CHAPTER

3

S-PARAMETERS, SIGNAL FLOW GRAPHS, AND OTHER MATRIX REPRESENTATIONS

3.1 S-PARAMETERS

In chapter 2 it was shown that a voltage waves propagating to the right and left could be represented, respectively, as

$$V^{+} = Ae^{jbd} \tag{3.1}$$

$$V^{-} = Be^{-jbd} \tag{3.2}$$

In the analysis of distributed circuits it is convenient to define normalized voltage variables, called *signal parameters* as follows

$$a = \frac{V^{+}}{\sqrt{Z_0}} \tag{3.3}$$

$$b = \frac{V^{-}}{\sqrt{Z_0}} \tag{3.4}$$

This definition results in $|a|^2$ equating to the power propagating to the right and $|b|^2$ equating to the power propagating to the left. In characterizing a linear device or a linear circuit, it is convenient to consider its response when subjected to an incident propagating signal. In such an experiment the circuit is called a *Device Under Test* (DUT). For a two port circuit the test can be visualized as starting with an infinitely long, lossless, transmission line with characteristic impedance Z_0 and inserting the two port DUT as shown in Figure 3.1. An incident signal from the left, designated a_1 results in a reflected signal, designated b_1, propagating back towards the left, and a transmitted signal, designated b_2, which emerges from the device and propagates to the right.

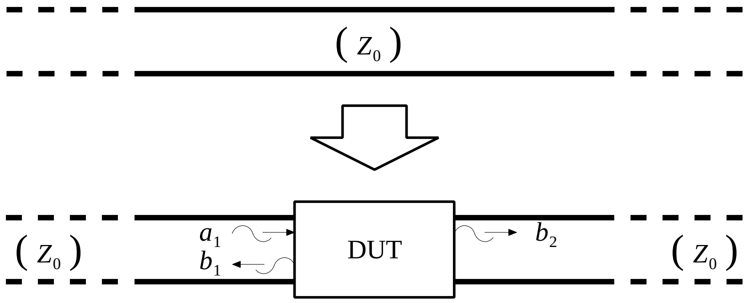


Figure 3.1 When a *Device Under Test* (DUT) is inserted in an infinitely long transmission line with an incident signal a_1 propagating to the right, a reflected signal, b_1, and transmitted signal, b_2, result.

The subscripts "1", and "2" refer to the left and right ports, respectively. If the device is linear then the stimulated signals are related to the incident signal by a constant. That is,

$$b_1 = S_{11}a_1 \text{ (incident signal in port 1 only)}$$
$$b_2 = S_{21}a_1 \text{ (incident signal in port 1 only)}$$

Likewise, if the device is hit with a signal from the right then again a reflected signal and a transmitted signal are generated. In this case

$$b_1 = S_{12}a_2 \text{ (incident signal in port 2 only)}$$
$$b_2 = S_{22}a_2 \text{ (incident signal in port 2 only)}$$

Since the device is linear, superposition holds and the combined response given two incident signals, one from left and one from right, can be written as

$$b_1 = S_{11}a_1 + S_{12}a_2 \tag{3.5}$$
$$b_2 = S_{21}a_1 + S_{22}a_2 \tag{3.6}$$

The parameters S_{ij} are called scattering parameters. This terminology conveys the idea that an incident signal impinges upon a target, the device under test, and signals are scattered away. For a two port device scattered signals can occur in only two directions, from ports 1 scattered signals propagate to the left and from port 2 scattered signals propagate to the right. The scattering parameters and incident and reflected signals can be represented in matrix form where the square matrix of s-parameters is called the *scattering matrix*.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{3.7}$$

The scattering matrix together with the underlying propagation environment, in this case transmission lines of characteristic impedance Z_0, uniquely characterize a two port device. Therefore, when a device is characterized by scattering parameters the reference characteristic impedance must also be given. Frequently, the characteristic impedance is 50Ω, which is often assumed unless a different value is explicitly specified.

The concept of a scattering matrix with scattering parameters has been developed for a two port device under test, but naturally extends its self to devices with any number of ports by assuming that transmission lines are connected to each port. Again the characteristic impedance of the lines must be known for the scattering matrix to define the behavior of the circuit.

$$\begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \begin{pmatrix} S_{11} & \cdot & \cdot & S_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ S_{n1} & \cdot & \cdot & S_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \tag{3.8}$$

Since a measurement environment consisting of an infinitely long transmission line is not possible, it is necessary to consider how one might practically measure the s-parameters. Each of the s-parameters are a ratio of scattered signal to the incident signal. Therefore, a situation must be created in which only one incident signal can exist. To do this requires a *matched source* and a *matched load*. To measure $|S_{11}|$ and $|S_{21}|$ the source is connected to the transmission line on the left and the matched load to the transmission line on the right as shown in Figure 3.2.

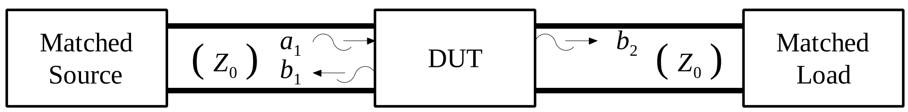


Figure 3.2 A matched source and a matched load simulate an incident signal on an infinitely long transmission.

The matched load insures that the scattered signal b_2 does not interact with the end of the line and produce a reflection that would propagate from the right towards the left and becomes a second, unwanted incident signal. That is, the matched load insures that $a_2 = 0$. Under this measurement condition the s-parameters are given by (3.9a) and (3.9b). If the set up is reversed then one can measure the other two s-parameters, i.e. (3.9c) and (3.9d).

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \tag{3.9a}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \tag{3.9b}$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \tag{3.9c}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \tag{3.9d}$$

It is possible and sometimes desirable to have a different reference impedance for each of the ports. This is illustrated in Figure 3.3 for a two port circuit.

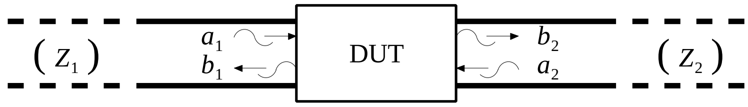


Figure 3.3 In general it is possible for the input transmission line and the output transmission line to be of different impedance.

In this case when a matched load is connected it must be appropriate for the particular line. And, as expected the values of the s-parameters may be different since the generating and scattered signals are propagating in a different environment. The normalization for the signal parameters is different for each of the ports. Again it is essential in characterizing a linear device to know both the s-parameter values and the reference impedance under which they were determined. To illustrate the determination of s-parameters a two-port device under test is created which consists of a shunt resistor. While this is a simple circuit its characterization will serve to illustrate the theory.

Example 3.1.1 *What are the s-parameters for the two port device (circuit) in Figure 3.4 where the input and output reference impedance is 50 W ?*

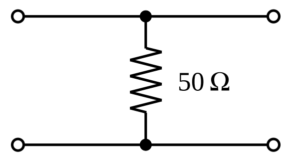


Figure 3.4 The Device Under Test (DUT) consists of a shunt 50 Ω resistor.

To determine S_{11}, a matched load is connected to port 2 as shown in Figure 3.5(a). An incident signal $a_1 = 1\big/\sqrt{Z_0}$ is assumed to be propagating into port 1, equating to an incident voltage $V_1^+ = 1$. Determination of S_{11} is equivalent to finding the input reflection coefficient, Γ_{IN}, which in turn can be found by computing the input impedance. This is accomplished in equations (3.10a) and (3.10b). The reflected voltage is $V_1^- = -1/3$, implying $b_1 = (-1/3)\big/\sqrt{Z_0}$, and therefore , S_{11} can be computed as the ratio of b_1 to a_1.

$$Z_{IN} = 25\Omega \tag{3.10a}$$

$$\Gamma_{IN} = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} = \frac{25-50}{25+50} = -\frac{1}{3} \tag{3.10b}$$

$$S_{11} = \frac{b_1}{a_1} = -\frac{1}{3} \tag{3.11}$$

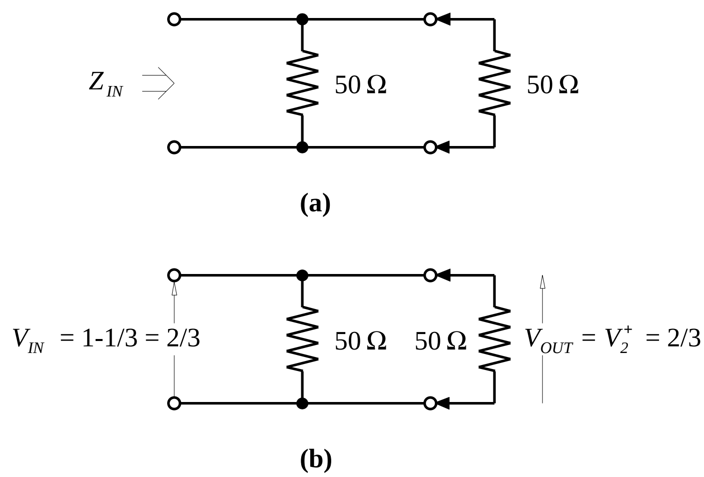


Figure 3.5 **(a)** $S_{11}$is calculated by first finding the input impedance with the load terminated in the reference impedance, 50 Ω in this case.
(b) The total input voltage is found by adding the incident voltage, 1, to the reflected voltage, $-1/3$.

The transfer s-parameter, S_{21}, can be found by realizing that the total input voltage equals $V_{IN} = V_1^+ + V_1^- = V_1^+\left(1+\Gamma_{IN}\right)$, and since $V_1^+ = 1$, $V_{IN} = 1+\Gamma_{IN} = 1-1/3 = 2/3$. Knowing this voltage one can find the voltage across the output which equals the voltage propagated away from port 2. Since $V_{OUT} = V_2^+ + V_2^-$, and, in this case, $V_{OUT} = V_{IN}$ and $a_2 = V_2^- \big/ \sqrt{Z_0} = 0$, consequently $V_2^+ = 2/3$ as illustrated in Figure 3.5(b). Therefore $b_2 = (2/3)\big/\sqrt{Z_0}$, and the transfer s-parameter $S_{21} = b_2/a_1 = 2/3$. Since the circuit is symmetric then $S_{11} = S_{22}$ and $S_{21} = S_{12}$ and the scattering matrix is given by

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \tag{3.12}$$

Example 3.1.2 *Find the s-parameters for the DUT in Figure 3.4 assuming an input and output reference impedance of 75 W..*

The procedure is similar to that used in the previous example. S_{11} is determined by connecting a matched load to port 2, this time equal to 75 Ω, and determining Z_{IN} and Γ_{IN} as illustrated in Figure 3.6(a). Assuming an incident signal $a_1 = 1\big/\sqrt{Z_0}$ propagate into port 1 which equates to an incident voltage $V_1^+ = 1$ implies that the reflected voltage $V_1^- = -3/7$, and the reflected signal $b_1 = (-3/7)\big/\sqrt{Z_0}$

$$Z_{IN} = \frac{50 \cdot 75}{50+75} = 30\Omega \tag{3.13a}$$

$$\Gamma_{IN} = \frac{30-75}{30+75} = -\frac{3}{7} \tag{3.13b}$$

$$S_{11} = \frac{b_1}{a_1} = -\frac{3}{7} \tag{3.14}$$

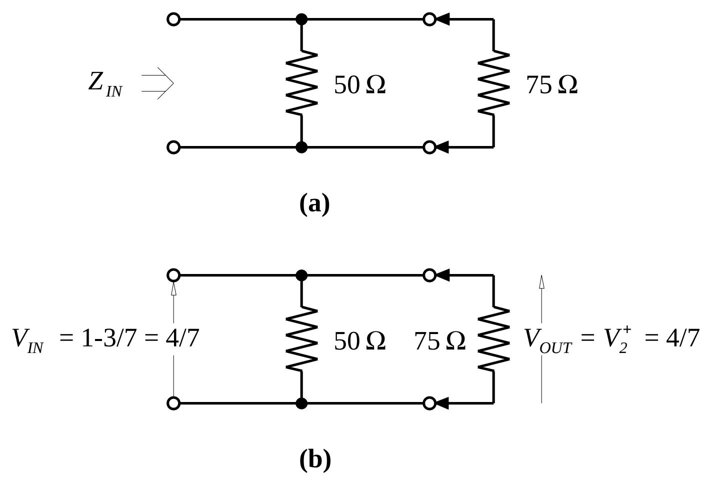


Figure 3.6 **(a)** The reflection coefficient is found by first determining the input impedance with the output terminated in the reference impedance, 75 Ω in this case.
(b) The total voltage at the input is the sum of the incident and reflected voltages.

The transfer s-parameter S_{21} is found in a similar manner to the previous example. First, the total input voltage is found which is used to determine the output voltage. The output voltage must equal the propagating voltage wave at port 2. Therefore, $b_2 = (4/7)\big/\sqrt{Z_0}$, and the transfer s-parameter $S_{21} = b_2/a_1 = 4/7$. Again the circuit is symmetric so $S_{11} = S_{22}$ and $S_{21} = S_{12}$. The scattering matrix is given by equation (3.15) which is observed to be different than that obtained above for the 50Ω input and output reference impedance.

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} -3/7 & 4/7 \\ 4/7 & -3/7 \end{pmatrix} \tag{3.15}$$

Example 3.1.3 *Find the s-parameters for the DUT in Figure 3.4 assuming an input reference impedance of 50 W and an output reference impedance of 75 W..*

To determine S_{11} a matched load is connected to port 2, this time equal to 75 Ω. The determination of Z_{IN} and Γ_{IN} are determined as shown below. An incident signal $a_1 = 1\big/\sqrt{Z_1}$ is assumed to propagate into port 1. This equates to an incident voltage $V_1^+ = 1$ which results in a reflected voltage $V_1^- = -1/4$, and a reflected signal $b_1 = (-1/4)\big/\sqrt{Z_1}$. Thus, we see that

$$Z_{IN} = \frac{50 \cdot 75}{50 + 75} = 30\Omega \tag{3.16a}$$

$$\Gamma_{IN} = \frac{30 - 50}{30 + 50} = -\frac{1}{4} \tag{3.16b}$$

$$S_{11} = \frac{b_1}{a_1} = -\frac{1}{4} \tag{3.17}$$

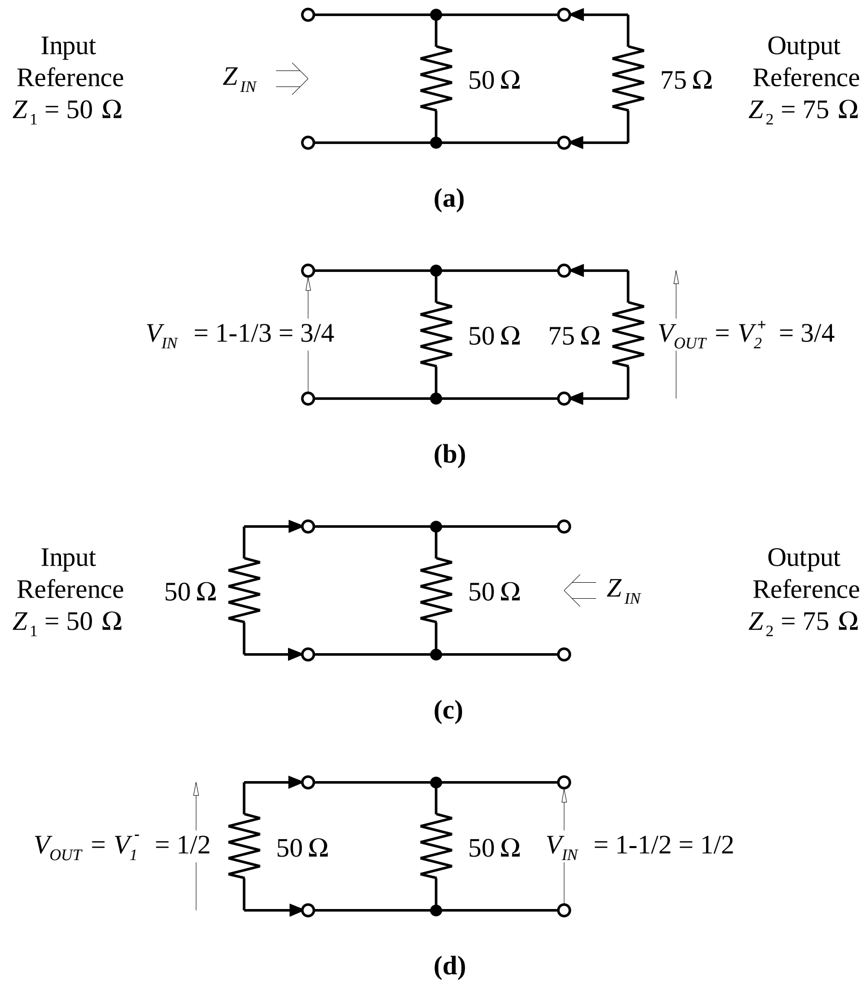


Figure 3.7 **(a)** The procedure starts by attaching a load equal to the output reference impedance and determining the input impedance and reflection coefficient.
(b) The total input voltage is next calculated (incident + reflected) and the output voltage determined.
(c) Now an incident signal is assumed to impinge on the left hand side of the DUT and hence the process is repeated from that point of view.
(d) The total voltage on the left hand side of the DUT is calculated and the resulting output voltage (right hand side) is determined.

Determination of the transfer s-parameter S_{21} is determined by first finding the total input voltage and then using it and the circuit configuration to determine the output or exiting voltage. In this case $b_2 = (3/4)/\sqrt{Z_1}$, and S_{21} equals

$$S_{21} = \frac{b_2}{a_1} = \frac{\dfrac{(3/4)}{\sqrt{Z_2}}}{\dfrac{1}{\sqrt{Z_1}}} = \left(\frac{3}{4}\right)\sqrt{\frac{Z_1}{Z_2}} = \left(\frac{3}{4}\right)\sqrt{\frac{50}{75}} = \left(\frac{3}{4}\right)\sqrt{\frac{2}{3}} = \frac{\sqrt{3}}{2\sqrt{2}} \qquad (3.18)$$

Symmetry no longer exists because of the different input and output reference impedance and one therefore must calculate S_{22} and S_{12} separately. To this end let an incident signal $a_2 = 1/\sqrt{Z_2}$ propagate into port 2. This equates to an incident voltage $V_2^+ = 1$. For this calculation port 2 now becomes the input and port 1 the output. The calculation of Z_{IN} and Γ_{IN} are illustrated below. The reflected voltage $V_2^- = -1/2$, and the reflected signal $b_2 = (-1/2)/\sqrt{Z_2}$ are used to determine S_{22}.

$$Z_{IN} = 25\Omega \tag{3.19a}$$

$$\Gamma_{IN} = \frac{25-75}{25+75} = -\frac{1}{2} \tag{3.19b}$$

$$S_{22} = \frac{b_2}{a_2} = -\frac{1}{2} \tag{3.20}$$

Determination of the transfer s-parameter S_{12} follows from the total input voltage which is used to determine the output voltage which equate the voltage wave exiting port 1. Therefore, $b_1 = (1/2)/\sqrt{Z_1}$, and S_{12} is given by equation (3.21). The scattering matrix is shown below and has values different from those obtained earlier. Notice that still $S_{12} = S_{21}$. This illustrates that a circuit may be *asymmetric (in this case because a different impedance is used for the input and output reference) but still reciprocal*.

$$S_{12} = \frac{b_1}{a_2} = \frac{\dfrac{(1/2)}{\sqrt{Z_1}}}{\dfrac{1}{\sqrt{Z_2}}} = \left(\frac{1}{2}\right)\sqrt{\frac{Z_2}{Z_1}} = \left(\frac{1}{2}\right)\sqrt{\frac{75}{50}} = \left(\frac{1}{2}\right)\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2\sqrt{2}} \tag{3.21}$$

$$\mathbf{S} = \begin{pmatrix} -1/4 & \sqrt{3}/2\sqrt{2} \\ \sqrt{3}/2\sqrt{2} & -1/2 \end{pmatrix} \tag{3.22}$$

Example 3.1.4 *Find the s-parameters $(ref = Z_0)$ for a circuit consisting of a quarter wave transmission line with characteristic impedance Z_1.*

The calculation of S_{11} is illustrated below.

$$Z_{IN} = \frac{Z_1^2}{Z_0} \tag{3.23a}$$

$$\Gamma_{IN} = \frac{\dfrac{Z_1^2}{Z_0} - Z_0}{\dfrac{Z_1^2}{Z_0} + Z_0} = \frac{Z_1^2 - Z_0^2}{Z_1^2 + Z_0^2} \tag{3.23b}$$

$$b_1 = \left(\frac{Z_1^2 - Z_0^2}{Z_1^2 + Z_0^2}\right) \cdot \frac{1}{\sqrt{Z_0}} \tag{3.24}$$

$$S_{11} = \frac{b_1}{a_1} = \left(\frac{Z_1{}^2 - Z_0{}^2}{Z_1{}^2 + Z_0{}^2} \right) \tag{3.25}$$

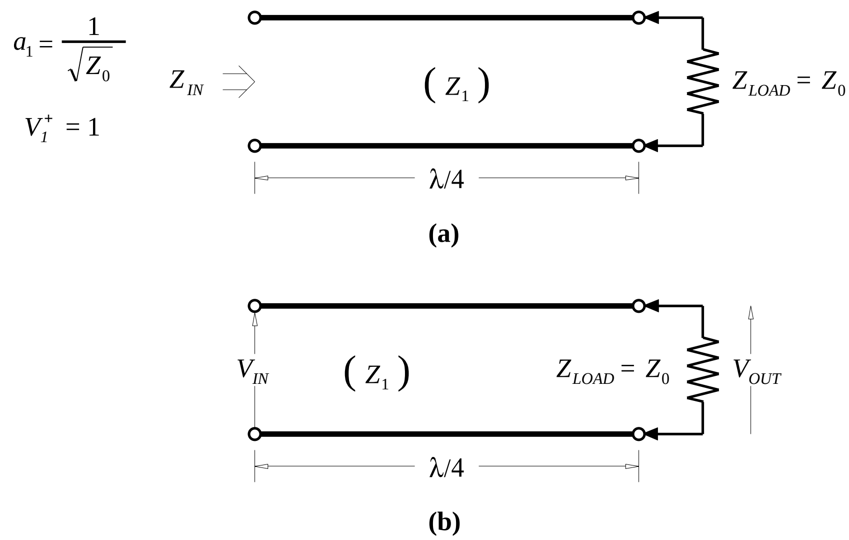


Figure 3.8 **(a)** The input impedance and reflection coefficient is found by connecting a load equal to the reference impedance.
(b) S_{21} is found by determining the total input voltage and then determining the output voltage.

The calculation of S_{21} is illustrated below. Using the voltage transfer ratio for quarter wave transmission lines permits the output voltage to be found from the total input voltage.

$$V_{IN} = 1 + \Gamma_{IN} = 1 + \frac{Z_1{}^2 - Z_0{}^2}{Z_1{}^2 + Z_0{}^2} = \frac{2Z_1{}^2}{Z_1{}^2 + Z_0{}^2} \tag{3.26}$$

$$V_{OUT} = -j\frac{Z_0}{Z_1} \cdot V_{IN} = -j\frac{Z_0}{Z_1}\left(\frac{2Z_1{}^2}{Z_1{}^2 + Z_0{}^2} \right) = \frac{-2j \quad Z_0 \quad Z_1}{Z_1{}^2 + Z_0{}^2} \tag{3.27}$$

$$b_2 = \left(\frac{-2j \quad Z_0 \quad Z_1}{Z_1{}^2 + Z_0{}^2} \right) \cdot \frac{1}{\sqrt{Z_0}} \tag{3.28}$$

$$S_{21} = \frac{b_2}{a_1} = \frac{-2j \quad Z_0 \quad Z_1}{Z_1{}^2 + Z_0{}^2} \tag{3.29}$$

$$\mathbf{S} = \frac{1}{Z_1{}^2 + Z_0{}^2} \cdot \begin{pmatrix} Z_1{}^2 - Z_0{}^2 & -2j \quad Z_0 \quad Z_1 \\ -2j \quad Z_0 \quad Z_1 & Z_1{}^2 - Z_0{}^2 \end{pmatrix} \tag{3.30}$$

3.2 SIGNAL FLOW GRAPHS

Signal and Scattering parameters can be described using a graphical technique called *Signal Flow Graphs* (SFG). In this technique variables which are related via a multiplicative constant are represented as *nodes* connected by *paths* with assigned values called parameters. The assigned value is equal to the multiplication constant. Consequently, $y = cx$ would be represented as Figure 3.9(a) and $w = c_1 x + c_2 y$ could be represented as Figure 3.9(b).

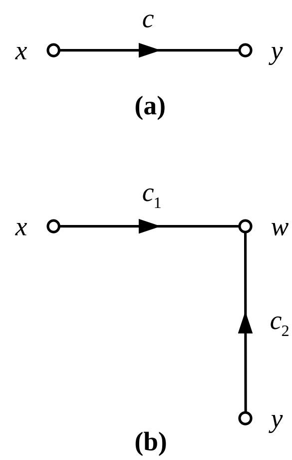


Figure 3.9 **(a)** A signal flow graphs consists of node or variable (for example x and y), and paths which show variable dependency and proportionality constants such as "c".

(b) SFG representation of $w = c_1 x + c_2 y$.

This technique can be used to represent the relationship of signals via their scattering parameters. That is, the signal parameters would be represented by nodes and the s-parameters would be represented by the paths. A two port device is described by the equations (3.5) and (3.6) restated below or by the SFG shown in Figure 3.10. When signal flow graphs are used to represent the signal, there will always be two nodes associated with each circuit port. *One node represents the entering signal and the second represents the exiting signal*. In signal flow graphs if two nodes are connected by a single path with a value of unity then the nodes are equivalent.

$$b_1 = S_{11} a_1 + S_{12} a_2 \tag{3.5}$$
$$b_2 = S_{21} a_1 + S_{22} a_2 \tag{3.6}$$

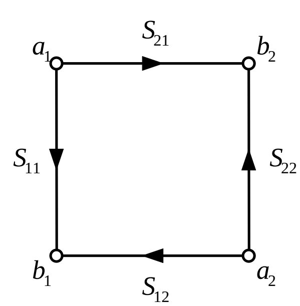
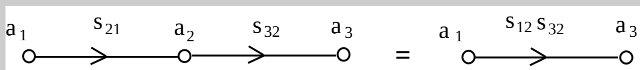


Figure 3.10 A signal flow graph representation of the scattering matrix

Signal flow graphs are sometimes useful in analyzing microwave circuits because it is usually possible do describe the flow of signals in a circuit, but may be difficult to solve a complex set of boundary conditions for various sources and loads. Generally, one is interested in obtaining the relationship between various signals. This is obtained by determining transfer ratios between signal variables, e.g. the ratio of the reflected signal on the load to the exiting signal at the source. Such ratios can be obtained from signal flow graphs by considering the denominator signal variable to be independent variable (i.e. independent node) and the numerator signal variable to be a dependent variable (dependent node) and to reduce the SFG until only one path exists between the two nodes. In order to perform such a reduction, it is necessary to understand only 4 fundamental rules. These are discussed below.

SERIES RULE

S-parameters for paths in series can be combined into one path by multiplying the s-parameters.

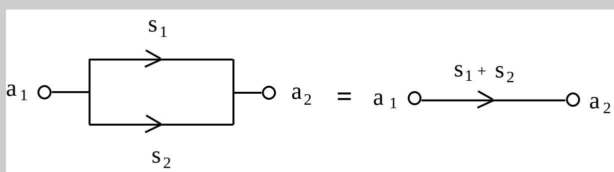


Proof:

$$a_2 = S_{21} a_1$$
$$a_3 = S_{32} a_2$$
$$a_3 = (S_{32} \qquad S_{21}) \qquad a_2$$

PARALLEL RULE

S-parameters for multiple paths connecting the same two nodes can be combined into a single path by adding the s-parameters.

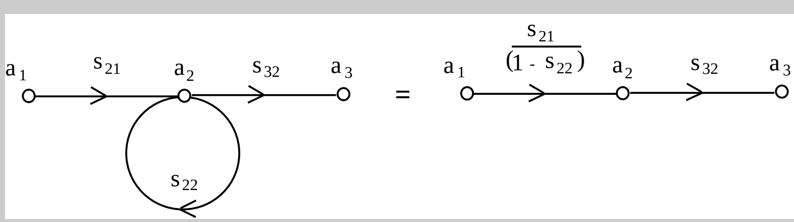


Proof:

$$a_2 = S_1 a_1 + S_2 a_1$$
$$a_2 = (S_1 + S_2) a_1$$

RECURSION RULE

A path with s-parameter "*S*" which returns to the same node can be eliminated if all other paths into the node are divided by (1-*S*)

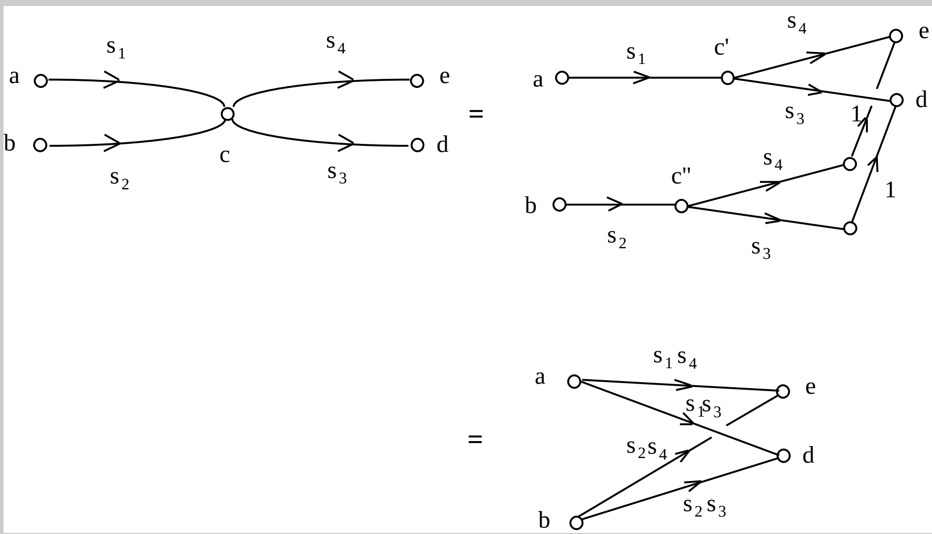


Proof:

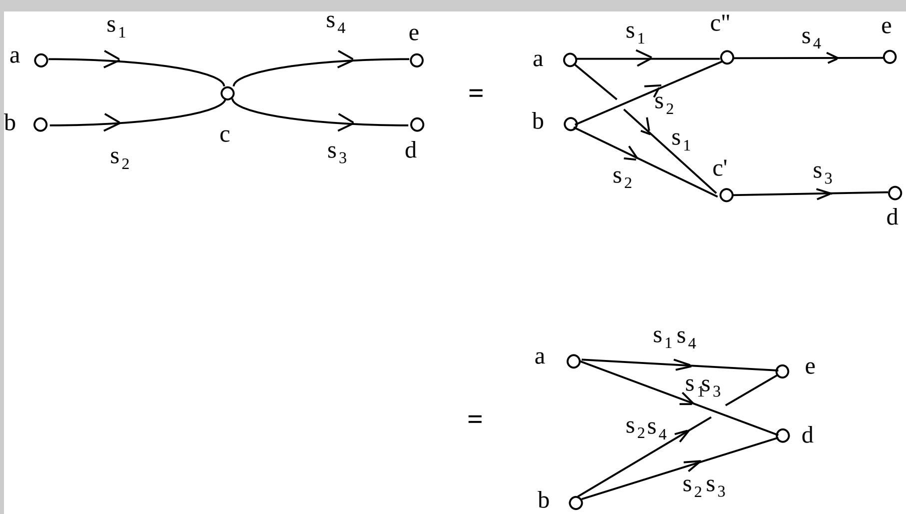
$$a_2 = S_{21}a_1 + S_{22}a_2$$
$$(1 - S_{22})a_2 = S_{21}a_1$$
$$a_2 = \frac{S_{21}}{(1 - S_{22})} \cdot a_1$$

NODE SPLITTING RULE

A node with multiple input or output paths can be split into multiple equivalent nodes with each of the separate input paths connected to the new nodes provided that each new node contains all of the output paths. Similarly, the nodes will be equivalent if each new node connects to each of the original output paths provided that each new node is connected to all of the original input paths.



or



Proof:

$$
\begin{aligned}
c &= S_1 a + S_2 b \\
d &= S_3 c \\
e &= S_4 c \\
d &= S_3 (S_1 a + S_2 b) \\
e &= S_4 (S_1 a + S_2 b)
\end{aligned}
$$

$$
\begin{aligned}
d &= (S_3 S_1) a + (S_3 S_2) b \\
e &= (S_4 S_1) a + (S_4 S_2) b
\end{aligned}
$$

The signal flow graph representation of a *source* can be derived by examining the scattering parameters of a voltage and impedance circuit when connected to a transmission line as illustrated in Figure 3.11. Assume a forward traveling voltage wave Ae^{jbd} impinges upon the load, which in this case

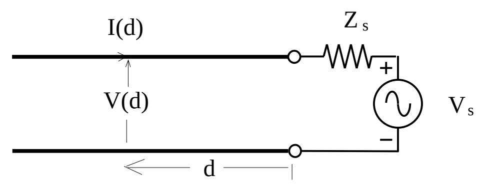


Figure 3.11 The SFG representation of a source is found by connecting a source to a transmission whose characteristic impedance is the reference impedance for the s-parameters.

happens to include an active element equal to Vs. The total voltage and current on the line equals (3.30a) and (3.30b), where B is an unknown constant. The constant A in front of the Ae^{jbd} is assumed to be

$$V(d) = Ae^{jbd} + Be^{-jbd} \tag{3.30a}$$

$$I(d) = \frac{\left(Ae^{jbd} - Be^{-jbd}\right)}{Z_o} \tag{3.30b}$$

known if the incident voltage is given. Applying Kirchoff's Laws at the point $d = 0$ reveals the relationship between V(0) and I(0) that must exist at that point in the circuit to insure that Kirchoff's Laws are satisfied. Such constraints are sometimes called *boundary conditions*. Substituting V(0) from equation (3.31a) and I(0) from equation (3.31b) into equation (3.31c) results in a relationship between A and B which can be represented as a SFG.

$$V(0) = A + B \ , \tag{3.31a}$$

$$I(0) = \frac{(A - B)}{Z_o}, \tag{3.31b}$$

$$V(0) = Z_s I(0) + V_s \tag{3.31c}$$

$$B = \left(\frac{Z_s - Z_o}{Z_s + Z_o} \right) A + \left(\frac{Z_o}{Z_s + Z_o} \right) V_s \tag{3.32a}$$

$$B = \Gamma_s A + \left(\frac{Z_o}{Z_s + Z_o} \right) V_s \tag{3.32b}$$

If one lets *a* be the signal parameter representing the incident signal, and *b* be the signal parameter for the reflected signal then $a = A/\sqrt{Z_o}$, and $b = B/\sqrt{Z_o}$ and equation (3.32b) becomes

$$b = \Gamma_s + b_s \tag{3.33a}$$

where

$$b_s = \left(\frac{Z_o}{Z_s + Z_o} \right) \frac{V_s}{\sqrt{Z_o}} . \tag{3.33b}$$

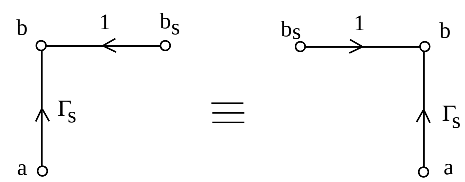


Figure 3.12 SFG representation of a source consists of an, independent node "b_s"

From previous work we it is clear that b_s is the normalized voltage that is initially launched by the source V_s onto the line with characteristic impedance Z_o. That is, the signal parameter b_s represents a voltage which has been divided by two series impedance of Z_s and Z_o. Since b_s does not depend upon the incident signal it is an independent node. The signal flow diagram for a source is shown in figure 3.12

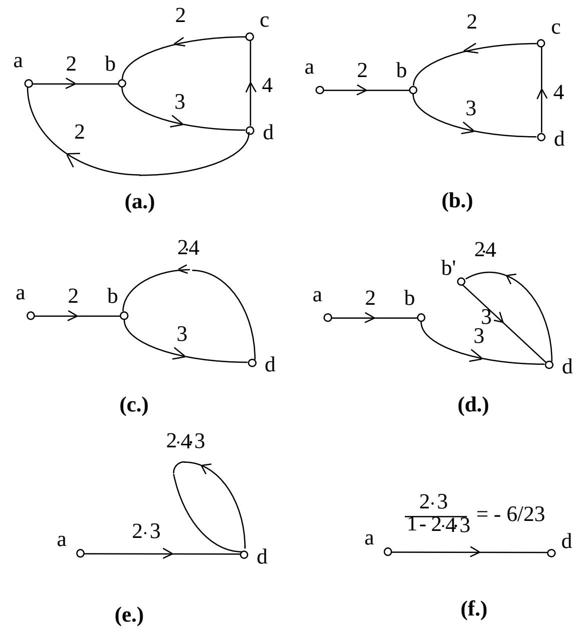


Figure 3.13 **(a.)** SFG to find transfer function d/a . **(b.)** Paths into the independent node "a" eliminated **(c.)** after applying the series rule.**(d.)** after applying the node splitting rule.(**e.)** after series rule applied twice.(**f.)** final SFG=transfer ratio obtained by applying recursion rule.

To illustrate the technique for signal flow graph reduction an example is considered. Assume that it is desired to find the transfer ratio d/a for the SFG illustrated in Figure 3.13a. To determine this ratio the node "a" is taken to be an independent variable (or node) and node "d" is considered to be a dependent variable (or node). The path from d to a can then be dropped since if "a" is known then it can be treated as an independent node and therefore inputs into "a" are not required for its definition. As illustrated in Figure 3.13c the path from "d" to "c" and "c" to "b" can be combined according to the *series rule*. Applying the *node splitting* rule to the node b results in Figure 3.13d. Two sets of paths can be combined using the *series* rule to get Figure 3.13e. Applying the *recursion rule* results in Figure 3.13f and therefore - d/a= -6/23. These principles can often be used effectively to analyze microwave circuits. In order to reinforce the skills it is useful to consider an ac circuits for which the results may be checked by conventional lumped element analysis techniques. .

Example 3.2.5.. *Calculate the power delivered to the load for the circuit shown in Figure 3.14 using conventional techniques.*

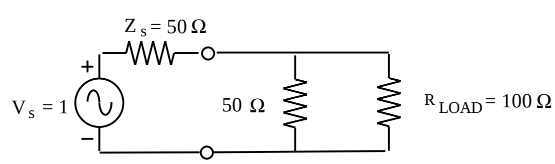


Figure 3.14 Example circuit to analyze using circuit reduction and SFG techniques.

The voltage across the load is seen to be 2/5 v. The power absorbed by the load is therefore,

$$P_{LOAD} = \frac{(2/5)^2}{100} = 1/625 watts \tag{3.34}$$

Example 3.2.6.. *Analyze the circuit in Figure 3.14 using SFG and s-parameters with a input and output impedance reference of 50W.*

The circuit can be viewed as consisting of a source, a load, and an intervening two port circuit (shunt resistor) as illustrated in Figure 3.15.

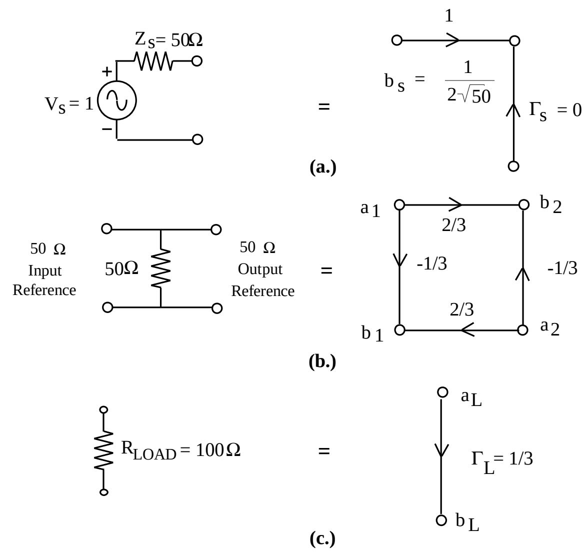


Figure 3.15 **(a.)** The SFG for the source.**(b.)** SFG for the two port (shunt 50Ω resistor) **(c.)** The SFG for the load.

It is desired to calculate the transfer ratio a_L / b_s since that will reveal the amount of power incident upon the load. The reflected power can be determined using the incident power together with the load reflection coefficient. The node b_s will be treated as an independent node (it was one anyway since it has no input paths). The path in Figure 3.16a involving Γ_s drops out since the associated s-parameter is zero. Additionally, the several lower paths in Figure 3.16a drop out since the arrows are opposite to give the simplified SFG in Figures 3.16b and 3.16c. The series and recursion rules can be applied to get the reduced SFG shown in Figures 3.16d. Therefore, $a_L = (3/5)b_s$ and since $b_s = 1/2\sqrt{50}$ then $a_L = 3/10\sqrt{50}$ and $|a_L|^2 = 9/5000$ which is the incident power impinging upon the load. The reflected signal is $b_L = \Gamma_L a_L$ and thus the reflected power is therefore $|b_L|^2 = |\Gamma_L|^2 |a_L|^2$, and substitution of $\Gamma_L = 1/3$ together with the value of a_L yields $|b_L|^2 = 1/5000$. The power delivered to (absorbed by) the load is given by equation (3.35c) which is the same result obtained above by conventional ac circuit analysis.

$$P_L = |a_L|^2 - |b_L|^2 \tag{3.35a}$$

$$= \left(\frac{9}{5000}\right) - \left(\frac{1}{5000}\right) \tag{3.35b}$$

$$P_L = \frac{1}{625} \tag{3.35c}$$

Example 3.2.7.. *Analyze the circuit in Figure 3.14 using SFG and s-parameters with a 50 input and a 100W output impedance reference.*

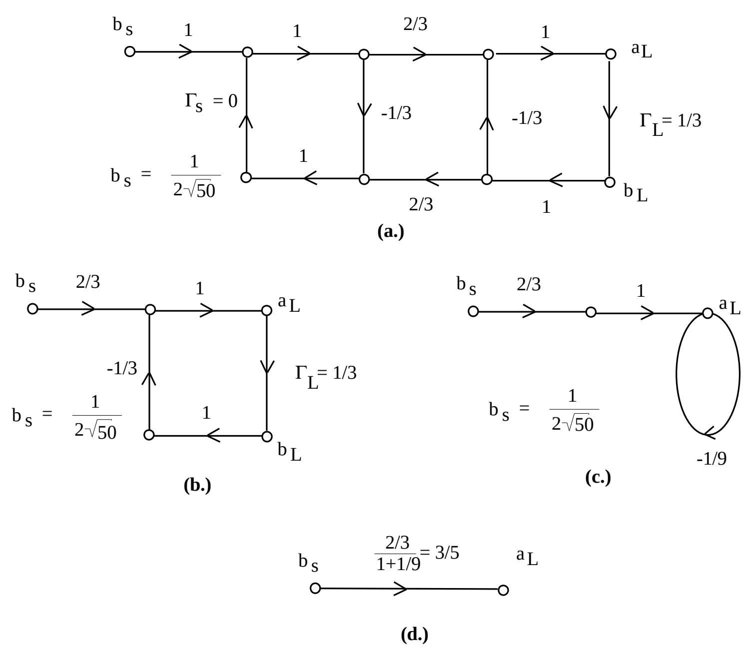


Figure 3.16 **(a.)** The SFG for the total circuit. **(b.)** The simplified SFG **(c.)** the SFG after application of the series rule. (**d.)** The SFG reduced to determine the transfer ratio between the source and the load.

Since the input reference is the same as before the SFG representation of the source remains the same. However, the load and the two port parameters change as illustrated in the figures below.

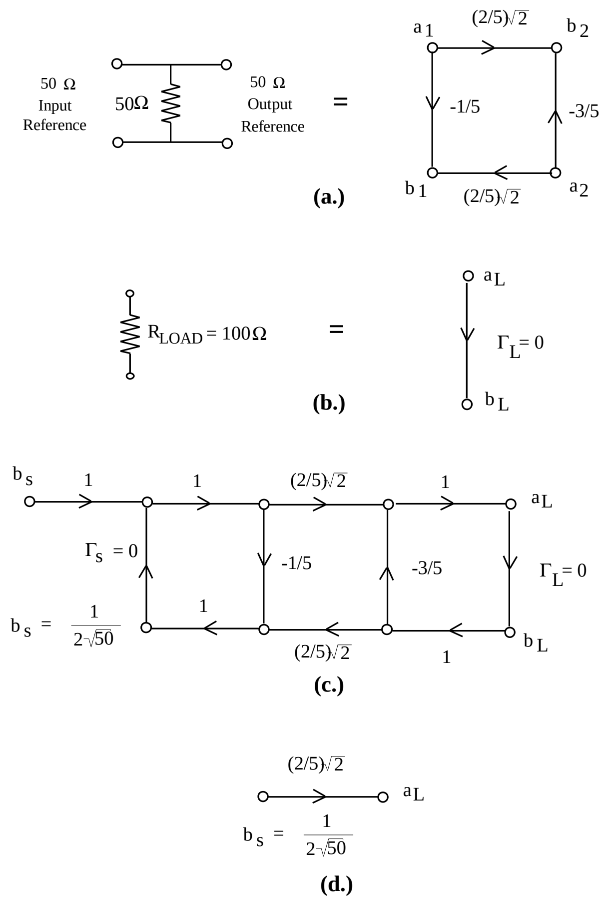


Figure 3.17 **(a.)** The SFG for the load with output reference of 100 Ω..**(b.)** The SFG for the two port circuit with the 50 Ω input and 100 Ω output reference**(c.)** The SFG for the total circuit **(d.)** and the reduced SFG. .

It is necessary to calculate the transfer ratio a_L / b_s since that will reveal the amount of power incident upon the load. In this case it will also be the power delivered to the load since the load reflection coefficient is zero for a 100Ω reference. That is, the load is matched to the 100Ω reference impedance which was used to compute the s-parameters (see previous section). The node b_s is an independent node. The paths involving Γ_s and Γ_L drop out since the associated s-parameters are zero. Again the other paths drop out since the arrows are opposite directions. Therefore, $a_L = (2/5)\sqrt{2} b_s$ and since $b_s = 1/(2\sqrt{50})$ then $|a_L|^2 = 8/5000$ and which is also the total power, i.e.,

$$P_L = \frac{1}{625} \tag{3.35}$$

3.3 S-PARAMETER PROPERTIES.

A *Symmetric* Circuit is an electrical device or combination of elements whose physical characteristics are exactly the same as viewed from each of its ports (terminals). An individual ideal resister, inductor, capacitor, transmission line, etc. are examples of a symmetric component. Combinations of basic elements can result in either a symmetric or asymmetric circuit. The circuits in

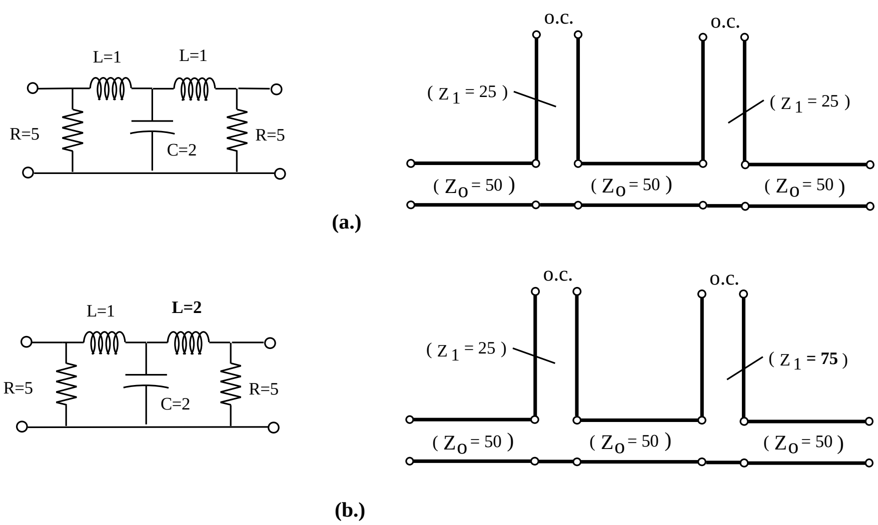


Figure 3.18 **(a.)** Examples of a lumped and distributed element *symmetric* circuits **(b.)** Examples of a lumped and distributed element as*ymmetric* circuits where the circuit topology (configuration of element types) is symmetric but the element values are asymmetric.

The circuits inFigure 3.18a are examples of symmetric circuits. One can see that the ports could be interchanged arbitrarily and the circuits would remain the same. On the other hand the circuits in Figure 3.18b illustrate examples of asymmetric circuits, i.e., the ports can not be interchanged. In this case the circuit topology is symmetric, i.e., the configuration of element types is the same but the element values violate symmetry. The circuit in figure 3.19 is clearly asymmetric since the pattern of element types is not even symmetric.

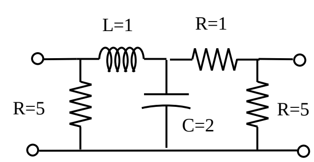


Figure 3.19 An example of an asymmetric circuit where the circuit topology (configuration of element types) is asymmetric.

For a *symmetric* two port

$$S_{11} = S_{22} \text{ and } S_{21} = S_{12} \tag{3.36}$$

A *reciprocal circuit* is one for which

$$S_{21} = S_{12} \tag{3.37}$$

Note that a symmetric circuit is automatically a reciprocal circuit but not visa versa. A circuit composed of symmetric components is guaranteed to be reciprocal even though the combination may end up being asymmetric. Hence, for the three figures above, all of the circuits are reciprocal. However only the circuits in Figure 3.18a are symmetric.

<u>Theorem</u>: If a circuit is passive then $\left|S_{ij}\right| \leq 1$,, , and $\left|S_{ii}\right| \leq 1$,

<u>Theorem</u>. The Scattering Matrix for a symmetric circuit has the following properties

1.) The diagonal terms are equal
2.) The off diagonal terms are pair wise equal

An example of a scattering matrix for a symmetric circuit would be

$$\begin{pmatrix} .25 & -j.1 & 0 \\ -j.1 & .25 & -j.2 \\ 0 & -j.2 & .25 \end{pmatrix} \tag{3.38}$$

<u>Theorem</u>. The Scattering Matrix for a reciprocal circuit has the property that the off diagonal terms are pair wise equal, i.e., $\mathrm{S_{ij} = S_{ji}\, for i \neq j}$

An example of a scattering matrix for a reciprocal circuit would be

$$\begin{pmatrix} ,1 & -j.1 & 0 \\ -j.1 & j.3 & j.2 \\ 0 & j.2 & -.4 \end{pmatrix} \tag{3.39}$$

<u>Theorem</u>. The Scattering Matrix for a passive, lossless, reciprocal circuit is a Unitary Matrix, i.e., $\mathrm{S \cdot S^{t*} = I}$, where I=identity matrix

<u>Theorem</u> Eigenvalue for passive scattering matrix all have magnitude less than or equal to one

<u>Theorem</u> Circulator must be non-reciprocal circuit

Example 3.3.1.. Calculate the S-parameters for the circuit shown in figure 3.19 assuming that the units are nH, pF, Ω, and that the operating frequency is 4 GHz.

Successive impedances are calulated to determine reflection coefficient which gives S_{11} and S_{22}. Repeated application of the voltage divider rule results in the S-parameters S_{21} and S_{12}.

```
%script file ex3_3_1.m
%Illustrate calculation of S-parameters
%

%           1         2         3
%    --- o-----L=1--------R=1------o  ---
%   |       |         |         |        |
%  R=50    R=5       C=2       R=5      R=50
%   |       |         |         |        |
%    --- o------------------------o  ---
%

% Units
nH=1e-9;
pF=1e-12;
GHz=1e9;

f=4*GHz; w=2*pi*f;
Zo=50;L=1*nH;C=2*pF;
XL=w*L; XC=-1/(w*C);

Z1= 5*50/(5+50);
Z2=Z1+1;
Z3=j*XC*Z2/(j*XC+Z2);
Z4=j*XL+Z3;
Z5= 5*Z4/(5+Z4);
S11=(Z5-Zo)/(Z5+Zo)

V1=1+S11;  V2=V1*Z3/Z4;  V3=V2*Z1/Z2;
S21=V3

ZZ1=5*50/(5+50);
ZZ2=ZZ1+j*XL;
ZZ3=j*XC*ZZ2/(j*XC+ZZ2);
ZZ4=1+ZZ3;
ZZ5=5*ZZ4/(5+ZZ4);
S22=(ZZ5-Zo)/(ZZ5+Zo)

VV3=1+S22;  VV2=VV3*ZZ3/ZZ4;  VV1=VV2*ZZ1/ZZ2;
S12=VV1
```

S11 = -0.8304 + 0.0299i

S21 = 0.0036 - 0.0309i

S22 = -0.8243 - 0.0090i

S12 = 0.0036 - 0.0309i

Example 3.3.2.. Calculate the S-parameters for a single element circuit consisting of only a series impedance, Z_{series} or of only a shunt impedance, Z_{shunt}

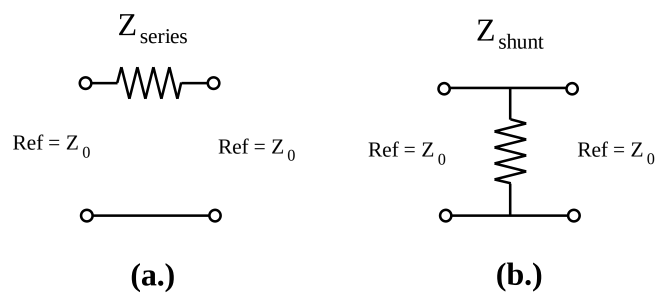


Figure 3.20 **(a.)** Series impedance **(b.)** Shunt impedance

For the *series* circuit:

$$Z_{in} = Z_{series} + Z_0$$
$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_{series}}{(Z_{series} + 2Z_0)}$$
$$V_1 = 1 + S_{11} = \frac{2(Z_{series} + Z_0)}{(Z_{series} + 2Z_0)}$$
$$S_{21} = V_1 \frac{Z_0}{Z_{series} + Z_0} = \frac{2Z_0}{(Z_{series} + 2Z_0)}$$
$$\mathbf{S} = \frac{1}{Z_{series} + 2Z_0}\begin{pmatrix} Z_{series} & 2Z_0 \\ 2Z_0 & Z_{series} \end{pmatrix}$$

Special Case $Z_{series} = 0 \Rightarrow \mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{I}$ and $Z_{series} = \infty \Rightarrow \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are left to the reader to verify

The fact that **S** is unitary for a lossless impedance, $Z_{series} = jX$, also left to reader

For the *shunt* circuit:

$$\mathrm{Z_{in}} = \frac{\mathrm{Z_{shunt} Z_0}}{\mathrm{Z_{shunt} + Z_0}}$$
$$\mathrm{S_{11}} = \frac{\mathrm{Z_{in} - Z_0}}{\mathrm{Z_{in} + Z_0}} = \frac{-\mathrm{Z_0}}{2\mathrm{Z_{shunt}} + \mathrm{Z_0}}$$
$$\mathrm{S_{21}} = \mathrm{V_1} = 1 + \mathrm{S_{11}} = \frac{2\mathrm{Z_{shunt}}}{2\mathrm{Z_{shunt}} + \mathrm{Z_0}}$$
$$\mathbf{S} = \frac{1}{2Z_{shunt} + Z_0}\begin{pmatrix} -Z_0 & 2Z_{shunt} \\ 2Z_{shunt} & -Z_0 \end{pmatrix}$$

Special Case $Z_{shunt} = 0 \Rightarrow \mathbf{S} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$ and $Z_{shunt} = \infty \Rightarrow \mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are left to the reader to verify

The fact that **S** is unitary for a lossless impedance, $Z_{shunt} = jX$, also left to reader

3.4 TRANSMISSION MATRIX REPRESENTATION.

When two port circuits are cascaded it is often convenient to organized the signal parameters so that the input (port 1) signal parameters, a_1, b_1 are grouped together and output (port 2) parameters, a_2, b_2 are grouped together. This can be accomplished by starting with signal parameters arranged to emphasize their scattering characteristic.

$$b_1 = S_{11}a_1 + S_{21}a_2 \tag{3.40}$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \tag{3.41}$$

Rearranging the second equation gives $a_1 = (b_2 - S_{22}a_2)/S_{21}$ which can be substituted into the first equation to give $b_1 = (-\Delta_S a_2 + S_{11}b_2)/S_{21}$, where the determinant of the Scattering matrix is $\Delta_S = S_{22}S_{11} - S_{21}S_{12}$. These equations can then be represented as the following matrix equation (3.42) where one notes that the signal parameters are organized so that signals propagating to the left are the top entry of the vector as illustrated below Figure 3.21

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \frac{1}{S_{21}} \begin{pmatrix} -\Delta_S & S_{11} \\ -S_{22} & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \tag{3.42}$$

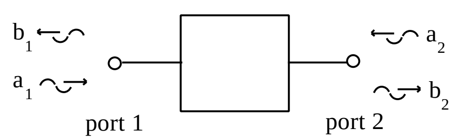
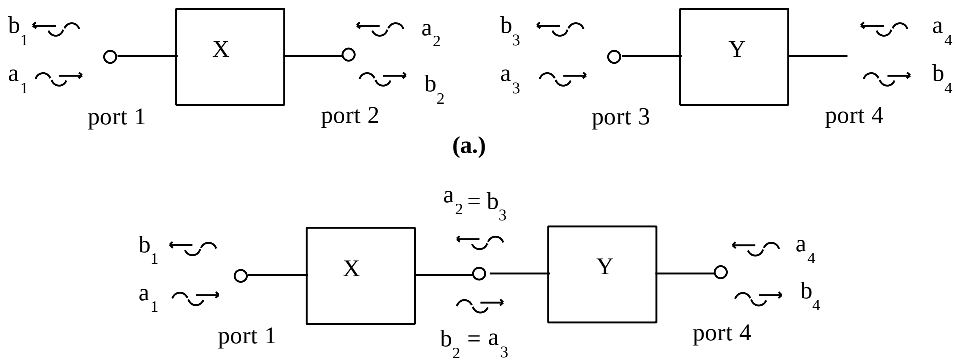


Figure 3.21 The signal vectors are configured with the left propagating terms as the top entry.

The value of the transmission representation can be seen when considering two circuits which are to be cascaded together to form a new combined circuit. Let X and Y represent the <u>transmission matrices</u> for the respective circuits shown in Figure 3.22a. Connecting the circuits yields the composite circuit illustrated in Figure 3.22b.

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = X \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad \begin{pmatrix} b_3 \\ a_3 \end{pmatrix} = Y \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} a_4 \\ b_4 \end{pmatrix}$$



(b.)

Figure 3.22 (a.) Circuits with a transmission matrices X and Y. **(b.)** Composite circuit formed by cascading the circuit X and Y.

Since $\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = X \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = X \begin{pmatrix} b_3 \\ a_3 \end{pmatrix} = XY \begin{pmatrix} a_4 \\ b_4 \end{pmatrix}$ then when a circuit with transmission matrix X is cascaded with a circuit with transmission Y then the resulting circuit has a transmission matrix Z equal to XY. The transmission matrix for the new cascaded circuit equals the matrix product of the individual circuit transmission matrices. If the transmission matrix for a circuit is known then one can determine the scattering matrix by rearranging the equations so that the scattered signals are the dependent variable (usually on left of equations). If the transmission matrix R equals $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ then $\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ and

$$b_1 = r_{11} a_2 + r_{12} b_2 \tag{3.43a}$$

$$a_1 = r_{21} a_2 + r_{22} b_2 \tag{3.43b}$$

Solving for b_2 in the second equation gives $b_2 = (a_1 - r_{21} a_2) / r_{22}$ and substitution of this into the first equation gives $b_1 = \left[r_{12} a_1 + (r_{11} r_{22} - r_{21} r_{12}) a_2 \right] / r_{22}$ which can be summarized as the following matrix equation, where the determinant notation is used, i.e., $\Delta_R = r_{11} r_{22} - r_{21} r_{12}$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{r_{22}} \begin{pmatrix} r_{12} & \Delta_R \\ 1 & -r_{21} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{3.44}$$

In summary

$$R=\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}=\frac{1}{s_{21}}\begin{pmatrix} -\Delta_S & s_{11} \\ -s_{22} & 1 \end{pmatrix}$$

$$S=\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}=\frac{1}{r_{22}}\begin{pmatrix} r_{12} & \Delta_R \\ 1 & -r_{21} \end{pmatrix}$$

where $\Delta_S = \det(S) = s_{11}s_{22} - s_{21}s_{12}$, and
$\Delta_R = \det(R) = r_{11}r_{22} - r_{21}r_{12}$

(3.45)

Example 3.4.1.. Find the S-parameters for the circuit illustrated in Figure 3.22 assuming that the units are nH, pF, Ω, and that the operating frequency is 4 GHz.

Use single impedance formulas for S-parameters for single element circuits then compute trasmission matrix, multiply to model cascading of component circuits, and convert result back to S-parameters. Calculation script files are shown below

```
%  script file ex3_4_1.m
%  Illustrate calculation of S-parameters
%  using tramission matrix multipication
%

%    Ref    1         2         3         4         5     Ref
%    --- o-----o  o--L=1--o  o-----o  o--R=1--o  o-----o  ---
%   |       |                   |                   |        |
%  R=50    R=5                 C=2                 R=5      R=50
%   |       |                   |                   |        |
%    --- o-----o  o-------o  o-----o  o-------o  o-----o  ---
%

% Units
nH=1e-9;
pF=1e-12;
GHz=1e9;


f=4*GHz; w=2*pi*f;
Zo=50;
L=1*nH;C=2*pF;
XL=w*L; XC=-1/(w*C);


Zshunt=5;       %Circuit #1
S1 = (1/(2*Zshunt+Zo))*[-Zo 2*Zshunt;2*Zshunt -Zo];
R1=S2R(S1);

Zseries=j*XL;   %Circuit #2
S2 = (1/(Zseries+2*Zo))*[Zseries 2*Zo; 2*Zo Zseries];
R2=S2R(S2);
```



```
Zshunt=j*XC;    %Circuit #3
S3 = (1/(2*Zshunt+Zo))*[-Zo 2*Zshunt;2*Zshunt -Zo];
R3=S2R(S3);

Zseries=1;      %Circuit #4
S4 = (1/(Zseries+2*Zo))*[Zseries 2*Zo; 2*Zo Zseries];
R4=S2R(S4);

Zshunt=5;       %Circuit #5
S5 = (1/(2*Zshunt+Zo))*[-Zo 2*Zshunt;2*Zshunt -Zo];
R5=S2R(S5);


Rtotal=R1*R2*R3*R4*R5
Stotal=R2S(Rtotal)
```

Calculation results:

Rtotal =
 -3.1018 -21.8404i -4.0544 -26.4145i
  2.7911 +26.3642i   3.7332 +31.9437i

Stotal =
 -0.8304 + 0.0299i   0.0036 - 0.0309i
  0.0036 - 0.0309i -0.8243 - 0.0090i

---------------------------------- Suopporting Functions ---

```
function S=R2S(R);

%Takes an R matrix and converts it into a Scattering Matrix (S)
%

% (b1) (r11  r12)(a2)    (b1) (s11  s12)(a1)
% (  )=(        )(  )    (  )=(        )(  )
% (a1) (r21  r22)(b2)    (b2) (s21  s22)(a2)
%

%   (s11  s12)    1  (r12   det(R))
%   (        ) = --- (             )
%   (s21  s22)   r22 (1     -r21  )


S11=R(1,2)/R(2,2);

S12=det(R)/R(2,2);

S21=1/R(2,2);

S22=-R(2,1)/R(2,2);

S=[S11,S12;S21,S22];
```

```
function R=S2R(S);

%Takes an S matrix and converts it into a Transmission Matrix (R)
%

% (b1) (r11  r12)(a2)    (b1) (s11  s12)(a1)
% (  )=(         )(  )    (  )=(         )(  )
% (a1) (r21  r22)(b2)    (b2) (s21  s22)(a2)
%

%   (r11  r12)    1  (-det(S)  s11)
%   (         ) = --- (            )
%   (r21  r22)   s21 (  -s22    1 )


R11=-det(S)/S(2,1);

R12=S(1,1)/S(2,1);

R21=-S(2,2)/S(2,1);

R22=1/S(2,1);

R=[R11,R12;R21,R22];
```

Example 3.4.2.. *Suppose a circuit with s-parameters* $\{u_{11}, u_{21}, u_{12}, u_{22}\}$ *is cascaded with a circuit with s-parameters* $\{v_{11}, v_{21}, v_{12}, v_{22}\}$. *Find the s-parameters for the combined circuit using signal flow graph reduction.*

The next example will solve the same problem using transmission matrix approach.. The signal flow graph for the combined circuit is shown in the figure below. The s-parameters for the combined circuit will be

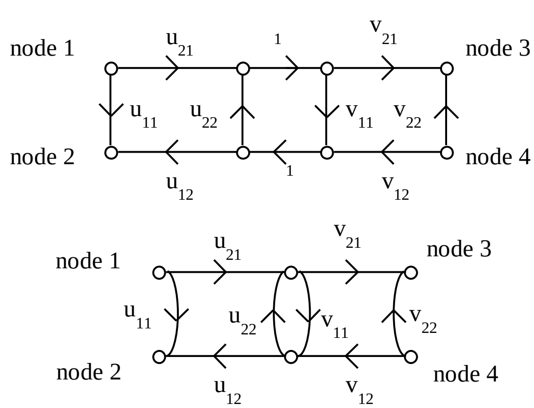


Figure 3.23. The SFG for two cascaded circuits

designated by $\{w_{11}, w_{21}, w_{12}, w_{22}\}$. Determination of the s-parameters for the combined circuit is best undertaken in steps. First w_{21} is determined by recognizing it as a transfer ratio between nodes “1” and “3” if Figure 3.23. With node “1” taken as an independent node then the SFG can be reduced as follows

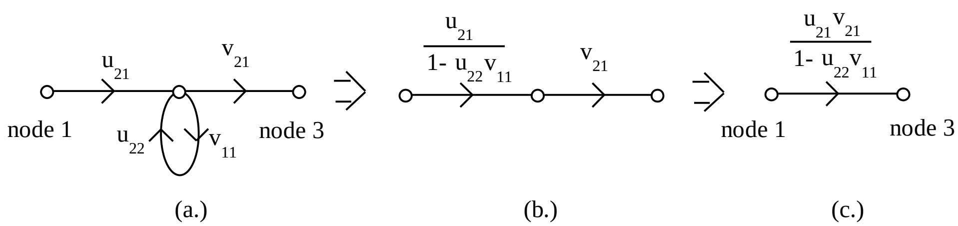


Figure 3.24 The SFG reduction to obtain the transfer ratio representing w_{21}

Therefore,

$$w_{21} = \frac{u_{21}v_{21}}{1 - u_{22}v_{11}} \tag{3.46}$$

In a similar manner the SFG of figure 3.22 can be reduced to determine w_{12} which is the transfer ratio from node “4” to node “2.” The reduction is illustrated below.

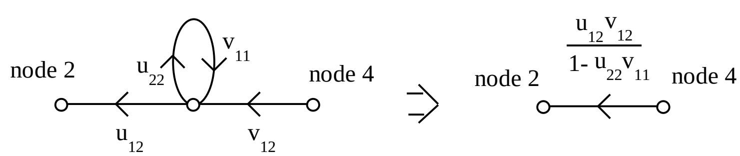


Figure 3.25 SFG reduction to determine w_{12}

$$w_{12} = \frac{u_{12}v_{12}}{1 - u_{22}v_{11}} \tag{3.47}$$

The s-parameter, w_{11}, is found from the transfer ration from node “1” to node “2.” The SFG is shown below

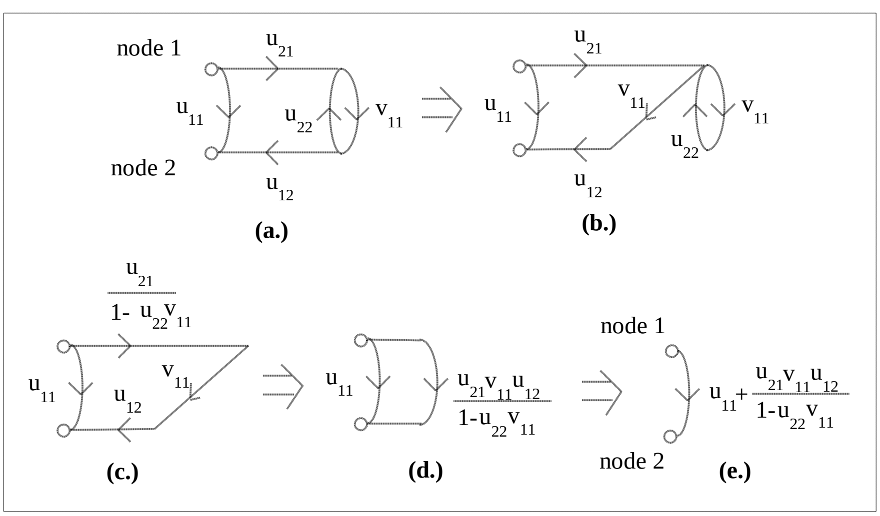


Figure 3.26 Sequence of SFG reductions to obtain w_{11}

$$w_{11} = u_{11} + \frac{u_{21}v_{11}u_{12}}{1 - u_{22}v_{11}} \tag{3.48}$$

In a similar manner w_{22} is found.

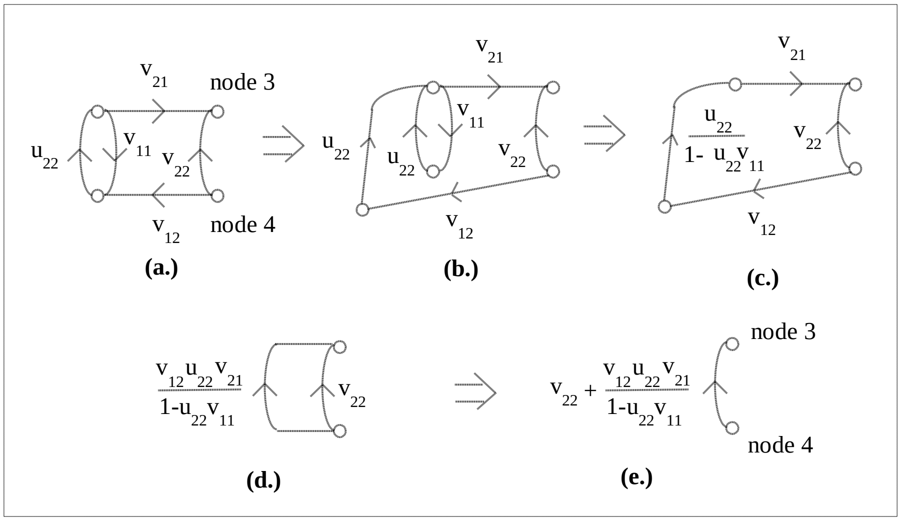


Figure 3.27 Sequence of SFG reductions to obtain w_{22}

$$w_{22} = v_{22} + \frac{v_{12}u_{22}v_{21}}{1 - u_{22}v_{11}} \tag{3.49}$$

Example 3.4.3.. Suppose a circuit with s-parameters $\{u_{11}, u_{21}, u_{12}, u_{22}\}$ is cascaded with a circuit with s-parameters $\{v_{11}, v_{21}, v_{12}, v_{22}\}$. Find the s-parameters for the combined circuit using a transmission matrix approach.

The two transmission matrices are

$$R_u = \frac{1}{u_{21}}\begin{pmatrix} -\Delta_u & u_{11} \\ -u_{22} & 1 \end{pmatrix}, \text{ and } R_v = \frac{1}{v_{21}}\begin{pmatrix} -\Delta_v & v_{11} \\ -v_{22} & 1 \end{pmatrix} \tag{3.50}$$

The transmission matrix for the combined circuit is

$$R_w = R_u R_v = \frac{1}{u_{21}v_{21}}\begin{pmatrix} -\Delta_u & u_{11} \\ -u_{22} & 1 \end{pmatrix}\begin{pmatrix} -\Delta_v & v_{11} \\ -v_{22} & 1 \end{pmatrix} \tag{3.51}$$

$$R_w = \frac{1}{u_{21}v_{21}}\begin{pmatrix} \Delta_u\Delta_v - u_{11}v_{22} & -\Delta_u v_{11} + u_{11} \\ u_{22}\Delta_v - v_{22} & 1 - u_{22}v_{11} \end{pmatrix} \tag{3.52}$$

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \frac{1}{r_{22}} \begin{pmatrix} r_{12} & \Delta_R \\ 1 & -r_{21} \end{pmatrix} \tag{3.53}$$

$$w_{11} = \frac{r_{12}}{r_{22}} = \frac{-\Delta_u v_{11} + u_{11}}{u_{21} v_{21}} \cdot \frac{u_{21} v_{21}}{1 - u_{22} v_{11}} = \frac{-\Delta_u v_{11} + u_{11}}{1 - u_{22} v_{11}} \tag{3.54}$$

$$w_{11} = u_{11} + \frac{u_{12} v_{11} u_{21}}{1 - u_{22} v_{11}} \tag{3.55}$$

$$w_{22} = -\frac{r_{21}}{r_{22}} = -\frac{\Delta_v u_{22} - v_{22}}{u_{21} v_{21}} \cdot \frac{u_{21} v_{21}}{1 - u_{22} v_{11}} = \frac{\Delta_v u_{22} - v_{22}}{1 - u_{22} v_{11}} \tag{3.56}$$

$$w_{22} = v_{22} + \frac{v_{21} u_{22} v_{12}}{1 - u_{22} v_{11}} \tag{3.57}$$

$$w_{12} = \frac{\Delta_R}{r_{22}} = \frac{u_{21} v_{21}}{1 - u_{22} v_{11}} \cdot \Delta_R \tag{3.58}$$

$$w_{12} = \frac{\Delta_R}{r_{22}} = \frac{u_{21} v_{21}}{1 - u_{22} v_{11}} \cdot \frac{(\Delta_u \Delta_v - u_{11} v_{22})(1 - u_{22} v_{11}) - (u_{22} \Delta_v - v_{22})(-\Delta_u v_{11} + u_{11})}{(u_{21} v_{21})^2} \tag{3.59}$$

$$w_{12} = \frac{1}{1 - u_{22} v_{11}} \cdot \frac{\Delta_u \Delta_v - u_{11} u_{22} v_{11} v_{22} + u_{11} u_{22} v_{12} v_{21} + v_{11} v_{22} u_{12} u_{21}}{u_{21} v_{21}} \tag{3.60}$$

$$w_{12} = \frac{1}{1 - u_{22} v_{11}} \cdot \frac{u_{12} u_{21} v_{12} v_{21}}{u_{21} v_{21}} \tag{3.61}$$

$$w_{12} = \frac{u_{12} v_{12}}{1 - u_{22} v_{11}} \tag{3.62}$$

$$w_{21} = \frac{1}{r_{22}} \tag{3.63}$$

$$w_{21} = \frac{u_{21} v_{21}}{1 - u_{22} v_{11}} \tag{3.64}$$

These are the same relationships obtained using the SFG approach.

3.5 IMPEDANCE AND ADMITTANCE MATRIX REPRESENTATION.

The figure below shows an N-port linear circuit. The voltage and current at each of the ports is related by the impedance or Z-matrix equation **V=ZI** which is shown in expanded form below.

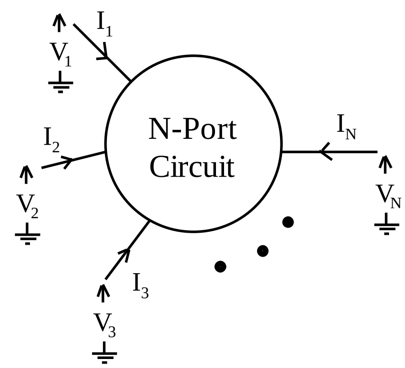


Figure 3.28. Voltage and current for an N-port circuit

$$\begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ V_N \end{bmatrix} = \begin{bmatrix} z_{11} & \cdot & \cdot & z_{1N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_{N1} & \cdot & \cdot & z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ I_N \end{bmatrix}$$

If each of the ports are connected to transmission lines then the total voltages and currents can be represented in terms of propagating voltage waves as $V_i = V_i^+ + V_i^-$ and $z_{0i} I_i = V_i^+ - V_i^-$, where z_{0i} is the characteristic impedance of the transmission line connected to the i th port of the circuit. In terms of signal parameters

$$V_i = (a_i + b_i) \cdot \sqrt{z_{0i}}$$
$$I_i = (a_i - b_i) / \sqrt{z_{0i}}$$

The matrix relationship between the total voltages and currents can now be represented as

$$\begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ V_N \end{bmatrix} = \begin{bmatrix} \sqrt{z_{01}} & 0 & \cdot & \cdot & 0 \\ 0 & \sqrt{z_{02}} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \sqrt{z_{0N}} \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_N \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_N \end{bmatrix} \right)$$

The formalism is facilitated by defining a diagonal matrix of characteristic impedance as follows

$$\mathbf{Z}_0 = \begin{bmatrix} z_{01} & 0 & \cdot & \cdot & 0 \\ 0 & z_{01} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & z_{01} \end{bmatrix} \Rightarrow \mathbf{Z}_0^{1/2} = \begin{bmatrix} \sqrt{z_{01}} & 0 & \cdot & \cdot & 0 \\ 0 & \sqrt{z_{02}} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \sqrt{z_{0N}} \end{bmatrix} \Rightarrow$$

$$\mathbf{Z}_0^{-1/2} = \begin{bmatrix} 1/\sqrt{z_{01}} & 0 & \cdot & \cdot & 0 \\ 0 & 1/\sqrt{z_{02}} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1/\sqrt{z_{0N}} \end{bmatrix}$$

Therefore,

$$\mathbf{V} = \mathbf{Z}_0^{1/2}(\mathbf{A}+\mathbf{B})$$
$$\mathbf{I} = \mathbf{Z}_0^{-1/2}(\mathbf{A}-\mathbf{B})$$

and substitution into **V=ZI** yields $\mathbf{Z}_0^{1/2}(\mathbf{A}+\mathbf{B}) = \mathbf{Z}(\mathbf{A}-\mathbf{B})\mathbf{Z}_0^{-1/2}$. Because the matrix $\mathbf{Z}_0$ is diagonal hence its multiplication is a commutative operation and $\mathbf{Z}_0(\mathbf{A}+\mathbf{B}) = \mathbf{Z}(\mathbf{A}-\mathbf{B})$. Manipulation of this to solve for **B** yields $\mathbf{B} = (\mathbf{Z}+\mathbf{Z}_0)^{-1}(\mathbf{Z}-\mathbf{Z}_0)\mathbf{A}$ which implies that the scatter matrix of s-parameters is

$$\mathbf{S} = (\mathbf{Z}+\mathbf{Z}_0)^{-1}(\mathbf{Z}-\mathbf{Z}_0) \tag{3.65}$$

It is interesting to note that while in general multiplication of square matrices is a non-commutative operation that in the particular expression in Equation (3.65) is equivalent to that of Equation (3.66)

$$\mathbf{S} = (\mathbf{Z}-\mathbf{Z}_0)(\mathbf{Z}+\mathbf{Z}_0)^{-1} \tag{3.66}$$

This follows from $\mathbf{I}+\mathbf{Z}^{-1}\mathbf{Z}_0 = \mathbf{I}+\mathbf{Z}_0\mathbf{Z}^{-1}$ which can be arranged as $\mathbf{Z}^{-1}(\mathbf{Z}+\mathbf{Z}_0) = (\mathbf{Z}+\mathbf{Z}_0)\mathbf{Z}^{-1}$ and taking the inverse of both sizes gives

$$(\mathbf{Z}+\mathbf{Z}_0)^{-1}\mathbf{Z} = \mathbf{Z}(\mathbf{Z}+\mathbf{Z}_0)^{-1} \tag{3.67}$$

In addition because $\mathbf{Z}_0$ is diagonal then

$$(\mathbf{Z}+\mathbf{Z}_0)^{-1}\mathbf{Z}_0 = \mathbf{Z}_0(\mathbf{Z}+\mathbf{Z}_0)^{-1} \tag{3.68}$$

Subtraction yields $(\mathbf{Z}+\mathbf{Z}_0)^{-1}\mathbf{Z} - (\mathbf{Z}+\mathbf{Z}_0)^{-1}\mathbf{Z}_0 = \mathbf{Z}(\mathbf{Z}+\mathbf{Z}_0)^{-1} - \mathbf{Z}_0(\mathbf{Z}+\mathbf{Z}_0)^{-1}$ and factorization gives $(\mathbf{Z}+\mathbf{Z}_0)^{-1}(\mathbf{Z}-\mathbf{Z}_0) = (\mathbf{Z}-\mathbf{Z}_0)(\mathbf{Z}+\mathbf{Z}_0)^{-1}$ which proves the assertion. In summary

$$\mathbf{S} = (\mathbf{Z}+\mathbf{Z}_0)^{-1}(\mathbf{Z}-\mathbf{Z}_0) = (\mathbf{Z}-\mathbf{Z}_0)(\mathbf{Z}+\mathbf{Z}_0)^{-1} \tag{3.69}$$

If the s-parameters are known then the Z-matrix for the circuit can be found by solving the above equation for Z, i.e.,

$$\mathbf{Z} = \mathbf{Z}_0(\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} = \mathbf{Z}_0(\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S}) \tag{3.70}$$

Because $\mathbf{Z}_0$ is a diagonal matrix it commutes under matrix multiplication and therefore can be placed at any location in the above matrix multiplication chain in Equation (3.70).

The admittance matrix Y is defined as $\mathbf{I} = \mathbf{YV}$ and therefore it follows that $\mathbf{Y} = \mathbf{Z}^{-1}$. If a diagonal matrix of characteristic admittances is defined as $\mathbf{Y}_o = \mathbf{Z}_0^{-1}$ then direct substitution shows that

$$\mathbf{S} = -(\mathbf{Y} + \mathbf{Y}_0)^{-1}(\mathbf{Y} - \mathbf{Y}_0) = -(\mathbf{Y} - \mathbf{Y}_0)(\mathbf{Y} + \mathbf{Y}_0)^{-1}$$

which can also be inverted to give the Y-matrix as a function of the scattering matrix, i.e.,

$$\mathbf{Y} = \mathbf{Y}_0(\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1} = \mathbf{Y}_0(\mathbf{I} + \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S})$$

Example 3.5.1.. Determine the S Z and Y matrix for the circuit below

$$V_1 = 50I_1 + 50(I_1 + I_2)$$
$$V_1 = 100I_1 + 50I_2$$
$$V_2 = 50I_1 + 50I_2$$
$$Z = \begin{pmatrix} 100 & 50 \\ 50 & 50 \end{pmatrix}$$

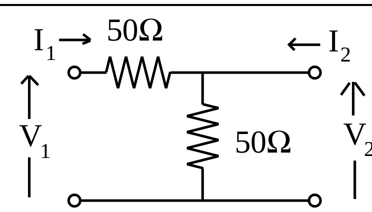


Figure 3.29 Find S Z and Y matrix

```
S=z2s(Z,50)
    0.2000    0.4000
    0.4000   -0.2000

Y=s2y(S,1/50)
    0.0200   -0.0200
   -0.0200    0.0400

Zchk=inv(Y)
   100    50
    50    50
```

--Supporting Functions--------------------------------

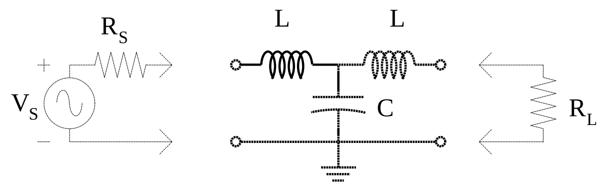
```
function y=S2Y(S,Yo)
% function converts S matrix to Y matrix
% with Yo=ref admittance

I=eye(size(S));
y=Yo*(I-S)*inv(I+S);


function y=S2Z(S,Zo)
% function converts S matrix to Z matrix
% with Zo=ref impedance

I=eye(size(S));
y=Zo*(I+S)*inv(I-S);
```

```
function s=Y2S(Y,Yo)
% function converts Y matrix to S matrix
% with Yo=ref admittance

I=eye(size(Y));
s=-(Y-Yo*I)*inv(Y+Yo*I);
```

```
function y=Z2S(Z,Zo)
% function converts Z matrix to S matrix
% with Zo=ref impedance

I=eye(size(Z));
y=(Z-Zo*I)*inv(Z+Zo*I);
```

3.6 PROBLEMS

1. (a.) Using the circuit reduction techniques illustrated in Example 3.3.1 calculate and display the magnitude of S_{21} for the filter below as a function of frequency from 50 MHz to 5 GHz., where L=3.98 nH, C=3.18 pF

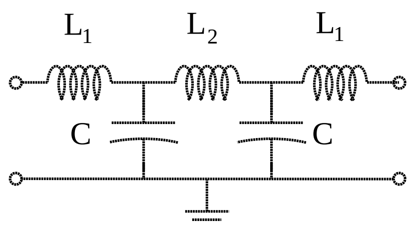


(b.) Repeat part (a.) by cascading R matrice and converting the final R matrix to and S-matrix as illustrated in Example 3.4.1.

(c.) If a 1 volt (rms), 50 ohm source is connected to the input of the circuit and and a 50 ohm load is connected to the output of the circuit what power is delivered to the circuit at 50MHz? 1GHz? 2 GHz? and 5 GHz?

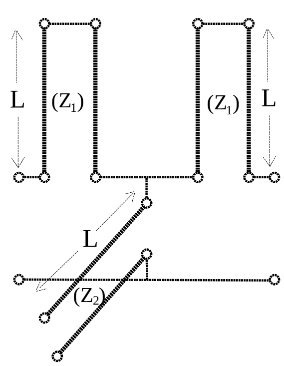
2. Repeat part (a.) of problem 1 where L=6.35 nH, C=1.75 pF.

3. (a.) Plot $|S_{11}|$ and $|S_{21}|$ on separate graphs for the following circuit as a function of frequency from 50 MHz to 5 GHz where L_1 = 2.46 nH, L_2 = 7.96 nH, and C=2.57 pF.

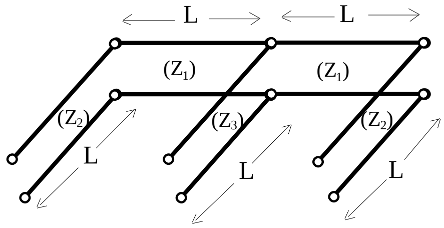


(b.) Plot $|S_{11}|$ and $|S_{21}|$ on the graphs in part (a.) for a circuit with parameters; L_1 = 6.79 nH, L_2 = 10.11 nH, and C=1.98 pF. Describe any qualitative difference observed between the plots of part (a.) and part (b.).

4. The circuit below consists of two short circuited stubs in series and one open circuited stub in shunt where L=λ/4 for a frequency of 4 GHz. Plot $|S_{11}|$ and $|S_{21}|$ for a frequency range of from 50 MHz to 10 GHz. Z_1 = 50 Ω, Z_2 = 25 Ω,



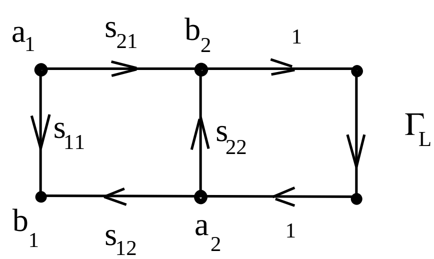
5. The circuit below consists of three open circuited stubs in shunt where L= $\lambda/4$ for a frequency of 4 GHz. Plot $|S_{11}|$ and $|S_{21}|$ for a frequency range of from 50 MHz to 10 GHz. $Z_1 = 50\ \Omega$, $Z_2 = 100\ \Omega$, $Z_3 = 25\ \Omega$,,



6. Because of electric field fringing effects at the end of an open circuit and open circuited stub often appears to be electrically longer than expected from its physical length. If the fringing capacitance for a 50 ohm open circuited line is .05 nH what is the effective increase in apparent line length for a frequency of 1 GHz and a phase velocity which is 66 % of the speed of light, c? Hint: An open circuited transmission line produces the same capacative reactance when its length satisfies the following equation for which the length can be solved.

$$\frac{1}{jZ_0 \tan b\lambda} = \frac{1}{jwC}$$

7. For the SFG shown below find the transfer ratio a_2 / a_1



3.7 REFERENCES

3.8 ANALYSIS TIPS

3.9 PROBLEMS