Specification Development

Mechanical Vibration and Shock Analysis

second edition – volume 5

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Foreword to Series

In the course of their lifetime simple items in everyday use such as mobile telephones, wristwatches, electronic components in cars or more specific items such as satellite equipment or flight systems in aircraft, can be subjected to various conditions of temperature and humidity, and more particularly to mechanical shock and vibrations, which form the subject of this work. They must therefore be designed in such a way that they can withstand the effects of the environmental conditions they are exposed to without being damaged. Their design must be verified using a prototype or by calculations and/or significant laboratory testing.

Sizing, and later and testing are performed on the basis of specifications taken from national or international standards. The initial standards, drawn up in the 1940s, were blanket specifications, often extremely stringent, consisting of a sinusoidal vibration, the frequency of which was set to the resonance of the equipment. They were essentially designed to demonstrate a certain standard resistance of the equipment, with the implicit hypothesis that if the equipment survived the particular environment it would withstand, undamaged, the vibrations to which it would be subjected in service. Sometimes with a delay due to a certain conservatism, the evolution of these standards followed that of the testing facilities: the possibility of producing swept sine tests, the production of narrow-band random vibrations swept over a wide range and finally the generation of wide-band random vibrations. At the end of the 1970s, it was felt that there was a basic need to reduce the weight and cost of on-board equipment and to produce specifications closer to the real conditions of use. This evolution was taken into account between 1980 and 1985 concerning American standards (MIL-STD 810), French standards (GAM EG 13) or international standards (NATO) which all recommended the *tailoring of tests*. Current preference is to talk of the *tailoring of the product to its environment* in order to assert more clearly that the environment must be taken into account from the very start of the project, rather than to check the behavior of the material *a* *posteriori*. These concepts, originating with the military, are currently being increasingly echoed in the civil field.

Tailoring is based on an analysis of the life profile of the equipment, on the measurement of the environmental conditions associated with each condition of use and on the synthesis of all the data into a simple specification, which should be of the same severity as the actual environment.

This approach presupposes a proper understanding of the mechanical systems subjected to dynamic loads and knowledge of the most frequent failure modes.

Generally speaking, a good assessment of the stresses in a system subjected to vibration is possible only on the basis of a finite elements model and relatively complex calculations. Such calculations can only be undertaken at a relatively advanced stage of the project once the structure has been sufficiently defined for such a model to be established.

Considerable work on the environment must be performed independently of the equipment concerned either at the very beginning of the project, at a time where there are no drawings available, or at the qualification stage, in order to define the test conditions.

In the absence of a precise and validated model of the structure, the simplest possible mechanical system is frequently used consisting of mass, stiffness and damping (a linear system with one degree of freedom), especially for:

– the comparison of the severity of several shocks (shock response spectrum) or of several vibrations (extreme response and fatigue damage spectra);

– the drafting of specifications: determining a vibration which produces the same effects on the model as the real environment, with the underlying hypothesis that the equivalent value will remain valid on the real, more complex structure;

– the calculations for pre-sizing at the start of the project;

– the establishment of rules for analysis of the vibrations (choice of the number of calculation points of a power spectral density) or for the definition of the tests (choice of the sweep rate of a swept sine test).

This explains the importance given to this simple model in this work of five volumes on “Vibration and Mechanical Shock”:

Volume 1 of this series is devoted to *sinusoidal vibration*. After several reminders about the main vibratory environments which can affect materials during their working life and also about the methods used to take them into account, following several fundamental mechanical concepts, the responses (relative and absolute) of a mechanical one-degree-of-freedom system to an arbitrary excitation are considered, and its transfer function in various forms are defined. By placing the properties of sinusoidal vibrations in the contexts of the real environment and of laboratory tests, the transitory and steady state response of a single-degree-of-freedom system with viscous and then with non-linear damping is evolved. The various sinusoidal modes of sweeping with their properties are described, and then, starting from the response of a one-degree-of-freedom system, the consequences of an unsuitable choice of the sweep rate are shown and a rule for choice of this rate deduced from it.

Volume 2 deals with *mechanical shock*. This volume presents the shock response spectrum (SRS) with its different definitions, its properties and the precautions to be taken in calculating it. The shock shapes most widely used with the usual test facilities are presented with their characteristics, with indications how to establish test specifications of the same severity as the real, measured environment. A demonstration is then given on how these specifications can be made with classic laboratory equipment: shock machines, electrodynamic exciters driven by a time signal or by a response spectrum, indicating the limits, advantages and disadvantages of each solution.

Volume 3 examines the analysis of *random vibration* which encompasses the vast majority of the vibrations encountered in the real environment. This volume describes the properties of the process, enabling simplification of the analysis, before presenting the analysis of the signal in the frequency domain. The definition of the power spectral density is reviewed, as well as the precautions to be taken in calculating it, together with the processes used to improve results (windowing, overlapping). A complementary third approach consists of analyzing the statistical properties of the time signal. In particular, this study makes it possible to determine the distribution law of the maxima of a random Gaussian signal and to simplify the calculations of fatigue damage by avoiding direct counting of the peaks (Volumes 4 and 5). The relationships, which provide the response of a one degree of freedom linear system to a random vibration, are established.

Volume 4 is devoted to the calculation of *damage fatigue*. It presents the hypotheses adopted to describe the behavior of a material subjected to fatigue, the laws of damage accumulation and the methods for counting the peaks of the response (used to establish a histogram when it is impossible to use the probability density of the peaks obtained with a Gaussian signal). The expressions of mean damage and of its standard deviation are established. A few cases are then examined using other hypotheses (mean not equal to zero, taking account of the fatigue limit, non-linear accumulation law, etc.). The main laws governing low cycle fatigue and fracture mechanics are also presented.

Volume 5 is dedicated to presenting the method of *specification development* according to the principle of tailoring. The extreme response and fatigue damage spectra are defined for each type of stress (sinusoidal vibrations, swept sine, shocks, random vibrations, etc.). The process for establishing a specification as from the life cycle profile of the equipment is then detailed taking into account the uncertainty factor (uncertainties related to the dispersion of the real environment and of the mechanical strength) and the test factor (function of the number of tests performed to demonstrate the resistance of the equipment).

First and foremost, this work is intended for engineers and technicians working in design teams responsible for sizing equipment, for project teams given the task of writing the various sizing and testing specifications (validation, qualification, certification, etc.) and for laboratories in charge of defining the tests and their performance following the choice of the most suitable simulation means.

Introduction

For many years mechanical environmental specifications have been taken directly from written standards, and this is often still the case today. The values proposed in such documents were determined years ago on the basis of measurements performed on vehicles which are now obsolete. They were transformed into test standards with very wide margins, and were adapted to the constraints of the testing facilities available at the time. A considerable number of tests taking the form of a swept sine vibration can therefore be found. These standards were designed more to verify resistance to the greatest stresses than to demonstrate resistance to fatigue. Generally speaking, the values proposed were extremely severe, resulting in the over-sizing of equipment.

Since the early 1980s, some of those standards (MIL-STD 810, GAM T13) have been upgraded, providing for drafting specifications on the basis of measurements taken under conditions in which the equipment is used. This approach presupposes an analysis of the life cycle profile of the equipment, by stipulating the various conditions of use (storage, handling, transport facilities, interfaces, durations, etc), and then relating characteristic measurements of the environment to each of the situations identified.

In this volume of the series a method for the synopsis of the collated data into specifications is presented. The equivalence criteria adopted are a reproduction of greatest stress and fatigue damage. This equivalence is obtained not from the responses of a real structure subjected to vibration, given that such a structure is unknown at the time of drafting specifications, but from the study of a single degree-of-freedom linear reference system. These criteria result in two types of spectra: extreme response spectra (ERS), similar to the older shock response spectra; and fatigue damage spectra (FDS).

Calculation of ERS is presented in Chapter 1 for sinusoidal vibrations (sine and swept sine) and in Chapter 2 for a random vibration.

Chapters 3 to 5 are devoted respectively to calculation of the FDS of sinusoidal and random vibrations and of shocks. Chapter 6 shows that ERSs and FDSs are insensitive to the choice of parameters necessary for their calculation.

Specifications may vary considerably, depending on the objectives sought. In Chapter 7 the main types of tests performed are recalled and, after a brief historical recapitulation, the current trend which recommends tailoring and taking into account the environment from the very beginning of the project is outlined.

Results of environmental measurements generally show a scattered pattern, due to the random nature of the phenomena. Moreover, it is well known that the resistance of parts obeys a statistical law and can therefore be described only by a mean value and a standard deviation. The stress–strength comparison can therefore only be drawn by the combination of two statistical laws, which results, when they are known, in a probability of failure, solely dependent, all other things being equal, on the ratio of the means of the two laws. For shocks and vibrations measured during an accident (environmental conditions which are not normal), ratio (k) is called the *uncertainty factor* or *safety factor* (Chapter 8).

In practice, the environment with its laws of distribution can be known, but the resistance of the equipment remains as yet unknown. Specifications give the values of the environment to be met, with a maximum tolerated probability of failure. The purpose of the test will therefore be to demonstrate the observance of that probability, namely that the mean resistance is at least equal to k times the mean environment. For understandable reasons of cost, only a very limited number of tests are performed, frequently only one. This small number simply makes it possible to demonstrate that the mean of the strength is in an interval centered on the level of the test, with a width dependent upon the level of confidence adopted and on the number of tests. To be sure that the mean, irrespective of its real position in the interval, is indeed higher than the required value, the tests must be performed to a greater degree of severity, something which is achieved by the application of a coefficient called the *test factor* (Chapter 10).

Certain items of equipment are used only after a long period in storage, during which their mechanical characteristics may have weakened through aging. For the probability of proper operability to be that which is required after such aging, much more must therefore be required of the equipment when new, at the time of its qualification, resulting in the application of another coefficient, called *the aging factor* (Chapter 9).

These spectra and factors form the basis of the method for drafting *tailored specifications* in four steps, as described in Chapter 11: establishment of the life cycle profile of the equipment, description of the environment (vibrations, shocks, etc.) for each situation (transport, handling, etc.), synopsis of the data thus collated, and establishment of the testing program.

The sensitivity of specifications developed with the method of equivalence of damages based on the different calculation parameters is studied in Chapter 12.

Chapter 13 provides a few other possible applications of the ERS and FDS, such as the comparison of the different types of vibrations (sinusoidal, random stationary or not, sine on random, shocks, etc.), the comparison between different standards or between standards and real environmental measures, the transformation of a large number of shocks into a specification of random vibration with similar severity, etc.

The Appendices show that the development method of specifications using ERSs and FDSs adds no additional hypothesis in relation to the PSD envelope method, which can lead to specifications which are too high in relation to the real environment if it is used without precaution. Contrary to this last method, the equivalence method of damages makes it possible to easily process more difficult cases, such as, for example, non-stationary vibrations, the establishment of a specification covering different types of vibrations with different application durations.

At the end of the book, a list of formulae combines the major relations established in the five volumes.

List of Symbols

The list below gives the most frequent definition of the main symbols used in this work. Some of the symbols can have another meaning locally, this will be defined in the text to avoid any confusion.

a	Threshold value of $z(t)$
aerf	Inverse error function
b	Parameter b of Basquin relationship $N\,\sigma^b = C$
c	Viscous damping constant
C	Basquin relationship constant $(N\,\sigma^b = C)$
D	Fatigue damage
e	Error
E	Exaggeration factor
$\overline{E}$	Mean of the environment
E_E	Expected environment
E_S	Selected environment
erf	Error function
ERS	Extreme response spectrum
$E(\ )$	Expectation of …
f	Frequency of excitation
f_M	Mean frequency
f_0	Natural frequency
FDS	Fatigue response spectrum
F_m	Maximum value of $F(t)$
$f(t)$	Frequency sweeping law
$F(t)$	External force applied to a system
g	Acceleration due to gravity
$G(\ )$	Power spectral density for $0 \le f \le \infty$
h	Interval (f/f_0)
H	Drop height
$H(\ )$	Transfer function
i	$\sqrt{-1}$
$I_0(\ )$	Zero order Bessel function
J	Damping constant
k	Stiffness or uncertainty coefficient
k_v	Aging coefficient
K	Constant of proportionality between stress and deformation
ℓ_{rms}	Rms value of $\ell(t)$
ℓ_m	Maximum value of $\ell(t)$
$\ell(t)$	Generalized excitation (displacement)
$\dot{\ell}(\ )$	First derivative of $\ell(t)$
$\ddot{\ell}(t)$	Second derivative of $\ell(t)$
m	Mass
MRS	Maximum response spectrum
M_n	Moment of order *n*

n	Number of cycles undergone by test-bar or material or Number of measurements or Number of tests
n_a^+	Mean number of up-crossings of threshold a with positive slope per second
n_0^+	Mean number of zero-crossings with positive slope per second (mean frequency)
n_p^+	Mean number of maxima per second
N	Number of cycles to failure or Mean number of envelope maxima per second or Number of peaks higher than a given threshold
N_a^+	Mean number of positive maxima higher than a given threshold for a given duration
P_v	Probability of correct operation related to aging
PSD	Power spectrum density
$p(\ )$	Probability density
$p(T)$	Probability density of first passage of a threshold during time T
$P(\ )$	Distribution function
q	$\sqrt{1-r^2}$
$q(u)$	Probability density of maxima
Q	Q factor (quality factor)
$Q(u)$	Probability that a maximum is higher than a given threshold
r	Irregularity factor
rms	Root mean square (value)
R	Extreme response spectrum
$\overline{R}$	Mean of strength
R_e	Yield stress
R_m	Ultimate tensile strength
R_U	Response spectrum with given up-crossing risk
s	Standard deviation
s_E	Standard deviation of environment
s_R	Standard deviation of resistance
SRS	Shock Response Spectrum
t	Time or Random variable of Student distribution law
t_s	Sweeping duration
T	Duration of vibration
T_F	Test factor
T_1	Time-constant of logarithmic frequency sweep
TS	Test severity
u	Ratio of threshold a to rms value z_{rms} of $z(t)$ or value of $u(t)$
u_{rms}	rms value of $u(t)$
u_m	Maximum value of $u(t)$
u_0	Threshold value of $u(t)$
URS	Up-crossing risk Response Spectrum
$u(t)$	Generalized response
v_i	Impact velocity
V_E	Variation coefficient of real environment

V_R	Variation coefficient of strength of the material
x_m	Maximum value of $x(t)$
$x(t)$	Absolute displacement of the base of a single degree -of-freedom system
$\dot{x}(t)$	Absolute velocity of base of a single degree-of-freedom system
$\ddot{x}(t)$	Absolute acceleration of the base of a single degree-of-freedom system
$\ddot{x}_{rms}$	rms value of $\ddot{x}(t)$
$\ddot{x}_m$	Maximum value of $\ddot{x}(t)$
$\ddot{y}_{rms}$	rms value of $\ddot{y}(t)$
$\ddot{y}(t)$	Absolute response acceleration of the mass of a one-degree-of-freedom system
z_{rms}	rms value of $z(t)$
$z_{a_{rms}}$	rms value of the response to a random vibration
z_m	Maximum value of $z(t)$
z_p	Peak value of $z(t)$
z_s	Relative response displacement to a sinusoidal vibration
$z_{s_{rms}}$	rms value of the response to a sinusoidal vibration
z_{sup}	The largest value of $z(t)$
$\dot{z}_{rms}$	rms value of $\dot{z}(t)$
$\ddot{z}_{rms}$	rms value of $\ddot{z}(t)$
$z(t)$	Relative response displacement of the mass of a single degree-of-freedom system with respect to its base
$\dot{z}(t)$	Relative response velocity
$\ddot{z}(t)$	Relative response acceleration
α	Risk of up-crossing
δ	Non-centrality parameter of the non-central t-distribution
Δf	Frequency interval between half-power points
ΔN	Number of cycles carried out between half-power points
Δt	Time spent between half-power points
ΔV	Velocity change
ε	Euler's constant (0.577 215 662 ...)
$\gamma(t)$	Incomplete gamma function
$\Gamma(\ )$	Gamma function
η	Dissipation (or loss) coefficient
φ	Phase
π	3.141 592 65 ...
π_0	Confidence level
σ	Stress
σ_a	Alternating stress
σ_D	Fatigue limit stress
σ_m	Mean stress
σ_{rms}	Rms stress value
σ_{max}	Maximum stress
ω_0	Natural pulsation $(2 \pi f_0)$
Ω	Pulsation of excitation $(2 \pi f)$
ξ	Damping factor

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Chapter 1

Extreme Response Spectrum of a Sinusoidal Vibration

1.1. The effects of vibration

Vibrations can damage a mechanical system as a result of several processes, among which are:

– the exceeding of characteristic instantaneous stress limits (yield stress, ultimate stress etc.);

– the damage by fatigue following the application of a large number of cycles.

In what follows we will consider the case of a single degree-of-freedom linear system only. This model will be used to characterize the relative severity of numerous vibrations. It will be assumed that, if the greatest stresses and damage due to fatigue generated in the system are equal, then these excitations are of the same severity in the model and, by extension, in a real structure undergoing such excitations.

Since it is only the largest stresses in a single degree-of-freedom standard model with mass-spring-damping that are of interest here, this is equivalent to consideration of extreme stress or extreme relative displacement, these two parameters being linked, for a linear system, by a constant:

$$\sigma_{max} = \text{constant } z_m \qquad [1.1]$$

1.2. Extreme response spectrum of a sinusoidal vibration

1.2.1. *Definition*

The *extreme response spectrum* (ERS) [LAL 84] (or *maximum response spectrum* (MRS)) is defined as a curve giving the value of the highest peak z_{sup} of the response of a linear one-degree-of-freedom system to vibration, according to its natural frequency f_0, for a given damping ratio ξ. The response is described here by the relative movement z(t) of the mass in relation to its support, and the coordinate axis refers to the quantity $(2\pi f_0)^2 z_{sup}$, by analogy with the shock response spectrum (Figure 1.1).

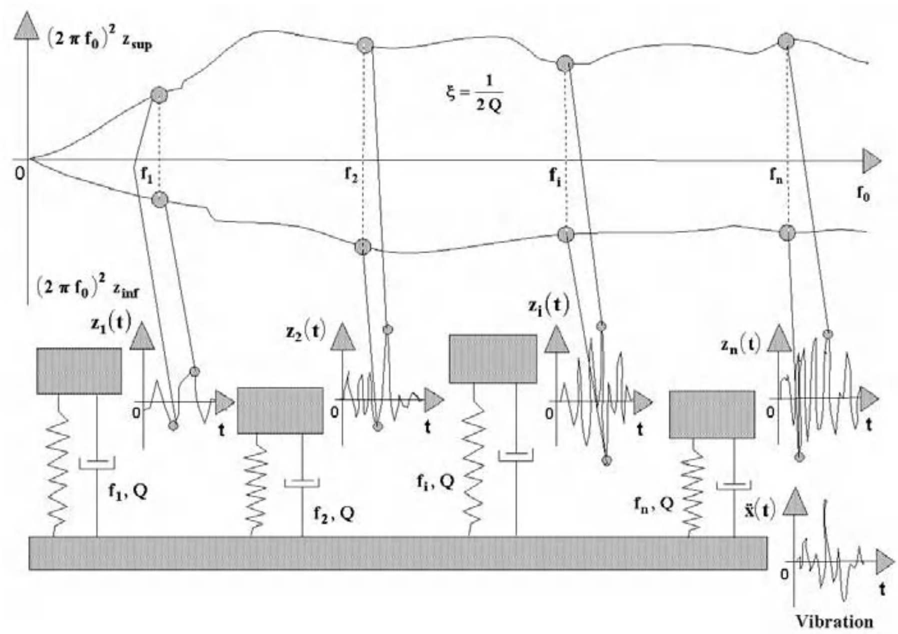


Figure 1.1. *ERS calculation model*

1.2.2. *Case of a single sinusoid*

A sinusoidal vibration can be defined in terms of a force, a displacement, a velocity or an acceleration.

1.2.2.1. *Excitation defined by acceleration*

Given an excitation defined by a sinusoidal acceleration of frequency f and amplitude $\ddot{x}_m$

$$\ddot{x}(t) = \ddot{x}_m \sin \Omega t$$

where $\Omega = 2 \pi f$. The response of a single degree-of-freedom linear system, characterized by the relative displacement $z(t)$ of the mass m with respect to the support, is expressed by:

$$z(t) = -\frac{\ddot{x}(t)}{\omega_0^2 \sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q f_0}\right)^2}} \qquad [1.2]$$

($\omega_0 = 2 \pi f_0$) and the highest response displacement (extremum) by

$$z_m = \pm \frac{\ddot{x}_m}{\omega_0^2 \sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q f_0}\right)^2}} \qquad [1.3]$$

The *extreme response spectrum* (ERS) is defined as the curve that represents variations of the quantity $R \equiv \omega_0^2 \left|z_m\right|$ as a function of the natural frequency f_0 of the system subjected to the sinusoid, for a given damping ratio ξ (or $Q = 1/2\,\xi$).

$$R \equiv \omega_0^2 \, z_m = \frac{\ddot{x}_m}{\sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q f_0}\right)^2}} \qquad [1.4]$$

NOTE: *The relative displacement is multiplied by ω_0^2 in order to obtain a homogenous parameter compatible with an acceleration (as with the shock response spectra; see Volume 2). The quantity $\omega_0^2 \, z_m$ is actually a relative acceleration ($\ddot{z}_m$) only when $\Omega = \omega_0$ (in sinusoidal mode) or more generally an absolute acceleration ($\ddot{y}_m$) when damping is zero.*

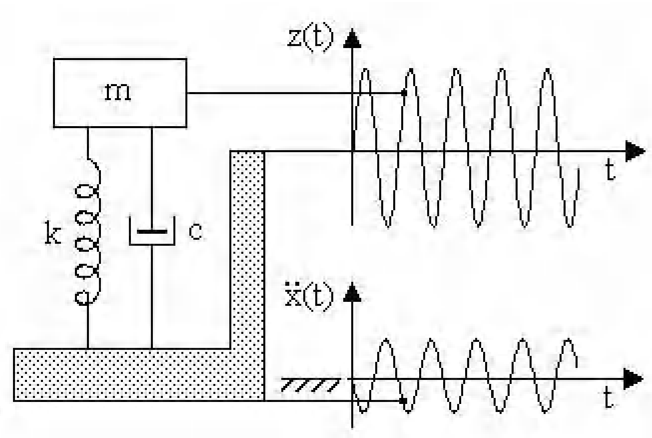


Figure 1.2. *Relative displacement response of a single degree-of-freedom linear system*

The spectrum corresponding to the largest negative values may also be considered. The positive and negative spectra are symmetrical. The positive spectrum has a maximum when the denominator $D = \left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f}{f_0}\right)^2$ has a minimum, i.e. for:

$$\frac{dD}{df_0} = 2\left[1 - \left(\frac{f}{f_0}\right)^2\right]\left[-2\,\frac{f}{f_0}\left(-\frac{f}{f_0^2}\right)\right] + \frac{1}{Q^2}\,2\,\frac{f}{f_0}\left(-\frac{f}{f_0^2}\right) = 0$$

yielding

$$f_0 = f\sqrt{\frac{2\,Q^2}{2\,Q^2 - 1}} = \frac{f}{\sqrt{1 - 2\,\xi^2}} \qquad [1.5]$$

and

$$R \equiv \omega_0^2 \ z_m = \frac{\ddot{x}_m}{\sqrt{\left[1-\left(1-\frac{1}{2\,Q^2}\right)\right]^2 + \frac{1}{Q^2}\left(1-\frac{1}{2\,Q^2}\right)}}$$

$$R \equiv \omega_0^2 \ z_m = \frac{\ddot{x}_m}{\sqrt{\left(1-1+\frac{1}{2\,Q^2}\right)^2 + \frac{1}{Q^2} - \frac{1}{4\,Q^4}}}$$

$$R \equiv \omega_0^2 \ z_m = \frac{\ddot{x}_m}{\sqrt{\frac{1}{4\,Q^4} + \frac{1}{Q^2} - \frac{1}{4\,Q^4}}}$$

$$R = \frac{\ddot{x}_m}{\frac{1}{Q}\sqrt{1-\frac{1}{4\,Q^2}}}$$

$$R = \frac{Q\ \ddot{x}_m}{\sqrt{1-\frac{1}{4\,Q^2}}} = \frac{\ddot{x}_m}{2\,\xi\,\sqrt{1-\xi^2}} \qquad [1.6]$$

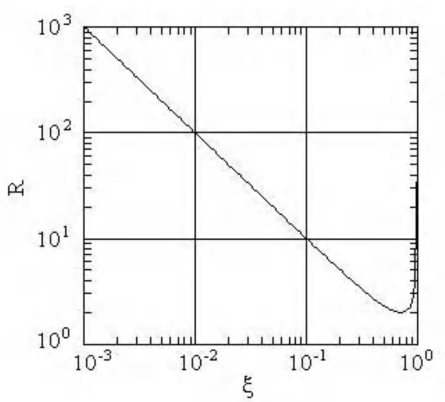


Figure 1.3. *Maximum of reduced ERS ($\ddot{x}_m$ = 1) versus damping ratio ξ*

Q	50	20	10	5	2	1
$\dfrac{Q}{\sqrt{1-1/4Q^2}}$	50.00275	20.00628	10.012525	5.02519	2.06559	1.1547

Table 1.1. *Reduced ERS for values of Q factor*

At a first approximation, it is completely reasonable to assume that $R \approx Q\,\ddot{x}_m$. When f_0 tends towards zero, R tends towards zero. When f_0 tends towards infinity, R tends towards $\ddot{x}_m$.

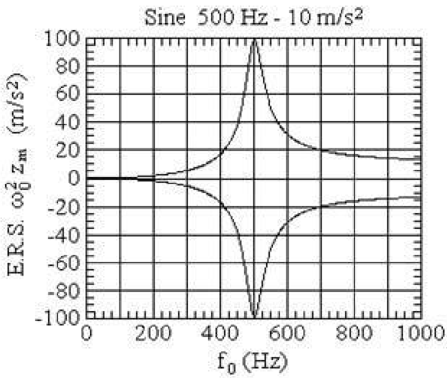


Figure 1.4. *Example of ERS of a sinusoid*

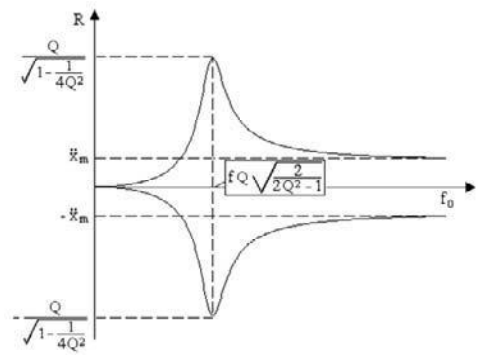


Figure 1.5. *ERS peak co-ordinates of a sinusoid*

1.2.2.2. *Reduced spectrum*

It is possible to trace this spectrum with reduced coordinates by considering the variations in the ratio $\dfrac{\omega_0^2\, z_m}{\ddot{x}_m}$ as a function of the dimensionless parameter $\dfrac{f_0}{f}$.

NOTE: *The reduced transfer function of a single degree-of-freedom linear system is defined by*

$$H\left(\frac{f}{f_0}\right)=\frac{1}{\sqrt{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{f}{f_0}\right)^2}}$$

where $H(\ )$ *is plotted as a function of the ratio* f/f_0, *whereas the extreme response spectrum shows the variations in the same expression as a function of* f_0/f.

1.2.3. *Case of a periodic signal*

If the stress can be represented by the sum of several sinusoids

$$\ddot{x}(t)=\sum_i \ddot{x}_{m_i}\ \sin\left(\Omega_i\ t+\varphi_i\right) \qquad [1.7]$$

the response of a linear system to only one degree-of-freedom is equal to

$$z(t)=-\sum_i \frac{\ddot{x}_{m_i}\ \sin\left(\Omega_i\ t+\varphi_i\right)}{\omega_0^2\ \sqrt{\left[1-\left(\frac{f_i}{f_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{f_i}{f_0}\right)^2}} \qquad [1.8]$$

and the extreme response spectrum is given by

$$R\equiv\omega_0^2\ z_m=\pm\sup\sum_i \frac{\ddot{x}_{m_i}\ \sin\left(\Omega_i\ t+\varphi_i\right)}{\omega_0^2\ \sqrt{\left[1-\left(\frac{f_i}{f_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{f_i}{f_0}\right)^2}} \qquad [1.9]$$

$$R\le\pm\sum_i \frac{\ddot{x}_{m_i}}{\omega_0^2\ \sqrt{\left[1-\left(\frac{f_i}{f_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{f_i}{f_0}\right)^2}} \qquad [1.10]$$

1.2.4. *General case*

The cases of an excitation defined by an acceleration, a velocity and a displacement can be brought together in the following general expression:

$$\omega_0^2 \, z_m \frac{E_m \, \Omega^\alpha}{\sqrt{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2+\left(\frac{f}{Q \, f_0}\right)^2}} \tag{1.11}$$

where

$$E_m = \begin{cases} \ddot{x}_m \\ v_m \\ x_m \end{cases} \qquad \alpha = \begin{cases} 0 & \text{for } \ddot{x}_m \\ 1 & \text{for } v_m \\ 2 & \text{for } x_m \end{cases}$$

If the excitation is a force,

$$R \equiv \omega_0^2 \, z_m = \frac{F_m}{m \sqrt{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2+\left(\frac{f}{Q \, f_0}\right)^2}} \tag{1.12}$$

1.3. Extreme response spectrum of a swept sine vibration

1.3.1. *Sinusoid of constant amplitude throughout the sweeping process*

1.3.1.1. *General case*

The extreme response spectrum is the curve giving the highest value (or lowest value) for the response $\omega_0^2 \, u(t)$ of a single degree-of-freedom linear system (f_0, Q) when f_0 varies. The upper value and lower value curves being symmetrical for a sine wave excitation, only one of them need be traced.

Given a sine wave excitation whose frequency is swept according to an arbitrary law, we will assume that the sweep rate is sufficiently slow that the response reaches a value very close to the steady state response. If the amplitude of the sinusoid remains constant and equal to ℓ_m during sweeping, the response of the system is, in the swept frequency interval f_1, f_2, equal to $Q\ \ell_m$ [GER 61] [SCH 81].

For frequencies f_0 located outside the swept range, at its maximum the response is equal to (Volume 1, [8.29]):

$$u_m = \frac{\ell_m}{\sqrt{\left[1-\left(\frac{f_1}{f_0}\right)^2\right]^2 + \frac{f_1^2}{Q^2\ f_0^2}}} \qquad [1.13]$$

for $f_0 \le f_1$ and to (Volume 1, [8.30]):

$$u_m = \frac{\ell_m}{\sqrt{\left[1-\left(\frac{f_2}{f_0}\right)^2\right]^2 + \frac{f_2^2}{Q^2\ f_0^2}}} \qquad [1.14]$$

for $f_0 \ge f_2$. These values are obtained for an extremely slow sweep.

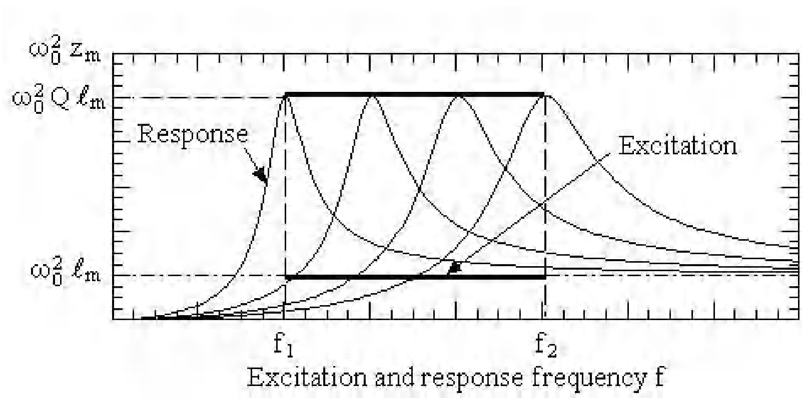


Figure 1.6. *Construction of the ERS for a swept sine vibration*

The value of u_m calculated in this way for $f_0 = f_1$ is the largest of all those which may be calculated for $f_0 \le f_1$ ranging between 0 and f_1. In the same way, for $f_0 \ge f_2$, it is the limit f_2 which gives the greatest value of u_m (Figure 1.6).

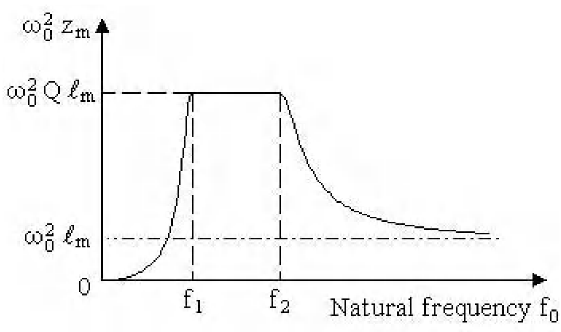


Figure 1.7. *ERS of swept sine vibration of constant amplitude between two frequencies*

The type of spectrum obtained in this way is shown in Figure 1.7 for a sweep between f_1 and f_2 and a sinusoid of amplitude ℓ_m. The spectrum increases from 0 to $Q\,\omega_0^2\,\ell_m$ at f_1 (if the sweep is sufficiently slow), remains at this value between f_1 and f_2, then decreases and tends towards the value $\omega_0^2\,\ell_m$ [CRO 68] [STU 67].

1.3.1.2. *Sweep with constant acceleration*

In this case, the generalized co-ordinates are equal to $\left|\ell_m\right| = \dfrac{\ddot{x}_m}{\omega_0^2}$ and $u_m = z_m$.

Between f_1 and f_2, the spectrum has as an ordinate

$$\omega_0^2\, z_m = Q\, \ddot{x}_m \qquad [1.15]$$

For $f_0 \le f_1$

$$\omega_0^2\, z_m = \frac{\ddot{x}_m}{\sqrt{\left[1 - \dfrac{f_1^2}{f_0^2}\right]^2 + \dfrac{f_1^2}{f_0^2\, Q^2}}} \qquad [1.16]$$

For $f_0 \geq f_2$

$$\omega_0^2\, z_m = \frac{\ddot{x}_m}{\sqrt{\left[1 - \dfrac{f_2^2}{f_0^2}\right]^2 + \dfrac{f_2^2}{f_0^2\, Q^2}}} \qquad [1.17]$$

1.3.1.3. *Sweep with constant displacement*

Here, $\left|\ell_m\right| = \dfrac{\Omega^2}{\omega_0^2}\, x_m$ yielding

for $f_1 \leq f_0 \leq f_2$

$$\omega_0^2\, z_m = Q\, \omega_0^2\, x_m \qquad [1.18]$$

for $f_0 \leq f_1$

$$\omega_0^2\, z_m = \frac{\Omega_1^2\, x_m}{\sqrt{\left[1 - f_1^2 / f_0^2\right]^2 + f_1^2 / f_0^2\, Q^2}} \qquad [1.19]$$

$(\Omega_1 = 2\,\pi\, f_1)$ and for $f_0 \geq f_2$

$$\omega_0^2\, z_m = \frac{\Omega_2^2\, x_m}{\sqrt{\left[1 - f_2^2 / f_0^2\right]^2 + f_2^2 / f_0^2\, Q^2}} \qquad [1.20]$$

$(\Omega_2 = 2\pi f_2)$.

1.3.1.4. *General expression for extreme response*

All these relationships may be represented by the following expressions:

if $f_1 \leq f_0 \leq f_2$

$$R = \omega_0^a\, E_m\, Q \qquad [1.21]$$

if $f_0 < f_1$

$$R = \frac{\Omega_1^a \; E_m}{\sqrt{\left(1 - h_1^2\right)^2 + h_1^2 / Q^2}} \qquad [1.22]$$

and if $f_0 > f_2$

$$R = \frac{\Omega_2^a \; E_m}{\sqrt{\left(1 - h_2^2\right)^2 + h_2^2 / Q^2}} \qquad [1.23]$$

where E_m and a characterize the vibration as indicated in the following table:

	$\mathbf{E_m}$	**a**
Acceleration	$\ddot{x}_m$	0
Velocity	v_m	1
Displacement	x_m	2

Table 1.2. *Parameters E_M and a according to the nature of the excitation*

1.3.2. *Swept sine composed of several constant levels*

In the case of a swept sine composed of several constant levels of amplitude ℓ_{m_j}, the extreme response spectrum is defined as the envelope of the separately plotted spectra of each sweep corresponding to a single level.

ℓ_{m_j} represents accelerations, velocities, displacements or a combination of the three.

It should be noted that, in the range (f_a, f_2) in Figure 1.8, the largest response occurs in the band (f_2, f_3) and not in (f_1, f_2), a range in which the resonators are excited at resonance ($u_{m1} = Q\,\ell_{m1}$).

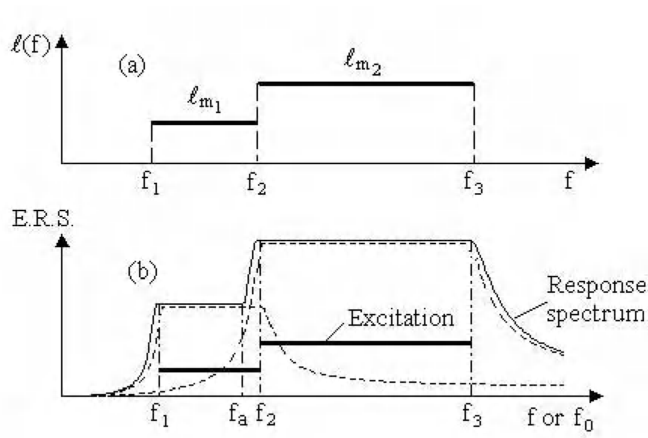


Figure 1.8. *ERS of a swept sine comprising several levels*

Example 1.1.

1. Figure 1.9 shows the extreme response spectrum of a swept sine vibration defined as follow:

Constant acceleration

20 to 100 Hz: ± 5 m s^{-2}

100 to 500 Hz: ± 10 m s^{-2}

500 to 1000 Hz: ± 20 m s^{-2}

t_b = 1200 s (from 20 to 1000 Hz)

Q = 10

The spectrum is plotted from 1 Hz to 2,000 Hz in steps of 5 Hz.

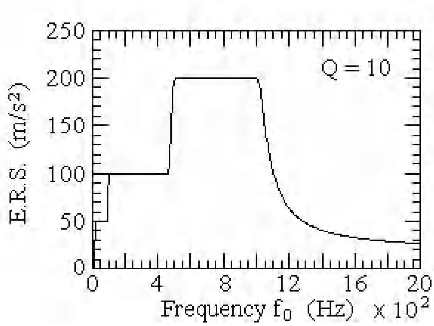


Figure 1.9. *Example of ERS of a swept sine vibration*

2. Let us consider a swept sine vibration with constant displacement:

   5 to 10 Hz: ± 0.050 m

   10 to 50 Hz: ± 0.001 m

   t_b = 1200 s

The ERS of this vibration, calculated for Q = 10 between 1 Hz and 200 Hz with 200 points (logarithmic step), is plotted in Figure 1.10.

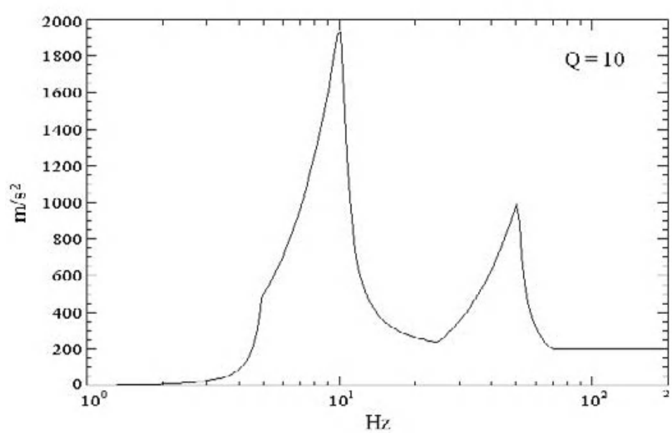


Figure 1.10. *ERS of a swept sine vibration at constant displacement*

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 2

Extreme Response Spectrum of a Random Vibration

The *extreme response spectrum* (ERS) (or *maximum response spectrum,* MRS) was defined as a curve that gives the value of the largest peak z_{sup} of the response of a single-degree-of-freedom linear system to any given vibration (a random acceleration $\ddot{x}(t)$ in this chapter), according to its natural frequency f_0, for a given ξ damping ratio. The response is described here in terms of the relative displacement $z(t)$ of the mass with respect to its support. By analogy with the shock response spectrum, the *y*-axis refers to the quantity $(2\ \pi\ f_0)^2\ z_{sup}$ [BON 77] [LAL 84]. The negative spectrum consisting of the smallest negative peak $(2\ \pi\ f_0)^2\ z_{inf}$ is also often plotted.

Due to the random nature of the signal, the choice of parameter used to characterize the largest response is not as simple as it is for a sinusoidal vibration. The definitions most commonly used are (for any given f_0):

– the largest value of the response on average over a given time T;

– the amplitude of the response equal to k times its rms value;

– the peak having a given probability of not being exceeded;

– the amplitude equal to k times the rms value of the response.

In the following sections we shall calculate the ERS in the above four cases. These four cases will be supplemented by some other relations from the statistical study of extreme values (Volume 3), as they can be useful for the sizing of the structures.

The cases should initially be distinguished as cases in which the vibration is characterized either by a time history signal or by a power spectral density.

2.1. Unspecified vibratory signal

When the signal is unspecified, and in particular when it is not stationary or Gaussian, it is not possible to determine a power spectral density. In such cases, each point of the ERS can only be obtained by direct numerical calculation of the response displacement $z(t)$ of a single-degree-of-freedom linear system to the excitation $\ddot{x}(t)$, and by noting the largest peak response observed (positive z_{sup} and/or negative z_{inf} or the greater of the two in absolute value) over the considered duration T (Figure 2.1).

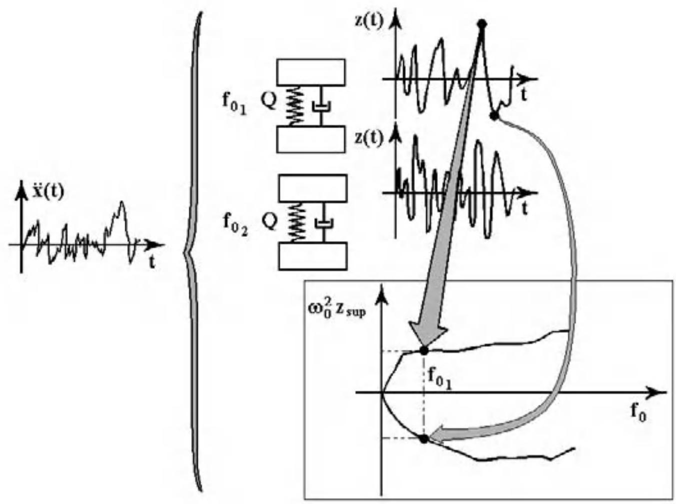


Figure 2.1. *Principle of ERS calculation*

This method is similar to that used to obtain a primary shock response spectrum, since the system's residual response at the end of the vibration is not considered.

If the duration is lengthy, the calculation is limited to a sample that is considered representative and of a reasonable duration for such a purpose. However, there is always a risk of a significant error being made, as the probability of finding the largest peak in another sample is not negligible (a risk related to the duration of the selected sample).

As far as possible, it is preferable to use the procedures set out in the following sections to limit the costs, the computing time being much longer when the duration T is greater.

Example 2.1.

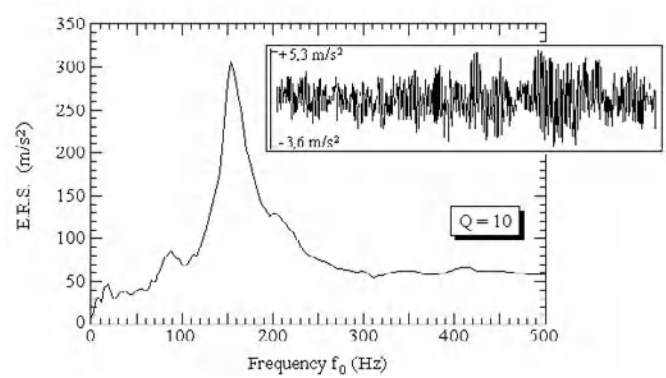


Figure 2.2. *ERS of a random vibration measured on a truck calculated from the time history signal*

NOTE: *Shock response spectra (Volume 2), extreme or with up-crossing risk response spectra and fatigue damage spectra defined in the following chapters must be calculated from a signal with a much greater sampling frequency than the one supported by Shannon theorem, approximately 10 times the maximum frequency of the desired spectrum. However, a sampled signal based on this theorem can be reconstructed to respect this requirement (Volume 1).*

2.2. Gaussian stationary random signal

2.2.1. *Calculation from peak distribution*

2.2.1.1. *General case*

When the distribution of instantaneous values of the stationary signal follows a Gaussian law, the response instantaneous values distribution is Gaussian itself. We can then calculate directly from its power spectral density (PSD) the probability density of the response *maxima* of each single-degree-of-freedom linear system, as well as the corresponding distribution functions of peaks ([6.64], Volume 3):

$$Q(u_0) = 1 - P(u_0) = \frac{1}{2}\left\{1 - \operatorname{erf}\left(\frac{u_0}{\sqrt{2\left(1-r^2\right)}}\right) + r\, e^{-\frac{u_0^2}{2}}\left[1 + \operatorname{erf}\left(\frac{r\, u_0}{\sqrt{2\left(1-r^2\right)}}\right)\right]\right\}$$

where $Q(u_0)$ is the probability that $u > u_0$ where $u = \dfrac{z_0}{z_{rms}}$, z_0 is a peak amplitude of a relative movement of response to the single-degree-of-freedom system considered and z_{rms} is the rms value of this relative movement.

In order to determine the ERS from this expression, we must calculate:

– the irregularity factor $r = \dfrac{n_0^+}{n_p^+}$ (Volume 3 [6.45]);

– the mean number of *positive maxima* per second n_p^+ of the relative displacement:

$$n_p^+ = \frac{1}{2\pi}\frac{\ddot{z}_{r.m.s.}}{\dot{z}_{r.m.s.}} = \left[\frac{\int_0^{+\infty} f^4\ G_z(f)\ df}{\int_0^{+\infty} f^2\ G_z(f)\ df}\right]^{\frac{1}{2}} \quad \text{(volume 3 [6.31]);}$$

– the mean frequency of the response:

$$n_0^+ = \frac{1}{2\,\pi}\frac{\dot{z}_{r.m.s.}}{z_{r.m.s.}} = \left[\frac{\int_0^{\infty} f^2\ G(f)\ df}{\int_0^{\infty} G(f)\ df}\right]^{\frac{1}{2}} \approx f_0 \quad \text{(Volume 3 [5.43] [5.50]);}$$

– the mean total number of response peaks higher than a threshold u_0 over chosen T duration:

$$N = n_p^+\ T\ Q_p\left(u_0\right); \qquad [2.1]$$

– the probability of the largest peak on average during T (or 1 / N);

– the amplitude z_0 of this peak, by consecutive iterations.

The method consists of setting a value of $Q(u_0)$ and determining the corresponding value of u_0. The largest peak during T (on average) corresponds roughly to the level u_0 which is only exceeded once (N = 1), yielding

$$Q(u_0) = \frac{1}{n_p^+ \, T} \qquad [2.2]$$

The level u_0 is determined by successive iterations. The distribution function $Q(u)$ being a decreasing function of u, two values for *u* are given such that:

$$Q(u_1) < Q(u_0) < Q(u_2) \qquad [2.3]$$

and, for each iteration, the interval (u_1, u_2) is reduced until, for example,

$$\frac{Q(u_1) - Q(u_2)}{Q(u_0)} < 10^{-2}$$

yielding, by interpolation,

$$z_s \approx z_0 = z_{rms} \left[(u_2 - u_1) \frac{Q(u_1) - Q(u_0)}{Q(u_1) - Q(u_2)} + u_1 \right] \qquad [2.4]$$

and

$$R = (2 \pi f_0)^2 \, z_s \approx (2 \pi f_0)^2 \, z_{rms} \left[(u_2 - u_1) \frac{Q(u_1) - Q(u_0)}{Q(u_1) - Q(u_2)} + u_1 \right] \qquad [2.5]$$

The peak obtained here is the largest peak, on average. It is therefore the average of results that would be obtained from the numeric calculation of the system's response and the histogram of peaks considering several signal samples.

Example 2.2.

Random vibration defined by:

100–300 Hz.......... 5 $(m\ s^{-2})^2/Hz$

300–600 Hz......... 10 $(m\ s^{-2})^2/Hz$

600–1 000 Hz........ 2 $(m\ s^{-2})^2/Hz$

Duration: one hour

The extreme response spectrum is plotted on Figure 2.3 for $5 \le f_0 \le 1500$ Hz and $Q = 10$.

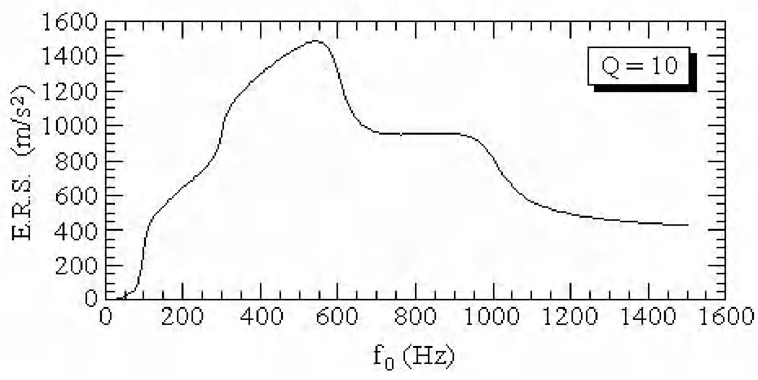


Figure 2.3. *ERS of a random vibration defined by its PSD*

NOTE 1: *Difference between extreme response spectrum and shock response spectrum*

Extreme response and shock response spectra both offer the greatest response of a single degree-of-freedom linear system according to its natural frequency, for a given Q factor, when submitted to vibration or the shock studied (we do not use the definition of ERSs from three times the rms value of the response). The calculation algorithm (Volume 2, Chapter 2) is the same.

In the case of long duration vibrations, this response occurs during the vibration: we only focus here on the primary spectrum.

In the case of shocks, on the other hand, the largest response peak can occur during or after the shock. We generally use the envelope spectrum of primary and residual spectra. It is the only difference between ERSs and SRSs.

NOTE 2: *Difference between extreme response spectra calculated from a signal and its PSD.*

When the random vibration is Gaussian, the ERS can be calculated from a signal of acceleration or from its PSD. With random vibration, the ERS has a statistic character: calculated from a PSD, it gives the largest peak on average over T duration. When it is directly obtained from a signal based on time, it represents the largest peak for this signal sample and duration.

In addition, the ERS is obtained from the peak probability density of response, when the PSD is used, whereas we establish a range histogram (from which we could deduct a peak histogram) with a signal according to time. In order to standardize the methods, the distribution law of Dirlik's ranges (Volume 4, Chapter 4) could be used.

Examples 2.3.

1– *The theoretical PSD in Figure 2.4 is a case in which there is significant difference between Rice's density probability of peaks and Dirlik's probability density of half-ranges; see Example 4.3, Volume 4). And yet, ERSs calculated from Rice and Dirlik hypotheses are similar (Figure 2.5).*

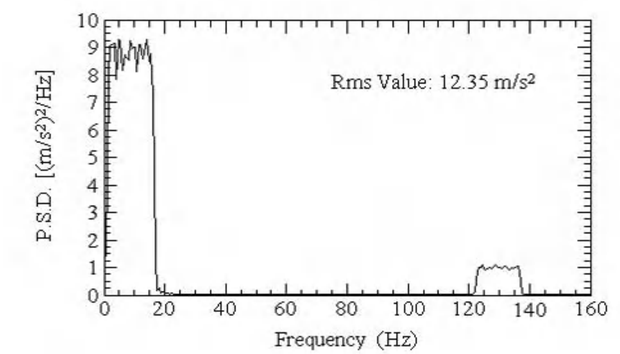


Figure 2.4. *Power spectral density of Example 4.3, Volume 4*

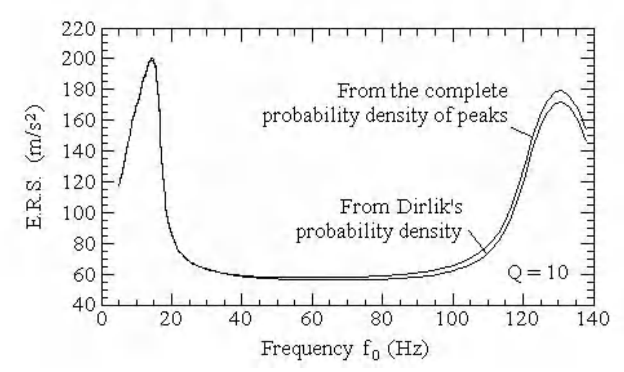


Figure 2.5. *Extreme response spectra calculated from the PSD of Figure 2.4*

2– Vibration measured in a plane

Consider a vibration measured in a plane, where the PSD is given in Figure 2.6. The comparison of ERSs calculated from Rice's peak probability density and the density of Dirlik ranges shows that these spectra are very close (Figure 2.7).

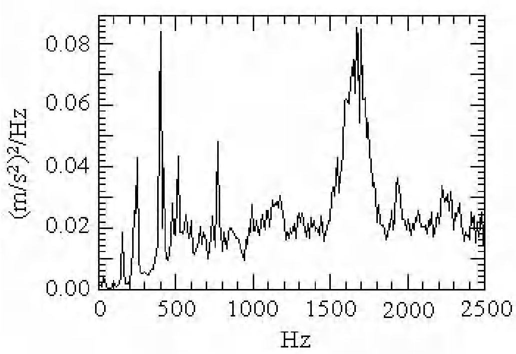


Figure 2.6. *PSD of a vibration measured in a plane*

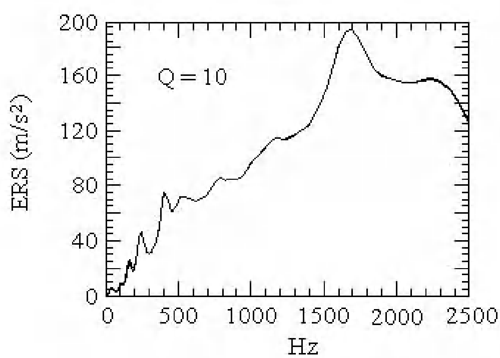


Figure 2.7. *ERS of the plane vibration calculated from Rice and Dirlik probability densities*

2.2.1.2. *Case of a narrow band response*

We consider the assumption where the vibration $\ddot{x}(t)$ is Gaussian and of zero mean. If the relative displacement response $z(t)$ and its derivative $\dot{z}(t)$ are independent functions, the mean number per second of crossing a given level a with positive slope can be written (Volume 3)

$$n_a^+ = n_0^+ \; e^{-\frac{a^2}{2\, z_{rms}^2}}$$

or, over duration T:

$$N_a^+ = n_0^+ \; T \; e^{-\frac{a^2}{2\, z_{rms}^2}}$$

The largest level during this length of time T is that which is only exceeded once:

$$N_a^+ = 1 = n_0^+ \; T \; e^{-\frac{a^2}{2\, z_{rms}^2}}$$

yielding level a:

$$a = z_{rms} \sqrt{2 \, \ln n_0^+ \, T} \qquad [2.6]$$

At the frequency f_0 (and for the selected value of Q), the extreme response spectrum has as amplitude:

$$R = \left(2 \, \pi \, f_0\right)^2 \; a$$

$$R = \left(2 \, \pi \, f_0\right)^2 \; z_{rms} \sqrt{2 \, \ln n_0^+ \, T} \qquad [2.7]$$

This result can also be obtained from the distribution function [6.64] in Volume 3.

The approximation is acceptable when the irregularity factor r is higher than 0.6. We can then consider that the distribution of *maxima* follows Rayleigh's law approximately.

NOTE: *The extreme response spectrum can be plotted separately for a positive and a negative, as a function of* f_0 *(as the shock response spectrum).*

One curve could also be plotted by considering up-crossings of threshold $|a|$. *In this case,* n_a^+ *must be replaced by* $n_a = 2 \, n_a^+$ *in the above relations (and* n_0^+ *by* $n_0 = 2 \, n_0^+$*). Having selected level a, uncertainty related to the random nature of the* *phenomenon does not relate to the amplitude a which can be exceeded over duration T, but to the possibility of obtaining level a in a time shorter than T. The duration T being given, level a is that which is observed on average over this duration.*

2.2.2. *Use of the largest peak distribution law*

The relations described in Chapter 7, Volume 3 could be used. Although more complex, they lead to results close to the preceding ones for the higher levels, in general.

The probability density of the largest peak can thus be considered for one length of time T, which has as its expression ([7.25], Volume 3):

$$p_0(u)\, du = -d\left[\exp\left(-n_0^+\, T\, e^{-\frac{u^2}{2}}\right)\right]$$

for mode ([7.45], Volume 3)]:

$$m = \sqrt{2\, \ln n_0^+\, T} = \frac{a}{z_{rms}}$$

(this is equation [2.6]) and for mean:

$$a = z_{rms}\left[\sqrt{2\ \ln\ n_0^+\ T} + \frac{\varepsilon}{\sqrt{2\ \ln\ n_0^+\ T}}\right] \qquad [2.8]$$

where ε is Euler's constant, equal to 0.577 215 665...

NOTE:

1. Expression [2.8] is an asymptotic limit of the relation ([7.28], Volume 3):

$$a = z_{r.m.s.}\sqrt{\frac{\pi}{2}}\left[\frac{N_p}{1!\sqrt{1}} - \frac{N_p\left(N_p - 1\right)}{2!\sqrt{2}} + \frac{N_p\left(N_p - 1\right)\left(N_p - 2\right)}{3!\sqrt{3}} - \cdots + \left(-1\right)^{N_p + 1}\frac{1}{\sqrt{N_p}}\right]$$

($N_p = n_0^+\, T$). But the error is acceptable: lower than 3% for $N_p > 2$ and lower than 1% for $N_p > 50$.

2. It was furthermore established ([7.39], Volume 3) that the standard deviation of this distribution, equal to

$$s = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ln n_0^+ T}},$$

is all the weaker when $n_0^+ T$ *is larger: a slight error is consequently made by taking the mean* $\overline{u_0}$ *as an estimate of the largest peak. Figure 2.8 shows that at a first approximation* $\overline{u_0}$ *can be replaced by m, the variation being equal to* $\dfrac{\varepsilon}{\sqrt{2 \ln n_0^+ T}}$

and the error e, given by

$$e = \frac{\dfrac{\varepsilon}{\sqrt{2 \ln n_0^+ T}}}{\sqrt{2 \ln n_0^+ T} + \dfrac{\varepsilon}{\sqrt{2 \ln n_0^+ T}}}$$

$$e = \frac{\varepsilon}{\varepsilon + 2 \ln n_0^+ T} \qquad [2.9]$$

This error decreases quickly when $n_0^+ T$ *increases (Figure 2.9).*

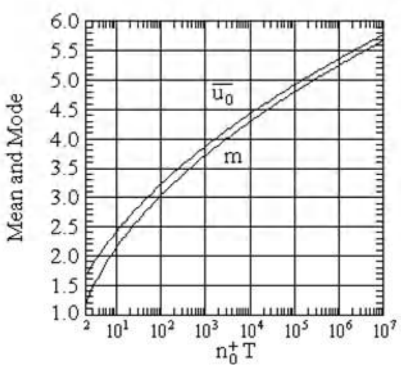


Figure 2.8. *Mean and mode of the largest peak over duration T*

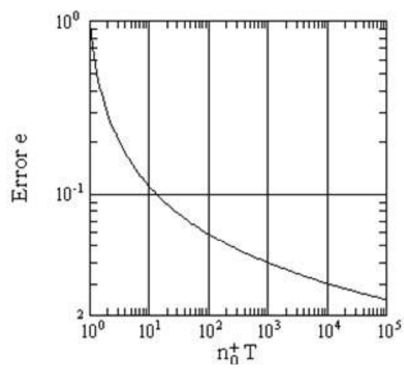


Figure 2.9. *Error made by considering the mode of the law of the largest peaks instead of the mean*

3. Relation [2.7] is an approximation of [2.8].

4. The mean and median of the largest peak distribution law are not identical, except at the limit for $n_0^+ T$ leaning toward infinity. The probability of finding a larger peak than the average of this law is higher than 40%.

2.2.3. *Response spectrum defined by k times the rms response*

2.2.3.1. *General expression*

The assumption is made that the distribution of instantaneous values of the response is Gaussian. Each point of the spectrum represents the response which has a constant fixed probability not to be exceeded.

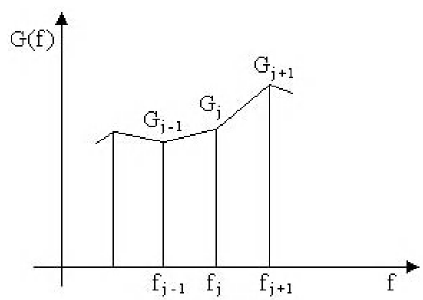


Figure 2.10. *Decomposition of the PSD in straight line segments for calculation of rms response displacement*

Given a PSD calculated from an acceleration $\ddot{x}(t)$ (Figure 2.10), it is possible to determine the rms value of the response displacement z_{rms} of a single-degree-of-freedom linear system, with a natural frequency f_0 and a given damping ratio ξ, from ([8.79], Volume 3),

$$z_{rms}^2 = \frac{\pi}{4\,\xi\,(2\pi)^4\,f_0^3}\sum_{j=1}^{n} a_j\,G_j$$

or, if the PSD is made up of horizontal straight line segments, by ([8.87], Volume 3):

$$z_{rms}^2 = \sum_{i=1}^{n}\frac{G_i}{4\,\xi\,\omega_0^3}\left[\frac{\xi}{\alpha}\ln\frac{h^2+\alpha\,h+1}{h^2-\alpha\,h+1}+\arctan\frac{2\,h+\alpha}{2\,\xi}+\arctan\frac{2\,h-\alpha}{2\,\xi}\right]_{h_i}^{h_{i+1}}$$

The response spectrum is obtained by plotting

$$R = k\left(2\pi f_0\right)^2 z_{rms} \qquad [2.10]$$

as a function of f_0, where ξ is given [BAN 78]. The constant k is chosen so as to be able to affirm, to a given probability P_0, that the maximum response is lower, for a frequency f_0, than the ordinate of the spectrum [BAD 70]. The probability P_0 is maintained constant whatever the value of f_0. Value 3 is often retained, making it possible to guarantee the probability that the response will be higher than that from ERS, which is equal to 0.135% if the distribution of instantaneous values is Gaussian.

Example 2.4.

White noise between 200 Hz and 1,000 Hz, of PSD equal to 1 $(ms^{-2})^2$/Hz

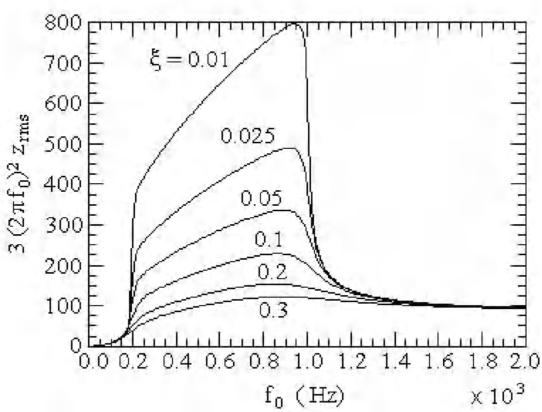


Figure 2.11. *ERS giving 3 times the rms value of the response, as a function of the damping ratio*

Figure 2.11 shows the ERS plotted for:

– f_0 variable between 1 Hz and 2,000 Hz;

– ξ respectively equal to 0.01, 0.05, 0.10, 0.20 and 0.30;

– k = 3.

2.2.3.2. *Approximate expressions*

The response relative displacement of a single-degree-of-freedom linear system (f_0, Q) submitted to white noise can be approximated by:

$$z_{rms} = \sqrt{\frac{Q\ G_{\ddot{x}0}}{4\ \omega_0^3}} \qquad [2.11]$$

$$z_{rms} = \left[\frac{G_{\ddot{x}}}{64\ \pi^3\ f_0^3\ \xi}\right]^{1/2} \qquad [2.12]$$

where $G_{\ddot{x}0}$ is the PSD value (of the signal of acceleration) at frequency f close to frequency f_0 and where $\omega_0 = 2\ \pi\ f_0$.

The rms absolute response acceleration can also be approximated by

$$\ddot{y}_{rms} = \sqrt{\frac{1+Q^2}{Q}\ \frac{\omega_0\ G_{\ddot{x}0}}{4}} = \sqrt{\frac{1+Q^2}{Q}\ \frac{\pi\ f_0\ G_{\ddot{x}0}}{2}} \qquad [2.13]$$

$$\ddot{y}_{rms} = \left[\frac{\pi\ f_0\ \left(1+4\ \xi^2\right)\ G_{\ddot{x}}}{4\ \xi}\right]^{1/2} \qquad [2.14]$$

NOTE: *If damping is small, the absolute response acceleration of the one-degree-of-freedom system is approximately equal to:*

$$\omega_0^2\ z_{rms} = \sqrt{\frac{Q\ \omega_0\ G_{\ddot{x}0}}{4}} = \sqrt{\frac{\pi\ Q\ f_0\ G_{\ddot{x}0}}{2}} \qquad [2.15]$$

the expression sometimes called the "Miles relation" [MIL 54].

When the PSD G(f) of the "input" acceleration is approximately constant around the frequency of resonance, between half-power points, the response rms value can be evaluated with the help of [BAN 78] [FOS 82] [SHO 68]:

$$z_{rms} = \frac{1}{(2\ \pi\ f_0)^2}\ \sqrt{\frac{\pi}{2}\ f_0\ Q\, G_{\ddot{x}_0}} \qquad [2.16]$$

If the PSD varies little around f_0, this relation gives an approximate value of z_{rms} acceptable even for a formed noise. The value $k = 3$ is often retained for the estimate of the extreme peaks; K. Foster chooses $k = 2.2$ for the study of fatigue failure [FOS 82]. The choice of a constant value for k is often criticized, for there is no reason to take a particular value whether 3, 4 or 5, as a large occasional peak is able to start a fissure which the smaller stresses will then make worse [BHA 58] [GUR 82] [LEE 82] [LUH 82]. The ERS $\omega_0^2 \, z_{rms}$ is also sometimes defined for $k = 1$.

The ERS is calculated:

– either in an exact way from the rms value of the PSD response determined from the transfer function [STA 76];

– or from the approximate relation [2.16] [SCH 81]. First of all, the Q factor is chosen according to acquired experience with the material in question (5 to 15 in general) or, more generally, we retain the conventional value Q = 10.

Each point of the PSD is used to evaluate $\omega_0^2 \, z_{rms}$ using

$$R_i = k \sqrt{\frac{\pi}{2} f_{0_i} \, Q \, G_i}$$

while proceeding as indicated in Figure 2.12 (calculation of R_i at each frequency to obtain the ERS from the PSD).

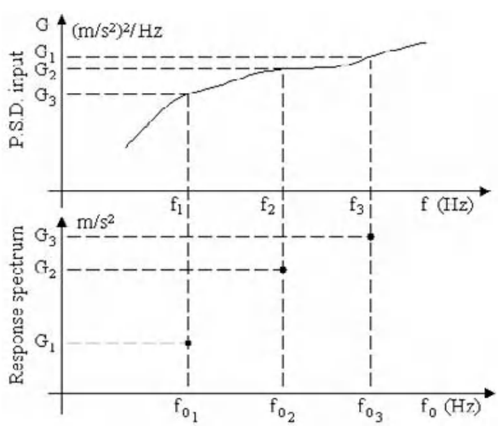


Figure 2.12. *Simplified calculation of the ERS from a PSD*

The extreme response spectra are often used to compare the relative severity of random vibrations, sinusoidal vibrations (swept or not swept) and of shocks, which is a difficult operation without this tool [BOI 61] [HAT 82]. We saw how to obtain them in the case of sinusoidal vibrations.

The analog of the ERS for shocks is the shock response spectrum. These spectra make it possible to plot $z_{rms}(f_0)$ very easily for use with other applications [FOS 82].

The ERS can be used, like the SRS (Volume 2), for the estimate of the response of systems with several degrees-of-freedom (complex structures), by carrying out calculations for each mode and by recombining the modes [SCH 81].

Validity of the approximation

These relations only involve the response of a single degree-of-freedom linear system.

In theory, the relation [2.11] is established for white noise. In practice, it remains usable as long as:

– the PSD experiences little variation between half-power points (interval Δf centered around the natural frequency and with width equal to f_0 / Q). The above relations can also give a sufficient approximation;

– the natural frequency f_0 is far enough from the PSD frequency boundaries. The response rms value can only be calculated if the natural frequency of the single degree-of-freedom system is located in the frequency domain where the PSD is defined.

The error is significant at PSD edges and in frequency ranges where it presents great variations.

The precision obtained from the expression $R = 3 \sqrt{\frac{\pi}{2} f_{0_i} Q G_i}$ is much better whenever the Q factor is larger. The precision is also a function of the position of f_{0_i} with respect to the boundaries f_1 and f_2 of the PSD. This remark can be illustrated using the following example.

Example 2.5.

Figures 2.13 and 2.14 show a PSD and its ERS plotted for $Q = 10$ under the following conditions:

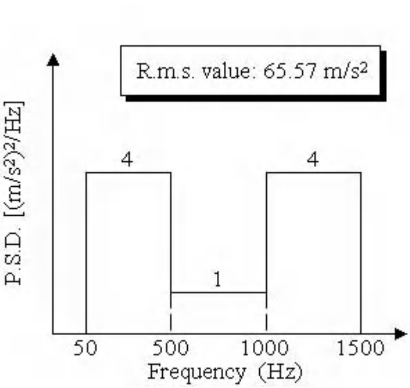


Figure 2.13. *Example of PSD*

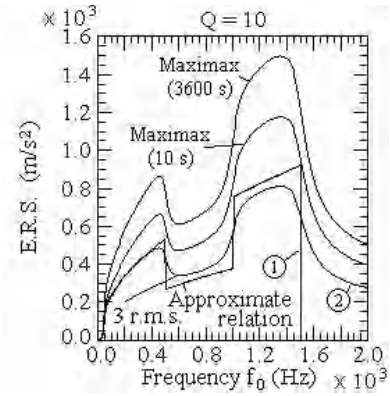


Figure 2.14. *Comparison between the ERS obtained from different assumptions*

1. With the approximate relation $R = 3\sqrt{\frac{\pi}{2} f_0\ Q\ G}$ where G is the value of the PSD at the frequency $f = f_0$.

2. From the exact rms value of the response $\omega_0^2\ z_{rms}$ of a single-degree-of-freedom system, multiplied by 3.

3. From the largest peak (on average) of the response of a single-degree-of-freedom system over a duration T equal to 10 s.

4. As in 3, but over duration T of 3600 s.

It is noted that:

– for this value of Q, the approximation is not excellent (curves 1 and 2) in the definition range of the PSD and it is poor on the right-hand side;

– the spectrum of the extreme values is definitely larger than three times the rms value, even for small values of T.

Example 2.6.

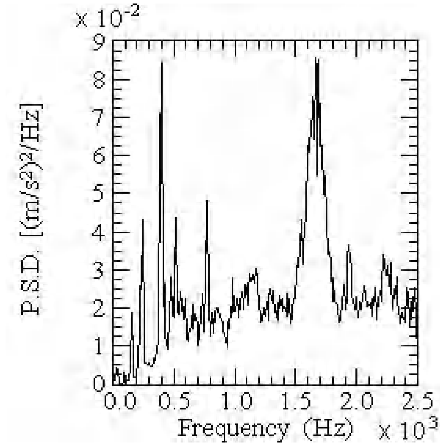


Figure 2.15. *PSD of a vibration measured on an aircraft*

Another example is that of a vibration measured on an aircraft. The extreme response spectra are calculated from this PSD (Figure 2.15) with:

1. the approximate relation $3\sqrt{\frac{\pi}{2} f_0 \, Q \, G}$;

2. $3\,\omega_0^2\, z_{rms}$ (z_{rms} being the exact rms value);

3. the largest peak for a duration $T = 1$ hour;

for $Q = 50$ (Figure 2.16) and for $Q = 5$ (Figure 2.17).

It is noted that for:

– $Q = 50$, the approximation is good, the spectra with $3\,\omega_0^2\, z_{rms}$ being in addition very much lower than the curve showing the largest peak;

– $Q = 5$, the three spectra are appreciably different.

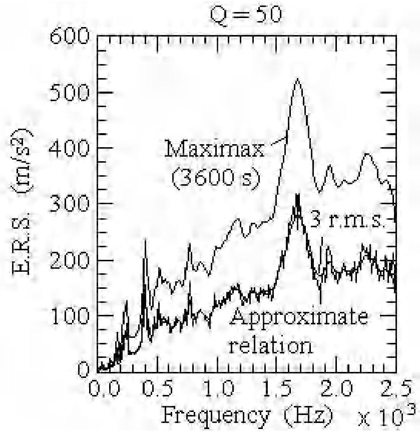


Figure 2.16. *ERS of vibration measured on an aircraft (Q = 50)*

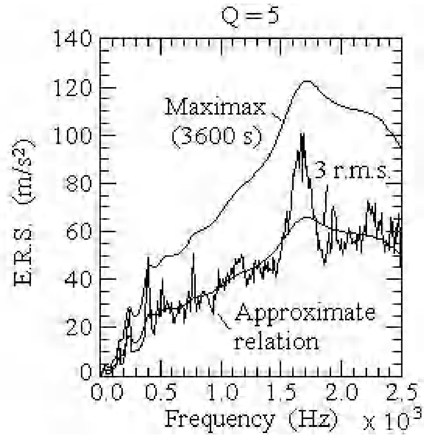


Figure 2.17. *ERS of vibration measured on an aircraft (Q = 5)*

NOTE: *S.P. Bhatia and J.H. Schmidt [BHA 58] propose a method for severity comparison of the vibrations and shocks based on curves similar to the ERS and using a fatigue criterion. The authors distinguish three zones on the S–N curve (Figure 2.18):*

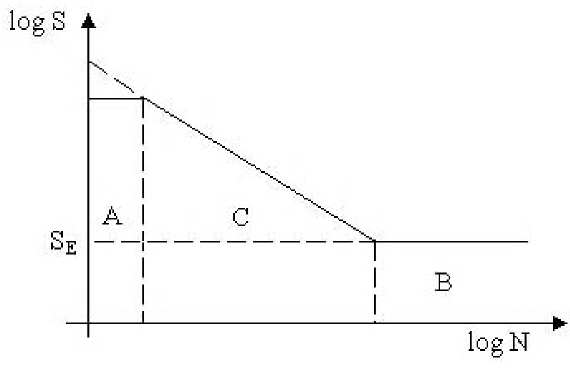


Figure 2.18. *S–N curve in three zones*

– area A where the vibratory environments are of short duration, so that a comparison of the stresses to the static properties is possible. The authors define the stresses as equal to 3 times the rms value and obtain a curve analogous to an extreme response spectrum;

– area B where the stresses are compared with the fatigue limit. Here, the authors multiply the sine and random stresses by a factor

$$K_1 = \frac{R_e}{\sigma_D} = \frac{admissible\ yield\ stress}{fatigue\ limit}; \qquad [2.17]$$

– area C. A sinusoidal stress σ_F equivalent to the idea of the damage (calculated with Miner's rule) is sought here. The sine environment is modified by multiplying it by

$$K_2 = \frac{R_e}{\sigma_F} \qquad [2.18]$$

and the random environment by

$$K_3 = \beta \frac{R_e}{\sigma_F} \qquad [2.19]$$

where β is a constant function of the parameter b chosen for the fatigue damage equivalence

$$\beta = \left[0.683 + 0.2712^{\eta\ b} + 0.04333^{\eta\ b} \right]^{1/\eta\ b} \qquad [2.20]$$

(η, equal to about 0.8 to 1.5 depending on each case, is a function of the non-linearities [LAM 80]). With this formulation, the multiplication by 3 of the rms value of the random vibration is not necessary.

Example 2.7.

Consider a noise defined as a constant PSD between 10 Hz and 1,000 Hz. The exact rms value of the single degree-of-freedom linear system response is drawn according to its natural frequency in Figure 2.19, as well as the approximate rms value of relation [2.11], for Q = 10.

We can observe that the approximate expression returns acceptable results, except at PSD edges. Figure 2.20 shows the relative error made in relation to the natural frequency.

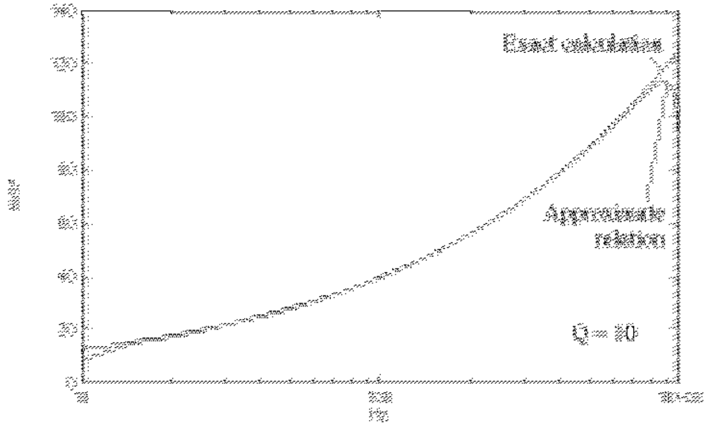


Figure 2.19. *Comparison of rms values $\omega_0^2 z_{sup}$ of the single-degree-of-freedom system response calculated exactly and from the approximate relation, based on its natural frequency, for Q = 10*

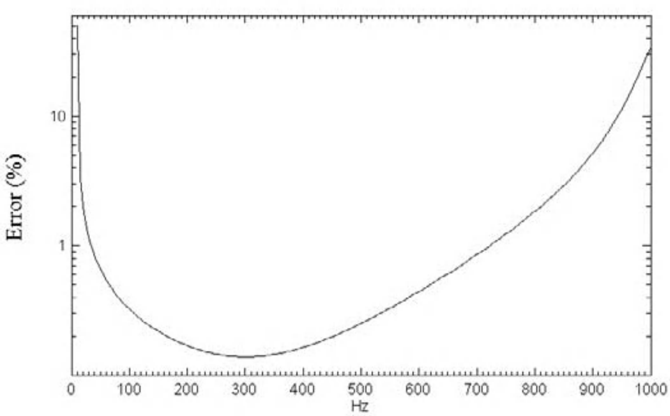


Figure 2.20. *Relative error made with the approximate relation*

Example 2.8.

In this example, the PSD (Figure 2.21) is made up of a wide band constant PSD noise with much larger amplitude between 200 Hz and 500 Hz.

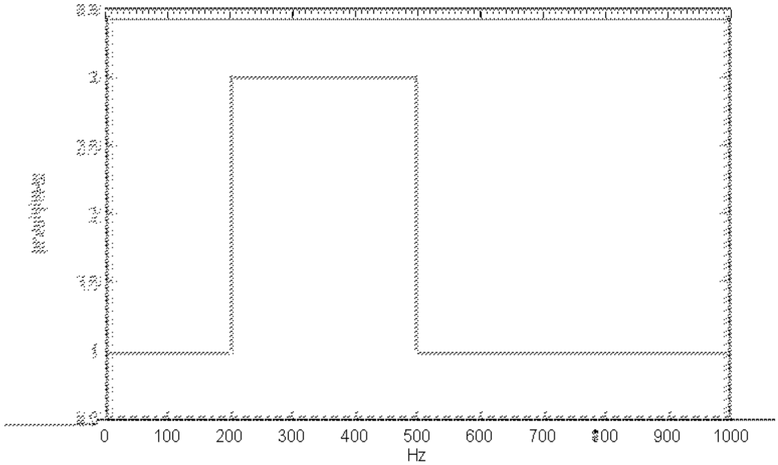


Figure 2.21. *Constant wide-band PSD with larger amplitude between 200 Hz and 500 Hz*

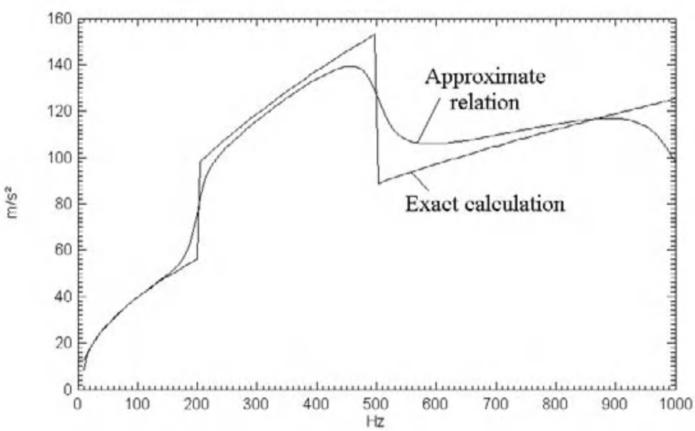


Figure 2.22. *Comparison of rms values $\omega_0^2 z_{rms}$ from the single degree-of-freedom system response calculated exactly and from the approximate relation, according to its natural frequency, for $Q = 10$*

The approximate relation returns its worst results at PSD edges and intermediate band (200 Hz and 500 Hz) edges (Figures 2.22 and 2.23).

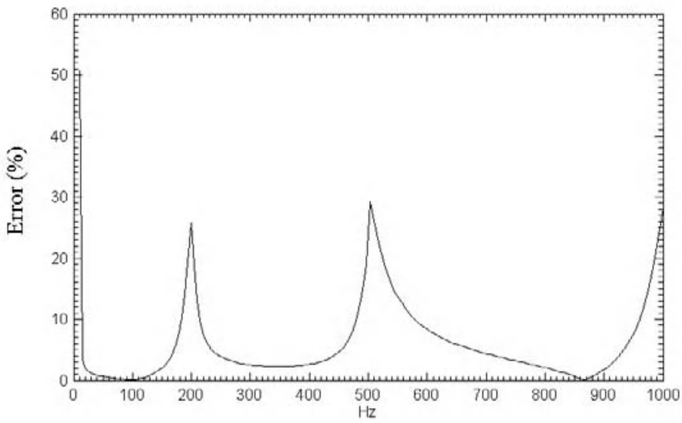


Figure 2.23. *Relative error made with the approximate relation (Q = 10)*

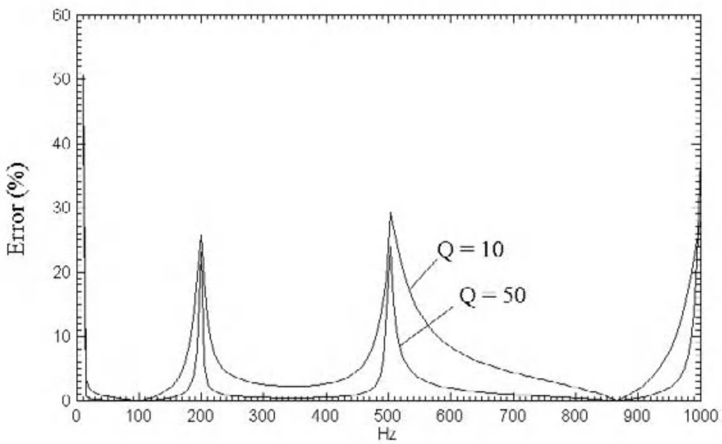


Figure 2.24. *Comparison of errors calculated for Q = 10 and Q = 50*

The error is lower when the Q factor is greater (Figure 2.24), since the interval between the half-power points is narrower.

Example 2.9.

The PSD is made up of a constant PSD noise between 0 and 1,000 Hz in which a very narrow line between 500 Hz and 510 Hz is superimposed (Figure 2.25).

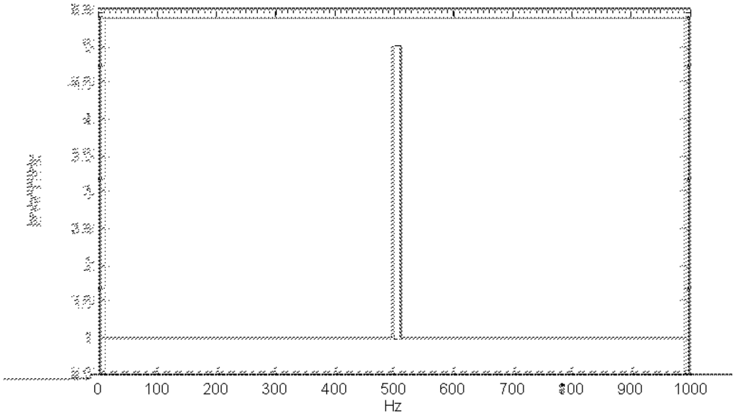


Figure 2.25. *PSD of a narrow band (centered in the 500 Hz frequency)on wide band noise*

We always find the error at the edges of the spectrum; it is also very significant around the peak (Figures 2.26 and 2.27).

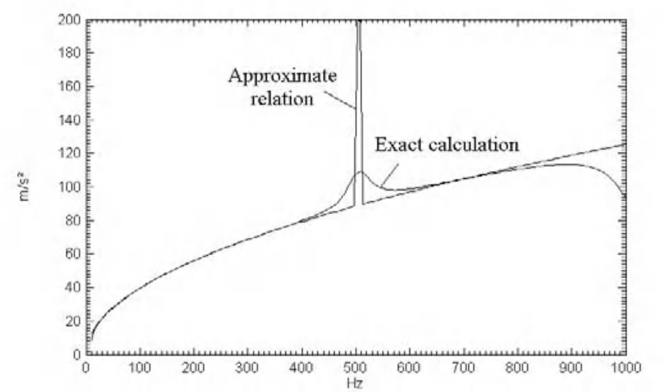


Figure 2.26. *Comparison of rms values $\omega_0^2 z_{rms}$ from the single degree-of-freedom system response calculated exactly and from the approximate relation, based on its natural frequency, for Q = 10*

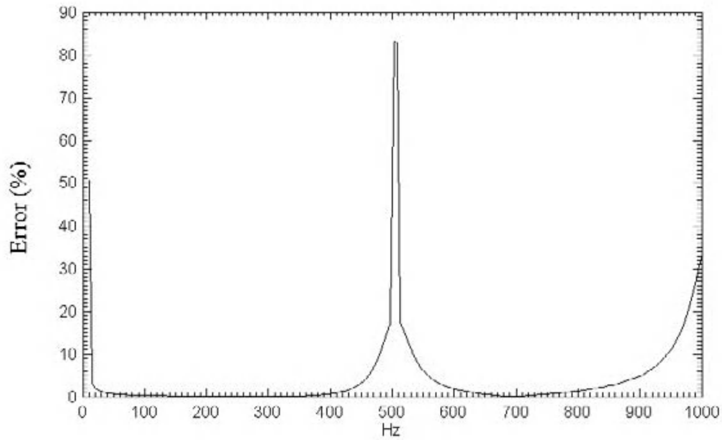


Figure 2.27. *Relative error made with the approximate relation*

Example 2.10.

In this last example, we have chosen the case of a real environment, a vibration measured in an aircraft, with a PSD that is represented in Figure 2.28.

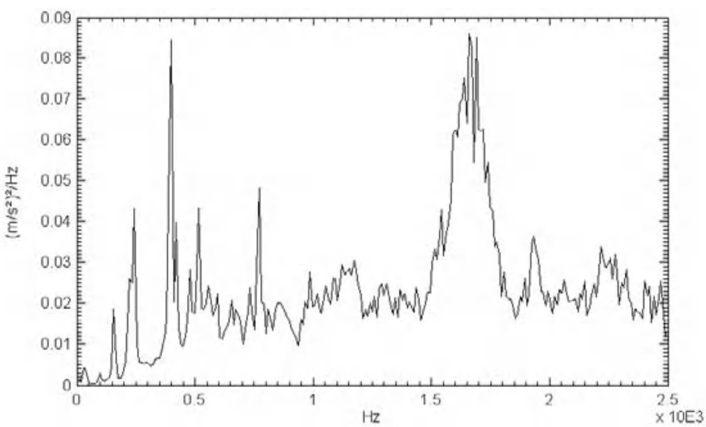


Figure 2.28. *Power spectral density of an aircraft vibration*

The approximate relation returns an rms value curve much more uneven than the exact relation. In fact, it follows the form of the PSD very closely, whereas the exact relation leads to smoothing (Figure 2.29).

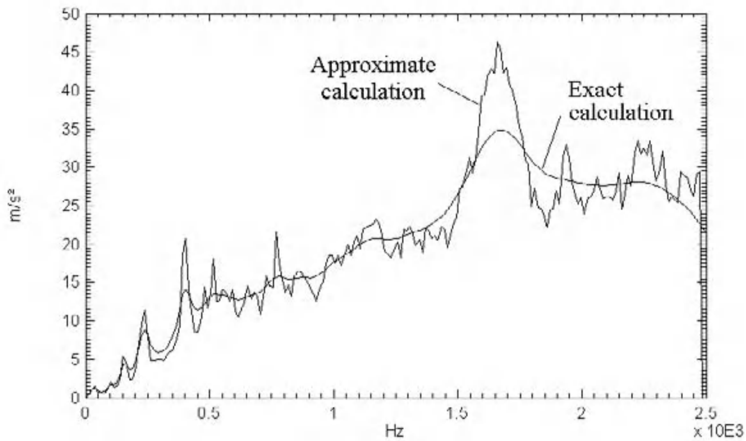


Figure 2.29. *Comparison of rms values $\omega_0^2 z_{rms}$ from the single degree-of-freedom system response calculated exactly and from the approximate relation, according to its natural frequency, for $Q = 10$*

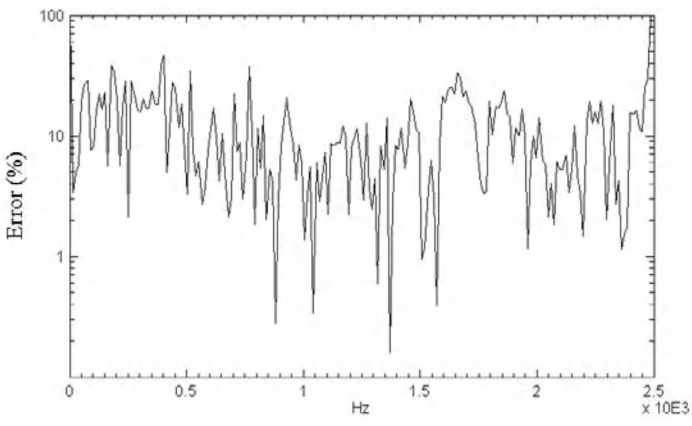


Figure 2.30. *Relative error made with the approximate relation*

The error is relatively significant (higher than 10%) over a large part of the interval of PSD definition (Figure 2.30).

2.2.4. *Other ERS calculation methods*

Relations providing the first up-crossing of the single degree-of-freedom system response experiencing random vibration established in the different hypotheses in Chapter 10 of Volume 3 could be used to calculate the ERS (see Table 2.2 of the current chapter).

2.3. Limit of the ERS at the high frequencies

The amplitude [2.7] of the ERS is a function of the rms value of the displacement response and its mean frequency. However

$$\omega_0^4 \; z_{rms}^2 = \omega_0^4 \int_{f_1}^{f_2} G_z(f)\, df = \int_{f_1}^{f_2} \frac{G_{\ddot{x}}(f)\, df}{\left(1 - \dfrac{f^2}{f_0^2}\right)^2 + 4\,\xi^2 \dfrac{f^2}{f_0^2}} \qquad [2.21]$$

At high frequencies of the ERS, when $f_0 >> f_2$,

$$\omega_0^4 \; z_{rms}^2 \approx \int_{f_1}^{f_2} G_{\ddot{x}}(f)\, df = \ddot{x}_{rms}^2 \qquad [2.22]$$

from which, by comparison with [2.21] and [2.22], the following can be deduced

$$G_z = \frac{G_{\ddot{x}}}{\omega_0^4} \qquad [2.23]$$

By definition, the mean frequency has as an expression

$$f_M^2 = \frac{\int_{f_1}^{f_2} f^2 \; G_z(f)\, df}{z_{rms}^2}$$

From which according to [2.22] and [2.23]:

$$f_M^2 \approx \frac{\int_{f_1}^{f_2} f^2 \; G_{\ddot{x}}(f)\, df}{\omega_0^4 \; z_{rms}^2} \approx \frac{\int_{f_1}^{f_2} f^2 \; G_{\ddot{x}}(f)\, df}{\ddot{x}_{rms}^2}\,. \qquad [2.24]$$

When the natural frequency becomes large compared to the upper limit of the definition range of the PSD input, the mean frequency of the relative response displacement tends towards that of the excitation. As a consequence, it is independent of the natural frequency of the system under excitation. Thus, expression [2.7] of the ERS has as a limit

$$\omega_0^2 \, z_{sup} \approx \ddot{x}_{rms} \sqrt{2 \ln f_M T} \qquad [2.25]$$

The general property of the shock response spectra which at high frequency tends towards the largest value of the excitation is thereby verified.

2.4. Response spectrum with up-crossing risk

In the preceding sections, the largest peak observed on average over a time T (amplitude z_s) was considered. Examination of the largest peak distribution shows that, the scatter being small, this average is sufficient as an estimate for the comparison of severity of several vibrations or for the writing of test specifications (since we are interested only in the relative position of the curves). However, should a design office need to dimension a material starting from this result, it would run the risk of disregarding peaks which have a non-negligible probability of being higher over the duration T (Volume 3, Chapter 7). For this purpose, it is preferable to choose a value with a low risk of being up-crossed.

2.4.1. *Complete expression*

The probability that a peak of the response of a single degree-of-freedom system of natural frequency f_0 may be larger than a value u_0 is given by (equation [6.64] Volume 3):

$$Q(u_0) = \frac{1}{2} \left\{ 1 - \operatorname{erf}\left(\frac{u_0}{\sqrt{2\left(1 - r^2\right)}} \right) + r \, e^{-\frac{u_0^2}{2}} \left[1 + \operatorname{erf}\left(\frac{r \, u_0}{\sqrt{2\left(1 - r^2\right)}} \right) \right] \right\}$$

r = factor of irregularity, calculated from moments of the signal PSD;

u = response relative displacement amplitude of a reduced maximum (amplitude to rms value ratio: $u = z / z_{rms}$);

erf() is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} \, d\lambda$.

For a signal lasting T made up of $n_p^+ \, T$ peaks (n_p^+ = average number of peaks by second), therefore there are N peaks larger than u_0, with

$$N = n_p^+ \, T \, Q(u_0)$$

We have seen that, in order to calculate the ERS, we look for peak u_0 that has only been up-crossed once by formulating N = 1:

$$Q(u_0) = \frac{1}{n_p^+ \, T}$$

u_0 value is obtained numerically by iterations.

Similarly, it is possible to find peak u_0 only exceeded n times from

$$Q(u_0) = \frac{n}{n_p^+ \, T}$$

where n can, for example, be equal to 10^{-2} or 10^{-3}.

When the natural frequency f_0 varies, we can draw a spectrum defined by

$$R_U(f_0) = \omega_0^2 \, u_0 \, z_{rms}$$

where $\omega_0 = 2 \pi f_0$. We will call the URS (up-crossing risk spectrum) the spectrum $R_U(f_0)$ that is obtained in this way. Here, each point of the spectrum has a different probability of occurrence. $Q(u_0)$ is the up-crossing risk.

Example 2.11.

This example is meant to illustrate the differences that may exist between ERS and URS. These spectra were calculated from the "transport by aircraft" vibration PSD in Figure 2.31.

URSs are drawn for a risk that is respectively equal to 0.01 and 0.001, duration of the vibration chosen is one hour (Figure 2.32), then five minutes (Figure 2.33).

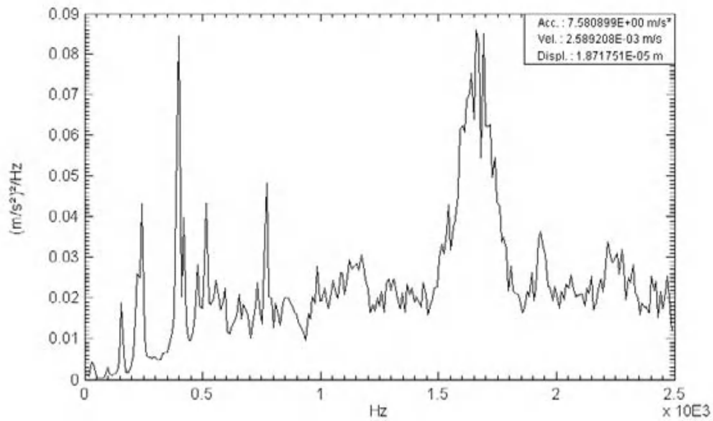


Figure 2.31. *PSD of a "transport by aircraft" vibration*

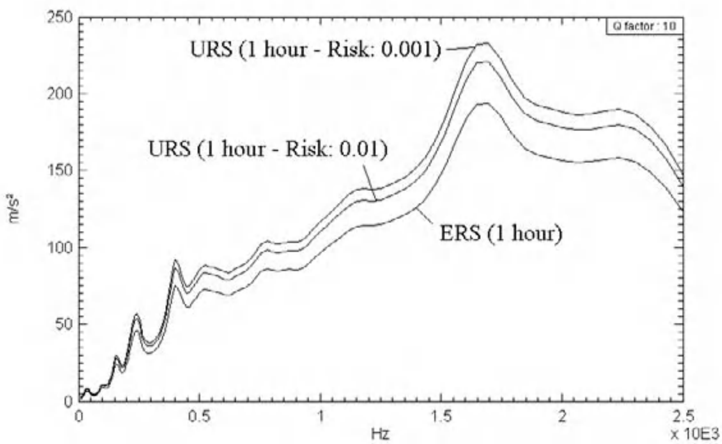


Figure 2.32. *Comparison of ERS and USR calculated for two up-crossing risk values (duration one hour)*

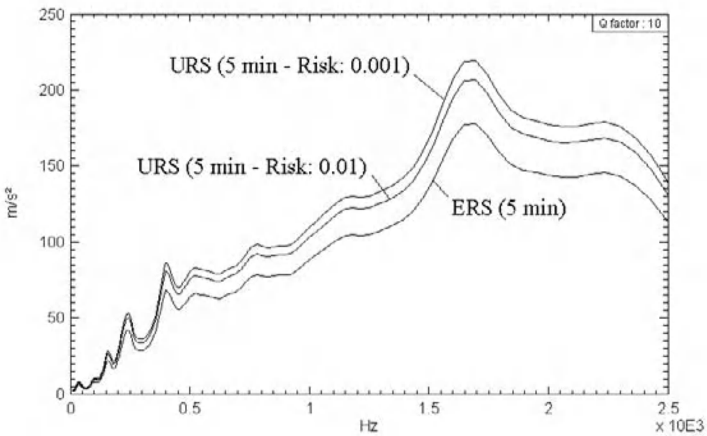


Figure 2.33. *Comparison of ERS and URS calculated for two up-crossing risk values (duration five minutes)*

We can observe that:

– sizing a structure with the ERS would lead to slightly under sizing it;

– the difference between URS drawn for risk equal to 0.01 and 0.001 is not very big;

– the URS, as the ERS, varies with duration (or more precisely, the number of cycles): it is more probable to find a high value in the response when duration is greater (see section 6.1). This variation is slow however (we can verify in this example that the gap was small between five minutes and one hour).

2.4.2. *Approximate relation*

From relation [7.54] (Volume 3), we have:

$$R_U = \left(2\,\pi\,f_0\right)^2 z_{rms} \sqrt{-2\,\ln\left[1-\left(1-\alpha\right)^{1/n_0^+ T}\right]} \qquad [2.26]$$

where:

– α is the accepted risk of up-crossing, i.e. the probability of finding a peak higher than the R_U value in $n_0^+ T$ peaks (for instance, the value $\alpha = 0.01$ can be acceptable);

– n_0^+ is the mean frequency of the response of the single-degree-of-freedom system with natural frequency $f_0 . n_0^+$ is equal to f_0, since expression [7.54] (Volume 3) was obtained on the assumption of a narrow band noise (here the response of the one-degree-of-freedom system).

The URS can be expressed as a function of the previously defined ERS according to

$$R_U = R \sqrt{\frac{-\ln\left[1-(1-\alpha)^{1/n_0^+ T}\right]}{\ln\left(n_0^+ T\right)}} \qquad [2.27]$$

Relation [2.26] can be simplified when $\alpha << 1$, from [7.55], Volume 3:

$$R_U = \left(2 \pi f_0\right)^2 z_{rms} \sqrt{2 \ln\left(\frac{n_0^+ T}{\alpha}\right)} \qquad [2.28]$$

yielding

$$R_U = R \sqrt{1 - \frac{\ln \alpha}{\ln n_0^+ T}} \qquad [2.29]$$

The curves in Figure 2.34, which show the variations of the ratio $\frac{R_U}{R}$ as a function of $n_0^+ T$ for $\alpha = 10^{-4}$, 10^{-3}, 10^{-2} and 0.1 respectively, show that this factor cannot be disregarded.

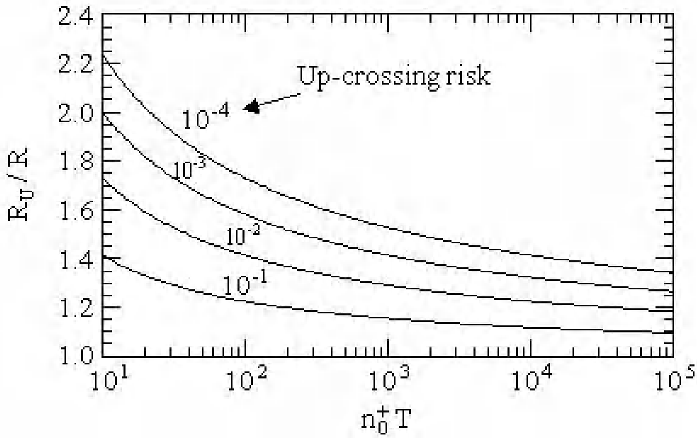


Figure 2.34. *Ratio URS / ERS*

The resulting spectrum from this approximate relation is generally identical to that determined using the complete formulation (except when r is very small compared to 1).

Example 2.12.

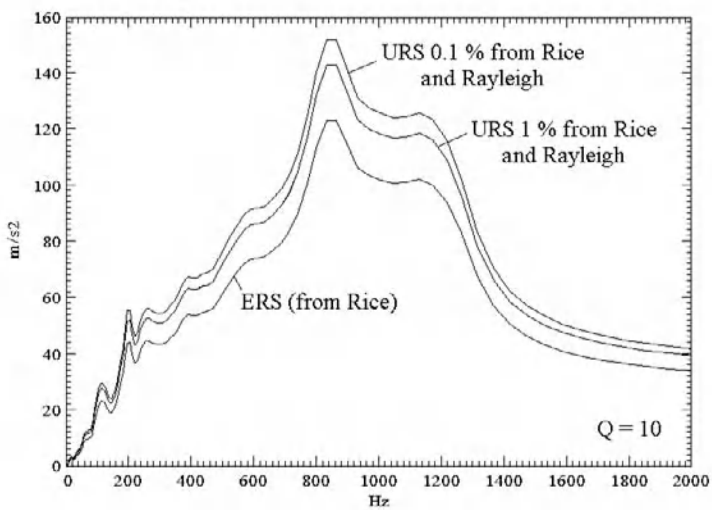


Figure 2.35. *URS examples*

Aircraft vibration

The ERS is calculated by iterations for Q = 10 and T = 10 min with the above method (section 2.2.1.1).

The URS is calculated in the same up-crossing risks conditions 1% and 0.1%:

– with the Rayleigh hypothesis [2.26];

– from Rice's distribution of peaks (section 2.4.1).

These URSs are merged (Figure 2.35) for two risk values.

2.4.3. *Calculation in a hypothesis of independence of threshold overshoot*

This same result can be obtained by assuming independence of the threshold up-crossings a, acceptable for high levels (which is related to this problem since only the highest of levels are of interest to us). In this case, the up-crossings are roughly distributed according to Poisson's law of average

$$E(T) = \frac{1}{n_a^+} \qquad [2.30]$$

and of standard deviation

$$s = \frac{1}{n_a^+} \qquad [2.31]$$

with the distribution function

$$P(T_0) = 1 - e^{-n_a^+ T_0} \qquad [2.32]$$

Suppose that $E(T) = T$, or that T is the average time, then

$$T = \frac{1}{n_a^+}$$

yielding

$$a = z_{rms} \sqrt{2 \ln n_0^+ T} \qquad [2.33]$$

The probability that this level is reached in a shorter time than T is:

$$P(T) = 1 - e^{-\frac{n_a^+}{n_a^+}} = 1 - \frac{1}{e} = 0.632$$

If we want to determine the level a that corresponds to a probability P that is smaller than the preceding one, we proceed as follows. Let $P = P_0$ be the chosen level of probability and $T = E(T) - k\ s(T)$ the corresponding duration:

$$T = \frac{1}{n_a^+} - \frac{k}{n_a^+} = \frac{1-k}{n_a^+}$$

$$P_0 = 1 - e^{-n_a^+ \frac{1-k}{n_a^+}} = 1 - e^{k-1}$$

$$k = 1 + \ln\left(1 - P_0\right)$$

yielding

$$T = \frac{1 - 1 - \ln\left(1 - P_0\right)}{n_a^+} = -\frac{\ln\left(1 - P_0\right)}{n_a^+} \qquad [2.34]$$

$$n_a^+\ T = -\ln\left(1 - P_0\right) = n_0^+\ T\ e^{-\frac{a^2}{2\, z_{rms}^2}}$$

$$\ln\left(\frac{-\ln\left(1 - P_0\right)}{n_0^+\ T}\right) = -\frac{a^2}{2\ z_{rms}^2}$$

and

$$a = z_{rms}\ \sqrt{2\ \left\{\ln\left(n_0^+\ T\right) - \ln\left[-\ln\left(1 - P_0\right)\right]\right\}} \qquad [2.35]$$

Example 2.13.

Figure 2.36 shows on a comparative basis the variations of $\frac{a}{z_{rms}} = \sqrt{2 \left\{\ln\left(n_0^+ \ T\right) + 4.6\right\}}$ for $P_0 = 0.01$ (curve 1) and of $\frac{a}{z_{rms}} = \sqrt{2 \ \ln\left(n_0^+ \ T\right)}$ (curve 2) as a function of $n_0^+ \ T$.

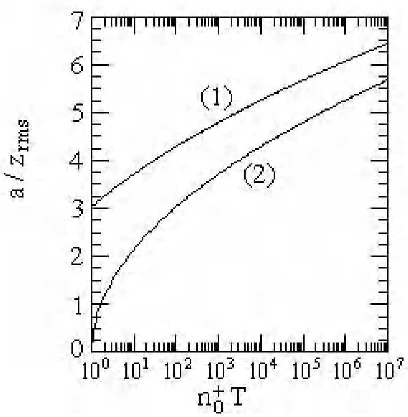


Figure 2.36. *Largest peak, on average, over a duration T*

2.4.4. *Use of URS*

The URS can be used for two types of applications:

– for dimensioning a structure: it gives the highest value of the response which can be obtained with a given probability of up-crossing, chosen *a priori* low (for example 1%);

– to demonstrate that the random vibration studied generates a higher response than a shock (the shock test will then not be carried out). In this case, the URS is calculated with a high risk of up-crossing (99% for example). If this URS is higher that the SRS of the shock, it is thus shown that the random vibration is more severe than the shock with a high probability.

Example 2.14.

We want to compare the severity of a shock and of a random vibration. Figure 2.37 shows that with a probability of 99% a peak of the response of a one degree-of-freedom system excited by the random vibration is larger than the largest response peak created by the shock (comparison of the URS with 99% and the SRS).

If we had to size a structure so that it would resist the random vibration, we would have to consider instead a URS calculated for a low up-crossing risk, for example 1%, that would make it possible to consider the largest stress triggered by this environment.

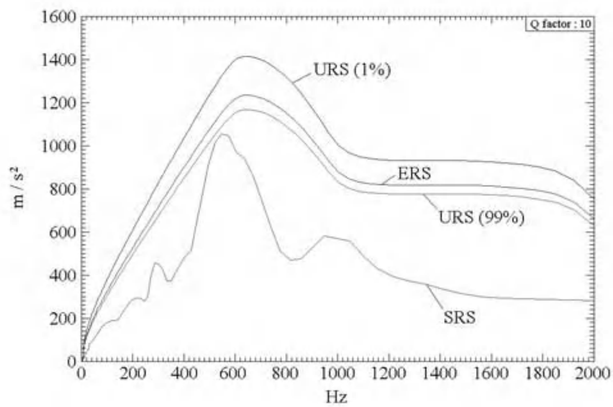


Figure 2.37. *SRS of a shock compared to the ERS and URS of a random vibration calculated for a 1% and 99% risk*

2.5. Comparison of the various formulae

There are several methods making it possible to evaluate the greatest value of the response of a one-degree-of-freedom system. We propose comparing, with the help of an example, the results obtained with the methods presented in this chapter with those obtained from the various formulae of Chapter 10 (Volume 3).

Example 2.15.

Consider a random vibration defined by a PSD of constant value $G = 1\ (\text{m s}^{-2})^2/\text{Hz}$ between 1 Hz and 2000 Hz, applied during one hour to a single-degree-of-freedom linear system of natural frequency $f_0 = 100$ Hz and damping ratio $\xi = 0.05$ $(Q = 10)$.

rms response displacement: $u_{rms} = 1.0036\ 10^{-4}$ m.

Average frequency: $n_0^+ = 99.87$ Hz.

Number of positive peaks per second: $n_p^+ = 150.47$.

$N_p = n_p^+ T$ if the parameter r is arbitrary, $N_p = n_0^+ T$ if $r \approx 1$. Here, $r = 0.664$.

Table 2.1 recapitulates the values of the responses calculated with the data from the above example using the relations established in this chapter.

Parameter	**Relation**[1]	**Displacement (m) (10^{-4})**	ω_0^2 **× displacement (m s^{-2})**
rms response	V 3 [8.62]	1.0036	39.62
3 times the rms response	V 3 [8.62]	3.011	118.86
$z_{rms}\sqrt{2\ln\left(n_0^+ T\right)}$	V 3 [5.58] V 3 [7.45]	5.076	200.41
Average of largest peaks $z_{rms}\left[\sqrt{2\ln\left(n_0^+ T\right)}+\dfrac{\varepsilon}{\sqrt{2\ln\left(n_0^+ T\right)}}\right]$ (assumption of a narrow band noise)	V 3 [7.29]	5.19	204.93
Average of largest peaks $z_{rms}\left[\sqrt{2\ln\left[r\ N_p\right]}+\dfrac{\varepsilon}{\sqrt{2\ln\left[r\ N_p\right]}}\right]$ and $N_p = n_p^+ T$	V 3 [7.56]	5.19	204.93

1. Volume and relationship number.

Parameter		**Relation**[2]	**Displacement (m) (10^{-4})**	$\boldsymbol{\omega_0^2}$ **× displacement ($m\ s^{-2}$)**
$z_{rms} \sqrt{2\left\{\ln\left(n_0^+ T\right) - \ln\left[-\ln\left(1 - P_0\right)\right]\right\}}$ with $P_0 = 10^{-3}$		V 3 [7.16]	5.721	225.85
Peak amplitude having a probability equal to 10^{-3} (calculated from the maximum response distribution)		V 3 [6.64]	3.62	142.83
The largest peak on average over duration T		V 5 [2.2]	5.08	200.41
Peak amplitude having a probability equal to 10^{-3} (distribution of the largest peaks of the response)		V 3 [7.26]	6.30	248.82
Average of the largest peaks + 3 standard deviations (r close to 1)		V 3 [7.29] V 3 [7.39]	5.954	235.07
Average of the largest peaks + 3 standard deviations (any r)		V 3 [7.56] V 3 [7.61]	5.987	236.35
The largest peak with a risk α of being exceeded	$\alpha = 0.01$	V 5 [2.29]	5.92	233.72
	$\alpha = 0.001$		6.30	248.70

Table 2.1. *Comparative table of various formulations of the largest peak, for one example*

It should be noted that the response calculated by taking 3 times the rms value can lead to a value much smaller than the mean of the largest peaks. This result can be reversed for an extremely short duration.

Table 2.2 gathers the results deduced from the calculations of first crossing presented in Chapter 10 of Volume 3 starting out with the same data. The displacement is obtained by seeking in the quoted relations the value of v in such a way that the probability of a first up-crossing of this threshold is, over the duration T, equal to $P = 1 - \frac{1}{n_p^+ T}$ for type B and D up-crossings, equal to $P = 1 - \frac{1}{N\ T}$ for those of type E (the mean number N T of peaks of envelope being given by [10.55] of Volume 3).

2. Volume and relationship number.

Method	Barrier	Relation N°	Displacement (10^{-4} m)	ω_0^2 × displacement (m s^{-2})
Independent threshold crossings	Type B	V3 [10.5]	7.33	289.23
Independent threshold crossings	Type D	V3 [10.7]	6.72	265.46
Maxima of the independent response	Type D	V3 [10.30]	6.72	265.46
Independent threshold crossings of the envelope of maxima	Type E	V3 [10.35]	6.35	250.84
Independent envelope maxima (Crandall)	Type E	V3 [10.49]	7.07	278.93
Independent envelope maxima (Aspinwall)	Type E	V3 [10.59]	6.91	272.99
Markov process (Mark)	Type B	V3 [10.66]	7.33	289.23
Markov process (Mark)	Type D	V3 [10.66]	7.43	293.19
Markov process (Mark)	Type E	V3 [10.67]	7.25	286.06
Two state Markov process	Type B	V3 [10.84]	6.62	261.50
Two state Markov process	Type D	V3 [10.88]	6.72	265.46
Two state Markov process	Type E	V3 [10.95]	6.51	257.18
Mean clump size	Type B	V3 [10.106] (modified)	6.13	242.08
Mean clump size	Type D	V3 [10.106]	6.22	245.65
Mean clump size (2 states)	Type D	V3 [10.112]	6.22	245.65

Table 2.2. *Comparative table of the various formulations of the first passage of a threshold for one example*

2.6. Effects of peak truncation on the acceleration time history

In order to evaluate the incidence of a truncation of the acceleration signal peaks applied to the base of the single-degree-of-freedom system used as a reference for the calculation of the ERS, we take the example from Volume 3 (section 2.12) which was presented with the same aim of calculating the PSD.

2.6.1. *Extreme response spectra calculated from the time history signal*

The ERS determined under the same conditions of truncation are plotted in Figure 2.38. A peaks truncation beyond $3\ \ddot{x}_{rms}$ affects the spectra weakly. The ERS is still weakly deformed if the peaks higher than $2.5\ \ddot{x}_{rms}$ are truncated. From this, the ERS are no longer acceptable. We will see (section 4.6) that the fatigue damage spectra are less sensitive to this effect.

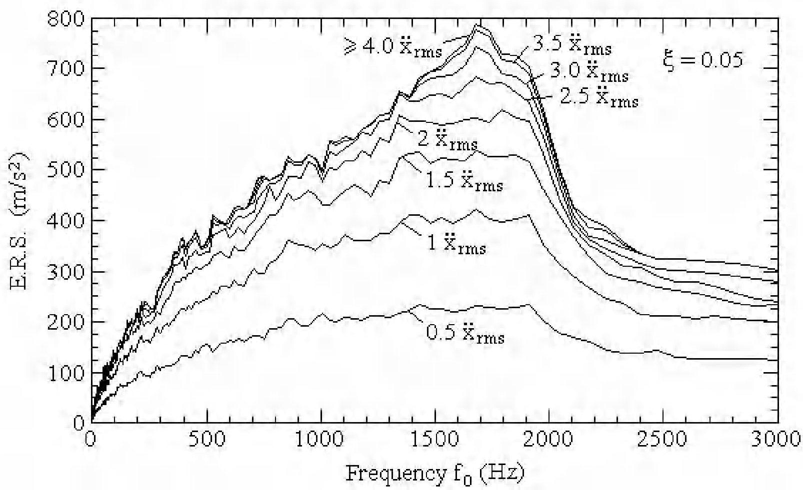


Figure 2.38. *ERS of truncated vibratory signals*

2.6.2. *Extreme response spectra calculated from the power spectral densities*

Note that in Figure 2.39 the incidence of truncation is negligible beyond $2.5\ \ddot{x}_{rms}$.

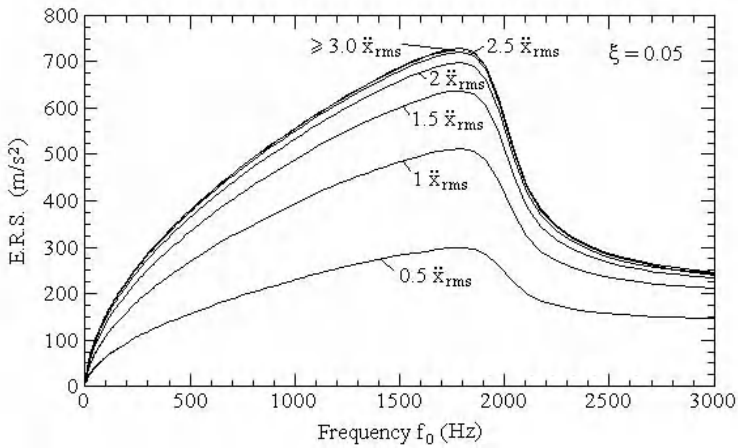


Figure 2.39. *ERS calculated from PSD of truncated signals*

2.6.3. *Comparison of extreme response spectra calculated from time history signals and power spectral densities*

So long as truncation is higher than approximately $2.5\ \ddot{x}_{rms}$, the two spectra follow each other quite well; the ERS obtained from the PSD is however much smoother – it is a mean spectrum in the statistical sense.

The ERS calculated directly from the time history signal, by sorting the response peaks, is deterministic. It is one of the spectra whose mean is calculated using the PSD.

When truncation is less than $2.5\ \ddot{x}_{rms}$, the ERS calculated from the time history signal tends to be lower than the ERS resulting from the PSD. Figures 2.40 and 2.41 show the ERS of the non-truncated or truncated (to $0.5\ \ddot{x}_{rms}$) signals calculated on the basis of their signal and their PSD.

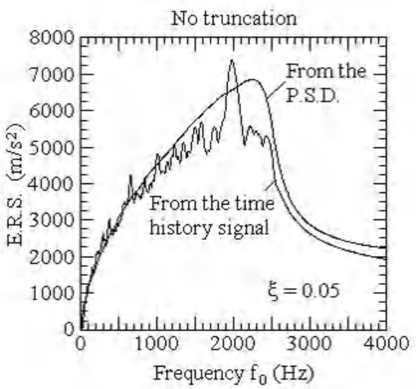


Figure 2.40. *Comparison of the ERS. calculated from a non-truncated signal and from its PSD*

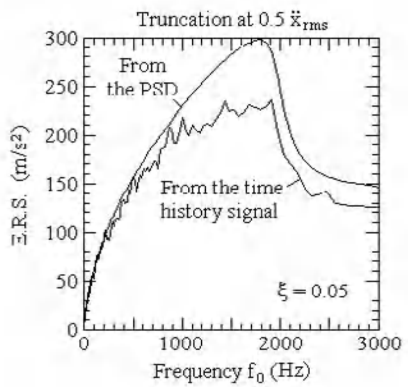


Figure 2.41. *Comparison of the ERS calculated from a truncated signal and from its PSD*

NOTE: *A complementary study carried out under strictly similar conditions, by replacing the white noise by a narrow band (between 600 Hz and 700 Hz, $G = 5\ (m\ s^{-2})^2/Hz$) superimposed on a white noise (between 10 Hz and 2,000 Hz, $G = 1\ (m\ s^{-2})^2/Hz$), led to similar results.*

2.7. Sinusoidal vibration superimposed on a broad band random vibration

2.7.1. *Real environment*

The real environment is not always simple and can be a combination of several random and sinusoidal vibrations.

Propeller planes equipped with constant rotation speed motors and helicopters for example generate vibrations made of wide band random noise onto which sinusoidal lines with very low frequency variation are superimposed.

Example 2.16.

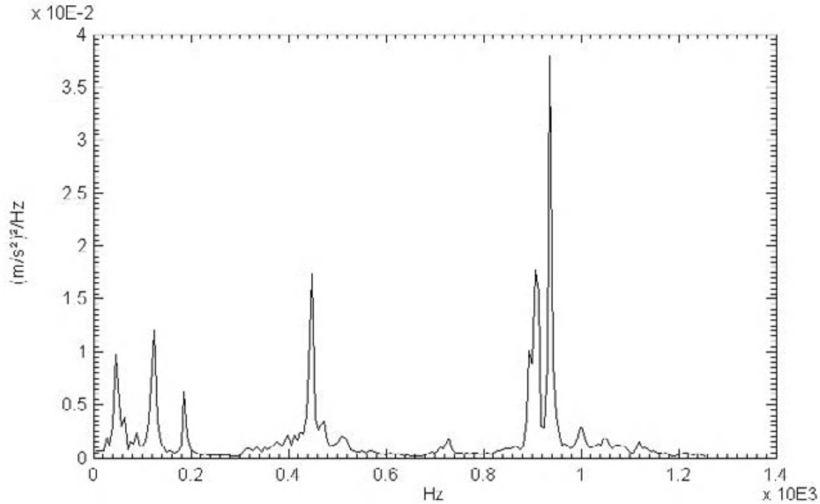


Figure 2.42. *PSD example of a measured vibration in a propeller plane*

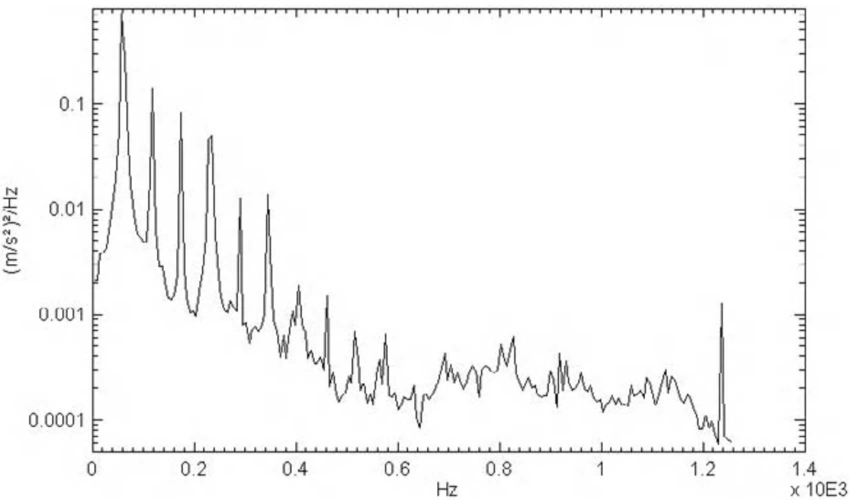


Figure 2.43. *Other PSD example of a measured vibration in a propeller plane*

Example 2.17.

Standard GAM EG 13B Propeller plane, close to motors [GAM 87].

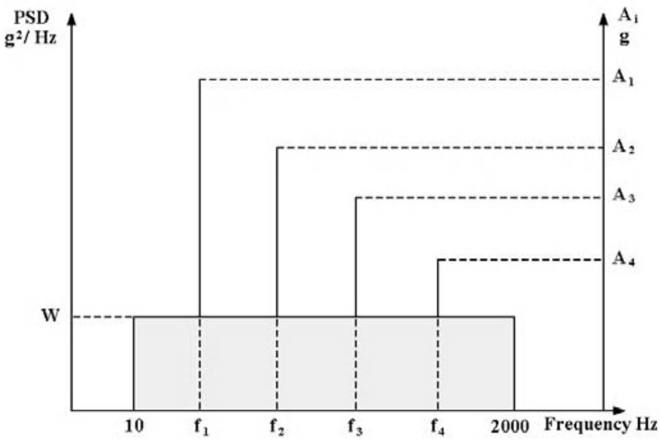


Figure 2.44. *GAM EG 13 Booklet 42: Propeller plane zone L (close to motors) f_1 = frequency of motor rotation, f_2, f_3 and f_4 = harmonics*

2.7.2. *Case of a single sinusoid superimposed to a wide band noise*

First of all let us consider the case of a vibration made of a sinusoidal line (frequency f_s and amplitude $\ddot{x}_m$) superimposed on a broad band random vibration of constant PSD G_0. Let:

– z(t) be the relative response displacement of a single-degree-of-freedom linear system (f_0 , Q) subjected to this excitation;

– $z_s(t)$ be the maximum relative response displacement of the same system to the sinusoid alone;

– $z_{a_{rms}}$ be the rms value of the relative response displacement to the random vibration alone.

2.7.2.1. *Probability density of peaks*

It is shown that the peak distribution of the response $z(t)$ has as its probability density [RIC 44]

$$p(z) = \frac{z}{z_{a\,rms}^2}\, e^{-\frac{z^2+z_s^2}{2\, z_{a\,rms}^2}}\, I_0\left(\frac{z\, z_s}{z_{a\,rms}^2}\right) \qquad [2.36]$$

where I_0 is a zero order Bessel function:

$$I_0\left(\frac{z\, z_s}{z_{a\,rms}^2}\right) = \sum_{n=0}^{\infty}\left(\frac{z\, z_s}{2\, z_{a\,rms}^2}\right)^{2n} \frac{1}{(n\,!)^2} \qquad [2.37]$$

Relation [2.36] is established by assuming that in the presence of the purely random vibration, the response of the single-degree-of-freedom system is a narrow band response and that the peak distribution of the response follows Rayleigh's distribution. Let $u = \dfrac{z}{z_{a\,rms}}$, $a = \dfrac{z_s}{z_{a\,rms}}$, where z_s is the maximum value of $z_s(t)$. This gives, in reduced form

$$p(u) = u\, e^{-\frac{u^2+a^2}{2}}\, I_0(a u) \qquad [2.38]$$

with

$$I_0(a\, u) = \sum_{n=0}^{\infty}\left(\frac{a\, u}{2}\right)^{2n} \frac{1}{(n\,!)^2} \qquad [2.39]$$

With similar notations, if σ is the generated stress ($\sigma = K\, z$), the probability density of peaks of $\sigma(t)$ is such that

$$p(\sigma) = \frac{\sigma}{\sigma_{a\,rms}^2}\, e^{-\frac{\sigma^2+\sigma_s^2}{2\, \sigma_{a\,rms}^2}}\, I_0\left(\frac{\sigma\, \sigma_s}{\sigma_{a\,rms}^2}\right) \qquad [2.40]$$

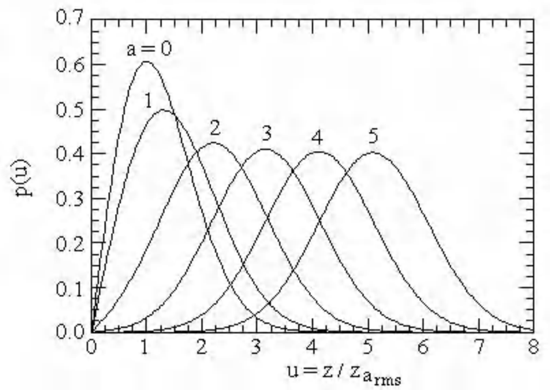


Figure 2.45. *Probability density of peaks of a sine wave plus wide band random noise*

2.7.2.2. *Distribution function of peaks*

By definition, the distribution function of peaks is expressed as

$$P(z \le z_1) = \int_0^{z_1} p(z)\,dz \qquad [2.41]$$

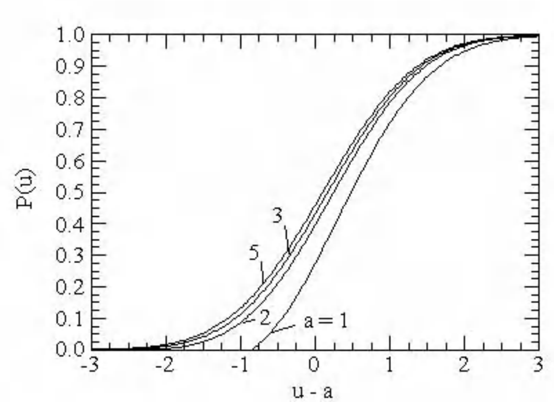


Figure 2.46. *Distribution function of peaks of a sine wave plus wide band random noise*

The probability of a peak being larger than z_1 is equal to

$$Q(z_1) = 1 - P(z_1) = \int_{z_1}^{\infty} p(z)\,dz \qquad [2.42]$$

or, in reduced form

$$Q(u_1) = \int_{u_1}^{\infty} p(u)\ du \tag{2.43}$$

$$Q(u_1) = \int_{u_1}^{\infty} u\ e^{-\frac{u^2+a^2}{2}} \sum_{n=0}^{\infty} \left(\frac{a\ u}{2}\right)^{2\,n} \frac{1}{(n\,!)^2}\ du \tag{2.44}$$

$$Q(u_1) = \sum_{n=0}^{\infty} \left(\frac{a}{2}\right)^{2\,n} \frac{e^{-\frac{a^2}{2}}}{(n\,!)^2} \int_{u_1}^{\infty} u^{2\,n+1}\ e^{-\frac{u^2}{2}}\ du \tag{2.45}$$

$$Q(u_1) = \sum_{n=0}^{\infty} 2 \left(\frac{a^n}{n\,!}\right)^2 e^{-\frac{a^2}{2}} \int_{u_1}^{\infty} \left(\frac{u}{2}\right)^{2\,n+1} e^{-\frac{u^2}{2}}\ du \tag{2.46}$$

By making $t = \frac{u^2}{2}$ in the integral, we can go back to an incomplete function of gamma of the form

$$\gamma(1+x) = \int_0^{u_1} t^x\ e^{-t}\ dt$$

If $\Gamma(1+x)$ is the gamma function ($\Gamma(1+x) = \int_0^{\infty} t^x\ e^{-t}\ dt$), and if we set

$$\gamma_1(1+x) = \Gamma(1+x) - \gamma(1+x)$$

we then have

$$Q(u_1) = \sum_{n=0}^{\infty} \left(\frac{a^n}{n\,!}\right)^2 \frac{1}{2^n}\ e^{-\frac{a^2}{2}}\ \gamma_1(1+n) \tag{2.47}$$

The function $\gamma(1+n)$ can be approximated by using

$$\gamma(1+n) = n\,! - e^{-x}\ W(n, t) \tag{2.48}$$

where [SPE 92]

$$W(n,t) = \begin{cases} 1 & \text{if } n = 0 \\ 1 + t & \text{if } n = 1 \\ n\,! + t^n + \displaystyle\sum_{i=1}^{n-1} \frac{n\,!}{(n-i)!} t^{n-i} & \text{for } n > 1 \end{cases} \tag{2.49}$$

yielding

$$\gamma_1(1+n) = e^{-t}\ W(n, t) = e^{-\frac{u_1^2}{2}}\ W\left(n, \frac{u_1^2}{2}\right) \tag{2.50}$$

combining [2.47] and [2.50], we obtain

$$Q(u_1) = \sum_{n=0}^{\infty} \left(\frac{a^n}{n\,!}\right)^2 \frac{1}{2^n}\ e^{-\frac{a^2}{2}}\ e^{-\frac{u_1^2}{2}}\ W\left(n, \frac{u_1^2}{2}\right) \tag{2.51}$$

$$Q(u_1) = e^{-\frac{a^2 + u_1^2}{2}} \sum_{n=0}^{\infty} \left(\frac{a^n}{n\,!}\right)^2 \frac{1}{2^n}\ W\left(n, \frac{u_1^2}{2}\right) \tag{2.52}$$

Particular cases

If $a_0 = \dfrac{a}{\sqrt{2}}$

$$Q(u_1) = e^{-\frac{a^2 + u_1^2}{2}} \sum_{n=0}^{\infty} \left(\frac{a_0^n}{n\,!}\right)^2\ W\left(n, \frac{u_1^2}{2}\right) \tag{2.53}$$

If $a_0 = 0$, i.e. in the absence of sinusoidal vibration, we find that

$$Q(u_1) = \int_{u_1}^{\infty} u\, e^{-\frac{u_1^2}{2}}\, du \qquad [2.54]$$

$$Q(u_1) = e^{-u_1^2/2} \qquad [2.55]$$

NOTE: *When a becomes large and when u is itself large, the distribution behaves like any other normal law, mean* z_s *and standard deviation* $z_{a\,rms}$.

2.7.2.3. *Extreme response*

General case

The number of peaks larger than u_1 over duration T is

$$N_T = n_p^+\, T\, Q(u_1) \qquad [2.56]$$

if n_p^+ is the mean number of peaks per second. The largest peak, on average, is not found more than once over the duration T. Let $N_T = 1$; then

$$Q(u_1) = \frac{1}{n_p^+\, T} \qquad [2.57]$$

The value of u_1 satisfying this relation is obtained by successive iterations as in the case of a purely random vibration (section 2.2.1). Knowing that the function $Q(u)$ is decreasing, we choose two values of u such as

$$Q(u_a) < Q < Q(u_b)$$

and, for each iteration, we reduce the interval until, for example, u_1 satisfies

$$\frac{Q(u_a) - Q(u_b)}{Q(u_a)} < 10^{-2}$$

By interpolation, we obtain

$$z_{sup} = z_{a\,rms} \left[\left(u_b - u_a\right) \frac{Q(u_a) - Q(u_1)}{Q(u_a) - Q(u_b)} + u_a \right] \qquad [2.58]$$

The extreme spectrum of response is thus

$$R = \left(2\,\pi\,f_0\right)^2 z_{sup} \qquad [2.59]$$

Calculation of n_0^+ and n_p^+

The rms value of the relative response displacement is equal to

$$z_{rms}^2 = z_{a\,rms}^2 + z_{s\,rms}^2 \qquad [2.60]$$

The rms velocity and rms acceleration are both given by [CRA 67] [RIC 48] [STO 61] respectively

$$\dot{z}_{rms}^2 = \dot{z}_{a\,rms}^2 + \left(2\,\pi\,f_s\right)^2 \dot{z}_{s\,rms}^2 \qquad [2.61]$$

and

$$\ddot{z}_{rms}^2 = \ddot{z}_{a\,rms}^2 + \left(2\,\pi\,f_s\right)^4 \ddot{z}_{s\,rms}^2 \qquad [2.62]$$

yielding the mean frequency of the composite signal

$$n_{0_{RS}}^+ = \frac{1}{2\,\pi}\,\frac{\dot{z}_{rms}}{z_{rms}} \qquad [2.63]$$

and the mean number of positive peaks per second

$$n_{p_{RS}}^+ = \frac{1}{2\,\pi}\,\frac{\ddot{z}_{rms}}{\dot{z}_{rms}} \qquad [2.64]$$

Example 2.18.

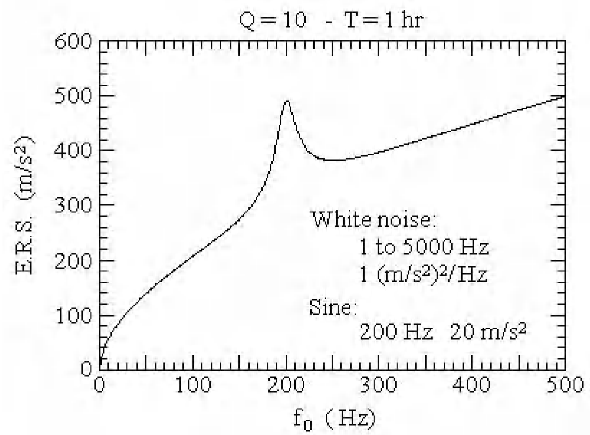


Figure 2.47. *ERS of a vibration made up of a sine wave plus wide band random noise*

Approximations

1. The mean number per second of crossings of a given threshold z_0 with positive slope is

$$n_{z_0}^{+} = \frac{1}{2\,\pi}\,\frac{\dot{z}_{rms}}{z_{rms}}\,e^{-\frac{z_0^2}{2\,z_{rms}^2}} = n_{0RS}^{+}\,e^{-\frac{z_0^2}{2\,z_{rms}^2}} \qquad [2.65]$$

If T is the duration of the vibration,

$$e^{-\frac{z_0^2}{2\,z_{rms}^2}} = \frac{n_{0RS}^{+}}{n_{z_0}^{+}} = \frac{n_{0RS}^{+}\,T}{n_{z_0}^{+}\,T} \qquad [2.66]$$

yielding

$$z_0^2 = 2\,z_{rms}^2\,\ln\frac{n_{0RS}^{+}\,T}{n_{z_0}^{+}\,T} \qquad [2.67]$$

The largest peak corresponds to the level $z_{sup} = z_0$ crossed only once:

$$n_{z\,sup}^{+} T = 1$$

yielding

$$z_{sup} = z_{rms} \sqrt{2 \ln n_{0RS}^{+} T} \qquad [2.68]$$

and

$$R = \left(2 \pi f_0\right)^2 z_{rms} \sqrt{2 \ln n_{0RS}^{+} T} \qquad [2.69]$$

2. If n_0^+ is solely the mean frequency of the random vibration, and if $n_0^+ T$ is higher than approximately 1000, the following approximate relation can be used

$$R \approx \left(2 \pi f_0\right)^2 z_{a_{rms}} \left(a + \sqrt{2 \ln n_0^+ T}\right) \qquad [2.70]$$

This approximation is better than the preceding one. Let ρ be the ratio of this value to that of the ERS of a narrow band noise ($R = \left(2 \pi f_0\right)^2 z_{rms_{NB}} \sqrt{2 \ln n_0^+ T}$), which gives us:

$$\rho = \frac{z_{a_{rms}} \left[\sqrt{2 \ln n_0^+ T} + a\right]}{z_{rms_{NB}} \sqrt{2 \ln n_0^+ T}} \qquad [2.71]$$

However:

$$\frac{z_{a_{rms}}}{z_{rms_{NB}}} \approx \frac{z_{a_{rms}}}{\sqrt{z_{a_{rms}}^2 + z_s^2/2}} = \frac{1}{\sqrt{1 + \frac{1}{2}\left(\frac{z_s}{z_{a_{rms}}}\right)^2}} \qquad [2.72]$$

$$\frac{z_{a_{rms}}}{z_{rms_{NB}}} \approx \frac{1}{\sqrt{1 + a^2/2}} \qquad [2.73]$$

and

$$\rho \approx \frac{1}{\sqrt{1 + a^2/2}} \left(1 + \frac{a}{\sqrt{2 \ln n_0^+ T}}\right) \qquad [2.74]$$

This relationship, which can be also expressed as, when $u_1 = \sqrt{2 \ln n_0^+ T} + a$,

$$\rho \approx \frac{1}{\sqrt{1 + a^2/2}} \frac{u_1}{\sqrt{2 \ln n_0^+ T}} \qquad [2.75]$$

can be used to determine the characteristics of a white noise + narrow band noise composite vibration equivalent to a white noise + sine vibration.

3. S.O. Rice [RIC 44] shows that, if $a\,u >> 1$ and if $|u - a| << a$,

$$P(u) \approx \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{u-a}{\sqrt{2}}\right) - \frac{1}{2\,a\,\sqrt{2\,\pi}} \left[1 - \frac{u-a}{4\,a} + \frac{1 + (u-a)^2}{8\,a^2}\right] \exp\left[-\frac{(u-a)^2}{2}\right]$$

[2.76]

(the erf function is related here to the error function E_1 as defined in Volume 3, Appendix A4.1). The approximation is correct if $1 < a < 4$ and $u > 5$ or $a \geq 4$ and $u \geq a$.

2.7.3. *Case of several sinusoidal lines superimposed on a broad band random vibration*

It is supposed here that the sinusoidal components have zero dephasing at the initial start time.

In general, where the frequencies of the sinusoids can be arbitrary, and particularly close each to other, the ERS can be obtained numerically by considering a time history signal made up of the sum of the sinusoids and of a random signal generated from the broad band PSD.

If the frequencies of sinusoids are sufficiently spaced, the ERS of the composite signal is very close to the envelope of the ERS calculated using the preceding formulae for each sinusoid alone, superimposed on the random noise.

Example 2.19.

Broad band noise:

10 Hz with 2000 Hz

$G_0 = 1\ (\text{m s}^{-2})^2/\text{Hz}$

Sinusoids:

100 Hz, 300 Hz and 600 Hz

Amplitude $20\ \text{m s}^{-2}$

ERS plotted between 10 Hz and 1000 Hz, with a step of 5 Hz, for Q = 10 (Figure 2.48).

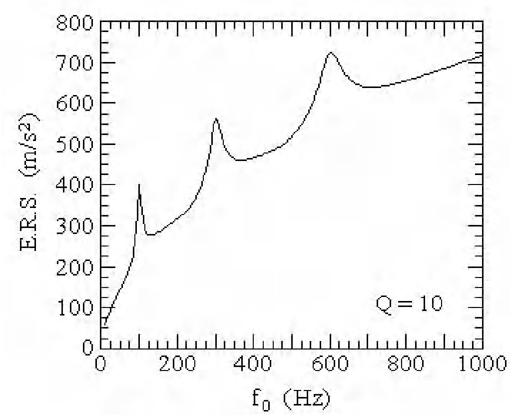


Figure 2.48. *ERS of a swept sine on broad band random vibration*

2.8. Swept sine superimposed on a broad band random vibration

2.8.1. *Real environment*

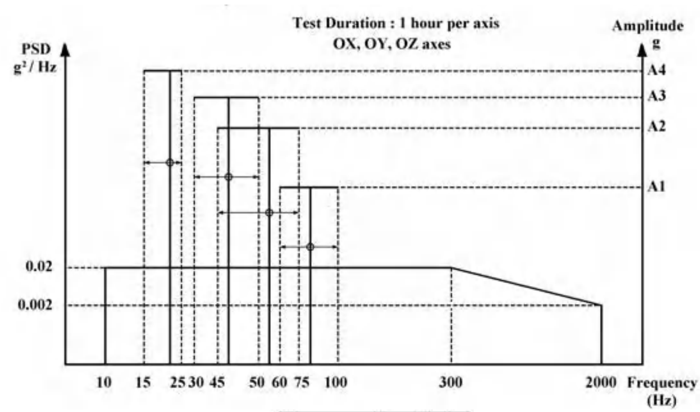
Vibration conditioning proposed in the standards (GAM EG 13, for example) to verify the behavior toward the vibratory environment of embedded materials are often made up of a wide band noise on which are superimposed:

– swept sinusoidal lines in the case of helicopters and for embedded material close to motors of propeller planes, when their rotation speed varies, as well as when it is constant, to take the misreading of this rotation speed and therefore the line frequency into consideration during the development of specifications;

– or swept narrow bands, obtained in the fuselage, wings and tail units of propeller planes because of the resonances of the device's structure.

Example 2.20.

GAM EG 13B, Booklet 42, Random vibrations. Helicopters, front fuselage, operation test [GAM 87].



	Frequency (Hz)	Amplitude (g)
A4	15 - 25	3,7
A3	30 – 50	3
A2	45 - 75	2,2
A1	60 - 100	2,2

Figure 2.49. *Helicopters, operation tests. Duration of test: one hour per axis. Two logarithmic sweep cycles per half hour (half an hour before and after an endurance test)*

2.8.2. *Case of a single swept sine superimposed to a wide band noise*

We shall consider the case of a sinusoidal vibration whose frequency varies as a function of time, in general in a linear way, superimposed on a broad band random vibration of constant PSD G_0. The sweeping duration in the selected interval is equal to the total duration of the composite vibration.

The calculation of the ERS is carried out by establishing the envelope of the ERS obtained with the formulae set out in the preceding section, by selecting n intermediate positions of the sinusoid in the swept frequency interval, each sinusoidal vibration having duration equal to the total duration of the vibration divided by the selected number n.

Example 2.21.

Wide band noise:

10 Hz to 2000 Hz

$G_0 = 1\ (\text{m s}^{-2})^2/\text{Hz}$

Swept sine:

100 Hz to 400 Hz

Amplitude $20\ \text{m s}^{-2}$

Linear sweep:

ERS plotted between 10 Hz and 1000 Hz, with a step of 5 Hz, for Q = 10 (Figure 2.50).

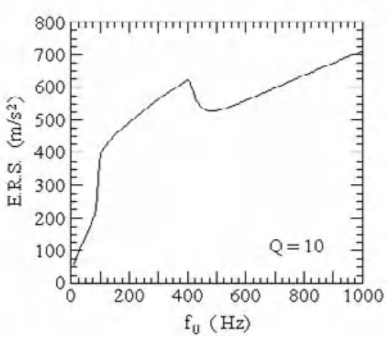


Figure 2.50. *ERS of a swept sine plus broad band random vibration*

2.8.3. *Case of several swept sines superimposed on a broad band random vibration*

The ERS is calculated by making the envelope for the ERS obtained for each intermediate situation of the frequency of the sinusoids in the swept ranges, according to the method set out in the preceding section.

2.9. Swept narrow bands on a wide band random vibration

2.9.1. *Real environment*

Vibrations collected in the fuselage of propeller planes are made up of the noise linked to the air flow and responses from the different structure modes of motor vibrations.

We consider that these vibrations can be assimilated into narrow bands superimposed onto a wide band noise. In the absence of a real available environment, the standards propose inclusive spectrums with swept narrow bands in the range of frequency covering the values observed in usual planes.

Example 2.22.

GAM EG 13B, Booklet 42. Central fuselage. Operation test [GAM 87].

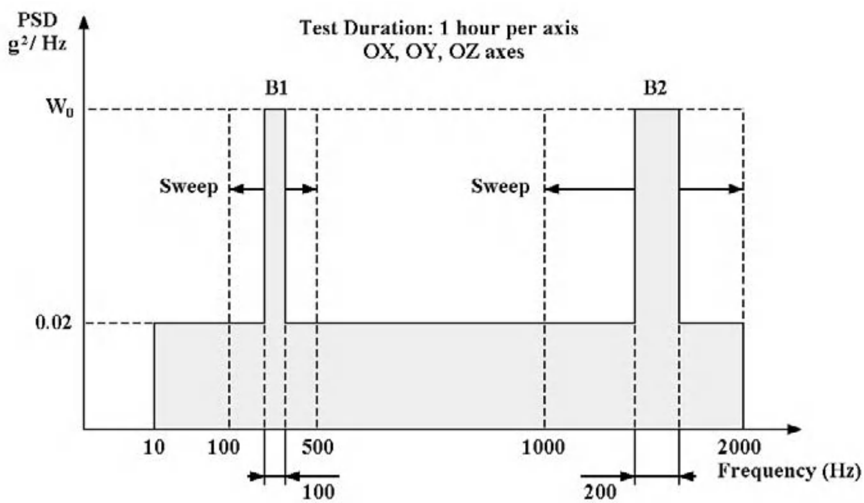


Figure 2.51. *Operation test (1 sweep per half hour)*

2.9.2. *Extreme response spectrum*

The standards sometimes specify random vibration tests defined by a broadband PSD of constant amplitude, on which one or more narrow bands of constant widths are superimposed, each central frequency moving according to time (linear law) in a specified interval.

Calculation of the ERS of such a vibration is carried out by taking the envelope of the ERS obtained as follows:

– the total sweeping duration T is divided into n intervals of duration T / n;

– in the middle of each time interval, the PSD made up of the wide-band noise and the narrow bands in their position at the precise moment is considered;

– the ERS is calculated for each PSD thereby defined, for a duration T / n, on the basis of the formulae in the preceding sections.

Example 2.23.

Random vibration made up:

– of a broad band noise from 10 Hz to 2000 Hz, with a PSD of amplitude $G_0 = 2\ (\text{m s}^{-2})^2/\text{Hz}$;

– of two narrow bands:

- one of width 100 Hz, whose central frequency varies linearly between 150 Hz and 550 Hz;
- one of width 200 Hz, swept between 1100 Hz and 1900 Hz.

The ERS is plotted between 10 Hz and 2,000 Hz for a damping ratio equal to 0.05.

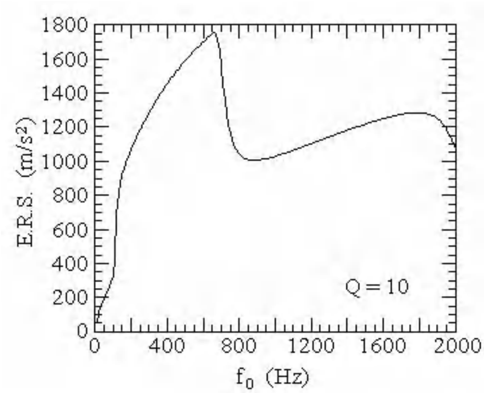


Figure 2.52. *ERS of a random vibration made up of two swept narrow bands superimposed on a wide band noise*

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 3

Fatigue Damage Spectrum of a Sinusoidal Vibration

3.1. Fatigue damage spectrum definition

The *fatigue damage spectrum* (FDS) of a vibration is obtained by tracing the fatigue damage experienced by a linear one degree-of-freedom system according to its natural frequency f_0, for given damping ratio ξ and for a given value of parameter b (this parameter comes from the Basquin law representing the Wöhler curve of the material constituting the structure).

Regardless of the signal studied (sinusoidal vibration, shock, random or composite vibration); the FDS can be obtained directly from the time history signal. The method consists of (Figures 3.1 and 3.2):

– numerically calculating relative response displacement of the mass in relation to its support;

– establishing a peak histogram, giving the number n_i of peaks according to their amplitude z_{p_i};

– using Miner's damage accumulation law (Volume 4):

$$D = \sum_i \frac{n_i}{N_i} \qquad [3.1]$$

in which N is the number of cycles leading to the fracture of a part under sinusoidal amplitude σ stress.

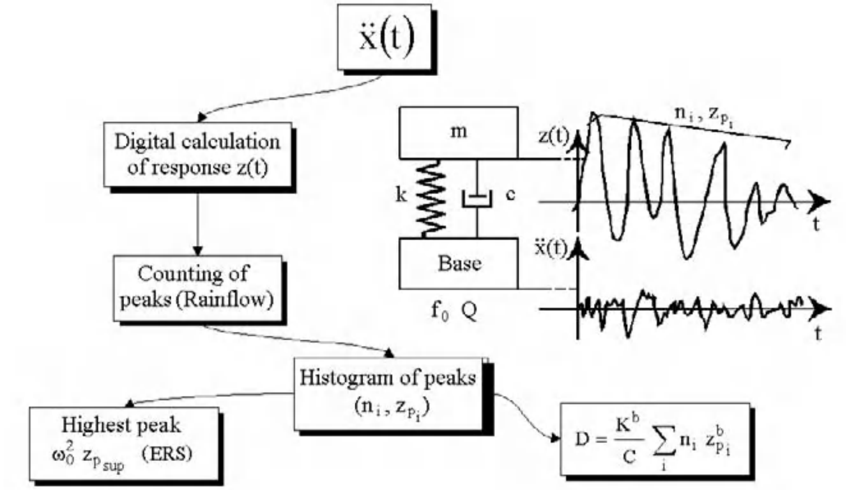


Figure 3.1. *Process of fatigue damage calculation from acceleration according to time*

The (experimental) curve that represents N variations according to σ is the Wöhler curve, which can be represented in an analytical way by the (Basquin) relation (Figure 3.3):

$$N \, \sigma^b = C \qquad [3.2]$$

b and C are constants characteristic of the material, function of the stressing mode (tension compression, torsion, bending, etc.), of the temperature, etc.

Since the system is supposed to be linear, the stress created in the elastic element is proportional to the relative displacement z_p corresponding to each extremum of $z(t)$ (the derivative $\dot{z}$ is actually zero in these points):

$$\sigma = K \, z_p$$

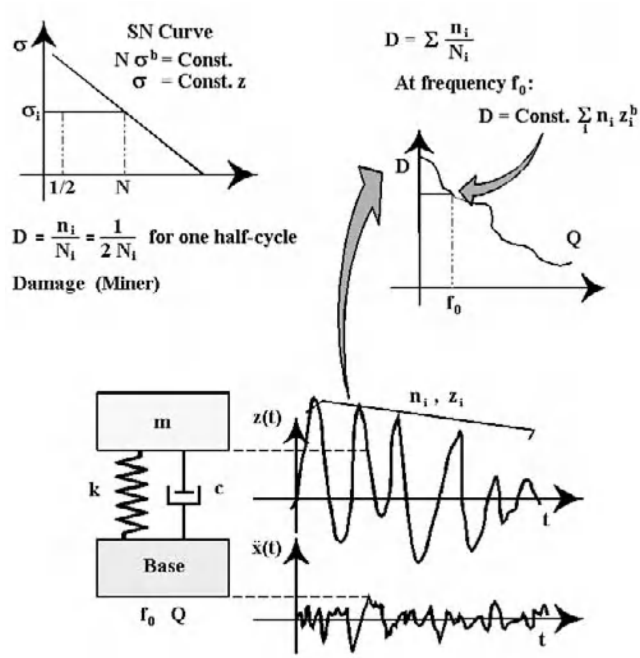


Figure 3.2. *Summary of the calculation principle of an FDS*

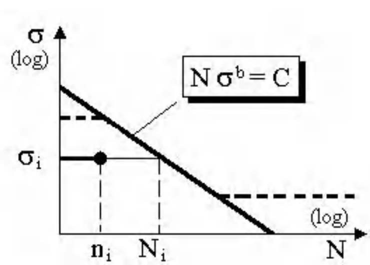


Figure 3.3. *Representation of the Wöhler curve*

The damage experienced by the system during the application of a half cycle of stress σ_i is:

$$\delta_i = \frac{1}{2 N_i} = \frac{\sigma_i^b}{2 C} \tag{3.3}$$

and, for n_i stress half cycles σ_i:

$$d_i = \frac{n_i}{2\ N_i} = \frac{n_i\ \sigma_i^b}{2\ C} = \frac{K^b}{2\ C} n_i\ z_{p_i}^b \qquad [3.4]$$

In these relations, N_i is the number of cycles leading at fracture at level σ_i, n_i is the number of half cycles counted at this level σ_i (which explains factor 2). If we have defined m classes of level z_{p_i} in the peak histogram, the total damage created can be written, according to the linear accumulation rule of Miner as:

$$D = \sum_{i=1}^{m} d_i = \sum_{i} \frac{n_i\ \sigma_i^b}{2\ C} \qquad [3.5]$$

yielding:

$$D = \frac{K^b}{2\ C} \sum_{i=1}^{m} n_i\ z_{p_i}^b \qquad [3.6]$$

3.2. Fatigue damage spectrum of a single sinusoid

Let us consider a sinusoidal stress $\ddot{x}(t) = \ddot{x}_m \sin 2\ \pi\ f\ t$ applied to a linear single-degree-of-freedom system for one length of time T.

The amplitude of each half-cycle is equal to

$$\left|z_m\right| = \frac{\ddot{x}_m}{\omega_0^2 \sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q\ f_0}\right)^2}}$$

and the damage can be written

$$D = \frac{n}{N} = \frac{n}{C} \sigma^b \qquad [3.7]$$

where $n = f\ T$ is the number of cycles applied.

$$D = \frac{K^b}{C}\, f\, T\, z_m^b \tag{3.8}$$

$$D = \frac{K^b}{C}\, f\, T \frac{\ddot{x}_m^b}{\omega_0^{2b}\left\{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q\, f_0}\right)^2\right\}^{b/2}} \tag{3.9}$$

Particular cases

If the test is carried out for resonance,

$$f = f_0 \sqrt{1 - \frac{1}{2\, Q^2}} \tag{3.10}$$

$$\left|z_m\right| = \frac{Q\, \ddot{x}_m}{\omega_0^2 \sqrt{1 - \frac{1}{4\, Q^2}}} \tag{3.11}$$

$$D = \frac{K^b}{C}\, f\, T \frac{Q^b\, \ddot{x}_m^b}{\omega_0^{2b}\left(1 - \frac{1}{4\, Q^2}\right)^{b/2}} = \frac{K^b}{C}\, f_0 \sqrt{1 - \frac{1}{2\, Q^2}}\; T \frac{Q^b\, \ddot{x}_m^b}{\omega_0^{2b}\left(1 - \frac{1}{4\, Q^2}\right)^{b/2}}$$

$$D \approx \frac{K^b}{C}\, f_0\, T \frac{Q^b\, \ddot{x}_m^b}{\omega_0^{2b}} \tag{3.12}$$

NOTE: *Precautions must be taken if we wish to carry out a fatigue test using a vibration whose frequency is equal to the resonance frequency. As the frequency f_0 varies when the part is damaged by fatigue (f_0 decreases in general); it is not easy to interpret the results of the tests by calculation, the amplitude of the applied stress being very sensitive to the value of the Q factor (or of the transfer function in the vicinity of resonance). It is therefore necessary to frequently verify the value of this frequency and consequently to readjust the frequency of the excitation. Another possibility could be to choose $f = \frac{f_0}{2}$, but the duration of test would be then much longer.*

If $f \ll f_0$

$$\left|z_m\right| \approx \frac{\ddot{x}_m}{\omega_0^2}$$

$$D \approx \frac{K^b}{C} f\ T \frac{\ddot{x}_m^b}{\omega_0^{2b}} \qquad [3.13]$$

In this range, the damage D is independent of Q.

If $\frac{f}{f_0} \to \infty$, $z_m \to 0$ and $D \to 0$.

At low frequencies, for $f_0 \ll f$, the damage D can be written, by making $\left(\frac{f}{f_0}\right)^{2b}$ a factor in the denominator,

$$D = \frac{K^b}{C} f\ T \frac{\ddot{x}_m^b}{(2\pi)^{2b} f_0^{2b} \frac{f^{2b}}{f_0^{2b}} \left\{\left[\left(\frac{f_0}{f}\right)^2 - 1\right]^2 + \left(\frac{f_0}{Q f}\right)^2\right\}^{b/2}} \qquad [3.14]$$

When $f_0 \to 0$,

$$D \to \frac{K^b}{C} \frac{T\ \ddot{x}_m^b}{(2\pi)^{2b} f^{2b-1}} \qquad [3.15]$$

For f_0 small, the damage D is independent of f_0 and Q.

The fatigue damage spectrum (FDS) is the curve giving the variations of D versus f_0, for given values of K, C, Q and b.

Example 3.1.

$\ddot{x}_m = 10 \text{ m s}^{-2}$

$f_{excit.} = 500 \text{ Hz}$

$T = 3600 \text{ s}$

$K = 6.3\ 10^{10} \text{ Pa m}^{-1}$

$C = 10^{80}$ (S.I.)

$b = 8$

$$D \approx \frac{\left(6.3\ 10^{10}\right)^8}{10^{80}} \frac{3600\ 10^8}{(2\,\pi)^{16}\ 500^{16-1}} \approx 4.96\ 10^{-36}$$

Example 3.2.

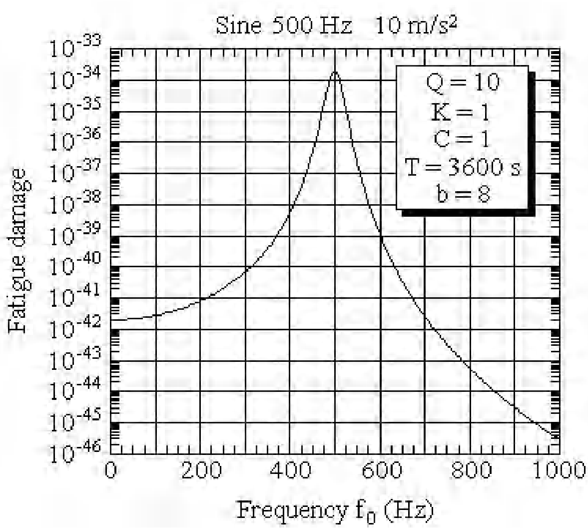


Figure 3.4. *FDS of a sinusoidal vibration*

3.3. Fatigue damage spectrum of a periodic signal

A periodic excitation can be expressed in the form of a sum of several sinusoids:

$$\ddot{x}(t) = \sum_i \ddot{x}_{m_i} \sin\left(\Omega_i + \varphi_i\right) \tag{3.16}$$

$$z_m = \max \sum_i \frac{\ddot{x}_{m_i} \sin\left(\Omega_i\, t + \varphi_i\right)}{\omega_0^2 \sqrt{\left[1 - \left(\frac{f_i}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f_i}{f_0}\right)^2}} \tag{3.17}$$

and

$$D = \frac{K^b}{C}\, f\, T\, z_m^b \tag{3.18}$$

3.4. General expression for the damage

In the same way as for the extreme response spectrum, with the same notations, a more general relationship appropriate for an excitation defined by an acceleration, a velocity or a displacement can be expressed:

$$D = \frac{K^b}{C}\, f_0\, T\, E_m^b\, \omega_0^{b(\alpha-2)} \frac{h^{\alpha b+1}}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \tag{3.19}$$

3.5. Fatigue damage with other assumptions on the S–N curve

It has been assumed up to now that the S–N curve was comparable to a straight line on a logarithmic scale, and that the influence of the fatigue limit (when it existed) could be disregarded.

3.5.1. *Taking account of fatigue limit*

Let us set σ_D as the fatigue limit. After calculation of the relative response displacement $z(t)$, we proceed as previously for the values z_m higher than $\frac{\sigma_D}{K}$ and we disregard the other values which, by definition, do not affect the damage.

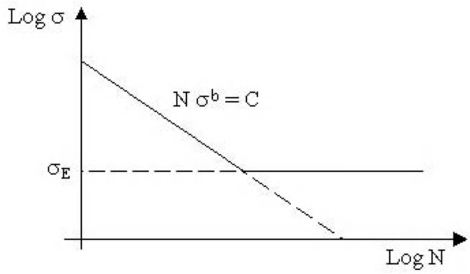


Figure 3.5. *S–N curve with fatigue limit*

3.5.2. *Cases where the S–N curve is approximated by a straight line in log–lin scales*

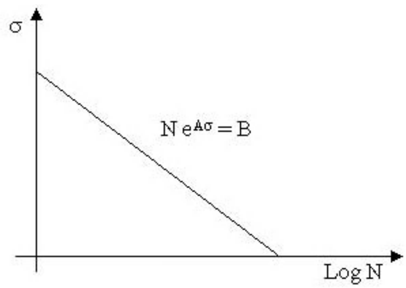


Figure 3.6. *Linear S–N curve in log–lin scales*

In this case, the S–N curve can be represented by the relationship

$$N \, e^{A \, \sigma} = B \tag{3.20}$$

yielding

$$D = \sum_i \frac{n_i}{N_i} = \frac{f \, T}{B} \, e^{A \, \sigma} \tag{3.21}$$

where the maximum stress σ_m is related to the relative displacement by $\sigma_m = K \, z_m$ and where

$$z_m = \frac{|\ddot{x}_m|}{4\,\pi^2\,f_0^2\,\sqrt{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2+\left(\frac{f}{Q\,f_0}\right)^2}} \qquad [3.22]$$

If, along these axes, the S–N curve has a fatigue limit stress σ_D, the calculation of the fatigue damage will be carried out, as in the preceding section, by removing the values of z_m lower than $\frac{\sigma_D}{K}$.

3.5.3. *Comparison of the damage when the S–N curves are linear in either log–log or log–lin scales*

Consider a S–N curve with two known points: (σ_1, N_1) and (σ_2, N_2).

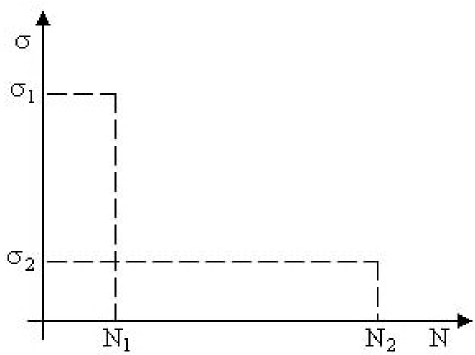


Figure 3.7. *Points of definition of a S–N curve*

If we represent the S–N curve by a straight line using logarithmic scales, we have a relationship of the form:

$$N\,\sigma^b = C$$

$$N_1\,\sigma_1^b = N_2\,\sigma_2^b \qquad [3.23]$$

yielding

$$b = \frac{\ln\left(N_2/N_1\right)}{\ln\left(\sigma_1/\sigma_2\right)} \qquad [3.24]$$

and

$$C = N_1\ \sigma_1^{\ln\left(N_2/N_1\right)/\ln\left(\sigma_1/\sigma_2\right)} \qquad [3.25]$$

If the S–N curve is represented by a straight line using log–lin scales, it follows a law of the form:

$$N\ e^{A\ \sigma} = B$$

yielding

$$N_1\ e^{A\ \sigma_1} = N_2\ e^{A\ \sigma_2} \qquad [3.26]$$

$$A = \frac{\ln\left(N_2/N_1\right)}{\sigma_1 - \sigma_2} \qquad [3.27]$$

and

$$B = N_1\ e^{\sigma_1\ \ln\left(N_2/N_1\right)/\left(\sigma_1-\sigma_2\right)} \qquad [3.28]$$

For N given, let us compare the fatigue–failure stress calculated on the basis of these two expressions; giving us $N_{\text{log, log}} > N_{\text{log, lin}}$ if

$$\frac{C}{\sigma^b} > \frac{B}{e^{A\ \sigma}} \qquad [3.29]$$

$$C\ e^{A\ \sigma} > B\ \sigma^b \qquad [3.30]$$

or if

$$e^{\sigma\ \ln\left(N_2/N_1\right)/\left(\sigma_1-\sigma_2\right)}\ \sigma_1^{\ln\left(N_2/N_1\right)/\ln\left(\sigma_1/\sigma_2\right)} > e^{\sigma_1/\left(\sigma_1-\sigma_2\right)}\ \sigma^{1/\ln\left(\sigma_1/\sigma_2\right)}$$

$$\frac{\sigma - \sigma_1}{\ln\left(\sigma/\sigma_1\right)} > \frac{\sigma_1 - \sigma_2}{\ln\left(\sigma_1/\sigma_2\right)} \qquad [3.31]$$

Example 3.3.

$\sigma_1 = 6$ and $\sigma_2 = 2$ (arbitrary unit system for this example).

$$\frac{\sigma - 6}{\ln(\sigma/6)} > \frac{4}{\ln 3} = 3.6410$$

For $\sigma = 8$, the first member is equal to 6.95 and, for $\sigma = 1$, to 2.79. There is equality for $\sigma = 2$. In fact, for any stress σ greater than σ_2, we have $N_{\text{log, log}} > N_{\text{log, lin}}$. Thus, the calculation carried out using the (log, log) assumption led to lifespans longer than the (log, lin) assumption. An interesting case is that for which σ_2 is equal to the fatigue limit. In this case, the inequality [3.29] is always seen to be verified.

3.6. Fatigue damage generated by a swept sine vibration on a single-degree-of-freedom linear system

3.6.1. *General case*

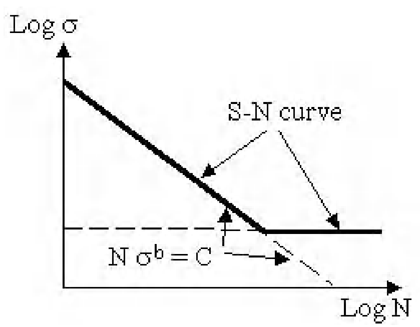


Figure 3.8. *Representation of the S–N curve by Basquin's law*

If we adopt Miner's rule and Basquin's representation ($N\,\sigma^b = C$) to describe the S–N curve, the fatigue damage *D* is expressed:

$$D = \sum_i \frac{n_i}{N_i} = \int_0^{t_b} \frac{dn}{N} \qquad [3.32]$$

with

$$N = \frac{C}{\sigma_{max}^b} \qquad [3.33]$$

(number of cycles to the failure at level σ). From the stress–relative displacement relationship

$$\sigma_{max} = K\ z_m \qquad [3.34]$$

and if t_b is the time at the end of sweep, $dn = f(t)\ dt$ (number of cycles during dt), $f(t)$ being the instantaneous frequency of the sinusoid at time t, then:

$$D = \frac{K^b}{C} \int_0^{t_b} f(t)\, z_m^b\ dt \qquad [3.35]$$

where z_m is the maximum response displacement (function of f), or

$$D = \frac{K^b}{C} \int_0^{t_b} f(t) \left[\left| \ell_m \right| H(f) \right]^b dt \qquad [3.36]$$

$H(f)$ being the transfer function of the system. If we can assume that the sweep rate is rather low, so that the response reaches a high percentage of the response to a steady state excitation (99 % for example) [CUR 71], the transfer function H of a single-degree-of-freedom system can be written:

$$H = \frac{1}{\sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f}{f_0}\right)^2}} \qquad [3.37]$$

yielding:

$$D = \frac{K^b}{C} \int_0^{t_b} f(t) \frac{|\ell_m|^b}{\left\{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f}{f_0}\right)^2\right\}^{b/2}} dt \qquad [3.38]$$

3.6.2. *Linear sweep*

3.6.2.1. *General case*

By hypothesis, the frequency f varies according to a law of the form $f = \alpha\, t + \beta$ when f_1 is the initial sweep frequency (at $t = 0$) and f_2 the final frequency (for $t = t_b$), $\alpha = \frac{f_2 - f_1}{t_b}$ and $\beta = f_1$ (Volume 1). Let $h = \frac{f}{f_0}$, then

$$dh = \frac{df}{f_0} = \frac{\alpha\, dt}{f_0} = \frac{f_2 - f_1}{f_0\, t_b} dt \qquad [3.39]$$

$$dt = \frac{f_0\, t_b}{f_2 - f_1} dh \qquad [3.40]$$

$$D = \frac{K^b}{C} \int_{h_1}^{h_2} \frac{f_0\, h\, |\ell_m|^b}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \frac{f_0\, t_b}{f_2 - f_1} dh \qquad [3.41]$$

when $h_1 = \frac{f_1}{f_0}$ and $h_2 = \frac{f_2}{f_0}$

$$D = \frac{K^b}{C} \frac{f_0^2\, t_b}{f_2 - f_1} \int_{h_1}^{h_2} \frac{h\, |\ell_m|^b}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} dh \qquad [3.42]$$

3.6.2.2. *Linear sweep at constant acceleration*

Case of a single level

$$\left|\ell_m\right| = \frac{\ddot{x}_m}{\omega_0^2} = \frac{\ddot{x}_m}{4\pi^2 f_0^2}$$

$$D = \frac{K^b}{C} \frac{f_0^2\, t_b\, \ddot{x}_m^b}{\left(4\pi^2 f_0^2\right)^b \left(f_2 - f_1\right)} \int_{h_1}^{h_2} \frac{h\, dh}{\left[\left(1-h^2\right)^2 + h^2/Q^2\right]^{b/2}} \qquad [3.43]$$

Swept sine excitation on several levels

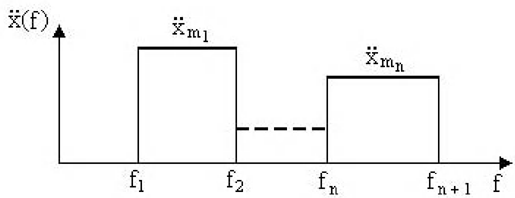


Figure 3.9. *Swept sine excitation at several levels*

If the swept sine excitation includes several levels with constant acceleration in the frequency band f_1, f_{n+1}, the damage D is calculated by summing the partial damages.

For the example in Figure 3.9, this would yield:

$$D = \frac{K^b}{C} \frac{t_b\, f_0^2}{\left(4\pi^2 f_0^2\right)^b \left(f_{n+1} - f_1\right)} \left\{ \ddot{x}_{m1}^b \int_{h_1}^{h_2} \frac{h\, dh}{\left[\left(1-h^2\right)^2 + \dfrac{h^2}{Q^2}\right]^{b/2}} + \cdots \right.$$

$$\left. + \ddot{x}_{mn}^b \int_{h_n}^{h_{n+1}} \frac{h\, dh}{\left[\left(1-h^2\right)^2 + \dfrac{h^2}{Q^2}\right]^{b/2}} \right\} \qquad [3.44]$$

Example 3.4.

Linear swept sine excitation at constant acceleration:

20 to 100 Hz: 5 m s^{-2}

100 to 500 Hz: 10 m s^{-2}

500 to 1000 Hz: 20 m s^{-2}

$t_b = 1200$ s

b = 10

Q = 10

K = 1

C = 1

Spectrum plotted from 1 Hz to 2000 Hz in steps of 5 Hz (Figure 3.10).

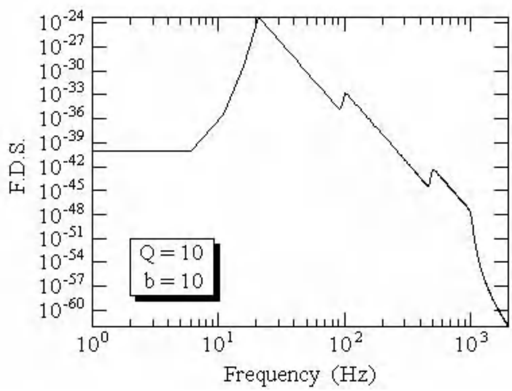


Figure 3.10. *FDS of a linear swept sine vibration at constant acceleration*

Approximated formulations

Because of its simplicity, the linear one-degree-of-freedom system is frequently used to evaluate and understand the effect of vibrations and shocks. It is often used as a model, for example, to choose the sweeping rate of a swept sine (Volume 1), to choose the minimum number of points of a PSD calculated from the of a specimen (Volume 3), to compare the severity of several vibrations and shocks (shock response spectrum, extreme response spectrum, fatigue damage spectrum etc.).

The response of the one degree-of-freedom system is primarily due to the frequency content of the excitation in the interval Δf between the half power points. It is particularly the case of fatigue damage created by a swept sine type vibration.

The relations expressing the damage include an integral that cannot be analytically calculated except for special cases. It can however be shown [MOR 65] [REE 60] that, when the frequency sweep includes a resonance frequency, the damage is mainly created by the cycles applied between the half-power points. We will consider this point later on. The error introduced by disregarding the other cycles does not exceed 3 per cent [MOR 65].

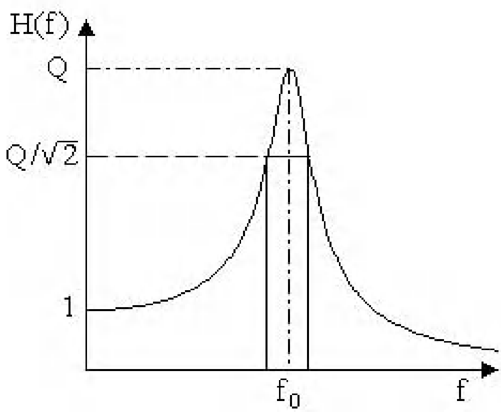


Figure 3.11. *Interval between half-power points*

It should be recalled that these points, located on either side of the natural frequency (Figure 3.11) are defined as the intersection of the curve representing the transfer function $H(f)$ and the horizontal $Q/\sqrt{2}$. Their values along the x-axis are $\sqrt{1-2\xi^2-2\xi\sqrt{1-\xi^2}}$ and $\sqrt{1-2\xi^2+2\xi\sqrt{1-\xi^2}}$ respectively.

It can then be shown [MOR 65] that:

$$I = \int_{h_1}^{h_2} \frac{h\,dh}{\left[\left(1-h^2\right)^2+\left(2\,\xi\,h\right)^2\right]^{b/2}} \approx \frac{\pi}{2} Q^{b-1} b^{-1/\sqrt{\pi}} \qquad [3.45]$$

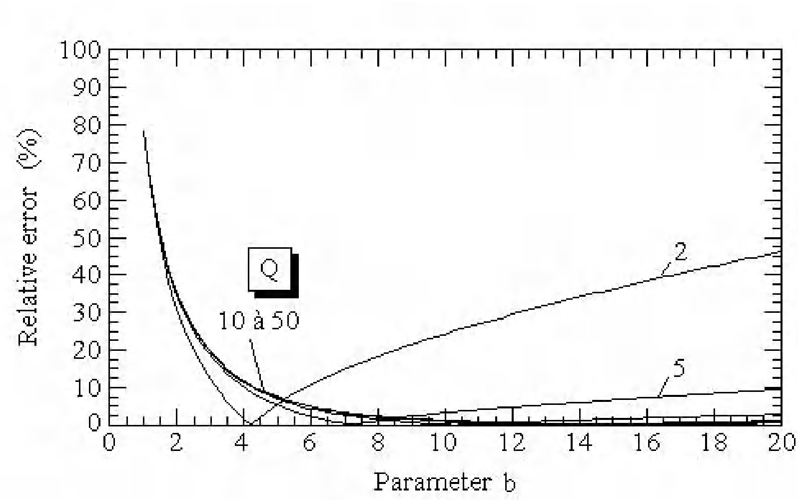


Figure 3.12. *Relative error related to the use of approximated relation [3.45]*

where

$$\left.\begin{aligned} h_1 &= \sqrt{1 - 2\,\xi^2 - 2\,\xi\,\sqrt{1-\xi^2}} \\ h_2 &= \sqrt{1 - 2\,\xi^2 + 2\,\xi\,\sqrt{1-\xi^2}} \end{aligned}\right\} \qquad [3.46]$$

This approximation is very good for $4 < b < 30$ and $\xi < 0.1$. Figure 3.12 shows the variations of the relative error $100 \left| \frac{I_{exact} - I_{approximated}}{I_{exact}} \right|$ versus parameter b, for different values of Q. This is used to determine the damage:

$$D = \frac{K^b}{C} \frac{f_0^2\, t_b\, \ddot{x}_m^b}{\left(4\,\pi^2\, f_0^2\right)^b \left(f_2 - f_1\right)} \frac{\pi}{2} Q^{b-1}\, b^{-1/\sqrt{\pi}} \qquad [3.47]$$

and the time until failure ($D = 1$):

$$t_b = \frac{2}{\pi} \frac{C}{K^b} \frac{\left(4\,\pi^2\, f_0^2\right)^b \left(f_2 - f_1\right)}{f_0^2\, \ddot{x}_m^b\, Q^{b-1}\, b^{-1/\sqrt{\pi}}} \qquad [3.48]$$

Another simplified method

M. Gertel [GER 61], returning to the fact that fatigue damage is mainly due to cycles between the half-power points (providing that the range covered by the sweep includes these points), defines a reduced transmissibility curve.

To do so (Figure 3.13), he plots the transfer function in the interval Δf around the resonance frequency, i.e. between the x-axis h_1 and h_2 [3.46], while placing $\frac{\left(h - h_1\right)}{\Delta f} = \frac{\left(h - h_1\right) Q}{f_0}$ (Δf= interval between the half-power points) on the x-axis

and the function $\frac{H(h)}{Q} = \frac{2\,\xi\,\sqrt{1-\xi^2}}{\sqrt{\left(1-h^2\right)^2 + \left(2\,\xi\,h\right)^2}}$ on the y-axis.

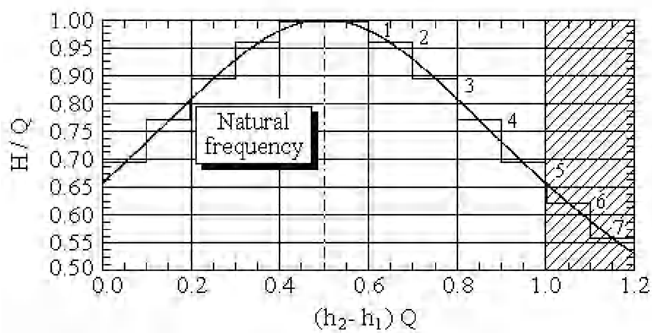


Figure 3.13. *Approximation of the transfer function between the half-power points [GER 61]*

All the transfer curves are very similar for these normalized notations as long as $Q > 5$ in this frequency interval. The interval $\frac{\Delta f . Q}{f_0}$ is then divided into 10 equal parts, grouped in pairs considering the curve to be roughly symmetrical in relation to the vertical of the natural frequency. The amplitudes of the five levels thus defined on these axes are shown in Table 3.1.

Level	H/Q
1	0.996
2	0.959
3	0.895
4	0.820
5	0.744

Table 3.1. *Values of the transfer function in the interval between the half-power points*

The fatigue damage can be calculated as follows:

$$D = \sum_i \frac{n_i}{N_i} = \sum_i \frac{n_i \, \sigma_i^b}{C} \tag{3.49}$$

$$D = \frac{K^b}{C} \sum_i \frac{\Delta N}{5} \, z_{m\,i}^b \tag{3.50}$$

where ΔN is the number of cycles completed in Δf. Here again, it is assumed that the sweep completely covers Δf.

$$D = \frac{K^b}{C} \frac{\Delta N}{5} \frac{\sum_i \left(4\,\pi^2 \, f_0^2 \, z_{m\,i}\right)^b}{\left(4\,\pi^2 f_0^2\right)^b} \tag{3.51}$$

where $4\,\pi^2 \, f_0^2 \, z_{mi}$ is given to each level by the values of H/Q from the above table, multiplied by the chosen Q factor and by the amplitude $\ddot{x}_m$ of the swept sine.

$$D = \frac{K^b}{C} \frac{\Delta N}{5\left(4\pi^2 f_0^2\right)^b} \sum \left[Q \left(\frac{H}{Q}\right) \ddot{x}_m \right]^b \tag{3.52}$$

$$D = \frac{K^b}{C} \frac{\Delta N}{5\left(4\,\pi^2 \, f_0^2\right)^b} \left[Q^b \, \ddot{x}_m^b \left(0.996^b + 0.959^b + 0.895^b + 0.82^b + 0.744^b\right)\right] \tag{3.53}$$

Example 3.5.

The simple system, with a natural frequency $f_0 = 100$ Hz and Q factor of 10, is subjected to a linear sweep between 10 Hz and 500 Hz with an amplitude $\ddot{x}_m = 10\ \text{m s}^{-2}$ and a duration of 30 minutes. It is assumed that the material used has a parameter b equal to 8 and, to simplify, that C = 1 and K = 1.

The number of cycles carried out in Δf is equal to:

$$\Delta N = \frac{f_0^2\ t_b}{Q\left(f_2 - f_1\right)} = \frac{100^2\ 1800}{10\,(500 - 10)}$$

$$\Delta N \approx 3.67\ 10^3 \text{ cycles}$$

$$D = \frac{3.67\ 10^3}{5\left(4\,\pi^2\,10^4\right)^8}\ 10^8\ 10^8\left(0.996^8 + 0.959^8 + 0.895^8 + 0.82^8 + 0.744^8\right)$$

$$D \approx 1.24\ 10^{-26}\left[0.97 + 0.72 + 0.41 + 0.20 + 0.09\right]$$

$$D \approx 2.98\ 10^{-26}$$

It can be seen that the last term in the brackets is already negligible, which justifies eliminating the subsequent ones (corresponding to levels located beyond the half-power points). Calculation of D from the same data using the simplified relationship [3.47] yields:

$$D = 3.03\ 10^{-26}$$

whereas calculating D by numerical integration from [3.43] gives:

$$D = 3.1\ 10^{-26} \text{ (for 300 integration points).}$$

Example 3.6.

A logarithmic swept sine test with amplitude of 10 m s^{-2}, lasting t_b = 10 minutes between 10 Hz and 1,000 Hz, applied to a linear mechanical one degree-of-freedom system with a natural frequency of 500 Hz and Q = 10.

Fatigue damage created for this system is equal to $3.6 \cdot 10^{-37}$ for b = 8 (b being the Basquin law exponent).

The interval between half-power points has the following value:

$$\Delta f = \frac{f_0}{Q} = \frac{500}{10} = 50 \text{ Hz}$$

The time spent between half-power points is equal to:

$$600 \frac{\ln \frac{525}{475}}{\ln \frac{1000}{10}} = 13{,}04 \text{ s}$$

We now consider a logarithmic swept sine of amplitude 10 m s^{-2} between 475 Hz and 525 Hz, lasting 13.04 s. It creates damage equal to $3.45 \cdot 10^{-37}$ in the same system, 95.83 % of the previous damage.

Fatigue damage spectra are compared in Figure 3.14 between 400 Hz and 600 Hz.

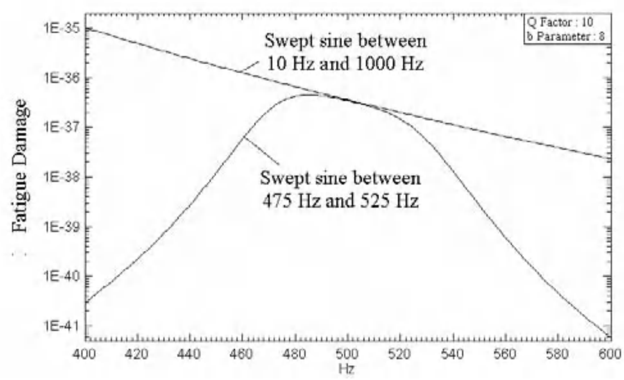


Figure 3.14. *Comparison of swept sine FDS between 10 Hz and 1000 Hz and between 475 Hz and 525 Hz (amplitude 10 m s^{-2}, Q = 10, b = 8)*

3.6.2.3. *Linear sweep at constant displacement*

$$\left|\ell_m\right| = \frac{\omega^2}{\omega_0^2} x_m$$

yielding

$$D = \frac{K^b}{C} \frac{f_0^2 \, t_b \, x_m^b}{f_2 - f_1} \int_{h_1}^{h_2} \frac{h^{2b+1} \, dh}{\left[\left(1 - h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \qquad [3.54]$$

Example 3.7.

Constant displacement

5 to 10 Hz: ±0.050 m $\qquad t_b = 1200$ s

10 to 50 Hz: ±0.001 m $\qquad Q = 10$

$b = 10$

$K = 1$

$C = 1$

Spectrum plotted between 1 Hz and 500 Hz in steps of 1 Hz (Figure 3.15)

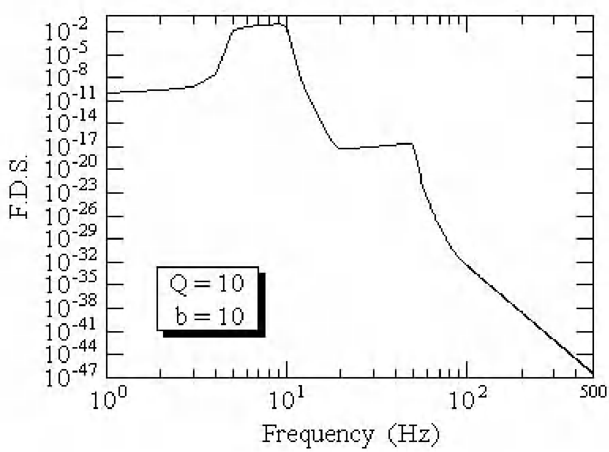


Figure 3.15. *FDS of a linear swept sine vibration at constant displacement*

NOTE: *When, during a test, part of the frequency band (in general the very low frequencies) is characterized by a constant displacement and the other part is characterized by a constant acceleration, the total damage is calculated for each resonance frequency* f_0*, by summing the damage created by each type of sweep, taking into account the time spent in each frequency band.*

3.6.3. *Logarithmic sweep*

3.6.3.1. *General case*

This sweep was defined by $f = f_1 \, e^{t/T_1}$. From [3.38],

$$D = \frac{K^b}{C} \int_0^{t_b} \frac{f(t) \left|\ell_m\right|^b \, dt}{\left\{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \frac{h^2}{Q^2}\right\}^{b/2}} \qquad [3.55]$$

Let $h = \dfrac{f}{f_0}$

$$dh = \frac{df}{f_0} = \frac{f_1}{T_1 \, f_0} e^{\frac{t}{T_1}} \, dt = \frac{f}{T_1 \, f_0} dt = \frac{h}{T_1} dt$$

$$D = \frac{K^b}{C} f_0 \, T_1 \int_{h_1}^{h_2} \frac{\left|\ell_m\right|^b \, dh}{\left[\left(1 - h^2\right)^2 + h^2/Q^2\right]^{b/2}} \qquad [3.56]$$

where $h_1 = \dfrac{f_1}{f_0}$ and $h_2 = \dfrac{f_2}{f_0}$.

3.6.3.2. *Logarithmic sweep at constant acceleration*

In this case, $\left|\ell_m\right| = \dfrac{\ddot{x}_m}{4 \pi^2 f_0^2}$, yielding:

$$D = \frac{K^b}{C} f_0 \frac{T_1 \ddot{x}_m^b}{\left(4\pi^2 f_0^2\right)^b} \int_{h_1}^{h_2} \frac{dh}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \qquad [3.57]$$

Example 3.8.

Same data as for the above example of linear sweep vibration at constant acceleration.

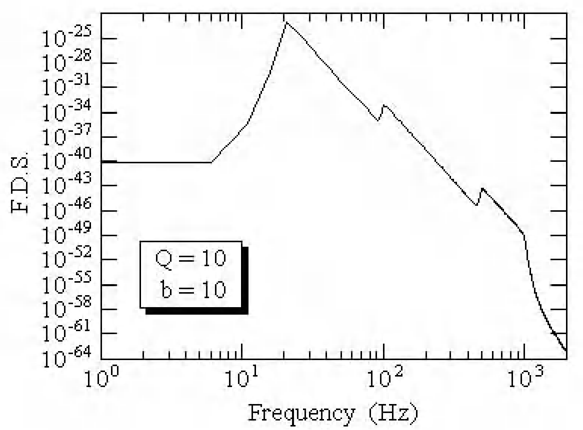


Figure 3.16. *FDS of a logarithmic swept sine vibration at constant acceleration*

3.6.3.3. *Logarithmic sweep at constant displacement*

We have

$$\left|\ell_m\right| = \frac{\Omega^2}{\omega_0^2} x_m = \frac{f^2}{f_0^2} x_m \text{, yielding}$$

$$D = \frac{K^b}{C} f_0 T_1 x_m^b \int_{h_1}^{h_2} \frac{h^{2b} dh}{\left[\left(1-h^2\right)^2 + h^2 / Q^2\right]^{b/2}} \qquad [3.58]$$

Example 3.9.

Same data as for the above example of linear sweep vibration at constant displacement.

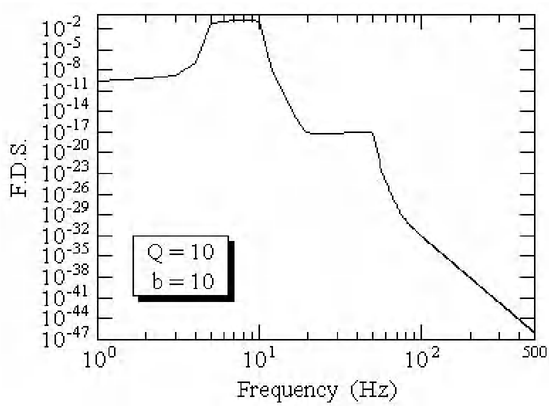


Figure 3.17. *FDS of a logarithmic swept sine vibration at constant displacement*

NOTE: *As above, the damage at a given natural frequency* f_0 *resulting from the application of a test defined by a swept sine excitation, swept partially at constant displacement and partially at constant acceleration, is equal to the sum of each of the two damages created separately by these two sweeps.*

Example 3.10.

For a test by a (logarithmic) swept sine excitation defined as follows:

1 Hz to 10 Hz: ±1 mm

10 Hz to 1000 Hz: ±4 m s^{-2}

Duration t = 1 hr

It should be noted that, although this is not an absolute rule, in such a case, the general arrangement is to have equal accelerations at the common frequency, i.e. 10 Hz in the example:

$$\left|\ddot{x}_m\right| = 4\,\pi^2\,f^2\,x_m$$

$$\left|\ddot{x}_m\right| = 4\ \pi^2\ 10^2\ 10^{-3}\ \text{m s}^{-2}$$

$$\left|\ddot{x}_m\right| \approx 4\ \text{m s}^{-2}$$

The time spent between 1 Hz and 10 Hz is:

$$t_1 = 3600\ \frac{\ln \frac{10}{1}}{\ln \frac{1000}{1}}\ \text{s}$$

$$t_1 = 1200\ \text{s}$$

3.6.4. *Hyperbolic sweep*

3.6.4.1. *General case*

As already seen, frequency varies in this case in relation to the time, according to the law $\frac{1}{f_1} - \frac{1}{f} = a\ t$, the constant a being such that, when $t = t_b$, $f = f_2$.

This yields $a = \frac{f_2 - f_1}{f_1\ f_2\ t_b}$. Let $h = \frac{f}{f_0}$; so that $dh = \frac{df}{f_0}$, or, since $f = \frac{f_1}{1 - a\ f_1\ t}$:

$$dh = \frac{f_1^2\ a\ dt}{f_0\ \left(1 - a\ f_1\ t\right)^2} \qquad [3.59]$$

or again:

$$dh = \frac{a\ f^2}{f_0}\ dt = a\ f_0\ h^2\ dt \qquad [3.60]$$

yielding:

$$D = \frac{K^b}{C\ a} \int_{h_1}^{h_2} \frac{\left|\ell_m\right|^b dh}{h\left[\left(1 - h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \qquad [3.61]$$

$$D = \frac{K^b}{C} \frac{f_1 f_2 t_b}{f_2 - f_1} \int_{h_1}^{h_2} \frac{\left|\ell_m\right|^b dh}{h\left[\left(1 - h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \tag{3.62}$$

Example 3.11.

Same data as for the case of linear sweep vibration.

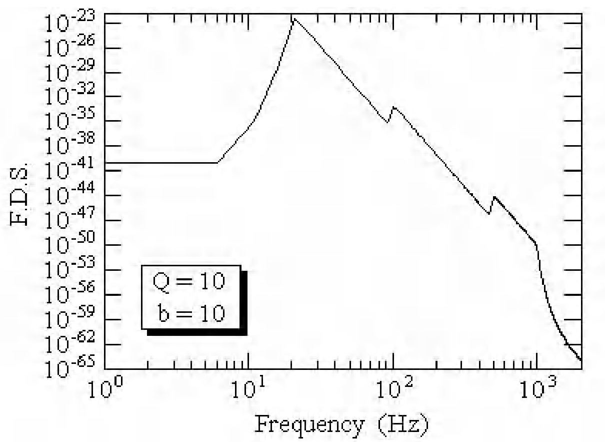


Figure 3.18. *FDS of a hyperbolic swept sine vibration at constant acceleration*

3.6.4.2. *Hyperbolic sweep at constant acceleration*

$$\left|\ell_m\right| = \frac{\ddot{x}_m}{4 \pi^2 f_0^2}$$

yielding:

$$D = \frac{K^b}{C} \frac{f_1 f_2 t_b}{f_2 - f_1} \left(\frac{\ddot{x}_m}{4 \pi^2 f_0^2}\right)^b \int_{h_1}^{h_2} \frac{dh}{h\left[\left(1 - h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \tag{3.63}$$

3.6.4.3. *Hyperbolic sweep at constant displacement*

$$\left|\ell_m\right| = \frac{\Omega^2}{\omega_0^2} x_m = \frac{f^2}{f_0^2} x_m$$

$$D = \frac{K^b}{C} \frac{f_1 \; f_2 \; t_b \; x_m^b}{f_2 - f_1} \int_{h_1}^{h_2} \frac{h^{2b-1} \, dh}{\left[\left(1 - h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \qquad [3.64]$$

Example 3.12.

Same data as for the case of linear sweep vibration.

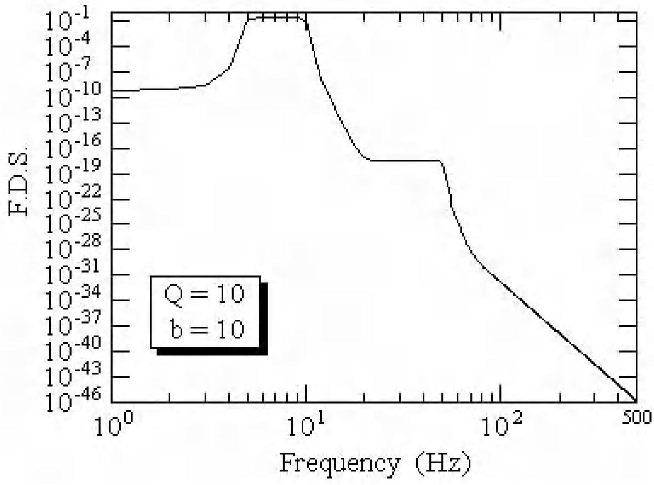


Figure 3.19. *FDS of a hyperbolic swept sine vibration at constant displacement*

3.6.5. *General expressions for fatigue damage*

In a general way, using the notations in section 1.3.1.4, the fatigue damage can be written in the form

$$D = \frac{K^b}{C} f_0 \, t_b \, E_m^b \, \omega_0^{b\,(a-2)} \int_{h_1}^{h_2} \frac{M(h)\, h^{a\,b-1}}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \, dh \qquad [3.65]$$

the function $M(h)$ being given in Table 3.2.

Sweep	$\mathbf{M(h)}$
Linear	$\frac{h^2}{h_2 - h_1}$
Exponential	$\frac{h}{\ln\left(h_2/h_1\right)}$
Hyperbolic	$\frac{h_1\, h_2}{h_2 - h_1}$

Table 3.2. *Expressions for functions characterizing the sweep*

For a swept sine excitation composed of n levels:

$$D = \frac{K^b}{C} f_0 \, t_b \, \omega_0^{b\,(a-2)} \sum_{i=1}^{n} E_{m\,i}^b \int_{h_i}^{h_{i+1}} \frac{M(h)\, h^{a\,b-1}}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{b/2}} \, dh \qquad [3.66]$$

where $M(h)$ is defined as in the above table, indices 1 and 2 being replaced by 1 and $n+1$.

NOTE: *All the above expressions for damage suppose that the sweep is sufficiently slow for the response to reach a very strong percentage of the response in steady state mode. Were that not the case, the same relations could be used by replacing* *the input acceleration* $\ddot{x}_m$ *by* $G\,\ddot{x}_m$ *(G being defined in section 9.2. Volume 1) or the displacement* x_m *by* $G\,x_m$ *(or, more generally,* ℓ_m *by* $G\,\ell_m$*).*

3.7. Reduction of test time

3.7.1. *Fatigue damage equivalence in the case of a linear system*

The above expressions show that, as much in pure sinusoidal mode as in all the other sweep modes, the damage D is proportional to:

– the duration of the test;

– the excitation amplitude $\ddot{x}_m$ or x_m to the power b.

It follows that, fatigue damage being equal, the test time can be decreased by increasing the levels according to following rules [MOR 76] [SPE 62]:

– for accelerations:

$$\ddot{x}_{m\,reduced} = \ddot{x}_{m\,\mathrm{real}} \left(\frac{T_{real}}{T_{reduced}} \right)^{1/b} ; \tag{3.67}$$

– for displacements:

$$x_{m\,reduced} = x_{m\,\mathrm{real}} \left(\frac{T_{real}}{T_{reduced}} \right)^{1/b} . \tag{3.68}$$

The ratio $E = \dfrac{\ddot{x}_{\mathrm{m\ reduced}}}{\ddot{x}_{\mathrm{m\ real}}}$ is called the "*exaggeration factor*" and $\left(\dfrac{T_{\mathrm{real}}}{T_{\mathrm{reduced}}} \right)$, the "*time reduction factor*".

For swept sine vibrations, it is important to check that the reduction of time does not lead to too fast a sweep, which as a consequence would have the effect of the response to a resonance to a value below the steady state response (or to an excessive increase in levels which would cause the equipment to work in a stress range too different from reality.

NOTE: *These results can be established directly from the ratio of stress / relative displacement proportionality*

$$\sigma_{max} = Const.\ z_m = Const.\ \ddot{x}_m \qquad [3.69]$$

σ_{max} = *maximum stress*

z_m = *maximum deformation*

$\ddot{x}_m$ = *excitation amplitude.*

and from Basquin's relationship:

$$N_{real}\ (Const.)^b\ \ddot{x}_{m\ real}^b = N_{reduced}\ (Const.)^b\ \ddot{x}_{m\ reduced}^b$$

Since the number of cycles N carried out is equal to the product $f\,T$ of the sinusoid frequency multiplied by the excitation duration, [3.67] can be easily found.

3.7.2. *Method based on fatigue damage equivalence according to Basquin's relationship*

There we take into account variation of material damping as a function of stress level according to Basquin's relationship.

From Basquin's relationship, we have

$$N_{reduced}\ \sigma_{max_{reduced}}^b = N_{real}\ \sigma_{max_{real}}^b$$

Knowing that $N = f\,T$ and by assuming that the stress is proportional to the relative displacement, this relationship is written

$$T_{reduced}\ z_{m_{reduced}}^b = T_{real}\ z_{m_{real}}^b \qquad [3.70]$$

It is assumed here that the specific damping energy is related to the stress by an equation with the form (Volume 1 [2.21]) [CUR 71]:

$$D = J\ \sigma^n \qquad [3.71]$$

where

J = constant of the material,

$n = 2$ for viscoelastic materials,

$n = 2.4$ for stress levels below 80 % of the fatigue limit,

$n = 8$ for stress levels higher than 80 % of the fatigue limit.

The total damping energy is related to specific energy by factors that depend on the geometry of the specimen and the stress distribution.

At a given resonance frequency, the Q factor is thus related to the relative response displacement z by an equation with the form:

$$Q = \alpha \, z^{2-n} \tag{3.72}$$

where α is a constant depending on the material, the geometry and the stress distribution. For a sinusoidal excitation at a frequency equal to the resonance frequency of the structure, the maximum response z_m is written:

$$z_m = \frac{Q \, \ddot{x}_m}{4\pi^2 \, f_0^2} \tag{3.73}$$

yielding $\sigma_{max} = \text{Const. } z_m = \text{Const. } Q \, \ddot{x}_m$

$$z_m = \text{Const. } \ddot{x}_m \, z_m^{2-n}$$

$$z_m^{n-1} = \text{Const. } \ddot{x}_m \tag{3.74}$$

combining [3.70] and [3.74], then

$$\frac{\ddot{x}_{m \text{ reduced}}}{\ddot{x}_{m \text{ real}}} = \left(\frac{T_{real}}{T_{reduced}} \right)^{\frac{n-1}{b}} \tag{3.75}$$

3.8. Notes on the design assumptions of the ERS and FDS

For a long time, it was proposed in the standards to reduce the test duration by increasing the vibratory levels starting from relation [3.67] and, as we shall see, for the random vibrations by [4.13]

$$\ddot{x}_{\text{rms reduced}} = \ddot{x}_{\text{rms real}} \left(\frac{T_{\text{real}}}{T_{\text{reduced}}} \right)^{\frac{1}{b}}$$

where the parameter b is generally *a priori* fixed at values close to 8 or 9.

The use of these relations implies the acceptance of almost all the design assumptions of the FDS. We saw in the preceding section that they arise directly from Basquin's law, and that it is implicitly assumed that stress σ is proportional to the deformation of the part considered, and is even proportional to the value of acceleration characterizing the applied vibration.

The calculation of the ERS and FDS is based on the response of a linear mechanical system with one degree of freedom with Q constant, which is the model of the shock response spectrum largely used today. The ERS is strictly the equivalent of the SRS for vibrations. This definition assumes that stress is proportional to relative displacement ($\sigma = K\, z$), which justifies the severity comparisons.

Moreover, the calculation of the FDS is carried out by considering that the S–N curve can be represented by Basquin's law $N\, \sigma^b = C$. The only additional assumption for the calculation of the FDS is that of the linear accumulation law of the damage.

This similarity of the assumptions is found in the results since by writing the equality of the damages (relations [3.19], [3.43], [3.57] of this volume or [4.41] of Volume 4) generated by two vibrations of comparable nature, but of different durations and amplitudes, expressions [3.67] and [4.13] are easily found.

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 4

Fatigue Damage Spectrum of a Random Vibration

4.1. Fatigue damage spectrum from the signal as function of time

Definition

The fatigue damage spectrum of a random vibration is obtained by plotting the variations of damage to a single-degree-of-freedom linear system versus its natural frequency f_0, for a given damping ratio ξ.

If the signal cannot be characterized by a power spectral density (if for example it is non-stationary or if it is not Gaussian), the fatigue damage can only be calculated by numerically determining the response of the one-degree-of-freedom system to the random vibration in question, and then by establishing a peak histogram of this response using a counting method (Volume 4).

We will assume that the fatigue damage is evaluated on the basis of the following assumptions:

– a single-degree-of-freedom linear system;

– an S–N curve represented by Basquin's relation ($N\ \sigma_p^b = C$);

– the peak stress being proportional to the maximum relative displacement of the system ($\sigma_p = K\ z_p$);

– the rainflow method used to count the response peaks (Volume 4, Chapter 3);

– Miner's rule used for damage accumulation (Volume 3, Chapter 2).

Let n_i be the number of half-cycles at amplitude z_{p_i}, the fatigue damage is then written (Volume 4, Chapter 4):

$$D = \frac{K^b}{2\,C} \sum_{i=1}^{m} n_i\, z_{p_i}^b$$

This calculation, repeated for several natural frequencies f_0 (and for ξ, b, K and C all given), makes it possible to plot the *fatigue damage spectrum* (FDS) $D(f_0)$ of the noise in question.

The spectrum obtained from a time history signal shows, in a deterministic way, the damage which would be created at each frequency f_0 by the sample signal. Even if the noise is stationary and ergodic, the spectrum calculated from another sample of the signal is different. On the basis of several samples, a mean spectrum $D\,(f_0, \xi)$ and its standard deviation $\sigma_D(f_0, \xi)$ could be determined. The fatigue damage estimate is particularly useful for vibrations of long duration. Except for particular cases where the necessary assumptions are not valid, it is preferable to calculate these statistical spectra by the method set out in the following section, which uses far less computing time.

If the vibrations are stationary and ergodic, it is possible to consider a single representative sample of shorter duration T and to calculate $D_T\left(f_0, \xi\right)$, then to estimate the damage over the total duration Θ by a rule of three:

$$D_\Theta = \frac{\Theta}{T} D_T \tag{4.1}$$

This method, which considers the signal as a function of time, can be used whatever the nature of the vibration (transitory, sinusoidal, random) and, for a random vibration, stationary or otherwise, ergodic or otherwise, with the proviso of one possible restriction relating to extrapolation on one duration Θ from D_T (risk of error related to the statistical aspect of the phenomenon).

Example 4.1.

Vibration measured on a truck; duration phase 10 hours.

Analysis conditions: Q = 10, b = 8, K = 1 and C = 1.

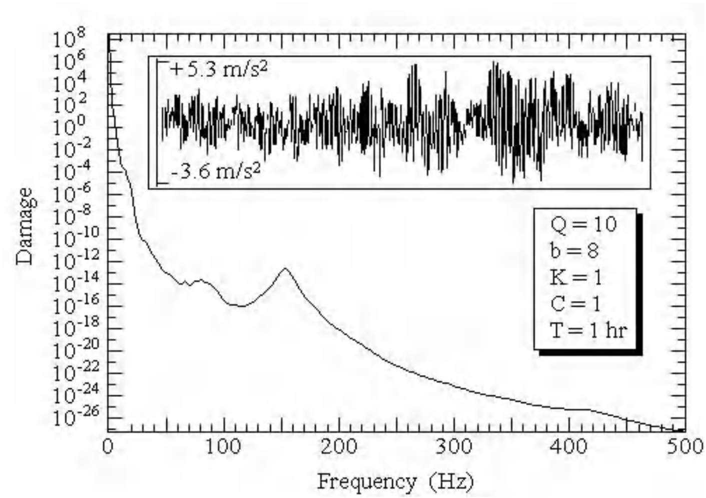


Figure 4.1. *FDS of a random vibration of a 'truck' calculated from the signal as a function of time*

4.2. Fatigue damage spectrum derived from a power spectral density

When random vibration is stationary and Gaussian, we know that we can obtain an analytical expression of the probability density of response peaks of the one-degree-of-freedom system, avoiding longer counting to establish the peak histogram.

The process includes the following major steps (Figure 4.2):

– calculation of the PSD of the vibratory signal (Volume 3, Chapter 4);

– calculation of the rms values z_{rms}, $\dot{z}_{rms}$, $\ddot{z}_{rms}$ of the relative displacement, velocity and acceleration of the response of the single-degree-of-freedom system on the basis of the PSD (Volume 3, Chapter 8);

– calculation, from these quantities, of the mean frequency and the mean number of peaks per unit time (Volume 3, Chapters 5 and 6);

– calculation of the irregularity factor r of the response (Volume 3, Chapter 9);

– determination of the peak probability density of the response (Volume 3, Chapter 6);

$$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}} e^{-\frac{u^2}{2(1-r^2)}} + \frac{u\,r}{2} e^{-\frac{u^2}{2}} \left[1 + \operatorname{erf}\left(\frac{u\,r}{\sqrt{2(1-r^2)}}\right)\right] \qquad [4.2]$$

where $u = z_p / z_{rms}$.

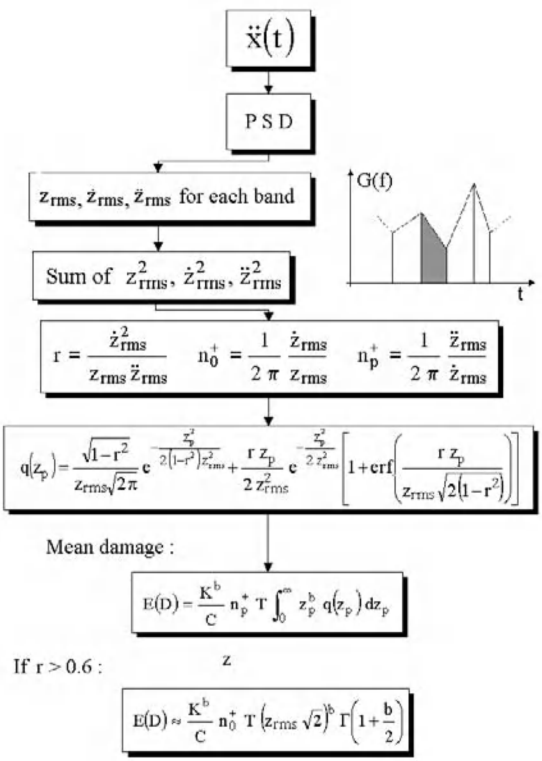


Figure 4.2. *Calculation principle of a Gaussian random vibration FDS*

This probability density $q(u)$ is the weighted sum of a Gaussian law and a Rayleigh law; its coefficients are a function of parameter r:

$$r = \frac{\int_0^\infty f^2\ G_z(f)\,df}{\sqrt{\int_0^\infty G_z(f)\,df\ \int_0^\infty f^4\ G_z(f)\,df}} = \frac{n_0^+}{n_p^+} \qquad [4.3]$$

n_0^+ is the mean frequency of the one-degree-of-freedom system response considered (average number of passings through zero with positive slope),

n_p^+ is the mean number of *maxima* per second,

$G_z(f)$ is the PSD of the relative response displacement,

erf( ) is the function of error, defined by:

$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\lambda^2}\,d\lambda \qquad [4.4]$$

Fatigue damage calculation is carried out from the expression (Volume 4, Chapter 4):

$$D = \frac{K^b}{C} n_p^+\ T\ z_{rms}^b \int_0^{+\infty} u^b\ q(u)\,du \qquad [4.5]$$

The mean damage inflicted on the system of natural frequency f_0 is then given by[1] (Volume 4, Chapter 4):

$$D = \frac{K^b}{C}\frac{n_p^+\ T}{z_{rms}} \int_0^{+\infty} z_p^b \left\{ \frac{\sqrt{1-r^2}}{\sqrt{2\pi}}\, e^{-\frac{z_b^2}{2(1-r^2)\,z_{rms}^2}} \right.$$

$$\left. + \frac{r\ z_p}{2\ z_{rms}}\, e^{-\frac{z_p^2}{2\,z_{rms}^2}} \left[1 + \operatorname{erf}\left(\frac{r\ z_p}{z_{rms}\ \sqrt{2(1-r^2)}} \right) \right] \right\} dz_p \qquad [4.6]$$

The fatigue damage spectrum $D(f_0)$ is obtained by varying f_0 for given values of ξ and b. For the severity comparisons of several vibrations and in order to write the specifications, the two constants K and C, in general unknown, are made equal to 1. The fatigue damage spectrum is then plotted by an approximate multiplying constant, which does not affect the results, as the vibrations being compared are supposedly applied to the same structure.

Example 4.2.

1. Random vibration defined by the following acceleration spectral density:

100 – 300 Hz:	5 $(m\ s^{-2})^2/Hz$	$\ddot{x}_{rms} = 69.28\ m\ s^{-2}$
300 – 600 Hz:	100 $(m\ s^{-2})^2/Hz$	$\dot{x}_{rms} = 3.61\ cm\ s^{-1}$
600 – 1000 Hz:	2 $(m\ s^{-2})^2/Hz$	$x_{rms} = 0.033\ mm$

Duration: 1 hour

The fatigue damage spectrum is plotted in Figure 4.3, for a mechanical system with a single-degree-of-freedom whose natural frequency f_0 varies between 5 Hz and 1500 Hz, with $Q = 10$. The parameters of material and structure are $b = 8$, $A = 10^9\ N\ m^{-3}$ ($C = A^b$), $K = 0.5 \times 10^{12}\ N\ m^{-3}$.

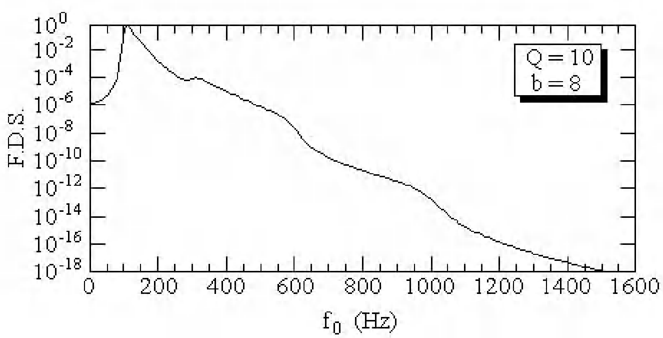


Figure 4.3. *FDS calculated from a PSD*

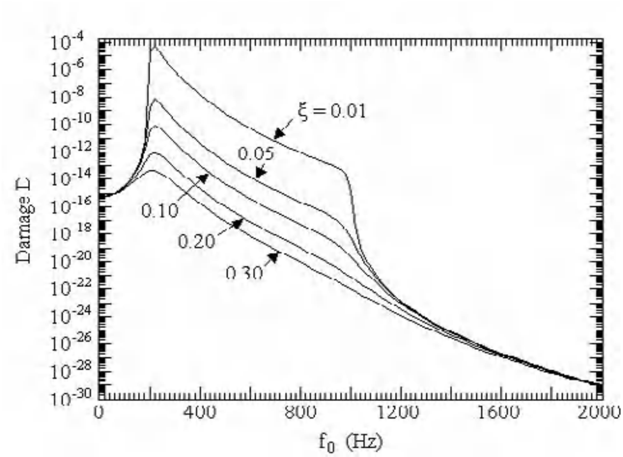


Figure 4.4. *Damping influence on the FDS of a random vibration*

2. The random vibration is a white noise between 200 Hz and 1000 Hz (amplitude 1 $(m\ s^{-2})^2/Hz$). The following values are used in the calculation:

$b = 10$

ξ respectively equal to 0.01, 0.05, 0.1, 0.2 and 0.3.

$K = 6.3 \times 10^{10}$ (Pa/m)

$C = 10^{80}$ (unit S.I.)

Duration: T = 1 hour

The fatigue damage spectrum of this noise is plotted between 1 Hz and 2000 Hz in Figure 4.4.

NOTE: *In Chapter 3 we saw the importance of the frequency interval between half-power points in the calculation of the damage created by a swept sine vibration. Damage produced by random vibration involves frequencies outside of this interval more. The damage is proportional to the rms relative displacement of the mass to the power b. We can, however, demonstrate (Volume 3, relation [8.72]) that responses of a one-degree-of-freedom system to white noise and narrow band noise defined in the frequency range Δf are only equal if the latter noise has a PSD amplitude twice as large. It results a ratio of damages equal to $2^{b/2}$. If for example b = 8, this ratio reaches a value of 16.*

Example 4.3.

We now compare the damage created by a random vibration defined between 10 Hz and 1000 Hz (PSD equal to 1 $(m\ s^{-2})^2$/Hz) and by a vibration with a PSD of similar amplitude in the interval between the half-power points of the one-degree-of-freedom system f_0 = 500 Hz, Q = 10. FDSs are drawn in Figure 4.5, for b = 8 and duration 1 hour.

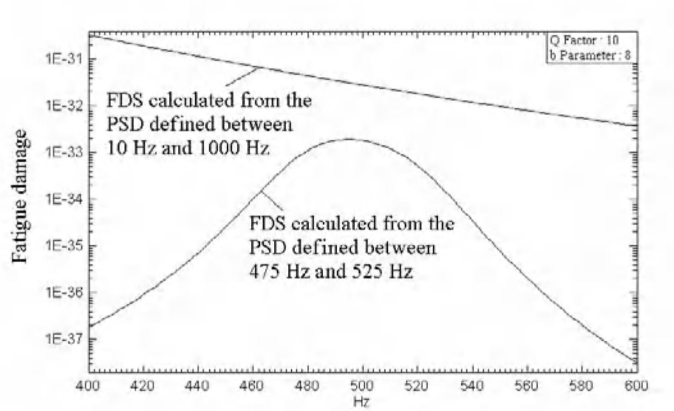


Figure 4.5. *Comparison of random vibration FDSs defined between 10 Hz and 1000 Hz and between 475 Hz and 525 Hz (amplitude 1 $(m\ s^{-2})^2$/Hz, Q = 10, b = 8)*

We can verify that damage ratio is equal to 16. The PSD between 475 Hz and 525 Hz only contributes approximately 6.25% to total damage.

4.3. Simplified hypothesis of Rayleigh's law

When the distribution of signal instantaneous values is Gaussian, we show that the probability density of *maxima* z_p of the z(t) response of a one-degree-of-freedom system follows a law in the form of expression [4.2].

The integrand of relation [4.5] can only be calculated numerically. For simplification purposes, we sometimes assume that the distribution of the response *maxima* can be assimilated to a Rayleigh law, or in other words r = 1 in relation [4.2]. With this hypothesis, the integral of [4.5] can be calculated analytically and we obtain:

$$D \approx \frac{K^b}{C} n_0^+ T \left(\sqrt{2}\, z_{rms}\right)^b \Gamma\left(1+\frac{b}{2}\right) \qquad [4.7]$$

also sometimes written according to the number of *maxima* per second:

$$D \approx \frac{K^b}{C} n_p^+ T \left(\sqrt{2}\, z_{rms}\right)^b \Gamma\left(1+\frac{b}{2}\right) \qquad [4.8]$$

and where $\Gamma(\,)$ is the gamma function that can be calculated with a serial development.

Theoretically, this approximation is only valid if parameter r (factor of irregularity) is close to 1. It was shown that the error made by assuming that Rayleigh's law remains lower than 4% regardless of $b \in [3, 30]$, for $r \geq 0.6$ (Volume 4).

For the conventional surtension value (Q = 10), we often consider that the response of the one-degree-of-freedom system is narrow band and consequently, that the approximation is valid. Experience shows that it is not always the case however.

Example 4.4.

Consider random Gaussian vibration measured in an aircraft, defined by a 5 s sample with 32,302 points, in the 6 Hz and 2,500 Hz frequency band (PSD in Figure 4.6).

The response relative displacement of a linear one-degree-of-freedom system with 10 Hz (Q = 10) natural frequency looks more like a sinusoid than a narrow band noise (Figure 4.7). And yet, small oscillations increase the number of peaks: in 5 seconds we count 50 passes through zero with positive slope (explaining a mean frequency equal to 10 Hz) and 205 peaks. Parameter r calculated from these two quantities (relation [4.3]) is equal to 0.244, very far from 1. These small peaks have very little influence on damage.

In practice, we can observe, in a more or less significant way, that parameter r is generally small at low frequency, and then increases to reach values that are close to unit at the high frequency. Figure 4.8 shows the variations of this parameter in the case of the aircraft vibration studied in this example.

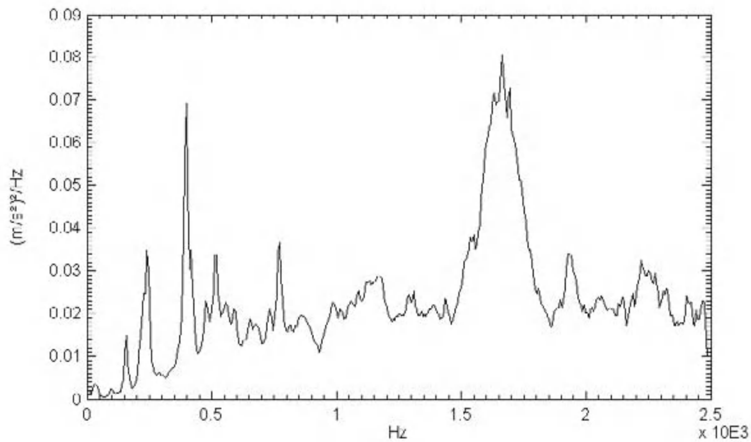


Figure 4.6. *Aircraft vibration PSD studied in this example*

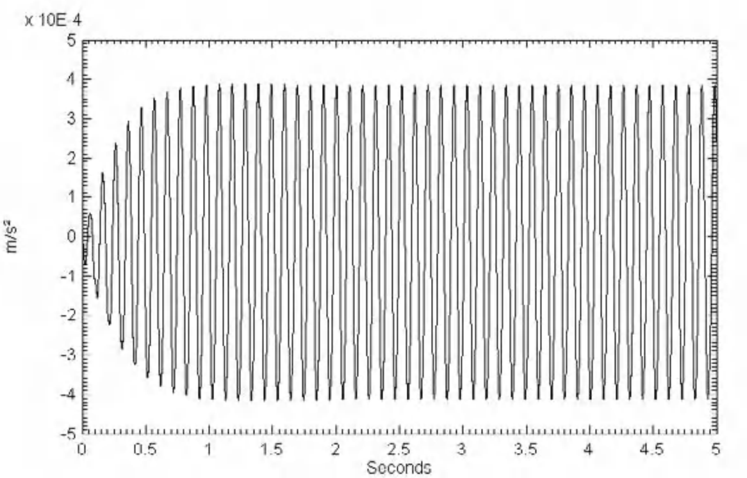


Figure 4.7. *Response $(2\,\pi f_0)^2\,z(t)$ of the 10 Hz (Q = 10) natural frequency system to the aircraft vibration*

In the same figure, the curve giving parameter r calculated from the PSD of this environment is drawn using the rms values of the relative displacement, relative velocity and relative acceleration. We can verify compatibility of results, knowing that, as with damage, the PSD gives a mean curve compared here to a specific distribution curve (relative to the sample of the signal studied).

The Rayleigh approximation is therefore only theoretically correct for this example, for natural frequencies higher than 145 Hz for which r is higher than 0.6.

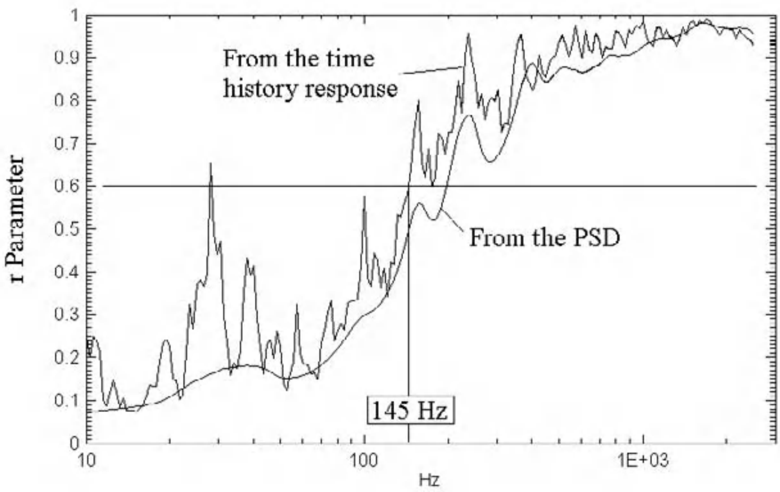


Figure 4.8. *Parameter r according to natural frequency for aircraft vibration, calculated from the PSD and response based in time (Q = 10)*

Using the average number of peaks tends to overestimate damage since each small peak is counted as a half-cycle starting at zero and coming back to it. At low frequency, the higher amplitude FDS is the one obtained with Rayleigh's law associated with the average number of peaks, followed by the FDS deducted from the complete probability density, then by the FDS calculated with Rayleigh's law and mean frequency (Figure 4.9). In the case of the response in the example shown in Figure 4.7, the FDS returning the most realistic damage is the one with the smallest amplitude.

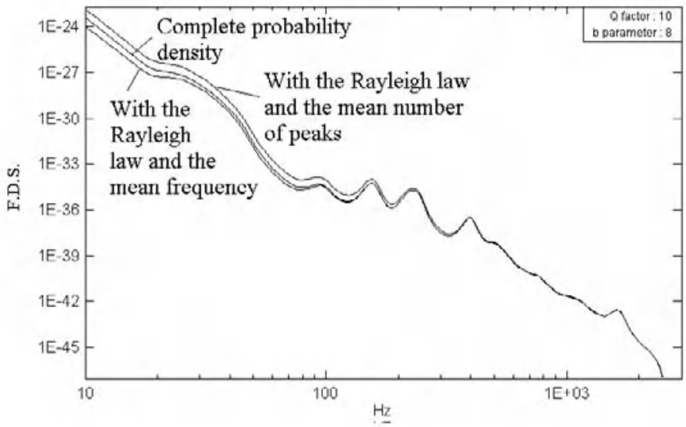


Figure 4.9. *FDS calculated from the PSD with complete probability density of peaks, Rayleigh's law and the mean number of peaks [4.8] and mean frequency [4.7]*

Example 4.5.

A random vibration, lasting 1.25 second, defined by 2000 points, measured in a truck. The frequency content of the PSD of this noise goes up to 600 Hz, but the 1 Hz to 250 Hz band is the only one that is significant (Figure 4.10).

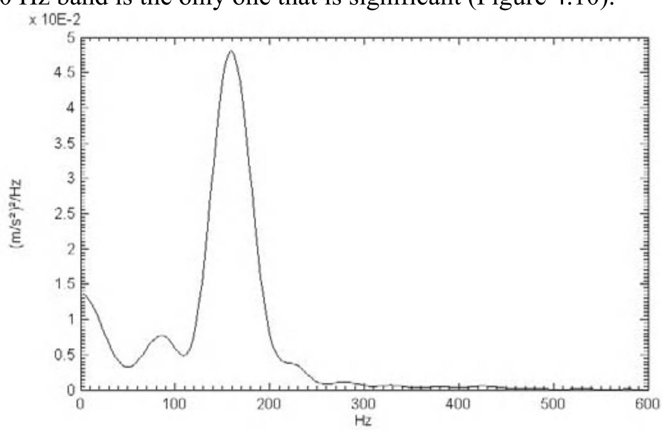


Figure 4.10. *PSD of a "truck" vibration*

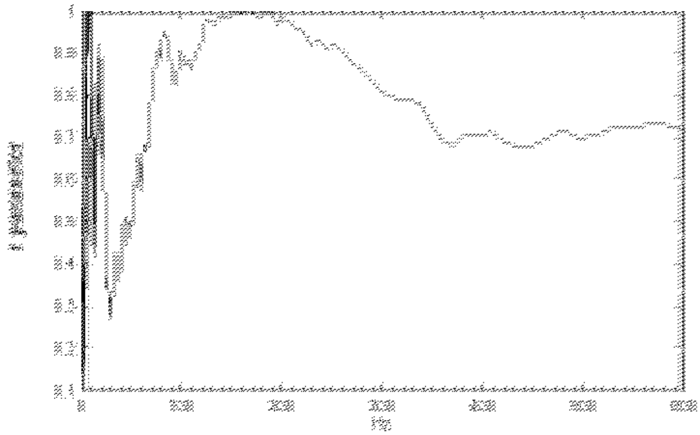


Figure 4.11. *Parameter r according to natural frequency for the truck vibration calculated from the response according to time (Q = 10)*

Parameter r approaches 1 when the frequency increases in the range where the frequency content is significant (Figure 4.11). It subsequently diminishes.

NOTE:

Approximate expression of damage

We have seen in Chapter 2 that the ERS could be calculated in an approximate way from the rms value of the response of a one-degree-of-freedom system in the hypothesis of a white noise excitation [2.11]:

$$z_{rms} = \sqrt{\frac{Q\ G_{\ddot{x}0}}{4\ \omega_0^3}}$$

In an analog way, fatigue damage could be calculated from this relation and [4.7], when using the hypothesis where $n_0^+ \approx f_0$*:*

$$D \approx \frac{K^b}{C}\, n_0^+\ T\ \left(\frac{Q\ G_{\ddot{x}0}}{2\ \omega_0^3}\right)^{b/2}\ \Gamma\left(1 + \frac{b}{2}\right) \qquad [4.9]$$

This expression can also be written as

$$D \approx \left[\frac{K^b}{C} \frac{\Gamma\left(1+\frac{b}{2}\right)}{\left(8\,\pi\,\omega_0^2\right)^{b/2}} \right] f_0\, T \left(\frac{G_{\ddot{x}0}}{\xi\, f_0}\right)^{b/2} = A\, f_0\, T \left(\frac{G_{\ddot{x}0}}{\xi\, f_0}\right)^{b/2} \qquad [4.10]$$

If the vibration is characterized by a velocity (instead of an acceleration), knowing that the velocity PSD ($G_{V\,0}$) is equal to the acceleration PSD ($G_{\ddot{x}0}$) divided by $\left(2\,\pi\, f\right)^2$, by considering the PSD value for $f = f_0$ expression [4.10] can be written as:

$$D \approx \left[\frac{K^b}{C} \frac{\Gamma\left(1+\frac{b}{2}\right)}{\left(8\,\pi\right)^{b/2}} \right] f_0\, T \left(\frac{G_{V\,0}}{\xi\, f_0}\right)^{b/2} = B\, f_0\, T \left(\frac{G_{V\,0}}{\xi\, f_0}\right)^{b/2} \qquad [4.11]$$

Apart the constant (B), we find here the "damage potential spectrum" defined by A.G. Piersol and G.R. Henderson (HEN 95] in the form

$$DP(f_0) = f_0\, T \left(\frac{G_{\ddot{x}0}}{\xi\, f_0}\right)^{b/2} \qquad [4.12]$$

Comments in section 2.2.3.2 involving the validity of expression [4.9] also apply here.

4.4. Calculation of the fatigue damage spectrum with Dirlik's probability density

When the FDS is calculated from a signal based on time, strain cycle counting can be carried out with the help of the Rainflow method providing a histogram of domains or a peak histogram. It is preferable to use the peak histogram. When we use a PSD, the Rice method leads to a probability density of strain peaks. There is a difference in principle that led T. Dirlik to propose a semi-empirical probability density of domains (Volume 4, Chapter 4).

In practice, considering this law will not result in a significant difference with the FDS calculated with the Rice formulation.

Example 4.6.

Using Example 2.3 from Chapter 2 again, we can observe that FDSs calculated using the hypotheses of Rice and Dirlik from the PSDs in Figures 2.4 and 2.6 are respectively very close (Figures 4.12 and 4.13).

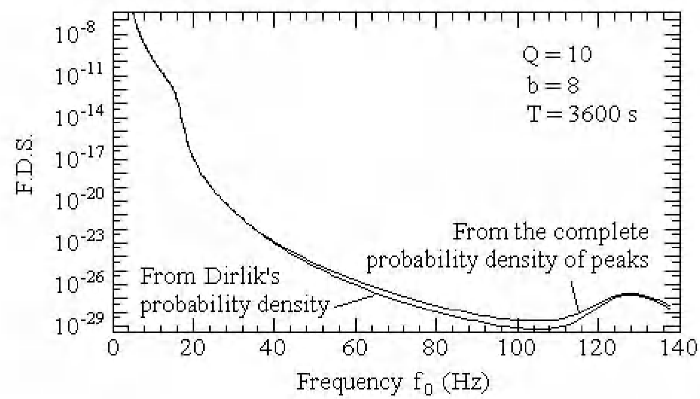


Figure 4.12. *Fatigue damage spectra calculated from the PSD of Figure 2.4*

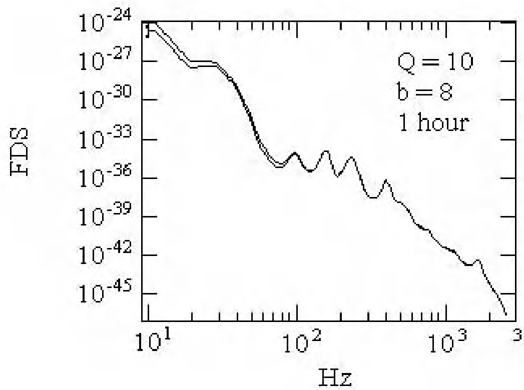


Figure 4.13. *Fatigue damage spectra calculated from the PSD of Figure 2.6 and Rice and Dirlik's probability densities*

4.5. Reduction of test time

4.5.1. *Fatigue damage equivalence in the case of a linear system*

Expression [4.6] shows that the mean damage is proportional to the time during which the random vibration is applied and to its rms value to the power b, since the response rms displacement z_{rms} is proportional to the rms value of the excitation $\ddot{x}_{rms}$. As for the sinusoidal case, this gives us

$$\ddot{x}_{rms\ reduced} = \ddot{x}_{rms\ real} \left(\frac{T_{real}}{T_{reduced}} \right)^{\frac{1}{b}} \qquad [4.13]$$

By considering the levels of power spectral density G, then:

$$\frac{G_{reduced}}{G_{real}} = \left(\frac{T_{real}}{T_{reduced}} \right)^{2/b} \qquad [4.14]$$

It is assumed that in this equation damping remains constant whatever the stress level may be.

4.5.2. *Method based on a fatigue damage equivalence according to Basquin's relationship taking account of variation of natural damping as a function of stress level*

It was shown (Volume 3, Chapter 8) that the relative rms response displacement of a single-degree-of-freedom linear mechanical system with natural frequency f_0 and quality factor Q subjected a random white noise at level G_0 can be expressed as:

$$z_{rms}^2 = \text{Const.} \left(f_0\ Q\ G_0 \right) \qquad [4.15]$$

yielding:

$$\sigma_{rms} = \text{Const.}\ z_{rms} = \text{Const.} \left(f_0\ Q\ G_0 \right)^{1/2}$$

By combining [3.72] and $\sigma_{rms} = \text{Const.}\ z_{rms}$, then

$$Q = \text{Const.}\ \sigma_{rms}^{2-n}$$

yielding

$$\sigma_{rms} = \text{Const.}\ \left(f_0\ G_0\right)^{1/2}\ \left(\sigma_{rms}^{2-n}\right)^{1/2}$$

$$\sigma_{rms} = \text{Const.}\ \left(f_0\ G_0\right)^{1/n} \qquad [4.16]$$

Since the peak levels of the stress are proportional to σ_{rms}, this yields, by eliminating σ_{rms} between $N\ \sigma_{rms}^b = \text{Const.}$ and [4.16]:

$$N\left(f_0\ G_0\right)^{b/n} = \text{Const.}$$

yielding, since $N = f\ T$,

$$\frac{G_{reduced}}{G_{real}} = \left(\frac{T_{real}}{T_{reduced}}\right)^{\frac{n}{b}} \qquad [4.17]$$

or

$$\frac{\ddot{x}_{rms\ reduced}}{\ddot{x}_{rms\ real}} = \left(\frac{T_{real}}{T_{reduced}}\right)^{\frac{n}{2b}} \qquad [4.18]$$

A.J. Curtis recommends taking n = 2.4 and b = 9. MIL - STD - 810 suggests taking $\frac{b}{n} = 4$. AIR 7304 also uses this value, without specifying its origin. For guidance, Table 4.1 can be used to compare these various equations applied to the same example. Let us calculate *the factor of exaggeration* which is used to derive from a real time T_{real} a test time $\frac{T_{real}}{4}$ with b = 9 and n = 2.4.

The results obtained for random vibrations are very similar.

		Reduction rule	Exaggeration factor
Q constant	Sinusoidal and Random	$\frac{\ddot{x}_{\text{rms reduced}}}{\ddot{x}_{\text{rms real}}} = \left(\frac{T_{\text{real}}}{T_{\text{reduced}}}\right)^{\frac{1}{b}}$	1.17
Q not constant	Sine	$\frac{\ddot{x}_{\text{m reduced}}}{\ddot{x}_{\text{m real}}} = \left(\frac{T_{\text{real}}}{T_{\text{reduced}}}\right)^{\frac{n-1}{b}}$	1.24
	Random	$\frac{\ddot{x}_{\text{rms reduced}}}{\ddot{x}_{\text{rms real}}} = \left(\frac{T_{\text{rms}}}{T_{\text{reduced}}}\right)^{\frac{n}{2b}}$	1.20

Table 4.1. *Examples of exaggeration factors (time reduction factor of 4)*

NOTE:

1. *Whereas the last procedure described imposes the values of n and b, the first leaves the choice of a value for b open. The exaggeration factor is observed to vary enormously depending on the value chosen. Let us choose, for instance, a ratio* $\frac{T_{real}}{T_{reduced}} = 4$ *and let us calculate the exaggeration factor for* $3 \le b \le 20$:

b	***3***	***6***	***10***	***15***	***20***
$E = \frac{\ddot{x}_{rms\ reduced}}{\ddot{x}_{rms\ real}}$	*1.59*	*1.26*	*1.15*	*1.10*	*1.07*

Table 4.2. *Influence of the parameter b on the exaggeration factor*

It is therefore important to choose a value for b as close as possible to the real value and, if in doubt, to use a default value.

2. *Representing the S–N curve by an analytical expression of the form* $N\ \sigma^b = Const.$, *which has the advantage of simplifying the calculations, results in the fatigue limit being disregarded, and on the assumption that each and every level contributes to the fatigue damage.*

It is obvious that, for stresses below the fatigue limit, the application of the exaggeration factor calculated in this way leads to conservative tests (but this fatigue limit, well known for steels, is not available for many materials).

3. *The practical limit on reduction of the test time is the requirement that the excitation level should not be increased above the ultimate strength. It is therefore often acceptable to limit the exaggeration factor to the ratio of the ultimate strength and the fatigue limit, to the order of 2 to 3 for most materials. Strictly applied, this rule can lead to very large time reduction factors. For an exaggeration factor E equal to 2, depending on the value given to b, the following maximum values are obtained from equation [4.13]:*

b	4	6	8	10	14
$\dfrac{T_{real}}{T_{test}}$	16	64	256	1024	16384

Table 4.3. *Influence of parameter b on test time reduction*

A badly controlled time reduction can result in the creation of non-existent problems, for several reasons:

– creation of maximum stresses exceeding the ultimate stress limit that are never reached by real levels;

– creation of shocks in equipment containing clearance that would not appear at the real levels (or which would be smaller);

– the damage equivalence is obtained assuming linearity of the structure, which can not be the case in practice. As a result, the error in the exaggeration factor is all the greater whenever the stress level is higher, and as a consequence the test time is shorter;

– finally, the first remark shows the influence of parameter b on the determination of the exaggeration factor E. A small error in the choice of b leads to an error in E which increases with the reduction factor.

Time reduction is a device, the use of which should be limited to a minimum. When it is used, the reduction factor should not be too high. In the event of problems occurring during the test, before modifying the specimen it is necessary to test times so that they are closer to real times (with, of course, consequently reduced a). Certain standards [AVP 70] limit the use of the accelerated tests to low levels, long duration environments such as those encountered in transportation and also recommend not applying a higher level during the test than the most severe level of the environment to which the equipment is to be subjected.

4.6. Truncation of the peaks of the "input" acceleration signal

The example in section 2.12 of Volume 3 and section 2.6 of this volume that evaluates the influence of a peak truncation on the PSD and the extreme response spectrum is re-used below for the fatigue damage spectra, calculated under the following conditions:

$Q = 10$	$f_{min} = 5$ Hz	$K = 1$	200 points
$b = 8$	$f_{max} = 3000$ Hz	$C = 1$	Logarithmic step

4.6.1. *Fatigue damage spectra calculated from a signal as a function of time*

Figure 4.14 shows the FDS calculated directly from the truncated signals.

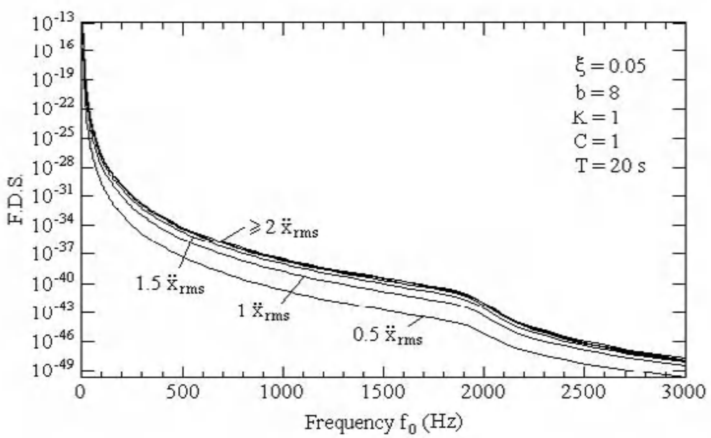


Figure 4.14. *FDS of truncated vibratory signals*

It should be noted that the FDS are very similar as long as the truncation level remains higher or equal than $2\ \ddot{x}_{rms}$.

4.6.2. *Fatigue damage spectra calculated from power spectral densities*

The FDS calculated from the PSD of the truncated signals are plotted in Figure 4.15.

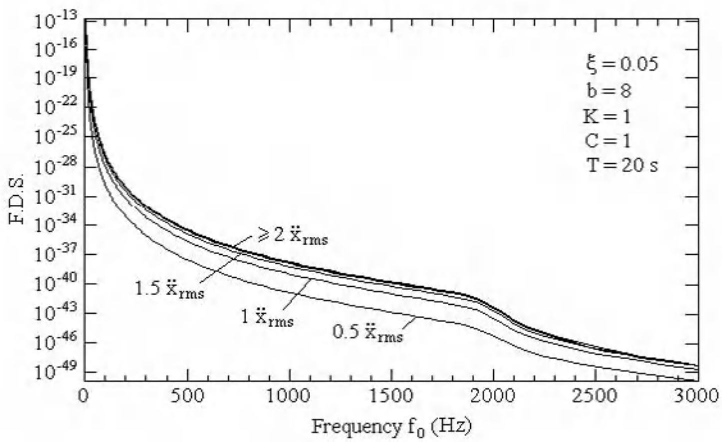


Figure 4.15. *FDS calculated from PSD of truncated vibratory signals*

Truncation has little effect beyond $2\ \ddot{x}_{rms}$. This result is easily comparable to that obtained by direct calculation of the FDS from the signal as a function of time.

4.6.3. *Comparison of fatigue damage spectra calculated from signals as a function of time and power spectral densities*

Whatever this level, the spectra are very similar. The effects of truncation appear to be similar and are therefore largely unaffected by the calculation method.

Figures 4.16 and 4.17 show the FDS calculated according to the two methods: signals that are at first not truncated and which are then truncated at $0.5\ \ddot{x}_{rms}$.

A complementary study based on narrow band noise superimposed on white noise led to the same conclusions.

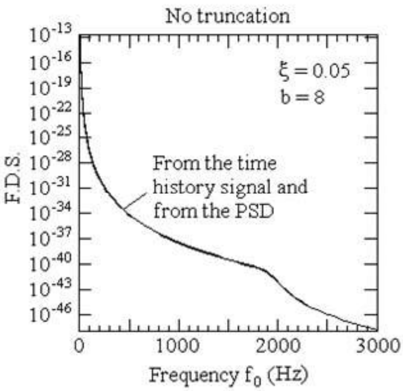


Figure 4.16. *Comparison of the FDS calculated from a non-truncated signal as a function of time and from its PSD*

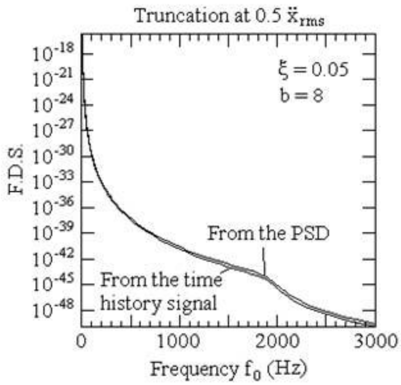


Figure 4.17. *Comparison of the FDS calculated from a truncated signal as a function of time and from its PSD*

4.7. Sinusoidal vibration superimposed on a broad band random vibration

4.7.1. *Case of a single sinusoidal vibration superimposed on broad band random vibration*

Let σ_s be the sinusoidal stress and σ_a the random stress. The fatigue damage is given by $D = \int_0^\infty \frac{dn}{N}$. Knowing that $N\,\sigma^b = C$ and that $dn = n_p^+\,T\,q(\sigma)\,d\sigma$, by following the notations of section 2.7, then

$$D = \frac{n_p^+\,T}{C}\int_0^\infty \sigma^b\,q(\sigma)\,d\sigma \tag{4.19}$$

$$D = \frac{n_p^+\,T}{C}\int_0^\infty \frac{\sigma^{b+1}}{\sigma_{a\,rms}^2}\,e^{-\frac{\sigma^2+\sigma_s^2}{2\,\sigma_{a\,rms}^2}}\,I_0\left(\frac{\sigma\,\sigma_s}{\sigma_{a\,rms}^2}\right)d\sigma \tag{4.20}$$

or

$$D = \frac{K^b}{C}\,\frac{n_p^+\,T}{z_{a\,rms}^2}\int_0^\infty z^{b+1}\,e^{-\frac{z^2+z_s^2}{2\,z_{a\,rms}^2}}\,I_0\left(\frac{z\,z_s}{z_{a\,rms}^2}\right)dz \tag{4.21}$$

If $u = \dfrac{z}{z_{a\ rms}}$ and $a = \dfrac{z_s}{z_{a\ rms}}$, the damage can be written as

$$D = \frac{K^b}{C}\, n_p^+\, T\, z_{a\ rms}^b \int_0^\infty u^{b+1}\, e^{-\frac{u^2+a^2}{2}}\, I_0(u\, a)\, du \qquad [4.22]$$

From [4.22] and equations in section 2.7 [SPE 61],

$$D = \frac{K^b}{C}\, n_p^+\, T\, z_{a\ rms}^b \int_0^\infty u^{b+1}\, e^{-\frac{u^2+a^2}{2}} \sum_{n=0}^{\infty} \left(\frac{a}{2}\right)^{2n} \frac{u^{2n}}{(n!)^2}\, du \qquad [4.23]$$

$$D = \frac{K^b}{C}\, n_p^+\, T\, z_{a\ rms}^b \sum_{n=0}^{\infty} \left(\frac{a}{2}\right)^{2n} \frac{1}{(n!)^2}\, e^{-\frac{a^2}{2}} \int_0^\infty u^{2n+b+1}\, e^{-\frac{u^2}{2}}\, du \qquad [4.24]$$

$$D = \frac{K^b}{C}\, n_p^+\, T\, z_{a\ rms}^b\, 2^{b+1} \sum_{n=0}^{\infty} \frac{a^{2n}}{(n!)^2}\, e^{-\frac{a^2}{2}} \int_0^\infty \left(\frac{u}{2}\right)^{2n+b+1} e^{-\frac{u^2}{2}}\, du \qquad [4.25]$$

However

$$\Gamma(1+x) = 2^{x+1} \int_0^\infty \left(\frac{u}{2}\right)^{2x+1} e^{-\frac{u^2}{2}}\, du \qquad [4.26]$$

Here, $2\,x + 1 = 2\,n + b + 1$, i.e. $x = n + b/2$, and $\Gamma\left(1 + n + \dfrac{b}{2}\right) = \left(n + \dfrac{b}{2}\right)!$

$$D = \frac{K^b}{C}\, n_p^+\, T \left(\sqrt{2}\, z_{a\ rms}\right)^b e^{-\frac{a^2}{2}} \sum_{n=0}^{\infty} \left(\frac{a^n}{n!}\right)^2 \frac{1}{2^n} \left(n + \frac{b}{2}\right)! \qquad [4.27]$$

From [4.21] [LAM 88],

$$D = \frac{K^b}{C}\, n_p^+\, T\, \left(\sqrt{2}\ z_{a\ rms}\right)^b\, \Gamma\left(1+\frac{b}{2}\right)\, {}_1F_1\left(-\frac{b}{2}, 1, -a_0^2\right) \qquad [4.28]$$

where $a_0^2 = \dfrac{a^2}{2}$ ($a_0 = \dfrac{z_{s\ rms}}{z_{a\ rms}}$) and ${}_1F_1(\ )$ is the hypergeometric function such that

$$ {}_1F_1\left(\alpha, \delta, \pm x\right) = \sum_{j=0}^{\infty} \frac{(\alpha)_j\ (\pm x)^j}{(\delta)_j\ j\,!} \qquad [4.29]$$

with

$$(\alpha)_j = \alpha\,(\alpha+1)\cdots(\alpha+j-1)$$

$$(\delta)_j = \delta\,(\delta+1)\cdots(\delta+j-1)$$

By setting $\alpha \equiv -\dfrac{b}{2}$, $\delta \equiv 1$ and $-x \equiv -a_0^2$, then

$$ {}_1F_1\left(-\frac{b}{2}, 1, -a_0^2\right) = 1 + \sum_{j=0}^{\infty} p_j \qquad [4.30]$$

where

$$p_j = \left(p_{j-1}\right)\left(-\frac{b}{2} - 1 + j\right)\frac{\left(-a_0^2\right)}{j^2} \qquad [4.31]$$

and $p_0 = 1$.

Example 4.7.

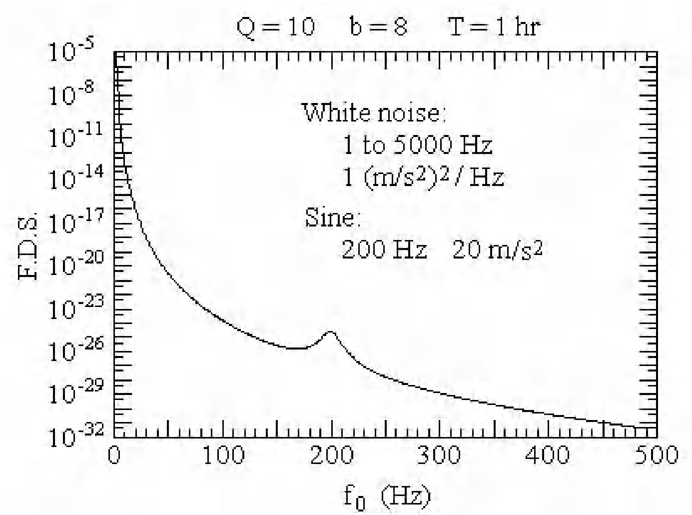


Figure 4.18. *FDS of an excitation made up of a sine wave plus wide band random noise*

The sinusoidal line/narrow band random distinction is not always easy to make, but the effects on the structure can be very different.

Example 4.8.

Consider:

– a sinusoidal vibration with 400 Hz frequency and 40 m s^{-2} (amplitude superimposed on a wide band random noise: 10 Hz to 2,000 Hz, 2 (m s^{-2}))^2/Hz,

– a narrow band noise centered on 400 Hz (4 Hz bandwidth) with the same rms value of the sine (leading to a PSD with a 200 (m s^{-2})^2/Hz amplitude) superimposed on a wide band random vibration: 10 Hz to 2,000 Hz, 2 (m s^{-2})^2/Hz.

The FDSs of these two vibrations are very close, except around 400 Hz, where the FDS of the wide band + narrow band vibration is greater (Figures 4.19 and 4.20). This result can be explained because rms values being equal, the narrow band noise presents higher peaks than the sine: 5 to 6 times the rms value with random, twice the root of the rms value for sine.

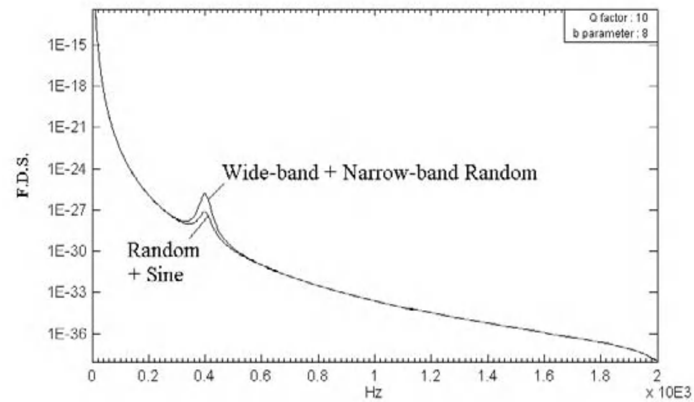


Figure 4.19. *Vibration FDS: random + fixed sine (40 m s^{-2})), random + narrow band. Duration 1 h (Q = 10, b = 8)*

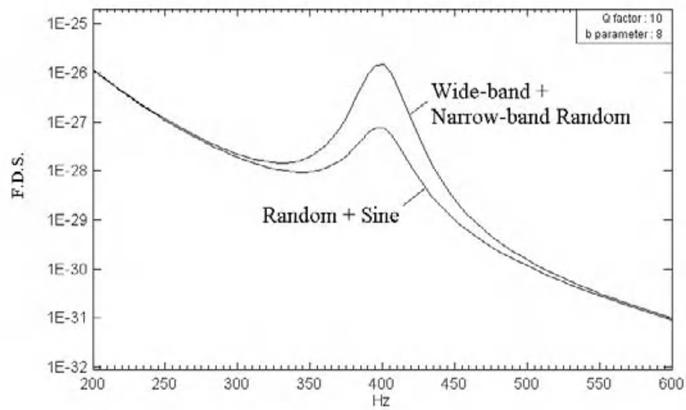


Figure 4.20. *Vibration FDS: random + fixed sine (40 m s^{-2}), random + narrow band. FDS between 200 Hz and 600 Hz, duration 1 h (Q = 10, b = 8)*

The ERS of the wide band plus narrow band random is always larger (Figure 4.21).

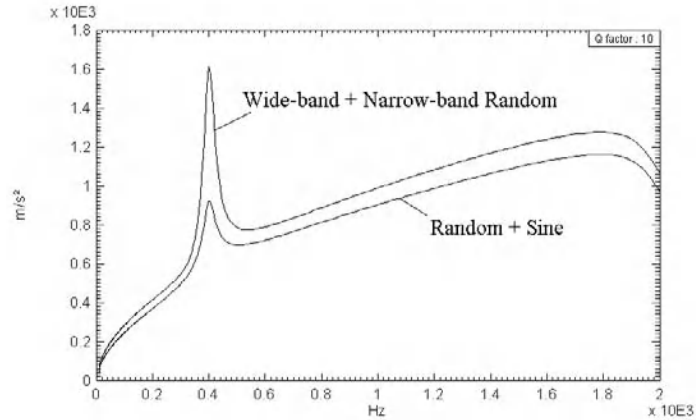


Figure 4.21. *Vibration ERS: wide band random + sine (40 m s^{-2}), wide band random + narrow band. Duration 1 h (Q = 10, b = 8)*

In order to obtain the same 400 Hz FDS values in these calculation conditions, we must set the amplitude of the sinusoid at 78 m s^{-2}. With this amplitude, FDSs are very close (Figure 4.22), but ERSs still present some differences (Figure 4.23).

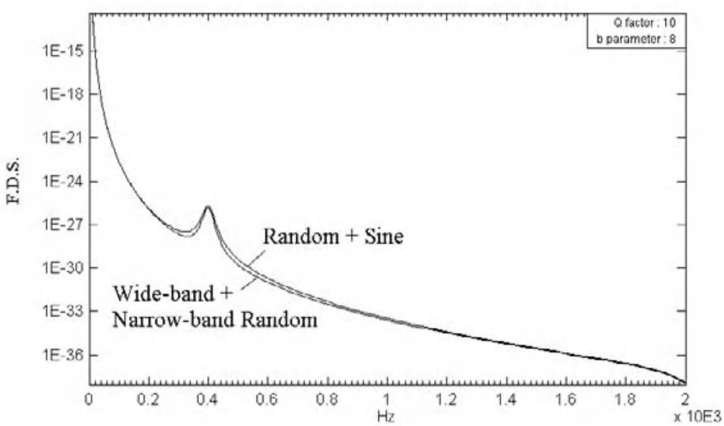


Figure 4.22. *Vibration FDS: wide band random + sine (75 m s^{-2}), wide band random + narrow band. Duration 1 h (Q = 10, b = 8)*

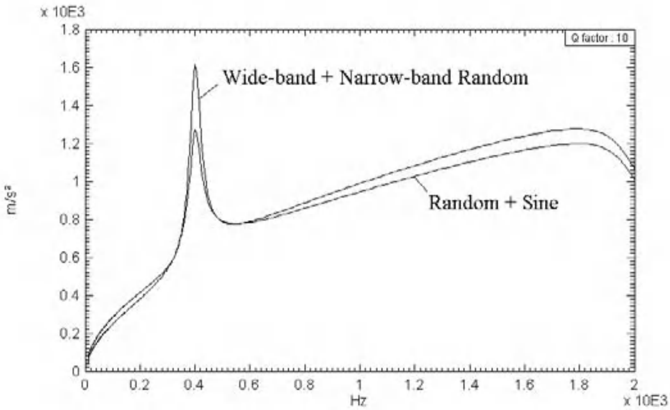


Figure 4.23. *Vibration ERS: wide band random + sine (75 m s^{-2}), wide band random + narrow band. Duration 1 h (Q = 10, b = 8)*

Example 4.9.

Consider the case where the amplitude of the narrow band noise (20 (m s^{-2})^2/Hz in this example), is smaller than previously in relation to the amplitude of wide band noise (2 (m s^{-2})^2/Hz here). FDSs become very close (Figure 4.24), whereas ERSs remain slightly different (Figure 4.25).

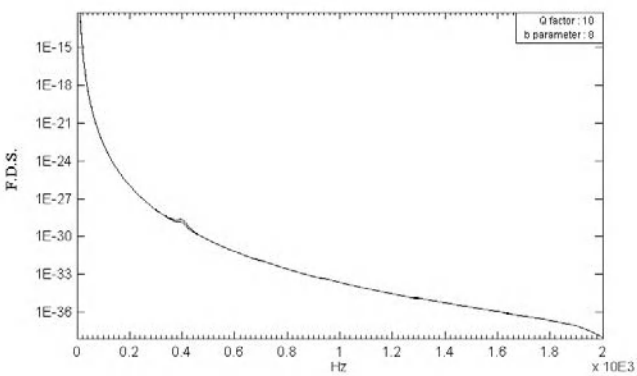


Figure 4.24. *Vibration FDS: wide band random + sine (20 m s^{-2}), wide band random + narrow band. Duration 1 h (Q = 10, b = 8)*

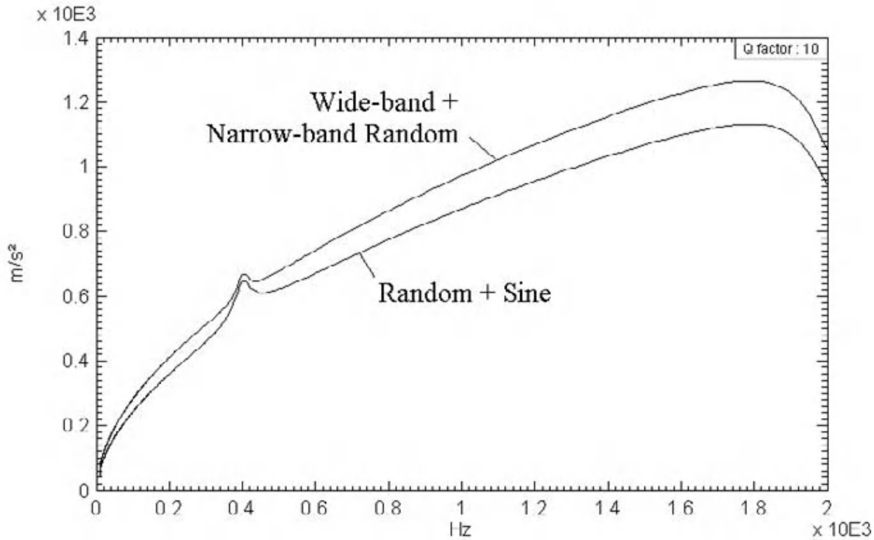


Figure 4.25. *Vibration ERS: wide band random + sine (20 m s^{-2}) wide band random + narrow band. Duration 1 h (Q = 10, b = 8)*

The gap increases with frequency: the number of peaks increases with frequency, the probability of finding a larger peak increases.

4.7.2. *Case of several sinusoidal vibrations superimposed on a broad band random vibration*

In general, the FDS is calculated from the signal as a function of time, obtained by summing the sinusoids and a random signal whose PSD defines the broad band noise (section 2.7.3).

When the frequencies are sufficiently spaced, the FDS of the composite signal is close to the envelope of the FDS calculated using the preceding formulae for each sinusoid superimposed on the random noise.

Example 4.10.

Wide band noise:

10 Hz to 2000 Hz

$G_0 = 1\ (\text{m s}^{-2})^2/\text{Hz}$

Sinusoids:

100 Hz, 300 Hz and 600 Hz

Amplitude: $20\ \text{m s}^{-2}$

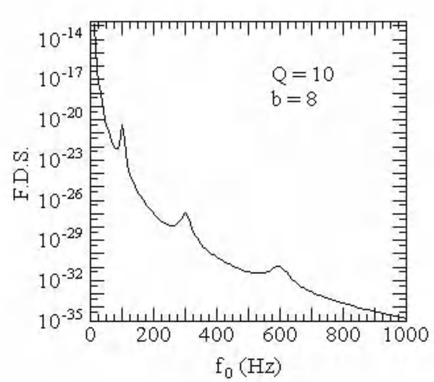


Figure 4.26. *FDS of an excitation composed of three sinusoids on a wide band noise*

FDS plotted between 10 Hz and 1000 Hz, in steps of 5 Hz, for T = 1 hr, Q = 10, b = 8, K = 1 and C = 1 (Figure 4.26).

4.8. Swept sine superimposed on a broad band random vibration

4.8.1. *Case of one swept sine superimposed on a broad band random vibration*

The FDS is calculated by summing the FDS obtained from each sinusoid of intermediate frequency regularly distributed throughout the swept range, superimposed on the random noise, for a duration equal to that of the total sweep divided by the chosen number of sinusoids (see section 2.8.2).

Example 4.11.

Wide band noise:

10 Hz to 2000 Hz

$G_0 = 1\ (\text{m s}^{-2})^2/\text{Hz}$

Swept sine vibration:

100 Hz to 400 Hz

Amplitude: $20\ \text{m s}^{-2}$

Linear sweep

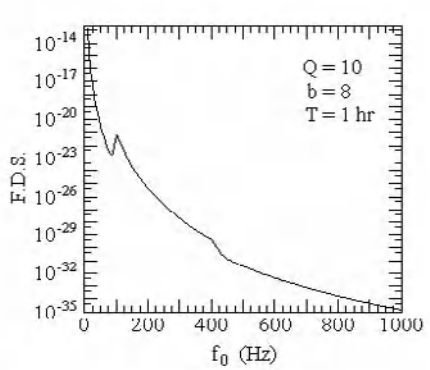


Figure 4.27. *FDS of a swept sine plus broad band random excitation*

FDS plotted between 10 Hz and 1000 Hz, in steps of 5 Hz, for T = 1 hr, Q = 10, b = 8, K = 1 and C = 1 (Figure 4.27).

4.8.2. *Case of several swept sines superimposed on a broad band random vibration*

The swept bands are, generally, sufficiently distant, for any given time, and as a first approximation the FDS is the envelope of the FDS calculated with only one of the sinusoids superimposed on the noise.

The FDS of the entire signal is obtained by summing the envelopes calculated in this way each time.

4.9. Swept narrow bands on a broad band random vibration

The calculation of the FDS of this vibration is carried out by summing the FDS obtained from the PSDs defined while following the procedure set out in section 2.9.2.

Example 4.12.

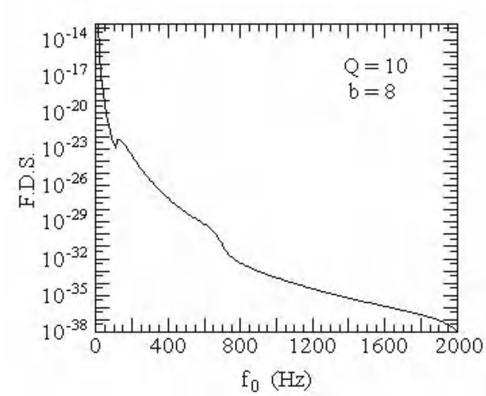


Figure 4.28. *FDS of a random vibration made up of two swept narrow bands superimposed on a wide band noise*

Returning once more to Example 2.23 in section 2.9, with a random vibration consisting of:

– a broad band noise from 10 Hz to 2000 Hz, with a PSD amplitude $G_0 = 2\ (\text{m s}^{-2})^2/\text{Hz}$;

– two narrow bands:

- one of width 100 Hz, whose central frequency varies linearly between 150 Hz and 550 Hz,

- the other of width 200 Hz swept between 1100 Hz and 1900 Hz.

The FDS is plotted (Figure 4.28) between 10 Hz and 2000 Hz for a damping factor equal to 0.05, a parameter b equal to 8, a duration of 1 hour, K = 1 and C = 1.

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 5

Fatigue Damage Spectrum of a Shock

5.1. General relationship of fatigue damage

Damage due to fatigue undergone by a linear system with a single-degree-of-freedom can be calculated from the time history signal describing the shock on the basis of the same formulae used to calculate damage due to a random vibration (Volume 4, section 4.2 and Volume 5, section 4.1). In the latter case, the response of the system is calculated for the time during which the vibration is applied, disregarding the residual response after the end of the vibration, which is of very short duration compared to that of the vibration itself.

In the case of shocks, which are by definition of short duration, this residual response is important, particularly at low frequencies where it can constitute the essential element of the histogram peaks. For calculation purposes it is necessary:

– to numerically determine the relative response displacement $z(t)$ generated by the shock;

– to establish a peak histogram of $z(t)$ (number n_i of maxima and minima having a given amplitude z_{m_i}, in absolute terms);

– to associate with each stress extremum $\sigma_i = K\, z_{m_i}$ a factor for elementary damage

$$d_i = \frac{1}{2\, N_i} \qquad [5.1]$$

where N_i is the number of cycles to failure under alternate sinusoidal stress of amplitude σ_i, according to Basquin's law $N_i \, \sigma_i^b = C$, yielding total damage for all the peaks listed in the histogram (Miner's law):

$$D = \sum_i d_i = \sum_i \frac{n_i}{N_i} \qquad [5.2]$$

$$D = \frac{K^b}{2\,C} \sum_i n_i \, z_{m_i}^b \qquad [5.3]$$

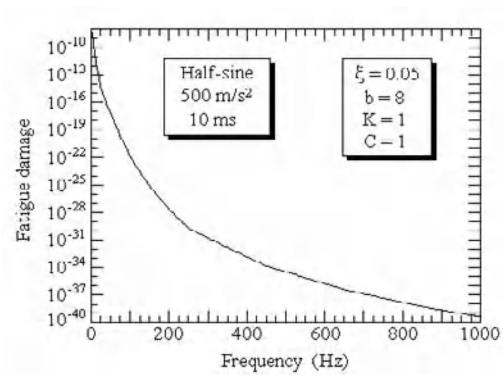


Figure 5.1. *FDS of a half-sine shock*

If the same shock is repeated p times at sufficiently large time intervals, for the residual response to return to zero before the arrival of the following shock, the total damage is expressed as:

$$\Delta = p \, D \qquad [5.4]$$

If this condition is not satisfied, the response to shock i is calculated by taking as initial conditions the relative velocity and displacement created by shock i-1 at the time of arrival of shock i.

NOTE: *Some experimental studies carried out on steels confirm that the fatigue life observed is:*

– *not very different in conventional fatigue and under shocks [BOU 85];*

– *close to that predicted by Miner's rule [TAN 63].*

5.2. Use of shock response spectrum in the impulse zone

Section 3.1 of Volume 2 examines which three zones can be distinguished in a shock response spectrum. The first, at low frequency, is the impulse zone, in which the response is primarily residual and depends only on the velocity change associated with the shock. This response is of the form

$$z(t) = Z\ e^{-\xi\,\omega_0\,t}\ \sin\omega_0\ \sqrt{1-\xi^2}\ t \qquad [5.5]$$

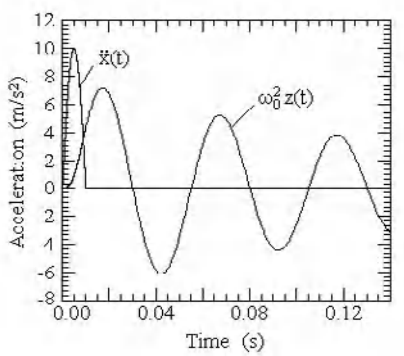


Figure 5.2. *Impulse response to a half-sine shock*

The amplitude of the rth extremum (Figure 5.2) is

$$z_r = (-1)^{r-1}\ Z\ \exp\left[-\xi\,(2\,r-1)\,\frac{\pi}{2}\right] \qquad [5.6]$$

For $r = 1$, $z_1 = Z\ \exp\left(-\frac{\pi\,\xi}{2}\right)$. Following the above assumptions, the fatigue damage created by the shock can be written

$$D = \frac{K^b}{2\,C}\sum_i n_i\ z_{r\,i}^b$$

where n_i is the number of half-cycles (or peaks) of amplitude z_{r_i}. The response being of the damped oscillatory type, there is only one half-cycle of amplitude z_{r_i}, yielding:

$$D = \frac{K^b}{2\,C} \sum_{i=1}^{\infty} z_{r\,i}^b = \frac{K^b}{2\,C}\, Z^b \sum_{r=1}^{\infty} \exp\left[-\frac{\pi}{2}\,\xi\, b\,(2\,r-1)\right] \qquad [5.7]$$

At the natural frequency f_0, the shock response spectrum plotted for the same value ξ of the damping factor is equal to $S_{RS} = \omega_0^2\, z_1$ (since the spectrum is calculated by considering the largest peak of the response, the first of the residual response here). By deduction, it follows that:

$$Z = \frac{S_{RS}}{\omega_0^2} \exp\left(\frac{\pi\,\xi}{2}\right) \qquad [5.8]$$

and that [MAI 59]

$$D = \frac{K^b}{2\,C}\left(\frac{S_{RS}}{\omega_0^2}\right)^b \exp\left(\frac{\pi\,\xi\,b}{2}\right) \sum_{r=1}^{\infty} \exp\left[-\frac{\pi\,\xi\,b\,(2\,r-1)}{2}\right] \qquad [5.9]$$

In this last relation, the sum can be calculated by noting that, if $a = \dfrac{\pi\,\xi\,b}{2}$,

$$\sum_{r=1}^{\infty} e^{-a\,(2\,r-1)} = e^a \left[e^{-2\,a} + e^{(-2\,a)^2} + \cdots\right] \qquad [5.10]$$

$$\sum_{r=1}^{\infty} e^{-a\,(2\,r-1)} = e^a\, \frac{e^{-2\,a}}{1 - e^{-2\,a}} = \frac{1}{2\ \mathrm{sh}\ a} \qquad [5.11]$$

yielding damage

$$D = \frac{K^b}{2\,C}\left(\frac{S_{RS}}{\omega_0^2}\right)^b \exp\left(\frac{\pi\,\xi\,b}{2}\right) \frac{1}{2\ \mathrm{sh}\left(\dfrac{\pi\,\xi\,b}{2}\right)} \qquad [5.12]$$

Let:

$$M = \frac{\exp\left(\frac{\pi\,\xi\,b}{2}\right)}{4\,\text{sh}\left(\frac{\pi\,\xi\,b}{2}\right)} \qquad [5.13]$$

then

$$D = \frac{K^b}{C}\left(\frac{S_{RS}}{\omega_0^2}\right)^b M \qquad [5.14]$$

Figure 5.3 gives the variations of M with damping factor ξ, for b ranging between 2 and 14.

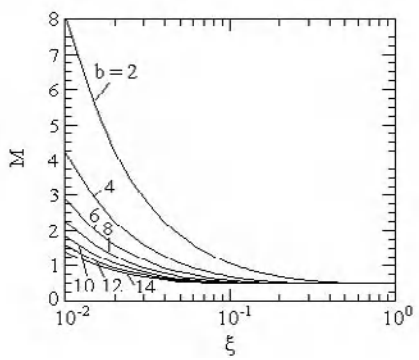


Figure 5.3. *Coefficient M in relation to damping factor, for various values of parameter b*

5.3. Damage created by simple shocks in static zone of the response spectrum

In the static zone of the spectrum, the response $\omega_0^2\, z(t)$ oscillates around the acceleration signal describing the shock and moves ever closer to the acceleration signal whenever the natural frequency f_0 of the system is larger (except for shocks which at the beginning of the signal have a zero rise time (infinite slope), like the rectangle pulse or the initial peak sawtooth pulse).

At a first approximation, the response peaks can be considered to be those of the shock itself, and the residual response has a negligible amplitude.

Example 5.1.

Consider a half-sine shock of amplitude $50\ \mathrm{m\ s^{-2}}$ and duration 10 ms. Figures 5.4 and 5.5 show the responses of systems with natural frequencies equal to 500 Hz and 1,000 Hz and for a damping factor $\xi = 0.05$

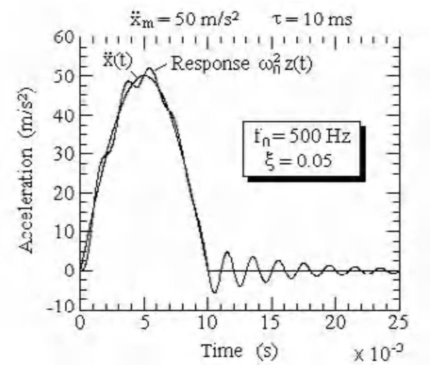


Figure 5.4. *Response at 500 Hz*

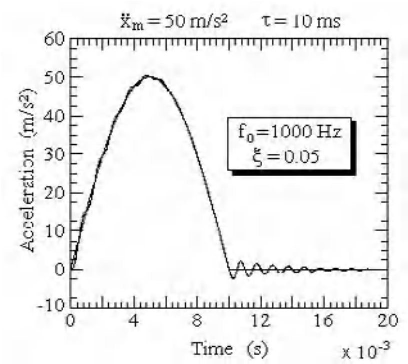


Figure 5.5. *Response at 1000 Hz*

This property can be used to calculate the fatigue damage associated with this type of simple shock. In the static zone, each shock applied to the system leads to damage for only one half-cycle, which is equal, when $\ddot{x}_m$ is the shock amplitude, to

$$D \approx \frac{K^b}{C} z_m^b \qquad [5.15]$$

At high frequencies, the shock response spectrum $S_{RS} = \omega_0^2\ z_m$ tends towards $\ddot{x}_m$, yielding

$$D \approx \frac{K^b}{C} \frac{\ddot{x}_m^b}{\omega_0^{2\,b}} \qquad [5.16]$$

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 6

Influence of Calculation: Conditions of ERSs and FDSs

This chapter shows the influence of calculation parameters for extreme response spectra and fatigue damage spectra.

6.1. Variation of the ERS with amplitude and vibration duration

Calculations made to obtain an ERS assume that the reference one-degree-of-freedom system is linear. Because of this, the ERS is directly proportional to the signal amplitude at each frequency.

Duration does not occur in the ERS calculation of sinusoidal vibrations (if it is large enough for a permanent system to be established).

The transitory response q_T has an amplitude equal to 1/N of its first peak after a number n of cycles such that:

$$\frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}} = \frac{1}{n}\ln\,N$$

or (Volume 1, Chapter 3):

$$n = \frac{\sqrt{1-\xi^2}}{2\,\pi\,\xi}\,\ln N \qquad [6.1]$$

For small ξ, it becomes:

$$n \approx \frac{\ln N}{2\,\pi\,\xi} \qquad [6.2]$$

hence:

$$n \approx \frac{Q\,\ln N}{\pi} \qquad [6.3]$$

Example 6.1.

For Q = 10, the number of cycles necessary for the transitory response to have a lower or equal to 1/100 of its first peak amplitude is equal to:

$$n = \frac{\sqrt{1-0.05^2}}{2\,\pi\,0.05}\ln 100$$

or $n \approx 15$ cycles.

Duration does not intervene in the case of a swept sine vibration either, as long as the sweep rate is slow enough for the system to have time to respond to its maximum value. It is generally the case for all tests defined to measure the resonance frequencies or for specifications extracted from standards.

In the case of random vibrations, the ERS is proportional to the rms value of the excitation and varies as the square root of its PSD. The probability of finding a larger peak in the response of a one-degree-of-freedom system to a random vibration increases with the signal duration. The ERS involved is therefore a function of the vibration duration. However, this variation quickly becomes insensitive, since time only occurs in the form of the square root of a logarithm [2.7]:

$$ERS = (2\,\pi\,f_0)^2\; z_{rms}\,\sqrt{2\ln n_0^+\,T} \qquad [6.4]$$

Example 6.2.

Figure 6.1 shows variations of the ERS value ratio to the rms value of the one-degree-of-freedom system response, where z_{rms} is multiplied by $\omega_0^2 = (2\pi f_0)^2$ according to the mean number of response cycles (product of the mean frequency n_0^+ by duration).

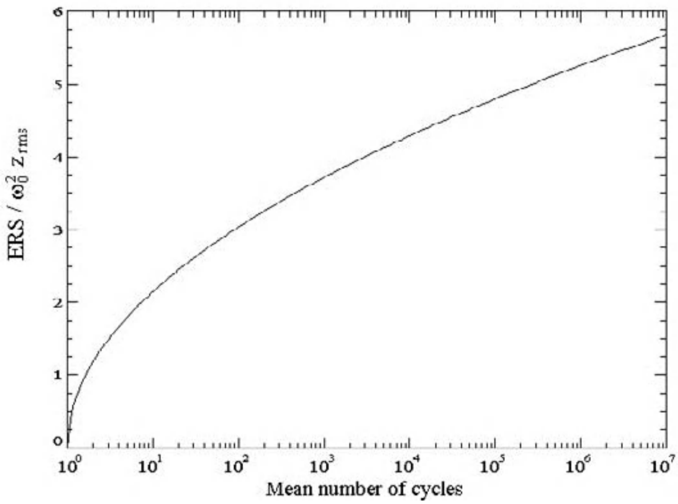


Figure 6.1. *Influence of the mean number of response cycles on the ERS*

ERSs obtained from the PSD of the "aircraft" vibration (Figure 6.2), calculated for duration equal to 1 hour, 25 hours, 50 hours and 100 hours respectively are drawn in Figure 6.3.

We can observe that:

– the difference between ERSs is low when duration is doubled;

– as the number of cycles increases with natural frequency, the gap between ERSs is more significant at high frequencies.

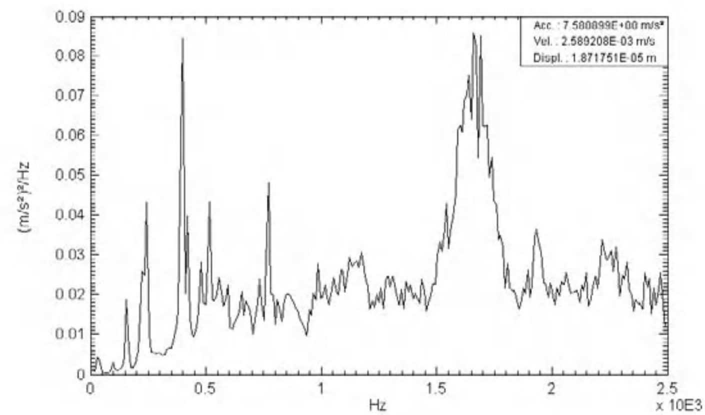


Figure 6.2. *PSD of an "aircraft transport" vibration*

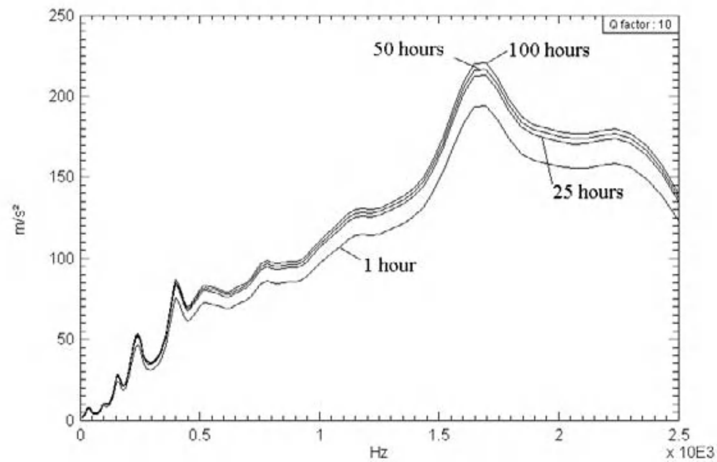


Figure 6.3. *ERS of an aircraft vibration for different durations*

Figure 6.4 shows theoretical variations (Rayleigh distribution of response peaks) of the highest peak (divided by the rms value of the response) on average (ERS) and for a given up-crossing risk (URS), according to the number of cycles. At a natural frequency of 2,000 Hz, 14 hours of vibrations are needed on average to reach 6 times the rms value (10^8/2000/3600).

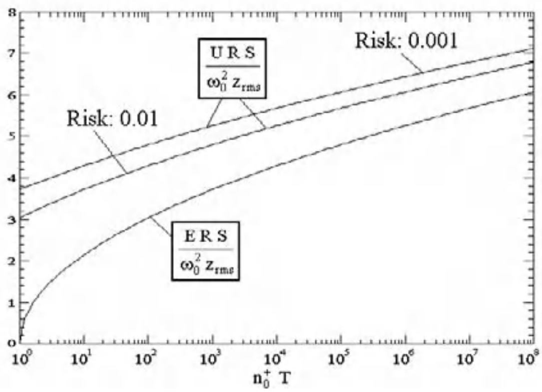


Figure 6.4. *Influence of the mean number of response cycles on ERSs and URSs*

6.2. Variation of the FDS with amplitude and duration of vibration

The Basquin relation $N\,S^b = C$ shows that fatigue damage varies with stress *S* at power b, in other words, since the structure is assumed to be linear, with the vibration amplitude at power b. For a random vibration, damage is proportional to the rms value at power b, or PSD at power b/2.

Fatigue damage is directly proportional to the number of applied cycles, i.e. to the duration of vibration, regardless of its nature (sinusoidal or random).

These properties can be found directly in expressions of damage created by these different types of vibrations (previous chapters).

6.3. Should ERSs and FDSs be drawn with a linear or logarithmic frequency step?

A logarithmic frequency step is better adapted to the analysis of dynamic mechanical phenomena. With a limited total number of points, it makes it possible to describe the low frequency part of the spectrum better, and to obtain a regular distribution of points in a logarithmic representation based on the axis of frequency. This choice is well adapted for vibrations of road environment.

Example 6.3.

The “truck” vibration FDS (Figure 6.5) was consecutively calculated and traced with a linear step and with a logarithmic step (Figure 6.6).

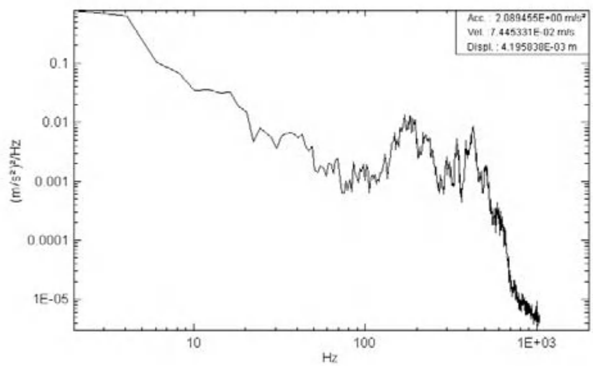


Figure 6.5. *PSD of a vibration measured in a truck*

Even with a large number of points (200 points), the curve definition with linear step is not enough at low frequencies (up to 4 Hz). This frequency range grows when the number of points decreases.

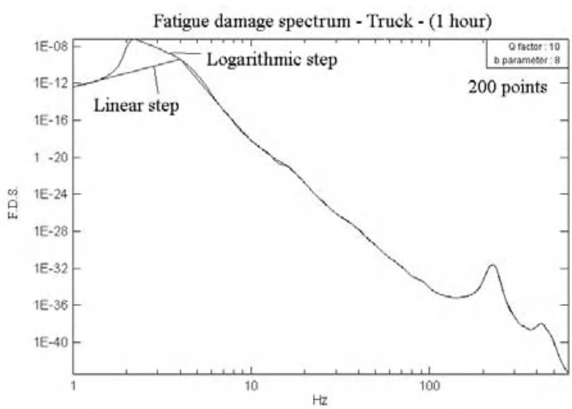


Figure 6.6. *Influence of the type of step (linear or logarithmic) on FDS*

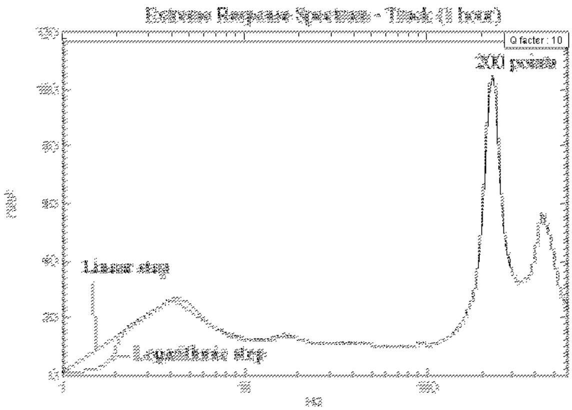


Figure 6.7. *Influence of the type of step (linear or logarithmic) on ERS*

This drawback can be insignificant however if the structure submitted to the vibration has no resonance in the range of frequencies involved.

Beyond 4 Hz, the type of distribution of frequency points has little influence.

ERS is only affected between 1 Hz and 2 Hz (Figure 6.7) in this example. The previous remarks on FDS also apply here.

6.4. With how many points must ERSs and FDSs be calculated?

ERSs and FDSs curves are:

– not very sensitive to the number of PSD definition points, and thus to details and small variations of this PSD;

– relatively smooth.

Because of this, it is not necessary to calculate them with a large number of points. We can use 200 points for safety purposes, knowing that 100 points are often enough.

However, in the case of vibrations with a content rich in low frequencies (truck transport for example), the number of PSD points must be sufficient to describe this frequency range correctly (knowing that distribution of PSD points is linear).

Example 6.4.

Figures 6.8 and 6.9 show FDSs and ERSs of aircraft vibration (PSD, 256 points, Figure 6.2) calculated with 50, 100, 200 and 400 points. We can observe that the curve defined by 50 points is different from the other three (insufficient peak sampling).

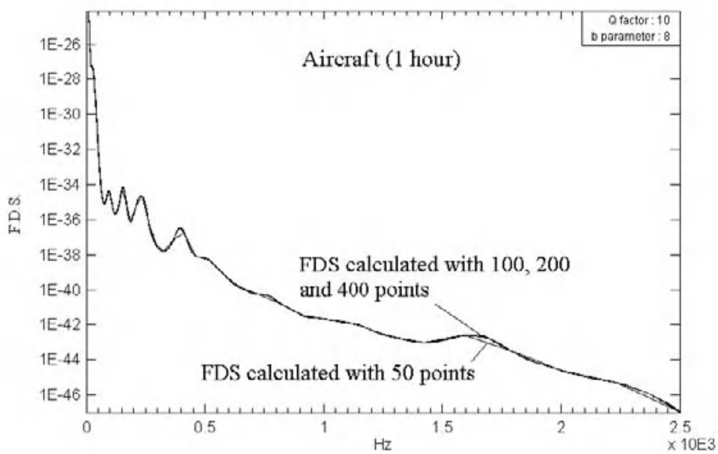


Figure 6.8. *Influence of the number of FDS calculation points*

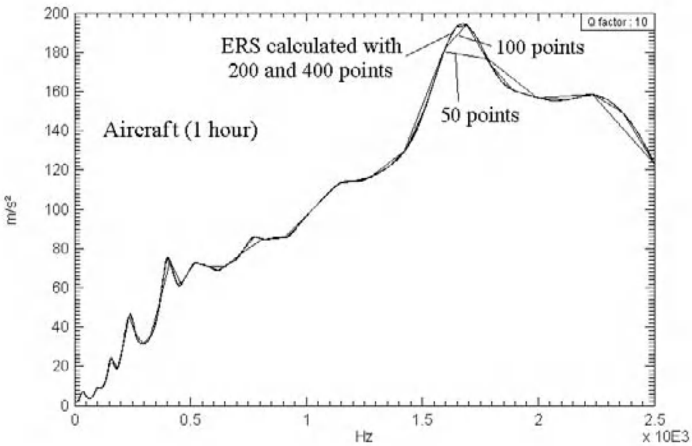


Figure 6.9. *Influence of the number of ERS calculation points*

6.5. Difference between ERSs and FDSs calculated from a vibratory signal according to time and from its PSD

Regardless of the nature of the vibratory signal studied, it is still possible to calculate ERSs and FDSs from a signal based on time (Chapters 2 and 4). It is generally the only method possible, except for Gaussian stationary random vibrations. In this case, these spectra can be obtained from the PSD of a signal sample.

When both calculation possibilities exist, what are the differences between the resulting spectra? Which method is better?

The advantage of calculating from the signal is to be able to process any type of vibration, stationary or not, whether it is a shock, a purely random or composite vibration, made up of one or more sine superimposed to a wide band random, or other, vibration.

The disadvantages of this calculation mode involve the random character of the signal:

– ERS (and FDS) thus determined is that of the sample of the signal involved. At each frequency, we have one of the possible values among those where the mean is the ERS (or FDS) point obtained from the PSD. Another chosen sample in a different time interval from the same recording, even if it is stationary, will statistically lead to a different ERS.

– For the ERS, the result is a function of the sample duration, the probability of finding a given amplitude peak in the response increasing with this duration.

– Calculation is relatively intensive: it requires a large sampling frequency of approximately 7 to 10 times the maximum spectrum frequency (section 6.8) and a large enough sample to be statistically representative, leading to large computer files and longer calculation times.

ERS and FDS calculation in the case of Gaussian instantaneous values distribution is much quicker. The PSD used is only defined in 512 or 1,024 points and its determination only requires signal sampling with a frequency of 2 times the maximum signal frequency (when filtered by a low-pass filter before its digitization).

ERSs and FDSs deducted from a PSD have a statistical character:

– the ERS gives each natural frequency the greatest response that can be observed, on average[1], in a vibratory signal lasting T. This definition is sufficient to make comparisons of severity and to determine test specifications;

– the FDS indicates the mean damage created by a random vibratory signal lasting T. We can also evaluate the standard deviation of damage.

When the signal is correctly sampled (at least 7 times the maximum frequency of the acceleration signal):

– the resulting FDSs from the signal and its PSD are extremely close;

– ERSs are close, but still separate. The difference is greater than for FDSs. The ERS provides the largest response peak. Statistically, the amplitude of this peak can vary with the sample chosen and its duration. This phenomenon can particularly be observed at high frequency; the ERS from a PSD tends toward a value close to ([2.25]):

$$\mathrm{ERS}_{\infty} = \ddot{x}_{rms} \sqrt{2 \ln(f_{m\ddot{x}}\ T)} \qquad [6.5]$$

(T = duration of the vibration, $f_{m\ddot{x}}$ = mean frequency of the input PSD, $\ddot{x}_{rms}$ = rms value of the analyzed vibration), whereas the ERS determined from the signal tends toward the greatest peak amplitude of the signal sample involved.

The FDS is not as sensitive to this random character because it is calculated by considering all the peaks in the response.

With the power of current computational tools, we can be tempted to calculate ERSs and FDSs from the signal in all cases, by carrying out all calculations without an initial analysis to separate the different phases or events.

Even though calculation times are acceptable today, this method is data intensive.

Indeed, in this case we use the complete vibratory signal that must be digitized with a large sampling frequency, higher than the one used to calculate a PSD (section 6.8).

1. We could focus on the value with a given risk of not being exceeded: it is the URS.

In order to calculate a PSD with a correct statistical error (good practice requires that we do not exceed 0.15), a signal sample of a few dozen seconds is all that is needed. We can even show that the statistical error, even if it is very large, has no influence on these spectra (sections 6.6 and 6.7).

Even though it can take some time, prior analysis can make it possible to isolate all specific events according to their characteristics, and to choose the most appropriate calculation method. It is very useful in understanding the phenomena and the quality of their laboratory simulation.

Example 6.5.

An "aircraft" vibration characterized by a digitized signal sample with a much higher frequency than the maximum signal frequency (18.6 times 2500 Hz).

FDSs calculated from the signal and its PSD (duration 30 s, Q = 10, b = 10) between 5 Hz and 3500 Hz can be superimposed (Figure 6.10).

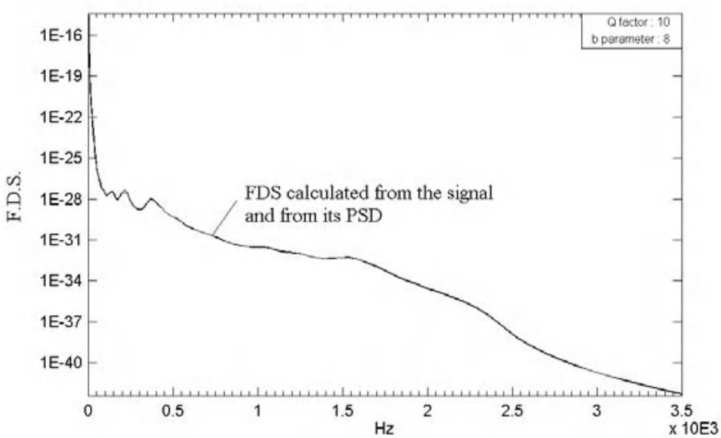


Figure 6.10. *FDS calculated from a signal according to oversampled time and from its PSD*

Corresponding ERSs (Figure 6.11) are very close up to 2,500 Hz, beyond this they differ slightly more: the largest peak of the processed signal sample is slightly higher than the mean value given by the ERS deducted from the PSD.

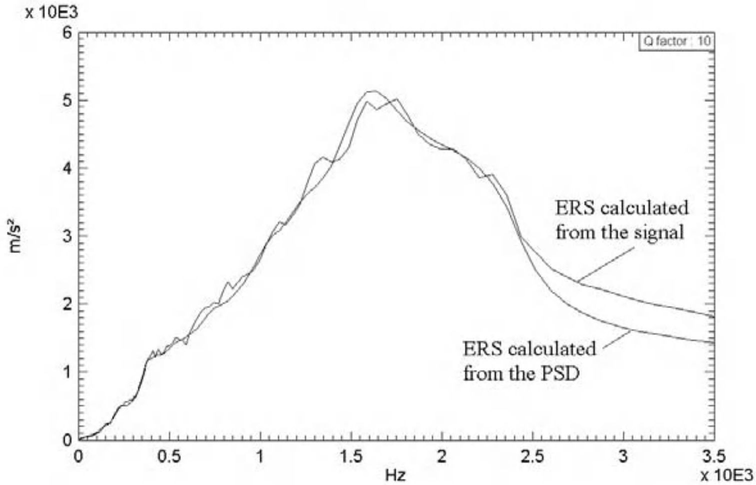


Figure 6.11. *ERS calculated from a signal according to oversampled time and from its PSD*

Influence of sample duration

Consider ERSs and FDSs of this same oversampled vibratory signal, calculated from samples with durations consecutively equal to 30 s, 25 s, 20 s, 15 s, 10 s, 5 s, 2 s and 1 s. FDSs are determined, to facilitate comparison, for the same duration of 30 s (by a rule of three).

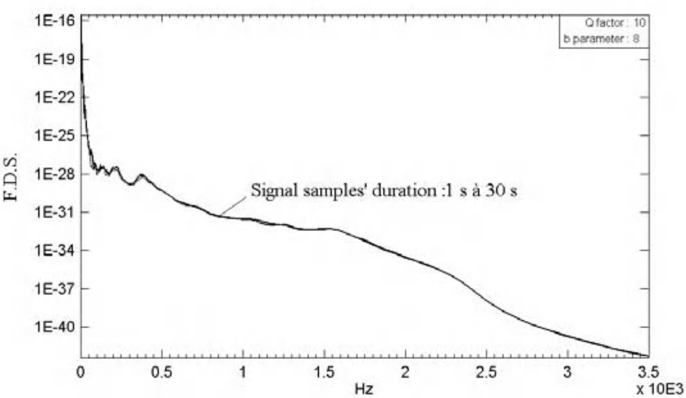


Figure 6.12. *Influence of the duration of the signal sample of the FDS*

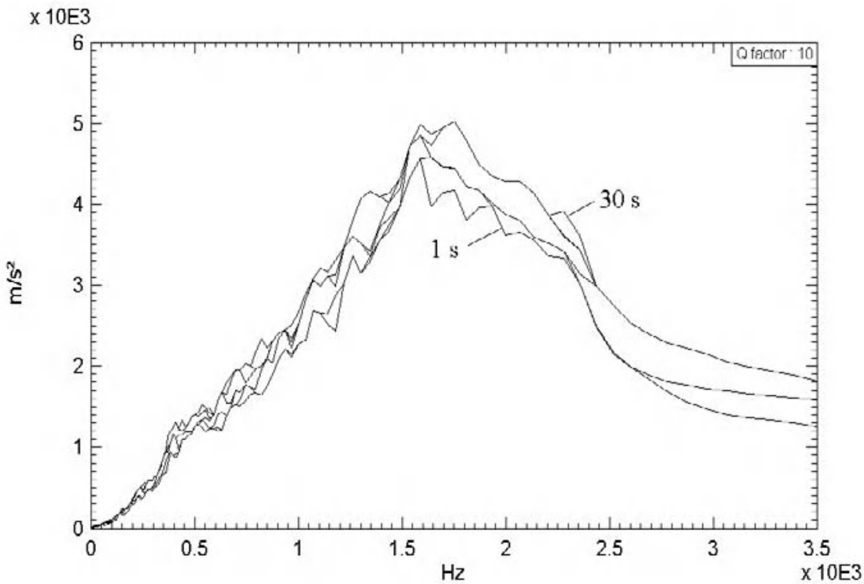


Figure 6.13. *Influence of the duration of the signal sample of the ERS*

We observe very little difference between FDSs (Figure 6.12). For statistical reasons already addressed, we can observe a slightly greater difference with ERSs, since the high probability of the presence of a peak is all the larger as the sample duration is longer.

We find these results when the signal is sampled with a frequency satisfying Shannon’s theorem (twice the maximum signal frequency).

Example 6.6.

Consider the accelerometric signal in Example 6.5 sampled with a frequency equal to 5,750 Hz (2.3 times 2,500 Hz).

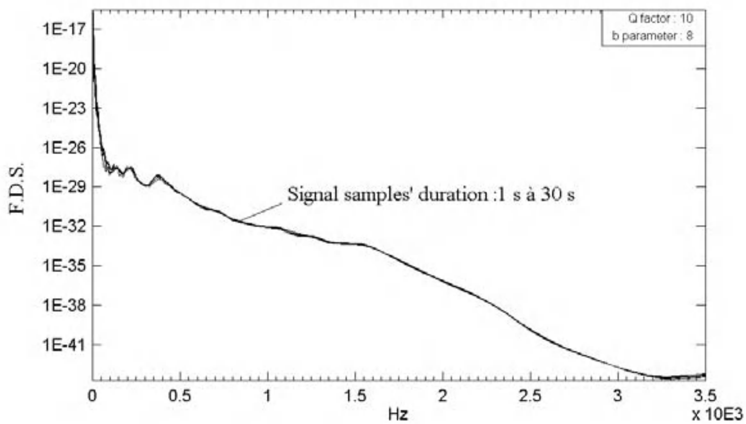


Figure 6.14. *Influence of the duration of the signal sample (digitized according to Shannon) on the FDS*

FDSs calculated for samples lasting between 1 s and 30 s are superimposed (Figure 6.14). We can observe the same gaps as before between ERSs (Figure 6.15).

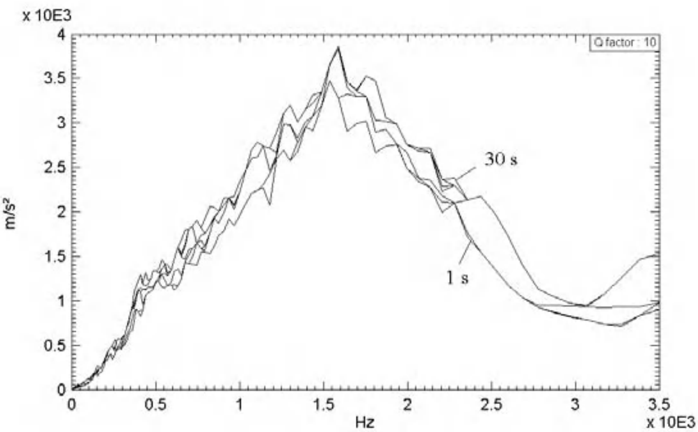


Figure 6.15. *Influence of the duration of the signal sample (digitized according to Shannon) on the ERS*

However, spectra calculated from sampled signals based on Shannon's theorem, and with a much greater frequency, are slightly different (Figures 6.16 and 6.17).

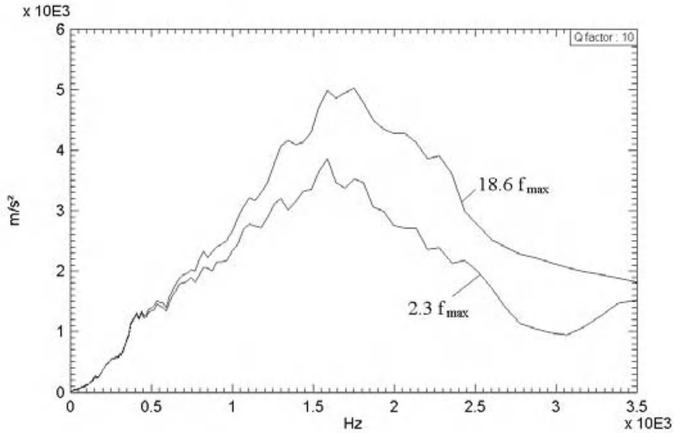


Figure 6.16. *Comparison of ERSs obtained from sampled signals with a frequency equal to 2.3 times and 18.6 times the maximum signal frequency*

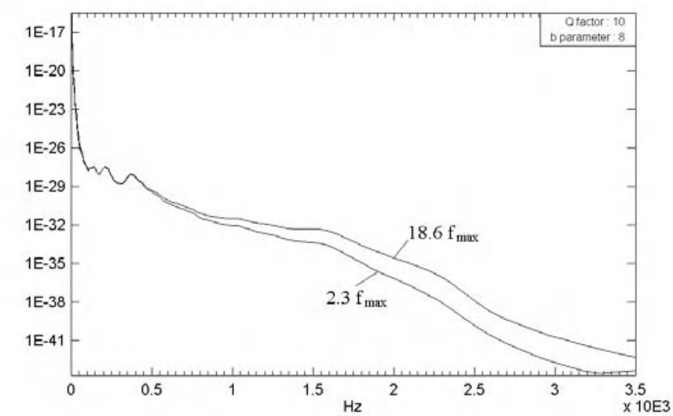


Figure 6.17. *Comparison of FDSs obtained from sampled signals with a frequency equal to 2.3 times and 18.6 times the maximum signal frequency*

We should note that a Shannon-based sample can lead to a larger ERS, as is shown in Figure 6.18, where temporal responses of a one-degree-of-freedom system, calculated from the same vibratory signal consecutively sampled with a frequency similar to Shannon's and with a frequency 9.3 times greater, are superimposed

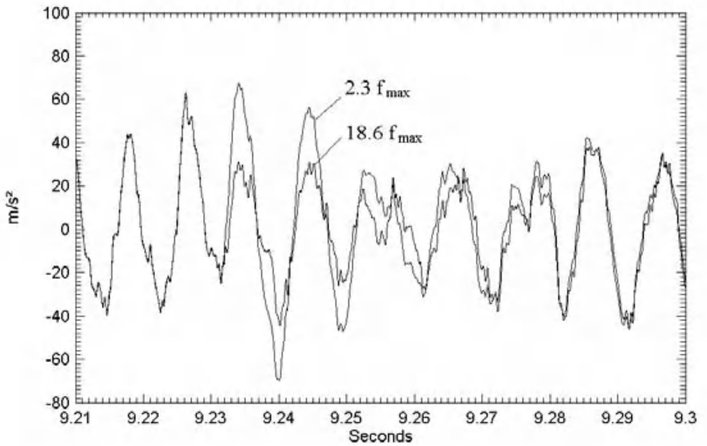


Figure 6.18. *Comparison of one-degree-of-freedom system responses obtained from sampled signals with a frequency equal to 2.3 times and 18.6 times the maximum signal frequency*

6.6. Influence of the number of PSD calculation points on ERS and FDS

The PSD of a vibration can be calculated with different values of the number of points, or the frequency interval between two consecutive points. The sample duration of the signal involved is set, and the choice of the number of points has an influence on the statistical error made during the PSD determination. This choice has no influence on FDSs and ERSs calculated from the PSD, except at low frequency, since the first PSD point has low frequency and consequently the frequency step is more or less low.

Example 6.7.

Figure 6.19 shows FDSs deducted from PSDs of the same signal (vibration of truck transport) and defined over 64, 256, 512, 1,024 and 2,048 points respectively. We can observe that the only influence of this number of points involves low frequencies, since the first PSD point has a different frequency.

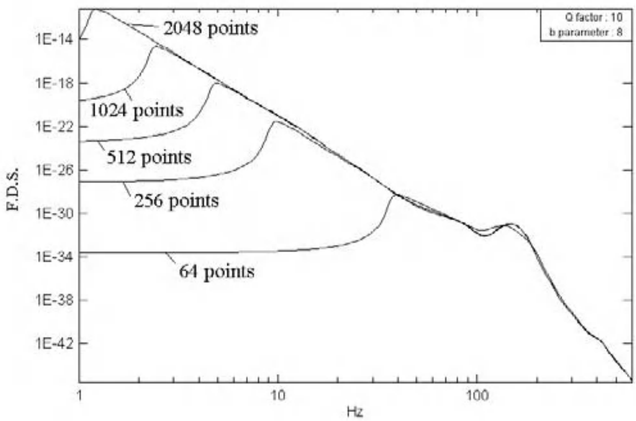


Figure 6.19. *Influence on FDS of the number of points of PSD definition*

Number of PSD points	Frequency step
64	12.19
256	3.05
512	1.52
1024	0.76
2048	0.38

Table 6.1. *PSD frequency steps in Figure 6.19*

Beyond approximately 30 Hz, FDSs are extremely close, even in the case of the PSD defined with 64 points. This property is used to establish a test specification with a low number of values from an FDS defined over 200 points. We can observe the same result at low frequency with ERSs. Beyond this range, these spectra are insensitive to the number of PSD definition points when higher than or equal to 256 (Figure 6.20).

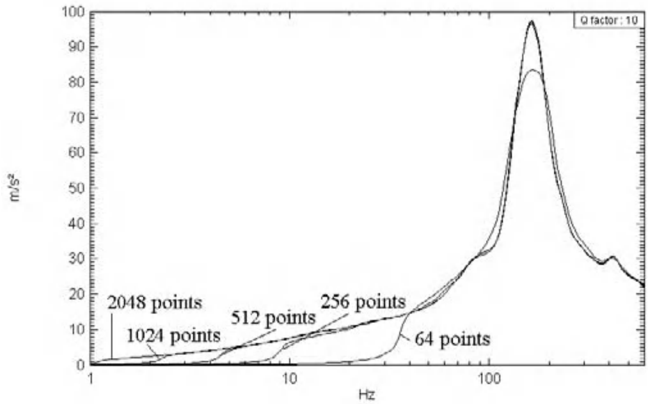


Figure 6.20. *Influence on ERS of the number of points of PSD definition*

Example 6.8.

Figures 6.21 and 6.22 give the PSD of an "aircraft" vibratory signal lasting 30 s (195,560 points) calculated with a number of points respectively equal to 256 and 1,024 frequency step: 12.6 Hz and 3.15 Hz, statistical error: 0.051 and 0.103).

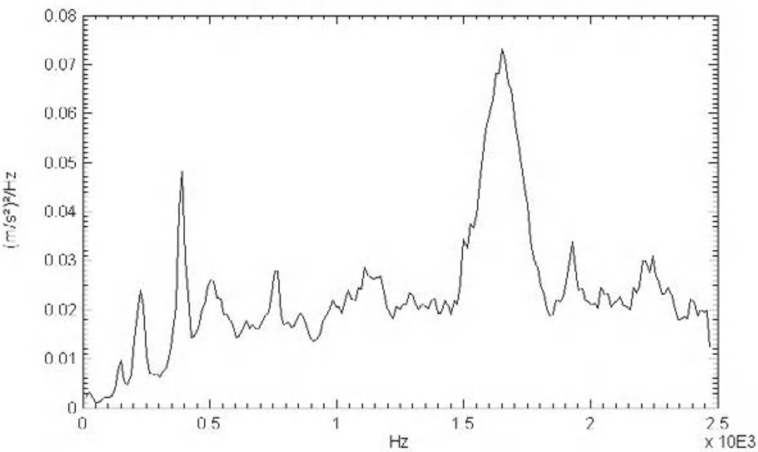


Figure 6.21. *"Aircraft" PSD calculated with 256 points (Δf = 12.6, ε = 0.051)*

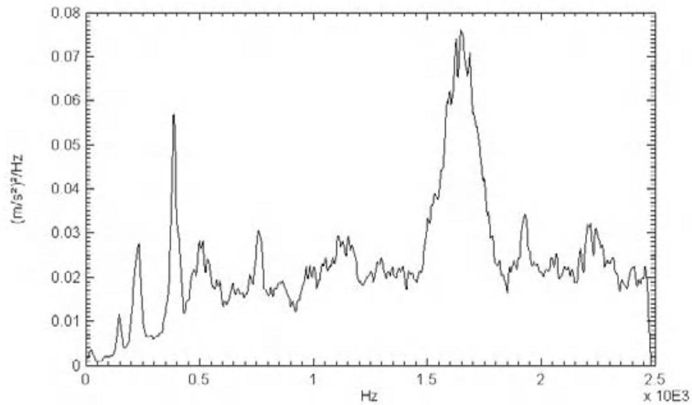


Figure 6.22. *"Aircraft" PSD calculated with 1024 points (Δf = 3.15 Hz, ε = 0.103)*

FDSs and ERSs calculated from these two PSDs (Q = 10, b = 8) are extremely close (Figures 6.23 and 6.24).

They are also very close to spectra obtained from a PSD of the same event calculated with a very strong statistical error, 36% (Figure 6.25).

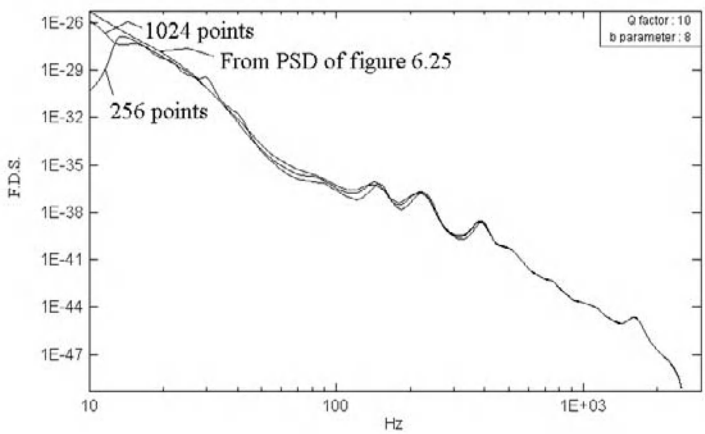


Figure 6.23. *Influence of the number of PSD points on FDS (Q = 10, b = 8, duration 30 s)*

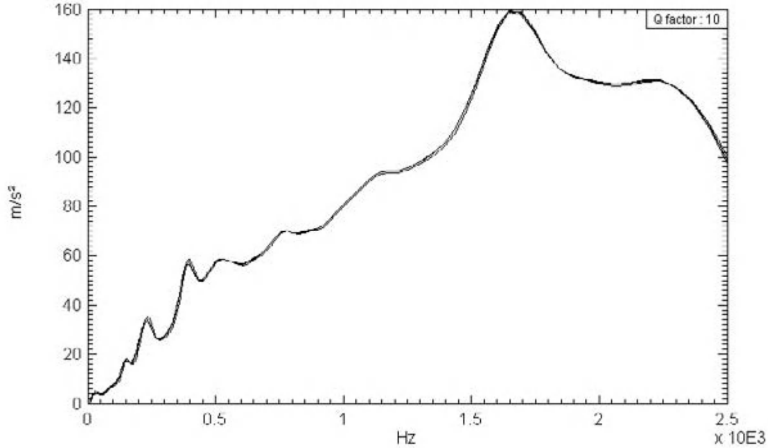


Figure 6.24. *Influence of the number of PSD points on ERS (Q = 10)*

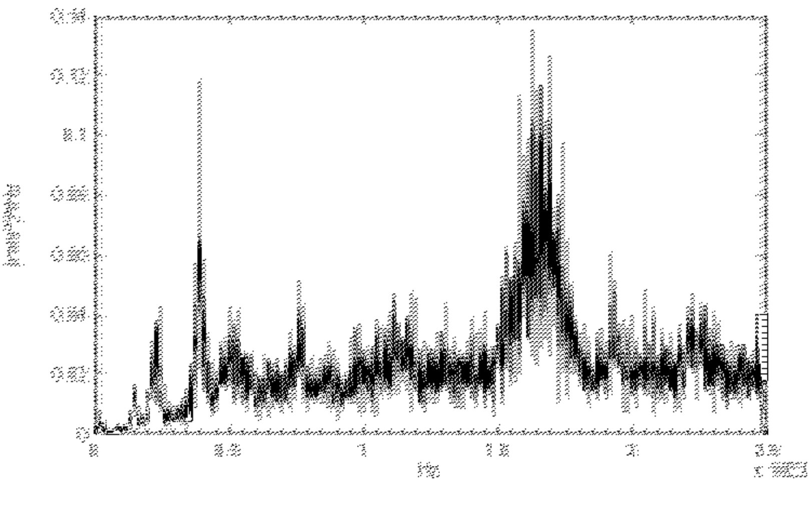


Figure 6.25. *PSD calculated with an error of approximately 36%*
(Δf = 1.58 Hz, 2,048 points)
(Figure 4.43, Volume 3)

6.7. Influence of the PSD statistical error on ERS and FDS

According to good engineering practices, the statistical error with which a PSD is calculated should generally be lower than 0.15. However, we may be unable to respect this value. Fortunately, we can observe that this error has very little influence on ERS and FDS, even if it is large (up to approximately 0.50).

Example 6.9.

In order to highlight this property, we have used PSDs presented in Figures 4.42 to 4.46 in Volume 3, calculated from the same vibration with a statistical error consecutively equal to 0.50, 0.36, 0.25, 0.18 and 0.13.

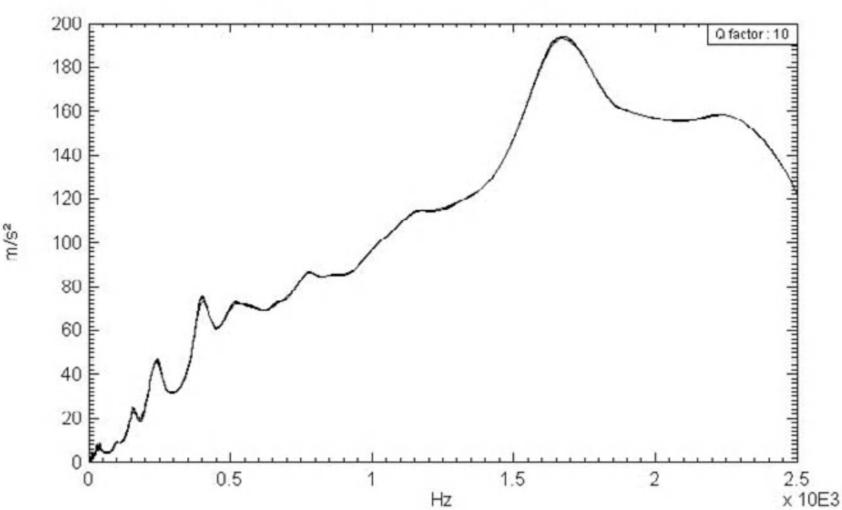


Figure 6.26. *ERS from a PSD calculated for several values of the statistical error (0.13, 0.18, 0.25, 0.36 and 0.50)*

ERS and FDS obtained from these PSDs, for Q = 10 and b = 8, are practically the same (Figures 6.26 and 6.27).

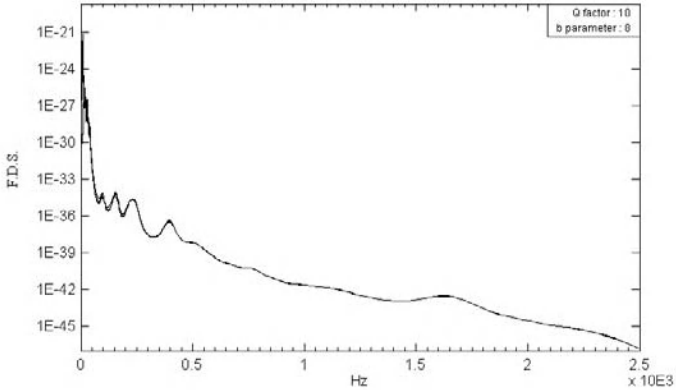


Figure 6.27. *FDS from a PSD calculated for several values of the statistical error (0.13, 0.18, 0.25, 0.36 and 0.50)*

6.8. Influence of the sampling frequency during ERS and FDS calculation from a signal based on time

The sampling frequency of vibratory signals is generally chosen to respect Shannon's theorem (twice the maximum signal frequency). In order to take into account a possible (or necessary) filtering before PSD calculation, the rule is often modified to favor 2.6 times the sampling frequency (Volume 1, section 1.5).

For the calculation of an SRS, we have seen (Volume 2, section 2.12) that the signal must be sampled with a frequency greater than 10 times the maximum SRS frequency (which can lead to an error of approximately 5%). The signal must in fact be defined with enough points for it not to be deformed, and the response of the one-degree-of-freedom system must have enough points to be able to pick up the amplitude of the largest peak with low error.

In the case of vibrations, these two requirements also exist *a priori*. In general, the ERS and FDS are calculated in the frequency range of the signal: the maximum frequency of these spectra is close to the greatest frequency contained in the signal. In this hypothesis, we will see that it is necessary to correctly represent the signal for the sampling frequency to be of approximately 7 times the maximum signal frequency.

If the one-degree-of-freedom system's natural frequency is greater than the maximum signal frequency, the response seems to become very close to the input signal. This comment can be verified by considering the relation linking the PSD $G_{\omega_0^2 z}$ of the response of a mechanical transfer function H(f) system and the excitation PSD ($G_{\ddot{x}}$):

$$G_{\omega_0^2 z} = \left|H(f)\right|^2 G_{\ddot{x}} \qquad [6.6]$$

The transfer function of a one-degree-of-freedom system is equal to 1 at its origin, remains close to 1, then increases and goes through a maximum when the frequency is equal to the resonance frequency, and decreases and tends towards zero when the frequency leans toward infinity (Figure 6.28).

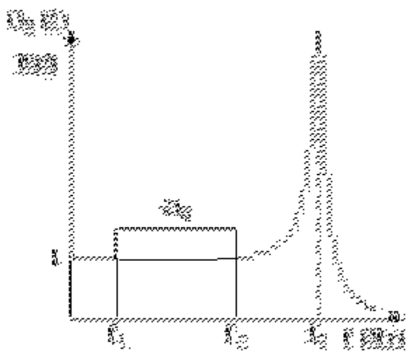


Figure 6.28. *Transfer function of a one dof system with a natural frequency that is large in relation to the maximum frequency of the random vibration*

If the excitation PSD is defined in an interval where the maximum frequency is sufficiently lower than the system's natural frequency, the response PSD, product of the excitation PSD by the square of the transfer function module, only contains the frequencies of the excitation.

The natural frequency is only found in the transitory part at the beginning of the response, in a very short period of time.

For random vibrations, in all cases, a sampling frequency approximately 7 times greater than the greatest signal frequency is enough to limit the error to 10 % (20 times for an error of approximately 5%).

ERS and FDS calculation from a signal based on time therefore requires a large number of points (and large files).

The use of a PSD is much more valuable: its calculation can be carried out with a sampled signal at 2 (or 2.6 times) its maximum frequency, as the number of points of PSD definition can be limited to 512 or 1024 (see section 6.6).

Example 6.10.

Random vibration is a signal created from a PSD defined between 10 Hz and f_{max} = 500 Hz, of amplitude G = 1 (m s^{-2})^2/Hz. This signal, lasting 30 s, is used for the calculation of an ERS and FDS between 1 Hz and 500 Hz.

The digitization frequency was chosen consecutively equal to twice, 4 times, 6.6 times and 40 times f_{max} (or 1,000 Hz, 2,000 Hz, 3,300 Hz and 20,000 Hz), this last very high value is considered as a reference.

We can verify that:

– the ERS calculated with a sampled signal based on Shannon's theorem (2 f_{max}) is very far from the ERS obtained with a signal sampled at 40 f_{max} (Figure 6.29). The error made is only lower than 10% if the sampling frequency is higher than 6.6 f_{max} (Figure 6.30);

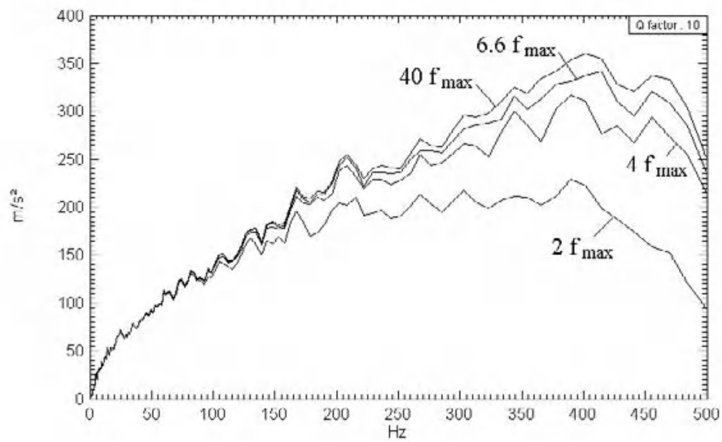


Figure 6.29. *ERSs calculated between 1 Hz and 500 Hz from signals sampled at 2 f_{max} (1,000 Hz), 4 f_{max} (2,000 Hz), 6.6 f_{max} (3,300 Hz) and 40 f_{max} (20,000 Hz)*

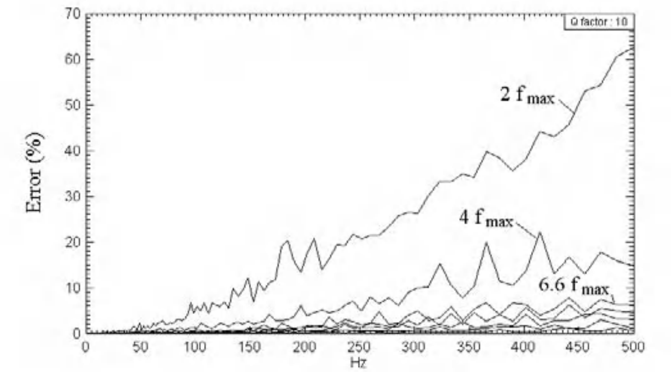


Figure 6.30. *Error made in ERS according to the sampling frequency of the signal*

– in order to obtain a 10% lower error with the FDS, we should have a sampling frequency higher than 10 kHz, or 20 times the maximum signal frequency (Figure 6.31). Because of the FDS sensitivity to the rms value of the vibration, it is possible to tolerate a more significant error on the FDSs, corresponding to approximately 7 f_{max} for example (Figure 6.32).

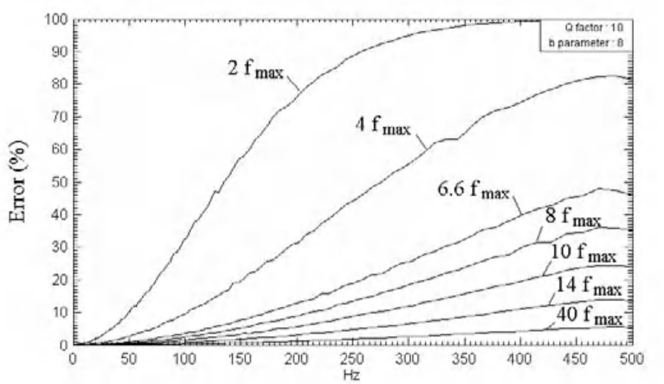


Figure 6.31. *Error made in FDS according to the sampling frequency of the signal*

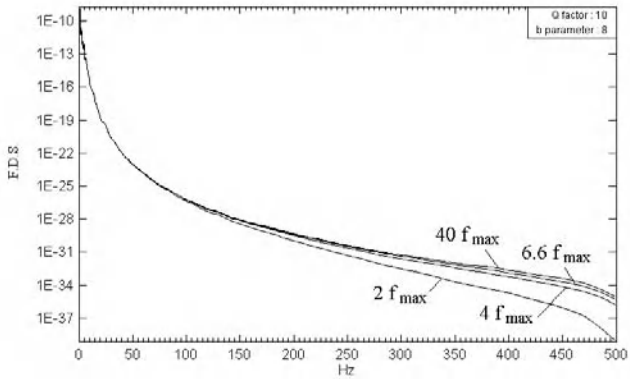


Figure 6.32. *FDSs calculated from sampled signals at 2 f_{max} (1,000 Hz), 4 f_{max} (2,000 Hz), 6.6 f_{max} (3,300 Hz) and 40 f_{max}, or 20,000 Hz (reference)*

In order to obtain an acceptable result, we must sample random vibrations with a frequency of approximately 7 times the maximum signal frequency, or 3.5 times more than the value supported by Shannon's theorem.

Example 6.11.

Consider the signal created from the PSD in Figure 6.33 by choosing a sampling frequency respectively equal to 1170 (twice the maximum PSD frequency), 2,340 and 4,680 points/s. This signal does not contain frequencies higher than 585 Hz, making the use of a low-pass filter, and thus the 2.6 factor, useless.

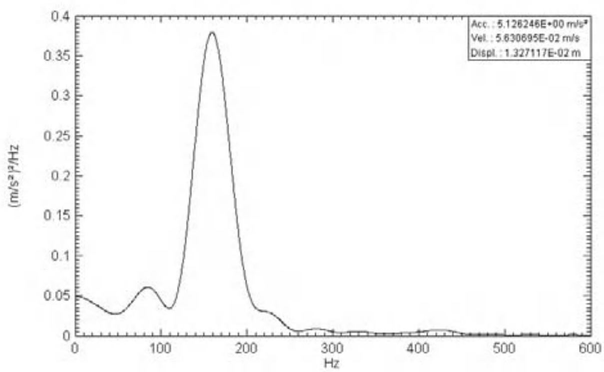


Figure 6.33. *PSD of a vibration measured during truck transport (Δf = 0.39 Hz, e = 0.13)*

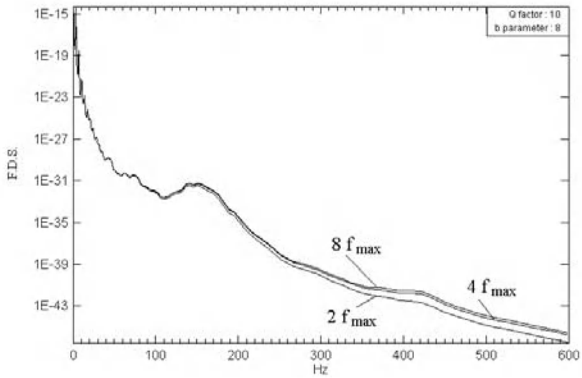


Figure 6.34. *Influence on the FDS of the sampling frequency of the vibratory signal (truck transport)*

Figure 6.34 shows FDSs calculated directly from these signals. We can observe that the application of Shannon's theorem does not provide a sufficient number of points to allow us to obtain the correct FDS values at high frequency. And yet, the spectrum of this vibration is practically equal to zero when 500 Hz is reached. The need for a largest number of points is also felt for calculating ERSs (Figure 6.35).

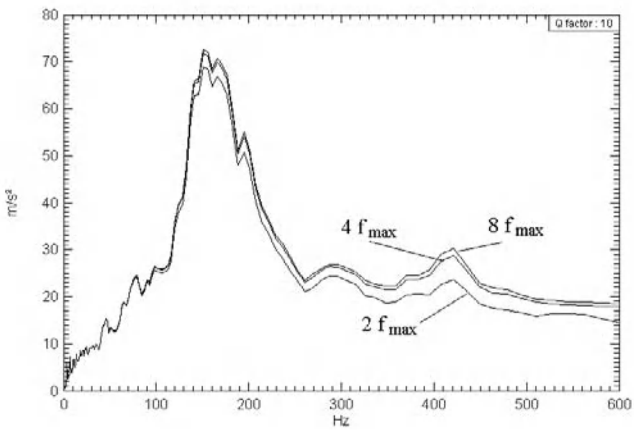


Figure 6.35. *Influence on the ERS of the sampling frequency of the vibratory signal (truck transport)*

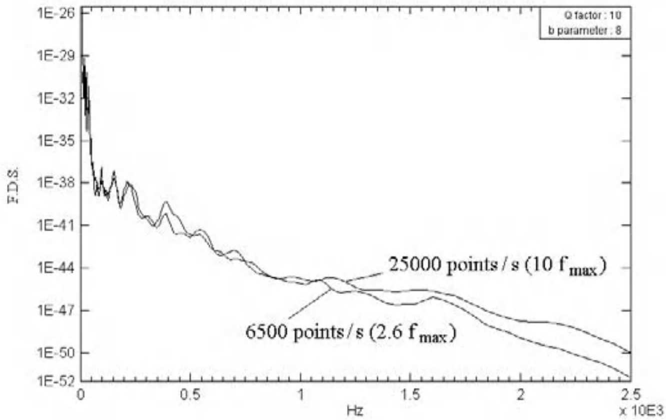


Figure 6.36. *Influence on the FDS of the sampling frequency of the vibratory signal (aircraft transport)*

The same observation can be made for signals from a vibration PSD with higher frequency. Figure 6.36 shows the gap between FDSs calculated from a sampled signal with frequencies respectively equal to 2.6 and 10 times the maximum PSD frequency. The difference is slightly greater at high frequency. It is found in ERSs (Figure 6.37).

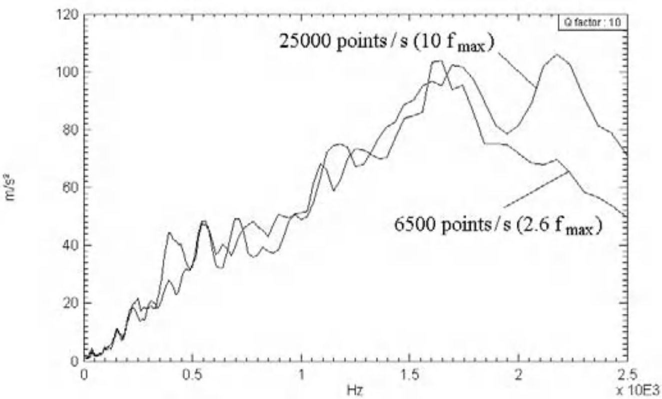


Figure 6.37. *Influence on the ERS of the sampling frequency of the vibratory signal (aircraft transport)*

This need is even more marked in the case of periodic or swept sine vibrations, alone or superimposed onto a random vibration.

Example 6.12.

The signal studied is made up of the sum of:

– a random vibration with a constant PSD between 10 Hz and 200 Hz and with 1.053 (m $s^{-2})^2$/Hz amplitude (14.14 m s^{-2} rms value);

– a logarithmic swept sine vibration between 10 Hz and 200 Hz, with amplitude of 20 m s^{-2} and lasting 200 s (1.3 bytes per minute scanning speed).

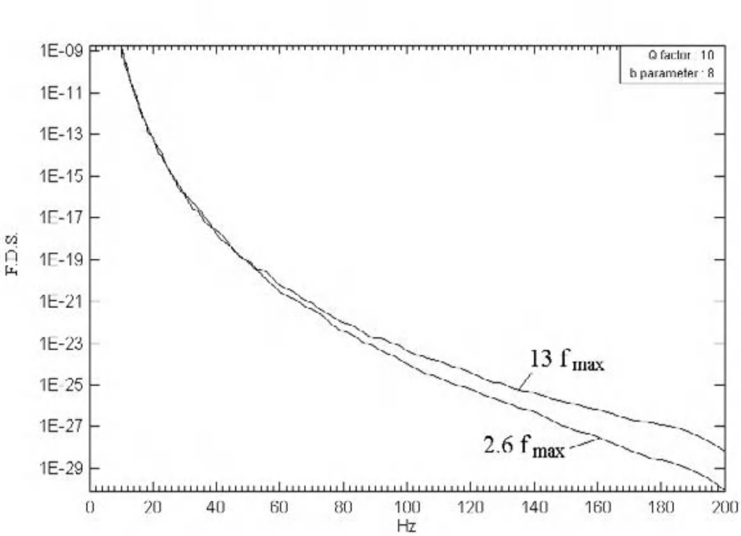


Figure 6.38. *FDS of a swept sine vibration superimposed to white noise, for two sampling frequencies*

Figures 6.38 and 6.39 show the significant differences at high frequency between FDSs and ERSs sampled at 520 points per second (2.6 times the maximum difference of the signal) and 2,600 points per second (13 f_{max}).

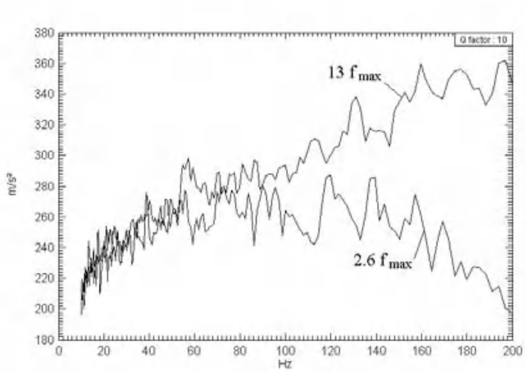


Figure 6.39. *ERS of a swept sine vibration superimposed to white noise, for two sampling frequencies*

These differences are justified when we simply imagine the signal of the swept sine vibration at the end of the sweeping process when it is sampled at 2.6 f_{max} (Figure 6.40). The number of points is not enough to correctly represent the sinusoidal cycles and especially their amplitude, to which FDSs and ERSs are sensitive.

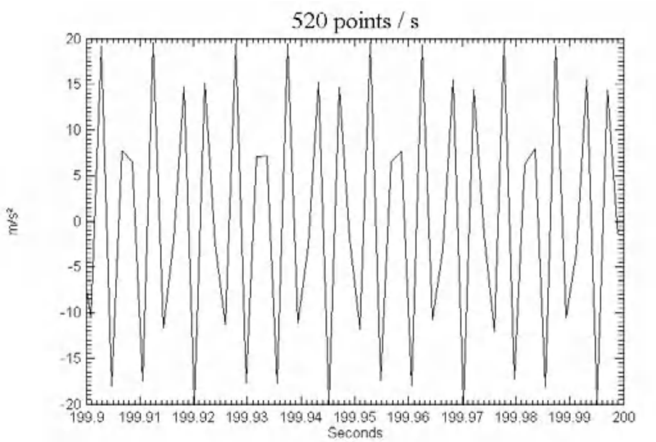


Figure 6.40. *Swept sine sampled at 2.6 times its maximum frequency (520 points per second)*

6.9. Influence of the peak counting method

When the FDS is directly determined from a signal based on time, it is necessary to calculate the relative response displacement of the mass of each one-degree-of-freedom system digitally, then to proceed to a count of *maxima* in order to establish a peak histogram giving the number of peaks at each amplitude. The FDS is then obtained with the help of the relation:

$$D = \frac{K^b}{2\,C} \sum_{i=1}^{m} n_i \; z_{p_i}^b \qquad [6.7]$$

There are many counting methods with some variations, the most widely used being the *rainflow* method (Volume 4, Chapter 3). We will compare, using an example, FDSs obtained with some of these methods:

– peak counting;

– counting of maximum peak between zero crossing;

– counting of level crossing;

– the *rainflow* counting method. We consider the results of this counting under the different forms of the results of a *rainflow* consecutively:

- peak histogram,
- the table of ranges supposedly recentered at zero,
- then the table of range considering the mean value of each range.

Example 6.13.

Consider a Gaussian stationary random vibration measured in an aircraft defined by a 5 s sample with 32 302 points, in the frequency band 6 Hz, 2500 Hz (PSD Figure 6.41). Duration is extrapolated at 1 hour.

In the peak counting method, we implicitly presume that each peak identified in the response signal is a half cycle starting and ending at zero. It is also the case during the operation of the peak table in the *rainflow* method.

Level crossings were used to determine signal peaks, by differentiating between crossings of two consecutive thresholds. Here again, peaks counted are considered as complete half-cycles.

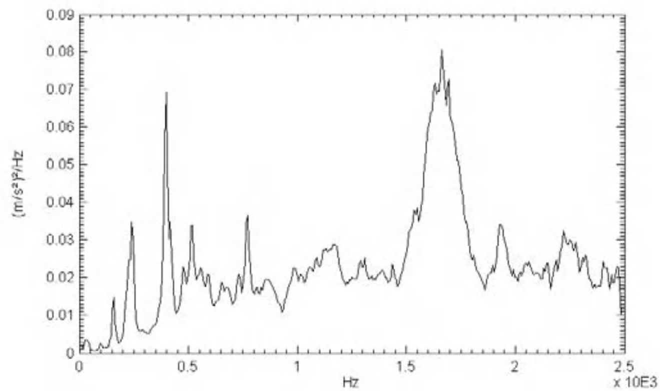


Figure 6.41. *PSD of aircraft vibration studied in this example*

With the counting method of the largest peak between two zero passings, we ignore all the other peaks between two zeros, regardless of the importance of the covered range.

The range counting method seems more interesting. It can be used by ignoring (or not) the mean value of ranges. The signal is then recentered (Figure 6.42), modifying stress values.

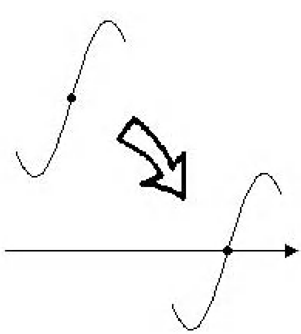


Figure 6.42. *Recentering of a range*

It seems better *a priori* to take the mean into account, for example by using modified Gerber [6.8] or Goodman relations [6.9]:

$$\sigma'_a = \frac{\sigma_a}{\left(1 - \frac{\sigma_m}{R_m}\right)} \quad [6.8]$$

$$\sigma'_a = \frac{\sigma_a}{\left[1 - \left(\frac{\sigma_m}{R_m}\right)^2\right]} \quad [6.9]$$

These expressions make it possible to determine the amplitude of a zero mean value stress (σ'_a) with the same severity as a non-zero mean stress σ_a σ_m. These relations involve the ultimate tensile stress R_m of the solicited material, usually unknown in our applications.

For this example, we assumed that the relative displacement of the highest peak of the response of the one-degree-of-freedom system is very close (90%) to the value leading to fracture, in order to better highlight the influence of the mean (Figure 6.43).

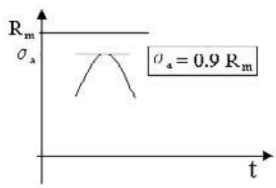


Figure 6.43. *The highest response peak reaches 90% of the stress leading to fracture*

All FDSs calculated in these conditions are very close (Figure 6.44), except in a small frequency interval between 10 Hz and 20 Hz. We find, in decreasing damage sequence, the FDSs obtained with:

– peaks;

– level crossings;

– ranges and means based on Goodman;

– peaks between two zero passings;

– ranges and means based on Gerber;

– ranges centered at zero.

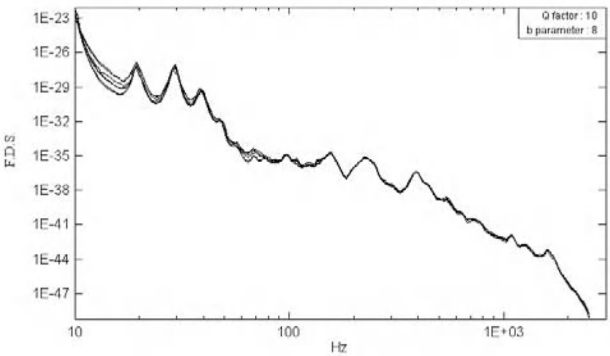


Figure 6.44. *Comparison of FDSs calculated from the different counting methods (b= 8, Q = 10)*

Apart from this small interval, we can say that, globally:

– the counting method has little influence;

– the consideration of the mean value of ranges is insignificant.

Example 6.14.

A non-stationary random vibration measured in a truck (Figure 6.45), defined over 1.25 seconds with 2,000 points.

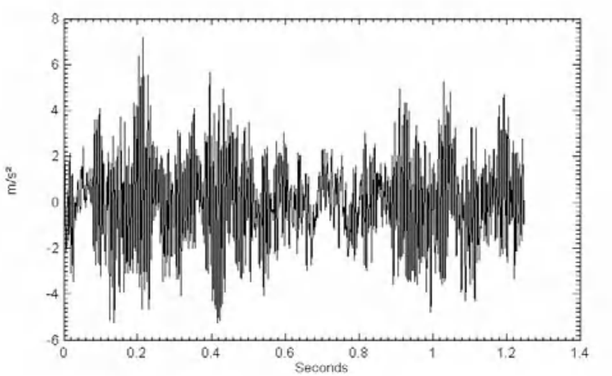


Figure 6.45. *Vibration measured on a truck (rms value 1.87 m s^{-2})*

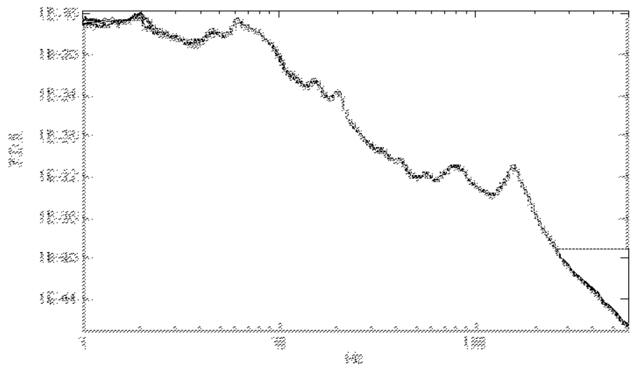


Figure 6.46. *Comparison of "truck" vibration FDSs calculated from the different counting methods (b= 8, Q = 10)*

FDSs calculated in the same conditions as in Example 6.13 are also very close, confirming our conclusions (Figure 6.46).

6.10. Influence of a non-zero mean stress on FDS

The effect of the consideration of the mean value of ranges for damage calculated was discussed in section 6.9.

Goodman and Gerber relations make it possible to replace, fatigue damage being equal, a composite σ_m, σ_a stress by a purely alternated (zero mean) stress σ'_a given by:

$$\sigma'_a = \frac{\sigma_a}{1 - \left(\frac{\sigma_m}{R_m}\right)^n} \qquad [6.10]$$

where n is a constant equal to 1 (Goodman) or to 2 (Gerber) depending on the case.

The alternating equivalent stress increases with the mean stress, and more so with the Goodman relation.

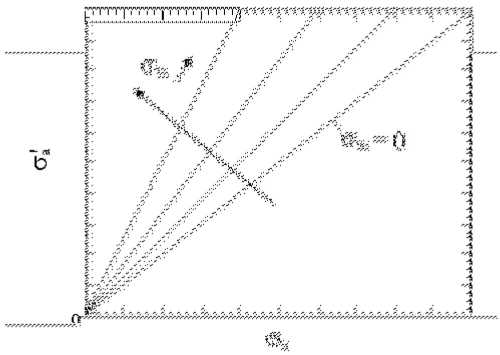


Figure 6.47. *Alternating equivalent stress according to Goodman based on alternating stress and mean stress*

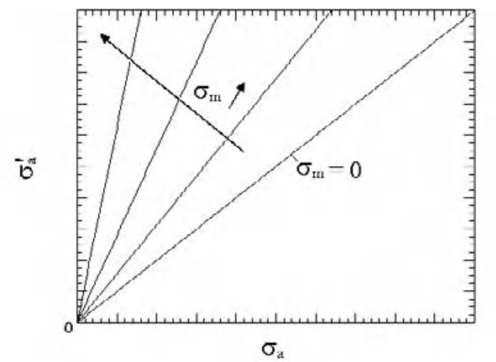


Figure 6.48. *Alternating equivalent stress according to Gerber based on alternating stress and mean stress*

We focus here on studying the influence of a static stress superimposed on the vibratory signal and particularly for a global constant stress Σ_a , and the influence of the mean stress relative value in relation to the alternating stress.

To simplify the methodology, we will consider a sinusoidal stress with amplitude σ_a that is superimposed onto a constant stress σ_m (Figure 6.49).

Or $\Sigma_a = \sigma_m + \sigma_a$.

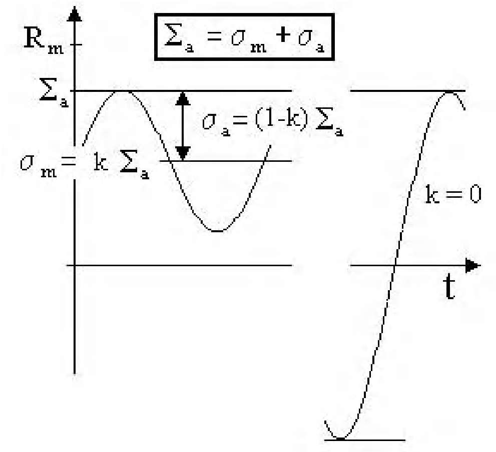


Figure 6.49. *Mean stress varies by retaining total stress*

Relation [6.10] involves ultimate tensile stress R_m. The effect of the mean stress can only be evaluated by taking the value of Σ_a into consideration in relation to this ultimate stress.

For this study, we will assume:

$$\sigma_m = k\,\Sigma_a \quad (0 \le k \le 1)$$

$$R_m = \alpha\,\Sigma_a \quad (\alpha > 1)$$

(R_m can only be higher than Σ_a, at the risk of instant fracture).

From these relations, it is easy to establish that:

$$\sigma'_a = \frac{1-k}{1-\left(\dfrac{k}{\alpha}\right)^n}\,\Sigma_a \tag{6.11}$$

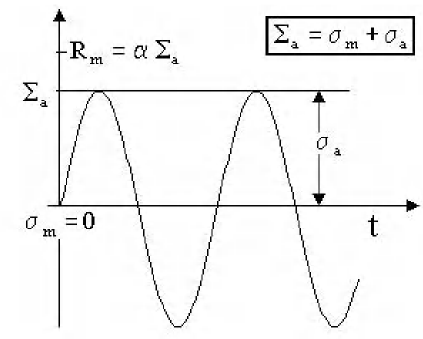


Figure 6.50. *Zero mean stress*

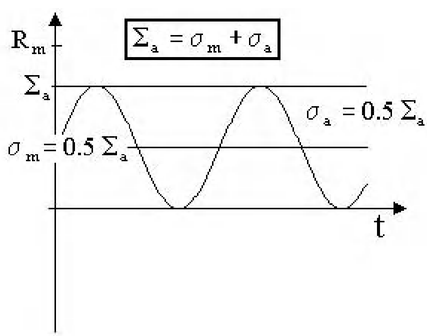


Figure 6.51. *Mean stress equal to alternating stress amplitude*

Consider three cases in which, for a constant value of the total stress Σ_a, the mean stress is either zero, or equal to the purely alternating stress σ_a, or very large in relation to σ_a. In each case, we will assume that R_m is consecutively equal to $2\, \Sigma_a$ and to $4\, \Sigma_a$.

Case no. 1: the mean stress is zero (Figure 6.50)

$$R_m = 2\, \Sigma_a \text{ and } R_m = 4\, \Sigma_a$$

Damage is a function of:

$$\boxed{(\sigma_a)^b = (\Sigma_a)^b}$$

Case no. 2: the mean stress is equal to the alternating stress (Figure 6.51)

$R_m = 2\,\Sigma_a$

$$\sigma'_a = \frac{0.5\,\Sigma_a}{1 - 0.5\,\Sigma_a / 2\,\Sigma_a}$$

$$(\sigma'_a)^b = (0.667\,\Sigma_a)^b$$

$R_m = 4\,\Sigma_a$

$$\sigma'_a = \frac{0.5\,\Sigma_a}{1 - 0.5\,\Sigma_a / 4\,\Sigma_a}$$

Damage is a function of:

$$\boxed{(\sigma'_a)^b = (0{,}57\,\Sigma_a)^b}$$

Case no. 3: the mean stress is equal to 9 times the alternating stress (Figure 6.52)

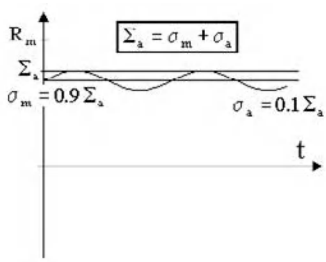


Figure 6.52. *Mean stress equal to 9 times the alternating stress amplitude*

$R_m = 2\,\Sigma_a$

$$\sigma'_a = \frac{0.1\,\Sigma_a}{1 - 0.1\,\Sigma_a / 2\,\Sigma_a}$$

$$(\sigma'_a)^b = (0.182\,\Sigma_a)^b$$

$$R_m = 4\,\Sigma_a$$

$$\sigma'_a = \frac{0.1\,\Sigma_a}{1 - 0.1\,\Sigma_a / 4\,\Sigma_a}$$

Damage is a function of:

$$(\sigma'_a)^b = (0.13\,\Sigma_a)^b$$

The equivalent stress decreases when the mean stress increases, all the more that the total stress is smaller compared to R_m.

This result can also be observed in the curves in Figures 6.53 and 6.54, respectively traced in the Goodman and Gerber hypothesis. They show the variations of the Σ_a coefficient in the expression of equivalent stress σ'_a according to k ($k = \sigma_m / \Sigma_a$), for different values of α ($\alpha = R_m / \Sigma_a$).

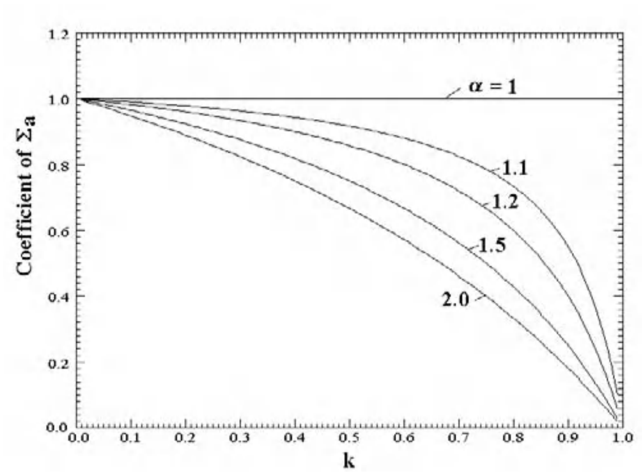


Figure 6.53. *The equivalent stress decreases when mean stress increases (k) (Goodman hypothesis)*

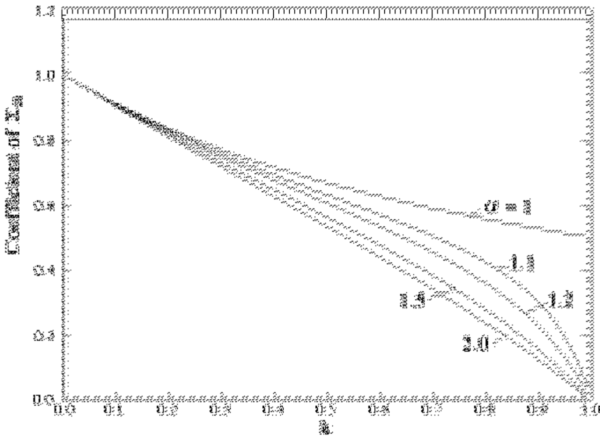


Figure 6.54. *The equivalent stress decreases when mean stress increases (k) (Gerber hypothesis)*

Figures 6.55 and 6.56 represent variations of this same parameter in relation to α, for many k values.

The equivalent stress decreases when total stress decreases in relation to R_m, all the more when the mean stress increases.

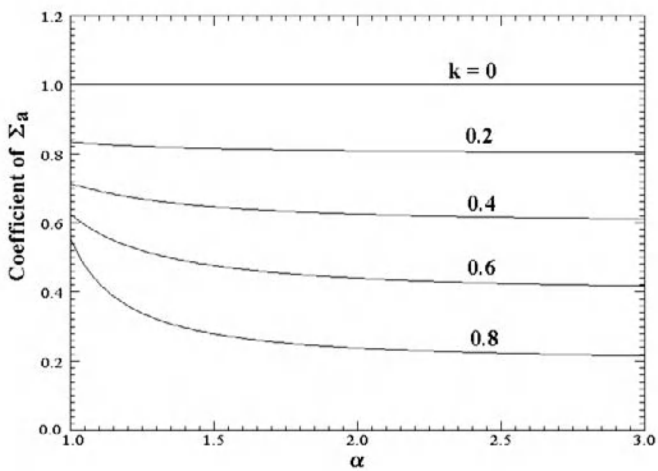


Figure 6.55. *The equivalent stress decreases when total stress decreases in relation to R_m (Goodman hypothesis)*

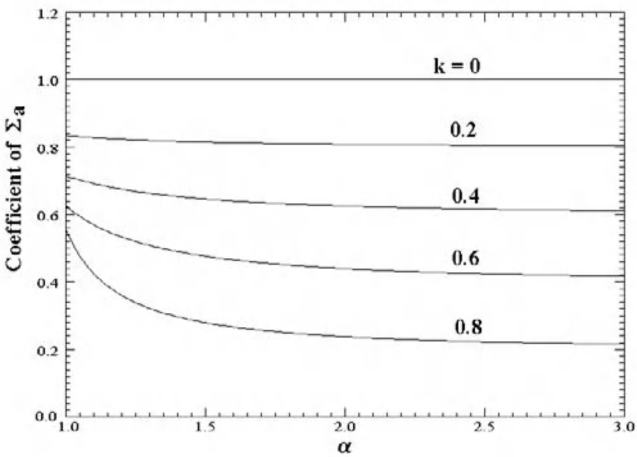


Figure 6.56. *The equivalent stress decreases when total stress decreases in relation to R_m (Gerber hypothesis)*

At constant total stress, damage is all the greater as mean stress is smaller.

FDS of a non-zero mean stress vibration

Contrary to the previous section, we assume that we add a mean stress with values different from the environmental stress, which remains the same. Total stress thus increases with the mean stress.

FDSs of a vibration measured on an aircraft were calculated by assuming the presence of a mean stress that could take three values under the following conditions.

k ($\sigma_m = k\,\Sigma_a$)	0.1	0.5	0.9
α ($R_m = \alpha\,\Sigma_a$)	1.1	1.1	1.1

In all cases, sum $\Sigma_a = \sigma_m + \sigma_a$ can obviously not exceed ultimate stress R_m. The rules of equivalence are from Goodman and Gerber.

FDSs are calculated for a Q factor equal to 10 and a b parameter of 8.

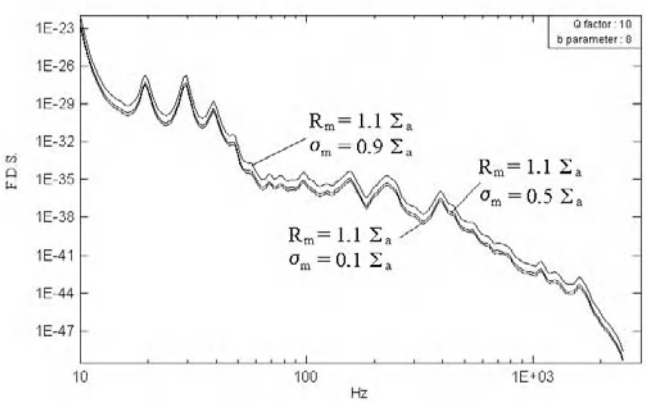


Figure 6.57. *FDS calculated for 3 values of the mean stress (10%, 50% and 90% of the total stress), in the Goodman hypothesis*

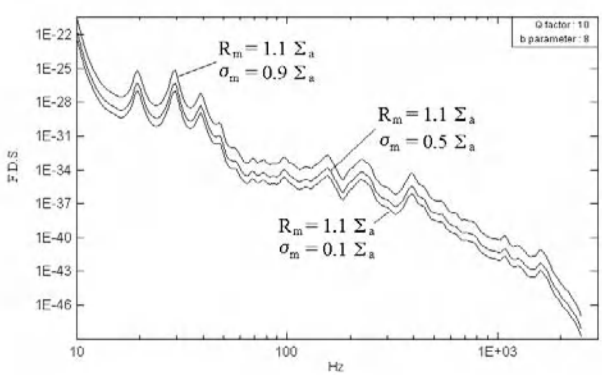


Figure 6.58. *FDS calculated for 3 values of the mean stress (10%, 50% and 90% of the total stress), in the Gerber hypothesis*

As expected, damage is all the greater as the mean stress is higher (Figures 6.57 and 6.58). The Goodman hypothesis is the one leading to the largest damage (Figures 6.59 to 6.61).

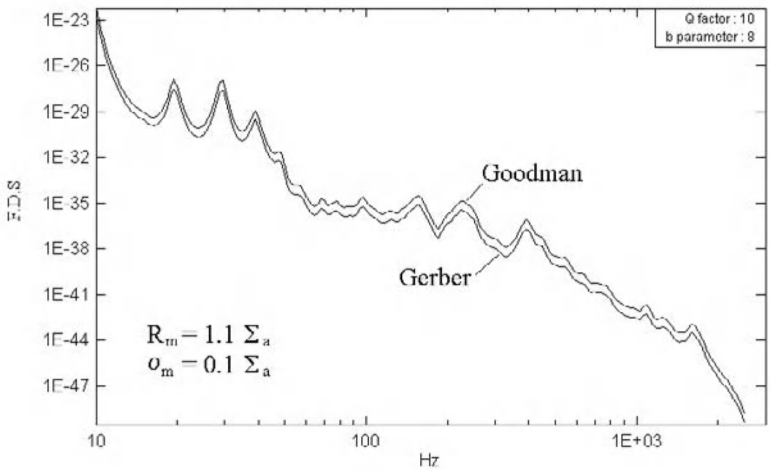


Figure 6.59. *FDSs calculated for a mean stress equal to 10% of total stress, in Goodman and Gerber hypotheses*

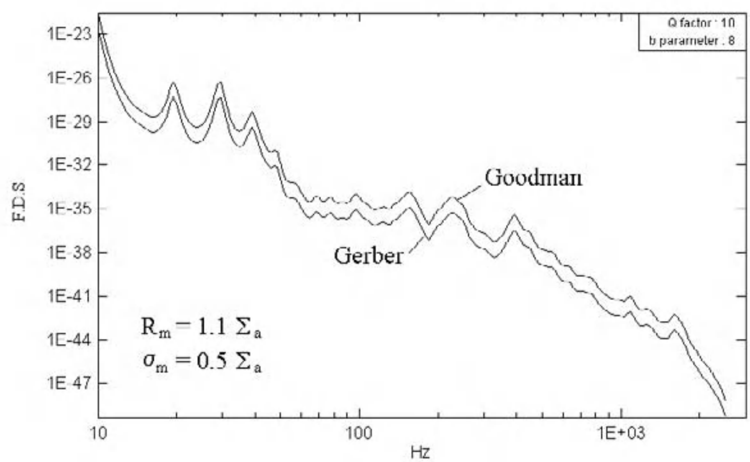


Figure 6.60. *FDSs calculated for a mean stress equal to 50% of total stress, in Goodman and Gerber hypotheses*

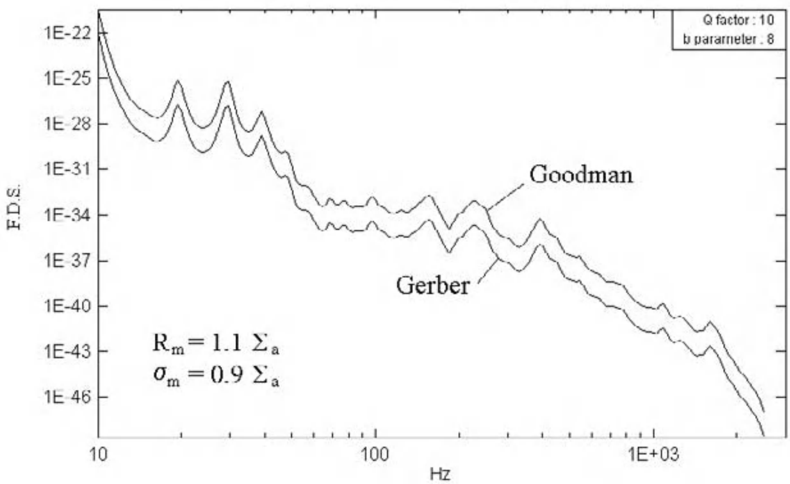


Figure 6.61. *FDSs calculated for a mean stress equal to 90% of total stress, in Goodman and Gerber hypotheses*

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 7

Tests and Standards

7.1. Definitions

7.1.1. *Standard*

A standard is a document which provides the technical specifications relative to a defined product or product class. In the case of the environmental conditions, these standards establish, for a given category of equipment, the tests to be performed, the severity and the procedures.

7.1.2. *Specification*

Specifications provide specific instructions which indicate how a specific task should be performed for a particular project. They may (or may not) be taken from a standard, but tend to become autonomous within the context of the specific project [DEL 69].

Specifications are established at the beginning of a project for the product design and are used during the tests to show that the product meets the requirements concerning its resistance to the environment.

They specify the type of environment to which the product will be exposed throughout its useful life cycle (random vibrations, mechanical shocks, etc.) and their severity (stress amplitude, duration, etc.).

7.2. Types of tests

The purpose of a test may vary according to the product's development phase:

– evaluation of characteristics of the equipment being tested;

– identification of dynamic behavior of the product being tested;

– evaluation of the strength of the product being tested;

– qualification;

– etc.

The specifications used vary according to the type of test. The terminology used is often disputed. A list of the most generally accepted definitions is given below.

7.2.1. *Characterization test*

This test is used to measure certain characteristics of an item of equipment or a material (Young's modulus, thermal constants, parameters that are characteristic of fatigue resistance, etc.).

7.2.2. *Identification test*

This test is used to measure certain parameters that are characteristic of the product's intrinsic behavior (e.g. transfer functions).

7.2.3. *Evaluation test*

This test is used to evaluate the behavior of all or part of the product, so that a solution can be chosen very early on in the pre-development phase.

This test does not necessarily simulate the real environment. It can be performed by using:

– the most severe real levels (or estimated levels that represent the real environment) so that the project can also be pre-evaluated [KRO 62];

– levels that are greater than the actual requirements of the project in order to obtain additional information and to acquire a certain confidence in the definition. The "over-stress" can be defined by a multiplying factor applied to the load's amplitude or to the test's duration. A successful test shows that the project has the desired resistance with a certain safety margin to take into account the variations due to each product's specific conditions of use [SUC 75];

– increasing levels until rupture so that the safety margins can be determined [KRO 62].

7.2.4. *Final adjustment/development test*

This test is used to take the product from the research stage through to its eventual contractual qualification [SUC 75].

The aim is to evaluate the performance of the product under environmental conditions and to determine whether this product is able to withstand such conditions [RUB 64]. This test is performed during the first development stages. It can show up failures such as performance degradation, deformation, intermittent running, structural rupture, or any other weak point that future studies will be able to improve. The levels to be applied are not necessarily those that result from the real environment; they are selected to locate the product's weak spots.

Development tests can also be used as a tool to evaluate several solutions and can assist when it comes to making a decision. These tests are used to improve the overall quality of the product until an appropriate level of confidence has been reached, i.e. the product can easily withstand the qualification test.

7.2.5. *Prototype test*

This test is performed on the first series product or on one of the first items from the production series. During this test all the product's performance and functional characteristics are checked. The equipment is also submitted to climatic and mechanical endurance tests.

7.2.6. *Pre-qualification (or evaluation) test*

This test is used to determine how well the product is able to resist either:

– real service conditions; the test must therefore simulate the real environment as well as possible (frequencies, levels, duration) [KRO 62], with a possible uncertainty (or safety) factor;

or

– qualification conditions when the specifications are drawn from general standard documents.

7.2.7. *Qualification*

An act by which an authorized authority recognizes, after verification, that an object has the necessary qualities needed to fulfill a specific function, as established in the technical specifications [SUC 75].

7.2.8. *Qualification test*

This is a set of tests used to demonstrate to the customer the quality and dependability of the equipment during its useful life cycle. In particular these tests show that the product may be submitted to the most severe service environment without being damaged [PIE 70].

According to its authors, this definition may or may not include reliability and safety tests. The test is performed on a specimen (usually only one) which represents the exact production configuration (series sample of prototype) with specifications that are defined according to the expected use [CAR 74] [SUC 75]. This specimen will not be used for normal service.

The qualification tests are carried out on each one of the equipments constituting the studied system. These are then integrated to make up the complete system being tested [STA 67]. For a number of technical reasons, additional vibration qualification tests are carried out on both the complete system being tested and on its components [BOE 63] to:

– eliminate artificial stress imposed by the fixtures while the different equipments are being tested;

– evaluate the interactions between sub-systems;

– evaluate connections, electric cables and many small components which are not submitted to sufficient stress during the individual qualification tests of the components;

– evaluate electrical parameter variations that result from the environment being applied.

7.2.9. *Certification*

The result of an assessment by which the State confirms that a given equipment complies with minimal technical characteristics that the State has defined as being prerequisites for the equipment use [SUC 75].

7.2.10. *Certification test*

This is a set of tests that are performed on a reference specimen which is used to show that the product being tested meets the certification requirements.

7.2.11. *Stress screening test*

This test is performed on all manufactured equipment, the aim of which is to find latent manufacturing faults so that faulty equipment can be eliminated and the product's overall reliability increased [DEV 86].

7.2.12. *Acceptance or reception*

A set of technical operations which enables the customer to check that the equipment he has received meet his requirements, generally expressed in the technical specifications.

Terms such as *receipt* ("recette" in French – to be avoided according to BNAE[1][SUC 75]), or *acceptance* or even *quality control* can also be found. In accordance with the general contract conditions that apply to government contracts, the terms given below are used in the following cases [REP 82]:

– acceptance when the test does not involve a transfer of property to the State;

– reception when the task involves a transfer of property to the State, in the context of an industrial contract.

7.2.13. *Reception test*

This test enables a customer to check that the equipment he receives has the characteristics within the tolerance limits established in the acceptance program.

These tests are generally not as thorough as the qualification tests and they are less severe. They are not destructive (except for specific cases such as "single-shot" or pyrotechnical equipment for example) and concern many or all of the specimens [PIE 70]. They should not reduce the utilization potential.

The test conditions are normally representative of average service conditions. The selected duration is equal to a fraction of the real duration. It may also be a very

1 Bureau de Normalisation de l'Aéronautique et de l'Espace (Office for Standardization in the field of Aeronautics and Space).

simple test without any direct relation to the real environment, which is performed to check a weak spot's resistance (e.g. a sine test where the level and frequency have been established during the development phase, to check the state of a weld, or highlight a manufacturing fault) [CAR 74]. These tests are generally known as "acceptance tests" in the USA. When they are performed according to the conditions described above, the NASA-GSFC (NASA Goddard Space Flight Center) calls them "flight tests".

7.2.14. *Qualification/acceptance test*

For products produced in several specimens, or even in one single specimen, the acceptance and qualification test objectives are sometimes combined into one single test which is known as a "qualification/acceptance" test. The product being tested is then put into service. The test is not considered to have damaged the specimen [PIE 70].

These tests are known as "qual-acceptance tests" in the USA (and "proto-flight tests" by the NASA-GSFC).

7.2.15. *Series test*

This test is performed on all products and it is used to check a certain number of functional characteristics, according to particular specifications. Unless otherwise stated, this does not involve testing under environmental conditions.

7.2.16. *Sampling test*

The purpose of these tests is to guarantee the consistency of manufacturing quality. They concern a percentage p of a product series (e.g. p = 10%) chosen at random from a manufacturing batch according to the indications given by a particular specification. They include all or part of the various tasks performed for the qualification tests.

7.2.17. *Reliability test*

The purpose of these tests is to determine the reliability of equipment under operating conditions. They are performed on a reasonable number of specimens. The tests should simulate the service conditions as closely as possible, with a duration that may last considerably longer than the real duration, in order to determine the mean time to rupture, the endurance limit or similar characteristics [SUC 75].

To reduce costs, equivalent processes are sometimes carried out. These consist of tests that are shorter in length but are greater in amplitude, based on criteria related to the degradation mode of the equipment concerned.

This list is in no way complete. It is obvious, for reasons of cost amongst other things, that all these tests cannot be performed on each product. As the general objective is to qualify or certify a product, these are the two types of tests that are most frequently encountered.

7.3. What can be expected from a test specification?

A specification should satisfy the following criteria [BLA 59] [KLE 65] [PIE 66]:

– If the equipment functions correctly during the test, there should be a very strong probability that it will function correctly in the real environment. This criterion means that the specification must be at least as severe as the real environment.

– If the equipment fails during the test there should be a very strong probability that it will fail in operation. This means that the test should be representative of real conditions but not excessively severe compared to the real environment.

– No other item of equipment built according to the same design should fail in operation.

A good test should therefore produce failures that would be observed in a real environment and should not cause failures that would not arise during operation [BRO 67] [FAC 72] (except in the case of testing to the limits conducted to evaluate margins and to determine weak spots). It should not be so severe as to lead to excessive derating (oversizing) of the equipment [EST 61].

7.4. Specification types

The tests can be:

– conducted *in situ* by placing the equipment in the real environment that it will encounter during its useful life. For instance, for transportation, the equipment is attached by its nominal attachment points to the chosen vehicle that should be driven under conditions representative of real conditions;

– conducted in the laboratory in simulation facilities. In this case, the specifications can be:

- derived from standards,
- defined from environmental data.

7.4.1. *Specification requiring in situ testing*

The equipment is placed in the real environment using all its interfaces with the medium (attachments, position, etc.). For transportation, the test is conducted by driving the vehicle under real conditions for a specified time.

Advantages

– Test highly representative (reproduction of impedances).

– No investment in simulation facilities.

– No problem in writing the test specifications.

Disadvantages

– Cost, especially if the equipment has a very long life cycle. In this case, the test time must be limited for practical reasons, making it impossible to verify the strength of the equipment over time.

– Impossibility of applying an uncertainty factor to vibration levels (the usefulness of such a factor will be discussed below).

– The real environment generally has a random character (varying over the environment and the equipment's characteristics). During an *in situ* test, the equipment is submitted to a sample of vibrations among others that are not necessarily the most severe.

– Practical impossibility of going to the limit to estimate the safety margins.

– Only the test phase is dealt with, without any indications as to dimensioning.

7.4.2. *Specifications derived from standards*

7.4.2.1. *History*

Early tests used to be carried out to ensure that the product could withstand a defined vibratory level which was not directly linked to the environment that it had to support during its useful life cycle. In the absence of a rational procedure, the specifications drawn up were strongly influenced by the writer's personal judgment, by the currently available test facilities, and later on, by the information from previous specifications, instead of by evaluating the available information in a purely scientific manner [KLE 65] [PIE 66].

A large number of measurements taken from numerous aircraft were compiled between 1945 and 1950 [KEN 49] and served as a basis for writing up certain standards (AF Specification 41065, specifications drawn up in 1954 using data measured, compiled and analyzed by North American Aviation [CRE 54] [FIN 51]).

One of the first methods used to convert this data into specifications involved grouping them into three main categories: structural vibrations, engine-created vibrations and resilient mounting assembly vibrations. The signals, as a function of time, were filtered via central frequency filters which varied continuously between a few Hertz and 2,000 Hz. The highest values of the response were plotted onto an amplitude (peak to peak displacement) – central frequency filter diagram. This procedure was carried out for different aircraft and different flight conditions, without taking into account the probability of their occurrence in real conditions. The authors [CRE 54], however, sorted the measurements in order to eliminate the conditions considered to be unrealistic (non-typical experimental aircraft flights, measurements taken from parts of the aircraft of little or no interest, etc.). This method was preferred to the PSD calculation method which uses an averaging procedure leading to spectra smoothing and possible masking of certain details considered to be important for developing the specification [ROB 86].

The North American Aviation [KEN 59] compilation did not make any distinction between transitory phases (climb, descent, turns) and permanent phases (cruising).

However, C.E. Crede, M. Gertel and R. Cavanaug [CRE 54] selected only the permanent phases.

The method at the time consisted of either:

– drawing the envelope of plotted points, made up of broken straight lines (on the logarithmic axes) that corresponded to a constant displacement or a constant acceleration effect depending on the frequency (Figure 7.1). This envelope was considered to be representative of the severest conditions that could be encountered in each of the three previously mentioned categories, without, however, mentioning how the loads were obtained;

– or carrying out a statistical analysis of the points plotted and defining a curve covering 95% of the data [BAR 65] [GER 61] [KEN 59].

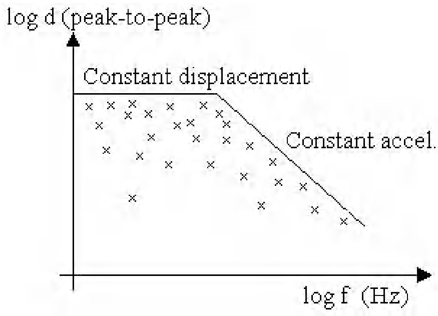


Figure 7.1. *Swept sine specification using measurement envelope of the highest filter responses*

This type of diagram was used for most of the first standards. Unfortunately, experimental data processed in this format was usually published without clearly defining the width of the Δf band of the analysis filters [MUS 60]. Some authors used a constant width of 200 Hz or sometimes 100 Hz in relation to the central frequency, others used a Δf width such as $\frac{\Delta f}{f}$ = constant (today it is difficult to find any trace of the value of this constant) [CRE 56].

The following points had to be defined in order for the curves to be applied to aeronautical equipment:

– either a sinusoidal vibration with a given amplitude, without attaching much importance to the frequency, had to be arbitrarily set (e.g. 25 Hz) [CZE 78];

– or, after looking for a specimen's resonance frequencies, a sine had to be fixed for the measured resonance(s), by selecting the sinusoid amplitude from the previous curve, even if the equipment was intended to operate in a random vibration environment [BRO 64].

The most severe vibrations applied in such a way to test the specimen under the harshest conditions (for its resonances) lead to maximum stress levels. However these conditions, although they may be encountered during service, are very unlikely to occur over a great length of time in any operational life cycle phase.

An endurance test of this type is similar to an accelerated test. The acceleration factor in terms of time was estimated to be equal to about 10 for a resonant specimen and 3.33 for a non-resonant specimen. These are considered to be conservative factors [KEN 49].

C.E. Crede and E.J. Lunney [CRE 56] note, however, that this procedure (fixed sine) is not recommended because it is difficult to detect all the significant resonances and because the vibration test becomes a non-standard test, influenced by the person piloting the test. These authors recommend:

– using a sine that is swept slowly in the direction of the increasing or decreasing frequencies;

– or a sine swept over the whole frequency range, first linearly, and then logarithmically [FAC 72].

The duration of the test was set either unconditionally or in a way that was representative of the real environment, or was reduced by increasing the equal fatigue damage levels as in the following equations (see section 4.4):

$$\frac{n_1}{n_2} = \left(\frac{\sigma_2}{\sigma_1}\right)^b \qquad [7.1]$$

or

$$\frac{\ddot{x}_{test}}{\ddot{x}_{real}} = \left[\frac{T_{real}}{T_{reduced}}\right]^b \qquad [7.2]$$

where b is related to the slope of the S–N curve [CRE 56] (n is the number of cycles performed at stress level s and $\ddot{x}$ the test or environment severity).

This method has often given rise to considerable problems during testing; design problems or unnecessary weight penalties [BRO 64].

Such standards were in force until 1955, and even much later in some sectors. A major breakthrough occurred in 1955 when it was discovered that there was a need to simulate the continuous aspect of measured vibration spectra on missiles, and thus to perform random vibration tests [CRE 56] [MOR 55]. Random vibrations were introduced into standards between 1955 and 1960. However, this met with much opposition.

C.G. Stradling [STR 60] considered, for instance, that the sinusoidal vibrations are easy to generate and the results can be interpreted relatively easily, but acknowledged that random vibrations represent the real environment better. However, he defended the sine tests with the following remarks:

– random vibrations measured at the point of input into an item of equipment often come from several sources simultaneously (acoustic and structural);

– vibrations in the environment are propagated and are filtered by structures, they are therefore similar to sinusoids [CRE 56];

– the acoustic energy is itself made up of several discrete frequencies corresponding to the structure's modal responses to the excitation, a sine test therefore approximates real vibrations better than it first seems;

– there are few applications, other than missiles, for which acoustic energy is really significant (these remarks date back to 1960);

– there are no structures that break up under random vibrations and not under sinusoidal vibrations (except, the author goes on to say, for tubes or relays which do not resist random vibrations so well);

– much more time is needed to prepare a random test than a sine test;

– the equipment (test installation) is more expensive;

– maintenance costs are higher (more complex means which therefore reduce reliability).

Many of these arguments, most of which originated in 1960, are no longer acceptable today. However, many arguments exist in favor of simulating real random phenomena as far as random vibration is concerned:

– an important feature of a test is to vibrate the equipment's mechanical elements at their resonance frequencies. With swept sinusoidal vibrations, the resonance frequencies are excited one after the other, whereas for random vibrations, they are all excited simultaneously, with a level that varies at random as a function of time [CRE 56]. Using fixed sinusoidal vibration means that interactions between several simultaneously excited modes can not be reproduced as in a real environment [HIM 57]. There is no equivalent sine test which can be used to study a response of a system with several degrees of freedom [KLE 61];

– in sinusoidal conditions and on resonance, the rms response acceleration is equal to Q times the excitation whereas in random conditions, the response is proportional to $\sqrt{Q}$;

– equipment can have several resonance frequencies, each one with a different Q factor. It is therefore difficult to select an amplitude for a single test which produces the same response as a random test for all the Q factor values. If the real excitation is random, a fixed sine test would tend to penalize those structures with relatively weak damping, which respond with a greater amplitude because, with sinusoidal vibration, the response varies as Q (this observation is not necessarily correct, as choosing the equivalence method may take this fact into account, provided that the exact values for Q are known);

– there are many rupture mechanisms, determined by the contact mechanisms, the type of equipment (tensile or brittle), etc. The equipment may therefore behave differently under random and sinusoidal stress [BRO 64];

– before comparing the random and sine test results, an equivalence method must be defined. This is a very controversial subject, as the criteria are generally established on the basis of the response of a single-degree-of-freedom system.

The spectral form of the noise then needed to be selected, with white noise often being chosen. However, in certain cases, the response of an intermediate structure can lead to a noise formed by “lines”.

The problem also remained of how this type of vibration could be generated in practice, as the means available at the time were limited as regarded power. Due to this, in 1957, M.W. Olson [OLS 57] suggested, for reasons of economy, using a swept straight band random vibration in the chosen frequency field, to replace the wide band random vibration by an equivalent one; as with the swept sine, the resonances are excited one after the other. The test is longer, but requires less power and maintains a statistical character.

Other authors have suggested performing a test made up of several sequential narrow frequency bands [HIM 57].

7.4.2.2. *Major standards*

There are many available standards. A survey carried out in 1975 by the L.R.B.A. (Laboratoire de Recherches Balistiques et Aérodynamiques: Ballistics and Aerodynamics Research Laboratory) [REP 75] showed that there were about 80 standards in the world (American, British, French, German and international) that defined mechanical shock and vibration tests. However, many of these standards are taken from previous documents, with some changes. Not all these standards have the same importance.

Among the most widely used standards in France, the following may be mentioned [CHE 81] [COQ 79] [GAM 76] [NOR 72]:

– AIR 7304 “*Environmental Test Conditions for Aeronautic Equipment: Electrical, Electronic and On-board Instruments*”;

– Standard GAM–T13–Inter-army specification titled “*General Testing of Electronic and Telecommunications Equipment*” written with the following objectives:

- to unify to the greatest extent possible the environmental requirements of the various departments of the Technical Agencies of the French General Delegation for Armament and to collect all these requirements in a single document;

- to have sufficiently accurate contractual documents;

- to facilitate the task of the engineers responsible for defining the tests applicable to the equipment they are responsible for designing;

- to ensure the best possible compatibility of the test methods and severities with those of international standards.

This document was replaced by GAM.EG 13 [GAM 86] which is associated with technical attachments written as guidelines for the users;

– MIL−STD standards;

The best known is MIL–STD 810 for mechanical and climatic environmental tests. Written under the auspices of the US Air Force, it is mainly used for cooperation and export programs.

The standards (MIL–STD 810 C, AIR 7304, etc.) specify arbitrary vibration levels calculated from real environment data. Generally, these levels cover conditions that can be encountered by a given vehicle relatively well.

It may be desirable to apply such standards in certain situations, such as those where:

– the conditions of use are not well known;

– there are no available measurements of the real environment, and such measurements cannot be approximated by measurements on a vehicle of the same category;

– it is desirable that the equipment be given a predefined strength under widely accepted standards, possibly international.

7.4.2.3. *Advantages and disadvantages*

Advantages

– As the standards are already available, there is no cost of establishment.

– The strength of equipment produced by different companies can be compared directly (providing the same standards were used).

– Equipment passing these standards can be carried on any type of vehicle of a given category without requiring additional tests.

Disadvantages

– Because of their arbitrary nature and their ample coverage of real levels, these standards may lead to oversizing the equipment (resulting in a possible increase in the design cost). Commonly used environmental specifications are rarely representative of actual service conditions. The levels, in general much too high [REL 63], are chosen arbitrarily to satisfy requirements for reproducibility and standardization [HAR 64] [SIL 65]. By their nature, the standard specifications lead to developing equipment which, in many cases, is not designed to withstand its real environment, but rather to withstand a conservative test established to simulate the environment [HAR 64].

– Unrepresentative failure mechanisms.

– Excessive development cost. Certain equipment produced in small production runs with state-of-the-art design techniques has a relatively high cost price. Requiring it to satisfy excessively broad environmental specifications can lead to a prohibitive cost. Other equipment, carried on launch vehicles or satellites, must have as low a weight as possible, prohibiting any oversizing.

– Increase in lead times. Being unnecessarily conservative in establishing margins can lead to redesigning the product, and excessively and unnecessarily increasing the cost [FAC 72].

7.4.3. *Current trend*

Until the beginning of the 1980s, standards stipulated test procedures and test levels. The choice of severity was left to the user. Most of the levels were derived from data on the real environment, but the user had no knowledge of the origin of the data or of the method used to analyze and transform it into a standard. To cover the large possible number of cases, the stipulated levels were generally far above the values encountered in a real operating environment and could therefore lead to extreme overtesting.

In more recent years (1980–1985), an important reversal of this trend has occurred internationally, leading the specification writer to increasingly use the real environment [LAL 85].

Already, certain standards such as MIL–STD 810 C had timidly left open the possibility of using the real environment "*if it was established that the equipment was subjected to an environment estimated to be different from that specified in the* *standard*" [COQ 81], but this was only for guidance. The emphasis was still placed on the arbitrary levels.

The turnaround occurred with publication of MIL–STD 810 D (adopted by NATO as STANAG[2]) and the work of GAM.EG 13. These new versions no longer directly stipulated vibration or heat levels according to the type of environment to be simulated, but recommended a preference for using real data. Lacking such data, the use of data acquired under similar conditions and estimated to be representative, or that of *default* values (fallback levels), obviously more arbitrary in character, was accepted in that order. This is known as *test tailoring*.

Today we talk more about *tailoring a product to its environment*, to confirm the need for taking the environmental conditions into account right from the beginning of the project and sizing the product according to its future use. This step may be integrated into project management stipulations, for instance as in the RG Aéro 00040 Recommendation. This point will be dealt with later.

7.4.4. *Specifications based on real environment data*

7.4.4.1. *Interest*

As early as 1957 R. Plunkett [PLU 57] suggested using measurements from the real environment. This was followed up by other authors such as W. Dubois [DUB 59], W.R. Forlifer [FOR 65], J.T. Foley [FOL 62] and E.F. Small [SMA 56].

Environmental tests should be based on the equipment's life cycle. When the conditions of use of the equipment being developed are well known, and if its life cycle can be divided into phases specifying, for example, the vehicle type or the storage conditions and duration of each phase, it may be preferable to develop special environmental specifications very similar to the real environments and to apply certain uncertainty factors [DEL 69] [SMA 56].

Taking real environment measurements into account leads to much less severe test levels, enabling a realistic simulation to be created, minimizing the risk of over-testing [TRO 72]. H.W. Allen [ALL 85] gives the example of a specification established on the basis of statistical analysis of 1839 flight measurements which resulted in lower specification levels, reduced by a factor of 1.23 (endurance test) to 3.2 (qualification test).

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2 STANdardization AGreement.

7.4.4.2. *Advantages and disadvantages*

Advantages

– Specifications very close to real levels (excluding the uncertainty factor) allow the equipment to be operated under conditions very close to live conditions. The equipment can therefore be designed with more realistic margins.

– The safety margins can be evaluated by testing to the limit.

Disadvantages

– The material defined is specific to the selected life profile; to a certain extent, any modification of the conditions of use will have to result in an examination of the new environment, a readjustment of the specifications and, possibly, an additional test.

– The cost of developing the specification is higher (but this expense is amply offset during the equipment's development stage). An early analysis of the conditions of use can avoid many disappointments all too often only noted later on.

– Comparison of the mechanical strength of different items of equipment (designed to different specifications) is more difficult.

– Since the aim is to provide the best possible simulation of the real environment, it is then necessary to be certain that the test facility correctly follows the specification. For heavy specimens, this may prove difficult because of mechanical impedance problems.

7.4.4.3. *Exact duplication of the real or synthesized environment*

The specifications can be written according to one of the two following concepts [BLA 67] [GER 66] [KLE 65] [PIE 66]:

– exact duplication of the real environment;

– simulation of the damaging effects of the environment.

At first view, it may appear preferable to faithfully reproduce the measured environment, for instance by using the real signal recorded on magnetic tape as a control signal for the test facility [TUS 73]. If the reproduction is faithful, the test obviously has the same severity characteristics as the real environment. With this approach, it is not necessary to have prior knowledge of the damaging characteristics of the environment for the equipment. In this sense, such an approach may be considered ideal.

Unfortunately, it is not practicable for a number of reasons among which [BLA 62] [BLA 67] [GER 66] [KLE 65] [PIE 66] [SMA 56] [TUS 67] [TUS 73] are:

– Faithful reproduction requires a unit scale factor between the duration of the real environment and the simulation test time. In the case of a long duration real environment, the method can be unrealistic.

– If the life cycle profile of the equipment specifies several different vibration environments, it is inconvenient and costly to reproduce each one sequentially in the laboratory.

– The severity of the environment is statistical in nature. Environmental data exhibit considerable statistical dispersion which can be attributed to differences in test conditions, the human factor and other causes that are not always obvious. It has been demonstrated that structures manufactured under the same conditions can have different transfer functions at high frequencies [PAR 61].

A particular recording (e.g. of a flight) cannot necessarily be considered representative of the most severe conditions possible (the worst case flight). Even if several recordings are available, the problem of the choice of sample to be used has to be solved.

– The vehicle generating the vibrations might not yet exist, may be new when the test is defined and the specifications are written, and few if any measurements might be available, whereas the specific purpose of the test is to test the equipment before installing it on board. The data are therefore measured on more or less similar vehicles, with a measurement point that is more or less where the equipment will be installed.

– The tests are often performed to qualify equipment that can be installed anywhere in a vehicle or a specific type of vehicle. The procedure would need the vibration response to be recorded at any point on the vehicle structure where the equipment could be installed, and all the signals collected would need to be reproduced.

– More generally, certain equipment may be used in several types of aircraft (for instance). It would be very difficult to reproduce each one of the possible environments.

– In the case of equipment with several attachment points, for which the vibration inputs are different, the problem of choosing the recording to be reproduced for the test arises.

– Faithful reproduction in the laboratory requires that the specimen's interaction with the test facility, or impedance matching, be the same as under real conditions, which is often difficult to achieve. Exact duplication does not therefore guarantee a perfectly representative test.

For these reasons, not all of which have the same importance, this method is not often used in the laboratory (except in the car industry). It has some applications for signals of short duration that are otherwise difficult to reliably analyze and simulate using any other process.

The second approach consists of defining a similar, but synthesized, vibration environment constructed from available data and producing the same damage as the real environment. The tests are expected to give results similar to those obtained in a real environment. It is attempted to reproduce the effects of the environment rather than the environment itself. This approach requires prior knowledge of the equipment failure mechanism when the equipment is subjected dynamic stresses.

The main advantage of the first approach is that it does not require any assumptions regarding failure modes and mechanisms. The main disadvantages are obviously the requirement for simulating all the features of the real environment, in particular the duration, and the difficulty of taking statistical aspects into account.

Searching a synthesized environment is a much more flexible method. Statistical analyses and time reductions can thus be performed, providing that an acceptable criterion of equivalent damage can be defined.

It is not necessary *a priori* to define the specifications in terms of vibration tests [KRO 62]. It could be considered, although this is not usually done, to propose a static test, to specify minimum natural frequencies for the equipment or an endurance limit, etc. These characteristics would be obtained from analyzing the effects of the real environment on the equipment. Generally an attempt is made to define an “idealized” vibratory environment having the same nature as the measured environment (random, sinusoidal, etc.) that has been selected as representative of the damage potential of the real environment [BLA 69], over a reduced period of time if possible.

7.5. Standards specifying test tailoring

Old standard documents and even some still active today, specify environmental values (accelerations, temperatures, etc.) to apply to components, based on their conditions of use. The proposed values are generally quite severe, and sometimes not well adapted to current needs.

More recent standards MIL–STD 810 in the USA, GAM.EG 13 in France and NATO require “test tailoring ”. They rely on a four-step methodology:

– writing and analyzing the life profile, specifying all the conditions of use of the material, taking note of each situation (storage, transport, etc.), and for each situation, in a qualitative way, establishing a list of all events requiring a specific representation (“*events*”) with duration and chronological order (for air transport, vibrations relating to driving on the runway, at take-off, landing, cruising, etc.);

– searching for real environmental data representative of each identified event, with enough data to take into consideration the variability inherent to these phenomena;

– synopsis of data to deduce a specification with realistic duration. This operation is vital because of the important number of measures collected in general: several situations made up of several events, each one described by several measures, with three axes for the vibratory environment;

– establishment of the test program.

This process makes it possible to develop a specification adapted to the use of the product, with reasonable and controlled margins, and to design the material without oversizing.

We must, however, have a method of data analysis which guarantees that the specification obtained has a level of severity at least equal to, but not in excess of, the real environment, and to decrease test time when the environment is lengthy.

We will focus here on the field of vibrations and mechanical shocks.

Three standards today require the test tailoring: the GAM.EG 13 standard, the MIL–STD 810 standard and the NATO (STANAG 430) standard.

7.5.1. *The MIL–STD 810 standard*

The American MIL–STD 810 (Environmental test methods and engineering guidelines) standard emerged from a first standard (Army Air Force Specification n 41065, “General Specification for Environmental Test of Equipment”) edited in 1945, replaced by the MIL–E–5272 (USAF) standard August 16th 1950 (Environmental Testing, Aeronautical and Associated Equipment) and renamed MIL–STD 810 (Military Standard) in June 1962.

Several editions have emerged since this date, the last one is version G (October 2008). Version D, published in July 1983, is the one that, for the first time, required test tailoring[3] according to the final purpose of the component.

The MIL–STD 810 standard describes the management, technology and technical roles in the environmental definition, and the process of test tailoring. It specifies that test methods should not be determined from envelopes or applied as rigid processes, but must be chosen and personalized to produce the most appropriate tests possible.

Tailoring is defined as a "*The process of choosing design characteristics/tolerances and test environments, methods, procedures, sequences and conditions, and altering critical design and test values, conditions of failure, etc., to take into account the effects of the particular environmental forcing functions to which materiel normally would be subjected during its life cycle. The tailoring process also includes preparing or reviewing engineering task, planning, test, and evaluation documents to help ensure realistic weather, climate, and other physical environmental conditions are given proper consideration throughout the acquisition cycle.*"

This standard particularly involves the:

– content of the life profile (interfaces, durations, material configuration, expected environment, number of occurrences, probability of appearance of environmental conditions, etc.);

– representativeness of data and their statistical value;

– representativeness of the test material;

– methods for mechanical, thermal and combined tests;

– management plan.

In June 1994, the William Perry (Secretary of Defense) report proposed abolishing purely military specifications which added no value, to reduce the cost of weapon systems and other material.

He considered that these standards to be an obstacle to the use of the most recent technology.

Defense used to be the priority of the development of technology, but that is no longer the case. In 1965, the DoD (Department of Defense) and manufacturing

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3 Transmitter: Aeronautical Systems Division, Wright-Patterson AFB (Ohio).

industry invested similar sums in research. By 1990, the manufacturing sector spent twice as much.

There was also the need to enable the diversification of companies who initially mainly (or in some cases only) manufactured products for the DoD, to make them able to produce at competitive cost.

He then demanded that:

– one third of military standards be abolished;

– a further third be abolished as soon as possible;

– civil standards should be used when they exist.

The MIL–STD 810 E standard was retained and became a national standard.

This American decision had consequences in Europe, particularly in Great Britain and France. The GAM.EG 13 standard is now fixed, and studies are only conducted in its technical appendices.

7.5.2. *The GAM.EG 13 standard*

The GAM.EG 13[4], standard sponsored by the DGA (Direction générale pour l'armement (Weaponry Branch)), with the support of the different DGA branches and industrial representatives, was implemented in November 1986 by a memorandum from the ministry director for standardization.

It replaced the GAM–T13 standard. It requires test tailoring and generally involves material and the specific containers in these materials intended for the military.

It does not apply to basic components such as screws, bolts, transistors, etc., or to general packaging.

Tailoring is defined as "*a concept leading to the study, development, completion and testing of material according to the real environment it is liable to encounter*".

4 GAM: guerre – air – mer (war – air – sea). EG: études générales (general studies).

It is mainly made up of:

– a first part combining all test methods. It presents the tailoring methodology in four steps, general testing and measurement conditions as well as the methodologies that apply to guarantee good reproducibility of tests in time and space;

– choice guides according to the specific use of the material (ground troops, air force, marines, missiles and satellites, joint ground material). In these documents, we can find fall back levels values to temporarily characterize some events from situations involved in these guides, when other data is not present;

– three technical appendices:

- the general mechanical appendix, which provides the mathematical tools to calculate test specifications by equivalence of extreme responses and fatigue damage (extreme response and fatigue damage spectra, uncertainty coefficient, test factor, etc.). This method is only proposed, without any obligation to use,

- an "environmental models and data" appendix providing measurements collected from a certain number of carriers. The problem with tailoring is to have the environmental data necessary to give information on the life profile of the material studied,

- an appendix representing the major methods of signal processing.

7.5.3. *STANAG 4370*

The NATO standard was implemented in 1986 by NATO members with the goal of "standardizing the operations or environmental tests and to note national acceptances (use of AECTPs in national or multinational projects for the development and acquisition of defense material for NATO applications)".

STANAG 4370 (Standardization Agreement) is made up of documents called AECTP (Allied Environmental Conditions and Test Publications):

– AECTP 100: Environmental Guidelines for Defence Materiel: Instructions relative to the organization of environmental tests of defense material;

– AECTP 200: Environmental conditions – definitions of environments;

– AECTP 300: Climatic tests;

– AECTP 400: Mechanical tests;

– AECTP 500: Electric tests.

AECTP 100:

– defines responsibilities and tasks of project directors, responsibilities and tests of environmental specialists;

– specifies the need for test tailoring, and the four-step methodology;

– provides a list of typical environments by situation (qualitatively).

"Environmental project tailoring" is defined as "*the process of assuring that materiel is designed, developed and tested to requirements which are directly derived from the anticipated service use conditions. A test program should normally reflect environmental stresses anticipated throughout the materiel's life cycle, and tests should be based on the anticipated environmental scenarios. The specified tests and their severities should be derived from the most realistic environments, either single or in combination. In particular, data obtained from real-world platforms as influenced by natural environmental conditions should be used to determine test criteria.*"

AECTP 200 is a guide which includes three main parts: climatic, mechanical and electric. All these parts are broken down into situations (ground transportation, aircraft transportation, etc.). For each of these situations, there are four sections:

– main characteristics of environments associated with the situation;

– potential damage for material by these environments;

– indications of the method of building a test program to simulate these environments;

– concrete examples of environments in the relevant situation.

AECTP 300 and 400 group climatic and mechanical methods. Each method is described considering national military and civilian documents.

The mechanical methods provide many typical severity examples (in contrast to the climatic methods).

The test methods presented are more numerous than in basic national documents.

7.5.4. *The AFNOR X50–410 standard*

Even though it involves environmental tests only marginally, we find it useful to cite standard AFNOR X50–410 that picks up the "Recommandation générale aéronautique (Aeronautical general recommendation)" RG Aéro 00040. This document involves project management. It particularly describes the major phases in a project, actions to take in each phase as well as the documents to provide.

The environmental conditions which the material must resist are among the material specifications. Subsequently, the validity of the material developed according to its specifications must be demonstrated, most often through tests.

The AFNOR X50–410 standard specifies the operations necessary in each phase of the project for taking the environment into account. Actions relative to the four step methodology described in the previously discussed standards are distributed and detailed in every phase of the project management process. The specifications of these actions are detailed in standard GAM.EG 13 [GUI 08].

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 8

Uncertainty Factor

8.1. Need – definitions

The safe behavior of a loaded structure can only be ensured if its mechanical strength is higher than the load applied. The k ratio between resistance and load is by definition the *safety factor* that we have in the case of accident risk or, if it involves material liable to fail during normal use, the *uncertainty coefficient*. This ratio must be higher than 1.

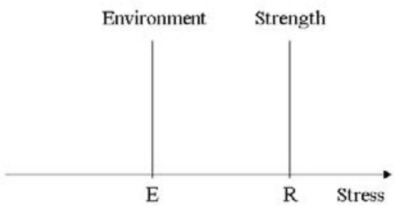

Figure 8.1. *Load applied to material and material strength*

In order to take account of all the uncertainties that exist when the test specifications are written (see section 8.2), the severities are often multiplied by such a factor also called the *guarantee factor* or *risk factor* or *conservatism factor,* etc., often considered as an *ignorance factor* [HOW 56]. The term *safety factor* is often also used with either the same meaning as above or to quantify the strength margins of equipment, with respect to its environment, but we prefer to use it for the case of accidental environments liable to affect the safety of people or goods.

This factor expresses the idea of a ratio between a certain value characterizing material strength and a value characterizing applied stress. Even if the experience of the specification writer is often taken into consideration, its numerial value is often chosen arbitrarily, generally after much discussion between equipment designers (who would like it to be as low as possible) and the specification writer (who is well aware of the limitations of his working data).

For non-critical components, we find in literature values of approximately 2 and 4 for elements which can involve the safety of people.

This factor can also be defined according to the precision with which the characteristics of loads and of the structure studied are known.

As an example, we can cite [AND 01]:

– The values proposed by J.P. Visodic [VIS 48], from his experience, functions of the knowledge of loads, allowable stresses of the material, properties of the material and environment in which the material is used (Table 8.1).

Safety Factor	Knowledge of loads	Knowledge of permitted stress	Knowledge of properties of material	Knowledge of environment
1.2–1.5	Accurate	Accurate	Well known	Controllable
1.5–2.0	Good	Good	Well known	Constant
2.0–2.5	Good	Good	Average	Ordinary
2.5–3.0	Average	Average	Less tried	Ordinary
3.0–4.0	Average	Average	Untried	Ordinary
3.0–4.0	Uncertain	Uncertain	/	Uncertain

Table 8.1. *Safety factors proposed by J.P. Visodic [SHI 01] [VIS 48]*

The values in this table are recommended for ductile materials, with the elastic or endurance limit for cyclic loads as reference. For brittle components, the author suggests twice the given value for ductile materials (reference: ultimate strength).

He proposes a safety factor of at least 2, multiplied by an impact factor of approximately 1.1 to 2 for impact problems.

– Safety factors proposed by R.L. Norton [NOR 96] (Table 8.2) taking into account similar criteria (knowledge of the material, stress-load model precision, surrounding conditions).

Safety Factor	k_1 Material properties (from tests)	k_2 Stress–load model accuracy	k_3 Service environnement
1–2	Well known characterized	Confirmed by testing	Same as material test conditions
3	Good approximation	Good approximation	Controlled room temperature
3	Fair approximation	Fair approximation	Moderately challenging
>5	Crude approximation	Crude approximation	Extremely challenging

Table 8.2. *Safety factors proposed by R.L. Norton [NOR 96]*

R.L. Norton recommends using the greatest of 3 values k_1, k_2 or k_3 for ductile materials and twice the value of the largest of these values for all brittle materials (reference: yield stress in the first case and ultimate strength in the second case).

– The safety factor defined by A.G. Pugsley [PUG 66] as the product of both factors:

$$k = k_1 \, k_2$$

where k_1 and k_2 are given in Tables 8.3 and 8.4 according to parameters A, B, C and D, E respectively characterizing:

A quality of material, workmanship, maintenance and service inspections;

B control over applied loads;

C accuracy of stress analysis, knowledge of experimental data or experience with similar parts;

D danger to people when a failure of the part occurs;

E financial impact when a failure of the part occurs.

These methodologies can be used when we are able to evaluate the maximum load and strength of the material easily, the load and strength being supposedly not widely dispersed, when a high safety factor is used (cables supporting an elevator car for example).

Parameter A	Parameter C	Parameter B			
		VG	G	F	P
A = VG	C = VG	1.10	1.30	1.50	1.70
	C = G	1.20	1.45	1.70	1.95
	C = F	1.30	1.60	1.90	2.20
	C = P	1.40	1.75	2.10	2.45
A = G	C = VG	1.30	1.55	1.80	2.05
	C = G	1.45	1.75	2.05	2.35
	C = F	1.60	1.95	2.30	2.65
	C = P	1.75	2.15	2.55	2.95
A = F	C = VG	1.50	1.80	2.10	2.40
	C = G	1.70	2.05	2.40	2.75
	C = F	1.90	2.30	2.70	3.10
	C = P	2.10	2.55	3.00	3.45
A = P	C = VG	1.70	2.15	2.40	2.75
	C = G	1.95	2.35	2.75	3.15
	C = F	2.20	2.65	3.10	3.55
	C = P	2.45	2.95	3.45	3.95

Table 8.3. *Factor k_1 according to parameters A, B and C*
(where the evaluation means: VG=Very good; G=Good; F=Fair; P4=Poor)

Parameter D (Danger)	Parameter E (Economic impact)		
	E = NS	E = S	E = VS
D=NS	1.0	1.0	1.2
D=S	1.2	1.3	1.4
D=VS	1.4	1.5	1.6

Table 8.4. *Factor k_2 according to parameters D and E*
(where the evaluation means: NS=Not Serious; S=Serious; VS=Very serious)

In practice however, the real environment, whatever its nature (sinusoidal, random or transient), is not strictly reproducible. Several consecutive measurements of the same phenomenon give statistically scattered results. The parameter describing the real environment characterizing a situation of the life profile must therefore be represented by a random variable with a probability density, its mean $\overline{E}$ and its standard deviation s_E. Most authors consider that the best distribution is a log-normal distribution. Others prefer a Gaussian or a Weibull distribution.

Similarly, not all the components of a piece of equipment have the same static, dynamic and fatigue strength. Their strength is distributed statistically by a curve with mean $\overline{R}$ and standard deviation s_R. Here again, the preferred distribution is log–normal, sometimes replaced by a Gaussian distribution. The Weibull law [PIE 92] or any other law [KEC 72] could also be used.

There is no longer a single load and a single strength where we can make the ratio, as these quantities must be represented by probability densities (Figure 8.2).

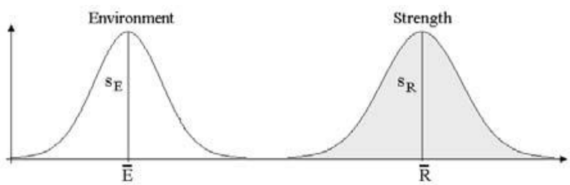

Figure 8.2. *Load and strength probability densities*

8.2. Sources of uncertainty

Uncertainties and approximations may exist for various reasons such as:

– measurement errors;

– relative dispersion of levels in the real environment. There are many sources of uncertainty to consider when predicting the vibration environments of aerospace systems [PIE 66] [PIE 74]. The main one is the characteristic variation of vibration levels at one point or another in complex structures, in particular at high frequencies.

If, for a given phase, the real environment is only defined by a single sample resulting from a short-duration measurement, an uncertainty factor can be applied to cover possible dispersion of this value. Depending on the author, this factor can vary between 1.15 and 1.5 [KAT 65]. As regards road vehicles, in order to compensate for uncertainties relating to variations in the vehicle's characteristics, the terrain, the speed, the time of year, the driver and the measurement point, G.R. Holmgren [HOL 84] suggests applying an uncertainty factor of 1.15 to the rms value. For the case of missile flight environment, he applies a higher factor, 1.4. If statistical results are used (frequency spectrum defined by a mean curve plus a few standard deviations), the factor is thought to be related to:

– dispersion in the equipment strength. For obvious cost related reasons, a single test is generally conducted. The strength of the selected specimen is defined by a statistical law of distribution. By chance, the test may be conducted on the sturdiest item of equipment. An uncertainty factor (e.g. 1.15) is therefore applied to the real levels to make sure that the weakest item of equipment will be capable of withstanding the environment;

– aging of the equipment (alteration of the strength over time, excluding fatigue and wear). Certain items of equipment are required to operate after a long storage period. To take this aging into account, a higher initial mechanical strength can be required of the equipment by increasing the test levels by a certain factor (such as 1.5). If the test is conducted on equipment that is already aged, a factor of one is obviously applied;

– the test method, which generally requires that tests are conducted sequentially on each axis, whereas the real vibration environment is a vector resulting from the three measured components. In some cases, a multiplier of 1.3 is applied to take into account the fact that each component is necessarily less than or equal to the modulus of the vector. [KAT 65];

– if the real spectral distribution differs from the specified smoothed spectral distribution, the specified spectrum may be exceeded locally [STE 81];

– the peak/rms value ratios found in flight may be higher than those existing during qualification tests, where the test facilities limit the value to 4.5 rms or lower if desired by the operator [STE 81].

NOTE: *If there are non-coupled resonances, simultaneously excited along two different axes, a rupture can occur in the real environment even though the axis-by-axis test may have been successful. Thus, a factor applied during the test does not necessarily solve the problem [CAR 65].*

There are, therefore, many reasons for applying an uncertainty factor. However, summing all the factors may lead to a large, unrealistic factor. The factors are also never all applied simultaneously.

The worth of a cautious approach, always advisable, in determining test specifications should not be exaggerated. H.D. Lawrence [LAW 61], discussing vibration isolating systems (suspension systems), considers that for structures, an uncertainty factor of 2 often leads to a dimension and weight penalty of around

20%. In the case of random vibrations, he feels that a factor of 2 commonly leads to penalties of 100%.

Such factors have, however, been applied in the past [CAR 74]:

– for acceptance tests: factors of 1 or 2 applied to the real environment depending on confidence in the data, with the test times preserved or increased to allow functional testing during the tests;

– for qualification tests: an increase in the acceptance test levels from 3 to 6 dB and of the test times from a factor of 3 to 5, resulting in a test that is globally four times more severe and five times longer than the mission.

Accurate representation is improved if these *ignorance* factors, all arbitrary by nature, can be eliminated through a better understanding of the real environment. This is made possible by performing the tests on equipment that is aged (where necessary) and on a three-axis test facility if no axis is predominantly excited over and above the others. Section 8.4 also provides a method of calculating an uncertainty factor that takes into account environment dispersions and equipment resistance, according to a given permitted maximum failure probability.

8.3. Statistical aspect of the real environment and of material strength

8.3.1. *Real environment*

Experience shows that loads measured in the real environment are random in nature [BLA 69] [JOH 53]. Spectrum analyses (PDSs, shock response spectra) are often given in relevant studies as statistical curves: spectra at 95%, 50%, etc. Some authors [SCH 66] note that the envelope spectrum is very close to the 95% spectrum.

W.B. Keegan [KEE 74] points out that the value of the mean amplitude plus two standard deviations is located between the 93% and 96% probability levels. He concludes that the best estimator of the amplitude at the 95% probability level is the mean plus two standard deviations.

8.3.1.1. *Distribution functions*

The mathematical form of most statistical load distributions is not known. The available data suggest that the median range of variation is either approximately normal, or log-normal or follows an extreme value distribution but the probabilities of high loads are poorly defined [BLA 62]. Table 8.5 summarizes the opinions of different authors on the choice of the best suited distribution. A number of authors choose a Normal distribution because it is the most common as an initial approximation and it is easier to use [SUC 75]. Many authors prefer a log–normal distribution, feeling that the Normal distribution is not a good model since it has negative values [KLE 65], [PIE 70].

The main difference between the normal and log–normal distributions is in the area of high values. The tail of the log–normal distribution curve indicates a higher probability of having large deviations from the median value [BAN 74].

Env.	**Analyze**	**Law**	**Author Reference**	**Observations**
Shocks	SRS	Log–normal Log–normal Log–normal	W.B. Keegan [KEE 74] J.M. Medaglia [MED 76] [LIP 60] W.O. Hugues [HUG 98]	Pyrotechnics shocks
Acoustic noises	PSD	Log–normal	W.B. Keegan [KEE 74]	Launch vehicles
Random Vibrations	PSD	Log–normal	A.G. Piersol [PIE 74]	Aircraft Launch vehicles
	PSD	Log–normal	W.B. Keegan [KEE 74]	Launch vehicles
	PSD	Log–normal	H.N. McGREGOR [GRE 61]	Launch vehicles
	PSD	Log–normal	C.V. Stahle [STA 67]	Launch vehicles
	PSD ERS	Log–normal	J.M. Medaglia [MED 76]	
	PSD	Log–normal	F. Condos [CON 62]	Launch vehicles
	PSD	Log–normal	R.E. Barret [BAR 64]	
	ERS	Log–normal	C.V. Stahle, H.G. Gongloff and W.B. Keegan [STA 75]	Extreme response
		Normal	J.W. Schlue [SCH 66]	

Table 8.5. *Distribution laws of spectra*

A Normal distribution can be sufficient when the volume of data is small and the uncertainty is large. A log–normal distribution appears to be relatively acceptable although experience shows that it often gives conservative probability estimates [KLE 65].

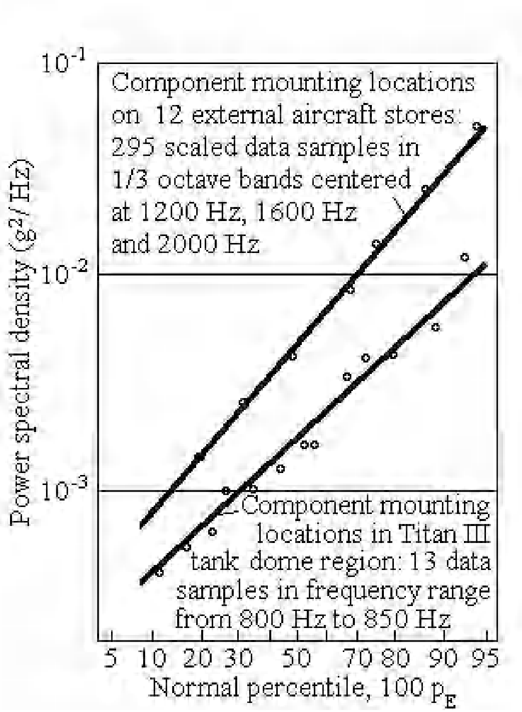

Figure 8.3. *Example of data with log–normal law [PIE 74]*

A.G. Piersol [PIE 74] chooses the above curves to show a good degree of agreement between experimental data and a log–normal distribution (Figure 8.3):

– vibration levels measured in a 50 Hz band in different locations on a Titan III tank dome;

– levels measured in 1/3 octave bands in different locations of an external aircraft store located beneath the aircraft, after correction on the basis of the differences in dynamic pressure and surface densities.

8.3.1.2. *Dispersions – variation coefficients observed in practice*

Some values

Analysis of data collected by A.G. Piersol [PIE 74] shows that the dispersion amplitude expressed in dB (characterized by the standard deviation s_y of random variable $y = 20 \log x$) of the rms vibration level within a narrow band tends to be located in the 5 – 8 dB range for conventional aerospace structures. This corresponds to a variation coefficient[1] for the rms values equal to:

$$V_E = \sqrt{e^{s_y^2/75.44} - 1} \qquad [8.1]$$

and between 0.63 and 1.16.

B58 landing:	$V_E = 0.87$ to 1.16 (6.5 dB to 8 dB) analyzed in octaves
Saturn V – Launch:	$V_E = 0.63$ to 1.16 (5 dB to 8 dB)
Titan III – External store:	$V_E = 0.63$ to 0.78 (5 dB to 6 dB)

NOTE: *These standard deviations correspond to levels situated across a very wide range.*

Example 8.1.

Standard deviation of 6 dB: the difference between the 97.5% and 50% vibration levels corresponds to a ratio of 4 to 1 for the rms values (16 to 1 for the PSDs).

The curves in Figure 8.4 show the distribution function for $s_y = 6$ dB and for two values of m (0 and 1). A factor approximately equal to 4 is found between the rms values read for $P = 97.5\,\%$ and $P = 50\,\%$.

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1 In fact, for a log-normal distribution of x, we have, when $y = \ln x$, $V = \left[e^{s_y^2} - 1\right]^{1/2}$.

When $y = 20 \log x$, y has as a ratio $\dfrac{20}{\ln 10}$ and $\left(\dfrac{20}{\ln 10}\right)^2 = 75.4447$ for s_y^2. Yielding

$$V = \sqrt{e^{s_y^2/(20/\ln 10)^2} - 1}$$

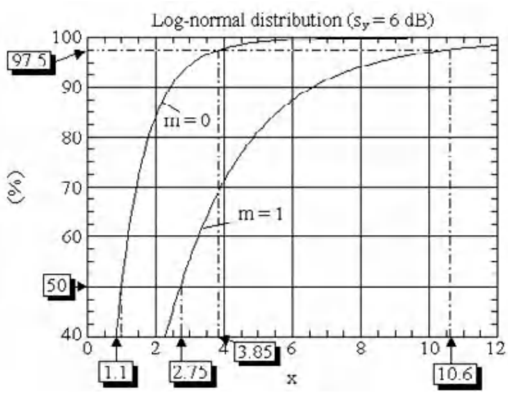

Figure 8.4. *Log–normal distribution function for two values of the mean and a standard deviation equal to 6 dB*

Other values can be found in relevant literature, such as coefficients of variation of 5% to 30% in shock and vibration amplitudes from 50% to 100% for launch vehicles [BRA 64] [LUN 56]. No upper limit for these loads was detected in flight [BLA 69].

Examples of variation coefficients calculated from measurements

The data given in the bibliography appear to show that the variation coefficient can reach values higher than one. Some examples of values calculated on spectra resulting from measurements taken on different carriers (missiles, fighter aircraft, trucks) are given later.

Shocks and vibrations measured on a missile

Analysis of these data showed a few interesting features:

– in the case of shocks, the coefficient $V_E = s_E / \overline{E}$ calculated from the shock spectra:

- varies substantially with the damping used in calculation of the spectra,

- differs according to the phenomenon measured,

- varies with the frequency. The highest values are observed at the frequencies of the peaks in the shock spectra,

- even for highly reproducible phenomena such as certain stage separation shocks, V_E is relatively large (between 0.1 and 0.5 for a very realistic damping of 0.07);

– for certain other shocks, V_E varies instead between 0.2 and 1 (in all cases $\xi = 0.07$);

– in the case of random vibrations, coefficient V_E calculated from the power spectral densities:

- can reach very high values of around 2,
- varies enormously with the frequency.

In spite of these high values, in the above examples, the ratio V_E calculated from the rms values of the spectral densities does not exceed a value of 0.9.

Vibrations with stores carried under fighter aircraft

The variation coefficients V_E relative to an external store, calculated in relation to the frequency, were analyzed based on:

– the power spectral densities of the signal;

– the corresponding extreme response spectra for Q = 10;

– the fatigue damage spectra for two values of parameter b (4 and 10).

It was observed that:

– the variation coefficient calculated from the PSDs varied approximately between 0.6 and 1.2 (the coefficient V_E calculated with the rms values was 0.33);

– the variation coefficient calculated from the extreme response spectra varied approximately between 0.3 and 0.4 and the limit at high frequencies was also around 0.33, which was consistent with the properties of the extreme response spectra in this range;

– the coefficient V_E calculated from the fatigue damage spectra differed according to the value chosen for parameter b. The higher the value of b, the higher the coefficient. This phenomenon, which might appear problematic, does not actually give rise to any difficulties, since the specification resulting from these calculations is not sensitive to the choice of b.

Vibrations relating to truck transportation

The dispersion displayed on the PSDs is lower than that above. It lies approximately between 0.2 and 0.4 in the relevant frequency range.

These observations clearly show that it is not realistic to choose an envelope value *a priori* for all the variation coefficients of the real environment. In view of the differences between phenomena and the variations observed according to the frequency, this envelope value would necessarily be very penalizing. It appears preferable, insofar as it is possible, to:

– individualize the variation coefficient for each type of environment;

– use real $V_E(f)$ curves rather than an envelope value.

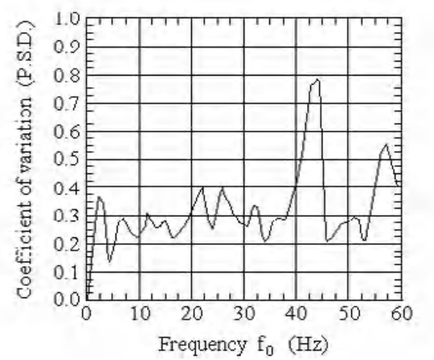

Figure 8.5. *Variation coefficient of 'truck' random vibration PSDs*

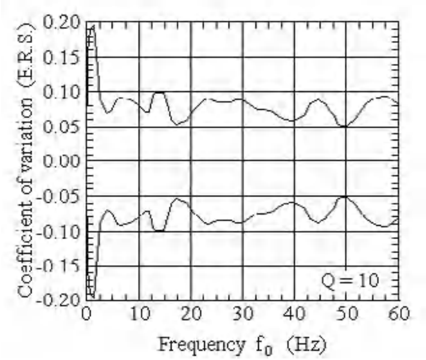

Figure 8.6. *Variation coefficient of 'truck' random vibration ERSs*

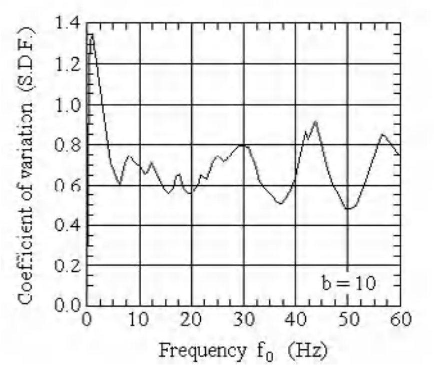

Figure 8.7. *Variation coefficient of 'truck' random vibration FDSs (b = 10)*

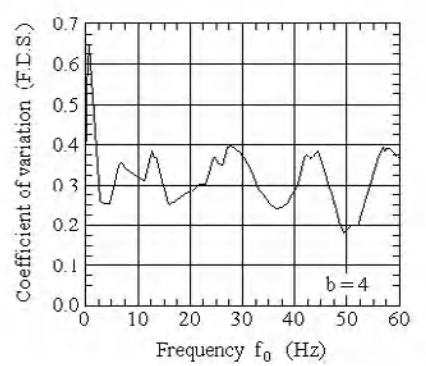

Figure 8.8. *Variation coefficient of 'truck' random vibration FDSs (b = 4)*

As an example, Figures 8.5 to 8.8 show the variations of the coefficient of variation in relation to the frequency of vibrations measured on the platform of a truck (10 measurements). The coefficient of variation was calculated on the basis of:

– the power spectral densities (Figure 8.5). The coefficient of variation calculated from the rms values was 0.04;

– the extreme response spectra plotted for $\xi = 0.05$ (Figure 8.6);

– the fatigue damage spectra ($\xi = 0.05$, Figure 8.7 for b = 10, Figure 8.8 for b = 4).

8.3.1.3. *Estimate of the variation coefficient – calculation of its maximum value*

In the above examples, the variation coefficient has been calculated from approximately ten samples. Although this number is low for statistical estimations, it is hardly ever reached in practice. As a result, it would be more correct to determine, for a given confidence level, the interval of confidence in which the variation coefficient may be located. So it would be more correct to determine, from measurements, the maximum value that the coefficient of variation can take with a given probability.

The non-central Student law

Let us consider a random variable x distributed according to a normal law and an independent random variable y distributed according to a χ^2 law with f degrees of freedom.

If δ is a constant, it can be shown that the variable

$$t = \frac{x + \delta}{\sqrt{\frac{y}{f}}} \qquad [8.2]$$

is distributed according to a non-central Student law [KAY 32]. When the non-centrality parameter δ equals zero, the familiar Student law is found.

The probability densities of the estimated mean and of the standard deviation of a normal random variable from n measurements are respectively distributed according to a normal law $\frac{1}{\sqrt{2\,\pi}}\,e^{-\frac{x^2}{2}}$ and a χ^2 law such as $\frac{1}{\Gamma\left(\frac{f}{2}\right)2^{f/2}}\,y^{\frac{f-2}{2}}\,e^{-\frac{y}{2}}$.

The probability density of the ratio of two random variables (standard deviation and mean) is obtained by integration of the product of the densities of these variables $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{2^{f/2}\,\Gamma\left(\frac{f}{2}\right)} y^{\frac{f-2}{2}} e^{-\frac{y}{2}}$, which also can be written [LAL 05]:

$$\frac{\sqrt{2\pi}}{2^{f/2}\,\Gamma\left(\frac{f}{2}\right)} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) y^{\frac{f-2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\right).$$

If we set $y = u^2$, knowing that p(y) dy = p(u) du, this product can be written:

$$\frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) u^{f-1} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\right) \qquad [8.3]$$

By integration of this expression for all the values of the standard deviation u between zero and infinity, we obtain the density

$$p(x) = \frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)} \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) u^{f-1} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\right) du \qquad [8.4]$$

i.e.

$$p(x) = \frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)} \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) u^{f-1}\, G'(u)\, du \qquad [8.5]$$

where

$$G'(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \qquad [8.6]$$

Knowing that $x = \frac{t\,u}{\sqrt{f}} - \delta$, we obtain, by writing the probability density as a function of the variable t, since $p(x)\,dx = p(t)\,dt$:

$$p(t) = \frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)\sqrt{f}} \int_0^{\infty} \left(\frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{u\,t}{\sqrt{f}}-\delta\right)^2}{2}} \right) u^f\, G'(u)\, du \qquad [8.7]$$

The distribution function of the t variable is obtained by integration of the density function p(t) between $-\infty$ and t_0:

$$P\left(t \le t_0\right) = \frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)\sqrt{f}} \int_0^{\infty} \left(\int_{-\infty}^{t_0} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{t\,u}{\sqrt{f}}-\delta\right)^2}{2}}\, dt \right) u^f\, G'(u)\, du$$

[8.8]

However

$$\int_{-\infty}^{t_0} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{t\,u}{\sqrt{f}}-\delta\right)^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{x^2}{2}}\, \frac{\sqrt{f}}{u}\, dx = \frac{\sqrt{f}}{u}\, G\left(\frac{t_0\,u}{\sqrt{f}} - \delta\right) \qquad [8.9]$$

from where [OWE 63]:

$$P\left(t \le t_0\right) = \frac{\sqrt{2\pi}}{2^{\frac{f-2}{2}}\,\Gamma\left(\frac{f}{2}\right)} \int_0^{\infty} G\left(\frac{t_0\,u}{\sqrt{f}} - \delta\right) u^{f-1}\, G'(u)\, du \qquad [8.10]$$

with

$$G\left(U\right) = \int_{-\infty}^{U} G'(\mu)\, d\mu \qquad [8.11]$$

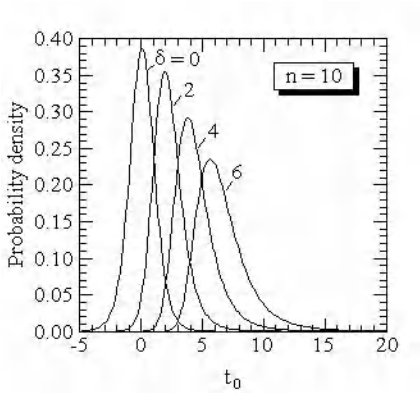

Figure 8.9. *Probability density of the non-central Student law*

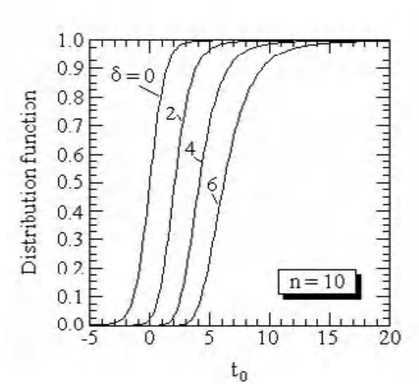

Figure 8.10. *Distribution function of the non-central Student law*

Figures 8.9 and 8.10 respectively show, as an example, the probability densities and the distribution functions of this law for n = 10, and δ equal to 0, 2, 4 and 6.

If the distribution function is integrated several times by parts, it can be put in a form easier to calculate numerically.

We obtain, for the odd values of f:

$$P(t \le t_0) = G\left(-\delta\sqrt{B}\right) + 2\,T\left(\delta\sqrt{B}, A\right) + 2\left(M_1 + M_3 + \cdots + M_{f-2}\right) \quad [8.12]$$

and for the even values:

$$P(t \le t_0) = G\left(-\delta\right) + \sqrt{2\pi}\left(M_0 + M_2 + \cdots + M_{f-2}\right) \quad [8.13]$$

where

$$A = \frac{t}{\sqrt{f}} \quad [8.14]$$

$$B = \frac{f}{f + t^2} \quad [8.15]$$

$$T\left(h, a\right) = \frac{1}{2\pi}\int_0^a \frac{e^{\left[-\frac{h^2}{2}\left(1+x^2\right)\right]}}{1 + x^2}\,dx \quad [8.16]$$

$$
\begin{array}{l|l}
M_{-1} = 0 & M_3 = \dfrac{2}{3} B \left(\delta A M_2 + M_1 \right) \\
M_0 = A \sqrt{B}\, G'\left(\delta \sqrt{B}\right) G\left(\delta A \sqrt{B}\right) & M_4 = \dfrac{3}{4} B \left(\dfrac{1}{2} \delta A M_3 + M_2 \right) \\
M_1 = B \left[\delta A M_0 + \dfrac{A}{\sqrt{2\pi}} G'(\delta) \right] & \cdots \\
M_2 = \dfrac{1}{2} B \left(\delta A M_1 + M_0 \right) & M_k = \dfrac{k-1}{k} B \left(a_k \delta A M_{k-1} + M_{k-2} \right)
\end{array}
\qquad [8.17]
$$

with

$$
a_k = \frac{1}{(k-2)\, a_{k-1}} \qquad [8.18]
$$

for $k \geq 3$ and $a_2 = 1$.

Application – distribution law of the coefficient of variation

Let us consider a Gaussian random variable x with a mean m and a standard deviation s. By definition, the coefficient of variation is equal to $V = \dfrac{s}{m}$.

Let us set μ and σ as the values of mean and of the standard deviation estimated from n measurements. An estimation of the coefficient of variation is given by $V_{measured} = \dfrac{\sigma}{\mu}$. It can be written:

$$
\frac{\sqrt{n}}{V_{measured}} = \frac{\sqrt{n}\,\mu}{\sigma} = \frac{\dfrac{\sqrt{n}\,(\mu - m)}{s} + \dfrac{\sqrt{n}\, m}{s}}{\dfrac{\sigma}{s}}
$$

$$
\frac{\sqrt{n}}{V_{measured}} = \frac{U + \dfrac{\sqrt{n}}{V}}{\sqrt{\dfrac{V_{measured}}{n-1}}} \qquad [8.19]
$$

where U is a Normal variable (everything happens as if we had a Normal law $N(\delta, 1)$ as the numerator).

By comparison with [8.2], it appears that $\frac{\sqrt{n}}{V_{measured}}$ is distributed according to a non-central t-distribution where f = n – 1 and $\delta = \frac{\sqrt{n}}{V}$ [DES 83] [JOH 40] [KAY 32] [OWE 63] [PEA 32].

This law has been tabulated [HAH 70] [LIE 58] [NAT 63] [OWE 62] [OWE 63] [RES 57].

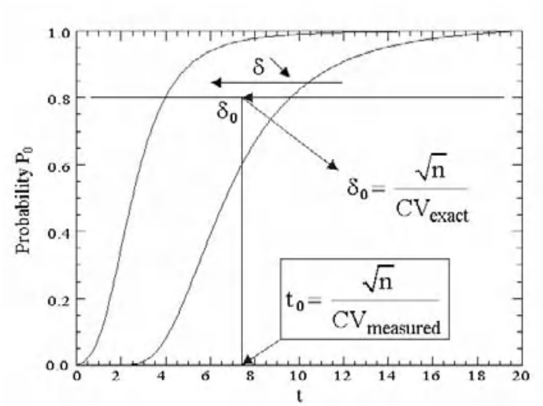

Figure 8.11. *Calculation principle of the maximum coefficient of variation from the distribution function of an off-centered Student distribution*

The variation coefficient can be estimated as follows:

– from the available tables. These cannot be accessed easily (outdated publications) and are difficult to use (interpolations required);

– from tables composed of random selection of values following a Gaussian law (Monte Carlo method) [COE 92] [GIR 97];

– from relations [8.12] to [8.18] and a numerical calculation. For a given n and t_0 value equal to $t_0 = \frac{\sqrt{n}}{V_{measured}}$, we search the value δ_0 of δ such that the probability is equal to a given probability $P_1 = P(t \leq t_0)$ (i.e. the probability that the true coefficient of variation is lower than its maximum value thus determined). $V_{max} = \frac{\sqrt{n}}{\delta_0}$ is then calculated from the δ value.

This step has the advantage of allowing a relatively simple semi-analytical calculation (the G(x) function can be evaluated from the error function and a series development). Tables 8.6 and 8.7 show, as an example, the maximum value of the variation coefficient in relation to a number n of measurements, at confidence levels of 90% and 80%, for different values of the variation coefficient estimated from n measurements.

Example 8.2.

If the variation coefficient evaluated from five measures is 0.10, we can read from Table 8.6 the value of the maximum variation coefficient 0.19 with a level of confidence of 90%.

	Estimated Variation Coefficient									
n	**0.05**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
2	0.3977	0.8009	1.8394	3.9758	13.043	/	/	/	/	/
3	0.1545	0.3120	0.6500	1.0519	1.5772	2.3303	3.5369	5.8287	11.962	84.088
4	0.1135	0.2287	0.4703	0.7401	1.0580	1.4511	1.9611	2.6588	3.6812	5.3339
5	0.0971	0.1953	0.3994	0.6217	0.8735	1.1686	1.5256	1.9716	2.5488	3.3292
6	0.0882	0.1773	0.3612	0.5589	0.7779	1.0276	1.3189	1.6669	2.0927	2.6280
7	0.0826	0.1659	0.3372	0.5196	0.7190	0.9422	1.1969	1.4930	1.8437	2.2671
8	0.0787	0.1580	0.3205	0.4925	0.6787	0.8845	1.1157	1.3797	1.6856	2.0453
9	0.0758	0.1521	0.3082	0.4725	0.6492	0.8426	1.0575	1.2995	1.5755	1.8943
10	0.0735	0.1476	0.2987	0.4450	0.6266	0.8107	1.0135	1.2395	1.4942	1.7842
11	0.0718	0.1439	0.2912	0.4350	0.6087	0.7855	0.9789	1.1927	1.4313	1.7002
12	0.0703	0.1409	0.2849	0.4266	0.5940	0.7650	0.9509	1.1550	1.3811	1.6336
13	0.0690	0.1384	0.2797	0.4266	0.5818	0.7479	0.9277	1.1239	1.3400	1.5795
14	0.0680	0.1365	0.2753	0.4195	0.5714	0.7335	0.9081	1.0978	1.3056	1.6345
15	0.0671	0.1345	0.2715	0.4133	0.5625	0.7210	0.8913	1.0756	1.2763	1.4965
16	0.0663	0.1329	0.2681	0.4080	0.5547	0.7102	0.8767	1.0563	1.2511	1.4638
17	0.0656	0.1315	0.2651	0.4032	0.5478	0.7007	0.8640	1.0394	1.2291	1.4353
18	0.0650	0.1302	0.2625	0.3990	0.5417	0.6923	0.8526	1.0245	1.2097	1.4104
19	0.0644	0.1291	0.2602	0.3953	0.5362	0.6848	0.8425	1.0112	1.1925	1.3882
20	0.0639	0.1280	0.2580	0.3919	0.5313	0.6780	0.8335	0.9993	1.1770	1.3685
25	0.0619	0.1241	0.2498	0.3788	0.5124	0.6520	0.7988	0.9539	1.1187	1.2941
30	0.0606	0.1214	0.2442	0.3698	0.4994	0.6343	0.7753	0.9234	1.0796	1.2447

Table 8.6. *Maximum variation coefficient as a function of the variation coefficient resulting from n measurements (confidence level of 90%)*

Example 8.3.

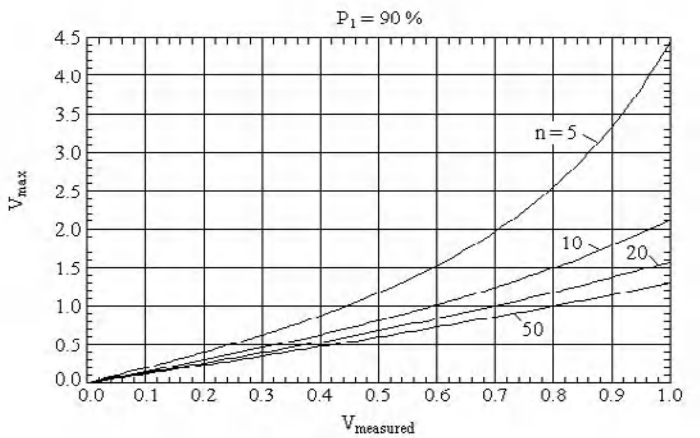

Figure 8.12. *Maximum variation coefficient as a function of the variation coefficient resulting from measurements (confidence level of 90%)*

Figure 8.12 shows the maximum variation coefficient calculated from relations [8.12] to [8.18], for an estimated variation coefficient resulting from a number of measurements equal to 5, 10, 20 and 50 respectively, and with a confidence level of 90%. This type of calculation can also be made to estimate the variation coefficient of the strength of a material, from n tests.

Table 8.8 was drawn up to underline the influence of the confidence level value. It gives, as a function of n, the maximum variation coefficient to be used when the estimated value equals 10%.

Comparison of methods of calculation

The values of the maximum variation coefficient, calculated by the expression of the non-central t-distribution are very close to the values deduced by selection, for values of n higher than about 2 (according to the P_1 value). An example is given in Table 8.9 for an estimated variation coefficient of 0.10.

	Estimated Variation Coefficient									
n	**0.05**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**
2	0.1972	0.3937	0.7882	1.2225	1.7553	2.4634	3.4826	5.1065	8.1408	15.913
3	0.1060	0.2126	0.4307	0.6615	0.9142	1.2006	1.5344	1.9340	2.4251	3.0466
4	0.0865	0.1734	0.3508	0.5364	0.7353	0.9530	1.1958	1.4711	1.7878	2.1578
5	0.0779	0.1563	0.3156	0.4814	0.6570	0.8461	1.0528	1.2813	1.5368	1.8253
6	0.0731	0.1465	0.2957	0.4501	0.6125	0.7857	0.9726	1.1763	1.4003	1.6483
7	0.0699	0.1402	0.2826	0.4296	0.5834	0.7464	0.9208	1.1089	1.3134	1.5370
8	0.0677	0.1357	0.2734	0.4151	0.5628	0.7185	0.8841	1.0615	1.2527	1.4599
9	0.0660	0.1323	0.2664	0.4042	0.5473	0.6977	0.8567	1.0261	1.2076	1.4029
10	0.0647	0.1296	0.2609	0.3956	0.5352	0.6813	0.8353	0.9986	1.1726	1.3589
11	0.0636	0.1275	0.2565	0.3887	0.5254	0.6681	0.8180	0.9764	1.1446	1.3237
12	0.0628	0.1257	0.2529	0.3830	0.5173	0.6872	0.8038	0.9582	1.1215	1.2948
13	0.0620	0.1242	0.2498	0.3782	0.5105	0.6480	0.7918	0.9428	1.1021	1.2706
14	0.0614	0.1229	0.2472	0.3740	0.5047	0.6402	0.7815	0.9297	1.0855	1.2500
15	0.0608	0.1218	0.2449	0.3704	0.4996	0.6333	0.7726	0.9183	1.0712	1.2322
16	0.0603	0.1208	0.2429	0.3673	0.4951	0.6273	0.7648	0.9083	1.0587	1.2166
17	0.0599	0.1200	0.2411	0.3645	0.4912	0.6220	0.7579	0.8995	1.0476	1.2029
18	0.0595	0.1192	0.2395	0.3620	0.4876	0.6173	0.7517	0.8916	1.0377	1.1906
19	0.0592	0.1185	0.2381	0.3597	0.4844	0.6130	0.7461	0.8845	1.0288	1.1796
20	0.0589	0.1179	0.2368	0.3577	0.4815	0.6091	0.7411	0.8781	1.0208	1.1697
25	0.0577	0.1154	0.2317	0.3498	0.4704	0.5941	0.7216	0.8533	0.9898	1.1314
30	0.0568	0.1137	0.2283	0.3443	0.4626	0.5837	0.7081	0.8362	0.9685	1.1052

Table 8.7. *Maximum variation coefficient as a function of the variation coefficient resulting from n measurements (confidence level of 80%)*

	Confidence level P_1 (%)				
n	**50**	**60**	**70**	**80**	**90**
2	0.1479	0.1902	0.2509	0.3937	0.8009
3	0.1200	0.1399	0.1677	0.2126	0.3120
4	0.1125	0.1267	0.1454	0.1734	0.2287
5	0.1091	0.1206	0.1352	0.1563	0.1953
6	0.1072	0.1170	0.1293	0.1465	0.1773
7	0.1059	0.1146	0.1254	0.1402	0.1659
8	0.1050	0.1129	0.1226	0.1357	0.1580
9	0.1043	0.1117	0.1204	0.1323	0.1521
10	0.1038	0.1106	0.1188	0.1296	0.1476
11	0.1034	0.1098	0.1174	0.1275	0.1439
12	0.1031	0.1092	0.1163	0.1257	0.1409
13	0.1029	0.1086	0.1154	0.1242	0.1384
14	0.1026	0.1081	0.1146	0.1229	0.1365
15	0.1024	0.1077	0.1138	0.1218	0.1345
16	0.1023	0.1073	0.1132	0.1208	0.1329
17	0.1021	0.1070	0.1127	0.1200	0.1315
18	0.1020	0.1067	0.1122	0.1192	0.1302
19	0.1019	0.1065	0.1117	0.1185	0.1291
20	0.1018	0.1062	0.1113	0.1179	0.1280
25	0.1014	0.1053	0.1098	0.1154	0.1241
30	0.1012	0.1047	0.1087	0.1137	0.1214

Table 8.8. *Maximum variation coefficient as a function of the confidence level (for a measured variation coefficient equal to 0.10)*

	Confidence Level P_1 (%)					
	50		80		90	
n	**MC method VC**	**Analytical VC**	**MC method VC**	**Analytical VC**	**MC method VC**	**Analytical VC**
2	0.1533	0.1478	0.4487	0.3937	0.9927	0.8009
3	0.1222	0.1200	0.2159	0.2126	0.3131	0.3120
4	0.1131	0.1125	0.1741	0.1734	0.2280	0.2287
5	0.1094	0.1091	0.1566	0.1563	0.1947	0.1953
6	0.1074	0.1072	0.1467	0.1465	0.1769	0.1773
7	0.1061	0.1059	0.1403	0.1402	0.1656	0.1659
8	0.1052	0.1050	0.1358	0.1357	0.1577	0.1521
9	0.1045	0.1043	0.1323	0.1323	0.1518	0.1549
10	0.1040	0.1038	0.1297	0.1296	0.1474	0.1476
15	0.1025	0.1024	0.1218	0.1218	0.1344	0.1345
20	0.1018	0.1018	0.1179	0.1179	0.1280	0.1280
25	0.1014	0.1014	0.1154	0.1154	0.1240	0.1241
30	0.1012	0.1012	0.1137	0.1137	0.1213	0.1214

Table 8.9. *Comparison of the maximum variation coefficient calculated from the analytical et Monte Carlo methods (for a measured variation coefficient equal to 0.10)*

8.3.2. *Material strength*

8.3.2.1. *Source of dispersion*

For a given lot of mechanical parts, static and dynamic tests show that properties such as the elastic limit, the ultimate strength and the fatigue limit have a random character and can only be evaluated statistically [BAR 77a]. The strength cannot be negative, zero or infinite, but its limit is never known with accuracy. In calculations, it is therefore made to vary between zero and infinity.

Variations between several parts of the same set can have various origins [GOE 60]:

– the position of the test-bar in the ingot;

– test direction (stress applied in relation to the axes of the ingot);

– properties when exposed to temperature (the effects differ after exposure to a temperature which is different from the ambient temperature, according to the method of heat treatment);

– variations due to the thickness of the metal plates.

Moreover, W.P. Goepfert [GOE 60] notes that in one lot, the properties' variations are of the same order of magnitude as in several lots for a given product. E.B. Haugen [HAU 65] quotes the following results (aluminum alloys):

– yield stress tends to be more important for very thin specimens or for very small sections (size effect);

– extruded shapes and rolled bars tend to display wider dispersions than do plates or sheets of same alloy and same heat treatment;

– tensile yield and compressive yield strength tend to display approximately the same degree of dispersion;

– cladding clearly modifies scatter for very thin specimens;

– dispersion is somewhat affected by heat treatment;

– alloy seems to be a minor factor in determining dispersion (for normal alloys).

Equipment manufactured from these materials has a resistance of random nature ([BLA 69] [FEL 59] [JOH 53] [STA 67]) related, of course, to the dispersion of the characteristics of materials, but also to that of the structures (tolerances of manufacture, manufacturing process, residual stresses, etc.). Much less information is available on these last sources of scatter.

8.3.2.2. *Distribution laws*

Experience shows that Gaussian, log–normal, extreme values [JUL 57] or Weibull [CHE 77] distribution laws can in most cases be used to represent a material's static strength [FEN 86] [STA 65]. Certain authors consider that the distribution is approximately Gaussian [CES 77] [STA 65], others that it is probably log–normal since a Gaussian distribution can exhibit negative values [LAM 80] [OSG 82]. A Gaussian distribution can often be an acceptable approximation [HAY 65]. A log–normal distribution, however, appears to give more realistic practical results in all cases [ALB 62] [JUL 57] [LAL 94] [PIE 92].

The left side of the strength distribution (Figure 8.13) is in general most useful (section 8.4). An approximation of a log–normal distribution by applying a Normal law tends to be on the conservative side for low values.

From a compilation of 262 test results, A.M. Freudenthal and P.Y. Wang [FRE 70] established an empirical relation giving the ultimate stress distribution of aeronautical structures (assuming that all the structures belonged to the same population, regardless of the structure and failure mode), based on a Weibull distribution:

$$F = 1 - \exp\left[-\left(\frac{\lambda}{0.96}\right)^{19}\right] \qquad [8.20]$$

where

$$\lambda = \frac{\text{Ultimate strength}}{\text{Mean value of the ultimate strength}} \qquad [8.21]$$

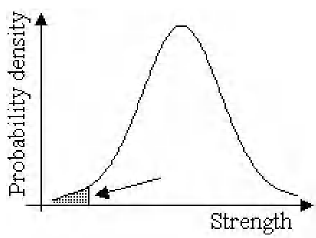

Figure 8.13. *Range of low values of probability density*

More detailed analyses by type of structure were made by other authors [CHE 70].

In the case of the numbers of cycles up to fracture by fatigue (i.e. of fatigue damage), it is generally considered that life expectancy has a log–normal law, which appears to be consistent with experimental data, and in particular with the variations of life expectancy, and is relatively easy to use [MAR 83] [WIR 80] [WIR 83a] (Volume 4, section 1.4.3). Other laws closer to the experimental results in certain domains of the S–N curve were proposed [LIE 78]. The Weibull law is sometimes preferred for physical and mathematical considerations.

8.3.2.3. *A few values of the variation coefficient*

Generally, dispersion measurements for static strength are smaller than those observed for loads (values representing the real environment) [CHO 66] [PIE 74]. As an example, Table 8.10 gives values of the strength variation coefficient of some materials taken from various publications (ratio of the standard deviation and the mean value).

It should be noted that these results generally come from measurements taken on low-size test-bars. C.O. Albrecht [ALB 62] notes that the very few values, it has seem to show that these variation coefficients are larger for specimens of large size.

Author	Material	V_R (%)	Test type
W. Barrois [BAR 77a and b]	Aerospace metallic materials	1 to 19	Tensile
T. Yokobori [YOK 65]	Cast iron Mild steel	8.8 5.1	Ultimate strength Ultimate tensile strength
R. Cestier J.P. Garde [CES 77]	Review of data from the Metals Handbook [MET 61] (3500 tests)	≤8 (Gauss)	Static strength
C.V. Stahle [STA 67]		21 (Log–normal)	
A.G. Piersol [PIE 74]		5 (Gauss)	
Laparlier Liberge [LAP 84]	Composites	1.8 to 5.6 according to temperature	Interlaminar shear
P.H. Wirsching [WIR 83b]		< 10	Traction
E.B. Haugen [HAU 65]		0.05 to 0.07	Ultimate strength Yield stress
R.E. Blake [BLA 62]	Metal structures Light alloy structures	3 to 30 3	Static strength
N.I. Bullen [BUL 56]	Wood Molded light alloy parts Glass (plate)	7 10 20 (Gauss)	
P. Albrecht [ALB 83]	Beams	6 to 22	

Table 8.10. *Examples of variation coefficients of strength*

The variation coefficient of the number of cycles up to fracture by fatigue can reach much higher values, exceeding 100%. Some examples can be found in section 1.4.3 of Volume 4.

8.4. Statistical uncertainty factor

8.4.1. *Definitions*

8.4.1.1. *Ratio of the smallest strength to the largest load*

From reasoning similar to that in section 8.1, the uncertainty coefficient can be defined as the relation between strength of the lowest element and the largest load with a given probability: mean $-\alpha$ standard deviations for strength, mean $+ \alpha$ standard deviations for load (Figure 8.14).

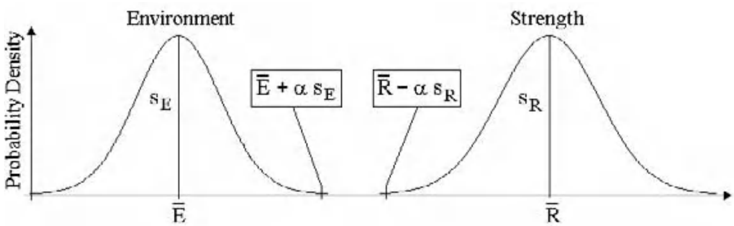

Figure 8.14. *Comparison between the largest load and the smallest strength with a given probability*

We must have at least:

$$\overline{R} - \alpha\, s_R = k_1\, (\overline{E} + \alpha\, s_E) \qquad [8.22]$$

or, if V_E is the variation coefficient of load ($V_E = \dfrac{s_E}{\overline{E}}$) and $V_R = \dfrac{s_R}{\overline{R}}$ is that of strength,

$$\overline{R}\,(1 - \alpha\, V_R) = k_1\, \overline{E}\,(1 + \alpha\, V_E) \qquad [8.23]$$

Hence the uncertainty (or safety) factor:

$$k_1 = \frac{\overline{R}}{\overline{E}}\, \frac{(1 - \alpha\, V_R)}{(1 + \alpha\, V_E)} \qquad [8.24]$$

Table 8.11 gives the probability of failure tolerated according to the value of parameter α.

α	0.50	1.64	2.33	3
Confidence level	0.5%	95%	99%	99.87%

Table 8.11. *Confidence level according to the number of standard deviations*

8.4.1.2. *Definition using reliability considerations*

If these distributions are known (type, mean, standard deviation and therefore coefficient of variation V, or ratio of the standard deviation to the mean), the probability P_0 of a specimen not surviving its environment can be calculated [LAL 87] [LAL 89]:

$$P_0 = \text{prob}(\text{Environment} > \text{Strength}) \tag{8.25}$$

The failure area is located on the graph under the probability density curves of the two distributions (Figure 8.15).

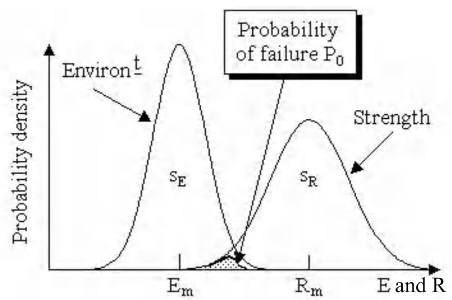

Figure 8.15. *Probability density of stress and strength*

For Gaussian and log–normal distributions of given standard deviations, it can be shown that P_0 depends solely on the ratio:

$$k = \frac{\text{Mean equipment strength } (\overline{R})}{\text{Mean real or specified environment } (\overline{E})} \tag{8.26}$$

This ratio $\frac{\overline{R}}{\overline{E}}$ is called the *uncertainty factor.* It depends on:

– variability of the environment characterizing the situation of the life cycle profile;

– variability of the ultimate equipment strength with respect to the situation under consideration.

The uncertainty factor is therefore determined for a given item of equipment and a given environment. The uncertainty factor can only be increased at the design stage, for a given excitation, by increasing the mean equipment strength (or decreasing the standard deviation of its distribution). k is also a function of the tolerated maximum failure probability.

The uncertainty factor can be calculated on the basis of the quantity directly representing the environment (e.g. the *static* acceleration incurred by equipment during propulsion of a launch vehicle) or from the results of an analysis. For a vibration, this factor is calculated for each frequency of the extreme response or fatigue damage spectra.

8.4.2. *Calculation of uncertainty factor*

8.4.2.1. *Case of Normal distributions*

In mechanics, reliability is the name given to the probability that the ultimate strength R of a part will be greater than the stresses E applied over a given duration (of the mission). It is assumed here that the curves characterizing the distributions of the environment and strength can be assimilated to Gaussian curves with the form

$$p(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2 s^2}} \qquad [8.27]$$

The probability that the stress resulting from the environment is below the ultimate strength of the equipment (reliability) is given by:

$$F = \text{prob}(E < R) = \text{prob}(R - E > 0) \qquad [8.28]$$

The failure probability is therefore equal to:

$$P_0 = 1 - F = \text{prob}(E > R) \qquad [8.29]$$

We shall now calculate the value of the uncertainty factor from a maximum specified failure probability

$$k = \frac{\overline{R}}{\overline{E}} \qquad [8.30]$$

If variables E and R have a Normal distribution, variable R – E also has a Normal distribution. Reliability F is given by

$$F = \int_0^{\infty} \frac{1}{s_{R-E}\sqrt{2\pi}}\, e^{-\frac{\left[x-\left(\overline{R-E}\right)\right]^2}{2\,s_{R-E}^2}}\, dx \qquad [8.31]$$

Considering the reduced centered variable $t = \frac{x-\left(\overline{R-E}\right)}{s_{R-E}}$, we obtain

$$F = \int_{-\overline{R-E}/s_{R-E}}^{\infty} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{t^2}{2}}\, dt \qquad [8.32]$$

i.e., because the Normal distribution is symmetric,

$$F = \int_{-\infty}^{\overline{R-E}/s_{R-E}} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{t^2}{2}}\, dt \qquad [8.33]$$

This expression can be written using the error function

$$\operatorname{erf}(x) = \int_0^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt \qquad [8.34]$$

yielding

$$F = \frac{1}{2} + \operatorname{erf}\left(\frac{\overline{R-E}}{s_{R-E}}\right) \qquad [8.35]$$

Furthermore, knowing that

$$\overline{R-E} = \overline{R} - \overline{E} \qquad [8.36]$$

and that

$$s_{R-E} = \sqrt{s_E^2 + s_R^2} \qquad [8.37]$$

by deduction, we obtain:

$$\operatorname{erf}^{-1}\left(F - \frac{1}{2}\right) = \frac{k - 1}{\sqrt{V_E^2 + k^2 V_R^2}} \qquad [8.38]$$

where V_E and V_R are the environment and strength coefficients of variation $\frac{s_E}{\overline{E}}$ and $\frac{s_R}{\overline{R}}$, respectively [BRE 70] [LIP 60] [RAV 69] [WIR 76]. Therefore, if

$$\text{aerf} = \operatorname{erf}^{-1}\left(F - \frac{1}{2}\right) = \operatorname{erf}^{-1}\left(\frac{1}{2} - P_0\right),$$

$$k = \frac{1 + \sqrt{1 - \left(1 - V_E^2\ \text{aerf}^2\right)\left(1 - V_R^2\ \text{aerf}^2\right)}}{\left(1 - V_R^2\ \text{aerf}^2\right)} \qquad [8.39]$$

(for $1 - V_R^2\ \text{aerf}^2 \neq 0$). For two Normal distributions characterized simply by their coefficients of variation V_E and V_R, we therefore arrive at a one-to-one relation between k and P_0. P_0 when given is equivalent to k and vice versa. As an example, Tables 8.12 and 8.13 give k as a function of V_E and V_R, for $P_0 = 10^{-6}$ and $P_0 = 10^{-3}$, respectively.

NOTE: *When the number of measurements used for the calculation of V_E or of V_R is small, the maximum value of this parameter for a given level of confidence can be calculated using the method described in section 8.3.1.3.*

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.00	1.105	1.235	1.399	1.614	1.906	2.328	2.990	4.177	6.930	20.31
0.05	1.238	1.266	1.35	1.486	1.683	1.963	2.376	3.032	4.214	6.963	20.34
0.10	1.475	1.496	1.560	1.674	1.849	2.110	2.507	3.149	4.320	7.059	20.43
0.15	1.713	1.732	1.790	1.895	2.059	2.309	2.694	3.325	4.485	7.214	20.58
0.20	1.951	1.969	2.826	2.128	2.289	2.535	2.916	3.542	4.697	7.420	20.78
0.25	2.188	2.207	2.264	2.367	2.529	2.776	3.158	3.786	4.943	7.669	21.03
0.30	2.426	2.445	2.504	2.609	2.774	3.025	3.414	4.049	5.216	7.954	21.33
0.35	2.664	2.683	2.744	2.852	3.022	3.280	3.678	4.324	5.507	8.267	21.68
0.40	2.902	2.922	2.984	3.097	3.273	3.539	3.947	4.609	5.813	8.604	22.06

Table 8.12. *Normal distributions* – $P_0 = 10^{-6}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.00	1.066	1.141	1.228	1.328	1.447	1.589	1.762	1.978	2.253	2.618
0.05	1.154	1.171	1.215	1.284	1.373	1.484	1.620	1.789	2.002	2.274	2.637
0.10	1.309	1.320	1.351	1.404	1.479	1.577	1.703	1.863	2.068	2.335	2.692
0.15	1.463	1.472	1.499	1.545	1.611	1.701	1.818	1.970	2.168	2.427	2.779
0.20	1.618	1.626	1.651	1.693	1.755	1.840	1.952	2.098	2.290	2.544	2.890
0.25	1.772	1.780	1.804	1.845	1.905	1.987	2.096	2.239	2.427	2.678	3.020
0.30	1.927	1.935	1.958	1.998	2.057	2.138	2.246	2.388	2.575	2.824	3.164
0.35	2.081	2.089	2.113	2.153	2.212	2.293	2.401	2.543	2.729	2.978	3.318
0.40	2.236	2.244	2.267	2.308	2.368	2.449	2.558	2.701	2.889	3.139	3.481

Table 8.13. *Normal distributions* – $P_0 = 10^{-3}$

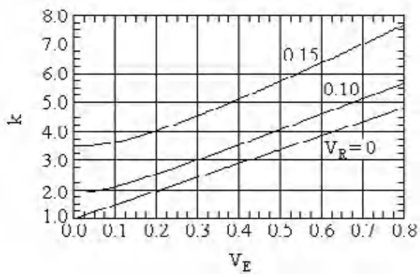

Figure 8.16. *Uncertainty factor Normal distributions –* $P_0 = 10^{-6}$

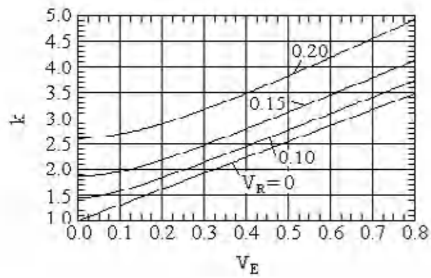

Figure 8.17. *Uncertainty factor Normal distributions –* $P_0 = 10^{-3}$

8.4.2.2. *Case of log–normal distributions*

As above, reliability is expressed as [KEC 68] [MAR 74] [RAV 78]:

$$F = \text{prob}(E < R) = \text{prob}(R - E > 0) \tag{8.40}$$

and failure probability as

$$P_0 = 1 - F = \text{prob}(E > R) \tag{8.41}$$

Variables E and R follow log–normal distributions in this case. This means that $\ln E$ and $\ln R$ have Normal distributions. Similarly $\ln R - \ln E$ has a Normal distribution. The relationship $F = \text{prob}(E < R)$ is equivalent to $F = \text{prob}[\ln(E) < \ln(R)]$, or to

$$F = \text{prob}[(\ln R - \ln E) > 0] \tag{8.42}$$

or

$$F = \text{prob}\left(\ln \frac{R}{E} > 0\right) \tag{8.43}$$

yielding, where t is the reduced centred variable $t = \dfrac{x - (\overline{\ln R - \ln E})}{s_{\ln R - \ln E}}$ and if $t_0 = \dfrac{\overline{\ln R - \ln E}}{s_{\ln R - \ln E}}$:

$$F = \int_{-\infty}^{t_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \qquad [8.44]$$

or

$$F = \frac{1}{2} + \mathrm{erf}\left(\frac{\overline{\ln \frac{R}{E}}}{s_{\ln \frac{R}{E}}} \right) \qquad [8.45]$$

It can be easily demonstrated that

$$\overline{\ln R - \ln E} = \overline{\ln R} - \overline{\ln E} \qquad [8.46]$$

$$s_{\ln R - \ln E} = \sqrt{s^2_{\ln E} + s^2_{\ln R}} \qquad [8.47]$$

The mean m and the standard deviation s of a log–normal distribution of a variable x are, moreover, related to the mean $m_{\ln}$ and the standard deviation $s_{\ln}$ of the corresponding Normal distribution by the equations:

$$\overline{x} = e^{\overline{\ln x} + \frac{s^2_{\ln x}}{2}} \qquad [8.48]$$

$$s^2_x = e^{(2\,\overline{\ln x} + s^2_{\ln x})} \left(e^{s^2_{\ln x}} - 1 \right) \qquad [8.49]$$

or, conversely,

$$s_{\ln x} = \sqrt{\ln\left(1 + \frac{s^2_x}{\overline{x}^2}\right)} \qquad [8.50]$$

and

$$\overline{\ln x} = \ln \overline{x}^2 - \frac{1}{2} \ln\left(\overline{x}^2 + s^2_x\right) \qquad [8.51]$$

yielding

$$\overline{\ln R - \ln E} = \ln k + \ln \sqrt{\frac{1 + V_E^2}{1 + V_R^2}} \qquad [8.52]$$

$$s_{\ln R - \ln E} = \sqrt{\ln\left[\left(1 + V_E^2\right)\left(1 + V_R^2\right)\right]} \qquad [8.53]$$

and

$$\operatorname{erf}^{-1}\left(F - \frac{1}{2}\right) = \operatorname{erf}^{-1}\left(\frac{1}{2} - P_0\right) = \frac{\ln k + \ln \sqrt{\dfrac{1 + V_E^2}{1 + V_R^2}}}{\sqrt{\ln\left[\left(1 + V_E^2\right)\left(1 + V_R^2\right)\right]}} \qquad [8.54]$$

Setting $\text{aerf} = \operatorname{erf}^{-1}\left(F - \frac{1}{2}\right)$, yields:

$$k = \exp\left\{\text{aerf} \sqrt{\ln\left[\left(1 + V_E^2\right)\left(1 + V_R^2\right)\right]} - \ln \sqrt{\frac{1 + V_E^2}{1 + V_R^2}}\right\} \qquad [8.55]$$

NOTE: *In this equation, the k factor depends solely on* V_E *and* V_R*, and due to its relation to* $aerf$*, on the failure probability* P_0*.*

When the failure mode under consideration is fatigue fracture, E is the fatigue damage created by the environment (fatigue damage spectrum at a given natural frequency) and R is the ultimate damage.

When the failure mode involves the exceeding of a stress limit (ultimate stress for instance), R is the limit and E, depending on the case, is the static acceleration applied to the equipment, the value of the ERS or of the SRS at a given frequency.

V_E \ V_R	0.00	0.05	0.08	0.10	0.15	0.20	0.25	0.30	0.35	0.40
0.00	1.00	1.270	1.467	1.615	2.055	2.615	3.323	4.215	5.332	6.724
0.05	1.267	1.399	1.568	1.706	2.133	2.689	3.399	4.295	5.419	6.819
0.10	1.599	1.693	1.833	1.955	2.361	2.912	3.626	4.536	5.680	7.107
0.15	2.010	2.092	2.218	2.332	2.726	3.278	4.006	4.940	6.119	7.591
0.20	2.514	2.592	2.715	2.828	3.222	3.786	4.540	5.512	6.742	8.280
0.25	3.127	3.207	3.331	3.447	3.855	4.443	5.235	6.260	7.560	9.184
0.30	3.867	3.950	4.081	4.203	4.634	5.259	6.102	7.196	8.584	10.32
0.35	4.75	4.840	4.980	5.111	5.574	6.247	7.156	8.335	9.830	11.69
0.40	5.796	5.893	6.046	6.188	6.692	7.423	8.412	9.693	11.31	13.33

Table 8.14. *Uncertainty factor for log–normal distributions –* $P_0 = 10^{-6}$

V_E \ V_R	0.00	0.05	0.08	0.10	0.15	0.20	0.25	0.30	0.35	0.40
0.00	1.000	1.168	1.284	1.368	1.603	1.881	2.206	2.586	3.029	3.542
0.05	1.166	1.244	1.341	1.417	1.642	1.915	2.238	2.617	3.059	3.573
0.10	1.354	1.406	1.482	1.546	1.752	2.013	2.331	2.708	3.150	3.665
0.15	1.568	1.610	1.673	1.730	1.919	2.170	2.481	2.856	3.299	3.818
0.20	1.808	1.846	1.903	1.955	2.133	2.376	2.683	3.058	3.504	4.027
0.25	2.076	2.111	2.166	2.216	2.388	2.627	2.933	3.309	3.760	4.292
0.30	2.373	2.407	2.460	2.509	2.679	2.917	3.226	3.607	4.066	4.608
0.35	2.698	2.732	2.785	2.835	3.005	3.246	3.559	3.948	4.417	4.973
0.40	3.053	3.088	3.142	3.191	3.365	3.611	3.931	4.330	4.812	5.384

Table 8.15. *Uncertainty factor for log–normal distributions –* $P_0 = 10^{-3}$

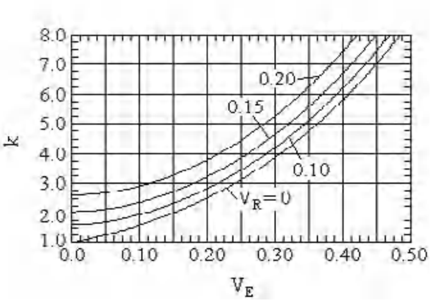

Figure 8.18. *Uncertainty factor Log–normal distributions –* $P_0 = 10^{-6}$

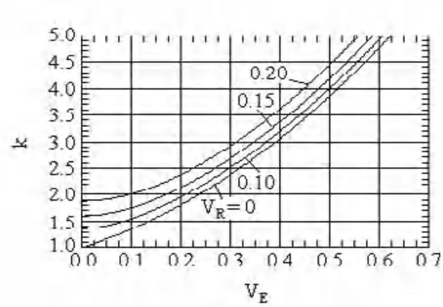

Figure 8.19. *Uncertainty factor Log–normal distributions –* $P_0 = 10^{-3}$

NOTE: *If one of the variation coefficients is estimated from a small number of measurements, the formula indicated in section 8.3.1.3 can be used to process the logarithms of the measured values (which are based on a Normal law), and* V_{max} *can be deduced for a given confidence level. However, it is not known how to simply calculate the variation coefficient V that appears in relationship [8.55] from* V_{max}. *The difficulty can be avoided using expression [8.50], which links the standard deviation* $s_{ln\,x}$ *to* V_x^2. *For a given number of measurements n and a level of confidence* P_0, *the maximum limit of the confidence interval including* $\sigma_{ln\,x}$, *or* $s_{ln\,x\,max}$, *can be calculated using the following formula*

$$s_{ln\,x\,max} = s_{ln\,x}\sqrt{\frac{n-1}{\chi^2_{\frac{1-P_0}{2},\,n-1}}} \qquad [8.56]$$

And the maximum value of V_x *can be deduced using*

$$V_{x\,max} = \sqrt{e^{s^2_{ln\,x\,max}} - 1} \qquad [8.57]$$

If the two variation coefficients are estimated from a small number of measurements, the logarithm of the values can be calculated, and the Normal laws can be used to evaluate the maximum value of the variation coefficients on one hand, and the uncertainty factor on the other.

8.4.2.3. *General case*

The probability that the stress (environment) is in the interval $E_0 - \frac{dE}{2}$, $E_0 + \frac{dE}{2}$ around a given E_0 value is equal to $p_E(E_0)\,dE$ (Figure 8.20).

The probability that resistance is lower than the applied stress (probability of failure) is [KEC 68] [LIG 79]:

$$P(R < E_0) = \int_{R_{Inf}}^{E_0} p_R(R)\,dR$$

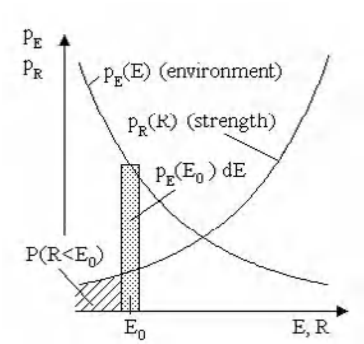

Figure 8.20. *First probability theory of failure*

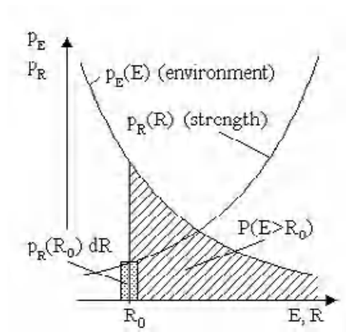

Figure 8.21. *Second probability theory of failure*

(E_{Inf}, E_{Sup}) and (R_{Inf}, R_{Sup}) are the limits to the definition intervals for the probability densities of the environment and strength $(-\infty, +\infty)$ for the Normal distributions and $(0, +\infty)$ for the log–normal distributions.

The failure probability due to the stresses in the interval dE around E_0 is thus

$$dP_0 = p_E(E_0)\,dE \int_{R_{Inf}}^{E_0} p_R(R)\,dR$$

yielding failure probability for all the values E of E_0 as:

$$P_0 = \int_{E_{Inf}}^{E_{Sup}} p_E(E)\left[\int_{R_{Inf}}^{E} p_R(R)\,dR\right] dE$$

A similar calculation based on the estimate of the probability that a given strength value R_0 is lower than the stress E led to the equivalent expression (Figure 8.21):

$$P_0 = \int_{R_{Inf}}^{R_{Sup}} p_R(R) \left[\int_{R}^{E_{Sup}} p_E(E)\, dE \right] dR$$

In general, calculation of the theoretical probability of failure P_0, or that of the uncertainty factor for a given P_0, can only be carried out numerically.

Tables 8.16 to 8.23 give the uncertainty factor (applicable to the mean of the environment) in the case of coupled Gaussian distributions, log–normal distributions and Weibull distributions, for $P_0 = 10^{-6}$ and $P_0 = 10^{-3}$ and various values of the variation coefficients V_E and V_R.

V_E \ V_R	**0.00**	**0.02**	**0.04**	**0.06**	**0.08**	**0.10**	**0.12**	**0.14**	**0.16**	**0.18**	**0.20**
0.00	1.000	1.100	1.210	1.332	1.466	1.615	1.778	1.958	2.156	2.374	2.614
0.05	1.238	1.265	1.337	1.437	1.558	1.698	1.856	2.033	2.229	2.446	2.686
0.10	1.475	1.496	1.553	1.640	1.751	1.884	2.038	2.213	2.408	2.626	2.868
0.15	1.713	1.731	1.784	1.867	1.976	2.109	2.264	2.441	2.641	2.864	3.113
0.20	1.951	1.969	2.021	2.104	2.214	2.350	2.509	2.692	2.900	3.132	3.391
0.25	2.188	2.206	2.259	2.344	2.458	2.598	2.764	2.956	3.172	3.416	3.687
0.30	2.426	2.445	2.499	2.587	2.705	2.852	3.025	3.226	3.454	3.709	3.995
0.35	2.664	2.683	2.739	2.831	2.954	3.107	3.290	3.500	3.740	4.009	4.310
0.40	2.901	2.921	2.980	3.075	3.204	3.365	3.556	3.778	4.030	4.313	4.629

Table 8.16. *Normal (environment) and log–normal (strength) distributions –* $P_0 = 10^{-6}$

V_E \ V_R	**0.00**	**0.02**	**0.04**	**0.06**	**0.08**	**0.10**	**0.12**	**0.14**	**0.16**	**0.18**	**0.20**
0.00	1.000	1.064	1.132	1.206	1.284	1.368	1.457	1.553	1.655	1.764	1.881
0.05	1.155	1.170	1.211	1.269	1.337	1.414	1.499	1.592	1.692	1.799	1.914
0.10	1.309	1.319	1.349	1.395	1.453	1.522	1.601	1.689	1.785	1.889	2.002
0.15	1.464	1.472	1.498	1.538	1.592	1.656	1.731	1.816	1.910	2.012	2.124
0.20	1.618	1.626	1.650	1.688	1.739	1.802	1.875	1.959	2.052	2.154	2.267
0.25	1.773	1.780	1.803	1.840	1.890	1.952	2.026	2.110	2.203	2.307	2.421
0.30	1.927	1.935	1.957	1.994	2.044	2.107	2.181	2.265	2.361	2.467	2.583
0.35	2.082	2.089	2.112	2.149	2.199	2.263	2.338	2.425	2.522	2.631	2.750
0.40	2.236	2.244	2.267	2.304	2.356	2.420	2.497	2.586	2.686	2.797	2.920

Table 8.17. *Normal (environment) and log–normal (strength) distributions –* $P_0 = 10^{-3}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.105	1.235	1.399	1.614	1.906	2.328	2.989	4.176	6.926	20.278
0.05	1.267	1.291	1.366	1.495	1.688	1.966	2.378	3.032	4.214	6.959	20.300
0.10	1.599	1.614	1.664	1.758	1.911	2.153	2.535	3.166	4.329	7.060	20.397
0.15	2.010	2.023	2.065	2.143	2.270	2.475	2.815	3.404	4.532	7.234	20.548
0.20	2.514	2.526	2.566	2.638	2.754	2.937	3.237	3.772	4.841	7.491	20.763
0.25	3.127	3.140	3.180	3.251	3.363	3.536	3.812	4.296	5.283	7.845	21.046
0.30	3.866	3.880	3.921	3.995	4.109	4.281	4.545	4.992	5.894	8.323	21.404
0.35	4.750	4.764	4.808	4.886	5.006	5.182	5.446	5.873	6.703	8.961	21.843
0.40	5.795	5.811	5.859	5.943	6.070	6.256	6.527	6.950	7.729	9.807	22.378

Table 8.18. *Log–normal (environment) and Normal (strength) distributions –* $P_0 = 10^{-6}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.066	1.141	1.228	1.328	1.447	1.589	1.763	1.978	2.253	2.618
0.05	1.166	1.180	1.222	1.288	1.375	1.486	1.622	1.790	2.002	2.275	2.638
0.10	1.354	1.363	1.390	1.436	1.503	1.596	1.717	1.873	2.076	2.341	2.696
0.15	1.568	1.575	1.596	1.633	1.688	1.765	1.871	2.012	2.200	2.452	2.797
0.20	1.808	1.814	1.833	1.865	1.914	1.982	2.075	2.203	2.376	2.613	2.943
0.25	2.076	2.082	2.099	2.129	2.174	2.237	2.323	2.440	2.600	2.822	3.136
0.30	2.373	2.378	2.395	2.424	2.467	2.527	2.609	2.719	2.870	3.079	3.378
0.35	2.699	2.704	2.721	2.749	2.792	2.851	2.930	3.036	3.181	3.381	3.666
0.40	3.054	3.059	3.076	3.105	3.148	3.206	3.285	3.389	3.530	3.724	4.000

Table 8.19. *Log–normal (environment) and Normal (strength) distributions –* $P_0 = 10^{-3}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.233	1.527	1.900	2.377	2.988	3.773	4.787	6.101	7.808	10.035
0.05	1.136	1.285	1.567	1.936	2.412	3.024	3.811	4.826	6.146	7.909	10.088
0.10	1.295	1.395	1.659	2.024	2.501	3.117	3.911	4.938	6.267	8.000	10.200
0.15	1.481	1.549	1.792	2.153	2.633	3.257	4.064	5.107	6.458	8.213	10.496
0.20	1.699	1.747	1.965	2.321	2.806	3.442	4.266	5.333	6.714	8.506	10.840
0.25	1.952	1.991	2.181	2.531	3.022	3.672	4.517	5.613	7.030	8.869	11.259
0.30	2.247	2.280	2.445	2.785	3.282	3.949	4.821	5.950	7.413	9.306	11.768
0.35	2.587	2.617	2.763	3.089	3.592	4.277	5.177	6.346	7.859	9.818	12.360
0.40	2.979	3.007	3.139	3.450	3.955	4.659	5.590	6.805	8.376	10.408	13.035

Table 8.20. *Weibull (environment) and Weibull (strength) distributions –* $P_0 = 10^{-6}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.105	1.225	1.361	1.516	1.692	1.895	2.126	2.392	2.697	3.048
0.05	1.104	1.152	1.257	1.387	1.538	1.713	1.914	2.144	2.410	2.714	3.066
0.10	1.223	1.248	1.331	1.449	1.594	1.765	1.964	2.193	2.458	2.752	3.114
0.15	1.357	1.374	1.436	1.541	1.678	1.844	2.040	2.268	2.532	2.837	3.189
0.20	1.507	1.521	1.570	1.661	1.789	1.949	2.141	2.368	2.632	2.938	3.291
0.25	1.676	1.687	1.728	1.807	1.925	2.079	2.268	2.496	2.756	3.063	3.419
0.30	1.864	1.874	1.910	1.980	2.088	2.234	2.419	2.642	2.906	3.213	3.573
0.35	2.073	2.082	2.115	2.178	2.278	2.417	2.596	2.816	3.080	3.390	3.752
0.40	2.303	2.312	2.342	2.401	2.495	2.627	2.801	3.018	3.280	3.592	3.958

Table 8.21. *Weibull (environment) and Weibull (strength) distributions –* $P_0 = 10^{-3}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.233	1.527	1.900	2.377	2.988	3.773	4.787	6.101	7.808	10.035
0.05	1.267	1.332	1.586	1.958	2.420	3.030	3.816	4.832	6.154	7.880	10.059
0.10	1.599	1.624	1.775	2.096	2.553	3.158	3.947	4.970	6.298	8.024	10.280
0.15	2.010	2.027	2.118	2.365	2.788	3.383	4.172	5.205	6.549	8.301	10.586
0.20	2.514	2.529	2.598	2.782	3.149	3.718	4.504	5.547	6.914	8.698	11.024
0.25	3.127	3.142	3.203	3.353	3.662	4.189	4.961	6.011	7.402	9.224	11.608
0.30	3.866	3.882	3.940	4.075	4.344	4.822	5.566	6.618	8.030	9.896	12.339
0.35	4.749	4.766	4.825	4.954	5.202	5.641	6.349	7.387	8.819	10.728	13.240
0.40	5.795	5.813	5.874	6.005	6.244	6.658	7.333	8.352	9.792	11.744	14.328

Table 8.22. *Log–normal (environment) and Weibull (strength) distributions –* $P_0 = 10^{-6}$

V_E \ V_R	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
0.00	1.000	1.105	1.225	1.361	1.516	1.692	1.895	2.126	2.392	0.180	3.048
0.05	1.166	1.188	1.272	1.394	1.543	1.716	1.916	2.146	2.411	2.716	3.066
0.10	1.354	1.365	1.410	1.499	1.627	1.789	1.981	2.207	2.469	2.772	3.122
0.15	1.568	1.576	1.607	1.670	1.773	1.914	2.094	2.311	2.567	2.867	3.214
0.20	1.808	1.815	1.840	1.890	1.973	2.095	2.257	2.416	2.709	3.003	3.318
0.25	2.076	2.083	2.104	2.148	2.219	2.325	2.471	2.661	2.898	3.183	3.523
0.30	2.373	2.379	2.399	2.439	2.504	2.600	2.733	2.909	3.133	3.410	3.742
0.35	2.699	2.704	2.724	2.762	2.823	2.913	3.037	3.204	3.418	3.683	4.008
0.40	3.053	3.059	3.079	3.116	3.175	3.261	3.381	3.539	3.744	4.003	4.320

Table 8.23. *Log–normal (environment) and Weibull (strength) distributions –* $P_0 = 10^{-3}$

8.4.2.4. *Influence of the choice of distribution laws*

Figures 8.22 and 8.23 allow a comparison between the uncertainty factor k obtained as a function of V_E, for several values of V_R, and the Normal distributions:

$$k = \frac{1}{1 - \text{aerf}^2\ V_R^2} + \sqrt{\frac{1}{\left(1 - \text{aerf}^2\ V_R^2\right)^2} - \frac{1 - \text{aerf}^2\ V_E^2}{1 - \text{aerf}^2\ V_R^2}}$$

and with log–normal distributions:

$$k = \sqrt{\frac{1 + V_R^2}{1 + V_E^2}}\ \exp\left\{\text{aerf}\ \sqrt{\ln\left[\left(1 + V_R^2\right)\left(1 + V_R^2\right)\right]}\right\}$$

(aerf corresponds to a given failure probability P_0). These curves show that:

– when V_E and V_R are very small, the results are very similar;

– the curves diverge very quickly as V_E and V_R increase;

– log–normal distributions lead to lower uncertainty factors than Normal distributions when V_E is small and to higher uncertainty factors when V_E is large (for a given value of V_R that is not zero).

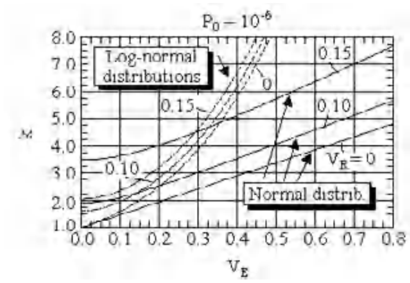

Figure 8.22. *Comparison of the uncertainty factors calculated from Normal and log–normal distribution, as a function of the variation coefficient of the environment ($P_0 = 10^{-6}$)*

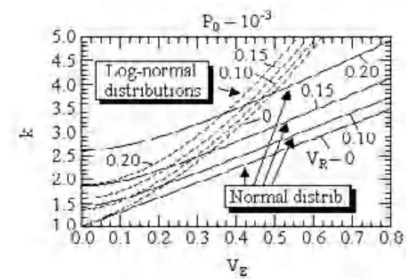

Figure 8.23. *Comparison of the uncertainty factors calculated from Normal and log–normal distributions, as a function of the variation coefficient of the environment ($P_0 = 10^{-3}$)*

Figures 8.24 and 8.25 show the results obtained as a function of V_R for a few values of V_E with both types of distributions.

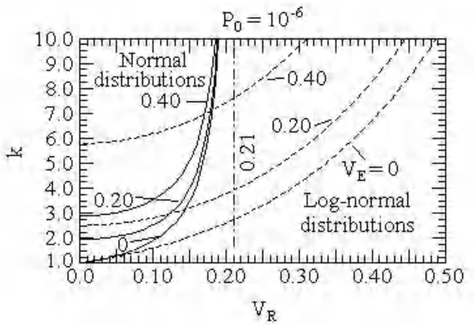

Figure 8.24. *Comparison of the uncertainty factors calculated from Normal and log–normal distributions as a function of the variation coefficient of strength ($P_0 = 10^{-6}$)*

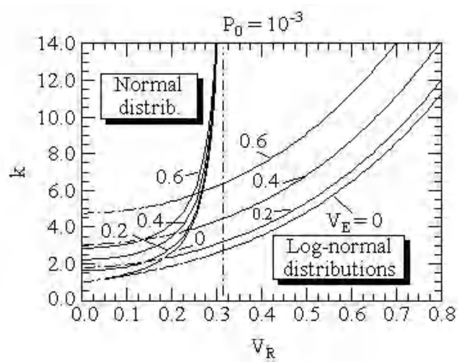

Figure 8.25. *Comparison of the uncertainty factors calculated from Normal and log–normal distributions as a function of the variation coefficient of strength ($P_0 = 10^{-3}$)*

It can be seen that:

– the comments made in relation to the above curves are of course confirmed;

– furthermore, there is a vertical asymptote in the case of Gaussian distributions. It can be shown that this asymptote occurs when:

$$1 - \text{aerf}^2\ V_R^2 = 0$$

$$V_R = \left|\frac{1}{\text{aerf}}\right| \qquad [8.58]$$

Thus, for each probability level P_0 there is a corresponding limit value V_R. Figure 8.26 below shows the variations of this limit value in relation to P_0 and Table 8.20 gives a few individual values.

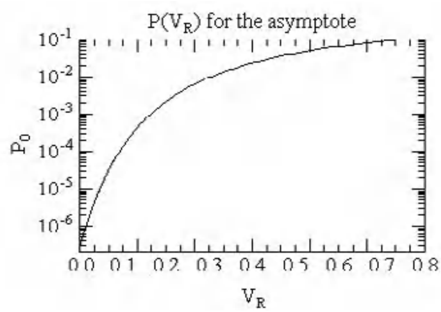

Figure 8.26. *Failure probability relating to the asymptote as a function of the variation coefficient of strength*

P_0	V_R
10^{-6}	0.21036
10^{-5}	0.23443
10^{-4}	0.26885
10^{-3}	0.32363
10^{-2}	0.42988
10^{-1}	0.78015

Table 8.24. *Failure probability relating to the asymptote as a function of the variation coefficient of strength*

These differences in behavior are due to the difference between the distributions at high values and to the negative values existing with a Normal distribution [JUL 57].

All these observations show that, in relation to the range of practical values of V_E and V_R, the chosen form of distribution does have an impact.

When the environment and strength distributions are not of a comparable nature, it may be seen that the uncertainty factors depend:

– less on the strength distribution as V_E gets larger (Figure 8.27). For the usual values of V_E and V_R (0.3 and 0.1 respectively), the factor is not very sensitive to the nature of this distribution;

– more on the strength distribution as V_R gets larger (Figure 8.28);

– more on the environment distribution as V_E gets larger (Figure 8.29);

– less on the environment distribution as V_R gets larger (Figure 8.30).

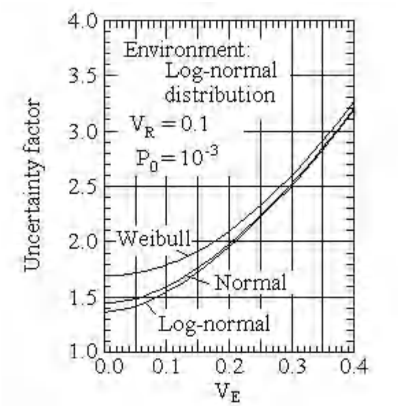

Figure 8.27. *Influence of the strength distribution*

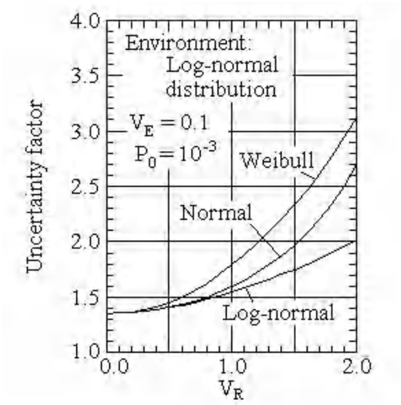

Figure 8.28. *Influence of the strength distribution*

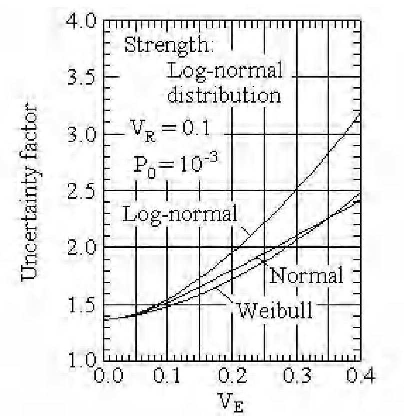

Figure 8.29. *Influence of the environment distribution*

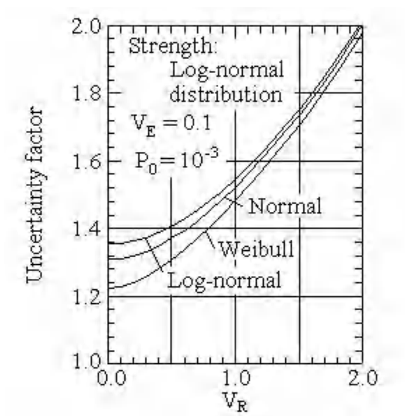

Figure 8.30. *Influence of the environment distribution*

8.4.3. *Calculation of an uncertainty coefficient when the real environment is only characterized by a single value*

The material may have to be sized to resist a maximum load value; this is the case, for example, for a significant drop during material handling, or the "static" propulsion acceleration of a satellite launcher.

In this situation, we can settle for considering that the given value is the mean of the distribution and that the standard deviation is zero. Strength variability is the only parameter considered.

Example 8.4.

Consider material that can experience an accidental drop from a set height of 9 meters. Suppose that the strength variability of this material is characterized by a variation coefficient of approximately 0.08 (log–normal distribution).

If we want to carry out drop tests to show that the material will withstand this drop with a probability equal to 10^{-6}, we must apply an uncertainty coefficient equal to:

$$k = 1.467$$

and thus the drop occurs from a height:

$$H = 13.2 \text{ m}$$

NOTE: *We will see in Chapter 10 that, if we only carry out one drop test, the demonstration of the material's behavior with the probability retained can only be given if the height is complemented by a second factor, the test factor. In the case of this example, this factor would be equal to*

$$T_F = 1.103$$

for a confidence level of 90% (Table 10.4). Hence the test specification:

$$H = 14.56 \text{ m}$$

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Chapter 9

Aging Factor

9.1. Purpose of the aging factor

Qualification tests demonstrate that, when tested, a piece of equipment is able to withstand the vibration conditions described in its life cycle profile. However, if this equipment is not used until several years after it has passed its qualification test, its strength may have decreased to such an extent that it will no longer be capable of withstanding its vibration environment as a result of aging.

One solution to this problem is to demand greater strength of new equipment, calculated so that after aging and deterioration, the residual strength is sufficient to withstand the vibration and shocks of the life cycle profile. This additional strength can be expressed by an aging factor applied when writing the specifications. It is calculated by making an assumption regarding deterioration of the equipment [LAL 89] over time.

9.2. Aging functions used in reliability

Let P_v be the probability of correct operation related to aging. P_v is a function of time that can be approximated by several functions such as [BON 71]:

– the Weibull function:

$$P_v = \exp\left[-\frac{(t-\lambda)^{\beta}}{\eta}\right] \tag{9.1}$$

where β, λ and η are constants;

– the Normal function:

$$P_v = \frac{1}{s\sqrt{2\pi}} \int_t^{\infty} \exp\left[-\frac{(t-m)^2}{2\,s^2}\right] dt \qquad [9.2]$$

m = mean s = standard deviation.

This function, which is easier to manipulate and includes fewer parameters to be determined, will be used below. However, this is not a limitation on the proposed method, which could be used with any relevant equation provided that the numerical data necessary for its definition are available.

Relation [9.2] can be written in different forms facilitating numerical calculations. Let:

$$u = \frac{t-m}{s} \qquad [9.3]$$

$$P_v = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} e^{-\frac{u^2}{2}}\, du \qquad [9.4]$$

If $u > 0$,

$$P_v = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{u} e^{-\frac{u^2}{2}}\, du \qquad [9.5]$$

and if $u < 0$

$$P_v = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{|u|} e^{-\frac{u^2}{2}}\, du \qquad [9.6]$$

yielding, in the general case,[1]

$$P_v = \frac{1}{2} - \mathrm{sgn}(u)\, \frac{1}{\sqrt{2\pi}} \int_0^{|u|} e^{-\frac{u^2}{2}}\, du \qquad [9.7]$$

1. sgn( ) = sign of ( ).

Knowing that the error function can be defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad [9.8]$$

and that

$$\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt \qquad [9.9]$$

the probability P_V can also be written in the form

$$P_V = \frac{1}{2}\left[1 - \mathrm{sgn}(u)\ \mathrm{erf}\left(\frac{|u|}{\sqrt{2}}\right)\right] \qquad [9.10]$$

Example 9.1.

Let us assume that $\frac{s}{m} = 0.1$ and that, after a time $t = 10$ years, the probability P_V is equal to 0.999. It is confirmed that, when $t = 0$, since $u = -\frac{m}{s}$,

$$P_V = \frac{1}{\sqrt{2\pi}} \int_{-\frac{m}{s}}^{\infty} e^{-\frac{u^2}{2}}\, du \qquad [9.11]$$

$$P_V = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{\frac{m}{s}} e^{-\frac{u^2}{2}}\, du \qquad [9.12]$$

P_V is very close to a single unit ($\frac{m}{s} = 10$). If $t = 10$ years, in order for $P_V = 0.999$, it is necessary that

$m = 14$ years

therefore

$s = 1.4$ years.

9.3. Method for calculating aging factor

Let P_0 be the probability of correct operation at time $t = 0$, which may be demonstrated by a test conducted with a specification based on coefficient k developed in section 8.4 (or in reference [LAL 87]).

If the equipment is used at time $t > 0$, the probability of correct operation P_t is equal to

$$P_t = P_0 \; P_v \tag{9.13}$$

and is lower than P_0 since $P_v < 1$. At the time t_u of equipment use, a probability P_t equal to the probability P_0 specified up until now for time $t = 0$, may be required. In such a case, to take account of aging and in line with the basic idea stated in section 9.1, the equipment must necessarily have a greater strength at time $t = 0$, corresponding to the probability of correct operation $P_0' > P_0$.

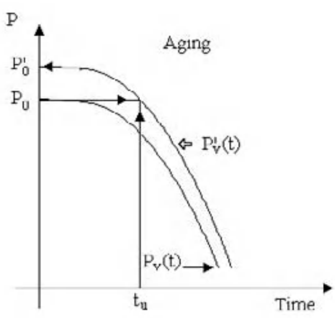

Figure 9.1. *Translation of the curve representing correct operation probability versus time*

Given an aging law $P_v(t)$, the method of calculating the coefficient for aging k_v consists therefore of the calculation of P_0' (for $t = 0$) in such a way that the curve $P_v'(t)$ deduced by vertical transformation passes from $P_v(t)$ through the point (t_u, P_0).

On the basis of [9.10], this yields, for $t = t_u$ and $u_u = \dfrac{t_u - m}{s}$,

$$P_v = \frac{1}{2}\left[1 - \mathrm{sgn}(u_u)\ \mathrm{erf}\left(\frac{|u_u|}{\sqrt{2}}\right)\right] \qquad [9.14]$$

and

$$P_0' = \frac{2\ P_0}{1 - \mathrm{sgn}(u_u)\ \mathrm{erf}\left(\frac{|u_u|}{\sqrt{2}}\right)} \qquad [9.15]$$

which can also be expressed as

$$P_0' = \frac{2\ P_0}{1 - \mathrm{sgn}(u_u)\sqrt{\frac{2}{\pi}}\ \int_0^{|u_u|} e^{-\frac{u^2}{2}}\ du} \qquad [9.16]$$

On the basis of this value P_0', it is known [LAL 87] how to calculate the factor, k', that is to be applied to the stress amplitude in order to demonstrate this probability and then to deduce the aging factor, k_v, that is solely related to the aging effects, by evaluating the ratio

$$k_v = \frac{k'}{k} \qquad [9.17]$$

(k being the uncertainty factor necessary to guarantee P_0).

NOTE: *There is a relatively obvious limit to this calculation. The probability P_0' increases with t_u. For sufficiently large t_u, we can numerically have $P_0' > 1$, which is of course absurd (Figure 9.2 where $P_0 = 0.9$).*

The condition $P_0' \le 1$ has as a consequence

$$2\ P_0 \le 1 - sgn(u_u)\ erf\left(\frac{|u_u|}{\sqrt{2}}\right) \qquad [9.18]$$

or, if $u < 0$ (useful case):

$$erf\left(\frac{|u_u|}{\sqrt{2}}\right) \geq 2\ P_0 - 1 \qquad [9.19]$$

a relation leading to a value limits of t_u.

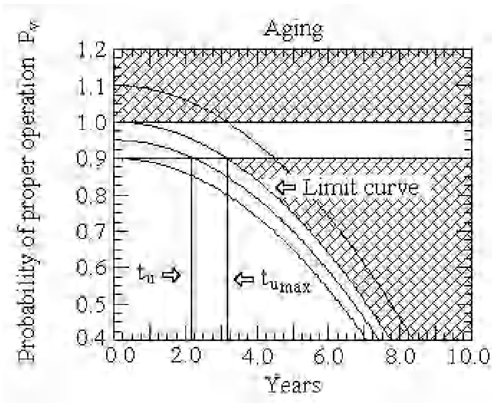

Figure 9.2. *The limit of transformation possibilities*

Example 9.2.

Let us go back to the previous example (Gaussian distribution, $m = 14$ years, $s = 1.4$ years), with $P_0 = 0.999$. Figure 9.3 shows the variations of P_v versus time. It can be seen that P_v remains very close to one until $t = 9$ years, when it decreases suddenly.

Figure 9.4 gives the variations of the aging factor k_v in relation to time (calculated for $V_E = 0.20$, $V_R = 0.08$, and log–normal distributions). This factor varies between 1 and 1.3 for $t < 9.671$ years. There is no solution if $t \geq 9.674$ years, for the reasons already mentioned above.

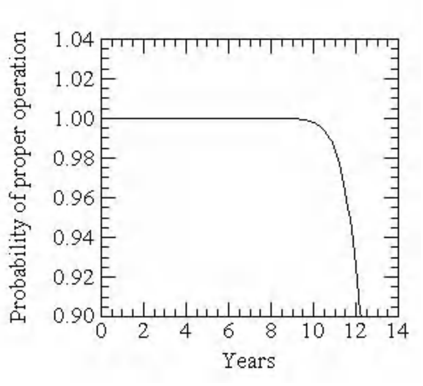

Figure 9.3. *Example of a curve of correct operation probability as a function of time*

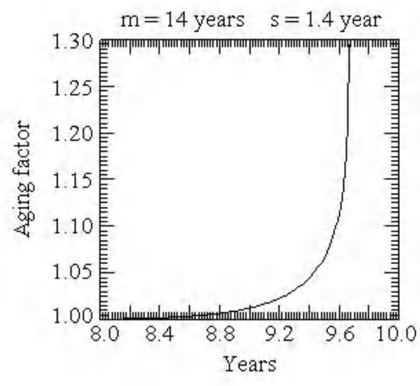

Figure 9.4. *Example of an aging coefficient as a function of the utilization duration to be guaranteed*

9.4. Influence of standard deviation of the aging law

Figure 9.5 shows $k_v(t)$ calculated under the same conditions as above for various values of s (and therefore of s / m).

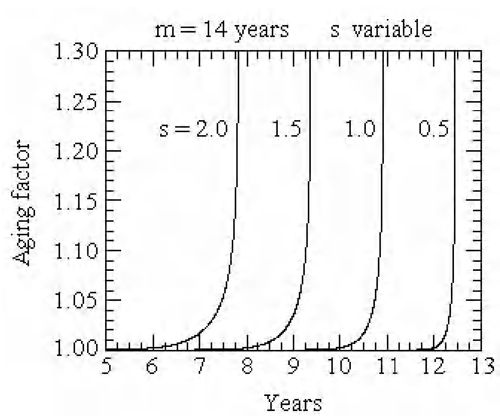

Figure 9.5. *Influence of the standard deviation of the aging law*

It may be seen that, for a given value of t_u, however small s (or s / m), k_v is smaller still.

9.5. Influence of the aging law mean

Following the same assumptions, Figure 9.6 shows the variations of $k_v(t)$ for $\frac{s}{m} = 0.1$ and m variable. The larger m is, the smaller k_v will be.

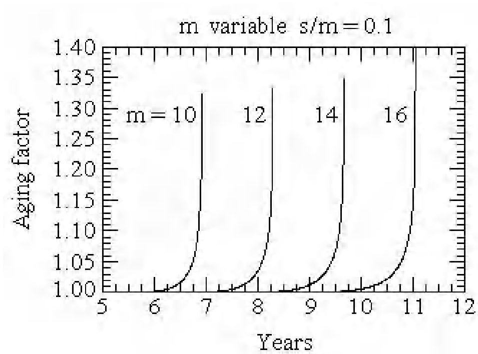

Figure 9.6. *Effect of the mean of the aging law*

NOTES:

1. *The aging factor must not be applied if the equipment has undergone accelerated aging before the tests.*

2. *This method of calculating k_v can be criticized since it makes an across the board allowance for a decrease over time for the performance of the mechanical system, without analyzing the real aging mechanisms which often integrate chemical processes.*

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 10

Test Factor

10.1. Philosophy

The purpose of the qualification test is to demonstrate that the equipment has at least the specified strength at the time of its design, i.e. the failure probability is less than or equal to P_0, or that the mean of the strength distribution is higher than k times the mean of the environment.

It was shown in preceding chapters that the reliability of an item depended solely on the ratio k (uncertainty factor) of the mean strength ($\overline{R}$) of the equipment to the environmental stress ($\overline{E}$) (for Normal or log–normal distributions of the given variation coefficients).

The stress of the environment is known, with its probability distribution, its mean value and its standard deviation. Assumptions can be made regarding the strength distribution of the equipment, and its variation coefficient (section 8.3.2), but the mean value is unknown.

If it were possible to conduct a very large number of tests, the *test severity* TS chosen would be k times the mean of the environment (withhold environment E_W). However, for obvious cost-related reasons, this test will, in general, only be conducted on very a small number of specimens, and often on a single specimen.

When the mean strength of a particular series is determined on the basis of various tests, all that is shown is that the mean lies within an interval located somewhere around the level of the test that leads to a fracture probability of 50%, the limits of which can be calculated with a given confidence level. In our case, if the tests carried out on level E_W are conclusive, we will be able to confirm that the mean is at least equal to a value taken in an interval centered on E_W. In a rather conservative manner and to simplify the explanation, the following will take E_W as the level which leads to a failure probability of 50%. The value of the mean shown by the test is then located statistically in an interval centered on E_W, the width of which would decrease if the number of tests were decreased, and would tend towards zero for an infinite number of tests. Let us return to the more realistic case of a small number of tests.

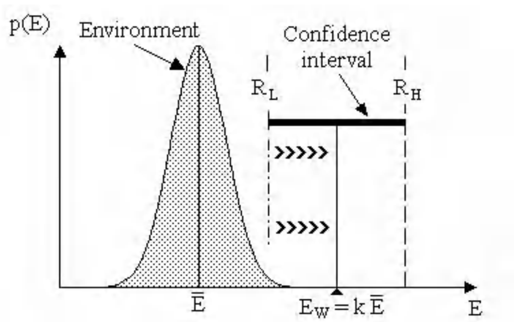

Figure 10.1. *Determination of test severity*

By conducting n tests, we will therefore seek to ensure that the mean $\overline{R}$ is such that [LAL 87]:

$$\overline{R} \geq k\,\overline{E}$$

10.2. Calculation of test factor

10.2.1. *Normal distributions*

Assuming that the tests are conducted with a severity equal to $k\,\overline{E}$. Since the number of tests is very limited, the value obtained for mean strength is only an estimate. The true value is actually located between two limits $\overline{R}_L$ and $\overline{R}_H$ [ALB 62]. For a Normal distribution, there is [ALB 62] [NOR 87]:

$$\overline{R}_L = \overline{R}_C - \frac{a'}{\sqrt{n}}\, s_R \qquad [10.1]$$

where:

$\overline{R}_L$ is the lower estimated limit of the true value;

$\overline{R}_C$ is the value calculated on the basis of the n tests;

s_R is the standard deviation of the distribution (in this case supposedly known).

a' is the probability factor for a given confidence level π_0. If V_R is known, the probability that a Normal random variable is included in an interval $(-a', a')$ is equal to $a' = N^{-1}\left(\dfrac{1+\pi_0}{2}\right)$ (N( ) being the normal law).

Example 10.1.

$a' = 1.64$ for $\pi_0 = 90\,\%$. This means that there is a 90% chance that the true mean strength is located between $\overline{R}_L$ and $\overline{R}_H$, a 5% chance that it is located below $\overline{R}_L$ and a 5% chance that it is located above $\overline{R}_H$.

NOTE:

The degree of confidence π_0 can be written

$$\pi_0 = \frac{1}{\sqrt{2\,\pi}} \int_{-a'}^{a'} e^{-u^2/2}\, du = 2\,\frac{1}{\sqrt{2\,\pi}} \int_0^{a'} e^{-u^2/2}\, du$$

It can therefore be expressed by using the previously defined (Volume 3, A4.1.1 section) error function $E_1(x)$:

$$\pi_0 = E_1\left(a'/\sqrt{2}\right)$$

yielding

$$a' = \sqrt{2}\; E_1^{-1}(\pi_0)$$

Upper bound R_H is useless, however, since we are simply attempting to demonstrate that $\overline{R} \geq k\,\overline{E}$ (Figure 10.2), the interval just needs one bound, for example between R_L and infinity, to be unilateral.

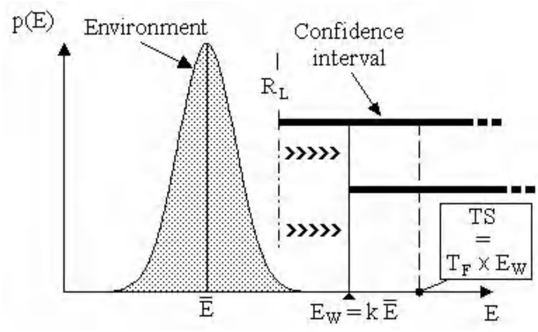

Figure 10.2. *Determination of test severity. Unilateral confidence interval*

In this case, relation [10.1] still applies, but with $a' = N^{-1}(\pi_0)$.

Example 10.2.

Probability $\pi_0 = 90\,\%$ is obtained for $a' \approx 1.28$. This means that there is a 90% chance that the true mean strength is higher than $\overline{R}_L$.

Table 10.1 gives a few values of a' calculated under these conditions.

a'	π_0	a'	π_0
0.00000	0.50000	2.32635	0.990000
0.25335	0.60000	2.50000	0.993790
0.50000	0.691463	2.575829	0.995000
1.00000	0.841345	3.00000	0.998650
0.52440	0.70000	3.090023	0.999000
0.67449	0.75000	3.50000	0.999767
0.84162	0.80000	3.71901	0.9999000
1.03643	0.85000	4.00000	0.999968
1.50000	0.933193	4.26489	0.9999900
1.28155	0.90000	4.50000	0.9999966
1.64485	0.95000	4.75342	0.999999
2.00000	0.977250	5.00000	0.99999971

Table 10.1. *Some values of the factor a' and the associated confidence level*

Actual mean strength can be between $\overline{R}_L$ and $\overline{R}_C$ and thus lower than the desired value $\overline{R}_C$, which is not satisfactory. For conservative purposes, we will consider that, in the absence of fracture, the strength demonstrated is equal to the lower bound $\overline{R}_L$ of the interval where the mean strength can be located from n tests divided by the mean load $\overline{E}$. In this case, the uncertainty coefficient is

$$k' = \frac{\overline{R}_L}{\overline{E}} \qquad [10.2]$$

Since $\overline{R}_L < \overline{R}$, the uncertainty coefficient demonstrated is lower than the expected value k and consequently reliability shown is lower than the desired value.

To guarantee k, we must increase the severity of the test by a factor T_F such that the lower bound of the new interval of confidence is equal to $\overline{R} = k\,\overline{E}$, or

$$T_F = \frac{\overline{R}_C}{\overline{R}_L} \qquad [10.3]$$

From [10.1], we have

$$\overline{R}_C = \overline{R}_L + \frac{a'}{\sqrt{n}}\, s_R = \overline{R}_L \left(1 + \frac{a'}{\sqrt{n}} \frac{s_R}{\overline{R}_L}\right) \qquad [10.4]$$

or, by setting $V_R = \frac{s_R}{\overline{R}_L}$ (variation coefficient, supposedly known)

$$\overline{R}_C = \overline{R}_L \left(1 + \frac{a'}{\sqrt{n}} V_R\right) \qquad [10.5]$$

Hence

$$T_F = 1 + \frac{a'}{\sqrt{n}} V_R \qquad [10.6]$$

It may be observed that this factor depends on the variation coefficient of the strength and the number of tests expected (for a given confidence level).

NOTE: *For a Gaussian distribution, the standard deviation over the mean ratio can be of approximately 3 or higher. Strictly speaking, a population should not be considered Gaussian if the variation coefficient is too large. Sizes that we study are generally positive (shock response spectra, fatigue damage spectra, etc.), so that, for values with a high probability of being positive, mean m and standard deviation s must be such that m–3s is positive, and thus the variation coefficient is lower than approximately 0.33 [JOH 40].*

Table 10.2 gives a few values of T_F as a function of V_R and n, calculated for a confidence level π_0 of 0.90.

		V_R								
		0	**0.05**	**0.08**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**
n	**1**	1	1.064	1.103	1.128	1.192	1.256	1.320	1.385	1.449
	2	1	1.045	1.073	1.091	1.136	1.181	1.227	1.272	1.317
	5	1	1.029	1.046	1.057	1.086	1.115	1.142	1.172	1.201
	10	1	1.020	1.032	1.041	1.061	1.081	1.101	1.122	1.142)
	20	1	1.014	1.023	1.029	1.043	1.057	1.072	1.086	1.100

Table 10.2. *Test factor with confidence level of 90% (Gauss law)*

NOTE: *The demonstration leading to this result is debatable, since it assumes that we know the standard deviation, whereas the variation coefficient is actually the parameter considered as known.*

A mathematically more correct relationship was established by L. Pierrat and J. Vanuxeem [PIE 09a] relying on:

– the search for an estimator of the mean distribution of an n size sample by the method of the likelihood maximum;

– the search of an approximation of the mean of this distribution;

– the search of an approximation of the variation coefficient of this distribution.

$$T_F \simeq 1 + \frac{V_R\, N^{-1}(\pi_0)}{\sqrt{n\left(1 + 2\, V_R^2\right)}} \qquad [10.7]$$

If V_R is small compared with one, the 2 V_R^2 term can be ignored; we then get relation [10.6] which is an upper bound.

The values in Table 10.2 are modified as follows (Table 10.3).

		V_R								
		0	**0.05**	**0.08**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**
n	**1**	1	1.064	1.102	1.127	1.188	1.247	1.302	1.354	1.402
	2	1	1.045	1.072	1.090	1.133	1.174	1.214	1.250	1.284
	5	1	1.029	1.046	1.057	1.084	1.110	1.135	1.158	1.180
	10	1	1.020	1.032	1.040	1.059	1.078	1.096	1.112	1.127
	20	1	1.014	1.023	1.028	1.042	1.055	1.068	1.079	1.090

Table 10.3. *Test factor at a confidence level of 90% (Normal distribution).Pierrat's expression*

Example 10.3.

Let us consider the example of a container protecting an item of equipment which is susceptible to a fall from a random height H characterized by five measurements. It is assumed that the stress created by the impact is proportional to the impact velocity, and that this velocity is distributed according to a Gaussian law.

H (m)	4.895	3.77	4.50	4.04	4.31
v_i (m s^{-1})	9.8	8.6	9.4	8.9	9.2

The mean of this distribution is $\overline{v}_i = 9.18\ \text{m s}^{-1}$ and the standard deviation is $s_{v_i} = 0.46\ \text{m s}^{-1}$, or a variation coefficient $V_E = \dfrac{s_{v_i}}{\overline{v}_i} \approx 0.05$. If the variation coefficient of the strength of the most sensitive part of the equipment is equal to $V_R = 0.08$, to obtain a failure probability of less 10^{-3}, the average strength of the material, expressed in impact velocity $\overline{R}$, must be higher than or equal to k $\overline{v}_i$ where k is taken from Table 8.13:

$$k = 1.373$$

therefore

$$\overline{R} = 1.373 \text{ x } 9.18 = 12.60 \text{ m s}^{-1}$$

The purpose of the test is to demonstrate that the equipment in the container can withstand, on average, an impact velocity of 12.60 ms^{-1} (i.e. a drop with a height of 8.1 m).

Bearing in mind that these calculations will be demonstrated in only one test, it is necessary to apply a test factor, the value of which is T_F = 1.046 at a confidence level of 90%, (Table 10.3). The drop test severity is then:

$$TS = T_F \ k \ \overline{v}_i = 1.102 \text{ x } 1.373 \text{ x } 9.18 \approx 13.89 \text{ m s}^{-1}$$

which corresponds to a drop of height $H \approx 9.83$ m.

NOTES:

1. *The variation coefficient of the distribution law for the impact velocity ($V_E \approx 0.05$) has been estimated from 5 measurements. To take this small number into account, it would be better to use the maximum limit of the interval where the true value of this variation coefficient may be located (at a confidence level of 90% for example), instead of the estimated value $V_E \approx 0.05$, which would result in $V_{E\,max} = 0.0971$ (Table 8.6).*

The value then obtained with the same method will be $k = 1.472$. Therefore

$$TS = T_F \ k \ \overline{v}_i = 1.102 \text{ x } 1.472 \text{ x } 9.18 \approx 14.89 \text{ m s}^{-1}$$

and $H \approx 11.30$ m.

It would also be possible to take the small number of measurements into account to estimate the variation coefficient of the strength. The maximum coefficient $V_{R\,max}$ calculated in this way would modify both the k value and the T_F value.

2. *The uncertainty factor has been determined in this example by considering that the representative parameter of the impact effect is the induced stress. If the structure of the container is distorted on impact, it is preferable to examine the thickness of the crushed material, which is proportional to the height of the drop. In these conditions, the estimated variation coefficient of the drop height is equal to 0.10, and the uncertainty factor k to 1.479, which leads to a test severity (determined by the height of the drop) of 7.01 m, as compared to 9.83 m. This example clearly shows that it is important to correctly select the parameter used to describe the environment, associated with the anticipated effects, otherwise significant errors will appear.*

10.2.2. *Log–normal distributions*

For a log–normal distribution, the lower limit of the interval in which the mean strength can be located is, similarly

$$\ln \overline{R}_L = \ln \overline{R}_C - \frac{a'}{\sqrt{n}} s_{\ln R} \qquad [10.8]$$

where $s_{\ln R}$ is the standard deviation of the log–normal distribution of R:

$$s_{\ln R} = \sqrt{\frac{\sum_j \left(\ln R_j - \ln \overline{R}\right)^2}{n}} \qquad [10.9]$$

yielding, as above,

$$\ln \overline{R}_L = \ln \overline{R}_C - \frac{a'}{\sqrt{n}} \ln\left(1 + V_R^2\right) \qquad [10.10]$$

and

$$\overline{R}_L = \overline{R}_C \exp\left(a' \sqrt{\frac{\ln\left(1 + V_R^2\right)}{n}} \right) \qquad [10.11]$$

The test factor is therefore equal to

$$T_F = \exp\left(a' \sqrt{\frac{\ln\left(1 + V_R^2\right)}{n}} \right) \qquad [10.12]$$

		V_R									
		0	**0.05**	**0.08**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**	**0.40**
n	**1**	1	1.064	1.103	1.128	1.192	1.256	1.320	1.384	1.449	1.513
	2	1	1.045	1.072	1.091	1.136	1.181	1.227	1.272	1.317	1.362
	5	1	1.029	1.046	1.057	1.086	1.115	1.143	1.172	1.201	1.229
	10	1	1.020	1.032	1.041	1.061	1.081	1.101	1.122	1.142	1.162
	20	1	1.014	1.023	1.029	1.043	1.057	1.072	1.086	1.100	1.115

Table 10.4. *Test factor for a confidence level of 90% (log–normal distribution)*

NOTE : *As with the Normal case, L. Pierrat and J. Vanuxeem [PIE 09b] propose a more rigorous relation in the form:*

$$T_F = \frac{exp\left[N^{-1}(\pi_0)\sqrt{ln\left(1+V_R^2 / n\right)}\right]}{\sqrt{1+V_R^2 / n}} \qquad [10.13]$$

if this factor is calculated from the mean of the distribution, or $T_F = exp\left[N^{-1}(\pi_0)\sqrt{ln\left(1+V_R^2 / n\right)}\right]$ *if it is calculated from the median.*

Table 10.5 gives a few values of T_F *calculated with expression [10.13] for* $\pi_0 = 0.90$.

		V_R									
		0	**0.05**	**0.08**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**	**0.40**
n	**1**	1	1.065	1.104	1.131	1.197	1.264	1.330	1.395	1.459	1.521
	2	1	1.046	1.073	1.092	1.139	1.186	1.233	1.280	1.327	1.373
	5	1	1.029	1.046	1.058	1.087	1.117	1.146	1.176	1.206	1.236
	10	1	1.020	1.033	1.041	1.061	1.082	1.103	1.124	1.145	1.166
	20	1	1.014	1.023	1.029	1.043	1.058	1.073	1.087	1.102	1.117

Table 10.5. *Test factor for a confidence level of 90% (log–normal distribution). Pierrat's expression*

10.2.3. *Weibull distributions*

L. Pierrat and J. Vanuxeem [PIE 09c] propose the relation:

$$T_F \simeq \left[1 + \frac{V_R\ N^{-1}(\pi_0)}{3\sqrt{n}} - \left(\frac{V_R}{3\sqrt{n}}\right)^2\right]^3 \qquad [10.14]$$

Table 10.6 gives a few values of T_F calculated with this expression for $\pi_0 = 0.90$.

		V_R									
		0	**0.05**	**0.08**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**	**0.35**	**0.40**
n	**1**	1	1.065	1.104	1.130	1.196	1.263	1.330	1.398	1.466	1.533
	2	1	1.046	1.073	1.092	1.138	1.185	1.232	1.279	1.327	1.375
	5	1	1.029	1.046	1.058	1.087	1.116	1.146	1.175	1.205	1.235
	10	1	1.020	1.033	1.041	1.061	1.082	1.103	1.123	1.144	1.165
	20	1	1.014	1.023	1.029	1.043	1.058	1.072	1.087	1.102	1.116

Table 10.6. *Test factor for a confidence level of 90% (Weibull distribution)*

10.3. Choice of confidence level

The choice of the confidence level is related to the choice of constant a'. Figures 10.3 and 10.4 show the influence of the choice of this parameter on the test factor for Normal and log–normal distributions (and for $n = 1$) from equations [10.6] and [10.12] respectively.

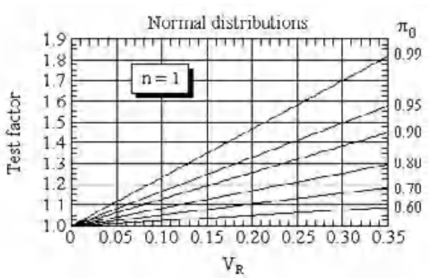

Figure 10.3. *Test factor for Normal distributions (n = 1)*

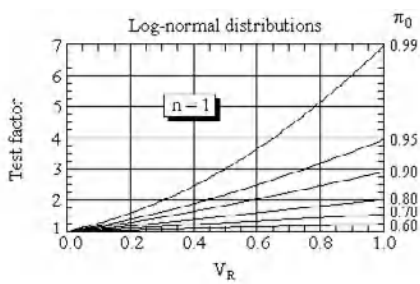

Figure 10.4. *Test factor for log–normal distributions (n = 1)*

It can be seen that:

– for π_0 given, T_F is smaller for a Normal distribution;

– for V_R given, T_F increases with π_0 (*a priori* obvious).

We generally use $a' = 1.64$ ($\pi_0 = 90\%$) in our calculations, which appears to us to be a reasonable value.

10.4. Influence of the number of tests n

Figures 10.5 and 10.6 show the variations of T_F as a function of V_R, for a few values of n between 1 and 20 ($\pi_0 = 0.90$). It can be observed that T_F varies little in relation to n, especially when V_R is small.

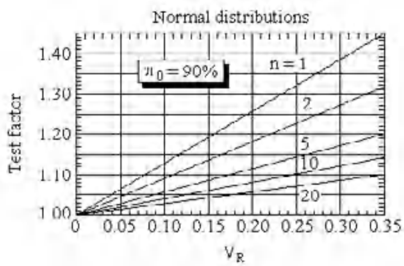

Figure 10.5. *Test factor for Normal distributions (confidence level of 90%)*

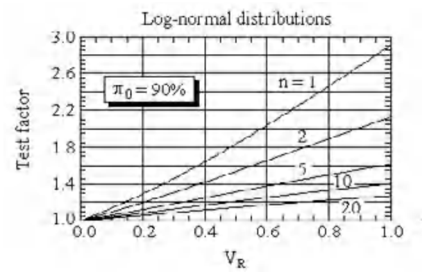

Figure 10.6. *Test factor for log–normal distributions (confidence level of 90%)*

Specification Development: Second Edition – Volume 5
Christian Lalanne

Chapter 11

Specification Development

Several national and international standards today require test tailoring, or in other words, to be able to write specifications from the life profile of the material studied and measurements of its real environment.

The goal of this chapter is to reveal the principle of this methodology and to show the value of the method using ERSs and FDSs which, without new assumptions, makes it possible to obtain specifications very close to the real environment rationally, with controlled margins, regardless of the nature of vibratory signals.

11.1. Test tailoring

All the procedures used to develop test specifications include a number of general stages [GEN 67] [KLE 65] [PAD 68] [PIE 66], which are summarized in Figure 11.1.

This approach, known as *tailoring,* is currently used in the following standards: GAM.EG13 and MIL–STD 810F. The specification determination process can be divided into four main stages:

1. analysis of the life cycle profile;

2. collection of data on the real environment;

3. synopsis of the data;

4. establishment of the test program.

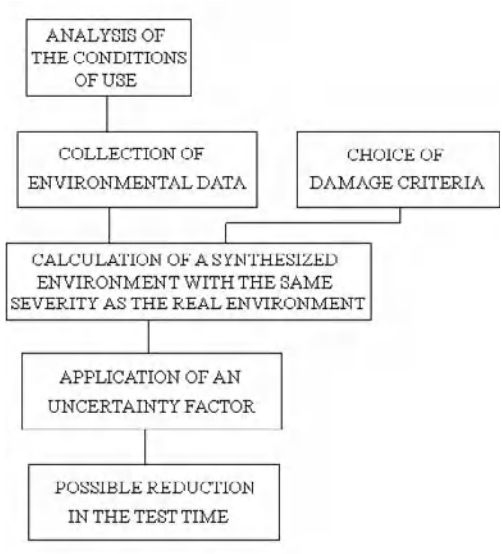

Figure 11.1. *General tailoring procedure*

11.2. Step 1: analysis of the life cycle profile. Review of the situations

It is assumed that it is possible to break the life cycle profile of the product down into elementary phases, called *situations*, such as storage, road transport, air transport by helicopter or fixed-wing aircraft, specifying the conditions that could affect the severity of the associated environments: speed and nature of the terrain for road transport, duration, position of the equipment, interfaces and assembly on the structure of the carrier vehicle, etc. [HAH 63].

The environmental parameters (vibration, heat, cold, acoustics, mechanical shock, etc.) that could degrade the equipment are listed qualitatively for each situation. Each situation corresponds to a particular phase of the life cycle profile. It is then subdivided into *sub-situations* or *events* wherever the environment can be considered as specific.

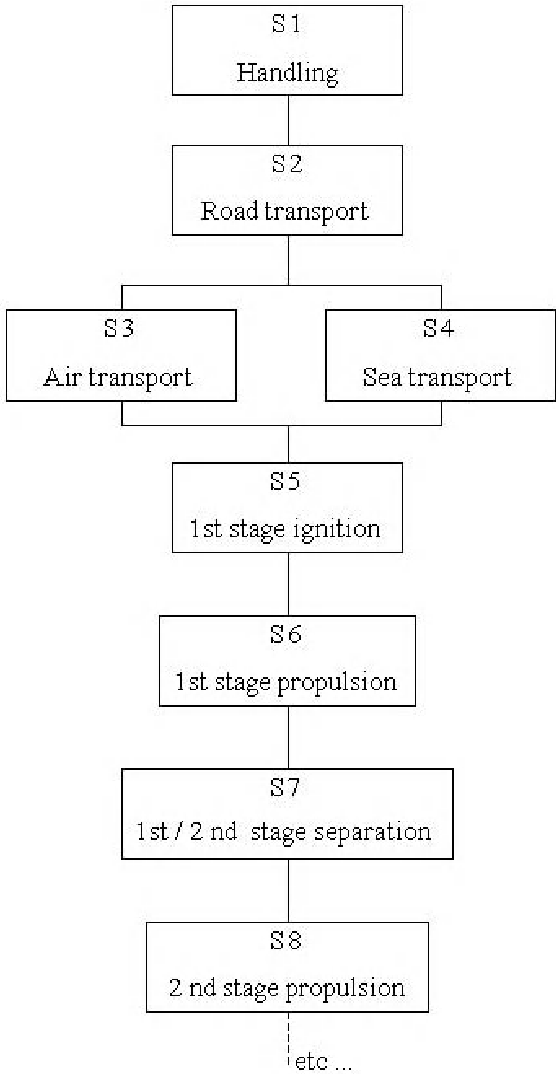

Figure 11.2. *Example of a simple life cycle profile (satellite)*

For instance, the air transport situation can include sub-situations for taxiing, takeoff, climbing, cruising, turning, descent, landing, etc. The duration of each sub-situation is specified. Figure 11.3 shows, as an example, the analysis of a life cycle profile limited to carriage under an aircraft [HAN 79].

Mission list	
Mission number	**Time (min)**
1	100
2	117
3	180
4	85
5	125
6	90
7	105
8	360
9	140

Maneuver list		
Number	**Type**	**Percent of average mission**
1	Takeoff	6.80
2	Flaps and Gear Down	6.80
3	IG Buffet Stall	0.16
4	Landing	6.80
5	360 °Roll	3.12
6	Straight and Level	5740
7	Throttle Sweep	0.68
8	Wind-up-turn	7.80
9	Pushover Nz= –1	0.008
10	Rolling Pullout	1.952
11	Side Slip	6.00
12	Pullout Nz= 6	0.48
13	Speedbrake extended	2.00

Figure 11.3. *Example of events for a given situation (fighter aircraft external store) [HAN 79]*

11.3. Step 2: determination of the real environmental data associated with each situation

The second step consists of quantitatively evaluating each of the environments identified above for each sub-situation of the life cycle profile. The available data can have several origins [PIE 74]:

– Measurements under real conditions (real vehicle, representative roads or runways) taken directly from the equipment. This is the most favorable case, but unfortunately the least usual. It presupposes that the equipment or a model has been mounted in a fairly representative way on the vehicle, and a complete series of measurements has been made;

– Measurements taken from the platform of the vehicle concerned. If the equipment is not placed directly on the platform (containerized equipment, for instance), it is necessary to transfer the vibrations. This operation may require modeling the intermediate structure or measuring the transfer functions;

– Measurements taken from another type of vehicle in the same category, but considered to be a good approximation of real values.

For a number of projects, measurements may have already been systematically taken to characterize each phase of the life cycle profile [KAC 68] [ROS 82] [SCH 66]. These measurements obtained from earlier projects can be used as a basis for writing up the specifications for a new piece of equipment;

– Measurements taken from studies of a new system following its manufacture are useful to verify the predictions, possibly determining the levels at specific points and supplying data for a more detailed failure analysis. These measurements will be used in turn for future projects.

To satisfy this need, databanks were created to contain the largest possible number of measurements recorded on vehicles under a variety of conditions [CAI 85] [COQ 81] [FOL 62] [FOL 65] [GEN 67]. The data may be stored in various forms in the databanks. The basic data generally consist of signals as a function of time. To decrease the space occupied by the data in the bank, these signals are processed where possible as spectra:

– shock response spectra for shocks;

– power spectral densities for stationary random vibrations.

Certain databanks (SANDIA [FOL 67]) also contain data expressed as probability densities. Such data give the percentage of the number of peaks in the signal for each acceleration level in specified frequency bands. These data can be expressed in a simplified form by indicating the maximum or average peak acceleration in relation to the frequency or any other statistical parameter (such as the level that has a 99% chance of not being exceeded):

– type-synopses, which are frequency analysis envelope curves of the signals measured on vehicles belonging to a particular category;

– prediction computations.

11.4. Step 3: determination of the environment to be simulated

A test must be defined from the data collected for each environment of each phase of the life cycle profile:

– It must have the same severity as in the real environment.

– It must be able to be conducted using standard test equipment. Actually, this requirement is not necessary at this point in time. However, it is preferable to take this need into account at an early stage, whenever possible, so as to rapidly identify any incompatibilities that might arise between required adjustments and the capabilities of the test facilities. Were such incompatibilities to be proved unacceptable, another simulation method might have to be proposed.

Rather than reproducing the exact data measured in the real environment on the test facility, it is preferable to develop a test with a synthetized environment with the same severity, reproducing the same damage as the real environment, in line with specific criteria (section 7.4.4.3).

11.4.1. *Need*

The qualities that can be expected of the equivalence method are multiple:

– Equivalence criteria are representative of real physical phenomena from the damage potential perspective [SMA 56].

It is very important to determine very early on the nature of the stresses that could lead to a failure. The damage modes could be multiple (creep, corrosion, fatigue, exceeding of a limit stress, fracture, etc.) [FEL 59]. It is, however, generally agreed that the two most frequent modes are the exceeding of a stress limit (elastic limit, ultimate strength) and fatigue damage [KLE 65] [KRO 62] [PIE 66] [RAC 69]. This task is also essential to determine an uncertainty factor [SCH 60].

– The data required for the calculation are readily available.

– The test time can be reduced.

– Several situations, several phases can be combined in a single test.

– Equivalence criteria are the same for all environments with the same structure (e.g. random vibrations, sinusoidal vibrations, mechanical shocks).

A working group organized by A.S.T.E. (Association pour le développement des Sciences et Techniques de l’Environnement[1]) undertook a study designed to identify the equivalence methods used in France for analyzing mechanical vibration and to compare the results obtained from each method. This study was submitted to the A.S.T.E. 1980 seminar [LAL 80] and was the subject of BNAE[2] Recommendations [MIS 81].

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1. Association for the development of environmental sciences and techniques.
2. Bureau de Normalisation de l'Aéronautique et de l'Espace.

11.4.2. *Synopsis methods*

The main methods identified in the study (for vibration) are as follows:

– power spectral density envelope;

– definition of a test with a standardized test time designed to demonstrate that the equipment is working below its endurance limit;

– elimination of non-constraining values (two variants);

– equivalence of the extreme response and fatigue damage spectra [LAL 84].

11.4.3. *The need for a reliable method*

The synopsis method of the measured data must be:

– reproducible: the results obtained by several specifiers using the same data must be very close and relatively insensitive to the analysis conditions;

– reliable: the synthesized values must be similar to the measured values (if there is no plus factor coefficient).

As part of a "round robin" comparative study carried out under the aegis of the CEEES[3] [RIC 90], an analog magnetic tape with acceleration data measured on the platform of a truck (three-axis accelerometer) during a 30 minute drive, was sent to several European laboratories in 1991, three of which were in France. Each laboratory analyzed the measurement data according to its usual method and established a test specification covering these 30 minutes of transportation. The drive included different road and speed conditions. The signals measured were random vibrations and a transient signal.

Most of the laboratories expressed the specification as a power spectral density and a shock. The PSDs obtained are given in Figure 11.4. The results are extremely diverse, with PSD values separated by a factor of up to 100, thereby confirming the need for a reliable method.

3. Committee of European Environmental Engineering Societies.

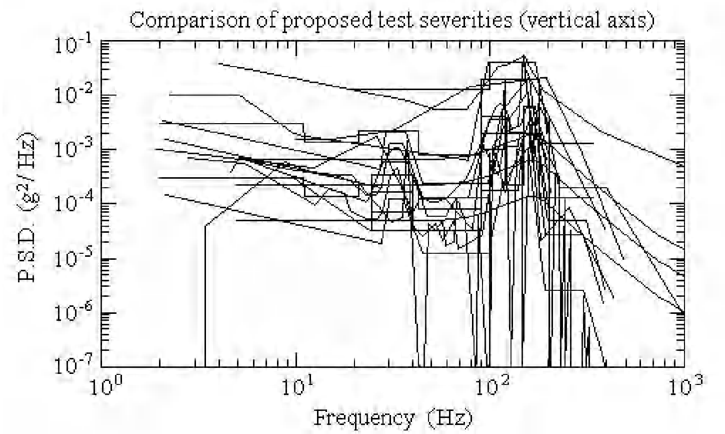


Figure 11.4. *Comparison of the "round robin" comparative study results [RIC 90]*

11.4.4. *Synopsis method using power spectrum density envelope*

Stationary random vibrations are generally represented by Power Spectral Densities (PSD). Let us consider a PSD characterizing a specific event, obtained by enveloping several PSDs, calculated from several measurements, following the application of an uncertainty factor as defined in section 8.4, if necessary. For reasons of convenience, the maximum number of PSD points is generally limited to approximately 10 in order to describe the specification obtained in the documents and to display the PSD on the control station during the test. In the past, this was necessary because analog control stations were used. Nowadays, the data can be directly transferred to a digital control system via a computerized system since these can manage a large number of PSD definition points.

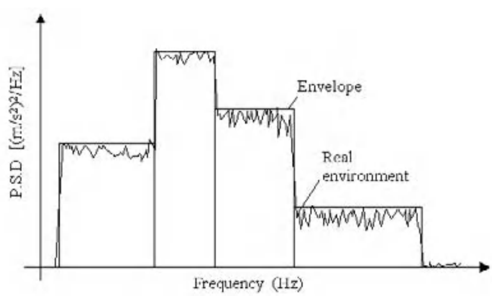


Figure 11.5. *Specification by PSD envelope*

The above specification is taken from the environmental PSD; its pattern is simplified by broken straight lines, generally horizontal (although not necessarily). This operation has at least two disadvantages:

– the result obtained is dependent on the test requester;

– as the trend is to widely envelope the reference spectrum, the rms value of the specification deduced from this envelope is very often much larger than the value of the initial PSD; a factor larger than two can be reached.

These effects can be minimized by reducing the specification application time using the guidelines given in section 4.4.

If $\ddot{x}_{rms}$ is the rms value of the vibration characterized by the reference PSD (real environment), and T_E the duration of the event considered; and if $\ddot{X}_{rms}$ is the PSD rms value obtained by enveloping, the duration T_R for applying the envelope PSD can be calculated using the following formula (section 4.4):

$$T_R = T_E \left(\frac{\ddot{x}_{rms}}{\ddot{X}_{rms}} \right)^b$$

in such a way that the fatigue damage to the equipment is accurately reproduced. It is advisable to check that the exaggeration coefficient $E = \frac{\ddot{X}_{rms}}{\ddot{x}_{rms}}$ is not too high (higher than 2 for example). If it is not the case this calculation can lead to the time being overly reduced and the instantaneous stress being too high when compared to the stress induced by the real vibration. It would then be necessary to re-shape the envelope by following the PSD more closely. Table 11.1 summarizes this method.

When applied under these conditions, this method requires one specification per event to be drawn up, and the number of events can be multiplied since there are generally several situations and several events per situation. In order to overcome this drawback, the method can be improved in the following ways (Table 11.2):

– by characterizing each event as above;

– by drawing up an envelope composed of broken straight lines for each PSD of each of the events under consideration. The rms value of each spectrum is then calculated, while ensuring that the exaggeration coefficients E_i are not too high;

– by superimposing the envelopes obtained in this way and by drawing an envelope (broken straight lines) of these curves. This final curve represents the specification that is sought;

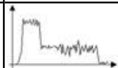
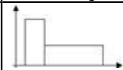
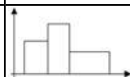
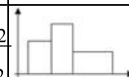
Life cycle profile		Real environment		Real duration specification			Reduced duration specification				Spe.
Event	Duration	PSD	rms acceleration	PSD envelope	rms acceleration	Duration	Exaggeration coefficient	PSD	rms acceleration	Duration	N°
E_1	T_{E_1}		$\ddot{x}_{rms1}$		$\ddot{X}_{rms1}$	T_{E1}	$E_1 = \dfrac{\ddot{X}_{rms1}}{\ddot{x}_{rms1}}$		γ_{rms1}	T_{R1}	1
E_2	T_{E_2}		$\ddot{x}_{rms2}$		$\ddot{X}_{rms2}$	T_{E2}	$E_2 = \dfrac{\ddot{X}_{rms2}}{\ddot{x}_{rms2}}$		γ_{rms2}	T_{R2}	2
…	…	…	…	…	…	…	…	…	…	…	…

Table 11.1. *First process envelope of specification development by PSD*

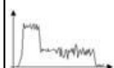
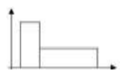
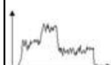
Life cycle profile		Real environment		Real duration specification				Reduced duration specification			
Ev.	Duration	PSD	rms accelerat.	PSD	rms accelerat.	Duration	Envelope	Exaggeration coefficient	Largest coef.	Basic duration	Test duration
E_1	T_{E1}		$\ddot{x}_{rms1}$		$\ddot{X}_{rms1}$	T_{E1}		$E_1 = \dfrac{\gamma_{rms}}{\ddot{X}_{rms1}}$	Largest value of E_i,	$T_{R1} = \dfrac{T_{E1}}{E_1^b}$	$T_R = \sum_i T_{Ri}$
E_2	T_{E2}		$\ddot{x}_{rms2}$		$\ddot{X}_{rms2}$	T_{E2}	rms value γ_{rms} Duration $T = \sum_i T_{E_i}$	$E_2 = \dfrac{\gamma_{rms}}{\ddot{X}_{rms2}}$	for comparison with the permitted value	$T_{R2} = \dfrac{T_{E2}}{E_2^b}$	

Table 11.2. *Second process of specification development by PSD envelope*

– by determining the reduced duration of each event from its real duration T_{E_i} and from the exaggeration coefficient E_i;

– the total duration to be associated with the specification (a single PSD) is the sum of the reduced durations.

Advantages

This method:

– can easily be implemented, with relatively few calculations;

– allows the duration to be reduced using a fatigue damage criterion;

– allows several events (or situations) to be summarized in only a single PSD.

Disadvantages

– The way the envelope is drawn, with broken straight lines, is very subjective; the results can be very different depending on the operator.

– The duration reduction is only determined by the envelope pattern. No duration is set *a priori* but, for the drawn envelope, it would appear necessary to reduce the duration to the value deduced from the exaggeration factor.

A survey recently performed in Europe has shown that the power spectrum density envelope method is often used (in the most simple form, i.e. without time reduction). Studies are currently being carried out in England to try to take into account distribution of the instantaneous values of the measured signal [CHA 92].

NOTE: *There are more complex methods of determining a specification from the PSDs of the real environment (for example the ITOP 1.1.050 method [ITO 06]).*

11.4.5. *Equivalence method of extreme response and fatigue damage*

The specification is determined by seeking a vibration of the same severity as the vibrations measured in the real environment. The comparison is carried out not on the real structure, whose dynamic behavior is generally not known at the time of the study, but rather on a simple mechanical model, a linear single-degree-of-freedom system, the natural frequency of which is varied in across a range broad enough to cover the resonance frequencies of the future structure.

This system does not claim to represent the real structure even if, at a first approximation, it can often give an initial idea of the responses. It is simply a reference system that makes it possible to compare the effects of several environments on a rather simple system, on the basis of mechanical damage criteria. The selected criteria are the greatest stress generated in the model and the fatigue damage, which allow the extreme response and fatigue damage spectra to be plotted.

It is assumed, then, that two vibrations which produce the same effects on this "standard system" will have same severity on the real structure under study, which is in general neither a single degree of freedom, nor a linear system. Various studies have shown that this assumption is not unrealistic (Volume 2, Chapter 2 for shocks, the study of resistance to fatigue for vibrations [DEW 86]).

The extreme response and fatigue damage spectra were defined in preceding chapters. The ASTE study previously referred to underlined the interest in a method based on these curves that might establish a synopsis of several vibratory environments.

The process of synopsis includes the following major phases:

– for each event:

- calculation of SRSs (shock) or ERSs and FDSs (vibration) from the measurement of features characterizing the event,

- calculation of the mean, standard deviation and the variation coefficient of spectra (or the envelope if the number of spectra is too small),

- calculation and application of an uncertainty coefficient;

– for each situation:

- sum of FDSs and ERS envelope for vibrations;

- SRS envelope for shocks;

– for the life profile:

- SRS envelope,

- ERS envelope,

- sum (serial situations) or envelope (parallel situations) of FDSs,

- search for the characteristics of a random vibration producing, at each natural frequency, the same fatigue damage as the accumulated environment of the life profile for a chosen duration,

- development of a shock specification from the SRS envelope,

- application of a test factor function of the number of tests that will be carried out,

- validation of duration reduction.

These actions are detailed in the following sections.

11.4.6. *Synopsis of the real environment associated with an event (or sub-situation)*

11.4.6.1. *Shock synopsis*

A mechanical shock can be described by signals in relation to time or by shock response spectra (SRS). In the first case, the shock response spectra of the signals are calculated (standard damping value $\xi = 0.05$). Then, for each shock event, for each frequency the spectra mean $\overline{E}$, the standard deviation s_E, the variation coefficient $V_E = \dfrac{\text{standard deviation}}{\text{mean}}$ (V_E is a function of frequency) and, by the mean + α, standard deviation α = 3 are calculated, for instance (Figure 11.6).

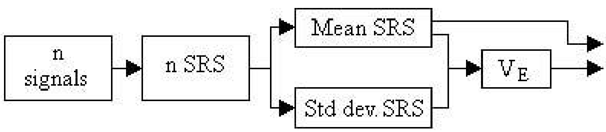


Figure 11.6. *SRS synopsis characterizing a shock event*

11.4.6.2. *Random vibrations*

Random vibrations are initially described by a signal as a function of time. They can be non-stationary because the rms value of the signal and/or its frequency content varies over time. It is not then mathematically correct to calculate a power spectral density, but it is still possible to determine ERSs (non-probabilistic spectra in this case) and FDS directly from the signal based on time.

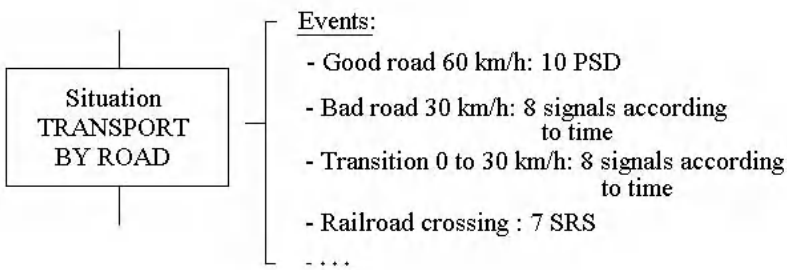


Figure 11.7. *Example of events in a situation*

Non-stationary signals

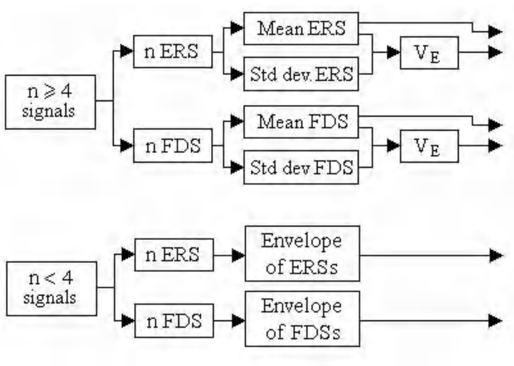


Figure 11.8. *Synopsis of measures characterizing a non-stationary random vibration*

ERS and FDS of each signal are calculated (Figure 11.8). If the number of measurements is sufficient, as regards shocks, the mean, the standard deviation, the variation coefficient V_E and the quantity $\overline{E} + 3\, s_E$ (or a value different from 3), for both the ERS and the FDS respectively, are then determined at each frequency.

If the number of measurements is too small for statistical calculation ($n < 4$), we also carry out envelope ERSs and FDSs respectively.

Stationary signals

If they are stationary, their power spectral densities are used.

For an analog method, from each PSD (Figure 11.9):

– if the number of measurements is enough to perform statistical calculations, an ERS and an FDS are calculated, then the mean, the standard deviation, the variation coefficient V_E and the quantity $\overline{E} + 3\, s_E$, for both the ERS and the FDS respectively;

– otherwise, we calculate and carry out the ERS and FDS envelope respectively.

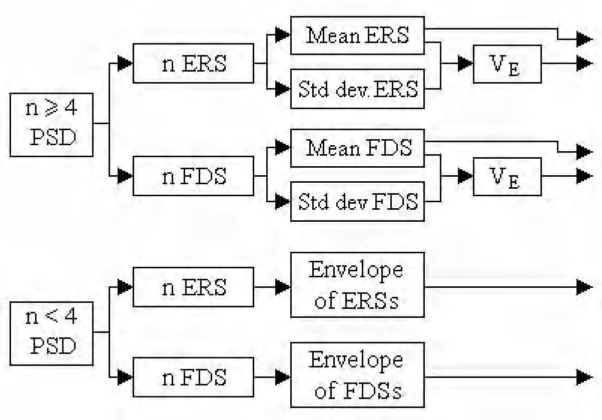


Figure 11.9. *PSDs synopsis of a stationary random vibration*

11.4.6.3. *Calculation parameters*

In order to calculate the spectra (SRS, ERS and FDS), it is necessary to set:

– The initial frequency and the final frequency. These frequencies must surround the known or assumed natural frequencies of the equipment. In case of uncertainty, a large frequency range must be considered, such as between 1 to 2000 Hz, for instance.

– The b parameter related to the slope of the S–N curve (Volume 4, Chapter 1).

> *Influence of the b parameter*
>
> The choice of the b parameter has little impact on the resulting specification (PSD) as long as the test time is not too short compared to that of the real environment. For an equal length of time, the rms value of the PSD obtained is practically independent of b. High values of b may, however, lead to somewhat more detailed spectra. The PSDs obtained for b small are on the contrary smoother.

– The Q factor, generally chosen by convention as equal to 10 ($\xi = 0.05$).

Influence of the Q factor

The impact of the Q factor is comparable to the impact of parameter b. The higher the value of Q, the more detailed the fatigue damage spectrum and therefore the PSD.

– The constants K and C involved in the calculation of the fatigue damage spectra. K is the proportionality constant between the stress and strain of a single-degree-of-freedom system ($\sigma = K\ z$). C is the constant of the Basquin equation describing the S–N curve ($N\ \sigma^b = C$). Since the vibrations of the real environment and those of the specification determined from this environment are applied to the same equipment (the equipment for which the specification is written), the values of these parameters are unimportant while the aim remains that of comparing spectra, and not that of evaluating the exact damage (or the life expectancy). Therefore, by convention, a value of 1 is used for K and C.

NOTE: *If the real environment is described by only a small number of spectra (lack of data), the sub-situation can be represented by the envelope spectrum of these spectra.*

11.4.6.4. *Application of uncertainty factor*

From a mean spectrum

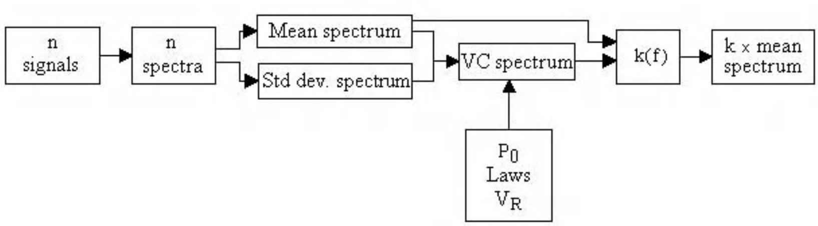


Figure 11.10. *Synopsis of spectra characteristic of an event*

The process of determining the coefficient k(f) is based on the value of the variation coefficient V_E of the environment and requires the following to be chosen:

– the maximum permissible failure probability P_0;

Choosing the failure probability

Since the variation coefficient V_E varies in relation to the frequency for each type of environment, it is not possible to set *a priori* both an uncertainty factor k and a failure probability P_0. When k is set, the failure probability P_0 varies and, conversely, when P_0 is set, k varies. It appears preferable to us to set the failure probability rather than k.

Few values are found in studies listed in the bibliography. In the case of preliminary work solely taking strength dispersion into account, it was suggested that the value of the dimensioning stress should be $\overline{R} - \alpha\, s_R$ where $\alpha = 2.3$, corresponding to a 1% risk for a Gaussian distribution [REP 55].

R.E. Blake [BLA 67] [BLA 69] considers that for Gaussian distributions and for missiles, the factor for reliability (calculated according to the stress–strength concept) should be selected between 99.9% and 99.995%.

Two values are used in what follows (as an example) for the failure probability:

– 10^{-3}, which could be chosen for specifications relative to the normal environment;

– 10^{-6} for the accidental environment.

– the nature of the environment and equipment strength distributions;

Choice of the distribution functions

Loads

For shock spectra and for power spectral densities (and therefore rms values), most authors agree that real phenomena are best represented by a log–normal distribution, a major drawback of a Gaussian distribution being the possibility of negative values (see Chapter 8). A log–normal distribution is therefore used.

Strengths

The studies listed in the bibliography also show that the central part of the strength distribution curves (stress, elastic limit, ultimate stress, ultimate endurance, etc.) can be satisfactorily represented by a normal or log–normal distribution.

A log–normal distribution is generally considered preferable for our problem:

– because Normal distributions exhibit negative values;

– a log–normal distribution is much more representative of the distribution curve ends, which are precisely the ranges that involve uncertainty factor computations.

Choice of the distribution functions

Loads

For shock spectra and for power spectral densities (and therefore rms values), most authors agree that real phenomena are best represented by a log–normal distribution, a major drawback of a Gaussian distribution being the possibility of negative values (see Chapter 8). A log–normal distribution is therefore used.

The log–normal distribution is widely considered to be a good approximation of fatigue damage (or the number of cycles to failure).

– a variation coefficient V_R chosen according to the materials included in the equipment under test.

Variation coefficient of strength

Extreme response

The value suggested in reference [CES 77] ($V_R = 0.08$) is an envelope of numerous published results. However, it should be noted that (section 8.3.1):

– larger values are sometimes found;

– the values mentioned concern tests on specimens. The variation coefficient relative to structures made of larger parts is undoubtedly larger [BAR 65].

Fatigue aspect

The variation coefficient for the number of cycles to fracture or for the damage to fracture Δ can be higher than 1. However, the values are often below 0.8 (specimens).

The uncertainty factor is then calculated on the basis of these data and the equations given in section 8.4, and subsequently that of the *withhold environment* $k\ \overline{E}$.

We follow the same methodology for each event (Figure 11.11). Each one is characterized, following this analysis, either by an SRS if it is a shock, or an ERS and FDS if it is a random (or sinusoidal) vibration.

If the environment is characterized by an $\overline{E} + 3\, s_E$ spectrum, the withhold selected environment E_W is given by

$$E_W = \frac{k}{1 + 3\ V_E} \left(\overline{E} + 3\ s_E\right) \qquad [11.1]$$

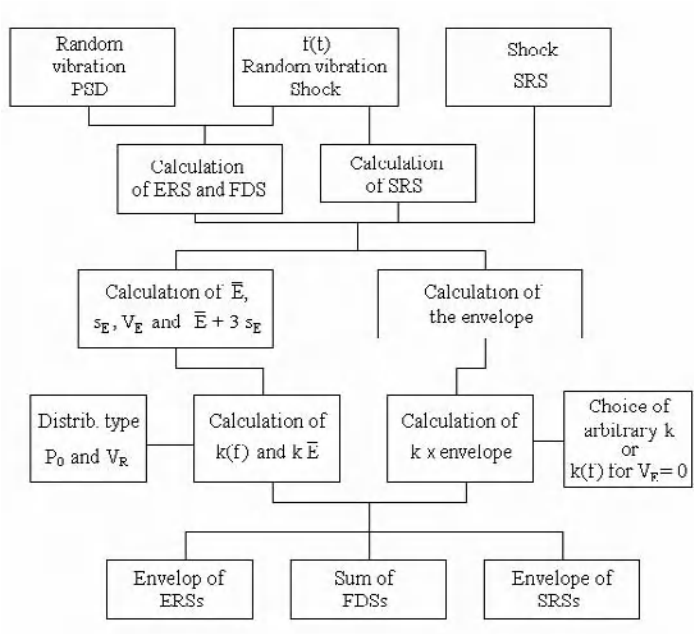


Figure 11.11. *Specification establishment process*

If the environment is characterized by an $\bar{E} + 3\ s_E$ spectrum, the selected withhold environment E_W is given by

$$E_W = \frac{k}{1 + 3\ V_E} \left(\bar{E} + 3\ s_E\right) \qquad [11.1]$$

From an envelope spectrum

If the spectrum representing an event in the real environment is an envelope spectrum, there are two possibilities:

– applying an uncertainty coefficient calculated according to the previous method, by considering that the variation coefficient V_E is equal to zero;

– using an arbitrary uncertainty coefficient (1.3 for example).

11.4.6.5. *Possible application of an aging factor*

Equipment stored for a long time before use ages and loses strength before it is subjected to the environment. This phenomenon can be taken into account by requiring more of new equipment, by stipulating a failure probability P'_0 lower than the value P_0 that the equipment must have when it is put to use. The way the aging factor is calculated in order to achieve this objective is explained in Chapter 9.

11.4.7. *Synopsis of a situation*

The sub-situations comprising a situation are all in series. A situation is therefore characterized by three curves (*withhold vibration environments* $\left[E_W\right]$):

– an extreme response spectrum enveloping the ERSs characterizing each event in the situation involved;

– a fatigue damage spectrum equal to the sum of all the FDSs corresponding to the vibration environments selected for each event;

– a shock response spectrum enveloping all the SRSs of the shock environments chosen for each event.

11.4.8. *Synopsis of all life profile situations*

The above spectra are then combined as follows.

11.4.8.1. *Parallel situations*

In this case, the equipment is only subjected to one or other of the environments. The envelopes of the ERSs, FDSs and SRSs of the parallel situations are therefore calculated in succession. The resulting curves are considered to be those of a situation in series interconnected with other situations.

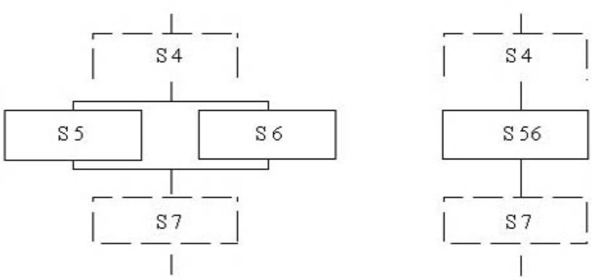


Figure 11.12. *Synopsis of parallel situations*

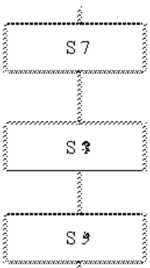


Figure 11.13. *Situations in series*

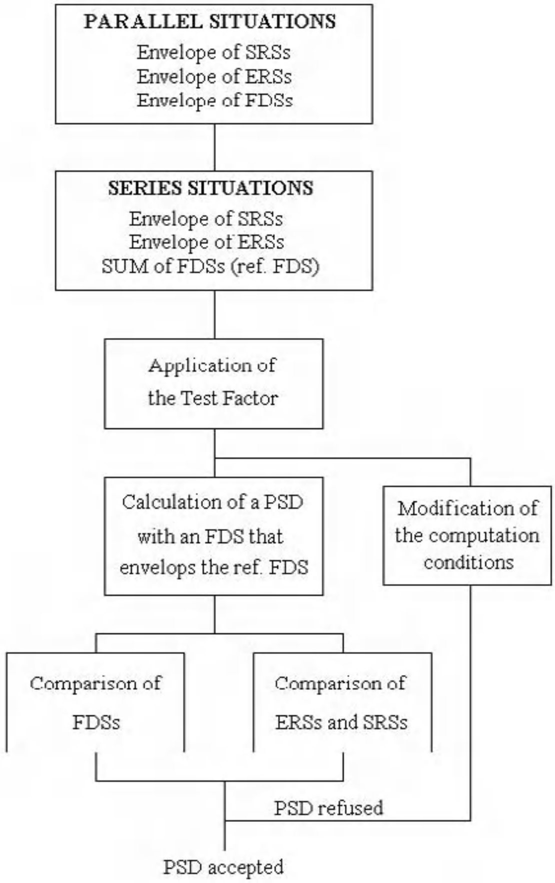


Figure 11.14. *Specification drafting procedure*

11.4.8.2. *Situations in series*

The equipment will be subjected to all the situations in the series. It is therefore necessary to:

– sum the FDSs characterizing each situation;

– calculate the envelope of the ERSs;

– calculate the envelope of the SRSs.

The entire life cycle profile can then be represented by three equivalent spectra.

11.4.9. *Search for a random vibration of equal severity*

Developing the specification will then consist of searching for the characteristics:

– of a random vibration defined by its PSD which, over a given duration[4], has an FDS that is very close to the one resulting from the previous synopsis (reference spectrum) (Figure 11.14);

– of a shock with an SRS that is very close to the life profile SRS. The specification can also be this spectrum if it is possible to simulate it in a test on shaker with control using the SRS.

11.4.9.1. *Method by matrix inversion*

Establishing the specification can be done in several ways:

– by searching for a PSD defined by line segments with any slope whatsoever. We consider the expression of the fatigue damage [4.43] (Volume 4) as:

$$D = \frac{K^b}{C} \frac{T}{(4\,\xi)^{b/2}\,(2\,\pi)^{3\,b/2}}\, f_0^{1-\frac{3\,b}{2}} \left(\sum_j a_j\, G_j\right)^{\frac{b}{2}} \Gamma\left(1+\frac{b}{2}\right)$$

It is possible:

– either to take all the points (N) defining the fatigue damage spectrum. This option leads to a (specification) PSD characterized by N points;

– or, to simplify the specification, choosing only a few points ($n < N$) of the fatigue damage spectrum, which will lead to a PSD itself defined by n points. In this case, the FDS of the PSD obtained may not be quite as close to the environment FDS (it is desirable that it remains an envelope).

On the basis of n couples, points f_{0_i}, D_i, n equations are obtained with the form:

$$D_i = \frac{K^b}{C} \frac{T}{(4\,\xi)^{b/2}\,(2\,\pi)^{3\,b/2}}\, f_{0_i}^{1-\frac{3\,b}{2}} \left(\sum_j a_{i,j}\, G_j\right)^{\frac{b}{2}} \Gamma\left(1+\frac{b}{2}\right) \qquad [11.2]$$

where ([8.80] of Volume 3)

$$a_{i,j} = \frac{{}^{j-1,j}\Delta I_1 - h_{j-1}\ {}^{j-1,j}\Delta I_0}{h_j - h_{j-1}} - \frac{{}^{j,j+1}\Delta I_1 - h_{j+1}\ {}^{j,j+1}\Delta I_0}{h_{j+1} - h_j} \tag{11.3}$$

$${}^{j,j+1}\Delta I_p = I_p\left(h_{i,j+1}\right) - I_p\left(h_{i,j}\right) \tag{11.4}$$

and

$$h_{i,j} = \frac{f_j}{f_{0_i}} \tag{11.5}$$

There is then a set of n linear equations between values G_j that can be expressed as follows in matrix form:

$$D = A\ G_{b/2}$$

($G_{b/2}$ = a matrix column, each term of which is equal to $G_j^{b/2}$), yielding:

$$G_{b/2} = A^{-1}\ D \tag{11.6}$$

hence the amplitudes G_j. The PSD thereby obtained is defined by n points f_{0_j}, G_j connected by straight line segments;

– by searching for a PSD defined by horizontal straight line segments (plateaux). We then examine the simplified relation [4.43] of Volume 4

$$D_i\left(f_{0_i}\right) = \frac{K^b}{C} \frac{f_{0_i}\ T\left(\sqrt{2}\right)^b}{\left[(2\pi)^4 f_{0_i}^3\right]^{b/2}}\ \Gamma\left(1+\frac{b}{2}\right)\left\{\frac{\pi}{4\xi}\sum_{j=1}^{n} G_j\left[I_0\left(h_{i,j+1}\right) - I_0\left(h_{i,j}\right)\right]\right\}^{b/2} \tag{11.7}$$

where

$$I_0 = \frac{\xi}{\pi\alpha}\ln\frac{h^2+\alpha h+1}{h^2-\alpha h+1} + \frac{1}{\pi}\left[\arctan\frac{2h+\alpha}{2\xi} + \arctan\frac{2h-\alpha}{2\xi}\right]$$

The G_j values are obtained in the same way by matrix inversion. In this case, however, the PSD is made up of straight line segments with amplitude G_j between two selected frequencies, for example, in the middle of the intervals $(f_{0_{i-1}}, f_{0_i})$ and $(f_{0_i}, f_{0_{i+1}})$.

There are different linear system matrix and resolution by inversion methods, among which are direct (pivot method: Gauss, Gauss–Jordan, etc.) and iterative (Gauss–Seidel, Jacobi, relaxation, etc.) methods.

Direct methods provide an exact solution. If they are used for calculating PSD amplitudes, they lead to digital inaccuracies appearing by PSD values close to zero at some frequencies or to a very disturbed PSD at high frequency, particularly when the number of points chosen for the PSD is large (for example between 100 and 200). Iterative methods such as the Gauss–Seidel method provide an approximate solution (the accepted error is set *a priori*) and provide better results, but there can be cases where there is no convergence.

NOTES:

1. *Relation [11.6] shows that the calculation of the FDS or, conversely, that of the PSD, is always carried out starting from matrix A, the coefficients of which are functions of parameter b, and the damping of each system to a single degree of freedom. As a result of this the specification obtained starting with an environment characterized by only one PSD is independent of:*

– parameter b chosen for the calculation of the FDS (if there is no time reduction);

– damping ξ. It might be imagined that ξ could be varied with a certain law related to the natural frequency in order to calculate the FDS. This variation does not affect the specification obtained.

In the more interesting case of a specification established after summing several fatigue damage spectra corresponding to various situations, these properties still remain valid in practice, even if they are no longer strictly speaking true.

2. *For certain frequencies, the inversion of the matrix giving the amplitudes of the equivalent PSD may lead to some unrealistic negative values. In this case, these values should be corrected by setting them to zero or to a very small value (for instance 10^{-3} $(m\ s^{-2})^2/Hz$, the lowest value displayable on control units). This correction leads to excessive damage in certain frequency bands.*

3. *A local difference of a factor of 10 between the FDSs has relatively little influence on the values of the PSD.*

4. *The choice of n points can be facilitated by plotting the reference spectrum in such a way that the important frequency ranges leading to peaks or valleys in the power spectral density will stand out. To this end, the ordinates of the spectrum are multiplied by a factor to eliminate the terms for* f_0 *in the expression of the damage. This factor can be evaluated more easily from the relationship ([4.41] of Volume 4):*

$$D = \frac{K^b}{C} n_0^+ \; T \; \left(\sqrt{2}\; z_{rms}\right)^b \; \Gamma\left(1 + \frac{b}{2}\right)$$

and from relationship [8.79] of Volume 3:

$$z_{rms}^2 = \frac{1}{8\;\xi\;\left(2\;\pi\;f_0\right)^3} \left(\sum_j a_j \;\; G_{\ddot{x}\,j}\right)$$

Knowing that $n_0^+ \approx f_0$*, we have*

$$D = \frac{K^b}{C} \; \frac{T}{\left(4\;\xi\right)^{b/2} \; \left(2\;\pi\right)^{3\,b/2}} \; f_0^{\,1-\frac{3\,b}{2}} \left(\sum_j a_j \;\; G_j\right)^{\frac{b}{2}} \; \Gamma\left(1 + \frac{b}{2}\right) \qquad [11.8]$$

The multiplicative factor of D must therefore be taken as equal to

$$M = f_0^{\,\frac{3\,b}{2}-1} \qquad [11.9]$$

The nearest representation of the PSD is obtained by noting on the y-axis

$$M^{2/b}\; D_{2/b} = f_0^{\left(3-\frac{2}{b}\right)} \; D_{2/b} \qquad [11.10]$$

($D_{2/b}$ *= column matrix each term of which is equal to* $D_j^{2/b}$ *)*

Example 11.1.

Vibratory environment measured on a helicopter.

Fatigue damage spectrum calculated for one hour (b = 8, Q = 10).

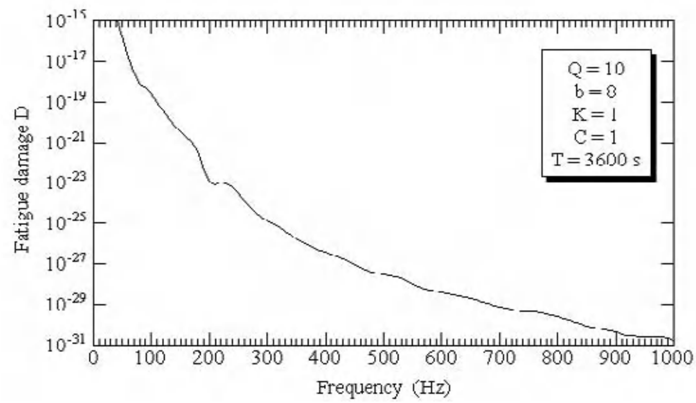


Figure 11.15. *Example of fatigue damage spectrum (helicopter)*

Modified plot

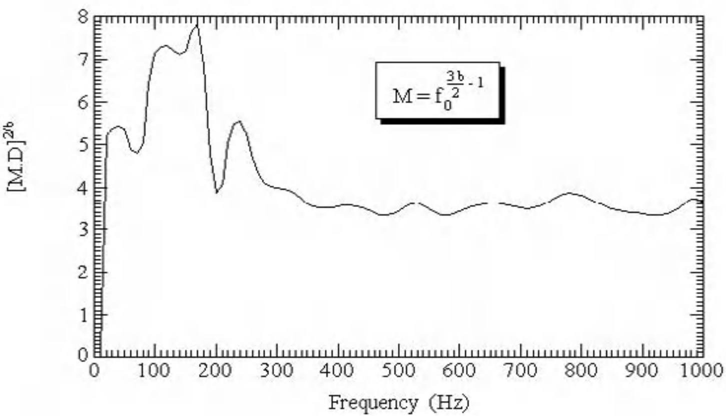


Figure 11.16. *Modified fatigue damage spectrum*

Resulting specification (PSD, 1 hour)

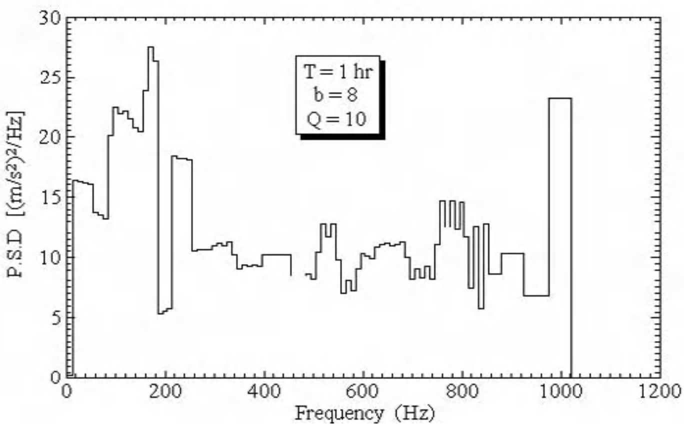


Figure 11.17. *Specification (PSD)*

11.4.9.2. *Method by iteration*

In order to avoid the pitfalls of matrix inversion the equivalent damage PSD can also be evaluated by consecutive iterations. The method consists of having a PSD *a priori* (frequencies chosen are all equal to the ones defining the life profile FDS points (reference), 1 point in n, or a certain number of points per octave, or even frequencies corresponding to points chosen in the FDS). This PSD is, for example, constant with amplitude equal to 0.1 (m s^{-2})^2/Hz. The comparison of the FDS of this PSD with the life profile FDS makes a first updating of PSD amplitudes based on the possible rule

$$\mathrm{PSD}_1(f_i) = \mathrm{PSD}_0(f_i)\left(\frac{\mathrm{FDS}_0(f_i)}{\mathrm{FDS}_1(f_i)}\right)^{\frac{2}{b}} \qquad [11.11]$$

After some iterations, calculation quickly converges toward a PSD with an FDS very close to the life profile FDS. The resulting PSD does not present any of the disadvantages observed with the matrix inversion.

11.4.10. *Validation of duration reduction*

On the basis of the PSD evaluated in this way, it is necessary to recalculate its ERS and FDS in order to estimate the quality of the specification obtained.

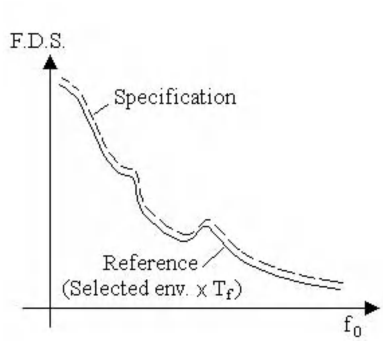


Figure 11.18. *Checking that the FDSs are equal*

The FDS is thus compared with the FDS of the complete life cycle profile. If the differences are too large, the number of definition points of the PSD or the frequency values of the selected points may be modified (Figure 11.18).

The ERS is compared with the life cycle ERS to evaluate the effect of the reduction in the test time with respect to the complete life cycle. In order to simplify the presentation, the notation ERS_{SP} will represent the extreme response spectrum of the specification, and ERS_{LP} that which results from the life cycle profile environment (reference). Several situations are possible:

– SRS > ERS_{SP} > ERS_{LP} (Figure 11.19). This is the ideal situation. The ERS_{SP} is greater than the ERS_{LP} due to the reduced duration, but is smaller than the SRS. (shock response spectrum): under test conditions, the equipment will not be submitted to instantaneous levels greater than those under the real environment. The specification comprises a random vibration and a shock defined on the basis of the SRS of the life cycle profile (simple type shock or SRS itself) according to the methods outlined in Volume 2 (Chapter 4);

– ERS_{SP} > SRS > ERS_{LP} (Figure 11.20). The ERS_{SP} is greater than the SRS. There can be larger stress peaks during the random vibration test than during the shock test. Two attitudes are possible:

- maintain the specification with its duration (reduced), and by so doing taking the risk that a problem may occur during the test which could be due to instantaneous stress levels to which the equipment would not normally be submitted in its operational life. This choice can be justified by the need to considerably reduce the test when the real environment duration is great. However, if an incident occurs during the test, this does not necessarily show that the equipment does not comply. An envelope spectrum shock of the ERS_{SP} at the beginning of the test might be envisaged in order to check that the equipment is able to withstand the stress to which it will be submitted artificially under vibration (without being damaged). There is no need to simulate the shock that corresponds to the SRS,

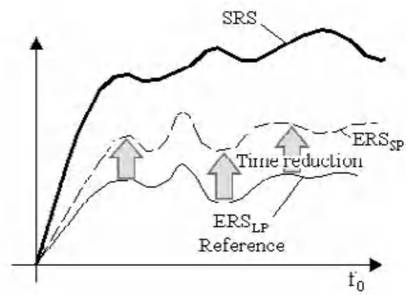


Figure 11.19. *Acceptable duration reduction*

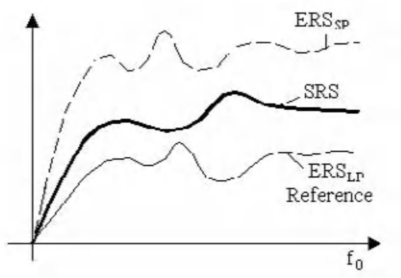


Figure 11.20. *Too important an exaggeration factor: test duration to be increased*

- select a greater duration in order to return to the previous case;

– ERS_{SP} > ERS_{LP} > SRS (Figure 11.21). Real environment shocks are fairly weak in amplitude compared to vibrations and cannot clear the time reduction. In this instance it is advisable not to reduce the duration by too much. It is also possible to start with an envelope spectrum shock of the ERS_{SP} for the same reasons as mentioned earlier. There is no need to perform a shock that covers the SRS of the real environment;

–ERS_{LP} > ERS_{SP} (Figure 11.22). The vibrations of one of the life cycle profile events are undoubtedly much stronger than the other vibrations, and do not last very long. The specification is influenced mainly by this event. If this specification is applied for a reduced time compared to the duration of the whole life cycle profile, but for a longer time than that of the overriding event, it will result in the test time being extended and therefore to the levels being reduced. In this case, the test duration must be reduced again until the two spectra are very close, the ERS_{SP} slightly enveloping if at all possible the ERS_{LP}.

NOTE: *These comparisons could be done from the URS of the specification instead of its ERS, particularly when the specification duration is very small [COL 07]. For common durations, these two spectra are very close.*

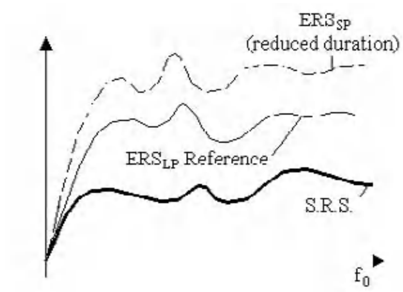


Figure 11.21. *Reduction of the duration will always present a risk in this case*

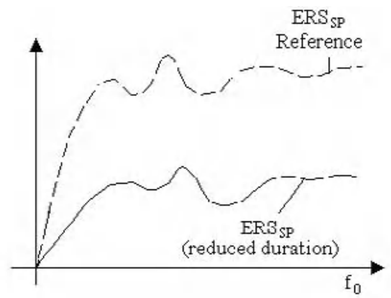


Figure 11.22. *Duration of test too long*

Example 11.2.

Consider a life profile made up of two situations, including truck transport lasting 20 hours (rms value: 3 m s^{-2}) and a missile flight lasting 5 minutes (rms value: 27.6 m s^{-2}). Durations and amplitudes are very different. PSDs are drawn in Figures 11.23 and 11.24.

The specification, established from the sum of the FDSs of these two vibrations (traced for Q = 10 and b = 8), was calculated for a duration corresponding to 5 hours, i.e. *a priori* with a time reduction of a factor of approximately 4 in relation to the total duration of both situations (20 hours + 5 minutes).

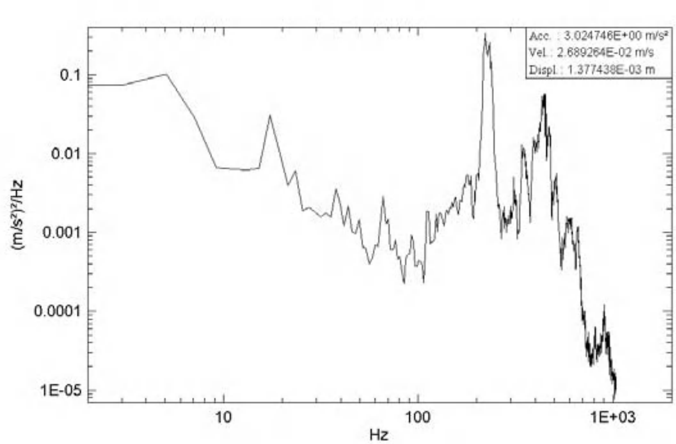


Figure 11.23. *PSD of the "truck" vibration*

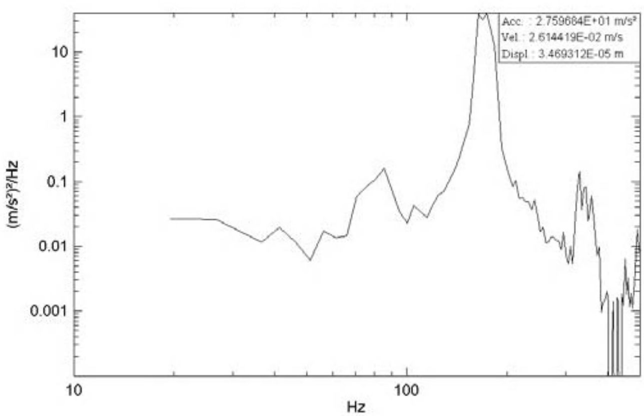


Figure 11.24. *PSD of the "missile flight" vibration*

We can verify that the FDS of the specification is very close to the FDS of both vibrations (Figure 11.25). Surprisingly, the reduced duration specification ERS is lower than the envelope ERS of life profile ERSs (Figure 11.26).

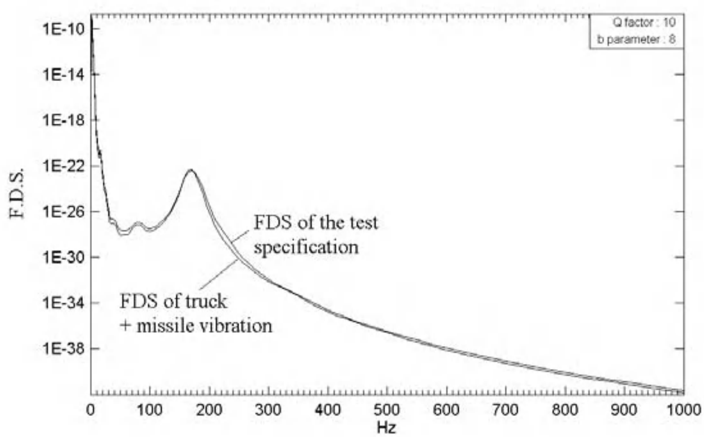


Figure 11.25. *FDS of specification (5 hours) and both vibrations*

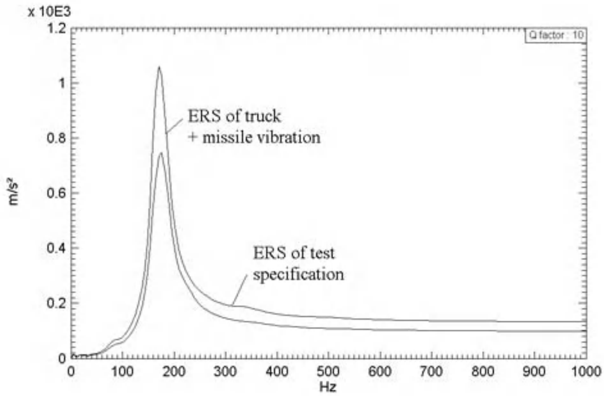


Figure 11.26. *ERS of the specification (5 hours) and both vibrations*

The comparison of both vibration FDSs studied shows that, even though much shorter, the missile flight is much more severe that the truck transport lasting 20 hours (Figure 11.27). Taking into account the relative values of the damage, the sum of both FDSs is practically identical to the missile flight FDS. Establishing a specification lasting 5 hours comes down to increasing the duration of the real environment largely dominated by the 5 minutes missile flight to 5 hours, leading to a decrease in the stresses and therefore the ERS.

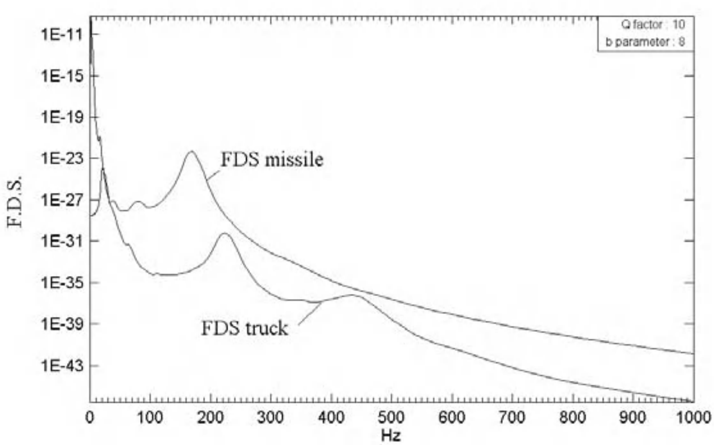


Figure 11.27. *Comparison of "truck" and "missile" vibration FDSs*

If we establish a specification lasting 5 minutes, we can verify that ERSs are very close (Figure 11.28).

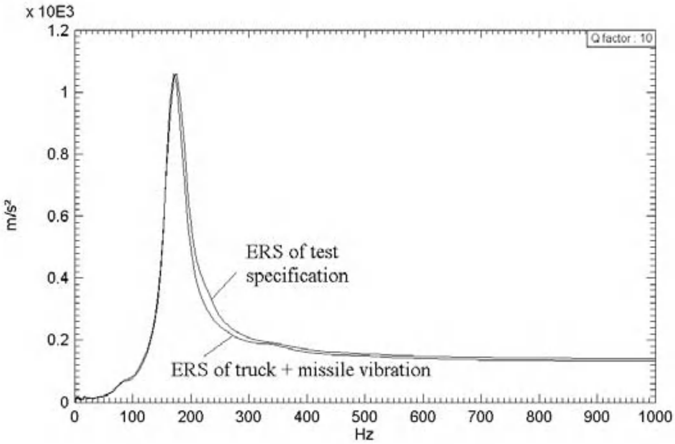


Figure 11.28. *Comparison of the envelope life profile ERS and the ERS of the specification lasting 5 minutes*

In the case of this very simple example, this result could be expected by comparing vibration rms values. In the general case of more complex profiles, the problem can often go unnoticed without this ERS analysis, which should always be done.

Example 11.3.

Let us consider a very simple life cycle profile consisting of two aircraft hours and three helicopter hours. Both of these environments are characterized by the PSD in Figure 11.29. Both of these PSDs have very similar rms values, but very different frequency content. The purpose is to establish a specification that covers this life cycle profile, without any coefficient.

Figure 11.30 shows the damage spectrum as being equal to the total of the FDSs for each of these situations, the specification damage spectrum obtained by calculating 31 points of the spectrum, and the corresponding PSD (31 levels), calculated for a reduced duration of one hour (rms value 10.1 m s^{-2}).

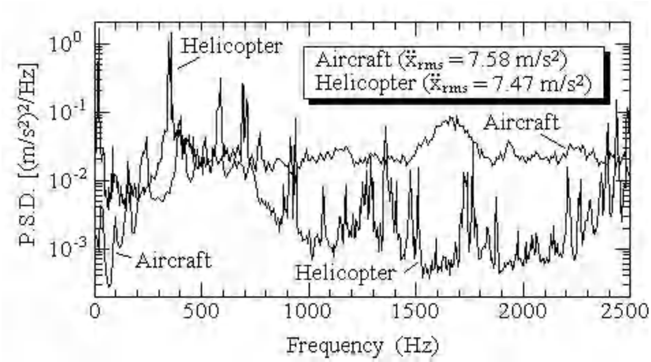


Figure 11.29. *PSD of an aircraft vibration and a helicopter vibration*

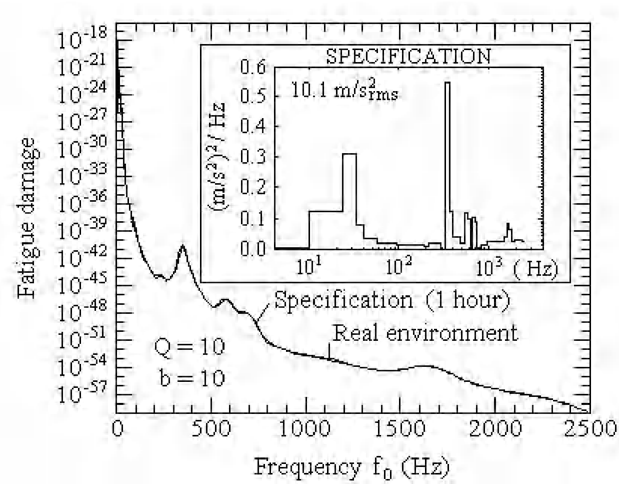


Figure 11.30. *FDS of the real environment and of the reduced duration specification*

Comparison of the ERS (Figure 11.31) shows the increase (small) of the instantaneous levels resulting from a reduction in time from five hours to one hour.

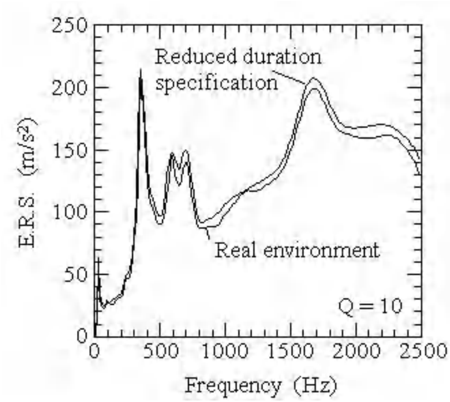


Figure 11.31. *ERS of the real environment and of the reduced duration specification*

11.5. Step 4: establishment of the test program

11.5.1. *Application of a test factor*

For obvious cost reasons, we generally only proceed to a single test, which does not demonstrate the behavior of all materials with the specified probability by itself, because of the variability of its strength. From the estimate of the variation coefficient of this strength, we must therefore apply an additional multiplying factor based on the number of tests planned, the "*test factor*".

The test factor is solely used to demonstrate that the material tested supports without damage the real environment raised by the uncertainty coefficient. For a given confidence level, this test factor depends on the number of tests to be conducted and the variation coefficient of the equipment strength (cf. Chapter 10).

It has been shown that for Gaussian distributions (environment and strength of the material), it is equal to [10.6]

$$T_F = 1 + \frac{a' \, V_R}{\sqrt{n}}$$

and, for log–normal distributions, to [10.12]

$$T_F = \exp\left[a' \frac{\sqrt{\ln\left(1 + V_R^2\right)}}{\sqrt{n}}\right]$$

This factor is to be applied to the representative environment, in order to deduce the test severity TS:

$$TS = T_F \; E_S = T_F \; k \; \overline{E} \qquad [11.12]$$

– for the amplitudes of static accelerations;

– for the amplitudes of the sinusoidal vibrations;

– for the amplitudes of the shocks or on the shock response spectra;

– for the extreme response spectra;

– for the fatigue damage spectra (the variation coefficient in this case being calculated from the number of cycles to failure).

In principle, the operation, related to the qualification strategy, should be implemented during this fourth step. We already have the PSD for random vibrations, calculated in step 3. Multiplying the FDS by T_F is the same as multiplying the corresponding PSD by $T_F^{2/b}$ and its rms value by $T_F^{1/b}$.

To simplify the process, the test factor is often taken into account, in practice, during step 3, before the PSD is sought.

11.5.2. *Choice of the test chronology*

It has been shown how it is possible to reduce all the vibrations associated with a life cycle profile to a single test (for each axis). Although the method allows this to be done without difficulty, this extremity is not always desirable for several reasons:

– the need to operate the equipment under test in the presence of vibrations associated with a particular situation;

– the need to reproduce a combined vibration/thermal environment specific to one of the situations;

– a preferred separation of vibrations that are very different in nature: low amplitude/long duration and high amplitude/short duration, such as the example of road driving (10 hours, 0.5 m s^{-2} rms) and the free flight of a missile (1 minute, 60 m s^{-2} rms).

This is why, in practice, the life cycle profile is divided into several sections. For each section a vibration and shock specification is calculated using the method described.

The establishment of a test program therefore entails establishing the chronology over which the tests are to be conducted (vibration, thermal, combined environments, static acceleration, etc.), attempting to satisfy both the requirement that tests should represent conditions well, and concerns over the cost of performance. For this purpose, the tests are sequenced insofar as possible, with the aim of limiting the number of changes of configuration (changes of test facility, change of axis). These operations are time-consuming, since they require the test to be stopped, the measuring equipment to be disconnected, the specimen disassembled, and then reassembled on another facility, the measuring equipment to be checked after making the new connections, etc.

In theory, this four-step process (including calculating an uncertainty factor and a test factor) applies to all types of environment (mechanical, climatic, electromagnetic, etc.). In order for it to be fully implemented in practice, it requires a method of data synthesis comparable to the equivalence method of extreme responses and damages used in mechanics.

Example 11.4.

The example chosen to illustrate this specification writing process is simply made up of four situations, including two which are parallel (Figure 11.32). We assume that the first situation (S1), truck transport, includes three necessary serial events: a railroad crossing, normal road with constant speed and bad road shock.

Shocks were the subject of 10 measures; random vibrations are in both cases characterized by 10 signal samples from 10 measurements done in similar conditions. It is obviously an ideal case, since the number of measurements available is generally much lower in practice.

The principle of the synopsis of a situation's data is shown considering the first situation (Figure 11.33).

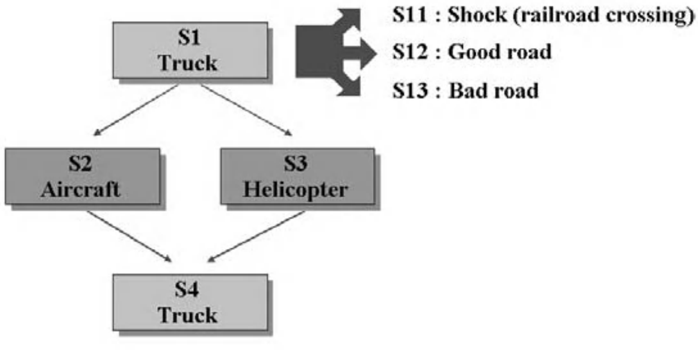


Figure 11.32. *Example of life profile with 4 situations, including 2 in parallel*

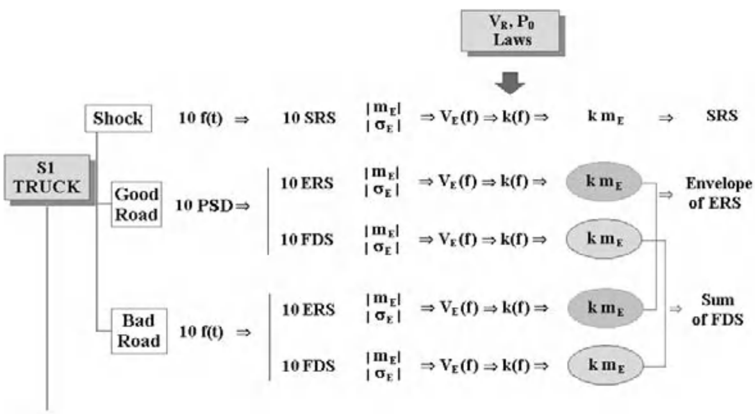


Figure 11.33. *Synopsis of data characterizing each event in a situation*

For shocks

Ten SRSs are calculated from the shock measurements. At each frequency, we determine:

– mean m_E and standard deviation σ_E of 10 SRS values;

– variation coefficient V_E and uncertainty coefficient k. This calculation involves the probability of maximum tolerated failure and requires an hypothesis on:

- SRS and material strength distribution laws (the hypothesis of log–normal distributions is most often accepted),

- the variation coefficient of the law of distribution for this strength (0.08 is an envelope value for most metallic materials);

– $k \cdot m_E$ product. The SRS obtained after this calculation is achieved at each frequency, statistically representing the shock studied.

For "good road" vibrations

We assume here that signal samples were picked up in recorded signal parts for which the rms value does not vary much, and thus that the signal is stationary. This hypothesis makes it possible to calculate, for each signal, a PSD that is used to obtain an SRS and an FDS considering the real duration of this stationary phase (much longer that the duration of signal samples of typically a few dozen seconds).

The 10 ERSs and 10 FDSs are then processed statistically like SRSs, according to the chosen probability of failure, and following a choice of distribution laws characterizing the environment and strength (log–normal for example). The strength variation coefficient has the same value for ERSs and SRSs, but is different for FDSs (and much greater, since it can exceed 100%).

For "bad road" vibrations

The methodology is the same as above, the only difference involves the mode of ERS and FDS calculation obtained directly from the signal based on time, as calculating a PSD from a non-stationary signal is not correct.

At the end of these operations, each of the events in the situation (three in situation S1, Figure 11.32), is characterized by:

– 1 SRS if a shock is involved (we assume in this life profile that it is only applied once and that there is no effect of fatigue);

– 1 ERS and 1 FDS if a vibration is involved.

Since all these events occur one after another and the effects of fatigue damage are cumulative, in order to characterize the situation involved, we have to sum up at each frequency of damages created for each event (FDSs defined by $k \cdot m_E$).

For shocks, at each frequency we retain the largest stress, leading to tracing the SRS envelope ($k \cdot m_E$) of the different types of shocks (only one in our example). Similarly, we proceed to the ERS envelope (good road and bad road).

Regardless of the number of events in the situation, the vibratory environment present in a situation is summarized by 3 curves: 1 SRS, 1 ERS and 1 FDS.

These same calculations are carried out for each life profile situation.

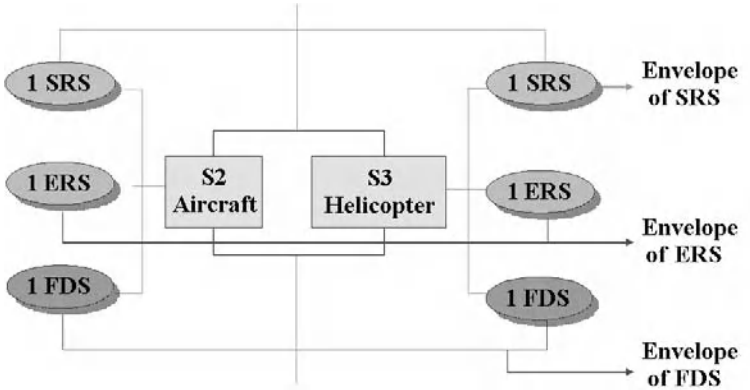


Figure 11.34. *Synopsis of parallel situations*

Synopsis of situations

For parallel situations S2 and S3, each one characterized by three spectra, we create (Figure 11.34):

– the SRS envelope (greatest stress created by each situation at each frequency);

– the ERS envelope (same reason);

– the FDS envelope: since situations are parallel, the material will only experience one of the two environments (aircraft or helicopter). We only retain, at each frequency, the largest damage created by each environment.

Both parallel situations are thus represented by three spectra, one SRS, one ERS and one FDS.

Serial situations

At this stage, situation S1, "equivalent" situation S2/S3 and situation S3 must now be synthesized (Figure 11.35).

Since all these situations are serial, the material consecutively experiences each corresponding environment. We should then make the sum of FDSs characterizing each situation, frequency to frequency. In addition, as with the events in a situation, we also consider SRS and ERS envelopes.

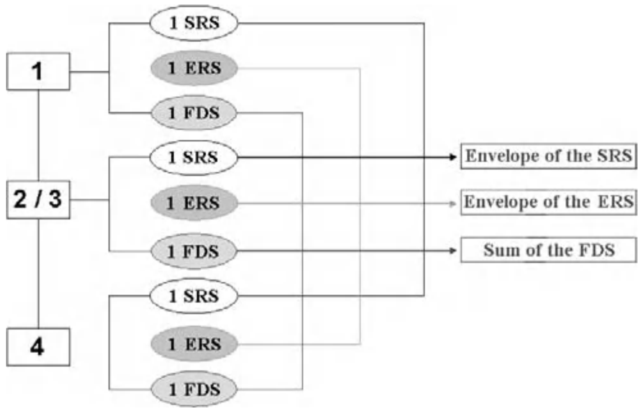


Figure 11.35. *Summary of serial situations*

The life profile can thus be characterized by 1 SRS, 1 ERS and 1 FDS. We have seen that, often and for different reasons, it is better to break up the life profile into a few distinct blocks, for example for electric testing, or for a combined vibration/thermal test in a specific vibratory environment.

This step is where we apply a test factor to the different spectra, with the purpose of considering the fact that only a small number (most often only one) of tests will be carried out to qualify the material (Figure 11.36).

This factor is based on the distribution law of resistance retained, the variation coefficient of this law (also already chosen for calculating the uncertainty coefficient), the number of tests that will be carried out and the confidence level chosen to demonstrate material behavior with the probability involved. Since the variation coefficient of strength is different for instant stresses and for fatigue damage, the test factor also has two distinct values, one for SRS and ERS and another for FDS.

The three spectra chosen after multiplication by the test factor are "reference spectra" used to establish specification.

The SRS is used to determine a shock specification, which can be expressed:

– either directly by the data in this SRS, if it can be carried out on exciter;

– or in the form of a simple shock to carry out on traditional shock machines.

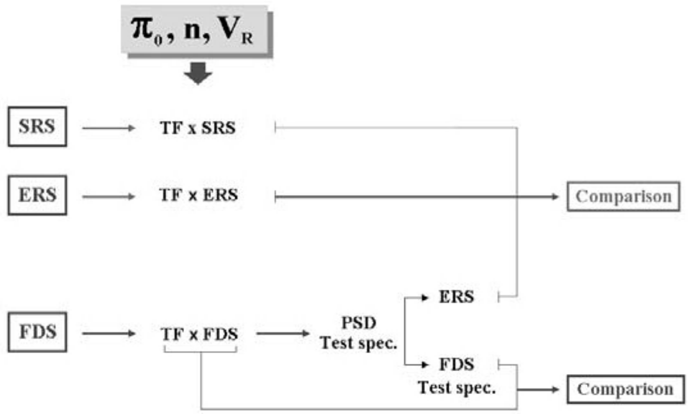


Figure 11.36. *Application of the test factor – calculation and validation of the specification*

The FDS is used to determine the characteristics of a random vibration (defined by its PSD) producing, at each natural frequency, the same fatigue damage over a chosen duration, equal or (generally) shorter than the real life profile vibration duration.

The reference ERS, specification ERS and SRS are used to validate the choice of the duration reduction factor as indicated in section 11.4.10.

11.6. Applying this method to the example of the "round robin" comparative study

Without consulting one another, three French laboratories completed the analysis using the method described above (excluding uncertainty and test factors). Each laboratory chose its own samples from the data and calculated PSDs under its own conditions. The PSDs were calculated in one case (Laboratory A) with a frequency step Δf equal to 0.5 Hz, whereas the other laboratories chose a step of 5 Hz. Similarly, each laboratory chose its own test time.

Table 11.3 below gives the rms values of a few samples along the three axes.

		rms value (m s^{-2})		
PSD n°	Speed (km h^{-1})	OX	OY	OZ
1	96	4.5	1.24	1.84
2	90	3.0	0.84	1.40
3	68	1.6	0.7	0.84
4	92	3.3	1.1	1.7
5	79	1.62	0.65	0.84
6	50	2.0	0.5	0.84
7	102	4.64	1.8	1.6
8	60	1.9	0.6	0.8
9	40	1.7	0.5	0.8
10	30	1.1	0.4	0.5

Table 11.3. *Rms values of the signal samples chosen for the "round robin" comparative study*

Table 11.4 gives the rms values of the specifications obtained for each axis by the three laboratories, with the corresponding test times (with neither uncertainty factor nor test factor).

	rms values (m s^{-2})		
Axis	Laboratory A	Laboratory B	Laboratory C
Test time (s)	600	180	360
OX	5.30	6.88	5.30
OY	1.54	1.78	1.49
OZ	2.00	2.54	2.06

Table 11.4. *Rms values of specifications established by 3 laboratories having applied the damage method (with different test times)*

To be able to make a valid comparison between these results, we brought all the specifications down to a test time of 600 s (a third of that of the real environment), correcting severities for equal fatigue damage (using the rules given in section 4.4). It can then be seen that these results are:

– very homogenous, in spite of the disparity of the initial processing;

– very similar to the vibrations measured in the real environment (Table 11.5).

	rms values (m s^{-2})		
Axis	Laboratory A	Laboratory B	Laboratory C
Test duration(s)	600	600	600
OX	5.3	5.9	5.05
OY	1.54	1.51	1.38
OZ	2	2.11	1.94

Table 11.5. *Rms values of specifications established by 3 laboratories from the damage method (with same test times)*

Figures 11.37 and 11.38 give a comparison of the ERSs and FDSs obtained by the three laboratories. Here again, it can be seen that the results agree with each other and with the real environment closely. The spectra plotted with the smallest frequency step are more detailed and less smooth.

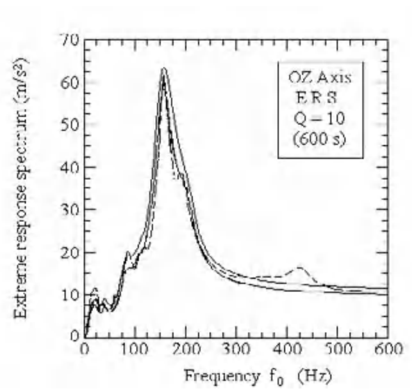


Figure 11.37. *ERS of the specifications of the three laboratories, for a test time of 600 s (axis OZ)*

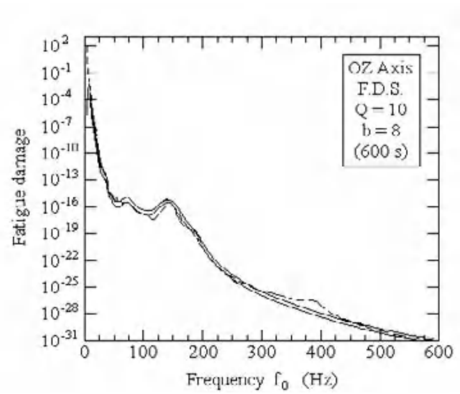


Figure 11.38. *FDS of the specifications of the three laboratories, for a test time of 600 s (axis OZ)*

11.7. Taking environment into account in project management

The previous section showed how specifications could be established by using measurements taken from the real environment. The method proposed here can be used for writing test specifications (test tailoring) and dimensioning specifications (tailoring the product to its environment). It is easy to understand why there is a need:

– to take into account environmental stress for the product and its sub-assemblies very early on in the project;

– to manage conversion of this data into specifications for the assemblies, sub-assemblies and equipment throughout the project in harmony with the way in which the product definition develops, and knowledge of the environment improves;

– to then follow-up the demonstration, illustrating whether the equipment satisfies these specifications.

The main purpose, which is to guarantee that the equipment completely withstands its real conditions of use, without an excessive margin, is in perfect accordance with the BNAE RG Aéro 00040 Recommendation [REC 91], one of the conditions being to develop a product that responds to the necessary minimum need rigorously.

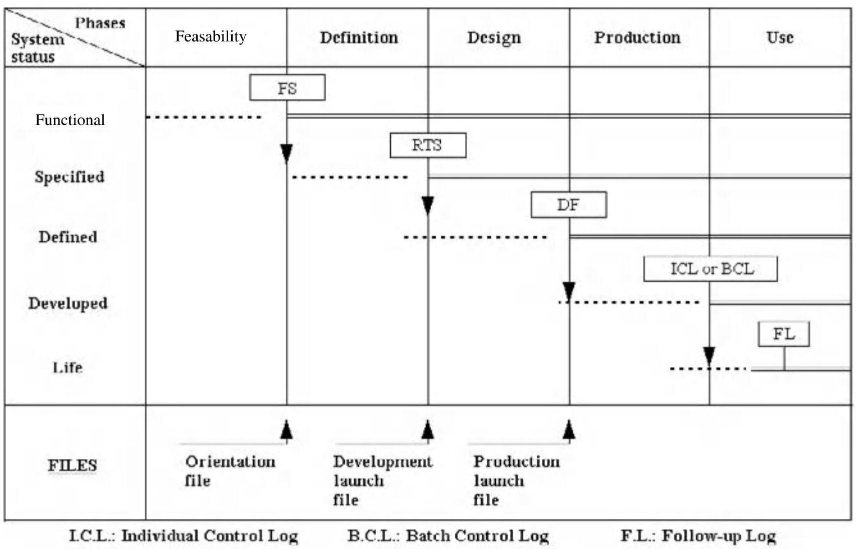


Figure 11.39. *Phases during the course of a project (R.G. Aéro 00040)*

The BNAE RG Aéro 00040 recommendation for the program management specification, which will become an AFNOR standard [REC 93], has therefore been completed to be integrated into this procedure. It breaks the project down into the following phases: feasibility, requirement definition, development, production, use and removal from service (Figure 11.39).

This recommendation suggests that a functional analysis be carried out at the beginning of the project, which would be used to identify service functions[5] for the product in the functional specifications (FS). Different technical solutions can then be contemplated and evaluated, so that a choice can be made in the requirement technical specification (RTS) during the requirement definition phase. The product's technical functions[6] are thus determined and detailed in the design phase leading to a definition file (DF) being drawn up. The design is validated by different actions, the results of which are recorded in the DBF (definition backup file) and the project enters the production phase. During this process, the system under study will have a different status at different times, e.g.: functional status, specified status, defined status, developed status and live status.

The product's environment may belong to different fields, which are defined as follows:

– normal field, for which the product's considered function must comply with the specified performances;

– limit field, for which the product's considered function may have down-graded performance, but still respects the safety requirements. This degradation must be reversible when the environment returns to normal field;

– extreme field, for which the product's considered function may be irreversibly down-graded, but still respects the safety requirements.

Integrating the tailoring process into the four-step project leads to the above-mentioned actions being divided up into all the different phases of the project. The tasks to be accomplished in each phase are summarized in Tables 11.8 (a), (b) and (c).

During the feasibility phase, work related to the environment is intended to characterize each of the agents for each of the identified events in each situation of the life cycle profile by a value or a spectrum that has a low probability of being exceeded (*awaited environment*). This value is determined from several measurements of the phenomenon (real environment or data base) or from calculations.

5. Service function (NF X 50-150): Action required of a product (or performed by the product) to satisfy an element of the requirement expressed by a given user.
6. Technical function (NF X 50-150): Action internal to the product (between its components), selected by the designer–producer, in a solution-finding context, to ensure the service functions.

In the case where each measurement of the considered environment agent should be characterized by a value, the awaited environment E_W is obtained by evaluating, from n measurements, the value that, with a given π_0 confidence level, has a probability P_0 of not being exceeded, on the basis of the quantity $\overline{E} + \alpha\, s_E$. The α constant is a function of P_0 and π_0.

If the environment agent is characterized by several spectra, the awaited environment corresponds to the spectrum obtained by calculating, for each frequency, the amount $\overline{E} + \alpha\, s_E$ from the spectrum values at this frequency.

When the distribution of points may be considered as Gaussian, the estimated mean $\overline{E}$ of the true value $\overline{E}_V$ is calculated using the following formula

$$\overline{E} = \frac{1}{n} \sum_{i=1}^{n} E_i \qquad [11.13]$$

and the estimated value s_E of the true standard deviation s_V from the following relationship

$$s_E = \sqrt{\frac{\sum_{i=1}^{n} \left(E_i - \overline{E}\right)^2}{n-1}} \qquad [11.14]$$

By definition, the awaited environment $\overline{E}_W$ is such that [OWE 63] [VES 72]:

$$\text{prob}\left[\int_{-\infty}^{\overline{E}+\alpha s_E} \frac{1}{s_E\sqrt{2\pi}}\, e^{-\frac{\left(E-\overline{E}\right)^2}{2\, s_E^2}}\, dE \le P_0\right] = \pi_0 \qquad [11.15]$$

If N( ) is the distribution function of the normal variable, this relation may be written as

$$\text{prob}\left[N\left(\frac{\overline{E} - \overline{E}_V + \alpha\, s_E}{s_V}\right) \le P_0\right] = \pi_0 \qquad [11.16]$$

or

$$\text{Prob}\left[N\left(\frac{\overline{E}-\overline{E}_V}{s_V}+\alpha\,\frac{s_E}{s_V}\right)\leq P_0\right]=\pi_0 \qquad [11.17]$$

Let u be the normal variable and $\chi^2(f)$ the variable χ^2 with $f = n - 1$ degrees of freedom:

$$\text{prob}\left[N\left(\frac{u}{\sqrt{n}}+\alpha\,\frac{\sqrt{\chi^2(f)}}{\sqrt{f}}\right)\leq P_0\right]=\pi_0 \qquad [11.18]$$

$$\text{prob}\left[\frac{u}{\sqrt{n}}+\alpha\,\frac{\sqrt{\chi^2(f)}}{\sqrt{f}}\leq u_{P_0}\right]=\pi_0 \qquad [11.19]$$

$u_{P_0} = P_0$ order fractile of u

$$\text{prob}\left[\frac{u-\sqrt{n}\;u_{P_0}}{\sqrt{\chi^2(f)}/\sqrt{f}}\leq -\alpha\sqrt{n}\right]=\pi_0 \qquad [11.20]$$

The variable $t_{f,\,\pi_0} = \dfrac{u-\sqrt{n}\;u_{P_0}}{\sqrt{\chi^2(f)}\;/\;\sqrt{f}}$ follows a Student distribution off-centered with f degrees of freedom and off-centering $-\sqrt{n}\;u_{P_0}$. Relation [11.20] may therefore be written as

$$\text{prob}\left[t_{f,\pi_0}\leq -\alpha\sqrt{n}\right]=\pi_0 \qquad [11.21]$$

whereby the value of α is:

$$\alpha=-\frac{1}{\sqrt{n}}\,t_{f,\,\pi_0} \qquad [11.22]$$

The values of this parameter in relation to f and π_0 are given in several publications [LIE 58] [NAT 63] [OWE 63] [PIE 96] [VES 72]. Table 11.6 provides some examples.

If the distribution is log–normal, the process is identical when replacing E by ln E in all of the calculations. The statistical value required is then determined by considering the amount $e^{\overline{\ln E} + \alpha\, s_{\ln E}}$.

	π_0											
	0.75				**0.90**				**0.95**			
n\\P_0	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**
3	1.464	2.501	3.152	4.396	2.602	4.258	5.310	7.340	3.804	6.158	7.655	10.552
4	1.256	2.134	2.680	3.726	1.972	3.187	3.957	5.437	2.619	4.416	5.145	7.042
5	1.152	1.961	2.463	3.421	1.698	2.742	3.400	4.666	2.149	3.407	4.202	5.741
6	1.087	1.860	2.336	3.243	1.540	2.494	3.091	4.242	1.895	3.006	3.707	5.062
7	1.043	1.791	2.250	3.126	1.435	2.333	2.894	3.972	1.732	2.755	3.399	4.641
8	1.010	1.740	2.190	3.042	1.360	2.219	2.755	3.783	1.617	2.582	3.188	4.353
9	0.984	1.702	2.141	2.977	1.302	2.133	2.649	3.641	1.532	2.454	3.031	4.143
10	0.964	1.671	2.103	2.927	1.257	2.065	2.568	3.532	1.465	2.355	2.911	3.981
15	0.899	1.577	1.991	2.776	1.119	1.866	2.329	3.212	1.268	2.068	2.566	3.520
20	0.865	1.528	1.933	2.697	1.046	1.765	2.208	3.052	1.167	1.926	2.396	3.295
30	0.825	1.475	1.869	2.613	0.966	1.657	2.080	2.884	1.059	1.778	2.220	3.064
40	0.803	1.445	1.834	2.568	0.923	1.598	2.010	2.793	0.999	1.697	2.126	2.941
50	0.788	1.426	1.811	2.538	0.894	1.560	1.965	2.735	0.961	1.646	2.065	2;863

Table 11.6. *Number of standard deviations corresponding to a given probability P_0 of not exceeding, at the confidence level π_0*

NOTE:

1. *The expected environment, the "envelope" of the real environment, could be determined using other methods such as, for example [PIE 96]:*

– a statistical curve of the same type calculated by evaluating the real distribution of points on the overall spectrum;

– an envelope of the curves, with possible smoothing. This curve may be associated with a non-overrun probability P_0, and a confidence level in direct relation to the selected value of P_0, through the relationship $\pi_0 = 1 - P_0^n$;

– a curve comprised, for each frequency, of the value that will exceed the next E value to be measured with a given confidence level. The spectrum distribution law may be Gaussian or, preferably, log–normal.

2. *The calculation of α may be performed by approximation using the relationships shown in section 8.3.1.3.*

In the feasibility phase, it is not possible to know the transfer functions that could be used to evaluate the environment on input of the sub-assemblies and equipment items. The expected environment determined here therefore applies only to the system.

Conversely, in the definition phase, it is possible to obtain a first assessment for these functions that will be used to define, for each assembly level, the specified environment values that must be indicated within the various specifications for technical requirements (per event, as for awaited environment).

The first synopsis will be performed at the beginning of the development phase, during the design process. After application of an uncertainty factor that assures a given probability of correct operation in an identified environment, the synopsis will move on to the withhold environment, determined according to the methodology presented in the previous sections (system, sub-assemblies and equipment items).

This process will be repeated before validation by the qualification tests, for writing the test severities, taking into account the most recent available data concerning the environment and applying the test factor calculated in relation to the planned test number and the selected confidence level. Table 11.7 summarizes, for each phase, the environment to be defined, the description mode and the assembly level to which it applies.

<table>
<tr><th>Phase</th><th colspan="2">Representing</th><th>Synthesis Level</th><th>Subassembly level</th></tr>
<tr><td>Feasibility</td><td colspan="2">Awaited environment
(Mean + α Standard deviations)</td><td>Per event</td><td>System</td></tr>
<tr><td>Definition</td><td colspan="2">Specified environment
(Mean + α standard deviations)</td><td>Per event</td><td>System
Sub-assemblies
Equipment items</td></tr>
<tr><td rowspan="2">Development</td><td>Design</td><td>Withhold env.
($k \times$ Mean of env.)</td><td>Situation synthesis</td><td rowspan="2">System
Sub-assemblies
Equipment items</td></tr>
<tr><td>Validation</td><td>Test severities
($T_F \times$ Withhold env.)</td><td>Situations synthesis</td></tr>
<tr><td colspan="5">The multiplying constant α assures that the real environment is lower, with a given P_0 probability, than the value or the awaited environment curve, for a given π_0 confidence level.</td></tr>
</table>

Table 11.7. *Description of environment in relation to each project phase*

Program phases	Environment values		Assembly level	Functions Concerned		Useful additional information	Remarks
	input	*output*		*serv.*	*techn.*		
Feasibility leading to the FS.	- Standard values arising from repertories, data bases. - Fallback levels. - Values resulting from calculation models. - Values measured for an event, a given situation.	- 1 value representing each event by signal type (transient, random), with the characteristic dispersion parameters (*awaited environment*).	System	X		- System life cycle profile. - Description of the service functions. - Hypotheses about behavior models (natural frequency ranges, Q factor, etc.). - Level of confidence of the confidence interval concerning the variation coefficient V_E.	For a given event, data will be described with values, and sets of values (spectrum, etc.) associated with a level of confidence and a statistical law either assumed or estimated on the basis of the measurements.
Definition leading to the RTS.	Above mentioned values updated by: - new measurements or estimates of value; - taking into account effects induced when selecting the design options; - indication of whether the value is comprised in the normal, limit or extreme field to which the value belongs.	Same as above. These values are *the specified environment values.*	All levels	X		- Life cycle profile of life, all assembly levels. - Estimated transfer functions. - Then, same as above.	Same as FS.

Table 11.8 (a). *Taking the environment into account in the feasibility and definition phases*

Program phases	Environment values		Assembly level	Functions Concerned		Useful additional information	Remarks
	Input	*output*		*serv.*	*Techn.*		
Design leading to the DF	Specified environment values	Dimension criteria - *Withhold environment values* used for calculations and simulation.	All levels		X	- Description of technical functions. - Additional hypotheses about the behavior models used for the synopsis (*b*, etc.). - Permitted failure probability. - Internal transfer functions. - Variation coefficient of the determinist equipment strength taken from directories or estimated with level of confidence associated with the frame.	- The environment values specified according to a situation or several situations will be summarized by grouping together several events or situations and the application of an uncertainty factor. They will lead to *withhold environment values.* - These values will allow design options to be selected
Validation of the design (leading to the DBF of the technical functions)	Same as Above	- *Severities of tailorized tests*	All levels		X	Same as above, plus: - Level of confidence of the confidence interval of the mean strength $\overline{R}$ of the performance in relation to the environment agent under consideration. - Number of identical items undergoing the same test.	Same as above, with application of a test factor. These values will be used, either for calculations and simulations, or to determine the severity of the tailored tests.

Table 11.8 (b). *Taking the environment into account in the development phase*

Program phases		Environment values		Assembly level	Functions concerned		Useful additional Information	Remarks
		Input	*output*		*serv.*	*techn.*		
Development	Validation of the development (leading to the DBF of the service functions)	Environment values specified in the RTS of the system	Same as above but for the service functions.	All levels	X		Same as above, but for service functions.	Same as above. The updated values will be compared to the corresponding values initially selected. If the result shows up an excess, the system will if necessary be updated with the values that can be deduced.
Production		Measurements characterizing some events in the production process	- Selected values summarized per significant event of the production process. - Severity of the stress screening and acceptance tests.	All levels		X	- Development of the manufacturing process . - Permitted failure probability according to the significant environmental configurations generated by the production process.	The values summarized by significant event of the production process will be compared to the summarized values of the same type as the life cycle profile. If the result shows up an excess, the system will be updated with the values that can be deduced.

Table 11.8 (c). *Taking the environment into account in the development and production phase*

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Chapter 12

Influence of Calculation: Conditions of Specification

Several parameters are involved *a priori* in the calculation of ERSs and FDSs and thus also in the calculation of a specification: the Q factor, parameter b, number of PSD points, etc. In this chapter, we study the influence of each of these parameters.

12.1. Choice of the number of points in the specification (PSD)

The number of PSD definition points deduced from an FDS cannot be higher than the number of FDS calculation points. It can be equal, but that is not really desirable. In fact, the specification thus calculated is intended:

– to be included in a document (for material sizing for example);

– or to control a test facility.

Using the same number of points as the number chosen for ERS and FDS calculation (between 100 and 200) would be too intensive and would present risks of errors.

It is generally possible to determine a PSD with a few dozen points (often less than 20 points) where the FDS follows reference FDS very closely (real environment after a synopsis). In this case, the PSD is smoothed (without a problem), but its rms value is still retained.

Example 12.1.

Consider FDSs of an "aircraft" vibration (200 points) and a specification (PSD) established with 40 points (Q = 10, b = 10). The specification here is calculated with line segments of unspecified slope. Both FDSs are very close (Figure 12.1).

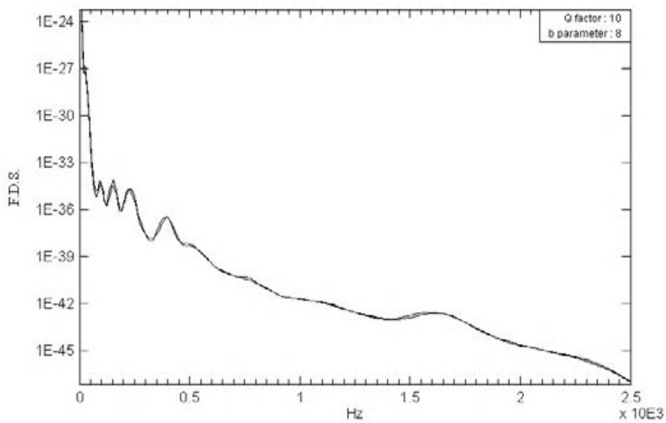

Figure 12.1. *FDS of an "aircraft" specification (PSD with unspecified slope segments) calculated with 40 points and FDS of the real environment*

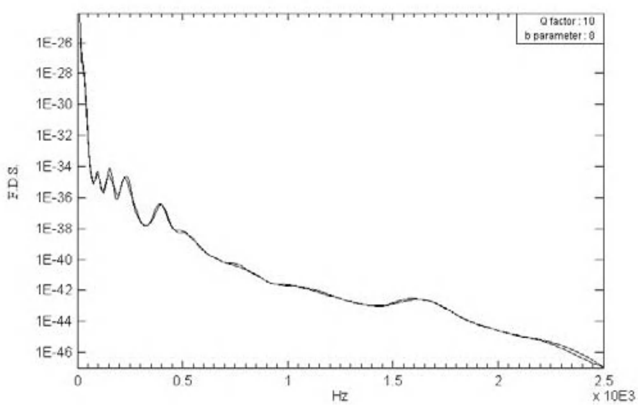

Figure 12.2. *FDS of an "aircraft" specification (PSD with horizontal segment) calculated with 40 points and FDS of the real environment*

It is not so easy to find the reference FDS from a PSD defined by horizontal line segments, especially when the number of levels is low (Figure 12.2).

Choosing a large number of points for PSD is not very useful. Specifications calculated with 100 points and 200 points are identical (Figure 12.3) for PSDs made up of line segments with unspecified slope.

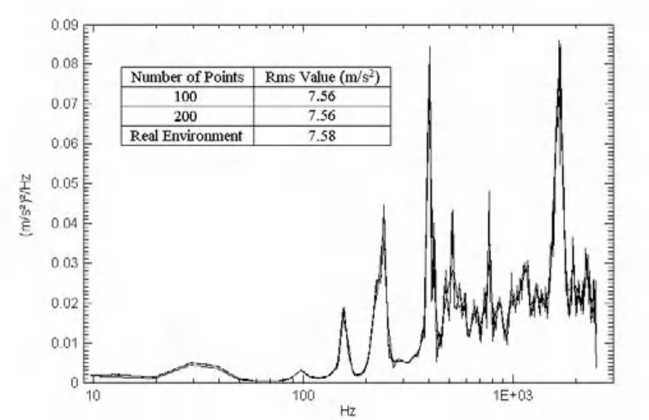

Figure 12.3. *Influence of the number of specification calculation points (PSD with unspecified slope segments)*

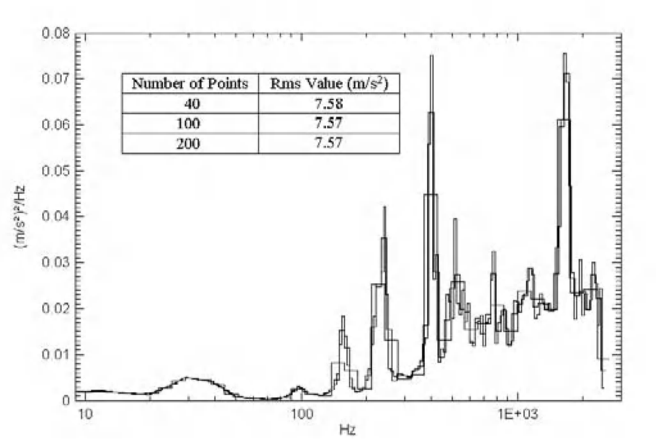

Figure 12.4. *Influence of the number of specification calculation points (PSD with horizontal segments)*

The choice of horizontal line segments also makes it possible to correctly follow the details of the original PSD with 100 points and 200 points. With 40 points, peaks are truncated, but larger (Figure 12.4). The rms value is retained.

12.2. Influence of Q factor on specification (outside of time reduction)

Q factor has no influence on the specification (PSD) obtained when it is defined by a large number of points. This property can be highlighted by considering the expression of damage created by a white noise random vibration. If the response can be considered as narrow band, fatigue damage is written as (Volume 4, Chapter 4):

$$D = \frac{K^b}{C} n_0^+ T (\sqrt{2}\, z_{rms})^b \, \Gamma\left(1 + \frac{b}{2}\right) \tag{12.1}$$

If $G_z(f)$ is the PSD of the relative response displacement and $G_{\ddot{x}}(f) = G_{\ddot{x}0}$ of the vibration, we have:

$$G_z(f) = H_{\ddot{x}z}^2 \, G_{\ddot{x}} \tag{12.2}$$

where:

$$\left|H_{\ddot{x}z}\right|^2 = \frac{1}{(2\pi f_0)^4 \left\{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(2\xi \frac{f}{f_0}\right)^2\right\}} \tag{12.3}$$

Hence:

$$z_{rms}^2 = \int_0^\infty G_z(f)\, df = \frac{G_{\ddot{x}}}{64\pi^3 f_0^3 \xi} = \frac{G_{\ddot{x}}}{8\,\omega_0^3\, \xi} \tag{12.4}$$

By transferring this value to the damage expression, it becomes:

$$D = \frac{K^b}{C} n_0^+ T \left(\frac{G_{\ddot{x}}}{4\,\omega_0^3\, \xi}\right)^{\frac{b}{2}} \Gamma\left(1 + \frac{b}{2}\right) \tag{12.5}$$

Knowing that $n_0^+ \approx f_0$, we can observe that damping occurs in a factor of $G_{\ddot{x}}$. The transfer of the PSD to damage, then the return from damage to the PSD involves the same factor. The numeric value of damping therefore has no effect.

The same comment can be made in the case of a PSD made up of line segments (Volume 4).

When the number of points is reduced (to a few dozen), the resulting PSD is smoother, all the more so because the surge is smaller. The rms value is still retained. ERSs and FDSs are still respected.

Example 12.2.

Consider a vibration measured on an aircraft (Figure 12.5).

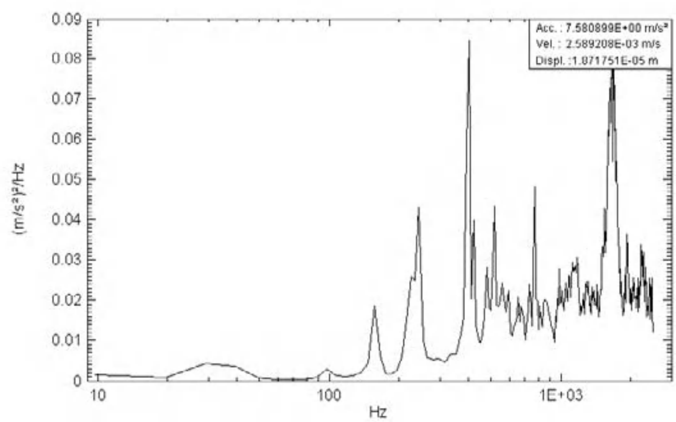

Figure 12.5. *PSD of a vibration measured on an aircraft (256 points)*

Figures 12.6 and 12.7 show FDSs and ERSs calculated from the PSD of this vibration for b = 8 and Q respectively equal to 5, 10 and 20 (duration 1 hour), as well as specification FDSs and ERSs.

The choice of bounds for the 40 PSD levels, made from automatic division of the frequency range (thus not optimized), is identical for all Q factor values in order to facilitate the comparison.

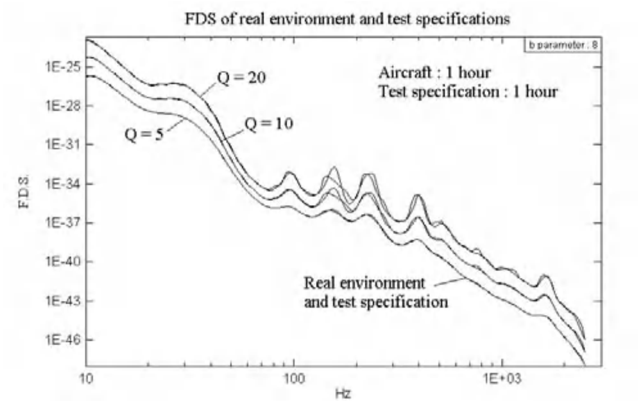

Figure 12.6. *Influence of Q factor on FDSs (real "aircraft" environment and specification)*

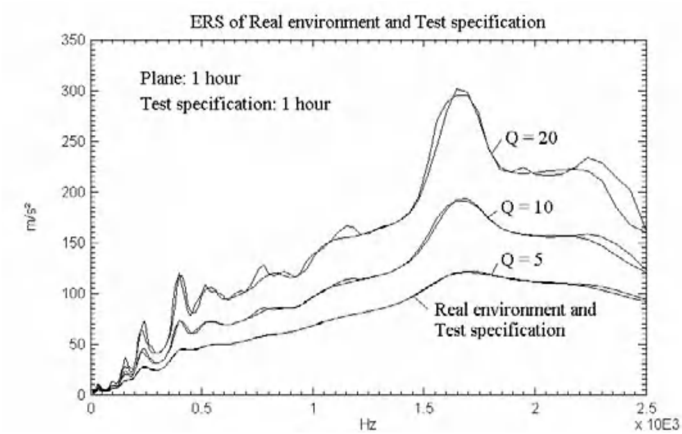

Figure 12.7. *Influence of Q factor on ERSs (real "aircraft" environment and specification)*

We can observe that rms values are very close to each other and in relation to the real vibration. With 40 points, the PSD is all the smoother as Q is smaller (Figure 12.8).

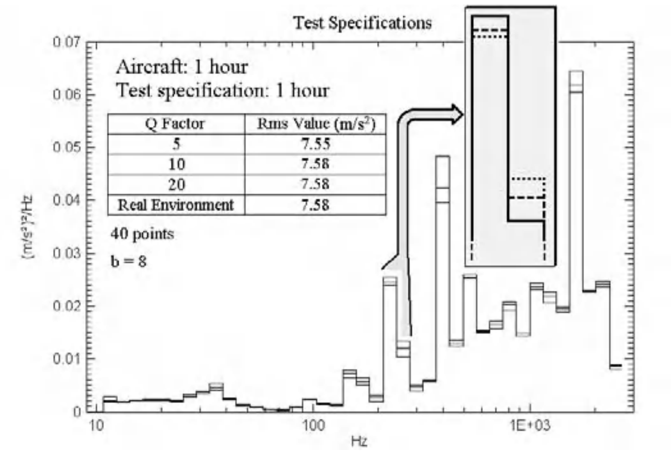

Figure 12.8. *Influence of Q factor on specification (PSD)*

With 100 points, the specification is closer to the original PSD (Figure 12.9).

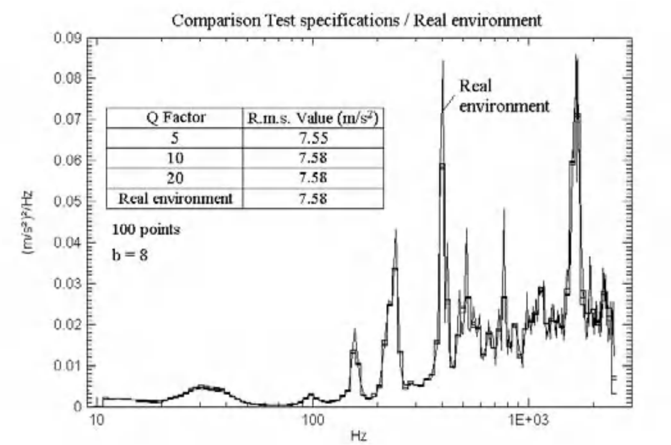

Figure 12.9. *Comparison of the PSD of the real "aircraft" environment and specifications obtained for several Q factor values*

NOTE: *Following specifications are defined by line segments with unspecified slope.*

With 200 points, specifications determined for Q = 10 and Q = 20 are merged with the reference PSD (Figures 12.10 and 12.11).

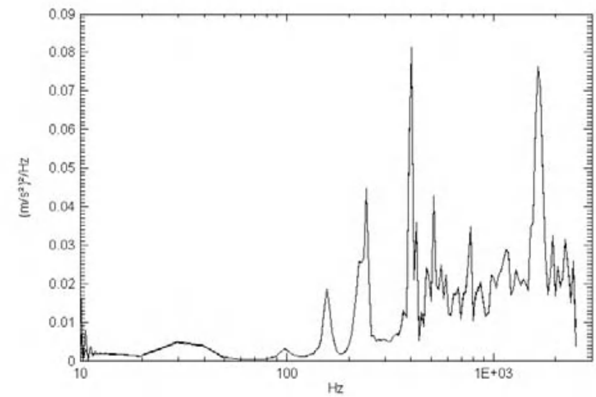

Figure 12.10. *"Aircraft" specifications calculated over 200 points with Q = 10 and Q = 20*

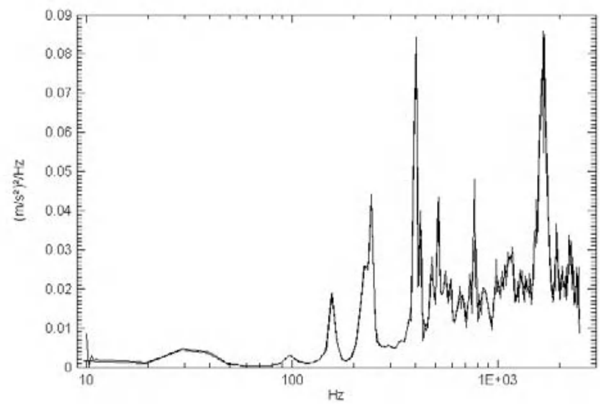

Figure 12.11. *PSD of the real "aircraft" environment and specification in 200 points calculated for Q = 20*

12.3. Influence of Q factor on specification when duration is reduced

The Q factor value chosen for calculations has no significant influence on the specification when duration is reduced.

The comments made in the absence of duration reduction strictly apply here.

Example 12.3.

An "aircraft" vibration lasting 10 hours. We study the specifications of a 1 hour duration established for b = 8 and a Q factor respectively equal to 5, 10 and 20.

Figure 12.12 shows FDSs of the reference PSD calculated for these Q factor values, as well as FDSs from specifications established with 40 points (automatic PSD frequency choice, not optimized).

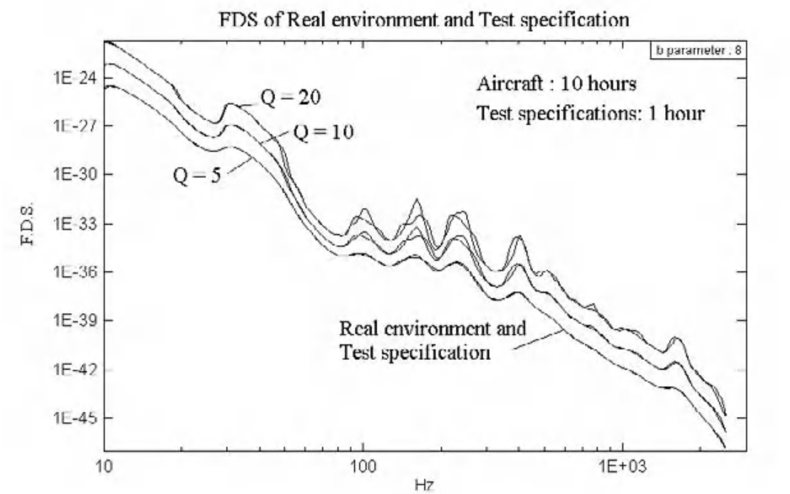

Figure 12.12. *Influence of Q factor on FDSs (real "aircraft" environment and reduced duration specification)*

Corresponding ERSs, shown in Figure 12.13, highlight the consequences of the increase in the levels resulting from duration reduction.

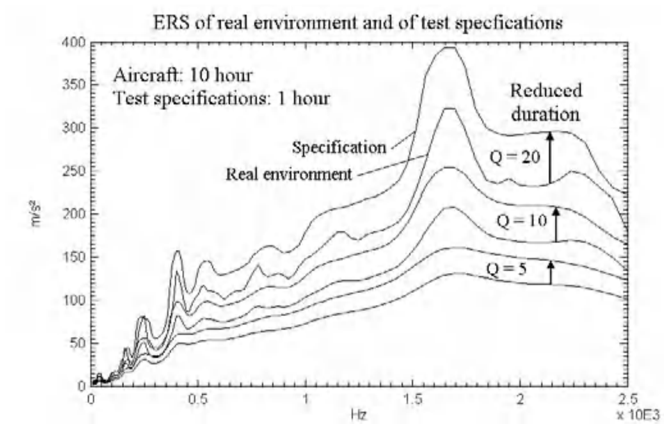

Figure 12.13. *Influence of Q factor on ERSs (real "aircraft" environment and reduced duration specification)*

Specifications (PSD) are represented in Figure 12.14.

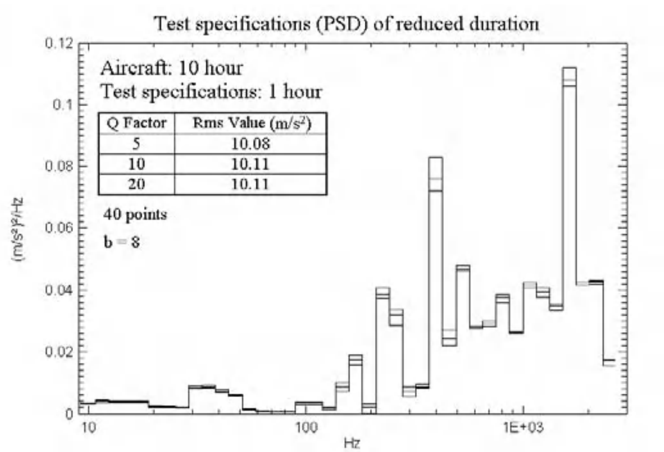

Figure 12.14. *Comparison of specifications (PSD) of reduced duration established for three Q factor values*

PSDs are all the more smooth as Q is smaller; the rms value is still retained. These PSDs follow the frequency content of the real environment PSD closely (Figure 12.15).

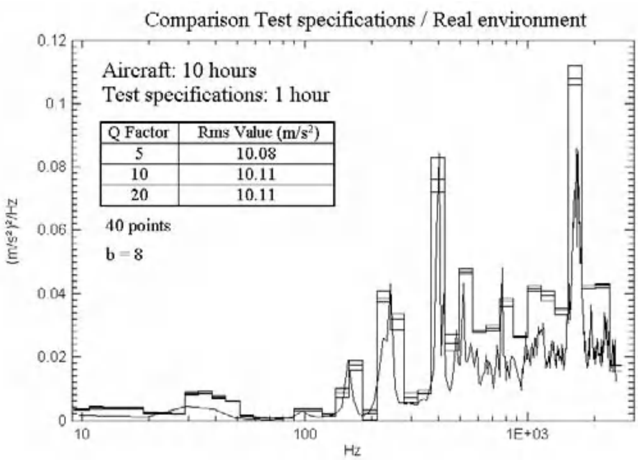

Figure 12.15. *Comparison of the PSD of the real "aircraft" environment and specifications obtained for several Q factor values*

Example 12.4.

Case of two situations in series

Figure 12.16. *Life profile with two situations*

The real environment here is made up of a truck vibration (lasting 24 hours) followed by an aircraft vibration (three hours) (Figure 12.16).

The specifications established from this very simple life profile with Q respectively equal to 5, 10 and 20 are almost identical (Figure 12.17). The small insignificant gaps between PSDs come from the frequency division with 40 levels which tends to smooth the PSD for small Q factor values.

The specifications correctly surround the PSD of the real "truck" and "aircraft" environment, with slightly larger amplitude at low frequency caused by more significant duration reduction for the truck (24 hours to 1 hour) than for the aircraft (three hours to one hour) at high frequency.

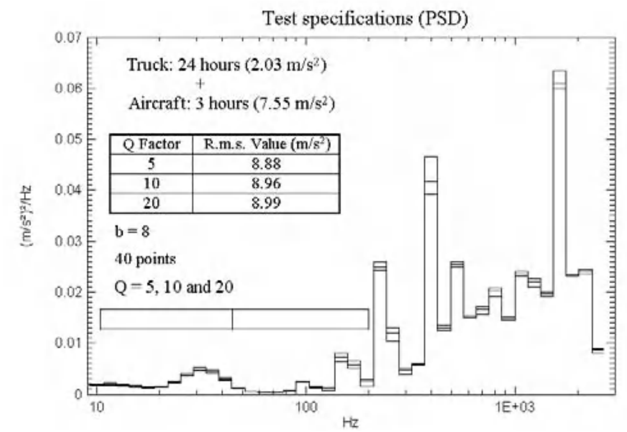

Figure 12.17. *Comparison of specifications (PSD) established for three Q factor values and covering truck transport and aircraft transport*

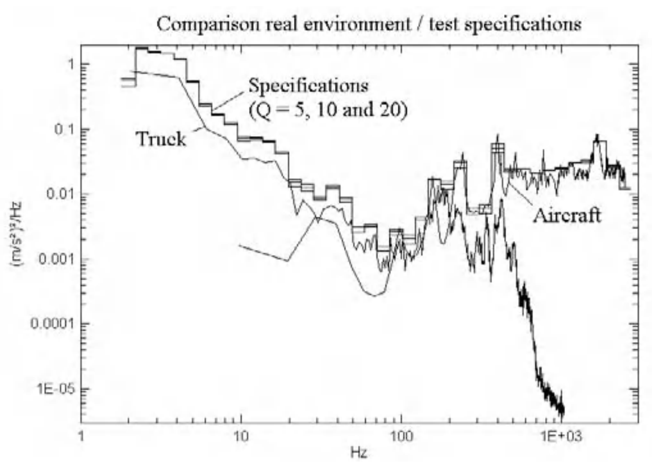

Figure 12.18. *Comparison of specifications established for three Q factor values with real environment PSD (truck and aircraft)*

12.4. Validity of a specification established for Q factor equal to 10 when the real structure has another value

In the absence of precise data on the dynamic behavior of the material involved, specifications are most often established from FDS and ERS calculated for a Q factor equal to 10. Later tests can show that the real Q factor has another value.

We have seen (sections 12.2 and 12.3) that the specification is not a function of the Q factor used. We can also verify with the help of ERSs and FDSs, that the effects of a specification established for Q = 10 (for example) in a structure with a Q factor equal to 20 are the same as those evaluated by direct application of the real environment in a system with a Q factor equal to 20.

Example 12.5.

From the real "aircraft" vibration defined by the PSD (Figure 12.5), we have calculated ERSs and FDSs for Q = 10, subsequently, we deduced a specification. ERS and FDS of this specification were recalculated for Q = 20 and compared to ERS/FDS directly calculated for Q = 20 from the original PSD.

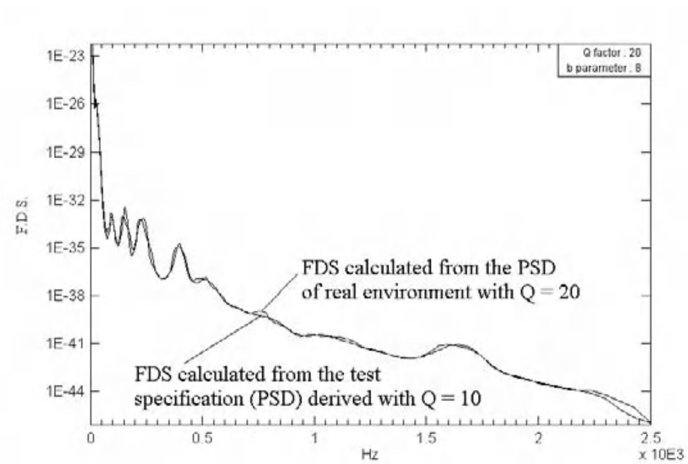

Figure 12.19. *FDS of the real "aircraft" environment, calculated for Q = 20, and of a specification established for Q = 10*

We can observe that the specification established for Q = 10 returns damage and extreme responses well if the real Q factor of the structure is different from 10.

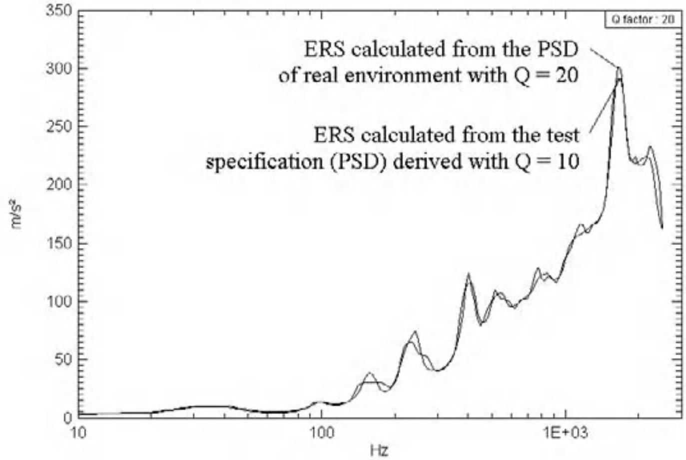

Figure 12.20. *FDS of the real "aircraft" environment, calculated for Q = 20, and of a specification established for Q = 10*

12.5. Advantage in the consideration of a variable Q factor for the calculation of ERSs and FDSs

In structures with several degrees of freedom, the Q factor varies at each mode. However, ERS and FDS are drawn for constant Q.

In order to take this variation into consideration, we could imagine using, to establish a specification, ERS and FDS calculated with variable Q factor according to the natural frequency, based on a law deemed representative.

The Q factor intervenes in the FDS calculation, but the reverse calculation of the specification (PSD) from tha FDS takes into account its value and its effect is exactly compensated in this last operation, at each natural frequency.This property was verified in the examples presented in sections 12.2 to 12.4. It remains true when Q varies according to the natural frequency during FDS calculation.

Example 12.6.

The following example shows FDSs of the "aircraft" vibration obtained for variable Q between 8 and 50 and, for comparison purposes, for Q = 10 in the entire frequency range. As we might expect, the FDS amplitude is larger when Q is larger than 10 (Figure 12.21).

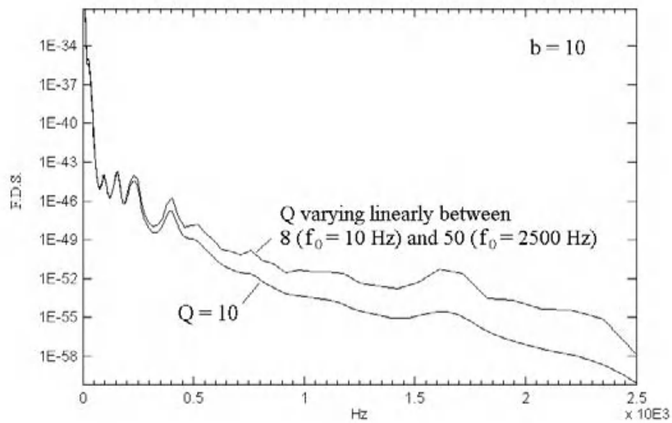

Figure 12.21. *FDS of "aircraft" vibration drawn for constant Q (equal to 10) and for variable Q according to the frequency*

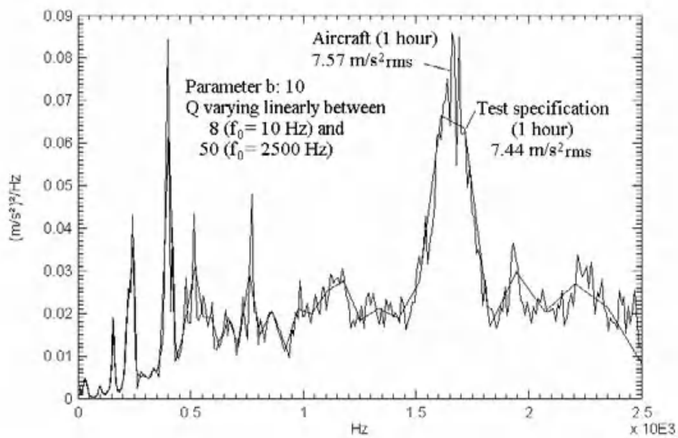

Figure 12.22. *Comparison of the specification established by considering variable Q according to frequency and the PSD of the real "aircraft" environment*

The specification determined from this variable Q FDS, defined here by line segments with unspecified slope, follows the real environment very closely

(Figure 12.22). FDSs of the real environment and specification are illustrated in Figure 12.23.

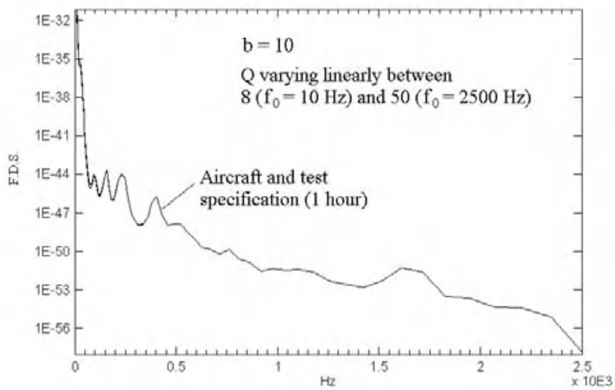

Figure 12.23. *Comparison of the FDS of a specification established considering variable Q according to frequency and the FDS of the real environment*

12.6. Influence of the value of parameter b on the specification

12.6.1. *Case where test duration is equal to real environment duration*

The FDS of a vibration has a different amplitude when parameter b is modified; the FDS is proportional to the PSD amplitude at power b/2. Reverse transformation making it possible to go from an FDS to a PSD is done by raising FDS amplitudes to power 2/b.

The resulting PSD is not sensitive to the value of parameter b chosen for intermediate FDS calculation. This property remains true in the more complex case where the PSD (specification) is the result of an FDS calculated by summation of damages of several situations of a life profile.

The specification is independent from the value of parameter b when test time is equal to the value of the real environment.

Example 12.7.

Consider a vibration measured on an aircraft and a specification determined from FDSs calculated from this environment for b consecutively equal to 4, 8 and 12.

Duration of the real environment and specification are presumed equal to 1 hour. In all cases, the specification (PSD) was calculated with 40 levels and still with the same frequency boundaries, in order to facilitate the comparison of results (Figure 12.24).

Figure 12.24. *Comparison of specifications calculated for b = 4, 8 and 12 from an "aircraft" environment (with no duration reduction)*

This choice of boundaries, made automatically and without optimization to follow the reference FDS as well as possible, explains the very slight differences observed.

We should mention that rms values are almost identical for the three specifications and equal to those of the real environment.

12.6.2. *Case where duration is reduced*

The PSD amplitude is increased according to the rule reduced by Basquin's law,

$$G_{\text{Reduced duration}} = G_{\text{Real duration}} \left(\frac{T_{\text{Real env.}}}{T_{\text{Specification}}} \right)^{\frac{2}{b}} \qquad [12.6]$$

The specification is a function of parameter b when duration is reduced.

Example 12.8.

An "aircraft" vibration. The FDS is calculated for 10 real environment hours and specification (Figure 12.25) is calculated for a test duration of 1 hour, with b respectively equal to 4, 8 and 12 (Q = 10, PSD in 40 points).

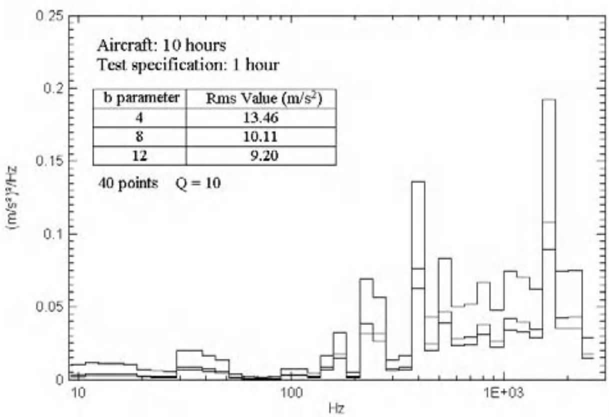

Figure 12.25. *Comparison of specifications calculated for b = 4, 8 and 12 from an "aircraft" environment (with duration reduction)*

PSD amplitude has a real variation with the value of parameter b used.

Example 12.9.

Case of two vibrations with different frequency contents (PSD) and durations. We assume that the life profile is made up of 2 situations in series, truck transport lasting 24 hours followed by aircraft for 3 hours. Specification is established for reduced duration of 1 hour, with b consecutively equal to 4, 8 and 12 (Figure 12.26).

The specification rms value increases when b decreases according to the reduced rule of Basquin's law. In this more complex case of 2 situations in series with very different characteristics, the low frequency part of the PSD, mainly from the truck transport lasting 24 hours, is increased to a greater extent by the duration reduction to 1 hour than the high frequency part, mainly corresponding to aircraft transport over 3 hours.

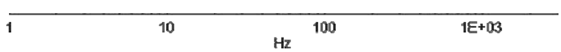

Figure 12.26. *Comparison of specifications calculated for b = 4, 8 and 12 from a life profile made up of two situations ("truck" and "aircraft")*

This result is highlighted in Figure 12.27, which shows the specification established for b = 8 superimposed onto the PSD of both real environments.

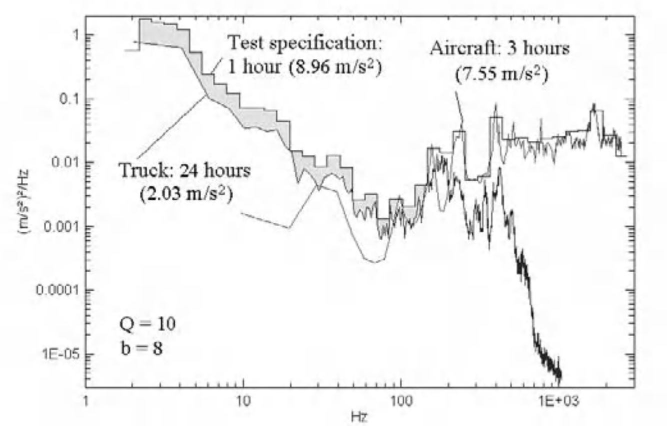

Figure 12.27. *Comparison of the specification covering truck transport followed by an aircraft transport with the PSDs of both real environments*

12.7. Choice of the value of parameter b in the case of material made up of several components.

This very real problem is not specific to the ERS/FDS method. The rules of duration reduction proposed in the standards already implicitly rely on an equivalence of fatigue damage and these rules mandate a specific parameter b value (often a value of 8 to 9 corresponding to aluminum alloys, standards involved usually intended for aeronautical material). In the ERS/FDS method, parameter b can be chosen based on need.

The choice is actually difficult to make in the case of multiple materials. There can be cases where we know *a priori* the material element that is most fragile, therefore the material constituting it and its corresponding parameter b.

The problem is more difficult when we have no information. Choosing the larger parameter b leads to an under-test of other components, which should obviously be avoided.

A conservative solution is to consider the smallest b value, leading to the increase of the largest PSD amplitude (specification). This methodology can be appropriate for sizing material. However, it leads a test to over-test the elements of a higher b material. This can be acceptable if we have no concern for the behavior of parts.

If we do have concerns, a solution could be to separate the problems and to carry out two tests. Let us consider equipment made up of a light alloy frame ($b = 9$) supporting electronic boards ($b = 4$).

If we ignore the most fragile element, to avoid an under-test (choice of $b = 9$) for the boards, or over-testing of the frame (for $b = 4$), two separate tests can be done:

– one defined for $b = 9$ to test the frame (test with boards representative of the mechanical behavior or real boards, without considering the test results involving them);

– the second one, defined for $b = 4$, on the boards themselves, placed in a structure made rigid respecting the fixed points.

The use of a rigid construction assumes that there is no dynamic interaction between the boards and frame.

There is no simple solution when the components cannot be tested separately, which is fortunately quite rare.

12.8. Influence of temperature on parameter b and constant C

The properties of mechanical strength of materials generally evolve with temperature, and particularly fatigue behavior. Constants involved in ERS and FDS calculations are as follows:

– constants b and C of Basquin's law ($N\sigma^b = C$);

– constant K with proportionality between the stress and strain;

– Q factor.

Until now, we have ignored the exact value of constants C and K (by making them equal to 1) by considering the ERSs and FDSs generally used for severity comparisons of several vibrations applied in a single structure.

The methodology can remain the same if all vibrations being compared are applied to a structure submitted to the same high temperature. In the case of the establishment of a specification, the test must then be carried out at this temperature. If there is reduction of the test time, the influence of parameter b is very important and we must recognize its value at the temperature considered. Little data is available in the literature.

The problem becomes more complicated when we want to define a specification covering the effects of several vibrations applied to very different temperatures, for which mechanical characteristics vary considerably. In this case, we can no longer ignore the exact value of constants (K, C, b and Q) involved in the calculation of ERSs and FDSs before their synopsis (envelope or sum). On a purely theoretical level, if these values were known, resulting ERSs and FDSs would enable severity comparisons and the development of specifications, whether at reduced duration or not, at the same temperature or not.

However, in practice the knowledge of these parameters at different temperatures is very difficult to obtain (very limited published data).

Example 12.10.

Figure 12.28 compares Wöhler curves obtained at two very different temperatures (20°C and 204°C) for titanium.

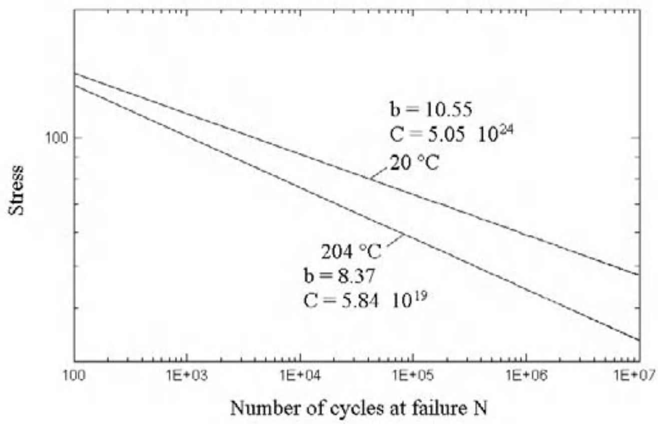

Figure 12.28. *Wöhler curves for titanium at 20°C and 204°C*

12.9. Importance of a factor of 10 between the specification FDS and the reference FDS (real environment) in a small frequency band

In order to simplify a specification, or to take into account a possible frequency variation of a peak in a small interval of frequency, we may have to smooth the FDS by choosing a larger envelope surrounding some peaks.

This methodology leads to higher damage, of a factor that may exceed 10 in small frequency bands. The consequences for the resulting specification are very limited.

Example 12.11.

Figure 12.29 shows the FDS of the "aircraft" vibration (200 points) and the FDS of a specification established with 40 horizontal levels in order to surround the peaks of the FDS between 100 and 400 Hz (without duration reduction).

The rms value of the specification does not exceed 2.7%, going from 7.56 m s^{-2} to 7.76 m s^{-2} (Figure 12.30). Smoothing consequences are very low.

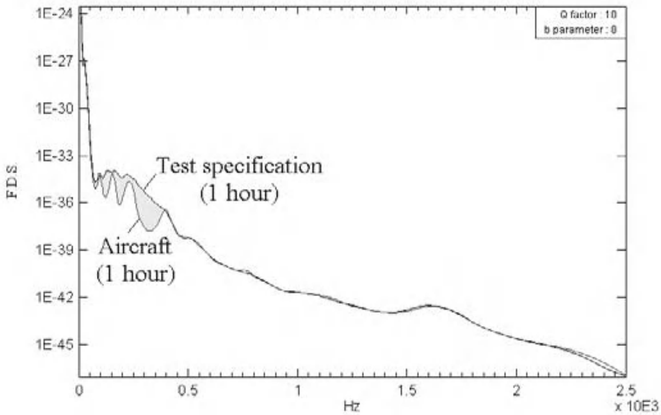

Figure 12.29. *FDS of a specification smoothing the real "Aircraft" environment FDS in a small interval of frequencies*

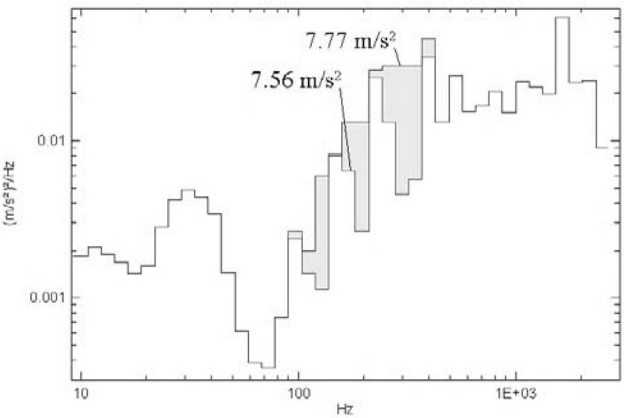

Figure 12.30. *Comparison of specifications established with and without smoothing of the real environment FDS in a small interval of frequencies*

12.10. Validity of a specification established by reference to a 1-dof system when real structures are multi-dof systems

Specifications are established by searching for a vibration producing the same effects (largest response, fatigue damage) as the real environment in a linear one degree-of-freedom system, with a natural frequency in a range which covers natural frequencies of the material involved.

In practice, we rarely find perfectly linear structures with a single-degree-of-freedom. It is therefore natural to question whether the specifications established are significant.

It is difficult to provide a general demonstration guaranteeing this equivalence. Several response elements can be raised however:

– some structures can be assimilated, in a first approximation, to a one degree-of-freedom system. It is particularly the case for suspended equipment and containers;

– in other cases, unless the first two modes are very close and highly coupled, the first mode is generally the one leading to the largest relative response displacement; it is therefore the one that is the source of the largest part of the damage;

– even though it is not a valid argument, we should still note that shock response spectra have been used for a long time to write specifications applied to complex structures, without negative effects;

– experimental studies carried out in order to validate the use of SRSs showed that maximum stresses generated in a complex structure submitted to two shocks with the same SRS do not differ by more than 20% if both shocks are similar (Volume 2) (with non-zero velocity change or oscillatory), and can be in a 2 to 3 ratio if the shocks are of a different nature. Calculations done in a numeric model for greatly varying nonlinearities have shown that this parameter did not modify equivalences significantly (even if the response values evolve significantly);

– a study [DEW 86] could demonstrate that it was possible to define a random vibration test with the same severity as a swept sine vibration with the help of FDS and ERS, as long as we know the values of parameter b and Q factor. The demonstration was carried out by comparing useful lifetimes of components from different generations assembled on electronic boards experiencing two types of excitations. The failure criterion was the appearance of a first electric problem (since the boards were powered during tests), or the acknowledgement of a mechanical problem (fracture of a component's leg). Tests that are supposed to be equivalent by equal FDSs have led to similar useful lifecycles (taking into account the dispersion usually observed in fatigue tests). This study carried out in 1986 was confirmed by tests done in 2002 on more recent components;

– this method has been used successfully for over 30 years in some industries.

Specification Development: Second Edition – Volume 5
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Chapter 13

Other Uses of Extreme Response, Up-Crossing Risk and Fatigue Damage Spectra

13.1. Comparisons of the severity of different vibrations

The problem of comparing the severity of several vibratory environments is often raised, whether it involves:

– several vibrations measured in different vehicles or in a single vehicle under different conditions of use (good road, bad road, etc.);

– several tests specified in a single standard or in different standards;

– specifications and measurements of the real environment.

The vibrations most frequently encountered in the real environment are random, but application durations can be very different. The problem increases when tests defined in standards are of different natures: how should a swept sine and random vibration, or a random vibration and a series of shocks be compared?

Extreme response spectra and fatigue damage spectra can be used to compare the severity of all these vibratory environments.

13.1.1. *Comparisons of the relative severity of several real environments*

Consider, for example, an aircraft environment and a helicopter environment, each characterized by an acceleration spectral density (Figure 11.29). These two vibrations have very similar rms values, but different frequency contents.

Figures 13.1 and 13.2 compare the extreme response and fatigue damage spectra of these real environments with each other, and with a standard covering these environments, assuming a duration of 3 hours for the helicopter and 2 hours for the aircraft. This standard proposes a swept sine defined as an acceleration of amplitude 1 *g* between 10 Hz and 600 Hz (duration 3 hours).

These spectra clearly show in which frequency range (i.e. the natural frequencies) the helicopter is the most severe. They also show that the standard chosen for this example is much too severe up to 600 Hz and too lax above that.

Example 13.1.

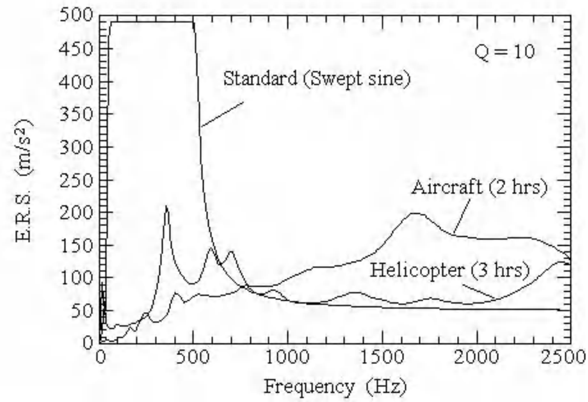


Figure 13.1. *Comparison of the severity of plane and helicopter vibrations and of the standard on the basis of their ERS*

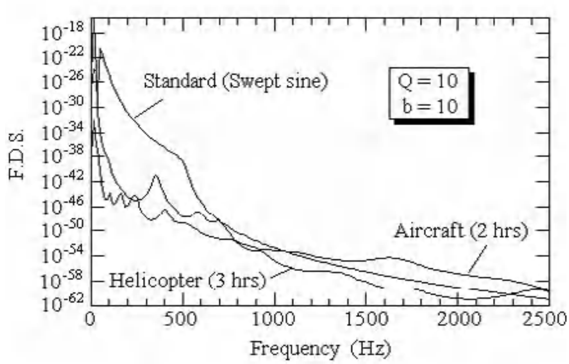


Figure 13.2. *Comparison of the severity of plane and helicopter vibrations and of the standard on the basis of their FDS*

13.1.2. *Comparison of the severity of two standards*

The spectra of Figures 13.3 and 13.4 are used to compare the severity of two standards, one defined by a random vibration and the other by a swept sine excitation:

– random vibration lasting 1 hour with PSD of:

20 Hz – 100 Hz: 1 $(\text{m s}^{-2})^2$/Hz

100 Hz – 500 Hz: 2 $(\text{m s}^{-2})^2$/Hz

– logarithmic swept sine lasting 3 hours such as:

10 Hz – 200 Hz: 10 m s^{-2}

200 Hz – 600 Hz: 60 m s^{-2}

Without these spectra, the comparison would not be easy.

Example 13.2.

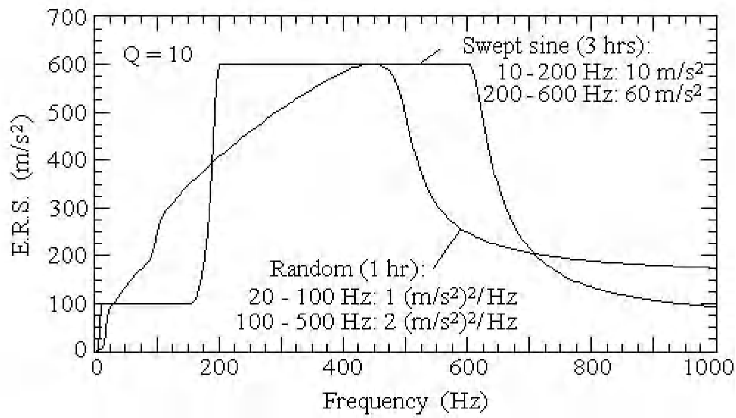


Figure 13.3. *Comparison of the severity of a random vibration test and of swept sine vibrations using their ERS*

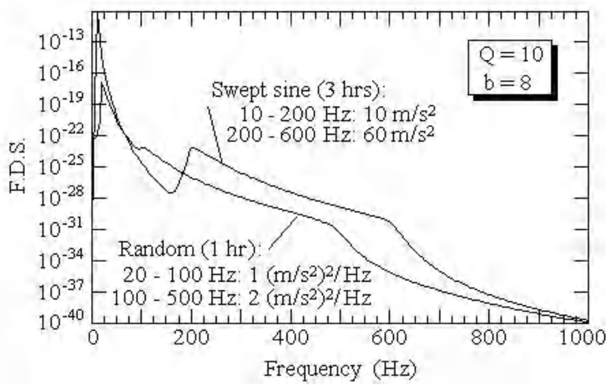


Figure 13.4. *Comparison of the severity of a random vibration test and swept sine vibrations using their FDS*

13.1.3. *Comparison of earthquake severity*

An earthquake can be compared to a shock phenomenon, and its severity is generally evaluated by using its shock response spectrum (keep in mind that the SRS was initially developed for the study of seismic effects). There are, however, documents describing seismic shocks by their power spectral density. We should not use this type of analysis, as the signal is not stationary. The PSD will, at most, give a qualitative indication of the frequency content.

Comparing seismic shocks is difficult from these two spectra. It is possible, however, to calculate an ERS or a URS with the PSD that can be compared directly to the SRS of the seismic shock.

It is possible to calculate the PSD corresponding to a given ERS or, conversely, to calculate the ERS of a given PSD. This correspondence should make it possible to solve the problem of comparing different standards presented in a different way. However, we must be very careful in this methodology.

13.2. Swept sine excitation – random vibration transformation

Although not at all recommended, it may be necessary to transform a test by swept sine excitation to a test by random vibrations. This operation is carried out as in the above section, starting out with the swept sine fatigue damage spectrum.

Example 13.3.

Figure 13.5 shows the fatigue damage spectrum of a (logarithmic) swept sine excitation for a duration of 3 hours:

$$10 \text{ Hz} - 200 \text{ Hz}: \; 10 \text{ m s}^{-2}$$

$$200 \text{ Hz} - 600 \text{ Hz}: \; 60 \text{ m s}^{-2}$$

and the equivalent random vibration (duration 2 hours) obtained for Q = 10 and b = 8. The reverse transformation is also possible (looking for a swept sine test equivalent to a random vibration defined by its PSD).

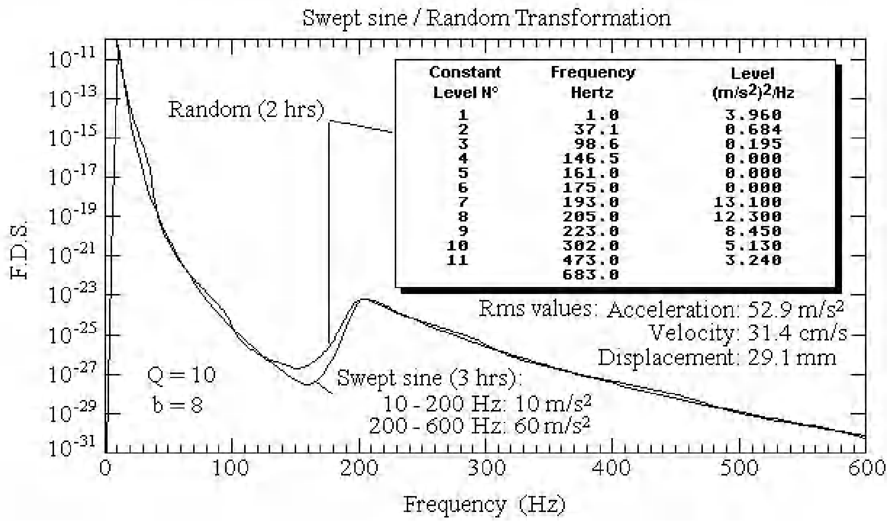


Figure 13.5. *Random vibration of the same severity (in terms of fatigue) as a swept sine vibration*

Example 13.4.

Figure 13.6 shows the power spectral density of a random vibration measured on the floor of a helicopter. It is assumed that the duration of this vibration was 1 hour.

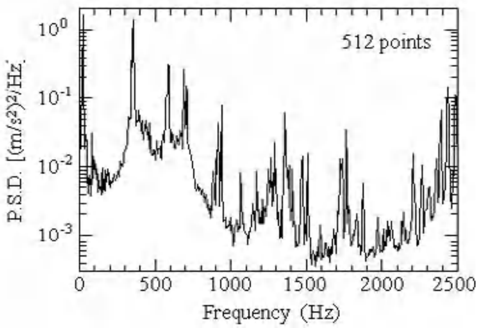


Figure 13.6. *PSD of a vibration measured on a helicopter*

The search for a swept sine type vibration, equivalent in terms of damage, leads to the test described in Table 13.1.

Frequency (Hz)	Acceleration peak (m s^{-2})	
	Duration: 1 hour	Duration: 162 s
1 – 20	0.1	0.14
20 – 330	2.0	2.7
330 – 360	15.0	20.5
360 – 550	7.3	10.0
550 – 730	9.3	12.7
730 – 970	5.5	7.5
970 – 1250	4.5	6.2
1250 – 1400	5.5	7.5
1400 – 2000	4.6	6.3

Table 13.1. *Swept sine equivalent in fatigue to one hour of helicopter vibrations*

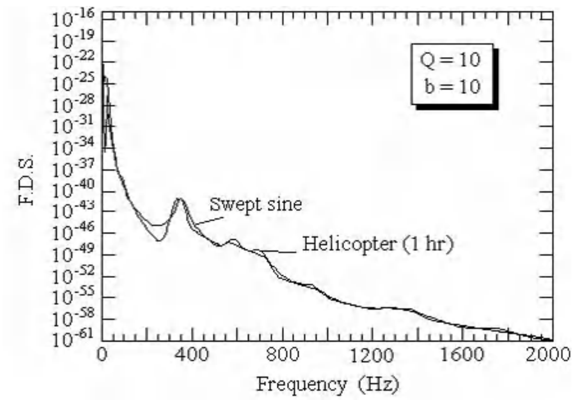


Figure 13.7. *FDS of the helicopter vibration (1 hour) and of the equivalent swept sine*

The calculations were made for Q = 10, b = 10, for a swept sine vibration time of 1 hour (duration of the random vibration), then 162 s.

Figure 13.7 shows the fatigue damage spectra of the real random vibration and of the equivalent swept sine vibration (duration 1 hour or 162 s). The relatively small difference between the spectra could still be reduced even more by increasing the number of levels.

Figure 13.8 shows the corresponding extreme response spectra. A large difference between the random and 1 hour swept sine spectra can be seen.

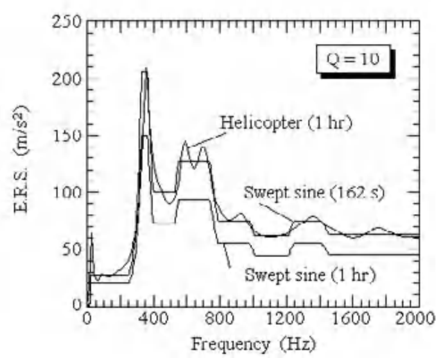


Figure 13.8. *ERS of the helicopter vibration (1 h) and of the equivalent swept sine of a duration of 1 h and 162 s*

For the extreme response spectra to be similar, it is necessary to multiply the amplitude of the spectrum (and the swept sine excitation) by a factor of about 1.364. To preserve the same damage by fatigue, it is necessary to divide the duration (1 hour) by $(1.364)^{10} \approx 22.22\ldots$, which amounts to a duration of 162 s. This duration is too short to allow the swept sine vibration to correctly excite the mechanical system. However, if the random vibrations were applied for several hours, an acceptable sweeping duration could be obtained.

This example shows one of the difficulties encountered in this type of problem: determination of an equivalence for the two criteria – extreme response spectra and fatigue damage spectra, leading to a very short swept sine test duration (or conversely, to a very long random vibration duration if the initial vibration is sinusoidal).

In addition, the result is very sensitive to the b and Q parameters. It may be noted here that the extreme response spectrum of a swept sine vibration varies proportionally in relation to Q. The ERS of a random vibration varies in relation to $\sqrt{Q}$. This property can easily be demonstrated in the case of the response of a linear single-degree-of-freedom system subjected to a white noise type random vibration with a PSD of G_0. The rms response is then:

$$\omega_0^2\ z_{rms} = \sqrt{\frac{\pi\, Q\, G_0}{2}\, f_0} \qquad [13.1]$$

and the extreme response is approximately equal to:

$$ERS \approx \sqrt{\frac{\pi\, Q\, G_0\, f_0}{2}}\sqrt{2\ \ln f_0\ T} \qquad [13.2]$$

(T = time during which the random vibration is applied).

Example 13.5.

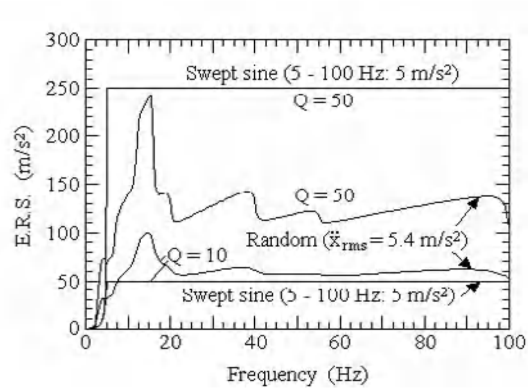


Figure 13.9. *Influence of Q factor on the ERS of random vibrations and swept sine vibrations*

Figure 13.9 illustrates this phenomenon. It shows the extreme response spectra for:

– a swept sine vibration of 5 m s^{-2} between 5 Hz and 100 Hz, for Q = 10 and Q = 50, respectively;

– a random vibration with an rms value of 5.42 m s^{-2}, characterized by the following PSD (Q = 10 and 50, duration 4 hours):

Frequency (Hz)	Amplitude [(m s^{-2})^2/Hz)]	rms value (m s^{-2})
3 – 4.5	0.781	
4.5 – 7	0.520	
7 – 12	1.041	
12 – 16	2.083	
16 – 20	0.520	5.416
20 – 40	0.260	
40 – 55	0.130	
55 – 100	0.091	

Table 13.2. *Values of PSD used in the example in Figure 13.9*

13.3. Definition of a random vibration with the same severity as a series of shocks

ERSs and FDSs can be used to replace one vibration with another of a different nature, with equal fatigue damage, by applying similar instantaneous stresses.

We can then for example find the characteristics of a swept sine vibration with the same severity as a random vibration, or conversely, the characteristics of a random vibration equivalent to a swept sine. We can also determine a random vibration (defined by its PSD) with similar severity as that of a large number of repeated shocks. This last operation can be interesting for studies in fatigue behavior under a large number of shocks, a random vibration was easier to reproduce than a large series of shocks.

All these equivalences are obviously possible, but require a few conditions:

– an obvious first condition supposes that the equivalent test can be achieved by usual test facilities: maximum force, displacement, as well as vibration duration, which should not be too long, or too short (this is sometimes a problem during the transformation of a random swept sine vibration with equal maximum stress);

– a second condition, for when the excitation nature changes (swept sine → random vibration, shocks → random vibration, etc.). It then becomes necessary to know the numeric values of parameter b, as well as Q factor [PER 03]. This last parameter is rarely known, there is therefore a significant limit for using these transformations;

– even if all these conditions are respected, and maximum stresses and fatigue damage are reproduced, we should not always replace a random vibration by a swept sine. In a random vibration test, we apply a load at each moment that includes a large spectrum and excite all resonance frequencies simultaneously, whereas a swept sine vibration excites resonances one after another.

Example 13.6.

This example is intended to illustrate the importance of the Q factor in shock equivalences – random vibrations. Here, we propose to find a PSD with the same severity as a shock repeated 20,000 times on a component.

The shock is illustrated in Figure 13.10, defined in 2,500 points. The equivalence was achieved by following the following process:

– FDS calculation of the shock applied 20,000 times and its SRS between 10 Hz and 2,000 Hz with a logarithmic step, for $b = 8$ and Q consecutively equal to 10 and 20;

– characteristic search of a PSD defined by 40 values with an FDS closer to shocks (for each Q value), as the duration of the test is chosen so that the random vibration ERS is close to the shock SRS, i.e. 20 hours (we search to generate on average during the vibration the largest peak created by one shock at each natural frequency; we should consider the random vibration URS if the criterion was to not generate peaks in the response higher than those observed under the shock).

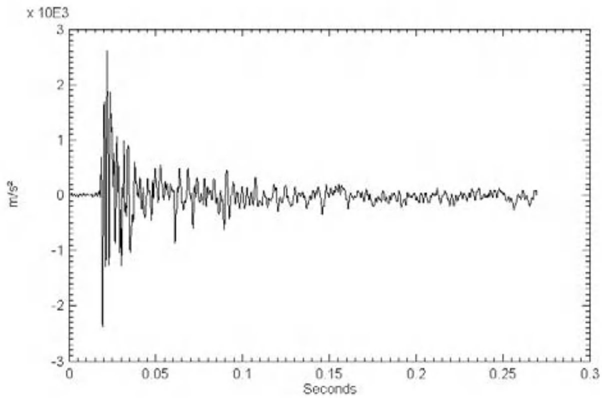


Figure 13.10. *Shock studied, applied 20,000 times*

Shock and equivalent random vibration FDS are illustrated in Figures 13.11 (for Q = 10) and 13.12 (for Q = 20).

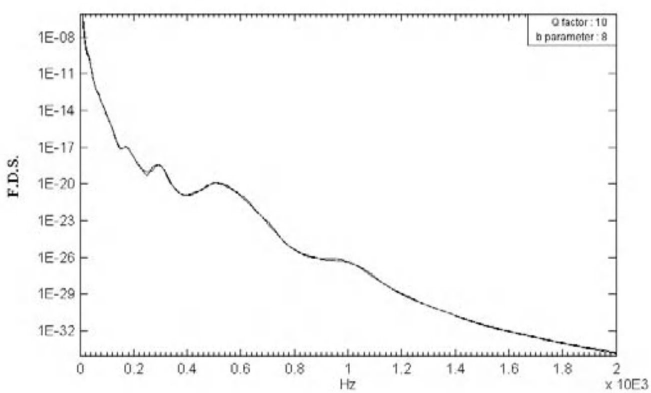


Figure 13.11. *FDS of 20,000 shocks and equivalent random vibration for Q = 10*

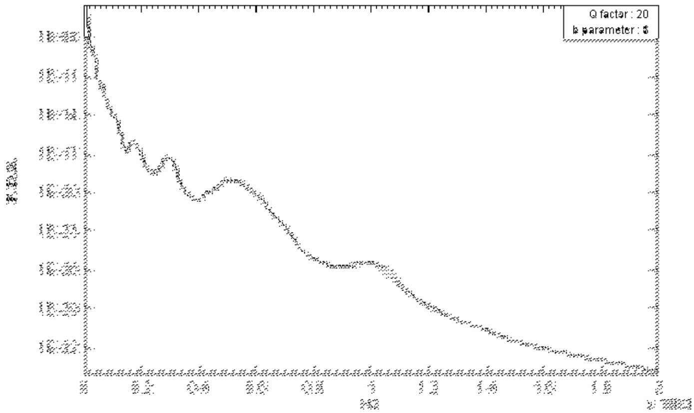


Figure 13.12. *FDS of 20,000 shocks and equivalent random vibration for Q = 20*

The ERSs of this vibration are compared to the shock SRS in Figures 13.13 (Q = 10) and 13.14 (Q = 20).

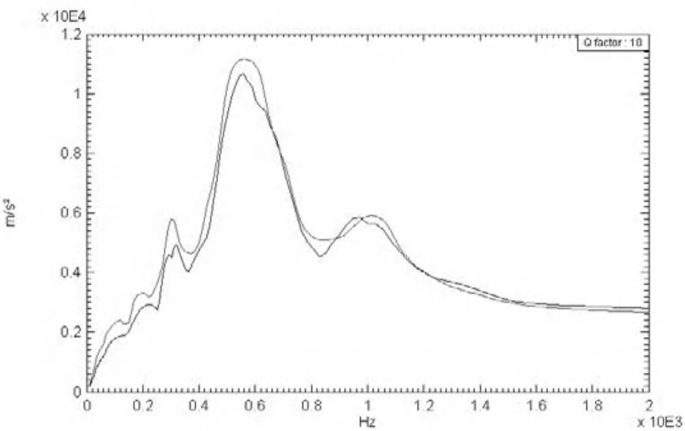


Figure 13.13. *Shock SRS of Figure 13.10 and ERS of equivalent random vibration (Q = 10)*

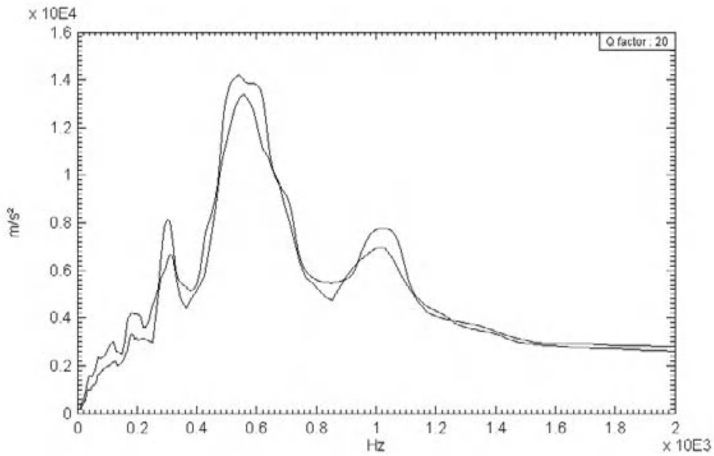


Figure 13.14. *Shock SRS of Figure 13.10 and ERS of equivalent random vibration (Q = 20)*

Figure 13.15 shows the PSDs of equivalent vibrations determined for these two Q factor values. We can observe a slight difference in results, the rms value goes from 398.3 m s^{-2} for Q = 10 to 345 m s^{-2} for Q = 20.

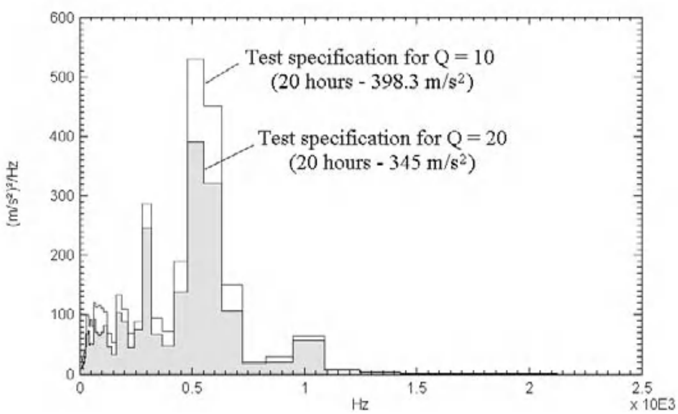


Figure 13.15. *PSDs of equivalent random vibration determined for Q = 10 and for Q = 20*

13.4. Writing a specification only from an ERS (or a URS)

For certain applications, it may be preferable to determine the characteristics of the specification (PSD) from an extreme response spectrum or a shock response spectrum.

The envelope ERS of ERSs from several life profile events can be used to calculate a test specification defined by a PSD when we can consider that the most important risk is failure caused by a stress that is too large.

The method consists of finding the characteristics of a PSD producing the same ERS as the life profile ERS, setting an arbitrary knowing that the ERS is a spectrum that is not sensitive to the duration of the vibration application (section 6.1).

It is preferable that the application duration of the specification thus defined should be small, or at maximum equal to the duration of the most severe event of the life profile, so that there is no risk of fatigue fracture that would not be representative. The goal of the test is simply to verify that the material withstands the strongest levels encountered in the real environment.

This is, for instance, the case for transformation of a seismic specification expressed as an SRS to be compared with another seismic specification defined by a PSD.

This procedure cannot be used unless:

– there are only parallel sub-situations and/or situations;

– fatigue damage is not a critical criterion.

Two methods are also possible here:

– matrix inversion;

– iteration.

13.4.1. *Matrix inversion method*

13.4.1.1. *Search for a specification from an ERS*

The specification is then calculated from an ERS as follows [LAL 88]. Knowing that the extreme response can be expressed as follows in its simplified form (by supposing that $n_0^+ \approx f_0$):

$$\mathrm{ERS} = \omega_0^2 \; z_{sup} \approx \omega_0^2 \; z_{rms} \; \sqrt{2 \; \ln(f_0 \; T)} \qquad [13.3]$$

where z_{rms} is the rms response displacement given by (Volume 3, [8.79])

$$z_{rms}^2 = \frac{\pi}{4\xi(2\pi)^4 \, f_0^3} \sum_{j=1}^{n} a_j \, G_j \qquad [13.4]$$

each line of the extreme response spectrum satisfies the following equation

$$\mathrm{ERS}_i \approx \sqrt{\frac{\omega_{0i} \ln(f_{0i} \; T)}{4\xi} \sum_{j=1}^{n} a_{i,j} \, G_j} \qquad [13.5]$$

For a PSD defined by horizontal straight line segments,

$$\mathrm{ERS}_i \approx \sqrt{\frac{\omega_{0i} \ln(f_{0i} \; T)}{4\xi} \sum_{j} G_j \left[I_0(h_{i,j+1}) - I_0(h_{i,j}) \right]} \qquad [13.6]$$

In its matrix form, this equation is

$$\mathrm{ERS}_2 = B \;\; G \qquad [13.7]$$

and therefore the values of $G(f)$.

If the PSD thereby determined is intended to be used as a specification to control a test, it must be kept in mind that the signal which will be delivered by the control unit, of a duration of about 30 s for a seismic shock, will not necessarily have the same ERS as that at origin. It can simply be confirmed that the mean of the ERSs of a great number of signals generated from the PSD would be close to the reference ERS.

Example 13.7.

Consider the acceleration signal measured during an earthquake represented in Figure 13.16. The SRS of this seismic shock, calculated for Q = 10, can be used to establish a specification expressed in the form of a random vibration that will produce the same extreme responses as the seismic shock. Figure 13.17 shows the PSD thus calculated for 30 s.

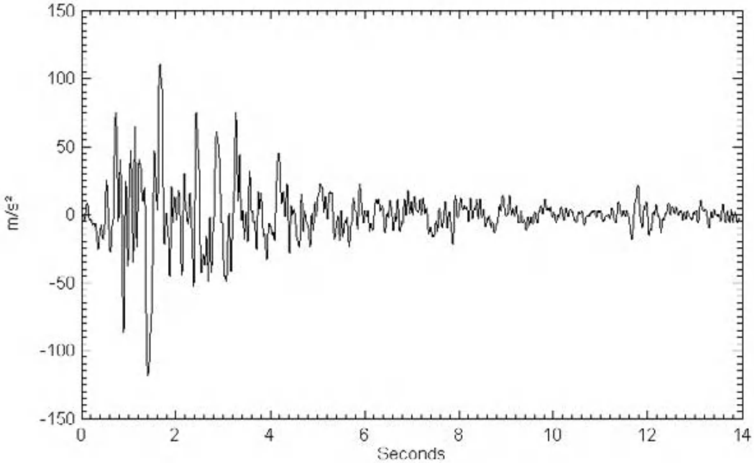


Figure 13.16. *Acceleration signal measured during a seism*

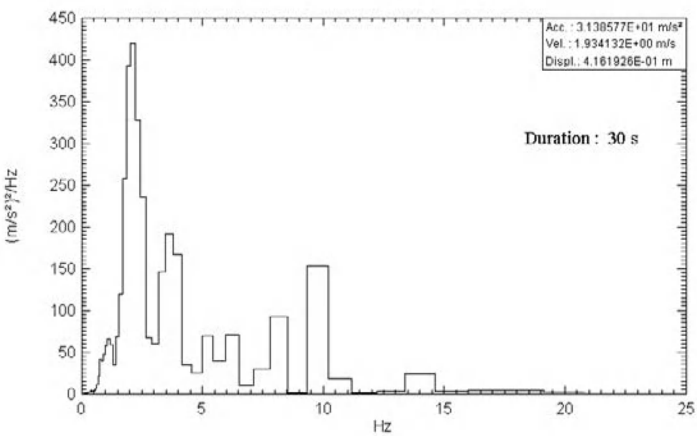


Figure 13.17. *Specification (PSD) lasting 30 s with an ERS very close to the SRS of the seismic shock*

The ERS of this specification is very close of the SRS of the seismic shock (Figure 13.18).

This transformation consists of replacing a non-stationary phenomenon by a stationary vibration producing the same ERS. The specification and seismic shock FDS are different because the duration was chosen *a priori*.

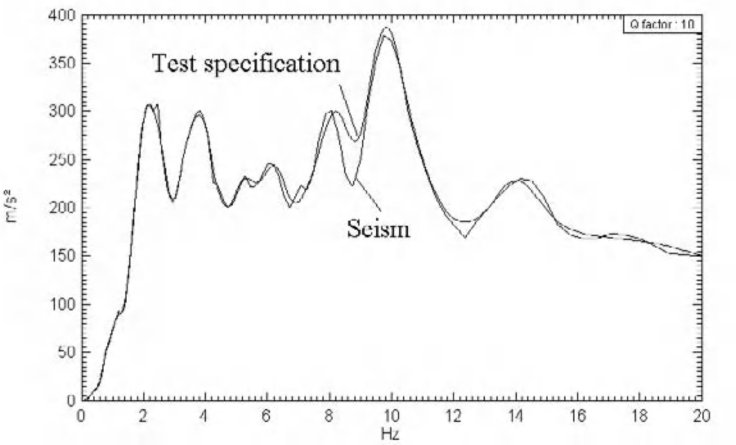


Figure 13.18. *Comparison of the specification ERS and seism SRS*

The random vibration defined from the ERS will produce on average in the same 30 s duration the same stress as the seismic shock. We could have searched for the PSD with the seismic shock SRS as a URS, with two possible approaches for the choice of the up-crossing risk:

– a very low risk that the response created by the random vibration would be always lower than the SRS;

– or on the contrary with a high risk to make sure that a higher response than indicated by SRS be produced.

In the same way as for the FDS, and for the same reasons, the extreme response spectra can be plotted in a modified form by multiplying the ordinates by

$$N = \frac{1}{\sqrt{f_0 \ \ln f_0 \ T}} \qquad [13.8]$$

Knowing that $n_0^+ \approx f_0$, the relationship [13.5] can in fact, after modifications, be expressed as,

$$ERS \approx \sqrt{\frac{\pi}{2\ \xi}}\ \sqrt{f_0\ \ln(f_0\ T)}\ \sqrt{\sum_i G_i\ \left[I_0(h_{i+1}) - I_0(h_i)\right]}$$

$$ERS \approx \frac{1}{N}\ \sqrt{\frac{\pi}{2\ \xi}}\ \sqrt{\sum_i G_i\ \left[I_0(h_{i+1}) - I_0(h_i)\right]} \qquad [13.9]$$

The modified spectrum is plotted by setting the quantity for N.ERS on the *y*-axis.

13.4.1.2. *Search for a specification from a URS*

The reasoning is the same from expression [2.27]:

$$URS = ERS \sqrt{\frac{-\ln\left[1-(1-\alpha)^{1/n_0^+ T}\right]}{\ln\left(n_0^+\ T\right)}} \qquad [13.10]$$

Relation [13.5] becomes, by retaining the hypothesis $n_0^+ \approx f_0$:

$$URS_i \approx \sqrt{\frac{-\omega_{0i}\ \ln\left[1-(1-\alpha)^{1/f_{0i}T}\right]}{4\xi}\ \sum_{j=1}^{n} a_{i,j}\, G_j} \qquad [13.11]$$

and [13.6]:

$$URS_i \approx \sqrt{\frac{-\omega_{0i}\ \ln\left[1-(1-\alpha)^{1/f_{0i}T}\right]}{4\xi}\ \sum_j G_j\left[I_0(h_{i,j+1}) - I_0(h_{i,j})\right]} \qquad [13.12]$$

expressions that can take an analog matrix form at [13.7].

Modified spectrum:

Relation [13.9] can still apply for the URS with, here:

$$N = \frac{1}{\sqrt{-\ln\left[1-(1-\alpha)^{1/f_{0i}T}\right]}} \qquad [13.13]$$

13.4.2. *Method by iteration*

Similarly to the FDS, an iteration process can be used to define a specification from an ERS, the correction of PSD amplitudes should be carried out according to relation:

$$PSD_1(f_i) = PSD_0(f_i)\left(\frac{ERS_0(f_i)}{ERS_1(f_i)}\right)^2 \qquad [13.14]$$

A few iterations are enough to obtain PSD amplitudes. More elaborate rules could be used to try to increase the speed of convergence.

13.5. Establishment of a swept sine vibration specification

It is also possible to determine a vibration test specification of the swept sine type on the basis of the reference fatigue damage spectrum. The method is in principle very close to that used for a random vibration specification.

Two strategies are also possible here:

– to choose n points among N points of the damage spectrum ($n \le N$). These points will be regarded as centered with respect to the definition intervals (f_j, f_{j+1}) of the swept sine (Figure 13.19). The terminal frequencies f_j and f_{j+1} arise therefore from the frequencies f_{0i} considered on the FDS;

– to choose the terminal frequencies of the intervals (f_j, f_{j+1}) *a priori*, then to read the values of the damage on the FDS at the frequencies $f_{0i} = \dfrac{f_j + f_{j+1}}{2}$.

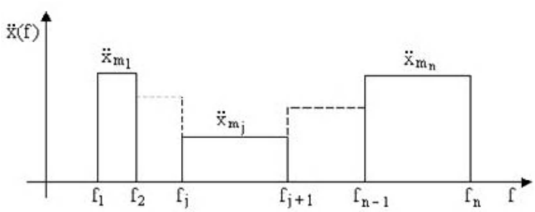


Figure 13.19. *Specification of swept sine vibration*

In every case, the swept sine therefore has the same number of levels as the number of points chosen for the FDS.

Relationship [3.66], which makes it possible to calculate fatigue damage at the natural frequency f_{0i} starting from a swept sine vibration composed of n levels, is, in the case of a sweeping logarithmic sweep of duration t_b, expressed as

$$\sum_{j=1}^{n} \frac{f_{0i}\ T_1\ \ddot{x}_{m\,j}^{b}}{(2\pi f_{0i})^b} \int_{h_j}^{h_{j+1}} \frac{dh}{\left[\left(1-h^2\right)^2+\frac{h^2}{Q^2}\right]^{\frac{b}{2}}} = D_i \qquad [13.15]$$

where $T_1 = \dfrac{t_b}{\ln \dfrac{f_n}{f_1}}$, $h_j = \dfrac{f_j}{f_{0i}}$, $h_{j+1} = \dfrac{f_{j+1}}{f_{0i}}$ $(1 \le j \le n)$ and $D_i = D(f_{0i})$ (value of the fatigue damage at the frequency f_{0i}).

This expression can be written in the form

$$\sum_j a_{i\,j}\ \ddot{x}_{m\,j}^{b} = D_i \qquad [13.16]$$

if we set

$$a_{ij} = \frac{f_{0i}\ T_1}{(2\pi f_{0i})^b} \int_{h_j}^{h_{j+1}} \frac{dh}{\left[\left(1-h^2\right)^2+\frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \qquad [13.17]$$

Relationship [13.16] has a matrix form

$$A\ \ddot{X}_b = D \qquad [13.18]$$

($\ddot{X}_b$ being a column matrix, each term of which is equal to the amplitude $\ddot{x}_{m\,i}$ to the power b). Yielding the amplitudes of the swept sine levels

$$\ddot{X}_b = A^{-1}\ D \qquad [13.19]$$

This calculation therefore requires matrix A to be built starting with the terms a_{ij} as defined by [13.17], to invert this matrix A and to multiply it by the column matrix D (array of the chosen damage values).

In linear sweeping, with the same notations, the damage created by a swept sine vibration is given by

$$\sum_{j=1}^{n} \frac{f_{0i}^2\, t_b\, \ddot{x}_{mj}^b}{(2\pi f_{0i})^b (f_n - f_1)} \int_{h_j}^{h_{j+1}} \frac{h\, dh}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} = D_i \qquad [13.20]$$

yielding, where

$$b_{ij} = \frac{f_{0i}^2\, t_{b1}}{(2\pi f_{0i})^b\, (f_n - f_1)} \int_{h_j}^{h_{j+1}} \frac{h\, dh}{\left[\left(1-h^2\right)^2 + \frac{h^2}{Q^2}\right]^{\frac{b}{2}}} \qquad [13.21]$$

$$B\, \ddot{X}_b = D \qquad [13.22]$$

and

$$\ddot{X}_b = B^{-1}\, D \qquad [13.23]$$

However, this process is in general highly inadvisable, for the following reasons:

– a swept sine vibration represents the effects of the vibrations measured in the real environment badly, these are of a rather broad band random nature and therefore simultaneously excite all resonances;

– the values of the amplitudes of the swept sine specification determined from a damage spectrum representing random vibrations are sensitive at the same time to the value of parameter b and the Q factor, which must therefore be correctly estimated.

If however this method must be used, it is important to select the sweep duration t_b in such a way that the extreme response spectrum of the specification obtained is close to the extreme response spectrum of the real environment (associated with the reference FDS).

Specification Development: Second Edition – Volume 5
Christian Lalanne

Appendices

A1. Comparison of hypotheses used in tailoring methods using PSD envelope and fatigue damage equivalence

A1.1. *Power spectral density envelope method*

Principle

This method consists of:

– tracing a simple envelope with several PSDs representing measures of a single event (aircraft transport, cruising phase) or several PSDs representative of events different from the real environment (aircraft transport, cruising, take-off and landing phases);

– reducing duration of the resulting specification in relation to the real environment. Standards propose a rule with the following form for this reduction:

$$\ddot{x}_{\text{Reduced duration}} = \ddot{x}_{\text{Real duration}} \left(\frac{\text{Real duration}}{\text{Reduced duration}} \right)^{\frac{1}{b}} \qquad \text{[A1.1]}$$

where parameter b varies, according to the standard, between 5 and 9. In this relation, $\ddot{x}$ may be the vibration amplitude (swept sine and sinusoidal vibration) or the rms value (random vibration). In this last case, we can involve amplitude G of the PSD. This amplitude varies with the square of the vibration's rms value. Hence:

$$G_{\text{Reduced duration}} = G_{\text{Real duration}} \left(\frac{\text{Real duration}}{\text{Reduced duration}} \right)^{\frac{2}{b}} \qquad \text{[A1.2]}$$

Parameter b involved in these expressions is set at different values based on the standards (Table A1.1);

Standard	**b**
DEF STAN 0035 (Part 5) [DEF 86]	5
Air 7304 (1972) [NOR 72]	8
Air 7306 [NOR 87]	8 (but can be adapted)
MIL–STD 810 F [MIL 97]	Random: 8, sine: 6 (can be adapted)
GAM.EG 13 [GAM 86]	Open choice

Table A1.1. *Some parameter b values mandated in the standards*

– tracing an SRS envelope of real environment shocks. The specification is either the envelope SRS or a signal based on time where the SRS is close to the envelope.

Implicit hypotheses

Different hypotheses are necessary for the establishment of the duration reduction rule in the standards. This rule comes from the law proposed by Basquin (*hypothesis 1*) to analytically represent the Wöhler curve in its linear part. It connects the number of cycles at failure of test bars of a given material and sinusoidal stress amplitude applied to it:

$$N \sigma^b = C \qquad \text{[A1.3]}$$

where b and C are constants characteristics of the material considered.

Thus, for two levels of stress, we have:

$$N_1 \sigma_1^b = N_2 \sigma_2^b \qquad \text{[A1.4]}$$

It is assumed that this relation still applies for numbers of cycles not leading to fracture. If $n_1 < N_1$ and $n_2 < N_2$, we then admit that there is even damage if:

$$n_1 \sigma_1^b = n_2 \sigma_2^b \qquad \text{[A1.5]}$$

The ratio of relations [A1.4] and [A1.5] leads to:

$$\frac{n_1}{N_1} = \frac{n_2}{N_2} \qquad [A1.6]$$

Here we find the definition of damage and their equality for two stress levels according to Miner (*hypothesis 2*).

Knowing that the number of cycles is proportional to vibration duration T (for a sinusoidal vibration with frequency f, $n = f\ T$), relation [A1.5] can be written as:

$$T_1\ \sigma_1^b = T_2\ \sigma_2^b \qquad [A1.7]$$

To obtain relation [A1.1], it is necessary to also assume that the amplitude of stress σ is proportional to the amplitude of applied acceleration (*hypothesis 3*):

$$\sigma = \alpha\ \ddot{x} \qquad [A1.8]$$

Relation [A1.5] can also be written for two other stress levels σ_3 and σ_4:

$$n_3\ \sigma_3^b = n_4\ \sigma_4^b \qquad [A1.9]$$

also leading to:

$$\frac{n_3}{N_3} = \frac{n_4}{N_4} \qquad [A1.10]$$

Using the sum of [A1.6] and [A1.10], we obtain:

$$\frac{n_1}{N_1} + \frac{n_3}{N_3} = \frac{n_2}{N_2} + \frac{n_4}{N_4} \qquad [A1.11]$$

The damage created by n_1 cycles at stress σ_1 and n_3 cycles at stress σ_3 is then equal to the damage produced by n_2 cycles at stress σ_2 and n_4 cycles at stress σ_4.

Total damage created by cycles at two levels of stress is equal to the sum of partial damages (*hypothesis 4*: damages add up in a linear way).

The calculation of an SRS relies on the following hypotheses:

– one-degree-of-freedom model (*hypothesis 5*);

– linear system.

The severity comparison of two shocks with the help of their SRS assumes that, if at a natural frequency (noted in abscissa) their two SRSs are equal, both shocks will have the same effects, not only on the one-degree-of-freedom system serving as reference, but also on the real structure, not necessarily linear or at one degree of freedom.

If we consider the relative displacements SRSs, where we read, in ordinate the largest relative displacement (multiplied by $(2\pi f_0)^2$) of the mass of the one degree-of-freedom system, this implies that:

– the relative displacement of the mass in relation to its support is proportional to the acceleration defining the excitation (*hypothesis 6*);

– the stress (representative of the severity) is proportional to relative displacement (*hypothesis 7*).

These two hypotheses are equivalent to hypothesis 3.

A1.2. *Method using ERSs and FDSs*

Hypotheses

The ERS has the same definition as the SRS: largest response (relative displacement) of a one degree-of-freedom linear system submitted to an unspecified vibration (random or sinusoidal). Hypotheses necessary for its calculation are strictly SRSs:

– reference one-degree-of-freedom system (*hypothesis a*);

– relative displacement of the mass in relation to its support is proportional to the acceleration defining the excitation (*hypothesis b*);

– the stress (representative of the severity) is proportional to relative displacement (*hypothesis c*).

Besides the ERS hypotheses, the FDS calculation assumes that:

– the Wöhler curve is represented by Basquin's law (*hypothesis d*);

– damage is defined according to Miner (*hypothesis e*);

– damage is linearly cumulative (*hypothesis f*).

A1.3. *Comparison of hypotheses*

In the case of the ERS/FDS method, the definition of a reduced duration test with equal fatigue damage is based on the expressions of damage deduced of Basquin's law. For sinusoidal vibrations, we then have:

$$D = \frac{K^b}{C} f\,T\,z_{max}^b = \text{Constant}\;N\,z_{max}^b \qquad [A1.12]$$

where relative displacement z_{max} is proportional to the stress.

It uses the same hypotheses as those considered more implicitly in the traditional method. It involves parameter b and has the same problems in the choice of its value in the case of structures made up of several materials.

This parameter is set at a certain value in standards, with the consequences of a systematic over-test (aluminum or steel structures for b = 5) or an under-test (electronic for b = 8), whereas it can *be chosen more relevantly with the ERS/FDS method.*

	PSD/SRS envelope method	**ERS/FDS/SRS method**
Basquin's law	Hypothesis 1	Hypothesis d
Definition of fatigue damage according to Miner	Hypothesis 2	Hypothesis e
Stress/acceleration proportionality	Hypothesis 3	Hypotheses b and c
One degree-of-freedom model	Hypothesis 5	Hypothesis a
Relative response displacement/acceleration proportionality	Hypothesis 6	Hypothesis b
Stress/relative response displacement proportionality	Hypothesis 7	Hypothesis c
Linearly cumulative damages	Hypothesis 4	Hypothesis f

Table A1.2. *Comparison of hypotheses*

A2. Limitations of the traditional PSD envelope method – advantage of the method using ERS and FDS

The establishment of a specification from PSDs can be made:

– by considering a “smoothed” PSD envelope of the real environment, with a risk of over-testing linked to the more or less large outline of the envelope;

– or, to avoid this problem, by only retaining the “raw” envelope of PSDs (largest curve at each frequency, with no margin).

These two methodologies can lead to errors. They assume PSDs of real environment vibrations can be theoretically calculated, which is not always the case. These different problems are addressed in the following examples.

A2.1. *Specification defined as white noise respecting the rms value of the real environment PSD*

Example A2.1.

In order to simplify the presentation, the real environment is simply described by a single PSD that we will consider as a “raw” envelope of two PSDs relative to aircraft and helicopter transport each lasting 2 hours.

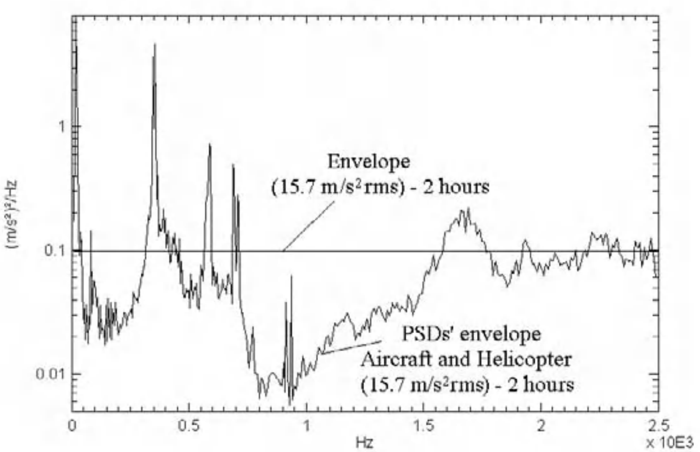


Figure A2.1. *PSD envelope of the “aircraft + helicopter” environment and simple specification with the same rms value*

The “specification” PSD is sometimes obtained by drawing a horizontal line with amplitude making it possible for the rms value of the “raw” envelope to be respected (Figure A2.1).

In what follows, we will compare severity differences between specification and original PSD with help from the only tool available based on mechanical criteria, ERSs and FDSs

ERSs and FDSs of the specification and real environment are drawn respectively in Figures A2.2 and A2.3. We can notice significant over-tests and under-tests at certain resonance frequencies.

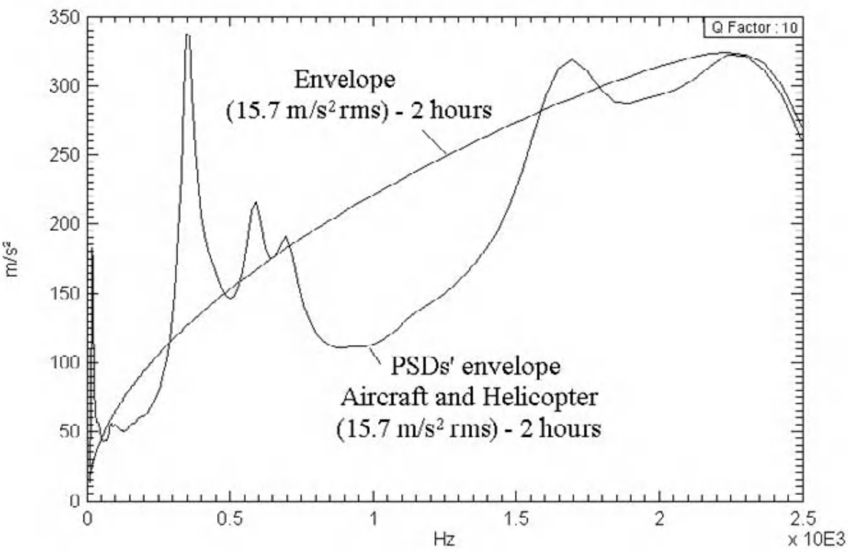


Figure A2.2. *ERS of the “aircraft + helicopter” vibration envelope and ERS of the specification with the same rms value*

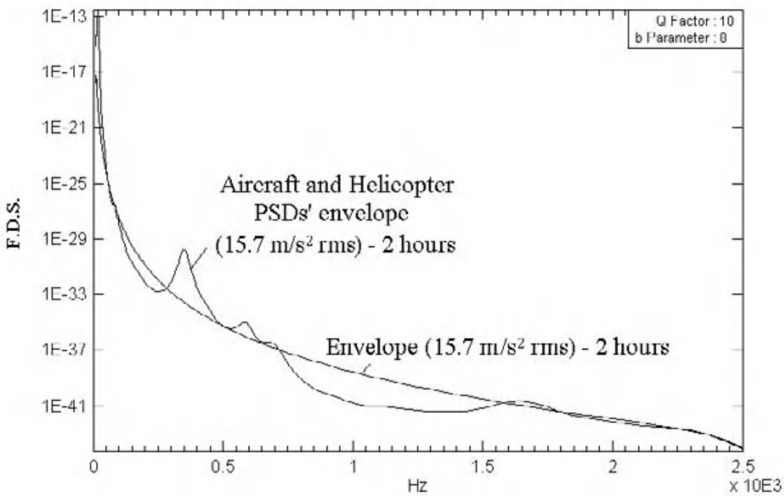


Figure A2.3. *FDS of the "aircraft + helicopter" vibration envelope and FDS of the specification with the same rms value*

A2.2. *Specification established by drawing several line segments enveloping PSDs of the real environment*

Another method may consist of tracing the envelope by respecting the appearance of real environment PSDs. This process is very subjective; several operators can have slightly different results, particularly concerning the rms value.

Example A2.2.

The envelope PSD in Figure A2.4 has an rms value equal to 27.6 m s^{-2}, much higher than that for the real environment (15.7 m s^{-2}).

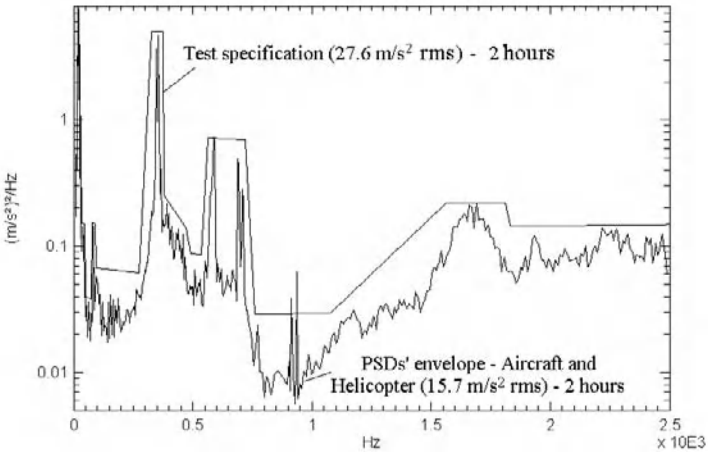


Figure A2.4. *Specification established by large PSD envelope of "aircraft + helicopter" vibrations*

The envelope drawn in Figure A2.5, closer to the real PSD, seems more accurate. It does, however, have a very large rms value (24.9 m s^{-2}).

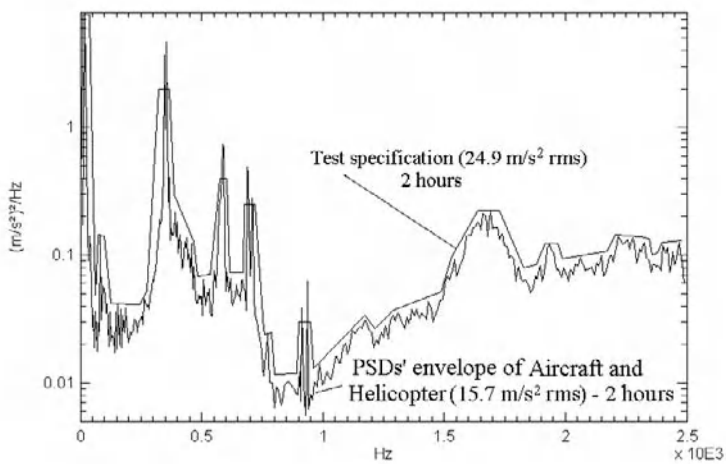


Figure A2.5. *Specification established by closer PSD envelope of "aircraft + helicopter" vibrations*

The ERS and FDS comparison of this last specification with the real environment confirms this over-test (Figures A2.6 and A2.7).

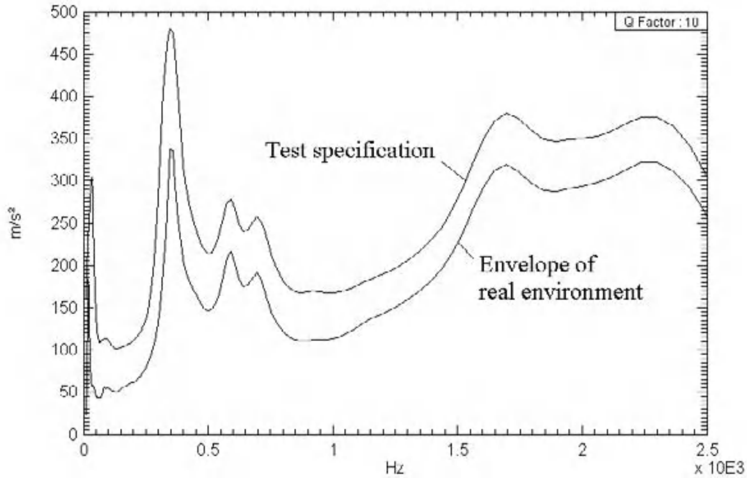


Figure A2.6. *ERS of "aircraft + helicopter" vibration envelope and specification ERS of Figure A2.5*

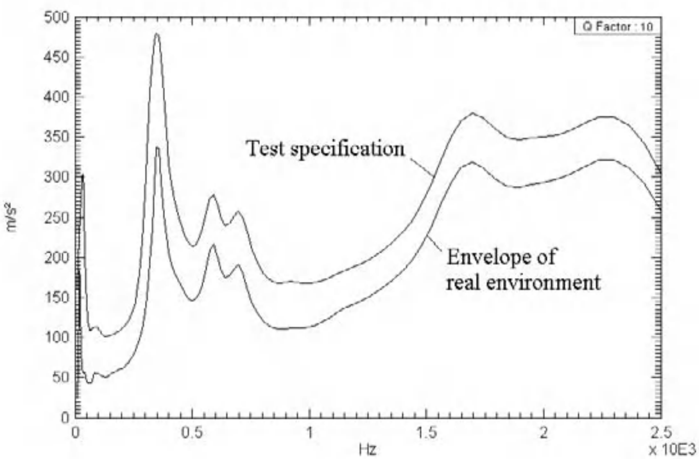


Figure A2.7. *FDS of "aircraft + helicopter" vibration envelope and specification FDS of Figure A2.5*

We can determine a smooth curve close to the reference PSD with a very close rms value. This work, needing to truncate peaks and give them adequate size, requires some experience to obtain a significant PSD defined using a small number of points.

The damage equivalence method easily leads to a specification with a severity that is very close to that of real vibratory environments. The duration choice can be guided by ERS comparison in order to reproduce stresses close to those experienced in real conditions on test materials.

Example A2.3.

The specification in Figure A2.8 was established from the sum of “aircraft” and “helicopter” vibration FDSs, for a 2-hour test duration.

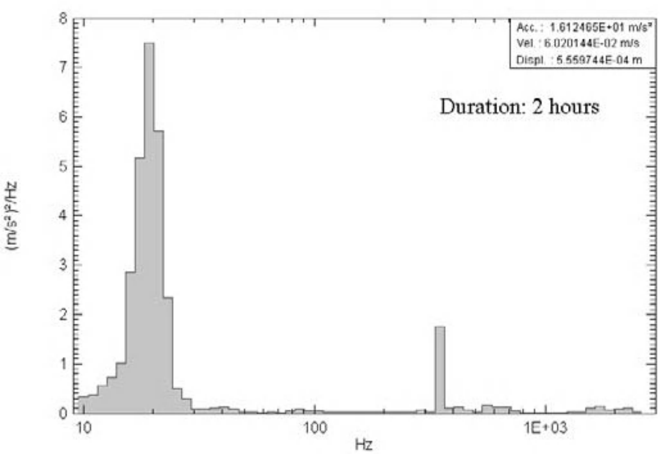


Figure A2.8. *Specification established with the damage equivalence method (duration: 2 hours) covering 2 hours of aircraft transport and 2 hours of helicopter transport*

ERSs and FDSs of this specification are compared to those of real “aircraft” and “helicopter” vibrations in Figures A2.9 and A2.10.

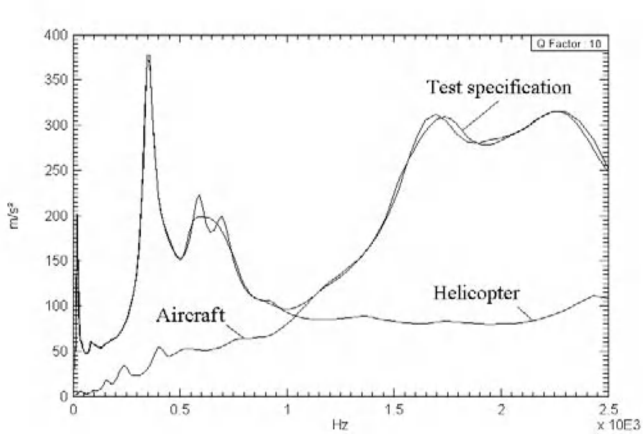


Figure A2.9. *ERS of "aircraft" and "helicopter" vibrations and specification ERS in Figure A2.8*

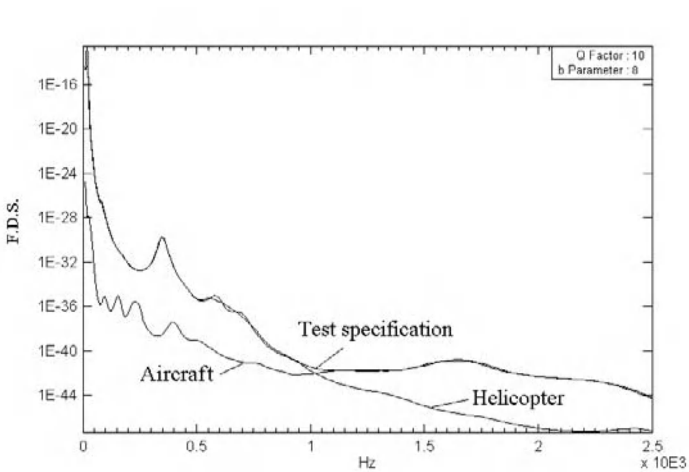


Figure A2.10. *FDS of "aircraft" and "helicopter" vibrations and specification FDS in Figure A2.8*

The PSDs envelope tracing is sometimes submitted to more precise rules. To establish a specification from calculated results (the method could also be used with measurements), R. Simmons [SIM 97] uses logarithmic axes and proposes the following conditions:

– the line segment slopes of the envelope PSD should be lower than 25 dB oct^{-1} or larger than –25 dB oct^{-1} (to specify more precisely according to the possibilities of the exciter used);

– frequency bands should be larger than 10 Hz;

– the PSD amplitude should not have values lower than those of a minimum spectrum;

– the narrow peaks should be truncated at mid height approximately (– 3 dB);

– the envelope must go down into the large "valleys";

– we must try to retain a global rms value of the specification lower than 1.25 times the rms value of the real environment.

Example A2.4.

To illustrate this methodology, R. Simmons [SIM 97] provides the example of Figure A2.11 where we find the reference PSD (from calculations), the PSD of the specification obtained by following the rules indicated and the minimum level to respect. We can verify that the rms value of the specification (9.31 m s^{-2}) does not exceed that of the reference PSD (7.45 m s^{-2}) by a factor higher than 1.25.

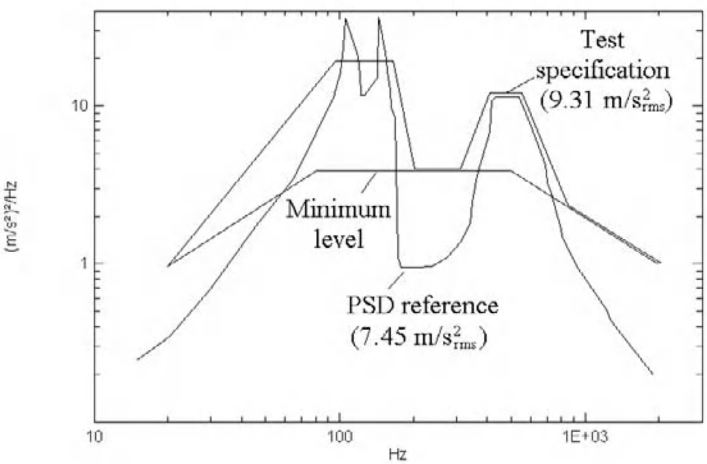


Figure A2.11. *Example of a specification established by respecting GSFC rules (NASA)*

Figures A2.12 and A2.13 enable us to compare ERSs and FDSs for this specification and the reference PSD.

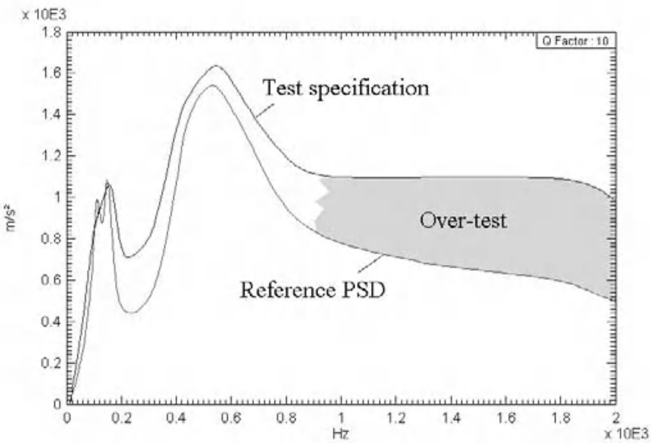


Figure A2.12. *ERS of the specification and reference PSD of Figure A2.11*

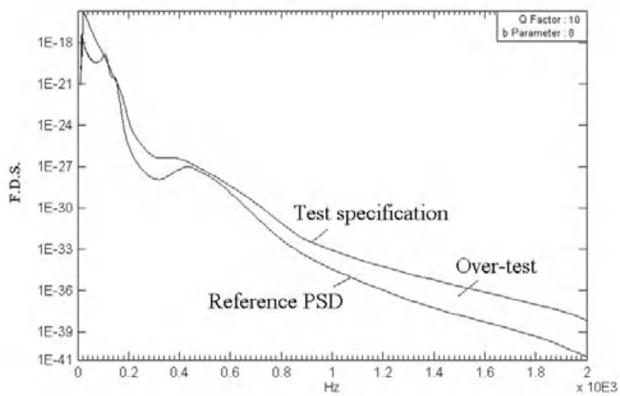


Figure A2.13. *FDS of the specification and reference PSD of Figure A2.11*

We can observe that:

– the increase in severity of the specification is very different based on the natural frequency of the tested material;

– the over-test is significant between 1,000 Hz and 2,000 Hz, since the maximum stress generated by the specification can exceed by a factor of approximately 1.7 that forecast by the initial calculation.

A2.3. ***"Raw" envelope of two (or more) PSD with very close frequency content, but with different amplitudes and durations.***

This a case that is quite common in practice, for which it seems the PSD envelope method can be used without apparent problems.

Producing an envelope of several PSDs obviously leads to the retention, at each frequency, of the highest amplitude PSD value. The resulting PSD necessarily has a higher rms value than that from each basic PSD (unless one of the PSDs is larger than all the others at each frequency).

Example A2.5.

Consider two PSDs characteristic of two phases of aircraft transport (Figure A2.14). The frequency ranges are very close. Durations are different and rms values respectively are 1.50 m s^{-2} and 1.71 m s^{-2}.

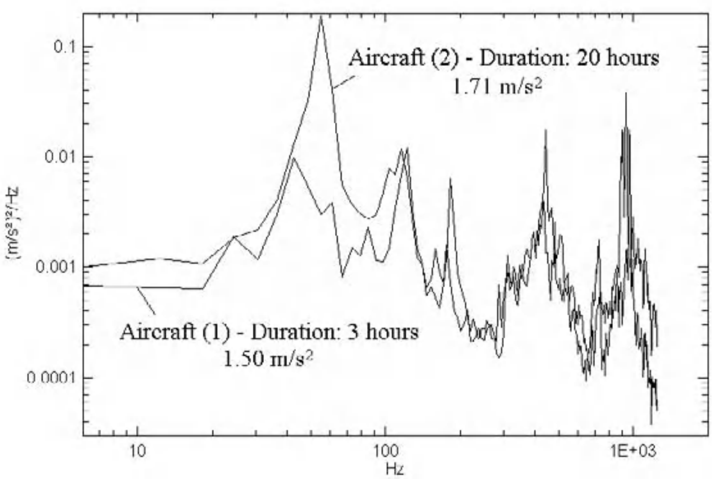


Figure A2.14. *PSD characteristic of two phases in an aircraft transport*

The envelope PSD is drawn in Figure A2.15. It is a strict envelope, without smoothing, to avoid any other artificial increase of the rms value, which is equal to 2.06 m s^{-2}.

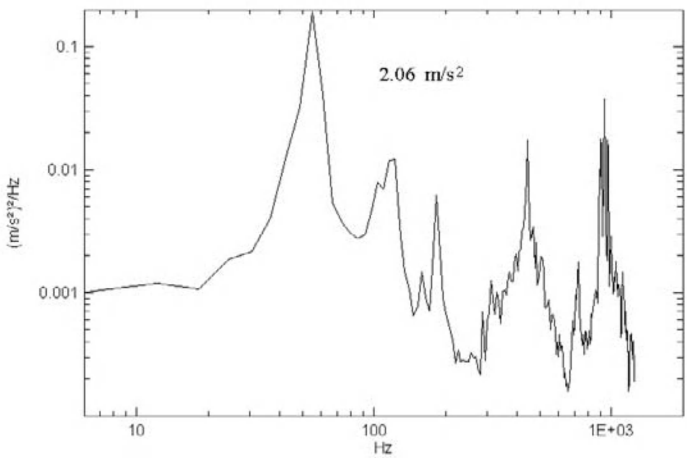


Figure A2.15. *PSD envelope of Figure A2.14*

What duration should this specification have: 23 hours, 20 hours, 3 hours or reduced duration, calculated with which reference duration?

In this example, it seems logical to retain the sum of durations for each vibration (23 hours) as the duration associated with the PSD envelope.

The damage method does not have this problem. An FDS is calculated from each PSD considering the corresponding vibration duration. The specification is then established from the sum of these FDSs, for a duration chosen and validated by ERSs (and SRSs).

The specification FDS by PSD envelope calculated for 23 hours is close to that of the specification established with the damage method over reduced duration of 1 hour (Figure A2.16). The same applies to corresponding ERSs (Figure A2.17).

A smoothing of PSD envelope according to usage would lead to greater severity (and thus to an over-test).

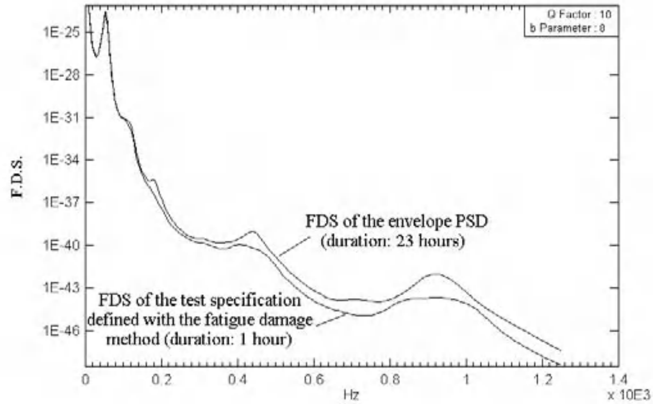


Figure A2.16. *FDS of envelope PSD (Figure A2.15) lasting 23 hours and specification established from the damage method (duration 1 hour)*

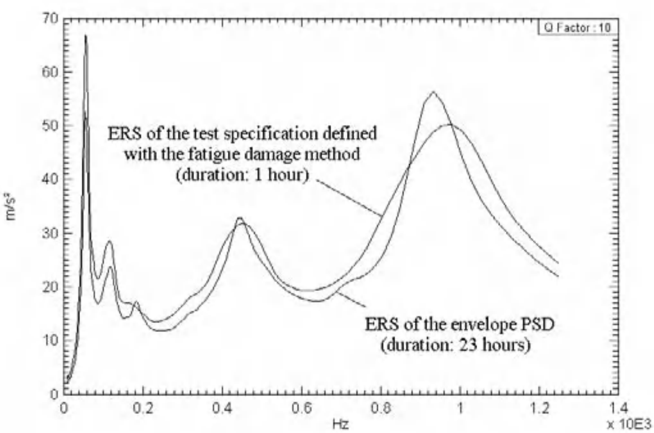


Figure A2.17. *ERS of the envelope PSD (Figure A2.15) lasting 23 hours and specification established from the damage method (duration 1 hour)*

From this last example, it is possible to attempt to reduce the specification duration established by PSD envelope of 23 hours to 1 hour with the usual rule ([A1.2] relation, in this case for b = 8). Figure A2.18 compares the result obtained to the specification calculated over 1 hour with the damage method.

The rms value of the specification calculated by PSDs envelope is equal to 3.05 m s^{-2}, whereas the one obtained by damage equivalence do not exceed 2.63 m s^{-2}. The ERS is consequently much greater (Figure A2.19) in some frequency bands. The envelope is very penalizing at certain frequencies.

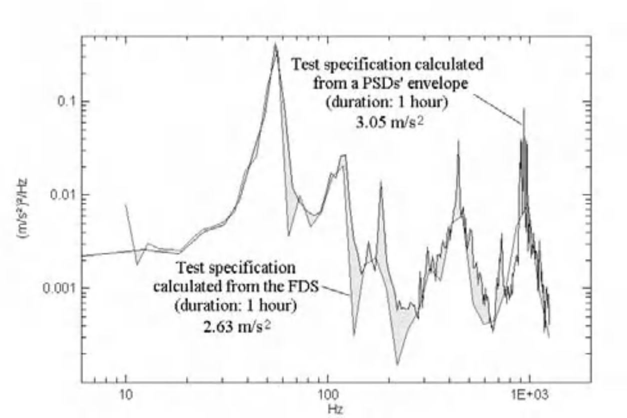


Figure A2.18. *Specifications established by PSD envelope and by the damage method for 1 hour*

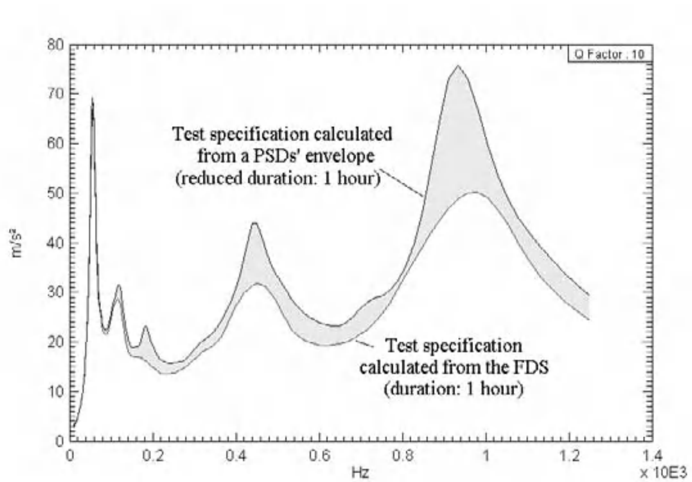


Figure A2.19. *ERS of specifications of Figure A2.18*

We can of course estimate that this margin is useful to cover the variation of the environment. But this over-test is not controlled. It is the result of the envelope process and can be very different depending on data used. It can be much greater if we try to smooth the PSD. It is better to start by defining a specification correctly, reproducing the effects of the real environment, and then to apply an uncertainty coefficient with a well identified value, as described in the damage method (not applied in all these examples) (Chapter 8) [LAL 01a] [LAL 01b].

A2.4. ***Two (or more) PSD envelopes with different frequency contents and different durations.***

The problem is even more obvious when we have to establish a specification covering two environments with very different frequency contents and durations.

Example A2.6.

This is the case of a life profile containing two situations in series, one corresponding to road transport (low frequencies) lasting 24 hours (rms value: 2.03 m s^{-2}), the other to aircraft transport (high frequencies) lasting 3 hours (7.55 m s^{-2}).

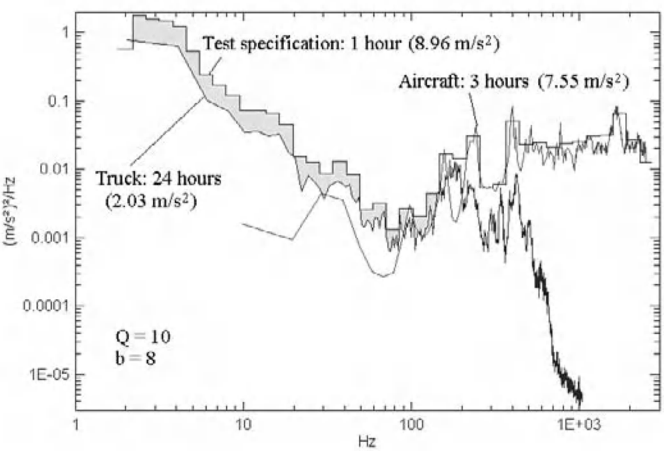


Figure A2.20. *Comparison of the specification established by the damage method (1 hour) and real environments (“truck” and “aircraft” transports)*

The calculation of a specification with the damage method is carried out from the FDS equal to the sum of FDSs for each vibration. In this example, duration was reduced to 1 hour. We can observe that the increase (in relation to the real environment) of the resulting specification (PSD) is much more significant at low frequency than at high frequency, because of a large reduction in time in this range (Figure A2.20).

A2.5. *The real environment is not stationary: its rms value varies according to time*

The vibrations measured can be non-stationary over a duration of varying length (speed change for a truck, turbulence on a flight, etc.). In such a case, it is not correct to calculate a power spectral density to represent the phenomenon and it is not possible to establish a specification by PSD envelope.

Example A2.7.

Figure A2.21 presents an obviously non-stationary signal, where we have calculated the global rms value over 5 seconds to provide a range, knowing that this rms value is not very significant because it varies over time. Figure A2.22 provides the variations, which are very pronounced.

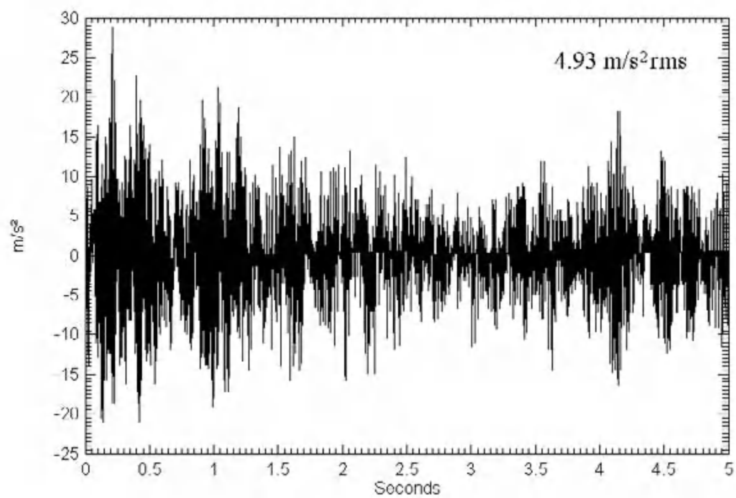


Figure A2.21. *Non-stationary vibration by variation of its rms value over time*

Duration here is limited to 5 s, but it could be much longer. The PSD calculation of such a signal is mathematically possible (Figure A2.23), even though it has no value (average of blocks with a different rms value for reasons not involving the random nature of the signal).

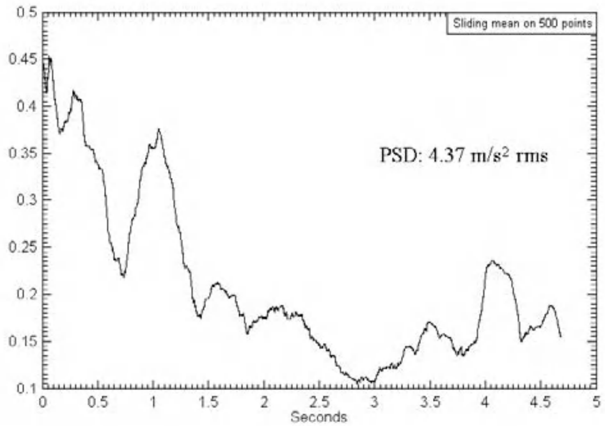


Figure A2.22. *Rms value according to of the non-stationary vibration of Figure A2.21*

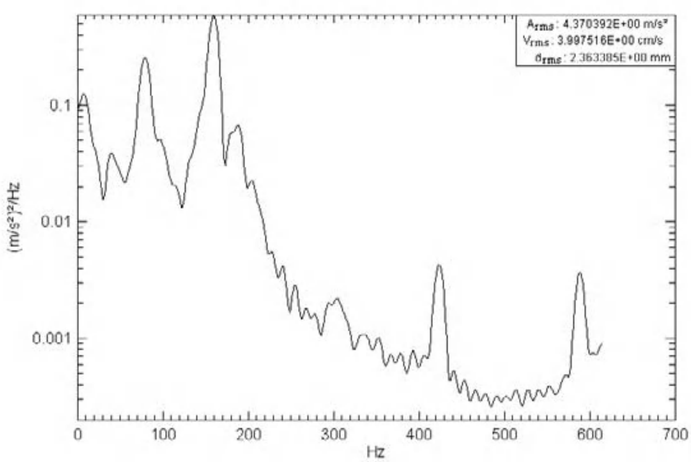


Figure A2.23. *PSD of the non-stationary vibration in Figure A2.21*

In order to show the error made by using this PSD as a specification, we compared it to that deduced from a fatigue damage spectrum directly calculated from the signal over time (Figure A2.24).

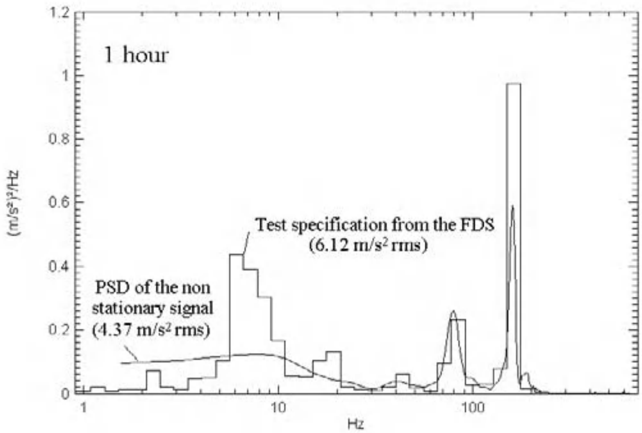


Figure A2.24. *Comparison of the non-stationary vibration PSD (Figure A2.23) and PSD established by damage equivalence from the signal based on time (Figure A2.21)*

Specifications obtained are very different, in frequency content and rms value simultaneously. The consequences of these differences are found in their ERSs and FDSs (Figures A2.25 and A2.26).

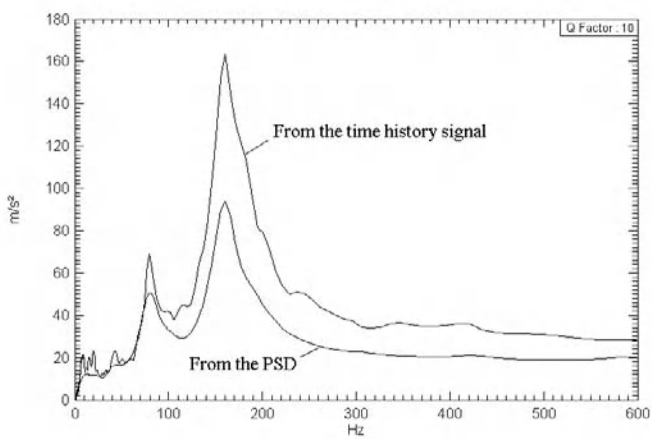


Figure A2.25. *PSD ERS of Figure A2.24*

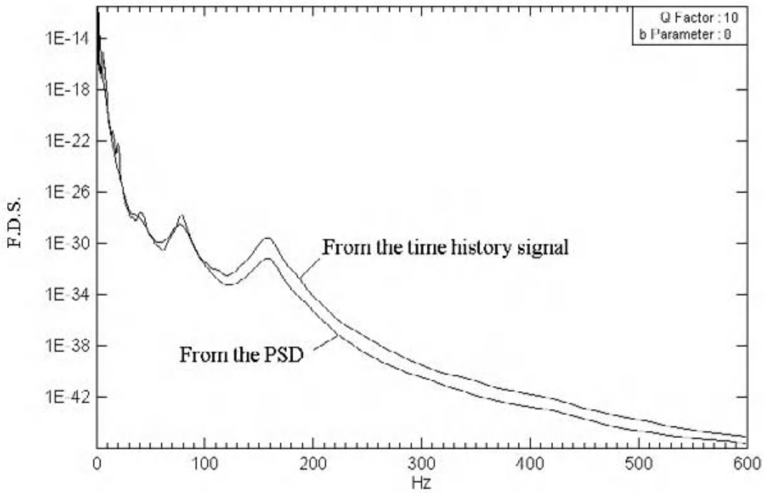


Figure A2.26. *PSD FDS of Figure A2.24*

The specification that would be the result of the signal PSD clearly underestimates the severity of the vibration regardless of the criterion (ERS or FDS).

In the case of non-stationary (or non-Gaussian) vibrations, ERSs and FDSs must be directly calculated from the signal according to time. The specification deduced from this FDS is expressed in the form of the PSD of a Gaussian stationary random vibration (control software to this day can only generate this type of vibration) which produces the same damage and the same maximum stress at each frequency as the real environment vibration.

A2.6. ***The real environment is not stationary: its frequency content varies over time, rms value being equal***

The non-stationarity of the vibration can also be linked to a variation of its frequency content according to time. This case is very pernicious since the rms value of the signal varies very little according to time: a simple drawing of the rms value over time (sliding mean on a constant number of points) can lead to considering the vibration as stationary, authorizing the calculation of its PSD and transformation into a specification, with a result that is obviously worthless.

Example A2.8.

This example was developed by creating a signal made up of a succession of basic signals (Figure A2.27) obtained from PSDs with the same rms value, made up of a constant level on which a peak, with similar characteristics but increasing frequency, moves (Figure A2.28). The duration, limited in this case to 45 s, could be much larger.

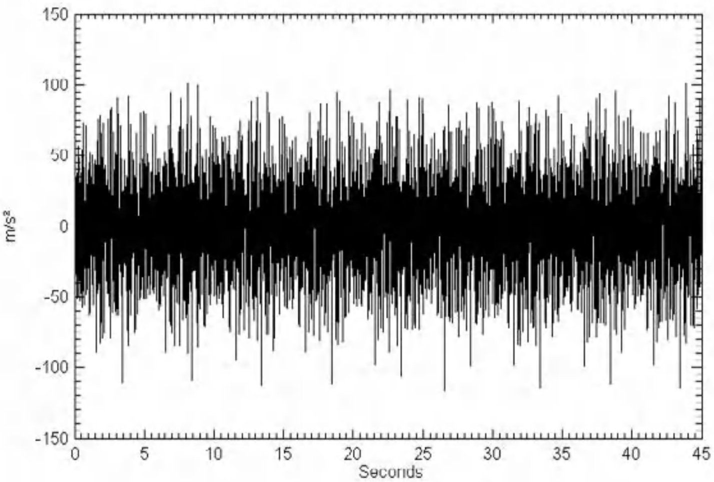


Figure A2.27. *Non-stationary random vibration with a frequency content that varies over time and constant rms value*

Since the rms value does not vary significantly over time (Figure A2.29), this signal could be considered *a priori* as stationary, in the absence of a more detailed analysis. This quick observation would authorize the calculation of a PSD and the establishment of an envelope specification of this spectrum. The resulting PSD would be the one in Figure A2.30.

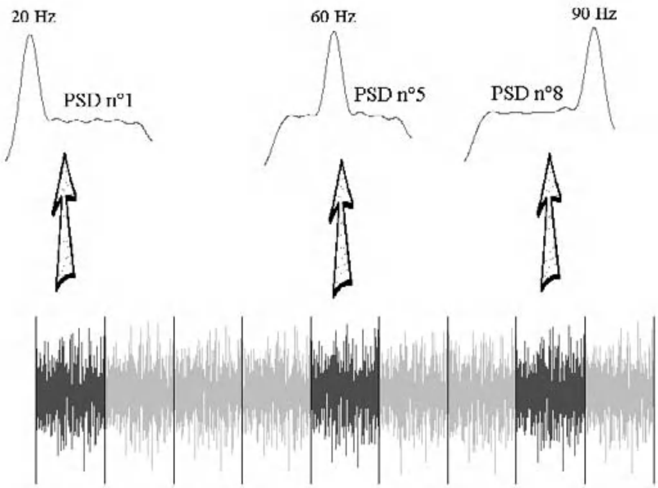


Figure A2.28. *Creation of a non-stationary random vibration with a frequency content that varies over time and constant rms value*

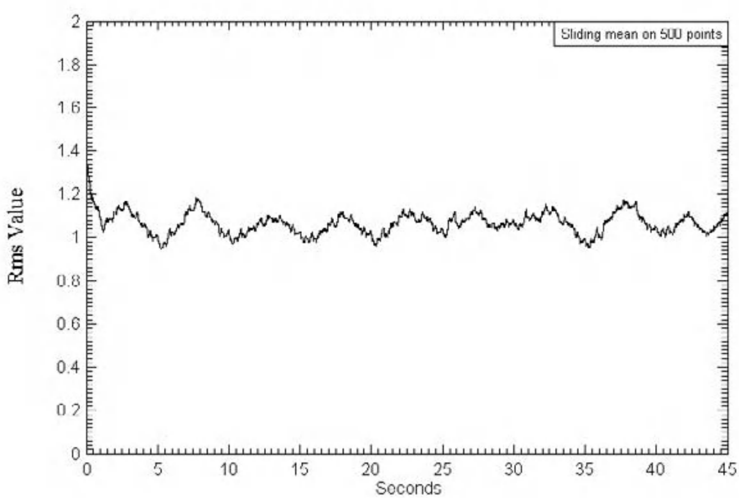


Figure A2.29. *RMS value based on time of the non-stationary vibration in Figure A2.27*

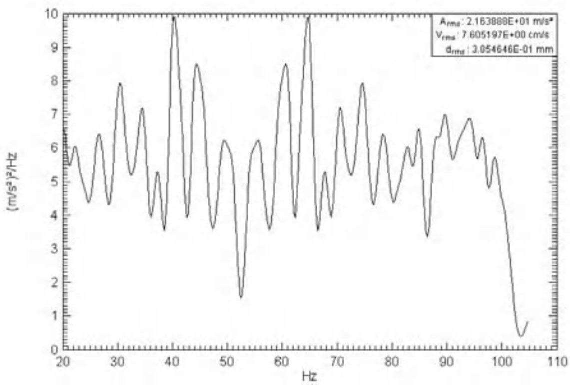


Figure A2.30. *PSD of the non-stationary random vibration in Figure A2.27*

ERS and FDS of the signal in Figure A2.27 were calculated and compared to the ERS and FDS of its PSD respectively (Figure A2.30). The examination of these curves (Figures A2.31 and A2.32) shows that the PSD would lead to a much too severe specification.

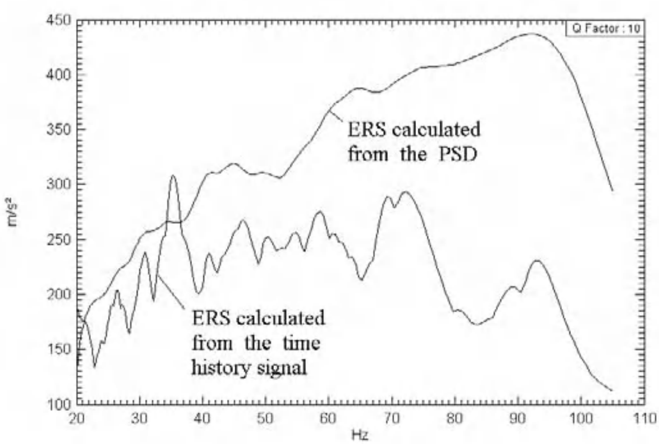


Figure A2.31. *Comparison of ERSs calculated from the signal based on time (Figure A2.27) and its PSD (Figure A2.30)*

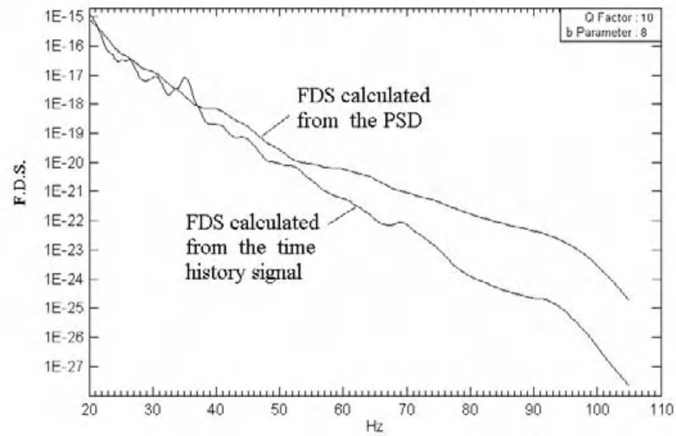


Figure A2.32. *Comparison of FDSs calculated from the signal based on time (Figure A2.27) and its PSD (Figure A2.30)*

Fatigue damage being equal, a specification established with the damage method for 45 s (equal to that of the real vibration) would lead to the application of smaller stresses in testing than those of the real environment (Figure A2.34).

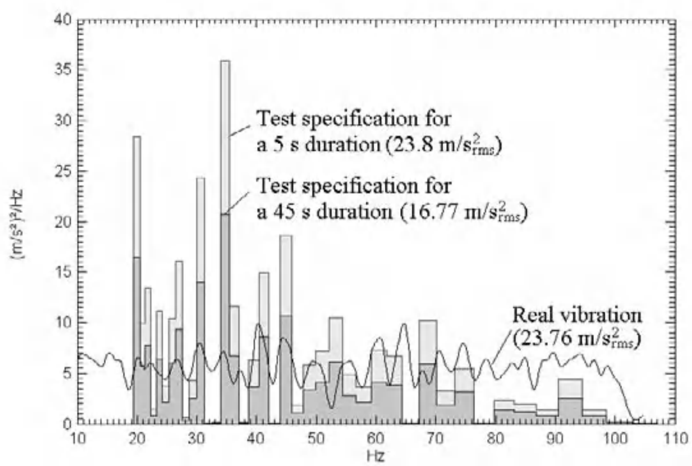


Figure A2.33. *Comparison of real vibration PSDs (Figure A2.27) and specifications established with the damage method, duration 5 s and 45 s*

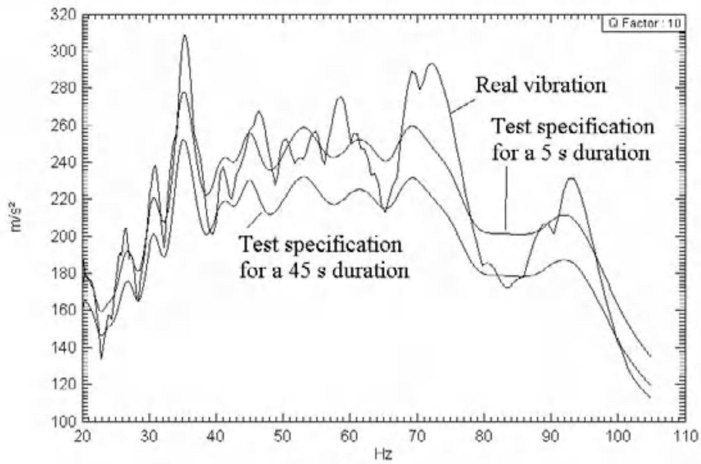


Figure A2.34. *Comparison of ERSs calculated from the signal over time (Figure A2.27) and specifications lasting 5 s and 45 s (Figure A2.33)*

In order for them to be in the same range, we would have to reduce duration to 5 s. The rms value of the 45 s specification is 16.77 m s^{-2} and the rms value of the 5 s specification at 23.8 m s^{-2}, very close to the real environment (23.76 m s^{-2}).

The PSD envelope method can lead to correct results in simple cases. It cannot pretend to be able to provide satisfactory results in many common cases in practice, in which the specification must cover vibrations with different durations with a frequency content that is sometimes close, but most often different, particularly when the test duration must be reduced for cost reasons.

In order to obtain believable specifications, it is better to only envelop PSDs representative of a specific event, which leads to a large number of tests when the life profile includes many situations (to multiply by 3, since there are three axes). In the case of non-stationary environments, calculating a PSD that would lead, if used, to incorrect specification, is wrong.

With identical hypotheses, the damage method seems to be much more flexible without representing the limitations above highlighted by the PSD envelope method.

It is based on (ERS and FDS) spectra, enabling:

– the harmonization of vibration and shock treatment by generalizing the use of the SRS model;

– the comparison of the severity of specifications and/or real environments from mechanical criteria (maximum stress, fatigue damage);

– the comparison of the severity of all types of vibrations: sine, swept sine, random, narrow band noise swept over white noise, swept sine lines over white noise, etc. References [CLA 98] and [RIC 01] provide examples.

It also presents the following advantages:

– even though it uses the same basic hypotheses for duration reduction, it makes it possible to define a better adapted test in the case of a specification covering a succession of vibrations with very different frequency contents and real durations, with greater choice and thus more tailored values of parameter b;

– all stationary, non-stationary, Gaussian or non-Gaussian vibrations can be considered;

– in fact, it makes it possible, in the absolute, to reduce a complete mission profile in a single random vibration specification and in a shock (per axis), even if, in practice, for different reasons (combined vibration/thermal tests, operation testing in a specific situation of the mission profile, etc.) we can be tempted to develop several specifications grouping a certain number of situations;

– existence of a validation process of duration reduction (section 11.4.10);

– insensitivity to Q factor value that must be introduced in calculations (sections 12.2 to 12.4);

– insensitivity to the value of parameter b chosen, in the absence of a duration reduction (section 12.6.1);

– reproduction capability of the results obtained regardless of the specifier, even when the analysis conditions (for example PSD calculation conditions) are different [RIC 93];

– possibility of including an uncertainty coefficient to protect against the variability of each real environment and material strength, function of a risk considered as admissible (Chapter 8);

– possibility of increasing the severity of tests to take account of the realization of only one test at the time of material qualification (experimental demonstration of the resistance of the material to its environment) (Chapter 10).

This method requires more work during the writing of specifications, but the investment is greatly worth it during development, because the resulting specifications make it possible to design material that is strictly sized according to the need, perfectly adapted to its future environment. This analysis work is made much easier, with the existence of computer tools in Windows or Unix, to carry out all spectra calculations and their combination automatically, all the way to the publication of the specification.

A3. Direct generation of a random signal from an FDS

The calculation of a specification from a life profile goes through a combination (sum or envelope) of vibration FDSs characteristic of each situation. The last step consists of determining the frequencies and amplitudes of a PSD after application of a test factor, which over a given time period will lead to the same fatigue damage, or to the same FDS.

This step is necessary, as random vibration tests are specified in a general way by the data of a PSD. This PSD is then transformed into a time history signal by the exciter control system to carry out the test.

And yet, it may seem preferable to avoid this intermediate transformation and, in the same manner as we control the exciters from an SRS, directly control the exciter from the FDS deduced from the life profile. The control system software must then directly calculate the driving signal of the exciter from the specified FDS.

The calculation method described below is based on the one which allows us to calculate a Gaussian time history signal from a PSD.

It consists of, after choosing the signal duration:

1. calculating at each FDS definition frequency, the amplitude of a sinusoid with the same frequency producing the same damage;

2. generating at each of these frequencies a sinusoid with the above determined amplitude and a random phase.

We show that we can obtain a normal distribution of instantaneous signal values when the phase is equal to [KNU 98]:

$$\varphi_m = 2\,\pi\,\sqrt{-2\,\ln r_1}\,\cos\left(2\,\pi\,r_2\right) \qquad \text{[A3.1]}$$

In this expression, r_1 and r_2 are two random numbers distributed in a rectangular manner in interval [0, 1]:

3. summing up all sinusoids;

4. calculating the FDS of the signal obtained;

5. iterating rules of three in the amplitude of sinusoids until the gap between the reference FDS and FDS of the signal obtained is low enough (operations 2 to 4).

Example A3.1.

In this example, we will consider a vibration measured in a truck. The FDS of this vibration, calculated in 200 points for a duration of 12.5 seconds with a logarithmic frequency step, for Q = 10 and b = 8, was used as a reference to fund a signal based on time with the same FDS with the help of the method described above.

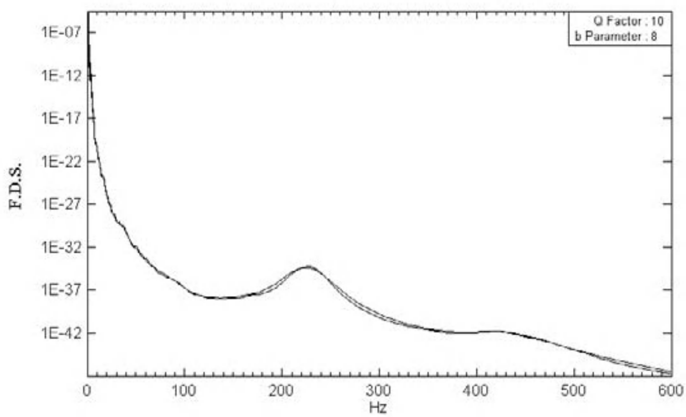


Figure A3.1. *Reference FDS and FDS of the reconstructed signal*

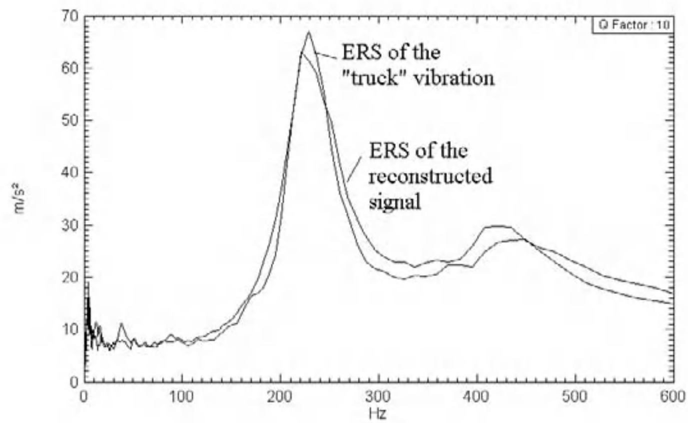


Figure A3.2. *Reference ERS and ERS of the reconstructed signal*

After 3 iterations, the FDS of the reconstructed signal is very close to the reference FDS (Figure A3.1). The same applies to ERSs (Figure A3.2). The rms value of the signal obtained (Figure A3.3) is 3.52 m s^{-2} (compared to 3.32 m s^{-2} for the original signal).

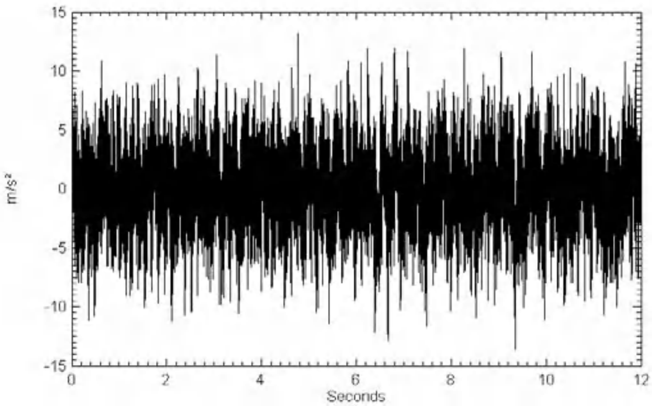


Figure A3.3. *Signal created from the reference FDS*

Its PSD is also very close to the PSD of the reference truck vibration (Figure A3.4).

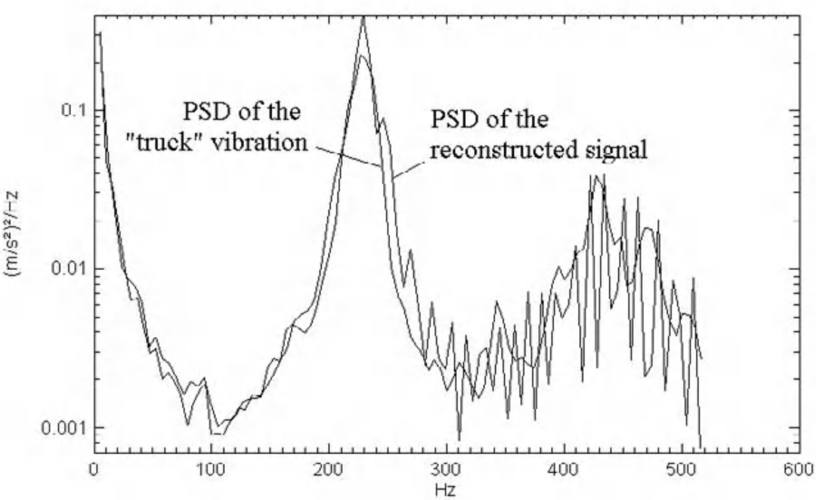


Figure A3.4. *Reference PSD and reconstructed signal PSD*

The choice of a logarithmic step for the FDS at high frequency leads to wider spaced sinusoids with visible effect on the PSD.

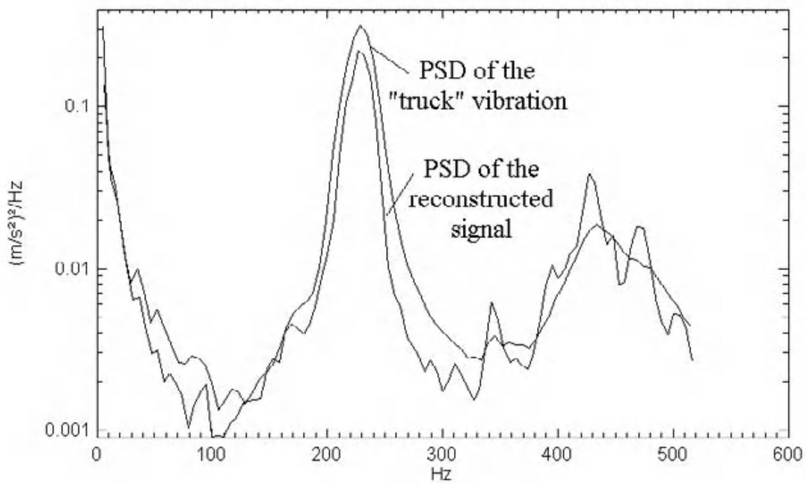


Figure A3.5. *Reference PSD and reconstructed signal PSD from an FDS with linear frequency step*

A linear frequency step slightly improves the appearance of the PSD in this range (Figure A3.5).

Specification Development: Second Edition – Volume 5
Christian Lalanne

Formulae

This list of formulae combines the most commonly used relations in the five volumes in the series with their reference, for easy access to their origin. Each one includes its reference in the volume from which it was taken preceded by the number of this volume (V1 to V5).

Natural frequency of a one-degree-of-freedom linear system

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad [V1.3.5]$$

Frequency of resonance [V1.Table 6.2]

Response	Frequency of resonance	Relative response amplitude
Displacement	$f_0\sqrt{1-2\xi^2}$	$\frac{1}{2\xi\sqrt{1-\xi^2}}$
Velocity	f_0	$\frac{1}{2\xi}$
Acceleration	$\frac{f_0}{\sqrt{1-2\xi^2}}$	$\frac{1}{2\xi\sqrt{1-\xi^2}}$

Critical damping

$$c_c = 2\sqrt{k\,m} \qquad [V1.3.100]$$

Relative damping

$$\xi = \frac{c}{2\sqrt{k\,m}} \qquad \text{[V1.3.101]}$$

***Q* Factor**

$$Q = \frac{f_0}{\Delta f} \qquad \text{[V1.6.109]}$$

$$Q = \frac{1}{2\,\xi} \qquad \text{[V1.6.81]}$$

Logarithmic decrement

$$\delta = \frac{1}{n}\ln\frac{q_{1_M}}{q_{(n+1)_M}} \qquad \text{[V1.3.127]}$$

$$\delta = \frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}} \qquad \text{[V1.3.136]}$$

or:

$$\xi = \frac{\delta}{\sqrt{\delta^2 + 4\,\pi^2}} \qquad \text{[V1.3.137]}$$

n number of cycles necessary to reach amplitude equal to 1/*N*-*n*th of the first peak

$$n = \frac{\sqrt{1-\xi^2}}{2\,\pi\,\xi}\ln N \qquad \text{[V1.6.58]}$$

Swept sine

$$\ddot{x}(t) = \ddot{x}_m \sin E(t) \qquad \text{[V1.8.1]}$$

Sweeping		Logarithmic	Linear
Law f(t)	f ↑	$f = f_1\, e^{t/T_1}$	$f = \alpha\, t + f_1$
	f ↓	$f = f_2\, e^{-t/T_1}$	$f = -\alpha\, t + f_2$
E(t)	f ↑	$E = 2\,\pi\, T_1 \left(f - f_1\right)$	$E = 2\,\pi\, t \left(\frac{\alpha\, t}{2} + f_1\right)$
	f ↓	$E = 2\,\pi\, T_1 \left(f_2 - f\right)$	$E = 2\,\pi\, t \left(-\frac{\alpha\, t}{2} + f_2\right)$
Constant		$T_1 = \frac{Q^2}{\eta\, f_0}$	$\alpha = \frac{f_2 - f_1}{t_b}$
η		$\frac{Q^2 \ln\left(f_2/f_1\right)}{f_0\, t_b}$	$\frac{Q^2}{f_0^2}\, \frac{f_2 - f_1}{t_b}$
Number of cycles in duration t_b between frequencies f_1 and f_2		$N_b = \frac{f_2 - f_1}{\ln \frac{f_2}{f_1}}\, t_b$	$N_b = \frac{f_1 + f_2}{2}\, t_b$
		$N_b = Q\;\Delta N\; \frac{f_2 - f_1}{f_0}$	$N_b = \frac{Q\;\Delta N}{2\; f_0^2} \left(f_2^2 - f_1^2\right)$
		$N_b = \frac{Q^2}{\eta\, f_0} \left(f_2 - f_1\right)$	$N_b = \frac{Q^2}{2\,\eta\, f_0^2} \left(f_2^2 - f_1^2\right)$
Sweep duration between frequencies f_1 and f_2		$t_b = Q\;\Delta t\;\ln \frac{f_2}{f_1}$	$t_b = \frac{Q\;\Delta t}{f_0} \left(f_2 - f_1\right)$
		$t_b = \frac{Q\, \Delta N}{f_0} \ln \frac{f_2}{f_1}$	$t_b = \frac{Q\;\Delta N}{f_0^2} \left(f_2 - f_1\right)$
		$t_b = \frac{Q^2}{\eta\, f_0} \ln \frac{f_2}{f_1}$	$t_b = \frac{Q^2}{\eta\, f_0^2} \left(f_2 - f_1\right)$
Time interval spent in band Δf		$\Delta t = \frac{Q}{\eta\, f_0}$	$\Delta t = \frac{Q}{\eta\, f_0}$

Table F.1. *Main expressions relative to swept sine (from Tables V1.9.3 and V1.9.4)*

Sweeping	**Logarithmic**	**Linear**
Number of cycles in interval Δf (between half power points) of a one-degree-of-freedom system	$\Delta N = \dfrac{f_0 \ N_b}{Q \ \left(f_2 - f_1\right)}$	$\Delta N = \dfrac{2 \ f_0^2 \ N_b}{Q \ \left(f_2^2 - f_1^2\right)}$
	$\Delta N = \dfrac{f_0 \ t_b}{Q \ \ln f_2 / f_1}$	$\Delta N = \dfrac{f_0^2 \ t_b}{Q \ \left(f_2 - f_1\right)}$
	$\Delta N = Q/\eta$	$\Delta N = Q/\eta$
Number of cycles to carry out between f_1 and f_0 (natural frequency)	$N_1 = \dfrac{Q \, \Delta N}{f_0} \left(f_0 - f_1\right)$	$N_1 = \dfrac{Q \, \Delta N}{2 \, f_0^2} \left(f_0^2 - f_1^2\right)$
	$N_1 = t_b \ \dfrac{f_0 - f_1}{\ln f_2 / f_1}$	$N_1 = \dfrac{t_b}{2} \ \dfrac{f_0^2 - f_1^2}{f_2 - f_1}$
	$N_1 = \dfrac{Q^2}{\eta \ f_0} \left(f_0 - f_1\right)$	$N_1 = \dfrac{Q^2}{2 \ \eta \ f_0^2} \left(f_0^2 - f_1^2\right)$

Table F.2. *Main expressions relative to swept sine (from Table V1.9.5)*

Sweeping	Logarithmic	Linear
Time t_1 between f_1 and f_0	$t_1 = \frac{Q\,\Delta N}{f_0} \ln \frac{f_0}{f_1}$	$t_1 = \frac{Q\,\Delta N}{f_0^2} \left(f_0 - f_1 \right)$
	$t_1 = t_b \frac{\ln f_0/f_1}{\ln f_2/f_1}$	$t_1 = t_b \frac{f_0 - f_1}{f_2 - f_1}$
	$t_1 = \frac{Q^2}{\eta\, f_0} \ln \frac{f_0}{f_1}$	$t_1 = \frac{Q^2}{\eta\, f_0^2} \left(f_0 - f_1 \right)$
Number of cycles to carry out between $f_1 = 0$ and f_0	$N_0 = Q\,\Delta N$	$N_0 = \frac{Q\,\Delta N}{2}$
	/	$N_0 = \frac{f_0^2}{2\, f_2} t_b$
	/	$N_0 = \frac{Q^2}{2\,\eta}$
Time t_0 between $f_1 = 0$ and f_0	/	$t_0 = \frac{Q\,\Delta N}{f_0}$
	/	$t_0 = \frac{f_0}{f_2} t_b$
	/	$t_0 = \frac{Q^2}{\eta\, f_0}$
Time spent between f_a and $f_c \in (f_1, f_2)$	$t_c - t_a = t_b \frac{\ln f_c/f_a}{\ln f_2/f_1}$	$t_c - t_a = t_b \frac{f_c - f_a}{f_2 - f_1}$
	$t_c - t_a = \frac{Q^2}{\eta\, f_0} \ln \frac{f_c}{f_a}$	$t_c - t_a = \frac{Q^2}{\eta\, f_0^2} \left(f_c - f_a \right)$
Sweep rate	$R_{om} = \frac{60 \ln f_2/f_1}{t_b \ln 2}$	$R = 60 \frac{f_2 - f_1}{t_b}$
	$R_{om} = \frac{60\, \eta\, f_0}{Q^2 \ln 2}$	$R = 60 \frac{\eta f_0^2}{Q^2}$

Table F.3. *Main expressions relative to swept sine (from Tables V1.9.5, V1.9.6 and V1.9.8)*

Displacements and velocities of a shock machine's table during the execution of simple form shocks

Table F.4 provides the velocity change, maximum displacement during shock, displacement at the end of shock and impact and rebound velocities, for the three main shapes of simple shocks, when executed on shock machines. These expressions are useful to carry out a first sizing of programmers when the ones provided by builders do not satisfy the need (Volume 2).

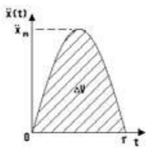
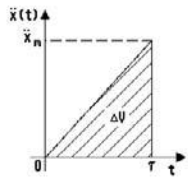
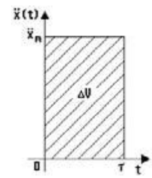
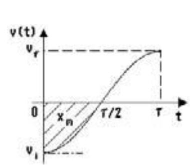
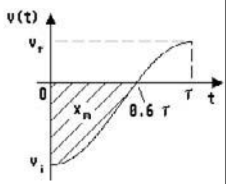
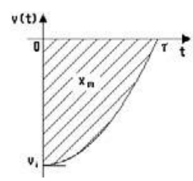
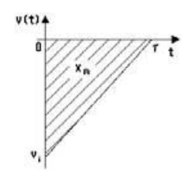
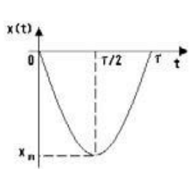
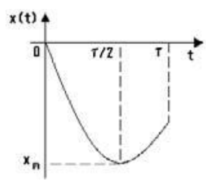
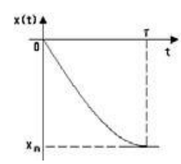
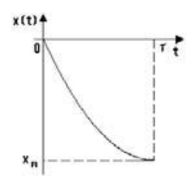
			
Impact with perfect rebound	Impact with rebound at 50 % of the initial velocity	Impact without rebound	Impact without rebound
			
			
$v_i = -\dfrac{\ddot{x}_m\ \tau}{\pi}$	$v_i = -\dfrac{4}{3}\ \dfrac{\ddot{x}_m\ \tau}{\pi}$	$v_i = -\dfrac{\ddot{x}_m\ \tau}{2}$	$v_i = -\ddot{x}_m\ \tau$
$v_R = -v_i = \dfrac{\ddot{x}_m\ \tau}{\pi}$	$v_R = -\dfrac{v_i}{2} = \dfrac{2}{3}\ \dfrac{\ddot{x}_m\ \tau}{\pi}$	$v_R = 0$	$v_R = 0$
$\Delta V = 2\ \dfrac{\ddot{x}_m\ \tau}{\pi}$	$\Delta V = 2\ \dfrac{\ddot{x}_m\ \tau}{\pi}$	$\Delta V = \lvert v_i \rvert = \dfrac{\ddot{x}_m\ \tau}{2}$	$\Delta V = \lvert v_i \rvert = \ddot{x}_m\ \tau$
$x_m = -\dfrac{\ddot{x}_m\ \tau^2}{\pi^2}$	$x_m \approx -\dfrac{\ddot{x}_m\ \tau^2}{2\ \pi}$	$x_m = -\dfrac{\ddot{x}_m\ \tau^2}{3}$	$x_m = -\dfrac{\ddot{x}_m\ \tau^2}{2}$

Table F.4. Practical conditions of execution of usual simple shape shocks *(from Tables V2.5.5, V2.5.9 and V2.5.11)*

Kinematics of the movement for symmetric pre-shock and post-shock

Tables F.5 to F.10 summarize expressions of parameters making it possible to evaluate the feasibility of a simple shape shock with an exciter, based on its shape, duration, its amplitude and pre- and post-shock characteristics (shape, amplitude and position) (Volume 2).

Half sine

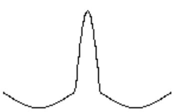
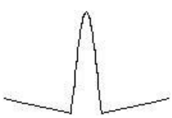
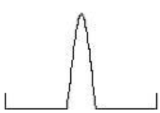
<table>
<tr><th rowspan="2">Half sine</th><th colspan="3">Symmetric pre- and post-shocks</th></tr>
<tr><th>Maximum velocity</th><th>Maximum displacement</th><th>Residual displacement</th></tr>
<tr><td>Half sine
</td><td rowspan="3">$v_m = \pm \frac{\ddot{x}_m \, \tau}{\pi}$</td><td>$x_m = -\frac{\ddot{x}_m \, \tau^2}{\pi} \left(\frac{1}{\pi} + \frac{1}{4 \, p} \right)$</td><td>$x_R = 0$</td></tr>
<tr><td>Triangles
</td><td>$x_m = -\frac{\ddot{x}_m \, \tau^2}{3 \, \pi^2} \left(3 + p + \frac{2}{p} \right)$</td><td>$x_R = 0$</td></tr>
<tr><td>Squares
</td><td>$x_m = -\frac{\ddot{x}_m \, \tau^2}{\pi^2} \left(1 + \frac{p}{2} + \frac{1}{2 \, p} - \frac{p^3}{24} \right)$</td><td>$x_R = 0$</td></tr>
</table>

Table F.5 Practical conditions of execution of usual simple shape shocks
(from Tables V2.5.5, V2.5.9 and V2.5.11)

TPS

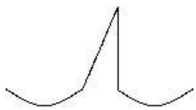
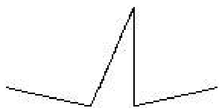
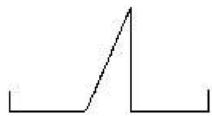
TPS	Symmetric pre- and post-shocks		
	Half sine 	Triangles 	Squares 
Pre- and post-shock durations	$\tau_1 = \frac{\pi\,\tau}{8\,p}$ (1)	$\tau_1 = \frac{\tau}{2\,p}$ $\tau_2 = \tau\left(\frac{1}{2\,p} - p\right)$	$\tau_1 = \frac{\tau}{2\,p}\left(\frac{1}{2} + p^2\right)$ $\tau_2 = \frac{\tau}{2\,p}\left(\frac{1}{2} - p^2\right)$ $\tau_3 = \frac{\tau}{4\,p}$
Maximum velocity	$v_m = \pm\frac{\ddot{x}_m\,\tau}{4}$		
Maximum displacement	$x_m = -\frac{\ddot{x}_m\,\tau^2}{2}\left(\frac{1}{3\sqrt{2}} + \frac{\pi}{32\,p}\right)$	$x_m = -\frac{\ddot{x}_m\,\tau^2}{12}\left(\sqrt{2} + p + \frac{1}{2\,p}\right)$	$x_m = -\frac{\ddot{x}_m\,\tau^2}{4}\left(-\frac{p^3}{6} + \frac{p}{2} + \frac{\sqrt{2}}{3} + \frac{1}{8\,p}\right)$
Residual displacement	$x_R = -\frac{\ddot{x}_m\,\tau^2}{12}$	$x_R = -\frac{\ddot{x}_m\,\tau^2}{12}\left(1 + p\right)$	$x_R = -\frac{\ddot{x}_m\,\tau^2}{4}\left(-\frac{p^3}{6} + \frac{p}{2} + \frac{1}{3}\right)$

Table F.6. *TPS – maximum velocity and displacement – residual displacement (from Table V2.7.2)*

1. τ_1 is the total duration of pre-shock (or post-shock if they are equal). τ_2 is the duration of the first pre-shock part when it is made up of two straight line segments (or last post-shock part). τ_3 is the total duration of post-shock when it is different from τ_1.

Square

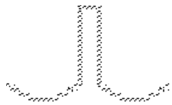
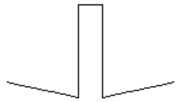
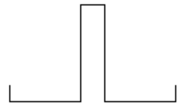
Square	Symmetric pre- and post-shocks		
	Half sine 	Triangles 	Squares 
Durations of pre- and post-shocks	$\tau_1 = \frac{\pi}{4\ p}\ \tau$	$\tau_1 = \frac{\tau}{p}$	$\tau_1 = \frac{\tau}{2\ p}$
Maximum velocity	$v_m = \pm \frac{\ddot{x}_m\ \tau}{2}$		
Maximum displacement	$x_m = -\frac{\ddot{x}_m\ \tau^2}{8}\left(1 + \frac{\pi}{2\ p}\right)$	$x_m = -\frac{\ddot{x}_m\ \tau^2}{2\ p}\left(\frac{1}{3} + \frac{p}{4}\right)$	$x_m = -\frac{\ddot{x}_m\ \tau^2}{8}\left(1 + \frac{1}{p}\right)$
Residual displacement	$x_R = 0$	$x_R = 0$	$x_R = 0$

Table F.7. *Maximum velocity and displacement – residual displacement (from Table V2.7.3)*

Kinematics of the movement for a pre-shock or post-shock alone

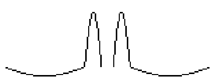
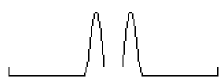
Half sine	Pre-shock or post-shock alone		
	Half sine 	Triangle 	Square 
Duration of pre-shock or post-shock	$\tau_1 = \frac{\pi\ \tau}{8\ p}$	$\tau_1 = \frac{4\ \tau}{\pi\ p}$ $\tau_2 = \frac{\tau}{\pi}\left(\frac{4}{p} - p\right)$	$\tau_1 = \frac{\tau}{\pi}\left(\frac{p}{2} + \frac{2}{p}\right)$ $\tau_2 = \frac{\tau}{\pi}\left(\frac{2}{p} - \frac{p}{2}\right)$
Maximum velocity	Pre-shock: $v_m = -\frac{2\ \ddot{x}_m\ \tau}{\pi}$ Post-shock: $v_m = \frac{2\ \ddot{x}_m\ \tau}{\pi}$		
Residual displacement	$\left\lvert x_R \right\rvert = \frac{\ddot{x}_m\ \tau^2}{\pi}\left(1 + \frac{1}{p}\right)$	$\left\lvert x_R \right\rvert = \frac{\ddot{x}_m\ \tau^2}{\pi}\left(\frac{2\ p}{3\ \pi} + 1 + \frac{8}{3\ \pi\ p}\right)$	$\left\lvert x_R \right\rvert = \frac{\ddot{x}_m\ \tau^2}{\pi^2}\left(\pi - \frac{p^3}{24} + p + \frac{2}{p}\right)$

Table F.8. *Half sine with pre or post-shock alone – maximum velocity and displacement – residual displacement (from Table V2.7.5)*

	Pre-shock or post-shock alone		
TPS	Half sine 	Triangle 	Square 
Duration of pre-shock or post-shock	$\tau_1 = \frac{\pi\, \tau}{4\, p}$	Pre-shock: $\tau_1 = \frac{\tau}{p}$ $\tau_2 = \tau \left(\frac{1}{p} - p \right)$ Post-shock: $\tau_1 = \frac{\tau}{p}$	Pre-shock: $\tau_1 = \frac{\tau}{2} \left(p + \frac{1}{2\, p} \right)$ $\tau_2 = \frac{\tau}{2} \left(\frac{1}{2\, p} - p \right)$ Post-shock: $\tau_1 = \frac{\tau}{2\, p}$
Maximum velocity	Pre-shock: $v_m = -\frac{\ddot{x}_m\, \tau}{2}$ Post-shock: $v_m = \frac{\ddot{x}_m\, \tau}{2}$		
Residual displacement	Pre-shock: $x_R = -\frac{\ddot{x}_m\, \tau^2}{2} \left(\frac{2}{3} + \frac{\pi}{8\, p} \right)$ Post-shock: $x_R = \frac{\ddot{x}_m\, \tau^2}{2} \left(\frac{1}{3} + \frac{\pi}{8\, p} \right)$	Pre-shock $x_R = -\frac{\ddot{x}_m\, \tau^2}{6} \left(p + 2 + \frac{1}{p} \right)$ Post-shock: $x_R = \frac{\ddot{x}_m\, \tau^2}{6} \left(1 + \frac{1}{p} \right)$	Pre-shock: $x_R = -\frac{\ddot{x}_m\, \tau^2}{24} \left(p^3 - 6\, p - 8 - \frac{3}{p} \right)$ Post-shock: $x_R = \frac{\ddot{x}_m\, \tau^2}{2} \left(\frac{1}{3} + \frac{1}{4\, p} \right)$

Table F.9. *TPS with pre or post-shock alone – maximum velocity and displacement – residual displacement (from Table V2.7.6)*

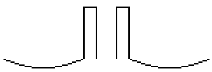
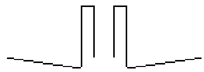
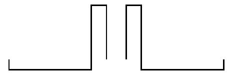
	Pre-shock or post-shock alone		
Square	Half sine 	Triangle 	Square 
Duration of pre-shock or post-shock	$\tau_1 = \dfrac{\pi\,\tau}{2\,p}$	$\tau_1 = \dfrac{2\,\tau}{p}$	$\tau_1 = \dfrac{\tau}{p}$
Maximum velocity	Pre-shock: $v_m = -\ddot{x}_m\,\tau$ Post-shock: $v_m = \dfrac{\ddot{x}_m\,\tau}{2}$		
Residual displacement	$\left\|x_R\right\| = \dfrac{\ddot{x}_m\,\tau^2}{2}\left(1 + \dfrac{\pi}{2\,p}\right)$	$\left\|x_R\right\| = \ddot{x}_m\,\tau^2\left(\dfrac{1}{2} + \dfrac{2}{3\,p}\right)$	$\left\|x_R\right\| = \dfrac{\ddot{x}_m\,\tau^2}{2}\left(1 + \dfrac{1}{p}\right)$

Table F.10. *Square with pre or post-shock alone – maximum velocity and displacement – residual displacement (from Table V2.7.7)*

Largest peak on average over a duration T

$$a = z_{rms}\sqrt{2\,\ln n_0^+\,T} \qquad [V3.5.58]$$

Value a with a probability P_0 of being reached during T

$$a = z_{rms}\sqrt{2\left\{\ln\left(n_0^+\,T\right) - \ln\left[-\ln\left(1 - P_0\right)\right]\right\}} \qquad [V5.3.8]$$

RMS values

$$\ddot{x}_{rms}^2 = \int_0^\infty G\left(f\right)\,df \qquad [V3.3.12]$$

$$v_{rms}^2 = \int_0^\infty \frac{G\left(f\right)}{\left(2\,\pi\,f\right)^2}\,df \qquad [V3.3.23]$$

$$x_{rms}^2 = \int_0^\infty \frac{G(f)}{(2\pi f)^4}\,df \qquad \text{[V3.3.24]}$$

Constant PSD in a frequency interval

Constant PSD between f_1 and f_2, $G(f) = G_0$:

$$\ddot{x}_{rms} = \sqrt{G_0\,(f_2 - f_1)} \qquad \text{[V3.3.25]}$$

$$v_{rms} = \frac{1}{2\pi}\sqrt{G_0\left(\frac{1}{f_1} - \frac{1}{f_2}\right)} \qquad \text{[V3.3.26]}$$

$$x_{rms} = \frac{1}{4\pi^2}\sqrt{\frac{G_0}{3}\left(\frac{1}{f_1^3} - \frac{1}{f_2^3}\right)} \qquad \text{[V3.3.27]}$$

PSD defined by a line segment with average gradient

Linear-linear axes

$$\ddot{x}_{rms} = \sqrt{\frac{(f_2 - f_1)(G_2 + G_1)}{2}} \qquad \text{[V3.3.33]}$$

$$v_{rms} = \frac{1}{2\pi}\sqrt{\frac{G_2 - G_1}{f_2 - f_1}\ln\left(\frac{f_2}{f_1}\right) + \frac{G_1}{f_1} - \frac{G_2}{f_2}} \qquad \text{[V3.3.34]}$$

$$x_{rms} = \frac{1}{4\pi^2 f_1 f_2}\sqrt{\frac{G_2 - G_1}{2}(f_1 + f_2) + \frac{1}{3}\left(\frac{G_1}{f_1} - \frac{G_2}{f_2}\right)\left(f_1^2 + f_1 f_2 + f_2^2\right)} \qquad \text{[V3.3.35]}$$

Linear-logarithmic axes

$$\ddot{x}_{rms} = \sqrt{\frac{e^b}{a}\left(e^{a f_2} - e^{a f_1}\right)} \qquad \text{[V3.3.36]}$$

where $a = \dfrac{\ln G_2 - \ln G_1}{f_2 - f_1}$ and $b = \dfrac{f_1 \ln G_2 - f_2 \ln G_1}{f_1 - f_2}$ (if $a \neq 0$, or if $G_2 \neq G_1$)

$$v_{rms}^2 = \frac{e^b}{4\pi^2} \int_{f_1}^{f_2} \frac{e^{a f}}{f^2} \, df \qquad \text{[V3.3.37]}$$

$$x_{rms}^2 = \frac{e^b}{16\pi^4} \int_{f_1}^{f_2} \frac{e^{a f}}{f^4} \, df \qquad \text{[V3.3.38]}$$

$$\int \frac{e^{a f}}{f^4} \, df = -\frac{e^{a f}}{3 f^3} - \frac{a}{6} \frac{e^{a f}}{f^2} - \frac{a^2}{6} \frac{e^{a f}}{f} + \frac{a^3}{6} \int \frac{e^{a f}}{f} \, df$$

Logarithmic-linear axes

$$a = \frac{G_2 - G_1}{\ln f_2 - \ln f_1} \text{ and } b = \frac{G_2 \ln f_1 - G_1 \ln f_2}{\ln f_1 - \ln f_2}$$

$$\ddot{x}_{rms}^2 = a \left(f_2 \ln f_2 - f_1 \ln f_1 \right) + \left(f_2 - f_1 \right) \left(b + a \right) \qquad \text{[V3.3.42]}$$

$$v_{rms}^2 = \frac{a}{4\pi^2} \left(\frac{\ln f_1}{f_1} - \frac{\ln f_2}{f_2} \right) + \left(\frac{1}{f_1} - \frac{1}{f_2} \right) \left(\frac{a + b}{4\pi^2} \right) \qquad \text{[V3.3.43]}$$

$$x_{rms}^2 = \frac{a}{48\pi^4} \left(\frac{\ln f_1}{f_1^3} - \frac{\ln f_2}{f_2^3} \right) - \frac{a + 3b}{144\pi^4} \left(\frac{1}{f_2^3} - \frac{1}{f_1^3} \right) \qquad \text{[V3.3.44]}$$

Logarithmic-logarithmic axes

If $b \neq -1$:

$$\ddot{x}_{rms} = \sqrt{\frac{f_2 G_2 - f_1 G_1}{b + 1}} \qquad \text{[V3.3.45]}$$

If $b = -1$:

$$\ddot{x}_{rms} = \sqrt{f_1 G_1 \ln \frac{f_2}{f_1}} \qquad \text{[V3.3.46]}$$

If $b \neq 1$:

$$v_{rms} = \frac{1}{2\pi}\sqrt{\frac{G_1}{(b-1)\,f_1}\left[\left(\frac{f_2}{f_1}\right)^{b-1} - 1\right]} = \frac{1}{2\pi}\sqrt{\frac{1}{b-1}\left(\frac{G_2}{f_2} - \frac{G_1}{f_1}\right)} \quad \text{[V3.3.47]}$$

If $b = 1$:

$$G = G_0 \frac{f}{f_1}$$

$$v_{rms} = \frac{1}{2\pi}\sqrt{\frac{G_1}{f_1}\ln\frac{f_2}{f_1}} \quad \text{[V3.3.48]}$$

If $b \neq 3$:

$$\begin{aligned} x_{rms} &= \frac{1}{4\,\pi^2}\sqrt{\frac{1}{b-3}\,\frac{G_1}{f_1^3}\left[\left(\frac{f_2}{f_1}\right)^{b-3} - 1\right]} \\ &= \frac{1}{4\,\pi^2}\sqrt{\frac{1}{b-3}\left(\frac{G_2}{f_2^3} - \frac{G_1}{f_1^3}\right)} \end{aligned} \quad \text{[V3.3.49]}$$

If $b = 3$:

$$x_{rms} = \frac{1}{4\pi^2}\sqrt{\frac{G_1}{f_1^3}\ln\frac{f_2}{f_1}} \quad \text{[V3.3.50]}$$

Slope in dB oct^{-1}:

$$R = 10\,\log_{10}(2)\,\frac{\log_{10}\,G_2/G_1}{\log_{10}\,f_2/f_1} \quad \text{[V3.3.52]}$$

$$R = 10\,\log_{10}(2)\,b \approx 3.01\,b \quad \text{[V3.3.53]}$$

If $\alpha = 10\log_{10}(2)$:

$$\ddot{x}_{rms}^2 = \frac{\alpha}{R+\alpha}\,\left(f_2\,G_2 - f_1\,G_1\right)$$

$R \neq -\alpha$:

$$\ddot{x}^2_{rms} = \frac{\alpha\ f_1\ G_1}{R+\alpha} \left[\left(\frac{f_2}{f_1} \right)^{\frac{R}{\alpha}+1} - 1 \right] \qquad [V3.3.54]$$

$$\ddot{x}^2_{rms} = \frac{f_2\ G_2}{\frac{R}{\alpha}+1} \left(1 - \frac{f_1\ G_1}{f_2\ G_2} \right) = \frac{f_2\ G_2}{\frac{R}{\alpha}+1} \left[1 - \left(\frac{f_1}{f_2} \right)^{1+\frac{R}{\alpha}} \right] \qquad [V3.3.55]$$

$$\ddot{x}^2_{rms} = \frac{\alpha\ G_2}{R+\alpha} \left[f_2 - f_1 \left(\frac{f_1}{f_2} \right)^{\frac{R}{\alpha}} \right] \qquad [V3.3.56]$$

$R \neq -\alpha$:

$$\ddot{x}^2_{rms} = \frac{f_1\ G_1}{\frac{R}{\alpha}-1} \left[1 - \left(\frac{f_2}{f_1} \right)^{1-\frac{R}{\alpha}} \right] = \frac{\alpha\ f_2\ G_2}{R+\alpha} \left[1 - \left(\frac{f_1}{f_2} \right)^{\frac{R}{\alpha}+1} \right] \qquad [V3.3.57, V3.3.58]$$

$R = -\alpha$:

$$\ddot{x}^2_{rms} = f_1\ G_1 \ln \frac{f_2}{f_1} = f_2\ G_2 \ln \frac{f_2}{f_1} \qquad [V3.3.59]$$

$R \neq \alpha$:

$$v^2_{rms} = \frac{\alpha}{4\ \pi^2\ (R-\alpha)}\ \frac{G_1}{f_1} \left[\left(\frac{f_2}{f_1} \right)^{\frac{R-\alpha}{\alpha}} - 1 \right]$$

$$= \frac{\alpha}{4\ \pi^2\ (R-\alpha)}\ \frac{G_2}{f_2} \left[1 - \left(\frac{f_1}{f_2} \right)^{\frac{R-\alpha}{\alpha}} \right] \qquad [V3.3.60]$$

$R = \alpha$:

$$v^2_{rms} = \frac{G_1}{4\,\pi^2\ f_1} \ln \frac{f_2}{f_1} = \frac{G_2}{4\,\pi^2\ f_2} \ln \frac{f_2}{f_1} \qquad [V3.3.61]$$

If $R \neq 3\alpha$:

$$x_{rms}^2 = \frac{\alpha \;\; G_1}{16 \;\; \pi^4 \;\; f_1^3 \;\; (R - 3 \;\; \alpha)} \left[\left(\frac{f_2}{f_1} \right)^{\frac{R-3\,\alpha}{\alpha}} - 1 \right]$$

$$= \frac{\alpha \;\; G_2}{16 \;\; \pi^4 \;\; f_2^3 \;\; (R - 3 \;\; \alpha)} \left[1 - \left(\frac{f_2}{f_1} \right)^{\frac{R-3\,\alpha}{\alpha}} \right] \qquad \text{[V3.3.62]}$$

$R = 3\alpha$:

$$x_{rms}^2 = \frac{1}{16\,\pi^4} \frac{G_1}{f_1^3} \ln \frac{f_2}{f_1} = \frac{1}{16\,\pi^4} \frac{G_2}{f_2^3} \ln \frac{f_2}{f_1} \qquad \text{[V3.3.63]}$$

PSD made up of several frequency intervals

$$\ddot{x}_{rms} = \sqrt{\sum_i \ddot{x}_{i\ rms}^2} \qquad \text{[V3.3.64]}$$

$$v_{rms} = \sqrt{\sum_i v_{i\ rms}^2} \qquad \text{[V3.3.65]}$$

$$x_{rms} = \sqrt{\sum_i x_{i\ rms}^2} \qquad \text{[V3.3.66]}$$

Statistical error (ε < 0.2)

$$\varepsilon \approx \frac{1}{\sqrt{T\ \Delta f}} \qquad \text{[V3.4.34]}$$

Confidence interval of PSD

$$\frac{\hat{G}(f)}{1+\varepsilon} < G(f) < \frac{\hat{G}(f)}{1-\varepsilon} \qquad \text{[V3.4.49]}$$

Number of threshold a_0 crossings per second

a(t) represents the signal of acceleration (instead of $\ddot{x}(t)$) in order to simplify the notations in the expressions of its first and second derivatives ($\dot{a}(t)$ and $\ddot{a}(t)$).

$$n_{a_0} = \frac{1}{\pi} \frac{\dot{a}_{rms}}{a_{rms}} e^{-\frac{a_0^2}{2 a_{rms}^2}} \qquad [V3.5.42]$$

Number of zero threshold crossings per second

$$n_0 = \frac{1}{\pi} \frac{\dot{a}_{rms}}{a_{rms}} \qquad [V3.5.43]$$

$$n_{a_0} = n_0 \, e^{-\frac{a_0^2}{2 a_{rms}^2}} \qquad [V3.5.44]$$

$$a_{rms}^2 = \int_0^\infty G(\Omega) \, d\Omega = R(0) \qquad [V3.5.45]$$

$$\dot{a}_{rms}^2 = \int_0^\infty \Omega^2 \, G(\Omega) \, d\Omega \qquad [V3.5.46]$$

$$\ddot{a}_{rms}^2 = \int_0^\infty \Omega^4 \, G(\Omega) \, d\Omega \qquad [V3.5.47]$$

$$n_{a_0} = 2 \, n_a^+ = \frac{1}{\pi} \left[\frac{\int_0^\infty \Omega^2 \, G(\Omega) \, d\Omega}{\int_0^\infty G(\Omega) \, d\Omega} \right]^{1/2} \exp\left(-\frac{a_0^2}{2 a_{rms}^2} \right) \qquad [V3.5.48]$$

Mean frequency (average number of zero crossings with positive slope)

$$n_0^+ = \left[\frac{\int_0^\infty f^2 \, G(f) \, df}{\int_0^\infty G(f) \, df} \right]^{\frac{1}{2}} \qquad [V3.5.50]$$

$$n_0^+ = \frac{1}{2 \pi} \sqrt{\frac{M_2}{M_0}} \qquad [V3.5.76]$$

Irregularity factor

$$r = \frac{\dot{a}_{rms}^2}{a_{rms} \ \ddot{a}_{rms}} = \frac{M_2}{\sqrt{M_0 \ M_4}} \qquad \text{[V3.6.6]}$$

Mean number of positive peaks per second

$$n_p^+ = \frac{1}{2 \pi} \sqrt{\frac{M_4}{M_2}} \qquad \text{[V3.6.13]}$$

$$n_p^+ = \frac{1}{2\pi} \frac{\ddot{a}_{rms}}{\dot{a}_{rms}} \qquad \text{[V3.6.30]}$$

$$n_p^+ = \left[\frac{\int_0^{+\infty} f^4 \ G(f) \ df}{\int_0^{+\infty} f^2 \ G(f) \ df} \right]^{\frac{1}{2}} \qquad \text{[V3.6.31]}$$

Average of largest peaks

Narrow band process

$$\overline{u_0} \approx \sqrt{2 \ \ln\left(n_0^+ \ T\right)} + \frac{\varepsilon}{\sqrt{2 \ \ln\left(n_0^+ \ T\right)}} \qquad \text{[V3.7.29]}$$

$\varepsilon \approx 0.577...$

Wide band process

$$\overline{u}_0 \approx \sqrt{2 \ \ln\left(r \ N_p\right)} + \frac{\varepsilon}{\sqrt{2 \ \ln\left(r \ N_p\right)}} \qquad \text{[V3.7.56]}$$

Standard deviation of largest peak distribution

Narrow band process

$$s_{u_0} \approx \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ \ln\left(n_0^+ \ T\right)}} \qquad \text{[V3.7.39]}$$

Wide band process

$$s^2 = \frac{\pi^2}{6} \frac{m^2}{\left(m^2 + 1\right)^2} \qquad [V3.7.61]$$

RMS value of the response of a one-degree-of-freedom linear system experiencing white noise

Absolute acceleration

$$\ddot{y}_{rms} = \left[\frac{\pi f_0 \left(1 + 4 \xi^2\right) G_{\ddot{x}}}{4 \xi} \right]^{1/2}$$

Relative displacement

$$z_{rms} = \left[\frac{G_{\ddot{x}}}{64 \pi^3 f_0^3 \xi} \right]^{1/2} = \left[\frac{Q G_{\ddot{x}}}{4 \omega_0^3} \right]^{1/2} \qquad [V3.8.22]$$

Width of a rectangular filter equivalent to a one-degree-of-freedom system

$$\Delta f \approx \frac{\pi}{2} \frac{f_0}{Q} \qquad [V3.9.46]$$

Basquin law

$$N \sigma^b = C \qquad [V4.1.13]$$

Gerber relationship

$$\sigma_a = \sigma'_a \left[1 - \left(\frac{\sigma_m}{R_m} \right)^2 \right] \qquad [V4.1.43]$$

Goodman relationship

$$\sigma_a = \sigma'_a \left(1 - \frac{\sigma_m}{R_m} \right) \qquad [V4.1.41]$$

Söderberg relationship

$$\sigma_a = \sigma'_a \left(1 - \frac{\sigma_m}{R_e}\right) \qquad [V4.1.42]$$

Miner's rule

$$D = \sum_{i=1}^{k} d_i = \sum_i \frac{n_i}{N_i} \qquad [V4.2.3]$$

Fatigue damage (Rayleigth distribution approximation)

$$D \approx \frac{K^b}{C} n_0^+ T \left(\sqrt{2}\, z_{rms}\right)^b \Gamma\left(1 + \frac{b}{2}\right) \qquad [V4.4.41]$$

Reduction of test time

Sinusoidal vibration

$$\ddot{x}_{m\ reduced} = \ddot{x}_m \left[\frac{t_b}{t_{b\ reduced}}\right]^{\frac{1}{b}} \qquad [V5.3.67]$$

$$\frac{\ddot{x}_{m\ reduced}}{\ddot{x}_{m\ real}} = \left(\frac{T_{real}}{T_{reduced}}\right)^{\frac{n-1}{b}} \qquad [V5.3.75]$$

Random vibration

$$\ddot{x}_{rms\ reduced} = \ddot{x}_{rms\ real} \left(\frac{T_{real}}{T_{reduced}}\right)^{\frac{1}{b}} \qquad [V5.4.13]$$

$$\frac{G_{reduced}}{G_{real}} = \left(\frac{T_{real}}{T_{reduced}}\right)^{2/b} \qquad [V5.4.14]$$

$$\frac{G_{reduced}}{G_{real}} = \left(\frac{T_{real}}{T_{reduced}}\right)^{\frac{n}{b}} \qquad [V5.4.17]$$

$$\frac{\ddot{x}_{\text{rms reduced}}}{\ddot{x}_{\text{rms real}}} = \left(\frac{T_{\text{real}}}{T_{\text{reduced}}} \right)^{\frac{n}{2\,b}} \qquad \text{[V5.4.18]}$$

Uncertainty coefficient

Normal distributions

$$k = \frac{1 + \sqrt{1 - \left(1 - V_E^2 \ \text{aerf}^2\right)\left(1 - V_R^2 \ \text{aerf}^2\right)}}{\left(1 - V_R^2 \ \text{aerf}^2\right)} \qquad \text{[V5.8.36]}$$

Log–normal distributions

$$k = \exp\left[\text{aerf} \ \sqrt{\left(1 + V_E^2\right)\left(1 + V_R^2\right)} - \ln \sqrt{\frac{1 + V_E^2}{1 + V_R^2}} \right] \qquad \text{[V5.8.52]}$$

Test factor

Normal distribution (strength)

$$T_F = \left(1 + \frac{N^{-1}(\pi_0) \ V_R}{\sqrt{n}} \right) \ (V_R \le 0.33) \qquad \text{[V5.10.6]}$$

Log–normal distribution

$$T_F = \exp\left(N^{-1}(\pi_0) \sqrt{\frac{\ln\left(1 + V_R^2\right)}{n}} \right) \qquad \text{[V5.10.12]}$$

Weibull distribution

$$T_F \simeq \left[1 + \frac{V_R \ N^{-1}(\pi_0)}{3\sqrt{n}} - \left(\frac{V_R}{3\sqrt{n}} \right)^2 \right]^3 \qquad \text{[V5.10.14]}$$

Bibliography

[ALB 62] C.O. Albrecht, "Statistical evaluation of a limited number of fatigue test specimens including a factor of safety approach", *Symposium on Fatigue of Aircraft Structures*, *ASTM*, STP no. 338, 150–66, 1962

[ALB 83] P. Albrecht, "S.N. fatigue reliability analysis of highway bridges, Probabilistic Fracture Mechanics and Fatigue Methods: Applications for Structural Design and Maintenance", *ASTM*, STP, 184–204, 1983

[ALL 85] H.W. Allen, "Modeling realistic environmental stresses on external stores", *Proceedings IES*, 392–9, 1985

[AND 01] D.O. Anderson, *Safety Factor*, Louisania Tech. University, 2001

[AVP 70] AVP 32, *Design Requirements for Guided Weapons*, Ministry of Technology, London, May 1970

[BAD 70] D. Badel, D. Bonnet and J.M. Bouquin, "Comparaison de la sévérité d'un essai de choc et d'un essai aux vibrations aléatoires au point de vue des contraintes maximales", *Mesures – Régulation – Automatisme*, Vol. 35, no. 12, 62–6, December 1970

[BAN 74] W.F. Bangs, "A comparison of acoustic and random vibration testing of spacecraft", *Proceedings IES*, 44–53, 1974

[BAN 78] B. Bang, B.B. Petersen and E. Morup, *Random Vibration Introduction*, Danish Research Centre for Applied Electronics, Elektronikcentralen, Horsholm, Denmark, September 1978

[BAR 64] R.E. Barret, "Statistical techniques for describing localized vibrating environments of rocket vehicles", *NASA – TND – 2158*, July, 1964

[BAR 65] R.L. Barnoski, "The maximum response of a linear mechanical oscillator to stationary and nonstationary random excitation", *NASA CR 340*, December, 1965

[BAR 77a] W. Barrois, "Fiabilité des structures en fatigue basée sur l'utilisation des résultats d'essais", *L'Aéronautique et l'Astronautique*, Part 1, 5, no. 66, 51–75, 1977

[BAR 77b] W. Barrois, "Fiabilité des structures en fatigue basée sur l'utilisation des résultats d'essais", *L'Aéronautique et l'Astronautique*, Part 2, 6, no. 67, 39–56, 1977

[BHA 58] S.P. Bhatia and J.H. Schmidt, "Evaluation of vibration specifications for static and dynamic material allowables", *The Shock and Vibration Bulletin*, no. 56, Part 2, 201–8, December 1958

[BLA 59] R.E. Blake, "A specification writer's viewpoint", *The Shock and Vibration Bulletin*, no. 27, Part 4, 91–4, 1959

[BLA 62] R.E. Blake, "A method for selecting optimum shock and vibration tests", *Shock, Vibration and Associated Environments Bulletin*, no. 31, 88–97, October 1962

[BLA 67] R.E. Blake, "Predicting structural reliability for design decisions", *Journal of Spacecraft and Rockets*, Vol. 4, no. 3, 392–8, March 1967

[BLA 69] R.E. Blake and W.S. Baird, "Derivation of design and test criteria", *Proceedings IES*, 128–38, 1969

[BOE 63] J.H. Boeckel, "The purposes of environmental testing for scientific satellites", *NASA TN D – 1900*, July, 1963

[BOI 61] B. Boissin, A. Girard and J.F. Imbert, "Methodology of uniaxial transient vibration test for satellites", *Recent Advances in Space Structure Design–Verification Techniques*, ESA–SP 1036, 35–63, October 1961

[BON 71] H. Bono, "Notions générales de fiabilité", *Stage ADERA Initiation à la fiabilité,* 1971

[BON 77] D. Bonnet and C. Lalanne, "Choix des essais – Analyse de l'environnement mécanique vibratoire réel en vue de l'élaboration des spécifications d'essais", *Journées ASTE*, 1977

[BOU 85] M. Bousseau, J. Clisson and Ch. Mas, "Fatigue par chocs – Dispositif expérimental – Résultats obtenus sur un acier au Nickel–Chrome–Molybdène", *Journal de Physique Colloque C5*, Vol. 46, no. 8, 681–8, August 1985

[BRA 64] J. Branger, Second seminar on fatigue and fatigue design, Tech. Rep. no. 5, Columbia University, Inst. for the Study of Fatigue and Reliability, June 1964

[BRE 70] W. Breyan, Effects of block size, stress level and loading sequence on fatigue characteristics of aluminum – alloy box beams, Effects of Environment and Complex Load History on Fatigue Life, ASTM, STP 462, 127–66, 1970

[BRO 64] J.T. Broch, "A note on vibration test procedures", *Bruël and Kjaer Technical Review 2*, 3–6, 1964

[BRO 67] J.T. Broch, "Essais en vibrations – Les raisons et les moyens", *Bruël and Kjaer, Technical Review no. 3,* 1967

[BUL 56] N.I. Bullen, "A note on test factors", *A.R.C.R (Aeronautical Research Council Report) R and M 3166*, September, 1956

[CAI 85] M.P. Caillot, "Banque de données ASTE – Un carrefour technologique", *La Revue des Laboratoires d'Essais*, no. 3, 9–11, April 1985

[CAR 65] R.F. Carmichael and D. Pelke, "Measurement, analysis and interpretation of F-5A 20mm gunfire dynamic environment", *The Shock and Vibration Bulletin*, 34, Part 4, 191–204, February 1965

[CAR 74] S.L. Carman, "Using fatigue considerations to optimize the specification of vibration and shock tests", *Proceedings IES*, 83–7, 1974

[CES 77] R. Cestier and J.P. Garde, "Etude des lois de distribution de la résistance mécanique des métaux. Interprétation de 3500 essais américains", *CEA/DAM/ICQ/CESTA – DO 631*, 18 July 1977

[CHA 92] D. Charles, "Derivation of environment descriptions and test severities from measured road transportation data"; "Part I", *Environmental Engineering*, Vol. 5, no. 4, 30, December 1992; "Part II", *Environmental Engineering*, Vol. 6, no. 1, 25–26, March 1993

[CHE 70] H.B. Chenoweth, "An indicator of the reliability of analytical structural design", *AIAA Journal of Aircraft*, Vol. 7, no. 1, 13–17, January–February 1970

[CHE 77] H.B. Chenoweth, "The error function of analytical structural design", *IES Proceedings*, 231–4, 1977

[CHE 81] E. Chevalier, , "Normalisation des essais en environnement des matériels", *Bulletin d'Information et de Liaison de l'Armement*, no. 67, 10/81, 56–67, 1981

[CHO 66] S.C. Choi and A.G. Piersol, "Selection of tests levels for space vehicle component vibration tests", *Journal of the Electronics Division*, ASQC, 4, no. 3, 3–9, July 1966

[CLA 98] G. Clark, T.R. Chesneau and M.P. Neale, "Determination of Test Severities for Munitions Carried in Tracked Vehicles", *I.E.S.T. Proceedings*, 200–7, 1998

[COE 92] "Coefficient de garantie", *GAM–EG–13. Essais Généraux en Environnement des Matériels*, Annexe générale Mécanique, Annexe 8, Ministère de la Défense, D.G.A, June 1992

[COL 07] B. Colin, *The Randomization of the MRS: A Response to Risk Management for Establishment of Tests Specifications and Design of the Equipments, in Terms of Extreme Values*, ASTELAB, Paris (in French), 2007

[CON 62] F. Condos, "Prediction of vibration levels for space launch vehicles", *Proceedings of the IAS National Meeting on Large Rockets*, October 1962

[COQ 79] M. Coquelet, *Recommandations pour la rédaction des clauses techniques d'essais en environnement des équipements aéronautiques*, September 1979

[COQ 81] M. Coquelet, "Progrès dans l'élaboration des programmes d'essais d'environnement mécanique", *AGARD Conference Proceeding*, Vol. CP 318, no. 3, 10–81, 1981

[CRA 67] H. Cramer and M.R. Leadbetter, *Stationary and Related Tochastic Processes*, John Wiley and Sons, New York, 1967

[CRA 68] S.H. Crandall, "Distribution of maxima in the response of an oscillator to random excitation", *M.I.T. Acoust. and Vib. Lab. Rept no. DSR 78867-3*, 1968

[CRA 70] S.H. Crandall, “First-crossing probabilities of the linear oscillator”, *J. Sound Vib.*, Vol. 12, no. 3, 285–99, 1970

[CRA 83] S.H. Crandall and W.Q. Zhu, “Random vibration: a survey of recent developments”, *Transactions of the ASME, Journal of Applied Mechanics*, Vol. 50, no. 4b, 953–62, December 1983

[CRE 54] C.E. Crede, M. Gertel and R.D. Cavanaugh, “Establishing vibration and shock tests for airborne electronic equipment”, *WADC Technical Report 54–272*, ASTIA AD 45 696, June 1954

[CRE 56] C.E. Crede and E.J. Lunney, “Establishment of vibration and shock tests for missile electronics as derived from the measured environment”, *WADC Technical Report 56–503*, ASTIA Document no. AD 118 133, 1 December 1956

[CRO 68] D.L. Cronin, “Response spectra for sweeping sinusoïdal excitations”, *The Shock and Vibration Bulletin*, no. 38, Part 1, 133–9, August 1968

[CUR 71] A.J. Curtis, N.G. Tinling, H.T. Abstein, *Selection and Performance of Vibration Tests*, The Shock and Vibration Information Center, United States Department of Defense, Washington, SVM 8-NRL, 1971

[CZE 78] A. Czechowski and A. Lenk, “Miner’s rule in mechanical tests of electronic parts”, *IEEE Transactions on Reliability*, Vol. R-27, no. 3, August 1978

[DEF 86] DEF-0035, Environmental Handbook for Defence Materiel, Ministry of Defence, UK, 31 October 1986.

[DEL 69] T.B. Delchamps, “Specifications: a view from the middle”, *The Shock and Vibration Bulletin*, 39, Part 6, 151–155, March 1969

[DES 83] A. Desroches and M. Nelf, “Introduction de la loi de probabilité du coefficient de variation dans les applications de la méthode résistance – contrainte”, *Revue de Statistiques Appliquées*, Vol. 31, no. 3, 17–26, 1983

[DEV 86] *Déterminage des matériels électroniques*, ASTE/AFQ, 1986

[DEW 86] J. De Winne “Equivalence of fatigue damage caused by vibrations”, *IES Proc.*, 227–34, 1986

[DUB 59] W. Dubois, “Random vibration testing”, *Shock, Vibration and Associated Environments Bulletin*, no. 27, Part 2, 103–12, June 1959

[EST 61] “The establishment of test levels from field data – Panel session”, *The Shock and Vibration Bulletin*, no. 29, Part IV, 359–76, 1961

[FAC 72] W.C. Fackler, *Equivalence Techniques for Vibration Testing*, The Shock and Vibration Information Center, SVM 9, 1972

[FEL 59] R.P. Felgar, “Reliability and mechanical design”, *Shock, Vibration and Associated Environments Bulletin*, no. 27, Part IV, 113–25, June 1959

[FEN 86] H. Fenech and A.K. Rao, “Statistical analysis of fatigue failure due to flow noise excitations”, *Journal of Vibration, Acoustics, Stress and Reliability in Design*, Vol. 108, no. 3, 249–54, July 1986

[FIN 51] Report: Final engineering report on investigation of vibration and shock requirements for airborne electronic equipment, North American Aviation, Inc., Report no. 120 X – 7, Contract no. AF 33 (038) – 7379, 30 March 1951

[FOL 62] J.T. Foley, "Environmental analysis", *IES Proceedings*, 427–31, 1962

[FOL 65] J.T. Foley, "Preliminary analysis of data obtained in the joint ARMY/AEC/SANDIA test of truck transport environment", *The Shock and Vibration Bulletin*, no. 35, Part 5, 57–70, 1965

[FOL 67] J.T. Foley, "An environmental research study", *IES Proceedings*, 363–73, 1967

[FOR 65] W.R. Forlifer, "Problems in translating environmental data into a test specification", *IES Proceedings*, 185–8, 1965

[FOS 82] K. Foster, "Response spectrum analysis for random vibration", *Proceedings IES, Designing Electronic Equipment for Random Vibration Environments*, 25–6 March 1982

[FRE 70] A.M. Freudenthal and P.Y. Wang, "Ultimate strength analysis of aircraft structures", *AIAA Journal of Aircraft*, Vol. 7, no. 3, 205–10, May–June 1970

[GAM 76] GAM–T13, *Essais généraux des matériels électroniques et de télécommunication*, Ministère de la Défense, Délégation Ministérielle pour l'Armement, Service Central des Télécommunications et de l'Informatique, June 1976

[GAM 86] GAM–EG–13, *Essais généraux en environnement des matériels*, Ministère de la Défense, Délégation Générale pour l'Armement, June 1986

[GAM 87] GAM EG13B (AIR 7306). *Essais de compatibilité à l'environnement climatique, mécanique, électrique, électromagnétique et spécial des matériels aéronautiques*, Ministère de la Défense, Délégation Générale pour l'Armement, April 1987

[GEN 67] M.G. Gens, "The environmental operations analysis function", *IES Proceedings*, 29–38, 1967

[GER 61] M. Gertel, Chapter 24. "Specification of laboratory tests", in C.M. Harris and C.E. Crede, *Shock and Vibration Handbook*, , Vol. 2–24, McGraw Hill Book Company, 1961

[GER 66] M. Gertel, "Derivation of shock and vibration test specifications based on measured environments", *Journal of Environmental Sciences*, December 1966, 14–19, or *The Shock and Vibration Bulletin*, vol 7, no. 31, Part II, 25–33, 1962

[GIR 97] D. Girardeau, "Estimation du coefficient de variation de l'environnement réel pour le calcul du coefficient de garantie", *ASTELAB 1997, Recueil de Conférences*, 27–35, 1997

[GOE 60] W.P. Goepfert, *Variation of Mechanical Properties in Aluminum Products*, Statistical Analysis Dept., Alcoa, Pittsburgh, Penn., 1960

[GRE 61] H.N. Mc Gregor et al., "Acoustic problems associated with underground launching of a large missile", *Shock, Vibration and Associated Environments Bulletin*, no. 29, Part IV, 317–35, June 1961

[GUI 08] *Guide for Tailoring Material to its Life Cycle Environment Profile Mechanical Environment*, MINDEF, France, 2008

[GUR 82] H. Gurien, "Random vibration testing and analysis of a large ceramic substrate assembly", *IES Proceedings*, 93–8, 1982

[HAH 63] P.G. Hahn, "Shock and vibration considerations in flight vehicle system design", *IES Proceedings*, 401–15, 1963

[HAH 70] G.J. Hahn, "Statistical intervals for a normal population, Part I. Tables, examples and applications", *Journal of Quality Technology*, Vol. 2, no. 3, 115–25, July 1970

[HAN 79] R.N. Hancock, "Development of specifications from measured environments", *Society of Environmental Engineers, S.E.E. Symposium*, Buntingford, Herts, UK, Vol. 1, 1–16, 1979

[HAR 64] W. Harvey, "Specifying vibration simulation", *Inst. Environ. Sci. Proc.*, vol. 5, 407–16, 1964

[HAT 82] A.E. Hatheway and C. Montano, "Analysis and test of ceramic substrates for packaging of leadless chip carriers", *IES Proceedings*, 49–51, 1982

[HAU 65] E.B. Haugen, *Statistical Strength Properties of Common Metal Alloys*, North American Aviation Inc., Space and Information Systems Division, SID 65–1274, 30 October 1965

[HAY 65] J.E. Hayes, Final report – Structural design criteria for boost vehicles bym statistical methods, North American Aviation Inc., Space and Information Systems Division, Tulsa, SID 64T – 290, 4 March 1965

[HEN 95] G.R. Henderson, A.G. Piersol, "Fatigue Damage Related Descriptor for Random Vibration Test Environments", *Sound and Vibration Magazine*, Vol. 29, No. 10, 20–4, October 1995

[HIM 57] H. Himelblau, "A comparison of periodic and random vibration problems", *Proc. 3rd Nat. Flt. Test Instr. Sym.*, ISA, 2-10-1/2-10-3, 1 May 1957

[HOL 84] G.R. Holmgren, "Simulation and testing techniques for mechanical shock and vibration environments", *Proceedings IES*, 340–4, 1984

[HOW 56] G.H. Howell, "Factors of safety and their relation to stresses loads materials", *Machine Design*, 76–81, 12 July, 1956

[HUG 98] W.O. Hughes and A.M. McNelis, "Statistical analysis of a large sample size pyroshock test data set", *NASA / TM – 1998 – 206621*, April 1998

[ITO 06] ITOP 1-1-050, *Development of Laboratory Vibration Test Schedules*, International Test Operations Procedure, Draft, October 12, 2006

[JOH 40] N.L. Johnson and B.L. Welch, "Applications of the non-central t-distribution", *Biometrika*, 31, 362–89, 1940

[JOH 53] A.I. Johnson, *Strength, Safety and Economical Dimensions of Structures*, Inst. of Building Statics and Structural Engineering, Royal Institute of Technology, Stockholm, Report 12, 1953

[JUL 57] O.G. Julian, "Synopsis of first progress report of Committee on factors of safety", *Journal of the Structural Division, Proceedings of the ASCE*, Vol. 83, ST 4, no. 1316, July 1957

[KAC 68] G. Kachadourian, "Spacecraft vibration: a comparison of flight data and ground test data", *The Shock and Vibration Bulletin*, no. 37, Part 7, 173–203, 1968

[KAT 65] H. Katz and G.R. Waymon, "Utilizing in-flight vibration data to specify design and criteria for equipment mounted in jet aircraft", *The Shock and Vibration Bulletin*, no. 34, Part IV, 137–46, February 1965

[KAY 32] A.T. McKay, "Distribution of the coefficient of variation and the extended t-distribution", *J.R. Stastist. Soc.*, 95, Vol. XCV, Part IV, 695–8, 1932

[KEC 68] D. Kececioglu and E.B. Haugen, "A unified look at design safety factors, safety margins and measures of reliability", *Annals of the Reliability and Maintenability Conference*, San Francisco, Calif., 14–17, 520–8, July 1968

[KEC 72] D. Kececioglu, "Reliability analysis of mechanical components and systems", *Nucl. Eng. design*, Vol. 19, 259–90, 1972

[KEE 74] W.B. Keegan, "A statistical approach to deriving subsystems specifications", *Proceedings IES*, 106–7, 1974

[KEN 49] D.C. Kennard, "Vibration testing as a guide to equipment design for aircraft", *WADC AF Technical Report*, no. 5847, June 1949, or *The Shock and Vibration Bulletin*, no. 11, February 1953

[KEN 59] D.C. Kennard, "Measured aircraft vibration as a guide to laboratory testing", *WADC AF Technical Report no. 6429*, May 1959

[KLE 61] G.H. Klein, "Defends random-wave vibration tests", *Product Engineering*, Vol. 32, no. 9, 28–9, 27 February 1961

[KLE 65] G.H. Klein and A.G. Piersol, "The development of vibration test specifications for spacecraft applications", *NASA CR - 234*, May 1965

[KNU 98] D.E. Knuth, *The Art of Computer Programming*, Third Edition, Addison-Wesley, Vol. 2, 1998

[KRO 62] R.C. Kroeger and G.J. Hasslacher, "The relationship of measured vibration data to specification criteria", *The Shock and Vibration Bulletin*, 31, Part 2, 49–63, March 1962

[LAL 80] C. Lalanne, "Utilisation de l'environnement réel pour l'établissement des spécifications d'essai. Synthèse sur les méthodes classiques et nouvelles", *ASTE, 7èmes Journées Scientifiques et Techniques*, 1980

[LAL 84] C. Lalanne, "Maximax response and fatigue damage spectra", *Journal of Environmental Sciences*, Part I, Vol. XXVII, no. 4, July–August 1984; Part II, Vol. XXVII, no. 5, September–October, 1984

[LAL 85] C. Lalanne, "Norme 810 D – Environnement réel et spécifications d'essais", *La Revue des Laboratoires d'Essais*, no. 3, 14–17, April 1985

[LAL 87] C. Lalanne, "Spécifications d'essais en environnement et coefficients de garantie", *CEA/CESTA/DT/EX/MEV 1089*, 18 November, 1987

[LAL 88] C. Lalanne, "Séismes: Comparaison de la sévérité des spécifications", *La Revue des Laboratoires d'Essais*, no. 17, 45, December 1988

[LAL 89] C. Lalanne, "Personalization and safety factor", *Institute of Environmental Sciences Proc.*, 1989, or "Coefficient de garantie", *Symposium ASTE 1989: La Norme Interarmées GAM EG13, Personnalisation des Essais en Environnement*, Paris, 7–8 June 1989

[LAL 94] C. Lalanne, "Methode d'élaboration des spécifications d'environnement mécanique", *CESTA/DT/EC no. 1015 / 94*, 28 November 1994

[LAL 01a] C. Lalanne, *Mechanical Environment Test Specification Development Method*, 4th edition, Centre d'Etudes Scientifiques et Techniques d'Aquitaine, 2001

[LAL 01b] C. Lalanne and H. Grzezkowiak, "Uncertainty Factor in the Process of Deriving Test Severity from Field Measurements", *I.E.S.T. Proceedings*, April 22–5, Paper N ° 32, 2001

[LAL 05] C. Lalanne and J.P. Cambou, "Estimation statistique du coefficient de variation de l'environnement ou de la résistance à partir de sa fonction de répartition", *ASTELAB, CNIT Paris La Défense,* 2005

[LAM 80] R.G. Lambert, "Criteria for accelerated random vibration tests", *Proceedings IES*, 71–5, May 1980

[LAM 88] R.G. Lambert, "Fatigue damage accumulation prediction for combined sine and random stresses", *Journal of the Environmental Sciences*, May–June 1988, Vol. 31, no. 3, 53–63, or *Proceedings IES*, 75–85, 1988

[LAP 84] M. Laparliere, *Etude de la dispersion des propriétés mécaniques de matériaux composites Carbone Epoxy*, Centre d'Essais Aéronautiques de Toulouse, France, Rapport M2-695600-PT-1, 3 July 1984

[LAW 61] H.C. Lawrence, "Prudent specification of random-vibration testing for isolators", *The Shock and Vibration Bulletin*, no. 29, Part IV, 106–12, June 1961

[LEE 82] P.Y. Lee, "Designing electronic equipment for random vibration environments", *IES Proceedings*, 43–7, 1982

[LIE 58] G.J. Lieberman, "Tables for one-sided statistical tolerance limits", *Industrial Quality Control*, 7–9, April 1958

[LIE 78] H.P. Lieurade, Comportement mécanique et métallurgique des aciers dans le domaine de la fatigue oligocyclique – Etude des phénomènes et application à la croissance des fissures, PhD Thesis, University of Metz, September 1978

[LIG 79] J.C. Ligeron, *La fiabilité en mécanique*, Desforges, Paris, 1979

[LIN 67] Y.K. Lin, *Probabilistic Theory of Structural Dynamics*, McGraw-Hill Book Company, 1967

[LIP 60] C. Lipson, “New concepts on... safety factors”, *Product Engineering*, Mid, Vol. 9, 275–8, September 1960

[LUH 82] H. Luhrs, “Random vibration effects on piece part applications”, *IES Proceedings*, 59–64, 1982

[LUN 56] B. Lundberg, “Discussion de l’article de F. Turner – Aspects of fatigue design of aircraft structures”, in A.M. Freudenthal, (ed.) *Fatigue in Aircraft Structures*, Academic Press, New York, 341–6, 1956

[MAI 59] R.M. Mains, “Damage accumulation in relation to environmental testing”, *The Shock and Vibration Bulletin*, no. 27, Part IV, 95–100, 1959

[MAR 66] W.D. Mark, “On false-alarm probabilities of filtered noise”, *Proceedings of the IEEE*, Vol. 54, no. 2, 316–7, February 1966

[MAR 74] C. Marcovici and J.C. Ligeron, *Utilisation des techniques de fiabilité en mécanique, Technique et Documentation*, 1974

[MAR 83] S.G. Martindale and P.H. Wirsching, “Reliability-based progressive fatigue collapse”, *J. Struct. Eng.*, Vol. 109, no. 8, 1792–811, August 1983

[MED 76] J.M. Medaglia, “Statistical determination of random vibration requirements for subassembly tests”, *The Shock and Vibration Bulletin*, no. 46, Part 4, 77–91, August 1976

[MET 61] *Metals Handbook – Properties and Selection*, Volume 1, 8th edition, American Society for Metals, Metals Park, Ohio, 1961

[MIL 54] J.W. Miles, “On Structural Fatigue Under Random Loading”, *Journal of the Aeronautical Sciences*, 753, November, 1954

[MIL 97] MIL-STD-810F, Test Method Standard for Environmental Engineering Considerations and Laboratory Tests, 1997.

[MIS 81] *Missiles – Essais de simulation de transport logistique et d’emport tactique – Environnement mécanique*, BNAé, Recommandations RE – Aéro 61210 à 61218, December 1981

[MOR 55] C.T. Morrow and R.B. Muchmore, “Shortcomings of present methods of measuring and simulating vibration environments”, *Journal of Applied Mechanics*, 367 71, September 1955

[MOR 65] R.E. Morse, “The relationship between a logarithmically swept excitation and the build-up of steady-state resonant response”, *The Shock and Vibration Bulletin*, no. 35, Part II, 231–62, 1965

[MOR 76] C.T. Morrow, “Environmental specifications and testing”, in C.M. Harris and C.E. Crede, *Shock and Vibration Handbook*, 2nd edition, 24–1/24–13, McGraw-Hill Book Company, 1976

[MUS 60] G.S. MUSTING and E.D. HOYT, *Practical and Theorical Bases for Specifying a Transportation Vibration Test*, Reed Reasearch, Inc., Wash., DC., Contract Nord 16687, Task 36, Project RR 1175 – 36, ASTIA – AD 285296, 25 February 1960

[NAT 63] M.G. Natrella, *Experimental Statistics*, National Bureau of Standards Handbook 91, 1 August 1963

[NOR 72] Norme AIR 7304, *Conditions d'essais d'environnement pour équipements aéronautiques: électriques, électroniques et instruments de bord*, Délégation Ministérielle pour l'Armement, DTCA, 1972

[NOR 87] Norme AFNOR NF X06-052, *Applications de la statistique*, Estimation d'une moyenne (variance connue), January 1987

[NOR 96] R.L. Norton, *Machine Design – An Integrated Approach*, Prentice Hall, New York, 1996

[OLS 57] M.W. Olson, "A narrow-band-random-vibration test", *The Shock and Vibration Bulletin*, no. 25, Part 1, 110, December 1957

[OSG 82] C.C. Osgood, *Fatigue Design*, Pergamon Press, 1982

[OWE 62] D.B. Owen, *Handbook of Statistical Tables*, Addison-Wesley Publishing Company, Inc., 1962

[OWE 63] D.B. Owen, *Factors for One-sided Tolerance Limits and for Variables Sampling Plans*, Sandia-Corporation Monograph, SCR-607, March 1963

[PAD 68] G.E. Padgett, "Formulation of realistic environmental test criteria for tactical guided missiles", *IES Proceedings*, 441–8, 1968

[PAR 61] H.J. Parry, *Study of Scatter of Structural Response and Transmissibility*, Lockheed California Co., Burbank, California, Report no. 15454, 1 December 1961

[PEA 32] E.S. Pearson, "Comparison of A.T. McKay's approximation with experimental sampling results", *J.R. Stastist. Soc*., 95, Vol. XCV, Part IV, 703–4, 1932

[PER 03] T. Percheron, *Study of a Specimen Structural Strength to Repetitive Shocks*, (in French), ASTELAB, Paris, France, 2003

[PIE 66] A.G. Piersol, "The development of vibration test specifications for flight vehicle components", *Journal of Sound and Vibration*, 4, no. 1, 88–115, July 1966

[PIE 70] A.G. Piersol and J.R. Maurer, *Investigation of Statistical Techniques to Select Optimal Test Levels for Spacecraft Vibration Tests*, Digitek Corporation, Report 10909801 – F, October 1970

[PIE 74] A.G. Piersol, "Criteria for the optimum selection of aerospace component vibration test levels", *IES Proceeding*, 88–94, 1974

[PIE 92] L. Pierrat, "Estimation de la probabilité de défaillance par interaction de deux lois de Weibull", *Revue de Statistique Appliquée*, Vol. XXXX, no. 4, 5–13, 1992

[PIE 96] A.G. Piersol, "Procedures to compute maximum structural responses from predictions or measurements at selected points", *Shock and Vibration*, Vol. 3, no. 3, 211–21, 1996

[PIE 09a] L. Pierrat, Probabilistic design based on the stress–strength method – Part 1: interaction between two different normal distributions, (text in French), Internal Report, LJ-Consulting, 2006 (to be published 2009).

[PIE 09b] L. Pierrat, Probabilistic design based on the stress–strength.method – Part 2: interaction between two different log–normal distributions, (text in French), Internal Report, LJ-Consulting, 2007 (to be published 2009).

[PIE 09c] L. Pierrat, Probabilistic design based on the stress–strength.method – Part 3: interaction between two Weibull distributions, (text in French), Internal Report, LJ-Consulting, 2008 (to be published 2009).

[PLU 57] R. Plunkett, “Problems of environmental testing”, *The Shock and Vibration Bulletin*, no. 25, Part 2, 67–9, December 1957

[PUG 66] A.G. Pugsley, *The Safety of Structures*, Edward Arnold Limited, London, 1966

[RAC 69] R.L. Racicot, Random vibration analysis – Application to wind loaded structures, Solid Mechanics, Structures and Mechanical Design Division Report no.30, PhD Thesis, Case Western Reserve University, June 1969

[RAV 69] M.K. Ravindra, A.D. Heaney and N.C. Lind, “Probalistic evaluation of safety factors”, *Symposium on Concepts of safety of structures and Methods of design,* International Association of Bridge and Structural Engineering, London, 35–46, 1969

[RAV 78] M.K. Ravindra and T.V. Galambos, “Load and resistance factor design for steel”, *Journal of the Structural Division, ASCE*, Vol. 104, no. ST 9, 1337–53, September 1978

[REC 91] Recommandation Générale pour la Spécification de Management de programme, *BNAé*, RG Aéro 000 40, June 1991

[REC 93] Recommandation générale pour la spécification de management de programme dans l’industrie aéronautique et spatiale, AFNOR X 50 – 410, August 1993

[REE 60] W.H. Reed, A.W. Hall and L.E. Barker, Analog techniques for measuring the frequency response of linear physical systems excited by frequency sweep inputs, NASA TN D 508, 1960

[REL 63] “The relationship of specification requirements to the real environment”, *The Shock and Vibration Bulletin*, no. 31, Part 2, 287–301, 1963

[REP 55] “Report on structural safety”, *Structural Engineer*, Vol. 33, 141–9, 1955

[REP 75] Répertoire des normes d’essais en environnement, LRBA, Note Technique E1 200 NT 1/SEM, January 1975

[REP 82] *Répertoire de concepts américains relatifs à l’organisation et aux méthodes concourant à la qualité*, Ministère de la Défense, DGA, Service de la Surveillance Industrielle de l’Armement, 1982

[RES 57] G.J. Resnikoff and G.J. Lieberman, *Tables of the Non-central t-distribution*, Stanford University Press, Stanford, California, 1957

[RIC 44] S.O. Rice, “Mathematical analysis of random noise”, *Bell System Technical Journal*, no. 23 (July 1944) and 24 (January 1945)

[RIC 48] S.O. Rice, “Statistical properties of a sine wave plus random noise”, *Bell System Technical Journal*, Vol. 27, 109–57, January 1948

[RIC 90] D.P. Richards, "A review of analysis and assessment methodologies for road transportation vibration and shock data", *Environmental Engineering*, Vol. 3, no. 4, 23–6, December 1990

[RIC 93] D.P. Richards and B.E. Hibbert, "A Round Robin exercise on road transportation data", *Proceedings I.E.S.*, 336–46, 1993

[RIC 01] D.P. Richards, "A Comparison of the Effects of Transportation with the Def Stan 00-35 Part 3 'Transportation of Material' Test Severity", *SEE Workshop 'Future Vibration & Shock Testing – A Computer Based Methodology'*, Stevenage, England, 15th May 2001

[ROB 86] W.B. Roberts, "Derivation of external store viration test spectra from flight data", *Journal of Environmental Sciences*, Vol. 29, no. 5, 22–5, September–October, 1986

[ROS 82] E.S. Rosenbaum, F.L. Gloyna, "Tomahawk cruise missile flight environmental measurement program", *The Shock and Vibration Bulletin*, no. 52, Part 3, 159–228, May 1982

[RUB 64] S. Rubin, "Introduction to dynamics", *Proceedings IES*, 3–7, 1964

[SCH 60] R.F. Schoof, "How much safety factor?", *Allis – Charmers Electrical Review*, First Quarter, 21–14, 1960

[SCH 66] J.W. Schlue, "The dynamic environment of spacecraft surface transportation", NASA, Technical Report, no. 32, 876, March 15, 1966

[SCH 81] J.H. Schmidt, "'Quick look' assessment and comparison of vibration specifications", *The Shock and Vibration Bulletin*, 51, Part 2, 73–9, May 1981

[SHI 01] J.E. Shigley and C.R. Mischke, *Mechanical Engineering Design*, 6th ed., Mc Graw Hill Inc., 2001

[SHO 68] *Shock and Vibration Technical Design Guide*, *Volume I: Methodology and design philosophy*, *Volume II: Analytical procedures*, *Volume III: Related technologies*, AD 844 559, Hughes Document no. FR 68-10-671, 1968

[SIL 65] A.J. Silver, "Problems in adding realism to standard specifications", *The Shock and Vibration Bulletin*, no. 34, Part 4, 133–6, February 1965

[SIM 97] R. Simmons, *Creating a Random Vibration Component Test Specification*, Goddard Space Flight Center, NASA, August 1997. Available at: http://femci.gsfc.nasa.gov/random/randomtestspec.html

[SMA 56] E.F. Small, "A unified philosophy of shock and vibration testing for guided missiles", *Proceedings IES*, 277–82, 1956

[SPE 61] H.R. Spence and H.H. Luhrs, "Structural fatigue under combined sinusoidal and random vibration", *Journal of the Acoustical Society of America*, Vol. 33, no. 8, 1, 1098–101, August 1961

[SPE 62] H.R. Spence and H.N. Luhrs, "Structural fatigue under combined random and swept sinusoidal vibration", *Journal of the Acoustical Society of America*, Vol. 34, no. 8, 1098–101, August 1962

[SPE 92] Spectre des réponses extrêmes d'un système à un degré de liberté soumis à une excitation gaussienne combinée à une raie sinusoidale, Ministère de la Défense, DGA, Essais Généraux en Environnement des Matériels, GAM EG 13, Annexe Générale Mécanique, Annexe 12, 1992

[STA 65] C.V. Stahle, "Some reliability considerations in specification of vibration test requirements for non recoverable components", *The Shock and Vibration Bulletin*, no. 34, Part 4, 147–52, February 1965

[STA 67] C.V. Stahle, "Estimate of effect of spacecraft vibration qualification testing on reliability", *The Shock and Vibration Bulletin*, no. 36, Part 7, 1–18, 1967

[STA 75] C.V. Stahle, H.G. Gongloff and W. Keegan, "Development of component random vibration requirements considering response spectra", *The Shock and Vibration Bulletin*, no. 6, Part 1, 60–1, October 1975

[STA 76] C.V. Stahle, H.R. Gongloff and W.B. Keegan, "Development of component random vibration requirements considering response spectra", *The Shock and Vibration Bulletin*, no. 46, Part IV, 57–75, August 1976

[STE 81] M. Steininger and G. Haidl, "Vibration qualification of external A/C stores and equipment", *AGARD Conference Proceedings*, Vol CP 318, no. 9, 12–81, 1981

[STO 61] J.P. Stoll, "Methods of vibration analysis for combined random and sinusoidal inputs", *The Shock and Vibration Bulletin*, no. 29, Part IV, 153–78, June 1961

[STR 60] C.G. Stradling, "How good are random-wave vibration tests?", *Product Engineering*, Vol. 31, no. 49, 5, 82–3, December 1960

[STU 67] The study of mechanical shock spectra for spacecraft applications, NASA – CR 91356, 1967

[SUC 75] M. Suchaud, La qualification d'un matériel technique et son contenu. Synthèse d'une enquête et d'une étude bibliographique de documents français et étrangers, Note Aérospatiale, DET no. 13 248/E/EXM, 1975

[TAN 63] S. Tanaka, "On cumulative damage in impulse fatigue tests", *Trans. of the ASME, Journal of Basic Eng.*, 85, 535–8, December 1963

[TRO 72] W.D. Trotter and D.V. Muth, "Combined-axis vibration testing of the SRAM missile", *The Shock and Vibration Bulletin*, Part 3, 39–48, 1972

[TUS 67] W. Tustin, "Vibration and shock tests do not duplicate service environment", *Test Engineering*, Vol. 18, no. 2, 18–20, August 1967

[TUS 73] W. Tustin, "Basic considerations for simulation of vibration environment", *Experimental Mechanics*, 390–6, September 1973

[VES 72] A. Vessereau, "Note sur les intervalles statistiques de dispersion", *Revue de Statistique Appliquée*, Vol. XX, no. 1, 67–87, 1972

[VIS 48] J.P. Visodic, , "Design Stress Factors," in Vol. 55, *Proceedings of the ASME*, ASME International, New York, May 1948

[WAN 45] M.C. Wang and G.E. Uhlenbeck, “On the theory of Brownian motion II”, *Reviews of Modern Physics*, Vol. 17, no. 2 and 3, 323–42, April–July 1945

[WIR 76] P.H. Wirsching and J.T.P. Yao, “A probabilistic design approach using the Palmgren–Miner hypothesis”, *Methods of Structural Analysis, ASCE*, Vol. 1, 324–39, 1976

[WIR 80] P.H. Wirsching, “Fatigue reliability in welded joints of offshore structures”, *Int. J. Fatigue*, Vol. 2, no. 2, 77–83, April 1980

[WIR 83a] P.H. Wirsching, “Probability-based fatigue design criteria for offshore structures”, *Final Project Report API-PRAC 81-15*, American Petroleum Institute, January 1983

[WIR 83b] P.H. Wirsching, Statistical summaries of fatigue data for design purposes, NASA – CR 3697, 1983

[YOK 65] T. Yokobori, “The strength, fracture and fatigue of materials”, in P. Noordhoff, *Strength, Fracture and Fatigue of Materials*, Groningen, Netherlands, 1965

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