(1)(a) $y = x^2 + 2x + 1$

2.1. DIFFERENTIATION

2.1.1 Differentiation of Powers

If $y = x^n$,

then $$\frac{dy}{dx} = nx^{n-1}$$

for all values of n. Th index n may be positive, negative, integral or fractional. For example, +2, -2, $-\frac{1}{2}$ and $+\frac{1}{2}$.
Also if $y = ax^n$

then $$\frac{dy}{dx} = nax^{n-1}$$

2.1.1.1 Examples

Differentiate the following functions with respect to x

(a) x^5

(b) $0.6x^7$

(c) $2x^{1.5}$

Solutions

Let "y" equal each function in turn.

(a) In this case n = 5 and a = 1 so that

$$\frac{dy}{dx} = 5\ (x^{5-1}) = 5x^4$$

(b) Here n = 7 and a = 0.6

$$\frac{dy}{dx} = 0.6\ (7x^{7-1}) = 4.2x^6$$

(c) For $y = 2x^{1.5}$, n = 1.5 and a = 2 so that

$$\frac{dy}{dx} = 2\ (1.5x^{1.5-1}) = 3x^{0.5}$$

2.1.2 Differentiation of a Sum of Functions

The differentiation of a sum of functions is equal to the sum of the individual differentiations of the functions.

In symbols, if $y = f_1(x) + f_2(x) + f_3(x)$,

then $$\frac{dy}{dx} = \frac{d}{dx}[f_1(x)] + \frac{d}{dx}[f_2(x)] + \frac{d}{dx}[f_3(x)]$$

2.1.2.1 Examples

Differentiate the following sums of functions with respect to x

(i) $y = 5x^3 + 6x^2 + 7$

(ii) $y = \sin x + \cos x$

Solutions

(i) $$\frac{dy}{dx} = \frac{d}{dx}(5x^3) + \frac{d}{dx}(6x^2) + \frac{d}{dx}(7)$$
$$= 15x^2 + 12x$$

(ii) $$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)$$
$$= \cos x - \sin x$$

2.1.3 Differentiation of a Product of Functions

Consider two functions of x, namely u(x) and v(x). Let y = u(x).v(x), that is, the product of the two functions.

Then $\frac{dy}{dx} = u(x)\frac{d}{dx}[v(x)] + v(x)\frac{d}{dx}[u(x)]$ or more simply

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

where u = u(x) and v = v(x).

2.1.3.1 Examples

Differentiate the following product of functions with respect to x

(i) $y = (x + 1)(x + 3)$

(ii) $y = (x + 1)^2 (x + 3)^3$

Solutions

(i) Let $u = x + 1$ and $v = x + 3$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1$

From above $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x + 1)(1) + (x + 3)(1)$

$= x + 1 + x + 3$

$= 2x + 4$

(ii) Let $u = (x + 1)^2$ and $v = (x + 3)^3$

$\frac{du}{dx} = 2(x + 1)$ and $\frac{dv}{dx} = 3(x + 3)^2$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x + 1)^2 [3(x + 3)^2] + (x + 3)^3 [2(x + 1)]$

$= 3(x + 1)^2 (x + 3)^2 + 2(x + 3)^2 (x + 1)$

2.1.4 Differentiation of a Quotient of Functions

Consider two functions of x, namely u(x) and v(x). Let $y = \frac{u(x)}{v(x)}$, that is the quotient of functions.

Then it can be shown that:

$$\frac{dy}{dx} = \frac{v(x)\frac{d}{dx}[u(x)] - u(x)\frac{d}{dx}[v(x)]}{[v(x)]^2}$$

or more simply

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

An easy way to recall this relation is to note tha v occurs first in the numeration and is squared in the denominator. This is emphasised below.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

2.1.4.1 Examples

Differentiate the following quotient of functions with respect to x

(i) $\frac{x}{x + 1}$ (ii) $\frac{x^2 + 2x + 1}{x^2 - 2x + 1}$

Solutions

Let y equal each function.

(i) $y = \frac{x}{x + 1}$

Let u = x; v = x + 1. Note that u and v are fixed in this case and cannot be interchanged as in the product rule.

$$\frac{du}{dx} = 1 \quad ; \quad \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2}$$

$$= \frac{x + 1 - x}{(x + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x + 1)^2}$$

(ii) $y = \frac{x^2 + 2x + 1}{x^2 - 2x + 1}$

$u = x^2 + 2x + 1$ and $v = x^2 - 2x + 1$

$\frac{du}{dx} = 2x + 2$ and $\frac{dv}{dx} = 2x - 2$

$$\frac{dy}{dx} = \frac{(x^2 - 2x + 1)(2x + 2) - (x^2 + 2x + 1)(2x - 2)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x + 2x^2 - 4x + 2 - 2x^3 - 4x^2 - 2x + 2x^2 + 4x + 2}{(x^2 - 2x + 1)^2}$$

$$= \frac{-4x^2 + 4}{(x^2 - 2x + 1)^2}$$

Note that usually a fair amount of simplification can be achieved in the numerator of the differential coefficient.

2.1.5 Differentiation of Trigonometric Functions

It is necessary to know the standard forms of the differential coefficients of trigonometric functions and apply the rules of differentiation in order to differentiate trigonometric functions.

Standard Forms

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = - \sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\text{cosec } x) = - \text{cosec } x \cot x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\cot x) = - \text{cosec}^2 x$

2.1.6 Differentiation of Exponential Functions

The differential coefficient of the exponential function can be found by the basic method of differential calculus, namely

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

where $y = e^x$

By this method it can be shown that

$$\boxed{\frac{dy}{dx} = e^x}$$

This function, e^x is the only mathematical function which when differentiated does not change.

The differential coefficient of

$$y = e^{ax}$$

is using rule 2.1.9 $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = ae^{ax}}$$

In general, if

$$y = e^{f(x)} \quad \text{then}$$

$$\boxed{\frac{dy}{dx} = f'(x)e^{f(x)}}$$

where $f'(x) = \frac{d}{dx}[f(x)]$

For example, if $y = e^{\frac{1}{2}bx^2 + x}$

then

$$\frac{dy}{dx} = \frac{d}{dx}[\tfrac{1}{2}bx^2 + x]\,e^{\frac{1}{2}bx^2 + x}$$

$$= (bx + 1)e^{\frac{1}{2}bx^2 + x}$$

2.1.7 Differentiation of $\log_e x$

Let $y = \log_e x$

Then by definition

$$x = e^y$$

Differentiate both sides with respect to y.

Then $\frac{dx}{dy} = e^y$

By inversion $\frac{dy}{dx} = \frac{1}{e^y}$

But $x = e^y$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{1}{x}}$$

where $y = \log_e x$

In general it can be show that if

$$y = \log_e f(x)$$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)}{f(x)}}$$

where $f'(x) = \frac{d}{dx}[f(x)]$

2.1.7.1 Example

From the differential coefficient of the following functions:

(a) $\log_e x^2$

(b) $\log_e (x^2 - 1)$

(c) $\log_e \sin x$

Solution

(a) Let $y = \log_e x^2$

where $f(x) = x^2$

then $\frac{dy}{dx} = (\frac{2x}{x^2}) = \frac{2}{x}$

Alternatively:

$y = \log_e x^2 = 2 \log_e x$

Then $\frac{dy}{dx} = 2 (\frac{1}{x}) = \frac{2}{x}$

(b) Let $y = \log_e (x^2 - 1)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2 - 1)}{x^2 - 1} = \frac{2x}{x^2 - 1}$$

(c) Let $y = \log_e \sin x$

Then $\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin x)}{\sin x}$

$= \frac{\cos x}{\sin x}$

$= \cot x$

2.1.8 Successive Differentiation

Consider the expression $x^3 + 3x^2 + 4$

Let $y = x^3 + 3x^2 + 4$

Then $\frac{dy}{dx} = 3x^2 + 6x$

Obviously $\frac{dy}{dx}$ is a function of x and can itself be differentiated with respect to x.

Then $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(3x^2 + 6x)$

$= 6x + 6$

$\frac{d}{dx}(\frac{dy}{dx})$ is written as $\frac{d^2y}{dx^2}$ (read "d squared y d x squared")

Then $\frac{d^2y}{dx^2} = 6x + 6$

Likewise $\frac{d^2y}{dx^2}$ is a function of x and can be differentiated with respect to x.

Then $\frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3} = 6$

Some functions can be differentiated successively indefinitely without the differential coefficient becomeing zero.

An example is: $y = \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{-1}{x^2}$, $\frac{d^2y}{dx^2} = \frac{2}{x^3}$, $\frac{d^3y}{dx^3} = \frac{-6}{x^4}$ and so on.

2.1.8.1 Example

Find the first three differential coefficients of the function

$(\frac{1}{2x + 1})$

Let $y = \frac{1}{2x + 1}$ $= (2x + 1)^{-1}$

Then $\frac{dy}{dx} = (-1)(2)(2x + 1)^{-2} = \frac{-2}{(2x + 1)^2}$

$$\frac{d^2y}{dx^2} = (2)(-2)(-2)(2x + 1)^{-3} = \frac{8}{(2x + 1)^3}$$

$$\frac{d^3y}{dx^3} = (8)(+2)(-3)(2x + 1)^{-4} = \frac{-48}{(2x + 1)^4}$$

2.1.9 Differentiation of a Function of a Function

Mathemtically sin x, e^x, log x and $x^2 + 1$ are all functions of x. However, consider such functions as:

$$\sin^2 (x^2 + 1), \quad e^{x^2}, \quad e^{\sin x} \quad \text{and} \quad \log_e \sin x$$

These functions are certainly functions of x but they contain two functions or are "function of a function" expressions.

Obviously a function $\log_e \sin x$ contains a logarithmic and and trigonometric function.

The differentiation of a "function of a function" expression can be difficult and a two-step differentiating process has been developed.

2.1.9.1 Consider the function of a function expression

$$y = \sin^3 (2x^2 - 1)$$

Let $u = 2x^2 - 1$ $\qquad \frac{du}{dx} = 4x$

then $y = \sin^3 u$ $\qquad \frac{dy}{du} = 3 \sin^2 u \cos u$

To combine these two differential coefficients the following relation is used:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Then $\frac{dy}{dx} = 4x \cdot 3 \sin^2 u \cos u$

But $u = 2x^2 - 1$

Hence $\frac{dy}{dx} = 12x \sin^2 (2x^2 - 1) \cos (2x^2 - 1)$

2.1.10 Differentiation of Implicit Functions

By definition an IMPLICIT function is one in which y is not expressed in terms of x EXPLICITLY.

The function $y = x^2 + x + 1$ is an EXPLICIT function, that is, y is expressed explicity as a function of x.

However, the equation

$$y^2 + 3xy + x^2 = 0$$

is an IMPLICIT function since y is not expressed in terms of x only. The equation implies that y is a function of x.

Some implicit functions can be made explicit by solving for y. For example,

$x^2 + y^2 = 9$ is an implicit function

and $y = \sqrt{9 - x^2}$ is an explicit function.

However, many implicit functions cannot be changed to explicit functions. For example,

$$y^2 + xy + x^2 = 0$$

To differentiate an implicit function use is made of the following relation:

If f(y) is a function of y and implicitly a function of x

$$\boxed{\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \frac{dy}{dx}}$$

For example, if $f(y) = y^2$

$$\frac{d}{dx}[y^2] = \frac{d}{dy}(y^2)\,\frac{dy}{dx}$$

$$= 2y\,\frac{dy}{dx}$$

2.1.10.1 Example

Differentiate the following implicit functions with respect to x.

(a) $x^2 + y^2 = 4$ (b) $y \log_e x = 2$

Solution

(a)

$$2x + \frac{d}{dx}(y^2) = 0$$

$$2x + \frac{d}{dx}(y^2)\,\frac{dy}{dx} = 0$$

$$2x + 2y\,\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

(b)

$$\frac{d}{dx}(y \log_e x) = \frac{d}{dx}(2)$$

Now $y \log_e x$ is a product of functions and must be differentiated as such.

Then

$$\frac{d}{dx}(y) \,.\, \log_e x + y\,\frac{d}{dx}(\log_e x) = 0$$

$$\frac{dy}{dx}\log_e x + y \,.\, \frac{1}{x} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} \,.\, \frac{1}{\log_e x}$$

2.2 INTEGRATION

2.2.1 Introduction

Integration is the other half of the story of the differential and integral calculus. The integral calculus has many fundamental and important applications in science and engineering, particularly electrical engineering. It would be no understatement to say that electrical engineering could not have developed without the integral calculus.

Integration may be approached in three different ways. Firstly, the integral of a function f(x) can be considered to be THE AREA UNDER THE CURVE of f(x). This approach allows a basic insight into the underlying principles of integration.

It is also the GEOMETRICAL INTERPRETATION of the integral of a function f(x) taken between two limits.

Secondly, the process of integration may be considered as a MATHEMATICAL PROCESS in its own right. The fundamental idea of integration is the synthesis of a large number of small quantitites to make a whole.

Thirdly, integration may be considered to be the reverse process of differentiation. Then integration is said to be ANTIDIFFERENTIATION. This is the extremely important connection between integration and differentiation. It means that given the differential coefficient of an unknown function, the function itself may be found by integrating the differential coefficient. This idea of integration as antidifferentiation leads up to the subject of differential equations which is of central importance in electrical engineering.

2.2.2 Notation

In words the $\int_a^b f(x)\,dx$ is THE DEFINITE INTEGRAL OF f(x) WITH RESPECT TO x FROM LIMIT a TO LIMIT b.

Evaluating the integral is the process of INTEGRATION. The function f(x) is called the INTEGRAND. The x = a to x = b are the BOUNDARIES or LIMITS of integration.

2.2.2.1 Example

Find the area under the curve of the function $f(x) = 3 \sin x + 10$ between the limits of integration $x = \pi$ and $x = 2\pi$.

Solution

Sketch this function as shown in Figure 2.1 and shade the required area.

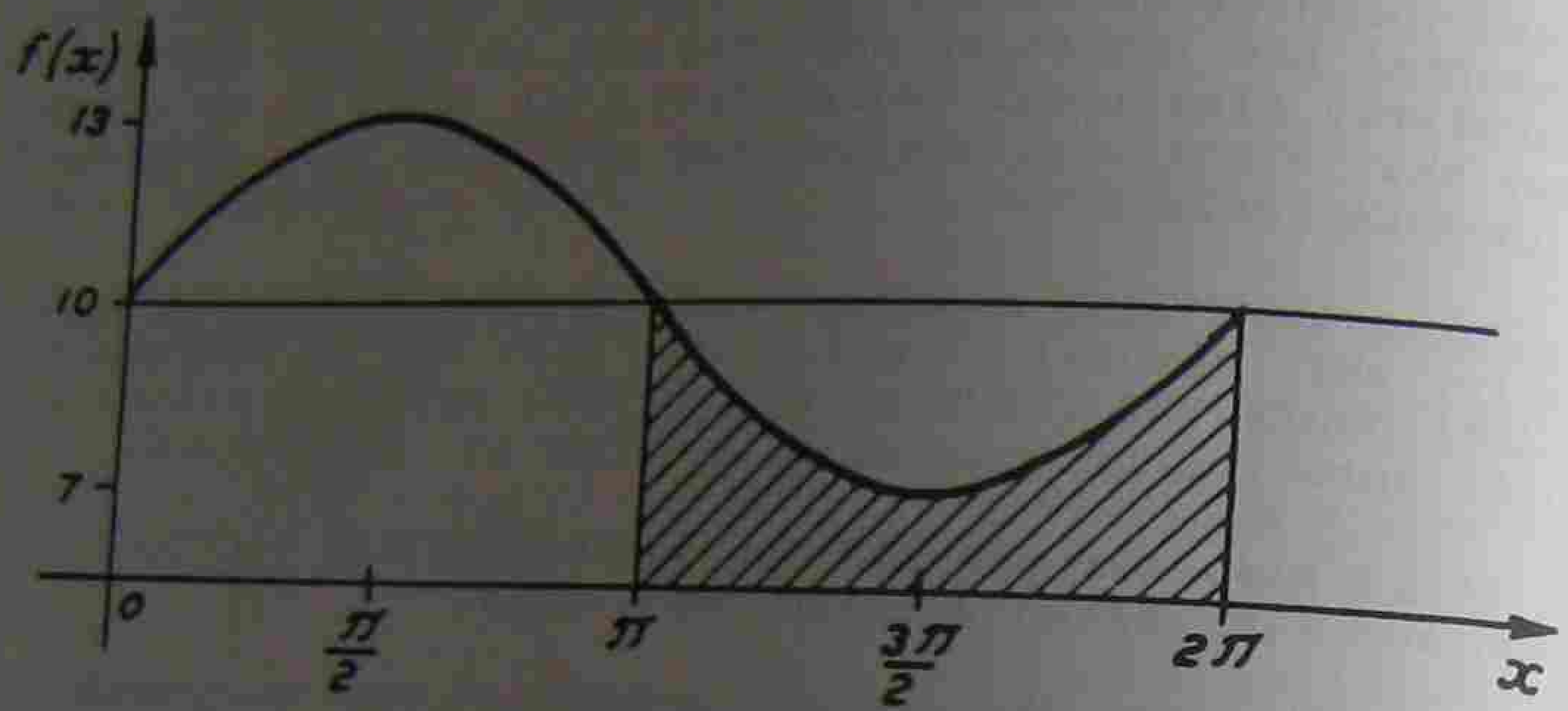


Figure 2.1 The required area under y = 3 sin x + 10 **between limits** π **and** 2π

The exact area under the curve is

$$A = \int_{\pi}^{2\pi} (3 \sin x + 10)\, dx$$

The evaluation of integrals will be discussed in more detail later in this unit.

However, this integral becomes

$$\int_{\pi}^{2\pi} (3 \sin x + 10)\, dx = [-3 \cos x + 10x]_{\pi}^{2\pi}$$

$$= -3 \cos 2\pi + 10(2\pi) - \{-3 \cos \pi + 10\pi\}$$

$$= -3 \times 1 + 20\pi + 3 \times (-1) - 10\pi$$

$$= -6 + 10\pi$$

$$= 10\pi - 6$$

This area may be evaluated as accurately as desired. Take

$$\pi = 3.1416$$

Then $A = 10 \times 3.1416 - 6$

$$= 31.416 - 6$$

$$\underline{A = 25.416 \text{ sq. units}}$$

2.2.3 Integration of a Power of x

$$\boxed{\int x^n\, dx = \left(\frac{1}{n+1}\right) x^{n+1} + C}$$

where $n \neq -1$

The index n may be positive, negative or fractional. Check the integration by differentiation.

$$\frac{d}{dx}\left[\left(\frac{1}{n+1}\right) x^{n+1} + C\right] = \frac{n+1}{n+1} x^{n+1-1}$$

$$= x^n$$

Note that if $n = -1$, $\frac{1}{n+1}$ becomes $\frac{1}{0}$ which is indeterminate. Hence, the special case $x = -1$ is not permissible in this standard integral.

2.2.3.1 Examples would be:

$$\int x^7\, dx = \frac{1}{7+1} x^{7+1} + C$$

$$= \frac{1}{8} x^8 + C$$

$$\int x^{\frac{1}{5}}\, dx = \frac{1}{\frac{1}{5}+1} x^{\frac{1}{5}+1} + C$$

$$= \frac{5}{6} x^{\frac{6}{5}} + C$$

$$\int x^{-3}\,dx = \frac{1}{-3 + 1}\,x^{-3 + 1} + C$$

$$= -\frac{1}{2}\,x^{-2} + C$$

$$\int x^{-\frac{1}{2}}\,dx = \frac{1}{-\frac{1}{2} + 1}\,x^{-\frac{1}{2} + 1} + C$$

$$= 2x^{\frac{1}{2}} + C$$

Other general examples would be:

$$\int ax^{n}\,dx = a\int x^{n}\,dx = \frac{ax^{n + 1}}{n + 1} + C$$

Note that a constant inside the integral sign may always be taken outside the integral sign.

$$\int (x^{n} + x^{m})\,dx = \frac{1}{n + 1}\,x^{n + 1} + \frac{1}{m + 1}\,x^{m + 1} + C$$

This means that when integrating a sum of functions, the funcitons may be integrated individually. For example:

$$\int x^{4} + 2x^{3}\,dx = \frac{1}{4 + 1}\,x^{4 + 1} + 2 \cdot \frac{1}{3 + 1}\,x^{3 + 1} + C$$

$$= \frac{1}{5}\,x^{5} + \frac{1}{2}\,x^{4} + C$$

2.2.4 The Standard Integral of $(ax + b)^{n}$ is

$$\boxed{\int (ax + b)^{n}\,dx = \frac{1}{a(n + 1)}\,(ax + b)^{n + 1} + C}$$

where $n \neq -1$

Again n may be positive, negative or fractional.

2.2.4.1 For example

$$\int (2x + 3)^{3}\,dx = \frac{1}{2\,(3 + 1)}\,(2x + 4)^{3 + 1} + C$$

$$= \frac{1}{8}\,(2x + 4)^{4} + C$$

$$\int (-3x + 2)^{-\frac{1}{3}}\,dx = \frac{1}{(-3)\,(-\frac{1}{3} + 1)}\,(-3x + 2)^{-\frac{1}{3} + 1} + C$$

$$= \frac{1}{(-3)\,(+\frac{2}{3})}\,(-3x + 2)^{+\frac{1}{3}} + C$$

$$= -\frac{1}{2}\,(-3x + 2)^{\frac{2}{3}} + C$$

$$\int (5x + 8)^{-2}\,dx = \frac{1}{5\,(-2 + 1)}\,(5x + 8)^{-2 + 1} + C$$

$$= -\frac{1}{5}\,(5x + 8)^{-1} + C$$

$$= \frac{1}{5\,(5x + 8)} + C$$

2.2.5 Integration of Trigonometric Functions

The standard integrals are:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln \sec x + C$$

$$\int \cot x \, dx = \ln \sin x + C$$

$$\int \cos nx \, dx = \frac{1}{n} \sin nx + C$$

$$\int \sin nx \, dx = -\frac{1}{n} \cos nx + C$$

From the differentiation of trigonometric functions, the following integrals are obtained:

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

2.2.6 Integration of Trigonometric Functions by Trigonometric Identities

By using trigonometric identities many trigonometric functions can be integrated.

Then since

$$\boxed{\cos 2x = 2\cos^2 x - 1}$$

$$\cos^2 x \, dx = \frac{\cos 2x + 1}{2}$$

Then

$$\int \cos^2 x \, dx = \int \frac{\cos 2x + 1}{2} \, dx$$

$$= \frac{1}{2} \int (\cos 2x + 1) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{2} \sin 2x + x\right) + C$$

$$= \frac{1}{4} \sin 2x + \frac{1}{2} x + C$$

Also since

$$\boxed{\cos 2x = 1 - 2\sin^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Then

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x\right) + C$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

In a similar way

$$\int \sin^2 2x\, dx = \int \frac{1 - \cos 4x}{2}\, dx$$

$$= \frac{1}{2} \int (1 - \cos 4x)\, dx$$

$$= \frac{1}{2} \{x - \frac{1}{4} \sin 4x\} + C$$

$$= \frac{1}{2} x - \frac{1}{8} \sin 4x + C$$

and $$\int \cos^2 2x = \int \frac{1 + \cos 4x}{2}\, dx$$

$$= \frac{1}{2} \int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2} (x + \frac{1}{4} \sin 4x) + C$$

$$= \frac{1}{2} x + \frac{1}{8} \sin 4x + C$$

Using the identity

$$\boxed{\sec^2 x = \tan^2 x + 1}$$

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$$

$$= \tan x - x + C$$

Other identities are

$$\boxed{\sin 2x = 2 \cos x \sin x}$$

$$\boxed{\operatorname{cosec}^2 x = \cot^2 x + 1}$$

Then

$$\int \cos x \sin x\, dx = \int \frac{1}{2} \sin 2x\, dx$$

$$= -\frac{1}{4} \cos 2x + C$$

$$\int \cot^2 x\, dx = \int (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\cot x - x + C$$

By use of the multiple angle formula, products of trigonometric functions can be integrated.

2.2.6.1 These are

$$\sin A \cos B = \frac{1}{2} \{\sin (A + B) + \sin (A - B)\}$$

$$\cos A \sin B = \frac{1}{2} \{\sin (A + B) - \sin (A - B)\}$$

$$\cos A \cos B = \frac{1}{2} \{\cos (A + B) + \cos (A - B)\}$$

$$\sin A \sin B = \frac{1}{2} \{\cos (A - B) - \cos (A + B)\}$$

Then

$$\sin 3x \cos 4x = \frac{1}{2} \{\sin (3x + 4x) + \sin (3x - 4x)\}$$

$$= \frac{1}{2} \{\sin 7x - \sin x\}$$

Then

$$\int \sin 3x \cos 4x\, dx = \int \frac{1}{2} \{\sin 7x - \sin x\}\, dx$$

$$= -\frac{1}{14} \cos 7x + \frac{1}{2} \cos x + C$$

2.2.7 Integration of Exponential Functions

From the derivative

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

$$\boxed{\int e^{ax}\, dx = \frac{1}{a} e^{ax} + c}$$

Likewise

$$\int be^{ax}\, dx = \frac{b}{a} e^{ax} + c$$

The following general relation is important:

$$\boxed{\int f^{1}(x)\, e^{f(x)}\, dx = e^{f(x)} + c}$$

where

$$f^{1}(x) = \frac{d}{dx} [\ f(x)\ ]$$

For example,

$$\int xe^{x^2}\, dx = \int \frac{d}{dx} \left(\frac{1}{2} x^2\right) e^{x^2}\, dx$$

$$= \frac{1}{2} e^{x^2} + c$$

2.2.8 Integration of Functions which Result in a Logarithmic Function

Since

$$\frac{d}{dx} [\ \ell n\ f(x)\ ] = \frac{f'(x)}{f(x)}$$

then

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ell n\ f(x) + C}$$

2.2.8.1 For example,

$$\int \frac{4x\, dx}{2x^2 + 3} = \ell n\ (2x^2 + 3) + C$$

$$\int \frac{x\, dx}{2x^2 + 3} = \frac{1}{4} \int \frac{4x}{2x^2 + 3}\, dx$$

$$= \frac{1}{4}\ \ell n\ (2x^2 + 3) + C$$

$$\int \frac{\sin x}{\cos x}\, dx = -\ \ell n\ \cos x + C$$

$$= \ell n\ (\cos x)^{-1} + C$$

$$= \ell n\ \frac{1}{\cos x} + C$$

$$= \ell n\ \sec x + C$$

2.2.9 Integration by Change of Variable

Let $I = \int x \sqrt{2x + 1}\ dx$

This expression $x \sqrt{2x + 1}$ cannot be integrated directly by any standard form.

let $u = \sqrt{2x + 1}$

then $u^2 = 2x + 1$

Also $2x = u^2 - 1$

$x = \frac{1}{2} (u^2 - 1)$

Then $\frac{dx}{du} = u$

$dx = u\ du$

Substitute in the above integral

then $$I = \int \frac{1}{2} (u^2 - 1)\ u\ .\ u\ du$$

$$= \frac{1}{2} \int (u^4 - u^2)\ du$$

$$= \frac{1}{2} \left\{\frac{1}{5} u^5 - \frac{1}{3} u^3\right\} + C$$

Now $$u = (2x + 1)^{\frac{1}{2}}$$

$$u^5 = (2x + 1)^{\frac{5}{2}}$$

$$u^3 = (2x + 1)^{\frac{3}{2}}$$

$$\therefore \quad I = \frac{1}{2} \left[\frac{1}{5} (2x + 1)^{\frac{5}{2}} - \frac{1}{3} (2x + 1)^{\frac{3}{2}} \right] + C$$

Another example would be:

$$I = \int \frac{x}{\sqrt{5 - x}}\ dx$$

$$= \int x\ (5 - x)^{-\frac{1}{2}}\ dx$$

Let $$u = 5 - x$$

$$\therefore \quad x = 5 - u$$

$$\frac{dx}{du} = -1$$

$$dx = (-\ du)$$

Then $$I = \int (5 - u)(u^{-\frac{1}{2}})(-\ du)$$

$$= \int (-\ 5u^{-\frac{1}{2}} + u^{\frac{1}{2}})\ du$$

$$= \left(\frac{-5}{\frac{1}{2}}\right) u^{\frac{1}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -\ 10\ u^{\frac{1}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -\ 10\ (5 - x)^{\frac{1}{2}} + \frac{2}{3} (5 - x)^{\frac{3}{2}} + C$$

2.2.10 Evaluation of the Definite Integral

It can be shown that in general

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)}$$

where $F(b)$ and $F(a)$ are substituted instances of the integrated function $F(x)$, that is

$$\int f(x)\ dx = F(x)$$

2.2.10.1 For example,

$$\int_2^4 x^2\ dx = \frac{1}{3} (4)^3 - \frac{1}{3} (2)^3$$

since $$\int x^2\ dx = \frac{1}{3} x^3$$

The usual notation is shown below.

$$\int_2^4 x^2 \, dx = \left[\frac{1}{3} x^3 \right]_2^4$$

$$= \frac{1}{3} (4)^3 - \frac{1}{3} (2)^3$$

$$= \frac{1}{3} (64 - 8)$$

$$= \frac{56}{3}$$

WORK TO BE FORWARDED FOR COMMENT

PART 1 - DIFFERENTIATION

1. Find the gradient to the curve $y = x^2 + \sin x$ where $x = \frac{\pi}{4}$ radians.

2. Find the gradient to the curve $y = x^3 - x^2 + 2$ when $x = 2$ and $x = -2$.

3. Find the differential coefficient of the following functions with respect to x:

(i) $y = x^{10}$

(ii) $y = x^3$

(iii) $y = mx^k$

(iv) $y = nx^{(s + 1)}$

(v) $y = (x + 4)^3$

(vi) $y = (x - 5)^2$

(vii) $y = 15(x + 4)^7$

(viii) $y = 2(x - 3)^4$

(ix) $y = 4x^{-3.2}$

(x) $y = 2.1x^{-\frac{1}{2}}$

(xi) $y = 3(x - 2)^{-\frac{1}{2}}$

(xii) $y = 3.2(2x + 3)^{\frac{1}{5}}$

4. (i) The eddy current loss of an electrical machine is given by:

$$P = k_1 f^2 t^2 B^2 V$$

where P = power

and f = frequency

k_1, t, B and V are all constants. Differentiate P with respect to f.

(ii) The hysteresis loss of an electrical machine is given by:

$$P = k_2 f B^{1.6} W$$

where P = power

and B = maximum flux density

k_2, f and W may be considered as constants. Determine $\frac{dP}{dB}$

5. Differentiate the following functions with respect to x.

(i) $y = 2x^{\frac{3}{2}} + x^{-\frac{3}{2}}$

(ii) $y = 7x^6 + 6x^5 + 4x^4 + 3x^2 + 2x + 1$

(iii) $y = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

(iv) $y = -x^{-2} + \frac{1}{x} + x^2$

(v) $y = ax^n + bx^m + c$

6. (i) The loss in an electrical machine is given by:

$$P = af + bf^2$$

where P = power

and f = frequency and "a" and "b" are constants

Differentiate P with respect to f.

(ii) The relation of induced e.m.f. E of a direct current machine to field current I_f in the windings was found by digital computer to be:

$$E = 0.58 + 681.5\, I_f - 461.8\, I_f^2 + 46.3\, I_f^3$$

This is the magnetisation curve of the d.c. machine.

Find $\frac{dE}{dI_f}$

7. Differentiate the following product of functions.

(i) $x^2(x^2 - 1)$

(ii) $(x + 2)(x - 7)$

(iii) $(x + 2)(x^2 + 4)$

(iv) $(x + 1)^{-\frac{1}{2}}(x - 5)$

(v) $(2x + 7)(4x^2 - 5)$

(vi) $(2x + 7)^3(4x^2 - 5)^2$

(vii) $(5x^2 + 6x + 3)(5x - 1)$

8. Differentiate the following quotient of functions.

(i) $\frac{x + 1}{x + 2}$

(ii) $\frac{x}{x + 1}$

(iii) $\frac{4 - x}{x - x^2}$

(iv) $\frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}} - 1}$

9. Differentiate the following trigonometric functions:

(i) $\sin(10x + 4) + \cos(7x + 1)$

(ii) $\tan^2 3\theta$

(iii) $\sec x \tan x$

(iv) $\operatorname{cosec}^4 (x^2 + 1)$

(v) $\cot 5x \sin 6x$

(vi) $i = 10 \sin 10t + 5 \sin 20t + 2.5 \sin 30t$

10. (a) The potential difference across an inductor of self-inductance L is:

$$v_L = L \frac{di}{dt}$$

If $i = 10 \sin(314t + 60^o)$ find the potential difference v_L.

(b) The self inductance of a rotor winding of a salient pole synchronous machine is:

$$L = L_0 + L_2 \cos 2\theta$$

where L_0 and L_2 are constant and θ is the angular position of the rotor. Find the rate of change of inductance with angular position.

11. Differentiate the following exponential, logarithmic and power functions:

(i) e^{-ax}

(ii) $e^{(x^2 + 2x)}$

(iii) $\log_e (x^2 + 2x + 3)$

(iv) $a^{x^2 + 1}$

(v) $x^{\sin x}$

12. The current growth in a resistive-inductive circuit from a suddenly applied battery e.m.f. is:

$$i = \frac{E}{R} (1 - e^{-\frac{R}{L} t})$$

where E, R and L are constants, t is time. Find the rate of change of current with respect to time.

13. Find the first two differential coefficients of the following functions:

(i) $\log_e (x^2 - 1)$

(ii) $\sin^2 x$

(iii) $e^{\frac{1}{2} x^2}$

(iv) a^x

14. Differentiate the following functions using the function of a function rule:

(a) $e^{-\sin x}$

(b) $\log \cot x$

(c) $\tan (\sin^2 \theta)$

WORK TO BE FORWARDED FOR COMMENT

PART 2 - INTEGRATION

1. Find the area under the curve:

$$y = 4 \sin x + 3$$

between the limits π and 2π

2. Integrate the following functions:

(i) x^5

(ii) $\frac{1}{3} x^{-\frac{1}{2}}$

(iii) $6x^{-2}$

(iv) $3x^{\frac{1}{5}}$

(v) $\frac{1}{2} x^{-\frac{1}{3}} + x^2$

(vi) $(2x + 3)^3$

(vii) $(1 + x)^{-4}$

(viii) $(3 - x)^{\frac{1}{2}}$

(ix) $(5x + 6)^{-\frac{1}{3}}$

(x) $(3x - 2)^{-\frac{3}{2}}$

3. Integrate the following trigonometric functions:

(i) $\sec^2 x$

(ii) $\tan x$

(iii) $\cos 3x$

(iv) $\sin 6x$

(v) $\frac{1}{5}\sin^2 2x$

(vi) $\cot x$

(vii) $\operatorname{cosec} x \cot x$

(viii) $\sec x \tan x$

(ix) $\frac{1}{5}\cos^2 5x$

(x) $\operatorname{cosec}^2 x$

(xi) $\tan^2 3x$

(xii) $\sin x \cos x$

(xiii) $\cot^2 x$

(xiv) $\sin 3x \cos 4x$

(xv) $\sin 6x \cos x$

(xvi) $\cos 3x \cos 5x$

(xvii) $\sin x \sin 3x$

4. Integrate the following exponential functions:

(i) e^{-3x}

(ii) $2e^{4x}$

(iii) $5e^{-\frac{1}{3}x}$

(iv) $3xe^{x^2}$

(v) $-3x^2e^{-x^3}$

(vi) $\cos x \, e^{\sin x}$

5. Integrate the following logarithmic functions:

(i) $\frac{x}{3x^2 + 2}$

(ii) $\frac{e^{ax}}{e^{ax} + 4}$

(iii) $\frac{\cos x}{\sin x + 4}$

(iv) $\frac{\sec^2 x + 1}{\tan x + x}$

6. Integrate the following by change of variable:

(i) $x\sqrt{5x + 4}$

(ii) $\frac{x}{\sqrt{3x + 4}}$

7. Evaluate the following integrals:

(i) $\int_{-3}^{2} \frac{1}{3} x^2 \, dx$

(ii) $\int_{1}^{4} \frac{1}{x} \, dx$

(iii) $\int_{0}^{\frac{\pi}{2}} \sin x \, dx$

(iv) $\int_{1}^{1.4} e^{-2x} \, dx$

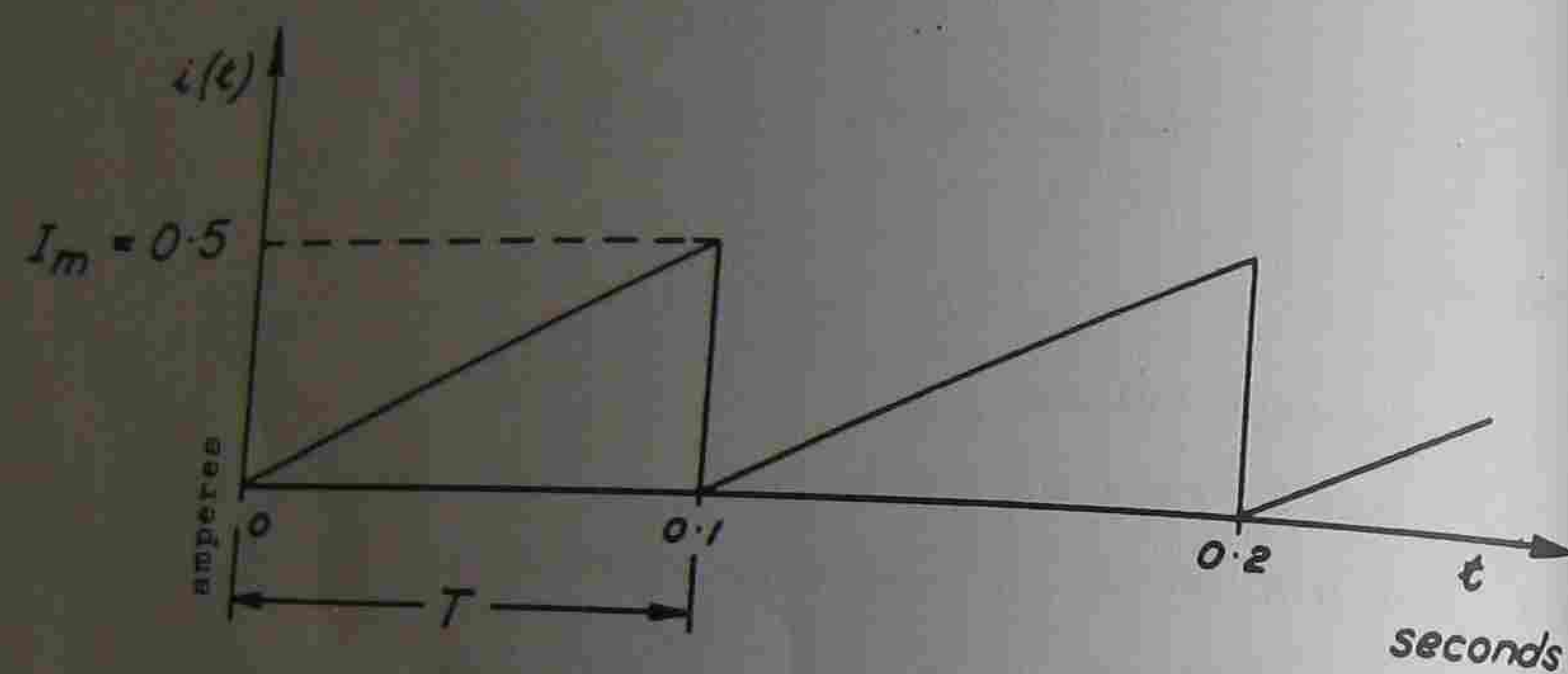
(v) $\int_{-4}^{2} (3x^2 + 4) \, dx$

AVERAGE AND R.M.S. VALUES

There are many physical applications of the definite integral in electrical engineering, but one of the most basic is the Root-Mean-Square (R.M.S.) value and the Direct Current for average value, usually referred to as the D.C. value. This may apply to a current or voltage wave.

3.1. AVERAGE VALUES

Consider a current wave as shown in Figure 3.1.



Example 3.1.1 Figure 3.1 A triangular wave of current

This current wave i(t) is periodic after every 0.1 seconds, that is, it repeats itself after each 0.1 seconds. In general, the time period is shown as T seconds. Hence, the average value needs only to be determined over one time period, as it will be the same for all time periods.

Then by definition:

$$I_{AVE} = \frac{1}{T}\int_0^T i(t)\,dt$$

where I_{AVE} = average or D.C. value of current.

In this particular case

$$i(t) = \left(\frac{I_m}{T}\right) t \quad \text{in the range } 0 \le t \le T$$

Then

$$I_{AVE} = \frac{1}{T}\int_0^T \left(\frac{I_m}{T}\right) t\,dt$$

$$= \frac{I_m}{T^2}\left[\frac{1}{2}t^2\right]_0^T$$

$$= \frac{I_m}{2}$$

Note that the integral

$$\int_0^T i(t)\,dt$$

is, in fact, the area under the graph so that by dividing by the period T the height of a rectangle is determined with base T. The area of the rectangle is the same as the area as given by the integral. In the above example then, the area of the triangle is

$$\frac{1}{2}\,T\,I_m$$

Then dividing by the base length T gives

$$I_{AVE} = \frac{\frac{1}{2}\,T\,I_m}{T} = \frac{1}{2}I_m$$

which is the same result given by the integral formula. This should be kept in mind as a check or short cut method. The above basic principle is illustrated in Figure 3.2.

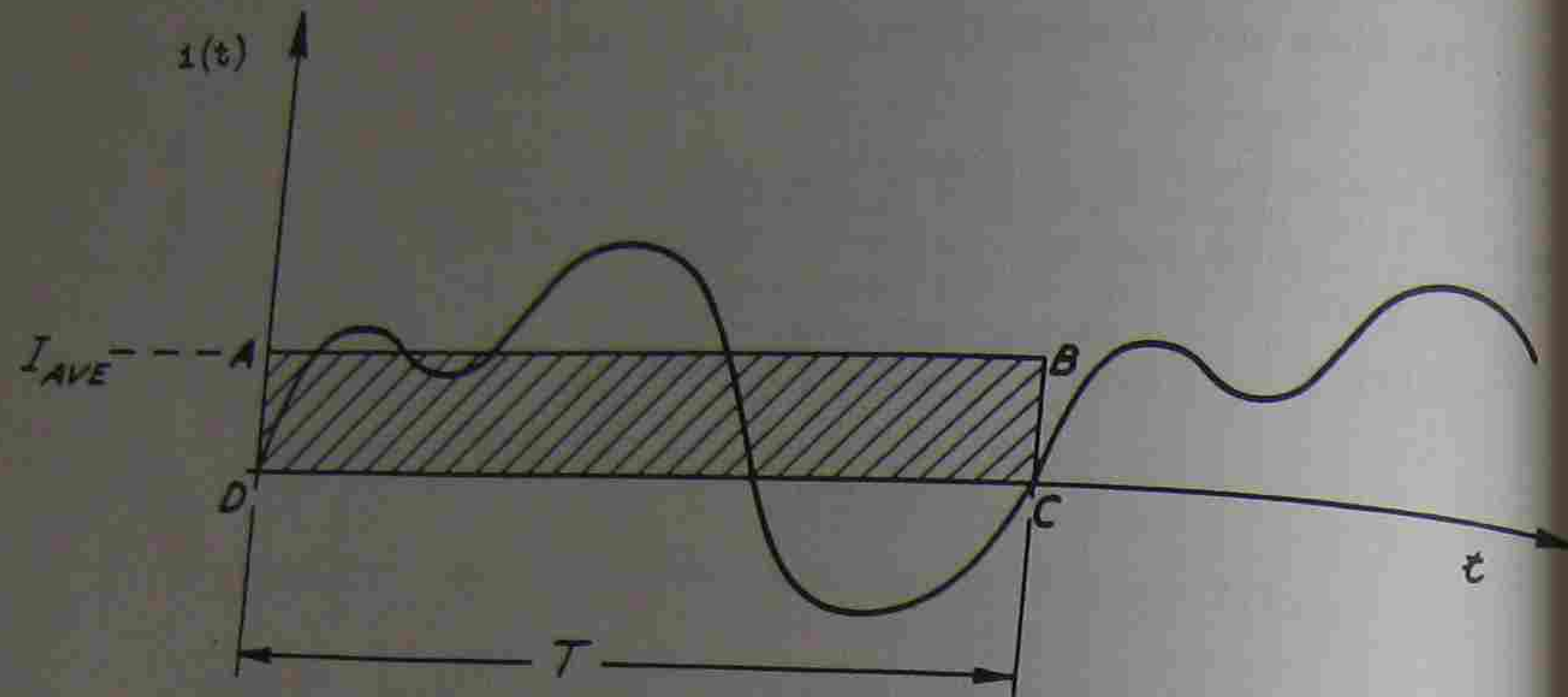


Figure 3.2 General case of the average value

For Figure 3.2

$$I_{AVE} = \frac{1}{T}\int_0^T i(t)\,dt = \frac{\text{AREA under } i(t) \text{ over period } T}{\text{BASE } T}$$

$$= \text{average height of } i(t)$$

and the integral

$$\int_0^T i(t)\,dt = \text{the area ABCD}$$

Note that the area below the X-axis would be subtracted from the area above the X-axis.

Example 3.1.2

Another case of particular interest is the rectified sine wave,

$$\boxed{i(\omega t) = I_m \sin \omega t} \qquad 0 \le \omega t \le \pi$$

and this is shown in Figure 3.3. Note for simplicity the variable of integration in this case has been changed from t to ωt.

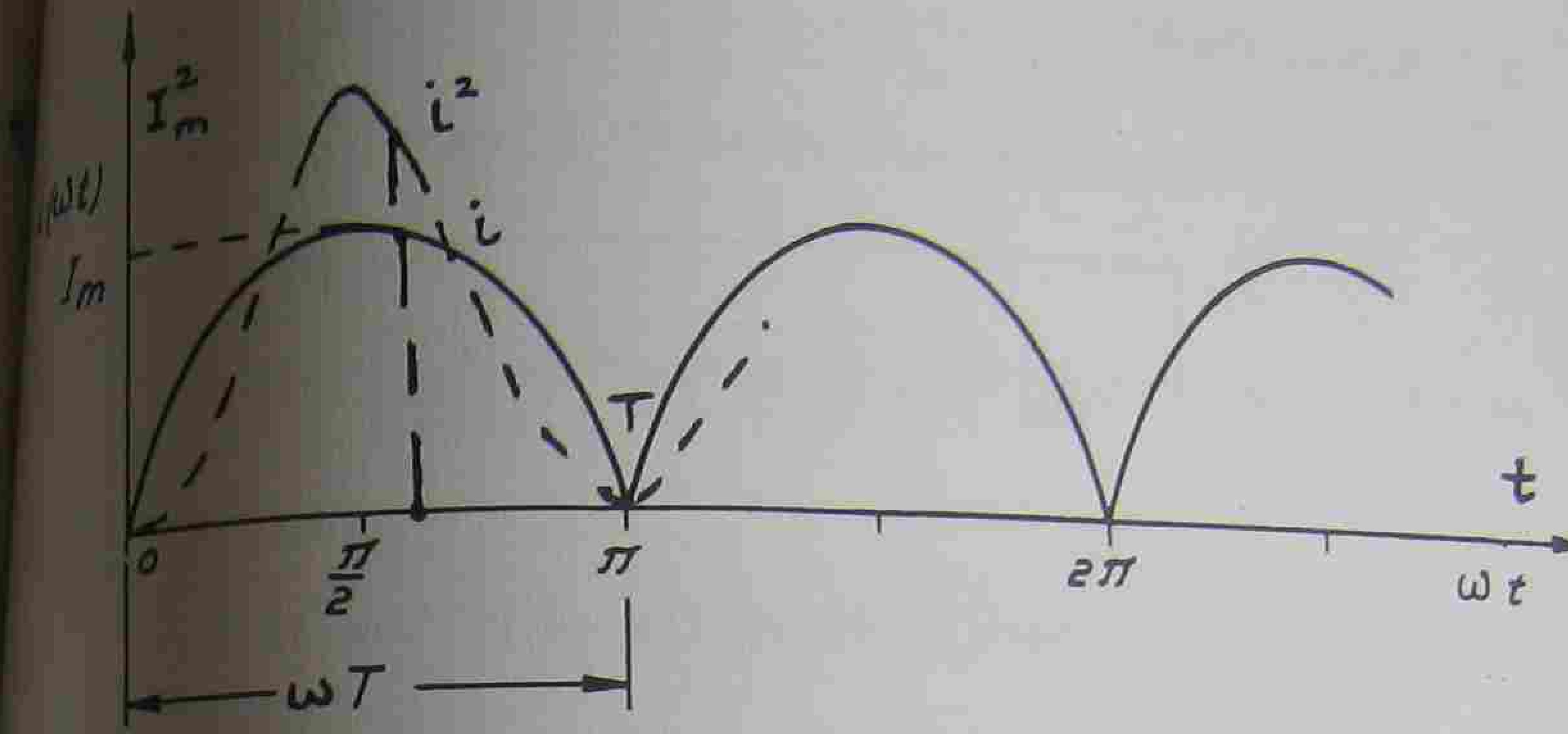


Figure 3.3. Rectified sine wave of current

In this case

$$I_{AVE} = \frac{1}{T}\int_0^T i(\omega t)\,dt$$

$$= \frac{1}{\pi}\int_0^\pi I_m \sin \omega t \; d(\omega t)$$

$$= \frac{I_m}{\pi}\left[-\cos \omega t\right]_0^\pi$$

Then

$$I_{AVE} = \frac{I_m}{\pi}\left[-\cos \pi + \cos 0°\right]$$

$$= \frac{I_m}{\pi}\left[1 + 1\right]$$

$$I_{AVE} = \frac{2\,I_m}{\pi}$$

$$= 0.636\; I_m$$

3.2 R.M.S. VALUES

For the Root-Mean-Square value of current, by definition

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T [i(t)]^2 \, dt}$$

This is usually more conveniently expressed as:

$$I_{RMS}^2 = \frac{1}{T}\int_0^T [i(t)]^2 \, dt = \frac{\text{AREA under } i^2 \text{ over period } T}{\text{BASE } T}$$

$$= \text{mean height of } i^2$$

After the integration is performed, the square root can be taken.

Example 3.2.1

Then for the <u>triangular wave form</u>

$$I_{RMS}^2 = \frac{1}{T}\int_0^T \left(\frac{I_m}{T}\right)^2 t^2 \, dt$$

$$\frac{I_m^2}{T^3}\left[\frac{1}{3}t^3\right]_0^T$$

$$I_{RMS}^2 = \frac{I_m^2}{3}$$

$$I_{RMS} = \sqrt{\frac{I_m^2}{3}}$$

$$I_{RMS} = \frac{I_m}{\sqrt{3}}$$

The result in itself is not important; but the method of arriving at a solution is. The R.M.S. current is the equivalent direct current which will give the same heating effect as the complex wave form. Since heating effect is $I_{DC}^2 R$,

that is, proportional to D.C. equivalent current squared, then the average is found of the SQUARE of the current wave form.

Example 3.2.2

For the <u>rectified sine wave</u> (or for an unrectified sine wave, as the result is the same)

$$I_{RMS}^2 = \frac{1}{\pi}\int_0^\pi I_m^2 \sin^2 \omega t \, d(\omega t)$$

Now $\quad \sin^2 \omega t = \frac{1 - \cos 2\omega t}{2}$

Then $\quad I_{RMS}^2 = \frac{I_m^2}{\pi}\int_0^\pi \left(\frac{1 - \cos 2\omega t}{2}\right) d(\omega t)$

$$= \frac{I_m^2}{2\pi}\left[\omega t - \frac{1}{2}\sin 2\omega t\right]_0^\pi$$

$$= \frac{I_m^2}{2\pi}\left[(\pi - 0) - (0 - 0)\right]$$

$$I_{RMS}^2 = \frac{I_m^2}{2}$$

$$I_{RMS} = \frac{I_m}{\sqrt{2}}$$

or the well-known result

$$I_{RMS} = 0.707\, I_m$$

3.3 R.M.S. VALUE OF A COMPLEX WAVEFORM

It can be show that a periodic waveform of complex shape may be expressed as a series of sine waves and cosine waves plus a possible d.c. component as indicated below:

Complex waveform Period $T = \frac{2\pi}{\omega}$ which is the period of the fundamental (first) harmonic.

$$i(t) = I_o + A_1 \cos \omega t + A_2\ 2\ \omega t + A_3 \cos 3\ \omega t$$
$$+ A_4 \cos 4\ \omega t + \ldots\ldots$$
$$+ B_1 \sin \omega t + B_2 \sin 2\ \omega t + B_3 \sin 3\ \omega t$$
$$+ B_4 \sin 4\ \omega t + \ldots\ldots$$

The R.M.S. value of this series is given by:

$$I_{RMS} = \sqrt{I_o^2 + \frac{1}{2}(A_1^2 + A_2^2 + A_3^2 + A_4^2 + \ldots.) + \frac{1}{2}(B_1^2 + B_2^2 + B_3^2 + B_4^2 + \ldots.)}$$

$$= \sqrt{(I^2_{DC} + A_1{}^2_{RMS} + A_2{}^2_{RMS} + \ldots + B_1{}^2_{RMS} + B_2{}^2_{RMS} + \ldots)}$$

$$= \sqrt{\text{Sum of (RMS)}^2 \text{ of Component Waveforms}}$$

Example 3.3.1

Determine the R.M.S. value for the periodic waveform given by the following equation:

$$i(t) = 10 + 15 \sin 100t + 5 \sin 200t \quad \text{amperes}$$

Solution:

$$I_{RMS} = \sqrt{10^2 + \frac{1}{2}(15^2 + 5^2)}$$

$$= \boxed{15 \text{ amperes}}$$

3.4 FORM FACTOR

The form factor of a waveform is the ratio of the effective (R.M.S.) value to the average value.

$$\text{FORM FACTOR} = \frac{\text{R.M.S. value}}{\text{Average value}}$$

$$= \frac{I_{RMS}}{I_{AVE}}$$

This factor gives some indication of the shape of a waveform and is of some use in industrial processes.

Example of form factors are given below:

Waveshape	Form Factor
Square wave (rectified)	1
Sine wave (rectified)	1.11
Triangular wave	1.15

Example 3.4.1

A voltage waveform is represented by the equation:

$$v(t) = 10 - 10 \sin \omega t \text{ volts}$$
$$= \text{D.C. component + a.c. component}$$

Determine the following:

(a) the average value;

(b) the R.M.S. vlaue

(i) by use of integration;
(ii) by use of formula.

(c) the form factor

Solution:

(a) $V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt$ (where T is the period of the wave $\omega T = 2\pi$)

$= \frac{1}{T}\int_0^T (10 - 10\sin\omega t)\,dt$

$= \frac{1}{T}\left[10t + \frac{10}{\omega}\cos\omega t\right]_0^T$

$= \frac{1}{T}\left[\left(10T + \frac{10}{\omega}\cos\omega T\right) - \left(0 + \frac{10}{\omega}\cos 0\right)\right]$

$= \frac{1}{T}\left[10T + \frac{10}{\omega}\cos 2\pi - \frac{10}{\omega}\cos 0\right]$

$=$ 10 volts = D.C. component of $v(t)$

Alternatively V_{AVE} = SUM OF AVE VALUES OF COMPONENT WAVEFORMS

Alternatively we have

$V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$ (where $\theta = \omega t$)

$= \frac{1}{2\pi}\int_0^{2\pi} (10 - 10\sin\theta)\,d\theta$

$= \frac{1}{2\pi}\left[10\,\theta + 10\cos\theta\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[(20\pi + 10\cos 2\pi) - (0 + 10\cos 0)\right]$

V_{AVE} = 10 volts

(b) (i) $V_{RMS} = \sqrt{\frac{1}{T}\int_0^T [\,v(t)\,]^2\,dt}$

Alternatively

$V^2_{RMS} = \frac{1}{2\pi}\int_0^{2\pi} [\,v(\theta)\,]^2\,d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} [\,10(1 - \sin\theta)\,]^2\,d\theta$

$= \frac{100}{2\pi}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\,d\theta$

$= \frac{50}{\pi}\int_0^{2\pi} \left[1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$

$= \frac{50}{\pi}\left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= \frac{50}{\pi}\left[\left(\frac{3}{2} \times 2\pi + 2 - 0\right) - (0 + 2 - 0)\right]$

$= 150$

V_{RMS} = 12.25 volts

(ii) $V_{RMS} = \sqrt{\text{SUM OF (RMS)}^2 \text{ OF COMPONENT WAVEFORMS}}$

$= \sqrt{10^2 + \left(\frac{10}{\sqrt{2}}\right)^2}$

$= \sqrt{150}$

V_{RMS} = 12.25 volts

(c) FORM FACTOR $= \frac{V_{RMS}}{V_{AVE}}$

$$= \frac{12.25}{10}$$

$$= \underline{1.225}$$

CIRCUIT TRANSIENTS

When a circuit is switched from one condition to another, the period when the currents and voltages are changing from one steady state condition to another steady state condition is referred to as the transient.

In this section we shall consider the circuit transients associated with series circuits containing resistance, inductance and capacitance. The linear differential equation with constant coefficients that describes the reaction to a circuit change has a two part general solution, the sum of the complimentary function and the particular function.

$$i = i_c + i_p$$

where $= \text{C.F.} + \text{P.I.}$

i_c = complimentary function (i.e. the transient) contains the arbitrary constants

i_p = particular integral function (i.e. the steady state component) or 'final' current

5.1 RESPONSE OF RL AND RC CIRCUITS TO DC VOLTAGES

5.1.1 RL CIRCUIT

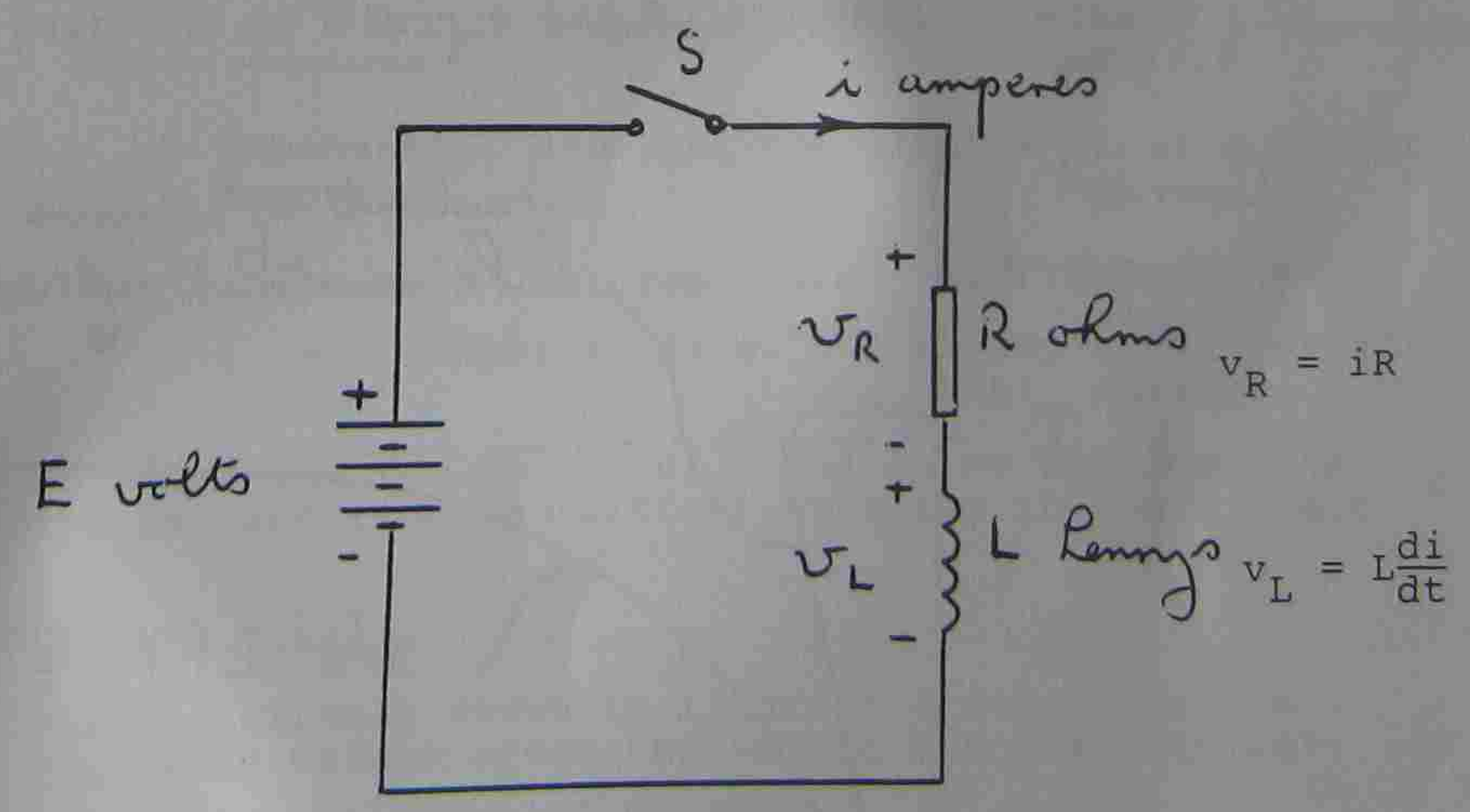


Figure 5.1

After closing switch S in figure 5.1 we have the following:

$$v_R + v_L = E$$

or $\boxed{iR + L\frac{di}{dt} = E}$

Initial condition
$v_L = E \quad v_R = 0$
$i = 0$ when $t = 0$
defines the arbitrary constant.

The solution to this first order differential equation is

$$\boxed{i = I(1 - e^{-\frac{R}{L}t})} \text{ amperes}$$

$iR + L\frac{di}{di} = E$

$i = I(1 - e^{-R/L\, t})$

where i = instantaneous value of current
I = final value of current = i_p
t = time after closing switch S.

$$i_c = -Ie^{-\frac{R}{L}t}$$

The solution of the differential equation is detailed in example 5.1.3.2

Time Constant

The time constant of a circuit gives an indication of the rate of response of the circuit to changing conditions.

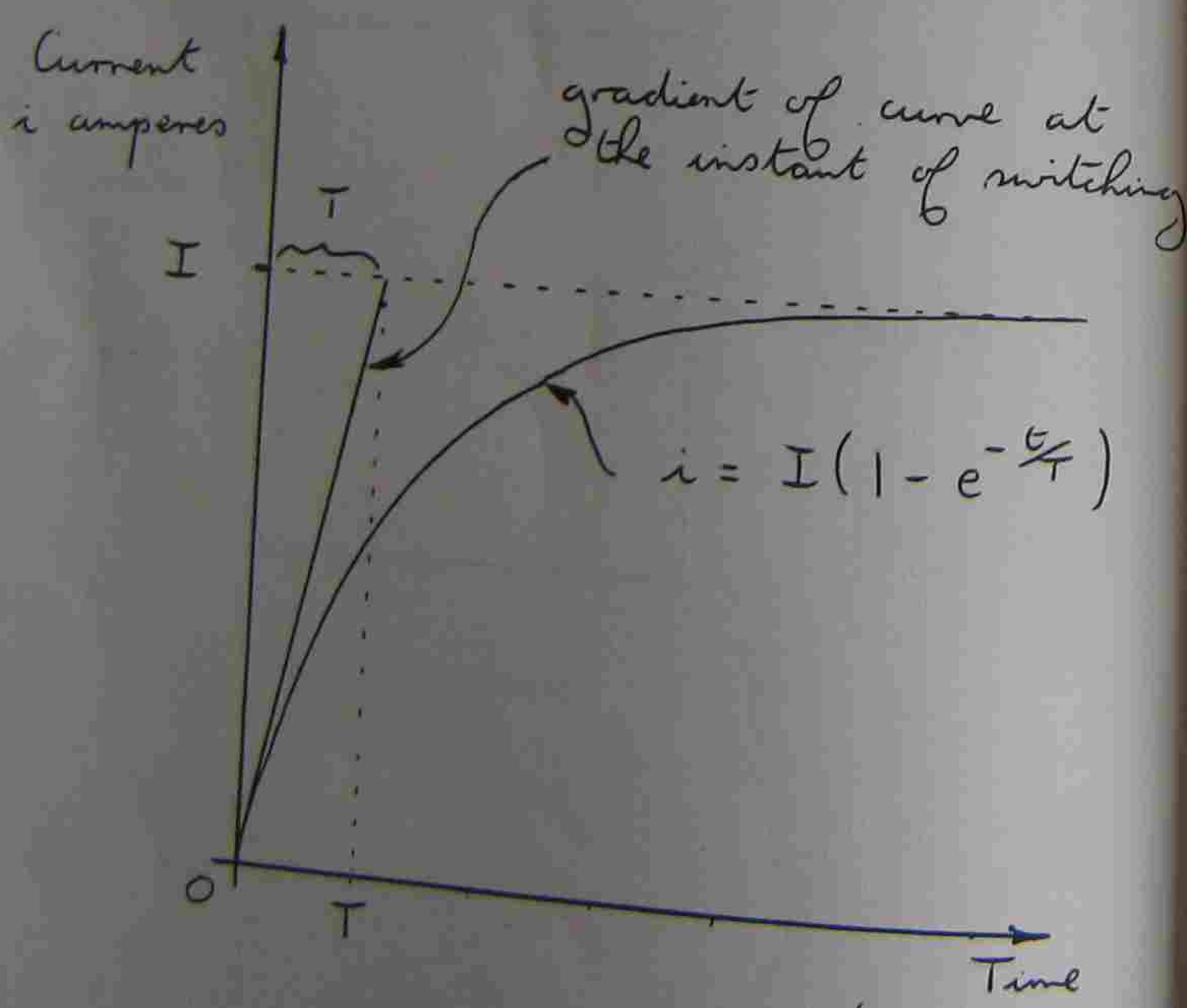


Figure 5.2

Time constant = T = measure of the initial rate of change of current.

For a circuit containing resistance and inductance the time constant is given by:-

$$\boxed{T = \frac{L}{R}} \text{ seconds}$$

Since $\frac{I}{T} = \frac{di}{dt}$ initially

$$= \frac{v_L}{L} = \frac{E}{L} = \frac{IR}{L}$$

thus $T = \frac{L}{R}$

The General Solution

The equation of the current in a circuit will depend upon the switching arrangements and will be of the following form:-

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p}$$ amperes. A, is the arbitrary constant

= Exponential decaying component + final constant component

where $i_c = Ae^{-\frac{t}{T}}$ amperes (i.e. the transient)

$i_p = I_p$ amperes (i.e. the steady state current or final current)

To determine the values of A and I_p we consider the initial and final conditions existing in the circuit.

Example 5.1.1.1

For the circuit shown in figure 5.3 determine the following values after the switch has been closed:-

(a) the final value of current;
(b) the initial value of current;
(c) the time constant of the circuit;
(d) the equation of the current;
(e) the initial rate of change of current.

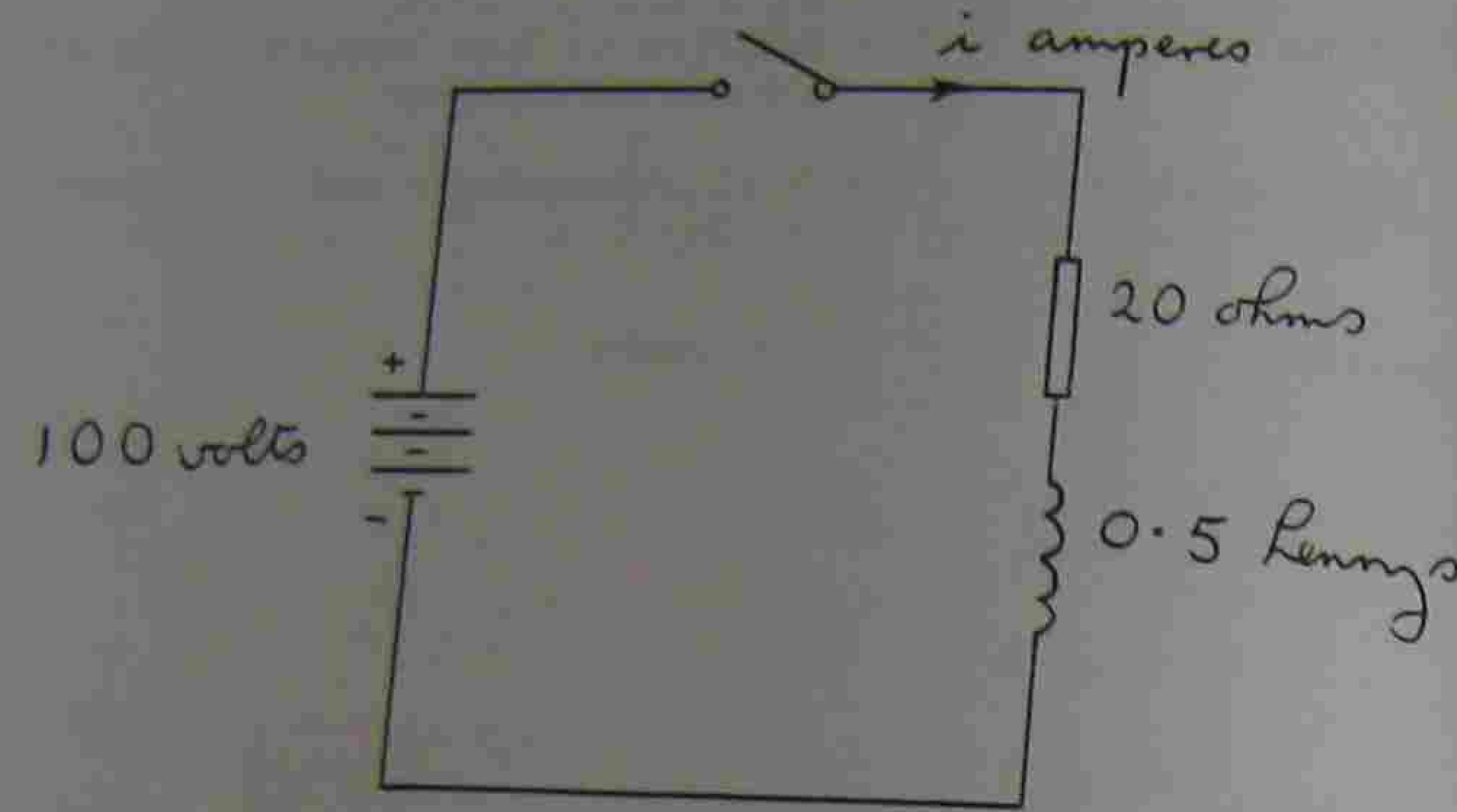


Figure 5.3

Solution:

(a) Final value of current $I = \frac{E}{R} = \frac{100V}{20\Omega} = \underline{5A}$

(b) Initial value of current = $\underline{0A}$

(c) Time constant = $\frac{L}{R} = \frac{0.5H}{20\Omega} = \underline{25ms}$

(d) $i = I(1 - e^{-\frac{t}{T}})$ amperes

$\therefore \underline{i = 5(1 - e^{-40t})}$ amperes.

(e) Rate of change of current = $\frac{di}{dt}$

$\frac{di}{dt} = -5 \times (-40)\ e^{-40t}$

$= 200e^{-40t}$ A/s

Initial rate of change of current

$= \underline{200\ A/s} = \frac{100V}{0.5H}$

$iR + L\frac{di}{dt} = E$

$t = 0\ i = 0 \therefore L\frac{di}{dt} = E$

$\frac{di}{dt} = \frac{E}{L} = \frac{100}{0.5H}$

Example 5.1.1.2

(a) Determine the equation of the current in figure 5.4 after switching to position ②. Assume that the steady state current had been attained in position ①.

(b) Sketch the current on suitable axes.

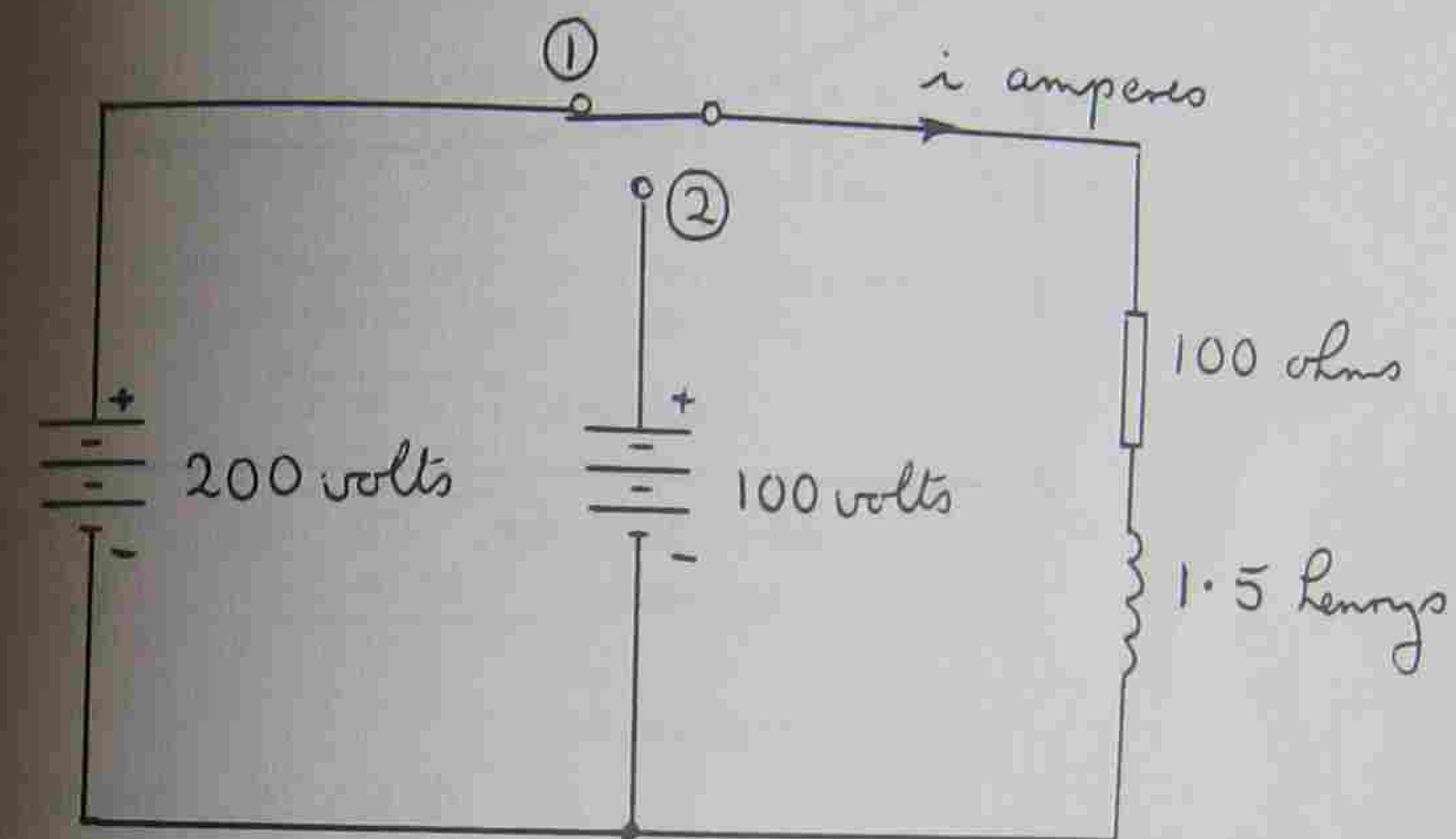


Figure 5.4

Solution

(a) In general:

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p} \text{ amperes.}$$

$T = \frac{L}{R} = \frac{1.5H}{100\Omega} = 15ms$

$i = Ae^{-66.67t} + I_p$ amperes

Initial current $= \frac{200V}{100\Omega} = 2$ amperes

Final current $= \frac{100V}{100\Omega} = 1$ ampere $= I_p$

i.e. $2 = A + I_p$

$\therefore A = I_p = 1$ ampere

$\underline{i = 1 + e^{-66.67t}}$ amperes

(b)

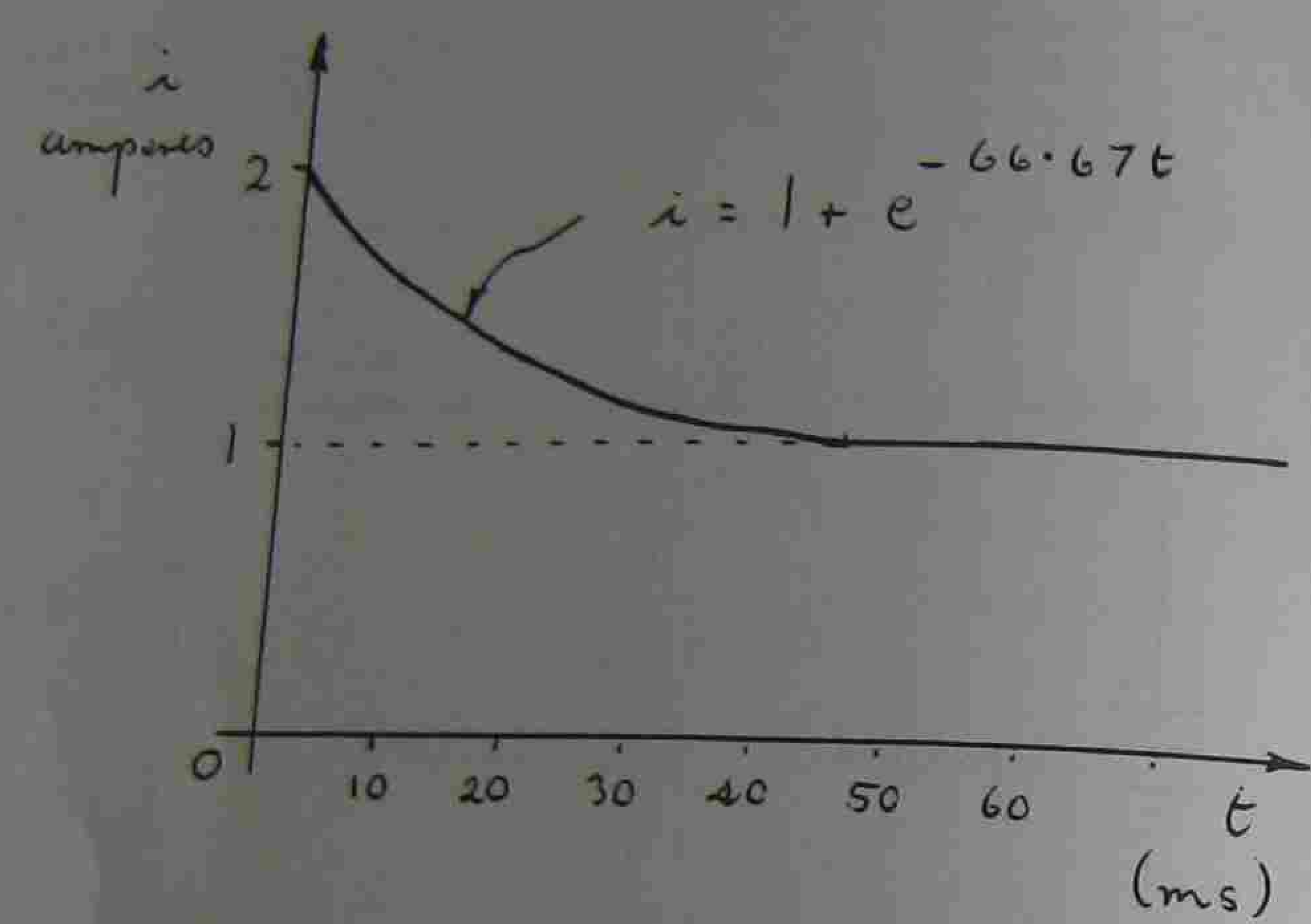
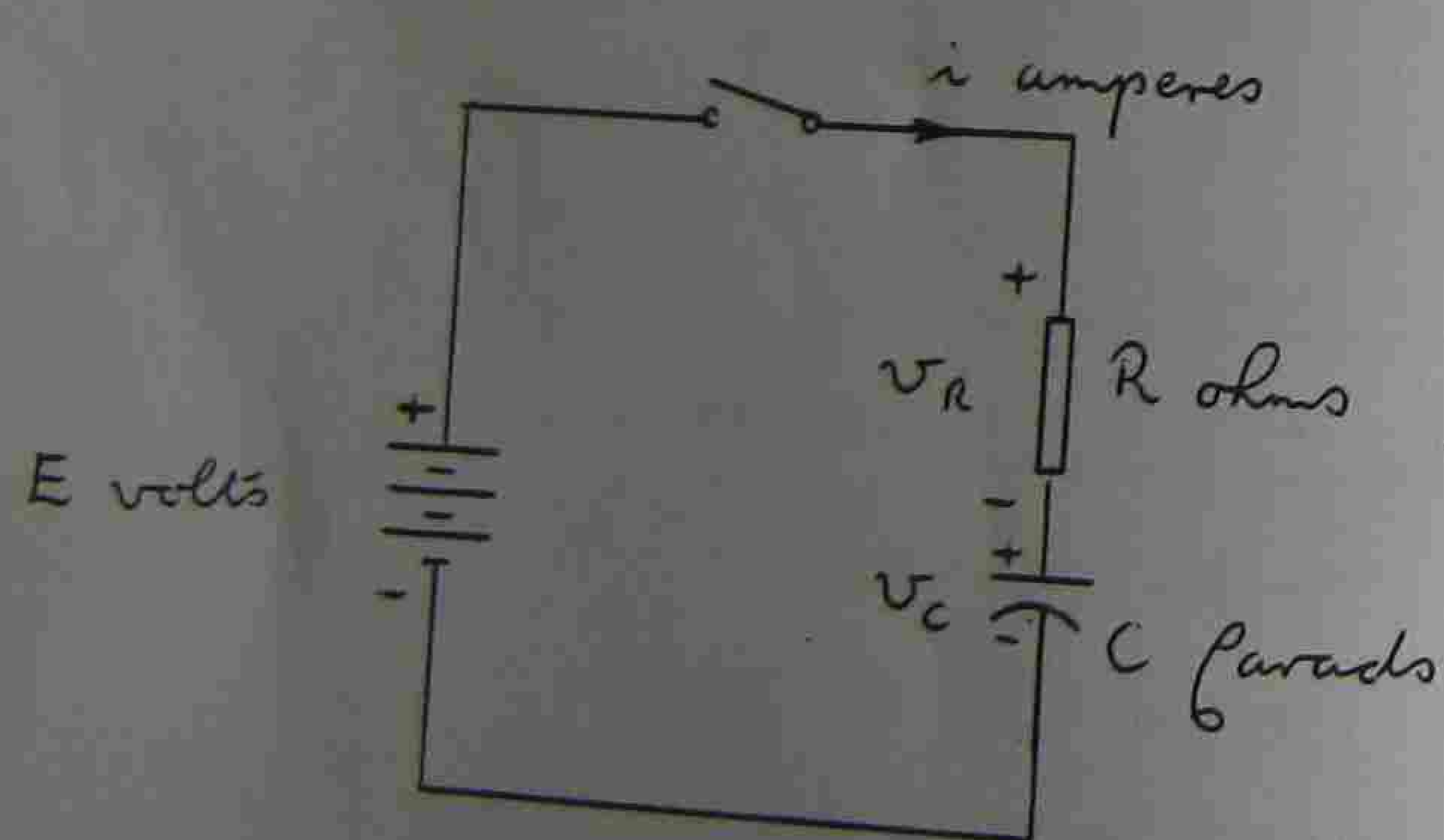


Figure 5.5

5.1.2 RC CIRCUIT



$$v_R = iR$$
$$v_c = \frac{q}{C}$$
$$v_c = \frac{1}{C}\int idt$$

Figure 5.6

After closing switch S in figure 5.6 we have the following:

$$v_R + v_c = E$$

or $iR + \frac{1}{C}\int idt = E$

Initial condition
$v_c = 0 \quad v_R = E$

$$i = \frac{E}{R} = I \text{ when } t = 0$$

Differentiate with respect to time.

$$R\frac{di}{dt} + \frac{1}{C}i = 0$$

The solution to this first order differential equation is

$$\boxed{i = Ie^{-\frac{t}{RC}}} \quad \text{amperes}$$

where
i = instantaneous value of current
I = initial value of current
t = time after closing switch S.

The time constant in a circuit containing resistance and capacitance is given by:

$$\boxed{T = RC} \quad \text{amperes.}$$

$$\boxed{i = Ie^{-\frac{t}{T}}} \quad \text{amperes}$$

General Solution

In the RC circuit there is no steady state current and therefore the particular function is zero.

i.e. $\boxed{i_p = 0}$

and $\boxed{i = i_c = A\ e^{-\frac{t}{T}}}$ amperes.

The value of A_1 is determined by the initial conditions.

Example 5.1.2.1

For the circuit shown in fig. 5.6 E is 200 volts, R is 1000 ohms and C is 100 microfarads. If the initial charge on the capacitor is 2 millicoulombs, determine the following after the switch S is closed:-

(a) the initial current;
(b) the time constant of the circuit;
(c) the equation of the current;
(d) the equation of the voltage across the capacitor;
(e) the time required after closing the switch for the capacitor voltage to be 100 volts;
(f) sketch the voltage across the capacitor.

Solution:

(a) Initial current $= \frac{E - v_c}{R}$

$$= \frac{200 - \frac{2 \times 10^{-3}}{100 \times 10^{-6}}}{1000} = \frac{(200 - 20)V}{1000\Omega}$$

A = 0.18 amperes

(b) Time constant $= T = RC$

$= 0.1$ seconds

(c) $i = Ae^{-\frac{t}{T}}$

$$\boxed{i = 0.18e^{-10t}} \text{ amperes.}$$

(d) Voltage across capacitor $= v_c = E - v_R$

$= E - iR$

$$v_c = 200 - 0.18e^{-10t} \times 1000$$

$$\boxed{v_c = 200 - 180e^{-10t}} \text{ volts}$$

(e) When $v_c = 100$

$$100 = 200 - 180e^{-10t}$$

$$180e^{-10t} = 200 - 100$$

$$e^{-10t} = \frac{100}{180}$$

$$-10t = \ln\left(\frac{100}{180}\right) \text{ seconds}$$

$t = 58.8$ ms

(f)

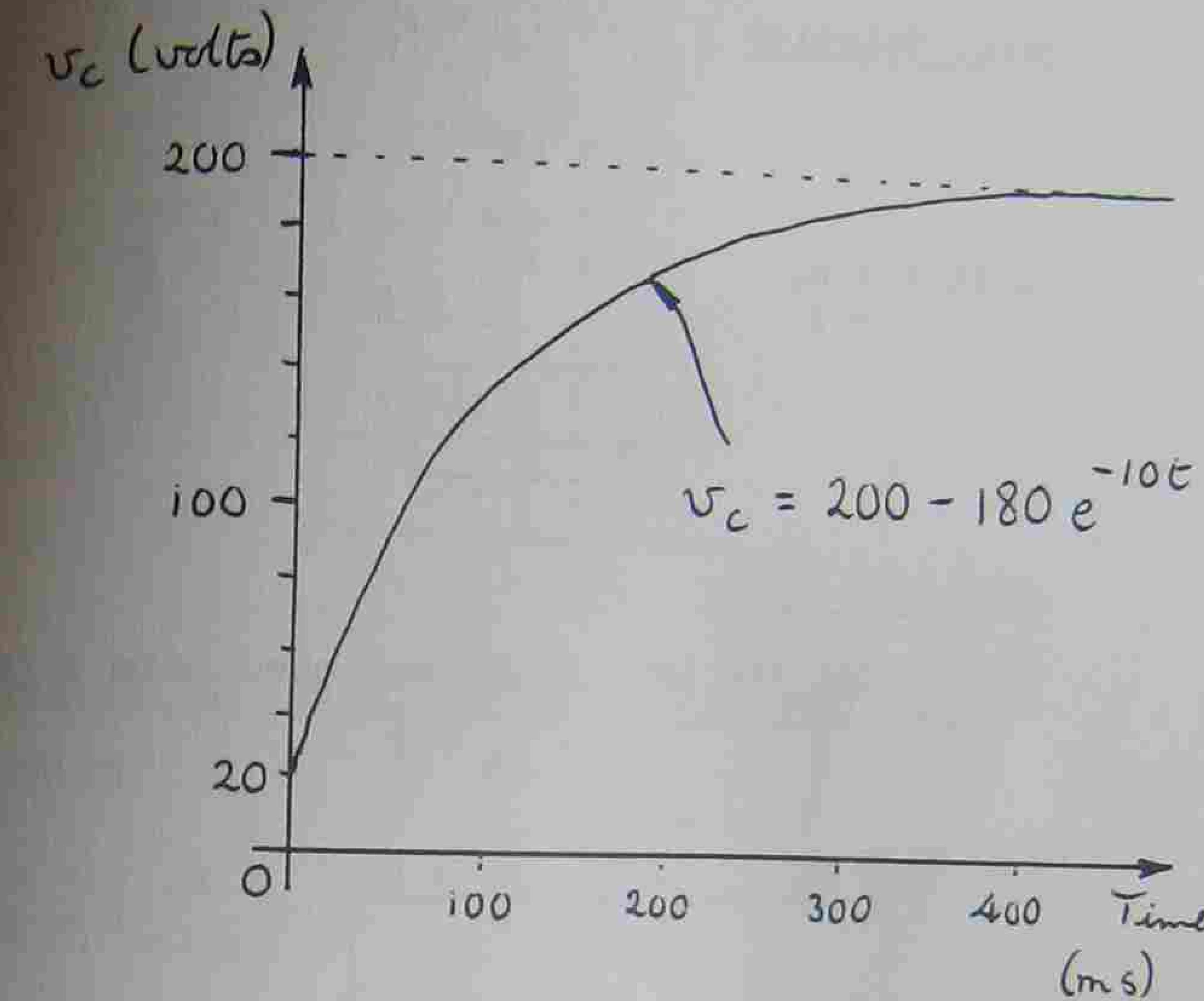


Figure 5.7

5.1.3 SOLVING SIMPLE FIRST ORDER DIFFERENTIAL EQUATIONS BY THE CF + PI METHOD

Example 5.1.3.1

Reconsider the CR CIRCUIT switched onto a DC SUPPLY

$$v_R + v_c = E$$

$$iR + \frac{1}{C}\int i\,dt = E$$

Solving DE $\boxed{R\frac{di}{dt} + \frac{1}{C}i = 0}$ At $t = 0$, $i = \frac{E}{R} = I$

PI $\quad i_p =$ final current $I_p = 0$

CF Subst. $i = Ae^{mt}$ A is the arbitrary constant.

$$0 = R\frac{di}{dt} + \frac{1}{C}i$$

$$= (Rm + \frac{1}{C})\, Ae^{mt}$$

$$0 = Rm + \frac{1}{C}$$

$$m = -\frac{1}{CR}$$

$$i_c = Ae^{-\frac{1}{CR}t} = Ae^{-\frac{t}{T}}$$

Total Solution $\quad i = i_c + i_p$

$$= Ae^{-\frac{t}{T}}$$

Subst $t = o$, $i = I$ $\quad A = I$

$$\boxed{i = Ie^{-\frac{t}{T}}}$$

Example 5.1.3.2

Reconsider the LR CIRCUIT switched onto a DC SUPPLY.

$$v_L + v_R = E$$

Solving DE $\quad L\frac{di}{dt} + Ri = E \quad$ At $t = o$, $i = o$

CF

Subst. $i = Ae^{mt}$ into DE with RHS = o
A is the arbitrary constant.

$$o = L\frac{di}{dt} + Ri$$

$$= (Lm + R)\,Ae^{mt}$$

$$o = Lm + R$$

$$m = -\frac{R}{L}$$

$$i_c = Ae^{-\frac{R}{L}t} = Ae^{-\frac{t}{T}}$$

PI

Since the RHS of the DE is a constant voltage E, the forcing function, then assume a solution of the form.

i_p = final current Ip

then $\quad \frac{di}{dt} = o$

Subst. into DE $\quad L(o) + R(I_p) = E$

$$RI_p = E$$

$$i_p = I_p = \frac{E}{R} = I$$

Total Solution

$$i = i_c + i_p$$

$$= Ae^{-\frac{t}{T}} + I$$

Subst. $t = o$, $i = o$. $\quad o = A + I$

$$A = -I$$

$$i = -Ie^{-\frac{t}{T}} + I$$

$$i = I(1 - e^{-\frac{t}{T}})$$

5.2 RESPONSE OF RL AND RC CIRCUITS TO AC VOLTAGES

5.2.1 RL CIRCUIT

$e = E_m \sin(wt + \Theta)$ volts

$v_R = iR$

$v_L = L\frac{di}{dt}$

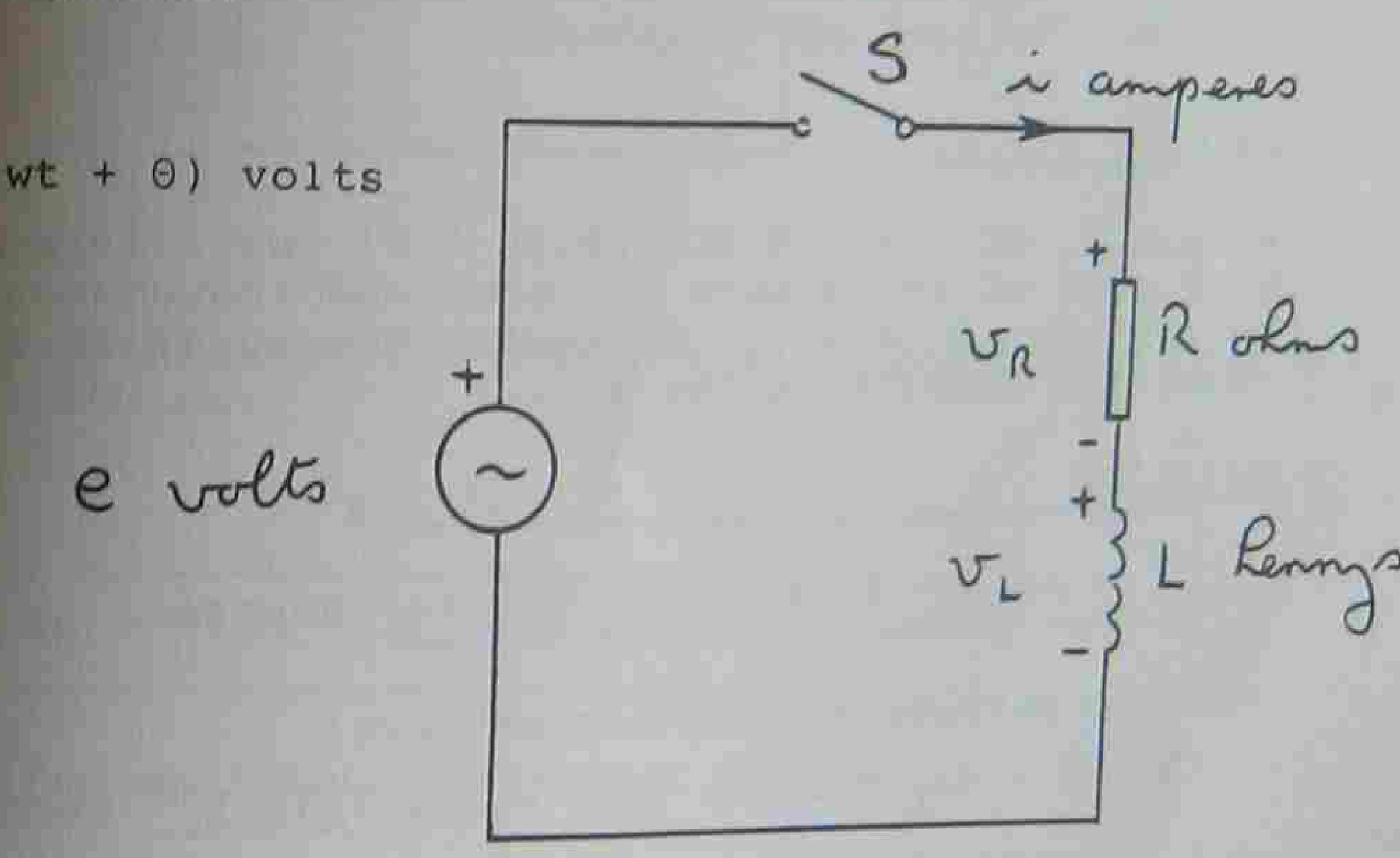


Figure 5.8

After closing the switch S in figure 5.8, we have the following:

$$v_R + v_L = e$$

$$\boxed{iR + L\frac{di}{dt} = E_m \sin(wt + \Theta)}$$

The solution to this first order differential equation has a complimentary function and a particular function.

$$i = i_c + i_p$$

where $i_c = Ae^{-\frac{t}{T}}$ amperes

$i_p = I_p \sin(wt + \theta - \varphi)$ amperes.

The constant A is determined by the conditions that apply at the time of switching and the particular function (i_p) is the steady state current in the circuit.

A General Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes}$$

= exponential decaying component + final sinusoidal component

Example 5.2.1.1

An e.m.f. of e = 100 sin (314t + θ) volts is applied to a coil of resistance 200 ohms and inductance 0.5 henrys when θ is 30°. Determine the equation of the resulting current in the coil.

Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes}$$

For the particular function we have:

$$\bar{Z} = R + jX_L = 200 + j157 = 254.3\underline{/38.13°} \text{ ohms}$$

$$\bar{E}_m = 100\underline{/30°} \text{ volts} \quad \text{note } \tan\varphi = \frac{X}{R}$$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{100\underline{/30°}}{254.3\underline{/38.13°}} = 0.3932\underline{/-8.13°} \text{ amperes.}$$

$$\therefore i_p = 0.3932 \sin(314t - 8.13°) \text{ amperes.}$$

Also, $T = \frac{L}{R} = \frac{0.5}{200}$ s = 2.5 ms

$$\therefore i_c = Ae^{-400t}$$

At t = 0; i = 0

$$\therefore 0 = A + 0.3932 \sin(-8.13°)$$

$$A = 0.0556$$

$$i = 0.0556e^{-400t} + 0.3932 \sin(314t - 8.13°) \text{ amperes.}$$

5.2.2 RC CIRCUIT

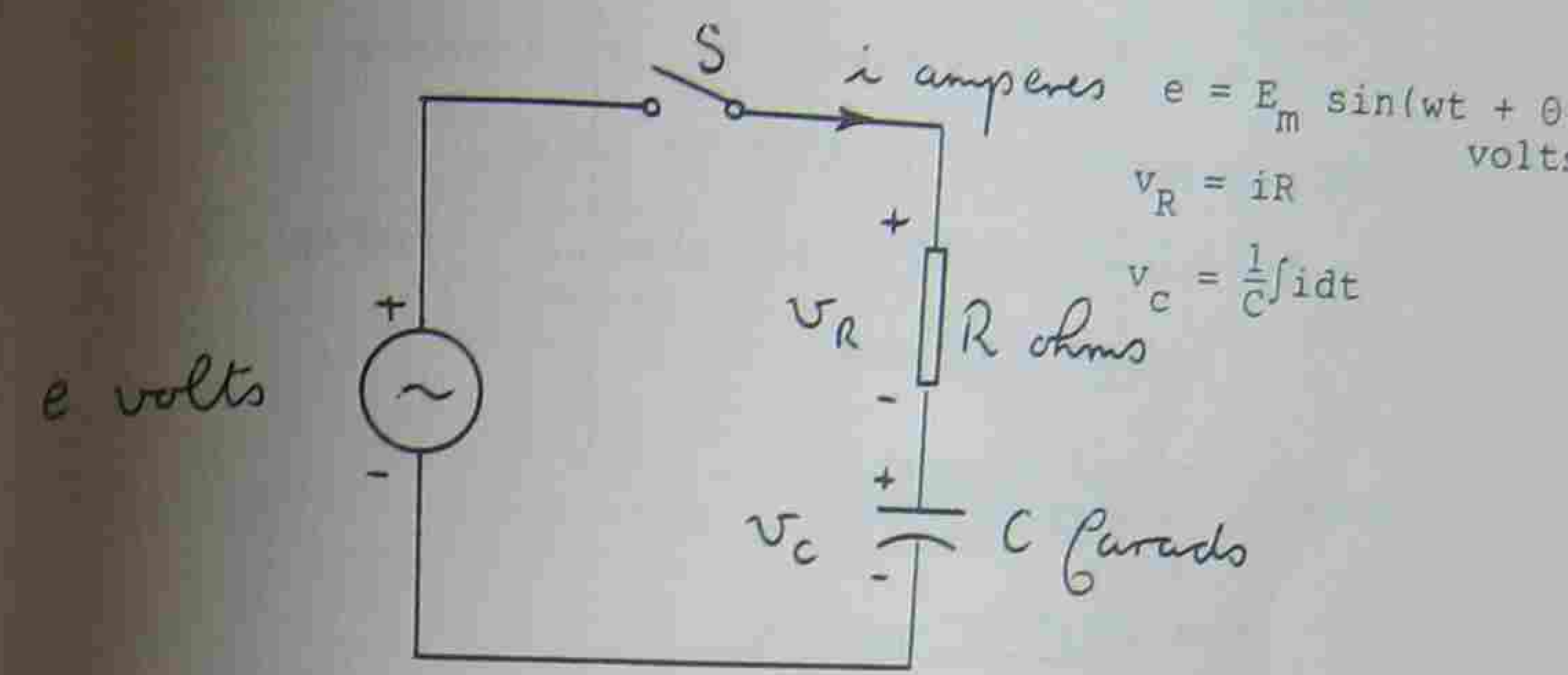


Figure 5.9

After closing the switch S in figure 5.9 we have the following:

$$v_R + v_c = e$$

$$\boxed{iR + \frac{1}{C}\int i\,dt = E_m \sin(wt + \theta)}$$

Differentiate with respect to time.

$$R\frac{di}{dt} + \frac{1}{C}i = E_m w \cos(wt + \theta)$$

The solution to this first order differential equation is the same as for the RL circuit.

$$i = i_c + i_p$$

where $i_c = Ae^{-\frac{t}{T}}$ amperes

$i_p = I_p \sin(wt + \theta - \varphi)$ amperes.

The General Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes.}$$

Maung Thura Tin Soe

Example 5.2.2.1

An e.m.f. of $e = 200 \sin(500t + \theta)$ volts is applied to an RC circuit where R = 300 ohms and C = 5 microfarads when θ is 60°. Determine the equation of the current in the circuit if the initial charge on the capacitor is 250 microcoulombs.

Solution:

$$i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi) \text{ amperes.}$$

For the particular function we have:

$$\bar{Z} = R - jX_C = 300 - j400 \text{ ohms} \qquad \tan\varphi = \frac{X}{R}$$

$$= 500\angle{-53.13°} \text{ ohms.}$$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{200\angle 60°}{500\angle{-53.13°}}$$

$$= 0.4\angle 113.13° \text{ amperes.}$$

Also $T = RC = 1.5 \times 10^{-3}$ s = 1.5 ms

$$\therefore \quad i = Ae^{-666.7t} + 0.4 \sin(500t + 113.13°)$$

Initial voltage across capacitor $= \frac{q}{C} = 50$ volts.

$$\text{Initial current} = \frac{E_m \sin 60° - 50}{R}$$

$$= 0.4107 \text{ amperes.}$$

$\therefore$ At t = 0; $0.4107 = A + 0.4 \sin(113.13°)$

$$\therefore \quad A = 0.0433$$

$$\boxed{i = 0.0433e^{-666.7t} + 0.4 \sin(500t + 113.13°)} \text{ amperes.}$$

5.3 TRANSIENTS IN SERIES RLC CIRCUITS - DC VOLTAGE

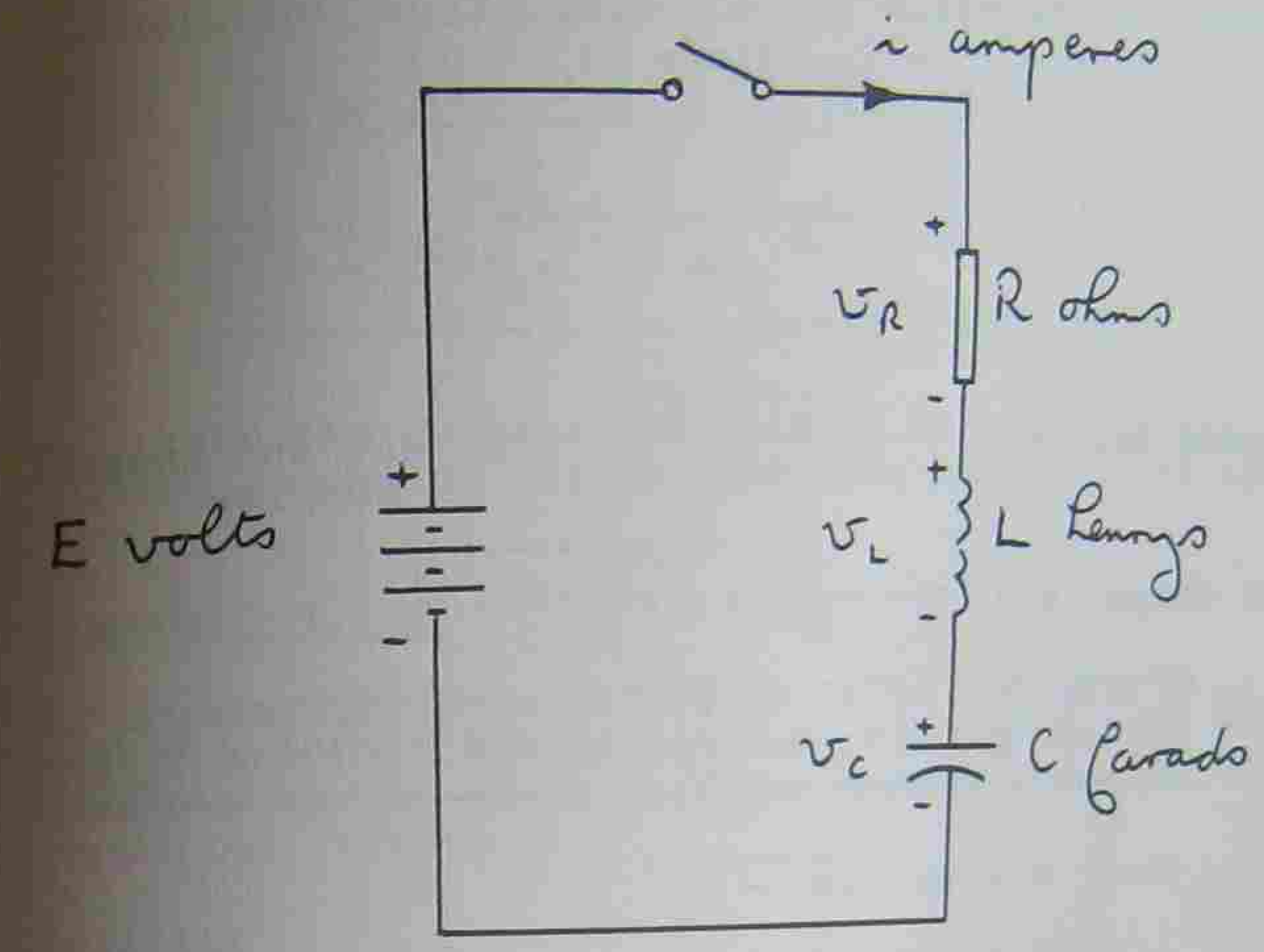


Figure 5.10

In figure 5.10 by applying Kirchhoff's Voltage Law after the switch is closed, we have:

$$v_R + v_L + v_C = E$$

$$v_R = Ri \ ; \quad v_L = L\frac{di}{dt} \ ; \quad v_C = \frac{q}{C} = \frac{1}{C}\int i\,dt$$

$$\therefore \quad L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E$$

Differentiate with respect to time.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\, i = 0$$

Divide throughout by L.

$$\boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}\, i = 0}$$

Compare form

$$\boxed{a\frac{d^2i}{dt^2} + b\frac{di}{dt} + ci = 0}$$

This is a second order differential equation and there are three possible solutions:

$$i = i_c + i_p \quad \text{since} \quad \boxed{i_p = 0}$$

$$i = i_c \quad \text{subst.} \quad i_c = Ae^{mt}$$

$$am^2 + bm + c = 0$$

Roots for $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real and unequal, equal, or complex.

By comparison $m = -\dfrac{R}{2L} \pm \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$

Let $\boxed{\alpha = \dfrac{R}{2L}}$; $\boxed{w_o = \dfrac{1}{\sqrt{LC}}}$; $\boxed{\beta = \sqrt{|\alpha^2 - w_o^2|}}$

then $m = -\alpha \pm \sqrt{\alpha^2 - w_o^2}$

$m = (-\alpha \pm \beta)$ or $(-\alpha \pm j\beta)$

Condition		General Solutions
1) Overdamped	$\alpha > w_o$	$i = e^{-\alpha t}(A_1 e^{\beta t} + A_2 e^{-\beta t})$
contains 2 arbitrary constants A_1 and A_2.		
2) Critically damped	$\alpha = w_o$	$i = e^{-\alpha t}(A_1 + A_2 t)$
3) Underdamped	$\alpha < w_o$	$i = e^{-\alpha t}(A_1 \sin\beta t + A_2 \cos\beta t)$
(w_o is the natural frequency)		

The current in the circuit may take the shape of the curves shown in figure 5.11.

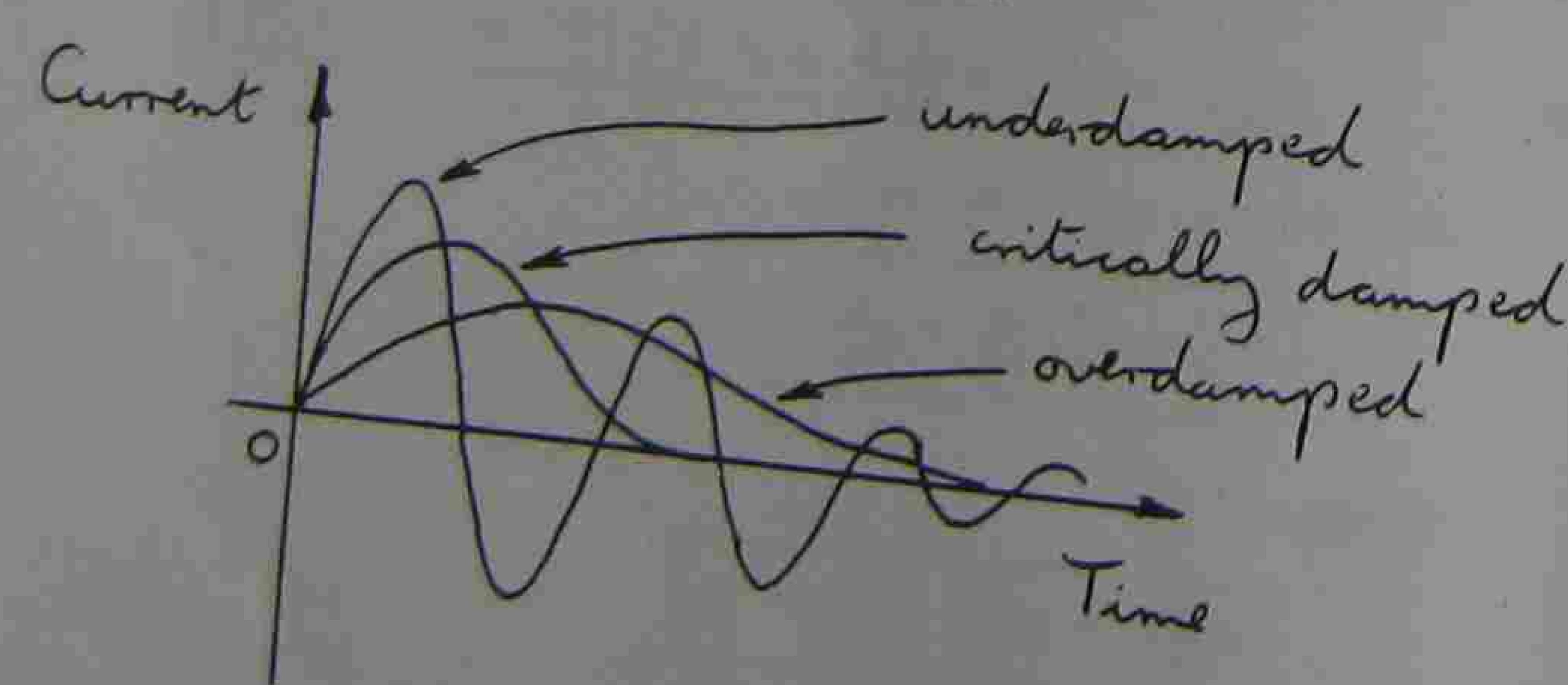


Figure 5.11

Case 1 - Overdamped ($\alpha > w_o$)

The solution to the second order differential equation is

$$i = A_1 e^{(-\alpha + \beta)t} + A_2 e^{(-\alpha - \beta)t}$$

$$\boxed{i = e^{-\alpha t}(A_1 e^{\beta t} + A_2 e^{-\beta t})}$$

To obtain the constants A_1 and A_2 we consider the initial values of current and rate of change of current.

Example 5.3.1

In figure 5.10 E is 100 volts, R is 500 ohms, L is 0.25 henrys and C is 100 microfarads. Determine the equation of the current if the initial charge on the capacitor is zero.

Solution

$$\alpha = \frac{R}{2L} = \frac{500}{2 \times 0.25} = 1000$$

$$w_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25 \times 100 \times 10^{-6}}} = 200$$

$$\beta = \sqrt{|\alpha^2 - w_o^2|} = 979.8$$

$\alpha > w_o$ therefore overdamped.

$$i = A_1 e^{(-\alpha + \beta)t} + A_2 e^{(-\alpha - \beta)t}$$

$$= A_1 e^{-20.2t} + A_2 e^{-1979.8t} \text{ amperes.}$$

At $t = 0$; $i = 0$ and $L\dfrac{di}{dt} = E = 100$ volts.

$\therefore \frac{di}{dt} = \frac{E}{L} = \frac{100}{0.25}$

$$\therefore A_1 + A_2 = 0$$

$$A_1 = -A_2$$

Hence $i = -A_2 e^{-20.2t} + A_2 e^{-1979.8t}$

$$\frac{di}{dt} = 20.2A_2 e^{-20.2t} - 1979.8A_2 e^{-1979.8t}$$

At t = 0 ; $\frac{di}{dt} = 20.2A_2 - 1979.8A_2 = \frac{100}{0.25}$

$A_2 = -0.2041$

$$\boxed{i = 0.2041e^{-20.2t} - 0.2041e^{-1980t}} \text{ amperes}$$

Case 2 - Critically Damped ($\alpha = w_o$)

The solution to the second order differential equation is

$$\boxed{i = e^{-\alpha t}(A_1 + A_2 t)}$$

The constants A_1 and A_2 are obtained by consideration of the initial conditions.

Example 5.3.2

In figure 5.10 E is 200 volts, R is 100 ohms, L is 0.5 henrys and C is 200 microfarads. Determine the equation of the current if the initial charge on the capacitor is 20 millicoulombs.

$$\alpha = \frac{R}{2L} = \frac{100}{1} = 100$$

$$w_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 200 \times 10^{-6}}} = 100$$

$\alpha = w_o$ therefore critically damped.

$$i = e^{-100t}(A_1 + A_2 t) \text{ amperes.}$$

At t = 0 ; i = 0 ; $v_c = \frac{q}{C} = \frac{20mC}{200\mu F} = 100$ volts

$L\frac{di}{dt} = E - v_c = 200 - 100 = 100$ volts

$$1(A_1 + A_2 \times 0) = 0$$

$$\therefore A_1 = 0$$

$$i = A_2 t\, e^{-100t} \text{ amperes.}$$

$$\frac{di}{dt} = A_2 e^{-100t} - 100\, A_2 t e^{-100t}$$

At t = 0 ; $\frac{di}{dt} = A_2 = \frac{100}{0.5} = 200$

$$\therefore \boxed{i = 200te^{-100t}} \text{ amperes.}$$

Case 3 - Underdamped ($\alpha < w_o$)

The solution to the second order differential equation is

$$\boxed{i = e^{-\alpha t}(A_1 \sin\beta t + A_2 \cos\beta t)}$$

Alternatively: $i = Ae^{-\alpha t}\sin(\beta t + \gamma)$

The constants A_1 and A_2 or A and γ are obtained by consideration of the initial conditions as shown in the overdamped and critically damped cases.

5.4 RESPONSE OF SERIES RLC CIRCUITS TO AC WAVEFORMS.

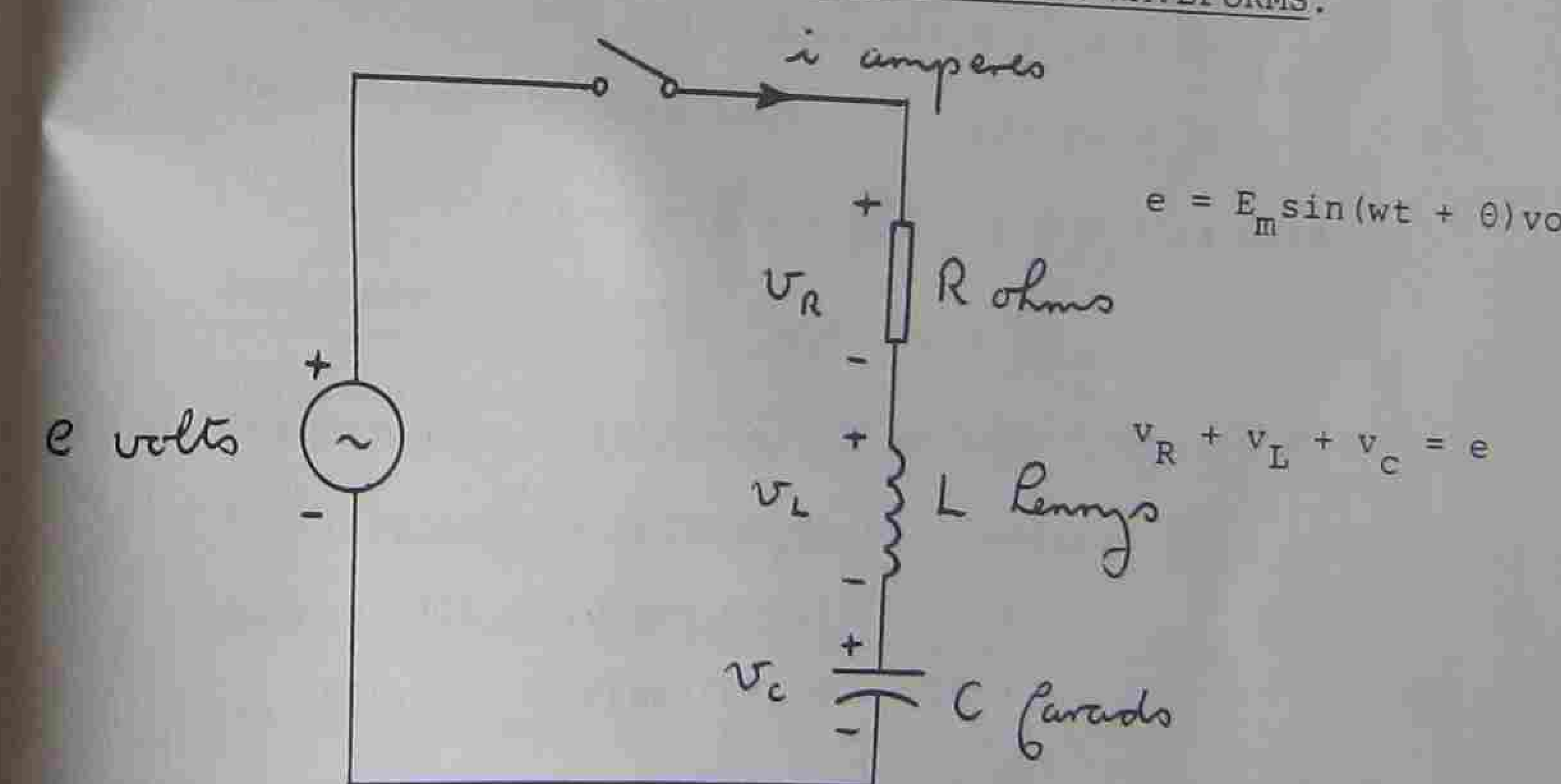


Figure 5.12

When a sinusoidal voltage is applied to a series RLC circuit as shown above, the resulting current in the circuit consists of a transient current and a current that exists after the transient has diminished to zero.

$$i_c = e^{-100t}(A_1\sin 100t + A_2\cos 100t) \text{ amperes.}$$

For the particular function:

$$\bar{Z} = R + j(X_L - X_c) = 100 + j(250 - 20)\ \Omega$$
$$= 100 + j230\ \Omega$$
$$= 250.8\underline{/1.161}\ \Omega$$

$$\bar{I}_p = \frac{100\underline{/1.571}}{250.8\underline{/1.161}} = 0.3987\underline{/0.4103} \text{ amperes.}$$

$$i_p = 0.3987\sin(500t + 0.4103) \text{ amperes.}$$

$$\therefore \quad i = e^{-100t}(A_1\sin 100t + A_2\cos 100t) + 0.3987\sin(500t + 0.4103) \text{ amperes.}$$

At the instant of closing the switch, $i = 0$ and $e = 100\sin 1.571$ i.e. $e = 100$ volts.

$$\therefore \quad 0 = A_2 + 0.3987\sin 0.4103$$

$$A_2 = -0.159$$

Also at $t = 0$; $L\frac{di}{dt} = e - v_c = 100$

$$\therefore \quad \frac{di}{dt} = 200 \text{ A/s}$$

In general, when we differentiate i with respect to time:

$$\frac{di}{dt} = -100e^{-100t}(A_1\sin 100t + A_2\cos 100t) + e^{-100t}(100A_1\cos 100t - 100A_2\sin 100t) + 500 \times 0.3987\cos(500t + 0.4103)$$

At $t = 0$; $\frac{di}{dt} = -100A_2 + 100A_1 + 182.8 = 200$

$$\therefore \quad A_1 = 0.013$$

$$\boxed{i = e^{-100t}(0.013\sin(100t) - 0.159\cos(100t)) + 0.3987\sin(500t + 0.4103) \text{ amperes.}}$$

The total current is given by the following:-

$$i = i_c + i_p$$

where i_c = complimentary function

i_p = particular function

For the above circuit we have

$$v_R + v_L + v_c = e$$

i.e. $$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e$$
$$= E_m\sin(wt + \theta)$$

Differentiate with respect to time.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = w\,E_m\cos(wt + \theta)$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{w\,E_m}{L}\cos(wt + \theta)$$

The solution to this equation is

$$i = i_c + i_p$$

5.4.1 Complimentary Function (i_c)

The complimentary function is the solution for current if the voltage applied is considered to be constant.

i.e. $$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \quad \text{subst. } i_c = Ae^{mt}$$

As before, there are three possible solutions:

(1) Overdamped: $i_c = e^{-\alpha t}(A_1e^{\beta t} + A_2e^{-\beta t})$

$$= A_1e^{(-\alpha + \beta)t} + A_2e^{(-\alpha - \beta)t}$$

(2) Critically damped: $i_c = e^{-\alpha t}(A_1 + A_2t)$

(3) Underdamped: $i_c = e^{-\alpha t}(A_1\sin\beta t + A_2\cos\beta t)$

$$= Ae^{-\alpha t}\sin(\beta t + \gamma)$$

... is that current that flows after the transient subsides and may be obtained by considering current and voltage within the frequency domain.

Impedance of circuit $= \bar{Z} = Z\underline{/\varphi}$ $\qquad \tan\varphi = \frac{X}{R}$

Applied e.m.f. $= \bar{E}_m = E_m\underline{/\Theta}$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{E_m\underline{/\Theta}}{Z\underline{/\varphi}}$$

$$= \frac{E_m}{Z}\underline{/\Theta - \varphi}$$

$$= I_p\underline{/\Theta - \varphi}$$

$$i_p = I_p \sin(wt + \Theta - \varphi)$$

Note: It is better to give Θ and φ in radians.

5.4.3 DETERMINATION OF CONSTANTS

$$\alpha = \frac{R}{2L}\ ; \quad w_o = \frac{1}{\sqrt{LC}}\ ; \quad \beta = \sqrt{|\alpha^2 - w_o^2|}$$

The constants A_1 and A_2 in the complimentary function may be obtained by consideration of the initial conditions for current (i) and rate of change of current $(\frac{di}{dt})$.

<u>Example 5.4.3.1</u>

In figure 5.12 R is 100 ohms, L is 0.5 henrys and C is 100 microfarads. If the switch is closed when e = 100sin(500t + 1.571) volts, determine the current in the circuit if the initial charge on the capacitor is zero.

<u>Solution</u>

$$\alpha = \frac{R}{2L} = 100$$

$$w_o = \frac{1}{\sqrt{LC}} = 141.4 \text{ rad/s}$$

$\alpha < w_o$ therefore underdamped. $\begin{cases} i_c = Ae^{mt} \\ m = -\alpha \pm j\beta \end{cases}$

$$\beta = \sqrt{|\alpha^2 - w_o^2|} = 100$$