(vi) *Increases reliability of supply* : The interconnected system increases the reliability of supply. If a major breakdown occurs in one station, continuity of supply can be maintained by other healthy stations.

Example 3.20. *A base load station having a capacity of 18 MW and a standby station having a capacity of 20 MW share a common load. Find the annual load factors and plant capacity factors of two power stations from the following data :*

Annual standby station output $= 7{\cdot}35 \times 10^6$ *kWh*
Annual base load station output $= 101{\cdot}35 \times 10^6$ *kWh*
Peak load on standby station $= 12$ *MW*
Hours of use by standby station/year $= 2190$ *hours*

Solution.

Installed capacity of standby unit

$$= 20 \text{ MW} = 20 \times 10^3 \text{ kW}$$

Installed capacity of base load plant

$$= 18 \text{ MW} = 18 \times 10^3 \text{ kW}$$

Standby station

$$\text{Annual load factor} = \frac{\text{kWh generated / annum}}{\text{Max. demand} \times \text{Annual working hours}} \times 100$$

$$= \frac{7{\cdot}35 \times 10^6}{(12 \times 10^3) \times 2190} \times 100 = \mathbf{28\%}$$

$$\text{Annual plant capacity factor} = \frac{\text{kWh output / annum}}{\text{Installed capacity} \times \text{Hours in a year}} \times 100$$

$$= \frac{7{\cdot}35 \times 10^6}{(20 \times 10^3) \times 8760} \times 100 = \mathbf{4{\cdot}2\%}$$

Base load station. It is reasonable to assume that the maximum demand on the base load station is equal to the installed capacity (*i.e.*, 18 MW). It operates throughout the year *i.e.*, for 8760 hours.

$$\therefore \quad \text{Annual load factor} = \frac{101{\cdot}35 \times 10^6}{(18 \times 10^3) \times 8760} = \mathbf{64{\cdot}2\%}$$

As the base load station has no reserves above peak load and it is in continuous operation, therefore, its capacity factor is also **64.2%**.

Example 3.21. *The load duration curve for a typical heavy load being served by a combined hydro-steam system may be approximated by a straight line; maximum and minimum loads being 60,000 kW and 20,000 kW respectively. The hydro power available at the time of minimum regulated flow is just sufficient to take a peak load of 50,000 kWh per day. It is observed that it will be economical to pump water from tail race to the reservoir by utilising the steam power plant during the off-peak periods and thus running the station at 100% load factor. Determine the maximum capacity of each type of plant. Assume the efficiency of steam conversion to be 60%.*

Solution. *OCBA* represents the load duration curve for the combined system as shown in Fig. 3.15. The total maximum demand (*i.e.*, 60,000 kW) is represented by *OC*, whereas the minimum demand (*i.e.*, 20,000 kW) is represented by *OD*.

Let
OE = Capacity of steam plant
EC = Capacity of hydro plant
Area *CHI* = The energy available from hydro plant in the low flow period.

Area FGB = The off-peak* period energy available from steam plant

Obviously, the energy of hydro plant represented by area $HEFI$ and available from reservoir has been supplied by steam power plant represented by area FGB. As steam electric conversion is 60%,

$$\therefore \quad \text{Area } HEFI = 0{\cdot}6 \times \text{Area } FGB \qquad \ldots (i)$$

But $$\text{Area } HEFI = \text{Area } CFE - \text{Area } CHI$$

$$= \frac{1}{2} xy - 50{,}000\,\dagger$$

Now $$\text{Area } FGB = \frac{1}{2} \times FG \times GB = \frac{1}{2}(24 - x)(40{,}000 - y)$$

Putting the various values in exp. (*i*), we get,

$$\frac{1}{2} xy - 50{,}000 = 0{\cdot}6\left[\frac{1}{2}(24 - x)(40{,}000 - y)\right]$$

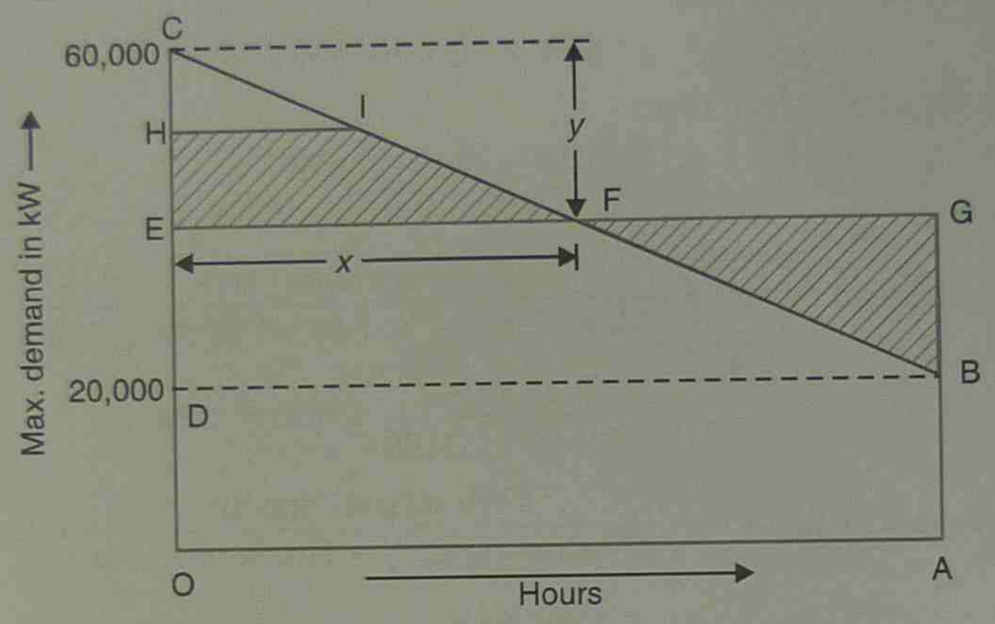

Fig. 3.15

or $0{\cdot}2\,xy + 12000\,x + 7{\cdot}2\,y - 3{,}38{,}000 = 0 \qquad \ldots (ii)$

Also from similar triangles CEF and CDB, we get,

$$\frac{y}{40{,}000} = \frac{x}{24}$$

$$\therefore \quad y = \frac{40{,}000\,x}{24} \qquad \ldots (iii)$$

Putting $y = 40{,}000\,x/24$ from exp. (*iii*) into exp. (*ii*), we get,

$$333\,x^2 + 24000\,x - 3{,}38{,}000 = 0$$

or $$x^2 + 72x - 1015 = 0$$

$$\therefore \quad x = \frac{-72 \pm \sqrt{5184 + 4060}}{2} = \frac{-72 \pm 96}{24} = 12$$

$\therefore$ Capacity of the hydro plant is

$$y\,(= EC) = \frac{40{,}000 \times 12}{24} = \mathbf{20{,}000\ kW}$$

Capacity of steam plant = $60{,}000 - 20{,}000 = \mathbf{40{,}000\ kW}$

Example 3.22. *The annual load duration curve for a typical heavy load being served by a steam station, a run-of-river station and a reservoir hydro-electric station is as shown in Fig. 3.16. The ratio of number of units supplied by these stations is as follows :*

---

* It is clear from load duration curve that the capacity of steam plant represented by area FGB is not being utilised efficiently. This steam energy can be used to pump water in tail race back to the reservoir.

† Because during minimum regulated flow, hydro energy supplied is 50,000 kWh.

Amount q deposited at the end of 2nd year becomes

$$= q\,(1+r)^{n-2}$$

Amount q deposited at the end of 3rd year becomes

$$= q\,(1+r)^{n-3}$$

Similarly amount q deposited at the end of $n-1$ year becomes

$$= q\,(1+r)^{n-(n-1)}$$

$$= q\,(1+r)$$

$\therefore$ Total fund after n years $= q\,(1+r)^{n-1} + q\,(1+r)^{n-2} + ... + q\,(1+r)$

$$= q\,[(1+r)^{n-1} + (1+r)^{n-2} + ... + (1+r)]$$

This is a G.P. series and its sum is given by :

$$\text{Total fund} = \frac{q(1+r)^n - 1}{r}$$

This total fund must be equal to the cost of replacement of equipment *i.e.*, $P - S$.

$\therefore$ $$P - S = q\frac{(1+r)^n - 1}{r}$$

or $$\text{Sinking fund, } q = (P-S)\left[\frac{r}{(1+r)^n - 1}\right] \qquad ...(i)$$

The value of q gives the uniform annual depreciation charge. The paraenthetical term in eq. (i) is frequently referred to as the "sinking fund factor".

$\therefore$ $$\text{Sinking fund factor} = \frac{r}{(1+r)^n - 1}$$

Though this method does not find very frequent application in practical depreciation accounting, it is the fundamental method in making economy studies.

Example 4.1. *A transformer costing Rs 90,000 has a useful life of 20 years. Determine the annual depreciation charge using straight line method. Assume the salvage value of the equipment to be Rs 10,000.*

Solution :

Initial cost of transformer, P = Rs 90,000

Useful life, n = 20 years

Salvage value, S = Rs 10,000

Using straight line method,

$$\text{Annual depreciation charge} = \frac{P-S}{n} = \text{Rs}\,\frac{90{,}000 - 10{,}000}{20} = \textbf{Rs 4000}$$

Example 4.2. *A distribution transformer costs Rs 2,00,000 and has a useful life of 20 years. If the salvage value is Rs 10,000 and rate of annual compound interest is 8%, calculate the amount to be saved annually for replacement of the transformer after the end of 20 years by sinking fund method.*

Solution :

Initial cost of transformer, P = Rs 2,00,000

Salvage value of transformer, S = Rs 10,000

Useful life, n = 20 years

Annual interest rate, r = 8% = 0·08

Annual payment for sinking fund,

$$q = (P - S)\left[\frac{r}{(1+r)^n - 1}\right]$$

$$= (2,00,000 - 10,000)\left[\frac{0 \cdot 08}{(1 + 0 \cdot 08)^{20} - 1}\right]$$

$$= 1,90,000\left[\frac{0 \cdot 08}{4 \cdot 66 - 1}\right] = \textbf{Rs 4153}$$

Example 4.3. *The equipment in a power station costs Rs 15,60,000 and has a salvage value of Rs 60,000 at the end of 25 years. Determine the depreciated value of the equipment at the end of 20 years on the following methods :*

(*i*) Straight line method ;

(*ii*) Diminishing value method ;

(*iii*) Sinking *fund method at 5% compound interest annually.*

Solution :

Initial cost of equipment, P = Rs 15,60,000

Salvage value of equipment, S = Rs 60,000

Useful life, n = 25 years

(*i*) *Straight line method*

$$\text{Annual depreciation} = \frac{P - S}{n} = \text{Rs}\ \frac{15,60,000 - 60,000}{25} = \text{Rs } 60,000$$

Value of equipment after 20 years

$$= P - \text{Annual depreciation} \times 20$$

$$= 15,60,000 - 60,000 \times 20 = \textbf{Rs 3,60,000}$$

(*ii*) *Diminishing value method*

Annual unit depreciation, $x = 1 - (S/P)^{1/n}$

$$= 1 - \left(\frac{60,000}{15,60,000}\right)^{1/25} = 1 - 0{\cdot}878 = 0{\cdot}122$$

Value of equipment after 20 years

$$= P(1 - x)^{20}$$

$$= 15,60,000\ (1 - 0{\cdot}122)^{20} = \textbf{Rs 1,15,615}$$

(*iii*) *Sinking fund method*

Rate of interest, $r = 5\% = 0{\cdot}05$

Annual deposit in the sinking fund is

$$q = (P - S)\left[\frac{r}{(1+r)^n - 1}\right]$$

$$= (15,60,000 - 60,000)\left[\frac{0 \cdot 05}{(1 + 0 \cdot 05)^{25} - 1}\right]$$

$$= \text{Rs } 31,433$$

$\therefore$ Sinking fund at the end of 20 years

$$= q\frac{(1+r)^{20} - 1}{r} = 31,433\ \frac{(1 + 0 \cdot 05)^{20} - 1}{0 \cdot 05} = \text{Rs } 10,39,362$$

Value of plant after 20 years = Rs (15,60,000 – 10,39,362) = **Rs 5,20,638**

4.5 Importance of High Load Factor

The load factor plays a vital role in determining the cost of energy. Some important advantages of high load factor are listed below :

(*i*) *Reduces cost per unit generated* : A high load factor reduces the overall cost per unit generated. The higher the load factor, the lower is the generation cost. It is because higher load factor means that for a given maximum demand, the number of units generated is more. This reduces the cost of generation.

(*ii*) *Reduces variable load problems* : A high load factor reduces the variable load problems on the power station. A higher load factor means comparatively less variations in the load demands at various times. This avoids the frequent use of regulating devices installed to meet the variable load on the station.

Example 4.4. *A generating station has a maximum demand of 50,000 kW. Calculate the cost per unit generated from the following data :*

Capital cost = Rs 95×10^6 ; *Annual load factor* = 40%

Annual cost of fuel and oil = Rs 9×10^6 ; *Taxes, wages and salaries etc.* = Rs $7{\cdot}5 \times 10^6$

Interest and depreciation = 12%

Solution :

Units generated/annum = Max. demand × L.F. × Hours in a year

= (50,000) × (0·4) × (8760) kWh = $17{\cdot}52 \times 10^7$ kWh

Annual fixed charges

Annual interest and depreciation = 12% of capital cost

= Rs $0{\cdot}12 \times 95 \times 10^6$ = Rs $11{\cdot}4 \times 10^6$

Annual Running Charges

Total annual running charges = Annual cost of fuel and oil + Taxes, wages etc.

= Rs $(9 \times 10^6 + 7{\cdot}5 \times 10^6)$ = Rs $16{\cdot}5 \times 10^6$

Total annual charges = Rs $(11{\cdot}4 \times 10^6 + 16{\cdot}5 \times 10^6)$ = Rs $27{\cdot}9 \times 10^6$

$$\therefore \quad \text{Cost per unit} = \text{Rs } \frac{27{\cdot}9 \times 10^6}{17{\cdot}52 \times 10^7} = \text{Re } 0{\cdot}16 = \textbf{16 paise}$$

Example 4.5. *A generating station has an installed capacity of 50,000 kW and delivers 220 × 10^6 units per annum. If the annual fixed charges are Rs 160 per kW installed capacity and running charges are 4 paise per kWh, determine the cost per unit generated.*

Solution :

Annual fixed charges = 160 × Plant capacity

= Rs $160 \times 50{,}000$ = Rs 80×10^5

Annual running charges = Rs $0{\cdot}04 \times 220 \times 10^6$ = Rs 88×10^5

Total annual charges = Rs $(80 \times 10^5 + 88 \times 10^5)$ = Rs 168×10^5

$$\text{Cost per unit} = \text{Rs } \frac{168 \times 10^5}{220 \times 10^6} = \text{Re } 0{\cdot}0764 = \textbf{7·64 paise}$$

Example 4.6. *A generating plant has a maximum capacity of 100 kW and costs Rs 1,60,000. The annual fixed charges are 12% consisting of 5% intererst, 5% depreciation and 2% taxes. Find the fixed charges per kWh if the load factor is (i) 100% and (ii) 50%.*

Solution :

Maximum demand = 100 kW

Annual fixed charges = Rs $0{\cdot}12 \times 1{,}60{,}000$ = Rs 19,200

Solution :

$$\text{kVA} = \frac{\text{kW}}{\cos\phi} = \frac{300}{0{\cdot}6} = 500 \text{ kVA}$$

$$\text{kW at 0·6 p.f.} = 300 \text{ kW}$$

$$\text{kW at 1 p.f.} = 500 \times 1 = 500 \text{ kW}$$

$\therefore$ Increased power supplied by the alternator

$$= 500 - 300 = \mathbf{200 \text{ kW}}$$

Note the importance of power factor improvement. When the p.f. of the alternator is unity, the 500 kVA are also 500 kW and the engine driving the alternator has to be capable of developing this power together with the losses in the alternator. But when the power factor of the load is 0·6, the power is only 300 kW. Therefore, the engine is developing only 300 kW, though the alternator is supplying its rated output of 500 kVA.

Example 6.2 *A single phase motor connected to 400 V, 50 Hz supply takes 31·7A at a power factor of 0·7 lagging. Calculate the capacitance required in parallel with the motor to raise the power factor to 0·9 lagging.*

Solution : The circuit and phasor diagrams are shown in Figs. 6.8 and 6.9 respectively. Here motor M is taking a current I_M of 31·7A. The current I_C taken by the capacitor must be such that when combined with I_M, the resultant current I lags the voltage by an angle ϕ where $\cos\phi = 0{\cdot}9$.

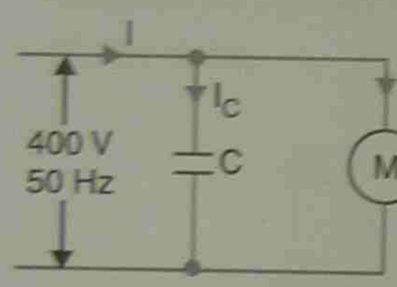

Fig. 6.8

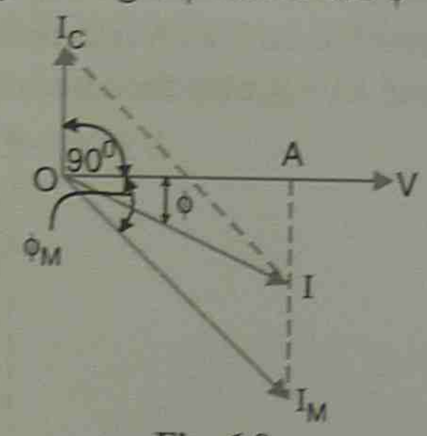

Fig. 6.9

Referring to the phasor diagram in Fig. 6.9,

$$\text{Active component of } I_M = I_M \cos\phi_M = 31{\cdot}7 \times 0{\cdot}7 = 22{\cdot}19\text{A}$$

$$\text{Active component of } I = I\cos\phi = I \times 0{\cdot}9$$

These components are represented by OA in Fig. 6.9.

$$\therefore \quad I = \frac{22{\cdot}19}{0{\cdot}9} = 24{\cdot}65\text{A}$$

$$\text{Reactive component of } I_M = I_M \sin\phi_M = 31{\cdot}7 \times 0{\cdot}714^* = 22{\cdot}6\text{A}$$

$$\text{Reactive component of } I = I\sin\phi = 24{\cdot}65\sqrt{1-(0{\cdot}9)^2}$$

$$= 24{\cdot}65 \times 0{\cdot}436 = 10{\cdot}75 \text{ A}$$

It is clear from Fig. 6.9 that :

$$I_C = \text{Reactive component of } I_M - \text{Reactive component of } I$$

$$= 22{\cdot}6 - 10{\cdot}75 = 11{\cdot}85\text{A}$$

But $$I_C = \frac{V}{X_C} = V \times 2\pi f C$$

or $$11{\cdot}85 = 400 \times 2\pi \times 50 \times C$$

$\therefore$ $$C = 94{\cdot}3 \times 10^{-6} \text{ F} = \mathbf{94{\cdot}3\ \mu F}$$

* $\sin\phi_M = \sqrt{1-\cos^2\phi_M} = \sqrt{1-(0{\cdot}7)^2} = 0{\cdot}714$

Example 6.4 *A 3-phase, 5 kW induction motor has a p.f. of 0·75 lagging. A bank of capacitors is connected in delta across the supply terminals and p.f. raised to 0·9 lagging. Determine the kVAR rating of the capacitors connected in each phase.*

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Solution :

Original p.f., $\cos \phi_1 = 0{\cdot}75$ lag ; Motor input, $P = 5$ kW

Final p.f., $\cos \phi_2 = 0{\cdot}9$ lag ; Efficiency, $\eta = 100$ % (assumed)

$\phi_1 = \cos^{-1}(0{\cdot}75) = 41{\cdot}41^\circ$; $\tan \phi_1 = \tan 41{\cdot}41^\circ = 0{\cdot}8819$

$\phi_2 = \cos^{-1}(0{\cdot}9) = 25{\cdot}84^\circ$; $\tan \phi_2 = \tan 25{\cdot}84^\circ = 0{\cdot}4843$

Leading kVAR taken by the condenser bank

$= P(\tan \phi_1 - \tan \phi_2)$

$= 5(0{\cdot}8819 - 0{\cdot}4843) = 1{\cdot}99$ kVAR

$\therefore$ Rating of capacitors connected in each phase

$= 1{\cdot}99/3 =$ **0·663 kVAR**

Example 6.5 *A 3-phase, 50 Hz, 400 V motor develops 100 H.P. (74·6 kW), the power factor being 0·75 lagging and efficiency 93%. A bank of capacitors is connected in delta across the supply terminals and power factor raised to 0·95 lagging. Each of the capacitance units is built of 4 similar 100 V capacitors. Determine the capacitance of each capacitor.*

Solution :

Original p.f., $\cos \phi_1 = 0{\cdot}75$ lag ; Final p.f., $\cos \phi_2 = 0{\cdot}95$ lag

Motor input, P = output/$\eta = 74{\cdot}6/0{\cdot}93 = 80$ kW

$\phi_1 = \cos^{-1}(0{\cdot}75) = 41{\cdot}41^\circ$

$\tan \phi_1 = \tan 41{\cdot}41^\circ = 0{\cdot}8819$

$\phi_2 = \cos^{-1}(0{\cdot}95) = 18{\cdot}19^\circ$

$\tan \phi_2 = \tan 18{\cdot}19^\circ = 0{\cdot}3288$

Leading kVAR taken by the condenser bank

$= P(\tan \phi_1 - \tan \phi_2)$

$= 80(0{\cdot}8819 - 0{\cdot}3288) = 44{\cdot}25$ kVAR

Leading kVAR taken by each of three sets

$= 44{\cdot}25/3 = 14{\cdot}75$ kVAR ... (*i*)

Fig. 6.11 shows the delta* connected condenser bank. Let C farad be the capacitance of 4 capacitors in each phase.

Phase current of capacitor is

$$I_{CP} = V_{ph}/X_C = 2\pi f C V_{ph}$$
$$= 2\pi \times 50 \times C \times 400$$
$$= 1{,}25{,}600\ C \text{ amperes}$$

$$\text{kVAR/phase} = \frac{V_{ph}\, I_{CP}}{1000}$$
$$= \frac{400 \times 1{,}25{,}600\ C}{1000}$$
$$= 50240\ C \quad \text{... (}ii\text{)}$$

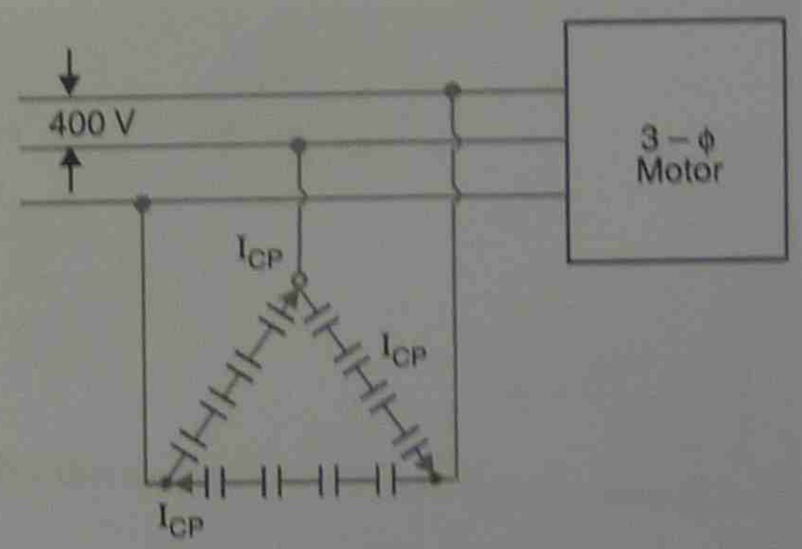

Fig. 6.11

* In practice, capacitors are always connected in delta since the capacitance of the capacitor required is one-third of that required for star connection.

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Assuming the area of X-section of neutral wire to be half that of the outer wire,

Volume of conductor material required

$$= 2{\cdot}5\, a_6\, l = 2{\cdot}5 \left(\frac{P^2 \rho\, l}{W\, V_m^2 \cos^2 \phi} \right) l = \frac{2{\cdot}5\, P^2 \rho\, l^2}{W\, V_m^2 \cos^2 \phi}$$

$$= \frac{2{\cdot}5}{\cos^2 \phi} \times \frac{P^2 \rho\, l^2}{W\, V_m^2}$$

$$= \frac{5K}{8 \cos^2 \phi} \qquad \left[\because K = \frac{4P^2 \rho\, l^2}{W\, V_m^2} \right]$$

Hence, the volume of conductor material required in this system is $5/8 \cos^2 \phi$ times that required in a 2-wire d.c. system with one conductor earthed.

7. Two phase, 4-wire a.c. system. As shown in Fig. 7.10, the four wires are taken from the ends of the two-phase windings and the mid-points of the two windings are connected together. This system can be considered as two independent single phase systems, each transmitting one half of the total power.

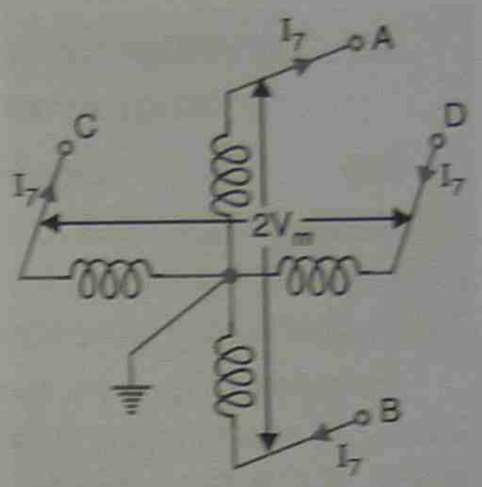

Fig. 7.10

Max. voltage between outers A and $B = 2V_m$

R.M.S. value of voltage $= 2V_m / \sqrt{2} = \sqrt{2}\, V_m$

Power supplied per phase (*i.e.*, by outers A and B) $= P/2$

Assuming p.f. of the load to be $\cos \phi$,

$$\text{Load current, } I_7 = \frac{P/2}{\sqrt{2}\, V_m \cos \phi} = \frac{P}{2\sqrt{2}\, V_m \cos \phi}$$

Let a_7 be the area of X-section of one conductor.

$$\text{Line losses, } W = 4 I_7^2 R_7 = 4 \left(\frac{P}{2\sqrt{2}\, V_m \cos \phi} \right)^2 \times \frac{\rho\, l}{a_7}$$

$$\therefore \quad W = \frac{P^2 \rho\, l}{2 a_7\, V_m^2 \cos^2 \phi}$$

$$\therefore \quad \text{Area of X-section, } a_7 = \frac{P^2 \rho\, l}{2\, W\, V_m^2 \cos^2 \phi}$$

$\therefore$ Volume of conductor material required

$$= 4\, a_7\, l$$

$$= 4 \left(\frac{P^2 \rho\, l}{2\, W\, V_m^2 \cos^2 \phi} \right) l = \frac{4P^2 \rho\, l^2}{2\, W\, V_m^2 \cos^2 \phi}$$

$$= \frac{1}{2 \cos^2 \phi} \times \frac{4P^2 \rho\, l^2}{W\, V_m^2}$$

$$= \frac{K}{2 \cos^2 \phi} \qquad \left[\because K = \frac{4P^2 \rho\, l^2}{W\, V_m^2} \right]$$

Hence, the volume of conductor material required for this system is $1/2 \cos^2 \phi$ times that of 2-wire d.c. system with one conductor earthed.

8. Two-phase, 3-wire system. Fig. 7.11 shows two-phase, 3-wire a.c. system. The third or neutral wire is taken from the junction of two-phase windings whose voltages are in quadrature with each other. Obviously, each phase transmits one half of the total power. The R.M.S. voltage between

tween metal fittings and the line conductor. The guard ring is contoured in such a way that shunt capacitance currents i_1, i_2 etc. are equal to metal fitting line capacitance currents i'_1, i'_2 etc. The result is that same charging current I flows through each unit of string. Consequently, there will be uniform potential distribution across the units.

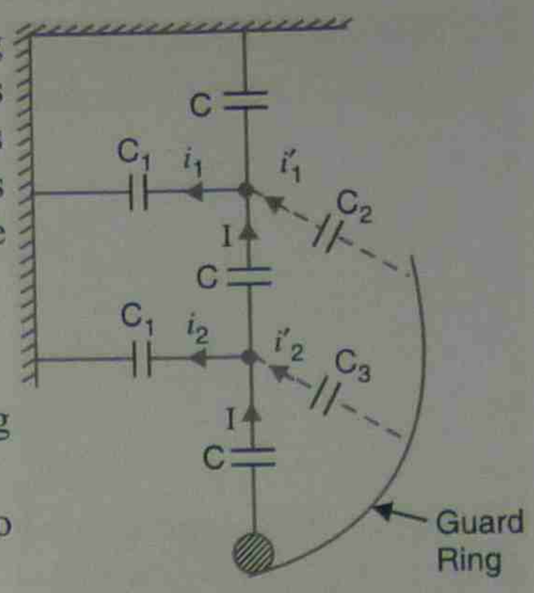

Fig. 8.13

8.9 Important Points

While solving problems relating to string efficiency, the following points must be kept in mind:

(*i*) The maximum voltage appears across the disc nearest to the conductror (*i.e.*, line conductor).

(*ii*) The voltage across the string is equal to phase voltage *i.e.*,

Voltage across string = Voltage between line and earth = Phase Voltage

(*iii*) Line Voltage = $\sqrt{3}$ × Voltage across string

Example 8.1. *In a 33 kV overhead line, there are three units in the string of insulators. If the capacitance between each insulator pin and earth is 11% of self-capacitance of each insulator, find (i) the distribution of voltage over 3 insulators and (ii) string efficiency.*

Solution. Fig. 8.14. shows the equivalent circuit of string insulators. Let V_1, V_2 and V_3 be the voltage across top, middle and bottom unit respectively. If C is the self-capacitance of each unit, then KC will be the shunt capacitance.

$$K = \frac{\text{Shunt Capacitance}}{\text{Self-capacitance}} = 0.11$$

Voltage across string, $V = 33/\sqrt{3} = 19{\cdot}05$ kV

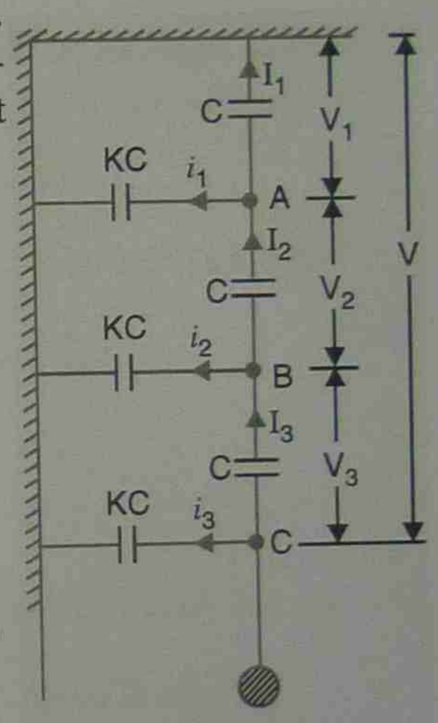

Fig. 8.14

At Junction A

$$I_2 = I_1 + i_1$$

or $$V_2 \omega C = V_1 \omega C + V_1 K \omega C$$

or $$V_2 = V_1 (1 + K) = V_1 (1 + 0{\cdot}11)$$

or $$V_2 = 1{\cdot}11\, V_1 \quad \ldots(i)$$

At Junction B

$$I_3 = I_2 + i_2$$

or $$V_3 \omega C = V_2 \omega C + (V_1 + V_2) K \omega C$$

or $$V_3 = V_2 + (V_1 + V_2) K$$

$$= 1{\cdot}11 V_1 + (V_1 + 1{\cdot}11\, V_1)\, 0{\cdot}11$$

$\therefore$ $$V_3 = 1{\cdot}342\, V_1$$

(*i*) Voltage across the whole string is

$$V = V_1 + V_2 + V_3 = V_1 + 1{\cdot}11\, V_1 + 1{\cdot}342\, V_1 = 3{\cdot}452\, V_1$$

or $$19{\cdot}05 = 3{\cdot}452\, V_1$$

$\therefore$ Voltage across top unit, $V_1 = 19{\cdot}05/3{\cdot}452 = \mathbf{5{\cdot}52\ kV}$

Voltage across middle unit, $V_2 = 1{\cdot}11\, V_1 = 1{\cdot}11 \times 5{\cdot}52 = \mathbf{6.13\ kV}$

Voltage across bottom unit, $V_3 = 1{\cdot}342\, V_1 = 1{\cdot}342 \times 5{\cdot}52 = \mathbf{7.4\ kV}$

(*ii*) String efficiency = $\dfrac{\text{Voltage across string}}{\text{No. of insulators} \times V_3} \times 100 = \dfrac{19{\cdot}05}{3 \times 7{\cdot}4} \times 100 = \mathbf{85.8\%}$

Example 8.2. *A 3-phase transmission line is being supported by three disc insulators. The potentials across top unit (i.e., near to the tower) and middle unit are 8 kV and 11 kV respectively. Calculate (i) the ratio of capacitance between pin and earth to the self-capacitance of each unit (ii)the line voltage and (iii) string efficiency.*

Solution. The equivalent circuit of string insulators is the same as shown in Fig. 8.14. It is given that $V_1 = 8$ kV and $V_2 = 11$ kV.

(*i*) Let K be the ratio of capacitance between pin and earth to self capacitance. If C farad is the self capacitance of each unit, then capacitance between pin and earth = KC.

Applying Kirchoff's current law to Junction A,

$$I_2 = I_1 + i_1$$

or $$V_2 \omega C = V_1 \omega C + V_1 K \omega C$$

or $$V_2 = V_1 (1 + K)$$

$\therefore$ $$K = \frac{V_2 - V_1}{V_1} = \frac{11 - 8}{8} = \mathbf{0{\cdot}375}$$

(*ii*) Applying Kirchoff's current law to Junction B,

$$I_3 = I_2 + i_2$$

or $$V_3 \omega C = V_2 \omega C + (V_1 + V_2) K \omega C$$

or $$V_3 = V_2 + (V_1 + V_2) K = 11 + (8 + 11) \times 0{\cdot}375 = 18{\cdot}12 \text{ kV}$$

Voltage between line and earth $= V_1 + V_2 + V_3 = 8 + 11 + 18{\cdot}12 = 37{\cdot}12$ kV

$\therefore$ Line Voltage $= \sqrt{3} \times 37{\cdot}12 = \mathbf{64{\cdot}28 \text{ kV}}$

(*iii*) String efficiency $= \dfrac{\text{Voltage across string}}{\text{No. of insulators} \times V_3} \times 100 = \dfrac{37{\cdot}12}{3 \times 18{\cdot}12} \times 100 = \mathbf{68.28\%}$

Example 8.3. *Each line of a 3-phase system is suspended by a string of 3 similar insulators. If the voltage across the line unit is 17·5 kV, calculate the line to neutral voltage. Assume that the shunt capacitance betwen each insulator and earth is 1/8th of the capacitance of the insulator itself. Also find the string efficiency.*

Solution. Fig. 8·15 shows the equivalent circuit of string insulators. If C is the self capacitance of each unit, then KC will be the shunt capacitance where $K = 1/8 = 0{\cdot}125$.

Voltage across line unit, $V_3 = 17{\cdot}5$ kV

At Junction A

$$I_2 = I_1 + i_1$$

$$V_2 \omega C = V_1 \omega C + V_1 K \omega C$$

or $$V_2 = V_1 (1 + K) = V_1 (1 + 0.125)$$

$\therefore$ $$V_2 = 1{\cdot}125\, V_1$$

At Junction B

$$I_3 = I_2 + i_2$$

or $$V_3 \omega C = V_2 \omega C + (V_1 + V_2) K \omega C$$

or $$V_3 = V_2 + (V_1 + V_2) K$$

$$= 1{\cdot}125\, V_1 + (V_1 + 1{\cdot}125\, V_1) \times 0.125$$

$\therefore$ $$V_3 = 1{\cdot}39\, V_1$$

Voltage across top unit, $V_1 = V_3/1{\cdot}39 = 17{\cdot}5/1{\cdot}39$

$= 12{\cdot}59$ kV

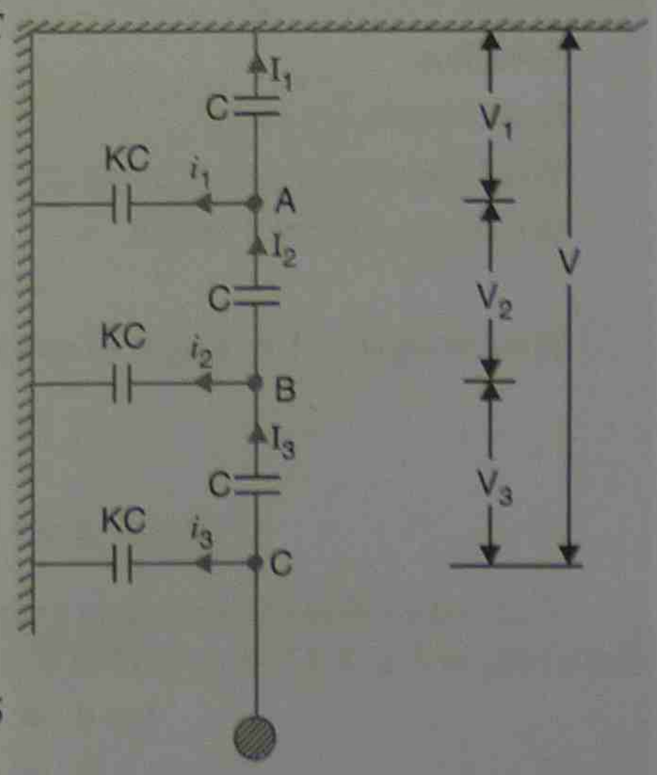

Fig. 8.15

When the conductor has wind and ice loading also, the following points may be noted :

(*i*) The conductor sets itself in a plane at an angle θ to the vertical where

$$\tan \theta = \frac{w_w}{w + w_i}$$

(*ii*) The sag in the conductor is given by :

$$S = \frac{w_t l^2}{2T}$$

Hence S represents the slant sag in a direction making an angle θ to the vertical. *If no specific mention is made in the problem, then slant slag is calculated by using the above formula.*

(*iii*) The vertical sag $= S \cos \theta$

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Example 8.17. *A 132 kV transmission line has the following data :*

Wt. of conductor = 680 kg/km ; *Length of span = 260 m*

Ultimate strength = 3100 kg ; *Safety factor = 2*

Calculate the height above ground at which the conductor should be supported. Ground clearance required is 10 metres.

Solution.

Wt. of conductor/metre run, $w = 680/1000 = 0{\cdot}68$ kg

Working tension, $T = \frac{\text{Ultimate strength}}{\text{Safety factor}} = \frac{3100}{2} = 1550$ kg

Span length, $l = 260$ m

$\therefore$ Sag $= \frac{w l^2}{8T} = \frac{0{\cdot}68 \times (260)^2}{8 \times 1550} = 3{\cdot}7$ m

$\therefore$ Conductor should be supported at a height of $10 + 3{\cdot}7 = \mathbf{13{\cdot}7\ m}$

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Example 8.18. *A transmission line has a span of 150 m between level supports. The conductor has a cross-sectional area of 2 cm^2. The tension in the conductor is 2000 kg. If the specific gravity of the conductor material is 9·9 gm/cm^3 and wind pressure is 1·5 kg/m length, calculate the sag. What is the vertical sag?*

Solution.

Span length, $l = 150$ m; Working tension, $T = 2000$ kg

Wind force/m length of conductor, $w_w = 1{\cdot}5$ kg

Wt. of conductor/m length, w = Sp. Gravity × Volume of 1 m conductor

$= 9{\cdot}9 \times 2 \times 100 = 1980$ gm $= 1{\cdot}98$ kg

Total weight of 1 m length of conductor is

$$w_t = \sqrt{w^2 + w_w^2} = \sqrt{(1{\cdot}98)^2 + (1{\cdot}5)^2} = 2{\cdot}48 \text{ kg}$$

$\therefore$ Sag, $S = \frac{w_t l^2}{8T} = \frac{2{\cdot}48 \times (150)^2}{8 \times 2000} = \mathbf{3{\cdot}48\ m}$

This is the value of slant sag in a direction making an angle θ with the vertical. Referring to Fig. 8.27, the value of θ is given by ;

$\tan \theta = w_w/w = 1{\cdot}5/1{\cdot}98 = 0{\cdot}76$

$\therefore \quad \theta = \tan^{-1} 0{\cdot}76 = 37{\cdot}23°$

$\therefore$ Vertical sag $= S \cos \theta$

$= 3{\cdot}48 \times \cos 37{\cdot}23° = \mathbf{2{\cdot}77\ m}$

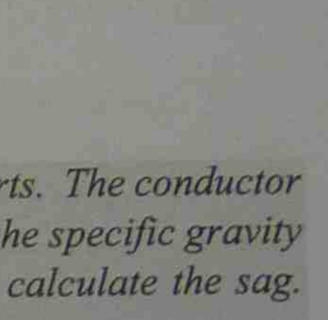

Fig. 8.27

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(*i*) *Single phase line*

$$\text{Inductance/conductor/m} = 2 \times 10^{-7} \log_e \frac{D_m}{D_s}$$

where $D_s = 0{\cdot}7788\, r$ and D_m = Spacing between conductors = d

(*ii*) *Single circuit 3-ϕ line*

$$\text{Inductance/phase/m} = 2 \times 10^{-7} \log_e \frac{D_m}{D_s}$$

where $D_s = 0{\cdot}7788\, r$ and $D_m = (d_1\, d_2\, d_3)^{1/3}$

(*iii*) *Double circuit 3-ϕ line*

$$\text{Inductance/phase/m} = 2 \times 10^{-7} \log_e \frac{D_m}{D_s}$$

where $D_s = (D_{s1}\, D_{s2}\, D_{s3})^{1/3}$ and $D_m = (D_{AB} \times D_{BC} \times D_{CA})^{1/3}$

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Example 9.1. *A single phase line has two parallel conductors 2 metres apart. The diameter of each conductor is 1·2 cm. Calculate the loop inductance per km of the line.*

Solution.

Spacing of conductors, $d = 2\text{ m} = 200\text{ cm}$

Radius of conductor, $r = 1{\cdot}2/2 = 0{\cdot}6\text{ cm}$

Loop inductance per metre length of the line

$= 10^{-7}(1 + 4 \log_e d/r)$ H

$= 10^{-7}(1 + 4 \log_e 200/0{\cdot}6)$ H

$= 24{\cdot}23 \times 10^{-7}$ H

Inductance Measurement using bridge

Loop inductance per km of the line

$= 24{\cdot}23 \times 10^{-7} \times 1000 = 24{\cdot}23 \times 10^{-4}\text{ H} = \mathbf{2{\cdot}423\text{ mH}}$

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Example 9.2. *A single phase transmission line has two parallel conductors 3 m apart, the radius of each conductor being 1 cm. Calculate the loop inductance per km length of the line if the material of the conductor is (i) copper (ii) steel with relative permeability of 100.*

Solution.

Spacing of conductors, $d = 300\text{ cm}$

Radius of conductor, $r = 1\text{ cm}$

Loop inductance $= 10^{-7}(\mu_r + 4 \log_e d/r)$ H/m

(*i*) With copper conductors, $\mu_r = 1$

$\therefore$ Loop inductance/m $= 10^{-7}(1 + 4 \log_e d/r)\text{ H} = 10^{-7}(1 + 4 \log_e 300/1)$ H

$= 23{\cdot}8 \times 10^{-7}$ H

Loop inductance/km $= 23{\cdot}8 \times 10^{-7} \times 1000 = 2{\cdot}38 \times 10^{-3}\text{ H} = \mathbf{2{\cdot}38\text{ mH}}$

(*ii*) With steel conductors, $\mu_r = 100$

$\therefore$ Loop inductance/m $= 10^{-7}(100 + 4 \log_e 300/1)\text{ H} = 122{\cdot}8 \times 10^{-7}$ H

Loop inductance/km $= 122{\cdot}8 \times 10^{-7} \times 1000 = 12{\cdot}28 \times 10^{-3}\text{ H} = \mathbf{12{\cdot}28\text{ mH}}$

Example 9.3. *Find the inductance per km of a 3-phase transmission line using 1·24 cm diameter conductors when these are placed at the corners of an equilateral triangle of each side 2 m.*

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$$\%\text{age Voltage regulation} = \frac{I R \cos \phi_R - I X_L \sin \phi_R}{V_R} \times 100 \qquad (\textit{for leading p.f.})$$

The following conclusions can be drawn from the above expressions :

(*i*) When the load p.f. is lagging or unity or such leading that $I R \cos \phi_R > I X_L \sin \phi_R$, then voltage regulation is positive *i.e.*, receiving end voltage V_R will be less than the sending end voltage V_S.

(*ii*) For a given V_R and I, the voltage regulation of the line increases with the decrease in p.f. for lagging loads.

(*iii*) When the load p.f. is leading to this extent that $I X_L \sin \phi_R > I R \cos \phi_R$, then voltage regulation is negative *i.e.* the receiving end voltage V_R is more than the sending end voltage V_S.

(*iv*) For a given V_R and I, the voltage regulation of the line decreases with the decrease in p.f. for leading loads.

2. Effect on transmission efficiency. The power delivered to the load depends upon the power factor.

$$P = V_R * I \cos \phi_R \quad \text{(For 1-phase line)}$$

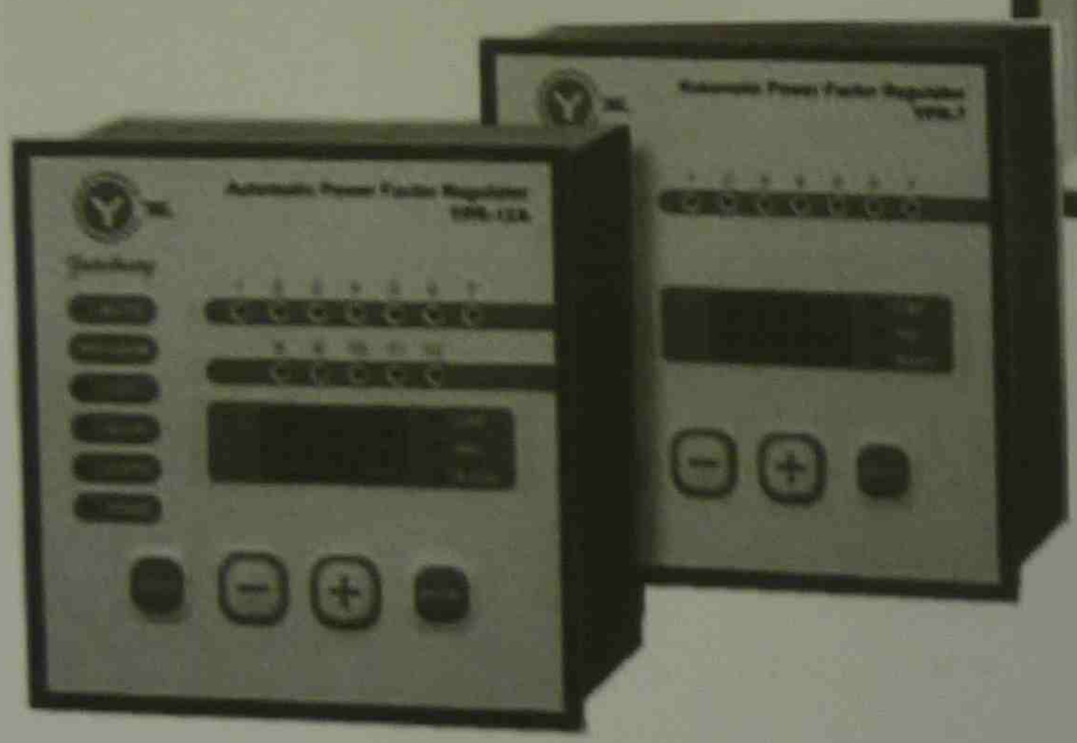
$$\therefore \quad I = \frac{P}{V_R \cos \phi_R}$$

$$P = 3 V_R I \cos \phi_R \quad \text{(For 3-phase line)}$$

$$\therefore \quad I = \frac{P}{3 V_R \cos \phi_R}$$

It is clear that in each case, for a given amount of power to be transmitted (P) and receiving end voltage (V_R), the load current I is inversely proportional to the load p.f. $\cos \phi_R$. Consequently, with the decrease in load p.f., the load current and hence the line losses are increased. This leads to the conclusion that transmission efficiency of a line decreases with the decrease in load p.f. and *vice-versa*.

Power Factor Meter


Power Factor Regulator

Example 10.1. *A single phase overhead transmission line delivers 1100 kW at 33 kV at 0·8 p.f. lagging. The total resistance and inductive reactance of the line are 10 Ω and 15 Ω respectively. Determine : (i) sending end voltage (ii) sending end power factor and (iii) transmission efficiency.*

Solution.

Load power factor, $\cos \phi_R = 0{\cdot}8$ lagging

Total line impedance, $\vec{Z} = R + j X_L = 10 + j\,15$

---

* $I_R = I_S = I$

As stated earlier, this method gives fairly correct results for lagging p.f. The reader will find that this method is used in the solution of some numericals.

Example 10.2. *What is the maximum length in km for a 1-phase transmission line having copper conductor of 0·775 cm^2 cross-section over which 200 kW at unity power factor and at 3300V are to be delivered ? The efficiecny of transmission is 90%. Take specific resistance as 1.725 μ Ω cm.*

Solution.

Receiving end power = 200 kW = 2,00,000 W

Transmission efficiency = 0·9

$\therefore$ Sending end power $= \dfrac{2,00,000}{0\cdot 9} = 2,22,222$ W

$\therefore$ Line losses = 2,22,222 – 2,00,000 = 22,222 W

Line current, $I = \dfrac{200\times 10^3}{3,300\times 1} = 60\cdot 6$ A

Let $R\ \Omega$ be the resistance of one conductor.

Line losses $= 2I^2R$

or $22,222 = 2\,(60\cdot 6)^2 \times R$

$\therefore$ $R = \dfrac{22,222}{2\times (60\cdot 6)^2} = 3\cdot 025\ \Omega$

Now, $R = \rho\, l/a$

$\therefore$ $l = \dfrac{Ra}{\rho} = \dfrac{3\cdot 025\times 0\cdot 775}{1\cdot 725\times 10^{-6}} = 1\cdot 36\times 10^6$ cm = **13·6 km**

Example 10.3. *An overhead 3-phase transmission line delivers 5000 kW at 22 kV at 0·8 p.f. lagging. The resistance and reactance of each conductor is 4 Ω and 6 Ω respectively. Determine : (i) sending end voltage (ii) percentage regulation (iii) transmission efficiency.*

Solution.

Load power factor, $\cos\phi_R$ = 0·8 lagging

Receiving end voltage/phase,* $V_R = 22,000/\sqrt{3} = 12,700$ V

Impedance/phase, $\vec{Z} = 4 + j\,6$

Line current, $I = \dfrac{5000\times 10^3}{3\times 12700\times 0\cdot 8} = 164$ A

As $\cos\phi_R = 0\cdot 8$ $\quad\therefore\ \sin\phi_R = 0\cdot 6$

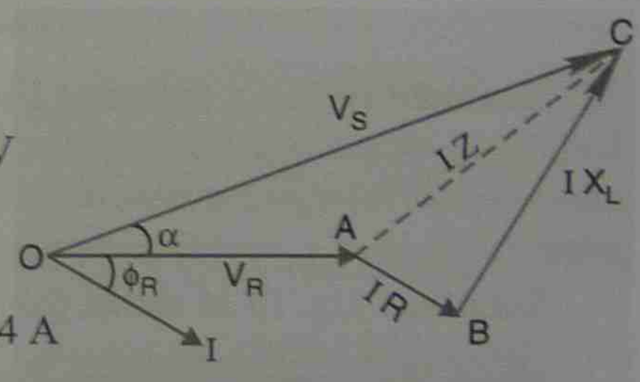


Fig. 10.6

Taking $\overrightarrow{V_R}$ as the reference phasor (see Fig. 10.6),

$\overrightarrow{V_R} = V_R + j\,0 = 12700$ V

$\vec{I} = I\,(\cos\phi_R - j\sin\phi_R) = 164\,(0\cdot 8 - j\,0\cdot 6) = 131\cdot 2 - j\,98\cdot 4$

(*i*) Sending end voltage per phase is

$\overrightarrow{V_S} = \overrightarrow{V_R} + \vec{I}\,\vec{Z} = 12700 + (131\cdot 2 - j\,98\cdot 4)\,(4 + j\,6)$

$= 12700 + 524\cdot 8 + j\,787\cdot 2 - j\,393\cdot 6 + 590\cdot 4$

$= 13815\cdot 2 + j\,393\cdot 6$

Magnitude of $V_S = \sqrt{(13815\cdot 2)^2 + (393\cdot 6)^2} = 13820\cdot 8$ V

* If not mentioned in the problem, star-connection is understood.

$$= \vec{I_R} + \vec{V_R}\frac{\vec{Y}}{2} + \frac{\vec{V_R}\vec{Y}}{2} + \frac{\vec{V_R}\vec{Y}^2\vec{Z}}{4} + \frac{\vec{Y}\vec{I_R}\vec{Z}}{2}$$

$$= \vec{I_R}\left(1 + \frac{\vec{Y}\vec{Z}}{2}\right) + \vec{V_R}\vec{Y}\left(1 + \frac{\vec{Y}\vec{Z}}{4}\right) \quad ...(xi)$$

Comparing equations (*x*) and (*xi*) with those of (*i*) and (*ii*), we get,

$$\vec{A} = \vec{D} = \left(1 + \frac{\vec{Y}\vec{Z}}{2}\right); \quad \vec{B} = \vec{Z}; \quad \vec{C} = \vec{Y}\left(1 + \frac{\vec{Y}\vec{Z}}{4}\right)$$

Also
$$\vec{A}\vec{D} - \vec{B}\vec{C} = \left(1 + \frac{YZ}{2}\right)^2 - ZY\left(1 + \frac{YZ}{4}\right)$$

$$= 1 + \frac{Y^2Z^2}{4} + YZ - ZY - \frac{Z^2Y^2}{4} = 1$$

(*iv*) **Long lines—Rigorous method.** By rigorous method, the sending end voltage and current of a long transmission line are given by :

$$V_S = V_R \cosh\sqrt{YZ} + I_R\sqrt{\frac{Z}{Y}}\sinh\sqrt{YZ}$$

$$I_S = V_R\sqrt{\frac{Y}{Z}}\sinh\sqrt{YZ} + I_R\cosh\sqrt{YZ}$$

Comparing these equations with those of (*i*) and (*ii*), we get,

$$\vec{A} = \vec{D} = \cosh\sqrt{YZ}; \quad \vec{B} = \sqrt{\frac{Z}{Y}}\sinh\sqrt{YZ}; \quad \vec{C} = \sqrt{\frac{Y}{Z}}\sinh\sqrt{YZ}$$

Incidentally

$$\vec{A}\vec{D} - \vec{B}\vec{C} = \cosh\sqrt{YZ} \times \cosh\sqrt{YZ} - \sqrt{\frac{Z}{Y}}\sinh\sqrt{YZ} \times \sqrt{\frac{Y}{Z}}\sinh\sqrt{YZ}$$

$$= \cosh^2\sqrt{YZ} - \sinh^2\sqrt{YZ} = 1$$

Example 10.16. *A balanced 3-phase load of 30 MW is supplied at 132 kV, 50 Hz and 0·85 p.f. lagging by means of a transmission line. The series impedance of a single conductor is (20 + j52) ohms and the total phase-neutral admittance is 315×10^{-6} siemen. Using nominal T method, determine: (i) the A, B, C and D constants of the line (ii) sending end voltage (iii) regulation of the line.*

Solution. Fig. 10.26 shows the representation of 3-phase line on the single phase basis.

Series line impedance/phase, $\vec{Z} = (20 + j\,52)\ \Omega$

Shunt admittance/phase, $\vec{Y} = j\,315 \times 10^{-6}$ S

(*i*) **Generalised constants of line.** For nominal *T* method, various constants have the values as under :

$$\vec{A} = \vec{D} = 1 + \vec{Z}\,\vec{Y}/2 = 1 + \frac{20 + j\,52}{2} \times j\,315 \times 10^{-6}$$

$$= 0{\cdot}992 + j\,0{\cdot}00315 = \mathbf{0{\cdot}992\ \angle 0{\cdot}18^\circ}$$

$$\vec{B} = \vec{Z}\left(1 + \frac{\vec{Z}\vec{Y}}{4}\right) = (20 + j\,52)\left[1 + \frac{(20 + j\,52)j\,315 \times 10^{-6}}{4}\right]$$

$$= 19{\cdot}84 + j\,51{\cdot}82 = \mathbf{55{\cdot}5\ \angle 69^\circ}$$

$$\vec{C} = \vec{Y} = \mathbf{0{\cdot}000315\ \angle 90^\circ}$$

Solution.

Receiving end voltage/phase, $V_R = 132 \times 10^3/\sqrt{3} = 76210$ V

Receiving end current, $I_R = \dfrac{50 \times 10^6}{\sqrt{3} \times 132 \times 10^3 \times 0{\cdot}8} = 273$ A

$\cos \phi_R = 0{\cdot}8; \quad \sin \phi_R = 0{\cdot}6$

Taking receiving end voltage as the reference phasor, we have,

$$\overrightarrow{V_R} = V_R + j\,0 = 76210 \angle 0^\circ$$

$$\overrightarrow{I_R} = I_R \angle -\phi_R = 273 \angle -36{\cdot}9^\circ$$

Sending end voltage per phase is

$$\begin{aligned}\overrightarrow{V_S} &= \vec{A}\ \overrightarrow{V_R} + \vec{B}\ \overrightarrow{I_R} \\ &= 0{\cdot}95 \angle 1{\cdot}4^\circ \times 76210 \angle 0^\circ + 96 \angle 78^\circ \times 273 \angle -36{\cdot}9^\circ \\ &= 72400 \angle 1{\cdot}4^\circ + 26208 \angle 41{\cdot}1^\circ \\ &= 72400 (\cos 1{\cdot}4^\circ + j \sin 1{\cdot}4^\circ) + 26208 (\cos 41{\cdot}1^\circ + j \sin 41{\cdot}1^\circ) \\ &= 72400 (0{\cdot}9997 + j\,0{\cdot}0244) + 26208 (0{\cdot}7536 + j\,0{\cdot}6574) \\ &= (72378 + j\,1767) + (19750 + j\,17229) \\ &= 92128 + j\,18996 = 94066 \angle 11{\cdot}65^\circ \text{ V}\end{aligned}$$

Sending end current, $\overrightarrow{I_S} = \vec{C}\ \overrightarrow{V_R} + \vec{D}\ \overrightarrow{I_R}$

$$\begin{aligned}&= 0{\cdot}0015 \angle 90^\circ \times 76210 \angle 0^\circ + 0{\cdot}95 \angle 1{\cdot}4^\circ \times 273 \angle -36{\cdot}9^\circ \\ &= 114 \angle 90^\circ + 260 \angle -35{\cdot}5^\circ \\ &= 114 (\cos 90^\circ + j \sin 90^\circ) + 260 (\cos 35{\cdot}5^\circ - j \sin 35{\cdot}5^\circ) \\ &= 114 (0 + j) + 260 (0{\cdot}814 - j\,0{\cdot}58) \\ &= j\,114 + 211 - j\,150 = 211 - j\,36\end{aligned}$$

Charging current, $\overrightarrow{I_C} = \overrightarrow{I_S} - \overrightarrow{I_R} = (211 - j36) - 273 \angle -36{\cdot}9^\circ$

$$= (211 - j\,36) - (218 - j\,164) = -7 + j\,128 = \mathbf{128{\cdot}2 \angle 93{\cdot}1^\circ\ A}$$

$$\% \text{ Regulation} = \frac{(V_S/A) - V_R}{V_R} \times 100 = \frac{94066/0{\cdot}95 - 76210}{76210} \times 100 = \mathbf{30\%}$$

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Example 10.18. *Find the following for a single circuit transmission line delivering a load of 50 MVA at 110 kV and p.f. 0·8 lagging :*

(i) sending end voltage (ii) sending end current (iii) sending end power (iv) efficiency of transmission. Given $A = D = 0{\cdot}98 \angle 3^\circ$; $B = 110 \angle 75^\circ$ ohm ; $C = 0{\cdot}0005 \angle 80^\circ$ siemen.

Solution.

Receiving end voltage/phase,

$$V_R = \frac{110}{\sqrt{3}} = 63{\cdot}5 \text{ kV}$$

Receiving end current, $I_R = \dfrac{50 \times 10^6}{\sqrt{3} \times 110 \times 10^3} = 262{\cdot}4$ A

Taking receiving end voltage as the reference phasor, we have,

$$\overrightarrow{V_R} = (63500 + j0)$$

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$$C_N = C_e + C_{eq}$$
$$= C_e + 3C_c$$

If V_{ph} is the phase voltage, then charging current I_C is given by ;

$$I_C = \frac{V_{ph}}{\text{Capacitive reactance per phase}}$$
$$= 2\pi f V_{ph} C_N$$
$$= 2\pi f V_{ph} (C_e + 3C_c)$$

11.15 Measurements of C_e and C_c

Although core-core capacitance C_c and core-earth capacitance C_e can be obtained from the empirical formulas for belted cables, their values can also be determined by measurements. For this purpose, the following two measurements are required :

(*i*) In the first measurement, the three cores are bunched together (*i.e.* commoned) and the capacitance is measured between the bunched cores and the sheath. The bunching eliminates all the three capacitors C_c, leaving the three capacitors C_e in parallel. Therefore, if C_1 is the measured capacitance, this test yields :

$$C_1 = 3\,C_e$$

or
$$C_e = \frac{C_1}{3}$$

Knowing the value of C_1, the value of C_e can be determined.

(*ii*) In the second measurement, two cores are bunched with the sheath and capacitance is measured between them and the third core. This test yields $2C_c + C_e$. If C_2 is the measured capacitance, then,

$$C_2 = 2C_c + C_e$$

As the value of C_e is known from first test and C_2 is found experminentally, therefore, value of C_c can be determined.

It may be noted here that if value of $C_N (= C_e + 3C_c)$ is desired, it can be found directly by another test. In this test, the capacitance between two cores or lines is measured with the third core free or connected to the sheath. This eliminates one of the capacitors C_e so that if C_3 is the measured capacitance, then,

$$C_3 = C_c + \frac{C_c}{2} + \frac{C_e}{2}$$
$$= \frac{1}{2}(C_e + 3C_c)$$
$$= \frac{1}{2} C_N$$

Example 11.17. *The capacitance per kilometre of a 3-phase belted cable is 0·3 μ F between the two cores with the third core connected to the lead sheath. Calculate the charging current taken by five kilometres of this cable when connected to a 3-phase, 50 Hz, 11 kV supply.*

Solution. The capacitance between a pair of cores with third core earthed for a length of 5 km is

$$C_3 = 0{\cdot}3 \times 5 = 1{\cdot}5\ \mu F$$

$$V_{ph} = \frac{11 \times 10^3}{\sqrt{3}} = 6351\ V\ ;\ f = 50\ \text{Hz}$$

As proved in Art 11.15, core to neutral capacitance C_N of this cable is given by :

$$C_N = 2C_3 = 2 \times 1{\cdot}5 = 3\ \mu F$$

∴ Charging current, $I_C = 2\pi f V_{ph} C_N$
$= 2\pi \times 50 \times 6351 \times 3 \times 10^{-6}$ A = **5·98 A**

Example 11.18. *The capacitances of a 3-phase belted cable are 12·6 μF between the three cores bunched together and the lead sheath and 7·4 μF between one core and the other two connected to sheath. Find the charging current drawn by the cable when connected to 66 kV, 50 Hz supply.*

Solution.

Here, $V_{ph} = \frac{66 \times 10^3}{\sqrt{3}} = 38105\ V$; $f = 50$ Hz ; $C_1 = 12{\cdot}6\ \mu F$; $C_2 = 7{\cdot}4\ \mu F$

Let core-core and core-earth capacitances of the cable be C_c and C_e respectively. As proved in Art. 11.15,

$C_1 = 3\,C_e$

∴ $C_e = C_1/3 = 12{\cdot}6/3 = 4{\cdot}2\ \mu F$

and $C_2 = 2C_c + C_e$

∴ $C_c = \frac{C_2 - C_e}{2} = \frac{7{\cdot}4 - 4{\cdot}2}{2} = 1{\cdot}6\ \mu F$

∴ Core to neutral capacitance is

$C_N = C_e + 3C_c = 4{\cdot}2 + 3 \times 1{\cdot}6 = 9\ \mu F$

Charging current, $I_C = 2\pi f V_{ph} C_N$
$= 2\pi \times 50 \times 38105 \times 9 \times 10^{-6}\ A$ = **107·74 A**

Example 11.19. *The capacitance per kilometre of a 3-phase belted cable is 0·18 μF between two cores with the third core connected to sheath. Calculate the kVA taken by 20 km long cable when connected to 3-phase, 50 Hz, 3300 V supply.*

Solution. The capacitance between a pair of cores with third core earthed for a length of 20 km is

$C_3 = 0{\cdot}18 \times 20 = 3{\cdot}6\ \mu F$, $V_{ph} = 3300/\sqrt{3} = 1905\ V$; $f = 50$ Hz

Core to neutral capacitance, $C_N = 2C_3 = 2 \times 3{\cdot}6 = 7{\cdot}2\ \mu$F

Charging current, $I_C = 2\pi f V_{ph} C_N$
$= 2\pi \times 50 \times 1905 \times 7{\cdot}2 \times 10^{-6}$ A $= 4{\cdot}3$ A

kVA taken by the cable $= 3\,V_{ph} I_C = 3 \times 1905 \times 4{\cdot}3 \times 10^{-3}$ kVA = **24·57 kVA**

TUTORIAL PROBLEMS

1. The capacitances per kilometre of a 3-phase cable are 0·63 μF between the three cores bunched together and the sheath and 0·37 μF between one core and the other two connected to the sheath. Calculate the charging current taken by eight kilometres of this cable when connected to a 3-phase, 50 Hz, 6600 V supply. **[4·31 A]**
2. The capacitances of a 3-core belted type cable are measured as detailed below :
   (*i*) Between three cores bunched together and sheath is 8 μF
   (*ii*) Between a conductor and the other two connected to the sheath together is 6 μF.
   Calculate the capacitance per phase. **[23/3 μF]**
3. A 3-core, 3-phase belted cable tested for capacitance between a pair of cores on single phase, with the third core earthed, gave a capacitance of 0·4 μF per km. Calculate the charging current for 15 km length of this cable when connected to 22 kV, 3-phase, 50 Hz supply. **[48 A]**

11.16 Current-Carrying Capacity of Underground Cables

The safe current-carrying capacity of an underground cable is determined by the maximum permissible temperature rise. The cause of temperature rise is the losses that occur in a cable which appear

In d.c. distribution calculations, one important point of interest is the determination of point of minimum potential on the distributor. The point where it occurs depends upon the loading conditions and the method of feeding the distributor. The distributor is so designed that the minimum potential on it is not less than 6% of rated voltage at the consumer's terminals. In the next sections, we shall discuss some important cases of d.c. distributors separately.

13.3 D.C. Distributor Fed at one End—Concentrated Loading

Fig. 13.5 shows the single line diagram of a 2-wire d.c. distributor $A\,B$ fed at one end A and having concentrated loads I_1, I_2, I_3 and I_4 tapped off at points C, D, E and F respectively.

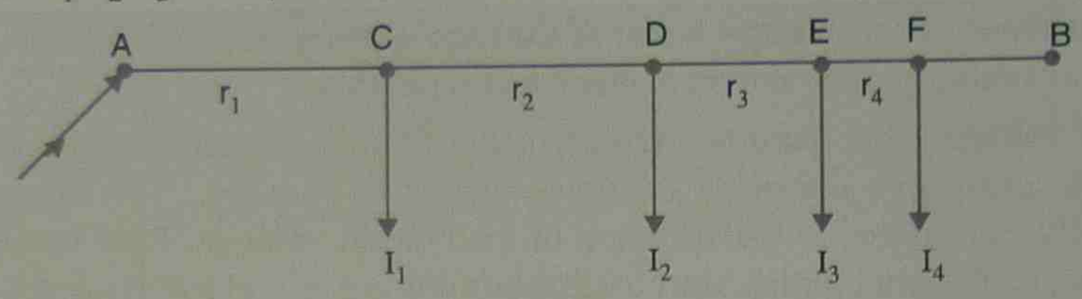

Fig. 13.5

Let r_1, r_2, r_3 and r_4 be the resistances of both wires (go and return) of the sections $A\,C$, CD, DE and EF of the distributor respectively.

Current fed from point A $= I_1 + I_2 + I_3 + I_4$

Current in section $A\,C$ $= I_1 + I_2 + I_3 + I_4$

Current in section CD $= I_2 + I_3 + I_4$

Current in section DE $= I_3 + I_4$

Current in section EF $= I_4$

Voltage drop in section $A\,C$ $= r_1\,(I_1 + I_2 + I_3 + I_4)$

Voltage drop in section CD $= r_2\,(I_2 + I_3 + I_4)$

Voltage drop in section DE $= r_3\,(I_3 + I_4)$

Voltage drop in section EF $= r_4\,I_4$

$\therefore$ Total voltage drop in the distributor

$= r_1\,(I_1 + I_2 + I_3 + I_4) + r_2\,(I_2 + I_3 + I_4) + r_3\,(I_3 + I_4) + r_4\,I_4$

It is easy to see that the minimum potential will occur at point F which is farthest from the feeding point A.

Example 13.1. *A 2-wire d.c. distributor cable AB is 2 km long and supplies loads of 100A, 150A,200A and 50A situated 500 m, 1000 m, 1600 m and 2000 m from the feeding point A. Each conductor has a resistance of 0·01 Ω per 1000 m. Calculate the p.d. at each load point if a p.d. of 300 V is maintained at point A.*

Solution. Fig. 13.6 shows the single line diagram of the distributor with its tapped currents.

Resistance per 1000 m of distributor $= 2 \times 0{\cdot}01 = 0{\cdot}02\ \Omega$

Resistance of section AC, R_{AC} $= 0{\cdot}02 \times 500/1000 = 0{\cdot}01\ \Omega$

Resistance of section CD, R_{CD} $= 0{\cdot}02 \times 500/1000 = 0{\cdot}01\ \Omega$

Resistance of section DE, R_{DE} $= 0{\cdot}02 \times 600/1000 = 0{\cdot}012\ \Omega$

Resistance of section EB, R_{EB} $= 0{\cdot}02 \times 400/1000 = 0{\cdot}008\ \Omega$

Referring to Fig. 13.6, the currents in the various sections of the distributor are :

$I_{EB} = 50$ A ; $I_{DE} = 50 + 200 = 250$ A

$I_{CD} = 250 + 150 = 400$ A ; $I_{AC} = 400 + 100 = 500$ A

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Example 13.28. *A 3-wire 500/250 V d.c. system has a load of 35 kW between the positive lead and the middle wire and a load of 20 kW between the negative lead and the middle wire. If there is a break in the middle wire, calculate the voltage between the outers and the middle wire.*

Solution. Fig. 13.53 (*i*) shows the arrangement before the disconnection of the middle wire. The voltages on the two sides of the middle wire are equal *i.e.* 250 V. Let R_1 be the resistance of load on the +ve side and R_2 be the resistance of load on the negative side.

$$R_1 = \frac{(250)^2}{35 \times 10^3} = 1{\cdot}785\ \Omega \quad ; \quad R_2 = \frac{(250)^2}{20 \times 10^3} = 3{\cdot}125\ \Omega \qquad [\because R = V^2/P]$$

When there is a break in the middle wire, two resistances R_1 and R_2 are put in series across 500 V as shown in Fig. 13.53 (*ii*).

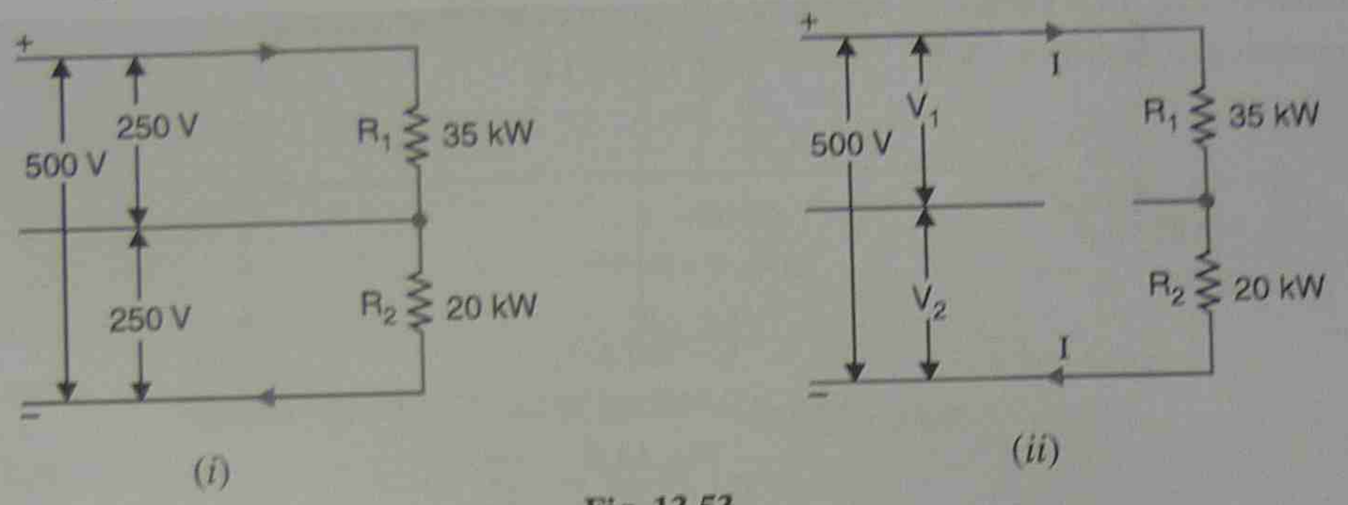

Fig. 13.53

∴ Circuit current, $I = \dfrac{500}{R_1 + R_2} = \dfrac{500}{1{\cdot}785 + 3{\cdot}125} = 101{\cdot}83$ A

∴ Voltage across +ve outer and middle wire,

$$V_1 = I R_1 = 101{\cdot}83 \times 1{\cdot}785 = \mathbf{181{\cdot}8\ V}$$

Voltage across −ve outer and middle wire,

$$V_2 = I R_2 = 101{\cdot}83 \times 3{\cdot}125 = \mathbf{318{\cdot}2\ V}$$

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Example 13.29. *A 3-wire, 500/250 V distributor is loaded as shown in Fig. 13.54. The resistance of each section is given in ohm. Find the voltage across each load point.*

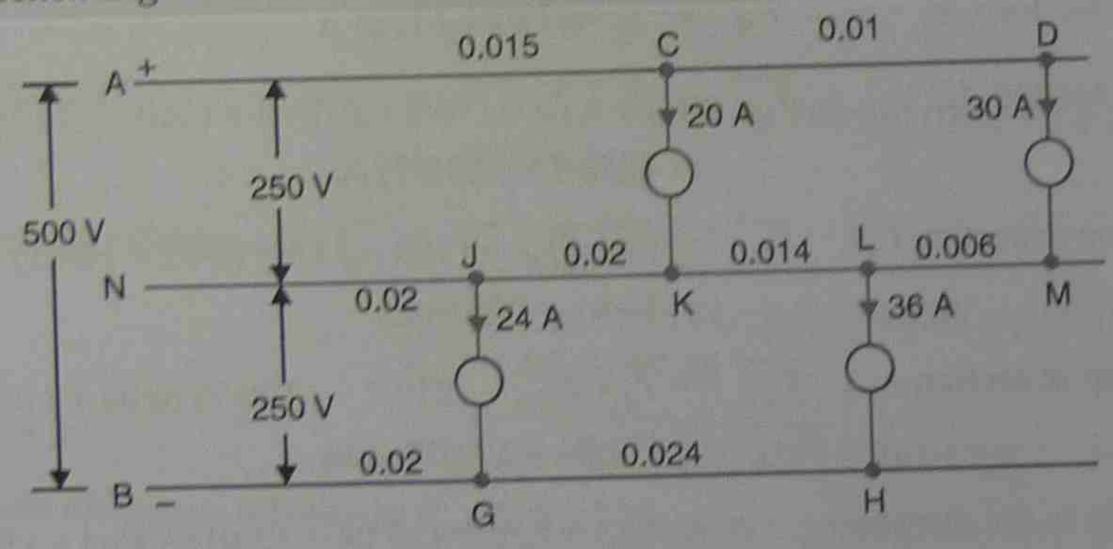


Fig. 13.54

Solution. From the current loading given in Fig. 13.54, we can find the magnitudes and directions of currents in the various sections by applying Kirchhoff's current law. Fig. 13.55 shows the magnitudes and directions of currents in the various sections.

referred to the voltage at the far end. The resistance and reactance per km (go and return) are 0·05 Ω and 0·1 Ω respectively. If the voltage at the far end is maintained at 230 V, calculate :

(*i*) *voltage at the sending end*

(*ii*) *phase angle between voltages at the two ends.*

Solution. Fig. 14.5 shows the distributor AB with C as the mid-point

Impedance of distributor/km $= (0{\cdot}05 + j\,0{\cdot}1)\ \Omega$

Impedance of section AC, $\vec{Z}_{AC} = (0{\cdot}05 + j\,0{\cdot}1) \times 1000/1000 = (0{\cdot}05 + j\,0{\cdot}1)\ \Omega$

Impedance of section CB, $\vec{Z}_{CB} = (0{\cdot}05 + j\,0{\cdot}1) \times 1000/1000 = (0{\cdot}05 + j\,0{\cdot}1)\ \Omega$

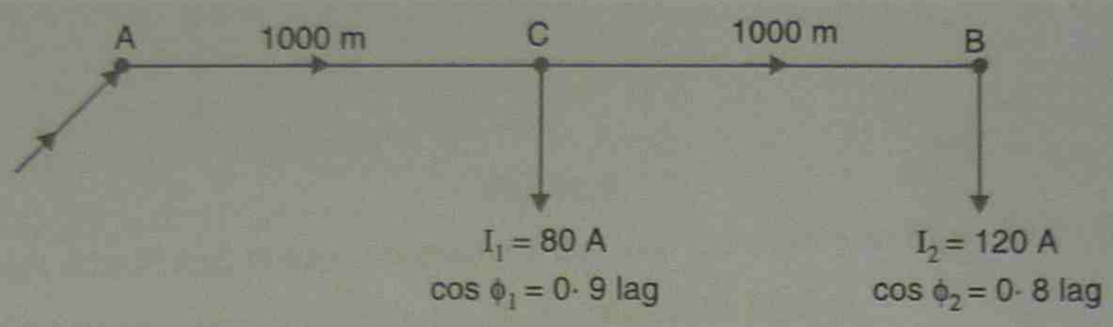


Fig. 14.5

Let the voltage V_B at point B be taken as the reference vector.

Then, $\overrightarrow{V_B} = 230 + j\,0$

(*i*) Load current at point B, $\vec{I}_2 = 120\,(0{\cdot}8 - j\,0{\cdot}6) = 96 - j\,72$

Load current at point C, $\vec{I}_1 = 80\,(0{\cdot}9 - j\,0{\cdot}436) = 72 - j\,34{\cdot}88$

Current in section CB, $\overrightarrow{I_{CB}} = \vec{I}_2 = 96 - j\,72$

Current in section AC, $\overrightarrow{I_{AC}} = \vec{I}_1 + \vec{I}_2 = (72 - j\,34{\cdot}88) + (96 - j\,72)$

$= 168 - j\,106{\cdot}88$

Drop in section CB, $\overrightarrow{V_{CB}} = \overrightarrow{I_{CB}}\,\overrightarrow{Z_{CB}} = (96 - j\,72)\,(0{\cdot}05 + j\,0{\cdot}1)$

$= 12 + j\,6$

Drop in section AC, $\overrightarrow{V_{AC}} = \overrightarrow{I_{AC}}\,\overrightarrow{Z_{AC}} = (168 - j\,106{\cdot}88)\,(0{\cdot}05 + j\,0{\cdot}1)$

$= 19{\cdot}08 + j\,11{\cdot}45$

$\therefore$ Sending end voltage, $\overrightarrow{V_A} = \overrightarrow{V_B} + \overrightarrow{V_{CB}} + \overrightarrow{V_{AC}}$

$= (230 + j\,0) + (12 + j\,6) + (19.08 + j\,11.45)$

$= 261.08 + j\,17.45$

Its magnitude is $= \sqrt{(261{\cdot}08)^2 + (17{\cdot}45)^2} = \mathbf{261{\cdot}67\ V}$

(*ii*) The phase difference θ between V_A and V_B is given by :

$$\tan\theta = \frac{17{\cdot}45}{261{\cdot}08} = 0{\cdot}0668$$

$$\therefore \quad \theta = \tan^{-1} 0{\cdot}0668 = \mathbf{3{\cdot}82^\circ}$$

Example 14.3. *A single phase distributor one km long has resistance and reactance per conductor of 0·1 Ω and 0·15 Ω respectively. At the far end, the voltage $V_B = 200$ V and the current is 100 A at a p.f. of 0·8 lagging. At the mid-point M of the distributor, a current of 100 A is tapped at a p.f.*

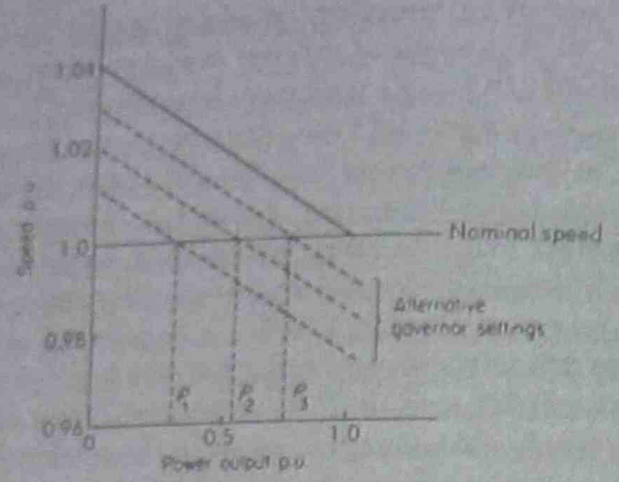


Figure 4.3 Effect of speeder gear on governor characteristics P_1, P_2, and P_3 = outputs at various settings but at same speed

where T_0 is the torque at speed N_0 and T the torque at speed N; k is a constant for the governor system. As the torque depends on both the main-valve position and the speed, $T = f(d, N)$.

There is a time delay between the occurrence of a load change and the new operating conditions. This is due not only to the governor mechanism but also to the fact that the new flow rate of steam or water must accelerate or decelerate the rotor in order to attain the new speed. In Figure 4.4 typical curves are shown for a turbogenerator which has a sudden decrease in the electrical power required, perhaps due to an external power network fault, and hence the retarding torque on the turbine shaft is suddenly much smaller. In the ungoverned

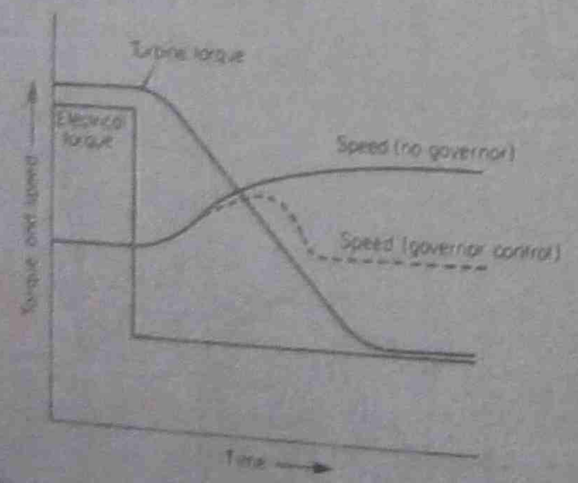


Figure 4.4 Graphs of turbine torque, electrical torque, and speed against time when the load on a generator suddenly falls

case the considerable time-lag between the load change and the attainment of the new steady speed is obvious. With the regulated or governed machine, due to the dead band in the governor mechanism, the speed–time curve starts to rise, the valve then operates, and the fluid supply is adjusted. It is possible for damped oscillations to be set up after the load change; this will be discussed in Chapter 8.

An important factor regarding turbines is the possibility of overspeeding when the load on the shaft is lost completely, with possible drastic mechanical breakdown. To avoid this, special valves are incorporated to automatically cut off the energy supply to the turbine. In a turbogenerator normally running at 3000 r.p.m. this overspeed protection operates at about 3300 r.p.m.

Example 4.1

An isolated 75 MVA synchronous generator feeds its own load and operates initially at no-load at 3000 r.p.m., 50 Hz. A 20 MW load is suddenly applied and the steam valves to the turbine commence to open after 0.5 s due to the time-lag in the governor system. Calculate the frequency to which the generated voltage drops before the steam flow meets the new load. The stored energy for the machine is 4 kW-s per kVA of generator capacity.

Solution

For this machine the stored energy at 3000 r.p.m.

$$= 4 \times 75\,000 = 300\,000 \text{ kW-s}$$

Before the steam valves start to open the machine loses $20\,000 \times 0.5 = 10\,000$ kW-s of the stored energy in order to supply the load.

The stored energy $\propto$ (speed)^2. Therefore the new frequency

$$= \sqrt{\frac{300\,000 - 10\,000}{300\,000}} \times 50 \text{ Hz}$$

$$= 49.2 \text{ Hz}$$

4.3 Control Loops

The machine and its associated governor and voltage-regulator control systems may be represented by the block diagram shown in Figure 4.5. The nature of the voltage control loop has been discussed already in Chapter 3; accurate machine representation is required, involving two-axis expressions, as indicated in Chapter 2.

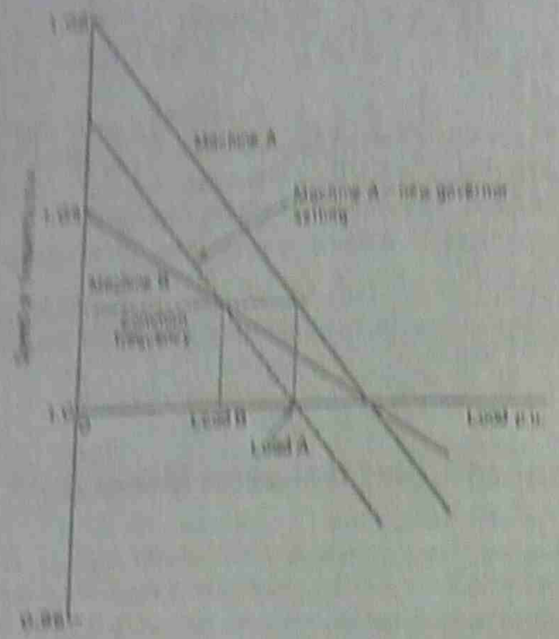


Figure 4.6 Two machines connected to an infinite busbar. The speeder gear of machine A is adjusted so that the machines load equally

Hence, (1) *the flow of power between two nodes is determined largely by the transmission angle*; (2) *the flow of reactive power is determined by the scalar voltage difference between the two nodes*.

These two facts are of fundamental importance to the understanding of the operation of power systems.

The angular advance of G_A (Figure 4.7) is due to a greater relative energy input to turbine A than to B. The provision of this extra steam (or water) to A is possible because of the action of the speeder gear without which the power outputs of A and B would be determined solely by the nominal governor characteristics. The following simple example illustrates these principles.

Example 4.2

Two synchronous generators operate in parallel and supply a total load of 300 MW. The capacities of the machines are 100 MW and 200 MW and both have governor droop characteristics of 4 per cent from no load to full load. Calculate the load taken by each machine, assuming free governor action.

Figure 4.7 Two generating stations linked by an interconnector of impedance $(R + jX)\Omega$. The rotor of A is in phase advance of B and $V_1 > V_2$

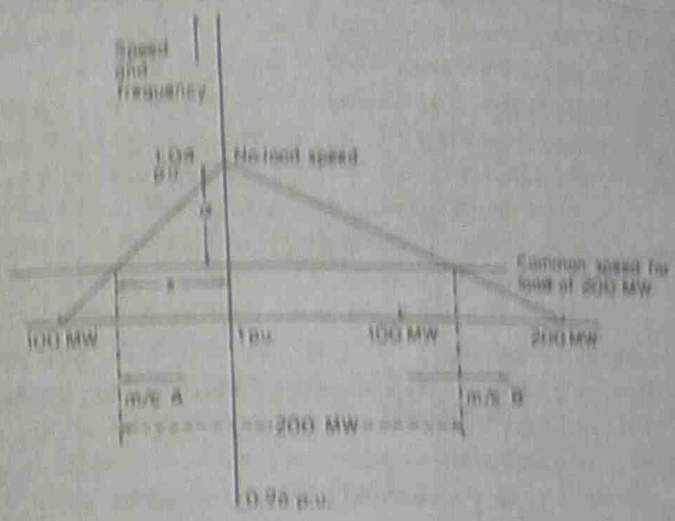


Figure 4.8 Speed–load diagram for Example 4.2

Solution

Let x megawatts be the power supplied from a 100 MW generator. Referring to Figure 4.8,

$$\frac{4}{100} = \frac{\alpha}{x}$$

For the 200 MW machine,

$$\frac{4}{200} = \frac{\alpha}{200 - x}$$

$$\frac{4x}{100} = \frac{800 - 4x}{200}$$

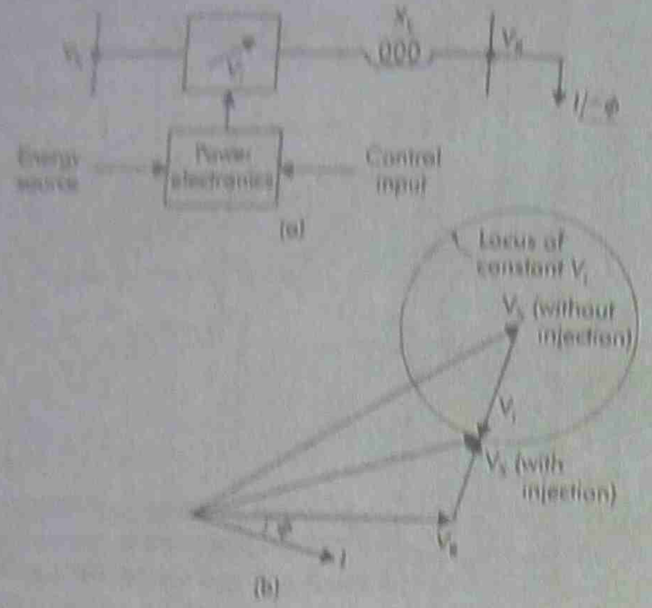


Figure 5.9 Principle of Universal Power Controller for series injection. (a) System diagram. (b) Phasor diagram

Figure 5.10. Here, t_s and t_r are fractions of the nominal transformation ratios, i.e. the tap ratio/nominal ratio. For example, a transformer of nominal ratio 6.6 to 33 kV when tapped to give 6.6 to 36 kV has a $t_s = 36/33 = 1.09$. V_1 and V_2 are the nominal voltages; at the ends of the line the actual voltages are $t_s V_1$ and $t_r V_2$. It is required to determine the tap-changing ratios required to completely compensate for the voltage drop in the line. The product $t_s t_r$ will be made unity; this ensures that the overall voltage level remains in the same order and that the minimum range of taps on both transformers is used.

(Note that all values are in per unit; t is the off-nominal tap ratio.)

Transfer all quantities to the load circuit. The line impedance becomes $(R+jX)/t_r^2$; $V_s = V_1 t_s$ and, as the impedance has been transferred, $V_2 = V_2 t_r$. The input voltage to the load circuit becomes $V_1 t_s / t_r$ and the equivalent circuit is as shown in Figure 5.10(c). The arithmetic voltage drop

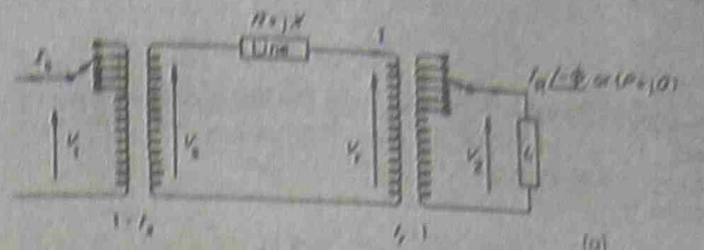
$$= (V_1 t_s / t_r) - V_2 \approx \frac{RP + XQ}{t_r^2 V_2}$$

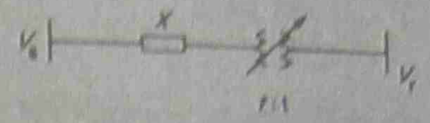
When $t_s = 1/t_r$,

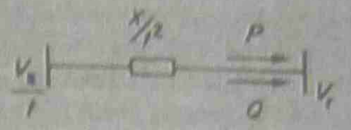
$$t_r^2 V_1 V_2 - V_2^2 = (RP + XQ) t_r^2$$

and

$$V_2 = \tfrac{1}{2}[t_r^2 V_1 \pm t_r (t_r^2 V_1^2 - 4(RP + XQ))^{\frac{1}{2}}] \qquad (5.7)$$







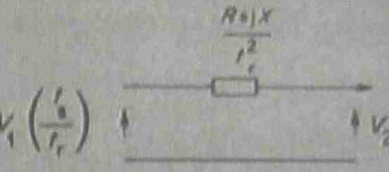


Figure 5.10 (a) Coordination of two tap-changing transformers in a radial transmission link (b) and (c) Equivalent circuits for dealing with off-nominal tap ratio. (b) Single transformer. (c) Two transformers

Hence if t_s is specified, there are two values of V_2 for a given V_1.

Example 5.3

A 13 kV line is fed through an 11/132 kV transformer from a constant 11 kV supply. At the load end of the line the voltage is reduced by another transformer of nominal ratio 132/11 kV. The total impedance of the line and transformers at 132 kV is (25 + j66) Ω. Both transformers are equipped with tap-changing facilities which are arranged so that the product of the two off-nominal settings is unity. If the load on the system is 100 MW at 0.9 p.f. lagging, calculate the settings of the tap-changers required to maintain the voltage of the load busbar at 11 kV. Use a base of 100 MVA.

direct-current component of the fault current the symmetrical r.m.s. value is modified by the use of multiplying factors such as the following:

- 8-cycle circuit breaker opening time, multiply by 1;
- 3-cycle circuit breaker opening time, multiply by 1.2;
- 2-cycle circuit breaker opening time, multiply by 1.4.

Consider an initially unloaded generator with a short circuit across the three terminals, as shown in Figure 7.2. The generated voltage per phase is E and therefore the short-circuit current is $[E/Z(\Omega)]A$, where Z is either the transient or subtransient impedance. If Z is expressed in per unit notation,

$$Z\,(\text{p.u.}) = \frac{I_{\text{FL}} Z(\Omega)}{E}$$

$$\therefore\ Z\,(\Omega) = \frac{EZ\,(\text{p.u.})}{I_{\text{FL}}}$$

Taking I_{FL} (full-load or rated current) and E as base values of voltage and current, the short-circuit current

$$I_{\text{s.c.}} = \frac{E}{Z\,(\Omega)} = \frac{EI_{\text{FL}}}{EZ\,(\text{p.u.})} = \frac{I_{\text{FL}}}{Z\,(\text{p.u.})} \tag{7.1}$$

Also, the three-phase short-circuit volt-amperes

$$= \sqrt{(3)} V_{\text{L}} I_{\text{s.c.}} = \frac{\sqrt{(3)} V_{\text{L}} I_{\text{FL}}}{Z\,(\text{p.u.})}$$

$$= \frac{\text{Base volt-amperes}}{Z\,(\text{p.u.})} \tag{7.2}$$

Hence the short-circuit level is immediately obtained if the impedance Z (p.u.) from the source of the voltage to the point of the fault is known.

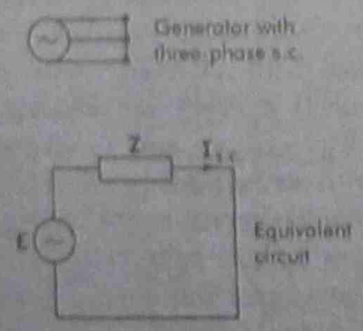


Figure 7.2 Voltage source with short circuit and equivalent circuit

Example 7.1

An 11.8 kV busbar is fed from three synchronous generators having the following ratings and reactances,

20 MVA, X' 0.08 p.u.; 60 MVA, X' 0.1 p.u.; 20 MVA, X' 0.09 p.u.

Calculate the fault current and MVA if a three-phase symmetrical fault occurs on the busbars. Resistance may be neglected. The voltage base will be taken as 11.8 kV and the VA base as 60 MVA.

Solution

The transient reactance of the 20 MVA machine on the above bases is $(60/20) \times 0.08$, i.e. 0.24 p.u., and of the 20 MVA machine $(60/20) \times 0.09$, i.e. 0.27 p.u. These values are shown in the equivalent circuit in Figure 7.3. As the generator e.m.f.s are assumed to be equal, one source may be used (Figure 7.3(c)). The equivalent reactance is

$$X_{\text{eq}} = \frac{1}{1/0.24 + 1/0.27 + 1/0.1} = 0.056\,\text{p.u.}$$

Therefore fault MVA

$$= \frac{60}{0.056} = 1071\,\text{MVA}$$

and fault current

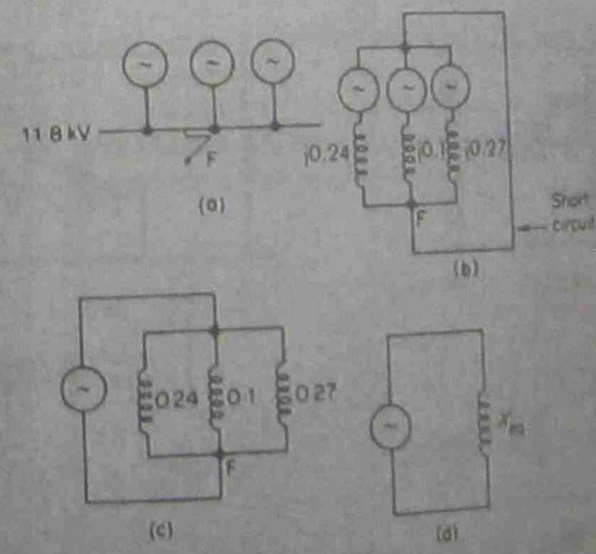


Figure 7.3 Line diagram and equivalent circuits for Example 7.1

$$= \frac{1071 \times 10^6}{\sqrt{(3)} \times 11\,800}$$
$$= 52\,402\,\text{A}$$

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Example 7.2

In the network shown in Figure 7.4, a three-phase fault occurs at point F. Calculate the fault MVA at F. The per unit values of reactance all refer to a base of 100 MVA. Resistance may be neglected.

Solution

The equivalent single-phase network of generator and line reactances is shown in Figure 7.4(b). This is replaced by the network shown in Figure 7.4(c) by the use of the delta-star transformation. A further transformation is carried out on the network in Figure 7.4(d) to give the final single equivalent reactance,

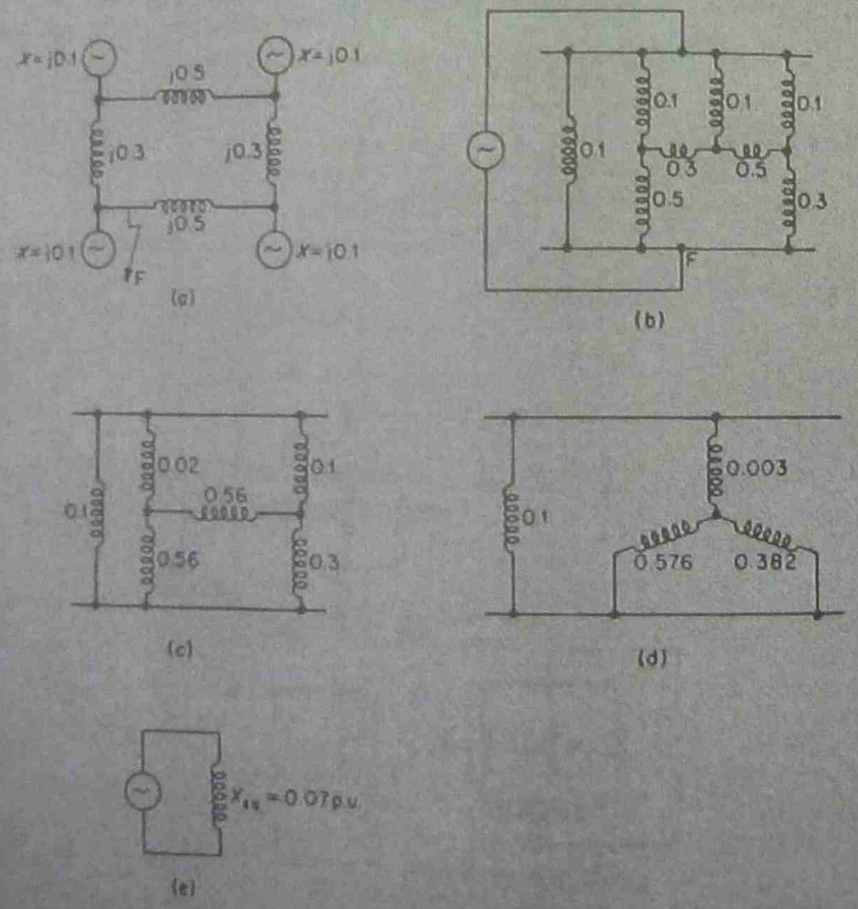


Figure 7.4 Line diagram and equivalent circuits for Example 7.2

$$X_{eq} = 0.07\,\text{p.u.}$$

The fault level at point F

$$= \frac{100}{0.07} = 1430\,\text{MVA}$$

This value is based on the assumption of symmetrical fault current. Allowing for a multiplying factor of 1.4, the fault level is 2000 MVA.

7.2.1 Current limiting reactors

The impedances presented to fault currents by transformers and machines when faults occur on substation or generating station busbars are low. To reduce the high fault current which would do considerable damage mechanically and thermally, artificial reactances are sometimes connected between bus sections. These current-limiting reactors usually consist of insulated copper strip embedded in concrete formers; this is necessary to withstand the high mechanical forces produced by the current in neighbouring conductors. The position in the circuit occupied by the reactor is a matter peculiar to individual designs and installations (see Figure 7.5(a) and (b)).

Example 7.3

A 400 kV power system contains three substations A, B, and C having fault levels (GVA) of 20, 20, and 30, respectively.

The system is to be reinforced by three lines each of reactance j5 Ω connecting together the three substations as shown in Figure 7.6(a). Calculate the new fault level at C (three-phase symmetrical fault). Neglect resistance.

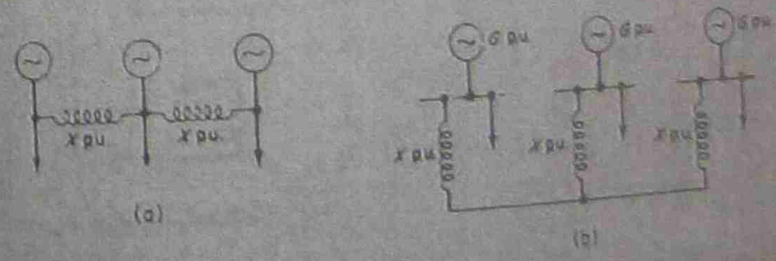


Figure 7.5 Connection of artificial reactors: (a) ring system; (b) tie-bar system. Transient reactance of machines, G p.u.; reactance of artificial reactors X p.u.

Example 8.1

A synchronous generator of reactance 1.5 p.u. is connected to an infinite busbar system (V = 1 p.u.) through a line and transformers of total reactance 0.5 p.u. The no-load voltage of the generator is 1.1 p.u. and the inertia constant $H = 5$ MW-s per MVA. All per unit values are expressed on the same base; resistance and damping may be neglected. Calculate the frequency of the oscillations set up when the generator operates at a load angle of 60° and is subjected to a small disturbance. The system frequency is 50 Hz.

Solution

The nature of the movement is governed by the sign of the quantity under the root sign in the equation for s_1 and s_2 (equation (8.4)). This changes when $K_3^2 = 4M(\partial P/\partial\delta)_1$ in this case $K_3 = 0$. The roots of the characteristic equations give the frequency of oscillation, when $\delta_0 = 60°$.

$$\left(\frac{\partial P}{\partial \delta}\right)_{60°} = \frac{1.1 \times 1}{2}\cos 60$$

$$= 0.275 \text{ p.u.}$$

$$s_1 \text{ and } s_2 = \pm j\sqrt{\left(\frac{\partial P}{\partial \delta}\right) \cdot \frac{1}{M}}$$

$$= \pm j\sqrt{\frac{0.275}{5 \times \left(\frac{1}{2\pi 50}\right)}}$$

$$= \pm j\sqrt{8.64}$$

Therefore frequency of oscillation

$$= 2.94 \text{rad/s} = \frac{2.94}{2\pi} \text{ Hz}$$

$$= 0.468 \text{ Hz}$$

and the periodic time

$$= \frac{1}{0.468} = 2.14 \text{ s}$$

The stability of a two-machine system

The terms M_1 and M_2 refer to machines 1 and 2 which are connected in parallel through an impedance.

The equations of motion are (for small changes, damping neglected)

$$M_1 s^2 \Delta\delta_1 + \left(\frac{\partial P_1}{\partial \delta_{12}}\right)\Delta\delta_{12} = 0$$

and

$$M_2 s^2 \Delta\delta_2 + \left(\frac{\partial P_2}{\partial \delta_{12}}\right)\Delta\delta_{12} = 0$$

As

$$\Delta\delta_1 - \Delta\delta_2 = \Delta\delta_{12}$$

then

$$s^2\Delta\delta_{12} + \left[\frac{(\partial P_1/\partial\delta_{12})}{M_1} - \frac{(\partial P_2/\partial\delta_{12})}{M_2}\right]\Delta\delta_{12} = 0$$

The characteristic equation has two roots,

$$s_1, s_2 = \pm j\sqrt{\frac{(\partial P_1/\partial\delta_{12})}{M_1} - \frac{(\partial P_2/\partial\delta_{12})}{M_2}}$$

Stability is assured if the quantity under the square root is positive. Hence the stability limit for small disturbances is not the same as the maximum power limit discussed in Chapter 3 and is, in fact, always larger. The difference, however, is never large, and when one machine is effectively an infinite busbar, the two limits coincide.

Effects of governor action

In the above analysis the oscillations set up with small changes in load on a system have been considered and the effects of governor operation ignored. After a certain time has elapsed the governor control characteristics commence to influence the powers and oscillations, as explained in Section 4.5. It is now the practice to represent both the excitation system and the governor system with the dynamic equations of the generator in the control-state space form, from which the eigenvalues of the complete system with feedback can be determined. Using well-established control design techniques, appropriate feedback paths and time constants can be established for a range of generating conditions and disturbances, thereby assuring adequate dynamic stability margins. Further information on these design processes can be found in advanced control textbooks. Here we will consider the simpler but important aspects of so-called steady-state stability.

8.4 Steady-State Stability—Practical Considerations

The steady-state stability limit is the maximum power that can be transmitted in a network between sources and loads when the system is subject to small disturbances. The power system is, of course, constantly subjected to small changes as load variations occur. To obtain the limiting value of power, small increments of load are added to the system; after each increment the generator excitations are adjusted to maintain constant terminal voltages and a load flow is carried out. Eventually, a condition of instability is reached.

The stability limits of synchronous machines have been discussed in Chapter 3. It was seen that provided the generator operates within the 'safe area' of the performance chart (Figure 3.15), stability is assured; usually, a 20 per cent margin of safety on the var absorb side is allowed and the limit is extended by the use of automatic voltage regulators. Often, the performance charts are not used directly and the generator-equivalent circuit employing the synchronous impedance is used. The normal operating load angle for modern machines is in the order of 60 electrical degrees, and for the limiting value of 90° this leaves 30° to cover the transmission network. In a complex system a reference point must be taken from which the load angles are measured; this is usually a point where the direction of power flow reverses.

The simplest criterion for steady-state synchronous stability is $(\partial P/\partial\delta) > 0$, i.e. the synchronizing coefficient must be positive. The use of this criterion involves the following assumptions:

1. generators are represented by constant impedances in series with the no-load voltages;
2. the input torques from the turbines are constant;
3. changes in speed are ignored;
4. electromagnetic damping in the generators is ignored;
5. the changes in load angle δ are small.

The degree of complexity to which the analysis is taken has to be decided, e.g. the effects of machine inertia, governor action, and automatic voltage regulators can be included; these items, however, greatly increase the complexity of the calculations. The use of the criterion $(\partial P/\partial\delta) = 0$ alone gives a pessimistic or low result and hence an inbuilt factor of safety.

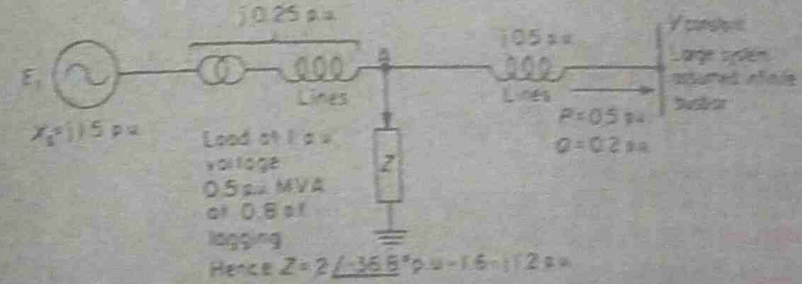
In a system with several generators and loads, the question as to where the increase of load is to be applied is important. A conservative method is to assume that the increment applies to one machine only, determine the stability, and then repeat for each of the other machines in turn. Alternatively, the power outputs from all but the two generators having the largest load angles are kept constant.

For calculations made without the aid of computers it is usual to reduce the network to the simplest form that will keep intact the generator nodes. The values of load angle, power, and voltage are then calculated for the given conditions, $\partial P/\partial\delta$ determined for each machine, and, if positive, the loading is increased and the process repeated.

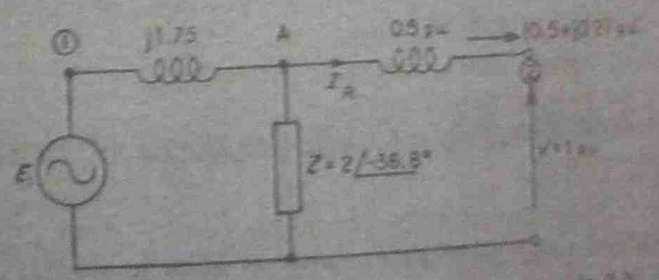
In a system consisting of a generator supplying a load through a network of lines and transformers of effective reactance, X_T, the value of $(dP/d\delta) = (EV/X_T)\cos\delta$, where E and V are the supply- and receiving-end voltages and δ the total angle between the generator rotor and the phasor of V. The power transmitted is obviously increased with higher system voltages and lower reactances, and it may be readily shown that line series capacitance increases the stability limit. The determination of $dP/d\delta$ is not very difficult if the voltages at the loads can be assumed to be constant or if the loads can be represented by impedances. Use can be made of the P–V, Q–V characteristics of the load if the voltages change appreciably with the redistribution of the power in the network; this process, however, is extremely tedious.

Example 8.2

For the system shown in Figure 8.3, investigate the steady-state stability



(a) Line diagram of system for Example 8.2



(b) Equivalent circuit for Example 8.2

Figure 8.3

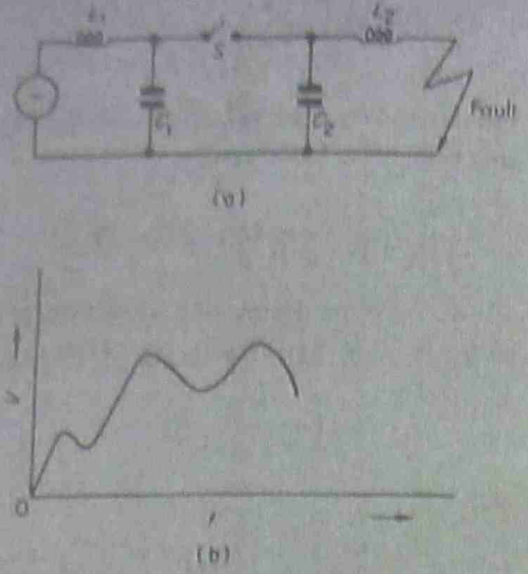


Figure 10.6 (a) System with double-frequency restriking transient. (b) Typical waveform of double frequency, although initial amplitude is low, rate of rise is high

$$\omega_1 = \frac{1}{\sqrt{(L_1 C_1)}} \quad \text{and} \quad \omega_2 = \frac{1}{\sqrt{(L_2 C_2)}}$$

Example 10.1

Determine the relative attenuation occurring in 5 cycles in the overvoltage surge set up on a 66 kV cable fed through an air-blast circuit breaker when the breaker opens on a system short circuit. The breaker incorporates resistance switching, i.e. an optimum resistance switched in across the contact gap on opening. The network parameters are as follows:

$$R = 7.8\,\Omega \qquad L = 6.4\,\text{mH} \qquad C = 0.0495\,\mu\text{F}$$

Solution

$$\text{Switching resistance} = \tfrac{1}{2}\sqrt{(L/C)} = 180\,\Omega$$

$$\alpha = \frac{1}{2}\cdot\frac{R}{L} = 610$$

$$\omega = \frac{1}{\sqrt{LC}} = 5.61 \times 10^4\ \text{rad/s}$$

i.e. transient frequency = 8.93 kHz.

Hence

$$\text{time for 5 cycles} = \frac{1}{8.93 \times 10^3} \times 5\,\text{s}$$

and

$$e^{-\alpha t} = \exp\left(-610 \times \frac{5}{8.93 \times 10^3}\right) = 0.712$$

The maximum theoretical voltage set up with the circuit breaker opening on a short-circuit fault would be

$$2 \times \frac{66\,000}{\sqrt{3}} \times \sqrt{2}$$

i.e. 107.5 kV, and in 5 cycles (i.e. 560 μs) this becomes $107.5 \times (1 - 0.712)$; i.e. 31 kV.

(Note that resistance switching lowers the current to be broken and raises the power factor so that the voltage is not at peak value at opening.)

10.2.3 Switching surges—interruption of capacitive circuits

The interruption of capacitive circuits is shown in Figure 10.7. At the instant of arc interruption, the capacitance is left charged to value v_m but half a cycle later the system voltage is $-v_m$, giving a gap voltage of $2v_m$. If the gap breaks down, an oscillatory transient is set up (as previously discussed), which can increase the gap voltage still further, as shown in Figure 10.7.

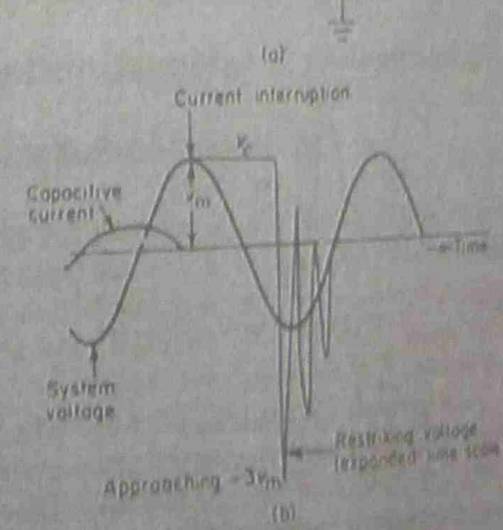


Figure 10.7 Voltage waveform when opening a capacitive circuit

The current in the lower electromagnet is induced by transformer action from the upper winding and sufficient displacement between the two fluxes results. This, however, may be adjusted by means of a reactor in parallel with the secondary winding.

The basic mode of operation of the induction disc is indicated in the phasor diagram of Figure 11.15. The torques produced are proportional to $\Phi_2 i_1 \sin\alpha$ and $\Phi_1 i_2 \sin\alpha$, so that the total torque is proportional to $\Phi_1 \Phi_2 \sin\alpha$ as Φ_1 is proportional to i_1 and Φ_2 to i_2.

This type of relay is fed from a current transformer (CT) and the sensitivity may be varied by the plug arrangement shown in Figure 11.14. The operating characteristics are shown in Figure 11.16. To enable a single characteristic curve to be used for all the relay sensitivities (plug settings) a quantity known as the current (or plug) setting multiplier is used as the abscissa instead of current magnitude, as shown in Figure 11.16. To illustrate the use of this curve (usually shown on the relay casing) the following example is given.

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Example 11.1

Determine the time of operation of a 1 A, 3 s overcurrent relay having a plug setting of 125 per cent and a time multiplier of 0.6. The supplying CT is rated 400 : 1 A and the fault current is 4000 A.

Solution

The relay coil current = (4000/400) × 1 = 10 A. The normal relay coil current is 1 × (125/100) = 1.25 A. Therefore the relay fault current as a multiple of the plug

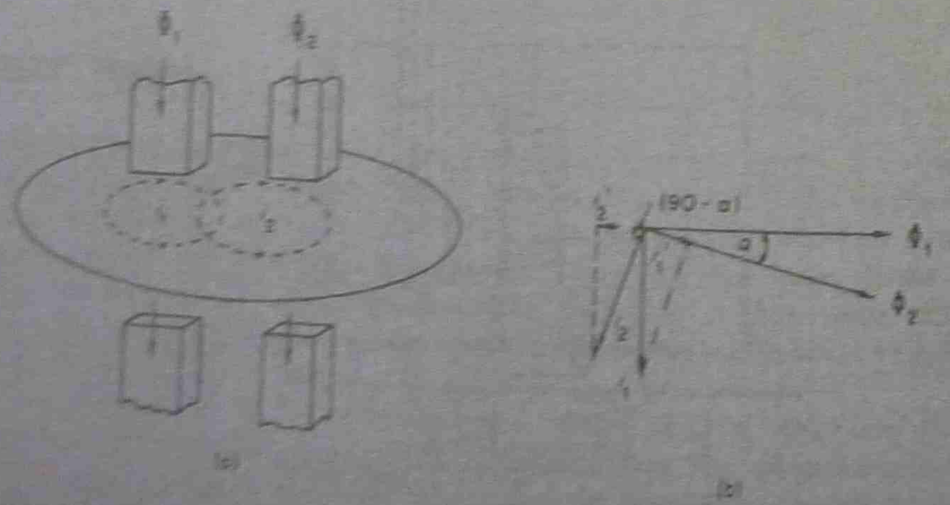

Figure 11.15 Operation of disc-type electromagnetic relay. (a) Fluxes. (b) Phasor diagram. i_1 and i_2 are induced currents in disc

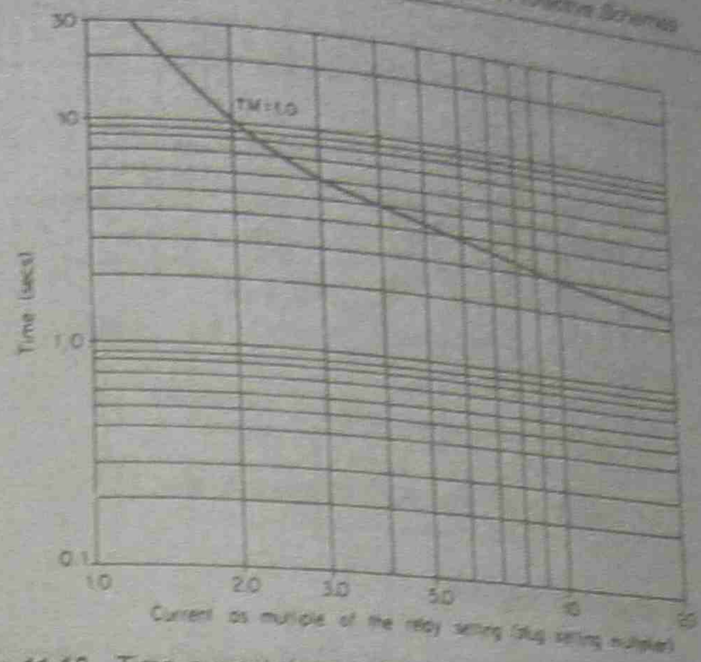

Figure 11.16 Time–current characteristics of a typical induction disc in terms of the plug-setting multiplier. TM = time multiplier. (Note: Time is along the vertical axis)

setting = (10/1.25) = 8. From the relay curve (Figure 11.16), the time of operation is 3.3 s for a time setting of 1. The time multiplier (TM) controls the time of operation by changing the angle through which the disc moves to close the contacts. The actual operating time = 3.3 × 0.6 = 2.0 s.

Induction-disc relays may be made responsive to power flow by feeding the upper magnet winding in Figure 11.14 from a voltage via a potential transformer and the lower winding from the corresponding current. As the upper coil will consist of a large number of turns, the current will lag the applied voltage by 90°, whereas in the lower (small number of turns) coil they are almost in phase. Hence, Φ_1 is proportional to V, and Φ_2 is proportional to I, and torque is proportional to $\Phi_1 \Phi_2 \sin\alpha$, i.e. to $\Phi_1 \Phi_2 \sin(90 - \phi)$, or $k' VI \cos\phi$ (where ϕ is the angle between V and I).

The direction of the torque depends on the power direction and hence the relay is directional. A power relay may be used in conjunction with a current-operated relay to provide a directional property.

Induction-cup relay

The operation is similar to the induction disc here two fluxes at right angles induce eddy currents in a bell-shaped cup which rotates and closes the control contacts. A four-pole relay is shown in Figure 11.17.

The magnitude and angle of the current supplied to the load at A may be calculated by the relations

$$I_a = \frac{P_a}{V_a F_{pa}} \underline{/\theta_a + \beta}, \quad \theta_a = \cos^{-1} F_{pa}$$

The current I_2 and the voltage at load B may then be calculated by the relations

$$\bar{I}_2 = \bar{I}_1 - \bar{I}_a$$

$$\bar{V}_b = \bar{V}_a - \bar{I}_2 Z_2$$

The procedure may be repeated until voltages at all load points have been determined.

The losses in the system may be obtained by reference to Equation (6-1), although not with accuracy because the loss term is usually small compared to the others.

$$P_t = P_a + P_b + P_c + P_d + \text{losses} \tag{6-1}$$

A determination of greater accuracy might be made by calculating the losses in each line section from known currents and circuit resistances.

It is obvious that this problem could also be worked by beginning at the extreme load end of the line if the conditions there were known or assumed. A calculation proceeding through this system from right to left would then give information as to the required conditions at the source station.

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EXAMPLE 6-1. Determine voltage condition on a system of the type shown in Fig. 6-7, assuming that

$V_a = 5000$ V

$P_1 = 600$ kW 0.8 power-factor lag

$Z_1 = 4 \underline{/70^\circ}\ \Omega$

$P_2 = 400$ kW 0.707 power-factor lag

$Z_2 = 2.5 \underline{/60^\circ}\ \Omega$

Solution.

a. At the source,

$$I_1 = \frac{P_1}{V_a \cos \varphi_1} = \frac{600{,}000}{5000 \times 0.8} = 150 \text{ A}$$

Considering $\bar{V}_a$ as reference,

$$\bar{V}_a = 5000 \underline{/0^\circ} \text{ V}$$

$$I_1 = 150 \underline{/-36.9^\circ} \text{ A}$$

$$I_1 Z_1 = 150 \underline{/-36.9^\circ} \times 4 \underline{/70^\circ} = 600 \underline{/33.1^\circ} \text{ V}$$

$$\bar{V}_b = \bar{V}_a - I_1 Z_1 = 5000 \underline{/0^\circ} - 600 \underline{/33.1^\circ}$$

$$= 5000 + j0 - (503 + j328)$$

$$= 4497 - j328$$

$$= 4510 \underline{/-4.2^\circ} \text{ V}$$

b. At A,

$$I_2 = \frac{P_2}{V_a \cos \varphi_2} = \frac{400{,}000}{4510 \times 0.707} = 125 \text{ A}$$

Since $\bar{V}_a = 4510 \underline{/-4.2^\circ}$ V,

$$I_2 = 125 \underline{/-45^\circ - 4.2^\circ} = 125 \underline{/-49.2^\circ} \text{ A}$$

$$I_2 Z_2 = 125 \underline{/-49.2^\circ} \times 2.5 \underline{/60^\circ} = 313 \underline{/10.8^\circ} = 307 + j58.7 \text{ V}$$

$$\bar{V}_b = \bar{V}_a - I_2 Z_2 = 4497 - j328 - (307 + j58.7)$$

$$= 4190 - j387$$

$$= 4210 \underline{/-5.3^\circ} \text{ V}$$

This method of solution could be carried on to all sections of the line if desired.

6-3. FOUR-TERMINAL NETWORKS; A, B, C, D CONSTANTS

a. General Four-Terminal Network. The general four-terminal network is described with the aid of Fig. 6-8. This network is described by two source-terminals between which there is a voltage V_s and through which a current I_s flows, as shown in the diagram. There are also

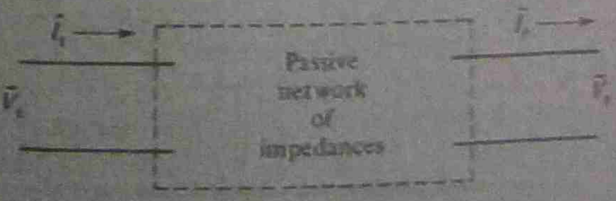

FIG. 6-8. The general 4-terminal network consisting of passive impedances connected in any fashion.

By Eq. (6-10),

$$B_{eq} = A_1B_2 + B_1D_2$$
$$= 1.0/\underline{0^\circ} \times 0 + 20/\underline{30^\circ} \times 1.0/\underline{0^\circ}$$
$$= 20/\underline{30^\circ}\ \Omega$$

By Eq. (6-11),

$$C_{eq} = C_1A_2 + D_1C_2$$
$$= 0 \times 1.0 + 1.0 \times 0.02/\underline{45^\circ}$$
$$= 0.02/\underline{45^\circ}\ \mathrm{U}$$

By Eq. (6-12),

$$D_{eq} = C_1B_2 + D_1D_2$$
$$= 0 \times 0 + 1.0 \times 1.0$$
$$= 1.0\ \underline{/0^\circ}$$

The same results could have been obtained by considering receiver-end conditions as $\bar{V}_r$ and $\bar{I}_r$, then solving the ladder network for $\bar{V}_s$ and $\bar{I}_s$ in terms of $\bar{V}_r$ and $\bar{I}_r$.

5. Two Four-Terminal Networks in Parallel. Two four-terminal networks may be connected in parallel as shown in Fig. 6-13. A new four-terminal network may be determined which will be equivalent in performance to the two networks in parallel. Note that

$$\bar{V}_{s1} = \bar{V}_{s2} = \bar{V}_s$$
$$\bar{V}_{r1} = \bar{V}_{r2} = \bar{V}_r$$

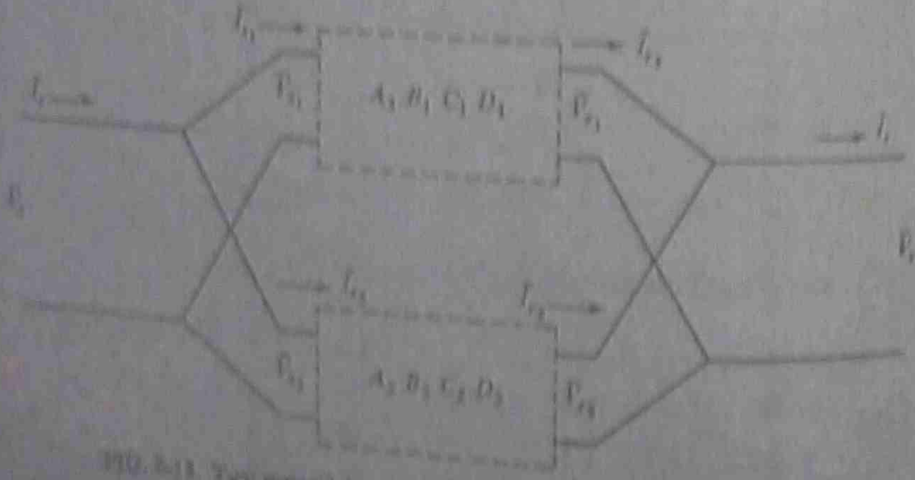

FIG. 6-13. Two general 4-terminal networks connected in parallel.

$$\bar{I}_s = \bar{I}_{s1} + \bar{I}_{s2}$$
$$\bar{I}_r = \bar{I}_{r1} + \bar{I}_{r2}$$

By a consideration of the above relations and by manipulations somewhat like those made for the tandem arrangement, it may be shown that the four-terminal network equivalent to that shown in Fig. 6-13 will have constants as follows:

$$A_{eq} = \frac{A_1B_2 + A_2B_1}{B_1 + B_2} \tag{6-13}$$

$$B_{eq} = \frac{B_1B_2}{B_1 + B_2} \tag{6-14}$$

$$C_{eq} = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \tag{6-15}$$

$$D_{eq} = \frac{B_2D_1 + B_1D_2}{B_1 + B_2} \tag{6-16}$$

613 EXAMPLE 6-3. Refer to Example 6-2. Consider that network 1 is connected in parallel with network 2. Determine the equivalent *ABCD* constants.

Solution.

By Eq. (6-13),

$$A_{eq} = \frac{A_1B_2 + A_2B_1}{B_1 + B_2} = \frac{1.0 \times 0 + 1.0 \times 20/\underline{30^\circ}}{20/\underline{30^\circ} + 0} = 1.0\ \underline{/0^\circ}$$

By Eq.(6-14),

$$B_{eq} = \frac{B_1B_2}{B_1 + B_2} = \frac{20/\underline{30^\circ} \times 0}{20/\underline{30^\circ} + 0} = 0\ \Omega$$

By Eq. (6-15),

$$C_{eq} = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}$$
$$= 0 + 0.02/\underline{45^\circ} + \frac{(1.0 - 1.0)(1.0 - 1.0)}{20/\underline{30^\circ} + 0}$$
$$= 0.02/\underline{45^\circ}\ \mathrm{U}$$

By Eq. (6-16),

$$D_{eq} = \frac{B_2D_1 + B_1D_2}{B_1 + B_2} = \frac{0 \times 1.0 + 20/\underline{30^\circ} \times 1.0}{20/\underline{30^\circ} + 0}$$
$$= 1.0\ \underline{/0^\circ}$$

It may be noted that these two networks in parallel have the identical characteristics of network 2 alone.

f. More Complicated Networks. The determination of the *ABCD* constants of complicated networks may involve considerable detailed computation. If the network can be broken up into a group of simpler four-terminal networks that are in series, in parallel, or in series-parallel arrangement, the relations so far presented will permit evaluation of the constants.

For some configurations it will be necessary to resort to general network analysis and make a solution through the application of Kirchhoff's voltage law and Kirchhoff's current law. For such determinations it is frequently desirable to use digital computers. The solutions of problems of this kind are beyond the interest of the present discussion. However, it is important to note that methods are available for calculating the *ABCD* constants of any generalized four-terminal network.

g. Experimental Determination of *ABCD* Constants. If a four-terminal network has been constructed and can be tested, its *ABCD* constants may be readily determined regardless of the complexity of the network. Two tests are sufficient: one in which the receiver terminals are *open-circuited* and one in which the receiver terminals are *short-circuited*.

When the receiver terminals are open-circuited (Fig. 6-14) I_r is 0 and Eqs. (6-2) and (6-3) reduce to

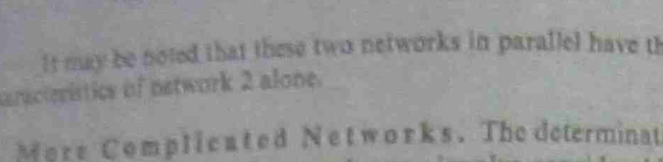

FIG. 6-14. A 4-terminal network. The receiver terminals are open-circuited for test, $\bar{I}_r = 0$.

$$\bar{V}_s = A\bar{V}_r$$

$$\bar{I}_s = C\bar{V}_r$$

The values of A and C may be determined from measured phasor values of $\bar{V}_s$, $\bar{I}_s$, and $\bar{V}_r$.

In the second test, the receiver terminals are short-circuited (Fig. 6-15) and measurements made of the phasor values of $\bar{V}_s$, $\bar{I}_s$, and $\bar{I}_r$. The constants B and D are then determinable from the relations

$$\bar{V}_s = B\bar{I}_r$$

$$\bar{I}_s = D\bar{I}_r$$

FIG. 6-15. A 4-terminal network. The receiver terminals are short-circuited for test, $\bar{V}_r = 0$.

EXAMPLE 6-4. Determine the *ABCD* constants of a network on which the following test results have been observed.

a. Receiver open-circuited, $\bar{I}_r = 0$:

$\bar{V}_s = 100 \angle 0°$ V

$\bar{V}_r = 70.7 \angle -45°$ V

$\bar{I}_s = 1.41 \angle -45°$ A

b. Receiver short-circuited, $\bar{V}_r = 0$:

$\bar{V}_s = 100 \angle 0°$ V

$\bar{I}_s = 2.0 \angle -90°$ A

$\bar{I}_r = 2.0 \angle -90°$ A

Solution:
From a,

$$\bar{V}_s = A\bar{V}_r$$

$$100\angle 0° = A \times 70.7\angle -45°$$

$$A = 1.41\angle 45°$$

and

$$\bar{I}_s = C\bar{V}_r$$

$$1.41\angle -45° = C \times 70.7\angle -45°$$

$$C = 0.02\angle 0° \ \mho$$

From b,

$$\bar{V}_s = B\bar{I}_r$$

$$100\angle 0° = B \times 2.0\angle -90°$$

$$B = 50\angle 90° \ \Omega$$

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exercises

1. Factorize
   (a) 3096
   (b) 1234
   (c) $2^4 - 1$
2. It was thought that $2^p - 1$ was prime when p is a prime. Shown that this is not true when $p = 11$
3. Find the gcd for 3096 and 1234.
4. Write the following decimal numbers in binary
   (a) 256_{10}
   (b) $2^4 - 1$
   (c) 549
   (d) 12.34

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In symbolic terms if p is a statement, say " it is raining" , then ¬p is its negation. That is ¬p is the statement "it is not raining". We summarize the truth or otherwise of the statements in a *truth table*, see table 2.1.

p	¬p
T	F
F	T

Table 2.1: Truth table for negation (¬)

In the truth table 2.1 the first row reads in plain English - "If p is true then ¬p is false" and row two "If p is false then ¬p is true'.

Conjunction ∧

Similarly, if p and q are statements, then p ∧ q is read as "p and q" . This (confusingly) is called the *conjunction* of p and q.

So if p is the statement " it is green" while q is the statement " it is an apple" then

p ∧ q is the statement "It is green and it is an apple "

We often write this in the shorter form:
If p=" it is green" and q = " it is an apple" then p ∧ q = "It is green and it is an apple "

Clearly this statement is true only when both p and q are true. If either of them is false then the compound statement is false. It will be helpful if we have a precise definition of ∧ and we can get one using a truth table.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2: The truth table for ∧

From table 2 we see that if p and q are both true then p ∧ q is also true. If p is true and q is false then p ∧ q is false.

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then the statement is true for every natural number n.

This is because, when the statement is true for $n = 1$, then according to 2, it will also be true for 2. But that implies it will be true for 3; which implies it will be true for 4. And so on. Hence it will be true for every natural number and thus is true for all n.

To prove a result by induction, then, we must prove parts 1, 2 and 3 above. The hypothesis of step 1

"The statement is true for $n = k$"

is called the induction assumption, or the induction hypothesis. It is what we assume when we prove a theorem by induction.

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Example

Prove that the sum of the first n natural numbers is given by this formula:

$$S_n = 1 + 2 + 3 + ... + n = n(n+1)/2$$

We will call this statement S_n, because it depends on n. Now we do steps 1 and 2 above.

1. First, we will assume that the statement is true for $n = k$ that is, we will assume that S_k is true so

   $$S_k = 1 + 2 + 3 + ... + k = k(k+1)/2$$

   Note this is the induction assumption.

2. Assuming this, we must prove that $S_{(k+1)}$ is also true. That is, we need to show:

   $$S_{(k+1)} = 1 + 2 + 3 + ... + (k+1) = (k+1)(k+2)/2$$

   To do that, we will simply add the next term $(k + 1)$ to both sides of the induction assumption,

   $$S_{(k+1)} = S_{(k+1)} + (k+1) =$$

   $$1 + 2 + 3 + ... + (k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2$$

   This is line 2, which is we wanted to show.

3. Next, we must show that the statement is true for $n = 1$. We have $S(1) = 1 = 1 \times 2/2$. The formula therefore is true for $n = 1$.

We have now fulfilled both conditions of the principle of mathematical induction. S_n is therefore true for every natural number.

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Example

We prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$. The proof is by mathematical induction.

1. Assume the result holds for $n = k$, that is $8^k - 3^k \mod 5 = 0$. Then $8^{k+1} - 3^{k+1} = 8 \times 8^k - 3 \times 3^k$.

2. Now the clever step

$$8^{k+1} - 3^{k+1} = 8 \times 8^k - 3 \times 3^k = 3 \times 8^k - 3 \times 3^k + 5 \times 8^k = 3 \times (8^k - 3^k) + 5 \times 8^k$$

   But $8^k - 3^k$ is divisible by 5 (by the induction hypothesis) and $5 \times 8k$ is obviously a multiple of 5. Therefore it follows that $(8^{k+1} - 3^{k+1})$ is divisible by 5. Hence, the result holds for $n = k + 1$.

3. The result holds for $n = 1$ because $8 - 3 = 5$ and so is divisible by 5.

So we have shown that the result holds for all n - by induction.

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Intersection

We can write the set of items that belong to both the set A and the set B as $A \cap B$. Formally $(x \in A) \wedge (x \in B) \Rightarrow (x \in A \cap B)$.

We call this the *intersection* of A and B or, less formally, A *and* B. In terms of the Venn diagram in figure 4.2 the two circles represent A and B while the overlap (in black) is the intersection. As examples

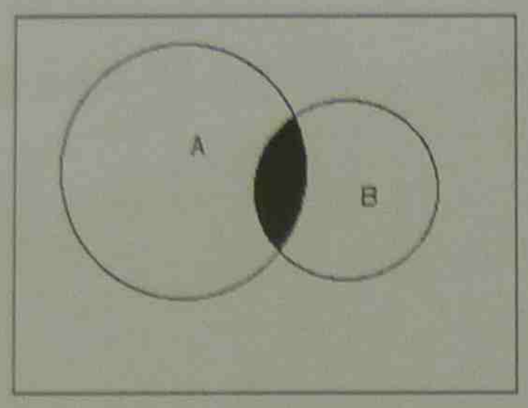

Figure 4.2: Venn diagram of $A \cap B$

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1. $\{1,2,3,4\} \cap \{ 3,4,5,6,7\} = \{ 3,4\}$. Notice $3 \in \{ 3,4\}$ while $1 \notin \{ 3,4\}$.
2. $\{1,2,3,4\} \cap \{ 13,14,15,16,27\} = \emptyset$.
3. {Abergail, Ann, Blodwin, Bronwin, Clair,}$\cap$ { Abergail, Bronwin, Gareth, Ian} = {Abergail, Bronwin, }.
4. In figure 4.2 we see $A \cap \bar{A} = \emptyset$ so A and $\bar{A}$ have nothing in common.
5. $A \cap B \subset B$ and $A \cap B \subset A$

Union:

We can write the set of items that belong to the set A or the set B or to both as $A \cup B$. Formally $(x \in A) \vee (x \in B) \Rightarrow x \in (A \cup B)$.

We call this the *union* of A and B or, less formally, A *or* B. The corresponding diagram is 4.3 Here the speckled area represents $A \cup B$

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Chapter 5

Counting

There are three types of people in this world: Those who can count, and those who can't.

Counting seem quite simple but this is quite deceptive, especially when we have complicated system. If you do not believe me have a look at the probability section. To make like a little simpler we lay down some rules.

Sets

If we have two sets A and B the number of item in the sets (the cardinality) is written $\|A\|$ and $\|B\|$. Then we can show that

$$\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|$$

This is fairly easy to see if you use a Venn diagram. For 3 sets

$$\|A \cup B\| = \|A\| + \|B\| + \|C\| - \|A \cap B\| - \|B \cap C\| - \|A \cap C\| + \|A \cap B \cap C\|$$

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Example

Let S be the set of all outcomes when two dice (one blue ; one green) are thrown. Let A be the subset of outcomes in which both dice are odd, and let B be the subset of outcomes in which both dice are even. We write C for the set of outcomes when the two dice have the same number showing.

How many elements are there in the following sets?

It is useful to have the set S set out as below

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The total number of items in n, so $n = n_1 + n_2 + \cdots n_k$ then there are

$$\frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

possible arrangements.

Suppose we have 3 white, 4 red and 4 black balls. They can be arranged in a row in

$$\frac{11!}{3!4!4!} = 11550$$

possible ways while the letters in WALLY can be arranged in

$$\frac{5!}{2!1!1!1!} = 60 \text{ ways}$$

Combinations

The number of ways of picking k items from a group of size n is written $\binom{n}{k}$ or (for the traditionalists) ${}_nC_k$. The definition is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

So the number of ways of picking 5 students from a group of 19 is

$$\binom{19}{5} = \frac{19!}{5!14!} = \frac{19 \times 18 \times 17 \times 16}{4 \times 3 \times 2 \times 2}$$

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Examples

1. Suppose you want to win the lottery. There are 49 numbers and you can pick 6. This can be done in

$$\frac{49!}{6!43!} = 13983816 \text{ ways}$$

so your chances of a win are 1/13983816.

2. How many ways can you pick 5 correct numbers in the lottery. There are $\binom{6}{5}$ ways to pick the 5 correct numbers and 49-6=43 ways of picking the remaining number. This gives 6×43 ways.

3. When we pick 3 correct numbers there are $\binom{6}{3}$ ways of picking the winning numbers and $\binom{43}{3}$ ways of picking the losing ones. This gives $\binom{6}{3} \times \binom{43}{3} = 20 \times 12341 = 246820$ ways in all.

5.0.7 Binomial Expansions

Now we have combinations we can examine a very useful result known as the binomial expansion. To start we can show that

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

In general we can prove that for an integer $n > 0$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n$$

or

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}b^i.$$

This can be done by induction, but there isis a page or so of algebra!

For example

$$(2+x)^5 = 2^5 + \binom{5}{1}2^4x + \binom{5}{2}2^3x^2 + \binom{5}{3}2^2x^2 + \binom{5}{4}2x^4 + x^5$$

or

$$(2+x)^5 = 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^2 + 5 \times 2x^4 + x^5$$

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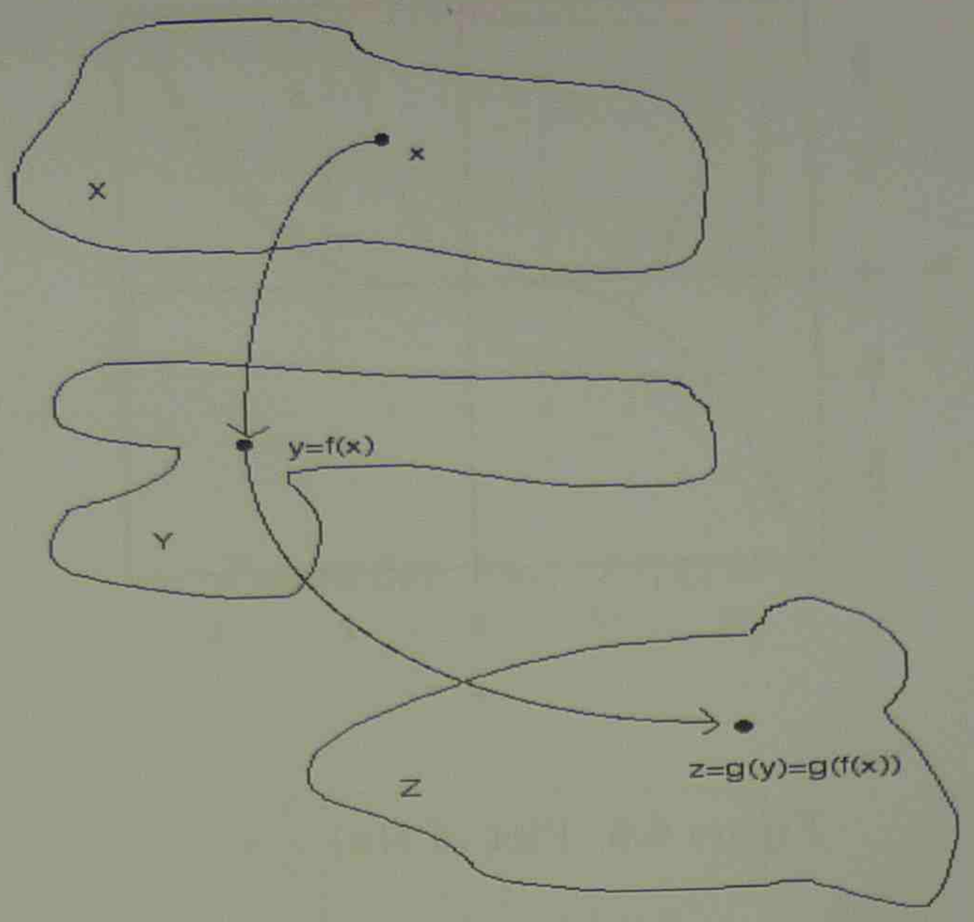

Figure 6.4: Composition of two functions f and g

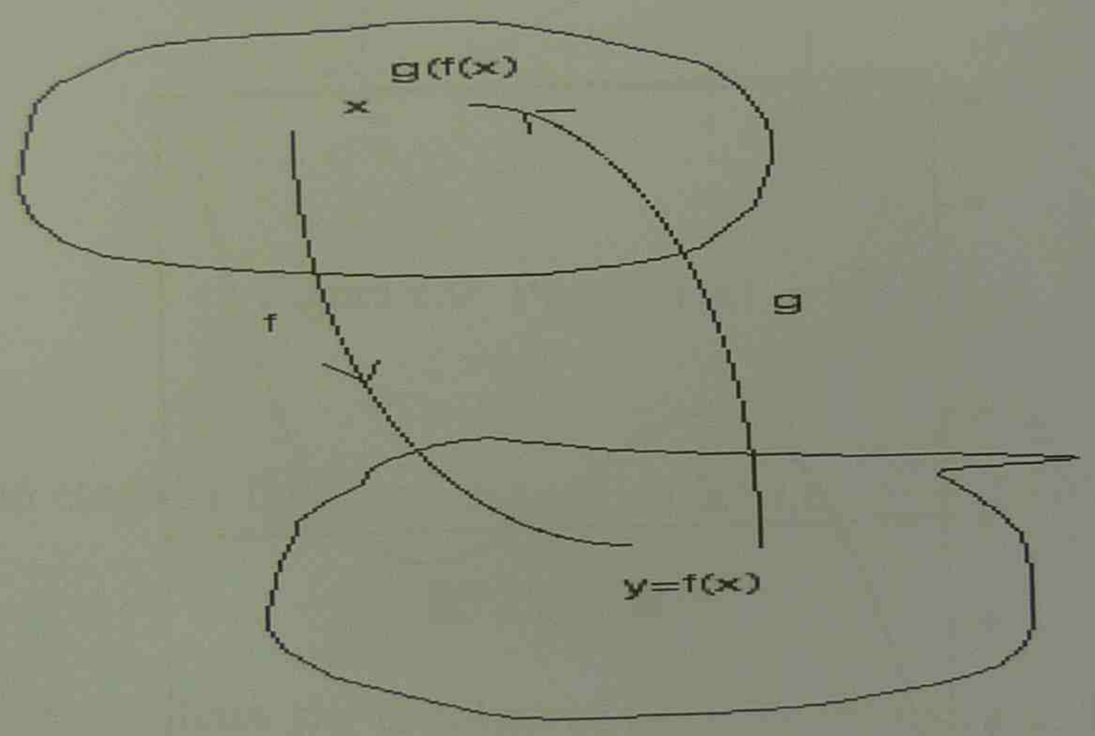

Figure 6.5: The inverse f and $g = f^{-1}$

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Examples

1. Suppose $f(x) = x^2$ and $g(y) = 1/y$ then $g(f(x)) = 1/x^2$. We of course have to take care about the definition if the range and the domain to avoid $x = 0$

2. When $f(x) = x^2$ and $g(x) = x^{1/2}$ g is the inverse function when f is defined on the positive reals.

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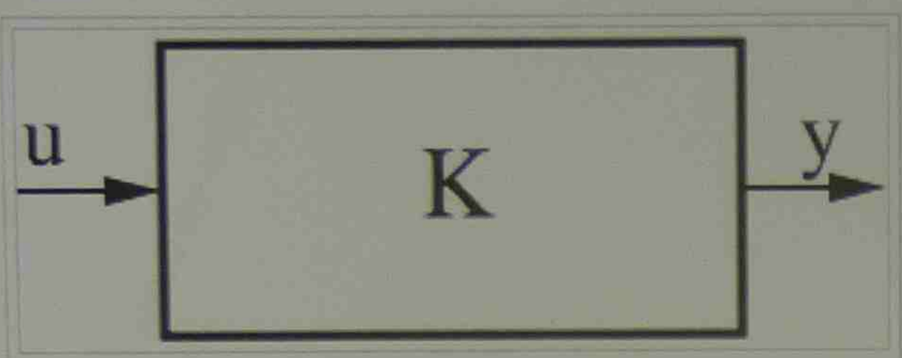
Block Diagrams

Block Diagram Representation

When designing or analyzing a system, often it is useful to model the system graphically. Block Diagrams are a useful and simple method for analyzing a system graphically. A "block" looks on paper exactly how it sounds:



A basic block diagram of system K, with input u(t) and output y(t).

If system K has a time-domain impulse response K(t), we can express y(t) as:

$$y(t) = u(t) * K(t)$$

Where the asterisk (*) denotes convolution. If system K has a Laplace-domain transfer function K(s), we show the relationship between the input and the output as:

$$Y(s) = U(s)K(s)$$

And if K is a state matrix, we can show the relationship between the input and output vectors as:

$$y = Kx$$

Systems in Series

When two or more systems are in series, they can be combined into a single representative system, with a transfer function that is the sum of the individual systems.

If we have two systems, F and G, we can put them in series with one another so that the output of system F is the input to system G. Now, we can analyze them depending on whether we are using our classical or modern methods.

Series Transfer Functions

If two or more systems are in series with one another, the total transfer function of the series is the product of all the individual system transfer functions.

Series State Space

If we have two systems in series (say system F and system G), where the output of F is the input to system G, we can write out the state-space equations for each individual system.

System 1:

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$$x'_F = A_F x_F + B_F u$$
$$y_F = C_F x_F + D_F u$$

System 2:

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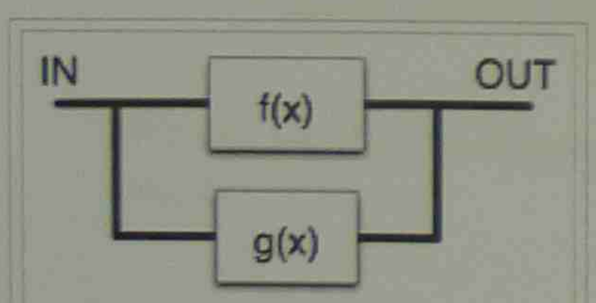
$$X'_G = A_G x_G + B_G y_F$$
$$Y_G = C_G x_G + D_G y_F$$

And we can write substitute these equations together form the complete response of system H, that has input u, and output y_G:

$$\begin{bmatrix} x_G \\ x_F \end{bmatrix} = \begin{bmatrix} A_G & B_G C_F \\ 0 & A_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} B_G D_F \\ B_F \end{bmatrix} u$$ **[Series state equation]**

$$\begin{bmatrix} y_G \\ y_F \end{bmatrix} = \begin{bmatrix} C_G & D_G C_F \\ 0 & C_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} D_G D_F \\ D_F \end{bmatrix} u$$ **[Series output equation]**

Systems in Parallel

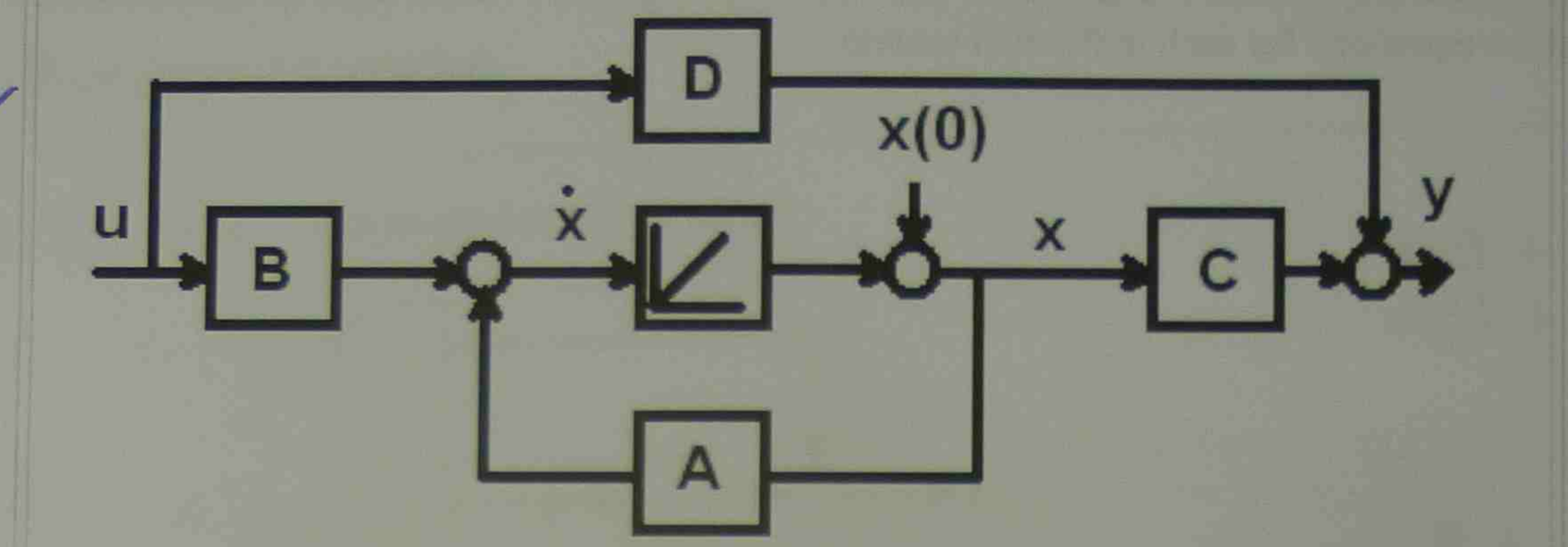


system f(x) in parallel with system g(x)

In practice, it is not common to see systems arranged in parallel. However, if you replace the node on the left with an adder, that combination is very common.

State Space Model

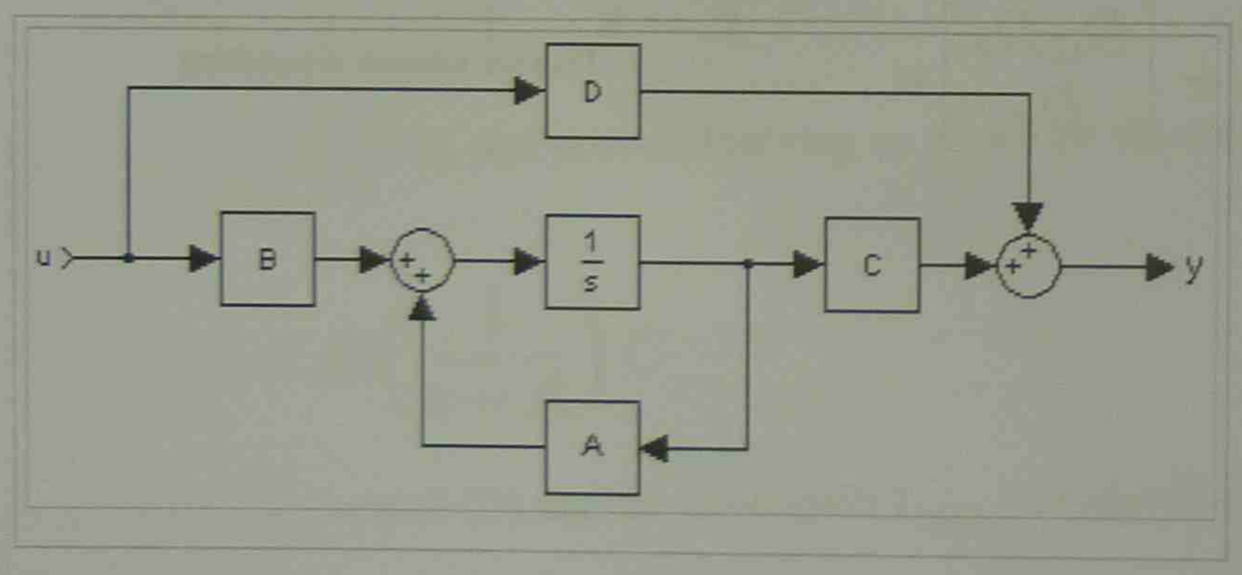
651 The state-space equations, with non-zero A, B, C, and D matrices conceptually model the following system:



In this image, the strange-looking block in the center is either an integrator, and can be represented in the transfer domain as:

$$\frac{1}{s} \text{ or } \frac{1}{z}$$

Depending on the time characteristics of the system. If we only consider continuous-time systems, we can replace the funny block in the center with an integrator:



In the Laplace Domain

The state space model of the above system, if A, B, C, and D are transfer functions A(s), B(s), C(s) and D(s) of the individual subsystems, and if U(s) and Y(s) represent a single input and output, can be written as follows:

$$\frac{Y(s)}{U(s)} = B(s)\left(\frac{1}{s - A(s)}\right)C(s) + D(s)U(s)$$

We will explain how we got this result, and how we deal with feedforward and feedback loop structures in the next chapter.

Adders and Multipliers

signal shows us how far off our output is from our desired input.

Example: State-Space Equation

In the previous chapter, we showed you this picture:

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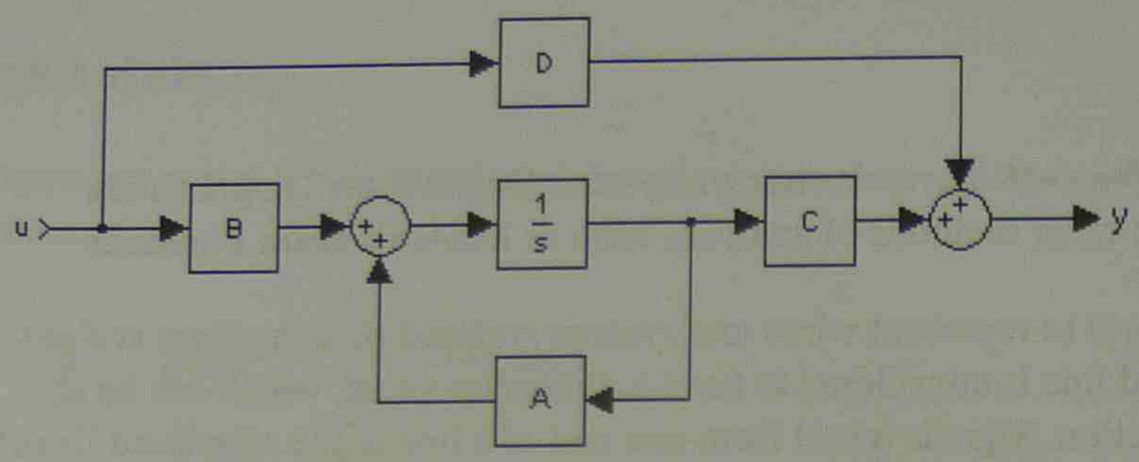

Now, we will derive the I/O relationship into the state-space equations. If we examine the inner-most feedback loop, we can see that the forward path has an integrator system, $\frac{1}{s}$, and the feedback loop has the matrix value A. If we take the transfer function only of this loop, we get:

$$T_{inner}(s) = \frac{\frac{1}{s}}{1 - \frac{1}{s}A} = \frac{1}{s - A}$$

Pre-multiplying by the factor B, and post-multiplying by C, we get the transfer function of the entire lower-half of the loop:

$$T_{lower}(s) = B\left(\frac{1}{s - A}\right)C$$

We can see that the upper path (D) and the lower-path T_{lower} are added together to produce the final result:

$$T_{total}(s) = B\left(\frac{1}{s - A}\right)C + D$$

Now, for an alternate method, we can assume that **x'** is the value of the inner-feedback loop, right before the integrator. This makes sense, since the integral of **x'** should be **x** (which we see from the diagram that it is. Solving for **x'**, with an input of **u**, we get:

$$x' = Ax + Bu$$

This is because the value coming from the feedback branch is equal to the value **x** times the feedback loop matrix A, and the value coming from the left of the adder is the input **u** times the matrix B.

If we keep things in terms of **x** and **u**, we can see that the system output is the sum of **u** times the feed-forward value D, and the value of **x** times the value C:

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- A is the sum of all individual loop gains
- B is the sum of the products of all the pairs of touching loops
- C is the sum of the products of all the sets of 3 touching loops
- D is the sum of the products of all the sets of 4 touching loops
- et cetera.

If the given system has no pairs of loops that touch, for instance, B and all additional letters after B will be zero.

Mason's Rule

GSY

If we have computed our delta values (above), we can then use **Mason's Gain Rule** to find the complete gain of the system:

$$M = \frac{y_{out}}{y_{in}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta}$$ **[Mason's Rule]**

Where M is the total gain of the system, represented as the ratio of the output gain (y_{out}) to the input gain (y_{in}) of the system. M_k is the gain of the k^{th} forward path, and Δ_k is the loop gain of the k^{th} loop.

Unstable Region	Right-Hand S Plane	$\sigma < 0$	Outside the Unit Circle, $\lvert z \rvert > 1$

Examples

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Example 1: First-Order System

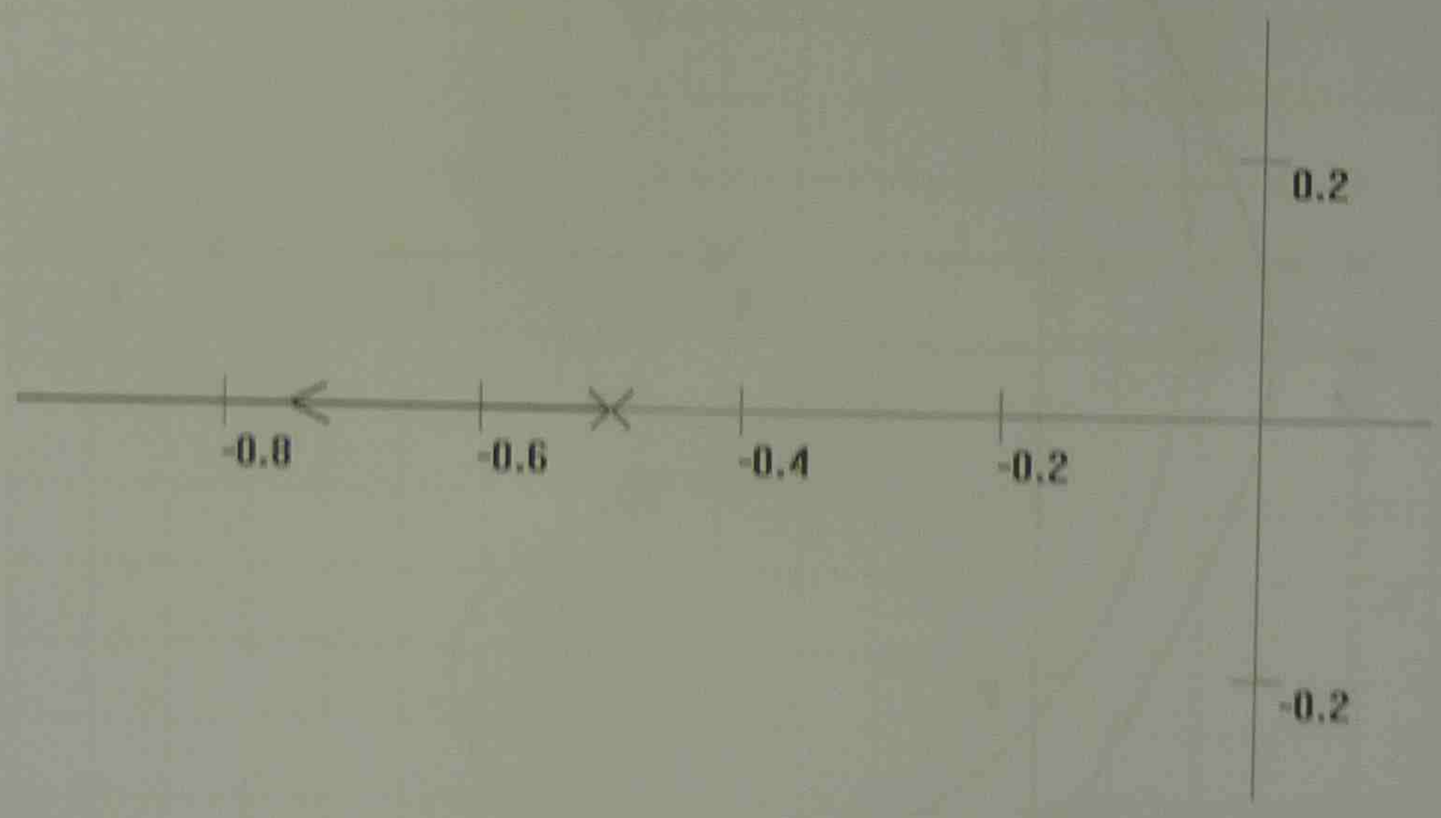
Find the root-locus of the closed-loop system:

$$T(s) = \frac{1}{1+2s}$$

If we look at the characteristic equation, we can quickly solve for the single pole of the system:

$$D(s) = 1 + 2s = 0$$
$$s = -\frac{1}{2}$$

We plot that point on our root-locus graph, and everything on the real axis to the left of that single point is on the root locus (from the rules, above). Therefore, the root locus of our system looks like this:



From this image, we can see that for *all values of gain* this system is stable.

Example 2: Third Order System

We are given a system with three real poles, shown by the transfer function:

$$T(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

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where the variables $x(t)$ and $f(t)$ of output and input are functions of time t, and a and b are real constants.

Frequently, it is sufficient to determine the dynamic behavior of the output variables $x(t)$. If the actual variables are required, the steady-state solutions at the operating point must be added, corresponding to x_0 and y_0 in Fig. 1.5. These can be found from the actual nonlinear equations by setting derivatives with respect to time to zero.

Example 1.5.1. Spring–Mass–Damper System

The linear case of the classical system in Fig. 1.6 is considered in Example 2.2.1. The differential equation (2.2) that describes the position $x(t)$ of the mass m in response to an external force $f(t)$ is

$$m\ddot{x} + c\dot{x} + kx = f$$

As discussed in Section 2.2, $-kx$ is the spring force on m, with spring constant k, and $-c\dot{x}$ the damping force, with damping coefficient c. In the linear case k and c are constant, and weight mg does not occur in the equation if $x = 0$ is chosen at the position of static equilibrium, where the weight is counterbalanced by a spring force.

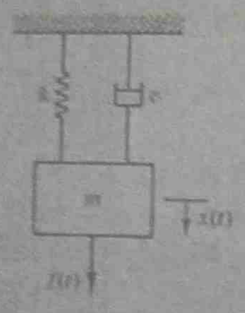

Figure 1.6 Spring–mass–damper system.

But often spring force does not change proportionally with x, and a nonlinear differential equation model is necessary, for example:

$$m\ddot{x} + c\dot{x} + kx^3 = f + mg$$

Weight mg is added to the downward force $f(t)$ and $x = 0$ at the position of zero spring force.

At a steady-state operating point where $f = f_0 =$ constant, the solution $x = x_0 =$ constant, so that $kx_0^3 = f_0 + mg$, or $x_0 = [(f_0 + mg)/k]^{1/3}$. To linearize the equation for small variations $\Delta x = x - x_0$, $\Delta f = f - f_0$ about the operating point, the nonlinear term $f(x) = kx^3$ is linearized according to (1.1) as

$$kx_0^3 + \left(\frac{df}{dx}\right)_{x_0} \Delta x = kx_0^3 + 3kx_0^2\,\Delta x$$

Then, since $(\Delta x)^{\cdot} = \dot{x}$, $(\Delta \dot{x})^{\cdot} = \ddot{x}$, the linear model is

$$m(\Delta\ddot{x}) + c(\Delta\dot{x}) + 3kx_0^2(\Delta x) + kx_0^3 = f_0 + \Delta f + mg$$

Since $kx_0^3 = f_0 + mg$, redefining x and f to represent the variations about x_0 and f_0 gives

$$m\ddot{x} + c\dot{x} + (3kx_0^2)x = f$$

If, say to determine physical clearances, the actual positions are needed, x_0 must be added to the solution of this linear differential equation.

For linear systems, the equations for actual variables and deviations are the same. For example, if $\ddot{x} + c\dot{x} + dx = y$, then substituting $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, where x_0 and y_0 are constant, yields

$$\Delta\ddot{x} + c\,\Delta\dot{x} + d\,\Delta x + dx_0 = \Delta y + y_0$$

But $dx_0 = y_0$ is the steady-state solution, so the equation in terms of Δx is the same as that for x.

In Chapter 12 the state space model will be introduced. It is a description of dynamic systems in terms of a set of first-order differential equations, written compactly in matrix form. This form of model is very powerful for the study of even very large systems. But in classical control theory, systems are commonly described by means of transfer functions. These will be defined by the use of Laplace transforms.

1.6 LAPLACE TRANSFORMS

Laplace transform theory is quite extensive, and it is therefore fortunate that only a small and isolated part of it is needed. The *Laplace transform* $F(s)$ of a function $f(t)$ is defined by

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt \tag{1.5}$$

For the present it suffices to consider the *Laplace variable* s simply as a complex variable with real part σ and imaginary part ω:

$$s = \sigma + j\omega \tag{1.6}$$

From (1.5), the transform changes a function of time into a function of this new variable s. The advantage will be found to be that differentiation and integration are changed into algebraic operations.

Important examples and theorems are discussed below. The solutions of the integrals, which need not be evaluated in the use that will be made of all these results, may be verified, if desired, from tables of integrals.

1. *Unit step function* $u(t)$: Shown in Fig. 1.7(a), this is a common test input to evaluate the performance of control systems. Its Laplace transform, from substitution of $f(t) = 1$ into (1.5), is

$$L[u(t)] = \int_0^\infty (1)e^{-st}\,dt = \frac{1}{s} \tag{1.7}$$

   For a step of magnitude A, $Au(t)$, the transform is A/s.

2. *Ramp function* At: This is also a common test input, and is shown in Fig. 1.7(b). With $f(t) = At$, integration by parts in equation (1.5) yields

Figure 1.9 Block in time (a) and Laplace (b) domain.

The representation of Fig. 1.9(a) is now replaced by that of Fig. 1.9(b), in terms of transforms and the transfer function. The following interpretation will often be applied:

$$C(s) = G(s)R(s) \tag{1.24}$$

In words, the transform $C(s)$ of the output equals the transfer function $G(s)$ times the transform $R(s)$ of the input.

*****

The assumption of zero initial conditions is very common in system analysis and design. It is natural for linearized equations, in which the variables are variations about operating point values. Furthermore, for linear systems the nature of the transient response is independent of the initial conditions: that is, whether or not the response is oscillatory, and if it is, whether the oscillations decay sufficiently fast. This is the type of information usually of most interest to the designer. At the same time, it is well to point out that the assumption of zero initial conditions in the definition of $G(s)$ is not a constraint. For linear systems the principle of superposition applies, which means that the total response is the sum of those to the input and to the initial conditions applied separately. Thus the initial conditions can be set to zero in determining the response to input $r(t)$, and $r(t)$ set to zero when calculating the response to initial conditions.

*****

The above in effect provides the scenario for calculating the response $c(t)$ of a system to a given input $r(t)$ with specified initial conditions. For example, using the dot notation for derivatives, if

$$\dddot{c} + a_2\ddot{c} + a_0 c = r \qquad c(0) = c_0, \quad \dot{c}(0) = \dot{c}_0 \tag{1.25}$$

then the transfer function is

$$G(s) = \frac{C_f(s)}{R(s)} = \frac{1}{s^3 + a_2 s + a_0} \tag{1.26}$$

and for a unit step input $r(t)$ [$R(s) = 1/s$] the transform of the output due to $r(t)$ is

$$C_f(s) = \frac{1}{s(s^3 + a_2 s + a_0)} \tag{1.27}$$

For the response to initial conditions, $r = 0$ in (1.25) and this equation is transformed, using (1.16) and (1.11) or Table 1.6.1, to

$$[s^3C_i(s) - sc_0 - \dot{c}_0] + a_2[sC_i(s) - c_0] + a_0C_i(s) = 0$$

Rearranging yields the output transform

$$C_i(s) = \frac{(s + a_2)c_0 + \dot{c}_0}{s^3 + a_2 s + a_0} \tag{1.28}$$

Hence the transform of the total output is

$$C(s) = C_f(s) + C_i(s) \tag{1.29}$$

*****

Inverse Laplace transformation of these transforms will give the corresponding responses in the time domain. For very simple cases these inverses may be available directly in a table of Laplace transform pairs such as Table 1.6.1, but it is now evident that techniques are needed to invert more complex expressions. This will be considered in detail and in a broader context in Chapter 3. But, in part to help motivate the derivation of transfer functions for physical systems in Chapter 2 by showing at least one of the uses to which they can be put, the simplest case of the technique to be used is introduced here.

Inverse Transformation by Partial Fraction Expansion

[Roots of denominator of $C(s)$ assumed real and distinct.] Examples will be used to demonstrate the technique.

Example 1.7.1

Determine the response to initial conditions $c(0) = 1$, $\dot{c}(0) = 2$ of the system

$$\ddot{c} + 7\dot{c} + 6c = r \tag{1.30}$$

For the response to initial conditions, the input r can be taken to be zero, and the transform is

$$[s^2C - sc(0) - \dot{c}(0)] + 7[sC - c(0)] + 6C = 0$$

or

$$C = \frac{(s + 7)c(0) + \dot{c}(0)}{s^2 + 7s + 6} = \frac{s + 9}{s^2 + 7s + 6}$$

For inverse transformation, the denominator is factored as $(s + 1)(s + 6)$, and $C(s)$ is expanded into partial fractions according to

$$C(s) = \frac{s + 9}{(s + 1)(s + 6)} = \frac{K_1}{s + 1} + \frac{K_2}{s + 6}$$

with a separate term for each real root factor in the denominator of $C(s)$. These terms are of the form $A/(s + a)$, for which Table 1.6.1 gives the inverse Ae^{-at}. The constants K_i are the *residues* at the corresponding roots. To find K_1, multiply both sides of the equation by its denominator $(s + 1)$:

$$(s + 1)C(s) = \frac{s + 9}{s + 6} = K_1 + \frac{K_2(s + 1)}{s + 6}$$

If s is permitted to approach the root -1 of $(s + 1)$, the second term on the right

residues and the residue K_1 is

$$K_1 = [(s+1)C(s)]_{s=-1} = \left(\frac{s+9}{s+6}\right)_{s=-1} = 1.6$$

Similarly K_2 at $s = -6$ is found in the same manner by multiplying through by $(s+6)$:

$$K_2 = [(s+6)C(s)]_{s=-6} = \left(\frac{s+9}{s+1}\right)_{s=-6} = -0.6$$

The partial fraction expansion is now

$$C(s) = \frac{1.6}{s+1} - \frac{0.6}{s+6}$$

By the linearity theorem, the inverse is the sum of the inverses of the terms, so that the transient response $c(t)$ is the sum of two decaying exponentials:

$$c(t) = 1.6e^{-t} - 0.6e^{-6t}$$

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Example 1.7.2

Find the response $c(t)$ of a system with transfer function

$$G(s) = \frac{2(s+3)}{(s+1)(s+6)} \tag{1.31}$$

to a decaying exponential input $r(t) = e^{-2t}$.

From Table 1.6.1, $R(s) = 1/(s+2)$, so that $C(s)$ and its partial fraction expansion can be written as follows:

$$C(s) = G(s)R(s) = \frac{2(s+3)}{(s+2)(s+1)(s+6)} = \frac{K_1}{s+2} + \frac{K_2}{s+1} + \frac{K_3}{s+6}$$

The residues are calculated next:

$$K_1 = [(s+2)C(s)]_{s=-2} = \left[\frac{2(s+3)}{(s+1)(s+6)}\right]_{s=-2} = -0.5$$

$$K_2 = [(s+1)C(s)]_{s=-1} = \left[\frac{2(s+3)}{(s+2)(s+6)}\right]_{s=-1} = +0.8$$

$$K_3 = [(s+6)C(s)]_{s=-6} = \left[\frac{2(s+3)}{(s+2)(s+1)}\right]_{s=-6} = -0.3$$

The partial fraction expansion is

$$C(s) = -\frac{0.5}{s+2} + \frac{0.8}{s+1} - \frac{0.3}{s+6}$$

so that the response sought is

$$c(t) = -0.5e^{-2t} + 0.8e^{-t} - 0.3e^{-6t}$$

The foregoing technique for calculating residues also applies when the roots

of the denominator of $C(s)$ are distinct but include complex conjugate pairs. But then the residues also include complex conjugate pairs. A vector-based method is preferable for these cases and for more complicated transforms generally. This is discussed in Chapter 3, where repeated roots are considered as well.

1.8 BLOCK DIAGRAM REDUCTION

The discussion of Section 1.7 appears to imply that if the transfer function relating input r and output c in a block diagram such as Fig. 1.3 is desired, a differential equation relating these two variables must be obtained first. Fortunately, this is not necessary. The transfer function can be derived instead by certain algebraic manipulations of those of the subsystems or blocks. Some examples will show this block diagram reduction technique and provide some useful results.

Example 1.8.1

Reduce the *cascade* or series connection of two blocks in Fig. 1.10 to a single block G as in Fig. 1.9(b). By definition,

$$C = G_2 M \qquad M = G_1 R$$

Hence, substituting the second into the first yields

$$C = G_2 G_1 R \qquad G = G_1 G_2 \tag{1.32}$$

By direct extension it follows that

> The overall transfer function of a series of blocks equals the product of the individual transfer functions.

Figure 1.10 Two-block cascade.

Example 1.8.2

The configuration in Fig. 1.11, equivalent to that in Fig. 1.3, is extremely common. By definition

$$C = G_1 M \qquad M = G_2 E \qquad E = R - B \qquad B = HC$$

Combining each pair yields

$$C = G_1 G_2 E \qquad E = R - HC$$

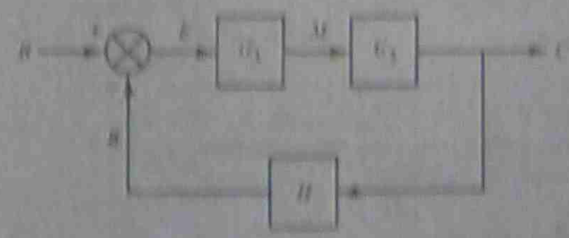


Figure 1.11 Standard feedback loop.

Eliminating E gives

$$C = G_1G_2R - G_1G_2HC$$

and rearranging this, the

$$\text{closed-loop transfer function } \frac{C}{R} = \frac{G_1G_2}{1 + G_1G_2H} \tag{1.33}$$

In words, and in somewhat generalized form, this important result states that

> The *closed-loop transfer function* of the standard loop equals the product of the transfer functions in the forward path divided by the sum of 1 and the loop gain function. The *loop gain function* is defined as the product of the transfer functions around the loop.

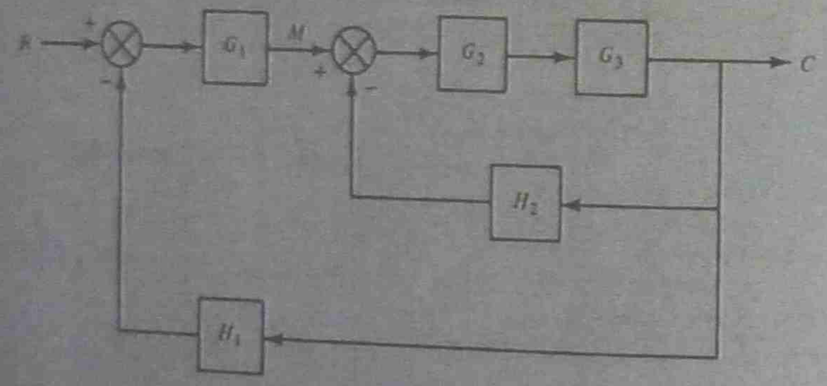
Note further that, since $C = G_1G_2E$,

$$\frac{E}{R} = \frac{1}{1 + G_1G_2H} \tag{1.34}$$

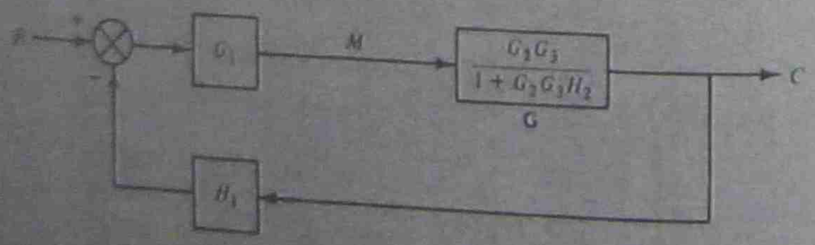
If $H = 1$, then $E = R - C$ is the system error, as in Fig. 1.3, and E/R is the input-to-error transfer function. It will permit the error response for a given input $r(t)$ to be found directly. The transfer function relating the input to any variable of interest can be found similarly.

Example 1.8.3

The configuration in Fig. 1.12(a), which includes a *minor feedback loop*, is very common in servomechanisms. Derivation of C/R by the approach of Example 1.8.2



(a)



(b)

Figure 1.12 Minor loop feedback.

would be laborious, but becomes simple if its result in (1.33) is used. It is applied first to reduce the minor feedback loop C/M to a single block, as shown in Fig. 1.12(b). But (1.33) applies again to this new loop and now yields the closed-loop transfer function

$$\frac{C}{R} = \frac{G_1G}{1 + G_1GH_1} = \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2G_3H_1} \tag{1.35}$$

Example 1.8.4

A two-input system is shown in Fig 1.13. The additional input D often represents a disturbance, such as a supply pressure variation in the level control example in Section 1.3. With the additional block L, the diagram models the effect of the disturbance on the system.

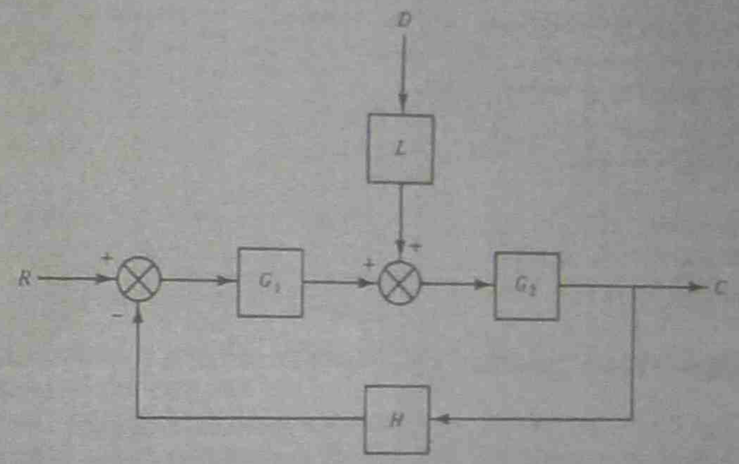


Figure 1.13 Two-input system.

For linear systems the principle of superposition applies, and the total output is the sum of the outputs due to each input separately. Thus the output due to R is found as before, and while finding that due to D, R is put equal to zero.

The rule of Example 1.8.2 applies when finding the response to D, but note that the product of the transfer functions in the forward path consists, aside from L, only of G_2. Note also that for $R = 0$ the minus sign for the feedback at R can be moved to the summing junction for D. Inspection now yields

$$\frac{C}{D} = \frac{G_2L}{1 + G_1G_2H} \tag{1.36}$$

Example 1.8.5

In Fig. 1.14 the two feedback loops interfere with each other. The rearrangements (a) and (b) are alternative first steps to make the result (1.33) again applicable. Verify that neither changes the system, and that applying (1.33) twice to (a) or (b) yields the closed-loop transfer function

$$\frac{C}{R} = \frac{G_1G_2}{1 + G_1H_2 + G_2H_1} \tag{1.37}$$

the writing of the equations, such as θ_1, the angle to the right as well as the left of J_1.

The approach of the preceding examples could be used, but an alternative is often convenient. Equations are written in order, starting at the input. To help visualize signs, assume that $\dot{\theta}_1 > \dot{\theta}_2$, $\dot{\theta}_2 > \dot{\theta}_3$. The shaft torque $k(\theta_1 - \theta_2)$ accelerates inertia J_1 and supplies the damping torque

$$k(\theta_1 - \theta_2) = J_1\ddot{\theta}_1 + c(\dot{\theta}_1 - \dot{\theta}_2)$$

It is the damping torque which in turn accelerates inertia J_2:

$$c(\dot{\theta}_1 - \dot{\theta}_2) = J_2\ddot{\theta}_2$$

Rearranging yields the differential equations

$$J_1\ddot{\theta}_1 + c\dot{\theta}_1 + k\theta_1 = c\dot{\theta}_2 + k\theta_2 \qquad (2.6)$$

$$J_2\ddot{\theta}_2 + c\dot{\theta}_2 = c\dot{\theta}_1$$

and the transformed equations

$$(J_1s^2 + cs + k)\theta_2(s) = cs\theta_1(s) + k\theta_1(s)$$

$$(J_2s^2 + cs)\theta_2(s) = cs\theta_1(s)$$

where $\theta(s)$ is the transform of $\theta_i(t)$. From the second equation,

$$\theta_2(s) = \left(\frac{J_2}{c}s + 1\right)\theta_1(s)$$

Substituting this into the first and rearranging gives the transfer function

$$\frac{\theta_2(s)}{\theta_1(s)} = \frac{k}{(J_1J_2/c)s^3 + (J_1 + J_2)s^2 + (J_2k/c)s + k} \qquad (2.7)$$

Example 2.2.4 Systems with Gears

Although not necessary, it is often convenient for analysis to replace a system with gears by a dynamically equivalent system without gears. Figure 2.5 shows the

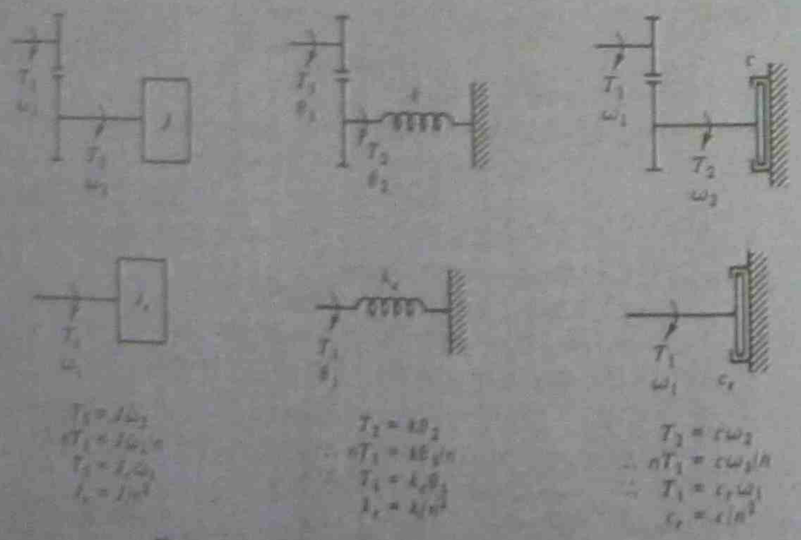


Figure 2.5 Equivalent systems without gears.

derivation of the equivalent inertia, spring, and damping, identified as J_1, k_1, and c_1, on the first shaft which can replace J, k, and c on the second shaft. In addition to the relation $T_2 = nT_1$ from Fig. 2.1, (2.1) is used, and the fact that $\dot{\theta}_1 = \omega_1$. *Note that in each case the equivalent element is obtained by dividing the original element by the square of the speed ratio n.*

Example 2.2.5 Nongeared Equivalent System

Using Fig. 2.5, the system in Fig. 2.6(a) can be replaced immediately by its nongeared equivalent in Fig. 2.6(b), where J_1, k_1, and c_1 are as given in Fig. 2.5.

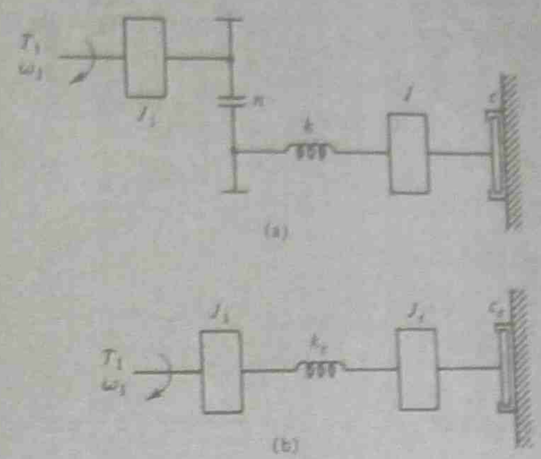


Figure 2.6 Example 2.2.5.

2.3 ELECTRICAL SYSTEMS: CIRCUITS

Figure 2.7 summarizes some important results for the modeling of electrical circuits. Figure 2.7(a) gives the voltage–current relations of the basic elements in the time domain and the Laplace domain, assuming zero initial conditions. In the general transformed relation $V = IZ$, Z is the impedance. In an *ideal* voltage source, the voltage difference across the terminals is independent of the current through them. In a *real* voltage source, the voltage v, across the terminals decreases with increasing current due to the voltage drop across the internal impedance of the source, which may be a resistance R.

The use of these results and those in Fig. 2.7(b) and (c) will be illustrated by application to the examples shown in Fig. 2.8. The transfer functions $E_o(s)/E_i(s)$ between inputs $e_i(t)$ and outputs $e_o(t)$ are desired. The current through the output terminals must be assumed to be negligibly small, because otherwise its value would affect e_o, and E_o/E_i would not be uniquely defined. In such a case a transfer function can be derived from differential equations written for the combination of the circuit and its load. This illustrates a key condition when subdividing a system into subsystem blocks.

> The division of systems into blocks must be such that each block does not *load*, that is, affect the output of, the preceding block.

4. *Lever mechanism:* For small enough angles from horizontal, the total motion y equals the sum of the motion due to x_1 with $x_2 = 0$ and that due to x_2 with $x_1 = 0$. It is useful to observe that the lever is a mechanical implementation of a summing junction in a block diagram, and is in fact often used for this purpose. If $a = b$, then $y = 0.5\,(x_1 - x_2)$ if the direction of x_2 is reversed. If input and output are available in mechanical form and are applied to x_1 and x_2, respectively, the linkage with $a = b$ and x_2 opposite implements the feedback loop as well as providing the system error $e = x_1 - x_2$.

5. *Gears:* This very common element is often identified in terms of its gear ratio n,

$$n = \frac{\text{speed of driving gear}}{\text{speed of driven gear}} = \frac{\omega_1}{\omega_2} = \frac{\dot\theta_1}{\dot\theta_2}\left(= \frac{\theta_1}{\theta_2} = \frac{\ddot\theta_1}{\ddot\theta_2}\right) \tag{2.1}$$

where $\omega_i = \dot\theta_i$ is the angular velocity (rad/sec) of the gear with diameter d_i. The relation $T_2 = nT_1$ between the torques arises because the two gears have a common contact force, and the torque equals this force times the gear radius.

Examples incorporating these elements are considered next.

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Example 2.2.1 Spring–Mass–Damper System

By Newton's law, $m\ddot x$ equals the resultant of all external forces on m in Fig. 2.2 in the downward direction. To help in determining the signs of the terms in such problems, it is useful to make any assumption concerning the motion: for example, that the mass is moving downward from $x = 0$. In that case the spring is stretched, so spring force kx is upward, and hence opposes downward acceleration. It therefore receives a minus sign on the right side of the equation for $m\ddot x$. Since the mass moves down, the damping force $c\dot x$ is upward, and this term must also have a minus sign. The external force $f(t)$ helps downward acceleration, and therefore has a plus sign. The resulting equation is

$$m\ddot x = -kx - c\dot x + f(t)$$

Rearranging gives the differential equation of motion in the usual form:

$$m\ddot x + c\dot x + kx = f(t) \tag{2.2}$$

It may have been observed that the effects of gravity do not appear, so that turning the system upside down will not affect the equation. This is done by choosing

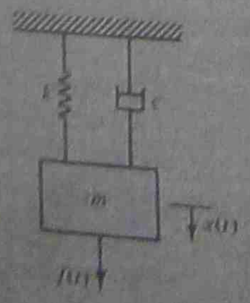

Figure 2.2 Spring–mass–damper system.

$x = 0$ at the position of static equilibrium, where the weight mg is counterbalanced by a spring force.

For a transfer function model, assume that the response $x(t)$ to $f(t)$ is desired. With zero initial conditions, the transform of (2.2) is

$$(ms^2 + cs + k)X(s) = F(s)$$

and the transfer function of interest is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{2.3}$$

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Example 2.2.2 Two-Mass System

The system in Fig. 2.3 can represent a dynamic absorber, where a relatively small mass m_1 is attached to a main mass m via spring k_1 and damper c_1 to reduce vibrations x due to force f. Assume, say, that m and m_1 both move to the right from the zero positions, m farther and faster than m_1. Then spring force $k_1(x - x_1)$ "opposes" m and "helps" m_1, and damper force $c_1(\dot x - \dot x_1)$ has the same effect. Hence the differential equations of motion become

$$m\ddot x = -kx - k_1(x - x_1) - c_1(\dot x - \dot x_1) + f$$

$$m_1\ddot x_1 = k_1(x - x_1) + c_1(\dot x - \dot x_1)$$

or

$$m\ddot x + c_1\dot x + (k + k_1)x = c_1\dot x_1 + k_1x_1 + f \tag{2.4}$$

$$m_1\ddot x_1 + c_1\dot x_1 + k_1x_1 = c_1\dot x + k_1x$$

For a transfer function model, the effect of f on x would be of interest. This requires the elimination of x_1, an algebraic operation if (2.4) are first transformed:

$$(ms^2 + c_1s + k + k_1)X(s) = (c_1s + k_1)X_1(s) + F(s) \tag{2.5}$$

$$(m_1s^2 + c_1s + k_1)X_1(s) = (c_1s + k_1)X(s)$$

Solving $X_1(s)$ from the second equation, substituting it into the first, and rearranging would give the transfer function X/F.

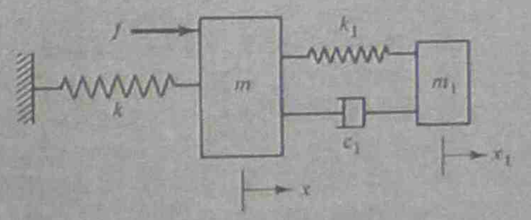


Figure 2.3 Dynamic absorber.

Example 2.2.3 Rotating Drive System

Figure 2.4 indicates a drive system, with c representing a friction coupling, and torsion spring k the twisting of a long shaft due to torque. Angle θ_1 is taken to be the input and θ_3 the output. Any other variables may be introduced to facilitate

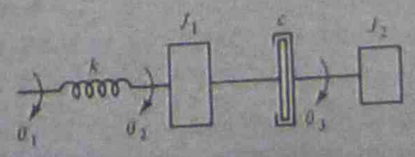


Figure 2.4 Rotating drive system.

Figure 2.8(c):

$$E_i = I\left(R_1 + R + \frac{1}{Cs}\right)$$

$$E_o = I\left(R + \frac{1}{Cs}\right)$$

where $\tau = RC \qquad \alpha = \dfrac{R}{R_1 + R}$

$$\frac{E_o}{E_i} = \frac{\tau s + 1}{(\tau/\alpha)s + 1} \qquad (2.8c)$$

Figure 2.8(d):

$$E_i = I\left(\frac{1}{(1/R_2) + C_2 s} + R\right)$$

since from Fig. 2.7(b) the equivalent impedance Z_t of the parallel impedances $Z_r = R_r$ and $Z_c = 1/(C_2 s)$ is given by

$$\frac{1}{Z_t} = \frac{1}{Z_r} + \frac{1}{Z_c} \quad \text{or} \quad Z_t = 1\Big/\left(\frac{1}{Z_r} + \frac{1}{Z_c}\right)$$

Hence

$$E_i = I\left(\frac{R_1 + R + RR_1C_2 s}{1 + R_1C_2 s}\right)$$

$$E_o = IR$$

where $\tau = R_1C_2 \qquad \alpha = \dfrac{R}{R_1 + R}$

$$\frac{E_o}{E_i} = \alpha\,\frac{\tau s + 1}{\alpha\tau s + 1} \qquad (2.8d)$$

Figure 2.8(e):

$$E_i = I\left(\frac{1}{(1/R_1) + C_1 s} + R_2 + \frac{1}{C_2 s}\right)$$

$$E_o = I\left(R_2 + \frac{1}{C_2 s}\right)$$

$$\frac{E_o}{E_i} = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2 + \tau_{12})s + 1} \qquad (2.8e)$$

where $\tau_1 = R_1C_1 \qquad \tau_2 = R_2C_2 \qquad \tau_{12} = R_1C_2$

Example 2.3.2 Bridged-*T* Network [Fig. 2.8(f)]

This network is often used in ac control systems, that is, systems in which signals are represented by modulation of an ac carrier. With e_o occurring inside a loop, it may be seen that the voltage-divider approach used above does not apply in equally straightforward fashion. But E_o/E_i can be obtained by the use of Kirchhoff's laws in Fig. 2.7(c), which could also have been used in Example 2.3.1. E_o/E_i will be derived by both the loop and node methods.

(a) *Loop method:* Kirchhoff voltage equations are written for each closed loop in terms of loop current variables:

$$E_i = \left(\frac{1}{Cs} + R_1\right)I_1 - \frac{1}{Cs}I_2 \qquad \text{(loop 1)}$$

$$0 = -\frac{1}{Cs}I_1 + \left(R_2 + \frac{2}{Cs}\right)I_2 \qquad \text{(loop 2)}$$

Since $E_o = E_i - I_2R_2$, this set is solved for I_2 by use of *Cramer's rule*:

$$I_2 = \begin{vmatrix} \frac{1}{Cs} + R_1 & E_i \\ -\frac{1}{Cs} & 0 \end{vmatrix} \Big/ \begin{vmatrix} \frac{1}{Cs} + R_1 & -\frac{1}{Cs} \\ -\frac{1}{Cs} & \frac{2}{Cs} + R_2 \end{vmatrix} = \frac{E_iCs}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2}$$

Then, from $E_o = E_i - I_2R_2$,

$$\frac{E_o}{E_i} = \frac{1 + 2R_1Cs + R_1R_2C^2s^2}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2} \qquad (2.8f)$$

(b) *Node method:* Here the Kirchhoff current equations are written in terms of voltage variables at each of the circuit nodes. In Fig. 2.8(f) the unknown node voltages are E_a $(=E_o)$ and E_b, and the equations are

$$\frac{E_a - E_i}{R_2} + \frac{E_o - E_b}{1/(Cs)} = 0 \qquad \text{(node } a\text{)}$$

$$\frac{E_b - E_a}{1/(Cs)} + \frac{E_b - E_i}{1/(Cs)} + \frac{E_b}{R_1} = 0 \qquad \text{(node } b\text{)}$$

Rearranging yields

$$\left(\frac{1}{R_2} + Cs\right)E_a - CsE_b = \frac{1}{R_2}E_i$$

$$-CsE_a + \left(\frac{1}{R_1} + 2Cs\right)E_b = CsE_i$$

Solution for E_a $(=E_o)$ by Cramer's rule yields E_o/E_i as in part (a), and in the case of this example in a somewhat more direct fashion.

Example 2.3.3 Ladder Network [Fig. 2.8(g)]

Using the node method, current equations are written for the circuit nodes a and b:

$$\frac{E_a - E_i}{L_1s} + \frac{E_a}{R_1} + \frac{E_a - E_b}{L_2s} = 0 \qquad \frac{E_b - E_a}{L_2s} + \frac{E_b}{R_2} = 0$$

Solution for $E_b = E_o$ gives

$$\frac{E_o}{E_i} = \frac{R_1R_2}{L_1L_2s^2 + (L_1R_1 + L_1R_2 + L_2R_1)s + R_1R_2}$$

2.4 ELECTROMECHANICAL SYSTEMS: TRANSFER FUNCTIONS OF MOTORS AND GENERATORS

Schematic diagrams of several arrangements of motors and generators are shown in Fig. 2.9. In all cases the motor load is assumed to consist of an inertia J and a damper with damping constant B. Motor shaft position θ and developed motor torque T are then related by

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t) \qquad T(s) = s(Js + B)\theta(s) \tag{2.9}$$

For convenience, the same variables are used in the time and Laplace domains. The identifier (t) or (s) will generally be omitted if it is evident from the context. Each of the systems in Fig. 2.9 will be considered in turn.

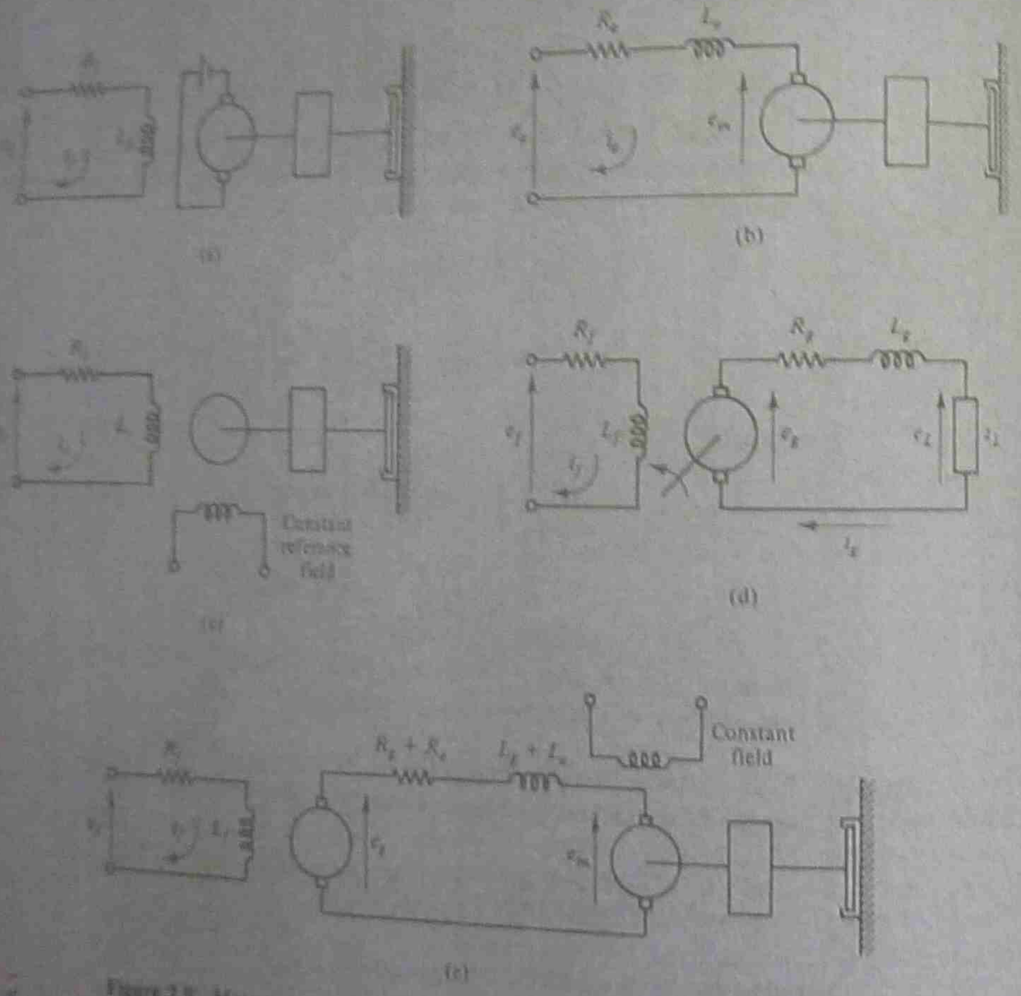


Figure 2.9 Motors and generators: (a) field-controlled dc motor; (b) armature-controlled dc motor; (c) two-phase ac servomotor; (d) dc generator; (e) motor–generator set.

Example 2.4.1 Field-Controlled DC Motor [Fig. 2.9(a)]

The equation for the field loop is

$$e_f = R_f i_f + L_f i_f \qquad E_f = (R_f + L_f s)I_f$$

With constant armature voltage, the developed motor torque T in (2.9) can be taken to be proportional to field current:

$$T = K_t i_f \qquad T = K_t I_f \qquad K_t = \text{motor torque constant}$$

Eliminating I_f and T between these transformed equations and (2.9) yields the desired transfer function between applied field voltage e_f and shaft position θ:

$$\frac{\theta}{E_f} = \frac{K_t/(R_f B)}{s(T_m s + 1)(T_f s + 1)} \tag{2.10}$$

where $T_m = J/B$ = motor time constant

$T_f = L_f/R_f$ = field time constant

Often $T_f \ll T_m$, and a satisfactory approximation in the operating range of interest is

$$\frac{\theta}{E_f} = \frac{K_t/(R_f B)}{s(T_m s + 1)} \tag{2.11}$$

Note that the transfer function was derived for the combination of the motor and its load. This load affects motor speed (i.e., it loads the motor), so a series connection of two blocks with individual transfer functions would be incorrect.

The factor s in the denominator of (2.10) and (2.11) should be noted. From Table 1.6.1, dividing $F(s)$ by s is equivalent to integrating $f(t)$. Thus the factor s represents the fact that a motor is basically an integrator: For a constant input e_f it has a shaft angle θ which increases at a constant rate, so θ is proportional to the integral of e_f.

Example 2.4.2 Armature-Controlled DC Motor [Fig. 2.9(b)]

The armature loop is described by

$$e_a = R_a i_a + L_a i_a + e_m \qquad E_a = (R_a + L_a s)I_a + E_m$$

Here the counter emf (electromotive force) voltage can be taken to be proportional to shaft speed,

$$e_m = K_v \dot{\theta} \qquad E_m = K_v s\theta$$

and the developed torque proportional to current i_a,

$$T = K_t i_a \qquad T = K_t I_a$$

Eliminating I_a, E_m, and T between these equations and (2.9) permits the desired transfer function to be arranged in the common form

$$\frac{\theta}{E_a} = \frac{1/K_v}{s(T_a T_m s^2 + (T_m + \gamma T_a)s + \gamma + 1)} \tag{2.12}$$

where $T_m = JR_a/(K_v K_t)$ = motor time constant

$T_a = L_a/R_a$ = armature time constant

$\gamma = BR_a/(K_v K_t)$ = damping factor

Example 2.4.3 Two-Phase AC Servomotor [Fig. 2.9(c)]

Fixed and variable magnitude ac voltages are applied to the reference and control fields, respectively. A 90° phase shift arranged between these voltages is made positive or negative depending on the desired direction of rotation. The control field is described by

$$e_c = R_c i_c + L_c \dot{i}_c \qquad E_c = (R_c + L_c s) I_c \tag{2.13}$$

The developed motor torque T can be taken to be proportional to i_c and to decrease proportionally with increasing speed, and is described by

$$T = K_c i_c - K_m \dot{\theta} \qquad T = K_c I_c - K_m s\theta \tag{2.14}$$

This dependence on speed is assumed to be the same under dynamic conditions as for the steady-state torque–speed motor characteristic curves indicated in Fig. 2.10. Eliminating T and I_c between these equations and (2.9) and rearranging gives the transfer function

$$\frac{\theta}{E_r} = \frac{K}{s(T_m s + 1)(T_c s + 1)} \tag{2.15}$$

where $T_m = J/(B + K_m)$ = motor time constant

$T_c = L_c/R_c$ = electrical time constant

$K = K_c/[R_c(B + K_m)]$ = motor constant

Often simplification to the form (2.11) is again satisfactory: $K/[s(T_m s + 1)]$.

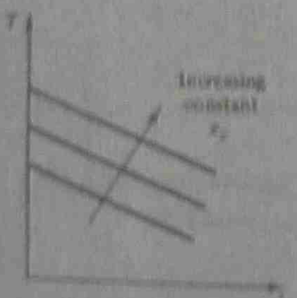


Figure 2.10 Ac motor characteristics.

Example 2.4.4 DC Generator [Fig 2.9(d)]

The field loop equation is

$$e_f = R_f i_f + L_f \dot{i}_f \qquad E_f = (R_f + L_f s) I_f$$

The developed generator voltage e_g can be assumed to be proportional to field current:

$$e_g = K_g i_f \qquad E_g = K_g I_f$$

The voltage e_L across the load is given by

$$E_L = Z_L I_g \qquad Z_L = \text{load impedance}$$

and the generator loop is described by

$$e_g = R_g i_g + L_g \dot{i}_g + e_L \qquad E_g = (L_g s + R_g + Z_L) I_g$$

Hence

$$\frac{E_g}{E_f} = \frac{K_g}{L_f s + R_f} \qquad \frac{E_L}{E_g} = \frac{Z_L}{L_g s + R_g + Z_L} \qquad \frac{E_L}{E_f} = \frac{E_L}{E_g}\frac{E_g}{E_f} \tag{2.16}$$

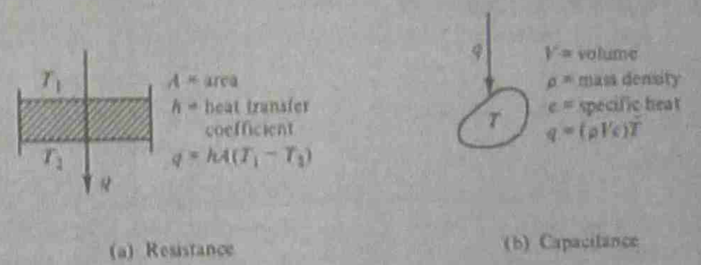
Example 2.4.5 Motor–Generator Set [Fig. 2.9(e)]

The generator serves as a rotating power amplifier. $\theta/E_f = (\theta/E_g)(E_g/E_f)$ is obtained directly by appropriate substitutions in (2.12) and (2.16).

2.5 THERMAL SYSTEMS

As in the preceding sections, thermal system elements are discussed first.

Thermal resistance [Fig. 2.11(a)]. A wall of area A separates regions with temperatures T_1 and T_2. The heat flow rate q, in units of heat per unit of time, say Btu/sec, is proportional to the temperature difference $T_1 - T_2$ and to the area A, and flows toward the lowest temperature. The constant of proportionality is the heat transfer coefficient h. In the case illustrated, this is an effective coefficient which combines the effects of heat convection at the surfaces and heat conduction through the wall. The equation would also represent heat convection across a single surface.



(a) Resistance (b) Capacitance

Figure 2.11 (a) Thermal resistance; (b) thermal capacitance.

For heat conduction through a wall with surface temperatures T_1 and T_2 and thickness d, the heat flow is proportional to the temperature gradient $(T_1 - T_2)/d$:

$$q = \frac{kA(T_1 - T_2)}{d} = \frac{k}{d} A(T_1 - T_2) \tag{2.17}$$

where k is the thermal conductivity. Hence the equivalent heat transfer coefficient is k/d.

To identify these relations in terms of a thermal equivalent of an electrical resistance, the equation in Fig. 2.11(a) is written as

$$T_1 - T_2 = qR_t \qquad R_t = \frac{1}{hA} \tag{2.18}$$

With $T_1 - T_2$ analogous to voltage drop v, and q to current i, R_t becomes the thermal resistance.

Thermal capacitance [Fig. 2.11(b)]. Let q be the net heat flow rate into a volume V of a material with mass density ρ and specific heat c (= heat required to raise the temperature of a unit mass by 1°). This net inflow q of heat per second must equal the change per second (i.e., the rate of change) of heat stored in V. Since the mass is ρV, the heat required for a 1° rise of temperature is ρVc, and hence the heat stored at a temperature T is ρVcT. Assuming ρ, V, and c to be constant, its rate of change is $\rho Vc\dot{T}$: hence the equation shown in Fig. 2.11(b).

From the equation $i = C\dot{v}$ in Fig. 2.7 for an electrical capacitance follows immediately the equivalent notion of a thermal capacitance C_t:

$$q = C_t\dot{T} \qquad C_t = \rho Vc \tag{2.19}$$

It is seen that C_t is the heat required for a 1° temperature rise.

Some examples of thermal systems follow.

Example 2.5.1 Process Control

Figure 2.12 shows a tank of volume V filled with an incompressible fluid of mass density ρ and specific heat c. Volume flow rates entering and leaving are f_i and f_o. T_i is the temperature of the inflow. It is assumed that the tank is well stirred, so that the outlet temperature equals the tank temperature T.

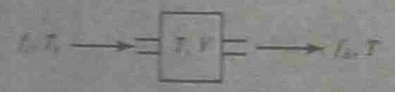


Figure 2.12 Process flow.

The tank is filled and the fluid incompressible, so $f_i = f_o$, and the mass flow rate entering and leaving is $f_i\rho$. Hence the heat inflow rate is $f_i\rho cT_i$, the outflow rate $f_i\rho cT$, and the net inflow rate $f_i\rho c(T_i - T)$. As explained, this must equal $V\rho c\dot{T}$, the rate of change of heat $V\rho cT$ stored in the tank, $V\rho c\dot{T} = f_i\rho c(T_i - T)$, or

$$\frac{V}{f_i}\dot{T} + T = T_i \tag{2.20}$$

The transform is $[(V/f_i)s + 1]T(s) = T_i(s)$, so that the following *simple lag* transfer function relates T_i and T:

$$\frac{T(s)}{T_i(s)} = \frac{1}{(V/f_i)s + 1} \tag{2.21}$$

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Example 2.5.2 Space Heating

In Fig. 2.13, let T be the difference with a constant ambient temperature. By (2.18), the heat loss q_o to ambient can be modeled by $q_o = T/R_t$, where R_t is the thermal resistance. If q_i is the heat inflow rate from an electrical heater, the net inflow $(q_i - T/R_t)$ must equal $C_t\dot{T}$, where C_t is the thermal capacitance. Hence the behavior is modeled by the differential equation

$$R_tC_t\dot{T} + T = R_tq_i \tag{2.22}$$

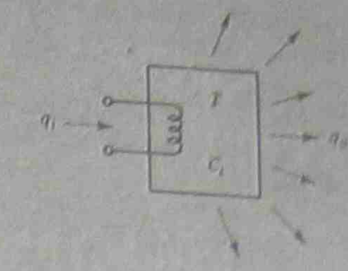


Figure 2.13 Space heating.

Therefore, the effect of heat flow q_i on the temperature T is approximated by the transfer function

$$\frac{T(s)}{Q_i(s)} = \frac{R_t}{R_tC_ts + 1} \tag{2.23}$$

As in Example 2.5.1 and the electrical RC circuit in Example 2.3.1, the behavior is described by a simple lag transfer function.

Example 2.5.3 Three-Capacitance System

Extending Fig. 2.12, in Fig. 2.14 allowance is made for heat loss to the ambient temperature T_a via surface area A_a with heat transfer coefficient h_a, and for the heating up, via surface A_m with heat transfer coefficient h_m, of an internal space or material with capacitance C_m and uniform temperature T_m. This temperature is measured by a sensor with a significant capacitance C_s, heated via a surface of area A_s and heat transfer coefficient h_s.

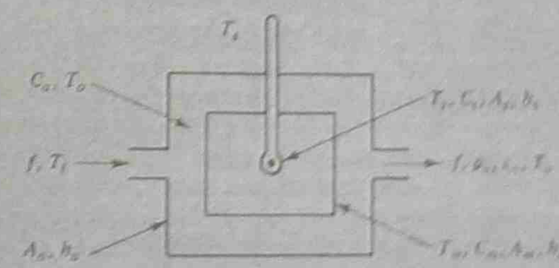


Figure 2.14 Three-capacitance system.

The most convenient approach is to equate the net heat flow rate to the rate of change of heat for each capacitance in turn:

$$\begin{aligned} C_s\dot{T}_s &= f\rho c_s(T_i - T_s) - A_sh_s(T_s - T_s) - A_mh_s(T_s - T_m) \\ C_m\dot{T}_m &= A_mh_s(T_s - T_m) \\ C_s\dot{T}_s &= A_sh_s(T_m - T_s) \end{aligned} \tag{2.24}$$

The second of these equations does not include the heat loss term $-A_sh_s(T_a - T_s)$ to the sensor, on the assumption that it is relatively negligible. The last equation immediately gives the transfer function relating T_m and its value as measured by the sensor:

$$\frac{T_s(s)}{T_m(s)} = \frac{1}{\tau_s s + 1} \qquad \tau_s = \frac{C_s}{A_sh_s} \tag{2.25}$$

Graphical Determination of Residues

To derive this important alternative to the analytical method in Section 1.7, it is recalled that in the partial fraction expansion

$$C(s) = \frac{K(s+2)}{s(s+1)(s+3)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3} \tag{3.3}$$

of the output transform, the residues are

$$K_1 = \left.\frac{K(s+2)}{(s+1)(s+3)}\right|_{s=0} \quad K_2 = \left.\frac{K(s+2)}{s(s+3)}\right|_{s=-1} \quad K_3 = \left.\frac{K(s+2)}{s(s+1)}\right|_{s=-3} \tag{3.4}$$

Figure 3.2 helps to interpret the typical factor $(s+a)\,|_{s=b}$ in these expressions in terms of a vector in the s-plane. The factor is

$$(s+a)\,|_{s=b} = b + a = b - (-a)$$

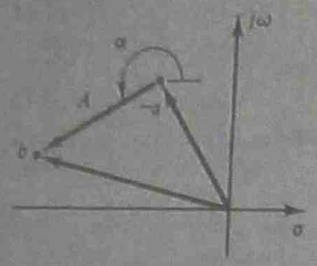


Figure 3.2 The vector $(s+a)$ for $s = b$.

Here b is a vector from the origin to point b in the s-plane, and $-a$ from the origin to point $-a$. The difference of these two vectors is that from $-a$ to b, and this vector can be represented by $Ae^{j\alpha}$, where A is the vector length and α the angle, measured positive counterclockwise from the direction of the positive real axis:

$$\begin{aligned}(s+a)\,|_{s=b} &= b + a = b - (-a) \\ &= (\text{vector from } -a \text{ to } b) = Ae^{j\alpha} = A\ \angle\alpha\end{aligned} \tag{3.5}$$

Here A and α may be found by measurement or calculation from the s-plane. Now a residue of the form of the K_i in (3.4) is

$$K_i = \left.\frac{K(s+a)}{(s+b)(s+c)}\right|_{s=-d} = \frac{K(A\ \angle\alpha)}{(B\ \angle\beta)(C\ \angle\gamma)} = \frac{KA}{BC}\ \angle\alpha - \beta - \gamma \tag{3.6}$$

To find K_1 in (3.4), vectors as shown in the upper half of Fig. 3.1(b) are drawn to the pole $s = 0$ of $C(s)$ from its zero at -2 and its other two poles at -1 and -3, and these give

$$K_1 = \frac{K(2\ \angle 0)}{(1\ \angle 0)(3\ \angle 0)} = \tfrac{2}{3}K\ \angle 0 = \tfrac{2}{3}K$$

The residue K_2 at the pole -2 is found from the vectors to this point, in the bottom half of Fig. 3.1(b). The vector from the pole at $s = 0$ to that at -1 is in the direction of the negative real axis, so has angle $\pm 180°$,

$$K_2 = \frac{K(1\ \angle 0)}{(1\ \angle 180)(2\ \angle 0)} = \tfrac{1}{2}K\ \angle -180 = -\tfrac{1}{2}K$$

Similarly,

$$K_3 = \frac{K(1\ \angle 180)}{(3\ \angle 180)(2\ \angle 180)} = \tfrac{1}{6}K\ \angle 180 - 360 = -\tfrac{1}{6}K$$

These values of K_1, K_2, and K_3 may be verified using the analytical method, and give

$$c(t) = K(\tfrac{1}{3} - \tfrac{1}{2}e^{-t} - \tfrac{1}{6}e^{-3t}) \tag{3.7}$$

The gain factor K in (3.1) to (3.3) will be called the root locus gain:

> The *root locus gain* factor of a transform or a transfer function is that which results if the coefficients of the highest powers of s in numerator and denominator are made equal to unity.

For, say,

$$C(s) = 2\,\frac{0.5s+1}{(s+3)(0.1s+1)} = \frac{2(0.5)}{0.1}\,\frac{s+2}{(s+3)(s+10)}$$

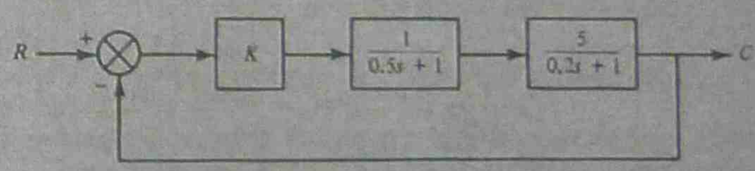
the root locus gain is $2(0.5/0.1) = 10$.

Reviewing the expressions for K_1, K_2, and K_3 reveals the following general rule:

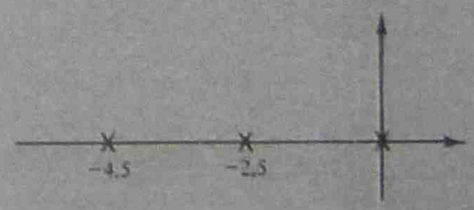
> The residue K_i at the pole $-p_i$ of $C(s)$ equals the root locus gain times the product of the vectors from all zeros of $C(s)$ to $-p_i$ divided by the product of the vectors from all other poles of $C(s)$ to $-p_i$.

Example 3.2.1

Figure 3.3(a) models a level control system of which the process consists of the two-tank system with control valve in Figs. 2.19 and 2.20 and the level in the second



(a)



(b)

Figure 3.3 Example 3.2.1.

and is to be controlled. Find the unit step response of the system for $K = 5$ lb/in.

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{45.36(5)}{(s)(s + 1.65)(s + 4)(s + 20.35)} = \frac{1.25}{s^3 + 26s^2 + 111.35s} = \frac{1.25}{(s + 2.5)(s + 4.5)}$$

and with $R(s) = 1/s$,

$$C(s) = \frac{1.25}{s(s + 2.5)(s + 4.5)} = \frac{K_0}{s} + \frac{K_1}{s + 2.5} + \frac{K_2}{s + 4.5}$$

The pole-zero pattern of $C(s)$ is Fig. 3.3(b) and the graphical residue rule yields

$$K_0 = \frac{1.25}{(2.5\ \angle 0)(4.5\ \angle 0)} = \frac{1}{9}$$

$$K_1 = \frac{1.25}{(2.5\ \angle 180°)(2\ \angle 0)} = -\frac{1}{4} \qquad (3.8)$$

$$K_2 = \frac{1.25}{(4.5\ \angle 180°)(2\ \angle 180°)} = \frac{1.25}{9}$$

$$c(t) = \frac{1}{9} - \frac{1}{4}e^{-2.5t} + \frac{1.25}{9}e^{-4.5t}$$

3.3 TRANSIENT RESPONSE, INCLUDING REPEATED AND COMPLEX POLES

In this section, transient responses are calculated for the cases of repeated real poles, a complex conjugate pair of poles, and combinations of distinct real poles and complex conjugate pairs.

Repeated Real Poles

Suppose that $C(s)$ has m poles at $-p_1$:

$$C(s) = \frac{Z(s)}{(s + p_1)^m(s + p_2)\cdots(s + p_n)} \qquad (3.9)$$

Then it can be shown that the partial fraction expansion must be written as follows:

$$C(s) = \frac{K_1}{(s + p_1)^m} + \frac{K_2}{(s + p_1)^{m-1}} + \cdots + \frac{K_m}{s + p_1} + \frac{K_{m+1}}{s + p_2} + \cdots + \frac{K_{m+n-1}}{s + p_n} \qquad (3.10)$$

The residues $K_{m+1}, \ldots, K_{m+n-1}$ are found as before, but for the repeated root the equation is

$$K_i = \frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[C(s)(s + p_1)^m\right]\Big|_{s=-p_1} \qquad i = 1, \ldots, m \qquad (3.11)$$

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Example 3.3.1

Find the response to a unit ramp input, $R(s) = 1/s^2$, of the system

$$G(s) = \frac{1}{Ts + 1} \quad \text{Simple lag} \qquad (3.12)$$

$$C(s) = G(s)R(s) = \frac{1}{s^2(Ts + 1)} = \frac{K_0}{s^2} + \frac{K_1}{s} + \frac{K_2}{s + 1/T}$$

K_0 and K_2 can be found in the usual way:

$$K_0 = \left[\left(s + \frac{1}{T}\right)\frac{1}{s^2(Ts + 1)}\right]_{s=-1/T} = -\frac{1}{Ts}\Big|_{s=-1/T} = T$$

$$K_2 = \left[s^2\frac{1}{s^2(Ts + 1)}\right] = \frac{1}{Ts + 1}\Big|_{s=0} = 1$$

But if to determine K_2 the equation is multiplied by its denominator s, the first term becomes K_0/s and tends to infinity for $s \to 0$. Equation (3.11) arises by first multiplying both sides of the equation by s^2:

$$s^2C(s) = K_0 + K_2s + \frac{K_2s^2}{s + 1/T}$$

K_0 is eliminated by differentiating both sides:

$$\frac{d}{ds}[s^2C(s)] = K_2 + K_2\frac{2s + 1/T) - s}{(s + 1/T)^2}$$

If now $s \to 0$, only K_2 remains on the right, so that, as in (3.11),

$$K_0 = \frac{d}{ds}(s^2C)\Big|_{s=0} = \frac{d}{ds}\left(\frac{1}{Ts + 1}\right)\Big|_{s=0} = -T$$

Hence

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s + 1/T}$$

Using Table 1.6.1, the transient response is

$$c(t) = t - T + Te^{-t/T}$$

Since the simple lag is so common, this response has been sketched in Fig. 3.4. The value T of the steady-state ($t \to \infty$) difference between r and c could have been found more directly by the final value theorem:

$$\lim_{t\to\infty}[r(t) - c(t)] = \lim_{s\to 0}s[R(s) - C(s)] = \lim_{s\to 0}s\left[\frac{T}{s(Ts + 1)}\right] = T$$

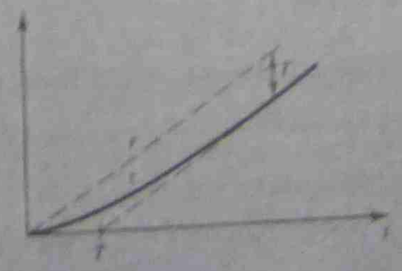
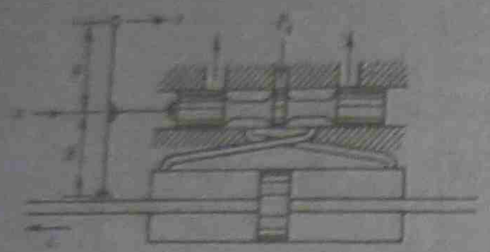


Figure 3.4 Simple lag ramp response

Example 4.2.2 Hydraulic Servo with Mechanical Feedback: Simple Model

In Fig. 4.3(a) the mechanical lever added to the hydraulic cylinder control in Fig. 2.26 acts as feedback because it causes piston motion c to affect valve position x. If, say, input r is moved to the right initially, the valve moves right, causing the piston to move left until the valve is again centered. Use of the simple valve-cylinder model in Fig. 2.27, which assumes small loads and neglects oil compressibility, now readily leads to the block diagram in Fig. 4.3(b).



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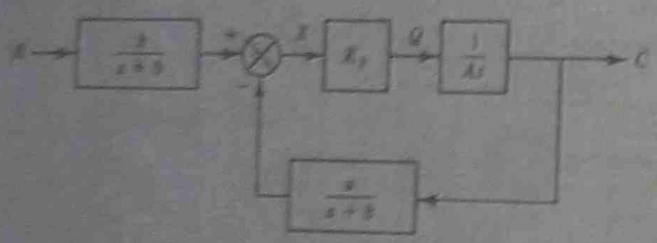


(b)

Figure 4.3 Hydraulic servo and block diagram.

Example 4.2.3 Simple Motor Position Servo

A simple motor position servo is shown in Fig. 4.4. The motor and its load are represented by the transfer function (2.11) for a field-controlled dc motor. The time constant $T_m = J/B$, where J is the inertia and B the damping constant. A potentiometer could be used to represent shaft position by a voltage, and an operational amplifier, discussed in Chapter 5, could serve as a summing junction to determine the error voltage E with the voltage R representing desired position. Alternatively, electrical bridge circuits could be used. The controller G_c may be a simple amplifier and generates a low-power output voltage V_c. Its power is raised in a power amplifier of which the output is applied to the motor.

Example 4.2.4 Servo with Velocity Feedback

In speed or position control servomechanisms such as Example 4.2.3, design for satisfactory performance is often complicated by a lack of adequate inherent damping

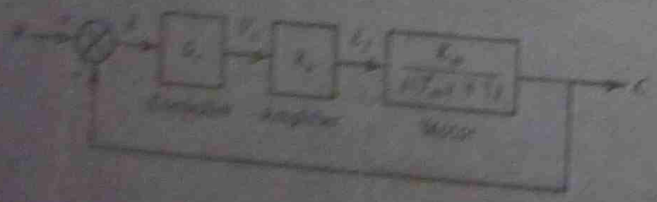


Figure 4.4 Motor position servo.

in motor and load. The difficulty of positioning a large inertia J rapidly without severe overshoot in response to a step input in Fig. 4.4 if the damping constant B is small can be appreciated intuitively. One possible solution is to install a mechanical damper on the motor shaft. However, a better and more elegant solution is possible by the use of feedback. A damping torque is a torque proportional to shaft speed $\dot{c}$ and in the opposite direction. Such a torque can also be generated by mounting a small tachometer-generator on the motor shaft to obtain a signal proportional to speed:

$$b = K_t\dot{c} \qquad B(s) = K_t s C(s) \tag{4.1}$$

and feeding this back negatively to the power amplifier input.

In the block diagram, this can be represented by a minor loop feedback path. In this case the minor loop is a velocity feedback, as shown in Fig. 4.5. Equation (1.35) gives the reduction of this type of diagram. Its importance may be judged from the availability of motors with integrally mounted tachometers on the shaft.

Example 4.2.5 Motor Position Servo with Load Disturbance Torque

To improve the model in Fig. 4.4 for a field-controlled dc motor position servo by making allowance for load disturbance torque T_l acting on the motor shaft, it is necessary to return to the motor equations in Example 2.4.1. Field voltage E_f and motor developed torque T are related to field current I_f by

$$E_f = (R_f + L_f s)I_f \qquad T = K_t I_f \tag{4.2}$$

where R_f and L_f are field resistance and inductance and K_t is the motor torque constant. If T_l is taken as positive in a direction opposite that of T, a net torque $(T - T_l)$ is available to accelerate motor and load inertia J and overcome their damping B:

$$T - T_l = J\ddot{c} + B\dot{c} \qquad T(s) - T_l(s) = Js^2C(s) + BsC(s)$$

Hence

$$\frac{C}{T - T_l} = \frac{1}{s(Js + B)} = \frac{1/B}{s(T_m s + 1)} \qquad T_m = \frac{J}{B} \tag{4.3}$$

Figure 4.4 is now modified to the diagram in Fig. 4.6. As was noted, often the field time constant $T_f = L_f/R_f \ll T_m$ and can be neglected. The factor $1/R_f$ can then be considered to be incorporated into K_t and $E_f = I_f$ assumed in the diagram for purposes of analysis. Figure 4.6 shows T_l as a second, disturbance, input to the block diagram. The reduction of such a diagram was discussed in Example

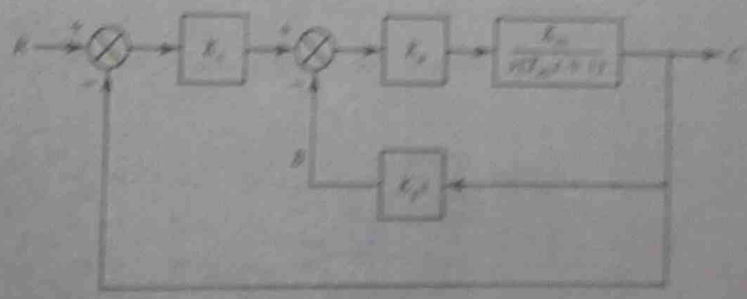


Figure 4.5 Servo with velocity feedback.

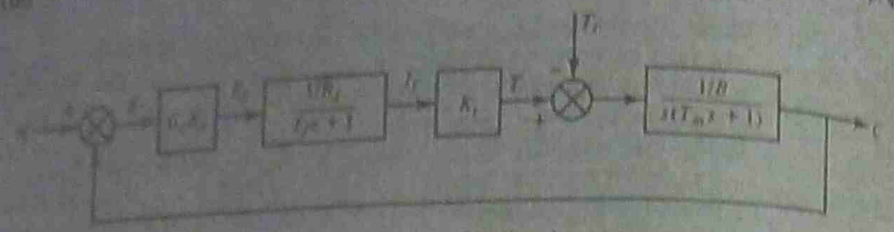


Figure 4.6 Position servo with disturbance torque.

1.8.4. Severe disturbance torques can arise, for example, due to wind in antenna positioning systems, or in steel rolling mill speed controls when slabs enter and leave the rolls.

4.3 DIRECT BLOCK DIAGRAM MODELING OF FEEDBACK SYSTEMS

In Section 4.2 feedback was realized by separate physical elements and the structure of the block diagram was easy to perceive. However, on many occasions the feedback is generated by the use of signals or physical elements which are an intrinsic part of the system. The precise nature of the feedback, or even its presence, may then be far from obvious. In such cases block diagrams can be derived directly from the system equations, and serve an important function in clarifying system behavior.

Example 4.3.1 Pneumatic Pressure Regulator

Figure 4.7 shows a schematic diagram of this very common device. Its purpose is to keep the pressure P_l to the load serviced by the controller constant, equal to a value set by manual adjustment, despite variations of the flow W_l required by the load. Physically, the action is that a reduction of P_l reduces the pressure against the bottom of the diaphragm. This permits the spring force to push it downward to increase valve opening x, and hence increase valve flow from a supply with constant pressure P_s. This increase serves to raise P_l back toward the set value.

Frequently, system equations can be represented by a variety of possible block diagrams, which are all mathematically correct but not all equally useful.

A *good block diagram* is one that clearly identifies the components and parameters in the feedback loop.

To obtain such a diagram, the system equations are written first. This is done as

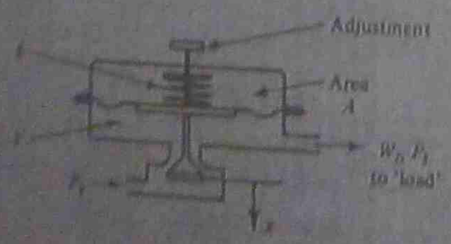


Figure 4.7 Pressure regulator.

in Chapter 2, for the parts of the system, taken in rather arbitrary order. Only after all have been written is consideration given to their combination into a good block diagram model on the basis of the physical operation of the system.

A linearized model is used, implying that the variables are deviations from operating point values. Hence the constant supply pressure P_s will not appear. For the weight flow W_v through the valve the linearized model (2.51) is used:

$$W_v = K_x x - K_p P_l \qquad W_v(s) = K_x X(s) - K_p P_l(s) \tag{4.4}$$

The net flow entering the volume below the diaphragm, equal to V at the operating point, is $(W_v - W_l)$. This flow must equal

$$W_v - W_l = C_p \dot{P}_l - pA\dot{x} \qquad W_v(s) - W_l(s) = C_p s P_l(s) - pAsX(s) \tag{4.5}$$

Here $C_p = V/(nRT)$ is the capacitance of V according to Fig. 2.17 and $C_p P_l$ the compressibility flow. The term $pA\dot{x}$ is the flow rate corresponding to the change of V, equivalent to, say, (2.57) (which is in terms of volume flow rates). The minus sign arises because Fig. 4.7 defines x as positive in the direction of decreasing volume below the diaphragm.

Force equilibrium on the moving parts, modeled as a spring–mass–damper system k, m, b, gives the equation

$$m\ddot{x} + b\dot{x} + kx = -AP_l \qquad (ms^2 + bs + k)X(s) = -AP_l(s) \tag{4.6}$$

This model is approximate because it neglects the flow forces on the poppet. The minus sign is needed because the pressure force AP_l on the bottom of the diaphragm acts in the direction of negative x.

To combine (4.4) to (4.6) into a block diagram, it is noted first that the output is the controlled variable P_l, and the input is the disturbance W_l, the unknown flow to the load. Thus, following convention, it is desirable to show W_l at the left and P_l at the right in the diagram. The feedback should then show how changes of P_l are used to make valve flow W_v "follow" W_l. Pressure P_l determines x via (4.6), but x and P_l together determine W_v, and for model simplicity it appears preferable to eliminate W_v between (4.4) and (4.5):

$$K_x X - K_p P_l - W_l = C_p s P_l - pAsX \qquad \text{or}$$

$$(C_p s + K_p)P_l = -W_l + (pAs + K_x)X \tag{4.7}$$

Figure 4.8 now shows the block diagram, obtained directly from (4.6) and (4.7), in classical form. Valve flow W_v could be shown at the expense of more loops and a less simple appearance.

Example 4.3.2 Motor with IR-Drop Compensation

Figure 4.9 shows the armature-controlled dc motor of Fig. 2.9(b) (Example 2.4.2) with a resistance R added in the armature loop, the voltage across which is fed

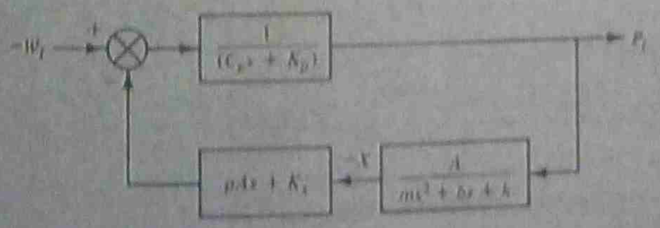
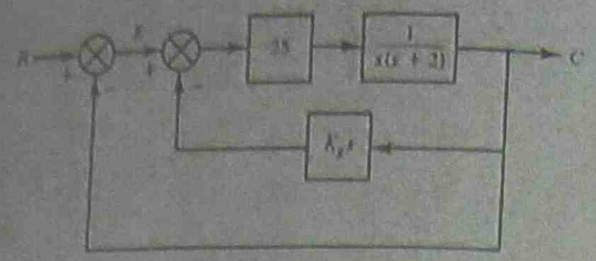


Figure 4.8 Block diagram for Fig. 4.7.



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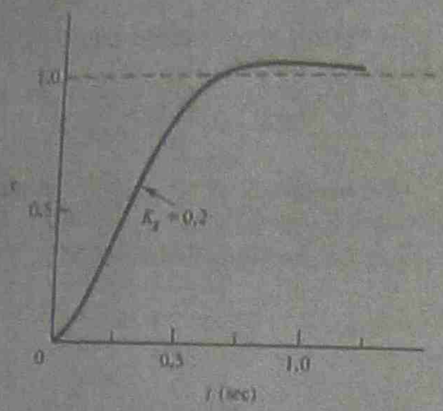

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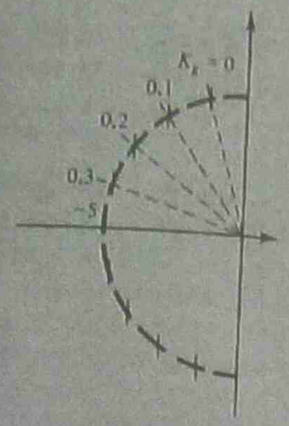

(c)

Figure 5.3 Example 5.2.1: velocity feedback.

is shown in Fig. 5.3(c). Except for large enough K_v, the poles lie on a circle of radius $\omega_n = 5$. The angle ϕ of the poles with the negative real axis depends on K_v according to $\zeta = \cos\phi = 0.2 + 2.5K_v$. The improvement due to velocity feedback is evident. The angle ϕ for $K_v = 0$ is not far from 90° and corresponds to a damping ratio of only 0.2.

However, the effect of this compensation on steady-state errors must also be considered. The loop gain function C/E shows that the system is still of type 1, so steady-state errors after step inputs remain zero. But the gain is $25/(2 + 25K_v)$, so the steady-state error following a unit ramp is $(0.08 + K_v)$ and increases with increasing velocity feedback gain K_v. Between $K_v = 0$ and $K_v = 0.2$ it changes from 0.08 to 0.28. Thus K_v should not be made larger than necessary to raise ζ to a satisfactory level.

5.3 SERIES COMPENSATION USING PID CONTROLLERS

If the series compensator $G_c(s)$ in Fig. 5.1(a) is a proportional plus integral plus derivative controller, its output m and input e are related by the equation

$$m = K_p e + K_i \int e\,dt + K_d \dot{e} \tag{5.3}$$

From Table 1.6.1, this implies the transfer function

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s \tag{5.4}$$

PID controllers are available off the shelf in a wide variety of pneumatic and electronic implementations. Alternatively, their action may be programmed in the form of a *control algorithm* on digital computers or microprocessors.

Not all terms need be present. P control, where $G_c(s) = K_c$, was used in Chapter 4, where its limitations were also examined.

PI control is extremely common in *process control*, a very large and important area of application of control systems. Process control is usually concerned with regulator systems, to maintain controlled variables such as temperatures, pressures, and levels at constant values despite disturbances and parameter variations. The water level control system in Fig. 4.1 is an example where the controller is most likely to be of the PI type.

The need for PI control to reduce steady-state errors was raised in both Examples 4.6.1 and 4.6.2, for plants consisting of one and two simple lags. With one simple lag, the high gains needed for small errors imply large actuating signals to the plant, so a high cost of control. With two simple lags, Fig. 4.18 shows excessive steady-state error even when a system damping ratio of 0.5 is allowed. To explain the basic control actions, the P control in these examples will be replaced with PI control and PD control.

Example 5.3.1 PI Control of Simple Lag Plant

In Fig. 5.1(a) let

$$G(s) = \frac{1}{s+1} \qquad G_c(s) = \frac{K_c(s+z)}{s} \tag{5.5}$$

The integral of the error as well as the error itself is used for control. The loop gain function

$$G_c G = \frac{K_c(s+z)}{s(s+1)}$$

shows immediately the essential improvement due to the addition of I control. The system has unity feedback and Table 4.5.1, on steady-state errors, applies. Thus:

> By adding I control, the system has been changed from type 0 to type 1, and hence now has zero steady-state error following a step input.

This improvement is due to the factor s in the denominator of $G_c(s)$. As explained in Section 4.5, for a step input a steady state cannot exist unless the input to the integrator, the error e, is zero. A direct physical reason will be given as well, when the pneumatic PI controller is discussed.

The closed-loop transfer function is

$$\frac{C}{R} = \frac{K_c(s+z)}{s^2 + (1+K_c)s + K_c z}$$

and design involves the choice of K_c and z. In Example 3.8.1 the effect of zero

2.3.1 Example 1 – Nonlinear Output Derivative Feedback

The Simulink diagram for this example is given in Figure 2.7, where the only nonlinearity is in the derivative of output feedback path. To plot the phase plane using Simulink the X-Y oscilloscope can be used or alternatively the output and its derivative can be fed to simout blocks for plotting after the simulation run has finished. The simout blocks must be put in the array mode and the time vector is available in tout. Often using the default integration algorithm in Simulink waveforms do not appear smooth so one may have to use an algorithm with a small fixed step size or limit the step size in an algorithm. Results from the X-Y scope can be read to the display accuracy. More accurate results can be obtained from the simout blocks but plotting the data may not be straightforward as one may have unequal vector lengths of data stored when a default integration algorithm is not used.

If the nonlinearity is replaced by a gain K then the characteristic equation is

$$s^2 + (K - 0.6)s + 1 = 0 \tag{2.10}$$

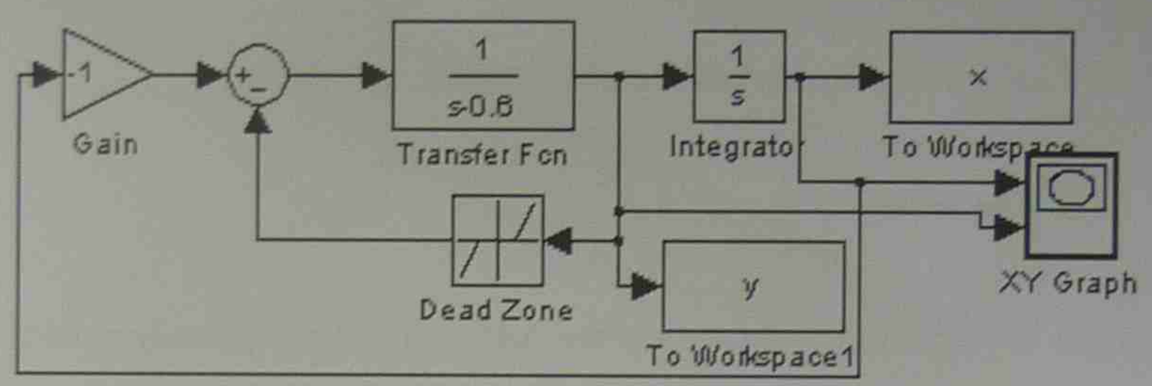

Figure 2.7 Simulink diagram for system with dead zone feedback

It is clear that for K < 0.6 the linear system is unstable and stable for K > 0.6. For the dead zone the gain is zero for a small sinusoidal signal but approaches unity for a large sinusoidal signal. Thus, one expects the effect of the dead zone nonlinearity to change the damping in some way such that the system is unstable for small signals and stable for large signals. This suggests the existence of a stable limit cycle. Figure 2.8 shows phase plane plots starting from (-1, 0) for the linear second order system with characteristic equation

$$s^2 + 2\zeta s + 1 = 0 \tag{2.11}$$

for various values of ζ, the damping ratio. For the unstable case with ζ = -0.05 the trajectory slowly spirals outwards and for the stable responses they converge to the origin in a spiral manner for $\zeta < 1$, when the origin is a focus and directly with no overshoot for $\zeta \geq 1$, when the origin is a node. The mathematical expressions for these curves are quite complicated, so to sketch them on a phase plane the method of isoclines or some special methods may be used [2.1]. The dead zone is set at levels of ±0.5, so that the trajectories describing the motion are those of a linear second order system with a damping ratio of -0.3 for $|x_2| < 0.5$ and 0.2 for $|x_2| > 0.5$. Phase plane plots have been obtained by simulation and are shown from initial conditions of (-8, 0) and (0.1, 0) in Figure 2.9. The resulting limit cycle is clearly shown in the figure.

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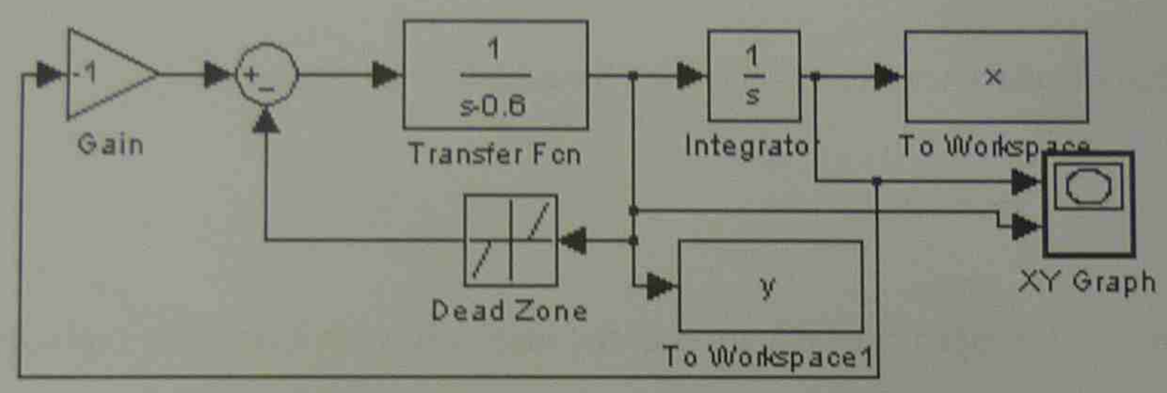

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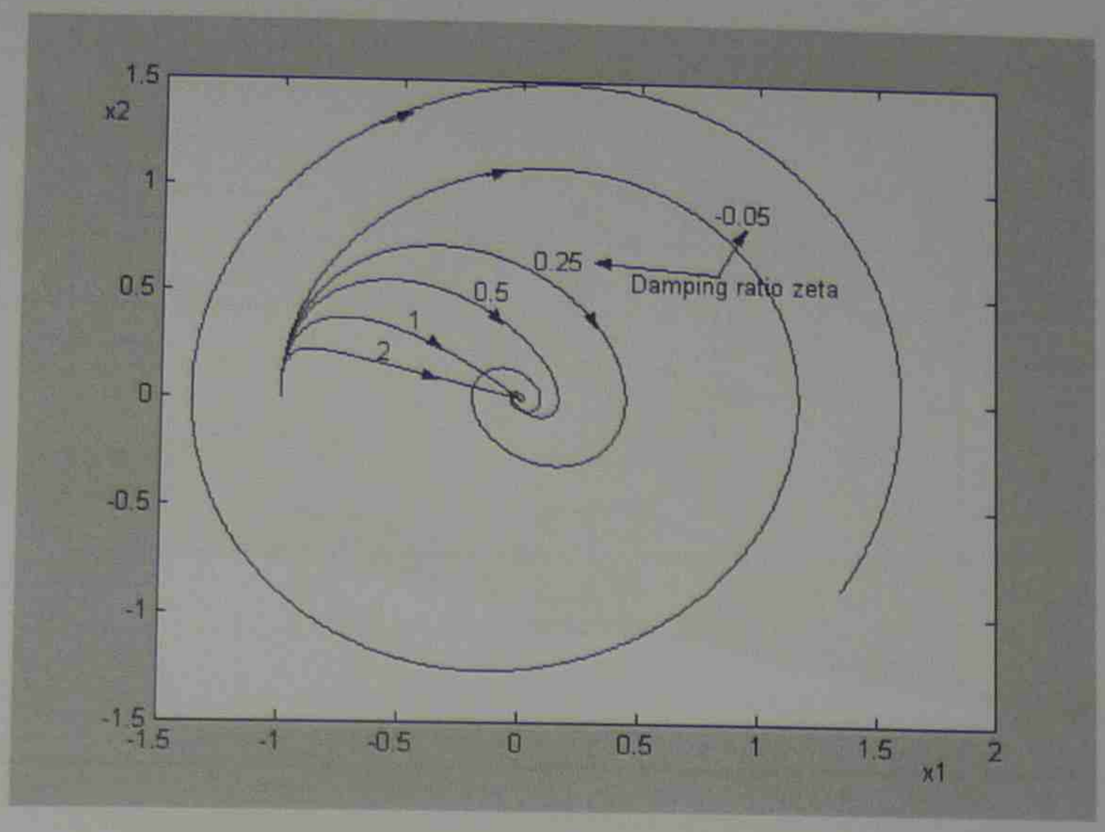

Figure 2.8 Phase plane plots for second order system

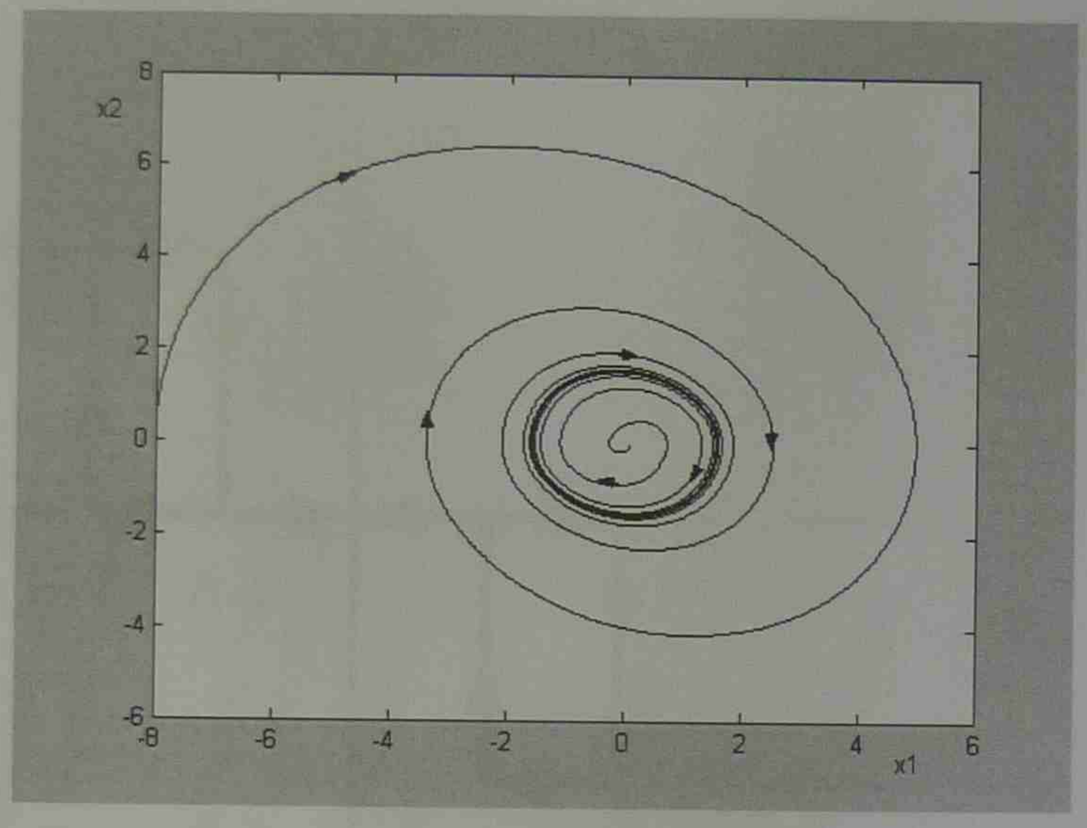

Figure 2.9 Phase plane plots for system of Figure 2.7

2.3.2 Example 2 – Relay Position Control

Consider a basic relay position control system with nonlinear velocity feedback having the Simulink diagram shown in Figure 2.10. The two relays in parallel simulate a relay with dead zone and hysteresis which has the characteristic shown in Figure 2.11, where for positive inputs the switch on level is $\delta + \triangle$ and the switch off level is $\delta - \triangle$. The relay parameters are set to give an output, h, of ± 1, and 0.

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2.3.3 Example 3 – Position Control with Torque Saturation

Figure 2.16 shows a Simulink diagram for a position control system with no viscous damping and with nonlinear effects due to torque saturation and Coulomb friction.

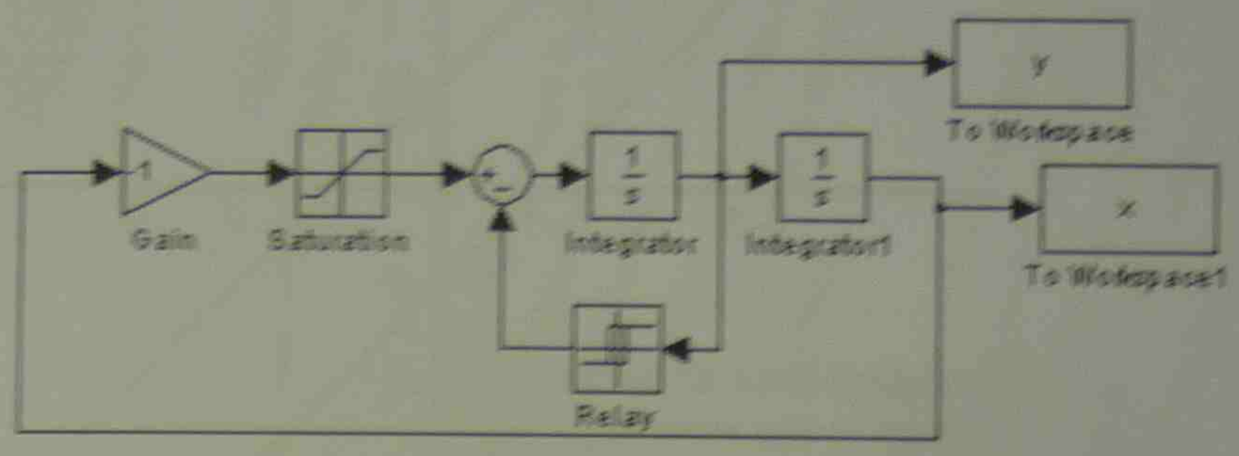

Figure 2.16 Simulink diagram of control system for Example 3

The differential equation of motion in phase variable form has

$$\dot{x}_2 = f_s(-x_1) - 0.5\,\mathrm{sgn}(x_2) \tag{2.17}$$

where f_s denotes the saturation nonlinearity and sgn the signum function, which is +1 for $x_2 > 0$ and –1 for $x_2 < 0$. There are six linear differential equations describing the motion in different regions of the phase plane. For x_2 positive, equation (2.17) can be written

$$\ddot{x}_1 + f_s(x_1) + 1/2 = 0 \tag{2.18}$$

so that for

(a) x_2 + ve, $x_1 < -2$, one has $\dot{x}_1 = x_2, \dot{x}_2 = 3/2$, a parabola in the phase plane.

(b) x_2 + ve, $-2 \le x_1 \le 2$, one has $\dot{x}_1 = x_2, \dot{x}_2 + x_1 + 1/2 = 0$, a circle in the phase plane.

(c) x_2 + ve, $x_1 > 2$, one has $\dot{x}_1 = x_2, \dot{x}_2 = -5/2$, a parabola in the phase plane.

Similarly for x_2 negative,

(d) x_2 – ve, $x_1 < -2$, one has $\dot{x}_1 = x_2, \dot{x}_2 = 3/2$, a parabola in the phase plane.

(e) x_2 – ve, $-2 \le x_1 \le 2$, one has $\dot{x}_1 = x_2, \dot{x}_2 + x_1 - 1/2 = 0$, a circle in the phase plane.

(f) x_2 – ve, $x_1 > 2$, one has $\dot{x}_1 = x_2, \dot{x}_2 = -3/2$, a parabola in the phase plane.

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level at the transmitting end and for converting the current back to voltage at the receiving end. Of course, current transmission is used because such a signal is independent of load variations other than accidental shunt conditions that may draw off some current. Thus, voltage-to-current and current-to-voltage converters are often required.

Digital Interface The use of computers in process control requires conversion of analog data into a digital format by integrated circuit devices called analog-to-digital converters (ADCs). Analog signal conversion is usually required to adjust the analog measurement signal to match the input requirements of the ADC. For example, the ADC may need a voltage that varies between 0 and 5 V, but the sensor provides a signal that varies from 30 to 80 mV. Signal conversion circuits can be developed to interface the output to the required ADC input.

2.2.4 Filtering and Impedance Matching

Two other common signal-conditioning requirements are filtering and matching impedance.

Often, spurious signals of considerable strength are present in the industrial environment, such as the 60-Hz line frequency signals. Motor start transients also may cause pulses and other unwanted signals in the process-control loop. In many cases, it is necessary to use high-pass, low-pass, or notch *filters* to eliminate unwanted signals from the loop. Such filtering can be accomplished by *passive* filters using only resistors, capacitors, and inductors; or *active* filters, using gain and feedback.

Impedance matching is an important element of signal conditioning when transducer internal impedance or line impedance can cause errors in measurement of a dynamic variable. Both active and passive networks are employed to provide such matching.

2.2.5 Concept of Loading

One of the most important concerns in analog signal conditioning is the loading of one circuit by another. This introduces uncertainty in the amplitude of a voltage as it is passed through the measurement process. If this voltage represents some process variable, then we have uncertainty in the value of the variable.

Qualitatively, loading can be described as follows. Suppose the open circuit output of some element is a voltage, say V_x, when the element input is some variable of value x. This element could be a sensor or some other part of the signal-conditioning circuit such as a bridge circuit or amplifier. *Open circuit* means that nothing is connected to the output. Loading occurs when we do connect something, a load, across the output, and the output voltage of the element drops to some value, $V_y < V_x$. Different loads will result in different drops.

Quantitatively, we can evaluate loading as follows. Thévenin's theorem tells us that the output terminals of any two terminal elements can be defined as a voltage source in series with an output impedance. Let's assume this is a resistance (the output resistance) to make the description easier to follow. This is called the Thévenin equivalent circuit model of the element.

FIGURE 2.2
The Thévenin equivalent circuit of a sensor allows easy visualization of how loading occurs.

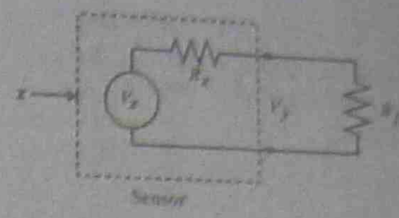


Figure 2.2 shows such an element modeled as a voltage V_x and a resistance R_x. Now suppose a load, R_L, is connected across the output of the element as shown in Figure 2.2. This could be the input resistance of an amplifier, for example. A current will flow and voltage will be dropped across R_x. It is easy to calculate that the loaded output voltage will thus be given by

$$V_y = V_x\left(1 - \frac{R_x}{R_L + R_x}\right) \tag{2.1}$$

The voltage that appears across the load is reduced by the voltage dropped across the internal resistance.

This equation shows how the effects of loading can be reduced. Clearly, the objective will be to make R_L much larger than R_x, that is, $R_L >> R_x$. The following example shows how the effects of loading can compromise our measurements.

EXAMPLE 2.1 An amplifier outputs a voltage that is ten times the voltage on its input terminals. It has an input resistance of 10 kΩ. A sensor outputs a voltage proportional to temperature with a transfer function of 20 mV/°C. The sensor has an output resistance of 5.0 kΩ. If the temperature is 50°C, find the amplifier output.

Solution

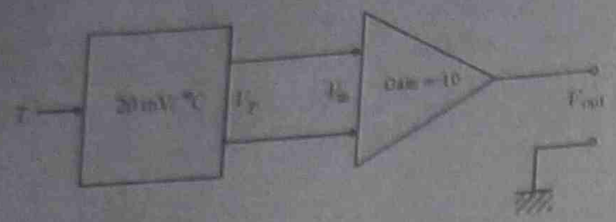
The naive solution is represented by Figure 2.3a. The unloaded output of the sensor is simply $V_T = (20\text{ mV/°C})50°\text{C} = 1.0\text{ V}$. Since the amplifier has a gain of 10, the output of the amplifier appears to be $V_{out} = 10V_{in} = (10)1.0\text{ V} = 10\text{ V}$. But this is wrong, because of loading!

Figure 2.3b shows the correct analysis. Here we see that there will be a voltage dropped across the output resistance of the sensor. The actual amplifier input voltage will be given by Equation (2.1).

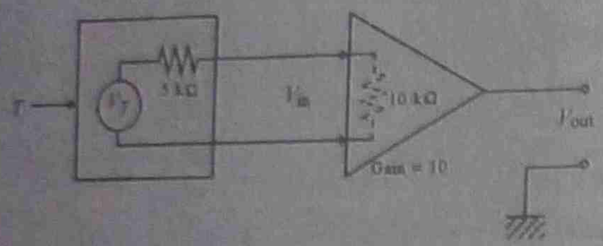
$$V_{in} = V_T\left(1 - \frac{5.0\text{ k}\Omega}{5.0\text{ k}\Omega + 10\text{ k}\Omega}\right)$$

where $V_T = 1.0$ V, so that $V_{in} = 0.67$ V. Thus, the output of the amplifier is actually $V_{out} = 10(0.67\text{ V}) = 6.7$ V.

This concept plays an important role in analog signal conditioning and is referred to many times in this and later chapters.



(a)



(b)

FIGURE 2.3
If loading is ignored, serious errors can occur in expected outputs of circuits and gains of amplifiers.

If the electrical quantity of interest is frequency or a digital signal, then loading is not such a problem. That is, if there is enough signal left after loading to measure the frequency or to distinguish ones from zeros, there will be no error. Loading is important mostly when signal amplitudes are important.

SSIVE CIRCUITS

Bridge and divider circuits are two passive techniques that have been extensively used for signal conditioning for many years. Although modern active circuits often replace these techniques, there are still many applications where their particular advantages make them useful.

Bridge circuits are used primarily as an accurate means of measuring changes in impedance. Such circuits are particularly useful when the fractional changes in impedance are very small.

Another common type of passive circuit involved in signal conditioning is for filtering out unwanted frequencies from the measurement signal. It is quite common in the industrial environment to find signals that possess high- and/or low-frequency noise as well as the desired measurement data. For example, a transducer may convert temperature information into a dc voltage, proportional to temperature. Because of the ever-present ac power lines, however, there may be a 60-Hz noise voltage impressed on the output that makes determination of the temperature difficult. A passive circuit consisting of a resistor and capacitor often can be used to eliminate both high- and low-frequency noise without changing the desired signal information.

2.3.1 Divider Circuits

The elementary voltage divider shown in Figure 2.4 often can be used to provide conversion of resistance variation into a voltage variation. The voltage of such a divider is given by the well-known relationship

$$V_D = \frac{R_2 V_x}{R_1 + R_2} \tag{2.2}$$

where V_x = supply voltage
R_1, R_2 = divider resistors

Either R_1 or R_2 can be the sensor whose resistance varies with some measured variable.

It is important to consider the following issues when using a divider for conversion of resistance to voltage variation:

1. The variation of V_D with either R_1 or R_2 is nonlinear; that is, even if the resistance varies linearly with the measured variable, the divider voltage will not vary linearly.
2. The effective output impedance of the divider is the parallel combination of R_1 and R_2. This may not necessarily be high, so loading effects must be considered.
3. In a divider circuit, current flows through both resistors; that is, power will be dissipated by both, including the sensor. The power rating of both the resistor and sensor must be considered.

EXAMPLE 2.2 The divider of Figure 2.4 has $R_1 = 10.0\ \text{k}\Omega$ and $V_x = 5.00$ V. Suppose R_2 is a sensor whose resistance varies from 4.00 to 12.0 kΩ as some dynamic variable varies over a range. Then find **(a)** the minimum and maximum of V_D, **(b)** the range of output impedance, and **(c)** the range of power dissipated by R_2.

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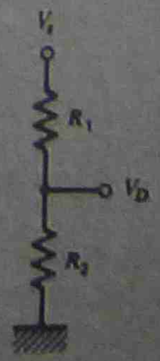
FIGURE 2.4
The simple voltage divider can often be used to convert resistance variation into voltage variation.



...ises for a stepping motor are being transmitted at 2000 Hz. Design a filter to reduce 60-...e noise, but reduce the pulses by no more than 3 dB.

...olution

...s us first find what voltage ratio corresponds to a 3-dB reduction. We remember that

$$P(\text{dB}) = 20 \log (V_{out}/V_{in})$$

...down by 3 dB" means that $P = -3$. Therefore,

$$(V_{out}/V_{in}) = 10^{-3/20} = 0.707$$

...u probably saw that coming. **The critical frequency is that frequency for which the ...put is attenuated by 3 dB.** Thus, in this case, $f_c = 2$ kHz. The effect on 60-Hz noise is ...ed using Equation (2.24), with $f = 60$ Hz.

$$V_{out}/V_{in} = \frac{(60/2000)}{[1 + (60/2000)^2]^{1/2}}$$

$$V_{out}/V_{in} = 0.03$$

..., we see that only 3% of the 60-Hz noise remains; that is, it has been reduced by 97%. ...Let's try a capacitor of 0.01 μF. Then Equation (2.22) shows that the resistor should ...e a value of 7.96 kΩ. Perhaps design criteria would allow use of either a 7.5-kΩ or 8.2-...resistor, because these are standard values.

Practical Considerations There are a number of issues that arise when design-...ing a system which will use the *RC* high-pass or low-pass filters. Consider the following ...s:

1. After the critical frequency is determined, the values of R and C are selected to ...fy Equation (2.22). In principle, any values of R and C can be employed that will sat-...isfy the critical frequency equation. A number of practical issues limit the selection, how-...ever. Some of these are:
   a. Very small values of resistance are to be avoided because they can lead to large ...currents, and thus large loading effects. Similarly, very large capacitance should ...be avoided. In general, we try to keep the resistance in the kΩ and above range, ...and capacitors in the μF or less range.
   b. Often, the exact critical frequency is not important, so that fixed resistors and ca-...pacitors of approximately the computed values can be employed. If exact values ...are necessary, it is usually easier to select and measure a capacitor, then compute ...and obtain the appropriate resistance using a trimmer resistance.
2. The effective input impedance and output impedance of the *RC* filter may have an ...effect on the circuit in which it is used because of loading effects. If the input impedance ...of the stage being fed by the filter is low, you may want to place a voltage follower (see ...Section 2.4) between the filter output and the next stage. Similarly, if the output impedance of the feeding stage to the filter is high, you may want to isolate the input of the filter with a voltage follower.
3. It is possible to cascade *RC* filters in series to obtain improved sharpness of the filter cutoff frequency. However, it is important to consider the loading of one *RC* stage by another. The output impedance of the first-stage filter must be much less than the input impedance of the next stage to avoid loading.

Apart from loading, cascading filters raises the filter equation by a power of the number cascaded. Thus, two high-pass filters of the same critical frequency would respond by Equation (2.24) squared.

EXAMPLE 2.13 A 2-kHz data signal is contaminated by 60 Hz of noise. Compare a single-stage and a two-stage *RC* high-pass filter for reducing the noise by 60 dB. What effect does each have on the data signal?

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Solution

An attenuation of 60 dB means that the output-to-input voltage ratio at 60 Hz will be

$$20 \log_{10}(V_{out}/V_{in}) = -60$$

$$V_{out}/V_{in} = 10^{-3}$$

For the single stage we use Equation (2.24) to find the critical frequency that will give this attenuation.

$$0.001 = \frac{(60/f_c)}{\sqrt{1 + (60/f_c)^2}}$$

Solving this for the critical frequency gives $f_c = 60$ kHz. To see what effect this has on the data signal, we evaluate Equation (2.24) at 2 kHz.

$$\frac{V_{out}}{V_{in}} = \frac{(2\text{ kHz}/60\text{ kHz})}{\sqrt{1 + (2\text{ kHz}/60\text{ kHz})^2}} = 0.033$$

Thus, only about 3.3% of the data signal is left!

Now, for the cascaded *RC* filter we have two *RC* high-pass filters in series, each with the same critical frequency. If the filters are independent—that is, if the output impedance of the first is much less than the input impedance of the second—the total response will be Equation (2.24) squared. Therefore, to get 60-dB reduction, we have

$$0.001 = \frac{(60/f_c)^2}{1 + (60/f_c)^2}$$

Solving this for the critical frequency gives $f_c = 1896$ Hz. The effect on the data signal is found by evaluating Equation (2.24) squared at 2 kHz with this critical frequency.

$$\frac{V_{out}}{V_{in}} = \frac{(2\text{ kHz}/1896\text{ Hz})^2}{1 + (2\text{ kHz}/1896\text{ Hz})^2} = 0.53$$

The sum can be scaled by proper selection of resistors. For example, if we make $R_1 = R_2 = R_3$, then the output is simply the (inverted) sum of V_1 and V_2. The average can be found by making $R_1 = R_3$ and $R_2 = R_1/2$.

MPLE Develop an op amp circuit that can provide an output voltage which is related to the input voltage by

$$V_{out} = 3.4V_{in} + 5$$

Solution

There are many ways to do this. One way is to use a summing amplifier with V_{in} on one input and 5 V on the other. The gains will be selected to be 3.4 and 1.0, respectively. The summing amplifier of Figure 2.31 is also an inverter, however, so the sign will be wrong. Thus a second amplifier will be used with a gain of -1 to make the sign correct. The result is shown in Figure 2.32. Selection of the values of the resistors is based on the general notion of keeping the currents in milliamperes.

2.5.3 Noninverting Amplifier

A noninverting amplifier may be constructed from an op amp, as shown in Figure 2.33. The gain of this circuit is found by summing the currents at the summing point S, and using the fact that the summing point voltage is V_{in} so that no voltage difference appears across the input terminals.

$$I_1 + I_2 = 0$$

where I_1 = current through R_1
I_2 = current through R_2

But these currents can be found from Ohm's law such that this equation becomes

$$\frac{V_{in}}{R_1} + \frac{V_{in} - V_{out}}{R_2} = 0$$

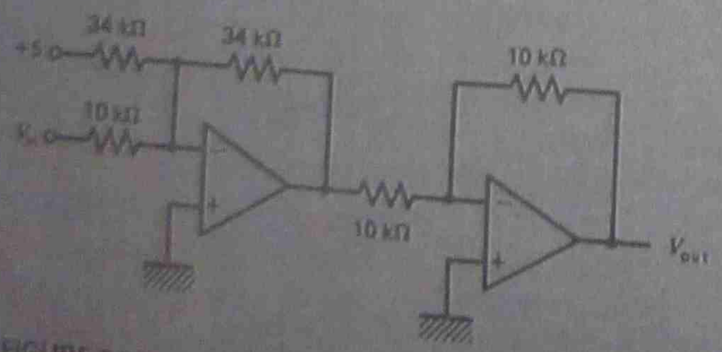
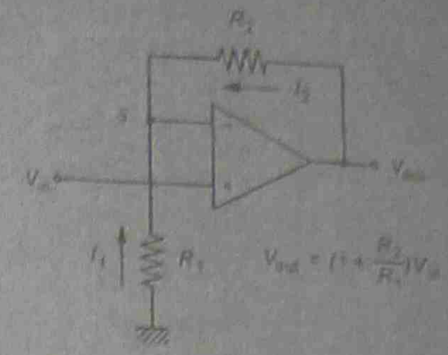


FIGURE 2.32
The op amp circuit for Example 2.18

FIGURE 2.33
A noninverting amplifier



Solving this equation for V_{out}, we find

$$V_{out} = \left[1 + \frac{R_2}{R_1}\right] V_{in} \tag{2.35}$$

Equation (2.35) shows that the noninverting amplifier has a gain that depends on the ratio of feedback resistor R_2 and the ground resistor R_1, but this gain can never be used for voltage attenuation because the ratio is added to 1. Because the input is taken directly into the noninverting input of the op amp, the input impedance is very high, since it is effectively equal to the op amp input impedance. The output impedance is very low.

EXAMPLE 2.19 Design a high-impedance amplifier with a voltage gain of 42.

Solution

We use the noninverting circuit of Figure 2.33 with resistors selected from

$$V_{out} = \left[1 + \frac{R_2}{R_1}\right] V_{in}$$

$$42 = 1 + \frac{R_2}{R_1}$$

$$R_2 = 41R_1$$

so we could choose $R_1 = \mathbf{1\ k\Omega}$, which requires $R_2 = \mathbf{41\ k\Omega}$.

2.5.4 Differential Instrumentation Amplifier

There are many instances in measurement and control systems in which the difference between two voltages needs to be conditioned. A good example is the Wheatstone bridge where the offset voltage, $\Delta V = V_a - V_b$, is the quantity of interest.

An ideal differential amplifier is one that provides an output voltage with respect to ground that is some gain times the difference between two input voltages:

$$V_{out} = A(V_a - V_b) \tag{2.36}$$

where A is the differential gain and both V_a and V_b are voltages with respect to ground.

Such an amplifier plays an important role in instrumentation and measurement.

A 15-V supply has been assumed for the divider resistance calculations. The 100-Ω resistor was selected to keep loading by the op amp circuit small. A trimmer (variable) resistor has been used so both loading of the divider by the op amp circuit and variation of the supply from exactly 15 V can be compensated for by adjusting until the bias is exactly 2.4 V.

The design could also be accomplished by a differential amplifier. If the 1.923 in the transfer equation is factored, we get

$$V_{out} = 1.923(V_{in} + 2.4)$$

So this is the equation of a differential amplifier with a gain of 1.923 and one input fixed at 2.4 V. A voltage follower would still be required on the input. (The reader should complete this design.)

EXAMPLE 2.25 Temperature is to be measured in the range of 250°C to 450°C with an accuracy of ±2°C. The sensor is a resistance that varies linearly from 280 Ω to 1060 Ω for this temperature range. Power dissipated in the sensor must be kept below 5 mW. Develop analog signal conditioning that provides a voltage varying linearly from −5 to +5 V for this temperature range. The load is a high-impedance recorder.

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Solution

Following the guidelines, let us first identify all the elements of the problem.

Measured Variable Parameter: Temperature

Range: 250 to 450°C

Accuracy: ±2°C

Noise: unspecified

Sensor Signal

Parameter: resistance

Transfer function: linear

Time response: unspecified

Range: 280 Ω to 1060 Ω, linear

Power: maximum 5 mW dissipated in sensor

Signal Conditioning

Parameter: voltage, linear

Range: −5 to +5 V

Input impedance: keep power in sensor below 5 mW

Output impedance: no problem, high-impedance recorder

The accuracy is ±0.8% at the low end and ±0.44% at the high end. Therefore, we will keep three significant figures to provide 0.1% on values selected.

The 5-mW maximum sensor dissipation means the current must be limited. To find the maximum current, we note that

$$P = I^2R$$
$$0.005 = I^2R$$
$$I = \sqrt{0.005/R}$$

The minimum current will thus occur at the maximum resistance,

$$I_{max} = \sqrt{0.005/1060} = 2.17 \text{ mA}$$

Thus, the design must always keep the sensor current below 2 mA.

Since the system must be linear, we should set up a linear equation between the sensor resistance and the output voltage. Then it is a matter of determining what circuits will implement the equation.

$$V_{out} = mR_s + V_0$$

We solve for m and V_0 by using the given information.

$$-5 = 280m + V_0$$
$$+5 = 1060m + V_0$$

Subtracting the first equation from the second gives

$$10 = 780m \quad \text{or} \quad m = 0.0128$$

Then, using this in the first equation,

$$-5 = 280(0.0128) + V_0$$
$$V_0 = -8.58$$

So the transfer function equation is

$$V_{out} = 0.0128R_s - 8.58$$

This can be provided by an inverting amplifier with the sensor resistor in the feedback, following by an inverting summer to get the signs correct. Figure 2.47 shows one possible solution. The fixed input voltage and input resistor of the first op amp have been selected to satisfy the 5-mW maximum power dissipation. This has been done by noting that the current through the sensor is just equal to the current through the input circuit. Thus, by using 1.00 kΩ and 1.00 V, the current will always be 1 mA and thus less than 2 mA, as required.

As in Example 2.24, trimmers are used in dividers so the fixed voltages can be adjusted to 1.00 and 8.58 V and thus account for supply voltage differences.

When most people think of a computer they visualize the common home and office *personal computer*, or PC. In fact, however, one of the most common applications of computers is for controlling something, and we seldom ever even see the computer. When we learn that our microwave oven has a microprocessor-based computer inside, it is there for the purpose of controlling the operation of the oven. So it is with the use of computers in automobiles, washing machines, airplanes, and a vast host of other examples.

Computers are digital electronic devices and so all the information they work with has to be digitally formatted. Therefore, if they are used to control a variable such as temperature, then the temperature has to be represented digitally. So that's why we need digital signal conditioning—to condition process-control signals to be in an appropriate digital format.

The second question is the topic of this chapter. Digital signal conditioning in process control means finding a way to represent analog process information in a digital format. In this chapter we review some common digital electronics principles and then study the way analog signals are converted into a digital format.

You should realize that there is no greater accuracy in using digital techniques to represent data; in fact, accuracy is usually lost. But digital data are much more immune from spurious influences that would cause subsequent inaccuracy, such as noise, amplifier gain changes, power supply drifts, and so on.

Use of computers in control systems is particularly valuable for a number of other reasons, however:

1. A computer can control multivariable process-control systems.
2. Nonlinearities in sensor output can be linearized by the computer.
3. Complicated control equations can be solved quickly and modified as needed.
4. Networking of control computers allows a large process-control complex to operate in a fully integrated fashion.

REVIEW OF DIGITAL FUNDAMENTALS

A working understanding of the application of digital techniques to process control requires a foundation of basic digital electronics. The design and implementation of control logic systems and microcomputer control systems require a depth of understanding that can be obtained only by taking several courses devoted to the subject. In this text, we assume a sufficient background that the reader can appreciate the essential features of digital electronic design and its application to process control. A summary of basic digital electronics concepts is presented in Appendix 2.

3.2.1 Digital Information

The use of digital techniques in process control requires that process variable measurements and control information be encoded into a digital form. Digital signals themselves are simply two-state (binary) levels. These levels may be represented in many ways, e.g., two voltages, two currents, two frequencies, or two phases. We speak, then, of the digital information as a high state (H or 1) or a low state (L or 0) on a wire that carries the digital signal.

Digital Words Given the simple binary information that is carried by a single digital signal, it is clear that multiple signals must be used to describe analog information. Generally, this is done by using an assemblage of digital levels to construct a binary number, often called a *word*. The individual digital levels are referred to as *bits* of the word. Thus, for example, a 6-bit word consists of six independent digital levels, such as 101011_2, which can be thought of as a six-digit base 2 number. An important consideration, then, is how the analog information is encoded into this digital word.

Decimal Whole Numbers One of the most common schemes for encoding analog data into a digital word is to use the straight counting of decimal (or base 10) and binary number representations. The principles of this process are reviewed in Appendix 2, together with octal and hexadecimal representations.

EXAMPLE 3.1 Find the base 10 equivalent of the binary whole number 00100111_2.

Solution

As in the base 10 system, zeros preceding the first significant digit do not contribute. Thus, the binary number is actually 100111_2, and so $n = 5$. To find the decimal equivalent, we use Appendix 2 and compute as follows:

$$N_{10} = a_5 2^5 + a_4 2^4 + \cdots + a_1 2^1 + a_0 2^0$$

$$N_{10} = (1)2^5 + (0)2^4 + (0)2^3 + (1)2^2 + (1)2^1 + (1)2^0 \qquad (A.2.1)$$

$$N_{10} = 32 + 4 + 2 + 1$$

$$N_{10} = 39$$

EXAMPLE 3.2 Find the binary equivalent of the base 10 number 47.

Solution

Starting the successive division, we get

720

$$\frac{47}{2} = 23 \text{ with a remainder of } \frac{1}{2} \text{ so that } a_0 = 1$$

then

$$\frac{23}{2} = 11 \text{ with a remainder of } \frac{1}{2} \text{ so that } a_1 = 1$$

then

$$\frac{11}{2} = 5 + \frac{1}{2} \qquad \therefore a_2 = 1$$

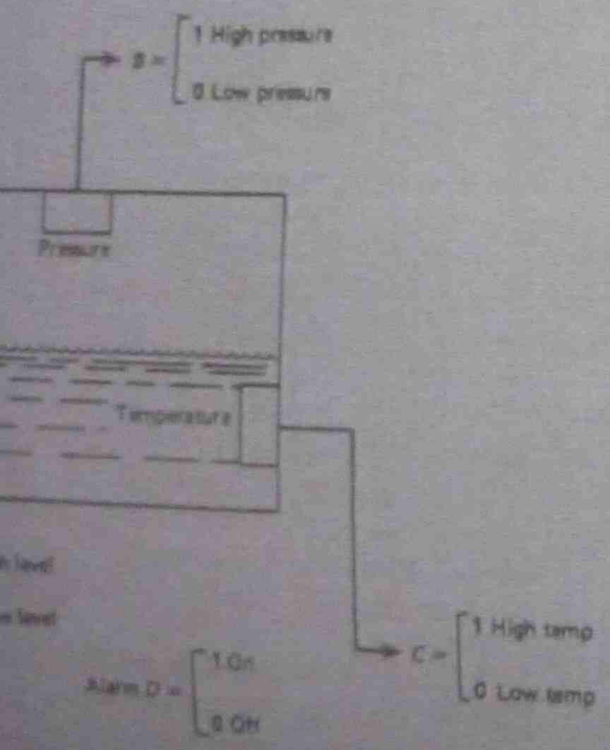
$$\frac{5}{2} = 2 + \frac{1}{2} \qquad \therefore a_3 = 1$$

$$\frac{2}{2} = 1 + 0 \qquad \therefore a_4 = 0$$

...to mathematically express the inputs, the logical evaluation, and the cor... ...*Boolean algebra* is a mathematical procedure that allows the combi... ...dual conditions in various logical operations by equations so that conclu... ...sion. For purposes of this text, we do not require expertise in Boolean ... operational familiarity with it that can be applied to a process-control en...

...particular problem in industry can be addressed using digital electronics, it ...d in terms that are amenable to the binary nature of digital techniques. Gen... ...complished by stating the problem in the form of a set of true/false-type ...must be applied to derive some desired result. These sets of conditions then ...form of one or more Boolean equations. We will see in Section 3.2.4 that ...son is in a form that is readily implemented with existing digital circuits. ...al approach of Boolean algebra allows us to write an analytical expression ...se stipulations. The fundamentals of Boolean algebra are summarized in

...sider a simple example of how a Boolean equation may result from a prac... ...consider a mixing tank for which there are three variables of interest: liquid ...and temperature. The problem is that we must signal an alarm when certain ...conditions occur among these variables. Referring to Figure 3.1, we de... ...pressure by *B*, and temperature by *C*, and assume that setpoint values have ...r each variable so that the Boolean variables are either **1** or **0** as the phys-



...ising Boolean applications to control

ical quantities are above or below the setpoint values. The alarm will be triggered when the Boolean variable D goes to the logic true state. The alarm conditions are

1. Low level with high pressure
2. High level with high temperature
3. High level with low temperature and high pressure

We now define a Boolean expression with AND operations that will give a $D = 1$ for each condition.

1. $D = \bar{A} \cdot B$ will give $D = 1$ for condition 1.
2. $D = A \cdot C$ will give $D = 1$ for condition 2.
3. $D = A \cdot \bar{C} \cdot B$ will give $D = 1$ for condition 3.

The final logic equation results from combining all three conditions so that if any is true, the alarm will sound ($D = 1$). This is accomplished with the OR operation

$$D = \bar{A} \cdot B + A \cdot C + A \cdot \bar{C} \cdot B \tag{3.2}$$

This equation would now form the starting point for a design of electronic digital circuitry that would perform the indicated operations.

3.2.4 Digital Electronics

The electronic building blocks of digital electronics are designed to operate on the binary levels present on digital signal lines. These building blocks are based on families of types of electronic circuits, as discussed in Appendix 2, that have their specific stipulations of power supplies and voltage levels of the **1** and **0** states. The basic structure involves the use of AND/OR logic and NAND/NOR logic to implement Boolean equations.

EXAMPLE 3.5 Develop a digital circuit using AND/OR gates that implements the equation developed in Section 3.2.3.

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Solution

The problem posed in Section 3.2.3 (with Figure 3.1) has a Boolean equation solution of

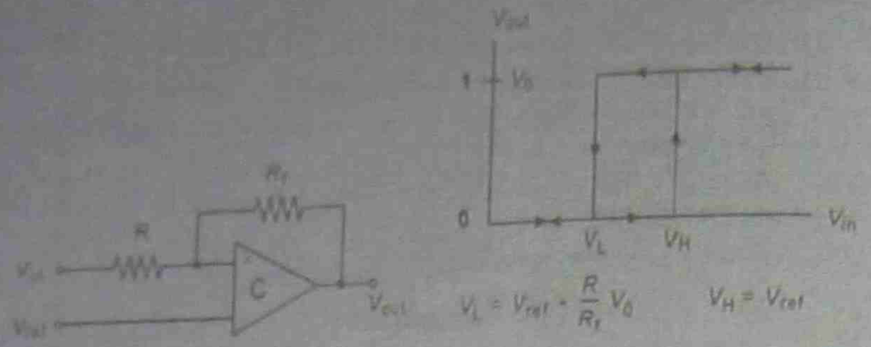
$$D = \bar{A} \cdot B + A \cdot C + A \cdot \bar{C} \cdot B \tag{3.2}$$

The implementation of this equation using AND/OR gates is shown in Figure 3.2. The AND, OR, and inverter are used in a straightforward implementation of the equation.

EXAMPLE 3.6 Repeat Example 3.5 using NAND/NOR gates.

Solution

One way to implement the equation in NAND/NOR would be to provide inverters after ev... ...ery gate, in effect, to convert the devices back to AND/OR gates. In this case, the circuit ...developed would look like Figure 3.2 but with an inverter after every gate. A second ap...



a) Hysteresis comparator circuit

b) Hysteresis comparator input-output relationship and equations

FIGURE 3.10
A generic DAC diagram, showing typical input and output signals.

Once having been driven high, the condition for the output to drop back to the low (0 V) state is given by the relation

$$V_{in} \leq V_{ref} - (R/R_f)V_0 \tag{3.4}$$

The deadband or hysteresis is given by $(R/R_f)V_0$, and is thus selectable by choice of the resistors, as long as this relation is satisfied. The response of this comparator is shown by the graph in Figure 3.10b. The arrows indicate increasing or decreasing input voltage.

EXAMPLE 3.8 A sensor converts the liquid level in a tank to voltage according to the transfer function (20 mV/cm). A comparator is supposed to go high (5 V) whenever the level becomes 50 cm. Splashing causes the level to fluctuate by ±3 cm. Develop a hysteresis comparator to protect against the effects of splashing.

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Solution

The nominal reference for the comparator occurs at 50 cm, which is $V_{ref} = (20\text{ mV/cm})(50\text{ cm}) = 1$ V. The splashing, however, causes a "noise" of $(20\text{ mV/cm}) \cdot (\pm 3\text{ cm}) = \pm 60$ mV. This is a total range of 120 mV. We need a deadband of at least 120 mV, but let us make it 150 mV for security. Thus, we have

$$(R/R_f)(5\text{ V}) = 150\text{ mV}$$
$$(R/R_f) = 0.03$$

If we make $R_f = 100\text{ k}\Omega$, then $R = 3\text{ k}\Omega$. Thus, use of these resistors, as shown in Figure 3.10, with a reference of 1 V will meet the requirement.

3.3.2 Digital-to-Analog Converters (DACs)

A DAC accepts digital information and transforms it into an analog voltage. The digital information is in the form of a binary number with some fixed number of digits. Especially when used in connection with a computer, this binary number is called a binary word or *computer word*. The digits are called *bits* of the word. Thus, an 8-bit word would be a binary number having eight digits, such as 10110110_2. A unipolar DAC converts a digital word into an analog voltage by scaling the analog output to be zero when all bits are zero and some maximum value when all bits are one. This can be mathematically represented by treating the binary number that the word represents as a *fractional* number. In this context, the output of the DAC can be defined using Equation (3.1) as a *scaling* of some reference voltage.

$$V_{out} = V_R[b_1 2^{-1} + b_2 2^{-2} + \cdots + b_n 2^{-n}] \tag{3.5}$$

where V_{out} = analog voltage output
V_R = reference voltage
$b_1 b_2 \ldots b_n$ = n-bit binary word

The minimum V_{out} is zero, and the maximum is determined by the size of the binary word because, with all bits set to one, the decimal equivalent approaches V_R as the number of bits increases. Thus, a 4-bit word has a maximum of

$$V_{max} = V_R[2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}] = 0.9375V_R$$

and an 8-bit word has a maximum of

$$V_{max} = V_R[2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-7} + 2^{-8}] = 0.9961V_R$$

An alternative equation to Equation (3.5) is often easier to use. This is based on noting that the expression in brackets in Equation (3.5) is really just the fraction of total counting states possible with the n bits being used. With this recognition, we can write

$$V_{out} = \frac{N}{2^n} V_R \tag{3.6}$$

where N = base 10 whole-number equivalent of DAC input

Suppose an 8-bit converter with a 5.0-V reference has an input of 10100111_2, or A7H. If this input is converted to base 10, we get $N = 167_{10}$ and $2^8 = 256$. From Equation (3.6), the output of the ADC will be

$$V_{out} = \frac{167}{256} 5.0 = 3.2617 \text{ volts}$$

EXAMPLE 3.9 What is the output voltage of a 10-bit ADC with a 10.0-V reference if the input is (a) 0010110101_2 = 0B5H, (b) 20FH? What input is needed to get a 6.5-V output?

so a 2-V reference is used.

A change of 0.1°C results in a voltage change of

$$(0.1°C)(0.02\ V/°C) = 2\ mV$$

so we need a word size where

$$0.002\ V = (2)(2^{-y})$$

Choose a size n that is the integer part of y plus 1. Thus, solving with logarithms, we find

$$y = \frac{\log(2) - \log(0.002)}{\log 2}$$

$$y = 9.996 \approx 10$$

so a 10-bit word is required for this resolution. A 10-bit word has a resolution of

$$V = (2)(2^{-10})$$

$$V = 0.00195\ V$$

which is better than the minimum required resolution of 2 mV.

Notice that the output actually changes from 1111111110_2 to 1111111111_2 at a voltage of

$$V_R(1 - 2^{-n}) = (10\ V)(1 - 2^{-10}) = 1.9980\ V$$

which corresponds to a temperature of (1.9980 V)/(0.02 V/°C) = 99.90°C. This means we are actually measuring temperature between 0.1°C and 99.9°C.

E Find the digital word that results from a 3.127-V input to a 5-bit ADC with a 5-V reference.

Solution

The relationship between input and output is given by Equation (3.9). Thus, we are to encode a fractional number of V_{in}/V_R, or

$$b_1 2^{-1} + b_2 2^{-2} + \cdots + b_5 2^{-5} = \frac{3.127}{5} = 0.6254$$

Using the method of successive multiplication defined in Section 3.2.2, we find

$$0.6254(2) = 1.2508 \quad \therefore b_1 = 1$$
$$0.2508(2) = 0.5016 \quad \therefore b_2 = 0$$
$$0.5016(2) = 1.0032 \quad \therefore b_3 = 1$$
$$0.0032(2) = 0.0064 \quad \therefore b_4 = 0$$
$$0.0064(2) = 0.0128 \quad \therefore b_5 = 0$$

so that the output is 10100_2.

Equation (3.9) can be written in a simpler fashion by expressing the fractional binary number as the fraction of counting states, as was done for the DAC. In this case the base 10 value of the digital output can be expressed as

$$N = \text{INT}\left(\frac{V_{in}}{V_R} 2^n\right) \tag{3.11}$$

where INT( ) means to take the integer part of the quantity in brackets. This is not a round-off, but rather a truncation, so that INT(3.3) = 3 and INT(3.99) = 3 also. The value of N is then converted to hex and/or binary to demonstrate the ADC output. In the previous example, we would have

$$N = \text{INT}\left(\frac{3.127}{5} 2^5\right) = \text{INT}(20.0128) = 20_{10}$$

or $14H = 10100_2$, as already found.

EXAMPLE 3.15 The input to a 10-bit ADC with a 2.500-V reference is 1.45 V. What is the hex output? Suppose the output was found to be 1B4H. What is the voltage input?

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Solution

We will use Equation (3.11) to find the solution to these questions. For the first part, we can form the expression

$$N = \text{INT}((1.45/2.5)2^{10})$$
$$N = \text{INT}(593.92)$$
$$N = 593$$
$$N = 251H$$

So the output of the ADC is 251H for a 1.45-V input. To get the voltage input for a 1B4H output, we solve Equation (3.11) for the voltage

$$V_{in} = \frac{N}{2^n} V_R$$

A conversion yields $1B4H = 436_{10}$.

$$V_{in} = (436/1024)2.50$$
$$V_{in} = 1.06445\ V$$

However, it is important to realize that any voltage from this to 1.06445 + 2.5/1024 = 1.06689 will give the same output, 1B4H. So the correct answer to the question is that the input voltage lies in the range 1.06445 to 1.06689 V.

Bipolar Operation A bipolar ADC is one that accepts bipolar input voltage for conversion into an appropriate digital output. The most common bipolar ADCs provide an

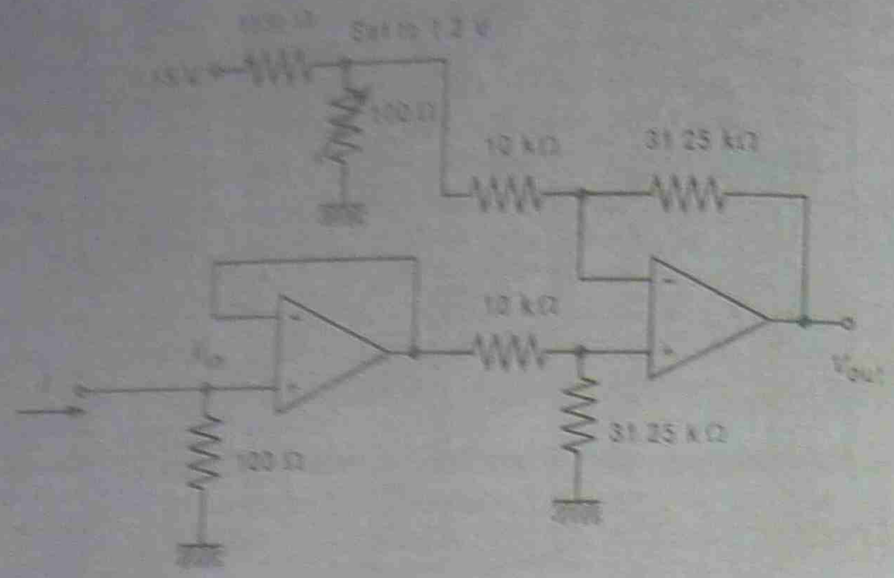

FIGURE 7.3
An op amp circuit to provide the signal-conditioning requirements of Example 7.1

size. The rest of this design would involve determining if the op amp can deliver enough current to drive the magnetic amplifier. If not, a booster would be required.

Motor Control Many motor control circuits are designed as packaged units that accept a low-level dc signal directly to control motor speed. If such a system is not available, it is possible to build circuits using amplifiers along with SCRs or TRIACs to perform this control. The nature of SCRs and TRIACs are presented in the next section. The basic elements of electrical motors and some words about their control are discussed later in this chapter.

7.3.2 Digital Electrical Signals

Conversions of digital signals to forms required by final control operations usually are carried out using systems already discussed in Chapter 3. We mention again, however, the basic elements of the output interface between computer and final control.

ON/OFF Control There are many cases in process control where the control algorithm is accomplished by simple commands to outside equipment to change speed, turn on (or off), move up, and so on. In such cases, the computer can simply load a latched output line with a 1 or 0 as appropriate. Then it is a simple matter to use this signal to close a relay or activate some other outside circuit.

DAC When the digital output must provide a smooth control, as it does in valve positioning, the computer must provide an input to a DAC that then determines an appropriate analog output. When a computer must provide outputs to many final control elements,

a data output module or system such as that described in Chapter 3 can be employed. These integrated modules contain channel addressing, DAC, and other required elements of a self-contained output interface system.

EXAMPLE 7.2 A 4-bit digital word is intended to control the setting of a 2-Ω dc resistive heater. Heat output varies as a 0–24 V input to the heater. Using a 10-V DAC followed by an amplifier and a unity-gain high-current amplifier, calculate (a) the settings from minimum to maximum heat dissipation, and (b) how the power varies with LSB changes.

Solution

a. The 4-bit word has a total of 16 states; thus, the DAC outputs voltage from 0 V for a 0000_2 input to 9.375 V for a 1111_2 input. If we use a gain of 2.56, then the heater input will be 0–24 V in 1.6-V steps. The heat dissipation is found from

$$P = \frac{V^2}{R} \tag{7.1}$$

Then, for the minimum, $P = 0$ because $V = 0$, and for the maximum

$$P_{max} = \frac{(24\ \text{V})^2}{2\ \Omega}$$

$$P_{max} = 288\ \text{W}$$

b. The variation in heating with voltage is not linear, because the power varies as the square of the voltage. The increase in power for a step change in voltage can be found by taking the difference in power between the two cases:

$$\Delta P = \frac{V^2}{R} - \frac{(V - \Delta V)^2}{R} = \frac{\Delta V(2V - \Delta V)}{R}$$

Because $\Delta V = 1.6$ V, we have

$$\Delta P = \frac{1.6(2V - 1.6)}{R} = 0.8(2V - 1.6)$$

The first bit change produces a power change of 1.28 W, and the last bit, a power change of 37.12 W.

Direct Action As the use of digital and computer techniques in process control becomes more widespread, new methods of final control have been developed that can be actuated directly by the computer. Thus, a stepping motor, to be discussed later, interfaces easily to the digital signals that a computer outputs. In another development, special integrated circuits are made that reside within the final control element and allow the digital signal to be connected directly.

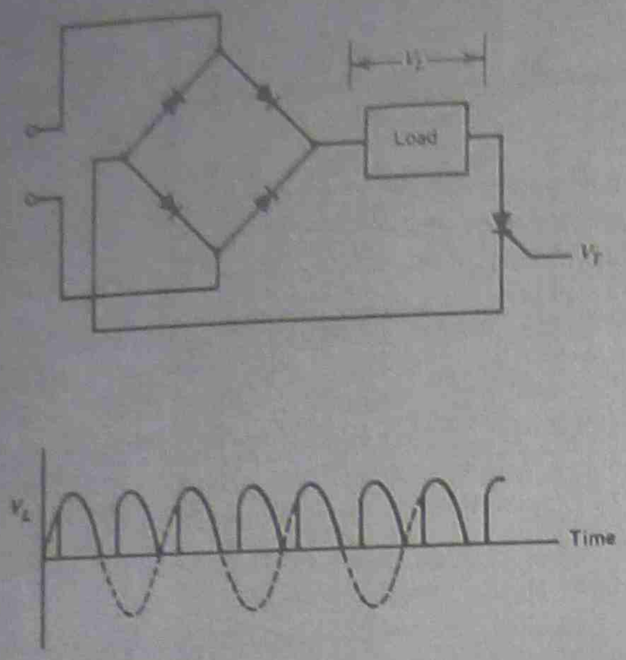

FIGURE 7.9
A full-wave SCR circuit. The effective rms dc voltage applied to the load is greater than that of the half-wave because the entire cycle is used.

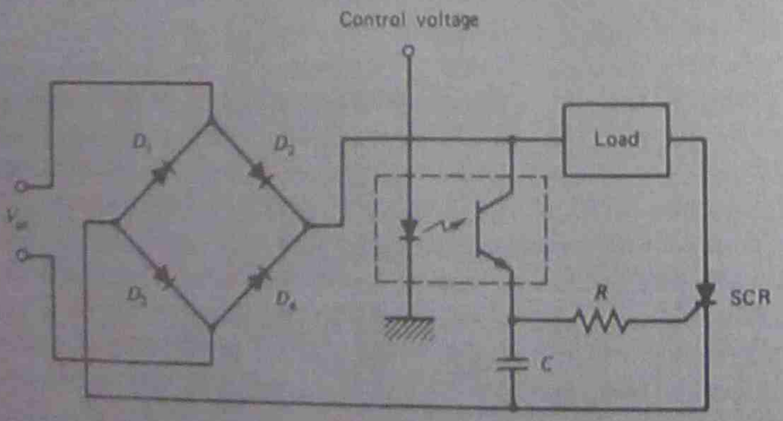

FIGURE 7.10
An optical coupler can be used to control triggering of the SCR from an external circuit.

EXAMPLE 7.3 An SCR with a 4.0-V trigger is used in the circuit of Figure 7.11 as a light-dimmer control. What resistance, R, should be used to provide approximately 10% to 90% on time?

Solution

Let's start by understanding each part of the circuit. Diode D_1, the 5-kΩ resistor, and the 10-V zener diode provide the source of about 10 V, but this goes to zero every cycle, as shown in Figure 7.12. This serves to charge the 0.12-μF capacitor through resistor R. When the ac voltage on the SCR is on the positive half of the cycle and the capacitor charge reaches the trigger value of 4 V, the SCR will go on, discharging the capacitor and turning on the light. When the SCR voltage goes to the negative half cycle, the SCR will turn off. Figure 7.12 shows the voltage waveforms in Figure 7.11 for a 50% on time setting of R.

When will the capacitor charge reach the trigger of 4.0 V. This can be found by the simple capacitor-charging equation, $V_c(t) = V_0(1 - e^{-t/RC})$, where V_0 is the zener voltage. In this case we are looking for a charge of 4.0 V, so

$$4.0 = 10(1 - e^{-t/RC})$$

Solving for t, we find $t = 0.511RC$. Thus, whatever R and C are, the SCR will trigger at the time given by this expression.

The period of a half cycle is 1/120 = 8.33 ms. Therefore, for a 10% on time we use $t = 0.833$ ms, and for a 90% on time we use $t = 7.5$ ms. Since the capacitor is given as 0.12 μF, we can find the required resistance for the two cases:

$$R_{min} = \frac{0.833 \text{ ms}}{(0.511)(0.12\ \mu\text{F})} = 13.6\ \text{k}\Omega$$

and

$$R_{max} = \frac{7.5 \text{ ms}}{(0.511)(0.12\ \mu\text{F})} = 122\ \text{k}\Omega$$

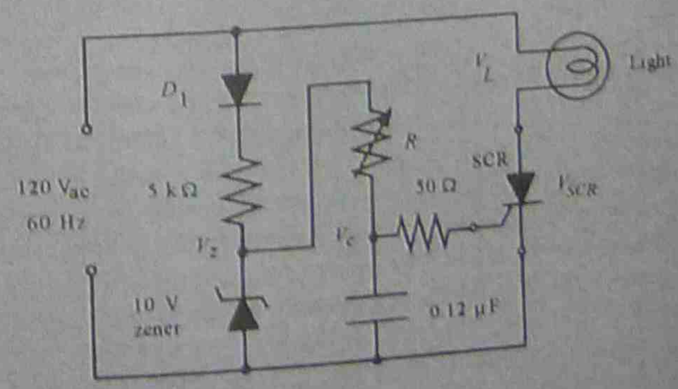

FIGURE 7.11
Circuit for Example 7.3.

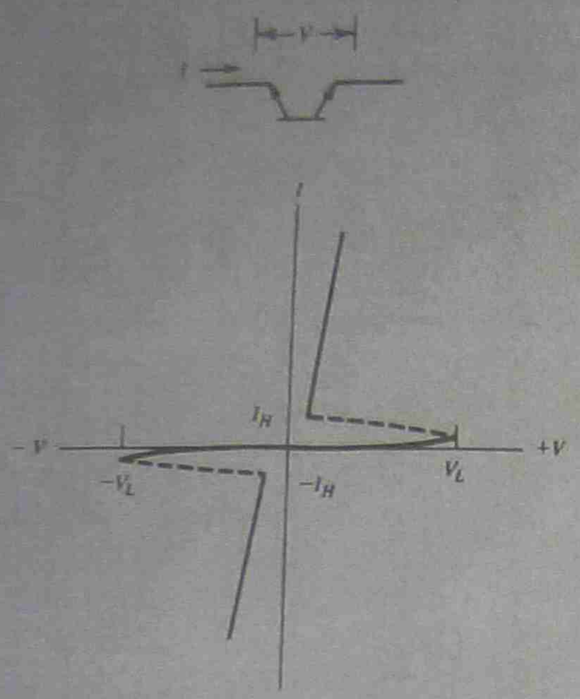

FIGURE 7.14
The DIAC has a bipolar breakover voltage, $\pm V_L$. After the voltage reaches $\pm V_L$, the DIAC conducts like a diode.

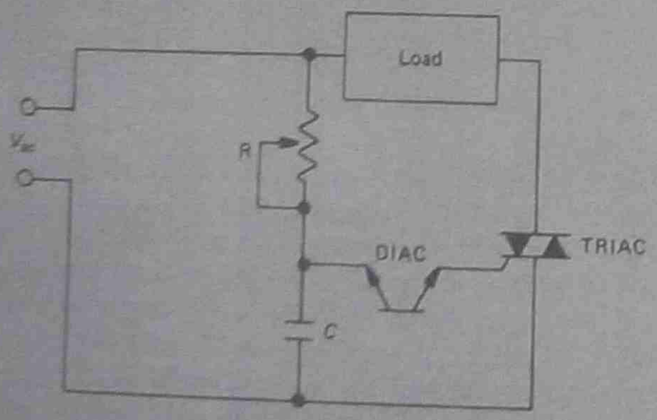

FIGURE 7.15
The TRIAC can be triggered from the ac line using a DIAC.

EXAMPLE 7.4 A DIAC is available that has a 28-V breakover voltage, V_L. Show how this can be used to make the light dimmer of Example 7.3 using a TRIAC. Again, we want approximately a 10% to 90% range of adjustment using R, with $C = 0.12\ \mu F$.

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Solution

The light dimmer can be implemented so that power is applied every cycle instead of every other cycle. We can use the circuit of Figure 7.15 with the load replaced by a light and the source given by the 120-V_{ac} rms, 60-Hz line. This is given by

$$V_{ac} = 169.7 \sin(120\pi t)$$

The first question is, When does the line voltage reach 28 so that it would even be possible to trigger the DIAC and thus the TRIAC? This can be found from

$$28 = 169.7 \sin(120\pi t)$$

Solving for t,

$$t = [\sin^{-1}(28/169.7)]/(120\pi) = 0.44 \text{ ms}$$

Since the half period is 8.33 ms, the light cannot be turned on before about 5% of the period has passed.

The next problem is to find the values of R that will provide a 10% and 90% on time, respectively. As before, this means the voltage must reach 28 V at 0.833 ms and 7.5 ms, respectively. Unfortunately, there is not such a simple relation between capacitor voltage and time as in Example 7.3, because the charging voltage is changing in time, that is, $169.7 \sin(120\pi t)$. An exact equation can be found, but let's try a simpler approach.

We can modify the circuit as shown in Figure 7.16 to provide a clamped ac voltage

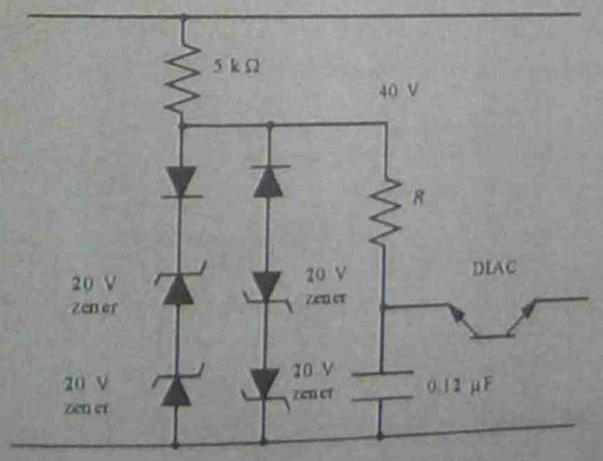

FIGURE 7.16
This modification of the triggering circuit eases calculation of a solution for Example 7.4.

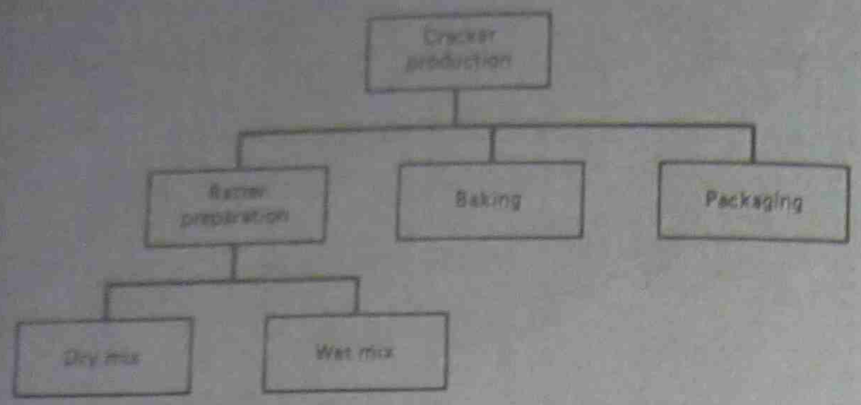

FIGURE 8.6
Objectives and subobjectives of a process.

duction we need metal specialists, and so on. Nevertheless, it is important for the control system specialist to study the industry and come to an understanding of the products, the process, and the objectives of the process.

Process Hardware With determination of the objectives of the process comes the design of hardware to implement these objectives. This hardware is closely tied to the nature of the industry, and its design must come from the joint efforts of process, production, and control personnel. For the control system specialist, the essential thing is to develop a good understanding of the nature of the hardware and its characteristics.

Figure 8.7 shows a pictorial representation of process hardware for a conveyor system. The objective is to fill boxes moving on two conveyors from a common feed hopper and material-conveyor system. A process-control system specialist may not have been involved in the development of this system. To develop the control system, he or she must study the hardware carefully and understand the characteristics of each element.

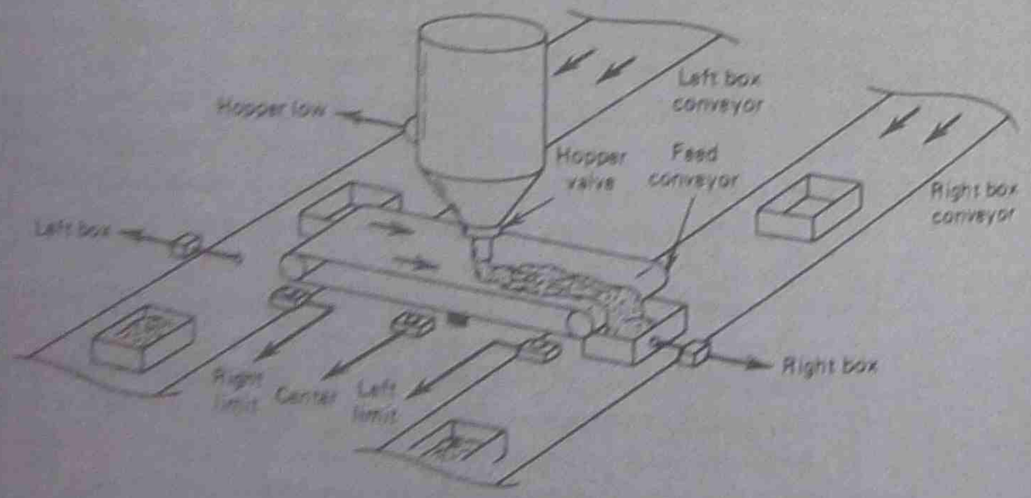

FIGURE 8.7
A discrete-control process.

In general, the specialist analyzes the hardware by considering how each part is related to the control system. There are really only two basic categories.

1. *Input devices* provide inputs *to* the control system. The operation of these devices is similar to the measurement function of continuous control systems. In the case of discrete-state process control, the inputs are two-state specifications, such as

   Limit switches: open or closed

   Comparators: high or low

   Push buttons: depressed or not depressed

2. *Output devices* accept output commands *from* the control system. The final control element of continuous control systems does the same thing. In discrete-state process control, these output devices accept only two-state commands, such as

   Light: on or off

   Motor: rotating or not rotating

   Solenoid: engaged or not engaged

EXAMPLE 8.2 Study the pictorial process of Figure 8.7. Identify the input and output devices and the characteristics of each device.

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Solution

A study of the system described in Figure 8.7 shows the following distribution of elements:

Input Devices (All Switches)

Right box present

Left box present

Feed conveyor right travel limit

Feed conveyor left travel limit

Hopper low

Feed conveyor center

Output Devices

Hopper valve solenoid

Feed stock conveyor motor off

Feed stock conveyor motor right

Feed stock conveyor motor left

Right box conveyor motor

Left box conveyor motor

The previous equation is in terms of the actual measured variable c, but the same equation can be expressed in terms of the measured indication b. It is necessary only to translate the measured minimum and maximum to b_{min} and b_{max}.

To express error as percent of span, it is necessary only to write both the setpoint and measurement in terms of percent of span and take the difference according to Equation (9.2). The result is

$$e_p = \frac{r - b}{b_{max} - b_{min}} \times 100 \tag{9.3}$$

where e_p = error expressed as percent of span

You can see the convenience of using a standard measured indication range like 4–20 mA, because the span is always 16 mA. Suppose we have a setpoint of 10.5 mA and a measurement of 13.7 mA. Then, without even knowing what is being measured, we know the error is

$$e_p = \frac{10.5 \text{ mA} - 13.7 \text{ mA}}{20 \text{ mA} - 4 \text{ mA}} \times 100$$

$$e_p = -20\%$$

A positive error indicates a measurement below the setpoint, and a negative error indicates a measurement above the setpoint.

EXAMPLE The temperature in Figure 9.1 has a range of 300 to 440 K and a setpoint of 384 K. Find the percent of span error when the temperature is 379 K.

Solution

The percent error is

$$e_p = \frac{r - b}{b_{max} - b_{min}} \times 100$$

$$e_p = \frac{384 - 379}{440 - 300} \times 100$$

$$e_p = 3.6\%$$

9.3.2 Variable Range

Generally, the variable under control has a range of values within which control is to be maintained. This range can be expressed as the minimum and maximum value of the variable or the nominal value plus and minus the spread about this nominal. If a standard 4–20-mA signal transmission is employed, then 4 mA represents the minimum value of the variable and 20 mA the maximum.

When a computer-based control system is used, the dynamic variable is converted to an n-bit digital signal. Often, the transformation is made so that all 0s are the minimum value of the variable and all 1s are the maximum value.

9.3.3 Control Parameter Range

Another range is associated with the controller output. Here we assume the final control element has some minimum and maximum effect on the process. The controller output range is the translation of output to the range of possible values of the final control element. This range also is expressed as the 4–20-mA standard signal, again with the minimum and maximum effects in terms of the minimum and maximum current.

Similarly, in computer-based control, the output will range over all states of the n-bit output. Generally, all 0s are the minimum output and all 1s the maximum. These numbers do not necessarily represent the minimum and maximum of the final control element, however. We may wish a valve never to be fully closed, for example; therefore, all 0s might represent some percentage of full open.

Often, the output is expressed as a percentage where 0% is the minimum controller output and 100% the maximum (obviously). Thus, in the example of Figure 9.1, the valve in the fully open position corresponds to a 100% controller signal output. Often, however, the minimum does *not* correspond to zero effect. For example, it may be that the steam flow should never be less than that flow which results with the valve half open. In this case, a 0% minimum controller corresponds to the flow rate with a half-open valve.

The controller output as a percent of full scale when the output varies between specified limits is given by

$$p = \frac{u - u_{min}}{u_{max} - u_{min}} \times 100 \tag{9.4}$$

where p = controller output as percent of full scale
u = value of the output
u_{max} = maximum value of controlling parameter
u_{min} = minimum value of controlling parameter

EXAMPLE 9.2 A controller outputs a 4–20-mA signal to control motor speed from 140–600 rpm with a linear dependence. Calculate (a) current corresponding to 310 rpm and (b) the value of (a) expressed as the percent of control output.

Solution

a. We find the slope m and intersect S_0 of the linear relation between current I and speed S, where

$$S_p = mI + S_0$$

Knowing S_p and I at the two given positions, we write two equations:

$$140 = 4m + S_0$$

$$600 = 20m + S_0$$

Solving these simultaneous equations, we get $m = 28.75$ rpm/mA and $S_0 = 25$ rpm. Thus, at 310 rpm we have $310 = 28.75I + 25$, which gives $I = 9.91$ mA.

b. Expressed as a percentage of the 4–20-mA range, this controller output is

L. Open hopper-feed valve
M. Test left limit switch
   1. If engaged, go to II.A
   2. If not, go to M

Note that the system cycles from step M to step A. The description is constructed by simple analysis of what events must occur and what the input and outputs must be to support these events.

EXAMPLE 8.3 Construct a narrative statement outline of the event sequence for the system shown in Figure 8.5. The objective is to fill bottles moving on a conveyor.

Solution

We assume that when a command is given to stop the continuous control system, the input valve is driven to the closed position. Then the sequence would be

I. Initialization (prefill of tank)
   A. Conveyor stopped, output valve closed
   B. Start the level-control system
      1. Operate for a sufficient time to reach the setpoint, or
      2. Add another sensor so that the system knows when the setpoint has been reached.
   C. When level is reached, stop the level control
   D. Go to the running phase

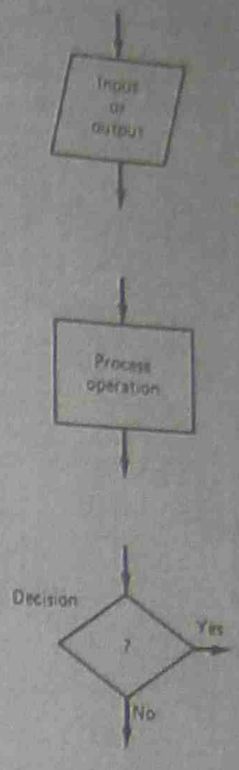
II. Running
   A. Start the bottle conveyor
   B. When a bottle is in position:
      1. Stop the conveyor
      2. Open the output valve
      3. Turn on a level-control system to keep the level constant during bottle fill
   C. When the bottle is full:
      1. Close the output valve
      2. Stop the level-control system
   D. Go to step II.A and repeat

Notice that hardware was added to the system when the event sequence was constructed. Hardware and software are often developed in conjunction, because the development of one demonstrates extra needs in the other.

Flowcharts of the Event Sequence It is often easier to visualize and construct the sequence of events if a flowchart is used to pictorially present the flow of events. Although there are many sophisticated types of flowcharts, the concept can be presented easily by using the three symbols shown in Figure 8.9.

The narrative statements are then simply reformatted into flowchart symbols. Often it is easier to express the sequence of events directly in terms of the flowchart symbols. Figure 8.10 shows part of the initialization phase of the conveyor system of Figure 8.7 expressed in the flowchart format.

FIGURE 8.9
Basic flowchart symbols



Binary-State Variable Descriptions Each event that makes up the sequence of events described by the narrative scheme corresponds to a *discrete state* of the system. Thus, it is also possible to describe the sequence of events in terms of the sequence of discrete states of the system. To do this simply requires that for each event the state, including both input and output variables, be specified.

The input variables cause the state of the system to change because operations within the system cause a change of one of the state variables; for example, a limit switch becomes engaged. The output variables, on the other hand, are changes in the system state that are caused by the control system itself.

The control system works like a look-up table. The input state variables with the output becomes like a memory address, and the new output state variables are the contents of that memory.

EXAMPLE 8.4 Define the state variable description for the process shown in Figure 8.11 and described by the following event sequence:

1. Fill the tank to level *A* from valve *A*

where $F(e_p)$ represents the relation by which the appropriate controller output is determined. In some cases, a graph of p versus e_p is also employed to aid in a definition of the control mode.

Reverse and Direct Action The error that results from measurement of the controlled variable may be positive or negative, because the value may be greater or less than the setpoint. How this polarity of the error changes the controller output can be selected according to the nature of the process.

A controller operates with *direct action* when an increasing value of the controlled variable causes an increasing value of the controller output. An example would be a level-control system that outputs a signal to an output valve. Clearly, if the level rises (increases), the valve should be opened (i.e., its drive signal should be increased).

Reverse action is the opposite case, where an increase in controlled variable causes a decrease in controller output. An example of this would be a simple temperature control from a heater. If the temperature increases, the drive to the heater should be decreased.

9.4 DISCONTINUOUS CONTROLLER MODES

This section discusses the various controller modes that show discontinuous changes in controller output as controlled variable error occurs. It is important that you understand these modes, both because of their frequent use in process control, and because they form the basis of the continuous modes to be discussed in Section 9.5.

9.4.1 Two-Position Mode

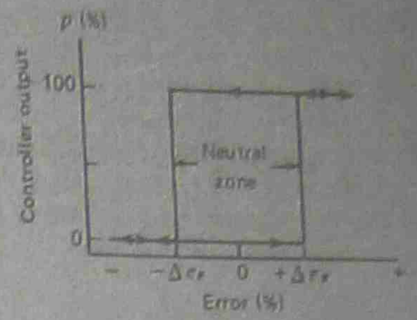
The most elementary controller mode is the ON/OFF, or two-position, mode. This is an example of a discontinuous mode. It is the simplest and the cheapest, and often suffices when its disadvantages are tolerable. Although an analytic equation cannot be written, we can, in general, write

$$p = \begin{cases} 0\% & e_p < 0 \\ 100\% & e_p > 0 \end{cases} \tag{9.6}$$

This relation shows that when the measured value is less than the setpoint, full controller output results. When it is more than the setpoint, the controller output is zero. A space heater is a common example. If the temperature drops below a setpoint, the heater is turned ON. If above the setpoint, it turns OFF.

Neutral Zone In virtually any practical implementation of the two-position controller, there is an overlap as e_p increases through zero or decreases through zero. In this span, no change in controller output occurs. This is best shown in Figure 9.3, which plots p versus e_p for a two-position controller. We see that until an increasing error changes by Δe_p above zero, the controller output will not change state. In decreasing, it must fall Δe_p below zero before the controller changes to the 0% rating. The range 2 Δe_p, which is referred to as the *neutral zone* or *differential gap*, is often purposely designed above a certain minimum quantity to prevent excessive cycling. The existence of such a neutral zone is an example of desirable hysteresis in a system.

FIGURE 9.3
Two-position controller action with neutral zone.



EXAMPLE 9.3 A liquid-level control system linearly converts a displacement of 2–3 m into a 4–20-1 mA control signal. A relay serves as the two-position controller to open or close an inlet valve. The relay closes at 12 mA and opens at 10 mA. Find (a) the relation between displacement level and current, and (b) the neutral zone or displacement gap in meters.

Solution

a. The relation between level and a current is a linear equation such as

$$H = KI + H_0$$

We find K and H_0 by writing two equations:

$$2 \text{ m} = K(4 \text{ mA}) + H_0$$
$$3 \text{ m} = K(20 \text{ mA}) + H_0$$

Solving these simultaneous equations yields $K = 0.0625$ m/mA and $H_0 = 1.75$ m, at the intersection of the linear relations.

b. The relay closes at 12 mA, which is a high level H_H of

$$H_H = (0.0625 \text{ m/mA})(12 \text{ mA}) + 1.75 \text{ m}$$
$$H_H = 2.5 \text{ m}$$

The low level H_L occurs at 10 mA, which is

$$H_L = (0.0625 \text{ m/mA})(10 \text{ mA}) + 1.75 \text{ m}$$
$$H_L = 2.375 \text{ m}$$

Thus, the neutral zone is $H_H - H_L = (2.5 - 2.375)$ m or **0.125 m.**

Applications Generally, the two-position control mode is best adapted to large-scale systems with relatively slow process rates. Thus, in the example of either a room heat-

usually requires a more complicated final control element, because it must have more than two settings. Figure 9.6 shows a graph of dynamic variable and final control element setting versus time for a hypothetical case of three-position control. Note the change in control element setting as the variable changes about the two trip points. On this graph the finite time required for the final control element to change from one position to another also is shown. Notice the overshoot and undershoot of the error around the upper and lower set-points. This is due to both the process lag time and the controller lag time, indicated by the finite time required for the control element to reach a new setting.

9.4.3 Floating-Control Mode

In the two previous modes of controller action, the output was uniquely determined by the magnitude of the error input. If the error exceeded some preset limit, the output was changed to a new setting as quickly as possible. In floating control, the specific output of the controller is not uniquely determined by the error. If the error is zero, the output does not change but remains (floats) at whatever setting it was when the error went to zero. When the error moves off zero, the controller output again begins to change. Actually, as with the two-position mode, there is typically a neutral zone around zero error where no change in controller position occurs.

Single Speed In the single-speed floating-control mode, the output of the control element changes at a fixed rate when the error exceeds the neutral zone. An equation for this action is

$$\frac{dp}{dt} = \pm K_F \qquad |e_p| > \Delta e_p \tag{9.11}$$

where $\frac{dp}{dt}$ = rate of change of controller output with time
K_F = rate constant (%/s)
Δe_p = half the neutral zone

If Equation (9.11) is integrated for the actual controller output, we get

$$p = \pm K_F t + p(0) \qquad |e_p| > \Delta e_p \tag{9.12}$$

where $p(0)$ = controller output at $t = 0$

which shows that the present output depends on the time history of errors that have previously occurred. Because such a history usually is not known, the actual value of p floats at an *undetermined* value. If the deviation persists, then Equation (9.12) shows that the controller saturates at either 100% or 0% and remains there until an error drives it toward the opposite extreme. A graph of single-speed floating control is shown in Figure 9.7a.

EXAMPLE 9.5 Suppose a process error lies within the neutral zone with $p = 25\%$. At $t = 0$, the error falls below the neutral zone. If $K = +2\%$ per second, find the time when the output saturates.

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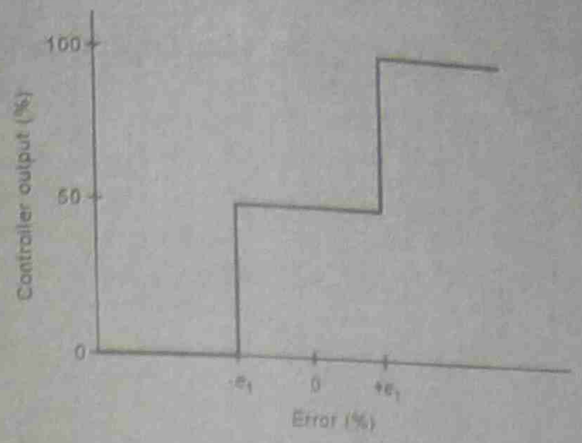

FIGURE 9.5
Three-position controller action.

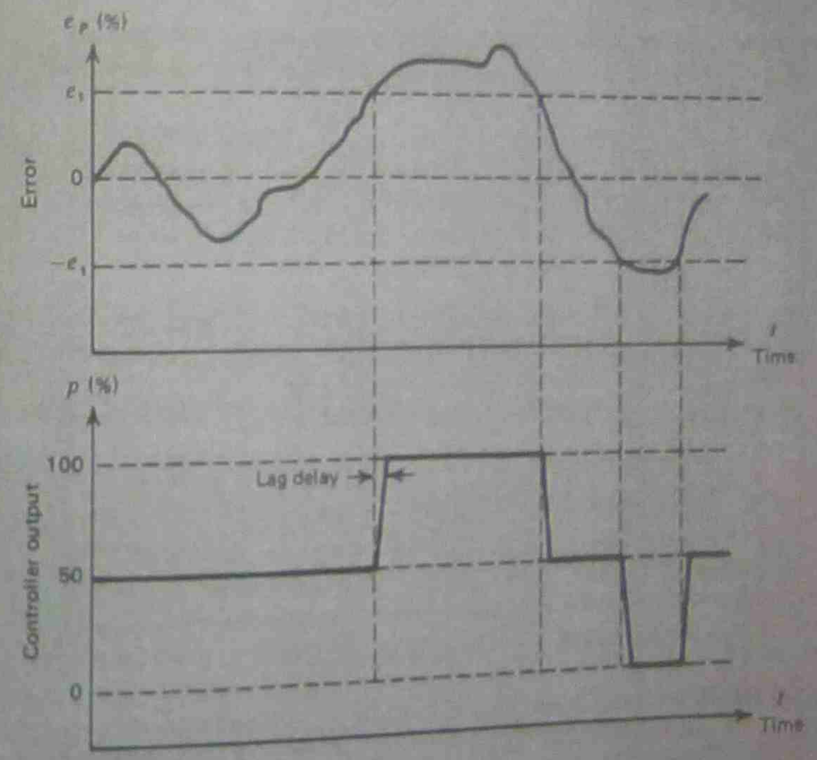

FIGURE 9.6
Relationship between error and three-position controller action, including the effects of lag

EXAMPLE 9.10 Let us combine everything and see how the error of Figure 9.22a produces an output in the three-mode controller with $K_P = 5$, $K_I = 0.7\ s^{-1}$, $K_D = 0.5$ s, and $p_I(0) = 20\%$. Draw a plot of the controller output.

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Solution
From Figure 9.22a, the error can be expressed as follows:

$$\begin{aligned} &0\text{–}1\ \text{s} \qquad e_p = t\% \\ &1\text{–}3\ \text{s} \qquad e_p = 1\% \\ &3\text{–}5\ \text{s} \qquad e_p = -\frac{1}{2}t + 2.5\% \end{aligned}$$

We must apply each of these spans to the three-mode equation for controller output:

$$p = K_P e_p + K_P K_I \int_0^t e_p\, dt + K_P K_D \frac{de_p}{dt} + p_I(0)$$

or

$$p = 5e_p + 3.5 \int_0^t e_p\, dt + 2.5 \frac{de_p}{dt} + 20$$

From 0–1 s, we have

$$p_1 = 5t + 3.5 \int_0^t t\, dt + 2.5 + 20$$

or

$$p_1 = 5t + 1.75t^2 + 22.5$$

This is plotted in Figure 9.24 in the span of 0–1 s. At the end of 1 s, the integral term has accumulated to $p_I(1) = 21.75\%$. Now, from 1–3 s, we have

$$p_2 = 5 + 3.5 \int_0^t (1)\, dt + 21.75$$

or

$$p_2 = 3.5(t - 1) + 26.75$$

This controller variation is shown in Figure 9.24 from 1–3 s. At the end of 3 s, the integral term has accumulated to a value of $p_I(3) = 28.75\%$. Finally, from 3–5 s, we have

$$p_3 = 5\left(-\frac{1}{2}t + 2.5\right) + 3.5 \int_3^t \left(-\frac{1}{2}t + 2.5\right) dt - \frac{2.5}{2} + 28.75$$

or

$$p_3 = -0.875t^2 + 6.25t + 21.625$$

This is plotted in Figure 9.24 from 3–5 s. After 5 s, the error is zero. Therefore, the output will simply be the accumulated integral response providing a constant output of $p_I = 32.25\%$.

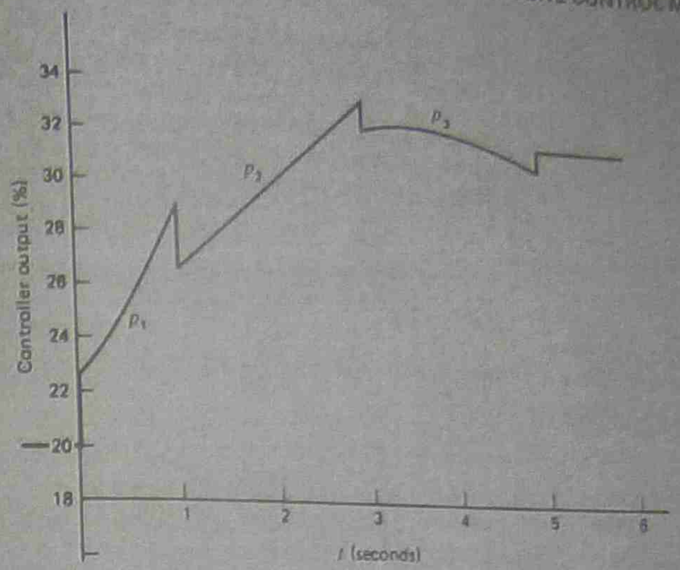

FIGURE 9.24
Solution for Example 9.10.

The examples used in this chapter are idealized in terms of the sudden way that errors change. In the real world, changes are not instantaneous, and therefore the sharp breaks in output, such as those shown in Figure 9.24, do not occur.

9.6.4 Special Terminology

There are a number of special terms used in process control for discussing the controller modes. The following summary defines some of these terms and shows how they relate to the equations presented in this chapter.

1. *Proportional band (PB)* Although this term was defined earlier, let us note again that this is the percentage error that results in a 100% change in controller output.
2. *Repeats per minute* This term is another expression of the integral gain for PI and PID controller modes. The term derives from the observation that the integral gain K_I has the effect of causing the controller output to change every unit time by the proportional mode amount. You also can see this by taking the derivative of the integral term in the controller equation. This gives a change in controller output Δp of

$$\Delta p = K_I K_P e_p\, \Delta t$$

Because $K_P e_p$ is just the proportional contribution, in a unit time interval $\Delta t = 1$, K_I just repeats the proportional term. For example, if $e_p = 0.5\%$ and $K_P = 10\%$,

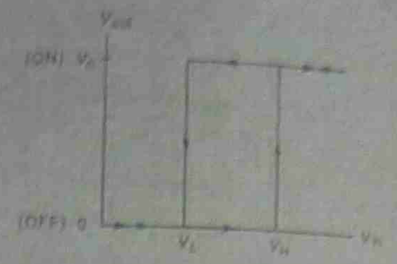

FIGURE 10.6
The circuit of Figure 10.5 shows the characteristic two-position response in terms of voltage.

switches states when the voltage on its input V_1 is equal to the setpoint value V_{sp}. Analysis of this circuit shows that the high (ON) switch voltage is

$$V_H = V_{sp} \tag{10.1}$$

and the low (OFF) switching voltage is

$$V_L = V_{sp} - \frac{R_1}{R_2} V_0 \tag{10.2}$$

Figure 10.6 shows the typical two-position relationship between input and output voltage for this circuit. The width of the neutral zone between V_L and V_H can be adjusted by variation of R_2. The relative location of the neutral zone is calculated from the difference between Equations (10.1) and (10.2).

The inverter resistance in Figure 10.5 can be chosen as any convenient value. Typically, it is in the 1- to 100-kΩ range.

EXAMPLE 10.1

Level measurement in a sump tank is provided by a transducer scaled as 0.2 V/m. A pump is to be turned on by application of +5 V when the sump level exceeds 2.0 m. The pump is to be turned back off when the sump level drops to 1.5 m. Develop a two-position controller.

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Solution

Let us use the circuit of Figure 10.5. The high and low trip voltages will be determined by the conditions of the problem. From these the values of the resistances can be determined.

$$V_H = (0.2 \text{ V/m})(2.0 \text{ m}) = 0.4 \text{ V}$$

and

$$V_L = (0.2 \text{ V/m})(1.5 \text{ m}) = 0.3 \text{ V}$$

This gives the following relations for the resistances and V_{sp}.

$$0.4 \text{ V} = V_{sp}$$

$$0.3 \text{ V} = V_{sp} - \frac{R_1}{R_2} V_0$$

Therefore, $V_{sp} = 0.4$ V and, from the second equation,

$$0.3 = 0.4 - (R_1/R_2)(5)$$

$$(R_1/R_2) = 0.02$$

Since there are two unknowns and only one condition, one unknown can be selected. Picking $R_1 = 5$ kΩ, for example, means that $R_2 = 250$ kΩ.

Reverse Action The two-position controller of Figure 10.5 can be made reverse acting by placing an inverter in the feedback and reversing the comparator. It is also possible to devise other circuits that automatically incorporate the reverse action.

Three-Position Figure 10.7 shows how a simple three-position controller can be realized with op amps and comparators. In this circuit it is assumed that the output of the comparators is 0 V for the OFF state and V_0 volts for the ON state. The summing amplifier also includes a bias voltage input V_B, which allows the three-position mode response to be biased up or down in voltage to suit particular needs. The inverter is needed to convert the sign of the inverting action of the summing amplifier.

Equation (9.10) presented the generic three-mode controller response. Equations (10.3) show that the circuit of Figure 10.7 has the same basic response.

$$\begin{aligned} V_{in} < V_{sp1} \qquad & V_{out} = V_B \\ V_{sp1} < V_{in} < V_{sp2} \qquad & V_{out} = V_B + \frac{R_3}{R_1} V_0 \\ V_{in} > V_{sp2} \qquad & V_{out} = V_B + \frac{R_3}{R_1} V_0 + \frac{R_3}{R_2} V_0 \end{aligned} \tag{10.3}$$

One difference between the circuit response and that of Equation (9.10) is that the output does not have to be symmetric (e.g., 0%, 50%, and 100%). The circuit response can have asymmetric output, as Example 10.2 shows. Figure 10.8 shows the response of this circuit for the particular case $V_B = 0$.

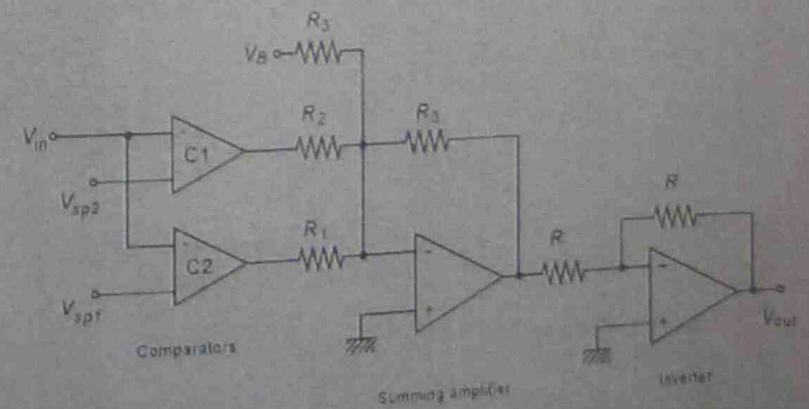

FIGURE 10.7
A three-position controller using two comparators and op amps.

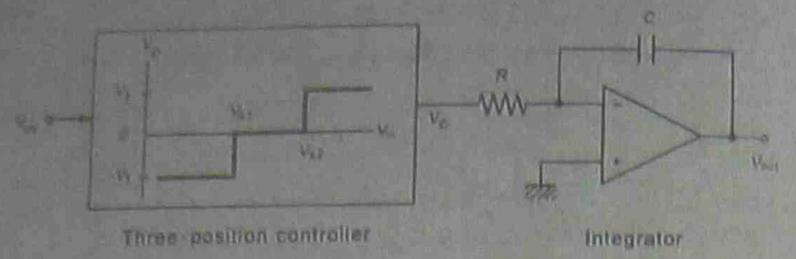

FIGURE 10.10
A floating-mode controller can be made from a three-position controller and an integrator.

cuit preceding the integrator. Remember that the output floats at whatever the latest value of output is when the input falls within the neutral zone.

EXAMPLE 10.3 A control signal varies from 0–5 V. A floating controller, such as that in Figure 10.10, has trip voltages of 2 V and 4 V, and a three-position controller has outputs of 0 and ±2 V. The integrator consists of a 1-MΩ resistor and a 1-μF capacitor. Plot the controller output in response to the input of Figure 10.11a.

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Solution

We can find the output by applying the relations of Equation (10.4) to the input voltage. The result is shown in Figure 10.11b. This result is arrived at as follows:

1. From 0–1 s, the output is zero (an assumed starting point) and remains so because the input is within the neutral zone.
2. From 1–3 s, the lower setpoint has been reached and the output is given by

$$V_{out} = -\frac{V_i}{RC}(t - 1) = -2(t - 1) \text{ volts}$$

   At $t = 3$ s, the output is −4 V.
3. From 3–4 s, the output remains at −4 because the input is in the neutral zone.
4. From 4–7 s, the input reaches the upper setpoint and the output becomes

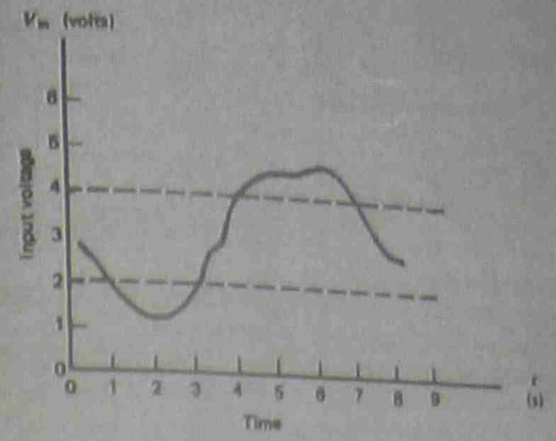
$$V_{out} = +\frac{V_i}{RC}(t - 4) - 4 \text{ V}$$

$$V_{out} = +2(t - 4) - 4 \text{ V}$$

   At $t = 7$ s, the input again falls within the neutral zone and the output becomes

$$V_{out} = +2$$

5. The output will remain at +2 until the input again hits a setpoint value.



a) Input error voltage

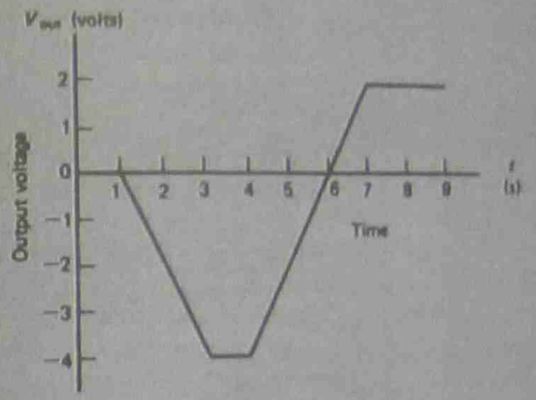


FIGURE 10.11
Input and output voltage for the floating controller of Example 10.3.

Proportional Mode Implementation of this mode requires a circuit that has a response given by

$$p = K_p e_p + p_0 \tag{9.14}$$

where p = controller output 0–100%
K_p = proportional gain
e_p = error in percent of variable range
p_0 = controller output with no error

If we consider both the controller output and error to be expressed in terms of voltage, we see that Equation (9.14) is simply a *summing amplifier*. The op amp circuit in Figure 10.12 shows such an electronic proportional controller. In this case, the analog electronic equation for the output voltage is

Generally, a voltage-to-current converter is used on the output to convert the output voltages to a 4–20-mA range of current signals to drive the final control element.

Integral Mode In the previous chapter we saw that the integral mode was characterized by an equation of the form

$$p(t) = K_I \int_0^t e_p(t)\,dt + p_I(0) \tag{9.17}$$

where $p(t)$ = controller output in percent of full scale
K_I = integration gain (s^{-1})
$e_p(t)$ = deviations in percent of full-scale variable value
$p_I(0)$ = controller output at $t = 0$

This function is easy to implement when op amps are used as the building blocks. A diagram of an integral controller is shown in Figure 10.14. The corresponding equation relating input to output is

$$V_{out} = G_I \int_0^t V_e\,dt + V_{out}(0) \tag{10.6}$$

where V_{out} = output voltage
$G_I = 1/RC$ = integration gain
V_e = error voltage
$V_{out}(0)$ = initial output voltage

The values of R and C can be adjusted to obtain the desired integration time. The initial controller output is the integrator output at $t = 0$. As we noted earlier, the integration time constant determines the *rate* at which controller output increases when the error is constant. If K_I is made too large, the output rises so fast that overshoots of the optimum setting occur and cycling is produced.

The actual value of G_I, and therefore of R and C, is determined from K_I and the input and output voltage ranges. One way to do this is to recognize that the integral gain says that an input error of 1% must produce an output that changes as K_I percent per second. Another way is to know that if an error of 1% lasts for 1 s, the output must change by K_I percent.

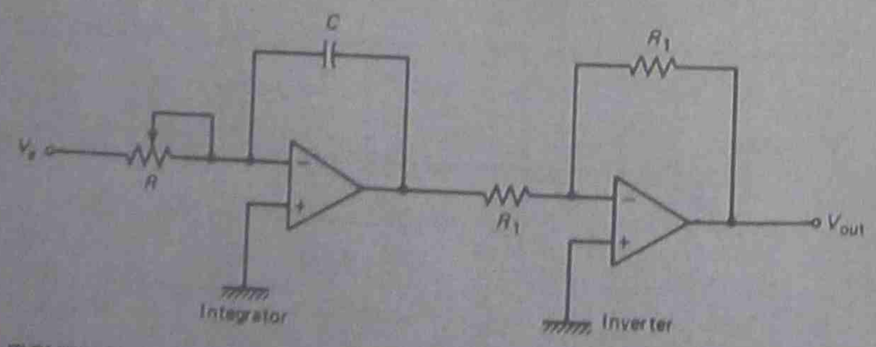


FIGURE 10.14
An op amp integral-mode controller.

Suppose we have an input range of 6 V, an output range of 5 V, and $K_I = 3.0\%/(\%\text{-min})$. Integral gain is often given in minutes because industrial processes are slow, compared to a time of seconds. This gain also is often expressed as integration time T_I, which is just the inverse of the gain, so $T_I = 3.33$ min.

We must first convert the time units to seconds. So

$$[3\%/(\%-\text{min})][1\text{ min}/60\text{ s}] = 0.05\%/(\%-\text{s})$$

An error of 1% for 1 s is found from

$$(0.01)(6\text{ V})(1\text{ s}) = 0.06\text{ V}-\text{s}$$

Furthermore, K_I percent of the output fusing the seconds expression for gain is

$$(0.0005)(5\text{ V}) = 0.0025\text{ V}$$

Therefore, the integral gain in terms of voltage must be

$$G_I = (0.0025\text{ V})/(0.06\text{ V-s}) = 0.0417\text{ s}^{-1}$$

The values of R and C can be selected from this.

EXAMPLE 10.6 An integral control system will have a measurement range of 0.4 to 2.0 V and an output range of 0 to 6.8 V. Design an op amp integral controller to implement a gain of $K_I = 4\%/(\%\text{-min})$. Specify the values of G_I, R, and C.

Solution

The input range is $2.0 - 0.4 = 1.6$ V, and the output range is 6.8 V. We must convert K_I to units of seconds

$$[4\%/(\%\text{-min})][1\text{ min}/60\text{ s}] = 0.0667\%/(\%\text{-s})$$

1% of the input for 1 s = $(0.01)(1.6\text{ V})(1\text{ s}) = 0.016\text{ V-s}$

0.0667% of the output = $(0.00067)(6.8\text{ V}) = 0.00454\text{ V}$

Thus, the gain is

$$G_I = (0.00454\text{ V}/0.016\text{ V}-\text{s}) = 0.283\text{ s}^{-1}$$

Because $G_I = 1/(R/C)$, we have

$$RC = 3.53\text{ s}$$

If we pick $C = 100\ \mu\text{F}$, then $R = \mathbf{35.3\ k\Omega}$

Derivative Mode The derivative mode is never used alone because it cannot provide a controller output when the error is zero (see Section 9.3.6). Nevertheless, we show here how it is implemented with op amps so it can be combined with other modes in the next section. The control mode equation was given earlier as

$$p(t) = K_D \frac{de_p}{dt} \tag{9.18}$$

A portable data logger has many of the same features as a fixed installation, such as a DAS computer, or an operator's console. Of course, some portable data loggers do not use a computer; they are merely strip-chart recorders or magnetic tape recorders.

In general, the computer-based portable data loggers have some mechanism for storing the logged data for later analysis. Possible recording media are

1. Printed output
2. Digitized strip-chart recording
3. Magnetic tape
4. Magnetic floppy disks
5. Networked data communication

In network data communications, the data logger may be connected to a local area network (LAN). The data then can be transmitted over the network to another, fixed computer installation. The data can be stored on mass-storage facilities that are part of the network.

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11.3.3 Supervisory Control

A natural extension of a computer data-logging system involves computer feedback on the process through automatic adjustment of loop setpoints. As various loads in a process change, it is often advantageous to alter setpoints in certain loops to increase efficiency or to maintain the operation within certain precalculated limits. In general, the choice of setpoint is a function of many other parameters in the process. In fact, a decision to alter one setpoint may necessitate the alteration of many other loop setpoints as interactive effects are taken into account. Given the number of loops, interactions, and calculations required in such decisions, it is more natural and expedient to let a computer perform these operations under program control.

Such a system is represented in Figure 11.6, where the effect is shown by the addition of a data-output system (DOS). Such a system assumes the controllers of analog loops have been designed to accept setpoint values as some properly scaled voltage. By proper switch addressing, the computer then outputs a signal through the multiplexer and DACs representing a new setpoint to a controller connected to that output line.

It might be helpful in understanding use of this type of control if a hypothetical example is given. Study for a moment the reaction process shown in Figure 11.7. The process specifications are

1. Reactants *A* and *B* combine such that one part *A* to two parts *B* produces one part *C*.
2. Volume production of *C* varies as the square root of the *A* and *B* flow rate product.
3. The operating temperature must be linearly decreased with *C* volume production rate.
4. For stability, the reaction must occur with the pressure maintained below a critical value.

A decision is made to increase production in this operation. The first step to accomplish this is to increase the flow rate of *A* by a change in setpoint. Let us see the consequences of this in the rest of the process.

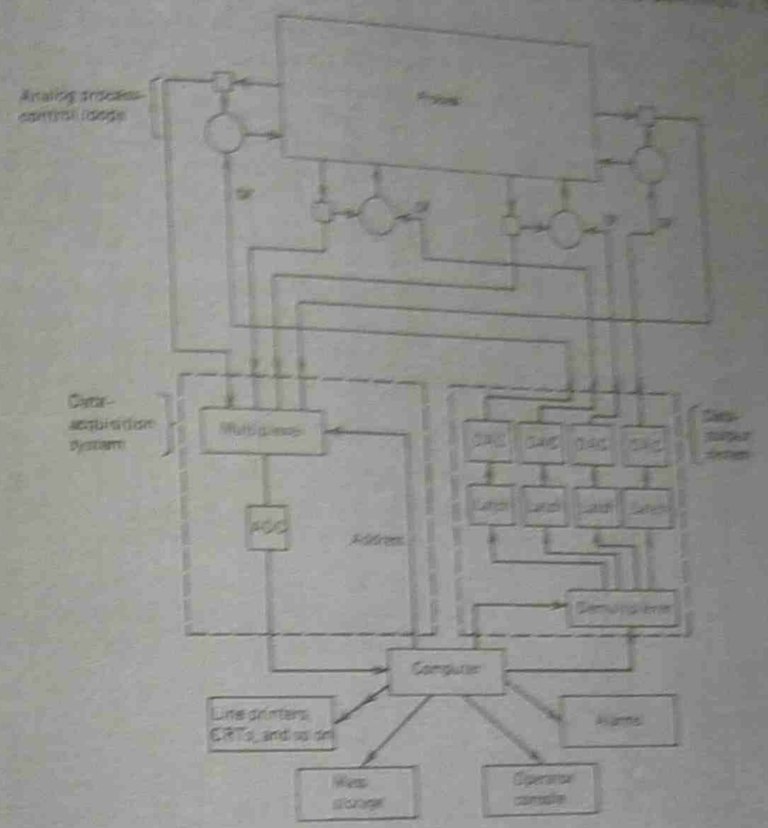

FIGURE 11.6
In computer supervisory control, the computer monitors measurements and outputs the loop setpoints.

1. The setpoint of *B* flow must be set to twice the *A* setpoint, keeping pressure below P_{max}.
2. The setpoint of *C* flow must be increased by the square root of the new *A* and *B* flow rates.
3. The temperature setpoint must be decreased by a proportion of the new *C* setpoint.

To accomplish this change in a purely analog system requires monitoring pressure constantly while the operations of the three steps are gradually performed and the new production rate is finally established. With each new setting of setpoint, we must wait until all parameters have adopted the new setpoints and wait for a safe pressure. To perform this manually requires constant human monitoring as the adjustments are made. In a supervisory control system, the computer performs these operations automatically while still performing other activities in the production.

A certain length of time is allotted for the converter to indicate that the conversion is complete. If such a response is not issued within this time, an error routine is initiated. Such a routine is necessary in case the ADC has failed and is unable to respond with a data sample. Without the error escape, the computer and control system would be "locked up" in a wait loop waiting for the EOC response from the ADC.

When a higher-level language is used, it is necessary that special software be provided showing the user how to construct the data-input process. This is often done with user-defined functions (UDFs). In these cases the program sequence of Figure 11.18 would be written in assembly language and called up from the higher-level language with some command such as Y = UDF(1).

Data-Acquisition Systems It is now common to employ a complete data-acquisition system on a printed circuit board that interfaces easily with popular computer systems, such as the PC or Macintosh computers. These boards typically provide for analog-to-digital conversion of input signals, digital-to-analog conversion of computer outputs, and a number of digital input and output lines.

A particularly significant feature of such systems is that they come with software routines for performing input and output using the on-board data input and output devices. This greatly simplifies the work of the software design, since data input and output are facilitated by calls to one of these routines.

Linearization In many cases the input binary number and the controlled variable are not linearly related. In such cases it is necessary to execute a program that will linearize the binary number so that it is proportional to the controlled variable value. There are two common approaches: equation inversion and table look-up.

Linearization by Equation When an equation is known that relates the value of the controlled variable and the binary number in the computer, an equation can be developed to determine the linearized value of the variable. For example, suppose that a transducer outputs a voltage related to pressure by

$$V = K[p]^{1/2} \tag{11.7}$$

The voltage is converted to a binary number DV by an ADC. Then it is also true that the binary number and pressure are still related by the square root:

$$DV \text{ varies as } [p]^{1/2} \tag{11.8}$$

What we want is a binary number that is linearly related to pressure. The way to get this is to square DV:

$$DP = DV * DV \text{ varies as } p \tag{11.9}$$

Thus, the program would input a sample DV and multiply it by itself. The resulting number would be linearly related to the pressure. Of course, there may have to be scale shifts and offsets before we have a number equal to the pressure, as the following example shows.

EXAMPLE 11.8 Pressure from 50 to 400 psi is converted to voltage by the relation

$$V = 0.385[p]^{1/2} - 2.722$$

This is input to an ADC with a 5.0-V reference, which provides 00H to FFH over the pressure range. An HLL uses an instruction $DV = UDF(1)$ to input the data from the ADC as a base 10 number DV that varies from 0 to 255 over the pressure range. Develop a linearization equation to give a quantity p in the program that is equal to the actual pressure.

Solution

We have enough information to work backward through the ADC, signal conditioning, and measurement. Thus, we know the voltage is related to DV by the ADC transformation

$$DV = (V/V_{ref})256 = (V/5)256$$

or

$$V = (5/256)\,DV$$

Then, using the known relation for V in terms of p,

$$0.385[p]^{1/2} - 2.722 = (5/256)\,DV$$

and solving for p,

$$p = (0.0507\,DV + 7.071)^2$$

This number p in the program is equal to the actual pressure value, 50 to 400 psi.

Linearization by Table Look-up There are many measurement processes where it is impossible to find a simple equation such as Equation (11.7) to relate the controlled variable and the binary number. Also, when the program must be written in assembly language, it may be difficult to evaluate even simple equations such as that of Example 11.8. In these cases it becomes much easier to use the look-up table approach.

This is really just what we humans do when we use thermocouple tables, for example. We measure the voltage, go to a table, and look up the temperature, and sometimes interpolate. It is the exact same thing with the software approach. The table of input values and corresponding physical variable values are stored in a table in memory. Following a measurement, the input value is looked up in the table and the correct measured value found.

Figure 11.19 shows a software approach for table look-up in flowchart format. It is assumed that the input values are stored in ascending value in N memory locations and the corresponding physical variable values are stored in the following N locations. Thus, if the input is found at the Ith memory location from the start of the table, then the actual variable value is found at the $N + I$ location.

Of course, there are many other methods of table construction and search. In many cases it is necessary to write interpolation routines using programming equations such as Equation (4.14) to refine the values between table values.

ence. The general intent of this chapter is merely to make the reader familiar with the general concepts.

The objectives of this chapter have been carefully selected to provide the reader with as much background as can be expected without a more extensive mathematical background. The emphasis is not on the theoretical development of process-control loop characteristics but on the interpretation and employment of the results of such theory. Many volumes are devoted to studies of loop stability, optimization techniques, multiloop DDC operations theory, and a host of other topics that could never be included in this brief chapter. In short, this chapter provides the process-control technologist with a general background in the practical considerations of process-control loop implementations.

12.2 CONTROL SYSTEM CONFIGURATIONS

We consider first the types of control-loop configurations that are encountered in typical industrial applications. The control-loop concept, as outlined in the previous chapters, is a correct one, but necessarily oversimplifies many of the actual configurations used in industry. The decision of which arrangement to select is made by the process designers based on the goal of the process relative to production requirements, and the physical characteristics of operations under control.

12.2.1 Single Variable

The elementary process-control loop (which has formed the basis for most of our discussion in previous chapters) is a *single-variable* loop. The loop is designed to maintain control of a given process variable by manipulation of a controlling variable, regardless of the other process parameters.

Independent Single Variable In many process-control applications, certain regulations are required regardless of other parameters in the process. In these cases a setpoint is established, controller action is started, and the system is left alone. Thus, in Figure 12.1, a flow-control system is used to regulate flow into a tank at a fixed rate determined by the setpoint. This system then makes adjustments in valve positions as necessary following a load change to maintain flow rate at the setpoint value.

Interactive Single Variable A second single-variable control loop, shown in Figure 12.1, regulates the temperature of liquid in the tank by adjustment of heat input. This also is a single-variable loop that maintains the liquid temperature at the setpoint value. Under nominal conditions, the flow into the tank is held constant and the temperature is also held constant, both at their respective setpoint values. Note, however, that a change in the setpoint of the flow-control system appears as a load change to the temperature-control system, because the fluid level in the tank or rate of passage through the tank must change. The temperature system now responds by resetting the heat flux to accommodate the new load and bring the temperature back to the setpoint.

We say, then, that these two loops *interact*. Almost any process where several variables are under control shows such interactive behavior. Any cycling or other instability of the flow-control loop causes cycling in the temperature system because of this interaction.

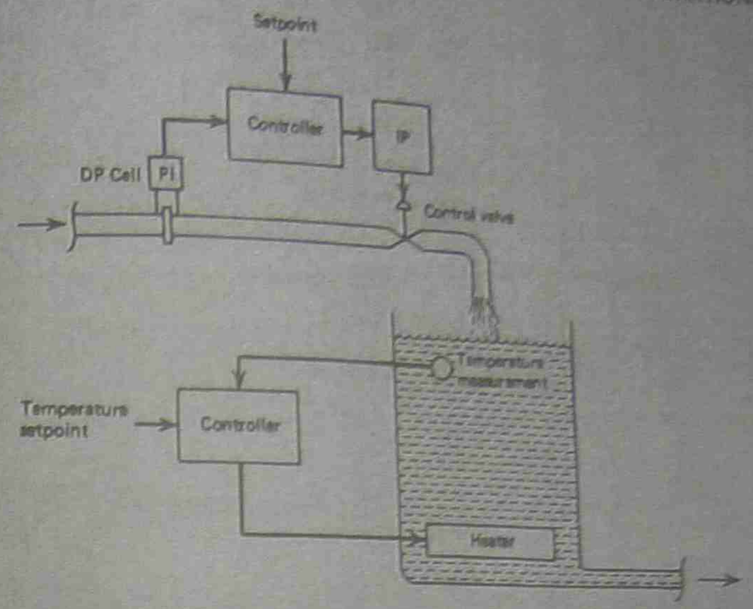

FIGURE 12.1
Two variable process-control loops that interact.

Compound Variable In some cases, a single process-control loop is used to provide control of the relationship between two or more variables. This can be accomplished by using measurements from, say, two sensors as input to the process controller. A signal conditioning system must scale the two measurements and add them prior to input to the controller for evaluation and action. The analysis of such systems can become quite complicated.

A common example is when the ratio of two reactants must be controlled. In this case, one of the flow rates is measured but allowed to float (that is, not regulated), and the other is both measured and adjusted to provide the specified constant ratio. An example of this system is shown in Figure 12.2. The flow rate of reactant *A* is measured and added, with appropriate scaling, to the measurement of flow rate *B*. The controller reacts to the resulting input signal by adjustment of the control valve in the reactant *B* input line.

EXAMPLE 12.1 In a compound control system, the ratio between two variables is to be maintained at 3.5 to 1. If each has been converted to a 0–5 V range signal, devise a signal conditioning system that will output a zero signal to the controller when the ratio is correct.

Solution
In this case, we use a summing amplifier (Section 2.5.2). Then the output is related to the input by

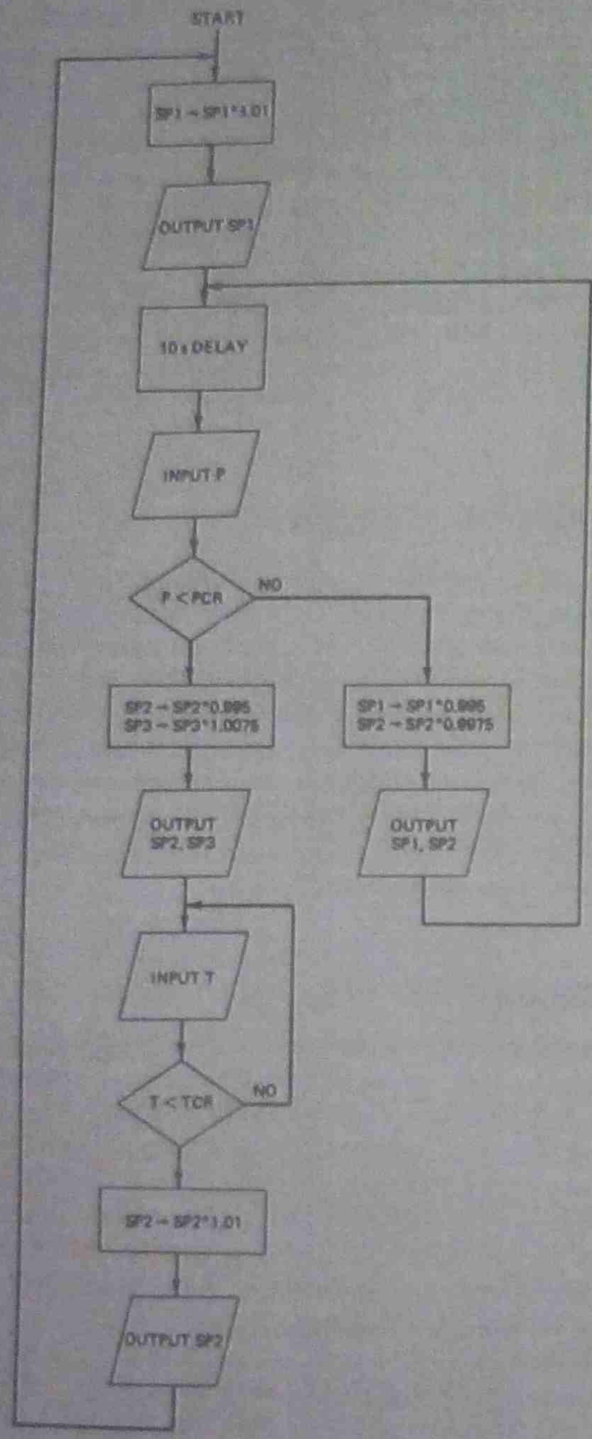

FIGURE 12.6
Flowchart for an interactive control problem.

12.3.2 Supervisory and Direct Digital Control

The computer is ideally suited to the type of control problem presented by the multivariable control system. The computer can make any necessary adjustments of system operating points in an incremental fashion, according to a predetermined sequence, while monitoring process parameters for interactive effects. The problem in such a system is determining the algorithm that the computer must follow to provide the control function of the setpoint-change sequence. In some cases, control equations are used. Usually, in complex interactions, these relations are not analytically known. In some cases *self-adaptive* algorithms are used, causing the computer to sequence through a set of operations and letting the result of one operation determine the next operation. As an example, if the temperature is slightly raised and the pressure rises, then drop the temperature, and so on. The computer can sequence through a myriad of such microadjustments of setpoints looking for the optimum adjustment path.

EXAMPLE 12.2

A process requires adjustment of setpoints to increase production. A particular sequence must be followed to provide the increase. SP1, SP2, and SP3 are the setpoints, P and PCR are the pressure and a critical pressure, respectively, and T and TCR are the temperature and critical temperature, respectively. Develop a flowchart that increases the setpoints as follows:

1. Increase SP1 by 1%.
2. Wait 10 s, test for pressure compared to critical.
3. If the pressure is less than critical, then
   a. decrease SP2 by $\frac{1}{2}$%
   b. increase SP3 by $\frac{3}{4}$%
   c. wait for T < TCR
   d. increase SP2 by 1%
   e. go to step 1
4. If the pressure is above critical,
   a. decrease SP1 by $\frac{1}{2}$%
   b. decrease SP2 by $\frac{1}{4}$%
   c. go to step 2

Solution

To implement this with either DDC or supervisory systems requires a flowchart from which the program instructions are developed. Such a flowchart is shown in Figure 12.6. The computer sequences through these steps in an optimum fashion according to the stipulations of the problem. If the sequence were performed manually, several operators would be required to monitor equipment and make necessary adjustments.

12.4 CONTROL SYSTEM QUALITY

When a manufacturing concept is to be implemented, the ultimate goal is to develop a product that satisfies present "design" criteria. If the product is crackers, they should have a certain color, flavor, salinity, size, and so forth. If it is gasoline, it should possess certain octane

duration is not a minimum in such a case. For that matter, the deviation itself usually is not a minimum either. Such a response is safe, however, in ensuring that no instabilities occur and that certain maximum deviations never occur.

Critically Damped Careful adjustment of the process-control loop brings about curve *B* of Figure 12.9. In this case, the duration is a minimum for a noncycling response. This is the optimum response for a condition where no overshoot or undershoot is desired in a setpoint change or no cycling, in general, is desired.

Underdamped The natural result of further adjustments of the process-control loop is cyclic response, where the deviation executes a number of oscillations about the setpoint. This is shown in Curve *C* of Figure 12.9. It is possible that this response gives minimum deviation and minimum duration in some cases. If the cycling can be tolerated, then such a response is preferred.

Two specialized measures of control are used when none of these conditions serves to define the measure of control desired in a process.

Quarter Amplitude When a process-control loop has a damped cyclic response to a disturbance, a criterion is sometimes used that is neither minimum deviation nor minimum duration. This measure of quality is found by adjusting the loop until the deviation from a disturbance is such that each deviation peak is down to one quarter of the preceding peak, as shown in Figure 12.10. In this case, the actual magnitude of the deviation is not included in the measure, nor is the time between each peak. In this sense, neither duration nor magnitude of the deviation is directly involved in a quarter-amplitude criterion.

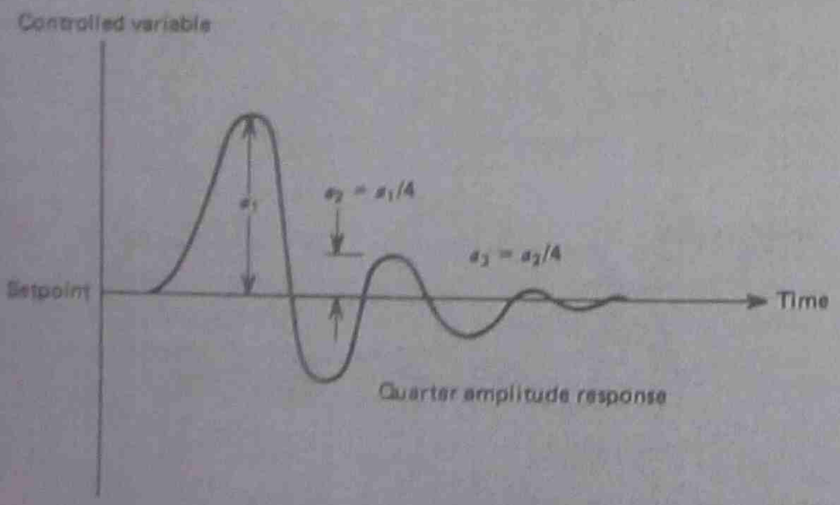

FIGURE 12.10
In one type of cyclic response, the system is adjusted to make each peak down to one quarter of the previous peak.

EXAMPLE 12.3 Suppose the deviation following a step-function disturbance is a 4.7% error in the controlled variable. If a quarter-amplitude criterion is used to evaluate the response, find the error of the second and third peak.

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Solution

The amplitude of each peak must be one quarter of the previous peak. In this case, the second peak is 4.7%/4, or 1.18%. The third peak is an error of 1.18%/4, or 0.30%.

Minimum Area In cases of cyclic or underdamped response, the most critical element is sometimes a combination of duration and deviation, which must be minimized. Thus, if minimum deviation occurs at one loop setting and minimum duration at another, then neither is optimum. One type of optimum measure of quality in these cases is to minimize the net area of the deviation as a function of time. In Figure 12.11 this is shown as the sum of the shaded areas. Analytically, this can be expressed as

$$A = \int |r - b| dt \tag{12.1}$$

where A = area of deviation
b = measured value
r = setpoint value

Another way of representing this is to use the full-scale percent error (Section 9.3.1) to write

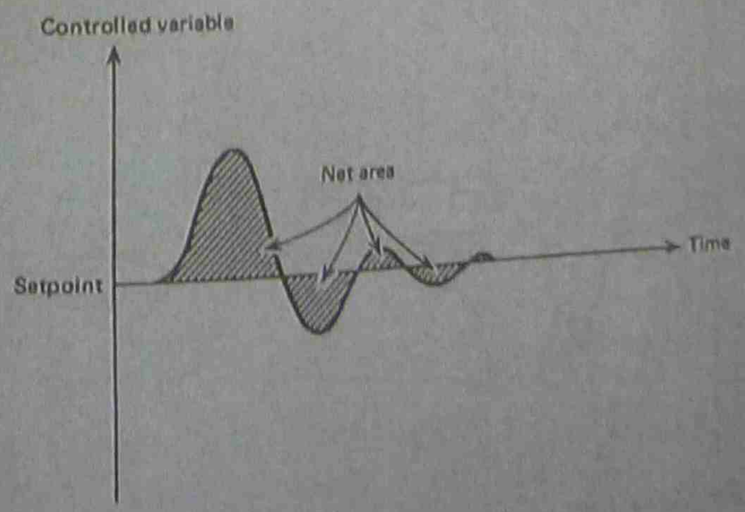
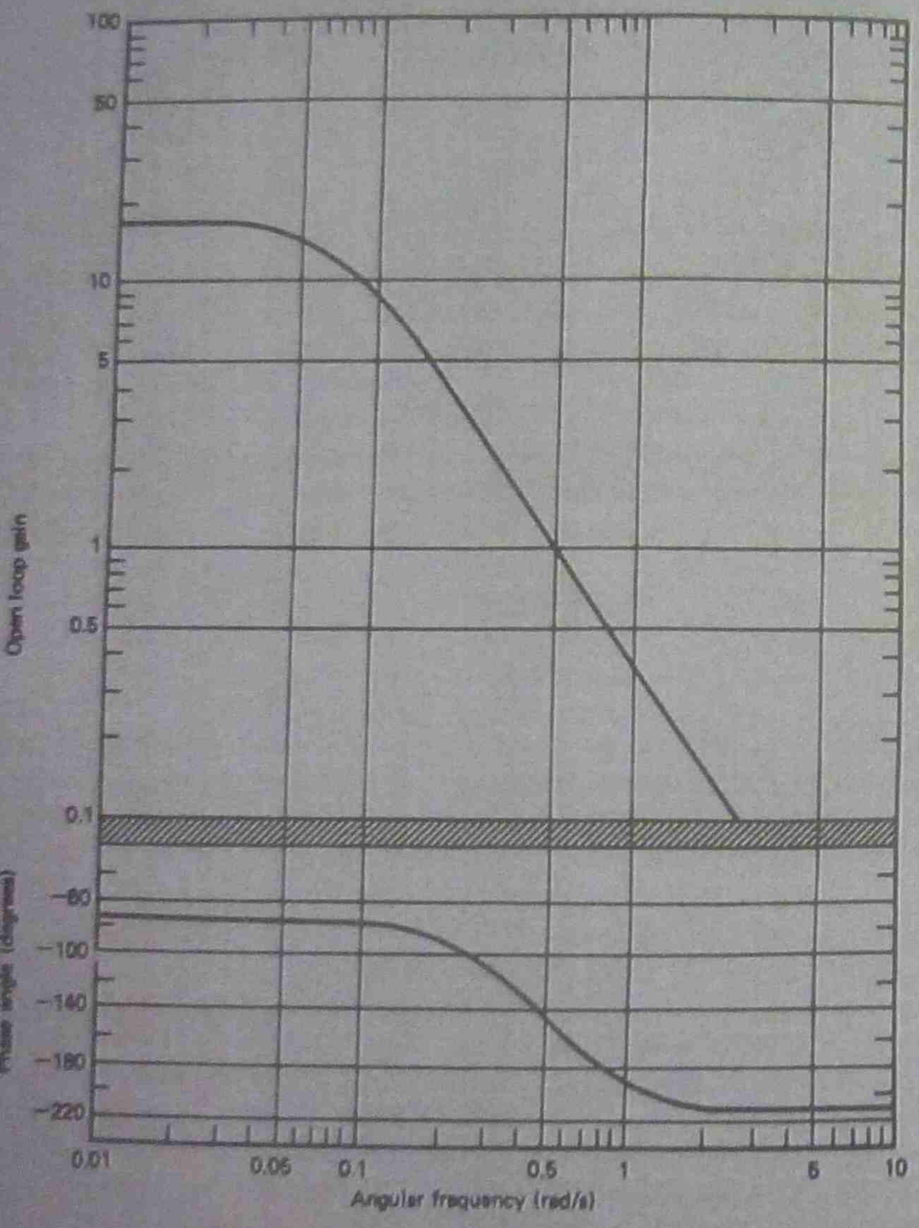


FIGURE 12.11
The minimum-area tuning response characteristics.

amplified by 2 and phase-shifted by 180° again becoming b_3 and then e_4. Thus, you can see that the error is actually growing! The control system is forcing the oscillating error to increase, instant by instant. Of course, in actuality, it happens smoothly, and the output would look something like Figure 12.8, for oscillating growth. If there is *any* frequency where such conditions for growth exist, the system is unstable and something like random noise will eventually set the system into growing oscillation. When a process-control installation is designed, one has the objective of regulating the controlled variable *without* instability



IGURE 12.16
Bode plot shows the open-loop gain and phase versus frequency of an applied sinusoid

in the loop. Stability can be ensured by designing the controller gains so that oscillation growth is never favorable, according to certain criteria.

12.5.2 Stability Criteria

We can derive stability criteria by determining just what conditions of system gain and phase lag can lead to an enhancement of error. We are led to two conclusions:

1. The gain must be greater than 1.
2. The phase shift must be −180° (lag).

Thus, if there is any frequency for which the gain is greater than one and the phase is −180°, then the system is unstable. From this argument, we develop two ways of specifying when a system is *stable*. These rules are as follows:

- **Rule 1** A system is stable if the phase lag is less than 180° at the frequency for which the gain is unity (one).
- **Rule 2** A system is stable if the gain is less than one (unity) at the frequency for which the phase lag is 180°.

The application of these rules to an actual process requires evaluation of the gain and phase shift of the system for all frequencies to see if Rules 1 and 2 are satisfied. This is easier to do if a plot of gain and phase versus frequency is used.

Bode Plot A particular type of graph, called a *Bode plot* or diagram, normally is used to plot the gain and phase of the control system versus frequency. In this case, the frequency is plotted along the abscissa (horizontal) on a log scale and expressed commonly as rad/s. Remember, if you want to know the actual frequency in Hz, you just divide the rad/s by 2π. Use of a log scale permits a large range of frequency to be displayed. The ordinate actually consists of two parts: Gain is plotted on a log scale in one part, and phase is plotted linearly as degrees in another part. Figure 12.16 shows the Bode plot with a gain/phase plotted for an example. We have labeled the gain as *open-loop gain*, since we do not include the effect of actually feeding that signal back into the system.

EXAMPLE 12.5 Determine whether the system of Figure 12.16 is stable according to the rules given earlier.

Solution

By Rule 1, we find the frequency for which the gain is unity (1). The system is stable if the phase lag is less than 180°. Unity gain occurs at 0.5 rad/s; this is $0.5/2\pi = 0.08$ Hz, or about five cycles per minute. Such frequencies are not uncommon in many industrial processes. For this frequency, the phase is about −140°, so the lag is less than 180°, and the system is **stable** under Rule 1. For Rule 2, we find the frequency for which the phase lag is 180° and check the gain, which should be less than one for stability. The phase lag is 180° at 0.8 rad/s, for which the gain is 0.5. Thus, the system is **stable** by Rule 2.

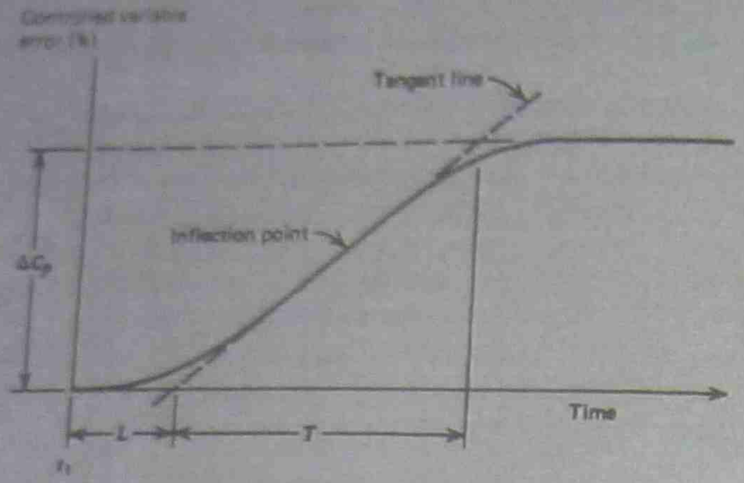


FIGURE 12.17
Process-reaction graph for loop tuning.

Proportional-Integral Mode When the controller mode is proportional-integral, the appropriate settings for proportional gain and integration time are

$$\left.\begin{aligned} K_P &= 0.9\frac{\Delta P}{NL} \\ 1/K_I = T_I &= 3.33L \end{aligned}\right\} \tag{12.12}$$

If the quarter-amplitude criterion is used, the gain is

$$\left.\begin{aligned} K_P &= \frac{\Delta P}{NL}\left[0.9 + \frac{1}{12}R\right] \\ T_I &= \left[\frac{30 + 3R}{9 + 20R}\right]L \end{aligned}\right\} \tag{12.13}$$

Three Mode For the three-mode controller, we find the appropriate proportional gain, integration time, and derivative time from

$$\left.\begin{aligned} K_P &= 1.2\frac{\Delta P}{NL} \\ T_I &= 2L \\ T_D &= 0.5L \end{aligned}\right\} \tag{12.14}$$

If the quarter-amplitude criterion is used, these equations are corrected by

$$\left.\begin{aligned} K_P &= \frac{P}{NL}(1.33 + R/4) \\ T_I &= 1\left[\frac{32 + 6R}{13 + 8R}\right]L \\ T_D &= L\left(\frac{4}{11 + 2R}\right) \end{aligned}\right\} \tag{12.15}$$

EXAMPLE 12.6 A transient disturbance test is run on a process loop. The results of a 9% controlling variable change give a process-reaction graph as shown in Figure 12.18. Find settings for three-mode action.

Solution

By drawing the inflection point tangent on the graph, we find a lag $L = 2.4$ min and a process reaction time of 4.8 min. The reaction rate is

$$N = \frac{\Delta C_P}{T} = \frac{3.9\%}{4.8\text{ min}} = 0.8125\%/\text{min}$$

The controller settings are found from Equations (12.14):

$$K_P = 1.2\frac{\Delta P}{NL} = 1.2\frac{9\%}{(0.8125)(2.4)}, \qquad \text{yielding } K_P = 5.54$$

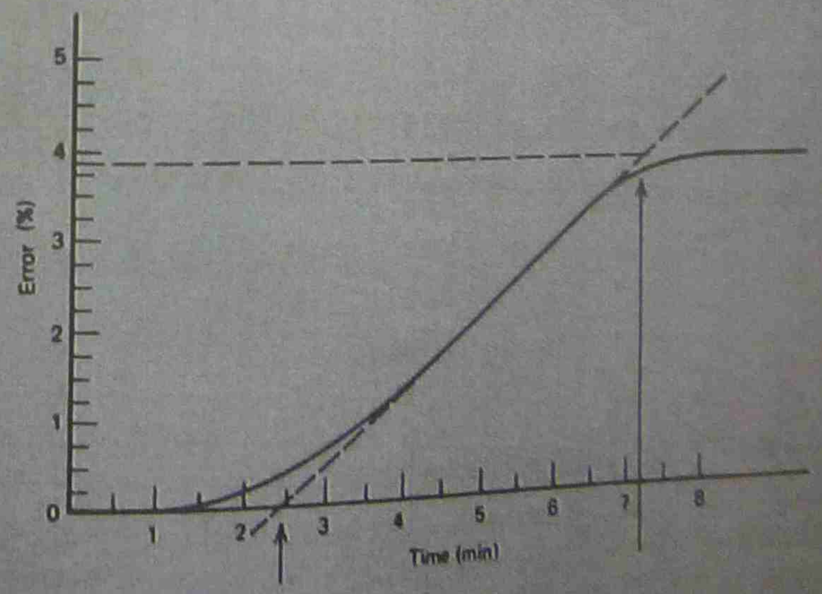


FIGURE 12.18
Process-reaction graph for Example 12.7.

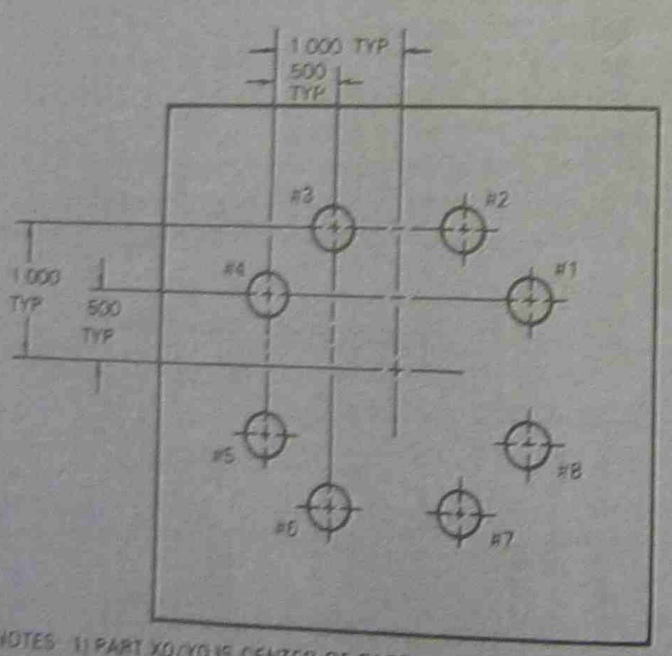
5–2, try to arrive at the coordinates to drill this part on an absolute positioning machine. The proper coordinates are as follows:

#1	X1.0000, Y0.5000	#5	X–1.0000, Y–0.5000
#2	X0.5000, Y1.0000	#6	X–0.5000, Y–1.0000
#3	X–0.5000, Y1.0000	#7	X0.5000, Y–1.0000
#4	X–1.0000, Y0.5000	#8	X1.0000, Y–0.5000

The same principles apply to the parts in Figures 5–1 and 5–2. The difference is that X0/Y0 is located at the center of the part in Figure 5–2. Notice that the signs of X and Y change as the coordinate locations move from quadrant to quadrant.

Figure 5–3 shows the same part as that in Figure 5–1 but delta dimensioned rather than datum dimensioned. Try to derive the proper coordinates to drill the holes in Figure 5–3, using an incremental positioning machine. The coordinates for the holes are as follows:

#1	X0.7500, Y1.7500
#2	X1.2500, Y–1.5000
#3	X1.0000, Y0.7500



NOTES 1) PART X0/Y0 IS CENTER OF PART
2) FOR INCREMENTAL MOVES, THE SPINDLE IS ASSUMED CENTERED OVER X0/Y0 AT THE START OF PROGRAM SEQUENCE

FIGURE 5–2

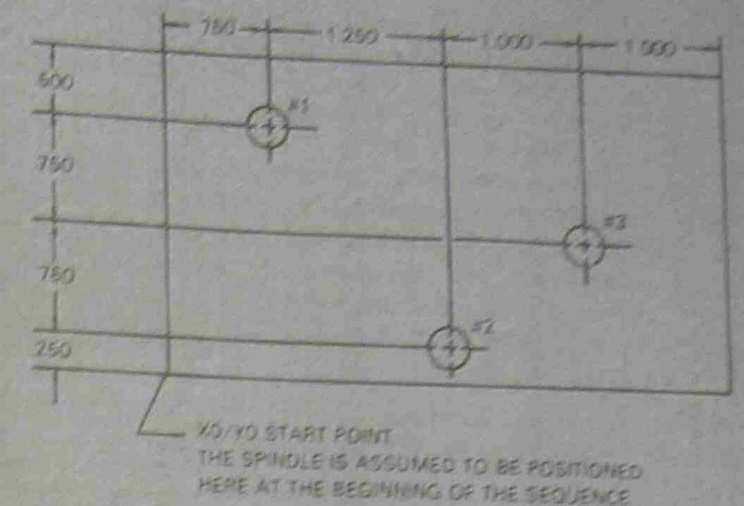


FIGURE 5–3

Notice that the sign of Y was negative when moving to hole #2. Since incremental positioning was being used, hole #1 became the X0/Y0 point for the movement to hole #2. With incremental drawings, it is necessary to add and subtract dimensions in order to correctly program the part, even when using delta dimensioned drawings.

Referring again to Figure 5–2, assume that an incremental positioning machine is to be used. Determine the coordinates necessary to drill the part. The correct coordinates are:

#1	X1.0000, Y0.5000	#5	X0.0000, Y–1.0000
#2	X–0.5000, Y0.5000	#6	X0.5000, Y–0.5000
#3	X–1.0000, Y0.0000	#7	X1.0000, Y0.0000
#4	X–0.5000, Y–0.5000	#8	X0.5000, Y0.5000

Even though this is a datum dimensioned drawing, it is often possible to program incrementally from it.

MILLING OPERATIONS

The system of coordinates presented thus far is used for centering a spindle over a particular location specified on a drawing. This means that when a coordinate location is given to the machine, the center of the spindle is sent to

```
(X,Y) = LOWER LEFT CORNER OF PART
TOOL CHANGE = X-1.0 Y-1.0
SPINDLE SPEED = 2500 RPM

1  X-1 Y-1.0  F #
2  TOOL 1          REM: 3/8 DRILL
3  X1.0 Y2    F #
4  DWELL           REM: DRILL HOLE
5  X1.25 Y1.5 F #
6  DWELL           REM: DRILL HOLE
7  X1.7 Y.5   F #
8  DWELL           REM: DRILL HOLE
9  X-1.0 Y-1.0  F #
10 END
```

FIGURE 6-3
Manual Data Language (MDL) program, inches, absolute positioning, for the part in Figure 6-1

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```
(X,Y) = LOWER LEFT CORNER OF PART
TOOL CHANGE = X-25 Y-25
SPINDLE SPEED = 2500 RPM

1  X-25 Y-25  F #
2  TOOL 1          REM: 9mm DRILL
3  X10 Y50    F #
4  DWELL           REM: DRILL HOLE
5  X30 Y35    F #
6  DWELL           REM: DRILL HOLE
7  X50 Y20    F #
8  DWELL           REM: DRILL HOLE
9  X-25 Y-25  F #
10 END
```

FIGURE 6-4
Manual Data Language (MDL) program, metric, absolute positioning, for the part in Figure 6-2

PROGRAM EXPLANATION—ABSOLUTE POSITIONING

(Refer to Figures 6-3 and 6-4.)

Notice the use of the term "REM" in the program. "REM" is used in the code for the word "remark." Remark statements are usually provided for by the computer manufacturer. They are ignored by the controller, some symbol or code is used to precede the statement (in this case the term "REM"). It is good practice to use remark statements in a program manuscript. They help not only the operator to determine what is happening in the program but also act as a reminder to the programmer when he or she is running the first part.

EVENT 1
- X,Y tool change coordinates. These coordinates position the spindle to the tool change location. The operator will install a 3/8-inch self-centering drill in the spindle (9-mm drill in the metric program).
- R—Instructs the machine to make the move at rapid traverse, which on this machine is 100 in./min.
- A—Tells the machine to use the absolute positioning mode.

A and R are programmed in, using a button on the MCU console. Once activated, rapid feedrate movement and absolute movement (positioning) remain in force until canceled by its complementary command. When writing a program manuscript, it is good practice to enter a rapid feedrate mode command and a positioning system command on every event with motion. If the program is entered in the MCU via the MCU keyboard, the programmer will be reminded to double check the positioning and feed modes for the correct setting at each affected event.

EVENT 2
- TOOL 1—Causes the machine to stop executing the program until the operator depresses the start button. This gives the operator time to safely install the tool without danger of machine movement. The remark (REM) notes that the operator will install a 3/8-inch self-centering drill in the spindle (9-mm drill in the metric program). This machine has no computer control of the spindle motor except the panic stop button, which kills power to the drive motors and the spindle. The operator would turn on the spindle motor to the correct speed at this time.

EVENT 3
- X,Y coordinates—To move from tool change to hole #1.
- R—Specifies rapid traverse.
- A—Specifies that absolute positioning is being used.

EVENT 4
- DWELL—Halts execution of the program until the start button is pushed. The operator then drills hole #1.

EVENT 5
- X,Y coordinates—To move from hole #1 to hole #2.
- R—Specifies rapid traverse.
- A—Specifies that absolute positioning is being used.

EVENT 6
- DWELL—Halts program execution. Hole #2 is drilled.

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```
%
O0611
(------------------------------------------------------)
(THIS PROGRAM USES ABSOLUTE POSITIONING)
(X/Y ORIGIN IS LOWER LEFT CORNER OF PART)
(PLACE 1/2 END MILL IN SPINDLE PRIOR TO START OF CYCLE)
(------------------------------------------------------)
(SET PARAMETERS TO RAPID - INCH INPUT - ABS. POS.)
N010G00G70G90
(AT PROG. STOP - LOWER SPINDLE AND CLAMP)
N020X-.25Y-.25
N030M00
(BEGIN ROUGH MILL CUT AT FEEDRATE)
N040G01X4.25F20.0
N050Y2.75
N060X-.25
N070Y-.25
(BEGIN FINISH MILL CUT)
N080X-.25Y-.25
N090Y2.75
N100X4.25
N110Y-.25
N120X-.25
(AT PROG. STOP UNCLAMP AND RAISE SPINDLE)
N130M00
(RETURN TO TOOL CHANGE LOCATION AND END CYCLE)
N140G00X-2.Y-1.5
N150M30
%
```

FIGURE 6-25

```
%
O0611
(------------------------------------------------------)
(THIS PROGRAM USES INCREMENTAL POSITIONING)
(CYCLE STARTS FROM TOOL CHANGE LOCATION)
(PLACE 1/2 END MILL IN SPINDLE PRIOR TO START OF CYCLE)
(------------------------------------------------------)
(SET PARAMETERS TO RAPID - INCH INPUT - INCR. POS.)
N010G00G70G91
(AT PROG. STOP - LOWER SPINDLE AND CLAMP)
N020X1.74Y1.24
N030M00
(BEGIN ROUGH MILL CUT AT FEEDRATE)
N040G01X4.52F20.0
N050Y3.02
N060X-4.52
N070Y-3.02
(BEGIN FINISH MILL CUT)
N080X.01Y.01
N090Y3.
N100X4.5
N110Y-3.
N120X-4.5
(AT PROG. STOP UNCLAMP AND RAISE SPINDLE)
N130M00
(RETURN TO TOOL CHANGE LOCATION AND END CYCLE)
N140G00X-1.75Y1.25
N150M30
%
```

FIGURE 6-26

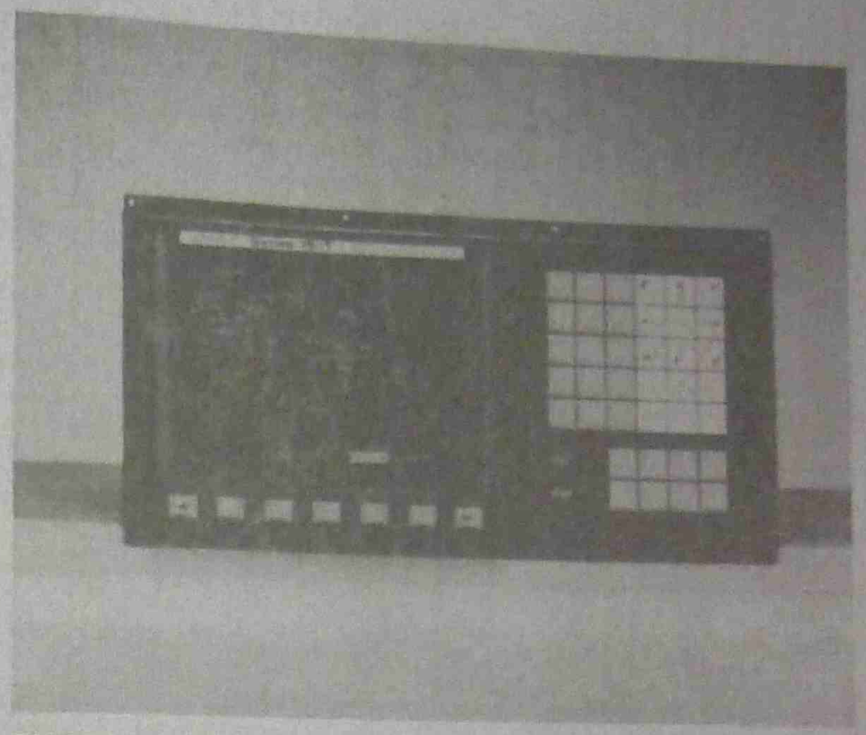


FIGURE 6-27
Fanuc series 15-T control

SUMMARY

The important concepts presented in this chapter are:

- Some procedure for tool change must be included in a program. For a manual tool change mill, a tool change location is used to safely position the spindle away from the part. The program must then be halted to allow the safe insertion of the tool. In Machinist Shop Language, the TOOL command is used to perform this function at the hole location; in word address format, an M06 is used.
- The spindle must be positioned safely out of the way at the end of the program to allow safe loading and unloading of the workpiece. This is accomplished in both the milling and drilling examples by sending the spindle back to its tool change location at the end of the program.

```
48 Z3 RA
49 X-3.26 Y2.26 RA
50 Z0 RA
51 Z-.62 FA
52 Y-2.26 FA
53 Z3 RA
54 X3.26 RA
55 Z0 RA
56 Z-.62 FA
57 Y2.26 FA
58 Z3 RA
59 X-3.26 RA
60 Z0 RA
61 Z-.62 FA
62 X-3.25 Y2.25 FA
63 Y-2.25 FA
64 Z3 RA
65 X3.25 RA
66 Z0 RA
67 Z-.62 FA
68 Y2.25 FA
69 TOOL 0
70 Z0 RA
71 X-12 YB RA
72 END
```

```
REM:RAPID TO #18
REM:RAPID TO BUFFER
REM:FEED TO DEPTH
REM:FEED TO #19

REM:RAPID TO #20
REM:RAPID TO BUFFER
REM:FEED TO DEPTH
REM:FEED TO #21

REM:RAPID TO #18
REM:RAPID TO BUFFER
REM:FEED TO DEPTH
REM:FEED TO #22
REM:FEED TO #23

REM:RAPID TO #24
REM:RAPID TO BUFFER
REM:FEED TO DEPTH
REM:FEED TO #25
REM:CANCEL OFFSET
REM:RETRACT QUILL
REM:TCH
```

FIGURE 7-3
Machinist Shop Language three-axis program, nonmetric, for the part in Figure 7-1

MACHINIST SHOP LANGUAGE

To use all three axes on the CNC machine, it is necessary to introduce some new commands.

TOOL—Used in Chapter 6 to call up a tool at tool change, TOOL is also used to assign the length of a tool into an offset register.

G—This is a preparatory function (G code). With three-axis operation, many different cycles may be used, each called up by a G code in both Machinist Shop Language and word address format. The G code G04 was used in the last chapter to cause a dwell to occur in word address format.

V—This means *variable*, and allows the programmer to assign values to specific things. V20, for example, specifies the feedrate for the Z axis when using G codes. V21 specifies the height off the surface of the workpiece at which the tool begins and ends its feedrate movements when using G codes. A listing of Machinist Shop Language G code and V code commands is contained in Appendix 2.


```
(0,Y0 = CENTER OF PART
TOOL CHANGE = X-300 Y200
TOOLS LIST:
TOOL 1 = 8mm COMB. DRILL
TOOL 2 = 25mm 4 FLUTE END MILL
TOOL 3 = 10mm 4 FLUTE END MILL
BUFFER HEIGHT: TOP OF PART FOR TOOL 1,2.54mm FOR
TOOLS 2 AND 3
CLEARANCE OVER CLAMPS: 75mm

 1 TOOL 1001
 2 Z100 A
 3 TOOL 1002
 4 Z75 A
 5 TOOL 1003
 6 Z87.5 A
 7 X-300 Y200 RA
 8 TOOL 1                        REM:8mm DRILL 1066 RPM
 9 V20 Z16
10 V21 2.54
11 G81
12 X-60 Y40 Z-13 RA             REM:DRILL #1
13 Z75 RA                       REM:RETRACT SPNDL
14 X60 RA                       REM:DRILL #2
15 Y-40 RA                      REM:DRILL #3
16 X0 Y0 RA                     REM:DRILL #4
17 X-60 Y-40 RA                 REM:DRILL #5
18 G80
19 TOOL 0                       REM:CANCEL OFFSET
20 Z0 RA                        REM:RETRACT SPNDL
21 X-300 Y200 RA                REM:TCH
22 TOOL 2                       REM:25mm E/M 425 RPM
23 FEED 172.5
24 X0 Y0 RA                     REM:RAPID TO #6
25 Z0 RA
26 Z-13 FA                      REM:FEED TO DEPTH
27 Y12.25 FA                    REM:FEED TO #7
28 X37.25 FA                    REM:FEED TO #8
29 Y-12.25 FA                   REM:FEED TO #9
30 X-37.25 FA                   REM:FEED TO #10
31 Y12.25 FA                    REM:FEED TO #11
32 X0 FA                        REM:FEED TO #7
33 TOOL 0                       REM:CANCEL OFFSET
34 Z0 RA                        REM:RETRACT SPNDL
35 X-300 Y200 RA                REM:TCH
36 TOOL 3                       REM:10.0 E/M 800 RPM
37 FEED 325.1
38 X0 Y0 RA                     REM:RAPID TO #6
39 Z0 RA                        REM:RAPID TO BUFFER
40 Z-13 FA                      REM:FEED TO DEPTH
41 Y20 FA                       REM:FEED TO #13
42 X45 FA                       REM:FEED TO #14
43 Y-20 FA                      REM:FEED TO #15
44 X-45 FA                      REM:FEED TO #16
45 Y20 FA                       REM:FEED TO #17
46 X0 FA                        REM:FEED TO #13
47 Y44.75 FA
48 Z75 RA
```

P4:
Note: .25167 in Z formula from P3 Z calculation.

X = .745 + .020(COS 37°) Z = −[.25167 − .020(SIN 37°)]

X = .76097 Z = −.23963

It is often necessary when determining coordinates to systematically solve a series of triangles in order to derive the necessary values for the final cutter solution, as this case demonstrates.

SUMMARY

The important concepts presented in this chapter are:

- Right-angle trigonometry is the mathematical science of solving right triangles.
- The sine of an angle equals the side opposite the angle divided by the hypotenuse of the triangle.
- The cosine of an angle equals the side adjacent to the angle divided by the hypotenuse of the triangle.
- The tangent of an angle equals the side opposite the angle divided by the side adjacent to the angle.
- The use of trigonometry is necessary for determining cutter offsets for linear and circular interpolation and for determining other part information from a blueprint.

REVIEW QUESTIONS

1. What is the sine of an angle? The cosine? The tangent?
2. What are the sine, cosine, and tangent of the triangle in Figure 8–11? What is angle B?
3. What are the coordinates of the holes in the part in Figure 8–12, assuming that the center is X0 Y0?
4. What are the coordinates of the four cutter locations indicated in Figure 8–13?

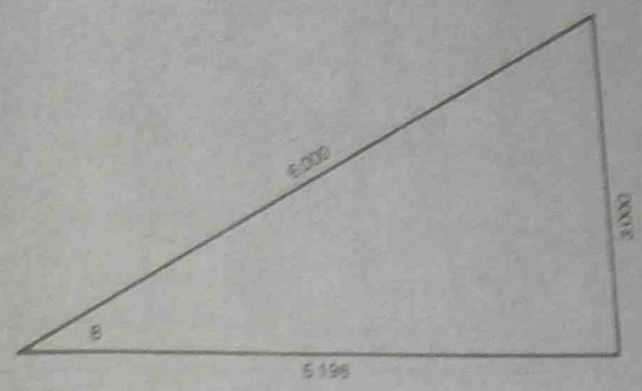


FIGURE 8–11
Triangle for review question #2

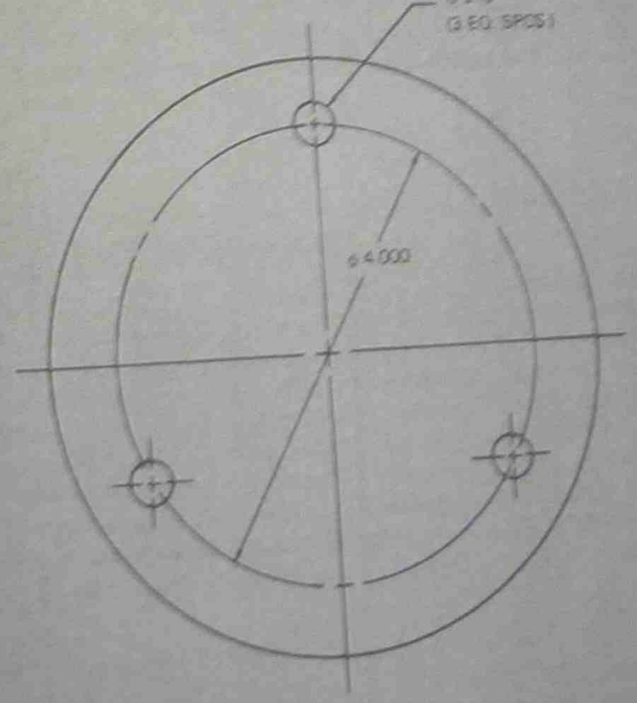
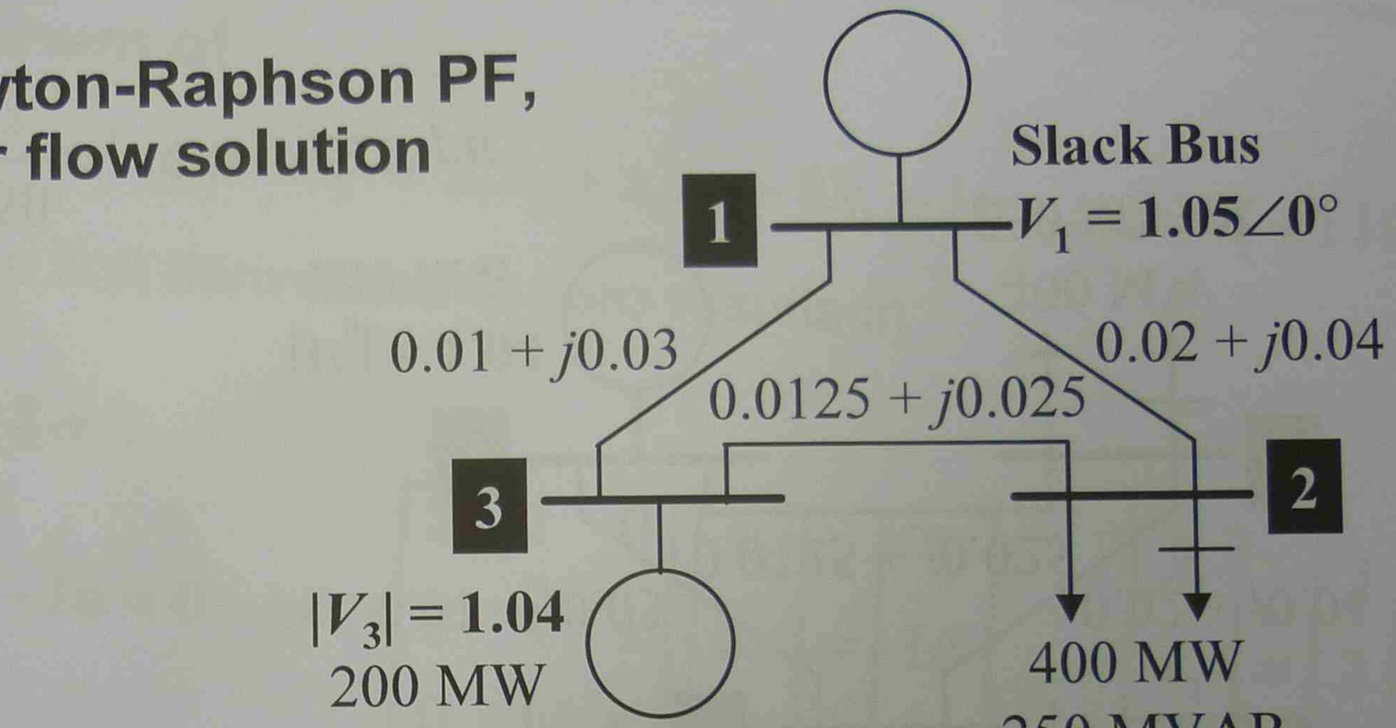


FIGURE 8–12
Part drawing for review question #3

Example

- **Using the Newton-Raphson PF, find the power flow solution**



$$y_{12} = 10 - j20 \ pu$$

$$y_{13} = 10 - j30 \ pu$$

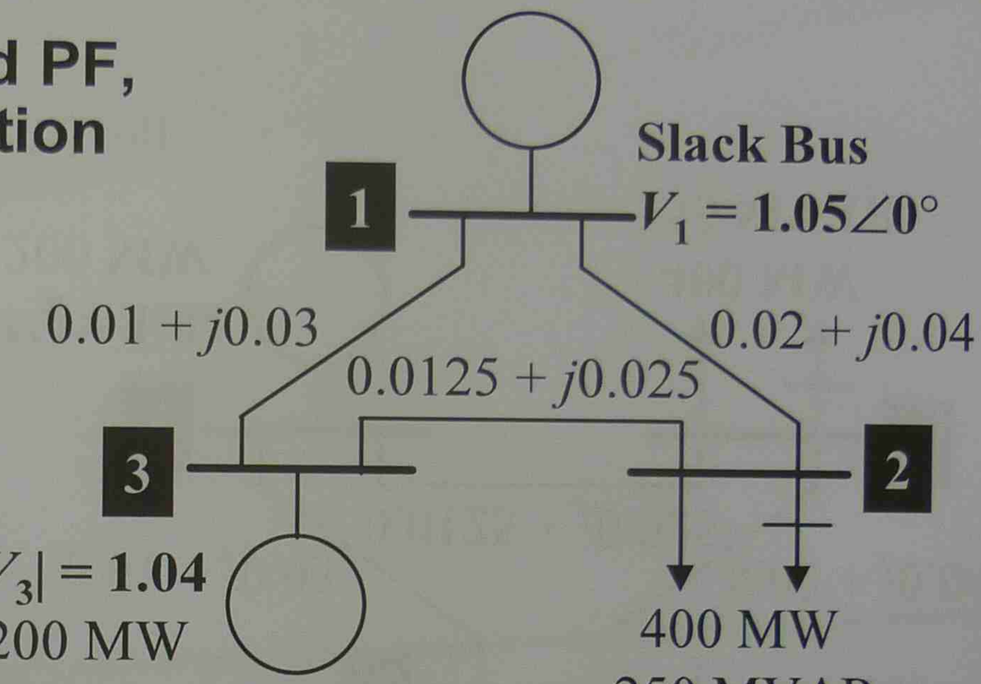
$$y_{23} = 16 - j32 \ pu$$

$$S_2^{sch} = -\frac{400 + j250}{100} = -4.0 - j2.5 \ pu$$

$$P_3^{sch} = \frac{200}{100} = 2.0 \ pu$$

Example

- **Using the fast decoupled PF, find the power flow solution**



$$y_{12} = 10 - j20 \ pu$$

$$y_{13} = 10 - j30 \ pu$$

$$y_{23} = 16 - j32 \ pu$$

$$S_2^{sch} = -\frac{400 + j250}{100} = -4.0 - j2.5 \ pu$$

$$P_3^{sch} = \frac{200}{100} = 2.0 \ pu$$

$$\frac{\partial f}{\partial x_2} = x_2 + 6x_3 - 32 = 0$$

Example

- **Find the minimum of**

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + x_2x_3 - 8x_1 - 16x_2 - 32x_3 + 110$$

  - evaluating the first derivatives to zero results in

$$\frac{\partial f}{\partial x_1} = 2x_1 + x_2 - 8 = 0$$

$$\frac{\partial f}{\partial x_2} = x_1 + 4x_2 + x_3 - 16 = 0 \quad or \quad \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \qquad \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\frac{\partial f}{\partial x_3} = x_2 + 6x_3 - 32 = 0$$

Example

- eliminating λ from the first two equations

$$\frac{16\lambda}{2x} = \frac{12\lambda}{2y} \rightarrow y = \frac{3}{4}x$$

- substituting for y in the third equation yields

$$(x-8)^2 + \left(\frac{3}{4}x - 6\right)^2 - 25 = 0$$

$$\frac{25}{16}x^2 - 25x + 75 = 0 \rightarrow x = 4 \quad \& \quad x = 12$$

$$\textit{extrema}: \quad (4,3),\ \lambda = 1 \quad \textit{and} \quad (12,9),\ \lambda = -3$$

$$\min \rightarrow \begin{matrix} x = 4 \\ y = 3 \end{matrix}$$

Example

- **Use the Kuhn-Tucker method to determine the minimum distance from the origin of the x-y plane to a circle described by**

$$(x-8)^2+(y-6)^2=25 \quad or$$

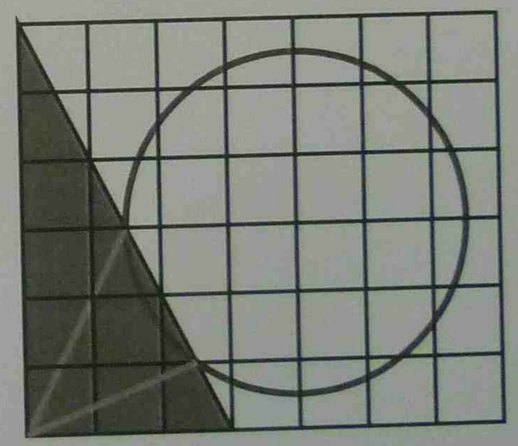
$$g(x,y)=(x-8)^2+(y-6)^2-25$$

constrained by

$$u(x,y)=2x+y\geq 12$$

- The minimum distance is obtained by minimizing the distance squared

$$f(x,y)=x^2+y^2$$



Example

- **Solve for the time-domain solution of the current**
  - for a faulted generator having the following characteristics

    $R = 0.125\ \Omega$ $\quad L = 10\ \text{mH}$ $\quad v(t) = 151 \sin(377\,t + \alpha)$
  - which will give

    (a) zero dc offset current,
    (b) maximum dc offset current

Example

- Consider a 500 MVA, 30 kV generator with no load and a constant excitation voltage of 400 V. A three-phase short circuit occurs at the terminals. Obtain the transient waveforms for the current in each phase and the field winding

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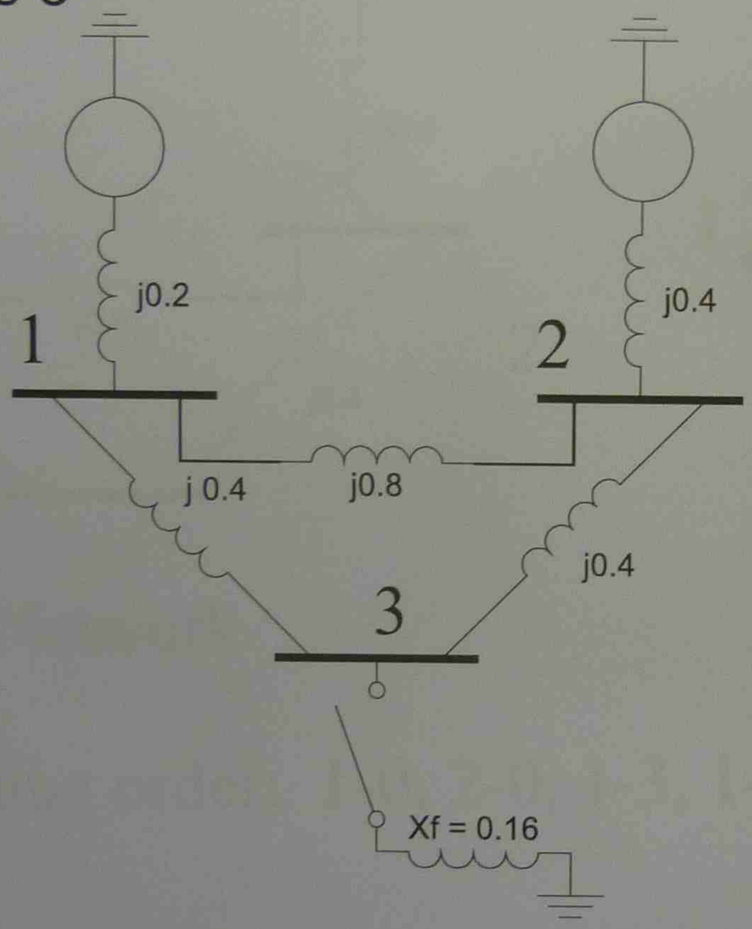
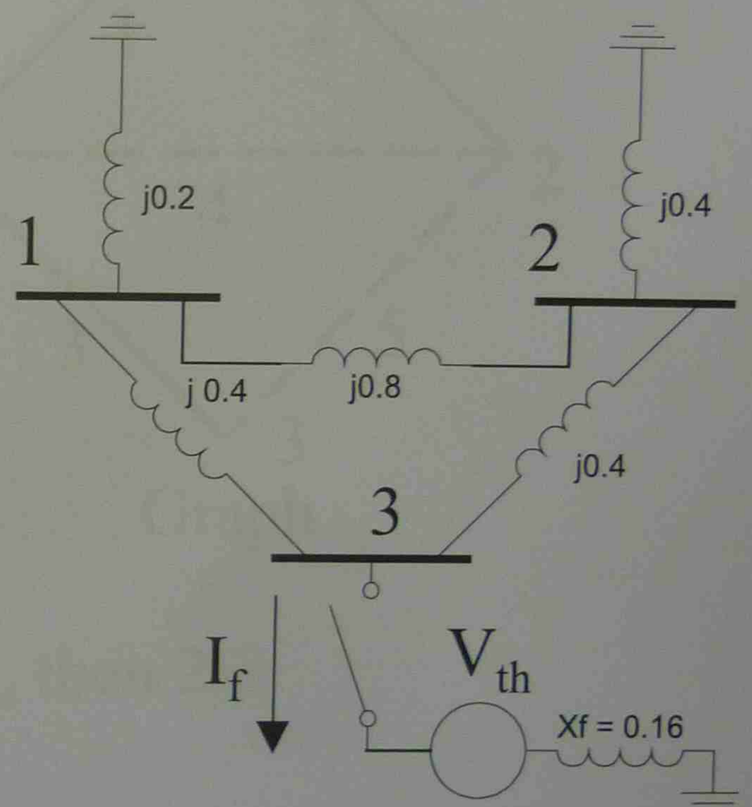
Determining the Transient Constants

- **Example**
  - A generator operating at no load, is set to deliver a terminal voltage of 1.0 pu
  - The generator is subjected to a fault on the terminals when phase a is at a voltage peak
  - Peak values of the current are recorded at the following times:
    - 0.0042 s : I = 8.7569
    - 0.0208 s : I = 6.7363
    - 0.3208 s : I = 2.8893
    - 0.3375 s : I = 2.8608
    - 5.0000 s : I = 1.1785
  - Determine the reactances and time constants

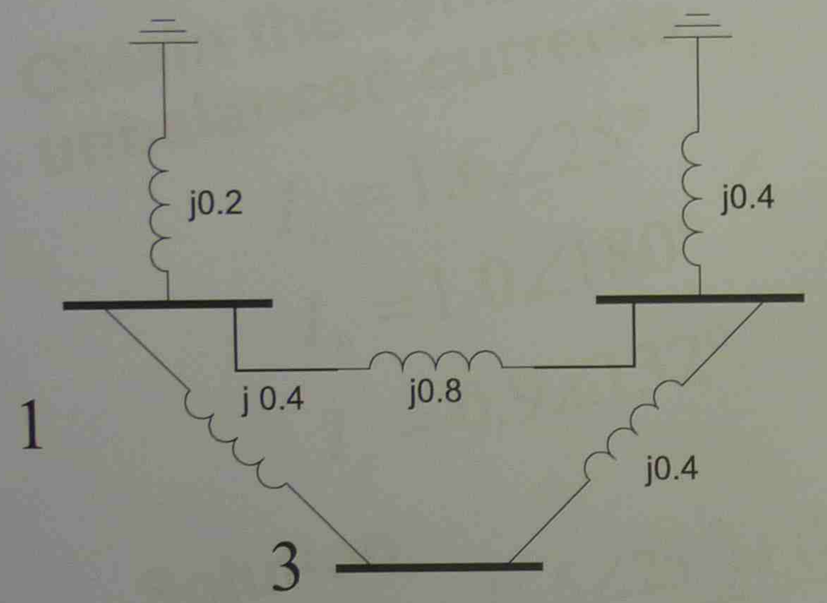
Example

- **3-phase fault with Z_f = j0.16 on**
  - bus 3

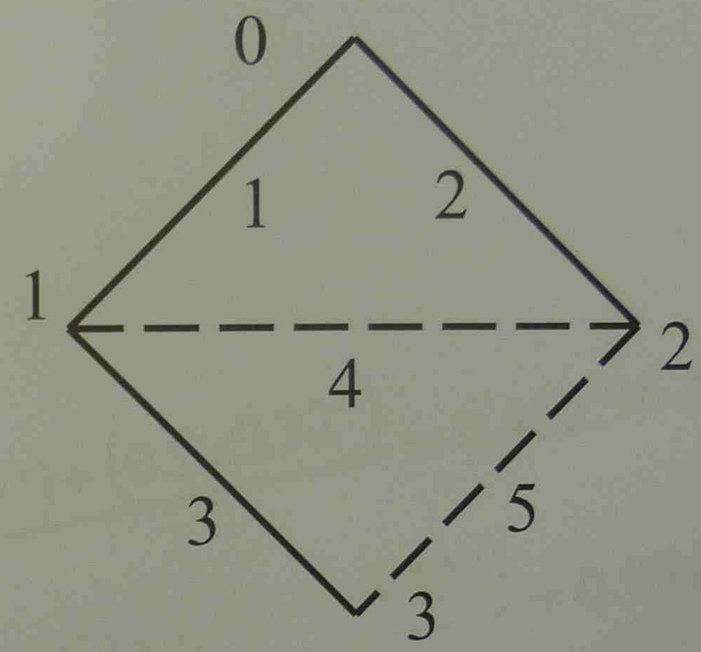




Example



Network



Graph

Line adding order: 1-0, 2-0, 1-3, 1-2, then 2-3

Example

- **Obtain the symmetrical components of a set of unbalanced currents**

$$I_a = 1.6\angle 25°$$

$$I_b = 1.0\angle 180°$$

$$I_c = 0.9\angle 132°$$

- **Solution**

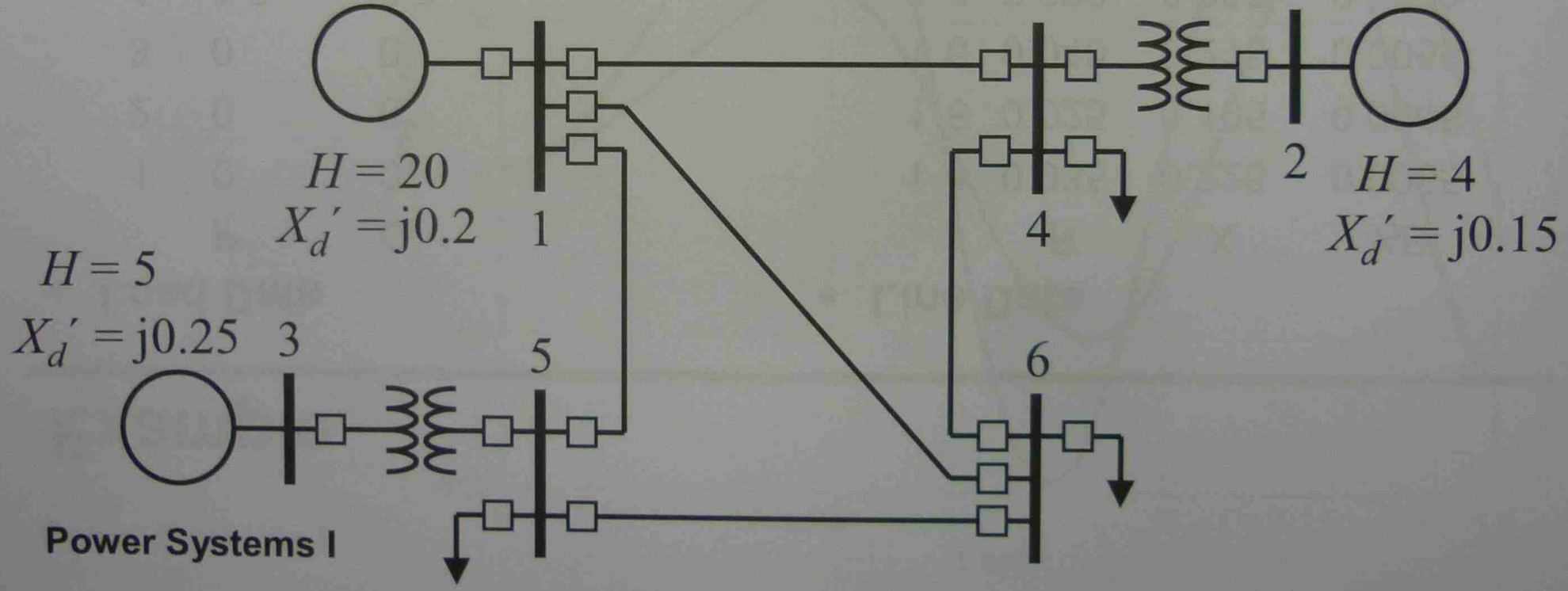
$$I_{a0} = \frac{(1.6\angle 25°) + (1.0\angle 180°) + (0.9\angle 132°)}{3} = 0.45\angle 96.5°$$

$$I_{a1} = \frac{(1.6\angle 25°) + a(1.0\angle 180°) + a^2(0.9\angle 132°)}{3} = 0.94\angle -0.1°$$

$$I_{a2} = \frac{(1.6\angle 25°) + a^2(1.0\angle 180°) + a(0.9\angle 132°)}{3} = 0.60\angle 22.3°$$

Example

- **Consider the 3 machine system below**
  - select generator #1 as the swing machine with a constant angle of 0 degrees
  - determine the system stability when a fault on the line 5-6 near bus 6 is cleared in 0.4 and 0.5 seconds



Example

- **Load Data**

	P	Q
1	0	0
2	0	0
3	0	0
4	1.0	0.7
5	0.9	0.3
6	1.6	1.1

- **Generator Data**

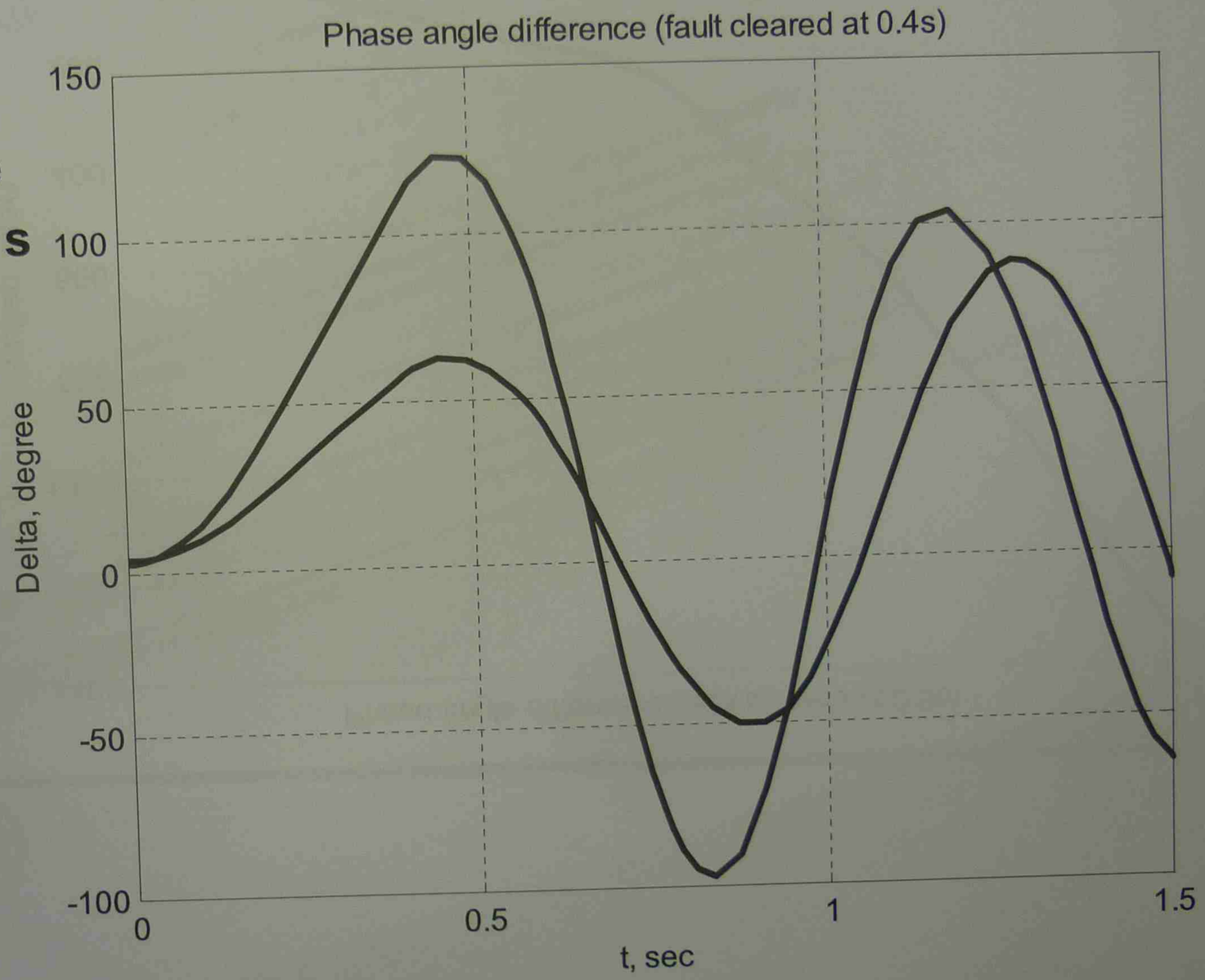
	V	P
1	1.06	---
2	1.04	1.5
3	1.03	1.0

- **Line Data**

		R	X	½B
1	4	0.035	0.225	0.0065
1	5	0.025	0.105	0.0045
1	6	0.040	0.215	0.0055
2	4	0.000	0.035	0.0000
3	5	0.000	0.042	0.0000
4	6	0.028	0.125	0.0035
5	6	0.026	0.175	0.0300

Example

Power angle / time
fault clearing in 0.4 s



Example

Power angle / time fault clearing in 0.5 s

The active and reactive power flows in opposite directions, P_{mk} and Q_{mk}, can be obtained in the same way, resulting in:

$$P_{mk} = U_m^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} + U_k U_m b_{km} \sin\theta_{km} \tag{3.5}$$

$$Q_{mk} = -U_m^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos\theta_{km} + U_k U_m g_{km} \sin\theta_{km} \tag{3.6}$$

From these expressions the active and reactive power losses of the lines are easily obtained as:

$$\begin{aligned} P_{km} + P_{mk} &= g_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= g_{km}|E_k - E_m|^2 \end{aligned} \tag{3.7}$$

$$\begin{aligned} Q_{km} + Q_{mk} &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}|E_k - E_m|^2 \end{aligned} \tag{3.8}$$

Note that $|E_k - E_m|$ represents the magnitude of the voltage drop across the line, $g_{km}|E_k - E_m|^2$ represents the active power losses, $-b_{km}^{sh}|E_k - E_m|^2$ represents the reactive power losses; and $-b_{km}^{sh}(U_k^2 + U_m^2)$ represents the reactive power generated by the shunt elements of the equivalent π-model (assuming actual transmission line sections, i.e. with $b_{km} < 0$ and $b_{km}^{sh} > 0$).

Example 3.1. *A 750 kV transmission line section has a series impedance of* $0.00072 + j0.0175$ *p.u., a total shunt impedance of 8.775 p.u., a voltage magnitude at the terminal buses of 0.984 p.u. and 0.962 p.u., and a voltage angle difference of* $22°$*. Calculate the active and reactive power flows.*

Solution The active and reactive power flows in the line are obtained by applying eqs. (3.3) and (3.4), where $U_k = 0.984$ p.u., $U_m = 0.962$ p.u., and $\theta_{km} = 22°$. The series impedance and admittances are as follows:

$$z_{km} = 0.00072 + j0.0175 \ p.u.$$

$$y_{km} = g_{km} + jb_{km} = z_{km}^{-1} = 2.347 - j57.05 \ p.u.$$

The π-model shunt admittances (100 MVA base) are:

$$b_{km}^{sh} = 8.775/2 = 4.387 \ p.u.$$

and

$$P_{km} = 0.984^2 \cdot 2.347 - 0.984 \cdot 0.962 \cdot 2.347 \cos 22° + 00.984 \cdot 0.962 \cdot 57.05 \sin 22° \ p.u.$$

$$Q_{km} = -0.984^2 \cdot (-57.05 + 4.39) - 0.984 \cdot 0.962 \cdot 57.05 \cos 22° - 00.984 \cdot 0.962 \cdot 2.347 \sin 22° \ p.u.$$

which yield

$$P_{km} = 2044 \ MW \qquad Q_{km} = 8.5 \ Mvar$$

In similar way one obtains:

$$P_{mk} = -2012\ MW \qquad Q_{mk} = -50.5\ Mvar$$

It should be noted that powers are positive when injected into the line. ♦

3.2 In-phase Transformers

The complex current I_{km} in an in-phase transformer is expressed as in eq. (2.13)

$$I_{km} = a_{km} y_{km} (a_{km} E_k - E_m)$$

The complex power, $S_{km} = P_{km} + jQ_{km}$, is given by

$$S_{km} = E_k I_{km}^* = y_{km}^* a_{km} U_k e^{j\theta_k} (a_{km} U_k e^{-j\theta_k} - U_m e^{-j\theta_m}) \tag{3.9}$$

Separating the real and imaginary parts of this latter expression yields the active and reactive power flow equations:

$$P_{km} = (a_{km} U_k)^2 g_{km} - a_{km} U_k U_m g_{km} \cos\theta_{km} - a_{km} U_k U_m b_{km} \sin\theta_{km} \tag{3.10}$$

$$Q_{km} = -(a_{km} U_k)^2 b_{km} + a_{km} U_k U_m b_{km} \cos\theta_{km} - a_{km} U_k U_m g_{km} \sin\theta_{km} \tag{3.11}$$

These same expressions can be obtained by comparing eqs. (3.9) and (3.2); in eq. (3.9) the term $jb_{km}^{sh} U_k^2$ is not present, and U_k is replaced by $a_{km} U_k$. Hence, the expressions for the active and reactive power flows on in-phase transformers are the same expressions derived for a transmission line, except the for two modifications: ignore b_{km}^{sh}, and replace U_k with $a_{km} U_k$.

Example 3.2. *A 500/750 kV transformer with a tap ratio of 1.050:1.0 on the 500 kV side, see Figure 2.4, has negligible series resistance and a leakage reactance of 0.00623 p.u., terminal voltage magnitudes of 1.023 p.u. and 0.968 p.u., and an angle spread of* 5.3°. *Calculate the active and reactive power flows in the transformer.*

Solution The active and reactive power flows in the transformer are given by eqs. (3.10) and (3.11), where $U_k = 1.023$ p.u., $U_m = 0.968$ p.u., $\theta_{km} = 5.3°$, and $a_{km} = 1.0/1.05 = 0.9524$. The series reactance and susceptance are as follows:

$$x_{km} = 0.00623 \text{ p.u.}$$

$$b_{km} = -x_{km}^{-1} = -160.51 \text{ p.u.}$$

The active and reactive power flows in opposite directions, P_{mk} and Q_{mk}, can be obtained in the same way, resulting in:

$$P_{mk} = U_m^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} + U_k U_m b_{km} \sin\theta_{km} \tag{3.5}$$

$$Q_{mk} = -U_m^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos\theta_{km} + U_k U_m g_{km} \sin\theta_{km} \tag{3.6}$$

From these expressions the active and reactive power losses of the lines are easily obtained as:

$$\begin{aligned} P_{km} + P_{mk} &= g_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= g_{km}|E_k - E_m|^2 \end{aligned} \tag{3.7}$$

$$\begin{aligned} Q_{km} + Q_{mk} &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}|E_k - E_m|^2 \end{aligned} \tag{3.8}$$

Note that $|E_k - E_m|$ represents the magnitude of the voltage drop across the line, $g_{km}|E_k - E_m|^2$ represents the active power losses, $-b_{km}^{sh}|E_k - E_m|^2$ represents the reactive power losses; and $-b_{km}^{sh}(U_k^2 + U_m^2)$ represents the reactive power generated by the shunt elements of the equivalent π-model (assuming actual transmission line sections, i.e. with $b_{km} < 0$ and $b_{km}^{sh} > 0$).

Example 3.1. *A 750 kV transmission line section has a series impedance of* $0.00072 + j0.0175$ *p.u., a total shunt impedance of 8.775 p.u., a voltage magnitude at the terminal buses of 0.984 p.u. and 0.962 p.u., and a voltage angle difference of* $22°$*. Calculate the active and reactive power flows.*

Solution The active and reactive power flows in the line are obtained by applying eqs. (3.3) and (3.4), where $U_k = 0.984$ p.u., $U_m = 0.962$ p.u., and $\theta_{km} = 22°$. The series impedance and admittances are as follows:

$$z_{km} = 0.00072 + j0.0175 \ p.u.$$

$$y_{km} = g_{km} + jb_{km} = z_{km}^{-1} = 2.347 - j57.05 \ p.u.$$

The π-model shunt admittances (100 MVA base) are:

$$b_{km}^{sh} = 8.775/2 = 4.387 \ p.u.$$

and

$$P_{km} = 0.984^2 \cdot 2.347 - 0.984 \cdot 0.962 \cdot 2.347 \cos 22° + 00.984 \cdot 0.962 \cdot 57.05 \sin 22° \ p.u.$$

$$Q_{km} = -0.984^2 \cdot (-57.05 + 4.39) - 0.984 \cdot 0.962 \cdot 57.05 \cos 22° - 00.984 \cdot 0.962 \cdot 2.347 \sin 22° \ p.u.$$

which yield

$$P_{km} = 2044 \ MW \qquad Q_{km} = 8.5 \ Mvar$$

In similar way one obtains:

$$P_{mk} = -2012\ MW \qquad Q_{mk} = -50.5\ Mvar$$

It should be noted that powers are positive when injected into the line. ♦

3.2 In-phase Transformers

The complex current I_{km} in an in-phase transformer is expressed as in eq. (2.13)

$$I_{km} = a_{km}y_{km}(a_{km}E_k - E_m)$$

The complex power, $S_{km} = P_{km} + jQ_{km}$, is given by

$$S_{km} = E_k I_{km}^* = y_{km}^* a_{km} U_k e^{j\theta_k}(a_{km}U_k e^{-j\theta_k} - U_m e^{-j\theta_m}) \qquad (3.9)$$

Separating the real and imaginary parts of this latter expression yields the active and reactive power flow equations:

$$P_{km} = (a_{km}U_k)^2 g_{km} - a_{km}U_k U_m g_{km}\cos\theta_{km} - a_{km}U_k U_m b_{km}\sin\theta_{km} \qquad (3.10)$$

$$Q_{km} = -(a_{km}U_k)^2 b_{km} + a_{km}U_k U_m b_{km}\cos\theta_{km} - a_{km}U_k U_m g_{km}\sin\theta_{km} \qquad (3.11)$$

These same expressions can be obtained by comparing eqs. (3.9) and (3.2); in eq. (3.9) the term $jb_{km}^{sh}U_k^2$ is not present, and U_k is replaced by $a_{km}U_k$. Hence, the expressions for the active and reactive power flows on in-phase transformers are the same expressions derived for a transmission line, except the for two modifications: ignore b_{km}^{sh}, and replace U_k with $a_{km}U_k$.

Example 3.2. *A 500/750 kV transformer with a tap ratio of 1.050:1.0 on the 500 kV side, see Figure 2.4, has negligible series resistance and a leakage reactance of 0.00623 p.u., terminal voltage magnitudes of 1.023 p.u. and 0.968 p.u., and an angle spread of* 5.3°. *Calculate the active and reactive power flows in the transformer.*

Solution The active and reactive power flows in the transformer are given by eqs. (3.10) and (3.11), where $U_k = 1.023$ p.u., $U_m = 0.968$ p.u., $\theta_{km} = 5.3°$, and $a_{km} = 1.0/1.05 = 0.9524$. The series reactance and susceptance are as follows:

$$x_{km} = 0.00623 \text{ p.u.}$$

$$b_{km} = -x_{km}^{-1} = -160.51 \text{ p.u.}$$

The active and reactive power flows can be expressed as

$$P_{km} = -0.9524 \cdot 1.023 \cdot 0.968 \cdot (-160.51) \sin 5.3^\circ \text{ p.u.}$$

$$Q_{km} = -(0.9524 \cdot 1.023)^2(-160.51) + 0.9524 \cdot 1.023 \cdot 0.968 \cdot (-160.51) \cos 5.3^\circ \text{ p.u.}$$

which yield

$$P_{km} = 1398 \text{ MW} \qquad Q_{km} = 163 \text{ Mvar}$$

The reader is encouraged to calculate P_{mk} and Q_{mk}. (The value of P_{mk} should be obvious.) ♦

3.3 Phase-Shifting Transformer with $a_{km} = 1$

The complex current I_{km} in a phase shifting transformer with $a_{km} = 1$ is as follows, see Figure 2.6:

$$I_{km} = y_{km}(E_k - e^{-j\varphi_{km}} E_m) = y_{km} e^{-j\varphi_{km}} (E_k e^{j\varphi_{km}} - E_m) \tag{3.12}$$

and the complex power, $S_{km} = P_{km} + jQ_{km}$, is thus

$$S_{km} = E_k I_{km}^* = y_{km}^* U_k e^{j(\theta_k + \varphi_{km})} (U_k e^{-j(\theta_k + \varphi_{km})} - U_m e^{-j\theta_m}) \tag{3.13}$$

Separating the real and imaginary parts of this expression, yields the active and reactive power flow equations, respectively:

$$\begin{aligned} P_{km} = {} & U_k^2 g_{km} - U_k U_m g_{km} \cos(\theta_{km} + \varphi_{km}) \\ & - U_k U_m b_{km} \sin(\theta_{km} + \varphi_{km}) \end{aligned} \tag{3.14}$$

$$\begin{aligned} Q_{km} = {} & -U_k^2 b_{km} + U_k U_m b_{km} \cos(\theta_{km} + \varphi_{km}) \\ & - U_k U_m g_{km} \sin(\theta_{km} + \varphi_{km}) \end{aligned} \tag{3.15}$$

As with in-phase transformers, these expressions could have been obtained through inspection by comparing eqs. (3.2) and (3.13): in eq. (3.13), the term $jb_{km}^{sh} U_k^2$ is not present, and θ_{km} is replaced with $\theta_{km} + \varphi_{km}$. Hence, the expressions for the active and reactive power flows in phase-shifting transformers are the same expressions derived for the transmission line, albeit with two modifications: ignore b_{km}^{sh} and replace θ_{km} with $\theta_{km} + \varphi_{km}$.

Example 3.3. *A Δ-Y, 230/138 kV transformer presents a 30° phase angle shift. Series resistance is neglected and series reactance is 0.0997 p.u. Terminal voltage magnitudes are 0.882 p.u. and 0.989 p.u., and the total angle difference is −16.6°. Calculate the active and reactive power flows in the transformer.*

- Uθ bus: U_k and θ_k are specified; P_k and Q_k are calculated

The Uθ bus, also called reference bus or slack bus, has double functions in the basic formulation of the power flow problem:

1. It serves as the voltage angle reference
2. Since the active power losses are unknown in advance, the active power generation of the Uθ bus is used to balance generation, load, and losses

In "normal" power systems PQ-buses or load buses are the far most common, typically comprising more than 80% of all buses.

Other possible bus types are P, U, and PQU, with obvious definitions. The use of multiple Uθ buses may also be required for certain applications. In more general cases, the given values are not limited to the specific set of buses (P, Q, U, θ), and branch related variables can also be specified.

Example 5.1. *Figure 5.1 shows a 5-bus network with four transmission lines and two transformers. Generators, with voltage control, are connected at buses 1, 3, and 5, and loads are connected at buses 4 and 5, and at bus 4 a shunt is also connected. Classify the buses according to the bus types PU, PQ and Uθ.*

Solution Buses 1, 3, and 5 are all candidates for PU or Uθ bus types. Since only one could be Uθ bus, we select (arbitrarily) bus 5 as Uθ. In a practical system usually a generator, or generator station, that could produce power within a large range is selected as reference or slack bus. It should be noted that even if a load is connected to bus 5 it can only be a PU or Uθ bus, since voltage control is available at the bus. The reference angle is set at bus 5, usually to 0. Bus 2 is a transition bus in which both P and Q are equal to zero, and this bus is consequently of type PQ. Bus 4 is a load bus to which is also connected a shunt susceptance: since shunts are modelled as part of the network, see next section, the bus is also classified as a PQ bus. ♦

5.2 Equality and Inequality Constraints

Eqs. (4.11) and (4.12) can be rewritten as follows

$$P_k = \sum_{m \in \Omega_k} P_{km}(U_k, U_m, \theta_k, \theta_m) \tag{5.1}$$

$$Q_k + Q_k^{sh}(U_k) = \sum_{m \in \Omega_k} Q_{km}(U_k, U_m, \theta_k, \theta_m) \tag{5.2}$$

where

$$Q_{km} = \frac{U_k^2 - U_k U_m \cos\theta_{km}}{x_{km}} \tag{6.38}$$

where x_{km} is the series reactance of the line.

The sensitivities between power flows P_{km} and Q_{km} and the state variables U and θ are for this approximation given by

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m \cos\theta_{km}}{x_{km}} \qquad \frac{\partial P_{km}}{\partial U_k} = \frac{U_m \sin\theta_{km}}{x_{km}} \tag{6.39}$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = \frac{U_k U_m \sin\theta_{km}}{x_{km}} \qquad \frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m \cos\theta_{km}}{x_{km}} \tag{6.40}$$

When $\theta_{km} = 0$, perfect decoupling conditions are observed, i.e.

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m}{x_{km}} \qquad \frac{\partial P_{km}}{\partial U_k} = 0 \tag{6.41}$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = 0 \qquad \frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m}{x_{km}} \tag{6.42}$$

As illustrated in Figure 6.3, in the usual range of operations (relatively small voltage angles), a strong coupling between active power and voltage angle as well as between reactive power and voltage magnitudes exists, while a much weaker coupling between reactive power and voltage angle, and between voltage magnitude and active power exists. Notice, however, that for larger angles this is no longer true. In the neighbourhood of $\theta_{km} = 90°$, there is strong coupling between P and U as well as between Q and θ.

Example 6.1. *A 750 kV transmission line has 0.0175 p.u. series reactance (the series resistance and the shunt admittance are ignored in this example). The terminal bus voltage magnitudes are 0.984 and 0.962 p.u. and the angle difference is 10°. Calculate the sensitivities of the active and reactive power flows with respect to voltage magnitude and phase angle.*

Solution The four sensitivities are calculated by using eqs. (6.39) and (6.40):

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m \cos\theta_{km}}{x_{km}} = \frac{0.984 \cdot 0.962 \cos 10°}{0.0175} = 54.1$$

$$\frac{\partial P_{km}}{\partial U_k} = \frac{U_m \sin\theta_{km}}{x_{km}} = \frac{0.962 \sin 10°}{0.0175} = 9.5$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = \frac{U_k U_m \sin\theta_{km}}{x_{km}} = \frac{0.984 \cdot 0.962 \sin 10°}{0.0175} = 9.4$$

$$\frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m \cos\theta_{km}}{x_{km}} = \frac{2 \cdot 0.984 - 0.962 \cos 10°}{0.0175} = 58.3$$

When no inequality constraints are violated, nothing is affected in the power flow equations, but if a limit is violated, the bus status is changed and it is enforced as an equality constraint at the limiting value. This normally requires a change in bus type: if, for example, a Q limit of a PU bus is violated, the bus is transformed into an PQ bus (Q is specified and the U becomes a problem unknown). A similar procedure is adopted for backing-off when ever appropriate. What is crucial is that bus type changes must not affect solvability. Various other types of limits are also considered in practical implementations, including branch current flows, branch power flows, active power generation levels, transformer taps, phase shifter angles, and area interchanges.

5.3 Problem Solvability

One problem in the definition of bus type (bus classification) is to guarantee that the resulting set of power flow equations contains the same number of equations as unknowns, as are normally necessary for solvability, although not always sufficient. Consider a system with N buses, where N_{PU} are of type PU, N_{PQ} are of type PQ, and one is of type Uθ. To fully specify the state of the system we need to know the voltage magnitudes and voltage angles of all buses, i.e. in total $2N$ quantities. But the voltage angle and voltage magnitude of the slack bus are given together with the voltage magnitudes of N_{PU} buses. Unknown are thus the voltage magnitudes of the PQ buses, and the voltage angles of the PU and the PQ buses, giving a total of $N_{PU} + 2N_{PQ}$ unknown states. From the PU buses we get N_{PU} balance equations regarding active power injections, and from the PQ buses $2N_{PQ}$ equations regarding active and reactive power injections, thus in total $N_{PU} + 2N_{PQ}$ equations, and hence equal to the number of unknowns, and the necessary condition for solvability has been fulfilled.

Similar necessary conditions for solvability can be established when other types of buses, such as P, U, and PQU buses, are used in the formulation of the power flow problem.

Example 5.2. *Consider again the 5-bus in Figure 5.1. Formulate the equality constraints of the system and the inequality constraints for the generator buses.*

Solution In this case $N = 5$, $N_{PQ} = 2$, $N_{PU} = 2$, and of course $N_{U\theta} = 1$. The number of equations are thus: $N_{PU} + 2N_{PQ} = 2 + 2 \cdot 2 = 6$, and these

The active power injection at bus k is thus given by

$$P_k = \sum_{m \in \Omega_k} x_{km}^{-1} \theta_{km} = \Big(\sum_{m \in \Omega_k} x_{km}^{-1} \Big) \theta_k + \sum_{m \in \Omega_k} (-x_{km}^{-1} \theta_m) \tag{6.61}$$

for $k = 1, 2, \ldots, N$, where N is the number of buses in the network. This can be put into matrix form as follows:

$$\mathbf{P} = \mathbf{B}' \theta \tag{6.62}$$

where

- $\mathbf{P}$ is the vector of the net injections P_k
- $\mathbf{B}'$ is the nodal admittance matrix with the following elements:

$$B'_{km} = -x_{km}^{-1}$$

$$B'_{kk} = \sum_{m \in \Omega_k} x_{km}^{-1}$$

- θ is the vector of voltage angles θ_k

The matrix $\mathbf{B}'$ in eq. (6.62) is singular, i.e. its determinant is equal to zero. This means that the system of equations in eq. (6.62) has no unique solution and that the rows of $\mathbf{B}'$ are linearly dependent. To make the system solvable, one of the equations in the system is removed, and the bus associated with that row is chosen as the angle reference, i.e. $\theta_{ref} = 0$. Since the network losses are ignored in this approximation a slack node is not required to compensate for these a priori unknown losses, but of course still an angle reference is needed.

In forming the matrix $\mathbf{B}'$, in-phase and phase-shifting transformers are treated like transmission lines. The phase-shifting transformers also contribute to the construction of the independent vector $\mathbf{P}$ with the Norton equivalent injections shown in Figure 6.6(c).

Example 6.2. *Consider the network given in Figure 6.7 in which the reference angle is $\theta_1 = 0$. Use the dc power flow method to calculate the power flows in the lines.*

Solution In this case, the elements of the matrix $\mathbf{B}'$ are calculated as

$$B_{22} = x_{21}^{-1} + x_{23}^{-1} = (1/3)^{-1} + (1/2)^{-1} = 5$$
$$B_{23} = -x_{23}^{-1} = -(1/2)^{-1} = -2$$
$$B_{32} = -x_{32}^{-1} = -(1/2)^{-1} = -2$$
$$B_{33} = x_{31}^{-1} + x_{32}^{-1} = (1/2)^{-1} + (1/2)^{-1} = 4$$

position.

$$z_{ij}\Big|_{i<j} = z_{ji} = \begin{cases} \text{not calculated if } r_{ij} = 0 \\ -\sum_{l=i+1}^{n} r_{il}\, z_{lj} \end{cases} \tag{7.35}$$

$$z_{ii} = \frac{1}{d_{ii}} - \sum_{l=i+1}^{n} r_{il}\, z_{lj} \tag{7.36}$$

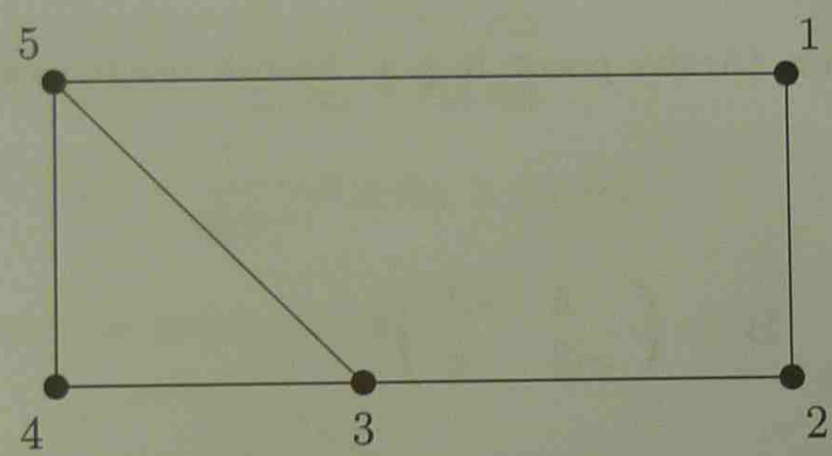


Figure 7.17. Sample network.

Example 7.1. *The whole procedure shall be demonstrated in single steps on the sample network given in Figure 7.17. With the arbitrarily chosen node numbering the admittance matrix* $\mathbf{Y}$ *has the following structure.*

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & 0 & 0 & y_{15} \\ y_{21} & y_{22} & y_{23} & 0 & 0 \\ 0 & y_{32} & y_{33} & y_{34} & y_{35} \\ 0 & 0 & y_{43} & y_{44} & y_{45} \\ y_{51} & 0 & y_{53} & y_{54} & y_{55} \end{bmatrix} \tag{7.37}$$

Solution During the triangular factorization of $\mathbf{Y}$ a fill-in element c_{25} is generated in the upper triangular matrix. The recursive solution of the

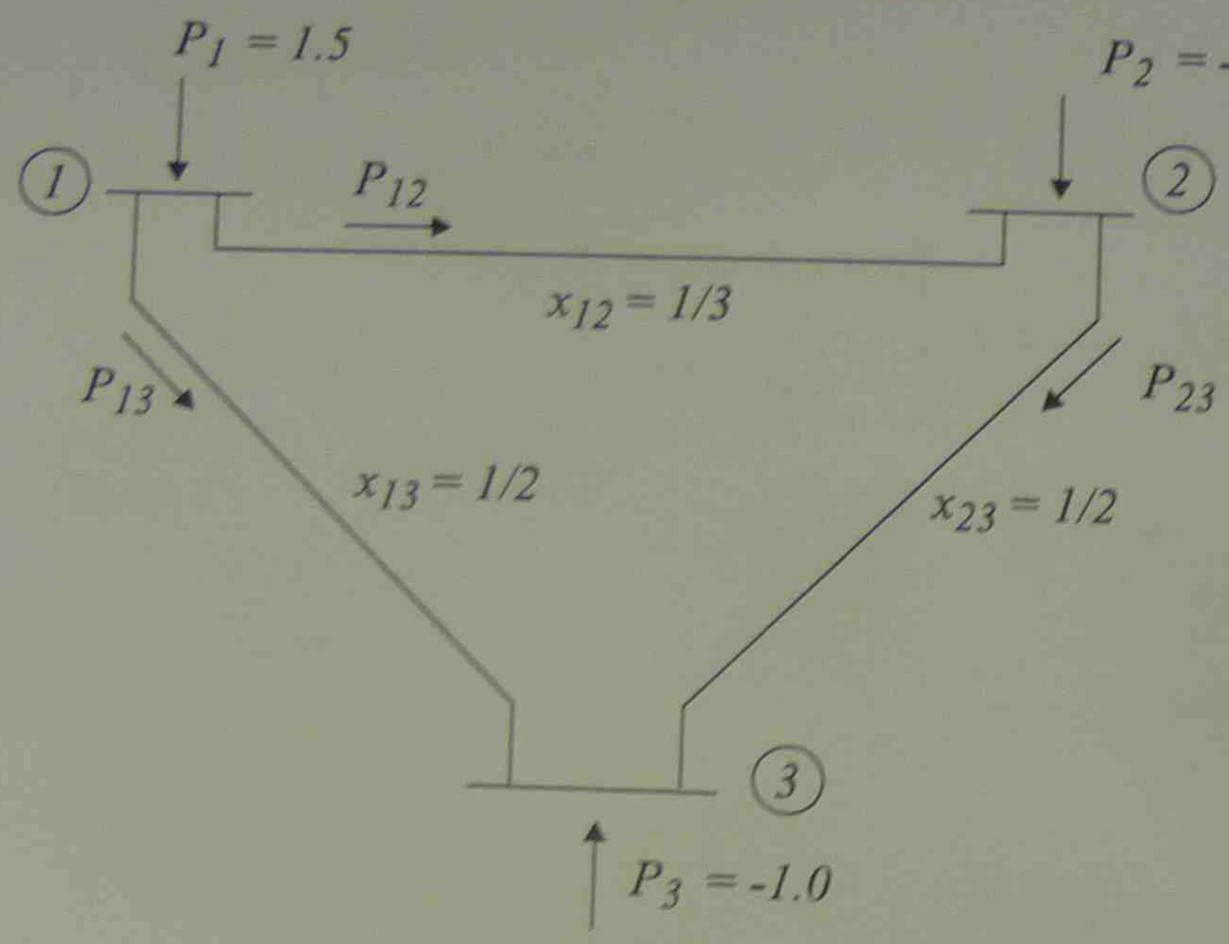


Figure 6.7. 3-bus network. (Active power in p.u.; branch reactances in p.u.)

and thus

$$\mathbf{B}' = \begin{pmatrix} 5 & -2 \\ -2 & 4 \end{pmatrix}$$

and

$$(\mathbf{B}')^{-1} = \begin{pmatrix} 1/4 & 1/8 \\ 1/8 & 5/16 \end{pmatrix}$$

The nodal voltage angles (in radians) can now easily be calculated

$$\theta = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = (\mathbf{B}')^{-1}\mathbf{P}$$

$$= \begin{pmatrix} 1/4 & 1/8 \\ 1/8 & 5/16 \end{pmatrix} \begin{pmatrix} -0.5 \\ -1.0 \end{pmatrix} = \begin{pmatrix} -0.250 \\ -0.375 \end{pmatrix}$$

The power flows in the transmission lines are according to the dc power flow model

$$P_{12} = x_{12}^{-1}\theta_{12} = 3 \cdot 0.25 = 0.75 \text{ p.u.}$$
$$P_{13} = x_{13}^{-1}\theta_{13} = 2 \cdot 0.375 = 0.75 \text{ p.u.}$$
$$P_{23} = x_{23}^{-1}\theta_{23} = 2 \cdot 0.125 = 0.25 \text{ p.u.}$$

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EEG13
Power system optimization

Assignment (50)

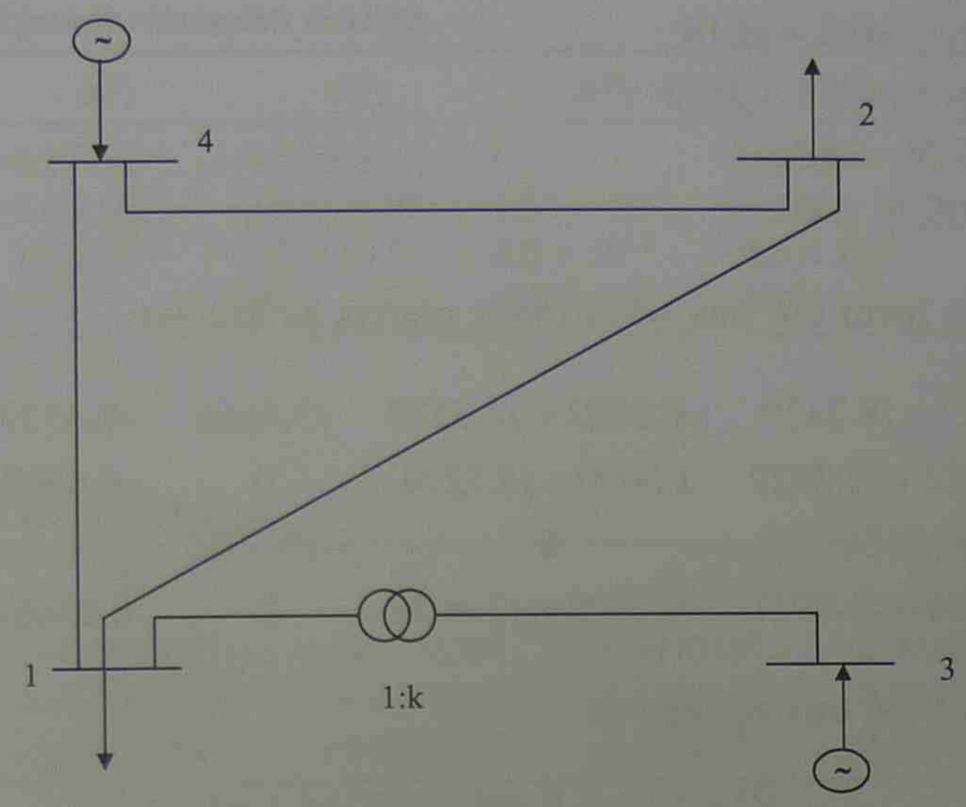

FIGURE 2.1 Four-bus power system

Step (5): Compute the elements in Jacobian matrix (2.37)–(2.44).

Step (6): Compute the corrected values of bus voltage, using equation (2.31).

Then compute the bus voltage:

$$V_i^{k+1} = V_i^k + \Delta V_i^k \tag{2.47}$$

$$\theta_i^{k+1} = \theta_i^k + \Delta\theta_i^k \tag{2.48}$$

Step (7): Return to Step (4) with new values of bus voltage.

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Example 2.1

The test example for power flow calculation, which is shown in Figure 2.1, is taken from reference [2].

The parameters of the branches are as follows:

$z_{12} = 0.10 + j0.40$
$y_{120} = y_{210} = j0.01528$
$z_{13} = j0.30, k = 1.1$
$z_{14} = 0.12 + j0.50$
$y_{140} = y_{410} = j0.01920$
$z_{24} = 0.08 + j0.40$
$y_{240} = y_{420} = j0.01413$

Buses 1 and 2 are PQ buses, bus 3 is a PV bus, and bus 4 is a slack bus. The given data are:

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$$\Delta F_i = \begin{bmatrix} \Delta P_i \\ \Delta V_i^2 \end{bmatrix} \tag{2.73}$$

$$J_{ij} = \begin{bmatrix} \dfrac{\partial \Delta P_i}{\partial e_j} & \dfrac{\partial \Delta P_i}{\partial f_j} \\ \dfrac{\partial \Delta V_i^2}{\partial e_j} & \dfrac{\partial \Delta V_i^2}{\partial f_j} \end{bmatrix} \tag{2.74}$$

It can be observed from equations (2.60)–(2.68) that the elements of the Jacobian matrix are the function of bus voltage, which will be updated through iterations. The element of the submatrix J_{ij} of the Jacobian matrix in equation (2.69) is related to the corresponding element in bus admittance matrix Y_{ij}. If $Y_{ij} = 0$, then $J_{ij} = 0$. Therefore, the Jacobian matrix in equation (2.69) is also a sparse matrix that is the same as the bus admittance matrix.

The steps of rectangular coordination-based Newton power flow solution are similar to those in the polar coordination-based algorithm, which was described in Section 2.2.2.

Example 2.2

For the same system as in Example 2.1, the Newton method with the rectangle coordinate system is used to solve power flow.

The bus admittance matrix is the same as in Example 2.1. Given the initial values of bus voltages:

$$e_1^0 = e_2^0 = e_3^0 = 1.0,$$
$$f_1^0 = f_2^0 = f_3^0 = 0.0,$$
$$e_4^0 = 1.05, \quad f_4^0 = 0.0$$

Computing the bus power mismatch and ΔV_i^2 with equations (2.52) and (2.55), we get

$$\Delta P_1^0 = P_{1s} - P_1^0 = -0.30 - (-0.02269) = -0.27731$$
$$\Delta P_2^0 = P_{2s} - P_2^0 = -0.55 - (-0.02404) = -0.52596$$
$$\Delta P_3^0 = P_{3s} - P_3^0 = 0.5$$
$$\Delta Q_1^0 = Q_{1s} - Q_1^0 = -0.18 - 0.23767 = -0.41763$$
$$\Delta Q_2^0 = Q_{2s} - Q_2^0 = -0.13 - (-0.14960) = 0.0196$$
$$\Delta V_3^{2(0)} = |V_{3s}|^2 - |V_3^0|^2 = 0.21$$

Computing the elements of the Jacobian matrix with equations (2.60) and (2.68), we get the following correction equation:

$$B'_{ij} = -\frac{1}{x_{ij}} \tag{2.125}$$

$$B'_{ii} = \sum_{j \neq i} \frac{1}{x_{ij}} \tag{2.126}$$

$$B''_{ij} = -\frac{x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{2.127}$$

$$B''_{ii} = -\sum_{j \neq i} B''_{ij} \tag{2.128}$$

where r_{ij}, x_{ij} are the resistance and reactance of branch ij, respectively.

For the BX version, the resistance is ignored during the calculation of B''. The elements of B', B'' are computed as

$$B'_{ij} = \frac{B_{ij}^2 + G_{ij}^2}{B_{ij}} \tag{2.129}$$

$$B'_{ii} = -\sum_{j \neq i} B'_{ij} \tag{2.130}$$

$$B''_{ij} = B_{ij} \tag{2.131}$$

$$B''_{ij} = -2B_{i0} - \sum_{j \neq i} B''_{ij} \tag{2.132}$$

Similarly, the BX version of the fast decoupled power flow model can also be simplified as

$$B'_{ij} = -\frac{x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{2.133}$$

$$B'_{ii} = \sum_{j \neq i} \frac{x_{ij}}{r_{ij}^2 + x_{ij}^2} \tag{2.134}$$

$$B''_{ij} = -\frac{1}{x_{ij}} \tag{2.135}$$

$$B''_{ii} = -\sum_{j \neq i} B''_{ij} \tag{2.136}$$

It is noted that the fast decoupled power flow algorithm may fail to converge when some of the major assumptions such as $r_{ij} \ll x_{ij}$ do not hold. In this situation, Newton power flow or decoupled power flow without major approximation is recommended.

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Example 2.3

In this example we solve the system in Example 2.1 with the decoupled PQ method.

First form the B', B'' matrices as below:

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We will get the necessary conditions for an extreme value of the objective function when we take the first derivative of the above function J with respect to each of the independent variables a, b, and c and set the derivatives equal to zero:

$$\frac{\partial J}{\partial a}=\sum_{k=1}^{n}2P_k^2(aP_k^2+bP_k+c-F_k)=0 \tag{4.5}$$

$$\frac{\partial J}{\partial b}=\sum_{k=1}^{n}2P_k(aP_k^2+bP_k+c-F_k)=0 \tag{4.6}$$

$$\frac{\partial J}{\partial c}=\sum_{k=1}^{n}2(aP_k^2+bP_k+c-F_k)=0 \tag{4.7}$$

From equations (4.5)–(4.7), we get

$$\left(\sum_{k=1}^{n}P_k^2\right)a+\left(\sum_{k=1}^{n}P_k\right)b+nc=\sum_{k=1}^{n}F_k \tag{4.8}$$

$$\left(\sum_{k=1}^{n}P_k^3\right)a+\left(\sum_{k=1}^{n}P_k^2\right)b+\left(\sum_{k=1}^{n}P_k\right)c=\sum_{k=1}^{n}(F_kP_k) \tag{4.9}$$

$$\left(\sum_{k=1}^{n}P_k^4\right)a+\left(\sum_{k=1}^{n}P_k^3\right)b+\left(\sum_{k=1}^{n}P_k^2\right)c=\sum_{k=1}^{n}(F_kP_k^2) \tag{4.10}$$

Coefficients a, b, and c can be obtained by solving equations (4.8)–(4.10).

Example 4.1

We collected some statistical data for a generating unit in one power plant. The capacity limits of the generator are

$$150\le P_G\le 200$$

Four data samples of unit fuel consumption are selected, i.e., 0.405, 0.379, 0.368, and 0.399 (Btu/MW·h), which correspond to power output of 150, 170, 185, and 200 (MW), respectively. The corresponding fuel consumptions are computed and listed in Table 4.1.

From Table 4.1, we get:

$$\sum_{k=1}^{n}P_k=150+170+185+200=705$$

$$\sum_{k=1}^{n}P_k^2=150^2+170^2+185^2+200^2=1.256\times10^5$$

$$\sum_{k=1}^{n}P_k^3=150^3+170^3+185^3+200^3=2.2619\times10^7$$

$$F_T^2 = \sum_{i=1}^{NG} F_i(P_{Gi}) + \gamma_1 P_L \tag{5.244}$$

where

γ_1, γ_2: The coefficient's of converting the system power loss and branch power violation to the fuel consumption, respectively.

The requirement of the two-stage economic dispatch will be

$$F_T^2 \le F_T^1 \tag{5.245}$$

where

F_T^1: The initial system total fuel consumption
F_T^2: The final system total fuel consumption

Example 5.1

The test example is the IEEE 30-bus system with some data change. The modified 30 bus system consists of 5 generation units, 21 loads, 41 transmission lines and transformers.

The fuel consumption functions of the generators are quadratic curves and are shown in Table 5.24. The two-stage economic dispatch results are shown in Tables 5.25 and 5.26.

Table 5.24 The fuel consumption function of generators for IEEE 30-bus system

Gen. No.	a	b	c
1	0.00984	0.33500	0.00000
2	0.00834	0.22500	0.00000
5	0.00850	0.18500	0.00000
11	0.00884	0.13500	0.00000
13	0.00834	0.22500	0.00000

where: $F_i = a_i P_{Gi}^2 + b_i P_{Gi} + c_i$

Table 5.25 The results of generation scheduling for IEEE 30-bus system

Gen. No.	Initial Point	Stage Two for ED
1	54.645	51.305
2	59.480	60.330
5	60.570	61.440
11	57.370	58.190
13	59.480	60.190

Section III: Symmetrical Fault in a Power System

- Calculation of Fault Current Using Impedance Diagram
- Calculation of Fault Current Using Z_{bus} Matrix

Calculation of Fault Current Using Impedance Diagram

Let us first illustrate the calculation of the fault current using the impedance diagram with the help of the following examples.

Example 6.1

Consider the power system of Fig. 6.8 in which a synchronous generator supplies a synchronous motor. The motor is operating at rated voltage and rated MVA while drawing a load current at a power factor of 0.9 (lagging) when a three phase symmetrical short circuit occurs at its terminals. We shall calculate the fault current that flow from both the generator and the motor.

We shall choose a base of 50 MVA, 20 kV in the circuit of the generator. Then the motor synchronous reactance is given by

$$X_m'' = 0.2 \times \frac{50}{25} = 0.4 \quad \text{per unit}$$

Also the base impedance in the circuit of the transmission line is

$$Z_{base} = \frac{66^2}{50} = 87.12 \quad \Omega$$

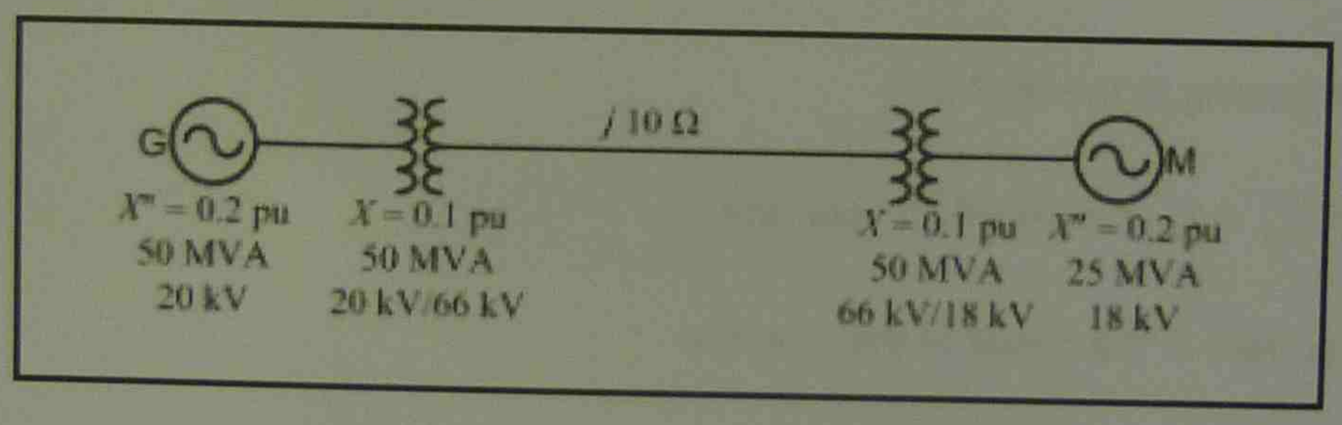

Fig. 6.8 A generator supplying a motor load though a transmission line.

Therefore the impedance of the transmission line is

$$X_{line} = j\frac{10}{87.12} = j0.1148 \quad \text{per unit}$$

The impedance diagram for the circuit is shown in Fig. 6.9 in which the switch *S* indicates the fault.

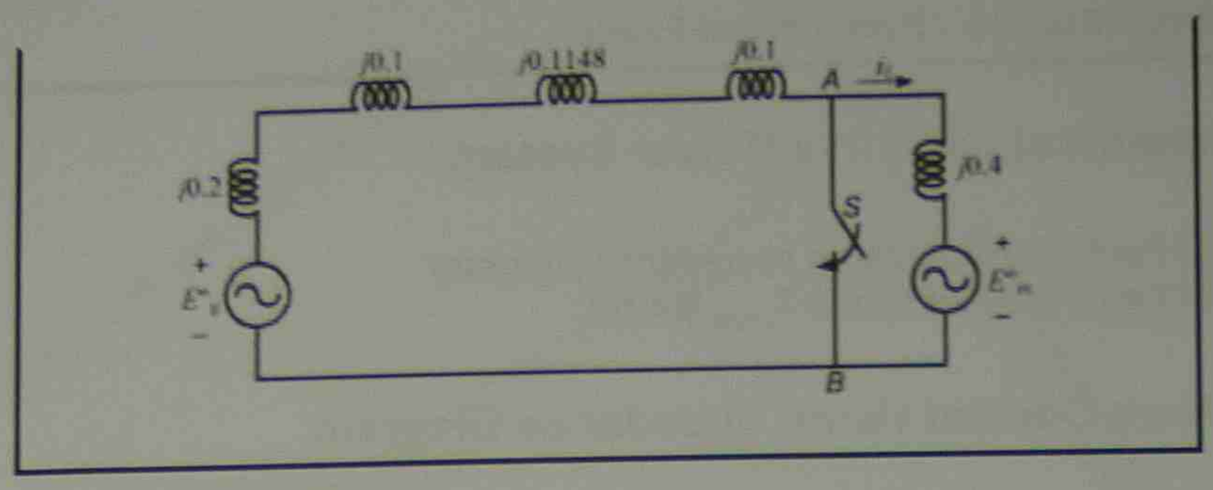

Fig. 6.9 Impedance diagram of the circuit of Fig. 6.8.

The motor draws a load current at rated voltage and rated MVA with 0.9 lagging power factor. Therefore

$$i_L = 1\angle -\cos^{-1}(0.9) = 0.9 - j0.4359 \text{ per unit}$$

Then the subtransient voltages of the motor and the generator are

$$E''_m = 1.0 - j0.4 \times i_L = 0.8256 - j0.36 \text{ per unit}$$

$$E''_g = 1.0 + j0.5148 \times i_L = 1.2244 + j0.4633 \text{ per unit}$$

Hence the subtransient fault currents fed by the motor and the generator are

$$I''_m = \frac{E''_m}{j0.4} = -0.9 - j2.0641 \text{ per unit}$$

$$I''_g = \frac{E''_g}{j0.5148} = 0.9 - j2.3784 \text{ per unit}$$

and the total current flowing to the fault is

$$I''_f = I''_g + I''_m = -j4.4425 \text{ per unit}$$

Note that the base current in the circuit of the motor is

$$I_{base} = \frac{50 \times 10^3}{\sqrt{3} \times 18} = 1603.8 \text{ A}$$

Therefore while the load current was 1603.8 A, the fault current is 7124.7 A.

Example 6.2

We shall now solve the above problem differently. The Thevenin impedance at the circuit between the terminals *A* and *B* of the circuit of Fig. 6.9 is the parallel combination of the impedances *j* 0.4 and *j* 0.5148. This is then given as

$$Z_{th} = j\frac{0.4 \times 0.5148}{0.4 + 0.5148} = j0.2251 \text{ per unit}$$

Since voltage at the motor terminals before the fault is 1.0 per unit, the fault current is

$$I_f'' = \frac{1.0}{Z_{th}} = -j4.4425 \text{ per unit}$$

If we neglect the pre-fault current flowing through the circuit, then fault current fed by the motor and the generator can be determined using the current divider principle, i.e.,

$$I_{m0}'' = \frac{I_f''}{j0.9148} \times j0.5148 = -j2.5 \text{ per unit}$$

$$I_{g0}'' = \frac{I_f''}{j0.9148} \times j0.4 = -j1.9425 \text{ per unit}$$

If, on the other hand, the pre-fault current is not neglected, then the fault current supplied by the motor and the generator are

$$I_m'' = I_{m0}'' - I_L = -0.9 - j2.0641 \text{ per unit}$$

$$I_g'' = I_{g0}'' + I_L = 0.9 - j2.3784 \text{ per unit}$$

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Section III: Symmetrical Fault in a Power System

- Calculation of Fault Current Using Impedance Diagram
- Calculation of Fault Current Using Z_{bus} Matrix

Calculation of Fault Current Using Impedance Diagram

Let us first illustrate the calculation of the fault current using the impedance diagram with the help of the following examples.

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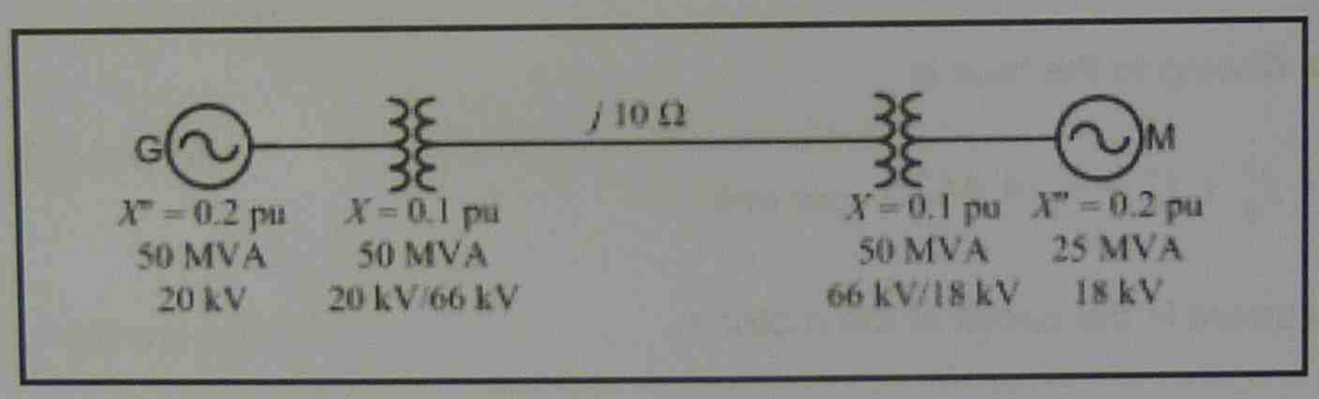

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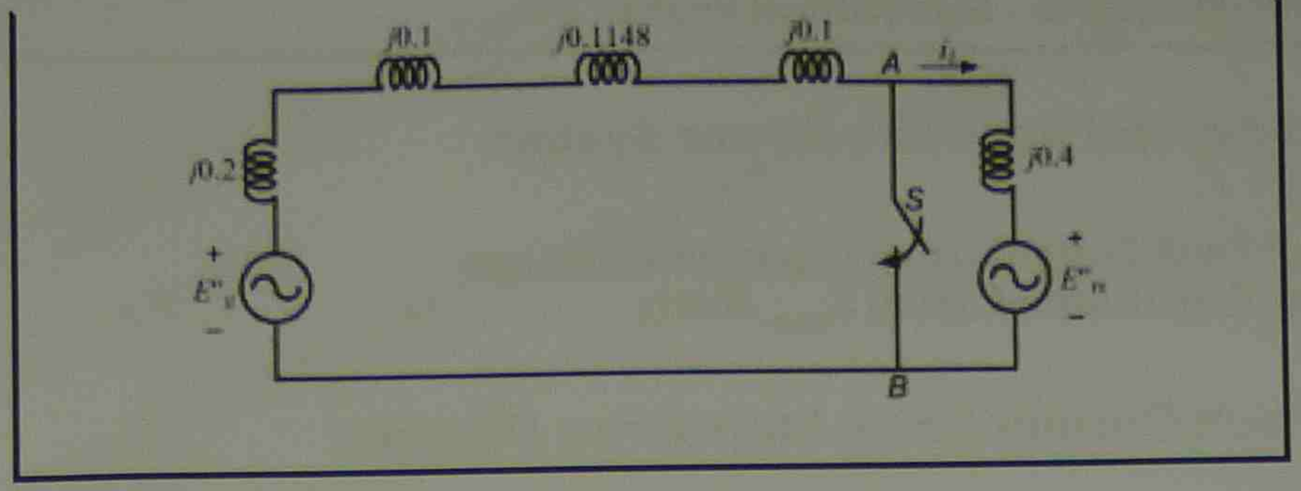
Therefore the impedance of the transmission line is

$$X_{line} = j\frac{10}{87.12} = j0.1148 \quad \text{per unit}$$

The impedance diagram for the circuit is shown in Fig. 6.9 in which the switch *S* indicates the fault.

Let a_7 be the area of X-section of one cond... $\sqrt{2}\,V_m \cos\phi \quad 2\sqrt{2}\,V_m \cos\phi$ ... conductor



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$$Z_{th} = j\frac{0.4 \times 0.5148}{0.4 + 0.5148} = j0.2251 \text{ per unit}$$

FAULT CURRENT COMPUTATION USING SEQUENCE NETWORKS

In this section we shall demonstrate the use of sequence networks in the calculation of fault currents using sequence network through some examples.

Example 8.4

Consider the network shown in Fig. 8.10. The system parameters are given below

Generator G : 50 MVA, 20 kV, $X'' = X_1 = X_2 = 20\%$, $X_0 = 7.5\%$

Motor M : 40 MVA, 20 kV, $X'' = X_1 = X_2 = 20\%$, $X_0 = 10\%$, $X_n = 5\%$

Transformer T_1 : 50 MVA, 20 kV Δ /110 kVY, $X = 10\%$

Transformer T_2 : 50 MVA, 20 kV Δ /110 kVY, $X = 10\%$

Transmission line: $X_1 = X_2 = 24.2\ \Omega$, $X_0 = 60.5\ \Omega$

We shall find the fault current for when a (a) 1LG, (b) LL and (c) 2LG fault occurs at bus-2.

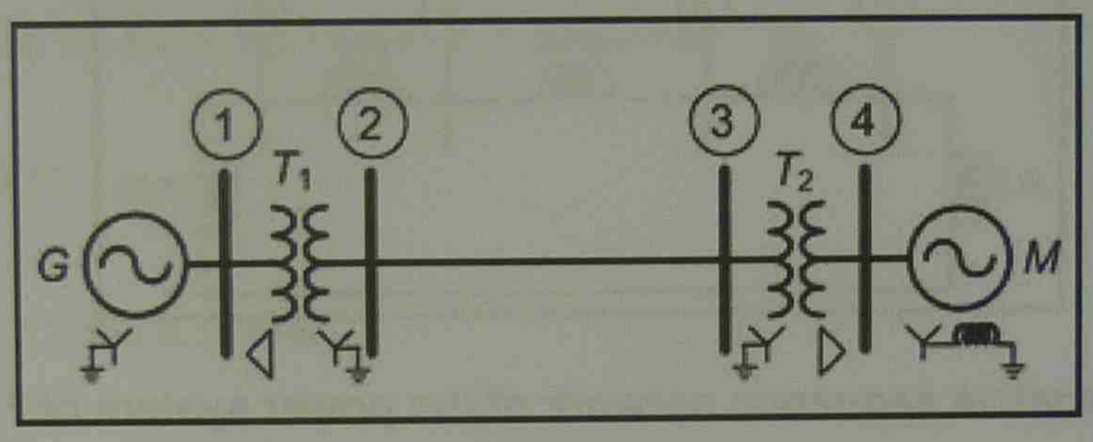


Fig. 8.10 Radial power system of Example 8.4.

Let us choose a base in the circuit of the generator. Then the per unit impedances of the generator are:

$$X_{G1} = X_{G2} = 0.2, \quad X_{G0} = 0.075$$

The per unit impedances of the two transformers are

$$X_{T1} = X_{T2} = 0.1$$

The MVA base of the motor is 40, while the base MVA of the total circuit is 50. Therefore the per unit impedances of the motor are

$$X_{M1} = X_{M2} = 0.2 \times \frac{50}{40} = 0.25, \quad X_{M0} = 0.1 \times \frac{50}{40} = 0.125, \quad X_n = 0.05 \times \frac{50}{40} = 0.0625$$

For the transmission line

Therefore the fault currents flowing in the line are

$$\begin{bmatrix} I_{fa} \\ I_{fb} \\ I_{fc} \end{bmatrix} = C^{-1} \begin{bmatrix} I_{fa0} \\ I_{fa1} \\ I_{fa2} \end{bmatrix} = \begin{bmatrix} 0 \\ 6.657\angle 137.11^\circ \\ 6.657\angle 42.89^\circ \end{bmatrix}$$

Furthermore the sequence components of bus-2 voltages are

$$V_{2a0} = -j0.0778 I_{fa0} = 0.2318$$
$$V_{2a1} = 1 - j0.18 I_{fa1} = 0.2318$$
$$V_{2a2} = -j0.18 I_{fa2} = 0.2318$$

Therefore voltages at the faulted bus are

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = C^{-1} \begin{bmatrix} V_{2a0} \\ V_{2a1} \\ V_{2a2} \end{bmatrix} = \begin{bmatrix} 0.6954 \\ 0 \\ 0 \end{bmatrix}$$

Example 8.5

Let us now assume that a 2LG fault has occurred in bus-4 instead of the one in bus-2. Therefore

$$X_1 = X_2 = j0.1667, \quad X_0 = j0.3125$$

Also we have

$$Z_{eq} = j0.1667 \| j0.3125 = j0.1087$$

Hence

$$I_{fa1} = \frac{1}{j0.1667 + Z_{eq}} = -j3.631 \text{ per unit}$$

Also

$$I_{fa0} = -I_{fa1} \frac{j0.1667}{j(0.1667 + 0.3125)} = j1.2631 \text{ per unit}$$

$$I_{fa2} = -I_{fa1} \frac{j0.3125}{j(0.1667 + 0.3125)} = j2.3678 \text{ per unit}$$

Therefore the fault currents flowing in the line are

2

The pn Junction

The diode is the simplest bipolar semiconductor device. It comprises p-type and n-type semiconductor materials brought together, usually after diffusion or implantation, to form a (step or abrupt) junction as shown in figures 2.1a and 2.3a.

A *depletion layer*, or alternatively a *space charge layer*, scl, is built up at the junction as a result of diffusion caused by the large carrier concentration gradients. The holes diffuse from the p-side into the n-side while electrons diffuse from the n-side to the p-side, as shown in figure 2.1b. The n-side, losing electrons, is charged positively because of the net donor charge left behind, while the p-side conversely becomes negatively charged. An electric potential barrier, ξ, builds up, creating a drift current which opposes the diffusion flow, both of which balance at thermo-dynamic equilibrium as shown in figure 2.1c. The electric field is a maximum at the junction and zero at the scl edges. There are no free carriers in the scl.

The zero external bias, built-in, junction potential or scl potential is given by

$$\Phi = \frac{kT_j}{q}\,\ell n\frac{N_A N_D}{n_i^2} \qquad \text{(V)} \qquad (2.1)$$

where q is the electron charge, 1.6×10^{-19} C
k is Boltzmann's constant, 1.38×10^{-23} J/K
T_j is the junction temperature, K.
Thus $\Phi = kT_j / q = 0.0259$ eV at room temperature, 300 K.

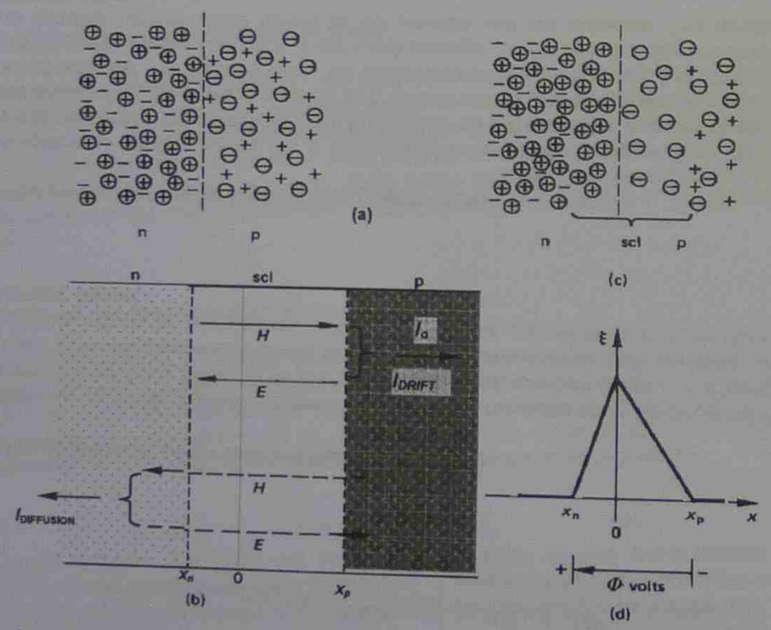


Figure 2.1. *The step junction: (a) the junction if carriers did not diffuse: ⊕ionised donors, ⊖ionised acceptors, + holes and - electrons; (b) electron and hole movements: ---- diffusion flow, —— drift flow; (c) ionised impurities and free carriers equilibrium distribution; and (d) scl electric field and voltage.*

BWW

One important feature of the pn junction is that current (holes) flows freely in the p to n direction when forward-biased, that is, the p-region is biased positive with respect to the n-region. Only a small leakage current flows in the reverse voltage bias case. This asymmetry makes the pn junction diode useful as a rectifier, exhibiting static current-voltage I-V characteristics as illustrated in figure 2.2.

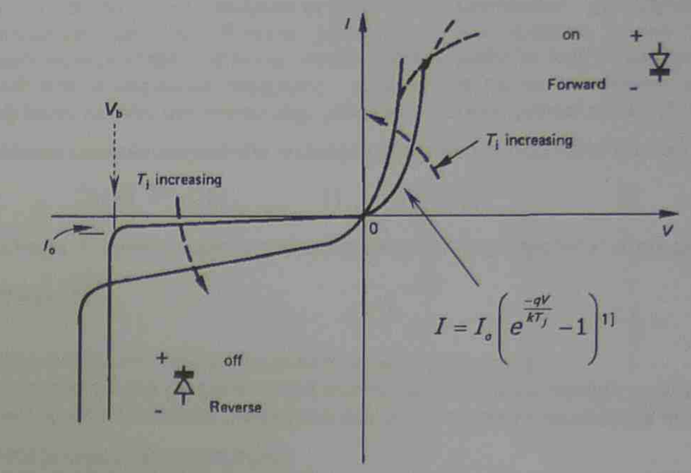


Figure 2.2. *Typical I-V static characteristics of a silicon pn junction diode, and the effects of junction temperature, T_j.*

Example 2.1: ***Built-in potential of an abrupt junction***

A silicon abrupt p-n junction has a p-type region of 2×10^{16} cm^{-3} acceptors and an n-type region containing 2×10^{16} cm^{-3} acceptors in addition to 10^{17} cm^{-3} donors.

Calculate

i. the thermal equilibrium density of electrons and holes in the p-type region, and both densities in the n-type region
ii. the built-in potential of the p-n junction at room temperature
iii. the built-in potential of the p-n junction at 400K, assuming the intrinsic concentration increases 300 fold over that at 300K

Solution

i. The thermal equilibrium densities, using $n_i^2 = p \times n$ from Chapter 1, are:

in the p-type region

$$p = N_A = 2 \times 10^{16}\ \text{cm}^{-3}$$

$$n = n_i^2/p = \frac{(1.5 \times 10^{10})^2}{2 \times 10^{16}\ \text{cm}^3} = 1.125 \times 10^3\ \text{cm}^{-3}$$

in the n-type region

$$n = N_D - N_A = 8 \times 10^{16}\ \text{cm}^{-3}$$

$$p = n_i^2/n = \frac{(1.5 \times 10^{10})^2}{8 \times 10^{16}\ \text{cm}^3} = 2.813 \times 10^3\ \text{cm}^{-3}$$

ii. The built-in potential at room temperature from equation (2.1) is

$$\Phi = \frac{kT_j}{q}\,\ell n\frac{p_n n_p}{n_i^2} = 0.0259\text{V} \times \ell n\left(\frac{2 \times 10^{16} \times 8 \times 10^{16}}{(1.5 \times 10^{10})^2}\right) = 0.766\text{V}$$

iii. The intrinsic carrier density is temperature dependant, and increases to $300 \times 1.5 \times 10^{10} = 4.5 \times 10^{12}$ at 400K.

Silicon carbide diodes have a higher temperature coefficient, typically +8mV/K.

The avalanche voltage increases with temperature, as does the reverse leakage current. The effects of temperature on the reverse bias characteristics are shown in figure 2.2. In the case of silicon carbide, increased temperature decreases the avalanche voltage and increases the leakage current.

The silicon temperature coefficient for avalanche is positive since the mean distance between collisions is reduced due to the increased thermal energy, which increases the vibrational amplitude. Higher electric fields are necessary for the carriers to gain sufficient kinetic energy for ionisation.

Equation (2.6) also indicates that the reverse bias current increases with increased junction temperature. This positive temperature coefficient does not generally result in thermal instability with silicon devices, provided sufficient heat sinking is employed on smaller devices.

Example 2.2: ***Diode forward bias characteristics***

A pn junction diode has a reverse saturation current of 100nA at a junction operating temperature of 28°C. What is the forward current when the forward bias voltage is 0.5V and the dynamic ac resistance at that current?

Solution

The diode current is given by equation (2.6), namely

$$I(v) = I_o[e^{-qv/kT_j} - 1] \quad \text{(A)}$$

where $V_T = q/kT_j = 0.0259\text{V}$ at 300K, 28°C.

$$I(0.5\text{V}) = 100\times10^{-9}\times[e^{v/0.0259} - 1]$$
$$= 26.1\text{A}$$

Differentiating the diode I-V equation with respect to voltage gives

$$\frac{di}{dv} = \frac{I_o}{V_T}e^{\frac{V_F}{V_T}} = \frac{1}{R_{ac}}$$
$$= \frac{100\times10^{-9}\text{A}}{0.0258\text{V}}\times e^{\frac{0.5\text{V}}{0.0258\text{V}}}$$
$$\Rightarrow R_{ac} = 1.0\text{m}\Omega$$

♣

2.4 Models for the bipolar junction diode

Semiconductor device electrical models are used extensively for power electronic circuit simulation. A basic piecewise-linear model is applicable to simple manual calculations, where the terminal I-V characteristics are empirically modelled based on ideal circuit elements. A more complex and accurate model is required for computer transient circuit analysis simulation. Such accurate models are based on the semiconductor physics of the device. Many power switching semiconductor device manufacturers provide values for the model parameters suitable for circuit simulation in the packages PSpice and SABER.

2.4.1 *Piecewise-linear junction diode model*

The pn junction diode is a unilateral device that, to a good approximation, conducts current in only one direction. Figure 2.4a shows a *piecewise-linear* (pwl) model of the diode that is suitable for static modelling in power electronic circuits. It includes a perfect diode, an on-state voltage source E_o, and a series resistor of resistance value R_o to account for the slope in the actual forward conduction characteristic. The forward I-V characteristic at a given temperature is given by

$$V_F(I_F) = E_o + I_F R_o \quad \text{for } V_F > E_o \quad \text{(V)} \qquad (2.8)$$

The model in figure 2.4a does not incorporate the static reverse characteristics of leakage and avalanche. These are shown in figure 2.4b, where V_b from equation (2.3) models the avalanche limit and $R_l\ (=V_b/I_o)$ gives linear leakage current properties for a given junction temperature. The three diode components in figure 2.4b are assumed ideal.

The model given by equation (2.8) is adequate for calculation of static balancing requirements of parallel and series connected diodes and thyristors, as considered in section 10.1 and the associated problems, 10.4, 10.5, and 10.9 to 10.12.

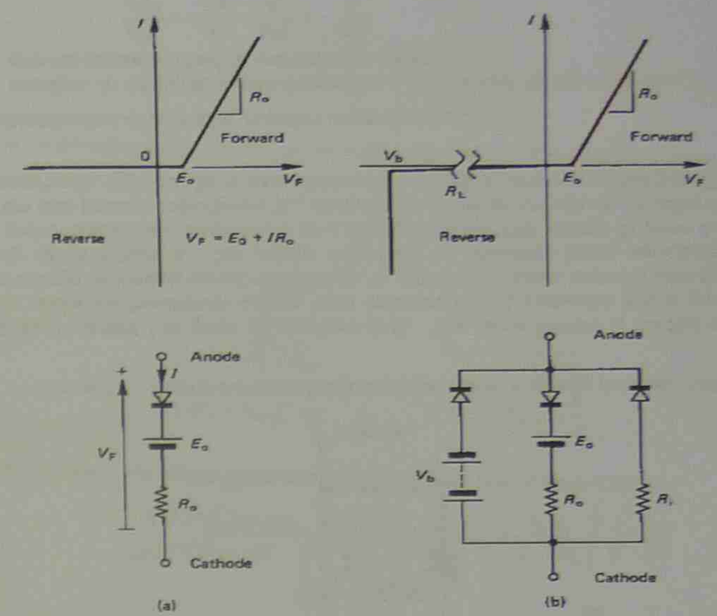


Figure 2.4. *Piecewise-linear approximations of junction diode characteristics: (a) ideal diode with an offset voltage and resistance to account for slope in the forward characteristic and (b) model including reverse bias characteristics.*

Example 2.3: ***Using the pwl junction diode model***

An approximation to the forward I-V characteristic of the diode shown in figure 2.4a, is given by $V_F(I_F) = 1.0 + 0.01\,I_F$. For a constant current of 45A for ⅔ of a cycle, calculate the diode

i. on-state voltage;
ii. mean power loss; and
iii. rms current.

Solution

i. The on-state voltage at 45A is given by

$$V_F(45\text{A}) = 1.0 + I_F\,0.01 = 1.0\text{V} + 45\text{A}\times0.01\Omega = 1.45\text{V}$$

ii. If the on-state duty cycle is $\delta = ⅔$, the average power loss is

$$\bar{P} = \delta\times V_F\times I_F = \tfrac{2}{3}\times1.45\text{V}\times45\text{A} = 43.2\text{W}$$

iii. The diode rms current is given by

$$I_{rms} = \sqrt{\delta}\times I_{dc} = \sqrt{\tfrac{2}{3}}\times45\text{A} = 36.7\text{A}$$

♣

Example 2.4: ***Static linear diode model***

A Schottky diode is used to half-wave rectify a square wave ±15V source in series with a 1Ω load resistor. If the diode model shown in figure 2.4b is modelled with R_o = 0.01 Ω, E_o = 0.2V, R_l = 1000Ω, and V_b = 30V, determine:

i. the diode model forward and reverse bias operating point equations for the series circuit
ii. the load current and diode voltage
iii. the rectifier losses (neglecting any recovery effects) and the load power dissipation
iv. estimate the power dissipated in the load if the source is ac with the same fundamental component as the square wave
v. what is the non-fundamental power dissipated with the square wave source?

diode assuming equations (2.3) and (2.5) are valid.
[15.3 V, 5.27×10^{16} /cc, 0.63μm]

2.3. An abrupt silicon

Example 2.5: *Space charge layer parameter values*

A 10μm thick p-type 2×10^{16} /cc silicon epitaxial layer is grown on an n^+ - type 1×10^{14} /cc silicon substrate, of area 1 cm^2, to form an abrupt pn junction.

Calculate the following PSpice parameter values, at room temperature:

i. zero bias junction potential, Φ;
ii. zero bias scl width, maximum electric field, charge, and junction capacitance, W_0, ξ_0, Q_o, C_{jo}; and
iii. avalanche breakdown voltage, V_b.

If the substrate is 150μm thick, for a 1000V reverse bias, calculate:

iv. scl width and penetration depth each side of the junction, W, x_n, x_p;
v. charge each side of the junction, maximum electric field, and the capacitance, Q, ξ_j, C_j.

Solution

i. From equation (2.1), the zero bias built-in voltage is

$$\Phi = \frac{kT_j}{q}\,\ell n\frac{N_A N_D}{n_i^2} = 0.0259\,\ell n\frac{2\times10^{16}\times 1\times10^{14}}{2.25\times10^{20}}$$
$$= 0.0259\times\ell n(8.89\times10^{9}) = 0.534\text{V}$$

ii. From equations (2.15), (2.20), (2.17), and (2.19)

$$W_0 = \sqrt{\left[\frac{2\Phi\varepsilon_s}{q}\times\left(\frac{1}{N_A}+\frac{1}{N_D}\right)\right]}$$

$$W_0 = \sqrt{\left[\frac{2\times0.53\,\text{V}\times11.8\times8.85\times10^{-12}}{1.6\times10^{-19}}\times\left\{\frac{1}{2\times10^{22}}+\frac{1}{1\times10^{20}}\right]\right.} = 2.65\mu\text{m}$$

$$\xi_0 = 2\,\Phi / W = 2\times0.534\,/\,2.65\mu\text{m}$$
$$\xi_0 = 0.40\,\text{MV/m}$$

$$Q_o = A\left[2\Phi q\varepsilon_s\frac{N_D N_A}{N_D+N_A}\right]^{\frac{1}{2}}$$

$$Q_o = 1\times10^{-4}\sqrt{2\times0.53\text{V}\times1.6\times10^{-19}\times11.8\times8.85\times10^{-12}\times\frac{2\times10^{42}}{2.01\times10^{22}}} = 4.21\,\text{nC}$$

$$C_{jo} = A\left[\frac{q\varepsilon_s}{2\Phi}\frac{N_D N_A}{N_D+N_A}\right]^{\frac{1}{2}}$$

$$C_{jo} = 1\times10^{-4}\times\left[\frac{1.6\times10^{-19}\times11.8\times8.85\times10^{-12}}{2\times0.53\text{V}}\times\frac{2\times10^{42}}{2.01\times10^{22}}\right]^{\frac{1}{2}}$$
$$= 3.95\times10^{-9}\text{F} = 3.95\,\text{nF}$$

iii. From equation (2.2), the estimated punch-through voltage is

$$V_{PT} = 7.67\times10^{-16}N_c W_c^2 = 7.67\times10^{-16}\times10^{14}\times(150)^2$$
$$= 1727\text{V}$$

That is, punch through occurs when the reverse bias is greater than the operating voltage, 1000V. If the diode is to breakdown due to avalanche then the avalanche breakdown voltage given by V_b (equation (2.3)) must be less than V_{PT}, 1727V.

$$V_b = 5.34\times10^{13}\,N_c^{-¾} = 5.34\times10^{13}\times(1\times10^{14})^{-¾}$$
$$= 1689\text{V}$$

iv. From equation (2.15) the scl width at -1000V reverse bias is

$$W_{scl} = W_0\sqrt{1-\frac{V}{\Phi}} = 2.65\mu\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$= 114.6\mu\text{m}$$

From equation (2.16) the scl penetration into each side of the junction at -1000V is

$$x_n = \frac{W_{scl}}{1+N_D/N_A} = \frac{114.6}{1+0.005} \qquad x_p = \frac{W_{scl}}{1+N_A/N_D} = \frac{114.6}{1+200}$$

$$x_n = 114.0\mu\text{m} \qquad x_p = 0.57\mu\text{m}$$

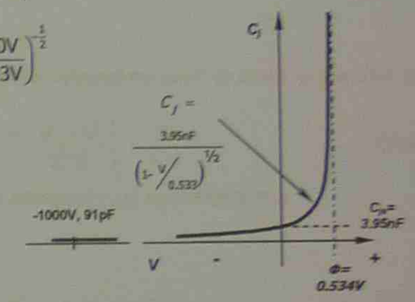
Note that when $N_A >> N_D$, $x_n \approx W_{scl}$, thus the lower the relative concentration of N_D, the deeper the scl penetration and the higher the portion of V supported in N_D. The junction scl can under these circumstances be analysed based on simplified equations – called one-sided junction equations.

v. The charge magnitude each side of the junction, shown in figure 2.6, is given by equation (2.17). The electric field at the junction is given by equation (2.20), while the junction capacitance at -1000V is given by equation (2.19):

$$Q_j = Q_0\sqrt{1-\frac{V}{\Phi}} = 4.2\text{nC}\times\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$= 182{\cdot}4\,\text{nC}$$

$$\xi_j = \xi_0\sqrt{1-\frac{V}{\Phi}} = 0.40\text{M}\times\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$\xi_j = 17.5\text{MV/m} \qquad (<\xi_b = 25\text{MV/m})$$

$$C_j = C_{jo}\left(1-\frac{V}{\Phi}\right)^{\frac{1}{2}} = 3.95\text{n}\times\left(1+\frac{1000\text{V}}{0.533\text{V}}\right)^{\frac{1}{2}}$$
$$C_j = 91\,\text{pF}$$



Reading list

See chapter 1 reading list.

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Yang, E. S., *Fundamentals of Semiconductor Devices*, McGraw-Hill, 1978.

Example 2.5: ***Space charge layer parameter values***

A 10μm thick p-type 2×10^{16} /cc silicon epitaxial layer is grown on an n$^+$ - type 1×10^{14} /cc silicon substrate, of area 1 cm^2, to form an abrupt pn junction.

Calculate the following PSpice parameter values, at room temperature:

i. zero bias junction potential, Φ;
ii. zero bias scl width, maximum electric field, charge, and junction capacitance, W_0, ξ_0, Q_o, C_{jo}; and
iii. avalanche breakdown voltage, V_b.

If the substrate is 150μm thick, for a 1000V reverse bias, calculate:

iv. scl width and penetration depth each side of the junction, W, x_n, x_p;
v. charge each side of the junction, maximum electric field, and the capacitance, Q, ξ_j, C_j.

Solution

i. From equation (2.1), the zero bias built-in voltage is

$$\Phi = \frac{kT_j}{q}\,\ell n\frac{N_A N_D}{n_i^2} = 0.0259\,\ell n\frac{2\times10^{16}\times 1\times10^{14}}{2.25\times10^{20}}$$
$$= 0.0259\times\ell n(8.89\times10^{9}) = 0.534\text{V}$$

ii. From equations (2.15), (2.20), (2.17), and (2.19)

$$W_o = \sqrt{\left[\frac{2\Phi\,\varepsilon_s}{q}\times\left(\frac{1}{N_A}+\frac{1}{N_D}\right)\right]}$$

$$W_o = \sqrt{\left[\frac{2\times0.53\,\text{V}\times11.8\times8.85\times10^{-12}}{1.6\times10^{-19}}\times\left\{\frac{1}{2\times10^{22}}+\frac{1}{1\times10^{20}}\right]\right.} = 2.65\mu\text{m}$$

$$\xi_{so} = 2\,\Phi / W = 2\times0.534 / 2.65\mu\text{m}$$
$$\xi_{so} = 0.40\,\text{MV/m}$$

$$Q_o = A\left[2\Phi q\varepsilon_s\frac{N_D N_A}{N_D+N_A}\right]^{\frac{1}{2}}$$

$$Q_o = 1\times10^{-4}\sqrt{2\times0.53\text{V}\times1.6\times10^{-19}\times11.8\times8.85\times10^{-12}\times\frac{2\times10^{42}}{2.01\times10^{22}}} = 4.21\,\text{nC}$$

$$C_{jo} = A\left[\frac{q\varepsilon_s}{2\Phi}\frac{N_D N_A}{N_D+N_A}\right]^{\frac{1}{2}}$$

$$C_{jo} = 1\times10^{-4}\times\left[\frac{1.6\times10^{-19}\times11.8\times8.85\times10^{-12}}{2\times0.53\text{V}}\times\frac{2\times10^{42}}{2.01\times10^{22}}\right]^{\frac{1}{2}}$$
$$= 3.95\times10^{-9}\text{F} = 3.95\,\text{nF}$$

iii. From equation (2.2), the estimated punch-through voltage is

$$V_{PT} = 7.67\times10^{-16}N_c W_c^2 = 7.67\times10^{-16}\times10^{14}\times(150)^2$$
$$= 1727\text{V}$$

That is, punch through occurs when the reverse bias is greater than the operating voltage, 1000V. If the diode is to breakdown due to avalanche then the avalanche breakdown voltage given by V_b (equation (2.3)) must be less than V_{PT}, 1727V.

$$V_b = 5.34\times10^{13}\,N_c^{-\frac{3}{4}} = 5.34\times10^{13}\times(1\times10^{14})^{-\frac{3}{4}}$$
$$= 1689\text{V}$$

iv. From equation (2.15) the scl width at -1000V reverse bias is

$$W_{scl} = W_o\sqrt{1-\frac{V}{\Phi}} = 2.65\mu\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$= 114.6\mu\text{m}$$

From equation (2.16) the scl penetration into each side of the junction at -1000V is

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$$x_n = 114.0\mu\text{m} \qquad x_p = 0.57\mu\text{m}$$

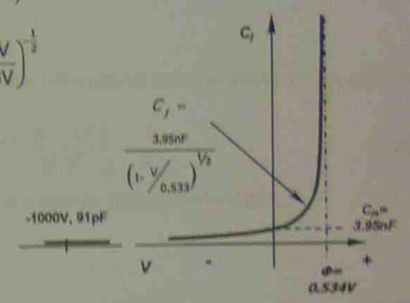
Note that when $N_A >> N_D$, $x_n \approx W_{scl}$, thus the lower the relative concentration of N_D, the deeper the scl penetration and the higher the portion of V supported in N_D. The junction scl can under these circumstances be analysed based on simplified equations – called one-sided junction equations.

v. The charge magnitude each side of the junction, shown in figure 2.6, is given by equation (2.17). The electric field at the junction is given by equation (2.20), while the junction capacitance at -1000V is given by equation (2.19):

$$Q_j = Q_o\sqrt{1-\frac{V}{\Phi}} = 4.2\text{nC}\times\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$= 182{\cdot}4\,\text{nC}$$

$$\xi_j = \xi_{so}\sqrt{1-\frac{V}{\Phi}} = 0.40\text{M}\times\sqrt{1+\frac{1000\text{V}}{0.533\text{V}}}$$
$$\xi_j = 17.5\,\text{MV/m} \qquad (<\xi_s = 25\text{MV/m})$$

$$C_j = C_{jo}\left(1-\frac{V}{\Phi}\right)^{-\frac{1}{2}} = 3.95\text{n}\times\left(1+\frac{1000\text{V}}{0.533\text{V}}\right)^{-\frac{1}{2}}$$
$$C_j = 91\,\text{pF}$$



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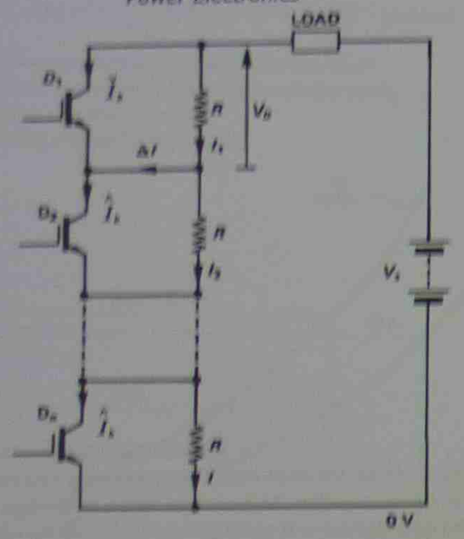


Figure 10.2. *Series IGBT string with resistive shunting for sustaining voltage equalisation in the off-state.*

By symmetry and Kirchhoff's voltage law, the total string voltage to be supported, V_s is given by

$$V_s = (n-1)\,I_2\,R + V_D \qquad \text{(V)} \qquad (10.4)$$

Eliminating ΔI, I_1, and I_2 from equations (10.1) to (10.4) yields

$$\hat{R} \le \frac{nV_D - V_s}{(n-1)\left(\hat{I}_s - \hat{I}_s\right)} \qquad \text{(ohms)} \qquad (10.5)$$

for $n \ge 2$.

Generally only the maximum leakage current at rated voltage and maximum junction temperature is specified. By assuming $\hat{I}_s = 0$, a conservative value of the maximum allowable resistance is obtained, namely

$$\hat{R} \le \frac{nV_D - V_s}{(n-1)\,\hat{I}_s} = \frac{n(1-k_s)V_D}{(n-1)\,\hat{I}_s} \qquad \text{(ohms)} \qquad (10.6)$$

The extent to which nV_D is greater than V_s, is termed the voltage sharing factor, namely

$$k_s = {}^{V_s}\!/_{nV_D} \le 1 \qquad (10.7)$$

As the number of devices is minimized the sharing factor approaches one, but equation (10.5) shows that undesirably the resistance for sharing decreases, hence losses increase.

The power dissipation of the resistor experiencing the highest voltage is given by

$$\hat{P}_R = V_D^2 / \hat{R} \qquad \text{(W)} \qquad (10.8)$$

If resistors of ± 100a per cent resistance tolerance are used, the worst case occurs when cell D_1 has a parallel resistance at the upper tolerance while all the others have parallel resistance at the lower limit. After using $V_D = (1+a)I_1R$ and $V_s = (n-1)(1-a)I_2R + V_D$ for equations (10.3) and (10.4), the maximum resistance is given by

$$\hat{R} \le \frac{n(1-a)V_D - (1+a)V_s}{(n-1)\left(1-a^2\right)\hat{I}_s} \qquad \text{(ohms)} \qquad (10.9)$$

for $n \ge 2$.

The maximum loss in a resistor is

$$\hat{P}_D = V_D^2 / \hat{R}(1-a) \qquad (10.10)$$

If the supply toleration is incorporated, then V_s in equations (10.6) and (10.9) is replaced by $(1+b)\times V_s$ where $100b$ is the supply percentage upper tolerance. This leads to a decreased resistance requirement, hence increased resistor power losses.

$$\hat{R} \le \frac{n(1-a)V_D - (1+a)(1+b)V_s}{(n-1)\left(1-a^2\right)\hat{I}_s} \qquad \text{(ohms)} \qquad (10.11)$$

The effects and importance of just a few per cent resistance or supply voltage tolerance on the maximum value for the sharing resistors and their power losses, are illustrated by the following example.

Example 10.1: ***Series device connection – static balancing***

Ten, 200 V reverse-blocking, ultra fast 35 ns reverse recovery diodes are to be employed in series in a 1500 V peak, string voltage application. If the maximum device reverse leakage current is 10 mA calculate the sharing factor, and for worst case conditions, the maximum value of sharing resistance and power dissipation.

- If 10 per cent tolerance resistors are employed, what is the maximum sharing resistance and its associated power rating?
- If a further allowance for supply voltage tolerance of ±5% is incorporated, what is the maximum sharing resistance and its associated power rating?

Solution

When $n = 10$, $V_D = 200$ V, $V_s = 1500$ V, and $\hat{I}_s = 10$ mA, the sharing factor is $k_s = 1500\text{V}/10\times200\text{V} = 0.75$. Equation (10.6) yields the maximum allowable sharing resistance

$$\hat{R} \le \frac{nV_D - V_s}{(n-1)\,\hat{I}_s} = \frac{10\times200\text{V} - 1500\text{V}}{(10-1)\times10\text{mA}} = 5.55\text{k}\Omega$$

The nearest (lower) preferred value, 4.7 kilohms, would be used.

Maximum resistor power losses occur when the diodes are continuously blocking. The maximum individual supporting voltage appears across the diode which conducts the least leakage current. Under worst case conditions this diode therefore supports voltage V_{Dx} hence maximum power loss $\hat{P}_D$ is

$$\hat{P}_D = V_D^2 / \hat{R}$$
$$= 200^2/4700\Omega = 8.5\text{ W}$$

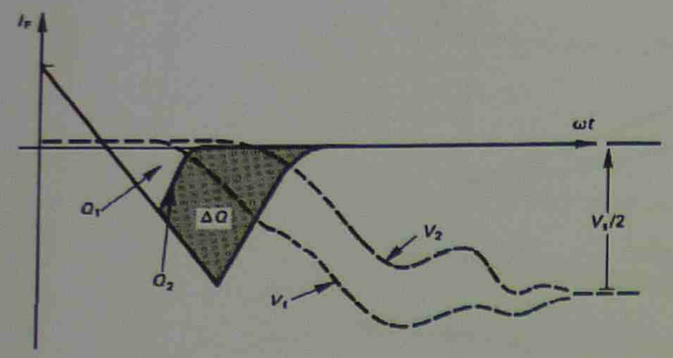


Figure 10.4. *Reverse recovery current and voltage for two mismatched series connected diodes.*

The total string voltage V_s, comprises the voltage across the fast-recovery device V_D plus the sum of the voltages across the slow n - 1 devices, V_{slow}. That is

$$V_s = V_D + (n - 1)V_{slow} \quad \text{(V)} \qquad (10.13)$$

The voltage across the slow devices is given by

$$V_{slow} = \frac{1}{n}\left(V_s - \Delta\hat{V}\right) \quad \text{(V)} \qquad (10.14)$$

where $\Delta\hat{V} = \Delta\hat{Q}/C$.

Eliminating V_{slow} from equations (10.13) and (10.14) yields

$$\breve{C} \geq \frac{(n-1)\,\Delta Q}{nV_D - V_s} = \frac{(n-1)\,\Delta Q}{n(1-k_s)V_D} \quad \text{(F)} \qquad (10.15)$$

This equation shows that as the number of devices is minimized, the sharing factor, k_s, which is in the denominator of equation (10.15), tends to one and the capacitance requirement undesirably increases.

Manufacturers do not specify the minimum reverse recovery charge but specify the maximum reverse recovery charge for a given initial forward current, reverse recovery *di/dt*, and temperature. For worst case design, assume $\breve{Q} = 0$, thus

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{nV_D - V_s} \quad \text{(F)} \qquad (10.16)$$

Sharing circuit design is complicated if the effects of reverse steady-state leakage current in ac thyristor blocking are taken into account.

Supply and sharing capacitance tolerances significantly affect the minimum capacitance requirement. Worst case assumptions for capacitance tolerances involve the case when the fastest recovering diode is in parallel with capacitance at its lower tolerance limit while all the other sharing capacitances are at their upper tolerance limit. Assuming the minimum reverse recovery charge is zero, then the minimum sharing capacitance requirement is

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{(1-a)(nV_D - V_s)} \quad \text{(F)} \qquad (10.17)$$

where -100a is the capacitor negative percentage tolerance and $n \geq 2$. Voltage sharing resistors help minimise capacitor static voltage variation due to capacitance variations.

If the supply tolerance is incorporated, then V_s in equations (10.16) and (10.17) are replaced by $(1+b)\times V_s$ where +100b is the supply percentage upper tolerance. This leads to an increased capacitance requirement, hence increased energy losses, $\frac{1}{2}CV_D^2$.

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{(1-a)(nV_D - (1+b)V_s)} \quad \text{(F)} \qquad (10.18)$$

Example 10.2: ***Series device connection – dynamic balancing***

The string of ten, 200 V diodes in worked example 10.1 is to incorporate capacitive reverse recovery transient sharing. Using the data in chapter 5, figure 5.9, specify a suitable sharing capacitance based on zero capacitance and supply tolerances (a = b = 0), ± 10 per cent capacitance tolerances (a = 0.1, b = 0), ± 5 per cent supply tolerance (a = 0, b = 0.05), then both tolerances (a = 0.1, b = 0.05). Estimate in each case the energy loss due to capacitor discharge.

Solution

Figure 5.9 shows that worst case reverse recovery conditions occur at maximum junction temperature, *di/dt*, and I_F, and a value of $\hat{Q}$ =6μC is appropriate.

The minimum possible sharing capacitance occurs when the capacitance and dc rail voltage are tightly specified. From equation (10.16)

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{nV_D - V_s} = \frac{(10\text{-}1)\times 6\mu\text{C}}{10\times 200 \text{ - } 1500\text{V}} = 108\text{nF}$$

The sharing capacitance requirement with 10% tolerance capacitors, is given by equation (10.17),

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{(1-a)(nV_D - V_s)} = \frac{(10 - 1)\times 6\mu\text{C}}{(1 - 0.1)\times(10\times 200\text{V-}1500\text{V})} = 0.12\mu\text{F @ 200Vdc}$$

A further increase in capacitance requirements results if the upper tolerance dc rail voltage is used. From equation (10.18)

$$\breve{C} \geq \frac{(n-1)\,\hat{Q}}{(1-a)(nV_D - (1+b)V_s)}$$

$$= \frac{(10 - 1)\times 6\mu\text{C}}{(1 - 0.1)\times(10\times 200\text{V} - (1+0.05)\times 1500\text{V})} = 0.14\mu\text{F @ 200Vdc}$$

In each tolerance case the next larger preferred capacitance value should be used, namely, 120nF, 120nF, and 150nF respectively.

$$pd = \left(1 - \frac{I_T}{nI_m}\right) \times 100 \quad = \left(1 - k_p\right) \times 100 \quad \text{per cent} \qquad (10.20)$$

where I_T = total current through the parallel arrangement
I_m = maximum allowable single device current rating
n = number of parallel devices
k_p = parallel sharing factor $= I_T / nI_m$

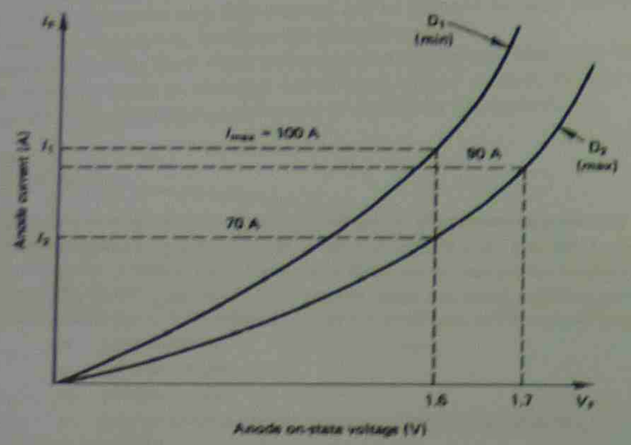


Figure 10.6. *Forward conduction characteristics of two unmatched devices.*

10.1.2ii - External forced current sharing

Derating does not account for effects such as layout and electrical and thermal impedance imbalance. The amount of derating is traded off against the extra cost involved in selecting devices with closer (matched) static characteristics.
Forced current sharing is applicable to both steady-state and transient conditions. For a current derating of less than 5 per cent it is usually cheaper to use forced sharing techniques rather than matched devices.
Figure 10.6 shows the maximum variation of *I-V* characteristics in devices of the same type. When parallel connected the maximum current is restricted to $I_m + I_2$, (= 100A+70A = 170A). The maximum current rating for each device is I_m (100A); hence with suitable forced sharing a combination in excess of $I_m + I_2$ should be possible. The resistive network in figure 10.7 is used for current sharing and in example 10.3 it is required that I_m, 100A, flows through D_1 and $(1-2\times pd)\times I_m > I_2$, (90A) flows through D_2, for a *pd* (5%) overall derating.
From Kirchhoff's voltage law in figure 10.7

$$V_1 + V_3 = V_2 + V_4$$
$$V_{D_1} + I_m R = V_{D_2} + \left(I_T - I_m\right) R \qquad (10.21)$$

From equation (10.20), rearranged for two devices, $n = 2$

$$I_T = 2 \times (1 - pd) I_m$$

Substituting for I_T in equation (10.21) gives

$$R = \frac{V_{D_2} - V_{D_1}}{2\ pd\ I_m} \quad \text{(ohms)} \qquad (10.22)$$

For n devices connected in parallel, equation (10.21) becomes

$$V_{D_1} + I_m R = V_{D_2} + \frac{\left(I_T - I_m\right)}{n-1} R \qquad (10.23)$$

which after substituting for I_T from equation (10.20), for maximum device voltage variation, gives

$$R = \frac{\hat{V}_D - \check{V}_D}{I_m} \frac{(n-1)}{n \times pd} \quad \text{(ohms)} \qquad (10.24)$$

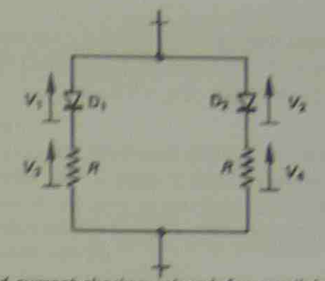


Figure 10.7. *Forced current sharing network for parallel connected devices.*

Although steady-state sharing is effective, sharing resistor losses can be high. The resistor losses in general terms for n parallel connected devices and a conduction duty cycle δ, are given by

$$P_r = \delta \left\{1 + \left(1 - \frac{n}{n-1} \times pd\right)^2\right\} I_m^2 R \quad \text{(W)} \qquad (10.25)$$

Example 10.3: ***Resistive parallel current sharing - static balancing***
For the two diodes shown in figure 10.6, with $\hat{I} = 100$A , what derating result when they are parallel connected, without any external sharing circuits?
The maximum current rating for each device is I_m, 100A; hence with suitable forced sharing a 190A combination should be possible. Using the network in figure 10.7 for current sharing, it is required that 100A flows through D_1 and 90A through D_2. Specify the per cent overall derating, the necessary sharing resistors and their worse case losses.

Solution
The derating for the parallel situation depicted in figure 10.6, without external sharing, is

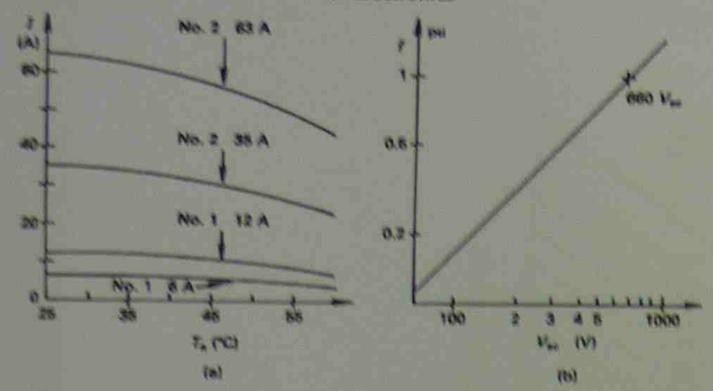


Figure 10.17. *Fuse derating with:*
(a) ambient temperature; (b) ac supply voltage; and (c) power factor.

Fuse protection in dc circuits represents greater difficulty than for ac circuits. No natural ac period current zeros exist and faults can result in continuous arcing, depending on the circuit L, R time constant, prospective fault, and supply voltage. Thus in dc applications the essential parameters are

- Peak let through current versus clearing time or clearing I^2t
- Applied voltage
- Prospective fault di/dt
- Time constant.

It may be possible in some applications to use an ac fuse in a dc circuit, before the rectification stage. Generally low voltage fuses are more effective than high voltage fuses. In high voltage transformer applications satisfactory protection may be afforded by transferring the fuse to the low voltage side. The fuse I^2t rating is transferred as with impedance transferring, that is, in the turns ratio squared.

$$I^2t_{fuse}^{primary} = \left(\frac{V_s}{V_p}\right)^2 \times I^2t_{semiconductor}^{secondary} \qquad (10.33)$$

Alternatively an mcb (miniature circuit breaker) may offer better protection in cases when the fault is more of an overload such that the current magnitude is limited. On overload, the mcb takes a longer time to clear than a fuse, thus the mcb is less prone to nuisance tripping.
Fuse protection is mainly applicable to more robust devices such as thyristors and diodes. Transistors (MOSFETs more readily than IGBTs) usually fail as a result of overcurrent before any fuse link can clear the fault.

Example 10.5: *AC circuit fuse design*

A fast acting fuse is connected in series with a thyristor in a 415 V ac, 50 Hz ac application. The average current in the thyristor is 30 A at a maximum ambient temperature of 45°C. The ratings of the thyristor are

$I_{T(AV)}$ = 45 A @ T_c = 85°C
I_{TRMS} = 80 A
I^2t = 5 k A^2s for 10 ms @ 125°C
I^2t = 20 k $A^2\sqrt{s}$
I_{TSM} = 1000 A for 10 ms @ 125°C and V_{RRM} = 0

The fault circuit inductance is 1.32 mH and the resistance is negligible. Using the figures 10.12 to 10.17, select a suitable fuse.

Solution

From figure 10.17a, the 35 A rms No. 2 fuse is rated at 30 A rms in a 45°C ambient.
From figure 10.15 the peak arc voltage for a type No. 2 fuse will be less than 1200 V, hence the thyristor voltage rating must be greater than 1200 V.
The short circuit or prospective rms symmetrical fault current is

$$I_{sc} = \frac{I_{sc}}{\sqrt{2}} = \frac{V_s}{X_L} = \frac{415\text{V}}{2\pi \times 50\text{Hz} \times 1.32\text{mH}} = 100\text{A}$$

Figure 10.13 gives a fuse peak let through current of 500 A, which is less than the thyristor peak current rating of 1 kA.
Figure 10.16 gives the fuse total I^2t of 300 A^2s and the total clearing time of t_c = 3.5 ms. Since the fuse clears in less than 10 ms (½ ac cycle), the thyristor re-applied V_{RRM} will be zero and an I_{TSM} = 1000 A rating is applicable. The total I^2t is corrected for voltage (415V ac) and power factor (0 pu) with f = 0.6 and c = 1.2 from figures 10.17b and c.

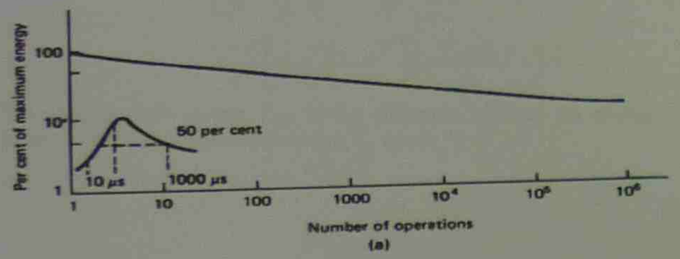
$$I^2t' = f \times c \times I^2t = 0.6 \times 1.2 \times 300\ \text{A}^2\text{s} = 216\ \text{A}^2\text{s}$$

which is significantly less than the thyristor I^2t rating of 5 kA^2s.
Since t_c is less than 10 ms, the $I^2\sqrt{t}$ rating of the thyristor is used.

Parallel operation is difficult and matched *I-V* characteristics are necessary. A feature of varistors often overlooked is deterioration. Figure 10.19a shows that at relatively low energy levels an infinite number of transients can be absorbed, while at rated absorbed energy only one fault is allowed. This single fault, lifetime, is defined as that energy level that causes a 10 per cent increase in clamping voltage level, for a specified current density.

Figure 10.19b shows that very high currents can be tolerated for short intervals. The lower the pulse number, the higher the allowable current.

The failure mode of the Zener diode and varistor is a short circuit. Subsequent high current flow may cause an explosion and disintegration of contacts, forming an open circuit. This catastrophic condition can be avoided by fuse protection.



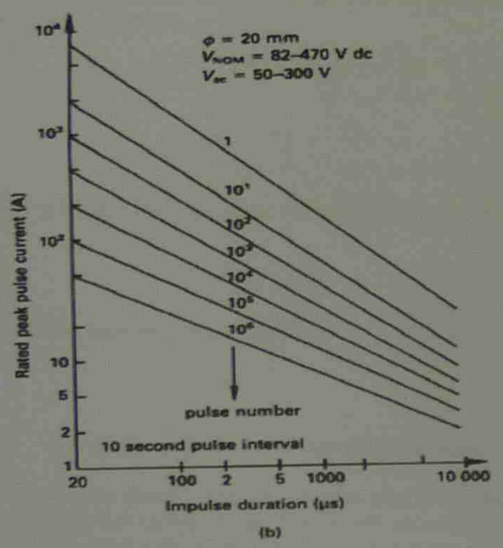


Figure 10.19. *Pulse lifetime ratings for a Zinc oxide varistor: (a) lifetime for fixed 10/100μs pulses and (b) lifetime number for variable-duration square-wave pulses.*

Table 10.1 A comparison of typical voltage transient suppressor characteristics

Suppressor type	Standby current (mA)	Peak current at 1 ms exp. (A)	Peak power at 1 ms (kW)	Peak energy (J)	Voltage clamping ratio at 10 A	Voltage range dc	Capacitance at 1 MHz (nF)
Silicon carbide varistor	5	-	-	50	4.6	15–300	-
Selenium	12	30	9	9	2.3	35–700	-
Metal oxide varistor	1	120	40	70	1.7	14–1200	2
Zener diode (5 W)	0.005	5.5	1.5	2	1.4	1.8–280	1

Example 10.6: *Non-linear voltage clamp*

Evaluate the current of a 1mA @ 250 V Zener diode when used to clamp at 340V dc. Calculate the percentage decrease in voltage-dependent resistance and the per unit increase in power dissipation, assuming $\alpha = 30$.

Solution

From $I = kV^{\alpha}$, equation (10.34)

i. $I_2 = I_1(V_2/V_1)^{\alpha} = 1$ mA $(340\text{V}/250\text{V})^{30} = 10.14$ A
The Zener diode will conduct 10.14A when clamping at 340 V (a 10,140 increase on the standby current of 1mA)

ii. From equation (10.36), $R = V^{1-\alpha}/k$ therefore

$$1-\frac{R_2}{R_1}=1-\left(\frac{V_2}{V_1}\right)^{1-\alpha}=1-\left(\frac{340\text{V}}{250\text{V}}\right)^{-29}=0.99987$$

The percentage decrease in resistance is 99.987 per cent.
The dynamic resistance decreases from (250 V / 1 mA) 250 kΩ to (340 V / 10.1 A) 33.5Ω. The incremental resistance (dv/di) reduces 10,000 to 1.

iii. $P = kV^{\alpha+1}$ (equation (10.37))

$$\frac{P_2}{P_1}-1=\left(\frac{V2}{V1}\right)^{31}-1=\left(\frac{340\text{V}}{250\text{V}}\right)^{31}-1=13793.5$$

The per unit power increase is 13,800.
The power increases from (250 V × 1 mA) 0.2 W at standby to (340 V × 10.14A) 3447.6 W when clamping at 340 V dc.

♣

$$\bar{I}_s = \frac{\delta V_s + \left(1-\frac{t_x}{T}\right)E}{R} - \frac{E}{R} = \left(\delta V_s - \frac{t_x}{T}E\right)\Big/R \quad \text{(A)} \qquad (13.37)$$

The input and output powers are related such that

$$P_{in} = V_s \bar{I}_s \qquad P_{out} = I_{rms}^2 R + E\bar{I}_s \qquad P_{in} = P_{out} \qquad (13.38)$$

from which the average input current can be evaluated.

Alternatively the average input current, which is the switch average current, is given by

$$\bar{I}_s = \bar{I}_{switch} = \frac{1}{T}\int_0^{t_x} i_s(t)\,dt$$

$$= \frac{1}{T}\int_0^{t_x} \frac{V_s - E}{R}\left(1 - e^{\frac{-t}{\tau}}\right)dt \qquad (13.39)$$

$$= \frac{V_s - E}{R}\left(\delta - \frac{\tau}{T}\left(1 - e^{\frac{-t_x}{\tau}}\right)\right) = \frac{V_s - E}{R}\delta - \frac{\tau}{T}\hat{I}$$

The average diode current $\bar{I}_{diode}$ is the difference between the average output current $\bar{I}_o$ and the average input current, $\bar{I}_s$, that is

$$\bar{I}_{diode} = \bar{I}_o - \bar{I}_s$$

$$= \frac{\tau}{T}\hat{I} - \frac{E\left(\frac{t_x}{T} - \delta\right)}{R} \qquad (13.40)$$

Alternatively, the average diode current can be found by integrating the diode current given in equation (13.33), as follows

$$\bar{I}_{diode} = \frac{1}{T}\int_{t_x}^{t_x+t_y}\left(-\frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \hat{I}e^{\frac{-t}{\tau}}\right)dt \qquad (13.41)$$

$$= \frac{\tau}{T}\hat{I} - \frac{E\left(\frac{t_x}{T} - \delta\right)}{R}$$

If E represents motor back emf, then electromagnetic energy conversion efficiency is given by

$$\eta = \frac{E\bar{I}_o}{P_{in}} = \frac{E\bar{I}_o}{V_s\bar{I}_s} \qquad (13.42)$$

The chopper effective input impedance is given by

$$Z_{in} = \frac{V_s}{\bar{I}_s} \qquad (13.43)$$

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Example 13.1: ***DC chopper with load back emf (first quadrant)***

A first-quadrant dc-to-dc chopper feeds an inductive load of 10 ohms resistance, 50mH inductance, and back emf of 55V dc, from a 340V dc source. If the chopper is operated at 200Hz with a 25% on-state duty cycle, determine, with and without (rotor standstill) the back emf:

i. the load average and rms voltages;
ii. the rms ripple voltage, hence ripple factor;
iii. the maximum and minimum output current, hence the peak-to-peak output ripple in the current;
iv. the current in the time domain;
v. the average load output current, average switch current, and average diode current;
vi. the input power, hence output power and rms output current;
vii. effective input impedance, (and electromagnetic efficiency for $E > 0$);
viii. sketch the output current and voltage waveforms.

Solution

The main circuit and operating parameters are

- on-state duty cycle $\delta = ¼$
- period $T = 1/f_s = 1/200\text{Hz} = 5\text{ms}$
- on-period of the switch $t_T = 1.25\text{ms}$
- load time constant $\tau = L/R = 0.05\text{H}/10\Omega = 5\text{ms}$

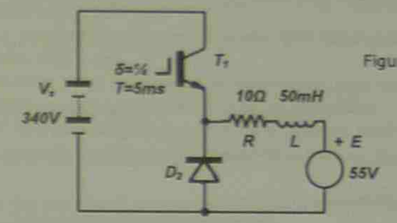

Figure Example 13.1. *Circuit diagram.*

i. From equations (13.2) and (13.3) the average and rms output voltages are both independent of the back emf, namely

$$\bar{V}_o = \frac{t_x}{T}V_s = \delta V_s$$

$$= ¼ \times 340\text{V} = 85\text{V}$$

$$V_r = \sqrt{\frac{t_x}{T}}V_s = \sqrt{\delta}\,V_s$$

$$= \sqrt{¼} \times 240\text{V} = 120\text{V rms}$$

ii. The rms ripple voltage hence ripple factor are given by equations (13.4) and (13.5), that is

$$V_r = \sqrt{V_{rms}^2 - V_o^2} = V_s\sqrt{\delta(1-\delta)}$$

$$= 340\text{V}\sqrt{¼ \times (1 - ¼)} = 147.2\text{V ac}$$

and

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$$i_o = \frac{V_s - E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \overset{v}{I}e^{\frac{-t}{\tau}}$$

$$i_o(t) = 28.5 \times \left(1 - e^{\frac{-t}{5ms}}\right) + 0.12e^{\frac{-t}{5ms}}$$

$$= 28.5 - 28.38e^{\frac{-t}{5ms}} \quad \text{(A)} \qquad \text{for} \quad 0 \le t \le 1.25\text{ms}$$

$$i_o = -\frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \hat{I}e^{\frac{-t}{\tau}}$$

$$i_o(t) = -5.5 \times \left(1 - e^{\frac{-t}{5ms}}\right) + 6.4e^{\frac{-t}{5ms}}$$

$$= -5.5 + 11.9e^{\frac{-t}{5ms}} \quad \text{(A)} \qquad \text{for} \quad 0 \le t \le 3.75\text{ms}$$

v. The average load current from equation (13.37) is

$$\bar{I}_o = {V_s - E}/{R}$$

$$= 85\text{V-}55\text{V}/{10\Omega} = 3\text{A}$$

The average switch current is the average supply current,

$$\bar{I}_i = \bar{I}_{switch} = \frac{\delta(V_s - E)}{R} - \frac{\tau}{T}\left(\hat{I} - \overset{v}{I}\right)$$

$$= \frac{¼ \times (340\text{V} - 55\text{V})}{10\Omega} - \frac{5\text{ms}}{5\text{ms}} \times (6.40\text{A} - 0.12\text{A}) = 0.845\text{A}$$

The average diode current is the difference between the average load current and the average input current, that is

$$\bar{I}_{diode} = \bar{I}_o - \bar{I}_i$$

$$= 3\text{A} - 0.845\text{A} = 2.155\text{A}$$

vi. The input power is the dc supply multiplied by the average input current, that is

$$P_{in} = V_s \bar{I}_i = 340\text{V} \times 0.845\text{A} = 287.3\text{W}$$

$$P_{out} = P_{in} = 287.3\text{W}$$

From equation (13.18) the rms load current is given by

$$\bar{I}_{o_{rms}} = \sqrt{\frac{P_{out} - E\bar{I}_o}{R}}$$

$$= \sqrt{\frac{287.3\text{W} - 55\text{V} \times 3\text{A}}{10\Omega}} = 3.5\text{A rms}$$

vii. The chopper effective input impedance is

$$Z_{in} = \frac{V_s}{\bar{I}_i}$$

$$= \frac{340\text{V}}{0.845\text{A}} = 402.4\ \Omega$$

The electromagnetic efficiency is given by equation (13.22), that is

$$\eta = \frac{E\bar{I}_o}{P_{in}}$$

$$= \frac{55\text{V} \times 3\text{A}}{287.3\text{W}} = 57.4\%$$

viii. The output voltage and current waveforms for the first-quadrant chopper, with and without back emf, are shown in the figure to follow.

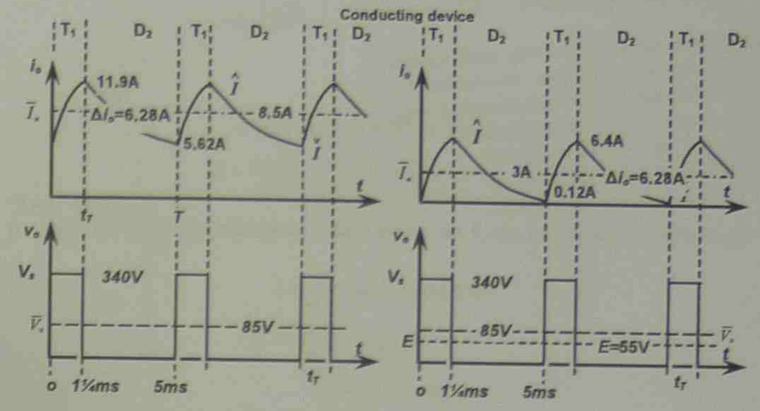

Figure Example 13.1. *Circuit waveforms.*

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Example 13.2: ***DC chopper with load back emf - verge of discontinuous conduction***

A first-quadrant dc-to-dc chopper feeds an inductive load of 10 ohms resistance, 50mH inductance, and back emf of 55V dc, from a 340V dc source. If the chopper is operated at 200Hz with a 25% on-state duty cycle, determine:

i. the maximum back emf before discontinuous load current conduction commences with $\delta = ¼$;
ii. with 55V back emf, what is the minimum duty cycle before discontinuous load current conduction; and
iii. minimum switching frequency at E=55V and t_T = 1.25ms before discontinuous conduction.

Solution

The main circuit and operating parameters are

- on-state duty cycle $\delta = ¼$
- period $T = 1/f_s = 1/200\text{Hz} = 5\text{ms}$
- on-period of the switch $t_T = 1.25\text{ms}$
- load time constant $\tau = L/R = 0.05\text{H}/10\Omega = 5\text{ms}$

First it is necessary to establish whether the given conditions represent continuous or discontinuous load current. The current extinction time t_x for discontinuous conduction is given by equation (13.24), and yields

$$t_x = t_T + \tau\,\ell n\left(1+\frac{V_s-E}{E}\left(1-e^{\frac{-t_T}{\tau}}\right)\right)$$

$$= 1.25\text{ms} + 5\text{ms}\times\ell n\left(1+\frac{340\text{V}-55\text{V}}{55\text{V}}\times\left(1-e^{\frac{-1.25\text{ms}}{5\text{ms}}}\right)\right) = 5.07\text{ms}$$

Since the cycle period is 5ms, which is less than the necessary time for the current to fall to zero (5.07ms), the load current is continuous. From example 13.1 part iv, with E=55V the load current falls from 6.4A to near zero (0.12A) at the end of the off-time, thus the chopper is operating near the verge of discontinuous conduction. A small increase in E, decrease in the duty cycle δ, or increase in switching period T, would be expected to result in discontinuous load current.

i. $\hat{E}$

The necessary back emf can be determined graphically or analytically.

Graphically:

The bounds of continuous and discontinuous load current for a given duty cycle, switching period, and load time constant can be determined from figure 13.5.

Using δ=¼, $T/\tau = 1$ with τ=5ms, and T = 5ms, figure 13.5 gives E/V_s=0.165. That is, $E = 0.165\times V_s = 0.165\times340\text{V} = 56.2\text{V}$

Analytically:

The chopper is operating too close to the boundary between continuous and discontinuous load current conduction for accurate readings to be obtained from the graphical approach, using figure 13.5. Examination of the expression for minimum current, equation (13.13), gives

$$\check{I} = \frac{V_s}{R}\frac{e^{\frac{\delta_x}{\tau}}-1}{e^{\frac{T}{\tau}}-1} - \frac{E}{R} = 0$$

Rearranging to give the back emf, E, produces

$$E = V_s\frac{e^{\frac{\delta_x}{\tau}}-1}{e^{\frac{T}{\tau}}-1}$$

$$= 340\text{V}\times\frac{e^{\frac{1.25\text{ms}}{5\text{ms}}}-1}{e^{\frac{5\text{ms}}{5\text{ms}}}-1} = 56.2\text{V}$$

That is, if the back emf increases from 55V to 56.2V then at that voltage, discontinuous load current commences.

ii. $\check{\delta}$

If equation (13.13) is solved for $\check{I} = 0$ then

$$\check{I} = \frac{V_s}{R}\frac{e^{\frac{\delta_x}{\tau}}-1}{e^{\frac{T}{\tau}}-1} - \frac{E}{R} = 0$$

Rearranging to isolate t_T gives

$$t_T = \tau\,\ell n\left(1+\frac{E}{V_s}\left(e^{\frac{T}{\tau}}-1\right)\right)$$

$$= 5\text{ms}\times\ell n\left(1+\frac{55\text{V}}{340\text{V}}\left(e^{\frac{5\text{ms}}{5\text{ms}}}-1\right)\right)$$

$$= 1.226\text{ms}$$

If the switch on-state period is reduced by 0.024ms, from 1.250ms to 1.226ms (24.52%), operation is then on the verge of discontinuous conduction.

iii. $\hat{T}$

If the switching frequency is decreased such that T=t_x, then the minimum period for discontinuous load current is given by equation (13.24). That is,

$$t_x = T = t_T + \tau\,\ell n\left(1+\frac{V_s-E}{E}\left(1-e^{\frac{-t_T}{\tau}}\right)\right)$$

$$T = 1.25\text{ms} + 5\text{ms}\times\ell n\left(1+\frac{340\text{V}-55\text{V}}{55\text{V}}\times\left(1-e^{\frac{-1.25\text{ms}}{5\text{ms}}}\right)\right) = 5.07\text{ms}$$

Discontinuous conduction operation occurs if the period is increased by 0.07ms.

In conclusion, for the given load, for continuous conduction to cease, the following operating conditions can be changed

- increase the back emf E from 55V to 56.2V
- decrease the duty cycle δ from 25% to 24.52% (t_T decreased from 1.25ms to 1.226ms)
- increase the switching period T by 0.07ms, from 5ms to 5.07ms (from 200Hz to 197.2Hz), with the switch on-time, t_T, unchanged from 1.25ms.

Appropriate simultaneous smaller changes in more than one parameter would suffice.

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Example 13.3: ***DC chopper with load back emf – discontinuous conduction***

A first-quadrant dc-to-dc chopper feeds an inductive load of 10 ohms resistance, 50mH inductance, and back emf of 100V dc, from a 340V dc source. If the chopper is operated at 200Hz with a 25% on-state duty cycle, determine:

i. the load average and rms voltages;
ii. the rms ripple voltage, hence ripple factor;
iii. the maximum and minimum output current, hence the peak-to-peak output ripple in the current;

$$= \frac{t_x - t_T}{T} V_s + \frac{T - t_x}{T} E = \left(\frac{t_x}{T} - \delta\right) V_s + \left(1 - \frac{t_x}{T}\right) E$$

$$\bar{V}_o = E - \delta V_s + \frac{t_x}{T}(V_s - E) \tag{13.64}$$

where the switch on-state duty cycle $\delta = t_T/T$ is defined in figure 13.7b.
The average output current is

$$\bar{I}_o = \frac{E - \bar{V}_o}{R} = \frac{\delta V_s - \frac{t_x}{T}(V_s - E)}{R} \tag{13.65}$$

The average output current can also be found by integration of the time domain output current i_o. By solving the appropriate time domain differential equations, the continuous load current i_o shown in figure 13.7a is defined by
During the **switch on-period**, when $v_o = 0$

$$L\frac{di_o}{dt} + Ri_o = E$$

which yields

$$i_o(t) = \frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) \qquad \text{for} \quad 0 \le t \le t_T \tag{13.66}$$

During the **switch off-period**, when $v_o = V_s$

$$L\frac{di_o}{dt} + Ri_o + V_s = E$$

which, after shifting the zero time reference to t_T, gives

$$i_o(t) = \frac{E - V_s}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \hat{I}e^{\frac{-t}{\tau}} \qquad \text{for} \quad 0 \le t \le t_x - t_T \tag{13.67}$$

$$\text{where} \quad \hat{I} = \frac{E}{R}\left(1 - e^{\frac{-t_x}{\tau}}\right) \quad \text{(A)}$$

$$\text{and} \quad \check{I} = 0 \quad \text{(A)} \tag{13.68}$$

After t_x, $v_o(t) = E$ and the load current is zero, that is

$$i_o(t) = 0 \qquad \text{for} \quad t_x \le t \le T \tag{13.69}$$

The output ripple current, for discontinuous conduction, is dependent of the back emf E and is given by equation (13.68),

$$I_{p-p} = \hat{I} = \frac{E}{R}\left(1 - e^{\frac{-t_x}{\tau}}\right) \tag{13.70}$$

The average switch current, $\bar{I}_{switch}$, can be derived by integrating the switch current given by equation (13.66), that is

$$\bar{I}_{switch} = \frac{1}{T}\int_0^{t_T} i_o(t)\,dt$$

$$= \frac{1}{T}\int_0^{t_T}\left(\frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right)\right)dt \tag{13.71}$$

$$= \frac{\delta E}{R} - \frac{\tau}{T}\hat{I}$$

The term $\hat{I} = I_{p-p}$ is the peak-to-peak ripple current, which is given by equation (13.70). By Kirchhoff's current law, the average diode current $\bar{I}_{diode}$ is the difference between the average output current $\bar{I}_o$ and the average switch current, $\bar{I}_{switch}$, that is

$$\bar{I}_{diode} = \bar{I}_o - \bar{I}_{switch}$$

$$= \frac{\delta V_s - \frac{t_x}{T}(V_s - E)}{R} - \frac{\delta E}{R} + \frac{\tau}{T}\hat{I} \tag{13.72}$$

$$= \frac{\tau}{T}\hat{I} - \frac{\left(\frac{t_x}{T} - \delta\right)(V_s - E)}{R}$$

The average diode current can also be found by integrating the diode current given in equation (13.67), as follows

$$\bar{I}_{diode} = \frac{1}{T}\int_0^{t_x - t_T}\left(\frac{E - V_s}{R}\left(1 - e^{\frac{-t}{\tau}}\right) + \hat{I}e^{\frac{-t}{\tau}}\right)dt$$

$$= \frac{\tau}{T}\hat{I} - \frac{\left(\frac{t_x}{T} - \delta\right)(V_s - E)}{R} \tag{13.73}$$

The power produced by the back emf source E is

$$P_E = E\bar{I}_o \tag{13.74}$$

The power delivered to the dc source V_s is

$$P_{V_s} = V_s\bar{I}_{diode} \tag{13.75}$$

Alternatively, the difference between the two powers is the power lost in the load resistor, R, that is

$$P_E = P_{V_s} + \hat{I}^2_{rms}R$$

$$I_{rms} = \sqrt{\frac{E\bar{I}_o - V_s\bar{I}_{diode}}{R}} \tag{13.76}$$

The efficiency of energy transfer between the back emf and the dc source is

$$\eta = \frac{P_{V_s}}{P_E} = \frac{V_s\bar{I}_{diode}}{E\bar{I}_o} \tag{13.77}$$

Example 13.4: ***Second-quadrant DC chopper – continuous conduction***

A dc-to-dc chopper capable of second-quadrant operation is used in a 200V dc battery electric vehicle. The machine armature has 1 ohm resistance and 1mH inductance.

i. The machine is used for regenerative braking. At a constant speed downhill, the back emf is 150V, which results in a 10A braking current. What is the switch on-state duty cycle if the machine is delivering continuous output current? What is the minimum chopping frequency for these conditions?

Example 17.2: ***Temperature effect on inductance***

The gapped pot core in example 17.1 is specified by a relative temperature coefficient of 1 x 10^{-6}/K. What is the expected inductance variation over the temperature range 25-55°C?

Solution
From example 17.1

$$\mu_e = \frac{A_L}{c} = \frac{250}{2.5} = 100$$

$$\Delta T = 55 - 25 = 30°\text{C}$$

From equation (17.44)

$$\frac{dL}{L} = \alpha_F \mu_e \Delta T$$
$$= 1\times10^{-6}\times100\times30$$
$$= 0.3 \text{ per cent inductance variation}$$

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17.3.7 Stored energy in inductors

The energy stored in the magnetic field is given by

$$E = \tfrac{1}{2}BH\breve{V}_e \quad \text{(J)} \qquad (17.45)$$

where $\breve{V}_e$ is the effective minimum volume. It can be shown that the stored magnetic energy is equivalent to the stored electrical energy, whence

$$E = \tfrac{1}{2}BH\breve{V}_e = \tfrac{1}{2}Li^2 \quad \text{(J)} \qquad (17.46)$$

For un-gapped cores, like a toroid, the ferrite effective volume, V_e, is equal to the minimum effective volume, $\breve{V}_e$. Inductors meeting this requirement may make the core size excessive. However the introduction of an air gap, ε, can reduce the core size significantly, since a significant amount of the energy can be stored in the gap volume. The minimum effective volume is now larger than the ferrite core effective volume, and is given by

$$\breve{V}_e = V_e + A_e\mu_e\varepsilon$$
$$= A_e\left(l_e + \mu_e\varepsilon\right) \quad (\text{m}^3) \qquad (17.47)$$

Figure 17.8 shows modified *B-H* characteristics for inductors with an air gap. The line curve o-b represents the core without an air gap, which results in the largest inductance. The energy stored in the core for a flux BA_e, in the linear portion of the curve, is the area of the shaded triangle 0-a-b, as defined by equation (17.45). When an air gap is introduced, the effective permeability falls as shown by the slope of line o-c. The figure

shows that as the air gap increases, the inductance decreases. It can be shown that the stored energy in the air gap and core is represented by the shaded area o-a-c, for a given flux, BA_e. It can be seen that the energy stored in the gap of length ε, although its length is much smaller than the core length, ℓ_c, its stored energy is much greater than that stored in the core.

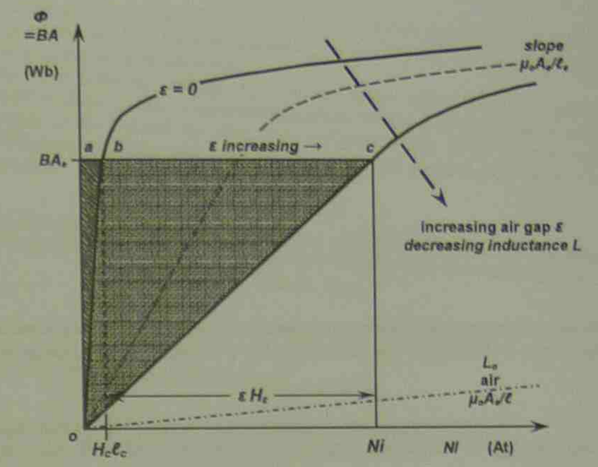

Figure 17.8. *Effects of air gap on inductance and stored energy.*

The total energy stored, E_T, in the magnetic circuit comprises the energy stored in the air gap, E_ε, plus the energy stored in the magnetic core material, E_{core}. It can be shown that these two energies are equal to the areas of the shaded triangles, o-b-c and o-a-b, respectively in figure 17.8. That is

$$E_T = E_{core} + E_\varepsilon$$
$$= \tfrac{1}{2}B_eH_eV_e + \tfrac{1}{2}B_eH_eV_e \qquad (17.48)$$

If leakage is neglected, then the air gap flux is the same as the core flux. If fringing is neglected then the area of the core at the air gap is the same as the area of gap. Then

$$E_T = \tfrac{1}{2}BH_eA_e\ell_e + \tfrac{1}{2}BH_eA_e\varepsilon$$
$$= \tfrac{1}{2}BA_e\left(H_e\ell_e + H_e\varepsilon\right) \qquad (17.49)$$

For a gapped core, as shown in figure 17.11

$$Ni = H_e\ell_e + H_e\varepsilon \qquad (17.50)$$

Therefore, on substituting equation (17.50) into equation (17.49) gives the total stored energy as

17.3.6 Inductance stability

Three factors affect inductance stability:

- *Parameter effects*
- *Time effects*
- *Temperature effects*

17.3.6i - Parameter effects
From the differential of equation (17.12)

$$\frac{dL}{L} = \frac{d\mu_e}{\mu_e} \tag{17.38}$$

while differentiating equation (17.16) yields

$$\frac{d\mu_e}{\mu_e^2} = \frac{d\mu_i}{\mu_i^2} \tag{17.39}$$

Substituting equation (17.39) into equation (17.38) gives

$$\frac{dL}{L} = \frac{d\mu_i}{\mu_i^2}\frac{A_L}{c} \tag{17.40}$$

The factor $d\mu_i / \mu_i^2$ is constant for a given temperature, hence any change in inductance is due to variations in A_L and c. Thus in order to increase the stability of an inductor in a given material with $\varepsilon << \ell_m$, it is necessary to increase the magnetic circuit air gap (to reduce the inductance factor A_L) or to select a bigger core (to increase the core permeance factor c).

17.3.6ii - Time effects
Initial permeability of a ferrite decreases with time under constant operating conditions, including constant temperature. A *disaccommodation factor, df*, independent of effective permeability, is introduced, which characterises the material such that the change in inductance is defined by

$$\frac{dL}{L} = df\ \mu_i \log_{10}\frac{t_1}{t_0} \tag{17.41}$$

This expression is based on the fact that permeability is proportional to the logarithm of time. The *df* increases slightly with temperature. Generally the *df* decreases, as shown in table 17.3:

- as the initial permeability increases for a given resistivity
- as resistivity decreases.

Example 17.1: ***Inductance variation with time***
A pot ferrite core with an effective permeability of 100 (A_L = 250) and a disaccommodation factor $df < 35 \times 10^{-6}$ has been in satisfactory operation for five weeks after production. What is the expected inductance variation after 10 years?

Table 17.3. Factors affecting the disaccommodation factor

			ρ(Ω cm)		
			10^5	500	≈ 20
μ_i			11-250	800-2000	4000
T_c		°C	450-300	250-170	145
df	$\times 10^{-6}$		50-10	20-2	3

Solution

From equation (17.41)

$$\frac{dL}{L} = df\ \mu_i \log_{10}\frac{t_1}{t_0} < 35\times10^{-6}\times100\times\log\frac{520\ \text{weeks}}{5\ \text{weeks}}$$

that is, $dL < 0.7$ per cent can be expected.

♣

17.3.6iii - Temperature effects

Figure 17.7 shows that between +5°C and +55°C the permeability μ_i variation as a function temperature is approximately linear for this ferrite. The temperature coefficient α is given by

$$\alpha = \frac{1}{\mu_{i1}}\frac{\Delta\mu_i}{\Delta T} \quad (K^{-1}) \tag{17.42}$$

where $\Delta\mu_i = \mu_{i2} - \mu_{i1}$ is the initial permeability variation over the temperature range $\Delta T = T_2 - T_1$.
In a magnetic circuit with an air gap and effective permeability, μ_e, the temperature coefficient of the core is reduced according to

$$\alpha_e = \alpha\frac{\mu_e}{\mu_i} = \alpha_F\mu_e \quad (K^{-1}) \tag{17.43}$$

The term $\alpha_F = \alpha_i / \mu_i$ is called the relative temperature coefficient. The relative inductance change between two temperatures can be determined by

$$\frac{dL}{L} = \alpha_F\mu_e\Delta T \tag{17.44}$$

For effective permeability $\mu_e < 80$, the temperature coefficient $\alpha_e = \mu_e\alpha_F$ should be increased by 10 to 30 × 10^6/K to account for the temperature influence of the winding.

This figure shows experimental families of curves of per unit core energy against magnetomotive force per unit length, for different air gaps. The resultant curves are ferrite type dependent and dimensionally independent. Hanna curves therefore allow the determination of minimum turns N and air gap ε, from the required inductance L and dc current I.

Three distinct energy levels are shown in the Hanna curves in figure 17.10.

i. At low dc currents (H) the per unit energy increases linearly with H. This region corresponds to the horizontal regions in figure 17.9b, where

$$L = \mu_\Delta c N^2 \quad \text{(H)} \tag{17.56}$$

and as H varies, μ_Δ is constant.

ii. In the mid energy region, the per unit energy can decrease with increased H. The incremental permeability decreases, causing L to decrease at a greater rate than the increase in the dc current squared, I^2. This region is characterised by the fall off in μ_Δ, hence inductance, as H increases as shown in figure 17.9b.

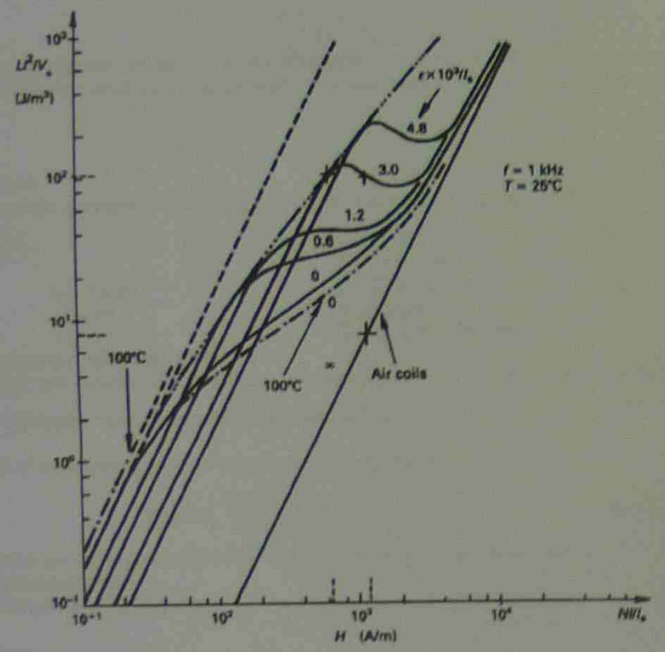

Figure 17.10. *Hanna curves, showing trajectories for different air gaps.*

iii. At high dc currents, the core material saturates, and μ_Δ tends to unity. An air core inductance results, where

$$L = L_a = c N^2 \tag{17.57}$$

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Example 17.3: ***Inductor design with Hanna curves***

A 20 μH, 10 A choke is required for a forward converter. The inductance must be constant for unidirectional currents to 10 A. An available E-core pair has the following effective parameters

$\ell_e = 0.11$ m, $A_e = 175 \times 10^{-6}$ m², $V_e = 19.3 \times 10^{-6}$ m³

and

$\mu_i = 2500$ @ 25°C and 3000 @ 100°C (from figure 17.7)

i. At a core temperature of 25°C, determine the required air gap and turns. Allow a 5 per cent decrease in inductance at rated conditions.
ii. Estimate the inductance at 20A dc.
iii. Calculate the inductance at 10A and 20A dc, both at 100°C.

Solution

i. Evaluate $\dfrac{LI^2}{V_e} = \dfrac{20\times10^{-6}\times10^2}{19.3\times10^{-6}} = 104$ J/m³

From figure 17.10, restricted to the constant-L region, 104 J/m³ corresponds to

(a) $\varepsilon/\ell_e = 3 \times 10^{-3}$

whence $\varepsilon = 3 \times 10^{-3} \times \ell_e = 3 \times 10^{-3} \times 0.11$

The required total air gap is 0.33 mm

(b) $H = 650$A/m

Since $H = NI/\ell_e$

$N = H\ell_e/I = 650 \times 0.11/10 = 7.15$ turns

Use 7 turns and a 0.33 mm total air gap.

ii. At 20 A, 25°C

$H = NI/\ell_e = 7 \times 20/0.11 = 1270$A/m

Two alternative design approaches may be used to estimate the inductance.

(a) The effective permeability, μ_e, before saturation can be evaluated from equation (17.17)

$$\frac{1}{\mu_e} = \frac{1}{\mu_a} + \frac{\varepsilon}{l_e} = \frac{1}{2500} + 3\times10^{-3}$$

that is

$$\mu_e \approx 300$$

From figure 17.9b, for μ_e = 300 it can be seen that the incremental permeability μ_Δ is constant, as required to 650A/m, then μ_Δ decrease as

Stage 10

The core loss, P_c, and the copper loss, P_{Cu}, are compared. If

(i) $P_{Cu} > P_c$

Either decrease the number of turns and increase the copper diameter. This will reduce the copper loss and increase B_{ac}, and hence P_c. Recalculate from stage 6.

or select a larger core, which will increase the copper window area, A_W, hence increasing the allowable wire diameter. Recalculate from stage 5.

(ii) $P_{Cu} < P_c$

Either increase the number of turns which will reduce d, B_{ac}, hence P_c, and then recalculate from stage 6.

or select a smaller core, which will require d to be reduced, and then recalculate from stage 5.

Proceed if $P_{Cu} \approx P_c$.

Stage 11

Update the value of total losses, P_L, and hence recalculate the power requirements and resultant efficiency.

Calculate the actual core temperature rise from equation (17.66), rearranged

$$\Delta T = 59\left(\frac{1000}{T_a + 273}\right)^{1.48} \times \left(\frac{P_L}{S_a}\right)^{0.86} \quad (K) \qquad (17.72)$$

where S_a is the heat dissipating area in cm^2 of the chosen core.

Example 17.6: Ferrite voltage transformer design

Consider the design requirements for the split rail push-pull smps shown in figure 15.7b, which is specified as follows

$V_o = 5$ V $V_{dc} = 48$ V ± 15 per cent
$I_o = 6$A $f = 20$kHz
$P_o = 20$ W $T_a = 25°C$, $\Delta T \leq 35$ K
$\eta = 97$ per cent

Solution

Based on the flowchart in figure 17.17 and the eleven stages outlined, design proceeds as follows.

Stage 1

The transformer must deliver 20 W plus losses associated with an output inductor and the pair of Schottky diodes in the output rectifier. The inductor loss is estimated at 4 per cent of the output power, 0.8 W, while the diode total loss is 0.6 V × 4 A = 2.4 W. Thus the transformer output power requirement P_s is 23.2 W (20 W + 0.8 W + 2.4 W). With a 97 per cent efficiency, the transformer input power, P_p, requirement is 1/97 per cent of 23.2 W, namely 23.9 W. The nominal primary current, I_p, at the nominal voltage, 24 V is

$$I_p = \frac{P_p}{V_p} = \frac{23.9\text{W}}{24\text{V}} = 1\text{A}$$

The maximum primary voltage, V_p, is 1.15×V_{dc}/2 = 27.6 V, since the 48 V supply is centre tapped and has + 15 per cent regulation. For worst case, it is assumed that the voltage drop across the switches is zero.

The transformer secondary voltage, V_{ac}, for the centre tapped full-wave rectifier circuit, must be large enough to overcome the diode voltage drop, V_d, and must allow for averaging of the nominal low duty cycle switching action of the primary input power. With pwm regulation each input switch operates for approximately 25 per cent of the time, thus

$$½V_s = 2\times(V_o + V_d)$$

where the ½ indicates that half of the secondary winding conducts at any one time, while the 2 approximates the pwm average on-time. Thus for $V_d = 0.6$ V and $V_o = 5$ V

$$V_{ac} = 4\times(5 + 0.6) = 22.4\text{V}$$

Stage 2

Extracting the transformer data from stage 1

$I_p = 1$A $I_s = 4$A
$V_p = 27.6$ V $V_{ac} = 22.4$ V
$P_p = 23.9$ W $P_s = 23.2$ W

$\eta = 97$ per cent, $f = 20$ kHz, $T_a = 25°C$, $\Delta T = 35$ K.

Stage 3

The total transformer power loss, P_L, from $P_s - P_p$, is 0.7 W.
Thus $P_c = P_{Cu} = 0.7/2 = 0.35$ W each.

Stage 4

The surface area requirement is calculated from equation (17.66)

$$S_a = 145\times\left[\frac{1000}{25 + 273}\right]^{1.48} \times \frac{0.7\text{W}}{35^{0.86}}$$

$= 16.1\ cm^2$ for a 35 K temperature rise

Example 3.1

The power we use at home has a frequency of 60 Hz. The period of this sine wave can be determined as follows:

$$T = \frac{1}{f} = \frac{1}{60} = 0.0166 \text{ s} = 0.0166 \times 10^3 \text{ ms} = 16.6 \text{ ms}$$

Example 3.2

The period of a signal is 100 ms. What is its frequency in kilohertz?

Solution

First we change 100 ms to seconds, and then we calculate the frequency from the period (1 Hz = 10^{-3} kHz).

$$100 \text{ ms} = 100 \times 10^{-3} \text{ s} = 10^{-1} \text{ s}$$

$$f = \frac{1}{T} = \frac{1}{10^{-1}} \text{ Hz} = 10 \text{ Hz} = 10 \times 10^{-3} \text{ kHz} = 10^{-2} \text{ kHz}$$

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Example 3.3

A sine wave is offset 1/6 cycle with respect to time 0. What is its phase in degrees and radians?

Solution

We know that 1 complete cycle is 360°. Therefore, 1/6 cycle is

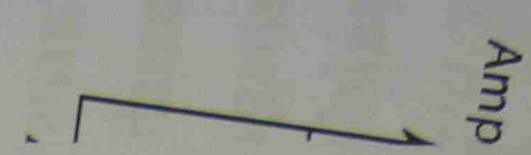
$$\frac{1}{6} \times 360 = 60° = 60 \times \frac{2\pi}{360} \text{ rad} = \frac{\pi}{3} \text{ rad} = 1.046 \text{ rad}$$

Example 3.7

The frequency domain is more compact and useful when we are dealing with more than one sine wave. For example, Figure 3.8 shows three sine waves, each with different amplitude and frequency. All can be represented by three spikes in the frequency domain.

Example 3.4

Figure 3.9 shows a periodic composite signal with frequency f. This type of signal is not typical of those found in data communications. We can consider it to be three alarm systems, each with a different frequency. The analysis of this signal can give us a good understanding of how to decompose signals.



Example 3.8

A nonperiodic composite signal has a bandwidth of 200 kHz, with a middle frequency of 140 kHz and peak amplitude of 20 V. The two extreme frequencies have an amplitude of 0. Draw the frequency domain of the signal.

Solution

The lowest frequency must be at 40 kHz and the highest at 240 kHz. Figure 3.15 shows the frequency domain and the bandwidth.

External noise

man made noise, Atmospheric noise, Space noise

Internal noise

Thermal noise

Power of generated noise

$$P_n = KT\Delta f$$

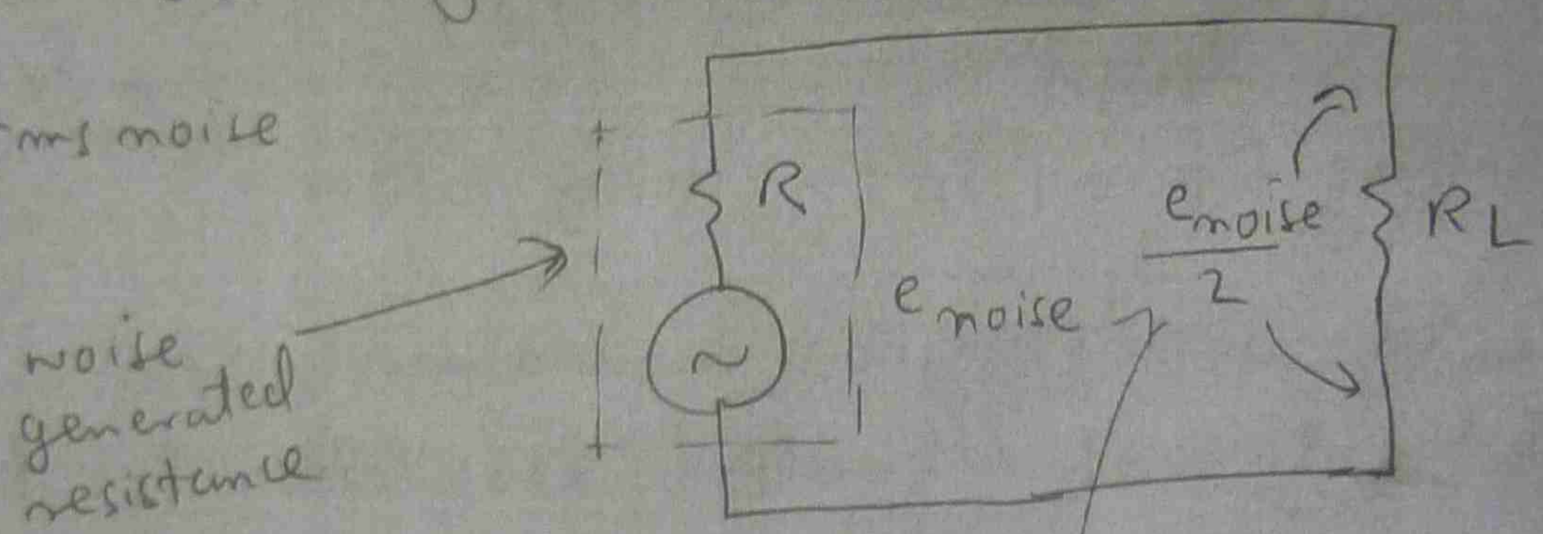
K = Bolt Zmann's constant $(1.38 \times 10^{-23} J/K)$

T = Resistor temperature (°K)

Δf = Frequency band width of ~~circle~~ the system.

$$e_n = \sqrt{4KT\Delta fR}$$

e_n = rms noise



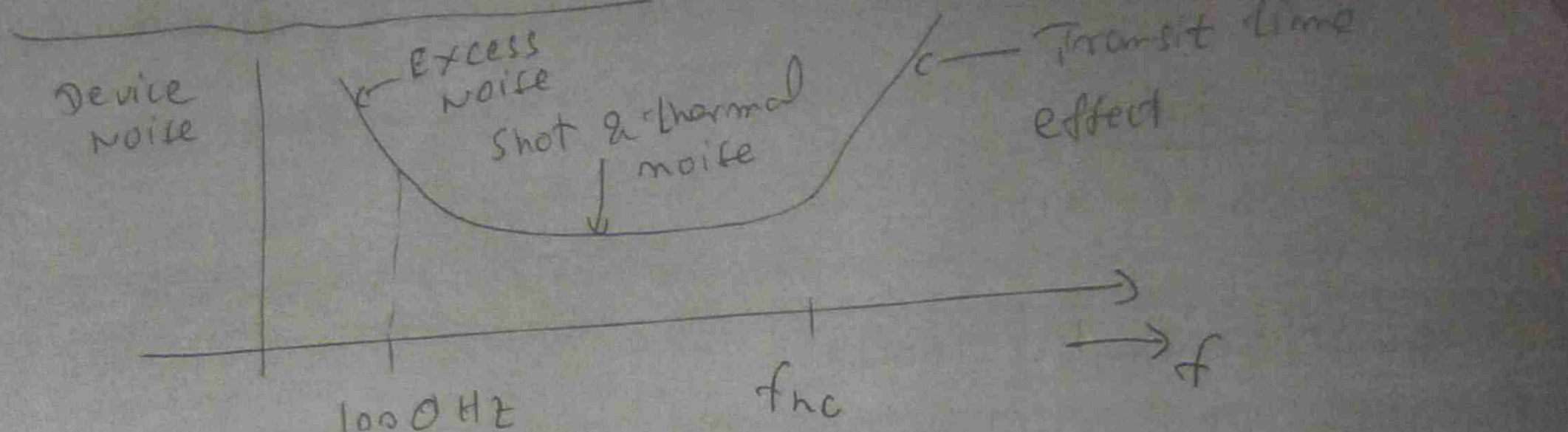
Ex 1-1

An amplifier operating over a 4 MHz bandwidth has a 100Ω input resistance. It is operating at 27°C has a voltage gain of 200, and has an input signal of 5 μV rms determine the rms output signals (desired & noise) assuming external noise can be disregarded.

$T = 27 + 273 = 300°K$ $\quad e_n = \sqrt{4KT\Delta fR} = \sqrt{4 \times 1.38 \times 10^{-23} \times 300 \times 4 \times 100}$

$= 2.57$ μV rms

Frequency Noise effect



Noise designation & calculation

$$\text{Signal to noise ratio} = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{S}{N} = \frac{P_S}{P_N}$$

$$\frac{S}{N} = 10 \log_{10} \frac{P_S}{P_N} \quad (dB)$$

$$\text{Noise figure } (NF) = 10 \log_{10} NR$$

NR = Noise Ratio

Ex 1-2

A transistor amplifier has a measured S/N of 10 at its input and 5 at its output. Calculate the transistor NF

$$NF = 10 \log \frac{S_i/N_i}{S_o/N_o} = 10 \log \frac{10}{5} = 10 \log 2$$

$$= 3 \text{ dB}$$

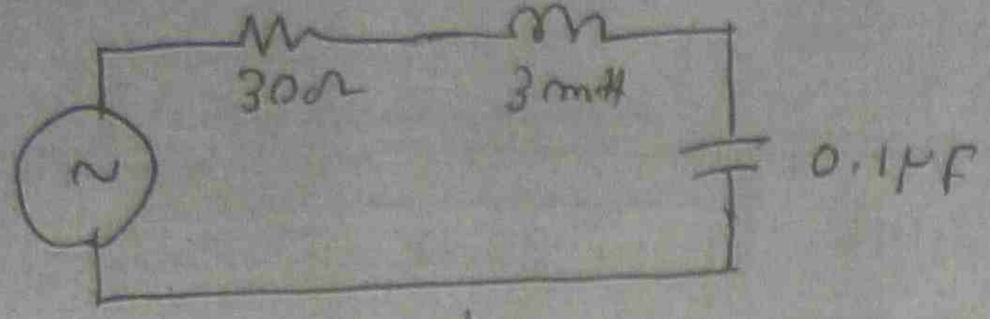
Ex 1-3

Two resistors 5 kΩ and 20 kΩ are at 27°C. Calculate for a 10 KHz bandwidth the thermal noise power and voltage.

(a) For each resistor.
(b) For their series combination
(c) For their parallel combination

Ex

Determine the resonant frequency for the circuit shown in Fig 1.14 calculate its impedence when $f = 12\,kHz$



$$f_r = \frac{1}{2\pi\sqrt{3mH \times 0.1\mu F}}$$

$$= 9.19\ kHz$$

at 12 kHz

$$X_L = 2\pi f L = 2\pi \times 12\,kHz \times 3mH = 226\Omega$$

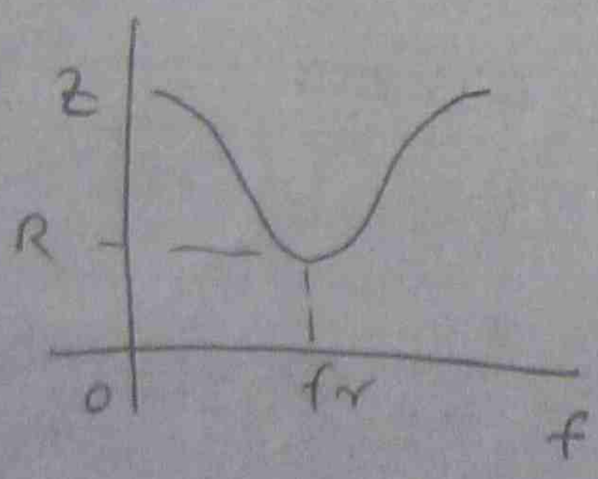
$$X_C = \frac{1}{2\pi f c} = \frac{1}{2\pi \times 12\,kHz \times 0.1\mu F} = 133\Omega$$

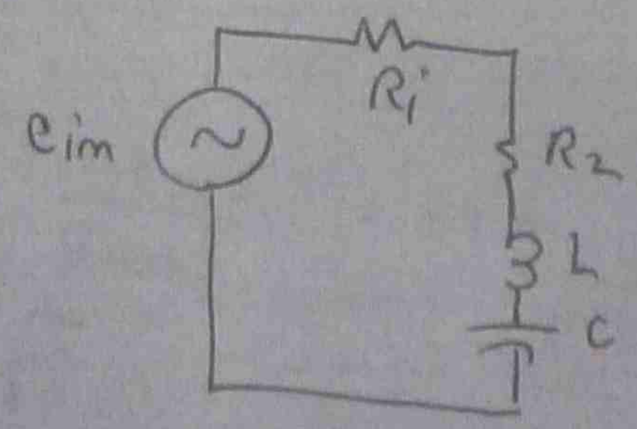
$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{30^2 + (226-133)^2}$$

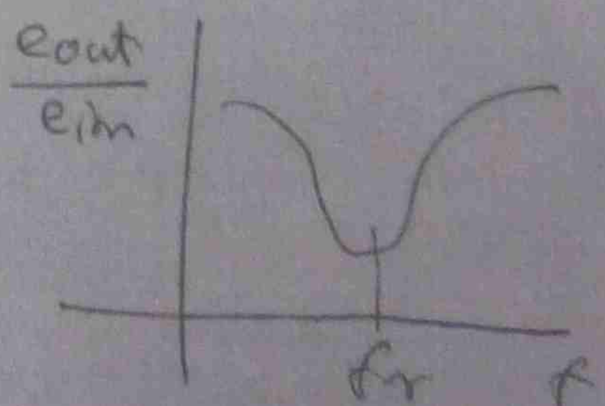
$$= 97.7\Omega$$

Ex determine f_r for the circuit shown in figure when $R_1 = 20\Omega$, $R_2 = 1\Omega$ $L = 1mH$, $C = 0.4\mu F$ $e_{in} = 50mV$. Calculate e_{out} at f_r and 12 kHz

Q40







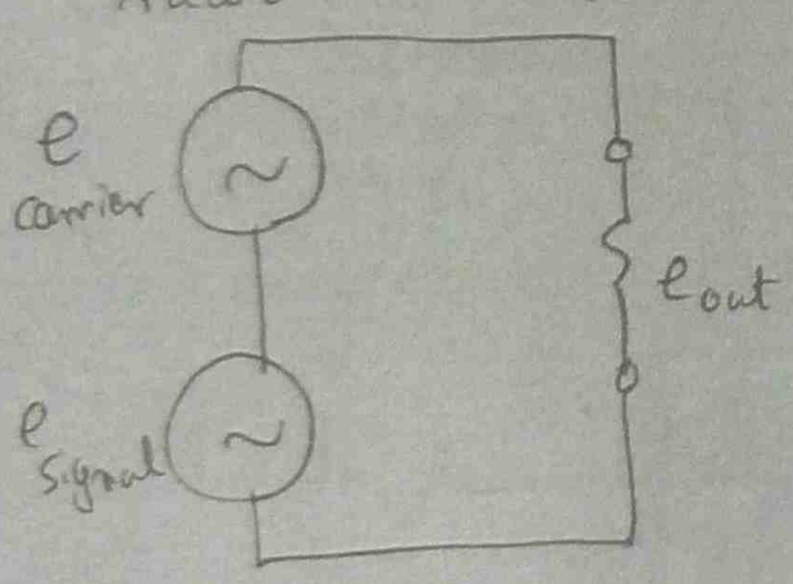
GAS Amplitude modulation transmission

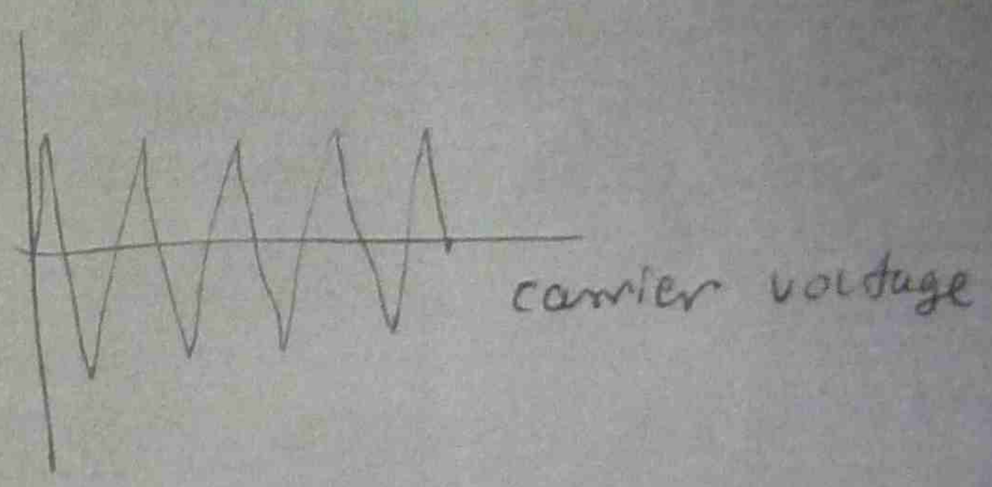
The process of impressing a low-frequency intelligence signal onto a higher frequency carrier signal.

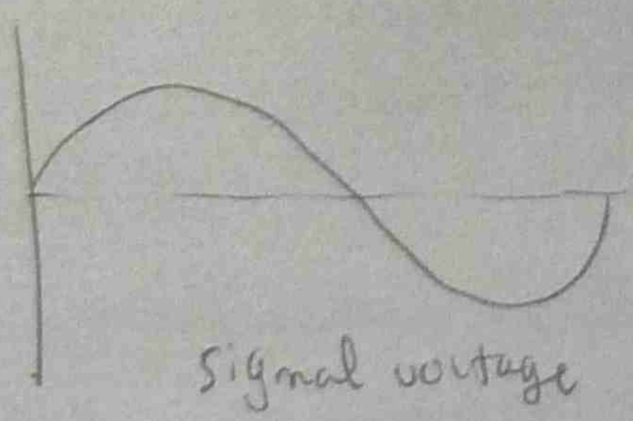
The reason for modulation

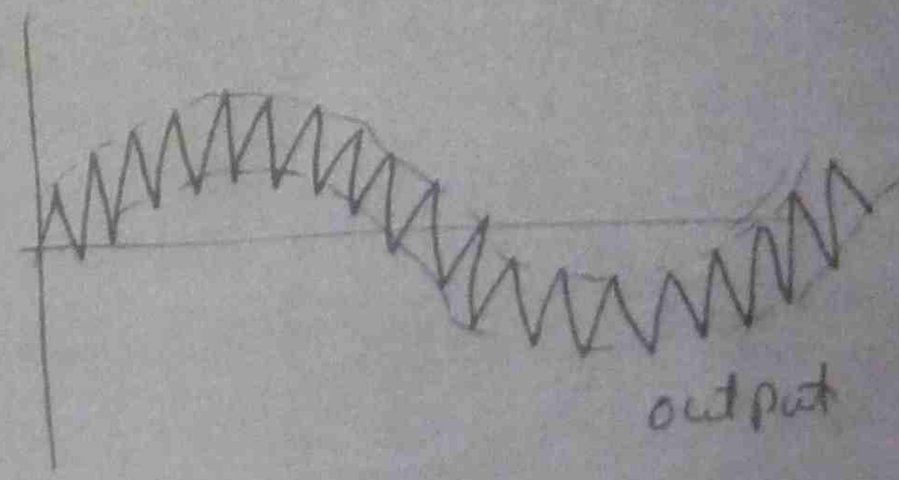
99| (a) Direct transmission of intelligence signals would result in catastrophic interference problems, since the resulting radio waves would all be at approximately the same frequency range

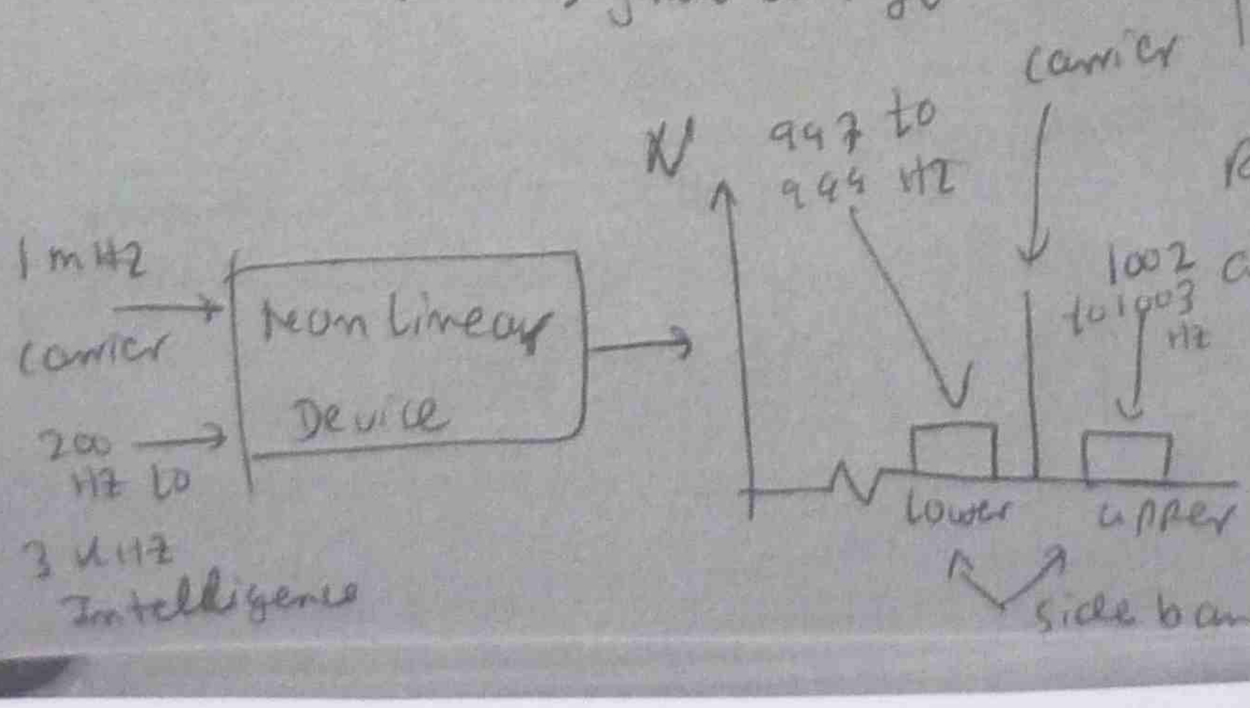
(b) most intelligible signals are at relatively low frequencies. Efficient transmission and reception of radio waves at low frequencies is not possible.











q(12

Ex Determine %m for the following conditions of the unmodulated carrier is 80V Peak to Peak (P-P)

	maximum P-P carrier (V)	minimum P-P carrier (V)
(a)	100	60
(b)	125	35
(c)	160	0
(d)	180	0
(e)	135	35

B < 2x unmodulated = 100 < 80x2

(a) $\%m = \frac{B-A}{B+A} \times 100 = \frac{100-60}{100+60} \times 100 = 25\%$

(b) $\%m = \frac{125-35}{125-35} \times 100 = 56.25\%$

B < 2x unmodulated
125 < 80x2

(c) $\%m = \frac{160-0}{160+0} \times 100 = 100\%$

B = 2x unmodulated

(d) ~~%m~~ B = 180

2x unmodulated = 2x 80 = 160 ∴ B > 2x unmodulated

Overmodulated

(e) B = 135

2x unmodulated = 2x 80 = 160 Hz

$\%m = \frac{135-35}{135+35} \times 100 = 58.8\%$

Ex A 500W carrier is to be modulated to a 90% level. Determine the total transmitted power

$m = 0.9$, $P_c = 500W$

$$P_t = P_c\left(1 + \frac{m^2}{2}\right)$$

$$= 500\left(1 + \frac{0.9^2}{2}\right) = 702.5W$$

Ex An AM broadcast station operates at its maximum allowed total output of 50kW and at 95% modulation. How much of its transmitted power is intelligence (side bands)?

To find $P_i = P_t - P_c$

$$P_c\left(1 + \frac{m^2}{2}\right)$$

Given $P_t = 50kW$, $m = 0.95$

$$\therefore P_t = P_c\left(1 + \frac{m^2}{2}\right)$$

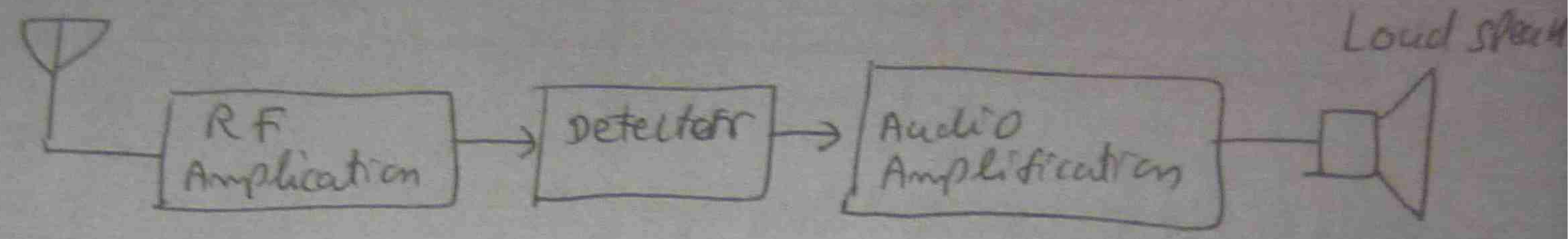
$$50kW = P_c\left(1 + \frac{0.95^2}{2}\right) \Longrightarrow P_c = \frac{50}{1 + \frac{0.95^2}{2}}$$

$$= 34.5kW$$

$$\therefore P_i = P_t - P_c = 50 - 34.5$$

$$= 15.5kW$$

Amplitude modulation reception



main characteristics of receiver

- Sensitivity ⟶ Ability to drive the output transducer to an acceptable level.
- Selectivity.

→ The input signal must be greater than the noise at the receiver's input.

→ Selectivity may be defined as the extent to which a receiver is capable of differentiating between the desired signal and disturbances at other frequencies

TRF selectivity

$$Q = \frac{f}{Bw}$$

Ex A TRF receiver is to be designed with a ~~signal~~ single tuned circuit using a 10 μH inductor.

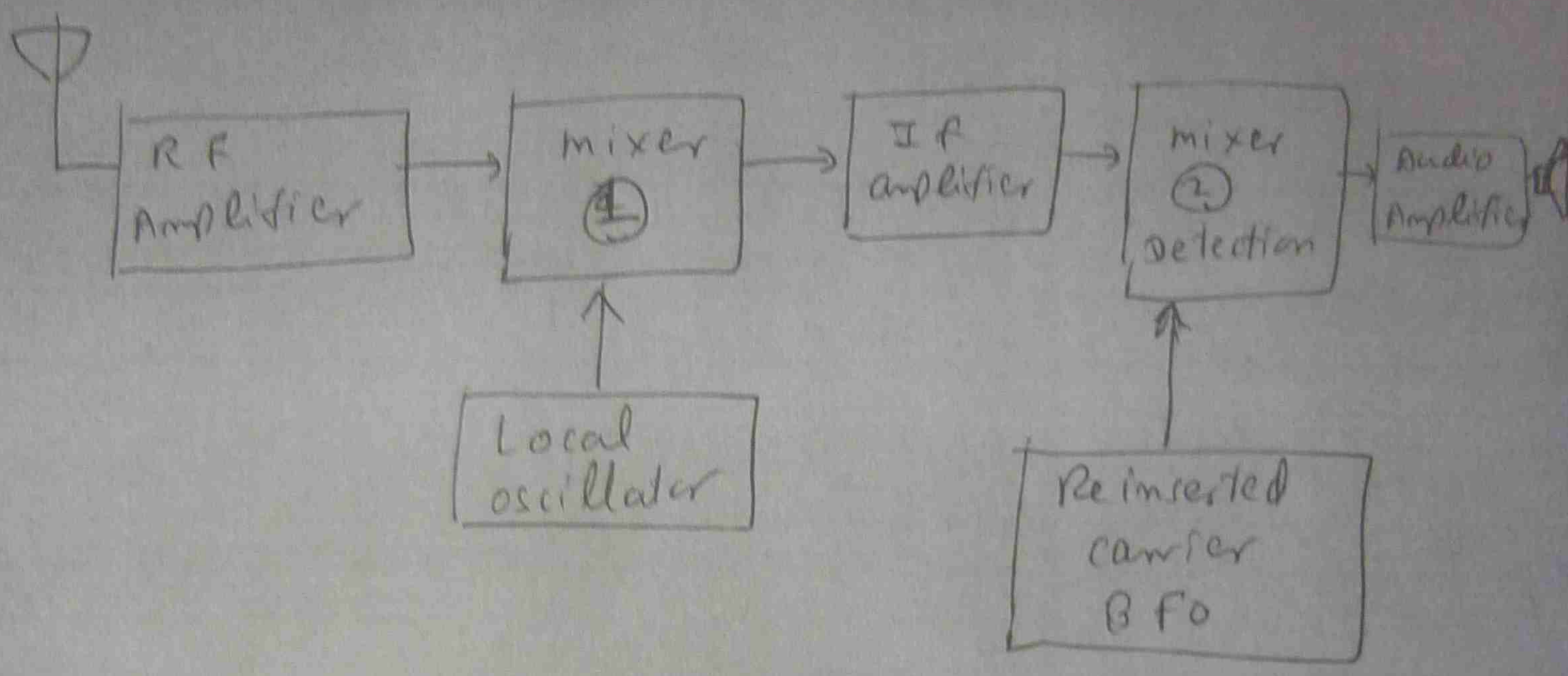
(a) Calculate the capacitance range of the variable capacitor required to tune from 550 to 1550 KHz

(b) The ideal 10 KHz BW is to occur at 1100 KHz Determine the required Q.

(c) Calculate the BW of this receiver at 550 KHz and 1550 KHz.

999

SSB Receiver



Pb The SSB receiver at the above diagram has outputs at 1 KHz and 3 KHz. The carrier used and suppressed at the transmitter was 2 MHz and the upper side band was utilized determine the exact frequencies at all stages for a 455 KHz IF frequency.

I RF Amp & first mixer input

Transmitter freq $\pm$ two outputs $\longrightarrow$ 2000 + 1 KHz = 2001 KHz
2000 + 3 KHz = 2003 KHz

II Local oscillator (mixer)

Transmitter freq + IF freq $\longrightarrow$ 2000 + 455 = 2455 KHz
↑ BFO

III First mixer output

Local oscillator (mixer) — (RF amp & first mixer)

rate of carrier frequency deviation.

10) Ph — An FM signal has a center frequency of 100mHz but is swinging between 100,001 mHz and 99.999 mHz at a rate of 100 times per second

Determine

(a) The Intelligence frequency fi
(b) The intelligence amplitude
(c) what happened to the intelligence amplitude if the frequency deviation changed to between 100,002 and 99.998 mHz.

(a) $f_i = 100mHz$

(b) No way to determine the actual amplitude of the intelligence signal. Every FM system has a different proportionality constant between the intelligence amplitude and the amount of deviation it causes.

(c) Doubling frequency deviation ⟶ doubling intelligence amplitude.

FM Analysis

For Phase modulation, the equation for instantaneous voltage

$$e = A \sin(\omega_c t + m_p \sin \omega_i t)$$

where e = instantaneous voltage
A = peak value of original carrier wave
ω_c = carrier angular velocity $2\pi f_c$
m_p = maximum phase shift caused by intelligence signal (rad)

Pb/ (a) Determine the permissible range of maximum modulation index for commercial fm that has 30Hz to 15kHz modulating frequencies.

(b) Repeat for a narrow band system that allows a maximum deviation of 10 kHz and 100Hz to 3kHz modulating frequencies.

(a) The maximum deviation in broadcast fm is 75KHz

$$m_f = \frac{\delta}{f_i} = \frac{75\,kHz}{30Hz} = 2500$$

$$\downarrow$$

$$\frac{75kHz}{15kHz} = 5$$

10kHz & 100Hz

(b) $m_f = \frac{10kHz}{100Hz} = 100$

100Hz to 3kHz

$$m_f = \frac{10kHz}{3kHz} = 3.33$$

Pb Determine the relative total power of carrier and side frequencies when $m_f = 0.25$ for a 10kW Fm transmitter.

$m_f = 0.25 \longrightarrow J_0 = 0.98$

$J_1 = 0.12$

Carrier Power = $(J_0)^2 \times$ Rating = $(0.98)^2 \times 10kW = 9.604kW$

Sideband = $(J_1)^2 \times$ Rating = $(0.12)^2 \times 10kW = 144W$

Total Power = carrier Power + 2 Sideband power

= $9.604 + 2 \times 144 \times 10^{-3} = 10kW$

Image rejection $dB = 20 \log \left[\frac{fi}{fs} - \frac{fs}{fi} \right] Q$

where fi = Image frequency
fs = desired signal frequency
Q = tuned circuit Q.

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pb / An AM broadcast receiver has two identical tuned circuits. The Q of these circuits is 60 and the IF frequency is 455 KHz and the receiver is tuned to a station at 680 KHz. Calculate the amount of frequency rejection.

desired signal (f_s)

fi = Image frequency = tuned freq + 2 × IF freq

= 680 + 2 × 455 = 1590 KHz

$f(s) = 680$ KHz

Image rejection (dB) $= 20 \log \left(\frac{fi}{fs} - \frac{fs}{fi} \right) Q$

$= 20 \log \left(\frac{1590}{680} - \frac{680}{1590} \right) 60$

$= 20 \log 114.6$

$= 41$ dB

Total suppression = 41 dB (or) 82 dB

Special techniques

Delayed AGC -

Automatic Gain Control - As soon as even a weak received signal is tuned, simple AGC provides some gain reduction.

Delay AGC → The AGC does not provide any gain reduction until some arbitrary signal level is exceeded

C.I.T

1004

Pb/ A digital transmission has an error probability of 10^{-4} and is 10^8 bits long. Calculate the expected number of error bits.

Average number of errors = m (Bit long) × error Probability

$= 10^8 \times 10^{-4}$

$= 10^4 = 10000$

$$C = Bw \log_2\left(1+\frac{S}{N}\right)$$

C = channel capacity (b/s)

Bw = Bandwidth (Hz)

S/N = Signal to noise power ratio.

Pb/ calculate the capacity of a telephone channel that has a S/N of 1023. Phone Bw = ~~3 Hz~~ 3 kHz

$C = Bw \log_2\left(1+\frac{S}{N}\right)$

$= 3\times10^3 \log_2(1+1023)$

$= 3\times10^3 \times 10 = 30{,}000$ bits.

Code Noise Immunity

Error occurs as a result of noise

Average number of errors = m × error Probability

Error Probability in digital system is the number of errors per total number of bits received.

Binary coding system has superior noise immunity.

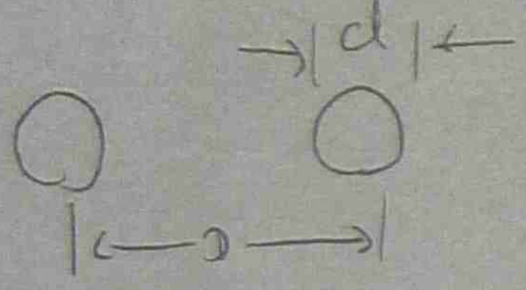
Week (13) - Transmission Lines

10/3 Characteristics impedance $Z_0 = \sqrt{\frac{L}{C}}$

Pb (1) A commonly used coaxial cable RG 8 A/U has a capacitance of 29.5 PF / ft and inductance of 73.75 mH / ft. Determine its characteristic impedance for a 1 ft section and for a length of 1 mile.

1 ft section $Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{73.75 \times 10^{-9}}{29.5 \times 10^{-12}}} = 50\Omega$

1 mile section $Z_0 = \sqrt{\frac{5280 \times 73.75 \times 10^{-9}}{5280 \times 29.5 \times 10^{-12}}} = 50\Omega$

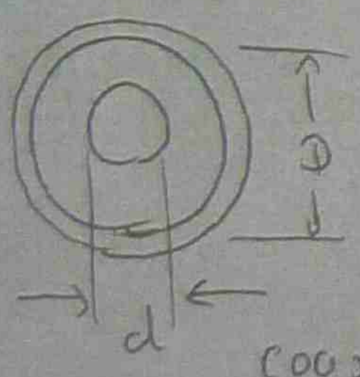


Parallel wires in air

$Z_0 = 120 \cosh^{-1} \frac{D}{d}$

for $D >> d$

$Z_0 \simeq 276 \log_{10} \frac{2D}{d}$



Coaxial

$Z_0 = \frac{138}{\sqrt{\varepsilon}} \log_{10} \frac{D}{d}$

curve is for $\varepsilon = 1.00$

ε = Dielectric constant

D = Inner diameter of outer conductor

d = outer diameter of inner conductor

10/4 Pb 2 Determine the characteristic impedance of

(a) A parallel wire line with $D/d = 2$ with air dielectric

(b) An air dielectric coaxial line with $D/d = 2.35$

$$t = \sqrt{LC}$$

$$v_p = \frac{D}{\sqrt{LC}}$$

v_p = velocity Propagation
D = Distance of travel
$\sqrt{LC}$ = time (t)

Delay line

Velocity of electric wave through vacuum 3×10^8 m/s
A transmission line decreases the velocity due to its inductance & capacitance

10/4
Pb 3

Determine the amount of delay and the velocity of Propagation introduced by a 1 ft section of RG 8A/U coaxial cable used as a delay line.

$$t = \sqrt{LC} = \sqrt{73.75 \times 10^{-9} \times 29.5 \times 10^{-12}} = 1.475 \times 10^{-9} \text{ s}$$

(or) 1.475 ns

The velocity of Propagation

$$v_p = \frac{D}{\sqrt{LC}} = \frac{1 \text{ ft}}{1.475 \text{ ns}} = 6.78 \times 10^8 \text{ ft/s}$$

(or) 2.07×10^8 m/s

Velocity factor $v_f = \frac{1}{\sqrt{\varepsilon}}$

where v_f = velocity factor
ε = relative dielectric constant

10/4

Pb 4 Determine the velocity factor for RG 8A/U cable by using the velocity of Propagation 2.07×10^8 m/s.

$$v_f = \frac{v_p}{3 \times 10^8} = \frac{2.07 \times 10^8}{3 \times 10^8} = 0.69$$

(or) $v_f = \frac{1}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{2.3}}$

1015

Pb 5 Determine the wave length of a 100 MHz signal in free space and while travelling through an RG 8A/u coaxial cable.

In free space

wave velocity 3×10^8 m/s

$$\lambda = \frac{V}{f} = \frac{3 \times 10^8 \text{ m/s}}{1 \times 10^8 \text{ Hz}} = 3 \text{ m}$$

In RG-8A/u cable. velocity of propagation = 2.07×10^8 m/s

$$\lambda = \frac{V}{f} = \frac{2.07 \times 10^8 \text{ m/s}}{1 \times 10^8 \text{ Hz}} = 2.07 \text{ m}$$

Reflected | Incident | Standing wave ratio | VSWR |

Smith Chart ⟶ 4269T | G042

From Page (119) 1022

Pb Calculate the power received at satellite given the following conditions

1. The power gain of the transmitting parabolic dish antenna is 30,000
2. The transmitter drives 2 uw of power in to antenna at carrier frequency 6.2 GHz
3. The satellite receiving antenna has power gain 30
4. The transmission path is 45000 km.

$$\left(\frac{P_R}{2uW}\right)_{dB} = 10\log_{10} \text{Power gain of antenna} + 10\log_{10} \text{Power gain} - (32.5 + 20\log_{10} \text{Transmission path} + 20\log_{10} \text{Carrier freq})$$

$$\left(\frac{P_R}{2uW}\right)_{dB} = 10\log_{10} 30{,}000 + 10\log_{10} 30 - (32.5 + 20\log_{10} 45000 + 20\log_{10} (210))$$

$$= 44.8 \text{ dB} + 14.8 \text{ dB} - (32.5 + 93.1 + 75.9) \text{ dB} = -141.9 \text{ dB}$$

$$\frac{P_R}{2uW} = \log^{-1} 14.19 \text{ dB} = \frac{1}{1.55 \times 10^{14}} \Rightarrow P_R = \frac{2 \text{ uW}}{1.55 \times 10^{14}} = 12.9 \text{ pW}$$

1017

Pb 2 — It is desired to build a $\lambda/2$ dipole to receive a 100 MHz broadcast. Determine the optimum length of the dipole.

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{100 \times 10^6} = 3\text{m}$$

~~Physical~~ Electrical length $= \frac{\lambda}{2} = \frac{3}{2} = 1.5\text{m}$

Length of antenna $= 0.95 \times$ Electrical length

$= 0.95 \times 1.5 = 1.43\text{m}$

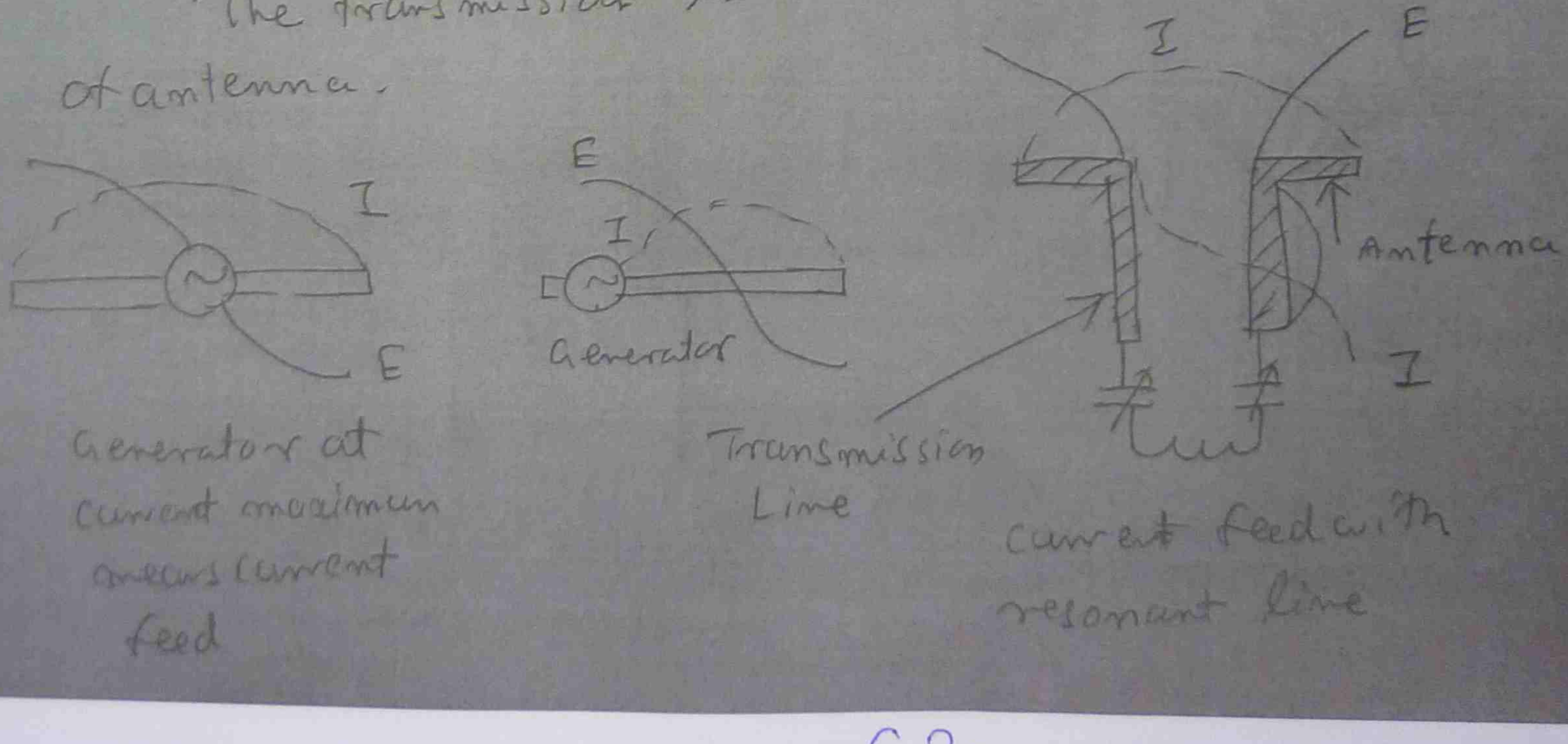
Antenna feed lines

If energy is applied at the geometrical centre of an antenna, the antenna is said to be center fed.

If energy is applied to the end of an antenna, it is known as an end fed antenna.

Resonant feed line

The transmission line is connected to the center of antenna.



Fiber attenuation and dispersion

P = optical power at distance l from transmitted power P_t

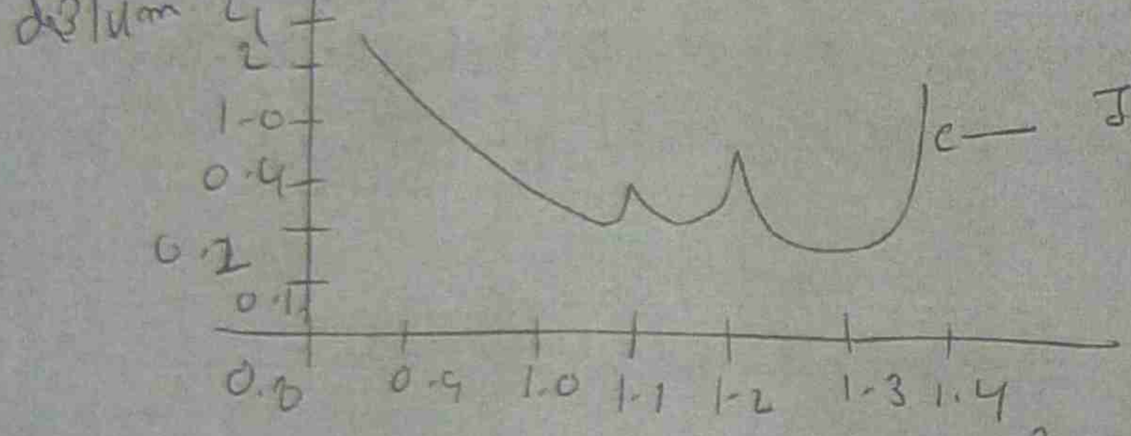
$$P = P_t \times 10^{-\frac{A}{10}}$$

A: fibre attenuation dB/km

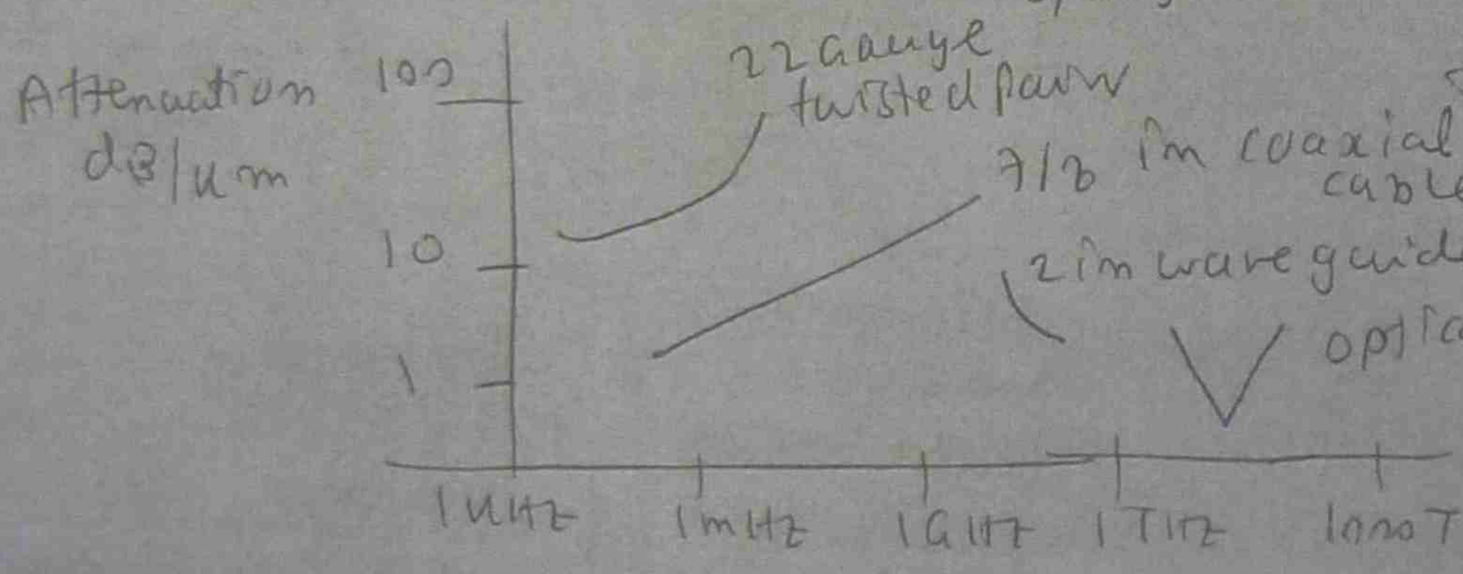
Pb (5) Calculate the optical power 50km from a 0.1mW source on a single mode fiber that has 0.25 dB/km loss.

$$P = P_T \times 10^{-\frac{A}{10}} = P_T \times 10^{-\left(\frac{0.25 \times 50}{10}\right)}$$

$$= 0.1 \times 10^{-3} \times 10^{-\frac{12.5}{10}} = 5.62\,\mu W$$



Attenuation vs wavelength for typical high quality fiber.



Dispersion → material dispersion — caused by slight variation of refractive index

→ waveguide dispersion — caused by a portion of the light energy travelling in the cladding

BW limit 20 mHz/km of length for multicore fibre

20% of energy contained

Systems ← Telementry
Fiber optic sensors
communications

Power budget

$$L = \frac{1}{5 B \Delta_t}$$

L = maximum fiber length (km)
B = maximum bit rate (mb/s)
Δ_t = dispersion (ns/km)

1025

Pb ⑥ / make a power budget analysis for a fiber optic system with the following characteristics.

Losses	Specifications
LED to fiber connection 5dB	LED power output 0.1mW
3 connectors 1-5dB each	detector sensitivity 2μW
six splices 0.5dB each	max bit rate - 5mb/s
10km of fiber 0.6dB/km	
Fiber to detector 6dB	

Total fiber dispersion 4ns/km

Sum of Total system attenuation = 5dB + 3(1.5dB) + 6(0.6dB) + 10 x 0.6dB + 6dB = 24.5dB

Assume 5dB safety factor total = 29.5dB

$$29.5\,dB = 10 \log \frac{P_t}{P_r}$$

$$\frac{P_t}{P_r} = \text{Log}^{-1} \frac{29.5}{10} = 891$$

$$P_r = \frac{P_t}{891} = \frac{0.1\,mW}{891} \approx 0.112\,\mu W$$

$\Delta t = 4 \times 10^{-3}$ μs/km

$= 0.004$

$$L = \frac{1}{5 B \Delta t} =$$

$$= \frac{1}{5 \times 5 \times 0.004} = 10\,km$$

System length equal the maximum cable length (10km) some testing

Since detector sensitivity is 0.1μW, the system should operate satisfactorily without repeater

will be necessary

TABLE 5.5 Overall Heat Loss Criteria

Design Heating Degree Days (65°F Daily Base)	Maximum Heat Loss (Btu/DD-ft²)	
	Conventional Buildings	Passively Solar Heated Buildings, Exclusive of Solar Wall
Less than 3000	8	7.6
3000–4000	8	6.6
4000–5000	7	5.6
5000–7000	6	4.6
Greater than 7000	5	3.6

Source: Balcomb et al. (1980).

tion. The equation used to determine the overall rate is

$$\frac{(U_{\text{average loss}} \times \text{F})_{\text{envelope}} + (\text{ACH} \times \text{volume})_{\text{infiltration}} \times 24\text{ h}}{\text{area heated floor area (ft}^2\text{)}} = \text{Btu/DD-ft}^2$$

One of the biggest unknowns in this procedure is the assumed rate of infiltration. A carefully designed and constructed small building can easily achieve a rate of 0.75 ACH, with increased values under average conditions. Tight burner installation, caulking of all cracks, and so on, rates of 0.33 ACH have been demonstrated.

EXAMPLE 5.1, PART C. This office building (Fig. 5.4) is passively solar heated, so the overall rate of heat loss, Btu/DD-ft², is determined by using the total $U \times A$ of the non-south-glass envelope. From Table 5.2, this is 42 + 25 + 72 = 139 Btu/h-°F.

To this must be added the effects of ventilation. From the same table, outdoor air is shown to be supplied at the rate of

30 cfm/person × 8 persons = 240 cfm

which can also be expressed as 180 cfm × 60 min/h = 9600 cf/h. Comparing this hourly rate to the volume of the heated box, we have

$$\frac{9600\ \text{ft}^3/\text{h}}{22{,}000\ \text{ft}^3} = 0.44\ \text{ACH}$$

The 0.4 ACH for ventilation is therefore

0.44 ACH × 22,000 ft³ × 0.018 = 174 Btu/h-°F

and the total rate of heat loss is

$$\frac{139\ \text{Btu/h-°F} + 174\ \text{Btu/h-°F} \times 24\ \text{h/day}}{1440\ \text{ft}^2} = 5.2\ \text{Btu/DD-ft}^2$$

This building is located in Eugene, Oregon, which corresponds most closely to the location of Salem, Oregon as listed in Appendix B, Table B.15, where we find DD65 in Salem = 4852 per year. From Table 5.5, the maximum recommended rate of heat loss for a passively solar heated building in a 3000 to 5000-DD climate is 5.6 Btu/DD-ft². This building's heat loss rate is under the maximum, enhancing energy conservation.

(d) Heating and Cooling Criteria for Components of the Envelope: Nonresidential Buildings. The preceding whole-building single-number criteria have several limitations. They do not satisfactorily account for differences in building size (and the accompanying differences in the likely rates of internal heat gains), nor do they distinguish between the thermal roles of roofs and walls, which can be quite different seasonally. The design guidelines for nonresidential buildings are taken from ASHRAE/IES Standard 90.1-1989, *Energy Efficient Design of New Buildings Except Low-Rise Residential Buildings*, published by the American Society of Heating, Refrigerating, and Air-Conditioning Engineers, Inc., Atlanta, GA. They are found in Appendix K, Section K.1.

(e) Heating and Cooling Criteria for Components of the Envelope: Residential Buildings. These design guidelines are taken from ASHRAE Standard 90A-1980, Energy Conservation in New Building Design, published by the American Society of Heating, Refrigerating, and Air-Conditioning Engineers, Inc., Atlanta, GA. As this book went to press, the newer BSR/ASHRAE Standard 90.2-1980, *Energy Efficient Design of New Low-Rise Residential Buildings*, was in the approval process. The newer standard should be used, once approval is final. The residential design criteria are found in Appendix K, Sections K.2 to K.4.

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EXAMPLE 5.1, PART D. A typical bay of the Oregon office building shown in Fig. 5.4 will be examined for compliance with ASHRAE/IES Standard 90.1-1989; see Appendix K for this standard.

STEP 1. *Select ACP table from Appendix K.* The climate of Eugene, Oregon, is very similar to Salem, Oregon, so ACP Table K.18 (page 1587) will be used.

STEP 2. *Select maximum allowable percentage fenestration.* Following the substeps:

(a) $\text{ILD} = \text{LPD} + \text{EPD} + \text{OLA}$

where LPD, lighting power density, will be considered as conforming to the maximum for office buildings from Table K.2, Part A (office area 1440 ft² (below 2000 ft²), = 1.9 W/ft²; where EPD, equipment power density, is taken from Table K.2, Part B, = 0.75 W/ft² for offices, and where OLA, occupant load allowance, will be left at 0 W/ft²; which allows for a built-in assumption of 0.6 W/ft²).

$$\text{ILD} = \text{LPD} + \text{EPD} + \text{OLA}$$
$$= 1.9 + 0.75 + 0 = 2.65\ \text{W/ft}^2$$

The electric lighting actually installed in the typical bay consists of six rows of indirect fluorescent luminaires, with a total of 24 fluorescent lamps (48 in.) at 35 W each (including the ballast), and 44 fluorescent lamps (48 in.) at 46 W each (including the ballast), so the total installed lighting wattage per bay is about 2860 W.

Actual power density

$$= \frac{2860\ \text{W lighting}}{1440\ \text{ft}^2\ \text{floor area}} = 1.99\ \text{W/ft}^2$$

Therefore, the ACP table will be entered in the ILD range 1.51 to 3.00.

(b) Select the PF (assumed projection shading factor). Trellises are used on all south-facing windows, but not on the north. These trellises are exactly as deep as the windows are high, and the bottom of the trellis is at the top of the window. Thus, for all south-facing windows, the PF is calculated as

$$\text{PF} = \frac{P_H}{H}$$

where P_H is here equal to H; therefore,

$$\text{PF} = 1.0$$

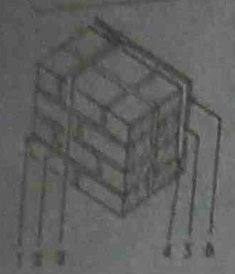
No projecting shading devices on north windows (97 ft²), combined with PF = 1.0 on south windows (97 + 132 ft²), indicates that an average PF for all windows of somewhat more than 0.5; therefore, continue into the ACP table in the PF range "0.5+."

(c) Select the shading coefficient of the window (SC). These windows are double glazed with clear glass, and interior white venetian blinds are provided below all light shelves. From Table 4.19, the SC can be assumed at 0.58. Therefore, continue into the ACP table in the SC_a range 0.500 to 0.50.

(d) Enter the vertical "fenestration" column. Unfortunately for this daylit building, no thermal credit can be taken for daylight in this climate using this prescriptive method. Therefore, only the "base case" columns are available.

(e) Select appropriate fenestration type. From Table 5.2, the U value of these double-glazed, aluminum frame (no thermal break), ¼-in. glass windows with a ½-in. airspace, product type C (commercial), is 0.74. Therefore, enter the vertical fenestration base case column in the U_a range 0.81 to 0.46. Finally, the maximum allowable % fenestration can be found at the intersection of the horizontal line of ILD 1.51 to 3.00, PF 0.5+, SC 0.500 to 0.50, and the vertical column of base case U_a 0.81 to 0.46:

TABLE 4.6 Coefficients of Transmission, U (Btu/h ft² F), of Solid Masonry Walls

Base Case Resistance: Between Furring, R_i	Base Case Resistance: At Furring, R_s	Construction	Resistance[a]: New Item 4 ("Construction")
			
0.17	0.17	1. Outside surface (15-mph wind)	0.17
1.60	1.60	2. Common brick, 8 in.	1.60
—	0.94	3. Nominal 1-in. × 3-in. vertical furring	—
1.01	—	4. Nonreflective air space, 0.75 in. (50 F mean; 10 F° temperature difference)	5.00
0.45	0.45	5. Gypsum wallboard, 0.5 in.	0.45
0.68	0.68	6. Inside surface (still air)	0.68
3.91	3.84	*Total thermal resistance, R*	$R_i = 7.90 = R_s$

Base Case

$$U = \frac{1}{3.91} = 0.256$$

New Item 4

$$U = \frac{1}{7.90} = 0.127$$

To adjust U values for the effect of furring strips, see the example given in Table 4.5. With 20% furring (typical of 1-in. × 3-in. vertical furring on masonry at 16-in. o.c.):

$U_{av} = 0.257$ (about 1% more heat loss)

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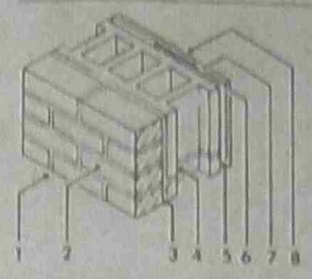
[a]Base case with furring strips and air space replaced with 1-in. expanded polystyrene extruded, smooth skin surface, 2.2 R.

TABLE 4.7 Coefficients of Transmission, U (Btu/h ft² F), of Masonry Cavity Walls

Base Case Resistance: Between Furring, R_i	Base Case Resistance: At Furring, R_s	Construction	Resistance[c]: New Items 3 and 7 ("Construction")
			
0.17	0.17	1. Outside surface (15-mph wind)	0.17
0.80	0.80	2. Common brick, 4 in.	0.80
1.10[a]	1.10[a]	3. Nonreflective air space, 2.5 in. (30 F mean; 10 F° temperature difference)	5.32[b]
0.71	0.71	4. Concrete block, three-oval core, stone and gravel aggregate, 4 in.	0.71
1.01	—	5. Nonreflective air space 0.75 in. (50 F mean; 10 F° temperature difference)	—
—	0.94	6. Nominal 1-in. × 3-in. vertical furring	—
0.45	0.45	7. Gypsum wallboard, 0.5 in.	0.11
0.68	0.68	8. Inside surface (still air)	0.68
4.92	4.85	*Total thermal resistance, R*	$R_i = R_s = 7.79$

Base Case

$$U_i = \frac{1}{4.92} = 0.203$$

New Items 3 and 7

$$U = \frac{1}{7.79} = 0.128$$

To adjust U values for the effect of furring strips, see the example given in Table 4.5; with 20% furring (typical of 1-in. × 3-in. vertical furring on masonry at 16-in. o.c.):

$U_{av} = 0.204$ (about 1% more heat loss)

[a]Interpolated from vertical air space values in Table 4.4.

[b]Calculated from vermiculite R values in Table 4.2.

[c]Resistance when furring strips and gypsum wallboard replaced with 0.625-in. plaster (sand aggregate) applied directly to concrete block and 2.5-in. air space filled with vermiculite insulation, 7–8.2 lb/ft³.

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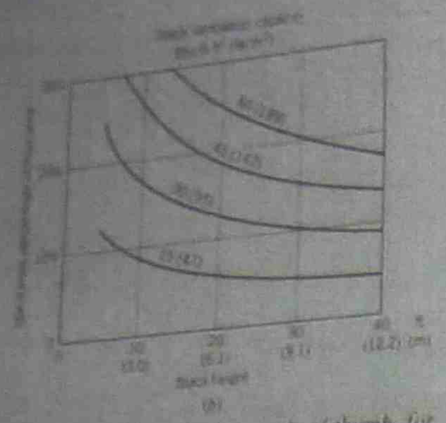

Fig. 5.8 (b) *Stack ventilation rule of thumb, for heat removed per unit floor area, and relationship area of stack and height of stack. Total stack area is expressed as a percent of total floor area. Note: Stack area refers to minimum area of inlets and of cross section through the vertical stack and of outlets. The figure assumes an internal temperature 3F° (1.7C°) above the exterior temperature (see also Fig. 4.7).*

cases, find the percentage of inlet area required, then multiply by the ratio

$$\frac{3F^\circ}{\text{actual } \Delta T}$$

to obtain required cross-ventilation areas for any specific temperature difference (see Example 5.1, Part H).

See Table 5.9 for a comparison of rule-of-thumb and detailed calculations for both cross and stack ventilation.

(e) Stack Ventilation. The inlet areas, expressed as percentage of total floor area, are related to stack height and the resulting heat removal in Fig. 5.8*b*. Remember that an equal (or greater) area of stack outlet openings, as well as at least an equal cross-sectional area through the vertical stack, are also required. The assumptions about indoor–outdoor temperature differences that were used to produce this guideline are explained in the figure.

Adjustment of the ΔT for this rule of thumb is similar to the procedure for cross ventilation. It requires multiplication of the required percentage of stack area by the ratio

$$\frac{3F^\circ}{\sqrt{\text{actual } \Delta T}}$$

to obtain the required stack ventilation areas for any specific temperature difference (see Example 5.3).

EXAMPLE 5.1, PART G. The office building in Fig. 5.4 is located in Eugene, Oregon, where monthly average wind velocities may be obtained from NOAA Local Climatological Data. For July, the hottest month, the average wind velocity is about 8 mph, from a nearly-due-north average direction. From Table 5.2 the inlet area (= outlet area also, in this case) is 16 ft². The inlet area, as a percentage of floor area, is therefore

$$16 \text{ ft}^2/1440 \text{ ft}^2 = 1\%$$

Entering Fig. 5.8 with 8-mph wind velocity and 1% inlet area, we find that somewhat less than 15 Btu/h-ft² heat gain can be removed by cross ventilation. In Example 5.1, Part F, we approximated a heat gain (excluding ventilation) of about 11 Btu/h-ft². Therefore, under average July conditions, the window area is probably adequate to remove heat gains and maintain an indoor temperature 3F° *above* outdoor temperature.

Since Eugene's average July daily high is 83 F, this would produce an interior daily high of 86 F. From Fig. 2.4*b*, such a temperature would require somewhat more than minimum air motion in order to feel comfortable. At Eugene's summer *design* temperature of 89 F, the interior temperature would be 92 F. For these reasons this building was designed to be night ventilated, relying on thermal mass rather than upon cross ventilation for hot-day comfort.

In milder weather, these windows can achieve substantial cooling because the temperature difference, ΔT, is more than the 3 F° assumed in Fig. 5.8. For example, when the temperature indoors is 75 F, outdoors 65 F, the fully opened windows with the 8-mph wind would result in a cooling rate of

$$\frac{\text{about 15 Btu/h-ft}^2 \times 10F^\circ}{3F^\circ} = \text{about 50 Btu/h-ft}^2$$

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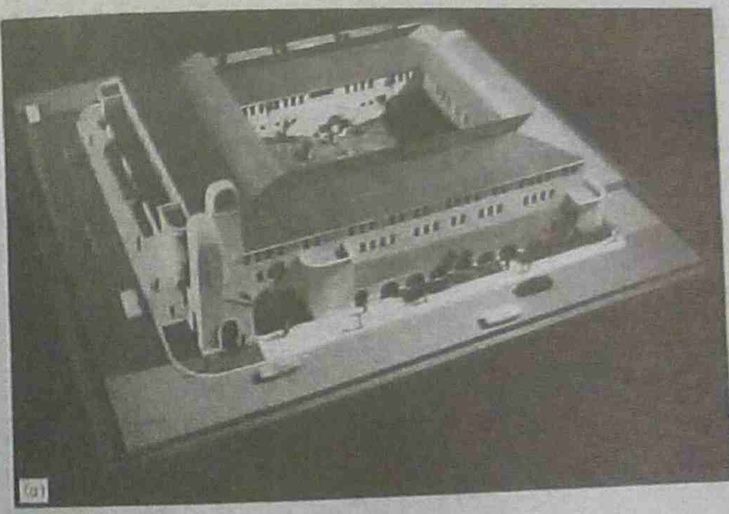
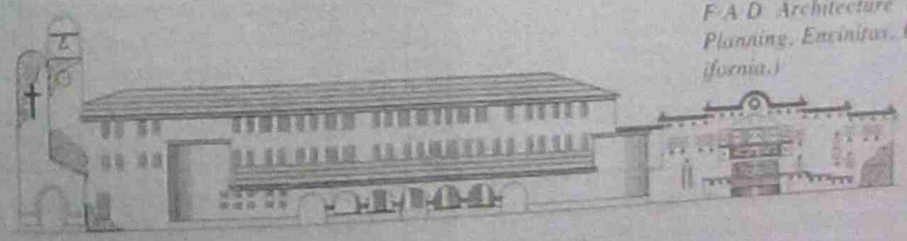
which is considerably more than the actual gain of 11 Btu/h-ft². Therefore, under such conditions the windows are only partially opened.

A large building utilizing both cross and fan-assisted stack ventilation is shown in Fig. 5.9. The St. Vincent de Paul/Joan Kroc center covers a block in downtown San Diego. Built to serve the homeless of that city, it can house 450 persons per night and serve 1500 meals a day. The 110,000-ft² structure relies on natural ventilation for the rooms on the second and third floors. There are openable, shaded windows in each room, and bell towers at the shelter's main entrance also enhance ventilation. One faces upwind to act as an intake for sea breezes that occur 70% of the time at a speed of between 8 and 11 mph; the other faces downwind and acts as a stack effect exhaust opening, supplemented by negative pressure from the breezes. A large central courtyard provides daylight, cross ventilation, and a secure outdoor place for the residents. The building also features advanced evaporative cooling equipment for the ground-floor facilities and a peaking/backup power plant that provides space heating and hot water when it operates.

(f) Night Ventilation of Thermal Mass. This procedure is shown in Fig. 5.10, in which climate data are related to two representative types of thermally massive building. For each type, the graphs show the daily Btu per square foot of floor area that can be stored.

The climate data (maximum summer design DB temperature and mean daily range) are given in Appendix A. These data also allow calculation of the *minimum* summer design DB temperature; that is, maximum design DB temperature minus mean daily range. This minimum temperature is of interest here because the thermal mass of the building will be lowered toward (but not quite to) it during night ventilation. For high daily range climates, the lowest temperature ob-





(b) WEST ELEVATION

Fig. 5.9 San Diego's facility for the homeless (a), *the St. Vincent de Paul/Joan Kroc Center, uses fan-assisted natural ventilation to cool the top two floors.* (b) *The west elevation shows the opening at the top of the bell tower that is an intake for the sea breezes; the other tower faces downwind, thus acts as an exhaust.* (c) *A diagram of the fresh air cooling system also shows the large central courtyard that encourages daylight and cross ventilation.* (*Courtesy of F A D Architecture and Planning, Encinitas, California.*)

Typical Residential Daily Internal Heat Gains from Appliances and Lighting

Appliance	Btu/day	kWh/day
	16,000	4.7
	10,900–15,700	3.2–4.6
	8,900–11,600	2.6–3.4
	6,500–11,300	1.9–3.3
	3,400–4,400	1.0–1.3
	2,400	0.7
	8,500–18,800	2.5–5.5
	13,700–27,400	4.0–8.0

Appliances	Btu/day
	15,700–27,400
	12,000–19,000
	12,000–27,000

...Presently, The SERI Solar Conservation Study, Brick House Publishing, Andover, Mass. ...the ventilation rate, the 1989 Handbook of Fundamentals, published by the American ...ating, Refrigerating, and Air-Conditioning Engineers, Inc., Atlanta, Ga.

...appliance's consumption per day; heat gain to house is less, depending on the amount of ...ed air.

...ual TV per household.

...ata for an average 1350-ft² house; rate per ft² floor area may be extrapolated.

...t load from water heater to house.

...3, PART 2. Calculate the January ...temperature for a typical bay of ...lding in Fig. 5.4. Assume that the ...ior January temperature over a 24-h ...d.

...n. The internal gains are from Ex... Part E: 9.9 + 8.5 = 8.4 Btu/h-ft², ... equipment, and electric lights. The ...ntal gains are

$$\frac{\text{Btu/ft}^2 \cdot \text{day}}{24\ \text{h}} = 2.4\ \text{Btu/h-ft}^2$$

...r solar gain, YS, from Appendix B, ... for Salem, Oregon, is 471 Btu/ft²-... of south glass.

$$\frac{\text{Btu/ft}^2 \cdot \text{day} \times \text{ft}^2\text{ south glass}}{24\ \text{h} \times \text{ft}^2} = 3.3\ \text{Btu/h-ft}^2$$

$$Q_i = 2.4 + 3.1 = 5.5\ \text{Btu/h-ft}^2$$

...*UA* for the typical bay (from Table 5.2 and Example 5.1, Part C) includes, for this calculation, all elements of the building envelope: (42 + 25 + 72 + 72 + 98 + 174 infiltration) = 483 Btu/h-°F, expressed per ft² floor area, 483/1440 = 0.34 Btu/h-°F-ft². Therefore,

$$T_b = 68\,\text{F} - \frac{5.5\ \text{Btu/h-ft}^2}{0.34\ \text{Btu/h-ft}^2} = 68 - 16 = 52\,\text{F}$$

(b) Degree Days. These data are published for each climate station and are calculated to various base temperatures (found in Appendix B). Heating degree days to a specified base temperature, such as HDD65, refer to average temperatures below the base temperature of 65 F. Cooling degree days to a specified base temperature, such as CDD65, refer to average temperatures above the base temperature of 65 F. For smaller buildings such as residences, heating degree days are in much wider usage than are cooling degree days, and many data sources simply list DD65; this notation is interchangeable with HDD65, and will be used for the remainder of this section. Until recently, degree days (DD) were always based on 65 F, because older, indifferently insulated buildings, with low internal gains, had a typical balance point of about 65 F. The combination of much higher levels of insulation and much more electric equipment has shoved the average building's balance point temperature downward; hence DD50, DD55, and DD60 are included with the traditional DD65 in Appendix B.

To derive heating DD for a particular climate and X base temperature DDX, each day's mean temperature (halfway between high and low) is subtracted from the base temperature; the result is the number of DDX for that day. If the mean temperature equals or exceeds the base temperature, no DDX are recorded. Then the DDX are totaled for an average year.

For example, assume that a day in Troy, New York, had a high of 60 F and a low of 34 F. The mean temperature was (60 − 34)/2 + 34 = 47 F.

65 − 47 = 18 DD65

60 − 47 = 13 DD60

55 − 47 = 8 DD55

50 − 47 = 3 DD50

Clearly, a building with a 65 F balance point will need more heat that day than will a building with a 50 F balance point temperature.

To convert DD conventional to DD SI, simply multiply DD conventional by 5/9:

DDSI = 0.56 DD conventional

To obtain the "DD balance point" needed to estimate your particular building's heating needs, interpolate between the various base DDs as required. (If your balance point is below 50 F, get lower DD base figures from your local weather station. Do not extrapolate!)

(c) Yearly Space Heating Energy. To estimate the energy needed over an average year for a building E, calculate

$$E = \frac{(UA \times \text{DD balance point} \times 24\ \text{h})}{kV}$$

where

E is in units of fuel consumed per year (e.g., therms of gas or kWh of electricity).

UA is the total heat loss rate, envelope + infiltration (Btu/h-°F, W/°C).

DD balance point is obtained as just described.

k is a factor that includes the effects of equipment efficiency at both full and potential capacity, of energy conservation devices, of energy-efficient buildings and equipment (see Table 5.14).

V is the heating value of the fuel, from Table 5.12.

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EXAMPLE 5.7. A residence in Springfield, Illinois, has a total heat loss rate UA of 544 Btu/h-°F and a balance point temperature of 55 F. It will have an induced draft, natural gas forced-air furnace, for which k = 0.8. For Springfield, DD55 = 3434. The average annual energy used for space heating is

$$E_{\text{therms}} = \frac{(544\ \text{Btu/h-°F} \times 3434\ \text{DD} \times 24\ \text{h/day})}{0.8 \times 100{,}000\ \text{Btu/therm}}$$

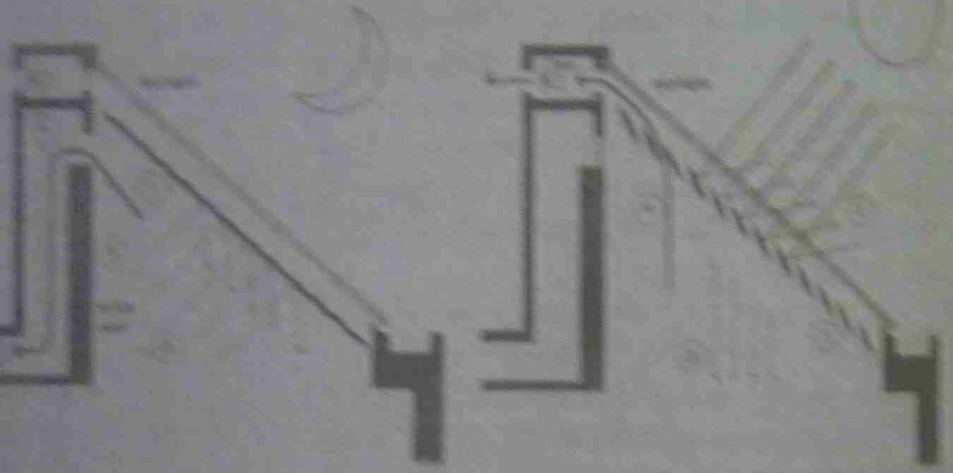
$$= 560\ \text{therms}$$

TABLE 5.14 **Heating Equipment Efficiency Factors, k**

Space Heating Equipment	k
Electric resistance heaters within the heated space	1.0
Gas-fired boiler, fully condensing	0.9
Gas furnace: forced-air, spark-ignited, and induced-draft	0.8
Oil-fired boiler, energy-efficient model	0.65
Gas furnace: forced-air, atmospherically vented	0.60

Source: Based on 1981 *Handbook of Fundamentals* published by American Society of Heating, Refrigerating, and Air-Conditioning Engineers, Inc., Atlanta, Ga.

*For use in predicting annual energy consumption for space heating. A detailed list of certified furnace and boiler efficiency ratings is available from the American Gas Association (1515 Wilson Blvd., Alexandria, VA 22209).



Winter Night

Sunny Summer Day

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sizes, duct cross sections influence the sizes and shapes of the vertical cores that serve multiuse buildings. An approximation of duct size can be obtained as follows.

1. Determine the quantity of air to be distributed through the largest duct, using Table 7.5, or the air changes per hour, from a calculation of night ventilation of thermal mass (e.g., Example 5.3, Part M). This will usually be expressed in cubic feet per hour (cfh).
2. Convert cfh to cfm:

$$\frac{\text{cfh} \times 1\text{ h}}{60\text{ min}} = \text{cfm}$$

3. Find the recommended velocity of the air within the duct from Table 6.18 (page 366), expressed in feet per minute (fpm).
4. The approximate required cross-sectional area of the duct A_d is then

$$A_d = \frac{\text{volume of air (cfm)}}{\text{velocity (fpm)}} \times 144\text{ in}^2/\text{ft}^2 \times \text{friction allowance}$$

where the friction allowances are

round ducts = 1.0 (may be neglected)

nearly square ducts (ratio of width to depth, 1:1):

small (< 1000 cfm) = 1.10

large (> 1000 cfm) = 1.05

flat rectangular ducts (ratio of width to depth, 1:5) = 1.25

EXAMPLE 7.1. Approximate the largest duct size required to serve the Oregon office building shown in Fig. 5.4, page 150. On the building's upper floor are five bays, each 144 ft², for a total of 720 ft².

duct. This duct must carry

$$0.25\text{ cfm/ft}^2 \times 720\text{ ft}^2 = 1800\text{ cfm}$$

(Part of this 1800 cfm is the outdoor air provided, at 160 cfm/bay × 5 bays = 800 cfm.)

Within this duct, an air velocity must be established. From Table 6.18, the recommended velocity for main ducts in public buildings is 1800 to 1300 fpm; for branch ducts, 800 to 600 fpm. The lower the velocity, the larger, quieter, and less fan-energy intensive will be the duct; assume a lower velocity midway between "main" and "branch" in this example: 800 fpm), because the main duct is so centrically exposed to the space it serves. The friction allowance for the nearly square, large central duct can be assumed as 1.05. This supply duct will therefore be, at maximum:

$$A_{d,s} = \frac{1800\text{ cfm} \times 144\text{ in}^2/\text{ft}^2 \times 1.05}{800\text{ fpm}} = 340\text{ in}^2$$

(The actual central supply duct size in this building's upper floor is 380 in² maximum. Compare the results above to the quicker sizing procedure of Example 5.1, Part L.)

(b) *Accessories.* Air diffusers and grilles (Fig. 7.42) are common to all these systems. The most popular locations for these accessories are ceilings, which are unobstructed by furniture and relatively unaffected by minor rearrangements in partitions. Also, the entire space above a suspended ceiling can be used as one huge return air duct; this arrangement is called plenum return. In a thermal zone where cooling is the prevalent condition, cool air introduced at the ceiling is ideal; being heavier, it sinks naturally upon entry. However, low wall or floor return air grilles will assure a better circulation of air. In a thermal zone where heating is the prevalent condition, the reverse locations are preferable—

mon light sources such as incandescent, fluorescent, mercury, etc., are not monochromatic, but produce light in many parts of the spectrum (see Fig. 18.6b), no single conversion factor between watts and lumens exists. Each source has its own luminous efficiency (lumens/watt), deter-

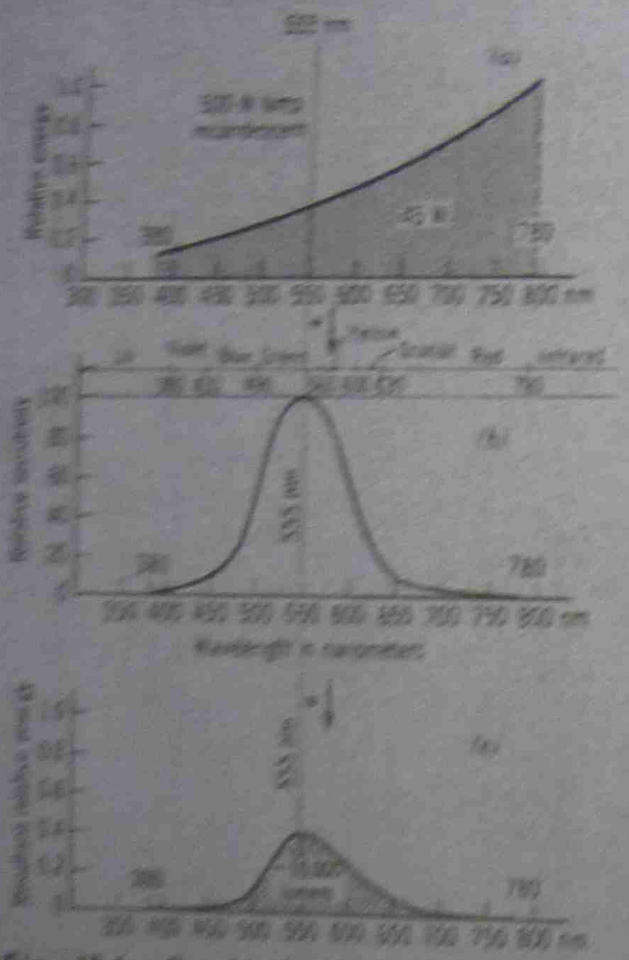

Fig. 18.6 Graphical demonstration of the method by which the unit of light flux is defined. (a) Shows the spectrum of the light produced by a 500- W incandescent lamp. It amounts to approximately 45 W, measured radiometrically. When filtered by the human eye whose spectral sensitivity curve is given in (b), this light power is perceived as shown in (c). This new light power curve is expressed in lumens and indicates the quantity of light, as perceived by the eye.

One lumen of luminous flux, uniformly incident on one square foot of area, produces an illuminance of one *footcandle* (fc). Illuminance is normally represented by the letter E. Restated, illuminance is the density of luminous power expressed in terms of lumens per unit area. If we were to consider a light bulb to be analogous to a sprinkler head, then the rate of water flow would be the lumens and the amount of water per unit time per square foot of floor area would be the footcandles. The acoustic analog is intensity in watts per area. In SI units, the area is expressed in square meters and the illuminance (illumination) in lux (lx). Thus the SI unit, lux, is smaller than the corresponding unit, footcandles, by the ratio of square feet to square meters. That is,

$$10.764 \text{ lux} = \text{one footcandle}$$

or multiply footcandle by 10.764 to obtain lux. These relations are shown in Fig. 18.5. Restating the above mathematically yields

$$\text{footcandles} = \frac{\text{lumens}}{\text{square foot of area}}$$

$$\text{fc} = \frac{\text{lum}}{\text{ft}^2} \tag{18.1}$$

and

$$\text{lux} = \frac{\text{lumens}}{\text{square meters of area}}$$

$$\text{lx} = \frac{\text{lm}}{m^2} \tag{18.2}$$

As an *approximation* (8% error)

$$10 \text{ lx} = 1 \text{ fc} \tag{18.3}$$

EXAMPLE 18.1. A 40-W, 430-mA (milliampere), 48-in. [122 cm] fluorescent tube produces 3200 lm. What is the illuminance on the floor of a 10-by-10 ft. room assuming 40% overall efficiency and uniform illumination?

$$\text{lx} = \frac{1280}{10 \times 10} \times (3.28 \text{ ft/m})^2 = 137.7$$

By approximation:

$$\text{lx} = 10 \cdot \text{fc} = 128$$

Illuminance at a *point* can be computed from intensity, as shown in Section 18.12 below.

18.8 Luminance, Exitance, and Brightness

An object is perceived because light coming from it enters the eye. The impression received is one of object *brightness*. This brightness sensation, however, is subjective, and depends not only on the object *luminance* (L) but also on the state of adaptation of the eye (see Sections 18.18 and 18.19). For this reason the physiological sensation is generally referred to in the literature as subjective or apparent brightness or simply *brightness*, whereas the measurable, reproducible state of object luminosity is its *luminance* (formerly "photometric brightness"). Luminance is normally defined in terms of intensity; it is the luminous intensity per unit *apparent* (projected) area of a primary (emitting) or secondary (reflecting) light source. Thus its units are candela terms such as stilb, apostilb, blondlambert, and candela per square in... avoided. In this book, the term lumin... be used except where it is specifically ... to refer to the physiological sensation ... in which case the terms *brightness*, ... *brightness*, or *apparent brightness* will ... Luminance has no readily conceivable ... cal or electrical analogy.

Another concept that the lighting ... will encounter is known as *luminous e*... simply as *exitance* and as the name im... scribes the total luminous flux dens... (exiting) a surface, irrespective of dir... viewer position. For instance, if a s... meter square emits one lumen, its lumi... ance is one lumen per square meter (1 ... 0.093 lm/ft². A surface that is a perfe... whether by emitting light diffusely o... light diffusely, is known as a *Lamb... face*. It is fairly simple to demonstr... matically that the luminance of such ... equals $1/\pi$ times its exitance. The imp... this relationship is its usefulness as ... mation. Although very few surfaces ... Lambertian, many are approximate... this relationship can be used as an ... accuracy approximation in many su...

TABLE 18.1 Lighting Units—Conversion Factors

Unit	Multiply	By	To Obtain
Illuminance (E)	Lux	0.0929	Footcandle
	Footcandle	10.764	Lux
Luminance (L)	cd/m²	0.2919	Footlambert
	cd/cm²	10000	cd/m²
	cd/in.²	1550	cd/m²
	cd/ft²	10.76	cd/m²
	millilambert	3.183	cd/m²
	Footlambert	3.4263	cd/m²
Intensity (I)	Candela	1.0	Candlepower

6	1.02	1.02	1.02	1.01
7	1.02	1.02	1.02	1.01
8	1.02	1.02	1.01	1.01
9	1.01	1.01	1.01	1.01
10	1.01	1.01	1.01	1.01

[a]Extracted from *IES Handbook* (1981), and reprinted with permission.

[b]For more precise data, for varying ρ_w see *IES Handbook* (1981).

ence, since all cavity ratio figures having determined one, the others

$$CCR = RCR \frac{h_{cc}}{h_{rc}} \quad (20.11)$$

$$FCR = RCR \frac{h_{fc}}{h_{rc}} \quad (20.12)$$

$$CCR = FCR \frac{h_{cc}}{h_{fc}} \quad (20.13)$$

See Table 20.3 and Figs. 20.38 and
step involves obtaining the effective
tance (ρ_{cc}) from Table 20.3. Note
reflectance remains as selected in
e fixtures are surface mounted or
en CCR = 0 and ρ_{cc} = selected ceil-
ce.

See Table 20.3 and Figs. 20.38 and
step involves obtaining the effective
ance ρ_{FC} as above in step 3, for ρ_{CC}
is the working plane, FCR = 0 and
ted floor reflectance.

Select CU from manufacturer's data.
nterpolation may be necessary for
ity reflectance (ρ_{CC}), which is be-
gures in the CU table. See Example
next section. Also, factors for CU
for ρ_{FC} other than 20% (standard in
are given in Table 20.4.

STEP 6. Calculate illuminance and number of fixtures or area per luminaire in the usual fashion.

Illustrative examples and shortcut methods are demonstrated below. CU coefficients are listed in Table 20.2.

20.33 Zonal Cavity Calculations: Illustrative Examples

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EXAMPLE 20.1. It is suggested that the reader photocopy Fig. 20.39 and fill it in as the solution progresses.

Given. Classroom: 6 m × 8 m × 3.70 m height, elementary school. Initial reflectances: ceiling 80%, entire wall 50%, floor 20%. (Note that the sketch on Fig. 20.39 can accommodate different reflectances for the upper, center, and lower wall sections.) Provide adequate lighting using fluorescent lamp fixtures. Assume yearly

tion in much of the space is normally considerable in schools because of the hours of use. It will frequently exceed this 500-lux level, but to demonstrate the calculation procedure, we must assume an absence of daylight. Lines 1, 2, and 3 of Fig. 20.39 and the room sketch can be filled in.

(b) Luminaire Selection. The criteria we have previously developed for a classroom situation requires an installation that will yield

1. Low direct glare (high VCP), because students spend a large proportion of their time in a heads-up position.
2. Low veiling reflections because much of the seeing task involves high reflectance materials, occasionally specular.
3. High efficiency and low energy use to meet IES/ASHRAE and most governmental requirements.
4. Minimum required maintenance, in view of the poor cleaning and maintenance situation that obtains in many schools.

Although ceiling height is sufficient to permit use of indirect lighting (e.g., luminaire 33 in Table 20.2) with all its distinct advantages, it is not chosen, because

5. The luminaire maintenance category is bad, and given the type of maintenance expected, light reduction would be serious.
6. Indirect lighting depends on a highly reflective ceiling, requiring yearly cleaning

light. We would se
an excessively dark
minance ratios here
Furthermore, fixtur
because of the up
the fixture caused
last. Mounting heig
[~9 ft] to permit e
row spacing. For
mended maximum
Work plane height

(c) Calculations

STEP 1. The requ
on the sketch in Fig

$h_{CC} = 1.0$
$h_{FC} = 0.75$

$\rho_C = 80\%$
$\rho_F = 20\%$

STEP 2. From equ

$RCR = 5h_{rc}$

$CCR = 5(1$

$FCR = 5(0$

STEP 3. From Tab
flectances:
For ρ_{CC} use $\rho_C = 0.8$, ρ
Therefore,

ρ_{CC}

STEP 4. For ρ_{FC} use
FCR = 1.09. Therefor