section	Φ (Wb)	A (m^2)	B (T)	H (At/m)	l (m)	Hl (At)
One continuous section	4×10^{-4}	2×10^{-3}			0.16	

a. The flux density B is

$$B = \frac{\Phi}{A} = \frac{4 \times 10^{-4}\ \text{Wb}}{2 \times 10^{-3}\ \text{m}^2} = 2 \times 10^{-1}\ \text{T} = 0.2\ \text{T}$$

Using the *B-H* curves of Fig. 11.24, we can determine the magnetizing force *H*:

$$H \text{ (cast steel)} = 170 \text{ At/m}$$

Applying Ampère's circuital law yields

$$NI = Hl$$

and
$$I = \frac{Hl}{N} = \frac{(170 \text{ At/m})(0.16 \text{ m})}{400 \text{ t}} = \mathbf{68 \text{ mA}}$$

(Recall that t represents turns.)

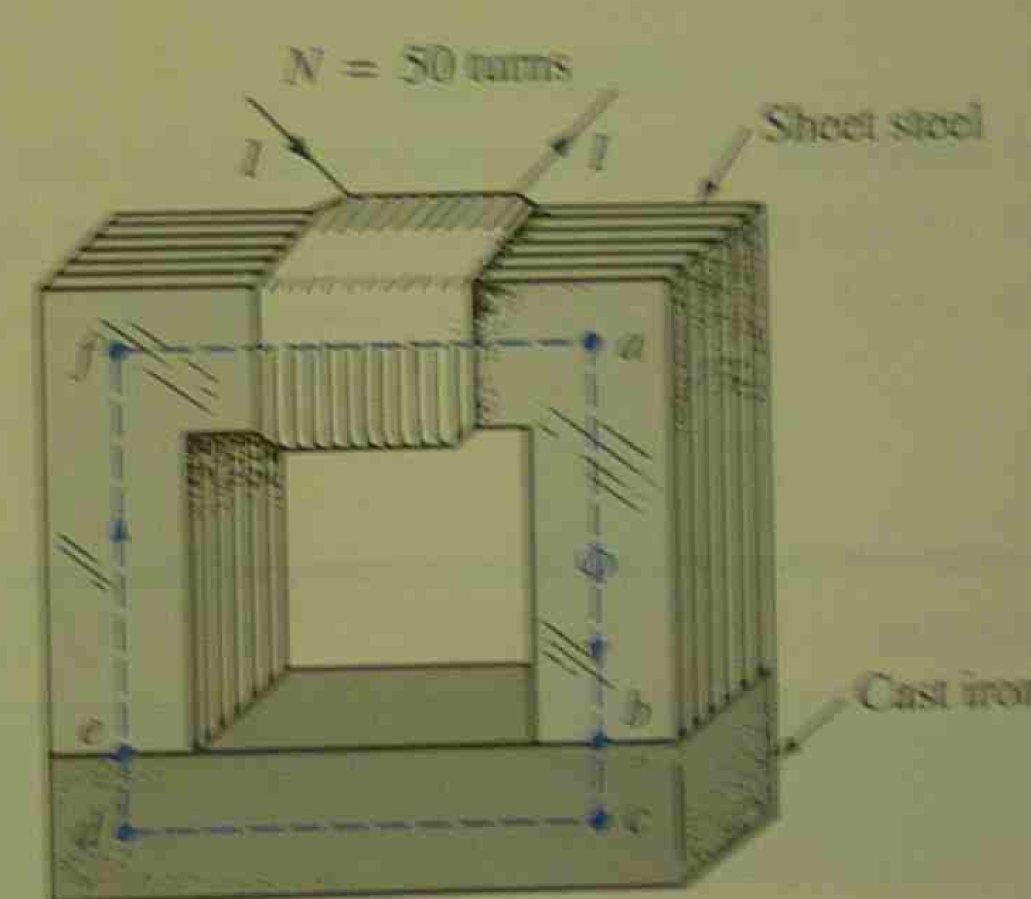
b. The permeability of the material can be found using Eq. (11.8):

$$\mu = \frac{B}{H} = \frac{0.2 \text{ T}}{170 \text{ At/m}} = \mathbf{1.176 \times 10^{-3} \text{ Wb/A·m}}$$

and the relative permeability is

$$\mu_r = \frac{\mu}{\mu_o} = \frac{1.176 \times 10^{-3}}{4\pi \times 10^{-7}} = \mathbf{935.83}$$

EXAMPLE 11.4 The electromagnet of Fig. 11.31 has picked up a section of cast iron. Determine the current *I* required to establish the indicated flux in the core.



$l_{ab} = l_{cd} = l_{ef} = l_{fa} = 4$ in.
$l_{bc} = l_{de} = 0.5$ in.
Area (throughout) = 1 in.^{2}
$\Phi = 3.5 \times 10^{-4}$ Wb

FIG. 11.31
Electromagnet for Example 11.4.

Solution: To be able to use Figs. 11.23 and 11.24, the dimensions must first be converted to the metric system. However, since the area is the same throughout, we can determine the length for each material rather than work with the individual sections:

$$l_{efab} = 4 \text{ in.} + 4 \text{ in.} + 4 \text{ in.} = 12 \text{ in.}$$

$$l_{bcde} = 0.5 \text{ in.} + 4 \text{ in.} + 0.5 \text{ in.} = 5 \text{ in.}$$

$$12 \text{ in.}\left(\frac{1 \text{ m}}{39.37 \text{ in.}}\right) = 304.8 \times 10^{-3} \text{ m}$$

$$5 \text{ in.}\left(\frac{1 \text{ m}}{39.37 \text{ in.}}\right) = 127 \times 10^{-3} \text{ m}$$

$$1 \text{ in.}^2\left(\frac{1 \text{ m}}{39.37 \text{ in.}}\right)\left(\frac{1 \text{ m}}{39.37 \text{ in.}}\right) = 6.452 \times 10^{-4} \text{ m}^2$$

The information available from the specifications of the problem has been inserted in Table 11.3. When the problem has been completed,

TABLE 11.3

Section	Φ (Wb)	*A* (m²)	*B* (T)	*H* (At/m)	*l* (m)	*Hl* (At)
efab	3.5×10^{-4}	6.452×10^{-4}			304.8×10^{-3}	
bcde	3.5×10^{-4}	6.452×10^{-4}			127×10^{-3}	

each space will contain some information. Sufficient data to complete the problem can be found if we fill in each column from left to right. As the various quantities are calculated, they will be placed in a similar table found at the end of the example.

The flux density for each section is

$$B = \frac{\Phi}{A} = \frac{3.5 \times 10^{-4}\ \text{Wb}}{6.452 \times 10^{-4}\ \text{m}^2} = 0.542\ \text{T}$$

and the magnetizing force is

$$H\ (\text{sheet steel, Fig. 11.24}) \cong 70\ \text{At/m}$$
$$H\ (\text{cast iron, Fig. 11.23}) \cong 1600\ \text{At/m}$$

Note the extreme difference in magnetizing force for each material for the required flux density. In fact, when we apply Ampère's circuital law, we will find that the sheet steel section could be ignored with a minimal error in the solution.

Determining Hl for each section yields

$$H_{efab} l_{efab} = (70\ \text{At/m})(304.8 \times 10^{-3}\ \text{m}) = 21.34\ \text{At}$$
$$H_{bcde} l_{bcde} = (1600\ \text{At/m})(127 \times 10^{-3}\ \text{m}) = 203.2\ \text{At}$$

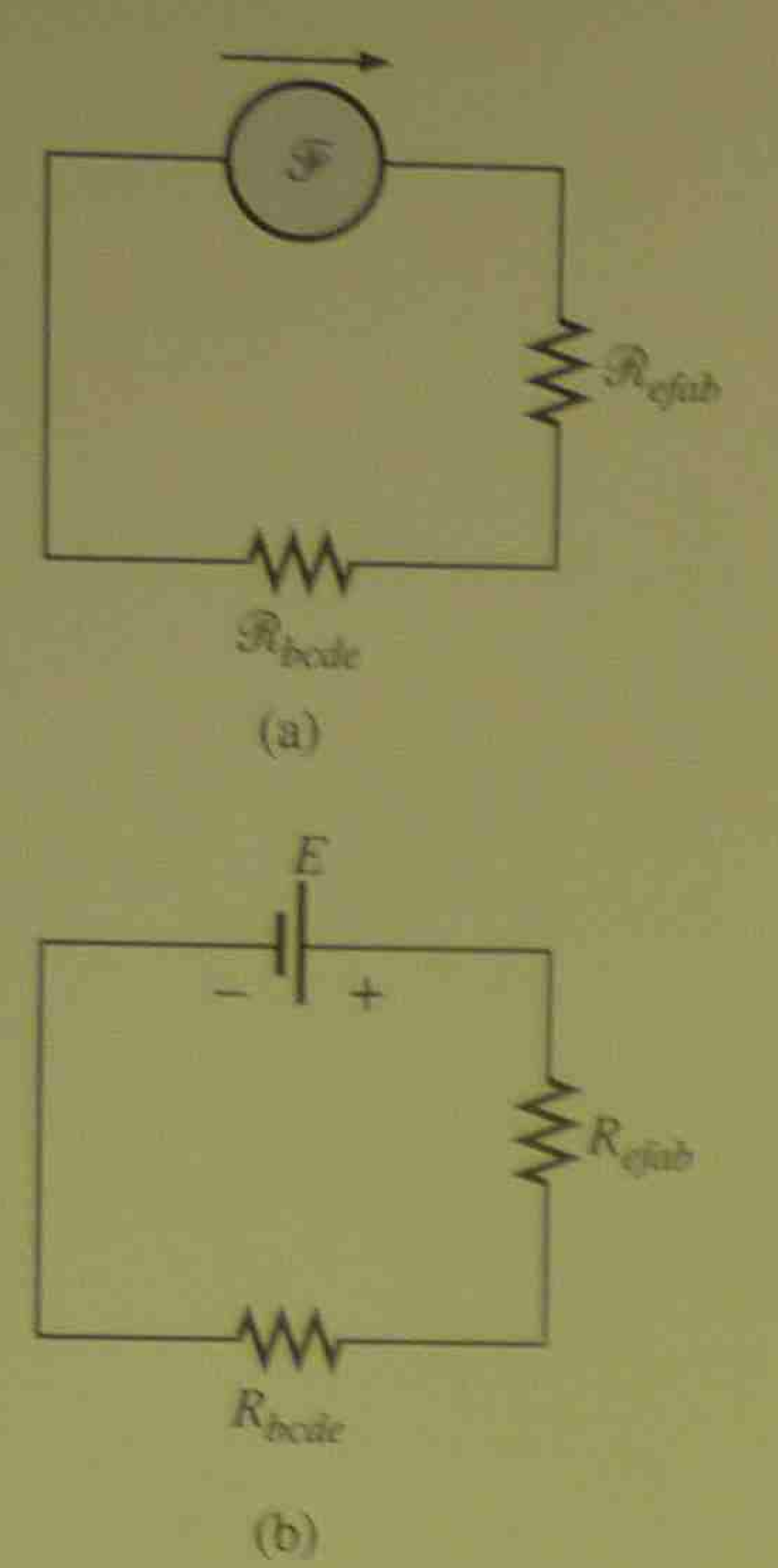


FIG. 11.32

(a) Magnetic circuit equivalent and (b) electric circuit analogy for the electromagnet of Fig. 11.31.

Inserting the above data in Table 11.3 will result in Table 11.4. The magnetic circuit equivalent and the electric circuit analogy for the system of Fig. 11.31 appear in Fig. 11.32.

TABLE 11.4

Section	Φ (Wb)	A (m²)	B (T)	H (At/m)	l (m)	Hl (At)
efab	3.5×10^{-4}	6.452×10^{-4}	0.542	60	304.8×10^{-3}	21.34
bcde	3.5×10^{-4}	6.452×10^{-4}	0.542	1600	127×10^{-3}	203.2

Applying Ampère's circuital law,

$$NI = H_{efab} l_{efab} + H_{bcde} l_{bcde}$$
$$= 21.34\ \text{At} + 203.2\ \text{At} = 224.54\ \text{At}$$

and

$$(50\ \text{t})I = 224.54\ \text{At}$$

so that

$$I = \frac{224.54\ \text{At}}{50\ \text{t}} = \mathbf{4.49\ A}$$

EXAMPLE 11.5 Determine the secondary current I_2 for the transformer of Fig. 11.33 if the resultant clockwise flux in the core is 1.5×10^{-5} Wb.

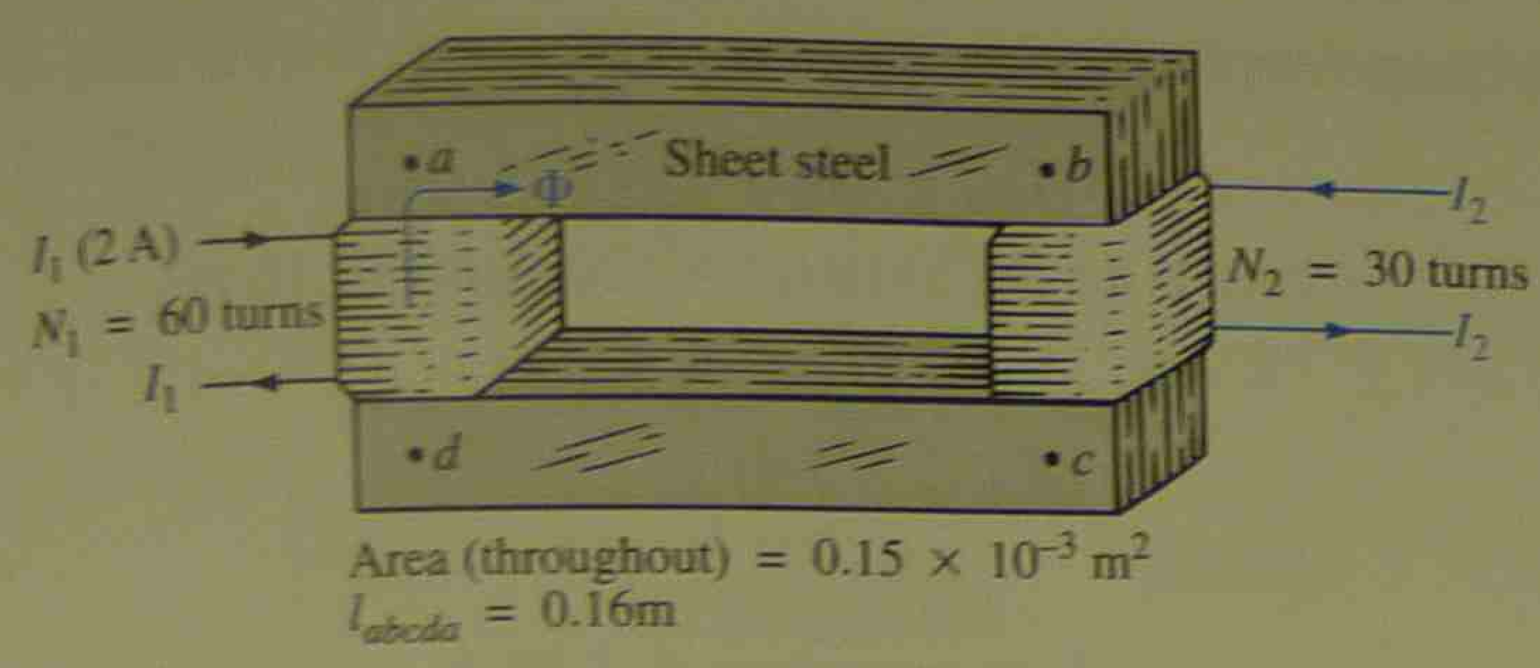


FIG. 11.33

Transformer for Example 11.5.

Solution: This is the first example with two magnetizing forces to consider. In the analogies of Fig. 11.34 you will note that the resulting flux of each is opposing, just as the two sources of voltage are opposing in the electric circuit analogy.

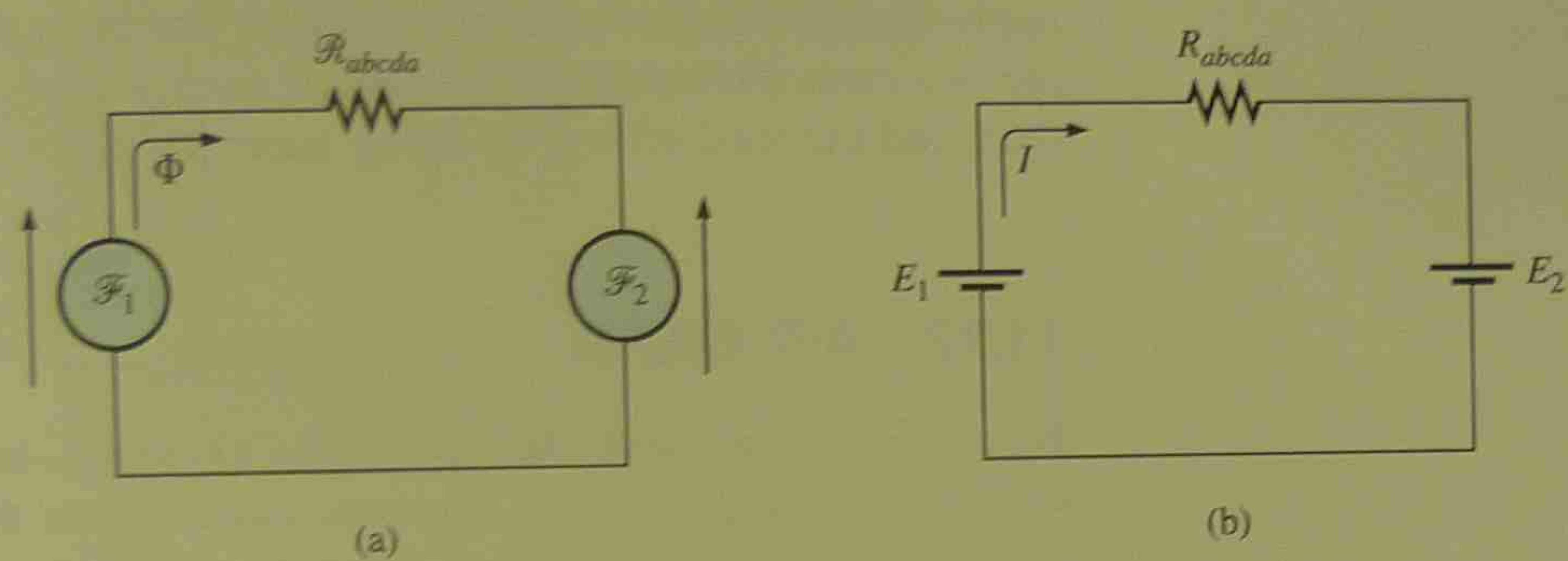


FIG. 11.34

(a) Magnetic circuit equivalent and (b) electric circuit analogy for the transformer of Fig. 11.33.

The structural data appear in Table 11.5.

TABLE 11.5

Section	Φ (Wb)	A (m²)	B (T)	H (At/m)	l (m)	Hl (At)
abcda	1.5×10^{-5}	0.15×10^{-3}			0.16	

The flux density throughout is

$$B = \frac{\Phi}{A} = \frac{1.5 \times 10^{-5}\ \text{Wb}}{0.15 \times 10^{-3}\ \text{m}^2} = 10 \times 10^{-2}\ \text{T} = 0.10\ \text{T}$$

and

$$H\ (\text{from Fig. 11.24}) \cong \frac{1}{5}(100\ \text{At/m}) = 20\ \text{At/m}$$

Applying Ampère's circuital law,

$$N_1I_1 - N_2I_2 = H_{abcda}l_{abcda}$$
$$(60 \text{ t})(2 \text{ A}) - (30 \text{ t})(I_2) = (20 \text{ At/m})(0.16 \text{ m})$$
$$120 \text{ At} - (30 \text{ t})I_2 = 3.2 \text{ At}$$

and
$$(30 \text{ t})I_2 = 120 \text{ At} - 3.2 \text{ At}$$

or
$$I_2 = \frac{116.8 \text{ At}}{30 \text{ t}} = \mathbf{3.89\ A}$$

For the analysis of most transformer systems, the equation $N_1I_1 = N_2I_2$ is employed. This would result in 4 A versus 3.89 A above. This difference is normally ignored, however, and the equation $N_1I_1 = N_2I_2$ considered exact.

Because of the nonlinearity of the *B-H* curve, *it is not possible to apply superposition to magnetic circuits;* that is, in the previous example, we cannot consider the effects of each source independently and then find the total effects by using superposition.

11.12 AIR GAPS

Before continuing with the illustrative examples, let us consider the effects an air gap has on a magnetic circuit. Note the presence of air gaps in the magnetic circuits of the motor and meter of Fig. 11.11. The spreading of the flux lines outside the common area of the core for the air gap in Fig. 11.35(a) is known as *fringing*. For our purposes, we shall neglect this effect and assume the flux distribution to be as in Fig. 11.35(b).

The flux density of the air gap in Fig. 11.35(b) is given by

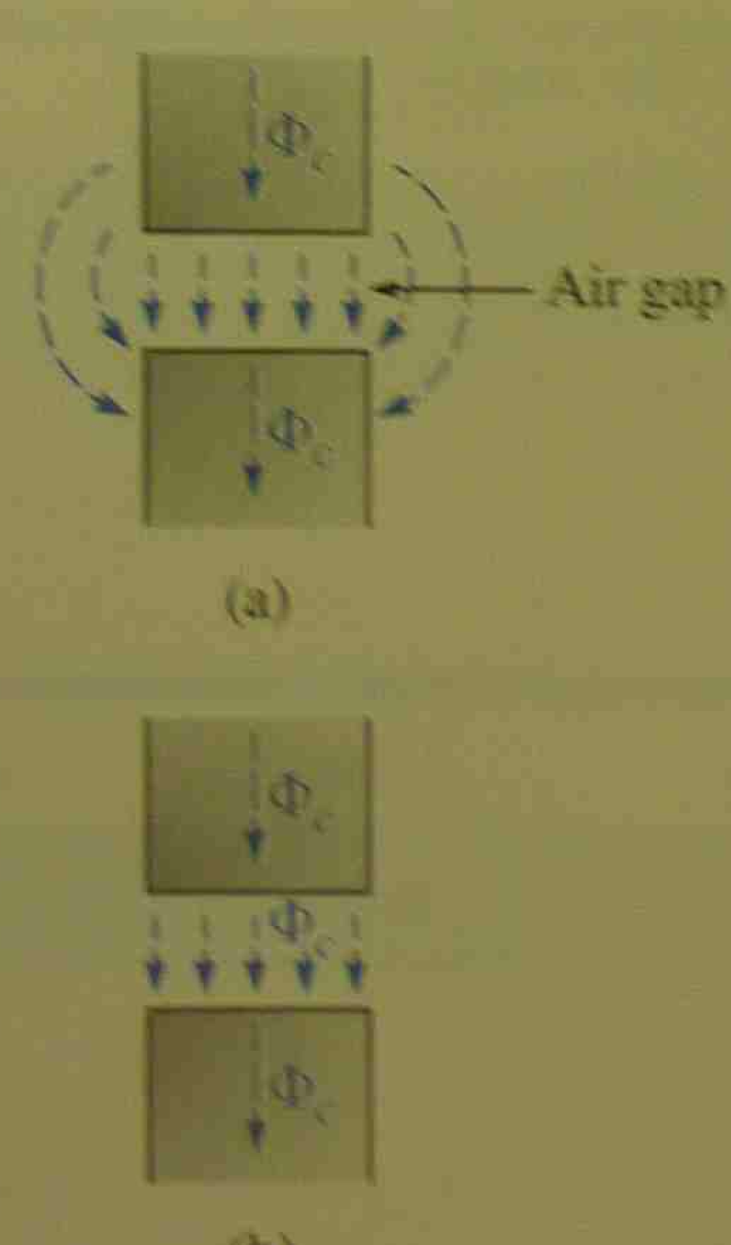


FIG. 11.35

$$B_g = \frac{\Phi_g}{A_g} \tag{11.13}$$

where, for our purposes,

$$\Phi_g = \Phi_{core}$$

and
$$A_g = A_{core}$$

For most practical applications, the permeability of air is taken to be equal to that of free space. The magnetizing force of the air gap is then determined by

$$H_g = \frac{B_g}{\mu_o} \tag{11.14}$$

and the mmf drop across the air gap is equal to $H_g l_g$. An equation for H_g is as follows:

$$H_g = \frac{B_g}{\mu_o} = \frac{B_g}{4\pi \times 10^{-7}}$$

and

$$H_g = (7.96 \times 10^5)B_g \qquad \text{(At/m)} \qquad (11.15)$$

EXAMPLE 11.6 Find the value of I required to establish a magnetic flux of $\Phi = 0.75 \times 10^{-4}$ Wb in the series magnetic circuit of Fig. 11.36.

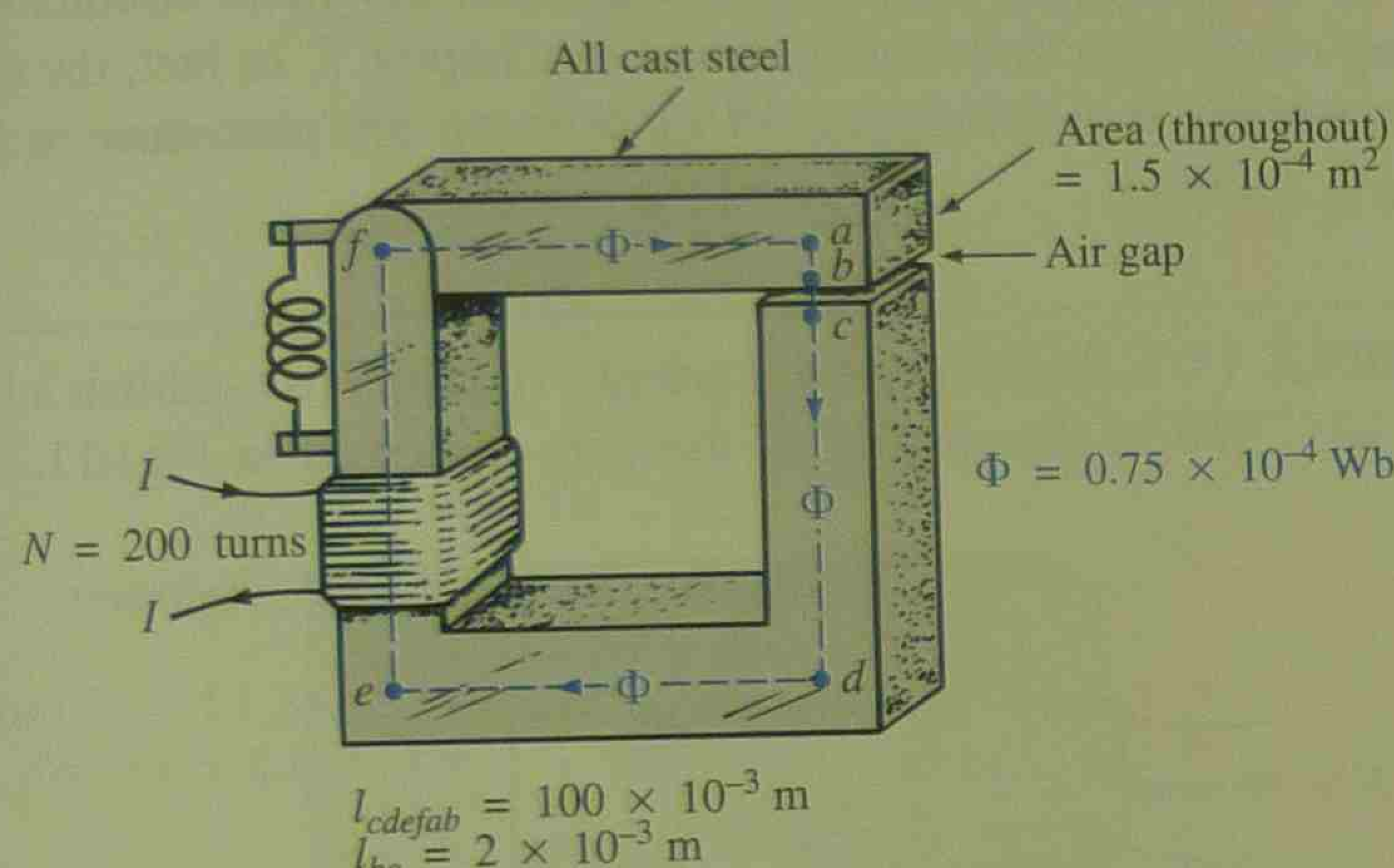


FIG. 11.36
Relay for Example 11.6.

Solution: An equivalent magnetic circuit and its electric circuit analogy are shown in Fig. 11.37.

The flux density for each section is

$$B = \frac{\Phi}{A} = \frac{0.75 \times 10^{-4}\ \text{Wb}}{1.5 \times 10^{-4}\ \text{m}^2} = 0.5\ \text{T}$$

From the *B-H* curves of Fig. 11.24,

$$H\ \text{(cast steel)} \cong 280\ \text{At/m}$$

Applying Eq. (11.15),

$$H_g = (7.96 \times 10^5)B_g = (7.96 \times 10^5)(0.5\ \text{T}) = 3.98 \times 10^5\ \text{At/m}$$

The mmf drops are

$$H_{core} l_{core} = (280\ \text{At/m})(100 \times 10^{-3}\ \text{m}) = 28\ \text{At}$$
$$H_g l_g = (3.98 \times 10^5\ \text{At/m})(2 \times 10^{-3}\ \text{m}) = 796\ \text{At}$$

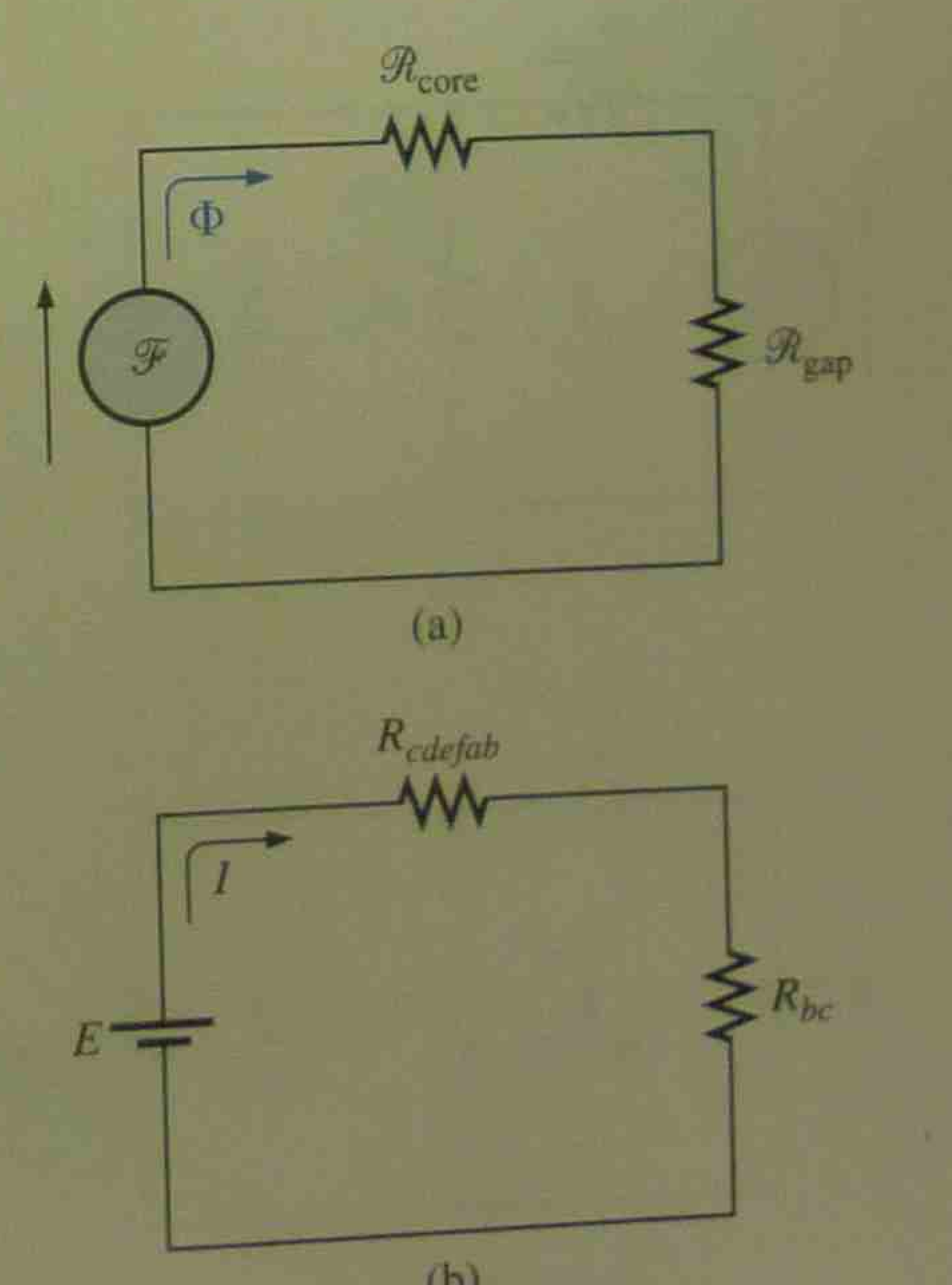


FIG. 11.37
(a) Magnetic circuit equivalent and (b) electric circuit analogy for the relay of Fig. 11.36.

Applying Ampère's circuital law,

$$NI = H_{core}l_{core} + H_g l_g$$
$$= 28\text{ At} + 796\text{ At}$$
$$(200\text{ t})I = 824\text{ At}$$
$$I = \mathbf{4.12\text{ A}}$$

Note from the above that the air gap requires the biggest share (by far) of the impressed NI due to the fact that air is nonmagnetic.

11.13 SERIES-PARALLEL MAGNETIC CIRCUITS

As one might expect, the close analogies between electric and magnetic circuits will eventually lead to series-parallel magnetic circuits similar in many respects to those encountered in Chapter 7. In fact, the electric circuit analogy will prove helpful in defining the procedure to follow toward a solution.

EXAMPLE 11.7 Determine the current I required to establish a flux of 1.5×10^{-4} Wb in the section of the core indicated in Fig. 11.38.

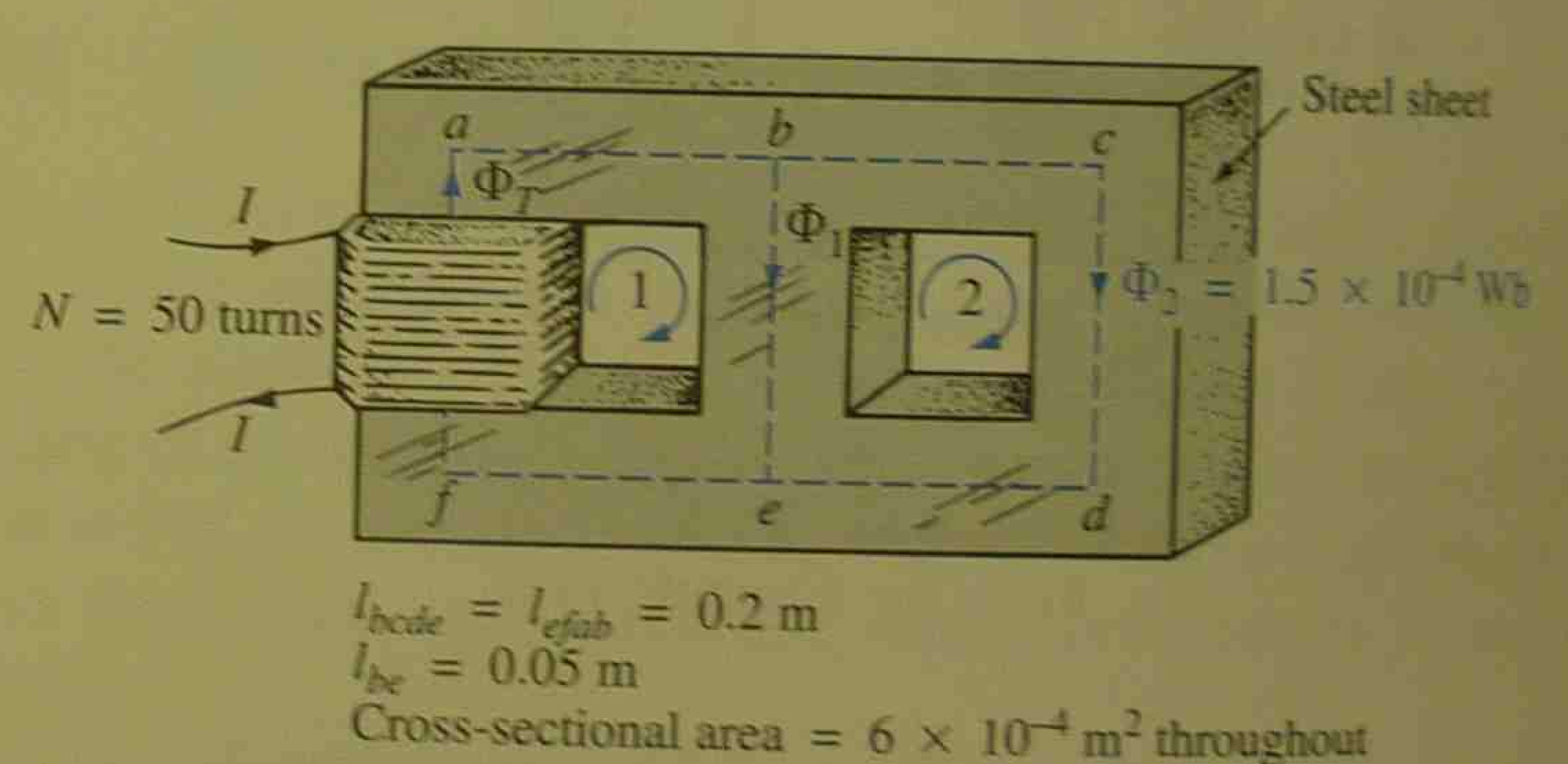


FIG. 11.38

Solution: The equivalent magnetic circuit and the electric circuit analogy appear in Fig. 11.39. We have

$$B_2 = \frac{\Phi_2}{A} = \frac{1.5 \times 10^{-4}\text{ Wb}}{6 \times 10^{-4}\text{ m}^2} = 0.25\text{ T}$$

From Fig. 11.24,

$$H_{bcde} \cong 40\text{ At/m}$$

Applying Ampère's circuital law around loop 2 of Figs. 11.38 and 11.39,

$$\Sigma_\circlearrowleft \mathcal{F} = 0$$
$$H_{be}l_{be} - H_{bcde}l_{bcde} = 0$$
$$H_{be}(0.05\text{ m}) - (40\text{ At/m})(0.2\text{ m}) = 0$$
$$H_{be} = \frac{8\text{ At}}{0.05\text{ m}} = 160\text{ At/m}$$

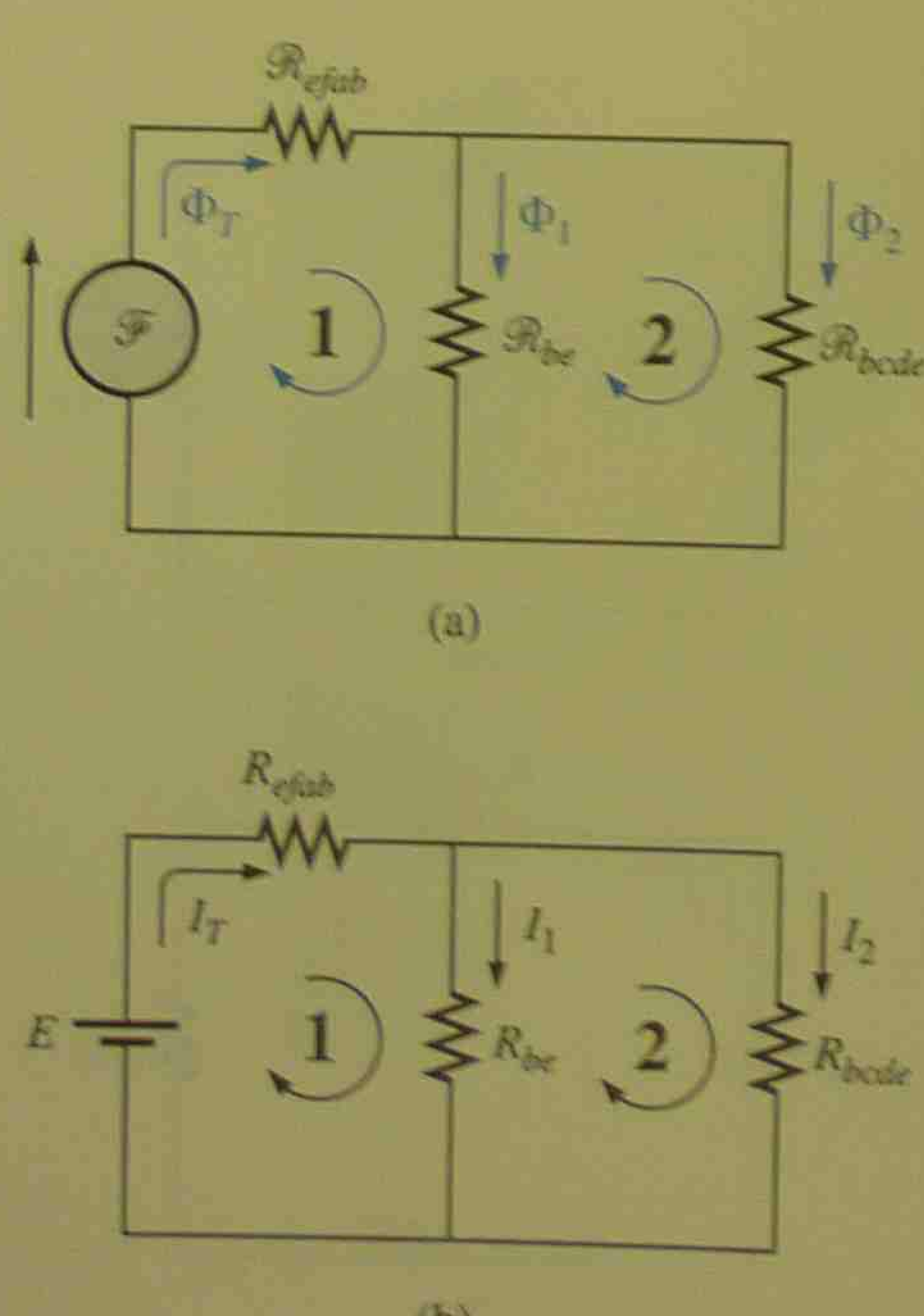


FIG. 11.39
(a) Magnetic circuit equivalent and (b) electric circuit analogy for the series-parallel system of Fig. 11.38.

From Fig. 11.24,

$$B_1 \cong 0.97 \text{ T}$$

and

$$\Phi_1 = B_1 A = (0.97 \text{ T})(6 \times 10^{-4} \text{ m}^2) = 5.82 \times 10^{-4} \text{ Wb}$$

The results are then entered in Table 11.6.

TABLE 11.6

Section	Φ (Wb)	A (m^2)	B (T)	H (At/m)	l (m)	Hl (At)
bcde	1.5×10^{-4}	6×10^{-4}	0.25	40	0.2	8
be	5.82×10^{-4}	6×10^{-4}	0.97 T	160	0.05	8
efab		6×10^{-4}			0.2	

The table reveals that we must now turn our attention to section *efab:*

$$\Phi_T = \Phi_1 + \Phi_2 = 5.82 \times 10^{-4} \text{ Wb} + 1.5 \times 10^{-4} \text{ Wb}$$
$$= 7.32 \times 10^{-4} \text{ Wb}$$
$$B = \frac{\Phi_T}{A} = \frac{7.32 \times 10^{-4} \text{ Wb}}{6 \times 10^{-4} \text{ m}^2}$$
$$= 1.22 \text{ T}$$

From Fig. 11.23,

$$H_{efab} \cong 400 \text{ At}$$

Applying Ampère's circuital law,

$$+NI - H_{efab} l_{efab} - H_{be} l_{be} = 0$$
$$NI = (400 \text{ At/m})(0.2 \text{ m}) + (160 \text{ At/m})(0.05 \text{ m})$$
$$(50 \text{ t})I = 80 \text{ At} + 8 \text{ At}$$
$$I = \frac{88 \text{ At}}{50 \text{ t}} = \mathbf{1.76 \text{ A}}$$

To demonstrate that μ is sensitive to the magnetizing force H, the permeability of each section is determined as follows. For section *bcde,*

$$\mu = \frac{B}{H} = \frac{0.25 \text{ T}}{40 \text{ At/m}} = 6.25 \times 10^{-3}$$

and

$$\mu_r = \frac{\mu}{\mu_o} = \frac{6.25 \times 10^{-3}}{12.57 \times 10^{-7}} = \mathbf{4972.2}$$

For section *be,*

$$\mu = \frac{B}{H} = \frac{0.97 \text{ T}}{160 \text{ At/m}} = 6.06 \times 10^{-3}$$

and

$$\mu_r = \frac{\mu}{\mu_o} = \frac{6.06 \times 10^{-3}}{12.57 \times 10^{-7}} = \mathbf{4821}$$

For section *efab,*

$$\mu = \frac{B}{H} = \frac{1.22\ \text{T}}{400\ \text{At/m}} = 3.05 \times 10^{-3}$$

and
$$\mu_r = \frac{\mu}{\mu_o} = \frac{3.05 \times 10^{-3}}{12.57 \times 10^{-7}} = \mathbf{2426.41}$$

11.14 DETERMINING Φ

The examples of this section are of the second type, where NI is given and the flux Φ must be found. This is a relatively straightforward problem if only one magnetic section is involved. Then

$$H = \frac{NI}{l} \qquad H \rightarrow B\ (B\text{-}H\ \text{curve})$$

and
$$\Phi = BA$$

For magnetic circuits with more than one section, there is no set order of steps that will lead to an exact solution for every problem on the first attempt. In general, however, we proceed as follows. We must find the impressed mmf for a *calculated guess* of the flux Φ and then compare this with the specified value of mmf. We can then make adjustments on our guess to bring it closer to the actual value. For most applications, a value within ±5% of the actual Φ or specified NI is acceptable.

We can make a reasonable guess at the value of Φ if we realize that the maximum mmf drop appears across the material with the smallest permeability if the length and area of each material are the same. As shown in Example 11.6, if there is an air gap in the magnetic circuit, there will be a considerable drop in mmf across the gap. As a starting point for problems of this type, therefore, we shall assume that the total mmf (NI) is across the section with the lowest μ or greatest $\mathcal{R}$ (if the other physical dimensions are relatively similar). This assumption gives a value of Φ that will produce a calculated NI greater than the specified value. Then, after considering the results of our original assumption very carefully, we shall *cut* Φ and NI by introducing the effects (reluctance) of the other portions of the magnetic circuit and *try* the new solution. For obvious reasons, this approach is frequently called the *cut and try* method.

EXAMPLE 11.8 Calculate the magnetic flux Φ for the magnetic circuit of Fig. 11.40.

Solution: By Ampère's circuital law,

$$NI = H_{abcda} l_{abcda}$$

or
$$H_{abcda} = \frac{NI}{l_{abcda}} = \frac{(60\ \text{t})(5\ \text{A})}{0.3\ \text{m}}$$
$$= \frac{300\ \text{At}}{0.3\ \text{m}} = 1000\ \text{At/m}$$

and
$$B_{abcda}\ (\text{from Fig. 11.23}) \cong 0.39\ \text{T}$$

Since $B = \Phi/A$, we have

$$\Phi = BA = (0.39\ \text{T})(2 \times 10^{-4}\ \text{m}^2) = \mathbf{0.78 \times 10^{-4}\ Wb}$$

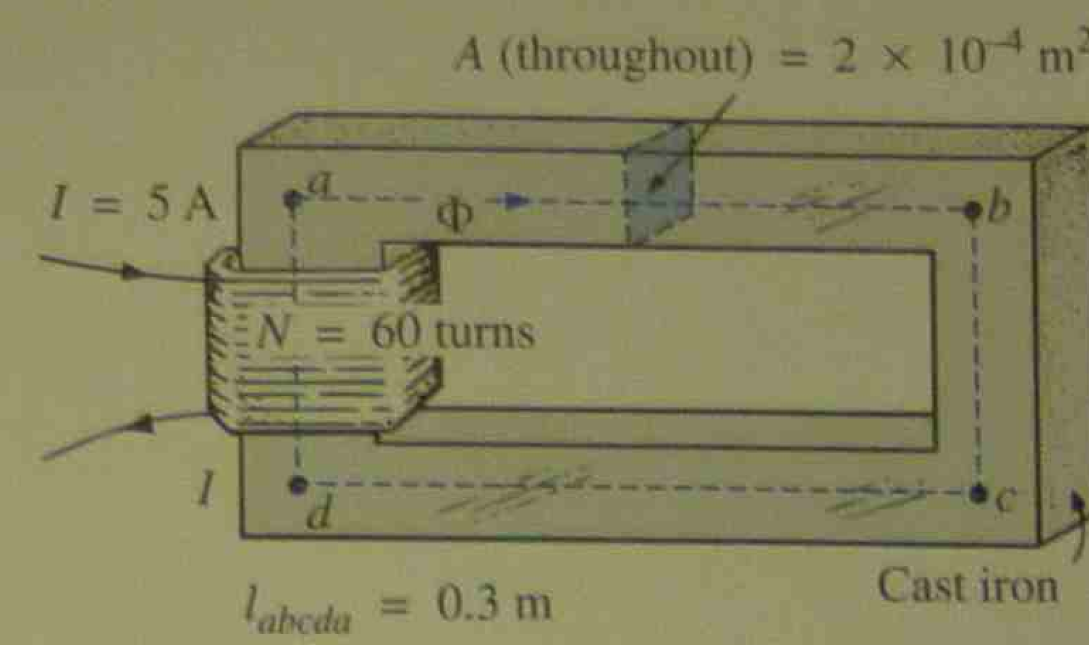


FIG. 11.40

EXAMPLE 11.9 Find the magnetic flux Φ for the series magnetic circuit of Fig. 11.41 for the specified impressed mmf.

Solution: Assuming that the total impressed mmf NI is across the air gap,

$$NI = H_g l_g$$

or
$$H_g = \frac{NI}{l_g} = \frac{400\ \text{At}}{0.001\ \text{m}} = 4 \times 10^5\ \text{At/m}$$

and
$$B_g = \mu_o H_g = (4\pi \times 10^{-7})(4 \times 10^5\ \text{At/m})$$
$$= 0.503\ \text{T}$$

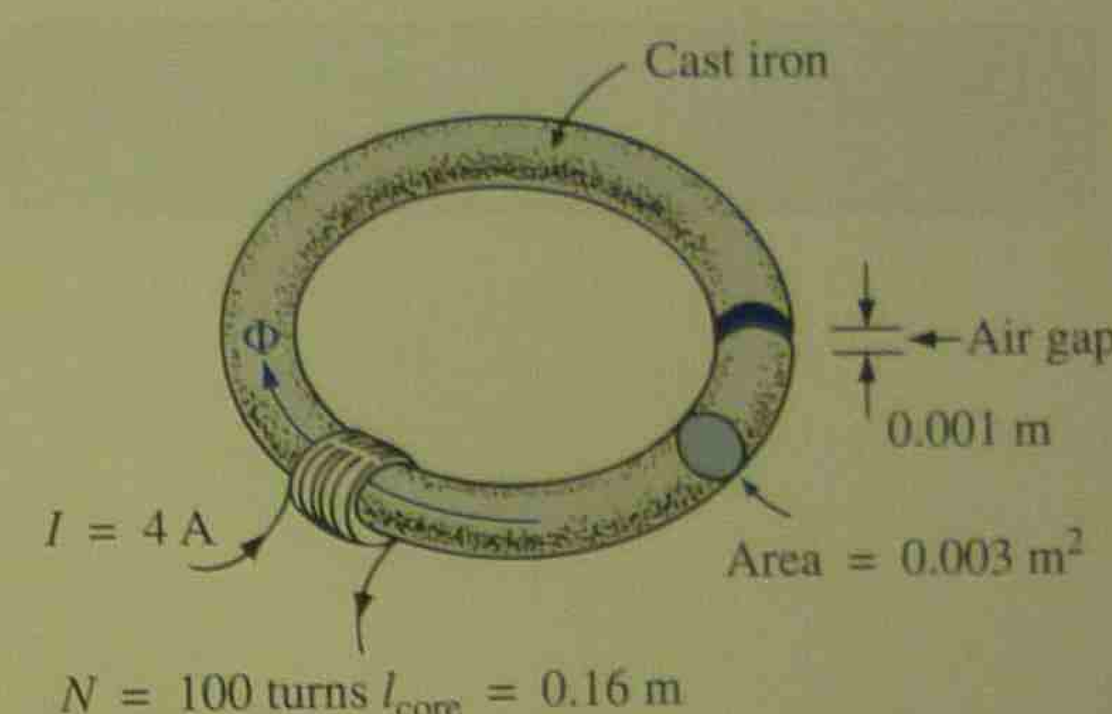


FIG. 11.41

The flux

$$\Phi_g = \Phi_{core} = B_g A$$
$$= (0.503\ \text{T})(0.003\ \text{m}^2)$$
$$\Phi_{core} = 1.51 \times 10^{-3}\ \text{Wb}$$

Using this value of Φ, we can find NI. The data are inserted in Table 11.7.

TABLE 11.7

Section	Φ (Wb)	A (m²)	B (T)	H (At/m)	l (m)	Hl (At)
Core	1.51×10^{-3} Wb	0.003	0.503 T	1500 (B-H curve)	0.16	
Gap	1.51×10^{-3} Wb	0.003	0.503 T	4×10^5	0.001	400

$$H_{core} l_{core} = (1500\ \text{At/m})(0.16\ \text{m}) = 240\ \text{At}$$

Applying Ampère's circuital law results in

$$NI = H_{core} l_{core} + H_g l_g$$
$$= 240\ \text{At} + 400\ \text{At}$$
$$NI = 640\ \text{At} > 400\ \text{At}$$

Since we neglected the reluctance of all the magnetic paths but the air gap, the calculated value is greater than the specified value. We must therefore reduce this value by including the effect of these reluctances.

Since approximately (640 At − 400 At)/640 At = 240 At/640 At ≅ 37.5% of our calculated value is above the desired value, let us reduce Φ by 30% and see how close we come to the impressed mmf of 400 At:

$$\Phi = (1 - 0.3)(1.51 \times 10^{-3}\ \text{Wb})$$
$$= 1.057 \times 10^{-3}\ \text{Wb}$$

See Table 11.8.

TABLE 11.8

Section	Φ (Wb)	*A* (m²)	*B* (T)	*H* (At/m)	*l* (m)	*Hl* (At)
Core	1.057×10^{-3}	0.003			0.16	
Gap	1.057×10^{-3}	0.003			0.001	

$$B = \frac{\Phi}{A} = \frac{1.057 \times 10^{-3}\ \text{Wb}}{0.003\ \text{m}^2} \cong 0.352\ \text{T}$$

$$H_g l_g = (7.96 \times 10^5) B_g l_g$$
$$= (7.96 \times 10^5)(0.352\ \text{T})(0.001\ \text{m})$$
$$\cong 280.19\ \text{At}$$

From *B-H* curves,

$$H_{core} \cong 850\ \text{At/m}$$
$$H_{core} l_{core} = (850\ \text{At/m})(0.16\ \text{m}) = 136\ \text{At}$$

Applying Ampère's circuital law yields

$$NI = H_{core} l_{core} + H_g l_g$$
$$= 136\ \text{At} + 280.19\ \text{At}$$
$$NI = \mathbf{416.19\ At} > 400\ \text{At} \qquad \text{(but within } \pm 5\% \text{ and therefore acceptable)}$$

The solution is, therefore,

$$\Phi \cong \mathbf{1.057 \times 10^{-3}\ Wb}$$

PROBLEMS

SECTION 11.3 Flux Density

1. Using Appendix G, fill in the blanks in the following table. Indicate the units for each quantity.

	Φ	*B*
SI	5×10^{-4} Wb	8×10^{-4} T
CGS	____	____
English	____	____

2. Repeat Problem 1 for the following table if area = 2 in.^{2}:

	Φ	B
SI	_____	_____
CGS	60,000 maxwells	_____
English	_____	_____

3. For the electromagnet of Fig. 11.42:
   a. Find the flux density in the core.
   b. Sketch the magnetic flux lines and indicate their direction.
   c. Indicate the north and south poles of the magnet.

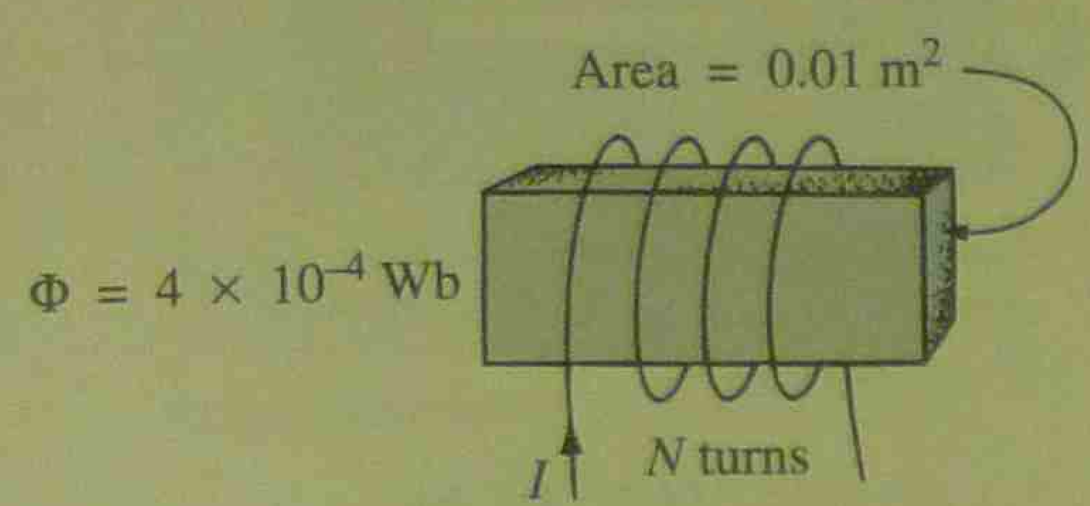


FIG. 11.42

SECTION 11.5 Reluctance

4. Which section of Fig. 11.43 [(a), (b), or (c)] has the largest reluctance to the setting up of flux lines through its longest dimension?

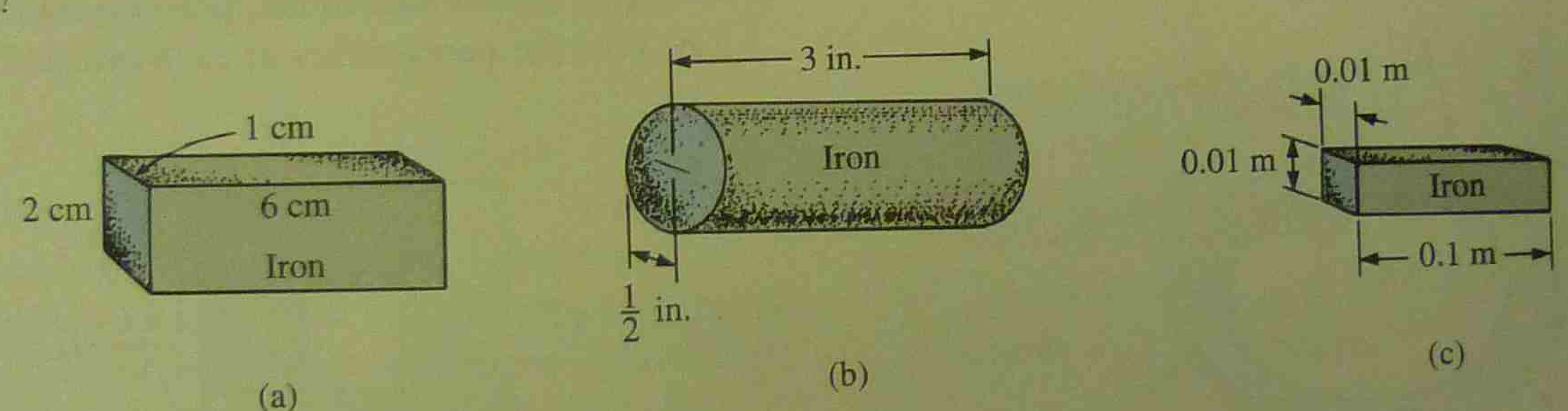


FIG. 11.43

SECTION 11.6 Ohm's Law for Magnetic Circuits

5. Find the reluctance of a magnetic circuit if a magnetic flux $\Phi = 4.2 \times 10^{-4}$ Wb is established by an impressed mmf of 400 At.

6. Repeat Problem 5 for Φ = 72,000 maxwells and an impressed mmf of 120 gilberts.

SECTION 11.7 Magnetizing Force

7. Find the magnetizing force H for Problem 5 in SI units if the magnetic circuit is 6 in. long.

8. If a magnetizing force H of 600 At/m is applied to a magnetic circuit, a flux density B of 1200×10^{-4} Wb/m^2 is established. Find the permeability μ of a material that will produce twice the original flux density for the same magnetizing force.

SECTION 11.8 Hysteresis

9. For the series magnetic circuit of Fig. 11.44, determine the current I necessary to establish the indicated flux.

10. Find the current necessary to establish a flux of $\Phi = 3 \times 10^{-4}$ Wb in the series magnetic circuit of Fig. 11.45.

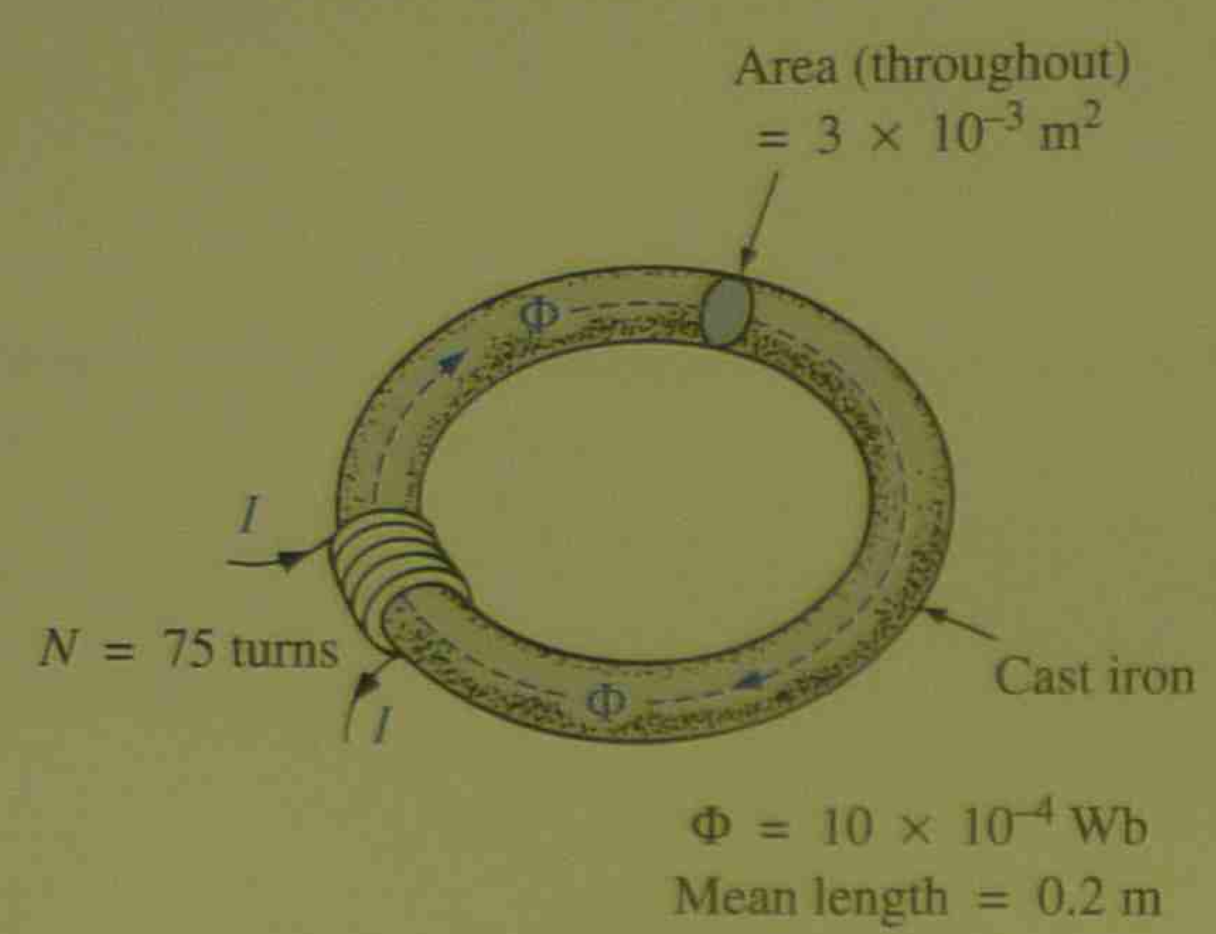


FIG. 11.44

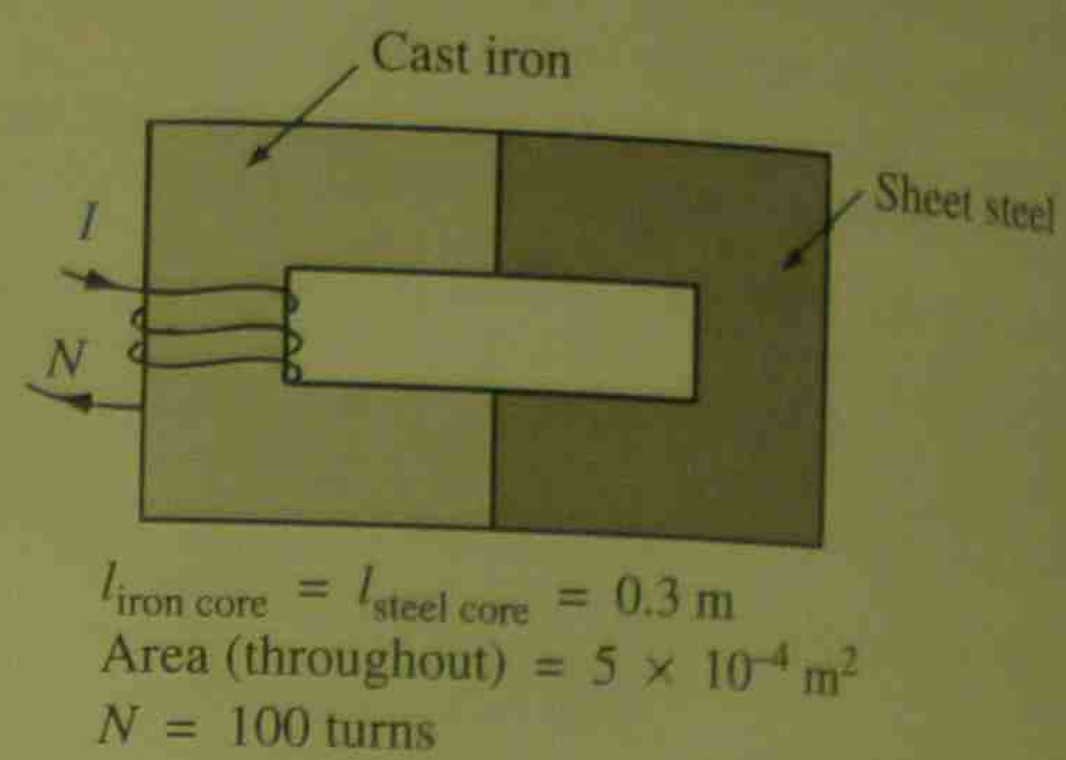


FIG. 11.45

11. a. Find the number of turns N_1 required to establish a flux $\Phi = 12 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 11.46.
    b. Find the permeability μ of the material.

12. a. Find the mmf (NI) required to establish a flux $\Phi = 80{,}000$ lines in the magnetic circuit of Fig. 11.47.
    b. Find the permeability of each material.

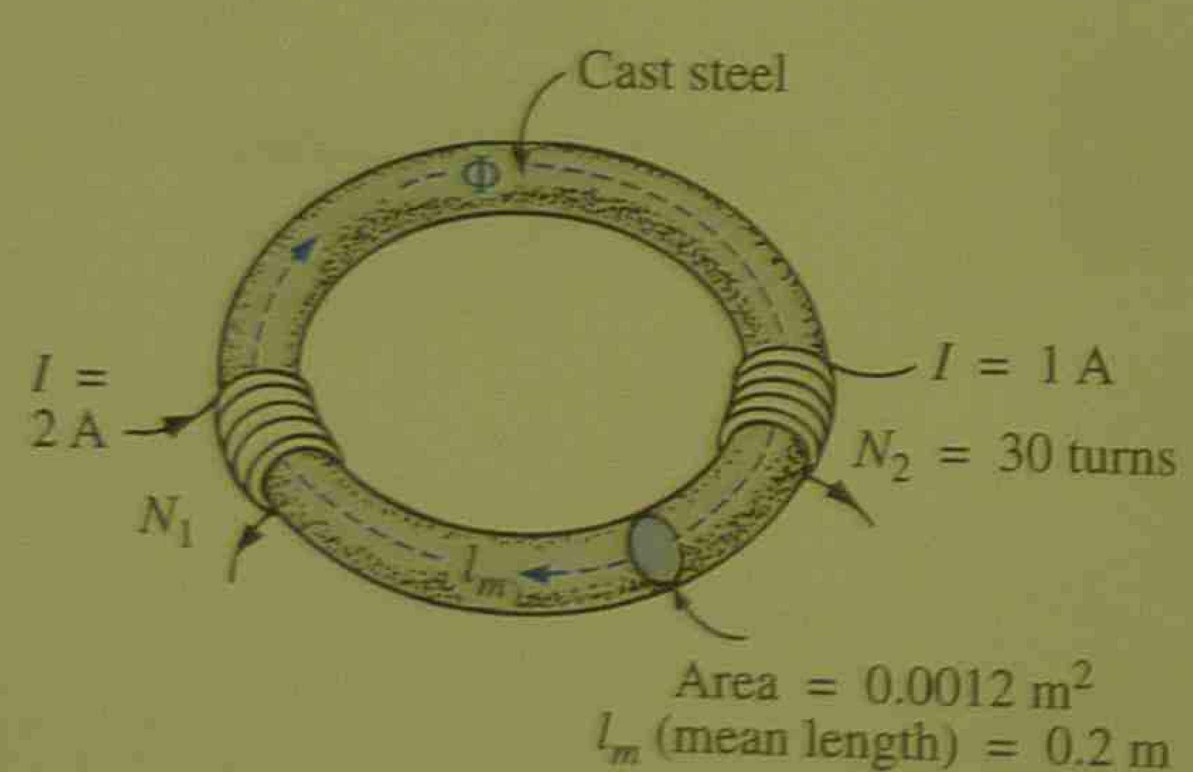


FIG. 11.46

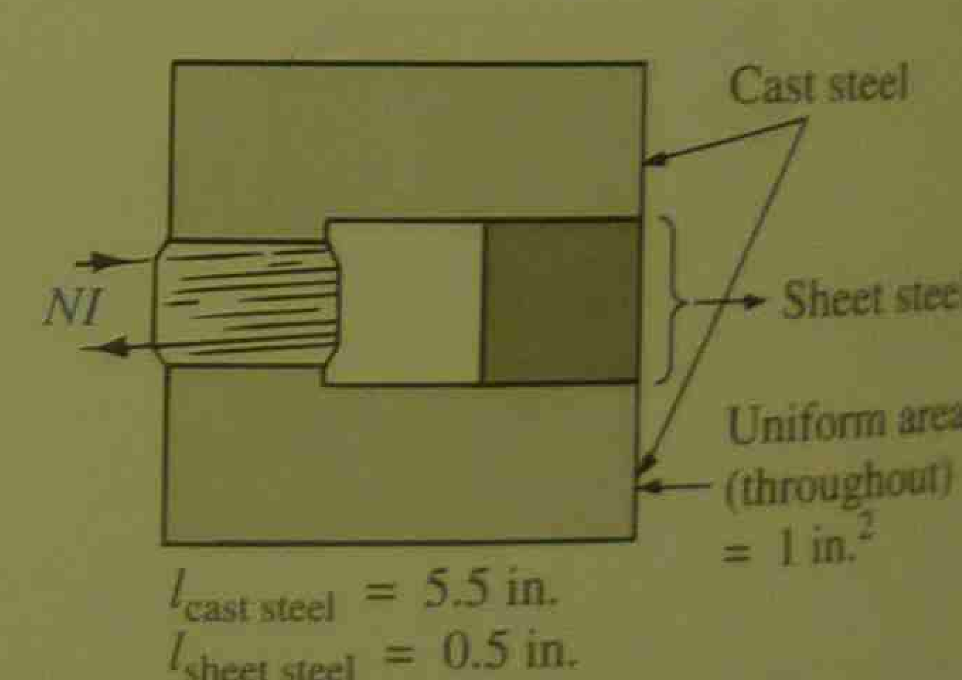


FIG. 11.47

*13. For the series magnetic circuit of Fig. 11.48 with two impressed sources of magnetic "pressure," determine the current I. Each applied mmf establishes a flux pattern in the clockwise direction.

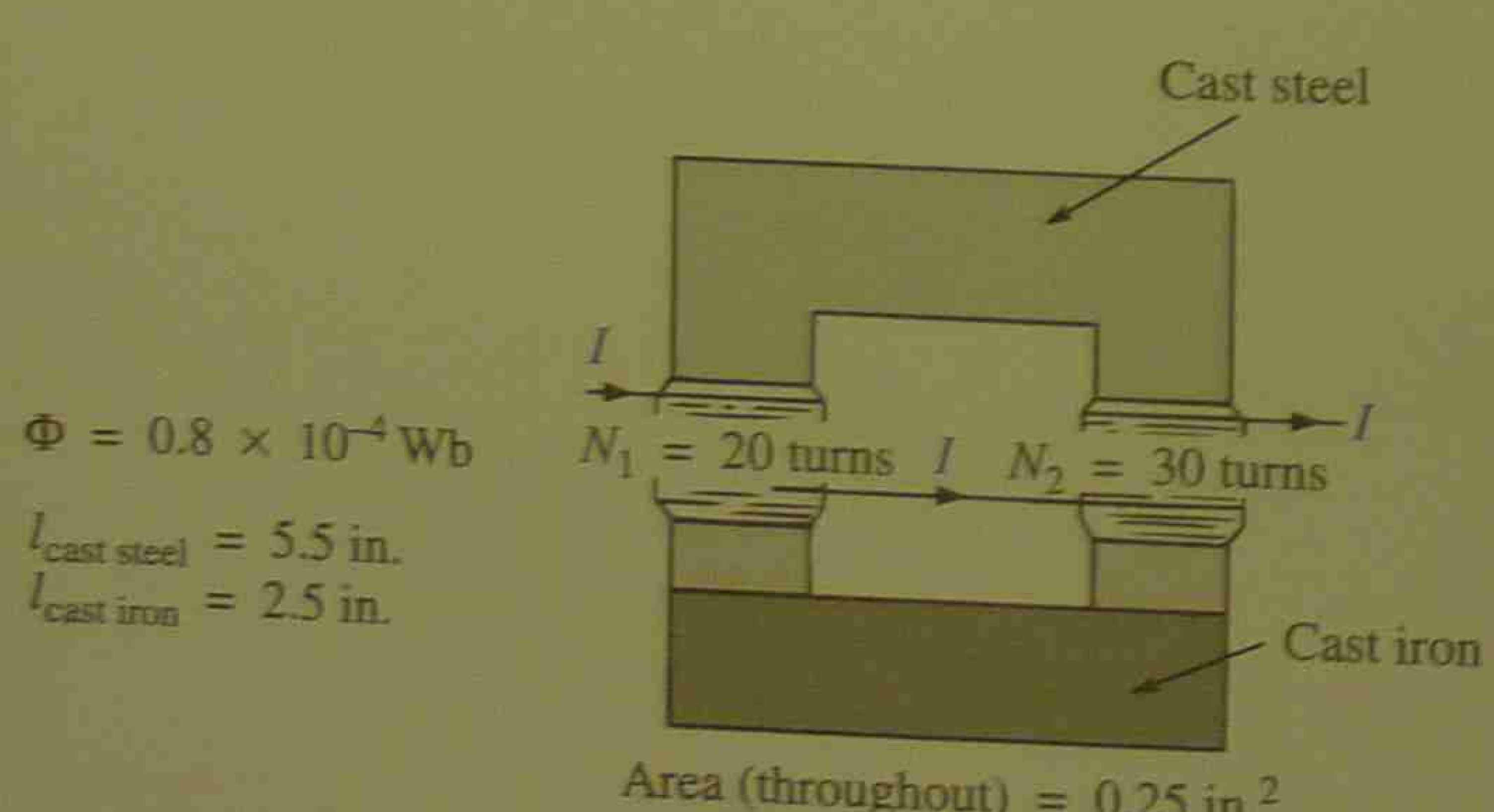


FIG. 11.48

SECTION 11.12 Air Gaps

14. a. Find the current I required to establish a flux $\Phi = 2.4 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 11.49.
   b. Compare the mmf drop across the air gap to that across the rest of the magnetic circuit. Discuss your results using the value of μ for each material.

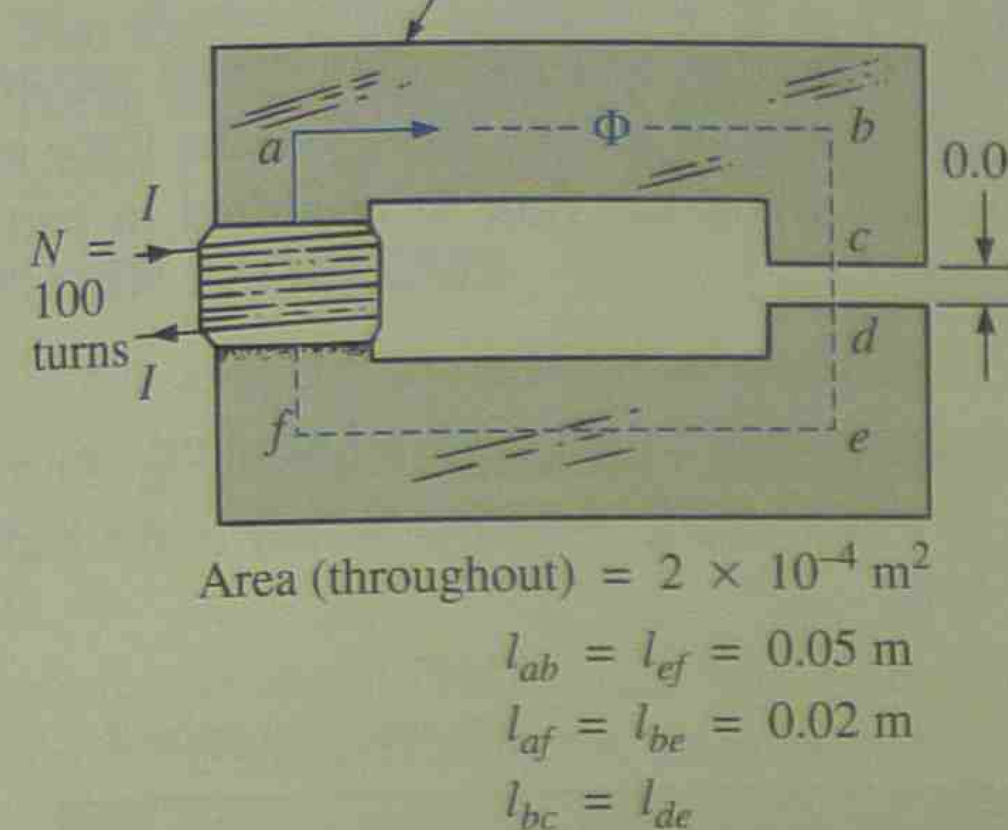


FIG. 11.49

*15. The force carried by the plunger of the door chime of Fig. 11.50 is determined by

$$f = \frac{1}{2} NI \frac{d\phi}{dx} \qquad \text{(newtons)}$$

where $d\phi/dx$ is the rate of change of flux linking the coil as the core is drawn into the coil. The greatest rate of change of flux will occur when the core is 1/4 to 3/4 the way through. In this region, if Φ changes from 0.5×10^{-4} Wb to 8×10^{-4} Wb, what is the force carried by the plunger?

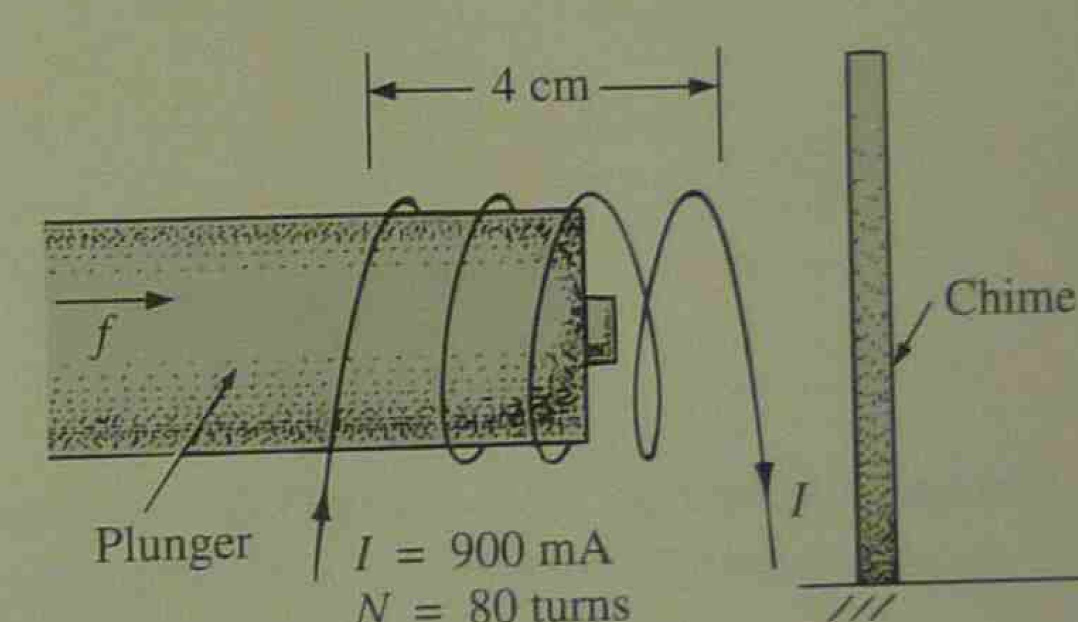


FIG. 11.50
Door chime.

16. Determine the current I_1 required to establish a flux of $\Phi = 2 \times 10^{-4}$ Wb in the magnetic circuit of Fig. 11.51.

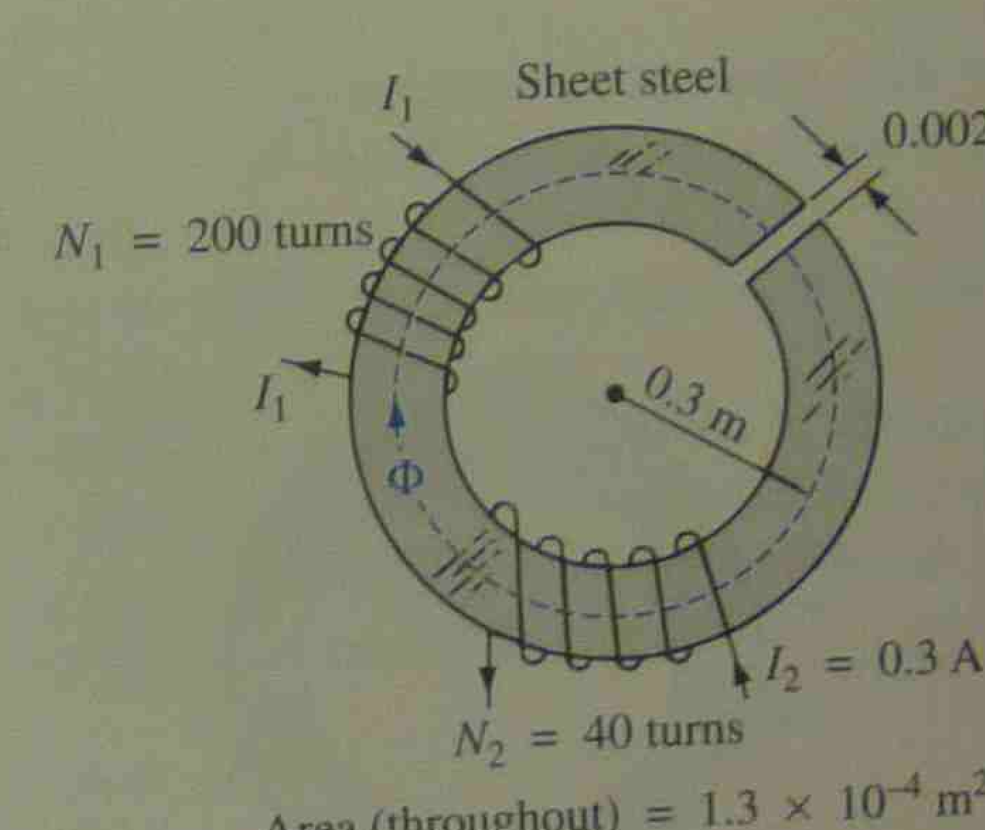


FIG. 11.51

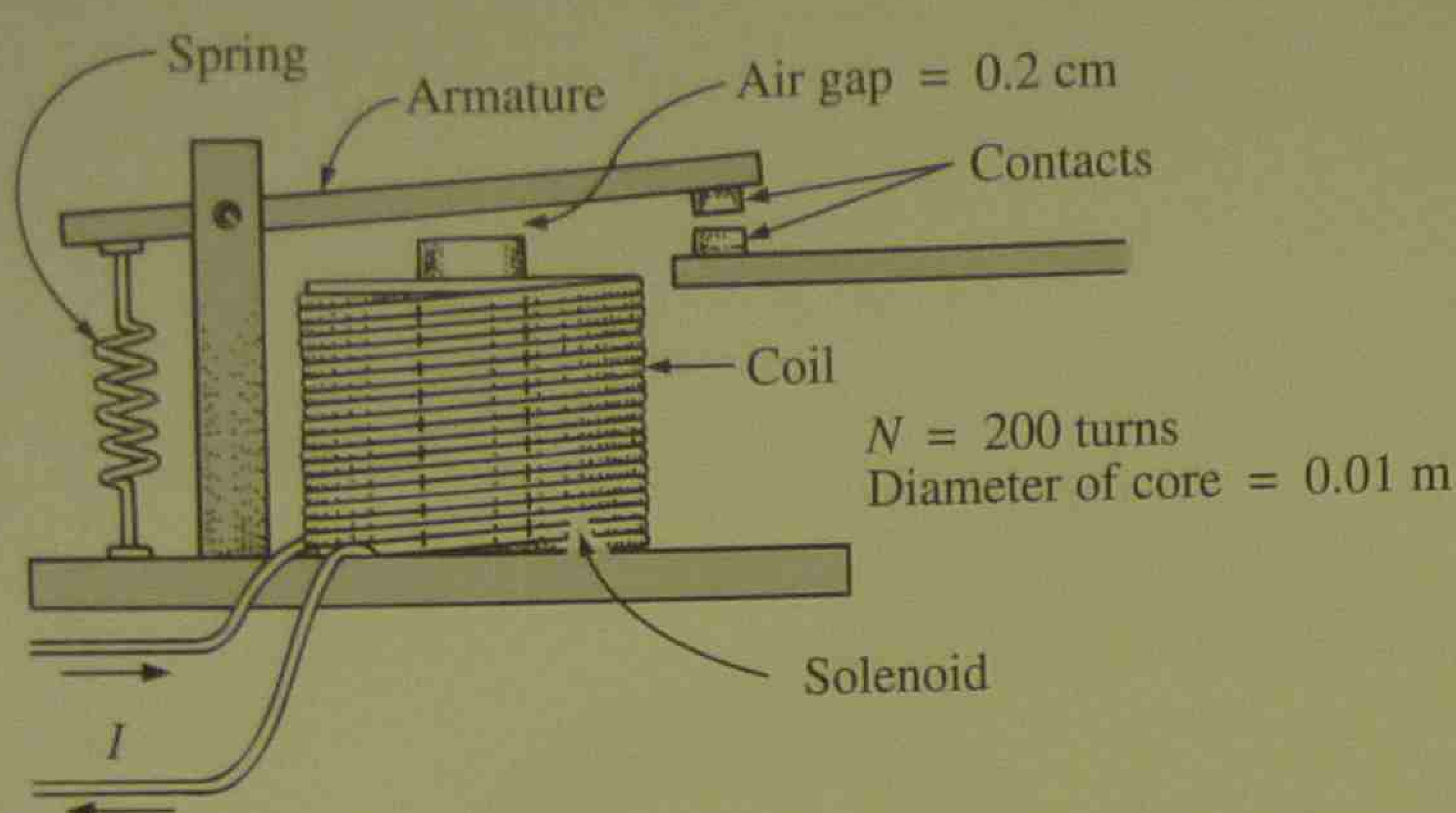


FIG. 11.52
Relay.

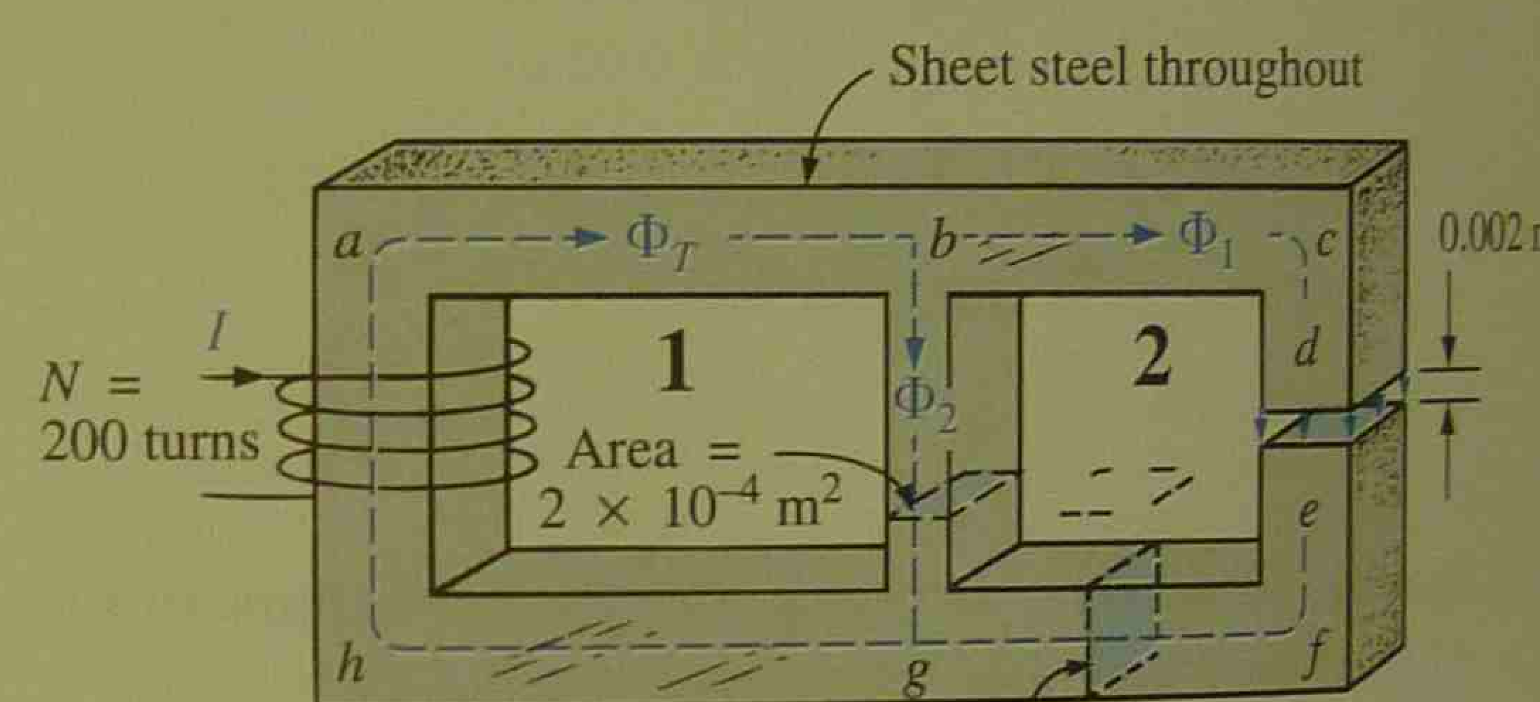
*17. a. A flux of 0.2×10^{-4} Wb will establish sufficient attractive force for the armature of the relay of Fig. 11.52 to close the contacts. Determine the required current to establish this flux level if we assume the total mmf drop is across the air gap.

b. The force exerted on the armature is determined by the equation

$$F \text{ (newtons)} = \frac{1}{2} \cdot \frac{B_g^2 A}{\mu_o}$$

where B_g is the flux density within the air gap and A is the common area of the air gap. Find the force in newtons exerted when the flux Φ specified in part (a) is established.

*18. For the series-parallel magnetic circuit of Fig. 11.53, find the value of I required to establish a flux in the gap $\Phi_g = 2 \times 10^{-4}$ Wb.



Area for sections other than $bg = 5 \times 10^{-4}\ m^2$
$l_{ab} = l_{bg} = l_{gh} = l_{ha} = 0.2$ m
$l_{bc} = l_{fg} = 0.1$ m, $l_{cd} = l_{ef} = 0.099$ m

FIG. 11.53

SECTION 11.14 Determining Φ

19. Find the magnetic flux Φ established in the series magnetic circuit of Fig. 11.54.

*20. Determine the magnetic flux Φ established in the series magnetic circuit of Fig. 11.55.

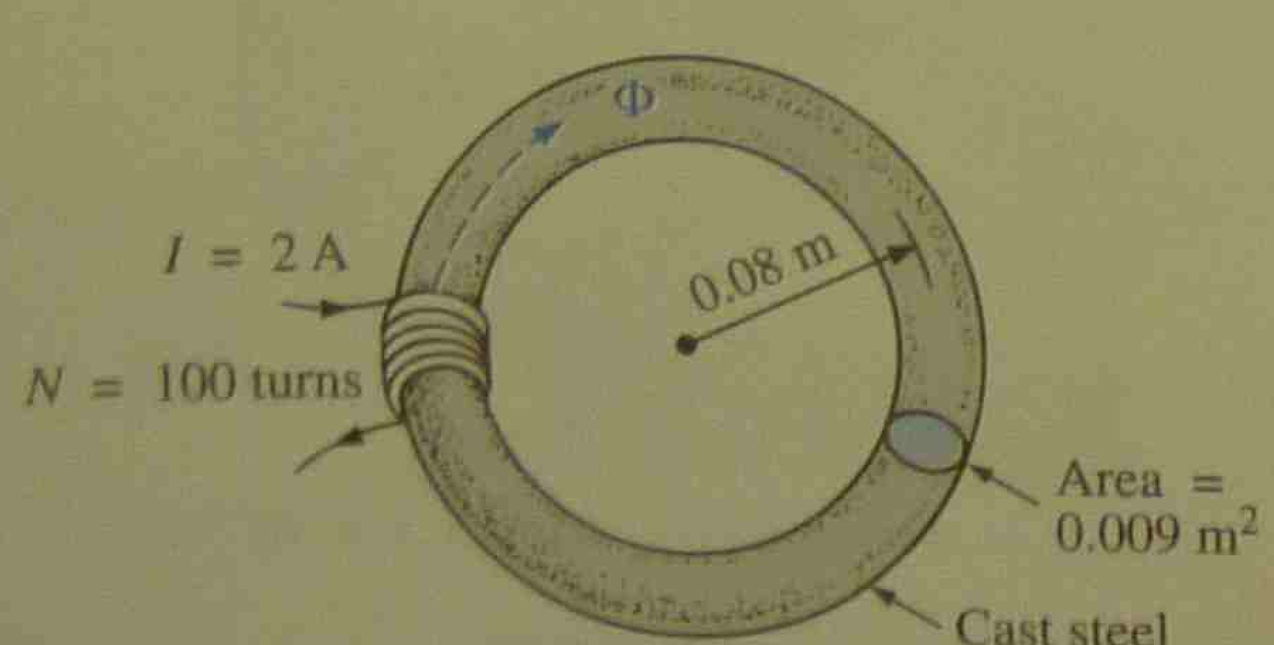
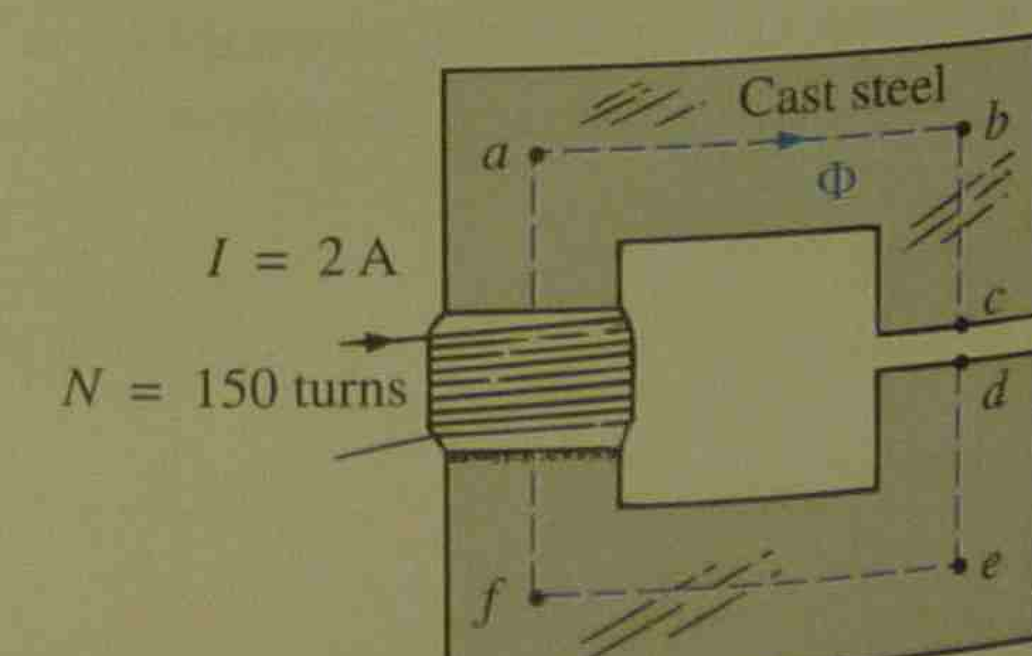


FIG. 11.54



$l_{cd} = 8 \times 10^{-4}$ m
$l_{ab} = l_{be} = l_{ef} = l_{fa} = 0.2$ m
Area (throughout) $= 2 \times 10^{-4}\ m^2$
$l_{bc} = l_{de}$

FIG. 11.55

*21. Note how closely the *B-H* curve of cast steel in Fig. 11.23 matches the curve for the voltage across a capacitor as it charges from zero volts to its final value.
   a. Using the equation for the charging voltage as a guide, write an equation for B as a function of H ($B = f(H)$) for cast steel.
   b. Test the resulting equation at $H = 900$ At/m, 1800 At/m, and 2700 At/m.
   c. Using the equation of part (a), derive an equation for H in terms of B ($H = f(B)$).
   d. Test the resulting equation at $B = 1$ T and $B = 1.4$ T.
   e. Using the result of part (c), perform the analysis of Example 11.3 and compare the results for the current I.

COMPUTER PROBLEMS

PSpice is not designed to perform an analysis of magnetic circuits, but BASIC can be employed if we use mathematical representations of the *B-H* curve of Fig. 11.23, as developed in Problem 21.

*22. Using the results of Problem 21, write a BASIC program to perform the analysis of a core such as that shown in Example 11.3. That is, let the dimensions of the core and the applied turns be input variables requested by the program.

*23. Using the results of Problem 21, develop a BASIC program to perform the analysis appearing in Example 11.9 for cast steel. A test routine will have to be developed to determine whether the results obtained are sufficiently close to the applied ampere-turns.

GLOSSARY

Ampère's circuital law A law establishing the fact that the algebraic sum of the rises and drops of the mmf around a closed loop of a magnetic circuit is equal to zero.

Diamagnetic materials Materials that have permeabilities slightly less than that of free space.

Domain A group of magnetically aligned atoms.

Electromagnetism Magnetic effects introduced by the flow of charge or current.

Ferromagnetic materials Materials having permeabilities hundreds and thousands of times greater than that of free space.

Flux density (B) A measure of the flux per unit area perpendicular to a magnetic flux path. It is measured in teslas (T) or webers per square meter (Wb/m^2).

Hysteresis The lagging effect between flux density of a material and the magnetizing force applied.

Magnetic flux lines Lines of a continuous nature that reveal the strength and direction of the magnetic field.

Magnetizing force (H) A measure of the magnetomotive force per unit length of a magnetic circuit.

Magnetomotive force ($\mathscr{F}$) The "pressure" required to establish magnetic flux in a ferromagnetic material. It is measured in ampere-turns (At).

Paramagnetic materials Materials that have permeabilities slightly greater than that of free space.

Permanent magnet A material such as steel or iron that will remain magnetized for long periods of time without the aid of external means.

Permeability (μ) A measure of the ease with which magnetic flux can be established in a material. It is measured in Wb/Am.

Relative permeability (μ_r) The ratio of the permeability of a material to that of free space.

Reluctance ($\mathscr{R}$) A quantity determined by the physical characteristics of a material that will provide an indication of the "reluctance" of that material to the setting up of magnetic flux lines in the material. It is measured in rels or At/Wb.

12

Inductors

12.1 INTRODUCTION

We have examined the resistor and the capacitor in detail. In this chapter we shall consider a third element, the *inductor,* which has a number of response characteristics similar in many respects to those of the capacitor. In fact, some sections of this chapter will proceed parallel to those for the capacitor to emphasize the similarity that exists between the two elements.

12.2 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

If a conductor is moved through a magnetic field so that it cuts magnetic lines of flux, a voltage will be induced across the conductor, as shown in Fig. 12.1. The greater the number of flux lines cut per unit time (by increasing the speed with which the conductor passes through the field), or the stronger the magnetic field strength (for the same traversing speed), the greater will be the induced voltage across the conductor. If

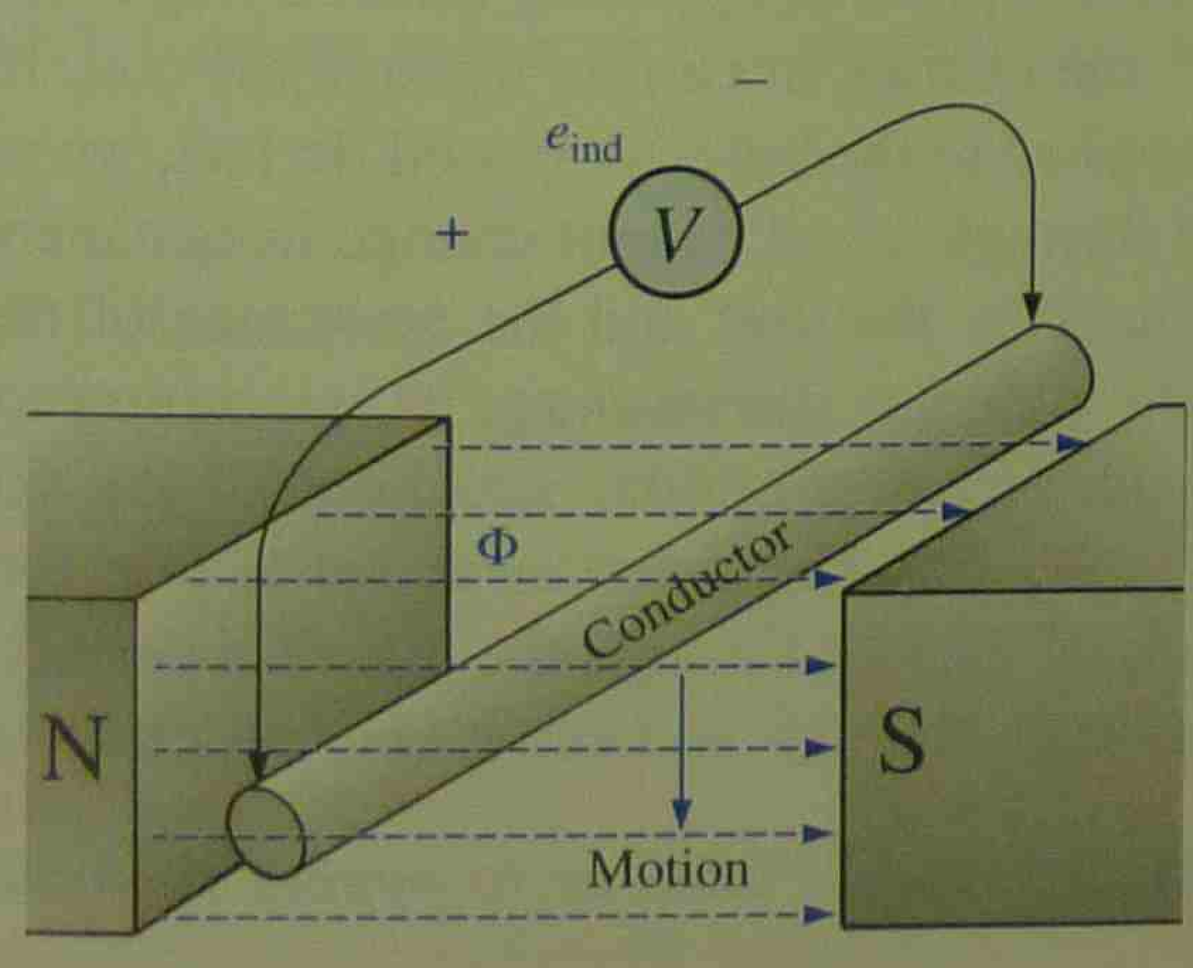


FIG. 12.1

the conductor is held fixed and the magnetic field is moved so that its flux lines cut the conductor, the same effect will be produced.

If a coil of N turns is placed in the region of a changing flux, as in Fig. 12.2, a voltage will be induced across the coil as determined by *Faraday's law:*

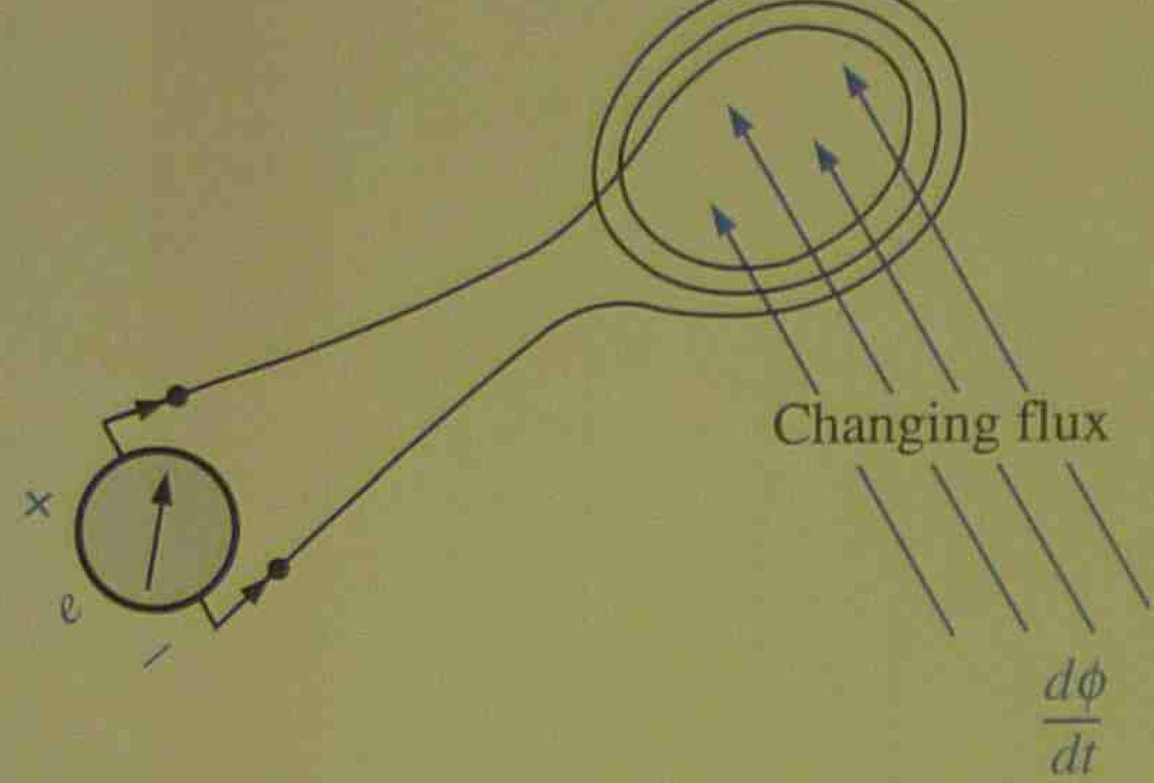


FIG. 12.2

$$e = N\frac{d\phi}{dt} \qquad \text{(volts, V)} \tag{12.1}$$

where N represents the number of turns of the coil and $d\phi/dt$ is the instantaneous change in flux (in webers) linking the coil. The term *linking* refers to the flux within the turns of wire. The term *changing* simply indicates that either the strength of the field linking the coil changes in magnitude or the coil is moved through the field in such a way that the number of flux lines through the coil changes with time.

If the flux linking the coil ceases to change, such as when the coil simply sits still in a magnetic field of fixed strength, $d\phi/dt = 0$, and the induced voltage $e = N(d\phi/dt) = N(0) = 0$.

12.3 LENZ'S LAW

In Section 11.2 it was shown that the magnetic flux linking a coil of N turns with a current I has the distribution of Fig. 12.3.

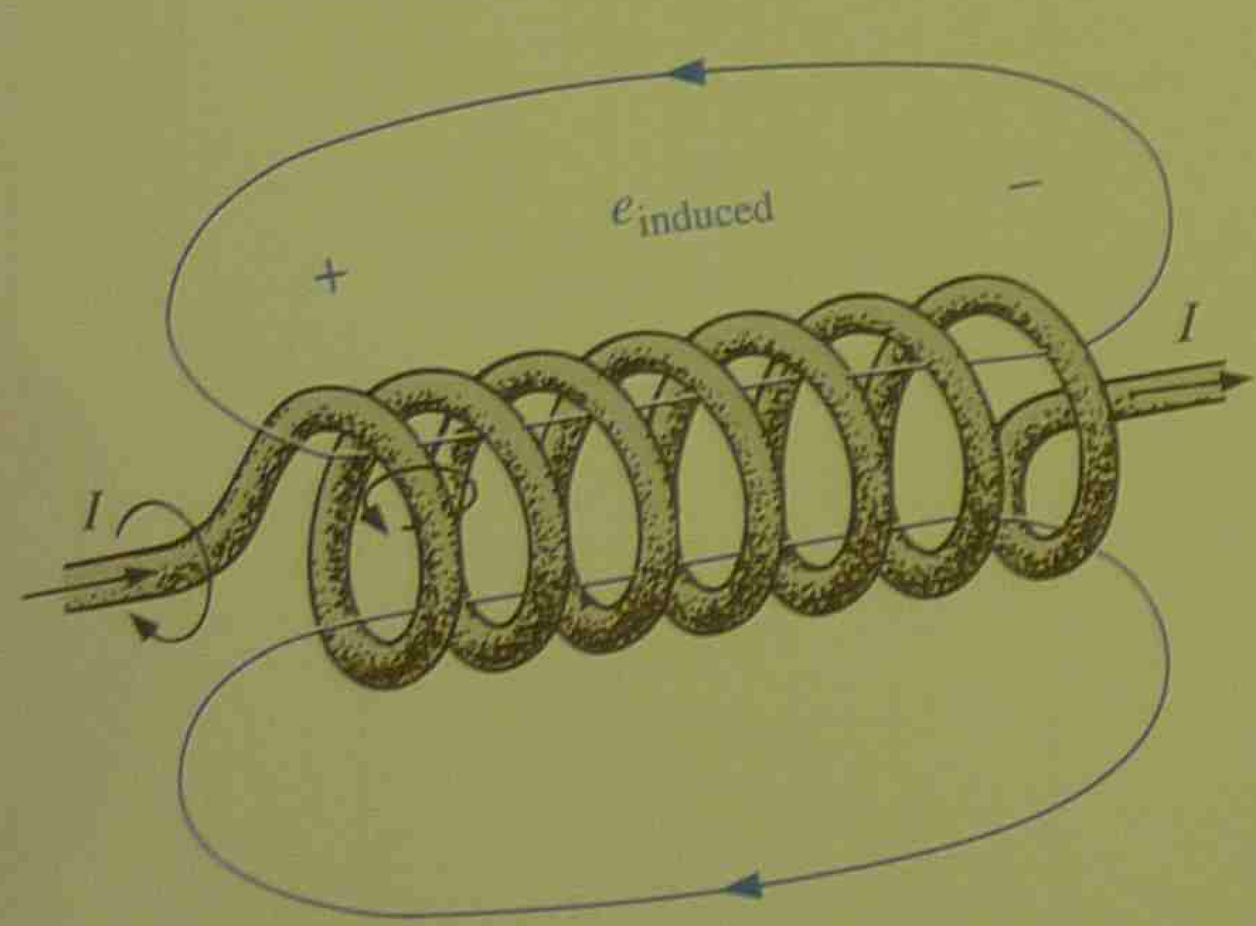


FIG. 12.3

If the current increases in magnitude, the flux linking the coil also increases. It was shown in Section 12.2, however, that a changing flux linking a coil induces a voltage across the coil. For this coil, therefore, an induced voltage is developed *across* the coil due to the change in current *through* the coil. The polarity of this induced voltage tends to establish a current in the coil which produces a flux that will oppose any change in the original flux. In other words, the induced effect (e_{ind}) is a result of the increasing current through the coil. However, the resulting induced voltage will tend to establish a current that will oppose the increasing change in current through the coil. Keep in mind that this is all occurring simultaneously. The instant the current begins to increase in magnitude, there will be an opposing effect trying to limit the change. It is "choking" the change in current through the coil. Hence, the term *choke* is often applied to the inductor or coil. In fact, we will find shortly that the current through a coil cannot change instantaneously. A period of time determined by the coil and the resistance of the circuit is required before the inductor discontinues its opposition to a momentary change in current. Recall a similar situation for the voltage across a capacitor in Chapter 10. The reaction above is true for increasing or decreasing levels of current through the coil. This effect is an example of a general principle known as *Lenz's law,* which states that

an induced effect is always such as to oppose the cause that produced it.

12.4 SELF-INDUCTANCE

The ability of a coil to oppose any change in current is a measure of the *self-inductance* L of the coil. For brevity, the prefix *self* is usually dropped. Inductance is measured in henries (H), after the American physicist Joseph Henry (Fig. 12.4).

Inductors are coils of various dimensions designed to introduce specified amounts of inductance into a circuit. The inductance of a coil varies directly with the magnetic properties of the coil. Ferromagnetic materials, therefore, are frequently employed to increase the inductance by increasing the flux linking the coil.

A close approximation, in terms of physical dimensions, for the inductance of the coils of Fig. 12.5 can be found using the following equation:

$$L = \frac{N^2 \mu A}{l} \quad \text{(henries, H)} \tag{12.2}$$

where N represents the number of turns, μ the permeability of the core (recall that μ is not a constant but depends on the level of B and H since $\mu = B/H$), A the area of the core in square meters, and l the mean length of the core in meters.

Substituting $\mu = \mu_r \mu_o$ into Eq. (12.2) yields

$$L = \frac{N^2 \mu_r \mu_o A}{l} = \mu_r \frac{N^2 \mu_o A}{l}$$

and

$$L = \mu_r L_o \tag{12.3}$$

where L_o is the inductance of the coil with an air core. In other words, the inductance of a coil with a ferromagnetic core is the relative permeability of the core times the inductance achieved with an air core.

Equations for the inductance of coils different from those shown above can be found in reference handbooks. Most of the equations are more complex than those just described.

EXAMPLE 12.1 Find the inductance of the air-core coil of Fig. 12.6.

Solution:

$$\mu = \mu_r \mu_o = (1)(\mu_o) = \mu_o$$

$$A = \frac{\pi d^2}{4} = \frac{(3.1416)(4 \times 10^{-3}\text{ m})^2}{4} = 12.57 \times 10^{-6}\text{ m}^2$$

$$L_o = \frac{N^2 \mu_o A}{l} = \frac{(100\text{ t})^2(4\pi \times 10^{-7}\text{ Wb/A}\cdot\text{m})(12.57 \times 10^{-6}\text{ m}^2)}{0.1\text{ m}}$$

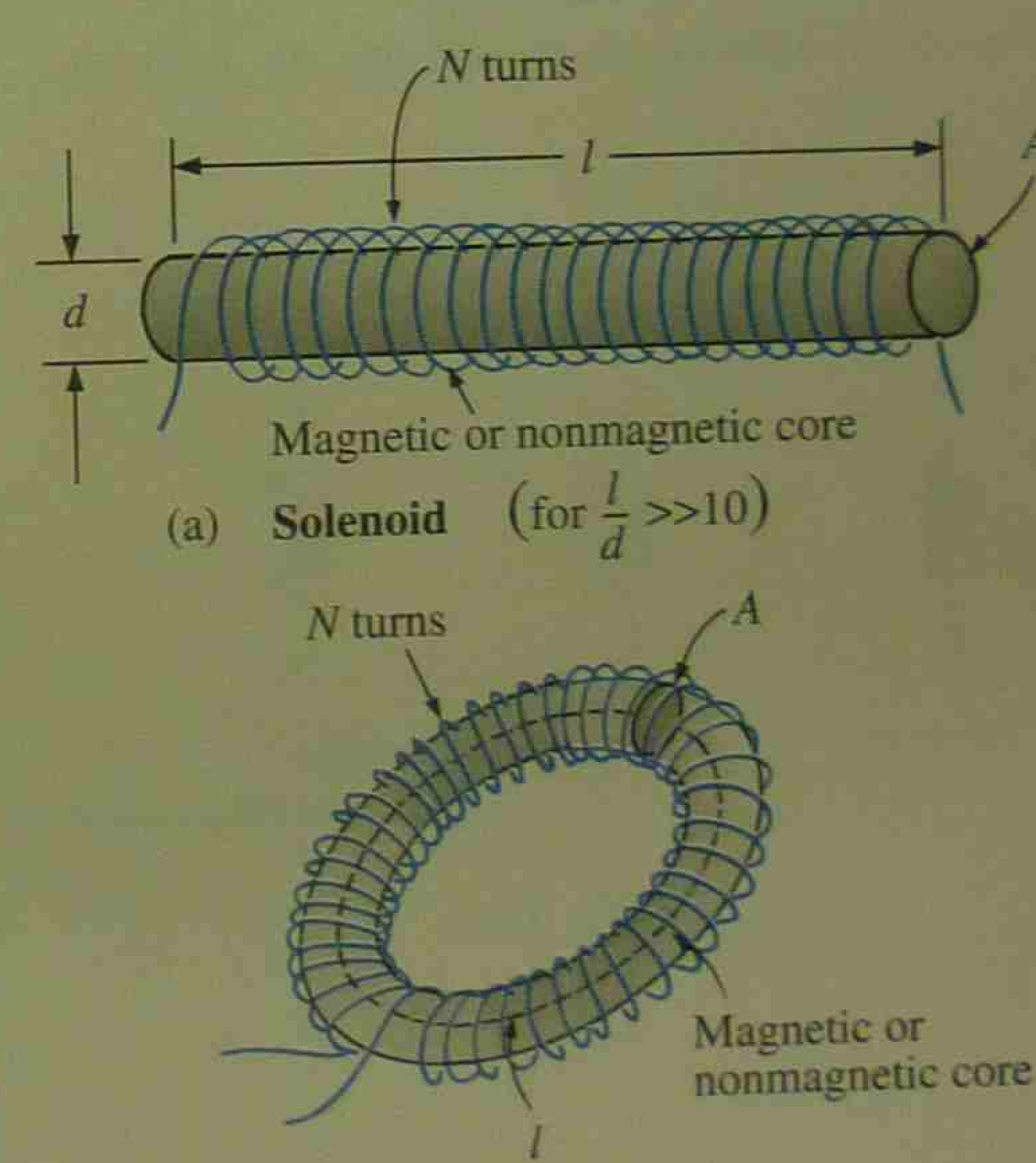
$$= \mathbf{1.58\ \mu H}$$

American (Albany, Princeton)
(1797–1878)
Physicist and Mathematician
Professor of Natural Philosophy, Princeton University

Courtesy of the Smithsonian Institution
Photo No. 59,054

In the early 1800s the title Professor of Natural Philosophy was applied to educators in the sciences. As a student and teacher at the Albany Academy, he performed extensive research in the area of electromagnetism. He improved the design of *electromagnets* by insulating the coil wire to permit a tighter wrap on the core. One of his earlier designs was capable of lifting 3600 pounds. In 1832 he discovered and delivered a paper on *self-induction.* This was followed by the construction of an effective *electric telegraph transmitter and receiver* and extensive research on the oscillatory nature of lightning and discharges from a *Leyden jar.* In 1845 he was appointed the first Secretary of the Smithsonian.

FIG. 12.4 JOSEPH HENRY



(a) **Solenoid** (for $\frac{l}{d} >> 10$)

(b) **Toroid**

FIG. 12.5

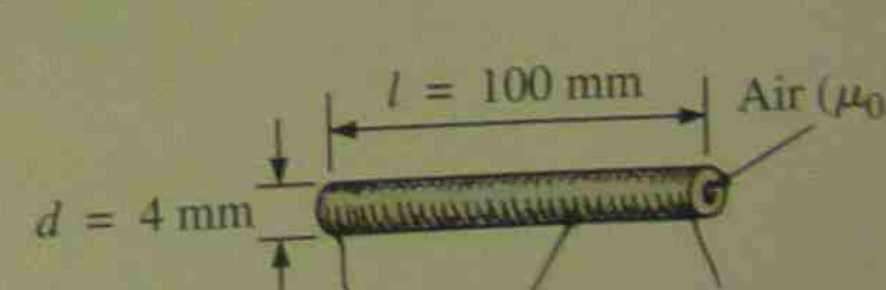


FIG. 12.6

EXAMPLE 12.2 Repeat Example 12.1, but with an iron core and conditions such that $\mu_r = 2000$.

Solution: By Eq. (12.3),

$$L = \mu_r L_o = (2000)(1.58 \times 10^{-6}\ \text{H}) = \mathbf{3.16\ mH}$$

12.5 TYPES OF INDUCTORS

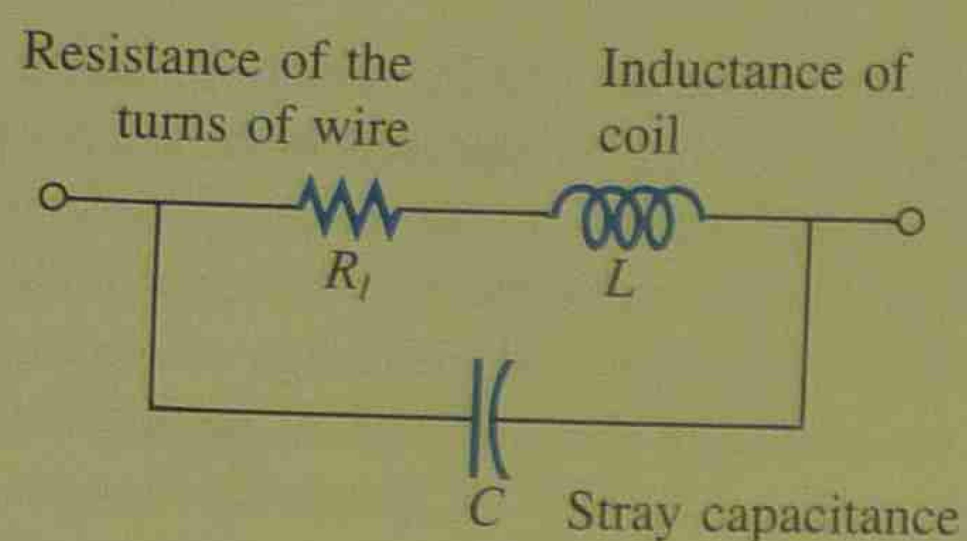


FIG. 12.7
Complete equivalent model for an inductor.

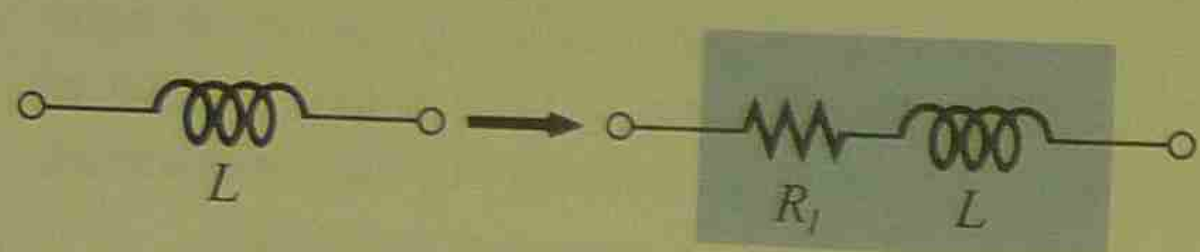


FIG. 12.8
Practical equivalent model for an inductor.

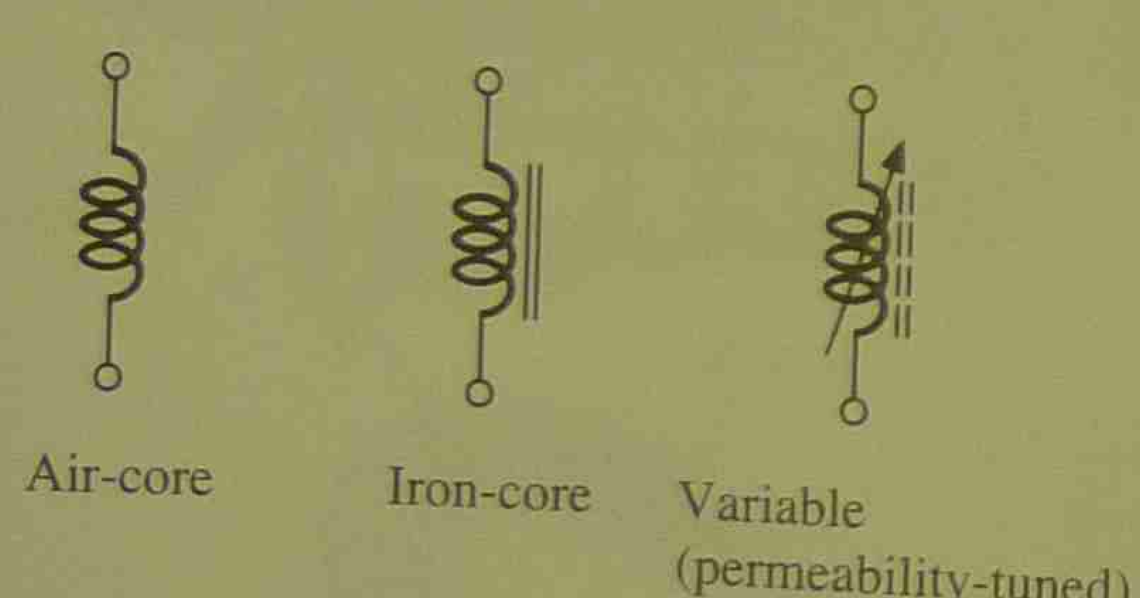


FIG. 12.9

Associated with every inductor are a resistance equal to the resistance of the turns and a stray capacitance due to the capacitance between the turns of the coil. To include these effects, the equivalent circuit for the inductor is as shown in Fig. 12.7. However, for most applications considered in this text, the stray capacitance appearing in Fig. 12.7 can be ignored, resulting in the equivalent model of Fig. 12.8.

The resistance R_l can play an important role in the analysis of networks with inductive elements. For most applications, we have been able to treat the capacitor as an ideal element and maintain a high degree of accuracy. For the inductor, however, R_l must often be included in the analysis and can have a pronounced effect on the response of a system (see Chapter 20, ''Resonance''). The level of R_l can extend from a few ohms to a few hundred ohms. Keep in mind that the longer or thinner the wire used in the construction of the inductor, the greater will be the dc resistance as determined by $R = \rho l/A$. Our initial analysis will treat the inductor as an ideal element. Once a general feeling for the response of the element is established, the effects of R_l will be included.

The primary function of the inductor, however, is to introduce inductance—not resistance or capacitance—into the network. For this reason, the symbols employed for inductance are as shown in Fig. 12.9.

All inductors, like capacitors, can be listed under two general headings: *fixed* and *variable*. The fixed air-core and iron-core inductors were described in the last section. The permeability-tuned variable coil has a ferromagnetic shaft that can be moved within the coil to vary the flux linkages of the coil and thereby its inductance. Several fixed and variable inductors appear in Fig. 12.10.

The primary reasons for inductor failure are shorts that develop between the windings and open circuits in the windings due to factors such as excessive currents, overheating, and age. The open-circuit condition can easily be checked with an ohmmeter (∞ ohms indication), but the short-circuit condition is harder because the resistance of many good inductors is relatively small and the shorting of a few windings will not adversely affect the total resistance. Of course, if one is aware of the typical resistance of the coil, it can be compared to the measured value. A short between the windings and the core can be checked by simply placing one lead of the meter on one wire (terminal) and the other on the core itself. An indication of zero ohms reflects a short between the two because the wire that makes up the winding has an insulation jacket throughout.

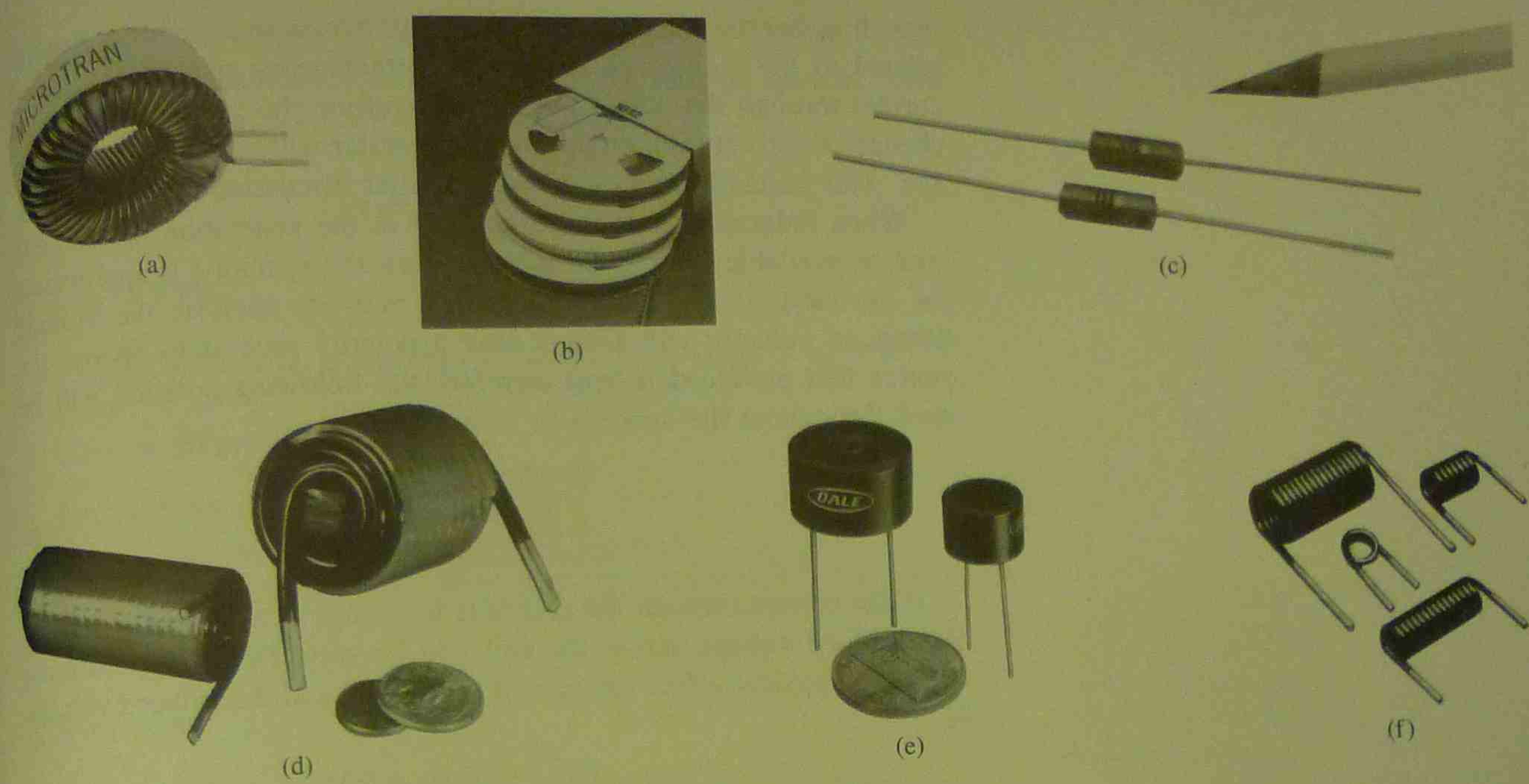


FIG. 12.10

Various types of inductors: (a) toroidal power inductor (1.4 μH to 5.6 mH) (courtesy of Microtan Co., Inc.); surface mount inductors on reels (0.1 μH through 1000 μH on 500-piece reels in 46 values) (courtesy of Bell Industries); (c) molded inductors (0.1 μH to 10 μH); (d) high current filter inductors (24 μH at 60 A to 500 μH at 15 A); (e) toroid filter inductors (40 μH to 5 H); (f) air core inductors (1 to 32 turns) for high frequency applications. (Parts (c) through (f) courtesy of Dale Electronics, Inc.)

12.6 INDUCED VOLTAGE

The inductance of a coil is also a measure of the change in flux linking a coil due to a change in current through the coil; that is,

$$L = N\frac{d\phi}{di} \quad \text{(H)} \qquad (12.4)$$

where N is the number of turns, ϕ is the flux in webers, and i is the current through the coil. The equation states that the larger the inductance of a coil (with N fixed), the larger will be the instantaneous change in flux linking the coil due to an instantaneous change in current through the coil.

If we write Eq. (12.1) as

$$e_L = N\frac{d\phi}{dt} = \left(N\frac{d\phi}{di}\right)\left(\frac{di}{dt}\right)$$

and substitute Eq. (12.4), we then have

$$e_L = L\frac{di}{dt} \quad \text{(V)} \qquad (12.5)$$

revealing that the magnitude of the voltage across an inductor is directly related to the inductance L and the instantaneous rate of change of current through the coil. Obviously, therefore, the greater the rate of change of current through the coil, the greater will be the induced voltage. This certainly agrees with our earlier discussion of Lenz's law.

When induced effects are employed in the generation of voltages such as available from dc or ac generators, the symbol e is appropriate for the induced voltage. However, in network analysis the voltage across an inductor will always have a polarity such as to oppose the source that produced it, and therefore the following notation will be used throughout the analysis to come:

$$\boxed{v_L = L\frac{di}{dt}} \tag{12.6}$$

If the current through the coil fails to change at a particular instant, the induced voltage across the coil will be zero. For dc applications, after the transient effect has passed, $di/dt = 0$, and the induced voltage is

$$v_L = L\frac{di}{dt} = L(0) = 0\text{ V}$$

Recall that the equation for the current of a capacitor is the following:

$$i_C = C\frac{dv_C}{dt}$$

Note the similarity between this equation and Eq. (12.6). In fact, if we apply the duality $v \rightleftarrows i$ (that is, interchange the two) and $L \rightleftarrows C$ for capacitance and inductance, each equation can be derived from the other.

The average voltage across the coil is defined by the equation

$$\boxed{v_{L_{av}} = L\frac{\Delta i}{\Delta t}} \quad \text{(V)} \tag{12.7}$$

where Δ signifies finite change (a measurable change). Compare this to $i_C = C(\Delta v/\Delta t)$, and the meaning of Δ and application of this equation should be clarified from Chapter 10. An example follows.

EXAMPLE 12.3 Find the waveform for the average voltage across the coil if the current through a 4-mH coil is as shown in Fig. 12.11.

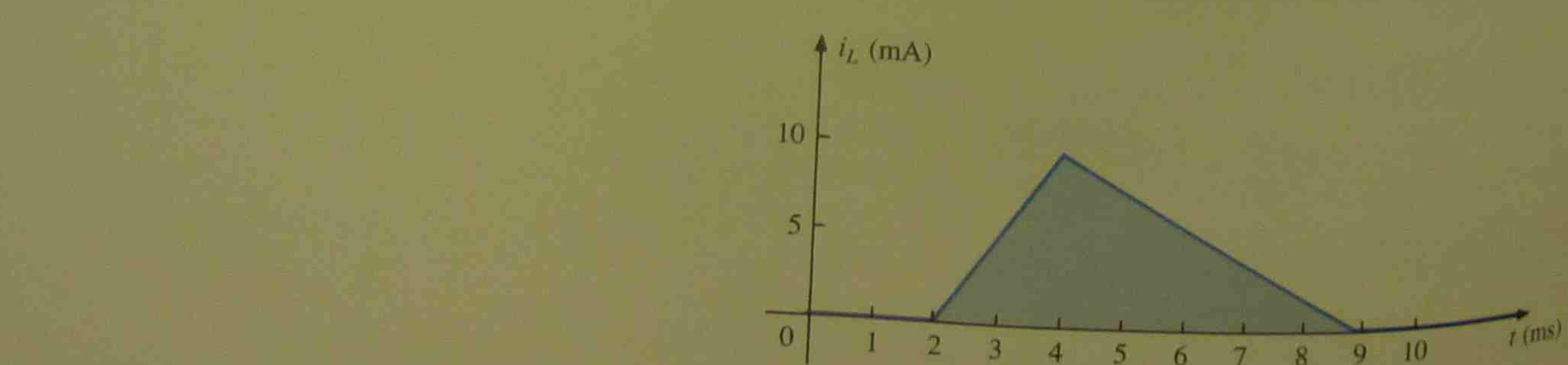


FIG. 12.11

Solution:

a. 0 to 2 ms: Since there is no change in current through the coil, there is no voltage induced across the coil; that is,

$$v_L = L\frac{\Delta i}{\Delta t} = L\frac{0}{\Delta t} = \mathbf{0}$$

b. 2 ms to 4 ms:

$$v_L = L\frac{\Delta i}{\Delta t} = (4 \times 10^{-3}\ \text{H})\left(\frac{10 \times 10^{-3}\ \text{A}}{2 \times 10^{-3}\ \text{s}}\right) = 20 \times 10^{-3}\ \text{V}$$
$$= \mathbf{20\ mV}$$

c. 4 ms to 9 ms:

$$v_L = L\frac{\Delta i}{\Delta t} = (-4 \times 10^{-3}\ \text{H})\left(\frac{10 \times 10^{-3}\ \text{A}}{5 \times 10^{-3}\ \text{s}}\right) = -8 \times 10^{-3}\ \text{V}$$
$$= \mathbf{-8\ mV}$$

d. 9 ms to ∞:

$$v_L = L\frac{\Delta i}{\Delta t} = L\frac{0}{\Delta t} = \mathbf{0}$$

The waveform for the average voltage across the coil is shown in Fig. 12.12. Note from the curve that

the voltage across the coil is not determined solely by the magnitude of the change in current through the coil (Δi), but by the rate of change of current through the coil (Δi/Δt).

A similar statement was made for the current of a capacitor due to a change in voltage across the capacitor.

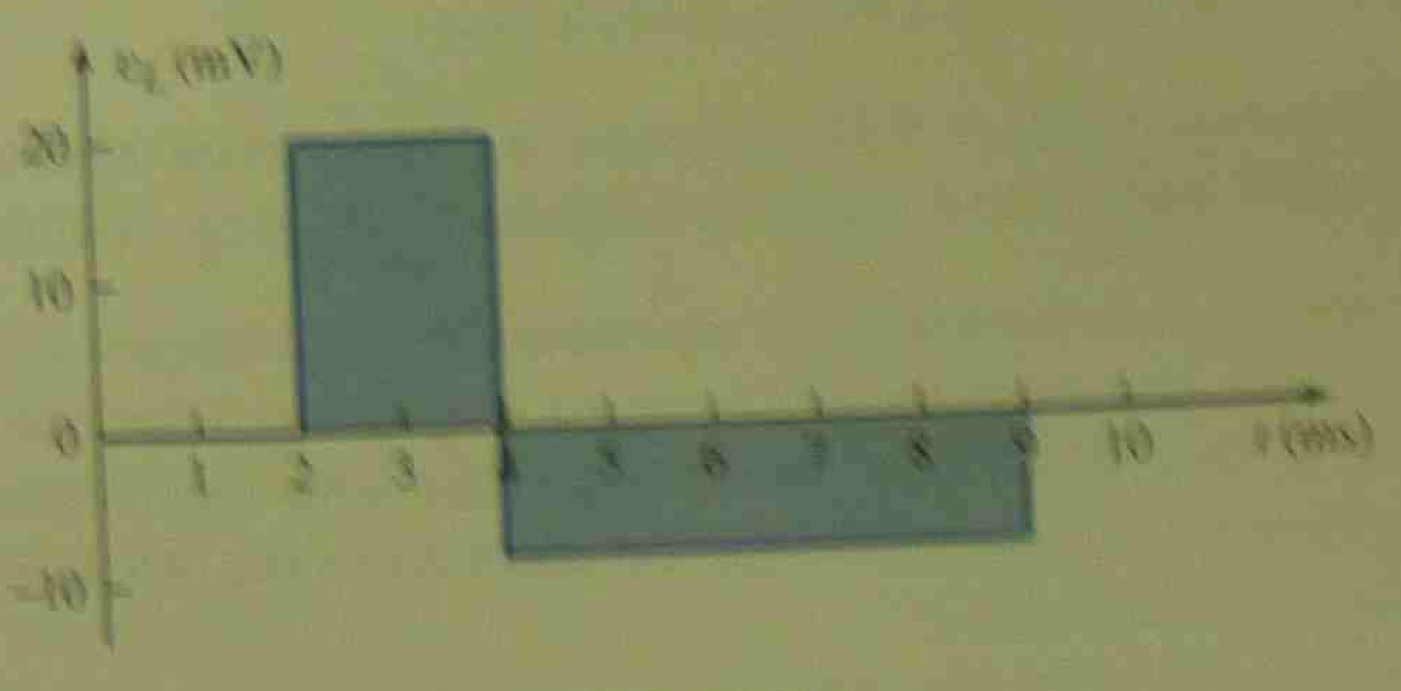


FIG. 12.12

A careful examination of Fig. 12.12 will also reveal that the area under the positive pulse from 2 ms to 4 ms equals the area under the negative pulse from 4 ms to 9 ms. In Section 12.13, we will find that the area under the curves represents the energy stored or released by the inductor. From 2 ms to 4 ms, the inductor is storing energy, whereas from 4 ms to 9 ms, the inductor is releasing the energy stored. For the full period zero to 10 ms, energy has simply been stored and released; there has been no dissipation as experienced for the resistive elements.

Over a full cycle, both the ideal capacitor and inductor do not consume energy but simply store and release it in their respective forms.

12.7 *R-L* TRANSIENTS: STORAGE CYCLE

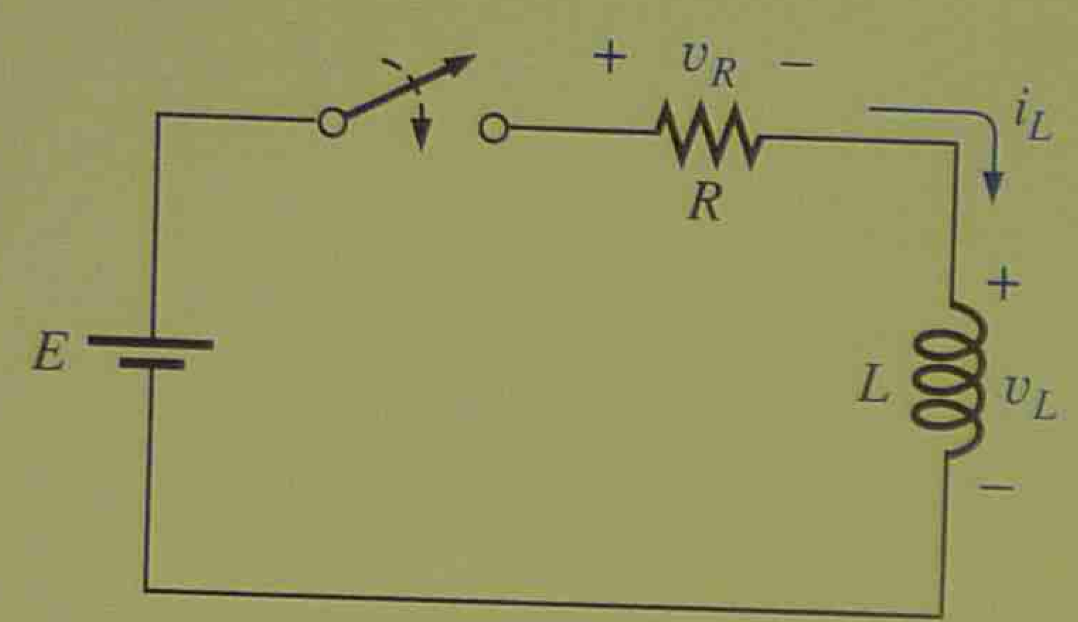


FIG. 12.13

The changing voltages and current that result during the storing of energy in the form of a magnetic field by an inductor in a dc circuit can best be described using the circuit of Fig. 12.13. At the instant the switch is closed, the inductance of the coil will prevent an instantaneous change in current through the coil. The potential drop across the coil, v_L, will equal the impressed voltage E as determined by Kirchhoff's voltage law since $v_R = iR = (0)R = 0$ V. The current i_L will then build up from zero, establishing a voltage drop across the resistor and a corresponding drop in v_L. The current will continue to increase until the voltage across the inductor drops to zero volts and the full impressed voltage appears across the resistor. Initially, the current i_L increases quite rapidly, followed by a continually decreasing rate until it reaches its maximum value of E/R.

You will recall from the discussion of capacitors that a capacitor has a short-circuit equivalent when the switch is first closed and an open-circuit equivalent when steady-state conditions are established. The inductor assumes the opposite equivalents for each stage. The instant the switch of Fig. 12.13 is closed, the equivalent network will appear as shown in Fig. 12.14. Note the correspondence with the earlier comments regarding the levels of voltage and current. The inductor obviously meets all the requirements for an open-circuit equivalent—$v_L = E$ volts, $i_L = 0$ A.

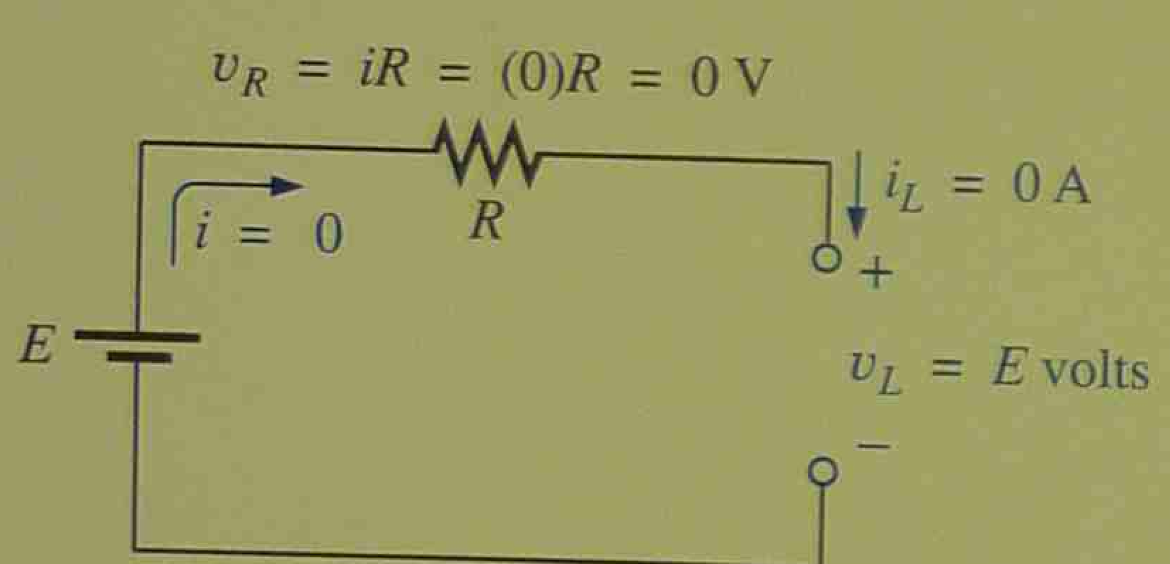


FIG. 12.14
Circuit of Fig. 12.13 the instant the switch is closed.

When steady-state conditions have been established and the storage phase is complete, the "equivalent" network will appear as shown in Fig. 12.15. The network clearly reveals that:

An ideal inductor ($R_l = 0\ \Omega$) assumes a short-circuit equivalent in a dc network once steady-state conditions have been established.

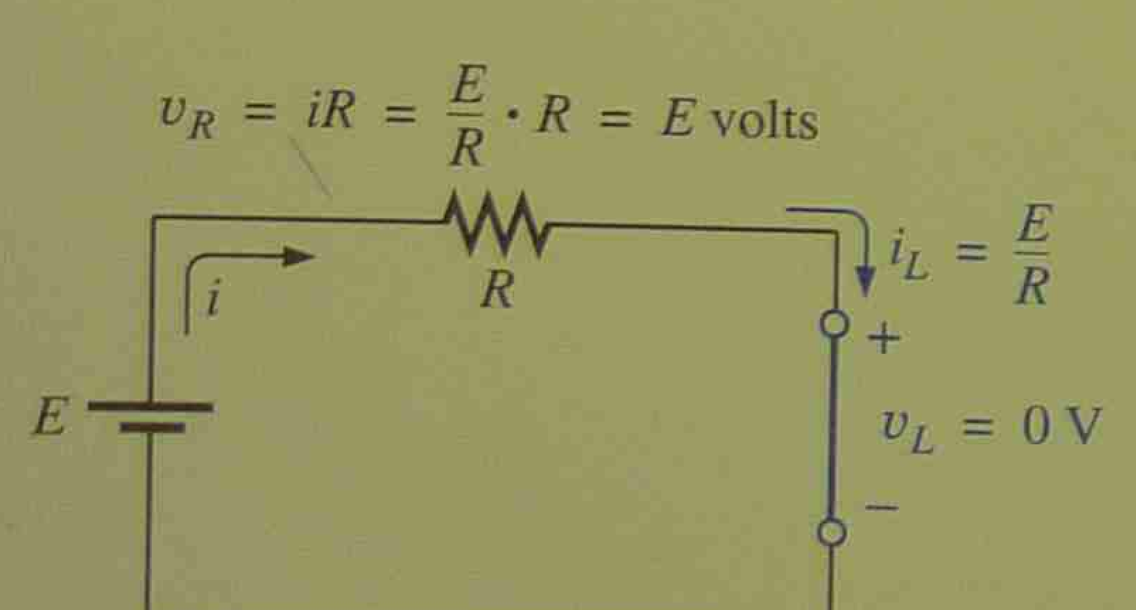


FIG. 12.15
Circuit of Fig. 12.13 under steady-state conditions.

Fortunately, the mathematical equations for the voltages and current for the storage phase are similar in many respects to those encountered for the *R-C* network. The experience gained with these equations in Chapter 10 will undoubtedly make the analysis of *R-L* networks somewhat easier to understand.

The equation for the current i_L during the storage phase is the following:

$$i_L = I_m(1 - e^{-t/\tau}) = \frac{E}{R}(1 - e^{-t/(L/R)}) \tag{12.8}$$

Note the factor $(1 - e^{-t/\tau})$, which also appeared for the voltage v_C of a capacitor during the charging phase. A plot of the equation is given in Fig. 12.16, clearly indicating that the maximum steady-state value of i_L is E/R, and that the rate of change in current decreases as time passes.

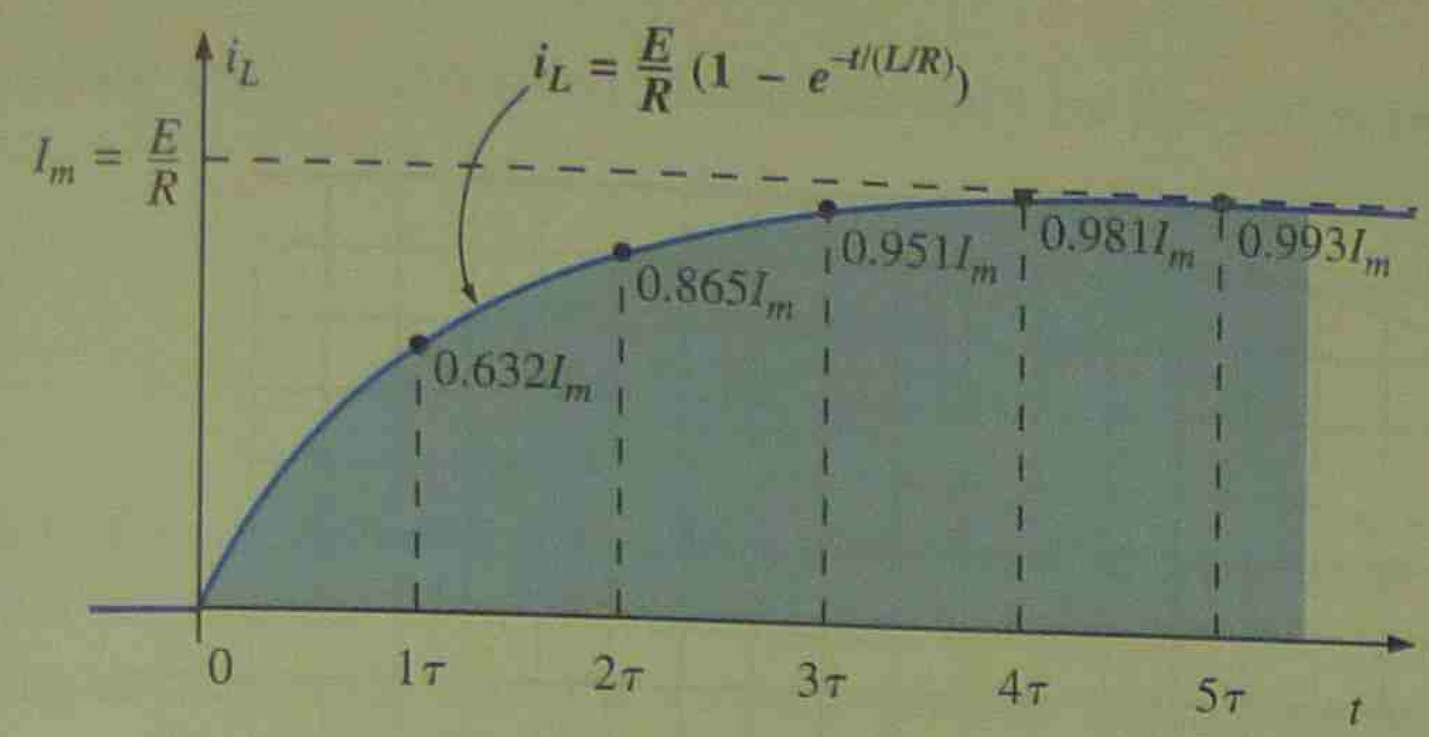


FIG. 12.16

The abscissa is scaled in time constants, with τ for inductive circuits defined by the following:

$$\tau = \frac{L}{R} \qquad \text{(seconds, s)} \tag{12.9}$$

The fact that τ has the units of time can be verified by taking the equation for the induced voltage

$$v_L = L\frac{di}{dt}$$

and solving for L:

$$L = \frac{v_L}{di/dt}$$

which leads to the ratio

$$\tau = \frac{L}{R} = \frac{\frac{v_L}{di/dt}}{R} = \frac{v_L}{\frac{di}{dt}R} \Rightarrow \frac{V}{\frac{IR}{t}} = \frac{\cancel{V}}{\frac{\cancel{V}}{t}} = t \quad \text{(s)}$$

Our experience with the factor $(1 - e^{-t/\tau})$ verifies the level of 63.2% after one time constant, 86.5% after two time constants, and so on. For convenience, Fig. 10.28 is repeated as Fig. 12.17 to evaluate the functions $(1 - e^{-t/\tau})$ and $e^{-t/\tau}$ at various values of τ.

If we keep R constant and increase L, the ratio L/R increases and the rise time increases. The change in transient behavior for the current i_L is plotted in Fig. 12.18 for various values of L. Note again the duality between these curves and those obtained for the R-C network in Fig. 10.31.

For most practical applications, we will assume that:

The storage phase has passed and steady-state conditions have been established once a period of time equal to five time constants has occurred.

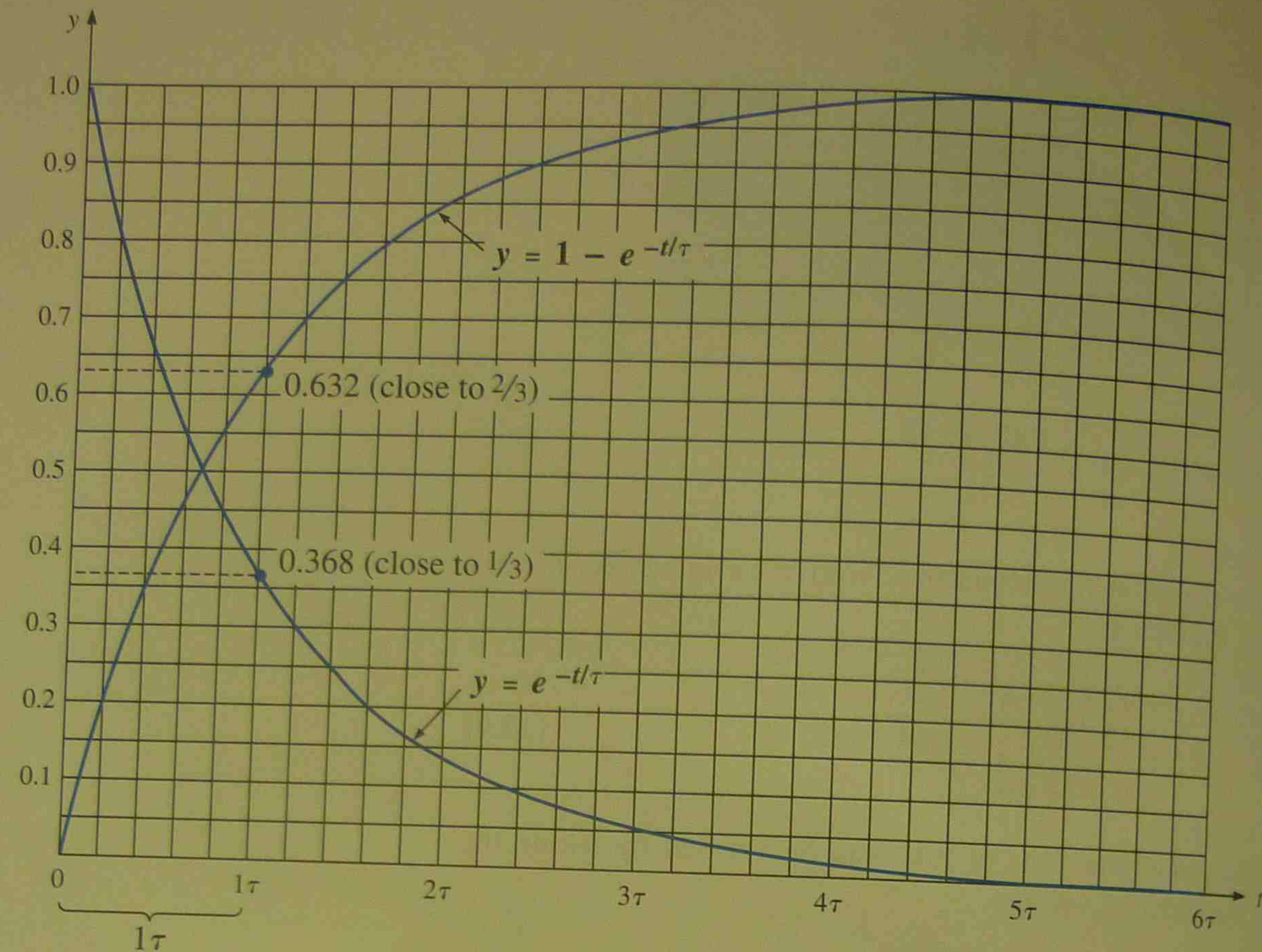


FIG. 12.17

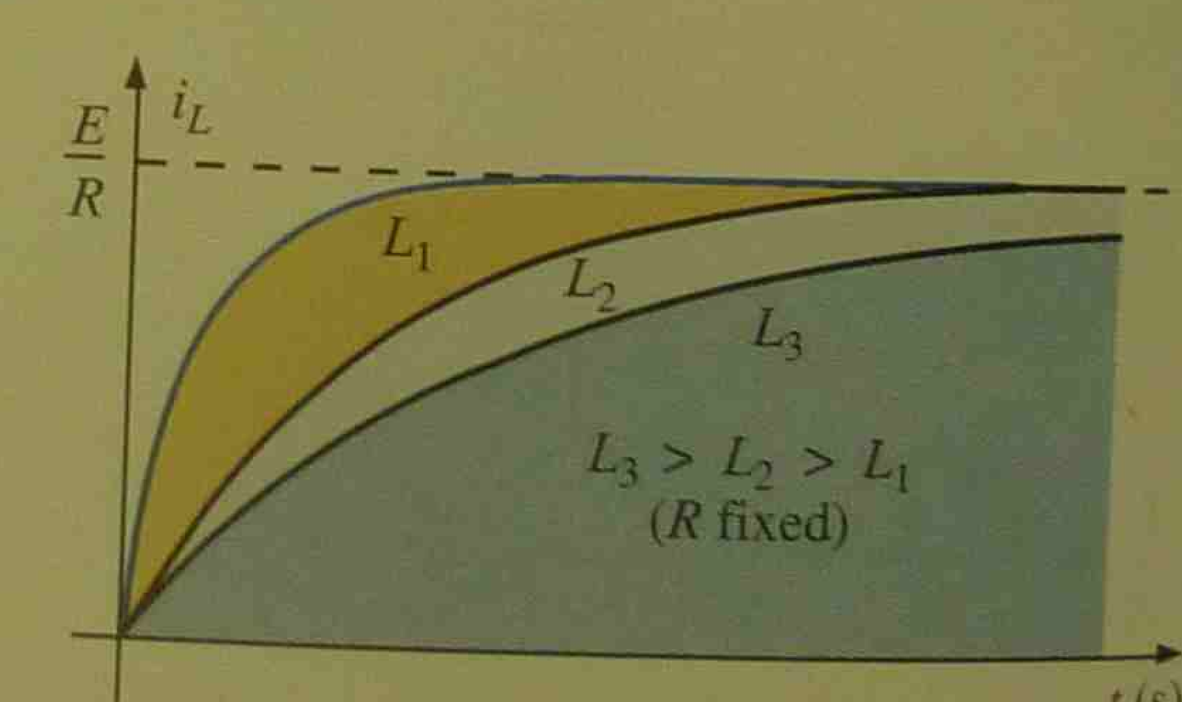


FIG. 12.18

In addition, since L/R will always have some numerical value even though it may be very small, the period 5τ will always be greater than zero, confirming the fact that

the current cannot change instantaneously in an inductive network.

In fact, the larger the inductance, the more the circuit will oppose a rapid buildup in current level.

Figures 12.15 and 12.16 clearly reveal that the voltage across the coil jumps to E volts when the switch is closed and decays to zero volts with time. The decay occurs in an exponential manner, and v_L during the storage phase can be described mathematically by the following equation:

$$v_L = Ee^{-t/\tau} \tag{12.10}$$

A plot of v_L appears in Fig. 12.19 with the time axis again divided into equal increments of τ. Obviously, the voltage v_L will decrease to zero volts at the same rate the current presses toward its maximum value.

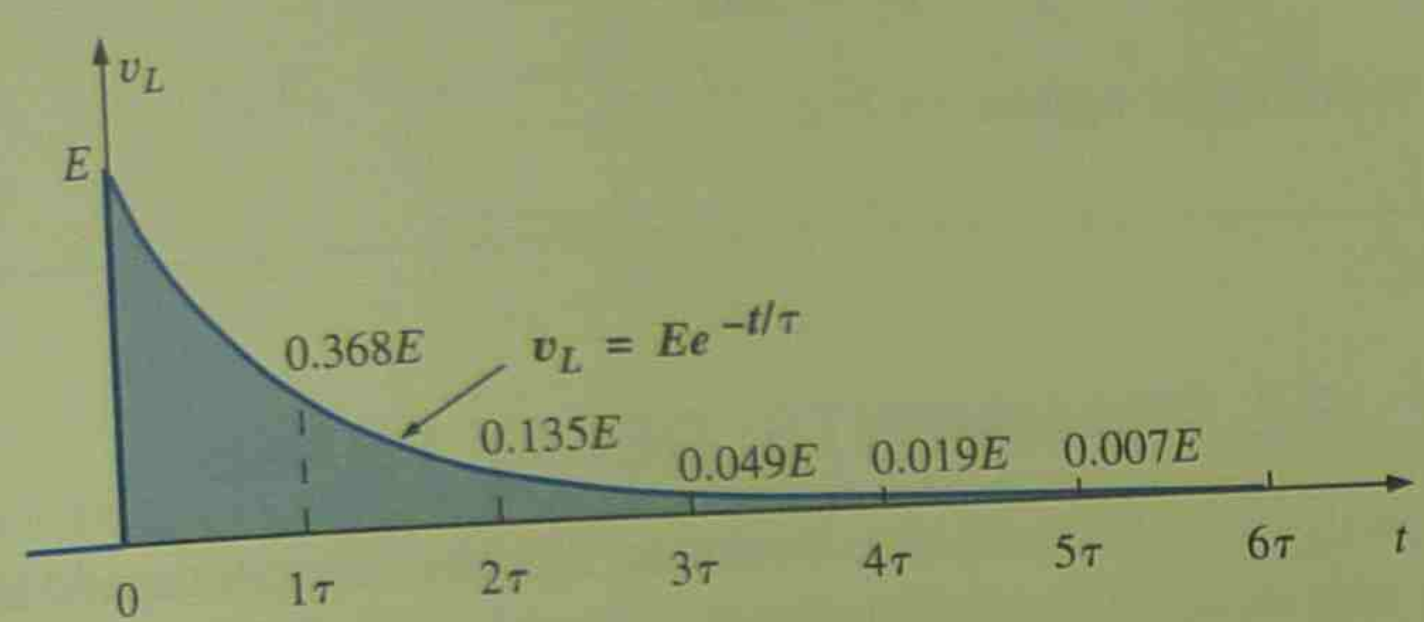


FIG. 12.19

In five time constants, $i_L = E/R$, $v_L = 0$ V, and the inductor can be replaced by its short-circuit equivalent.

Since
$$v_R = i_R R = i_L R$$

then
$$v_R = \left[\frac{E}{R}(1 - e^{-t/\tau})\right]R$$

and
$$v_R = E(1 - e^{-t/\tau}) \tag{12.11}$$

and the curve for v_R will have the same shape as obtained for i_L.

EXAMPLE 12.4 Find the mathematical expressions for the transient behavior of i_L and v_L for the circuit of Fig. 12.20 after the closing of the switch. Sketch the resulting curves.

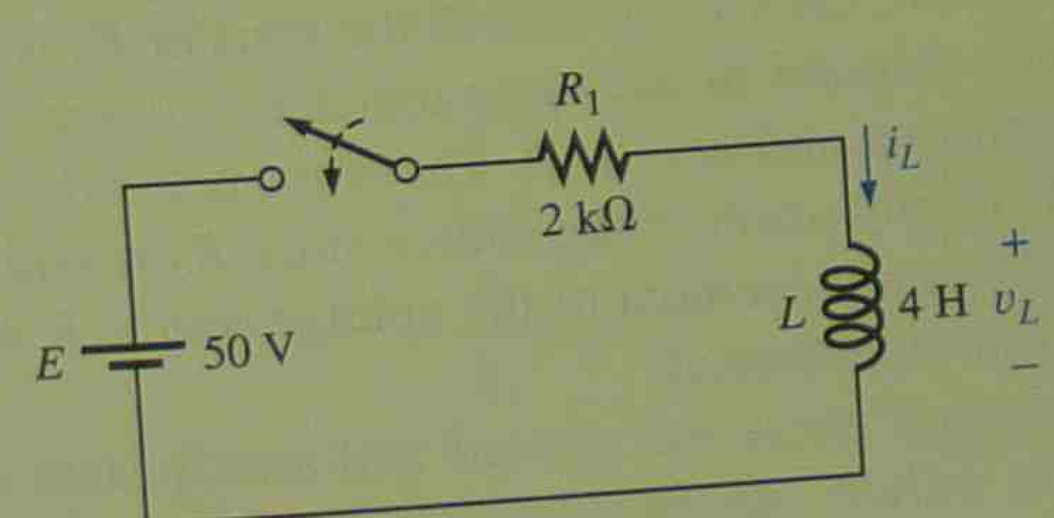


FIG. 12.20

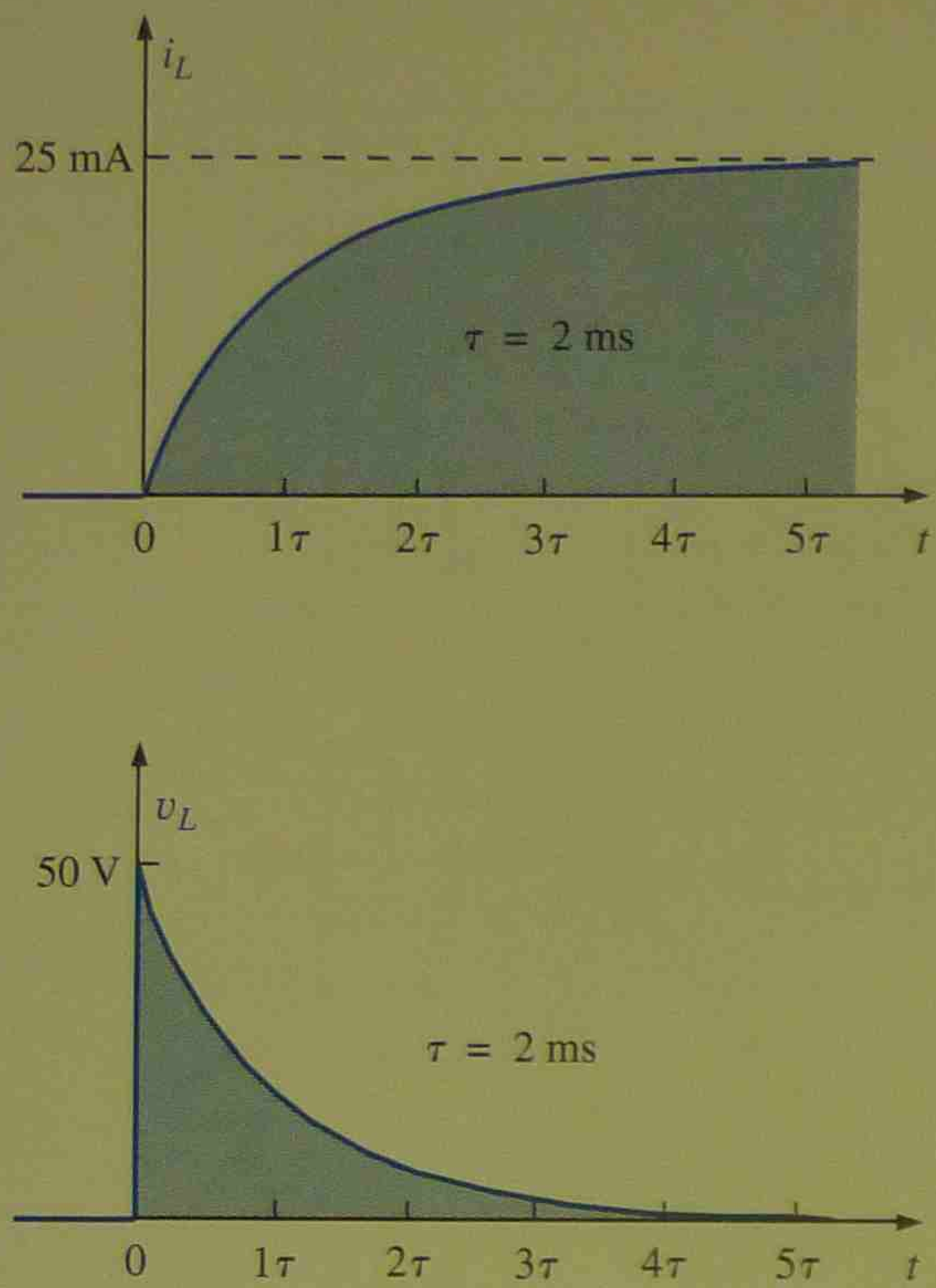


FIG. 12.21

Solution:

$$\tau = \frac{L}{R_1} = \frac{4\ \text{H}}{2\ \text{k}\Omega} = 2\ \text{ms}$$

By Eq. (12.8),

$$I_m = \frac{E}{R_1} = \frac{50}{2\ \text{k}\Omega} = 25 \times 10^{-3}\ \text{A} = 25\ \text{mA}$$

and

$$i_L = (25 \times 10^{-3})(1 - e^{-t/(2\times10^{-3})})$$

By Eq. (12.10),

$$v_L = 50e^{-t/(2\times10^{-3})}$$

Both waveforms appear in Fig. 12.21.

12.8 *R-L* TRANSIENTS: DECAY PHASE

In the analysis of *R-C* circuits, we found that the capacitor could hold its charge and store energy in the form of an electric field for a period of time determined by the leakage factors. In *R-L* circuits, the energy is stored in the form of a magnetic field established by the current through the coil. Unlike the capacitor, however, an isolated inductor cannot continue to store energy since the absence of a closed path would cause the current to drop to zero, releasing the energy stored in the form of a magnetic field. If the switch of Fig. 12.13 were opened quickly, a spark would probably occur across the contacts due to the rapid change in current from a maximum of E/R to zero amperes. The change in current di/dt of the equation $v_L = L(di/dt)$ would establish a high voltage v_L across the coil that would discharge across the points of the switch. This is the same mechanism as applied in the ignition system of a car to ignite the fuel in the cylinder. Some 25,000 V are generated by the rapid decrease in ignition coil current that occurs when the switch in the system is opened. (In older systems, the "points" in the distributor served as the switch.) This inductive reaction is significant when you consider that the only independent source in a car is a 12-V battery.

If opening the switch to move it to another position will cause such a rapid discharge in stored energy, how can the decay phase of an *R-L* circuit be analyzed in much the same manner as for the *R-C* circuit? The solution is to use a network such as that appearing in Fig. 12.22. When the switch is closed, the voltage across the resistor R_2 is E volts and the *R-L* branch will respond in the same manner as described above, with the same waveforms and levels. A Thevenin network of E in parallel with R_2 would simply result in the source since R_2 would be shorted out by the short-circuit replacement of the voltage source E when the Thevenin resistance is determined.

After the storage phase has passed and steady-state conditions are established, the switch can be opened without the sparking effect or rapid discharge due to the resistor R_2, which provides a complete path

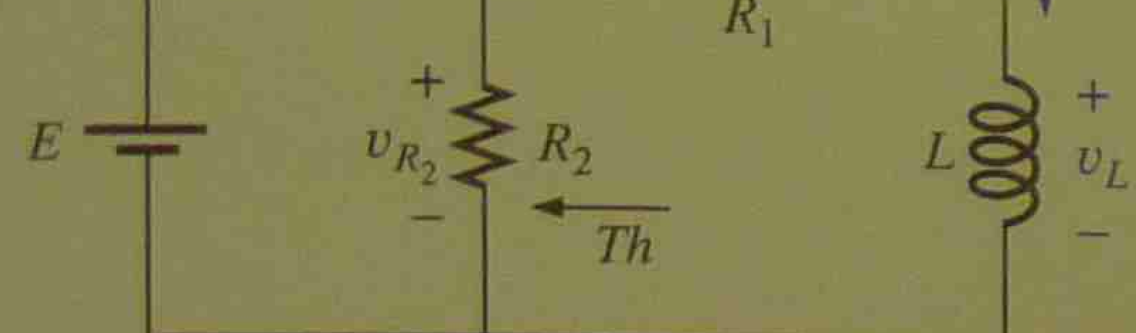


FIG. 12.22

for the current i_L. In fact, for clarity the discharge path is isolated in Fig. 12.23. The voltage v_L across the inductor will reverse polarity and have a magnitude determined by

$$v_L = v_{R_1} + v_{R_2} \qquad (12.12)$$

Recall that the voltage across an inductor can change instantaneously but the current cannot. The result is that the current i_L must maintain the same direction and magnitude as shown in Fig. 12.23. Therefore, the instant after the switch is opened, i_L is still $I_m = E/R_1$, and

$$v_L = v_{R_1} + v_{R_2} = i_1R_1 + i_2R_2$$

$$= i_L(R_1 + R_2) = \frac{E}{R_1}(R_1 + R_2) = \left(\frac{R_1}{R_1} + \frac{R_2}{R_1}\right)E$$

and

$$v_L = \left(1 + \frac{R_2}{R_1}\right)E \qquad (12.13)$$

which is bigger than E volts by the ratio R_2/R_1. In other words, when the switch is opened, the voltage across the inductor will drop instantaneously from E to $-[1 + (R_2/R_1)]E$ volts. The minus sign is a result of v_L having a polarity opposite to the defined polarity of Fig. 12.23.

As the inductor releases its stored energy, the voltage across the coil will decay to zero in the following manner:

$$v_L = V_ie^{-t/\tau'} \qquad (12.14)$$

with

$$V_i = \left(1 + \frac{R_2}{R_1}\right)E$$

and

$$\tau' = \frac{L}{R_T} = \frac{L}{R_1 + R_2}$$

The current will decay from a maximum of $I_m = E/R_1$ to zero, in the following manner:

$$i_L = I_me^{-t/\tau'} \qquad (12.15)$$

with

$$I_m = \frac{E}{R_1} \quad \text{and} \quad \tau' = \frac{L}{R_1 + R_2}$$

The mathematical expression for the voltage across either resistor can then be determined using Ohm's law:

$$v_{R_1} = i_{R_1}R_1 = i_LR_1$$

$$= I_me^{-t/\tau'}R_1$$

$$= \frac{E}{R_1}R_1e^{-t/\tau'}$$

and

$$v_{R_1} = Ee^{-t/\tau'} \qquad (12.16)$$

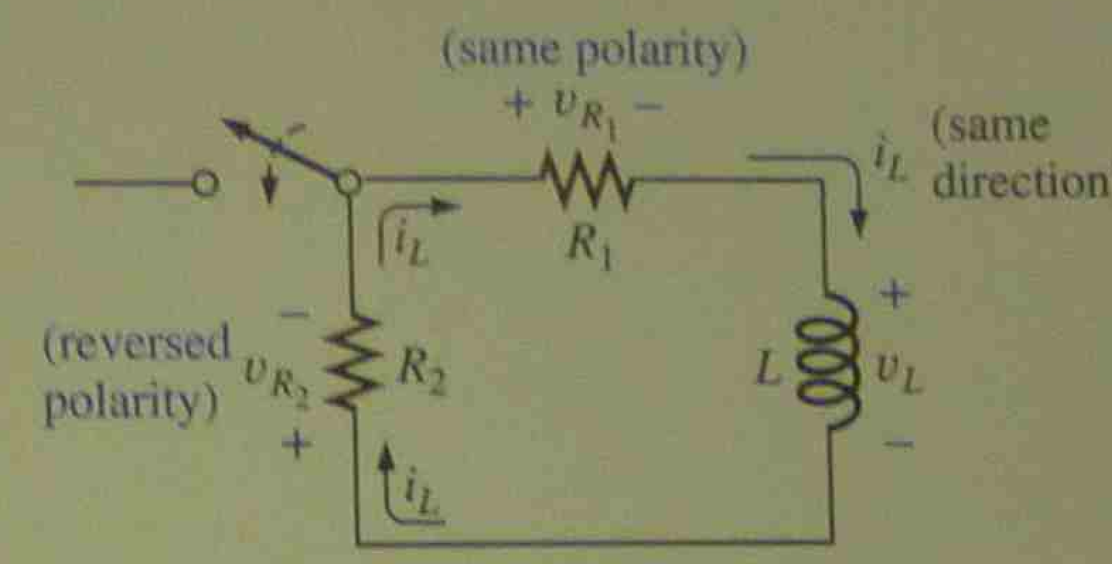


FIG. 12.23

Network of Fig. 12.22 the instant the switch is opened.

The voltage v_{R_1} has the same polarity as during the storage phase since the current i_L has the same direction. The voltage v_{R_2} is expressed as follows:

$$v_{R_2} = i_{R_2}R_2 = i_L R_2$$
$$= I_m e^{-t/\tau'} R_2$$
$$= \frac{E}{R_1} R_2 e^{-t/\tau'}$$

and
$$\boxed{v_{R_2} = \frac{R_2}{R_1} E e^{-t/\tau'}} \qquad (12.17)$$

with the polarity indicated in Fig. 12.23.

EXAMPLE 12.5 The resistor R_2 was added to the network of Fig. 12.21, as shown in Fig. 12.24.

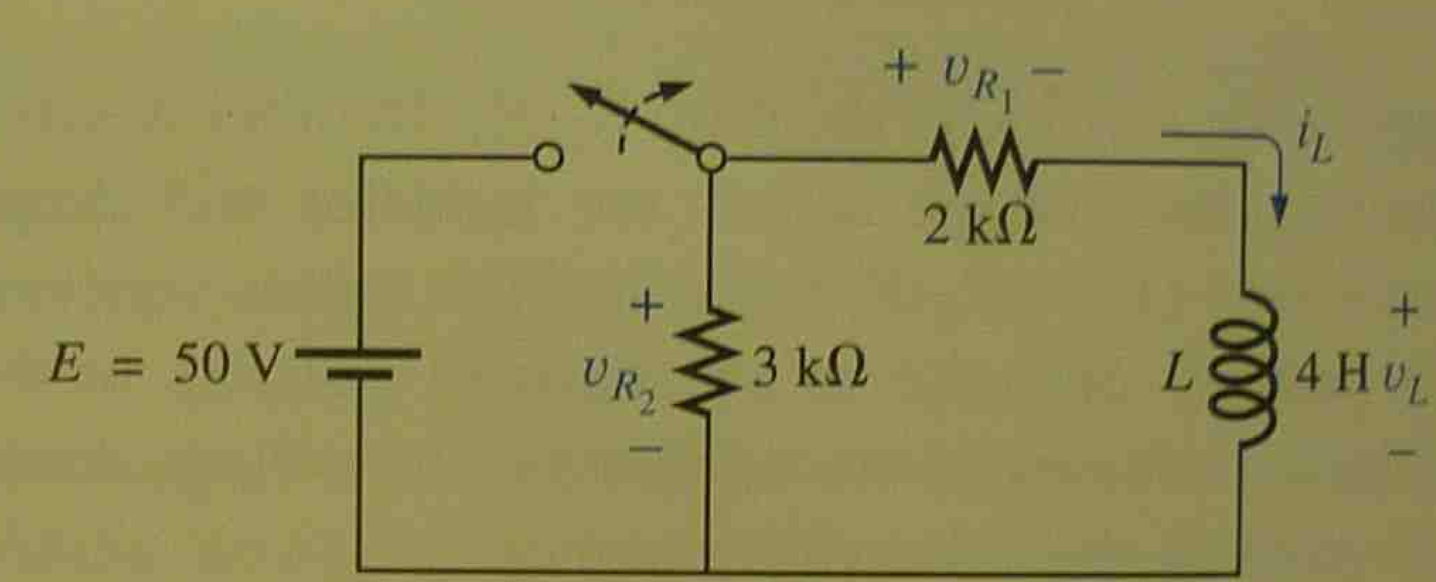


FIG. 12.24

a. Find the mathematical expressions for i_L, v_L, v_{R_1}, and v_{R_2} after the storage phase has been completed and the switch is opened.
b. Sketch the waveforms for each voltage and current for both phases covered by this example and Example 12.4 if five time constants pass between phases. Use the defined polarities of Fig. 12.22.

Solutions:

a. $$\tau' = \frac{L}{R_1 + R_2} = \frac{4\text{ H}}{2\text{ k}\Omega + 3\text{ k}\Omega} = \frac{4\text{ H}}{5 \times 10^3\ \Omega} = 0.8 \times 10^{-3}\text{ s}$$
$$= 0.8\text{ ms}$$

By Eq. (12.14),

$$V_i = \left(1 + \frac{R_2}{R_1}\right)E = \left(1 + \frac{3\text{ k}\Omega}{2\text{ k}\Omega}\right)(50\text{ V}) = 125\text{ V}$$

and
$$v_L = -V_i e^{-t/\tau'} = \mathbf{-125}e^{-t/(0.8\times10^{-3})}$$

By Eq. (12.15),

$$I_m = \frac{E}{R_1} = \frac{50\text{ V}}{2\text{ k}\Omega} = 25\text{ mA}$$

and
$$i_L = I_m e^{-t/\tau'} = \mathbf{(25 \times 10^{-3})}e^{-t/(0.8\times10^{-3})}$$

By Eq. (12.16),

$$v_{R_1} = Ee^{-t/\tau'} = \mathbf{50}e^{-t/(0.8\times10^{-3})}$$

By Eq. (12.17),

$$v_{R_2} = -\frac{R_2}{R_1}Ee^{-t/\tau'} = -\frac{3\ \text{k}\Omega}{2\ \text{k}\Omega}(50\ \text{V})e^{-t/\tau'} = -75e^{-t/(0.8\times10^{-3})}$$

b. See Fig. 12.25.

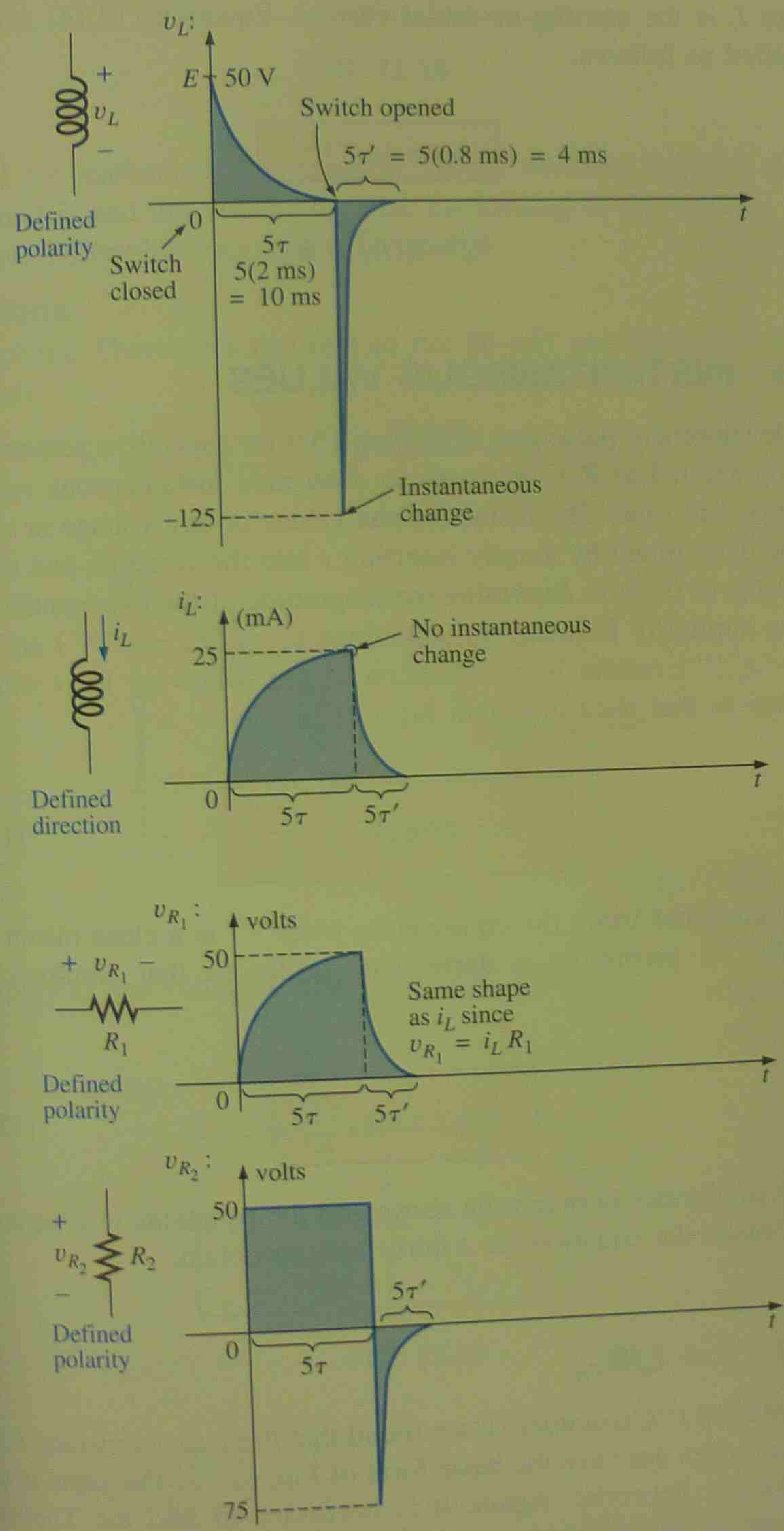


FIG. 12.25

In the preceding analysis, it was assumed that steady-state conditions were established during the charging phase and $I_m = E/R_1$, with $v_L = 0$ V. However, if the switch of Fig. 12.23 is opened before i_L reaches its maximum value, the equation for the decaying current of Fig. 12.23 must change to

$$i_L = I_i e^{-t/\tau'} \tag{12.18}$$

where I_i is the starting or *i*nitial current. Equation (12.14) would be modified as follows:

$$v_L = V_i e^{-t/\tau'} \tag{12.19}$$

with $$V_i = I_i(R_1 + R_2)$$

12.9 INSTANTANEOUS VALUES

The development presented in Section 10.9 for capacitive networks can also be applied to *R-L* networks to determine instantaneous voltages, currents, and time. The instantaneous values of any voltage or current can be determined by simply inserting t into the equation and using a calculator or table to determine the magnitude of the exponential term.

The similarity between the equations $v_C = E(1 - e^{-t/\tau})$ and $i_L = I_m(1 - e^{-t/\tau})$ results in a derivation of the following for t which is identical to that used to obtain Eq. (10.21):

$$t = -\tau \log_e\left(1 - \frac{i_L}{I_m}\right) \tag{12.20}$$

For the other form, the equation $v_C = Ee^{-t/\tau}$ is a close match with $v_L = Ee^{-t/\tau}$, permitting a derivation similar to that employed for Eq. (10.22):

$$t = -\tau \log_e \frac{v_L}{E} \tag{12.21}$$

The similarities between the above and the equations in Chapter 10 should make the equation for t fairly easy to obtain.

12.10 $\tau = L/R_{Th}$

In Chapter 10 ("Capacitors"), we found that there are occasions when the circuit does not have the basic form of Fig. 12.13. The same is true for inductive networks. Again, it is necessary to find the Thevenin equivalent circuit before proceeding in the manner described in this chapter. Consider the following example.

EXAMPLE 12.6. For the network of Fig. 12.26:

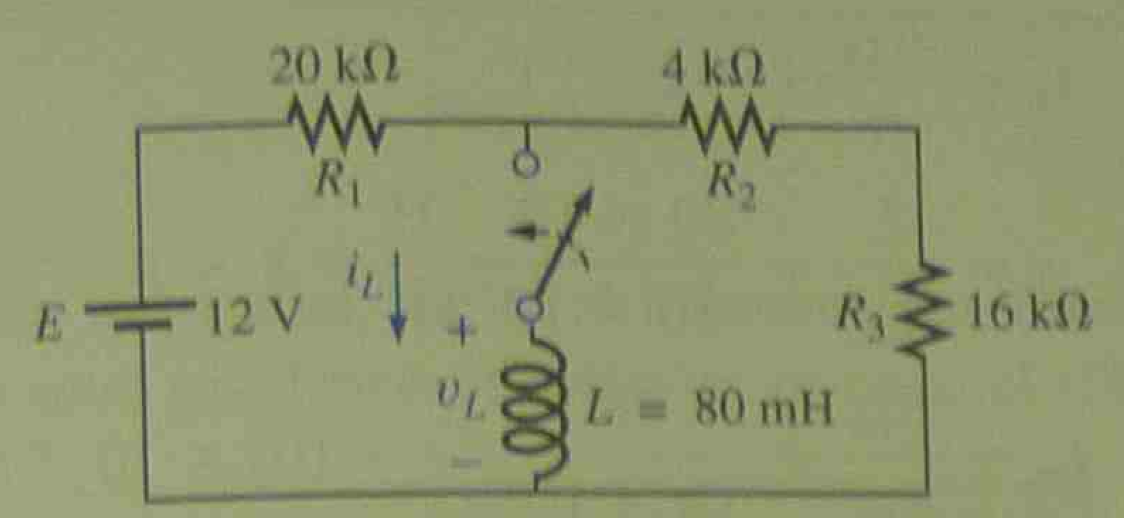


FIG. 12.26

a. Find the mathematical expression for the transient behavior of the current i_L and the voltage v_L after the closing of the switch.
b. Draw the resultant waveform for each.

Solutions:

a. Applying Thevenin's theorem to the 80-mH inductor (Fig. 12.27) yields

$$R_{Th} = \frac{R}{N} = \frac{20\ \text{k}\Omega}{2} = 10\ \text{k}\Omega$$

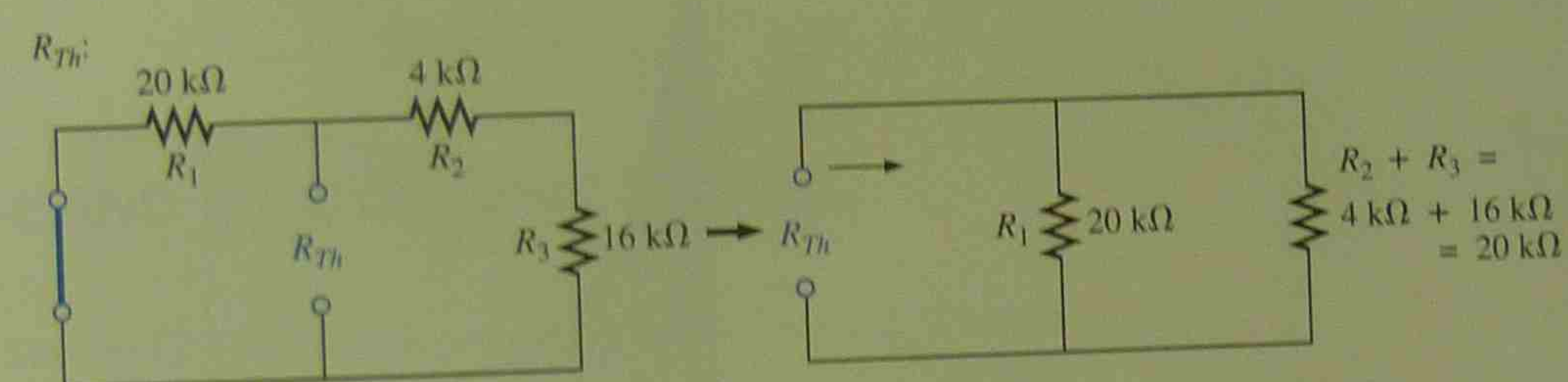


FIG. 12.27

Applying the voltage divider rule (Fig. 12.28),

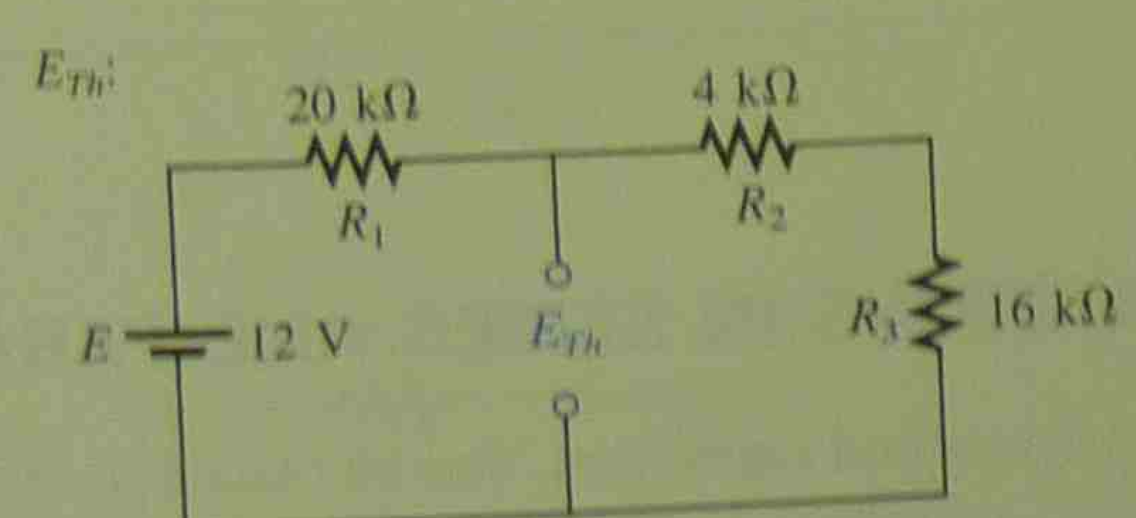


FIG. 12.28

$$E_{Th} = \frac{(R_2 + R_3)E}{R_1 + R_2 + R_3}$$
$$= \frac{(4\ \text{k}\Omega + 16\ \text{k}\Omega)(12\ \text{V})}{20\ \text{k}\Omega + 4\ \text{k}\Omega + 16\ \text{k}\Omega} = \frac{(20\ \text{k}\Omega)(12\ \text{V})}{40\ \text{k}\Omega} = 6\ \text{V}$$

Thevenin equivalent circuit:

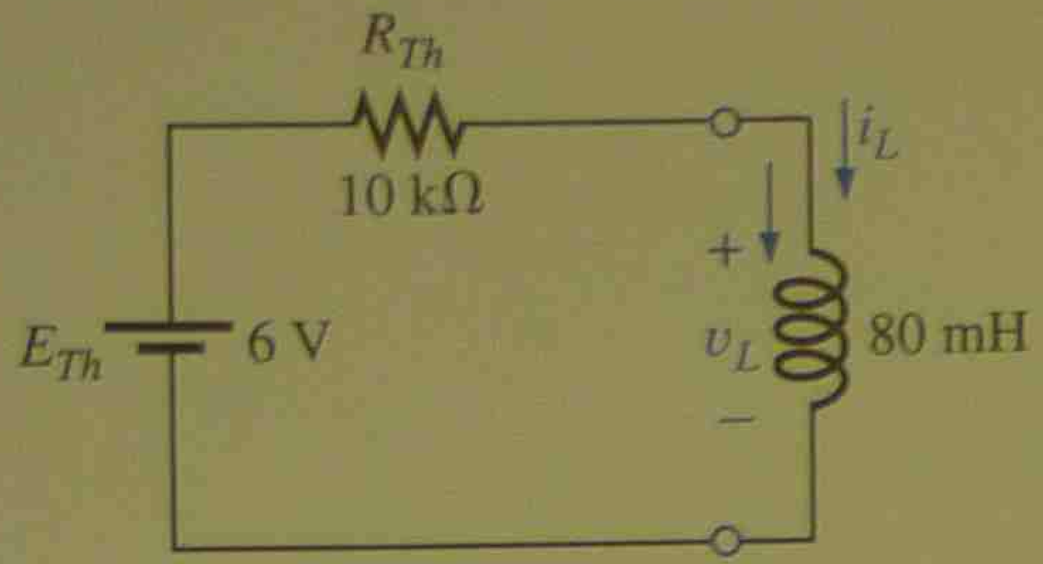


FIG. 12.29

The Thevenin equivalent circuit is shown in Fig. 12.29. Using Eq. (12.8),

$$i_L = \frac{E_{Th}}{R}(1 - e^{-t/\tau})$$

$$\tau = \frac{L}{R_{Th}} = \frac{80 \times 10^{-3}\ \text{H}}{10 \times 10^{3}\ \Omega} = 8 \times 10^{-6}\ \text{s}$$

$$I_m = \frac{E_{Th}}{R_{Th}} = \frac{6\ \text{V}}{10 \times 10^{3}\ \Omega} = 0.6 \times 10^{-3}\ \text{A}$$

and $$\mathbf{i_L = (0.6 \times 10^{-3})(1 - e^{-t/(8 \times 10^{-6})})}$$

Using Eq. (12.10),

$$v_L = E_{Th}e^{-t/\tau}$$

so that $$\mathbf{v_L = 6e^{-t/(8 \times 10^{-6})}}$$

b. See Fig. 12.30.

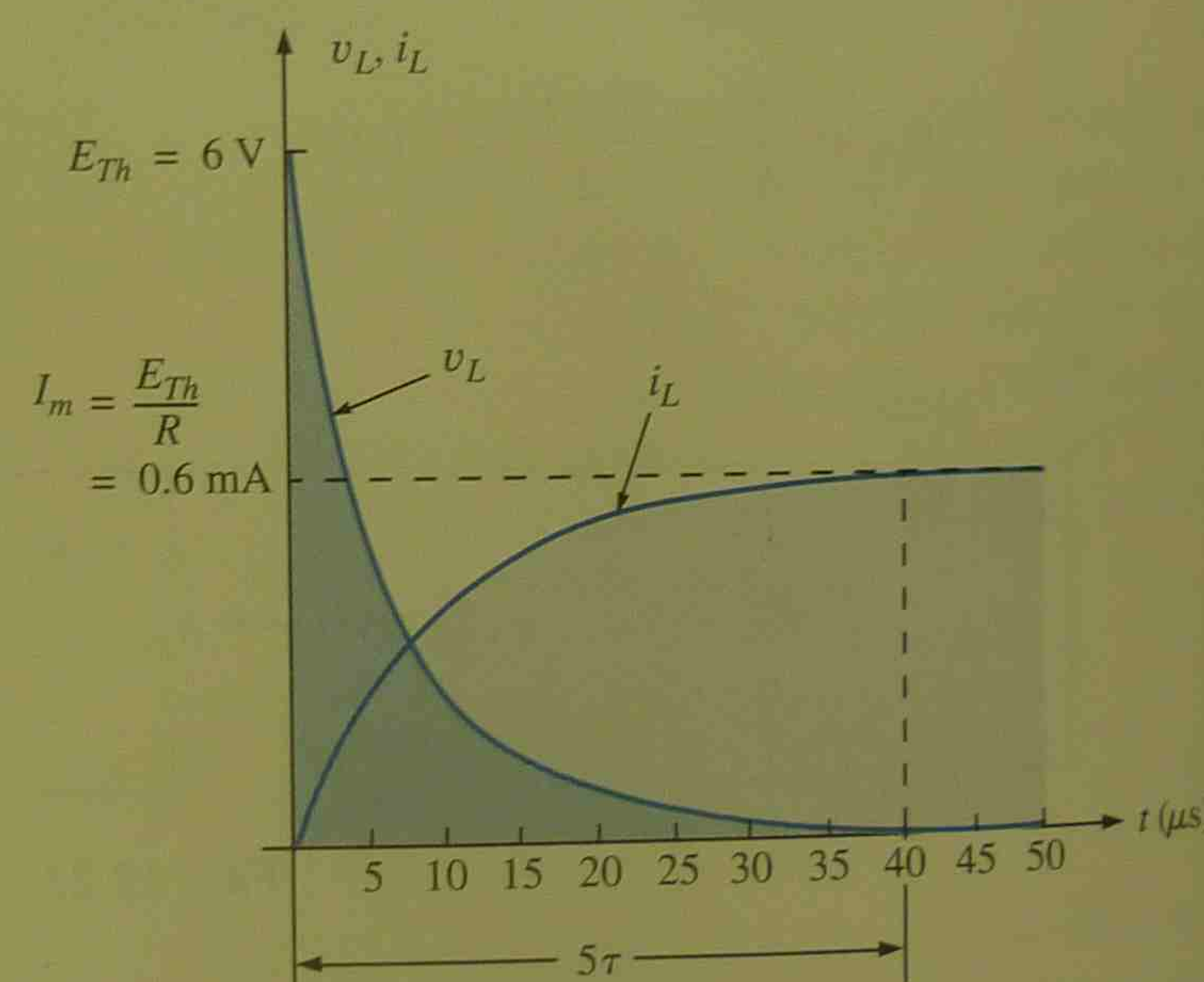


FIG. 12.30

12.11 INDUCTORS IN SERIES AND PARALLEL

Inductors, like resistors and capacitors, can be placed in series or parallel. Increasing levels of inductance can be obtained by placing inductors in series, while decreasing levels can be obtained by placing inductors in parallel.

For inductors in series, the total inductance is found in the same manner as the total resistance of resistors in series (Fig. 12.31):

$$L_T = L_1 + L_2 + L_3 + \cdots + L_N \tag{12.22}$$

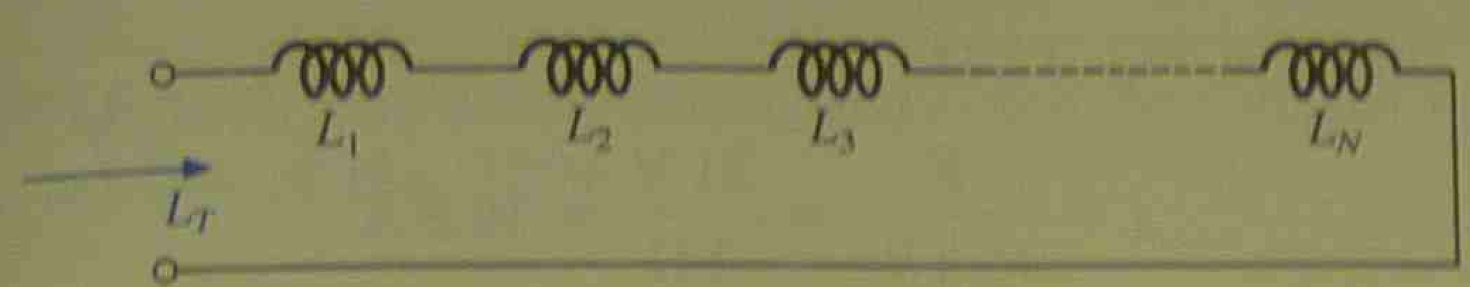


FIG. 12.31

For inductors in parallel, the total inductance is found in the same manner as the total resistance of resistors in parallel (Fig. 12.32):

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N} \tag{12.23}$$

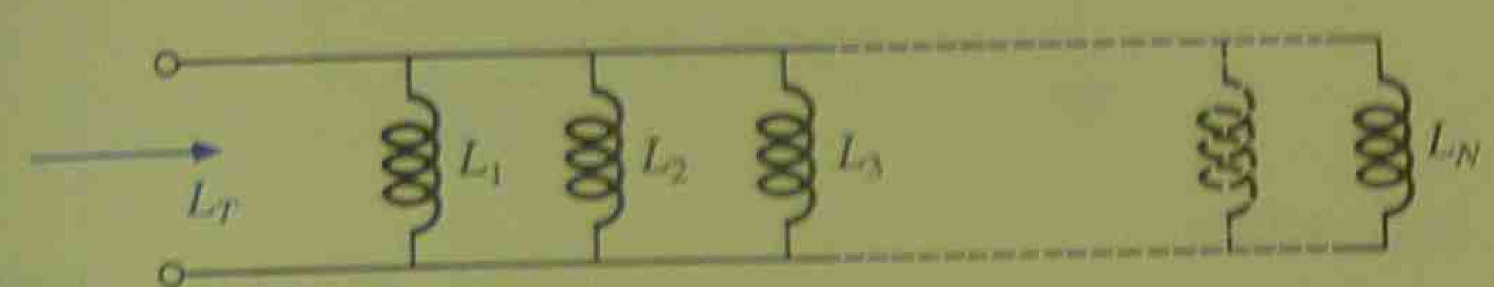


FIG. 12.32

For two inductors in parallel,

$$L_T = \frac{L_1 L_2}{L_1 + L_2} \tag{12.24}$$

12.12 *R-L* AND *R-L-C* CIRCUITS WITH dc INPUTS

We found in Section 12.7 that for all practical purposes, an inductor can be replaced by a short circuit in a dc circuit after a period of time greater than five time constants has passed. If in the following circuits we assume that all of the currents and voltages have reached their final values, the current through each inductor can be found by replacing each inductor by a short circuit. For the circuit of Fig. 12.33, for example,

$$I_1 = \frac{E}{R_1} = \frac{10}{2} = 5\text{ A}$$

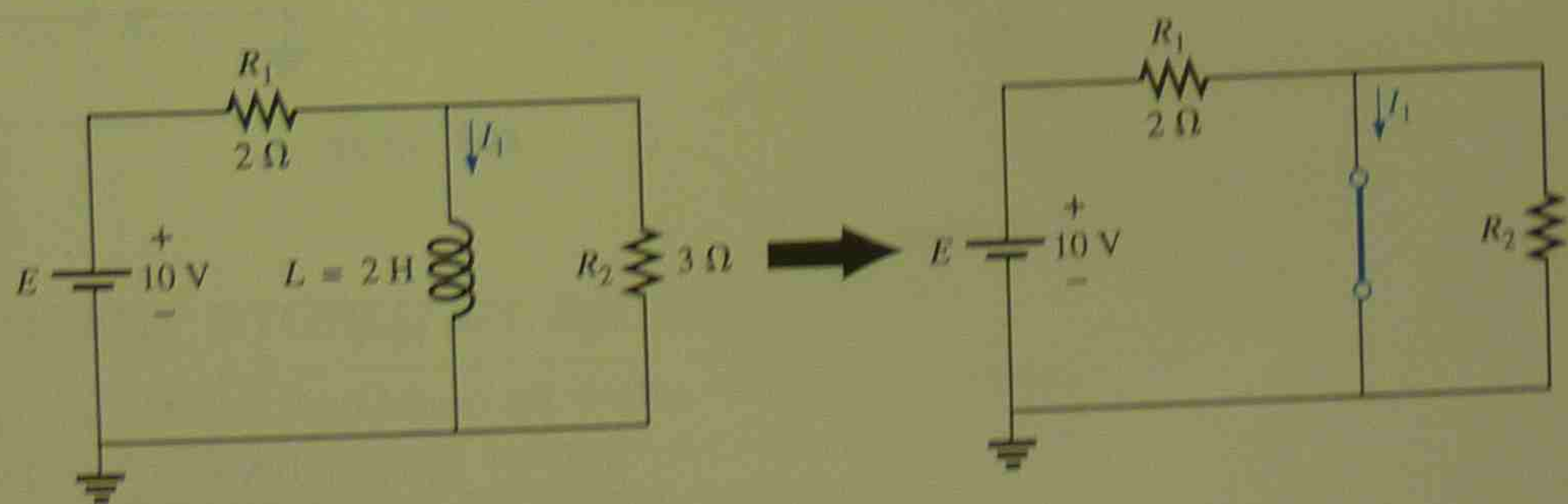


FIG. 12.33

For the circuit of Fig. 12.34,

$$I = \frac{E}{R_2 \| R_3} = \frac{21\ \text{V}}{2\ \Omega} = \mathbf{10.5\ A}$$

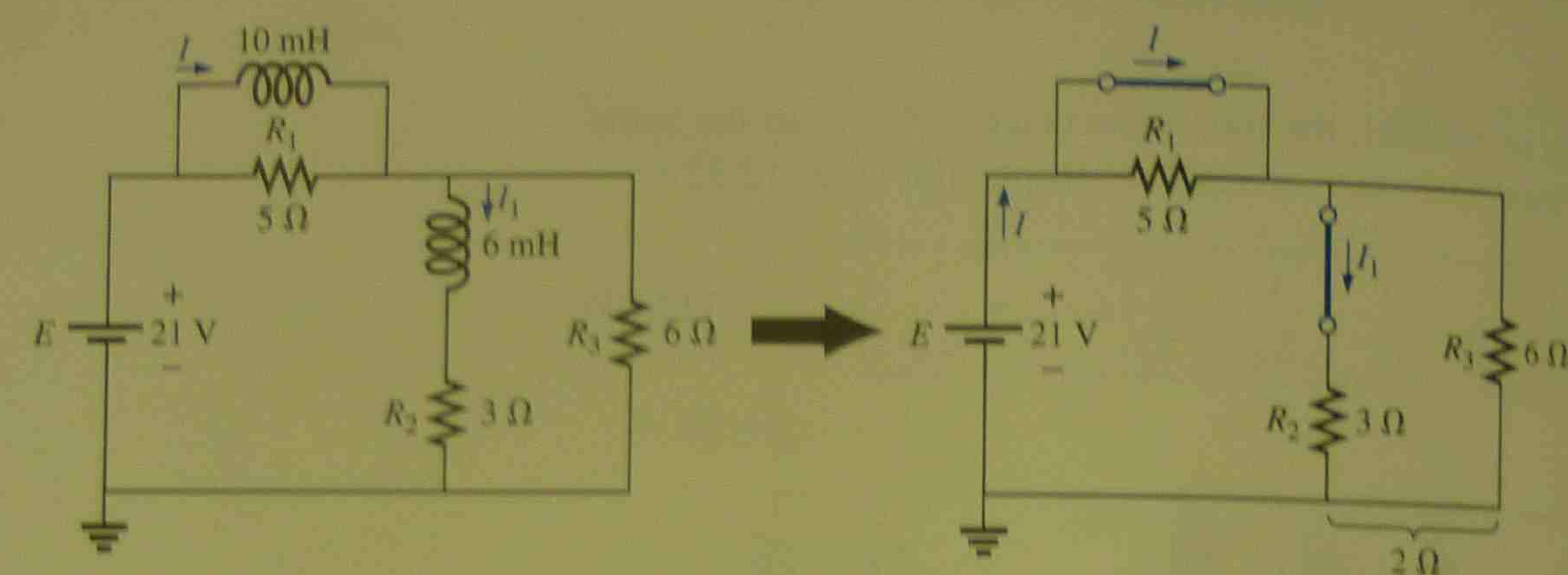


FIG. 12.34

Applying the current divider rule,

$$I_1 = \frac{R_3 I}{R_3 + R_2} = \frac{(6\ \Omega)(10.5\ \text{A})}{6\ \Omega + 3\ \Omega} = \frac{63\ \text{A}}{9} = \mathbf{7\ A}$$

In the following examples we will assume that the voltage across the capacitors and the current through the inductors have reached their final values. Under these conditions, the inductors can be replaced by short circuits, and the capacitors by open circuits.

EXAMPLE 12.7 Find the current I_L and the voltage V_C for the network of Fig. 12.35.

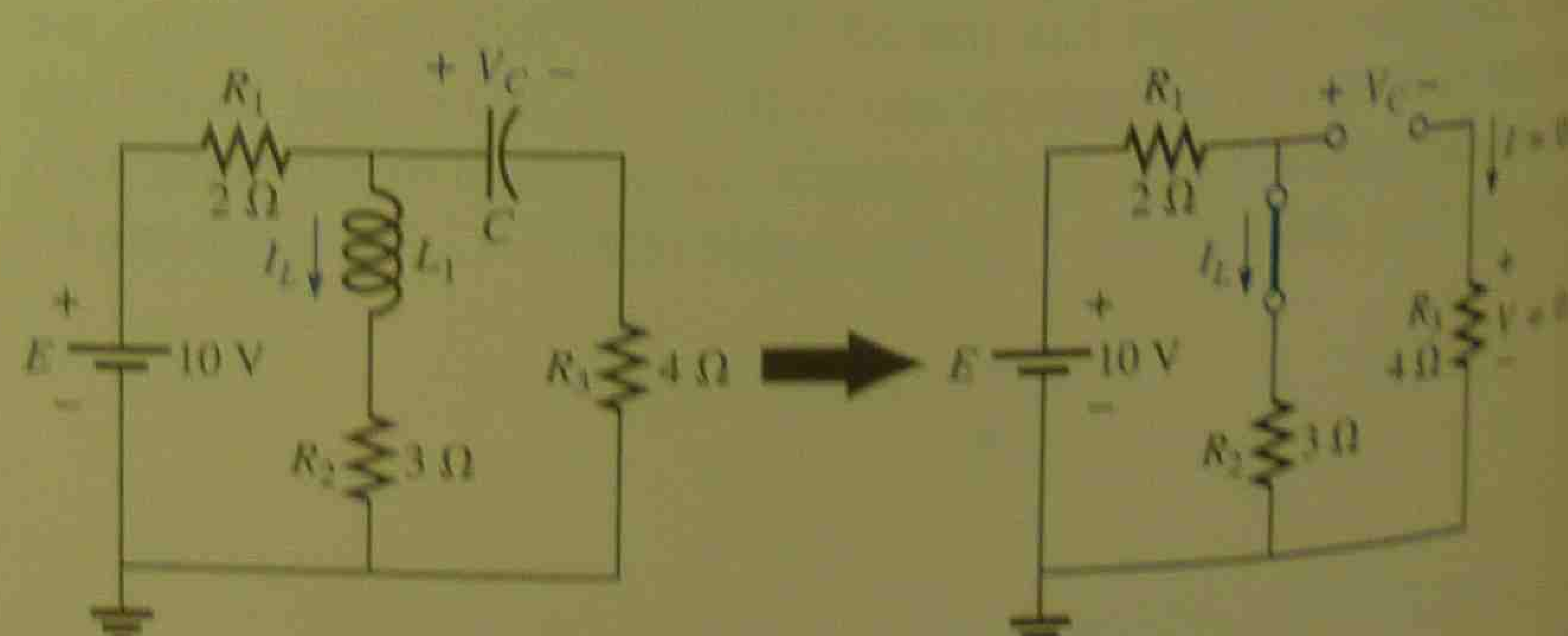


FIG. 12.35

Solution:

$$I_L = \frac{E}{R_1 + R_2} = \frac{10\ \text{V}}{5\ \Omega} = \mathbf{2\ A}$$

$$V_C = \frac{R_2 E}{R_2 + R_1} = \frac{(3\ \Omega)(10\ \text{V})}{3\ \Omega + 2\ \Omega} = \mathbf{6\ V}$$

EXAMPLE 12.8 Find the currents I_1 and I_2 and the voltages V_1 and V_2 for the network of Fig. 12.36.

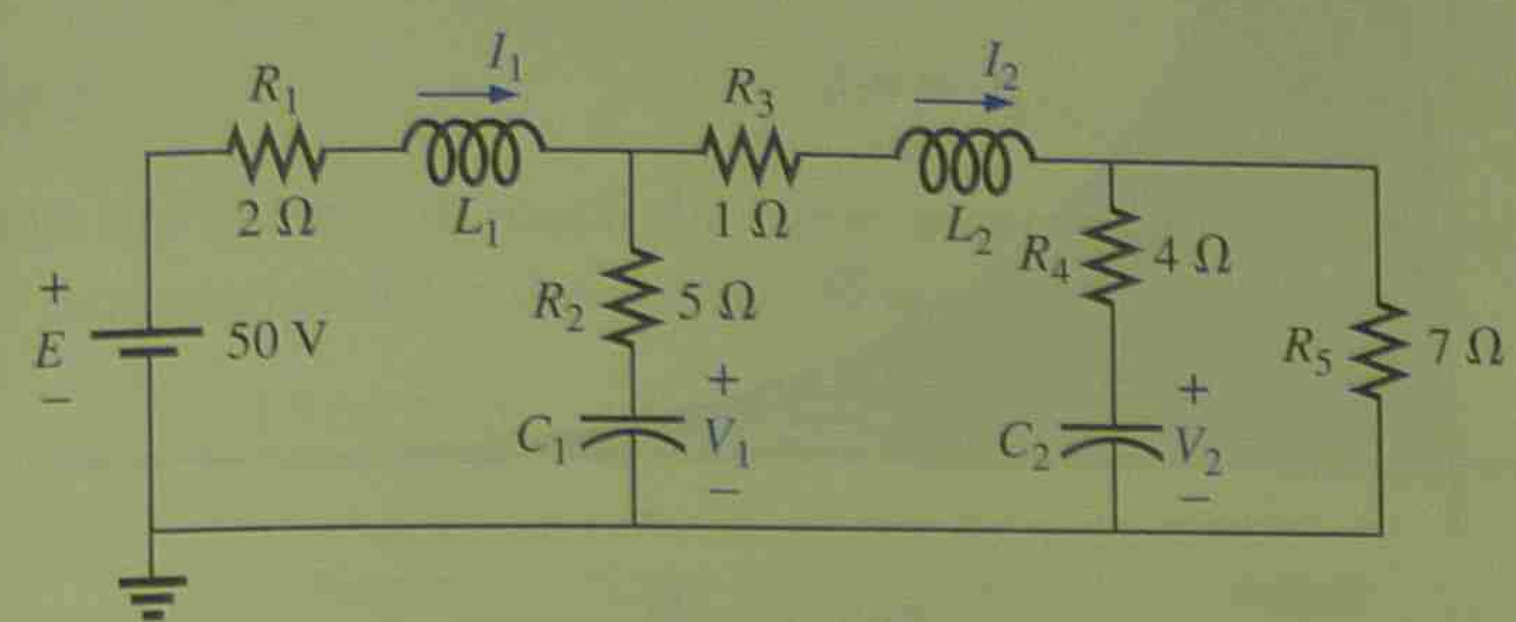


FIG. 12.36

Solution: Note Fig. 12.37:

$$I_1 = I_2$$

$$I_1 = \frac{E}{R_1 + R_3 + R_5} = \frac{50\text{ V}}{2\ \Omega + 1\ \Omega + 7\ \Omega} = \frac{50\text{ V}}{10\ \Omega} = \mathbf{5\ A}$$

$$V_2 = I_2R_5 = (5\text{ A})(7\ \Omega) = \mathbf{35\ V}$$

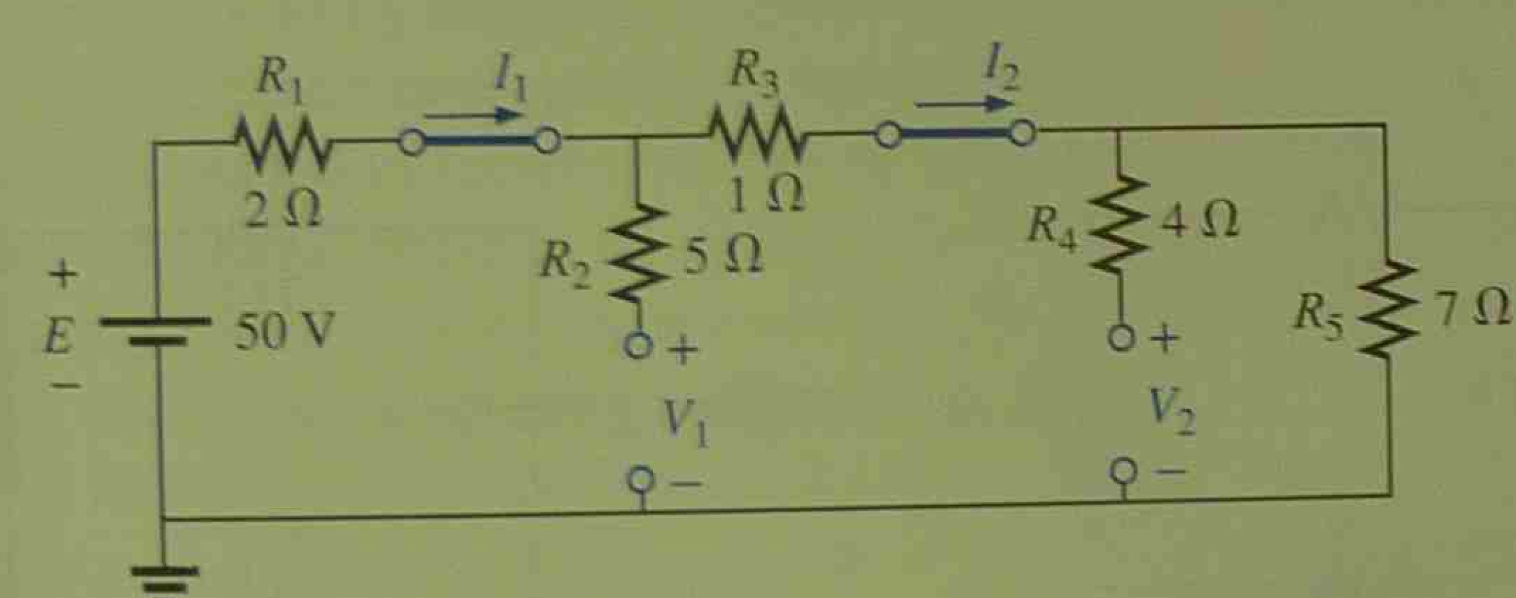


FIG. 12.37

Applying the voltage divider rule,

$$V_1 = \frac{(R_3 + R_5)E}{R_1 + R_3 + R_5} = \frac{(1\ \Omega + 7\ \Omega)(50\text{ V})}{2\ \Omega + 1\ \Omega + 7\ \Omega} = \frac{(8\ \Omega)(50\text{ V})}{10\ \Omega} = \mathbf{40\ V}$$

12.13 ENERGY STORED BY AN INDUCTOR

The ideal inductor, like the ideal capacitor, does not dissipate the electrical energy supplied to it. It stores the energy in the form of a magnetic field. A plot of the voltage, current, and power to an inductor is shown in Fig. 12.38 during the buildup of the magnetic field surrounding the inductor. The energy stored is represented by the shaded area under the

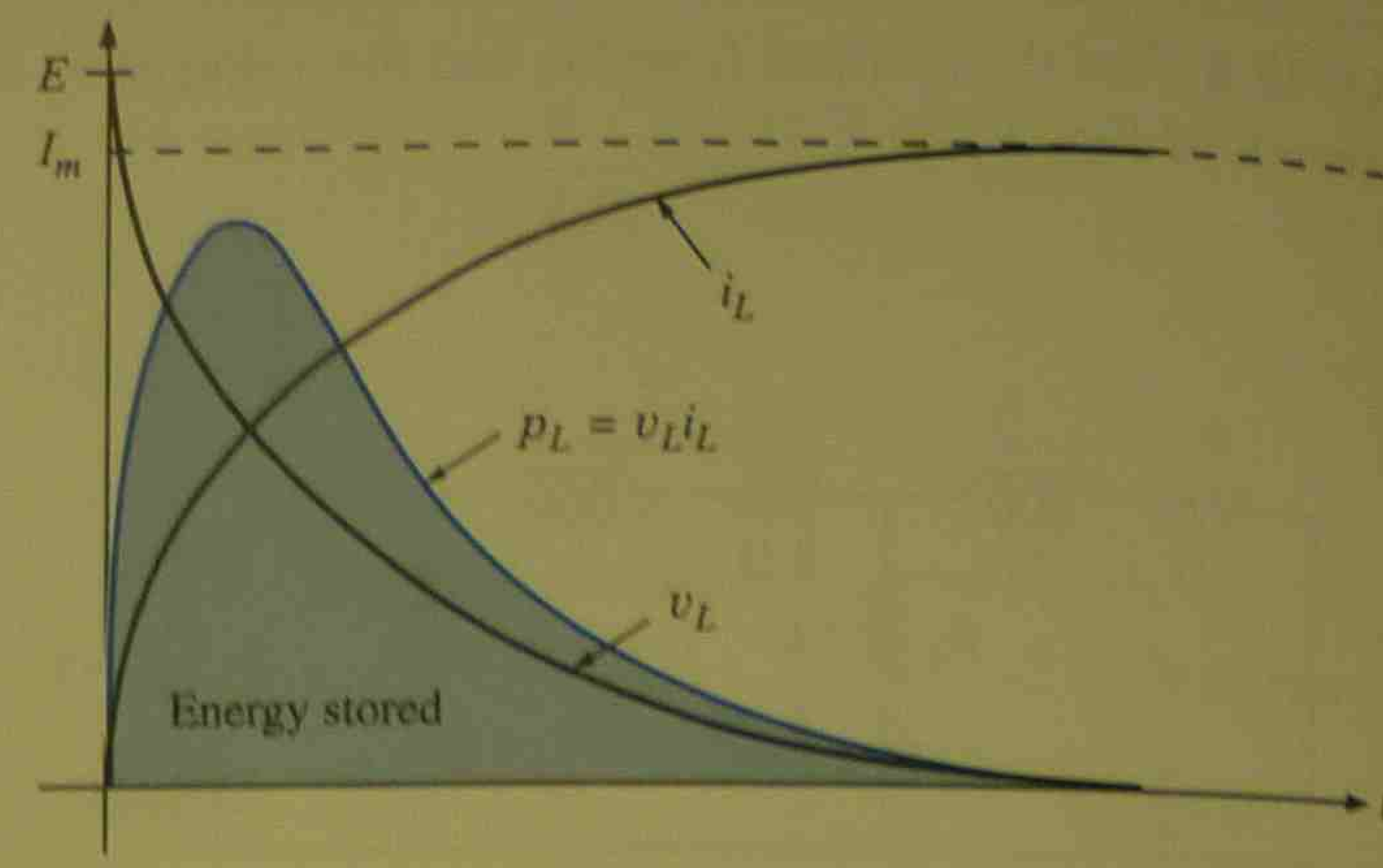


FIG. 12.38

power curve. Using calculus, we can show that the evaluation of the area under the curve yields

$$W_{\text{stored}} = \frac{1}{2} L I_m^2 \qquad \text{(joules, J)} \qquad (12.25)$$

EXAMPLE 12.9 Find the energy stored by the inductor in the circuit of Fig. 12.39 when the current through it has reached its final value.

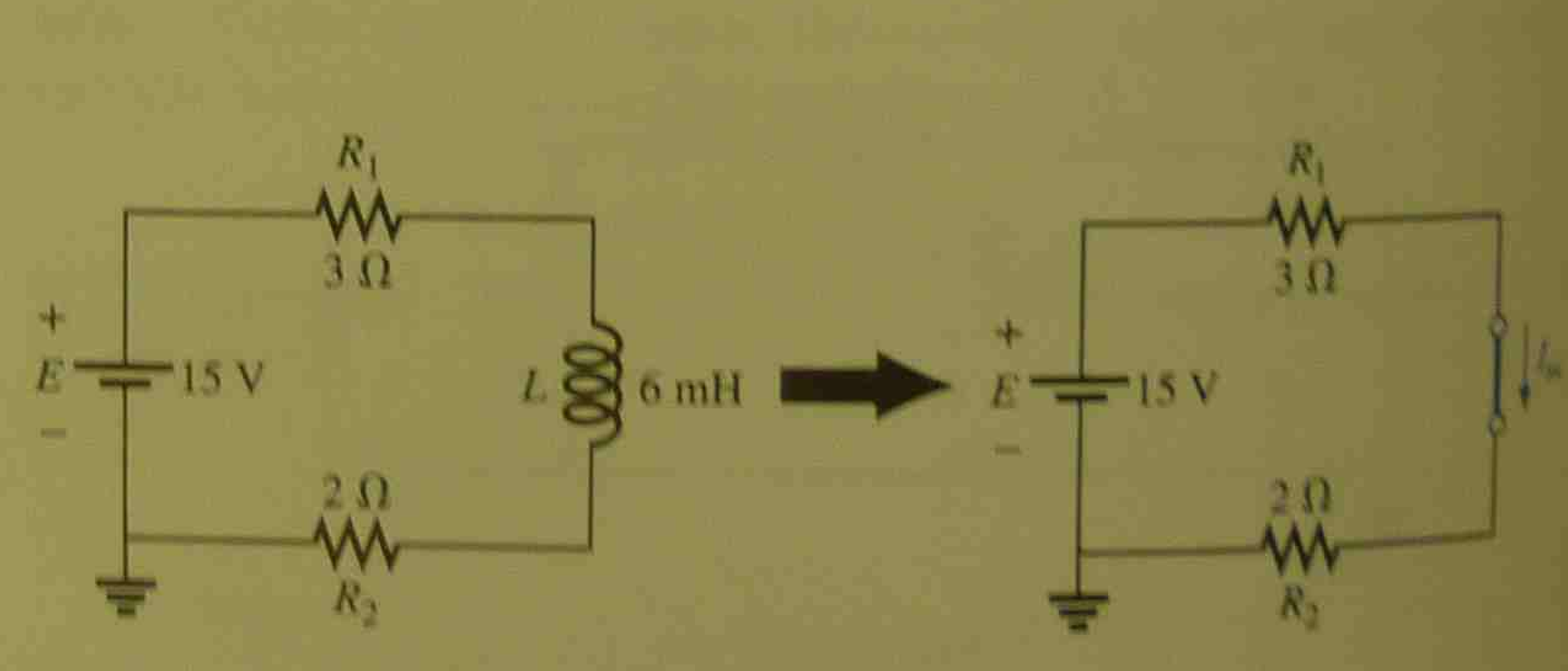


FIG. 12.39

Solution:

$$I_m = \frac{E}{R_1 + R_2} = \frac{15\text{ V}}{3\ \Omega + 2\ \Omega} = \frac{15\text{ V}}{5\ \Omega} = 3\text{ A}$$

$$W_{\text{stored}} = \frac{1}{2} L I_m^2 = \frac{1}{2}(6 \times 10^{-3}\text{ H})(3\text{ A})^2 = \frac{54}{2} \times 10^{-3}\text{ J}$$

$$= \mathbf{27\ mJ}$$

12.14 COMPUTER ANALYSIS

Both PSpice and BASIC can provide the transient response for an *R-L* circuit. In BASIC, the appropriate equations are employed to determine