CHAPTER
2

DISTRIBUTED-ELEMENT CIRCUIT ANALYSIS TECHNIQUES

2.1 TRANSMISSION LINES

A transmission line can be viewed as a multiple combination of small circuit segments shown below in Figure 2.1. The series inductance is due to magnetic field effects and the capacitance is due to electric field coupling between the lines. The losses in the transmission media are depicted by the series and the shunt resistors. These resistors represent the finite conductivity of the conductors and the dielectric insulator between the conductors, respectively. The constants *R*, *G*, *L*, and *C* are defined as per unit length circuit parameters and the resulting resulting circuit is referred to as a distributed model of a transmission line. The length of the transmission line segment is Δx.

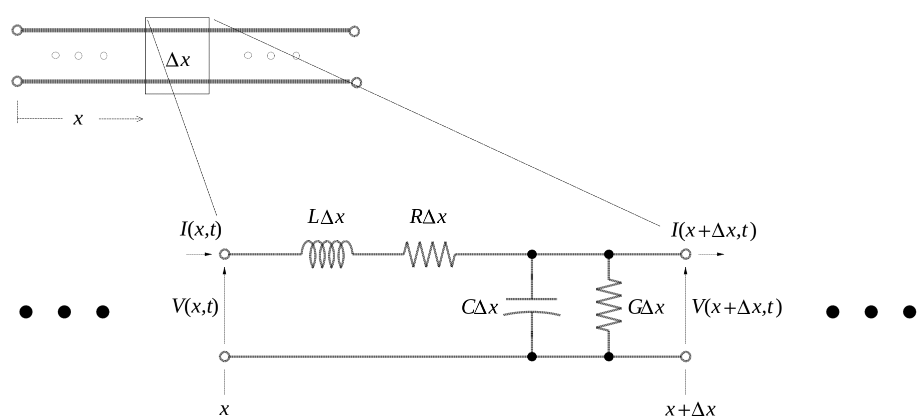


Figure 2.1 Distributed circuit model for a transmission line.

Applying Kirchoff's law to the series and shunt elements respectively, one gets

$$V(x,t) - V(x+\Delta x,t) = R\Delta x I(x,t) + L\Delta x \frac{\partial I(x,t)}{\partial t} \tag{2.1}$$

$$I(x,t) - I(x+\Delta x,t) = G\Delta x V(x+\Delta x,t) + C\Delta x \frac{\partial V(x+\Delta x,t)}{\partial t} \tag{2.2}$$

Dividing both sides of the above equations by Δx and taking the limit of both equations as $\Delta x \to 0$, results in

$$-\frac{\partial V(x,t)}{\partial x} = RI(x,t) + L\frac{\partial I(x,t)}{\partial t} \tag{2.3}$$

$$-\frac{\partial I(x,t)}{\partial x} = GV(x,t) + C\frac{\partial V(x,t)}{\partial t} \tag{2.4}$$

Sinusoidal steady state solutions of the voltage and current can be found by assuming the solutions to be $V(x,t) = V(x)e^{j\omega t}$ and $I(x,t) = I(x)e^{j\omega t}$, i.e., the voltage and current can be described as a phasor which is a complex vector rotating as a function of time. The amplitude and phase of the phasor is a function of x, the position on the transmission line. Substitution of these for the voltage and current in (2.3) and (2.4) yields

$$\frac{dV(x)}{dx} = -Z\,I(x) \tag{2.5}$$

$$\frac{dI(x)}{dx} = -YV(x) \tag{2.6}$$

where

$$Z = R + j\omega L \text{ and } Y = G + j\omega C$$

are known as the distributed impedance and addmittance, respectively. Since the time dependence is removed from the differential equations one is dealing with only the voltage and current phasors. Further differentiation of (2.5) and (2.6) resulted in two second order linear differential equations.

$$\frac{d^2V(x)}{dx^2} = YZV(x) \tag{2.7}$$

$$\frac{d^2I(x)}{dx^2} = YZ\,I(x) \tag{2.8}$$

Working with (2.7), the solution must bear the form of $e^{\pm\gamma x}$, where γ, known as the propagation constant is

$$\gamma = \sqrt{YZ} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \tag{2.9}$$

The parameter α is known as the attenuation constant and β is called the wave number. The general solution of the voltage phasor is then

$$V(x) = V^+e^{-\gamma x} + V^-e^{\gamma x} \tag{2.10}$$

where the + and - superscripts are chosen to indicate the propagation direction of the voltage wave. The current phasor can be derived from (2.10) and (2.5), i.e.

$$I(x) = I^{+}e^{-gx} - I^{-}e^{gx} = \frac{g}{R + jwL}\left(V^{+}e^{-gx} - V^{-}e^{gx}\right) \tag{2.11}$$

The the forward propagating voltage is related to the forward propagating current characteristic impedance which is

$$Z_c = \frac{R + jwL}{g} = \sqrt{\frac{R + jwL}{G + jwC}} \tag{2.12}$$

If the transmission line is lossless then R=G=0 and in that case the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{L}{C}} \tag{2.13}$$

and the propagation constant is given by $g = \sqrt{YZ} = \sqrt{(jwL)(jwC)} = 0 + jw\sqrt{LC}$ implying that

$$\alpha = 0 \text{ and } b = w\sqrt{LC} \tag{2.14}$$

The results can be understood by considering a sinusoidal signal propagating in the positive x direction is given by

$$A'\cos(wt - bx)$$

Note that for a fixed time that the wave repeats for different x positions along the line separated by intergal multiples of 2π/β. This repeating spacial distance is called the wave length and usually designated as λ. Therefore, λ=2π/β or the wave number is given by β = 2π/λ. The wave number "β" can be thought of as a spacial angular frequency analogous to ""ω which is a temporial angular frequency. If time is allowed to advance and then a position for which $wt - bx$ is a constant is called a point of constant phase. The velocity of a point of constant phase is called the phase velocity of the wave and equals ω/β = fλ.

The cosine trigometric function can be represented as the real part of a complex number given by

$$\mathrm{Re}\left\{A'e^{j(wt-bx)}\right\}$$

At a fixed x position the complex number, $A'e^{j(wt-bx)}$, is a rotating vector in the complex plane is referred to a phasor. It is convenient to think of the wave as complex and to omit the "Re" operator. In analyzing a circuit if the actual real values are required one only needs to take the real part of the phasor.
For the geometry shown in Figure 2.2 the voltage and current for a lossless transmission line are shown below where $\mathrm{A}'e^{j(wt-bx)}$ represents a forward traveling voltage phasor propagating in the positive x-direction, while $\mathrm{B}'e^{j(wt+bx)}$ represents a reverse traveling phasor, i.e. propagating in the negative x-direction.

$$\mathrm{V}(x) = \mathrm{A}'e^{j(wt-bx)} + \mathrm{B}'e^{j(wt+bx)}$$

$$\mathrm{V}(x) = e^{jwt}\left(\mathrm{A}'e^{-jbx} + \mathrm{B}'e^{jbx}\right)$$ (

Since the time dependence for a phasor is given by e^{jwt} which is multiplied by a complex number independent of time it is convenient to omit it and include it only if the explicit time behavior is required. Therefore a forward and reverse propagating phasor is represented, respectively, by $A'e^{-jbx}$ and $B'e^{jbx}$. If the explicit time dependence is required one multiplies the phasor by e^{jwt} and then take the real part of the resulting function. Using this convention the voltage pahsor on a line is given by

$$V(x) = A'e^{-jbx} + B'e^{jbx}$$

which is consistent with the results of (2.12) with γ = 0+jβ. Since x = L - d, where "d" is the length measured from the right side of the circuit and "L" is the length of the transmission line.

$$V(x) = A'e^{-jb(L-d)} + B'e^{jb(L-d)}$$

$$V(d) = A'e^{-jbL}e^{jbd} + B'e^{jbL}e^{-jbd}$$

If $A = A'e^{-jbL}$ and $B = B'e^{jbL}$ then

$$V(d) = Ae^{jbd} + Be^{-jbd} \tag{2.15}$$

The current is given by the two phasors

$$I(d) = \frac{Ae^{jbd} - Be^{-jbd}}{Z_0} \tag{2.16}$$

where the propagation constant is

$$b = w\sqrt{LC} = \frac{2p}{l} \tag{2.17}$$

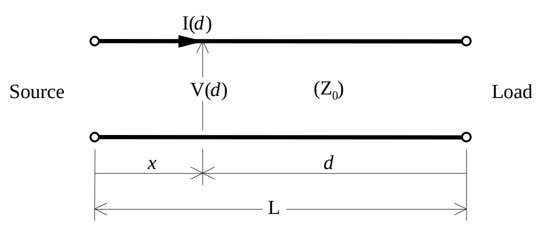


Figure 2.1 Transmission line geometry

A forward propagating voltage and current phasor are related by the characteristic impedance, Zo. A reverse propagating voltage and current phasor are related by the negative of the characteristic impedance, -Zo. The plus sign is required for the forward wave and the negative sign for the reverse wave so that the power associated with the propagating wave has the correct sign. When viewed at an arbitary point on the line the power associated with a forward wave would be positive indicating that power appears to be disipated by the right side of the circuit. On the other hand the power associated with a reverse wave would be negative indicating that power appears to be produced by the right side of the circuit, i.e., the right side of the circuit appears to be a source for the reverse wave.

2.2 IDEAL TRANSMISSION LINE CIRCUITS

Distributed-element circuits are those where the physical dimensions of one or more of the components affect the circuit performance. In many important circuits the distributed components can be represented in terms of transmission line models. Therefore, this section will consider the analysis of circuits which contain transmission lines as components. Initially lossless transmission lines will be considered, since they represent a simpler starting point and the techniques extend naturally to lossy lines. Such lines referred to as ideal transmission lines do not represent a serious restriction since microwave and RF circuits normally use low loss materials and circuit effects are rarely dominated by transmission line losses.

The analysis of transmission line circuits is a natural extension of lumped-element circuit analysis. Kirchoff's Current Law continues to be true at any node and Kirchoff's Voltage Law must hold around any loop. Voltage and current relations for lumped element resistances (Ohm's Law), inductive reactances, and capacitive reactances are augmented with those for transmission lines. Transmission lines are therefore treated like any other circuit element. Circuits are solved by using the current and voltage relationships to create a system of equations which can then be solved for branch currents and node voltages. Transmission line nodes exist at the input and the output and they may be combined with other elements in series and in parallel. Lumped-element (LE) and distributed-element (DE) circuits are shown below in Figure 2.1 and 2.2.

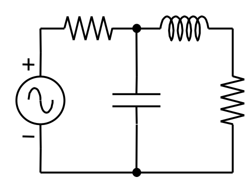


Figure 2.2 An example of a lumped element circuit in which the physical size and spacing of components are not factors in determining performance

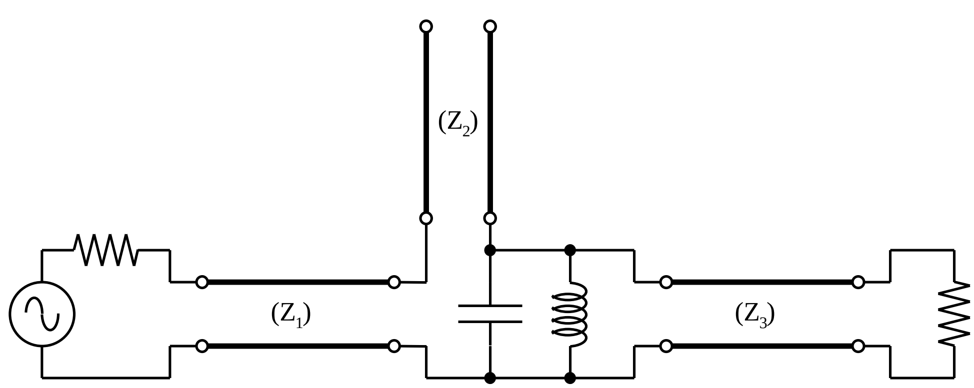


Figure 2.3 A distributed element circuit in which transmission lines as well as lumped elements are used as components

A simple distributed element circuit consisting of a source, a transmission line, and a load is illustrated in Figure 2.4. The load and source are assumed to be lumped-element components. It is desired to find the voltages and currents everywhere in the circuit. At each of the terminals of the transmission line Kirchoff's Laws together with Ohm's law can be applied to obtain Equations (2.18a) and (2.18b).

$$V_o = Z_o I_{in} + V_{in}\ ;\quad I_{in} = I(d = L)\ ;\quad V_{in} = V(d = L) \tag{2.18a}$$

$$V_{out} = Z_L I_{out} ; I_{out} = I(d = 0) ; V_{out} = V(d = 0) \tag{2.18b}$$

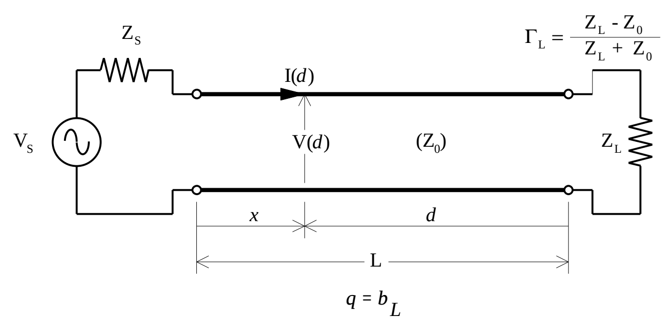


Figure 2.4 A source, a transmission line, and a load

At the load (d = 0) the reflection coefficient Γ_L is the ratio of these two waves, i.e. $\Gamma_L = B/A$ or $B = A\,\Gamma_L$ and the transmission line voltage and current can be expressed as shown in Equations (2.19a) and (2.19b). Since the source is on the left then we can view the forward propagating voltage wave as a

$$V(d) = A(e^{jbd} + \Gamma_L e^{-jbd}) \tag{2.19a}$$

$$I(d) = A(e^{j\beta d} - \Gamma_L e^{-j\beta d})/Z_0 \tag{2.19b}$$

stimulus which interacts with the load on the right to produce a reflected wave. At a point "d" along the line a *generalized reflection coefficient* can be defined as the ratio of reflected voltage, $A\Gamma_L e^{-jbd}$ to incident voltage, Ae^{jbd} . This reflection coefficient designated Γ can therefore be expressed as Equation (2.20a), and the total transmission line voltage and current is given by Equations (2.20b), and (2.20c).

$$\Gamma = \Gamma(d) = \Gamma_L e^{-j2bd} \tag{2.20a}$$

$$V(d) = Ae^{j\beta d}\left(1+\Gamma\right) \tag{2.20b}$$

$$I(d) = Ae^{j\beta d}\left(1-\Gamma\right)/Z_0 \tag{2.20c}$$

At the end of the line "d=0" the voltage is given by $V(0) = A(1+\Gamma)$ and the current is given by $I(0) = A(1-\Gamma)/Z_0$ which results in a ratio shown in Equation (2.21). The generalized and load reflection coefficient have a value that results in the proper voltage current ratio at the load. Equivalently, the load reflects just the right amount of voltage so that the ratio satisfies the boundary condition (i.e., Kirchoff's & Ohm's Laws) at the output node of the line. The actual magnitude of the voltage and current depends on the constant "A" which will now be shown to depend upon the input node boundary condition (again Kirchoff's Laws).

$$\frac{V(0)}{I(0)} = Z_o \frac{\left(1+\Gamma_L\right)}{\left(1-\Gamma_L\right)} = Z_L \tag{2.21a}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.21b}$$

Specifically at the input of the line $\mathrm{V(d=L)} = \mathrm{A}(e^{j\theta} + \Gamma_L e^{-j\theta})$ and $\mathrm{I(d=L)} = \mathrm{A}(e^{j\theta} - \Gamma_L e^{-j\theta})/Z_0$. Substitution of these equations into the input Kirchoff's Law condition yields Equation (2.22a). Factoring out the $\left(Z_s + Z_o\right)$ term results in Equation (2.22b) where the term Γ_s is given by Equation (2.22c)

$$\mathrm{V_s} = \mathrm{Z_s}\frac{\mathrm{A}}{\mathrm{Z_0}}(e^{j\theta} - \Gamma_L e^{-j\theta}) + \mathrm{A}(e^{j\theta} + \Gamma_L e^{-j\theta}) \tag{2.22a}$$

$$\mathrm{V_s} = \mathrm{A}\left(\frac{Z_s + Z_0}{Z_0}\right)e^{j\theta}\left(1-\Gamma_L\Gamma_s e^{-j2\theta}\right) \tag{2.22b}$$

$$\Gamma_s = \frac{Z_s - Z_o}{Z_s + Z_o} \tag{2.22c}$$

Γ_s formally appears as a source reflection coefficient, i.e., the reflection coefficient that would result from a load Z_s if an incident wave impinged upon it. It will shortly be shown that such an interpretation is physically correct. The constant "A" can now be determined as shown below and equals and the voltage and current at an arbitrary point "d" on the transmission line is given by

$$A = \frac{e^{-j\theta}}{\left(1-\Gamma_L\Gamma_s e^{-j2\theta}\right)}\left(\frac{Z_o}{Z_s + Z_o}\right)V_s$$

$$V(d) = \frac{e^{-j\theta}}{\left(1-\Gamma_L\Gamma_s e^{-j2\theta}\right)}\left(\frac{Z_o}{Z_s + Z_o}\right)V_s(e^{j\beta d} + \Gamma_L e^{-j\beta d}) \tag{2.23a}$$

$$I(d) = \frac{e^{-j\theta}}{\left(1-\Gamma_L\Gamma_s e^{-j2\theta}\right)}\left(\frac{1}{Z_s + Z_o}\right)V_s(e^{j\beta d} - \Gamma_L e^{-j\beta d}) \tag{2.23b}$$

While the voltage magnitude is proportional to V_s and depends upon Z_s in a complicated way the physical interpretation of the above results can be seen by using the relationship $1/(1-r)=1+r+r^2+r^3+\Lambda$ with $r=\Gamma_L\Gamma_s e^{-j2q}$. The voltage expression then becomes

$$V(d)=\left(\frac{Z_0}{Z_s+Z_0}\right)V_s e^{-jq}\left(1+\Gamma_L\Gamma_s e^{-j2q}+\Gamma_L^2\Gamma_s^2 e^{-j4q}+\Lambda\right)\times\left(e^{jbd}+\Gamma_L e^{-jbd}\right)$$

and

$$V(d)=\left(\frac{Z_0}{Z_s+Z_0}\right)V_s\left(\underset{(1)}{e^{j(bd-q)}}+\underset{(2)}{\Gamma_L\Gamma_s e^{j(bd-3q)}}+\underset{(3)}{\Gamma_L^2\Gamma_s^2 e^{j(bd-5q)}}+\Lambda\right)+$$

$$\left(\frac{Z_0}{Z_s+Z_0}\right)V_s\left(\underset{(1')}{\Gamma_L e^{-j(bd+q)}}+\underset{(2')}{\Gamma_L^2\Gamma_s e^{-j(bd+3q)}}+\underset{(3')}{\Gamma_L^3\Gamma_s^2 e^{-j(bd+5q)}}+\Lambda\right)$$

Each of the numbered terms can now be given a physical interpretation as follows:

(1) TERM: $[Z_0/(Z_s+Z_0)]V_s e^{j(bd-q)}$ represents a forward propagating wave. At the input node of the transmission line $d=L \Rightarrow bL=q$ and the voltage is thus seen to equal $[Z_0/(Z_s+Z_0)]V_s$ which can be interpreted as the source voltage V_s being divided by two series impedances of Z_s and Z_o (voltage divider rule). If one thinks of this wave as the one that is initially launched then the input impedance at the transmission line equals Z_o since no wave has yet reached the load and had a chance to be reflected back to the input. Consequently, the input impedance, $\mathrm{V_{in}}/\mathrm{I_{in}}$, is determined initially by only the characteristic impedance of the line. This voltage is referred to as the initially launched voltage or incident voltage and designate it by V_{inc}. Therefore, $V_{inc}=V_s[Z_0/(Z_s+Z_0)]$ and the forward propagating voltage at the load end is determined by substituting $d=0$ and equals $V_{inc}e^{-jq}$.

(1¢) TERM: $[Z_0/(Z_s+Z_0)]V_s\Gamma_L e^{-j(bd+q)}=V_{inc}\Gamma_L e^{-j(bd+q)}$ represents a reverse propagating wave. At the load, $d=0$ the voltage is the same as the initial wave *multiplied by the load reflection coefficient* Γ_L. This wave results in a voltage at the input node, $d=l$ or $bl=q$, equal to $V_{inc}\Gamma_L e^{-j2q}$

(2) TERM: $[Z_0/(Z_s+Z_0)]V_s\Gamma_L\Gamma_s e^{j(bd-3q)}=V_{inc}\Gamma_L\Gamma_s e^{j(bd-3q)}$ represents a forward traveling wave generated when wave 1' interacts with the source end of the line resulting in a reflected wave determined by the source reflection coefficient.

(2¢) TERM: $[Z_0/(Z_s+Z_0)]V_s\Gamma_L^2\Gamma_s e^{-j(bd+3q)}=V_{inc}\Gamma_L^2\Gamma_s e^{-j(bd+3q)}$ represents a reverse traveling wave resulting when wave 2 reaches the load end of the line.

The process of an incident wave resulting in repeated reflections can be represented by the following diagram:

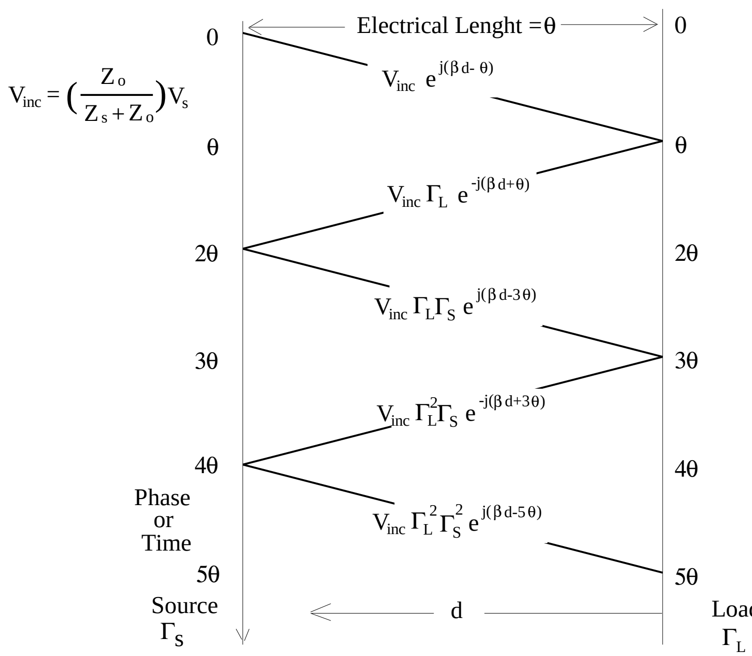


Figure 2.5 Illustration of multiple reflections generated by a source and load connected by a transmission line where V_{inc} = the initially launched voltage, and θ = the electrical length of the line.

Thus, the voltage expression above can be seen physically to be the steady state situation resulting from multiple reflected voltage waves on the line between the source and the load. The reflections occur because Kirchoff's Laws are required to be satisfied at both ends of the line.

Example 2.2.1.. *What is the voltage if* $Z_L = Z_0$ *, i.e., for a matched load?*

In this case $\Gamma_L = 0$ and substitution yields,

$$V(d) = \left(\frac{Z_o}{Z_s + Z_o}\right) V_s e^{j(bd-q)}$$

The initially launched wave V_{inc} has a magnitude of $V_{inc} = V_s\left[Z_0 / (Z_s + Z_0)\right]$. The phase term -θ appears since the phase at the load $(d = 0)$ since the load voltage will be out of phase by the electrical length of the line.

Example 2.2.2.. *What is the voltage on the line if* $Z_s = Z_o$ *?*

In this case $\Gamma_s = 0$ and

$$V(d) = \frac{1}{2} V_s e^{-jq}\left(e^{jbd} + \Gamma_L e^{-jbd}\right)$$
$$= \frac{1}{2} V_s\left(e^{j(bd-q)} + \Gamma_L e^{-j(bd+q)}\right)$$

There is an initially launched voltage of magnitude $V_{inc} = V_s / 2$ since the source voltage initially splits evenly between the source impedance and the line characteristic impedance (which is the input impedance for the initial wave). A second wave results since the load is not matched. The reflected wave at the load $(d=0)$ equals the incident wave multiplied by Γ_L. The reflected voltage has a term $e^{-j(bd+q)}$ which represents a wave propagating to the left. This is the same as the forward propagating exponential with "d" replaced by "-d." The phase delay, θ, at d=0 is the same for both exponential terms so the phase delay for the reflected signal continues to account for the total travel delay from the source.

It is often just as easy to analyze a transmission line circuit applying the specific boundary conditions and solving directly for the desired current or voltage. This approach often results in a better understanding of the physical behavior of the circuit and is illustrated in the next set of examples. Obviously, the results must agree with those obtained by direct substitution into the above equations.

Example 2.2.3.. *What is the voltage on the line if $Z_s = 0$?*

$$V(d) = A(e^{+jbd} + \Gamma_L e^{-jbd})$$
$$I(d) = A(e^{+jbd} - \Gamma_L e^{-jbd})/Z_0$$

$$V(L) = A(e^{+jq} + \Gamma_L e^{-jq}) = V_s$$
$$A = V_s \big/ (e^{+jq} + \Gamma_L e^{-jq})$$

$$V(d) = V_s(e^{+jbd} + \Gamma_L e^{-jbd}) \big/ (e^{+jq} + \Gamma_L e^{-jq})$$
$$I(d) = (e^{+jbd} - \Gamma_L e^{-jbd}) \big/ \left[(e^{+jbd} + \Gamma_L e^{-jbd})Z_0\right]$$

Example 2.2.4.. *Find the voltage transfer function ,(load voltage/input voltage) and input impedance for a quarter-wave transmission line.*

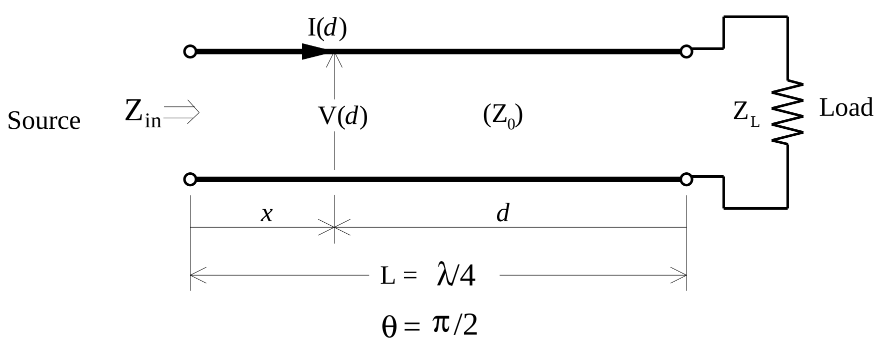


Figure 2.5 A quarterwave line with load

Referring to Figure 2.5 one sees that the total voltage at a distance "d" is $V(d) = A\left(e^{jbd} + \Gamma_L e^{-jbd}\right)$, and therefore at the input the total voltage is $V(l/4) = A\left(e^{jp/2} + \Gamma_L e^{-jp/2}\right) = (j - j\Gamma_L)$ which simplifies to $V(l/4) = jA(1-\Gamma_L)$. At the load $d=0$ and the total voltage equals $V(0) = A(1+\Gamma_L)$. The voltage transfer ratio is found by diving V(0) by V(λ/4) which simplifies to the results in Equation (2.24). This is

$$\frac{V(0)}{V(\,l/4)} = -j\,\frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\frac{V(0)}{V(\,l/4)} = -j\,\frac{Z_L}{Z_o} \tag{2.24}$$

a very useful result which is easy to remember since it is a simple ratio of the load impedance and the characteristic impedance. The -j term accounts for the phase shift naturally associated with a λ/4 line. The load impedance can be remembered as being in the numerator since for a short circuit load the voltage ratio must be zero.

The input impedance can be found by additionally considering the total current at an arbitrary distance "d" from the load, $I(d) = A(e^{jbd} - \Gamma_L e^{-jbd})/Z_0$ and specifically at the input where d=λ/4 the total current is $I(\,l/4) = A(e^{jp/2} - \Gamma_L e^{-jp/2})/Z_0 \;\; = A(1+\Gamma_L)/Z_0$ and therefore ,the impedance is given by the ratio of total voltage to total current which simplifies to give the results in Equation (2.25)

$$Z_{in} = \frac{V(\,l/4)}{I(\,l/4)} = \frac{1-\Gamma_L}{1+\Gamma_L}\,Z_o$$

$$Z_{in} = \frac{Z_o^2}{Z_L} \tag{2.25}$$

Power in a transmission line circuit is computed using the familiar expression $Power = \mathrm{Re}\{V \cdot I^*\} = \mathrm{Re}\{I \cdot V^*\}$. This represents actual power transmitted or lost at particular terminals of a circuit. The imaginary component, i.e., $\mathrm{Im}\{VI^*\}$, is called *reactive power* and represents power stored and exchanged (within a cycle) between magnetic and electric fields of the circuit. The time average of reactive power is zero. The expression above assumes that I and V have been defined as rms currents and voltages, which is an assumption which will always be made unless explicitly stated otherwise.

However, for transmission line circuits there are additional forms of power. Since voltage and current waves can exist on a line they represent a flow of power. The power associated with these waves is found by applying the above formula to the separate propagating components. To illustrate this consider the figure below.

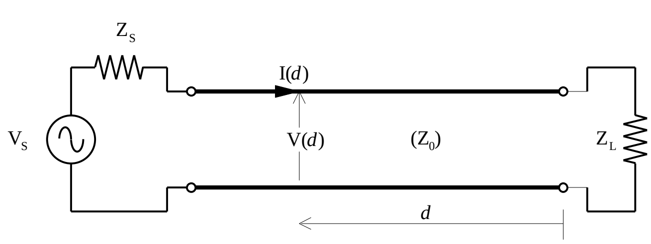


Figure 2.5 Power on the line is given by P(d)= $P^+(d)$ - $P^-(d)$.

The total voltage and current as seen earlier equals $V(d) = A(e^{jbd} + \Gamma_L e^{-jbd})$ and $I(d) = A(e^{jbd} - \Gamma_L e^{-jbd})/Z_0$ where $A = e^{-jq}\left(1 - \Gamma_L \Gamma_s e^{-j2q}\right)^{-1}\left[Z_0/(Z_s + Z_0)\right]V_s$. The forward voltage V^+, and current I^+ are given by $V^+ = Ae^{jbd}$ and $I^+ = Ae^{jbd}/Z_0$. The power associated with them is the forward propagating power P^+ and is given by Equation (2.26a). Similarly, the reverse propagating power P^- is given by Equation (2.26b). The net power or power delivered down the line is $P_{del} = P^+ - P^-$ which is shown in Equation (2.26c)

$$P^+ = \mathrm{Re}\left\{V^+ I^{+^*}\right\} = \mathrm{Re}\left\{\left(Ae^{jbd}\right)\left(\frac{A^*}{Z_o}\right)e^{-jbd}\right\} = \frac{|A|^2}{Z_o} \tag{2.26a}$$

$$P^- = |\Gamma_L|^2 \frac{|A|^2}{Z_o} \tag{2.26b}$$

$$P_{del} = \left(1 - |\Gamma_L|^2\right)\frac{|A|^2}{Z_o} \tag{2.26c}$$

Example 2.2.5.. *Compute the power using* $\mathrm{I(d)}^* \bullet V(d)$ *and compare with the previous results.*

The power associated with abstract terminals at a distance "d" along the line, designated P(d), is given by $P(d) = \mathrm{Re}\left\{I(d)^* V(d)\right\}$. Substitution of the expressions for I(d) and V(d) yields the results of Equation (2.27), where $\Gamma_L = \Gamma_R + j\Gamma_I$. .

$$\mathrm{I(d)}^* \cdot V(d) = \frac{|A|^2}{Z_o}\left(e^{-jbd} - \Gamma_L^* e^{jbd}\right)\left(e^{jbd} + \Gamma_L e^{-jbd}\right)$$

$$= \frac{|A|^2}{Z_o}\left[(1 - |\Gamma_L|^2) - \Gamma_L e^{-j2bd} + \Gamma_L^* e^{j2bd}\right]$$

$$\mathrm{I(d)}^* \cdot V(d) = \frac{|A|^2}{Z_o}\left[(1 - |\Gamma_L|^2) - 2j\left(\Gamma_I \cos 2bd + \Gamma_R \sin 2bd\right)\right] \tag{2.27}$$

Taking the real part of Equation (2.27) results in $P(d) = |A|^2\left(1 - |\Gamma_L|^2\right)/Z_0$ which is the net power or power delivered by the line, P_{del} , as seen above.

Example 2.2.6... *Calculate the forward and reverse propagating power and the power delivered to the load for the circuit shown in Figure 2.6*

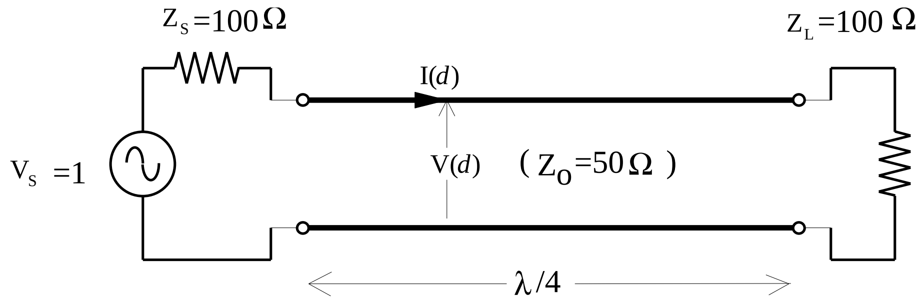


Figure 2.6 Quarterwave line with source and load

For this circuit substitution into the formulas yields $A = -j\dfrac{3}{10}$, therefore, $P^+ = \left(\dfrac{3}{10}\right)^2 \Big/ 50 = \dfrac{9}{5000} watts$. Since $\Gamma_L = 1/3$ then $P^- = |\Gamma_L|^2 P^+ = \left(\dfrac{1}{9}\right)\left(\dfrac{9}{5000}\right) = \dfrac{1}{5000}$. The power delivered to the load is

$$P_{del} = P^+ - P^- = \frac{9}{5000} - \frac{1}{5000} = \frac{1}{625} watts$$

The *Thevenin Equivalent Theorem* applies for transmission line circuits and the equivalent voltage and impedance is found using the familiar techniques. Given two terminals of a circuit, which can include any points along a transmission line, the equivalent circuit has the same voltage, current characteristics that would be observed by the original circuit at the terminals. The *Thevenin voltage equals the open circuit voltage* at the terminals and the *Thevenin impedance* is that seen by looking into the terminals (with voltage source shorted, and current sources opened). This is illustrated by the following.

Example 2.2.7... *Find the Thevenin Equivalent circuit using the output terminals of the transmission line circuit above*.

Having determined the equivalent circuit compute the power delivered to the load. The Thevenin Equivalent circuit is found by opening the terminals and observing the input impedance and open circuit voltage. This situation is illustrated in Figure 2.7.

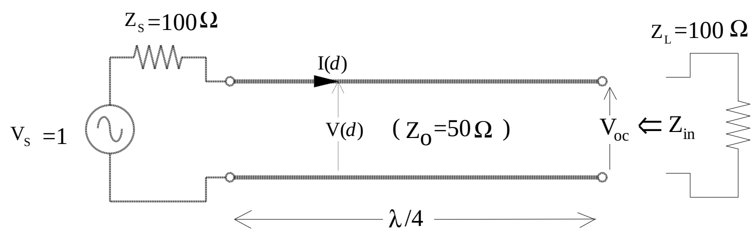


Figure 2.7 Determination of Thevenin Equivalent circuit for quarterwave line with source.

The input impedance can be found using the quarter wave relationship derived previously to get $Z_{\text{Thev}} = Z_0^2/100 = (50)^2/100 = 25\Omega$. The open circuit voltage is found using the voltage and current relationship for a transmission line, i.e., $V(d) = A(e^{jbd} + e^{-jbd})$ and $I(d) = A(e^{jbd} - e^{-jbd})/50$, since $\Gamma_L = 1$ for an open circuit load. At the input (d = λ/4) of the line $V(l/4) = 0$ and $I(l/4) = jA/25$. The current into the transmission line under these conditions would be 1/100. This determines the constant A, $A = -j(1/4)$ and the general voltage and current expression becomes $V(d) = -j(e^{jbd} + e^{-jbd})/4$, and $I(d) = -j(e^{jbd} - e^{-jbd})/200$. The open circuit voltageV_{oc}=V(0) = -j $\frac{1}{2}$. The Thevenin Equivalent circuit is therefore, Figure 2.8.

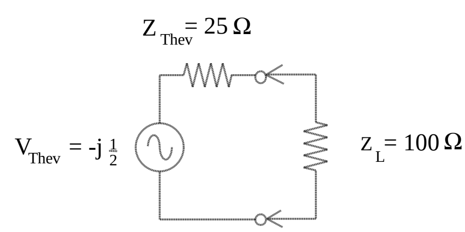


Figure 2.8 The Thevenin Equivalent circuit for the source and transmission line.

The power delivered from the transmission line to the load can now be found. The voltage across the load is $-j(1/2)100/(25+100) = -j2/5$-. The power delivered to the load is, therefore

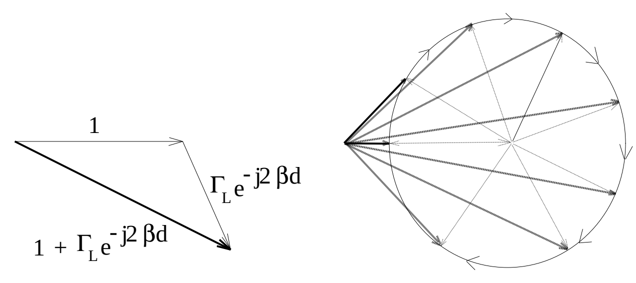
$$P_{LOAD} = \frac{|V_L|^2}{R_L} = \frac{(2/5)^2}{100} = \frac{1}{625} watts$$

which agrees with the results from above.

2.3 GRAPHICAL ANALYSIS OF TRANSMISSION LINES

The behavior of distributed circuits with respect to parameter changes is usually very important. The two parameters whose variation is of most interest is usually physical dimensions (E.G. length), and frequency (bandwidth). Interest in the first come about because of tolerance considerations in fabricating the circuits and interest in the second occurs because most circuits require more than a single frequency to operate or communicate. In the second case the range of frequencies is the bandwidth of the system. It is convenient to examine the effects of physical dimensions and frequency together for a distributed circuit because the critical parameter which determines performance is electrical length which compares physical length with wavelength. Of course, the wavelength of a propagating signal on a distributed circuit is a function of frequency.

Significant insights are possible by considering a transmission line with a load, Z_L. The load reflection coefficient is Γ_L and from Equation (2.5a) the total voltage at a distance "d" from the load equals $V(d) = Ae^{+jbd}\left(1+\Gamma_L e^{-j2bd}\right)$. The magnitude of this voltage is therefore, $|V(d)| = |A|\left|1+\Gamma_L e^{-j2bd}\right|$. This magnitude is illustrated in Figure 2.9a as the vector addition of two complex numbers, 1+j0, and $\Gamma_L e^{-j2bd}$. If the distance d is increased then the total voltage V(d) varies as shown in figure 2.9b. The vector representing the complex number $\Gamma_L e^{-j2bd}$ rotates in a clockwise fashion since negative angles are measured in a clockwise rotation and positive angles in a counter clockwise rotation. The total voltage increases and decreases as the tip of the vector traces out a circle or radius $|\Gamma_L|$. The maximum total voltage occurs when the vector $\Gamma_L e^{-j2bd}$ points to the right (0 degrees) and the minimum occurs when the vector points to the left (±180 degrees or ±π radians). A rectangular plot of the total voltage is illustrated in Figure 2.10 as a function of the distance d.



(b.)

(a.)

Figure 2.8 **(a.)** The total voltage on a transmission line as a vector sum. **(b.)** Illustration showing how the total voltage changes as the distance "d" is increased

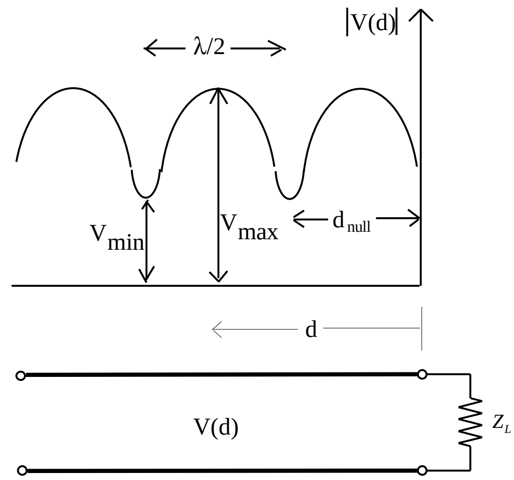


Figure 2.10 Plot illustrating variations in the total voltage magnitude $|V(d)|$

The voltage variation illustrated in figure 2.10 is called a *standing wave* pattern. It is an amplitude resulting from the combination of counter propagating voltages which remains stationary with respect to position on the line, and hence the name. The minimum amplitude is designated $V_{\min}$ and occurs in a null in the pattern. Similarly the maximum voltage amplitude is designated $V_{\max}$. The ratio of the maximum to minimum voltage on the line is called the Voltage Standing Wave Ration., $\mathrm{VSWR} = \mathrm{V}_{\max}/\mathrm{V}_{\min}$ which is a real number always greater than or equal to one. To emphasize that the VSWR is a ratio it is often numerical reported as "n:1", (spoken " n to one"), for example an antenna as a load at the end of transmission line may produce a VSWR on the line of 1.4:1. Often in this situation the load is referred to has having a VSWR which really means that it produces the VSWR when connected to the transmission line.

The maximum voltage is given by $V_{\max} = |A|(1+|\Gamma_L|)$ and while the minimum voltage is $V_{\max} = |A|(1-|\Gamma_L|)$. The ratio of these gives the expression for the VSWR shown in Equation (2.28a). Also, one can solve for the magnitude of the reflection coefficient which is observed to be only a function of the VSWR as seen in Equation (2.28b)

$$VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \tag{2.28a}$$

$$|\Gamma_L| = \frac{VSWR-1}{VSWR+1} \tag{2.28b}$$

In addition to the information provided by the VSWR, the locations of the nulls, which are usually much sharper than the peaks, provides useful information. The physical distance, Δd, between successive nulls occurs when the vector in figure 2.8 rotates through 180 degrees or 2π radians. This occurs when $2b \quad \Delta d = 2p$ and since $b = 2p/l$ then $\Delta d = l/2$. Therefore, *the null-to-null distance equals the half wavelength*. Based only on measurements of the total voltage magnitude the propagation wavelength and the magnitude of the load reflection coefficient can be determined. If the frequency is known then the propagation phase velocity, v is determined by $v = f\,l$. Since the load reflection coefficient is a complex number and only its magnitude has thus far been determined it remains to see how to calculate the angle for the load reflection coefficient. Assuming that the load reflection coefficient angle is θ then $\Gamma_L = |\Gamma_L| e^{+jq_L}$ and the distance to the first null is d_{null} then $\Gamma_L e^{-j2bd_{null}} = |\Gamma_L| e^{-jp}$ and $|\Gamma_L| e^{jq_L} e^{-j2bd_{null}} = |\Gamma_L| e^{-jp}$ which implies that $q_L = 2b\,d_{null} - p$ and with $b = 2p/\,l$ then $q_L = (4/d_{null} - 1)p$.

Example 2.3.8.. *When a 50 W is connected to an unknown load the following plot of voltage magnitudes are observed along the line for an operating frequency of 10 GHz. Determine the load impedance and the propagation velocity for the line.*

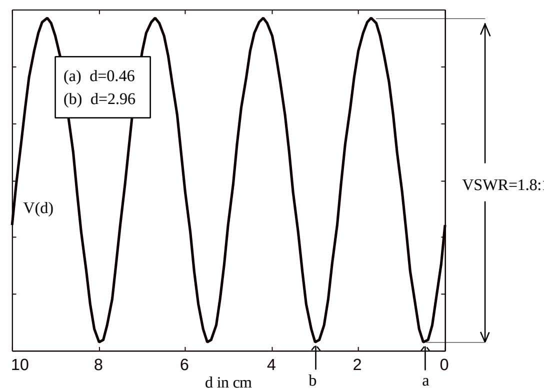


Figure 2.11 VSWR data for Example 2.3.8.

The problem is solved by first computing the wave length, lambda, and then computing the reflection coefficient observed at the first null, G. Note that the magnitude of G is found from VSWR formula and the angle is 180 degrees (or negative real number). The veloctiy is found since both the frequency (given) and the wavelength (calculated) are known.

```
dnull=0.46
lambda=2*(2.96-0.46)
lambda = 5
» Beta=2*pi/lambda
Beta =    1.2566


VSWR=1.8;
G=-(VSWR-1)/(VSWR+1)
G =   -0.2857


GL=G*exp(+j*2*Beta*dnull)
GL =  -0.1151 - 0.2615i


ZL=50*(1+GL)/(1-GL)
ZL =  35.0024 -19.9333i


» f=4e9;
» v=f*lambda
v =  2.0000e+010
```

Complex Geometry Theorem I:

A straight line in the complex plane is defined by $c^*z + cz^* = 2d$ *where c is a complex number of having magnitude 1 and d is a real number. The unit vector defined by the complex number c is perpendicular to the line and the real number d is equal to the line's distance from the origin.*

Proof:

Since $z = x + jy$ and $c = a + jb$ then $c^*z + cz^* = 2(ax + by) = 2d$ and $ax + by = d$ is a straight line. Since $|c| = |a + jb| = 1$ then $ax + by$ can be viewed as the dot product (inner product) between a unit vector $\hat{\mathbf{c}} = a\hat{\mathbf{i}}_x + b\hat{\mathbf{i}}_y$ and a position vector $\mathbf{r} = x\hat{\mathbf{i}}_x + y\hat{\mathbf{i}}_y$ and therefore $d = \mathbf{r} \cdot \hat{\mathbf{c}}$ and therefore d is the projected distance from the point (x,y) in the direction of $\hat{\mathbf{c}}$.

Complex Geometry Theorem II:

A circle in the complex plane is defined by $|z|^2 - c^*z - cz^* = b$ *where a is a complex number and b is a real number such that* $b + |c|^2 > 0$. *The center of the circle is given by the complex number a and the radius of the circle equals* $\sqrt{b + |c|^2}$.

Proof:

Adding $|c|^2$ to both sides of $|z|^2 - c^*z - cz^* = b$ gives $|z|^2 - c^*z - cz^* + |c|^2 = b + |c|^2$ and since $|z|^2 - c^*z - cz^* + |c|^2 = (z - c)(z - c)^* = |z - c|^2$ then $|z - c|^2 = b + |c|^2$ and $|z - c| = \sqrt{b + |c|^2}$ which is the equation for a circle with center located on the complex plane at the point "c" and having radius $\sqrt{b + |c|^2}$. Note that the equation requires that $b + |c|^2 > 0$

Looking at the relationship between Z and Γ one can consider how the reflection coefficient changes as the resistance alone varies. In this case $Z = R + jX_o$ where R can be viewed as a variable resistance while X_o is taken as a constant. The curve in the Z-plane is a straight line as shown in figure 2.12a. Using the complex geometry theorem I the equation for the constant reactance line would be $c^*z + cz^* = 2d$ where c=0+j1, or $-jZ + jZ^* = 2X_0$. Substitution of Z from the relationship $Z = Z_0\,(1 + \Gamma)/(1 - \Gamma)$ results in

$$-j\left(\frac{1+\Gamma}{1-\Gamma}\right)Z_0 + j\left(\frac{1+\Gamma^*}{1-\Gamma^*}\right)Z_0 = 2X_0$$

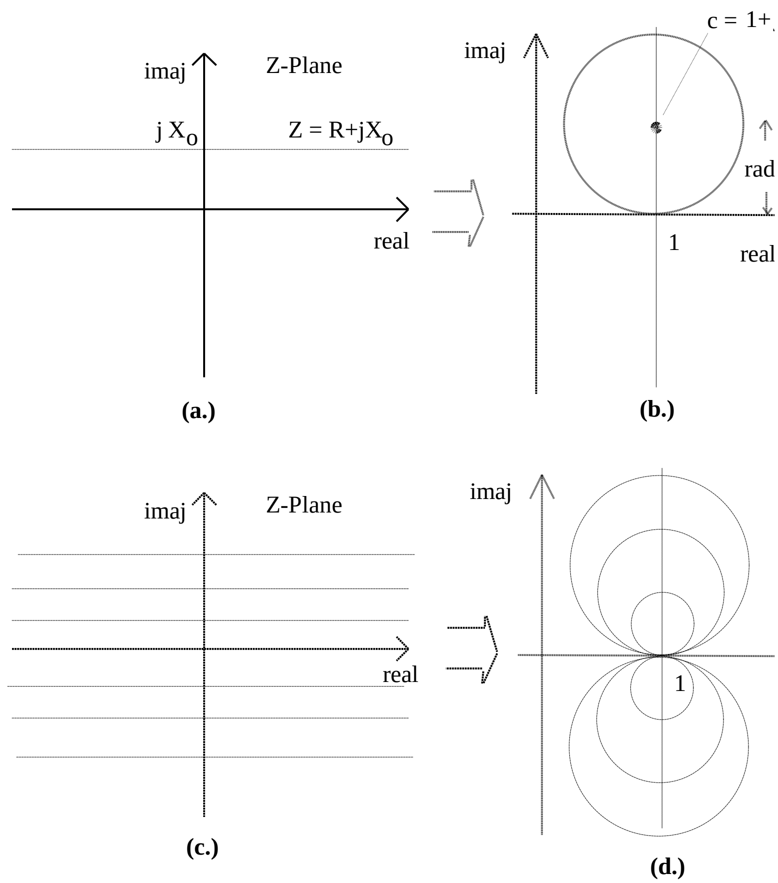


Figure 2.12 **(a.)** The complex impedance plane showing the trace of a constant reactance line **(b.)** the constant reactance curve in the Γ-plane **(c.)** multiple constant reactance line in Z-plane **(d.)** multiple constant reactance curves in Γ-plane

and clearing the denominator results in the Equation (2.29a) where the normalized reactance is given by $x_0 = X_0/Z_0$. This simplifies to give Equations (2.29c) which can be recognized from the complex geometry theorem II to represent a circle in the Γ-plane with center given by $c = 1 + j\,1/x_0$ and b=-1 implies that the radius=$\sqrt{|c|^2 + b} = \sqrt{1 + 1/x_o^2 - 1} = 1/|x_o|$. This is illustrated in figure 2.12b. Figures 2.12c and 2.12d illustrate the set of circle in the Γ-plane resulting from a set of constant reactance lines in the Z-plane.

$$(1+\Gamma)(1-\Gamma^*) - (1+\Gamma^*)(1-\Gamma) = j2x_0(1-\Gamma)(1-\Gamma^*) \tag{2.29a}$$

$$1+\Gamma-\Gamma^* - |\Gamma|^2 - \left(1-\Gamma+\Gamma^* - |\Gamma|^2\right) = j2x_0\left(1-\Gamma-\Gamma^* + |\Gamma|^2\right) \tag{2.29b}$$

$$|\Gamma|^2 - \left(1 - j\frac{1}{x_0}\right)\Gamma - \left(1 + j\frac{1}{x_0}\right)\Gamma^* = -1 \tag{2.29c}$$

In a similar way one can examine what happens to the constant resistance line in Z-plane when it is mapped into the Γ-plane. A constant resistance is presented by $Z = R_o + jX$ where X varies and R_o

remains constant. This is illustrated in figure 2.13a for several different choice for the constant R_o including several that are negative. The development will apply to both positive resistance loads as well as to negative resistance loads. Again the development begins using the Complex Geometry "Theorem I which describes a constant resistance line in the Z-plane, i.e., $Z + Z^* = 2R_0$. Substitution of $Z = Z_0\,(1+\Gamma)/(1-\Gamma)$ and $r_0 = R_0/Z_0$ together with simplification as before yields $2 - 2|\Gamma|^2 = 2r_0\,(1-\Gamma)(1-\Gamma^*)$ and $1 - |\Gamma|^2 = r_0 - r_0\Gamma - r_0\Gamma^* + r_0|\Gamma|^2$. Combining terms and division results in Equation (2.30) which is recognized from the complex Geometry Theorem II to be a circle

$$|\Gamma|^2 - \left(\frac{r_0}{1+r_0}\right)\Gamma - \left(\frac{r_0}{1+r_0}\right)\Gamma^* = \frac{1-r_0}{1+r_0} \tag{2.30}$$

where $b = (1-r_0)/(1+r_0)$ and $c = r_0/(1+r_0)$ and for both positive and negative values of r_0 one sees that $b + |c|^2 > 0$. The center of the circle is therefore given by $c = r_0/(1+r_0)$ and radius = $1/|1+r_0|$. Note that for $r_0 > -1$ that Center + Radius =1 i.e., the right hand side of the circles are tangent to point Γ=1+j0. For $r_0 < -1$ then Center - Radius=1 and the left hand side of the circles are tangent to the point Γ=1+j0. If $r_0 = -1$ then from $1 - r_0 = -r_0\Gamma - r_0\Gamma^* + (1+r_0)|\Gamma|^2$ it follows that $\Gamma + \Gamma^* = 2$ the curve is a straight line perpendicular to the real axis and crossing the real axis at the point Γ=1. This is illustrated in figure 2.13b.

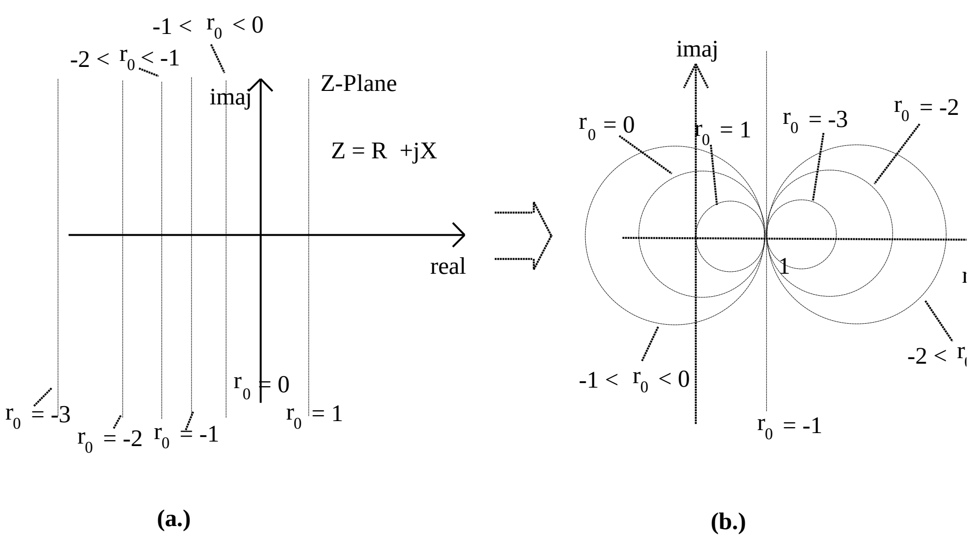


Figure 2.13 **(a.)** Example of seven constant resistance line in the S-plane. (b.) the seven constant resistance line in the Γ-plane.

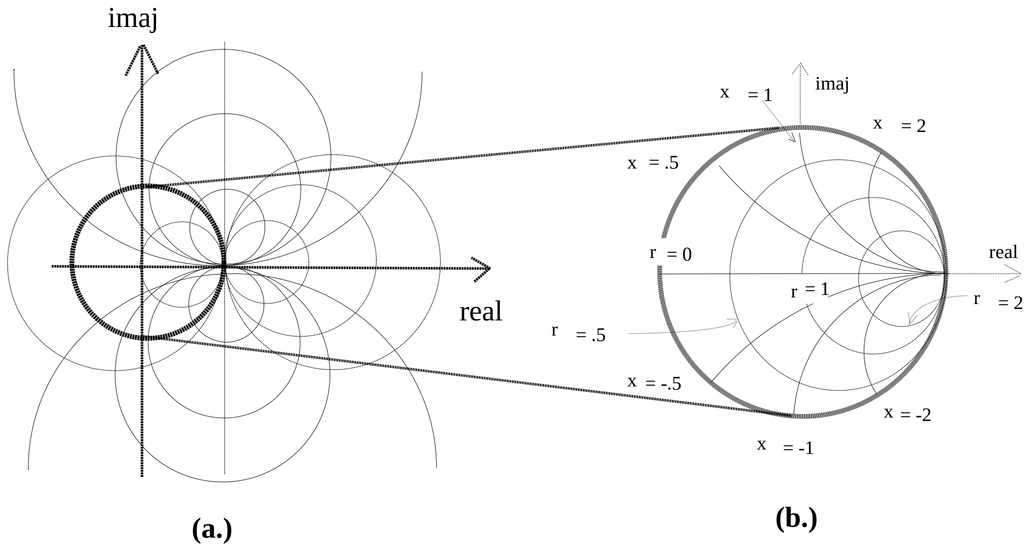


Figure 2.14 **(a.)** The Γ-plane with the circles of constant reactance and constant resistance displayed. **(b.)** The Passive Smith Chart or Unit Smith Chart (USC)

From the above analysis one can plot both the constant reactance and constant resistance circles on the Γ-plane as illustrated in figure 2.14a. The dark circle equates to a resistance of zero. Inside this circle the resistance is positive and outside of it the resistance is negative. Inside the reflection coefficient has a magnitude less than one and outside its magnitude is greater than one. The dark circleis the unit circle on the complex Γ-plane since $|\Gamma| = 1$. This representation of the Γ-plane with the circles is referred to as a Smith Chart named for its founder. The region in the dark circle is referred to as the Unit Smith Chart (USC) and represents passive loads, i.e., those having an impedance with a positive real part. The region outside of the USC equates to active load, i.e., those having an impedance with a negative real part. The USC region is shown expanded in figure 2.14b

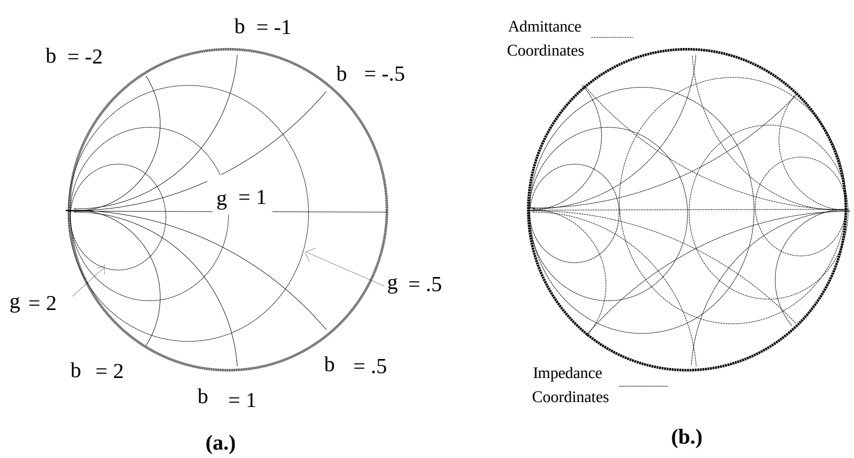


Figure 2.15 **(a.)** The USC showing circles of constant conductance and constant suseptance. **(b.)** A Y-Z Smith Chart shows both admittance and impedance circles

If one looks at the reflection coefficient in terms of an admittance $Y = 1/Z$ then the reflection coefficient becomes $\Gamma = -(Y - Y_0)/(Y + Y_0)$ which is of exactly the same form as the resistance formula except with a negative sign in the front. Therefore it follows immediately that a mapping from the complex admittance plane would result in lines of constant conductance and constant suseptance become circles on the Γ-plane. However the negative sign would mean that everything would be rotated by 180 degrees. This is illustrated in figure 2.15a for the USC where g represent conductance and b represents suseptance. Figure 2.15b illustrates the USC showing some of the admittance coordinates as well as the impedance coordinates. This type of representation is called a Y-Z Smith Chart. Note that *in all charts the bottom represents a capacitive load and the top an inductive load*

One now considers a series resonant circuit consisting of an inductor, capacitor, and resistor as a load and considers how the reflection coefficient would change as the frequency is increased from a low value to a high value. The reactance values will change from a net negative value to a net positive value as the frequency increases. The resistor, assuming that it is ideal, will have a value independent of frequency, i.e., it will be a constant. Hence the reflection coefficient, Γ, will follow a trace similar to that illustrated in Figure 2.16a. It will lie upon a circle of constant resistance an move from capacitive to inductive reactance, i.e., move from the lower half of the USC to the upper half. The real and imaginary axis are shown as a reminder that the Smith Chart is an overlay of impedance and admittance coordinates on the complex Γ-plane. Similarly, a shunt or parallel resonant consisting of an inductor, capacitor, and resistance will follow a constant suseptance trace moving from inductive which will dominate a lower frequencies to capacitive which will dominate at higher frequencies as illustrated in Figure 2.16b.

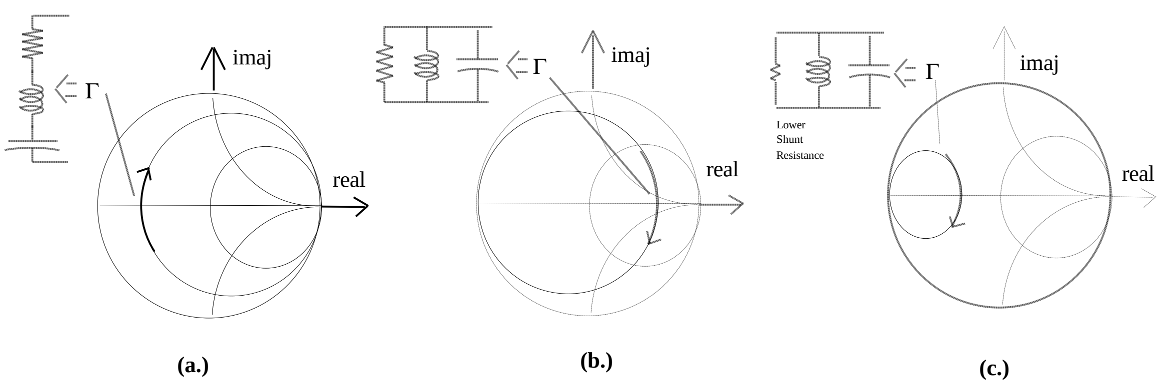


Figure 2.16 **(a.)** Plot of Γ with increasing frequency of a series resonant circuit, **(b.)** shunt or parallel resonant circuit with increasing frequency. **(c.)** Plot of the reflection coefficient of a shunt resonance circuit with a lower shunt resistance.

For the shunt resonator if the resistor is lower (conductance higher) then the reflection coefficient could follow a trace similar to that shown in Figure 2.16c. Therefore, it is clear that a shunt resonator always follows a path from top to bottom (inductive to capacitive) and is at resonance at the frequency where Γ crosses the real axis. The opposite is true for a series resonance.

In figure 2.16c the Q of the resonance is lower than the previous shunt circuit. Hence the radius of curvature of the trace gives an indication of the Q associated with the circuit's resonance. Often a circuit manifests more complicated behavior revealing multiple resonances. Such an example is illustrated in Figure 2.17a. In this example as the frequency is increased from a low to a higher value the circuits reflection coefficient (or impedance) first looks capacitive and then passes through a series resonance. As the frequency continues to increase the circuit looks inductive and then passes through a shunt resonance.

As the frequency is increased even higher the circuit passes through a series resonance again and remains inductive thereafter. The example of figure 2.17a illustrates the property that loops in the reflection coefficient trace proceed in a clockwise fashion as frequency increases. This usually makes it possible to examine data on a Smith Chart and deduce which point on the trace is the lower versus the higher frequency. This behavior of the reflection coefficient trace is due to of causality property of a physically realizable circuit. This follows from the fact that the reflection coefficient as a function of frequency is relates an reflected voltage to an incident voltage by $V^{-}(w) = \Gamma(w)V^{+}(w)$. If the incident voltage is viewed as a time signal whose Fourier Transform is $V^{+}(w)$ then $v^{+}(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} V^{+}(w)e^{jwt}dw$. Likewise the reflected time signal would be related to $V^{-}(w)$ by its Fourier transform, i.e.,

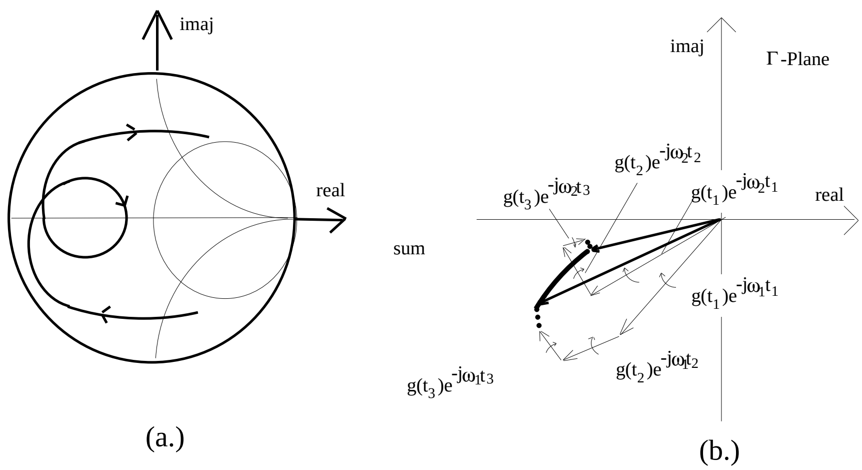


Figure 2.17 **(a.)** More complex circuits display multiple resonant behavior (2 series+1 shunt). **(b.)** Tendency towards clockwise rotation due to causality of physical circuits

$v^{-}(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} V^{-}(w)e^{jwt}dw$. Also $v^{-}(t) = g(t) \otimes v^{+}(t)$ where $\otimes$ represents the convolution operation, i.e., $v^{-}(t) = \int_{-\infty}^{+\infty} v^{+}(t-t)g(t)dt$. If the circuit is a physically realizable one then the principle of causality applies. This means that the signal output at a time "t" can only depend of signal inputs occurring only before time "t" and not afterwards. Therefore from the integral it must follow that g(τ) = 0 when τ<0 since otherwise the integral would have a contribution from $v^{+}(t)$ for times exceeding t. Also, $\Gamma(w) = \int_{-\infty}^{+\infty} g(t)e^{-jwt}dt$ and since g(t)=0 for negative times then $\Gamma(w) = \int_{0}^{+\infty} g(t)e^{-jwt}dt$. Notice that $\Gamma(w)$ is a sum (integral) of complex unit vectors, e^{-jwt}, rotating clockwise as a function of increasing frequency, ω, and which are scaled by an amplitude factor, g(t). Therefore a trace of $\Gamma(w)$ can be though of as a vector sum as illustrated in figure 2.17b.

Figure 2.18 illustrates a Γ plot of a transmission line with a load as a function of frequency. The radius of the circle is $|\Gamma_L|$. Again notice that the trace proceeds in a clockwise direction with increasing frequency.

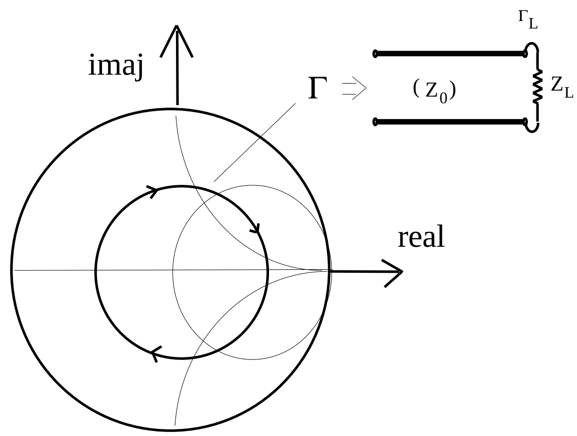


Figure 2.18 Reflection coefficient trace as a function of frequency for a transmission line connected to load impedance

Example 2.3.1.. *Find the reflection coefficient for a load consisting of a series combination 75 W resistance and 1 pF capacitor operating a 1 GHz where Zo= 50 W. What is reflection coefficient the load is connected to a l/2 line, l/4 line, or l/8 line.*

The calculations are easily computed using the above dirived relationships

f=1e9; ω=2*pi*f; C=1e-12; R=75; Zo=50;

X=-1/(ω*C); ZL=R+j*X; GL=(ZL-Zo)/(ZL+Zo) ⇒

$\boxed{\text{GL = 0.6948 - 0.3886i}}$

length=λ/2, ⇒ beta*length = pi ⇒ G=GL*exp(-2*jπ) ⇒

$\boxed{\text{G = 0.6948 - 0.3886i}}$

length=λ/4 ⇒ beta*length = pi/2 ⇒ G=GL*exp(-j2π/2) ⇒

$\boxed{\text{G = -0.6948 + 0.3886i}}$

length=λ/8 ⇒ beta*length = pi/4 ⇒ G=GL*exp(-j2π/4) ⇒

$\boxed{\text{G = -0.3886 - 0.6948i}}$

Example 2.3.2.. *A new load is created by making parallel connection of two 50 W lines each having their own load as illustrated below. What is the impedance. What is the reflection coefficient produced by the new load on a 75 W line with length l/4.*

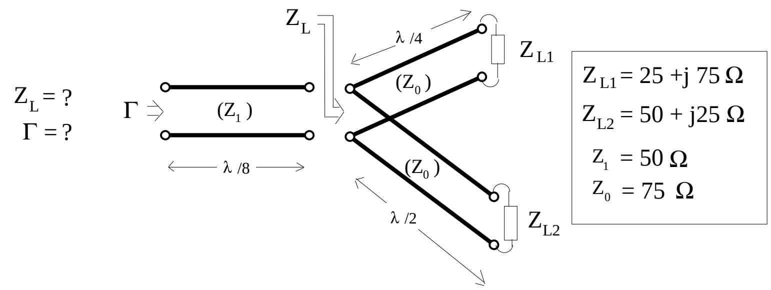


Figure 2.19 VSWR data for Example 2.3.2

ZL1=25+j*75;
Zo=50;
GL1=(ZL1-Zo)/(ZL1+Zo)
GL1 = 0.3333 + 0.6667i
G1=GL1*exp(-j*2*pi/2)
G1 = -0.3333 - 0.6667i
ZL1p=Zo*(1+G1)/(1-G1)
ZL1p = 10.0000 -30.0000i

ZL2=50+j*25;
GL2=(ZL2-Zo)/(ZL2+Zo)
GL2 = 0.0588 + 0.2353i
G2=GL2*exp(-j*2*pi)
G2 = 0.0588 + 0.2353i
» ZL2p=Zo*(1+G2)/(1-G2)
ZL2p = 50.0000 +25.0000i [Note same as ZL2 since line is lambda/2]

ZL=(ZL1p*ZL2p)/(ZL1p+ZL2p) [Parallel combination]

ZL = 22.4138 -18.9655i

Z1=75; [Characteristic impedance of line]
GL=(ZL-Z1)/(ZL+Z1)

GL = -0.4836 - 0.2888i

Example 2.3.3.. *For the load illustrated below compute the reflection coefficient as the frequency increases from .5 to 8 GHz..*

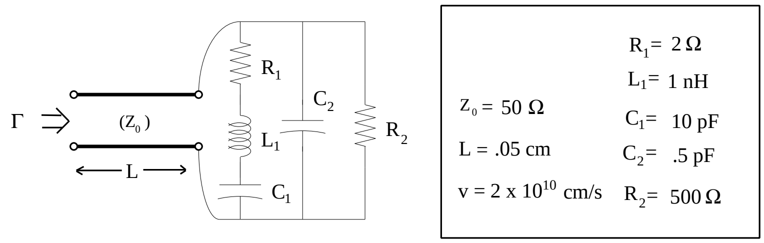
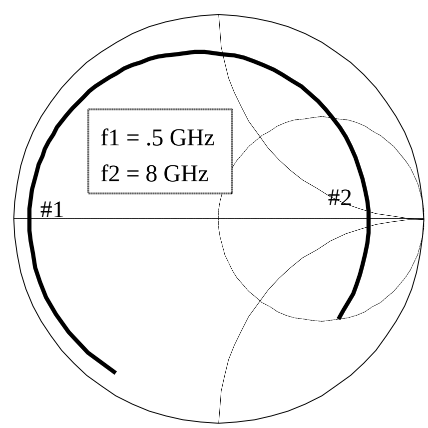


Figure 2.20 VSWR data for Example 2.3.3

```
%script file ex2_2_4.m
%Solution to exmaple 2.2.4
%

f=[.5:.1:8]*1e9;
w=2*pi*f;

L1=1e-9;
C1=10e-12;
R1=2;     |
C2=.5e-12;
R2=500;

v=2e10;
lambda=v./f;
beta=2*pi./lambda;
length=.05;
Zo=50;

XL1=w*L1;
XC1=-1./(w*C1);
Z1=R1+j*XL1+j*XC1;
Y1=1./Z1;

XC2=-1./(w*C2);
Z2=j*XC2;
Y2=1./Z2;

Y3=1/R2;

YL=Y1+Y2+Y3;
ZL=1./YL;
GL=(ZL-Zo)./(ZL+Zo);
G=GL.*exp(-j*2*beta*length);
plot(G)
hold on
usc1
axis off
```

function usc1

```
%over lays Unit smith chart on ploted complex data
plot(res2usc(0))
hold on
```



#1 = Series Resonance
#2 = Shunt or Parallel Resonance

Figure 2.21 Reflection coefficient trace for the circuit in Example 2.3.3


```
axis('square')
plot(res2usc(1),'c--')
  %plot(res2usc(1/3),'c--')   Comment out unwanted coordinates
  %plot(res2usc(.5))
  %plot(res2usc(.25))
  %plot(res2usc(2))
  %plot(res2usc(3),'c--')
  %plot(res2usc(4))
plot(x2usc(0),'c:')
plot(x2usc(1),'c:')
  %plot(x2usc(3),'c:')
  %plot(x2usc(1/3),'c:')
  %plot(x2usc(.5))
  %plot(x2usc(.25))
  %plot(x2usc(2))
  %plot(x2usc(4))
plot(x2usc(-1),'c:')
  %plot(x2usc(-3),'c:')
  %plot(x2usc(-1/3),'c:')
  %plot(x2usc(-.5))
  %plot(x2usc(-.25))
  %plot(x2usc(-2))
  %plot(x2usc(-4))
--------------------------------
function y=res2usc(r)
%generates constant resistance circle for Unit Smith Chart
theta=pi*[0:5:360]/180;
Ucir=exp(j*theta);
ctr=r/(1+r);
rad=1/abs(1+r);
y=ctr+rad*Ucir;

-------------------------------
function y=x2usc(x)
%generates constance reactance circle-segments for Unit Smith Chart
if x==0
  y=[-1+j*eps, 1+j*eps];
else
  ctr=1+j/x;
  rad=1/abs(x);
  maxangle=2*atan(x);
  if x>0
   theta=-pi/2-[0:maxangle/20:maxangle];
  elseif x<0
   theta=pi/2-[0:maxangle/20:maxangle];
  end
  y=ctr+rad*exp(j*theta);
end
```

2.4 TRANSMISSION LINE STUBS (l/4. etc.)

Transmission lines where the load is either an open circuit or a short circuit are of special importance. In the first case $Z_L = Z_{open} = \infty$ and the second $Z_L = Z_{short} = 0$, and the reflection coefficient becomes $\Gamma_{open} = 1$

and $\Gamma_{short} = -1$. The total voltage and currents at a location "d" from such loads are given by $V_{open}(d) = A(e^{+jbd} + e^{-jbd})$, $I_{open}(d) = A(e^{+jbd} - e^{-jbd})/Z_0$, and $V_{short}(d) = A(e^{+jbd} - e^{-jbd})$, $I_{short}(d) = A(e^{+jbd} + e^{-jbd})/Z_0$. Each of these expression can be simplified using Euler's identity to get $V_{open}(d) = 2A\cos bd$, $I_{open}(d) = j2A\sin bd / Z_0$, and $V_{short}(d) = j2A\sin bd$, $I_{short}(d) = 2A\cos bd / Z_0$. The impedance looking through a line of length d at an open or short load is given by $Z_{open}(d) = V_{open}(d)/I_{open}(d)$ and $Z_{short}(d) = V_{short}(d)/I_{short}(d)$ resulting in Equations 2.31a and 2.31b. This is illustrated in figure 2.22 which shows that the VSWR = ∞ Open and short circuited transmission line are often referred to an *open stub or shorted stub*.

$$Z_{open}(d) = -jZ_0 \cot bd \tag{2.31a}$$

$$Z_{short}(d) = jZ_0 \tan bd \tag{2.31b}$$

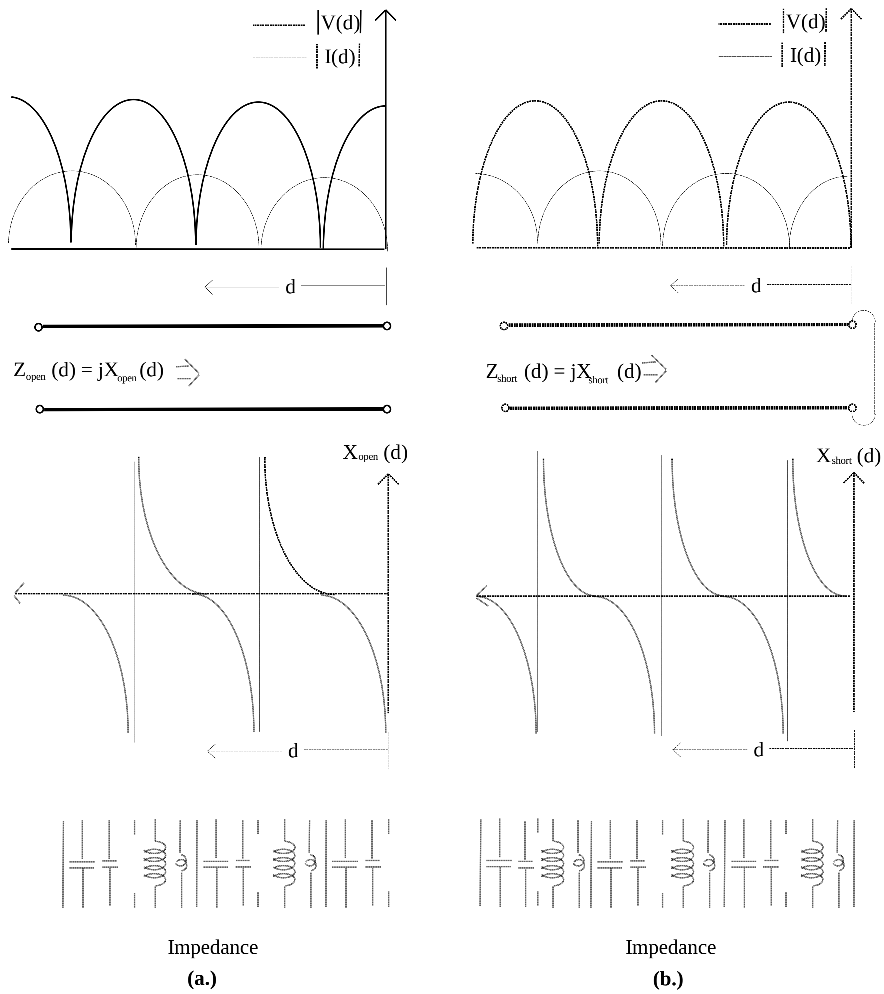


Figure 2.22 (a.) $|V(d)|$, $|I(d)|$, Reactance plots, and impedance type for an open circuited transmission line. (b.) same for a short circuit transmission line.

The reflection coefficient for an open circuited stub is $\Gamma = e^{-j2bd}$ and therefore data as a function of length, "d," or as a function of increasing frequency would produce a trace on the unit circle of Smith Chart display. This is illustrated in figure 2.23 which shows the impedance type associated with the point.

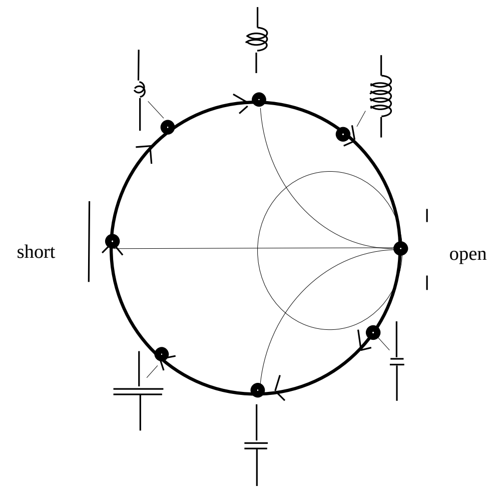


Figure 2.23 A reflection coefficient trace of an open stub on the USC follows the unit circle rotating in a clockwise direction starting from the right (open) value. Trace is a function of increasing length or increasing frequency.

Example 2.4.1.. A circuit consisting of two sources, one operating at 1 GHz and other at 2 ±.05 GHz are connected via a 50 Ω resistor to a transmission line which in turn is connected to a 50 Ω load. It is desired to suppress the 2 GHz signal using an open circuited stub. If a 2 GHz λ/4 open stub is connected in shunt with the load what suppression can be achieved as a worst case.? How does the characteristic impedance of the stub affect the power delivered to the load at the desired frequency?

This problem is easily examined using computer aided analysis. The solution below assumes a propagation velocity of 20 cm/ns for the direct line and stub (all impedances). A frequency of 2 GHz equates to a wavelength λ/4=5 cm which becomes the length for the stub, D1.

```
%script file ex2_3_1.m
%solution to example 2.3.1
%
RL=50;
Rs=50;
V=1;
for Zstub=20 :30:110;
  f=[1:.01:2.5]*1e9;
  w=2*pi*f;
  vo=2e10;
  vstub=2e10;
  beta=2*pi*f/vo;
  betastub=2*pi*f/vstub;
  Do=11;
  Dstub=2.5;
  Zo=50;
  ZZ=-j*Zstub*cot(betastub*Dstub);
  Y=1./ZZ;
  YL=1/RL;
  YT=Y+YL;
```

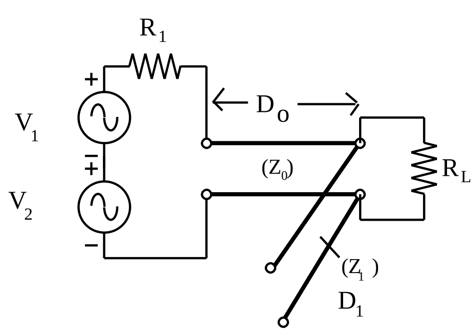


Figure 2.24 Example of a stub being used to suppress a signal

```
  ZT=1./YT;
  GT=(ZT-Zo)./(ZT+Zo);
  G=GT.*exp(-j*2*beta*Do);
  Zin=Zo*(1+G)./(1-G);
  Iin=V./(Rs+Zin);
  Vin=V*Zin./(Rs+Zin);
  P=conj(Iin).*Vin;
  Preal=real(P);
  P=10*log10(abs(Preal));
  Parray=[Parray;P];
end

Pmin=-80*ones(size(Parray));
plot(f*1e-9,max(Parray,Pmin))
axis([1 2.5 -80 0 ])
title('Power delivered to load vs Frequency')
xlabel('Frequency (GHz)')
ylabel('Power (dBm)')
Compare=[Parray(:,find(f==1e9)),Parray(:,find(f==1.95e9))];
Compare(:,1)-Compare(:,2)
```

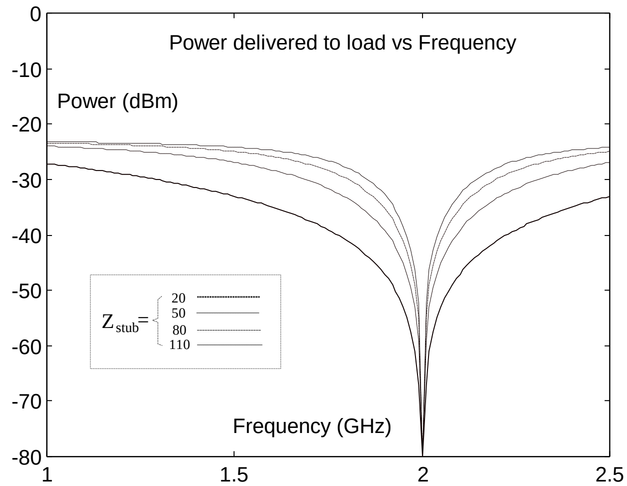


Figure 2.25 Plot showing power delivered to load as a function of frequency for a fixed stub

Power to load at desired freq compared to power to load at unwanted freq

Zo in Ω	P(1)/P(1.95) in dB
20	25.9702
50	21.1514
80	17.6748
110	15.1545

20 25.9702
80 17.6748

Example 2.4.2.. A λ/4 stub is used so that the voltage from a DC source (called a bias voltage) can be applied to the output of a transistor. Find the percentage of power that goes to the laod vs. the bias supply. for the ciruit below.

It is desirable for no DC power to reach the load so a series capacitor is used. It is also desireable for very little microwave power to propagate down the stub into the bias circuits so the source should appear as a low impedance.


```
%script file ex2_3_2.m
%solution to example 2.3.2
%
RL=50;
Rtran=200;
V=1;
Zo=50;   %Z1=Zo & Z2=Zo
Zstub=120;
f=1e9;
w=2*pi*f;
theta1=2*pi*3/8;  %beta1*D1
theta2=2*pi*7/16; %beta2*D2
thetastub=2*pi/4; %betastub*Dstub
Rbias=1;
Gbias=(Rbias-Zo)/(Rbias+Zo);
Gstub=Gbias*exp(-j*2*pi*thetastub);
Zstub=Zo*(1+Gstub)/(1-Gstub);

C=100e-12;
Xc=-1/(w*C);
Ztotal=RL+j*Xc;
GL=(Ztotal-Zo)/(Ztotal+Zo);
G2=GL*exp(-j*2*pi*theta2);
Z2=Zo*(1+G2)/(1-G2);

Zcomb=Zstub*Z2/(Zstub+Z2);
Gcomb=(Zcomb-Zo)/(Zcomb+Zo);
Gin=Gcomb*exp(-j*2*pi*theta1);
Zin=Zo*(1+Gin)/(1-Gin);
Iin=V./(Rtran+Zin);
Vin=V*Zin./(Rtran+Zin);
P=conj(Iin).*Vin;
Pin=real(P);
Pcomb=Pin;
V2magsqr=Pcomb*Zcomb;
Pstub=real(V2magsqr/Zstub);
P2=real(V2magsqr/Z2);
Loadfraction=10*log10(P2/Pin)      % dB
Stubfraction=10*log10(Pstub/Pin)   % dB
```

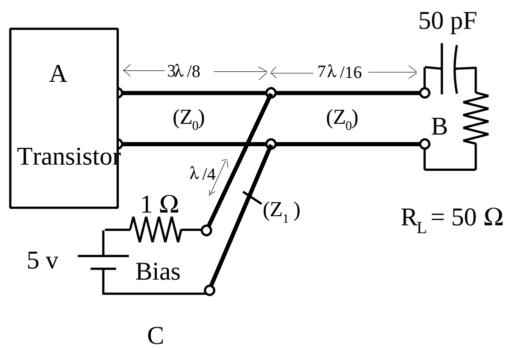


Figure 2.26 Plot showing power delivered to load as a function of frequency for a fixed stub

Fractional Power split between Load and Bias

Where Power Delivered	Power Ratio (dB)
To Load	-0.3135 dB
To Load	-0.3135 dB
To Bias Supply	-11.5714 dB

2.5 LOSSY TRANSMISSION LINES

In some cases it is desireable to consider the losses associated with propagation on transmission lines. This may occur when circuits involve longer lines, or for lines required to be fabricated with lossy materials, or for circuits operating at very high frequencies such as millimeter wave circuits. For a *low loss transmission line*, it is generally assumed that $R << w L$ and $G << w C$. With these assumptions one can determine the characteristic impedance Z_c the phase velocity of the signal using the expression for the propagation constant g derived in section 2.1. The low loss assumptionsimply that

$$Z_c \cong \sqrt{L/C} \tag{2.32}$$

which is the same as the characteristic impedance of a lossless transmission line. For the propagation constant, one should proceed as follows. From (2.9),

$$g = \sqrt{RG - w^2 LC + jw\left(RC + LG\right)} \tag{2.33}$$

With the low loss assumption, one can dismissed the RG term in the equation above since it is too small compared to other terms. Thus,

$$g \cong \sqrt{-w^2 LC + jw\left(RC + LG\right)} \tag{2.34}$$

Applying the binomial expansion, i.e. $(a - x)^n = a^n + na^{n-1}x + \dfrac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$, to (2.34).

$$g \cong \left(-w^2 LC\right)^{\frac{1}{2}} + \frac{1}{2}\left(-w^2 LC\right)^{-\frac{1}{2}} jw\left(RC + LG\right) + \frac{1}{8}\left(-w^2 LC\right)^{-\frac{3}{2}} w^2 \left(RC + LG\right)^2 + \cdots$$

or

$$g \cong jw\sqrt{LC} + \frac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) + \frac{j}{8} w\sqrt{LC}\left(\frac{R}{wL} + \frac{G}{wC}\right)^2 + \cdots \tag{2.35}$$

All terms after the second in the above equation is negligible based on the small loss assumptions. And, finally,

$$g \cong a + jb = \frac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) + jw\sqrt{LC} \tag{2.36}$$

The phase constant b is also the same as the wave number defined in a lossless transmission line, but now the attenuation constant is not zero. Since the wave is attenuated by the factor of $e^{-a x}$ as it traveled in the $+x$ direction,. One can readily see that

$$a = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}\left(RY_c + GZ_c\right) \tag{2.37}$$

The complete solution of the voltage wave that travels in the $+x$ direction is $V^{+}e^{-gx+jwt}$. The phase velocity is derived from the time displacement of the constant phase point, i.e. assuming the exponential term to be a constant, $-gx + jwt = u$, and taking the time derivative of it. This yields

$$v_p = \frac{dx}{dt} = \frac{jw}{g} \cong \frac{jw}{\frac{1}{2}\sqrt{LC}\left(\frac{R}{L}+\frac{G}{C}\right)+jw\sqrt{LC}} = \frac{1}{\frac{1}{2j}\sqrt{LC}\left(\frac{R}{wL}+\frac{G}{wC}\right)+\sqrt{LC}} \cong \frac{1}{\sqrt{LC}} \qquad (2.38)$$

The phase velocity of the signal in a low loss line is the same as that in a lossless line. It is customary to expressed the characteristic impedance in terms of the phase velocity and one of the circuit parameter as below.

$$Z_c = \sqrt{\frac{L}{C}} = v_p L = \frac{1}{v_p C} \qquad (2.39)$$

At high frequencies, the electromagnetic field can penetrate only a small distance into a conductor. In fact, the amplitude of the fields decay exponentially from its value at the surface of the conductor according to e^{-u/d_s}. Here, u is the normal distance into the conductor and d_s is the skin depth given below.

$$d_s = \sqrt{\frac{2}{w\, m_m\, s_m}} \qquad (2.40)$$

where w is the angular frequency of the signal, m_m is the permeability of the conductor, and s_m is the conductivity of the conductor. At one skin depth, i.e. $u = d_s$, the field strength is 37% of the surface value. At three skin depth, i.e. $u = 3d_s$, the field strength is down to only 5% of its surface value. For copper, a commonly used conductor in microwave transmission media, the skin depth at 100 MHz is 6.6 micron or 0.26 mils. At 10 GHz, the skin depth is 10 times smaller than that. Therefore, when considering the finite conductance of a lossy transmission line, the series resistance R in Figure 2.1 is usually expressed in terms of the surface resistance R_m, which is defined as

$$R_m = \frac{1}{s_m\, d_s} \qquad (2.41)$$

The exact formula for the series resistance of a particular transmission media also depends on the geometry of the media itself.

In Figure 2.1, the admittance Y of the shunt components can be expressed as

$$Y = jwC + G = jwC\left(1 - j\frac{G}{wC}\right)$$

The term G/wC, which causes the admittance Y to deviate from a pure imaginary value with non-zero dielectric conductivity, is called the *loss tangent* and conventionally denoted as "tan d". For a particular transmission media, the geometric factor for the conductance and capacitance of the dielectric is the same and can be eliminated from the loss tangent formula. In other words, $G = ks_d$ and $C = ke_d$, where k is the geometric factor, s_d and e_d is the conductivity and the permittivity of the dielectric respectively. Therefore,

$$\tan d = \frac{s_d}{w e_d} \tag{2.42}$$

and the dielectric conductance G in terms of the loss tangent is

$$G = w C \tan d \tag{2.43}$$

To account for attenuation due transmission line losses is reatively easy to accomadate since γ = α + jβ and the total voltage on the line is

$$V(d) = A\left(e^{gd} + \Gamma_L e^{-gd}\right)$$

The generalized reflection coefficient at a point "d" from the load is

$$\Gamma = \Gamma_L e^{-2gd}$$

The attenuation constant has units of inverse distance since when it is multiplied by "d" the exponent becomes a unitless number. This unitless number is given the honorary label, Neper, after the Napier the inventor of natural logarithms. Thus, α would be described in terms of Nepers/cm, Nepers/in indicating that when multiplied by the length in appropriate units that the resulting number is the correct exponential value. Howeve, loss is usually measured in terms of dB or dB per unit length and it is therefore important to understand how to convert to Nepers per unit length for calculation purposes. The voltage a point "d" on a matched line is compared with the voltage at a point "d+Δd" to get

$$\frac{V(d+\Delta d)}{V(d)} = \frac{Ae^{g(d+\Delta d)}}{Ae^{gd}} = e^{g\Delta d} = e^{a\Delta d} e^{jb\Delta d}$$

$$\frac{P(d+\Delta d)}{P(d)} = \left|\frac{V(d+\Delta d)}{V(d)}\right|^2 = e^{2a\Delta d}$$

$$\Delta P_{dB} = 10\log_{10}\left(\frac{P(d+\Delta d)}{P(d)}\right) = 20a\Delta d \log_{10} e$$

$$\frac{\Delta P_{dB}}{\Delta d}\Big/ a = 20\log_{10} e = 8.686$$

Therefore 1 Neper/unit length is seen to equal 8.68 dB/unit length. It is instructive to examine a line that has 1 dB of loss per wavelength.

Example 2.5.1.. Plot the total voltage magnitude on a rectangular plot and the reflection coefficient on a Smith chart for an open circuited stub with a 1dB per wavelength loss.

```
%script file ex2_4_1
%to illustrate attenuation

% alpha is 1 dB / wavelength
% beta is just 2*pi radians since
% d is measured as fractional
% wavelenght
d=[0:.01:3];
beta=2*pi;
alpha=1/8.686;
gamma=alpha+j*beta;
GL=1;
Vd=exp(gamma*d)+exp(-
gamma*d);
Vdmag=abs(Vd);
plot(d,Vdmag)

G=exp(-2*gamma*d);
figure
plot(G,'r');hold
```

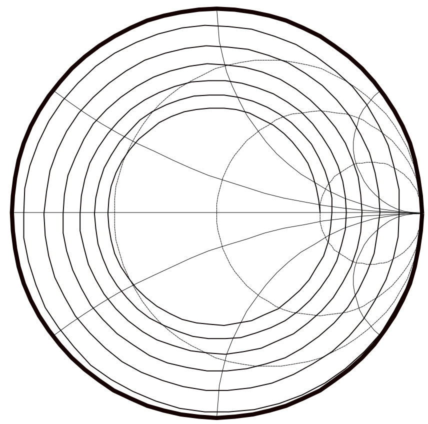


Figure 2.27 Plot showing reflection coefficient for loss open circuit stub for lengths up to 3 λ.

usc

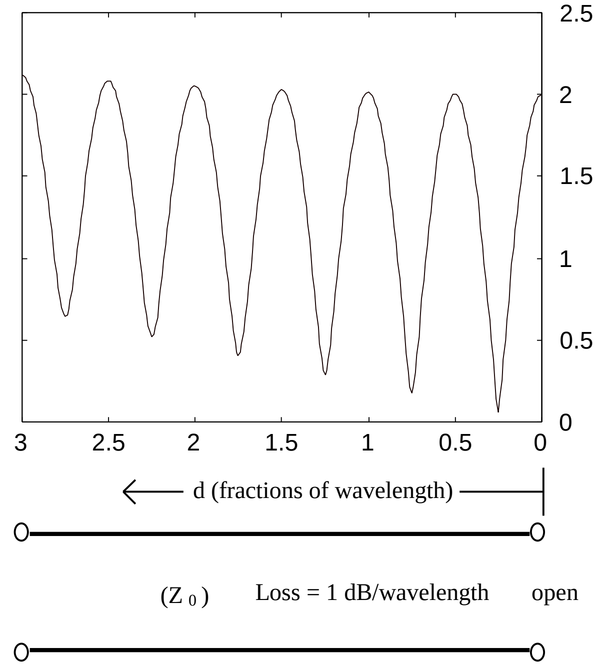


Figure 2.28 Rectangular plot of total voltage for loss open circuit stub with lengths up to 3 λ.

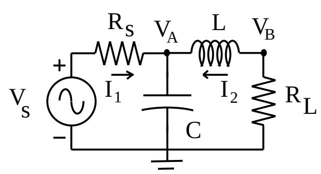
From Figure 2.28 it is noted that the VSWR changes as the measurement moves away from the open end of the stub. Measurements of the VSWR at multiple cycles from the open can be used to detemine the loss associated with the line. From the smith chart display one see that the trace makes 6 cycles equating to a length of 3λ. One dB per wavelength causes the reflection coefficient to spiral inward so that its radius has reduced from 1 to 1/2 for an open circuited stub.

2.6 PROBLEMS

1. The function $y=\cos(\omega t-\beta x)$ reprsents a propagating wave on the x-axis. If the frequency is 1 GHz and the wave-length is 20 cm plot y vs x for t=0, .2ns, .4ns, .6ns where the x-axis goes from 0 to 100 cm. Measure the movement of a wave crest for the different time intervals and calculate the phase velocity. Compare with the formula $v=f\lambda$.

2. A lossless transmission line has a distributive inductance of 1 nH/cm, a distributive capacitance of .5 pF/cm. What is the characteristic impedance of the line. What is the phase velocity. If a wave on this line has a frequency of 1 GHz what is its wavelength?

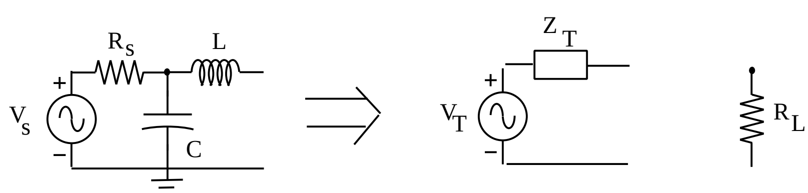
3. For the circuit shown use Kirchoff's voltage law (KVL) to write two equations in terms of the currents I_1 and I_2. Express the equations as a matrix equation $A\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = B$ where A is a 2x2 matrix and B is a 2x1 column matrix. Solve for the current matrix by inverting A. What is the power delivered to R_L?



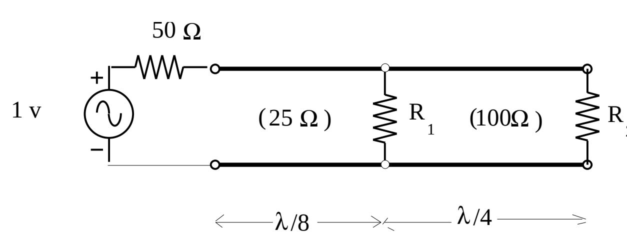
f = 1 GHz	L = 5 nH
V_S = 1 volt	C = 7 pF
R_S = 50 Ω	R_L = 50 Ω

4. For the circuit shown in problem 3 use Kirchoff's current law (KCL) to write two equations in terms of the voltages V_1 and V_2. Express the equations as a matrix equation $C\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = D$ where C is a 2x2 matrix and D is a 2x1 column matrix. Solve for the current matrix by inverting C. What is the power delivered to R_L?

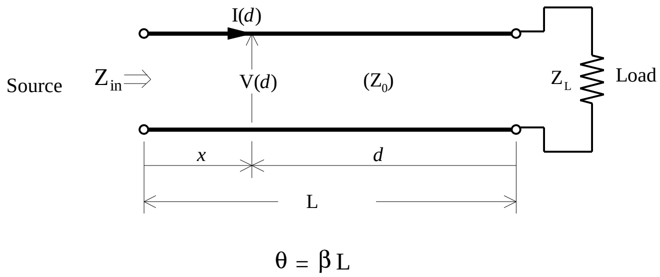
5. Solve the circuit in problem 3 by determining the Thevenin Equivalent for the circuit below an then connect the resistor R_L and determine the power delivered to it.



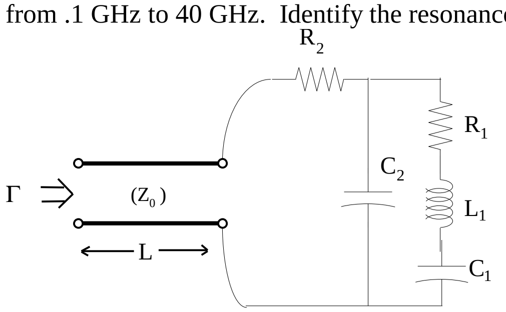
6. What is the power delivered to each of the the loads R_1 and R_2?



7. For an arbitary length, "L," of transmission line find the voltage transfer ratio (total voltage at load divided by total voltage at input. Express ratio in terms of electrical length, $\theta=\beta L$. Substitute special case where $L=\lambda/4$ and demonstrate results in chapter.



8. Plot the reflection coefficient with respect to Zo=50 Ω on the Unit Smith Chart for a frequency range from .1 GHz to 40 GHz. Identify the resonance points and characterize their type.



R_1= 10 Ω
L_1= 1 nH
C_1= 10 pF
C_2= 1 pF
R_2= 1 Ω

Z_0 = 50 Ω
L = .05 cm
v = 2 x 10^{10} cm/s

