Legendre functions and other orthogonal functions (1)

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ← Legendre's differential equation

General solution

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

$$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right] + c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 \cdots\right]$$

Recurrence formula

(1) $P_{n+1}(x) = \frac{2n+1}{n+1}\,x\,P_n(x) - \frac{n}{n+1}P_{n-1}(x)$

(2) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

Pb Given that $P_0(x) = 1$, $P_1(x) = x$ find (a) $P_2(x)$ and (b) $P_3(x)$

$$P_{(n+1)}(x) = \frac{2n+1}{n+1}\,x\,P_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

For $P_2(x)$, substitute $n=1$

$$\therefore P_{(1+1)}(x) = \frac{2\times1+1}{1+1}\,x\,P_1(x) - \frac{1}{1+1}P_{1-1}(x)$$

$$P_2(x) = \frac{3}{2}x \times x - \frac{1}{2}\underbrace{\cancel{P_0(x)}}_{1}$$

$$= \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$$

For $P_3(x)$ substitute $n=2$

$$P_{(2+1)}(x) = \frac{2\times 2+1}{2+1} x P_2(x) - \frac{2}{2+1} P_{2-1}(x)$$

$$P_3(x) = \frac{5}{3} \times x \times \left\{\frac{1}{2}(3x^2-1)\right\} - \frac{2}{3} P_1(x)$$

$$= \frac{5}{3} x \times \frac{(3x^2-1)}{2} - \frac{2}{3} \times x$$

$$= \frac{5}{6}(3x^2-1)x - \frac{2}{3}x$$

$$= \frac{2x}{3}\left[\frac{5}{2}(3x^2-1) - 1\right]$$

$$= \frac{2x}{3}\left[\frac{5(3x^2-1) - 2}{2}\right]$$

$$= \frac{2x}{3}\left[\frac{15x^2-5-2}{2}\right]$$

$$\frac{2x}{3}\left[\frac{15x^2-7}{2}\right]$$

Biomomial theorem

$$(P+v)^n = 1 + Pv + \frac{P(P-1)}{2!}v^2 + \frac{P(P-1)(P-2)}{3!}v^3$$

Generative function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x) P_n(x) t^{m+n}$$

Pb Prove that $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x) P_n(x)\,dx\right\} t^{m\ldots}$$

$$= \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n^2(x)\,dx\right\} t^{2\ldots}$$

③

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Squaring both sides

$$\frac{1}{1-2tx+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\, P_n(x)\, t^{m+n}$$

By integrating -1 to 1

$$\int_{-1}^{1} \frac{1}{1-2tx+t^2}\, dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{ \int_{-1}^{1} P_m(x)\, P_n(x)\, dx \right\} t^{m+n}$$

$$d(1-2tx+t^2) = \cancel{d1} - d2tx + \cancel{dt^2} =$$
$$= -2t\, dx$$

$$\therefore dx = -\frac{1}{2t}\, d(1-2tx+t^2)$$

$$\int_{-1}^{1} -\frac{1}{2t} \frac{d(1-2tx+t^2)}{1-2tx+t^2} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^{1} P_n^2(x)\, dx \right\} t^{2n}$$

$$-\frac{1}{2t} \ln(1-2tx+t^2) \Big|_{-1}^{1} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^{1} P_n^2(x)\, dx \right\} t^{2n}$$

LHS

$$-\left[\frac{1}{2\times t} \ln(1-2xt+t^2)\right] + \left[\frac{1}{2\times(t)} \ln(1-2\times t(-1)+(t)^2)\right]$$

$$-\frac{1}{2t} \ln(1-t)^2 + \frac{1}{2t} \ln(1+t)^2$$

$$-\frac{1}{2t} \times 2 \ln(1-t) + \frac{1}{2t} \times 2 \ln(1+t)$$

$$\cancel{-\frac{1}{2t}\left\{\ln 2(1-t)\right\} + \frac{1}{2t}\left\{\ln 2(1+t)\right\}}$$

$$\cancel{-\frac{1}{2t}\ln 2 \left\{\ln(1-t) + \ln(1+t)\right\}} \quad \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$$

$$+\frac{1}{2t}\cancel{\ln 2}\left\{\ln(1+t)-\ln(1-t)\right\}$$

$$\cancel{\frac{1}{}\ \ln\left(\frac{1+t}{1-t}\right)}$$

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)=\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

$$\boxed{\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)=\sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}}$$

$$\therefore \sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}=\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

$$\therefore \int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1}$$

#

Series of legendre polynomials

If $f(x) = \sum_{k=0}^{\infty} A_k P_k(x), \quad -1 < x < 1$

Then $A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) \, f(x) \, dx$

$\boxed{\text{Rodrigue's formula}}$ $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

Pb Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$

$$A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) \, f(x) \, dx$$

$$= \frac{2k+1}{2} \int_{-1}^{0} P_k(x) \, 0 \, dx + \frac{2k+1}{2} \int_{0}^{1} P_k(x) (1) \, dx$$

$$A_k = \frac{2k+1}{2} \int_{0}^{1} P_k(x) \, dx$$

$k=0 \rightarrow \quad A_0 = \frac{2 \times 0 + 1}{2} \int_0^1 P_0(x) \, dx \quad \frac{1}{2} \int_0^1 1 \, dx = \frac{1}{2} [x]_0^1$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

$k=1 \quad A_1 = \frac{2 \times 1 + 1}{2} \int_0^1 P_1(x) \, dx$

Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_0(x) = \frac{1}{2^0 \times 0!} \frac{d^0}{dx^0} (x^2-1)^0$$

$$= 1$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_1(x) = \frac{1}{2^1 \times 1!} \frac{d}{dx} (x^2-1) = \frac{1}{2} \times 2x$$

$$= x$$

$$\therefore A_1 = \frac{3}{2} \int_0^1 x\,dx = \frac{3}{2} \left[\frac{x^2}{2} \right]_0^1$$

$$= \frac{3}{2} \left[\frac{1-0}{2} \right] = \frac{3}{4}$$

$$P_2(x) = \frac{1}{2^2\, 2!} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{4 \times 2} \frac{d^2}{dx^2} (x^2-1)^2$$

$$\frac{d}{dx} (x^2-1)^2$$

$$= 2(x^2-1) \frac{d}{dx} x^2$$

$$\cancel{\frac{d^2}{dx^2} = 2 \times 2}$$

$$= 2(x^2-1) \times 2x$$

$$= 4(x^2-1)x$$

$$\frac{d^2}{dx^2}(\ ) = \frac{d}{dx} 4x^3 - 4x$$

$$= 4 \times 3x^2 - 4 = 4(3x^2-1)$$

$$= \frac{1}{4 \times 2} 4(3x^2-1) = \frac{3x^2-1}{2}$$

$$\therefore A_2(x) = \frac{2 \times 2 + 1}{2} \int_0^1 P_2(x)\,dx$$

$$= \frac{5}{2} \times \int_0^1 \frac{3x^2-1}{2}\,dx$$

$$= \frac{5}{2} \left[\frac{3}{2} \times \frac{x^3}{3} \Big|_0^1 - \left[\frac{x}{2}\right]_0^1 \right.$$

$$\frac{5}{2} \left[\frac{3}{2} \times \frac{1}{3} - \frac{1}{2} \right]$$

$$= \frac{5}{2} \times \left[\frac{1}{2} - \frac{1}{2} \right]$$

$$\frac{5}{2} \times 0 = 0$$

$$A_3 = \frac{2\times3+1}{2}\int^{1} P_3(x)\,dx$$

$$= \frac{7}{2}\int_{0}^{10} P_3(x)\,dx$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_3(x) = \frac{1}{2^3\times 3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{8\times3\times2}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3$$

$$\frac{d^3}{dx^3}(x^2-1)^3 = ? \qquad \frac{d}{dx}(x^2-1)^3 = 3(x^2-1)^{3-1}\frac{d}{dx}(x^2-1)$$

$$= 3(x^2-1)^2\times 2x$$

$$= 6x(x^2-1)^2$$

$$= 6x(x^4 - 2x^2 + 1) = 6x^5 - 12x^3 + 6x$$

$$\frac{d^2}{dx^2}(x^2-1)^3 = \frac{d}{dx}(6x^5 - 12x^3 + 6x)$$

$$= 6\times5\,x^4 - 12\times3\,x^2 + 6$$

$$= 30x^4 - 36x^2 + 6$$

$$\frac{d^3}{dx^3}(x^2-1)^3 = 30\times4\,x^3 - 36\times2\,x + 0$$

$$= 120x^3 - 72x$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \cancel{\frac{7}{2}}$$

$$P_3(x)\,dx = \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\left[120x^3 - 72x\right]$$

$$= \frac{5x^3 - 3x}{2}$$

$$A_3 = \frac{7}{2} \times \int_0^1 \frac{5x^3 - 3x}{2}\,dx = \frac{7}{2}\left[\frac{5}{2}\left(\frac{x^{3+1}}{3+1}\right)_0^1 - \frac{3}{2}\left(\frac{x^2}{2}\right)_0^1\right]$$

$$= \frac{7}{2}\left[\frac{5}{2}\left[\frac{x^3}{\cancel{3}4}\right]_0^1 - \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1\right]$$

$$= \frac{7}{2}\left[\frac{5}{2} \times \frac{1}{4} - \frac{3}{2} \times \frac{1}{2}\right]$$

$$= \frac{7}{2}\left[\frac{5}{8} - \frac{3}{4}\right] = \frac{7}{2}\left[\frac{\cancel{10}5 - \cancel{6}6}{8}\right]$$

$$= \frac{7}{2} \times \frac{-1}{8} = -\frac{7}{16}$$ #

$$f(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 (P_3(x)$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0\, P_2(x) + -\frac{7}{16} P_3(x)$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x)$$ #

Partial differential equation

Pb If $v = F(y-3x)$

Prove $\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$

$v = F(y-3x)$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} F(y-3x) = f'(y-3x)\frac{\partial}{\partial x}(y-3x)$$

$$= f'(y-3x)\left[\underbrace{\cancel{\frac{\partial}{\partial x}y}}_{0} - \frac{\partial}{\partial x}3x\right]$$

$$= -f'(y-3x) \times 3$$

$$= -3f'(y-3x) \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} f(y-3x) = F'(y-3x)\frac{\partial}{\partial y} \times (y-3x)$$

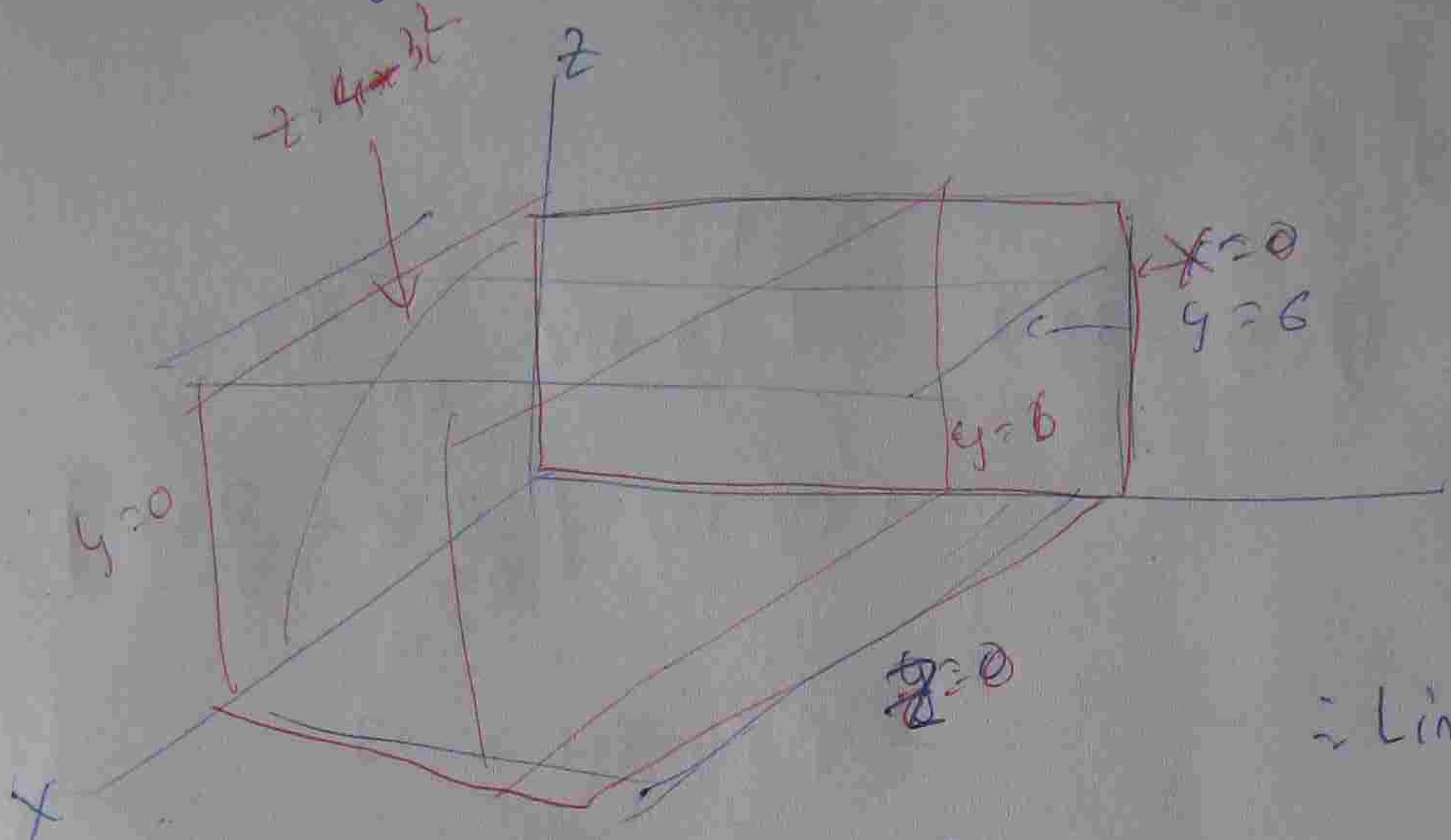
$$= f'(y-3x)\left[\frac{\partial y}{\partial y} - \underbrace{\cancel{\frac{\partial 3x}{\partial y}}}_{0}\right]$$

$$= f'(y-3x) \quad \text{--- (ii)}$$

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = -3f'(y-3x) + 3f'(y-3x)$$

$$= 0$$

Ph Find the volume of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x=0$, $y=0$, $y=6$, $z=0$



when $z=0$

$z = 4 - x^2$

$0 = 4 - x^2$

$\therefore x^2 = 4$

$x = \pm 2$

$y = 0 \to 6$

$\therefore$ Limit $z = 0 \longrightarrow 4 - x^2$

$y = 0 \longrightarrow 6$

$x = 0 \longrightarrow 2$

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} f(z)\,dz\,dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \underline{f(z)\,dz}\,dy\,dx$$

$z = 4 - x^2$

$$\int_{x=0}^{2}\int_{y=0}^{6}(4-x^2)\,dy\,dx$$

$$\int_{x=0}^{2}\Big[(4-x^2)y\Big]_{y=0}^{6}\,dx$$

$$\int_{x=0}^{2}(4-x^2)6\,dx$$

$$6\int_0^2(4-x^2)\,dx$$

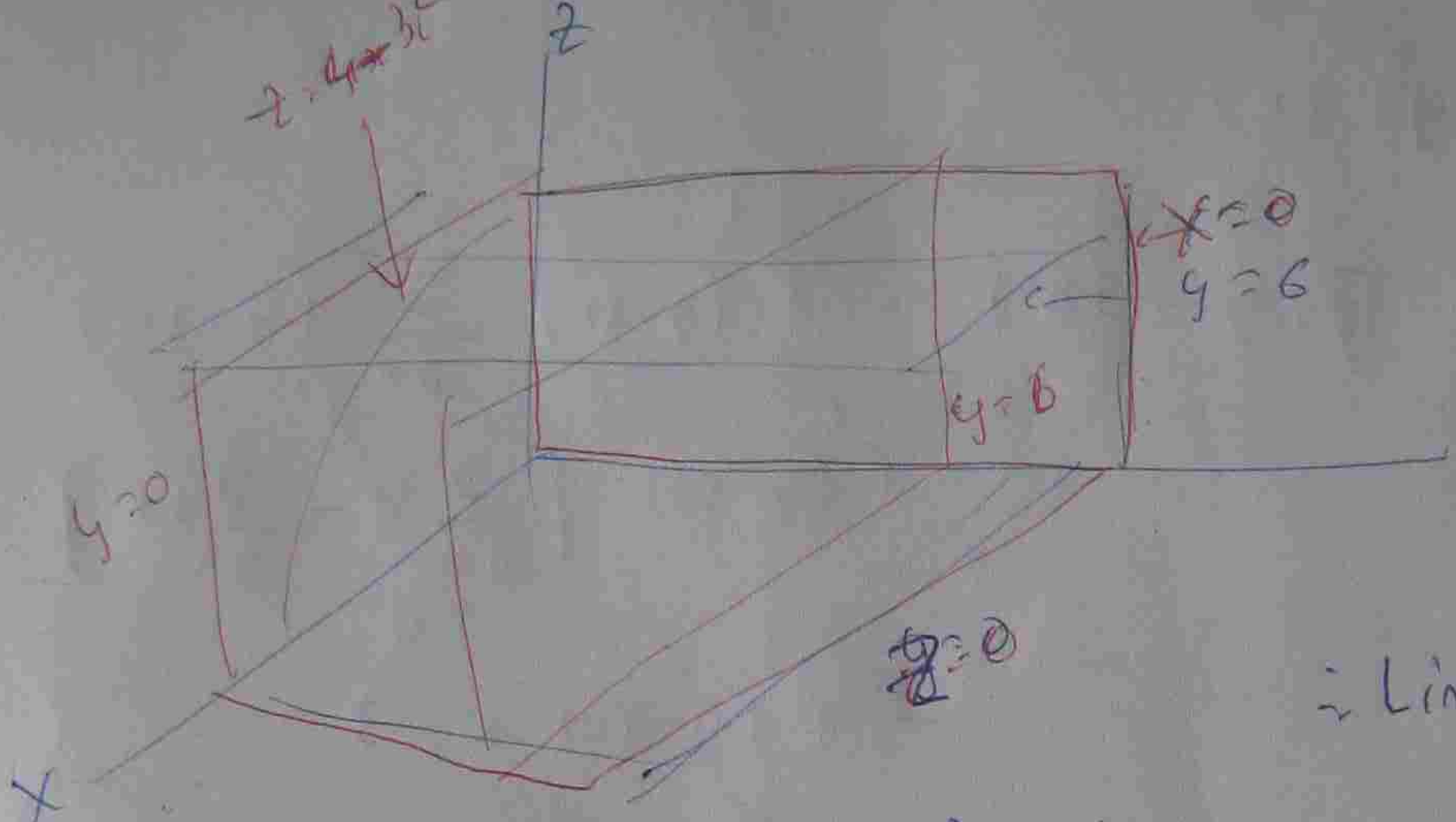
$$6\times\left\{4[x]_0^2-\left[\frac{x^3}{3}\right]_0^2\right\}$$

$$6\left\{4\times 2-\frac{2^3}{3}\right\}$$

$$6\left\{8-\frac{8}{3}\right\} = 6\times\frac{16}{3}$$

$$= 32$$

pb Find the volume of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x=0$ $y=0$ $y=6$ $z=0$



when $z=0$

$z = 4 - x^2$

$0 = 4 - x^2$

$\therefore x^2 = 4$

$x = \pm 2$

$y = 0 \to 6$

$\therefore$ Limit $z = 0 \longrightarrow 4 - x^2$

$y = 0 \longrightarrow 6$

$x = 0 \longrightarrow 2$

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} f(z)\, dz\, dy\, dx$$

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} \underline{\underline{f(z)\, dz}}\, dy\, dx$$

$z = 4 - x^2$

$$\int_{x=0}^{2}\int_{y=0}^{6} (4 - x^2)\, dy\, dx$$

$$\int_{x=0}^{2} \Big[(4 - x^2)y\Big]_{y=0}^{6} dx$$

$$\int_{x=0}^{2} (4 - x^2)6\, dx$$

$$6\int_{0}^{2} (4 - x^2)\, dx$$

$$6 \times \left\{ 4[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2 \right\}$$

$$6 \left\{ 4 \times 2 - \frac{2^3}{3} \right\}$$

$$6 \left[8 - \frac{8}{3} \right\} = 6 \times \frac{16}{3}$$

$= 32$

Matrices

Pb 1

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2+3 & 1+(-5) & 4+1 \\ -3+2 & 0+1 & 2+3 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 2-3 & 1-(-5) & 4-1 \\ -3-2 & 0-1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 3 \\ -5 & -1 & -1 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4\times2 & 4\times1 & 4\times4 \\ -3\times4 & 0\times4 & 2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & 16 \\ -12 & 0 & 8 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = r_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = r_2$$

$$\text{---------------------------}$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = r_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ | \\ r_n \end{bmatrix}$$

Pb2

$$A = \begin{pmatrix} 2 & 1\,4 \\ -3 & 0\;2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$A \times B = \begin{bmatrix} \text{Row 1} \times \text{colum 1} & \text{Row}_1 \text{ col 2} \\ \text{Row2} \times \text{colum 2} & \text{Row2 col}_2 \end{bmatrix}$

1st matrix × 2nd matrix

$$= \begin{bmatrix} (2\times 3 + 1\times 2 + 4\times 4) & (2\times 5 + 1\times(-1) + 4\times 2 \\ (-3\times 5 + 0\times -1 + 2\times 2) & (-3\times 5 + 0\times -1 + 2\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 17 \\ -1 & -11 \end{bmatrix}$$

Pb3

Determinant

$$\begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = ?$$

$$3 \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

$$= 3\,[2\times 2 - (-3)\times 1] + 2\,[1\times 2 - (4\times -3)] + 2\,[1\times 1 - 4\times 2]$$

$$= 3\,[4+3] + 2\,[2+12] + 2\,[1-8]$$

$$= 3\times 7 + 2\times 14 + 2\times(-7) = 35$$

Trace of matrix

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{pmatrix}$$

Trace of matrix $= 5+1+2 = 8$

Cofactor

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{bmatrix}$$

cofactor of 5

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{bmatrix}$$

Remove -

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

pb 4

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$$

matrix of cofactor A $(A_{jk}) = ?$ $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = (-1)^{1+1} \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+1} \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2\times 2 - (1)(-3) = 7$$

4

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+2}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} = -[1\times2-(4)(-3)]$$

$$= -[2+12]$$

$$= -14$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+3}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = [1\times1-4\times2]$$

$$= -7$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{2+1}\begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} = -[-2\times2-2\times1]$$

$$= -[-4-2]$$

$$= -(-6) = 6$$

$$A_{22} = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{2+2}\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} = +[3\times2-4\times2]$$

$$= +[6-8] = -2$$

$$\underset{\substack{\uparrow\ \ \uparrow \\ \text{row}\ \ \text{col}}}{A_{23}} = (-1)^{2+3}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = -1\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = -1[3-(4)(-2)]$$

$$= -1[3+8] = -11$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = 1\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} = 1[-2\times-3-2\times2]$$

$$= [+6-4]$$

$$= 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = -1 \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} = -1\left[3(-3) - 2\times1\right]$$

$$= -1[-9-2]$$

$$= 11$$

$$A_{33} = (-1)^{3+3} \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} = 3\times2 - (1)(-2)$$

$$= 6+2 = 8$$

cofactor

$$(A_{jk}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{bmatrix} & \begin{array}{l} \rightarrow \text{Row}_1 \\ \text{Row}_2 \\ \rightarrow \text{Row}_3 \end{array} \end{array}$$

Row1 → colum 1
Row2 — colum 2
Row3 → column 3

$$(A_{jk})^T = \begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}$$

Transpose of matrix

$$A^{-1} = \frac{(A_{jk})^T}{\det A}$$

$$\det A = ? \quad \det \begin{vmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{vmatrix}$$

$$= +7\begin{bmatrix} -2 & 11 \\ -11 & 8 \end{bmatrix} - 6\begin{bmatrix} -14 & 11 \\ -7 & 8 \end{bmatrix}$$

$$+ 2\begin{bmatrix} -14 & -2 \\ -7 & -11 \end{bmatrix}$$

$$\det A = 7\left[-2\times8 - (-11)11\right] - 6\left[-14\times8 - (-7)\times11\right] + 2\left[-14\times-11 - (-2)(-7)\right]$$

$$= 7\left[-16+121\right] - 6\left[-112+77\right] + 2\left[154-14\right]$$

$$\cancel{= 7\,[10\,5] - 6\times(-35) + 2\times 140}$$

$$= \cancel{28 +}$$

$$= \cancel{735} -$$

$$A^{-1} = \frac{(Aju)^T}{\det A}$$

$$\det A = \det \begin{vmatrix} \overset{+}{3} & \overset{-}{-2} & \overset{+}{2} \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$$

$$3\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$= 3\,[4 - (-3)\times 1] + 2\,[1\times 2 - 4(-3)] + 2\,[1\times 1 - 4\times 2]$$

$$= 3\,[4+3] + 2\,[2+12] + 2\,[1-8]$$

$$= 3\times 7 + 2\times 14 + 2\,(-7) =$$

$$= 21 + 28 - 14 = 35$$

$$\therefore A^{-1} = \frac{(Aju)^T}{\det A} = \frac{\begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}}{35} = \frac{1}{35}\begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & \frac{-2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{bmatrix}$$

Pb 5

Refer Pb 4 and solve the following equations

$$3x_1 - 2x_2 + 2x_3 = 10$$
$$x_1 + 2x_2 - 2x_3 = -1$$
$$4x_1 + x_2 + 2x_3 = 3$$

$$\begin{pmatrix} 3 & -2 & 2 \\ x_1 & 2 & -2 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix}$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix} \times A^{-1} = A^{-1} \begin{pmatrix} 16 \\ -1 \\ 3 \end{pmatrix}$$

According to Pb 4 $\quad A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{bmatrix}$

$$= \begin{pmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{pmatrix} \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix} = \begin{bmatrix} \frac{1}{5} \times 10 + \frac{6}{35}(-1) + \frac{2}{35} \times 3 \\ -\frac{2}{5} \times 10 + \left(-\frac{2}{35}\right)(-1) + \frac{11}{35}(+3) \\ -\frac{1}{5} \times 10 + \left(\frac{-11}{35}\right)(-1) + \frac{8}{35}(3) \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{35}\times 10 - \frac{6}{35} + \frac{6}{35} \\ -\frac{14}{35}\times 10 + \frac{2}{35} - \frac{11}{35} \\ -\frac{7}{35}\times 10 + \frac{11}{35} + \frac{24}{35} \end{bmatrix} = \begin{bmatrix} \frac{70}{35} - \cancel{\frac{6}{35}} + \cancel{\frac{6}{35}} \\ -\frac{140}{35} + \frac{2}{35} + \frac{38}{35} \\ -\frac{70}{35} + \frac{11}{35} + \frac{24}{35} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{-140+2+38}{35} \\ \frac{-70+11+24}{35} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$x_1 = 2,\ x_2 = 3,\ x_3 = 1$

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pb 6

Solve
$$2x_1 + 5x_2 - 3x_3 = 3$$
$$x_1 - 2x_2 + x_3 = 2$$
$$7x_1 + 4x_2 - 3x_3 = -4$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = m_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = m_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = m_3$$

$$x_1 = \frac{\begin{bmatrix} m_1 & a_{12} & a_{13} \\ m_2 & a_{22} & a_{23} \\ m_3 & a_{32} & a_{33} \end{bmatrix}}{\Delta}, \quad x_2 = \frac{\begin{bmatrix} a_{11} & m_1 & a_{13} \\ a_{21} & m_2 & a_{23} \\ a_{31} & m_3 & a_{33} \end{bmatrix}}{\Delta}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & m_1 \\ a_{21} & a_{22} & m_2 \\ a_{31} & a_{32} & m_3 \end{vmatrix}}{\Delta}$$

$$\cancel{x_3 = \frac{\begin{bmatrix} a_{11} & a_{12} & m \\ a_{21} & a_{22} & m \\ a_{23} & a_{32} & m \end{bmatrix}}{\Delta}}$$

$$\Delta = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 7 & 4 & -3 \end{bmatrix} = 2\begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 7 & -3 \end{bmatrix} + (-3)\begin{bmatrix} 1 & -2 \\ 7 & 4 \end{bmatrix}$$

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$2(2) - 5(-10) - 3(18)$

$4 + 50 - 54 = 0$

$\therefore$ No solution

Pb 7 solve

$$3x_1 - 2x_2 + 2x_3 = 10$$
$$x_1 + 2x_2 - 3x_3 = -1$$
$$4x_1 + x_2 + 2x_3 = 3$$

$$x_1 = \frac{\begin{bmatrix} m_1 & a_{12} & a_{13} \\ m_2 & a_{22} & a_{23} \\ m_3 & a_{32} & a_{33} \end{bmatrix}}{\Delta} \qquad x_2 = \frac{\begin{bmatrix} a_{11} & m_1 & a_{13} \\ a_{21} & m_2 & a_{23} \\ a_{31} & m_3 & a_{33} \end{bmatrix}}{\Delta}$$

$$x_3 = \frac{\begin{bmatrix} a_{11} & a_{12} & m_1 \\ a_{21} & a_{22} & m_2 \\ a_{31} & a_{32} & m_3 \end{bmatrix}}{\Delta}$$

$$\Delta = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = 3\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} - (-2)\begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

$$= 3(4+3) + 2(2+12) + 2(1-8)$$

$$= 3 \times 7 + 2 \times 14 - 2 \times 7 = 21 + 28 - 14 = 35$$

$$x_1 = \frac{\begin{bmatrix} 10 & -2 & 2 \\ -1 & 2 & -3 \\ 3 & 1 & 2 \end{bmatrix}}{35} = \frac{10\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} - (-2)\begin{bmatrix} -1 & -3 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}}{35}$$

$$= \frac{10[4+3] + 2[-2+9] + 2[-1-6]}{35}$$

$$= \frac{10 \times 7 + 14 - 14}{35} = 2$$

10

$$x_2 = \frac{\begin{vmatrix} 3 & 10 & 2 \\ 1 & -1 & -3 \\ 4 & 3 & 2 \end{vmatrix}}{35}$$

$$= \frac{3\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix} - 10\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix}}{35}$$

$$= \frac{3[-2+9] - 10[2+12] + 2[3+4]}{35}$$

$$= \frac{3\times 7 - 10\times 14 + 2\times 7}{35} = \frac{21 - 140 + 14}{35} = \frac{-105}{35}$$

$$= -3$$

$$x_3 = \frac{\begin{bmatrix} 3 & -2 & 10 \\ 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}}{35} = \frac{3\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix} + 10\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}}{35}$$

$$= \frac{3[6+1] + 2[3+4] + 10[1-8]}{35}$$

$$= \frac{3\times 7 + 2\times 7 - 70}{35} = \frac{35-70}{35} = \frac{-35}{35} = -1$$

$x_1 = 2$, $x_2 = -3$, $x_3 = -1$