CHAPTER
8

MATCHING NETWORKS

8.1 INTRODUCTION

This chapter examines an important class of networks known as matching circuits. A matched load has already been observed to be a critical element in the measurement of S-parameters. In this case "matched" meant that the load impedance equaled the reference impedance to simulate an infinitely long transmission line so that no reflections would occur that would otherwise contaminate the S-parameter measurement. As a result, one of the meanings of the term "matched" refers to a circuit or a load that produces no reflections. A second meaning comes from a concept in basic AC circuits. In this case the term “matched” refers to a circuit or load which receives maximum power from a source which may be the Thevenin equivalent representation of a complex configuration of sources and impedances. In the first case a matched load reduces the VSWR to 1, but may not result in the maximum power being delivered to the load. On the other hand, in the second case the matched load causes maximum power to be delivered to the load but the VSWR may be greater than 1. The maximum power transfer definition for the “matched” load will be examined first by considering the familiar AC circuit situation. In this case a load is connected to a source as illustrated in figure 8.1.

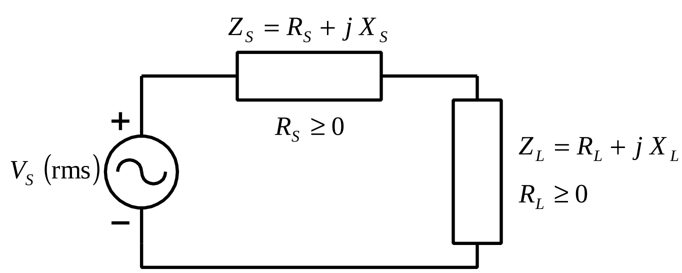

Figure 8.1 A load connected to a source with voltage, V_s ,and impedance, Z_s ,

In this case the power dissipated by the load, P_L , is given by

$$P_L = \frac{|V_s|^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

To maximize the value of P_L, on notes first that the load reactance, X_L, appears only in the denominator and if it is set equal to the negative of X_s then the fraction will be maximum with respect to the load reactance. Therefore, choosing $X_L = -X_s$ implies that

$$P_L\Big|_{X_L=-X_S} = \frac{|V_s|^2 R_L}{(R_s + R_L)^2}$$

The choice of an optimum value for the load resistance is a little more complicated since R_L appears in both the denominator and numerator of the expression. In this case the value of R_L resulting in maximum can be found by taking the derivative of the expression and setting it equal to zero. Therefore,

$$\frac{d}{dR_L}\left(P_L\Big|_{X_L=-X_S}\right) = \frac{|V_s|^2 (R_S - R_L)}{(R_s + R_L)^3} = 0$$

implies that $R_L = R_S$. And the maximum power equals

$$P_{LMAX} = \frac{|V_s|^2}{4R_s} \tag{8.1}$$

The maximum power is also referred to as the *available power* from the source. In this case notice that the load which results in maximum power transfer occurs when the load impedance is the conjugate to the source impedance, i.e., $Z_L = R_L + jX_L = R_S - jX_S = Z_s^*$. For this reason the matched load is sometimes referred to as a being a *conjugate match*. Often in the design of circuits a source or Thevenin Equivalent Source exists with an impedance that is unequal to the conjugate of the desired load. In this case it is desirable to insert a passive, lossless, reciprocal circuit (PLRC) between the source and load so that the circuit is conjugately matched. This is illustrated in figure 8.2.

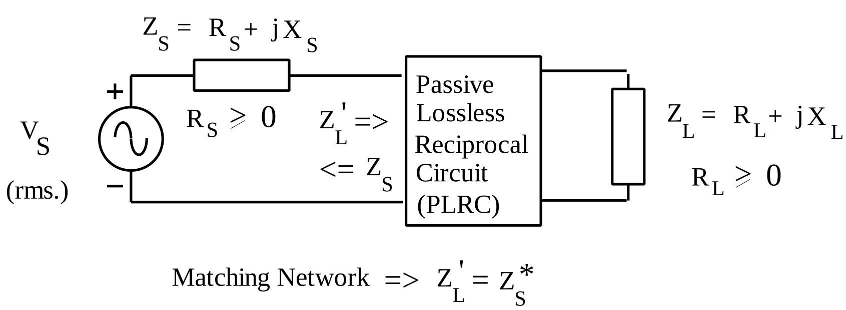

Figure 8.2 Illustrating a passive, lossless, reciprocal circuit insert between a load and a source to provide a conjugate match

In terms of reflection coefficients

$$\Gamma_L' = (Z_L' - Z_0)/(Z_L' + Z_0) = \left(Z_S^* - Z_0\right)/\left(Z_S^* + Z_0\right) = [(Z_S - Z_0)/(Z_S + Z_0)]^* = \Gamma_S^* \tag{8.2}$$

and conveniently a conjugate match means that $Z_L' = Z_S^*$ or equivalently, $\Gamma_L' = \Gamma_S^*$. The insertion of such a circuit means that maximum power is delivered to the input of the matching network, Z_L', and since the matching network is a lossless circuit then that same power must be delivered to the actual load. Equivalently, the available power from the source is delivered to the load because the real load is made to appear to the source as a conjugately matched load. Sometimes the matching network is called a transformer since it transforms the actual impedance into a value that matches the rest of the circuit.

Example 8.1.1. Prove the reflection coefficient conjugate match criterion using signal flow graphs and determine the maximum power (available power) in terms of the source reflection coefficient.

The circuit of figure 8.1 can be represented as the SFG shown in figure 8.3 where $b_s = V_s\sqrt{Z_0}/(Z_s + Z_0)$, $\Gamma_s = (Z_s - Z_0)/(Z_s + Z_0)$, and $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$.

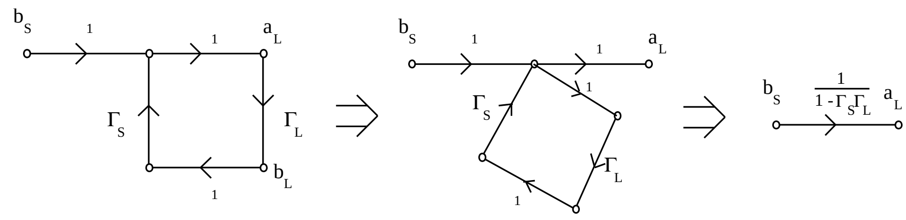

Figure 8.3 Signal flow graph representation of the circuit in figure 8.1.

The power delivered to the load is therefore,

$$P_L = \frac{|b_s|^2\left(1-|\Gamma_L|^2\right)}{|1-\Gamma_s\Gamma_L|^2}$$

and if $\Gamma_L = r_L e^{jq_L}$, $\Gamma_S = r_S e^{jq_S}$, where $r_L \geq 0$ and $r_S \geq 0$ then

$$P_L = \frac{|b_s|^2\left(1-r_L^2\right)}{1+r_L^2 r_S^2 - 2r_S r_S \cos(q_S + q_L)}$$

Maximizing P_L means that the cosine term should be as large as possible. This occurs when $q_L = -q_S$, and in that case

$$P_L = \frac{|b_s|^2\left(1-r_L^2\right)}{1+r_L^2 r_S^2 - 2r_S r_S}$$

Maximization with respect to r_L is implemented by setting the derivative of the above expression equal to zero, $\frac{dP_L}{dr_L} = 0$, which implies that $r_L = r_S$. Therefore the maximum power transfer to the load occurs when $\Gamma_L = r_L e^{jq_L} = r_S e^{-jq_S} = \Gamma_S^*$, which is the same result obtained earlier. The maximum power (or available power from the source) is found to be

$$P_{LMAX} = \frac{|b_S|^2}{1-r_S^2} = \frac{|b_S|^2}{1-|\Gamma_S|^2} \tag{8.3}$$

Direct substitution shows that

$$\frac{|b_S|^2}{1-|\Gamma_S|^2} = \frac{|V_S|^2}{4R_S} \tag{8.4}$$

thus verifying that the maximum power can be calculated from the reflection coefficient and initially launched signal or from the impedance and voltage parameters of the source.

Matching circuit are most often created by combining passive elements such as inductors, capacitors, transmission lines, and open-circuited or short-circuited transmission line stubs. It is often a good approximation at RF and microwave frequencies to assume that these elements are lossless. Therefore it is useful to consider some of the theoretical properties of passive lossless circuits and see their implications for matching networks. Recalling that the scattering matrix for a passive lossless circuit is a unitary matrix implies that

$$S \cdot S' = S \cdot S^{t*} = I \tag{8.5}$$

or

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.6}$$

In this case the complex transpose of the S matrix is the inverse of the S-matrix and from linear algebra it is well known that the inverse of a matrix commutes with its matrix under the operation of matrix multiplication. Therefore, it immediately follows that the following matrix relationship also holds.

$$S' \cdot S = S^{t*} \cdot S = I \tag{8.7}$$

The matrix equation (8.6) which results in the following set of equation

$$|S_{11}|^2 + |S_{12}|^2 = 1 \tag{8.8}$$

$$S_{11}S_{21}^* + S_{12}S_{22}^* = 0 \tag{8.9}$$

$$S_{21}S_{11}^* + S_{22}S_{12}^* = 0 \tag{8.10}$$

$$|S_{21}|^2 + |S_{22}|^2 = 1 \tag{8.11}$$

Equations (8.8) through (8.11) can be manipulated to provide a number of interesting facts about the s-parameters of two port passive lossless circuits.

Magnitude Theorem for 2-Port Passive Lossless Circuits: *The input and output s-parameters of a 2-port passive lossless circuit have equal magnitudes, and the forward and reverse transfer s-parameters have equal magnitudes, i.e.,* $|S_{11}| = |S_{22}|$ *and* $|S_{12}| = |S_{21}|$

Proof:

From equation (8.9) it follows that $S_{12} = -S_{11}S_{21}^*/S_{22}^*$ which can be substituted into equation (8.8) to get $|S_{11}|^2 + |S_{11}S_{21}^*/S_{22}^*|^2 = 1$ and multiplying by $|S_{22}|^2$ while noting that the magnitude operation eliminates the complex conjugate operation results in $|S_{11}|^2|S_{22}|^2 + |S_{11}|^2|S_{21}|^2 = |S_{22}|^2$ which factors to give $|S_{11}|^2\left(|S_{22}|^2 + |S_{21}|^2\right) = |S_{22}|^2$ and equation (8.11) implies $|S_{11}|^2 = |S_{22}|^2$ which means $|S_{11}| = |S_{22}|$ and the first part of the theorem is proven.

The second part of the theorem follows by combining equations (8.8) and (8.11) to get $|S_{11}|^2 + |S_{12}|^2 = |S_{21}|^2 + |S_{22}|^2$ and since $|S_{11}| = |S_{22}|$ it follows that $|S_{12}|^2 = |S_{21}|^2$ and therefore $|S_{12}| = |S_{21}|$.

Determinate Theorem for 2-Port Passive Lossless Circuits: *The determinate of a 2-port passive lossless circuit is the ratio of the output s-parameter divided by the complex conjugate of the input s-parameter and its magnitude equals 1, i.e.,* $\Delta = S_{22} / S_{11}^*$ *and* $|\Delta| = 1$.

Proof:

One can use equation (8.10) to get $S_{21} = -S_{22} S_{12}^* / S_{11}^*$, which can be substituted in the definition of the determinate, $\Delta = S_{11} S_{22} - S_{12} S_{21}$, to get $\Delta = S_{11} S_{22} + S_{12}\, S_{22} S_{12}^* / S_{11}^*$ which simplifies to $\Delta = S_{22} \left(|S_{11}|^2 + |S_{12}|^2 \right) / S_{11}^*$ and equation (8.8) results in $\Delta = S_{22} / S_{11}^*$ which proves the first part of the theorem.

The second part dealing with the magnitude of the determinate follows immediately from the first part and the magnitude theorem above, i.e., $\Delta = S_{22} / S_{11}^*$ implies $|\Delta| = \left| S_{22} / S_{11}^* \right|$ and $|\Delta| = |S_{22}| / \left| S_{11}^* \right| = |S_{22}| / |S_{11}|$. Since $|S_{22}| = |S_{11}|$ then $|\Delta| = 1$ and the second part of the theorem is proven.

Simultaneous Conjugate Match Theorem for a 2-Port Passive Lossless Circuit: *A 2-port passive lossless circuit inserted between a source and load can only be conjugately matched at its output if it is conjugately matched at its input, and vice versa.*

Proof:

The SFG describing this situation is shown below in figure 8.4

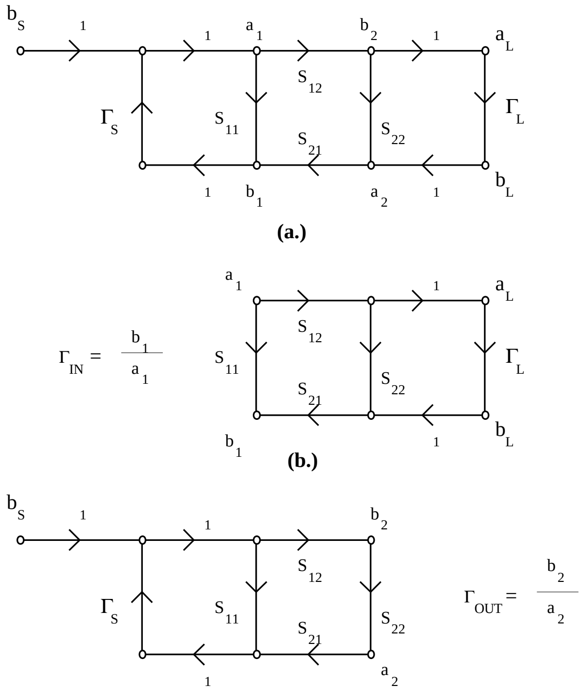

Figure 8.4 **(a.)** Signal flow graph showing a circuit inserted between a source and a load, and the **(b.)** input and **(c.)** output reflection coefficients

The input reflection coefficient is defined as the reflection coefficient that exists at the input of the circuit when the load is connected to the output. This is illustrated in figure 8.4 (b.). In this case

$$\Gamma_{IN} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \tag{8.12}$$

The output reflection coefficient is defined as the reflection coefficient that exists at the output of the circuit when the source is connected to the input. This is illustrated in figure 8.4 (c.) and

$$\Gamma_{OUT} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S} \tag{8.13}$$

Note that the input reflection coefficient is a function of the s-parameters of the circuit and the load reflection coefficient while the output reflection coefficient is a function of the s-parameters of the circuit and the source reflection coefficient

The theorem states that $\Gamma_S^* = \Gamma_{IN}$ if and only if $\Gamma_{OUT} = \Gamma_L^*$. Assuming that the input is conjugately matched one now must show that that the output is conjugately matched. Assuming $\Gamma_S^* = \Gamma_{IN}$ then equation (8.12) implies that

$$\Gamma_S = \left(\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right)^* = \frac{S_{11}^* - \Delta^*\Gamma_L^*}{1 - S_{22}^*\Gamma_L^*} \tag{8.14}$$

which can be substituted into equation (8.13) to get

$$\Gamma_{OUT} = \frac{S_{22} - \Delta\left(\dfrac{S_{11}^* - \Delta^*\Gamma_L^*}{1 - S_{22}^*\Gamma_L^*}\right)}{1 - S_{11}\left(\dfrac{S_{11}^* - \Delta^*\Gamma_L^*}{1 - S_{22}^*\Gamma_L^*}\right)} \tag{8.15}$$

which simplifies to

$$\Gamma_{OUT} = \frac{\left(S_{22} - \Delta S_{11}^*\right) - \left(|S_{22}|^2 - |\Delta|^2\right)\Gamma_L^*}{\left(1 - |S_{11}|^2\right) - \left(S_{22}^* - \Delta^* S_{11}\right)\Gamma_L^*} \tag{8.16}$$

Substituting the results previous theorem on 2-port passive, lossless circuits, i.e., $\Delta = S_{22}/S_{11}^*$ and $|\Delta| = 1$ results in

$$\Gamma_{OUT} = \frac{\left(1 - |S_{22}|^2\right)\Gamma_L^*}{\left(1 - |S_{11}|^2\right)} \tag{8.17}$$

and from equations (8.8) and (8.11) it follows that

$$\Gamma_{OUT} = \frac{|S_{21}|^2\,\Gamma_L^*}{|S_{12}|^2} \tag{8.18}$$

and the magnitude theorem for 2-port passive lossless circuits implies that $|S_{21}| = |S_{12}|$ and (8.17) becomes $\Gamma_{OUT} = \Gamma_L^*$ which proves the assertion. The reverse proof follows similarly and is left as an exercise.

Matching networks are often used in the design of active networks as an input, output, or interstage circuit as illustrated in figure 8.5. Matching networks are required because normally active circuits function optimally when their input and output impedances are different than Z_0. However the input and output to the combination circuit is normally Z_0. Therefore, input and output matching networks are for the purpose of creating a conjugate match between Z_0 and an arbitrary impedance, or equivalently, the matching network transforms the center of the Unit Smith Chart into reflection coefficient required for the input or output of an active circuit. On the other hand an interstage network is placed between two active circuits with the function of conjugately matching the output of the first circuit while simultaneously conjugately matching the input of the second active circuit. An interstage matching network transforms the optimum reflection coefficient required by the output of the first active circuit into the optimum reflection coefficient required to conjugately match the input of the second active circuit. In this case one expects that neither reflection coefficient will be located at the center of the USC as in the previously discussed input and output matching networks. As seen from the previous discussion, conjugate matching with a passive lossless network requires that both sides of the matching network be conjugately matched. Subsequent sections will illustrate specific matching networks.

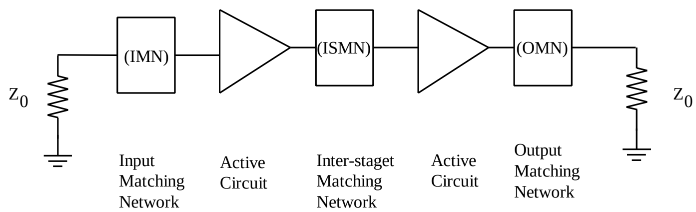

Figure 8.5 An illustration of matching networks being used with active circuits

Since matching networks are primarily designed as passive lossless networks it is worth examining the s-parameters of such circuits. Based on the magnitude theorem the s-parameters of a matching circuit would be given by $S_{11} = |S_{11}|e^{ja}$, $S_{22} = |S_{11}|e^{jb}$, $S_{21} = |S_{21}|e^{jg}$, and $S_{12} = |S_{21}|e^{jd}$. From equation (8.8) and (8.11) it follows that $|S_{21}| = \sqrt{1-|S_{21}|^2}$. From equations (8.9) the phase terms can be related, i.e.,

$$|S_{11}|\sqrt{1-|S_{21}|^2}\,e^{j(a-g)} + |S_{11}|\sqrt{1-|S_{21}|^2}\,e^{j(d-b)} = 0$$

$$e^{j(a-g)} = e^{j(d-b+np)} \quad \text{(n=odd integer)}$$

$$a - g = d - b + np$$

$$b = g + d - a + np \qquad \text{(n=odd integer)}$$

Therefore, the S-parameters of a passive, lossless network must be of the following form

$$\mathbf{S}_{PLN} = \begin{pmatrix} |S_{11}|e^{ja} & \sqrt{1-|S_{11}|^2}\,e^{jd} \\ \sqrt{1-|S_{11}|^2}\,e^{jg} & -|S_{11}|e^{j(g+d-a)} \end{pmatrix} \tag{8.19}$$

If in addition the circuit is reciprocal, as is usually the case, then $S_{12} = S_{21}$ or $d = g$ and the S-parameters circuit are of the form

$$\mathbf{S}_{PLRN} = \begin{pmatrix} |S_{11}|e^{ja} & \sqrt{1-|S_{11}|^2}\,e^{jg} \\ \sqrt{1-|S_{11}|^2}\,e^{jg} & -|S_{11}|e^{j(2g-a)} \end{pmatrix} \tag{8.20}$$

The determinate of the passive, lossless, reciprocal network equals

$$\Delta_{PLRN} = -e^{j2g} \tag{8.21}$$

Input and Output Matching Network Theorem: *When using a passive lossless reciprocal network (PLRN) as an input or output matching network to transform Z_0 to a predetermined impedance with reflection coefficient $|\Gamma|e^{jq}$, the S-parameters of the network are uniquely determined except for an arbitrary length of transmission line added to the Z_0 side of the matching circuit.*

Proof:

With out loss of generality one may assume that the PLRN is an output matching network and port 2 of the network will be connected to Z_0 and transform it to an impedance whose reflection coefficient is given by $|\Gamma|e^{jq}$. This is illustrated in figure 8.6, where one notes that $S_{11} = |\Gamma|e^{jq}$. From equation (8.20) the s-parameter matrix would be given by equation (8.22).

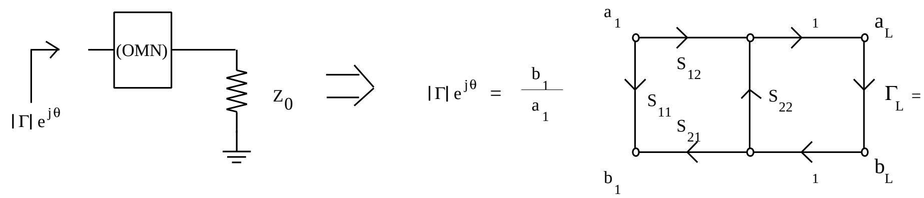

Figure 8.6 Illustration of an output matching network and its implication on the s-parameters

$$\mathbf{S}_{OMN} = \begin{pmatrix} |\Gamma|e^{jq} & \sqrt{1-|\Gamma|^2}\,e^{jg} \\ \sqrt{1-|\Gamma|^2}\,e^{jg} & -|\Gamma|e^{j(2g-q)} \end{pmatrix} \tag{8.22}$$

which can be represented in SFG form by figure 8.7.

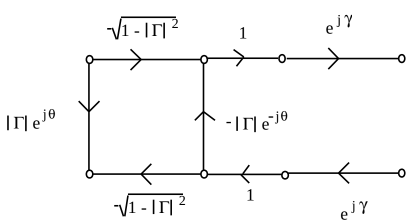

Figure 8.7 A SFG of an OMN showing that s-parameters are uniquely determining except for an arbitrary length of transmission line.

8.2 LUMPED-ELEMENT MATCHING NETWORKS

Suppose one desires to build an inter-stage matching network to be inserted between a source with impedance is $Z_S = 75 + j75\ \Omega$ and a load impedance of $Z_S = 75 + j50\ \Omega$. This is illustrated in figure 8.8(a.). Figure 8.8(b.) shows the source and load impedance reflection coefficients on the unit Smith Chart. Determining a lumped element matching network consists of finding segments of constant series resistance or shunt conductance circles that move from Z_L (or Γ_L) to Z_S^* (or Γ_S^*). Conceptually one can imagine starting the process by looking at the load and backing through the matching network until finally looking into the input side of the matching network. At that point one wishes to see the complex conjugate of the source impedance for maximum power transfer.

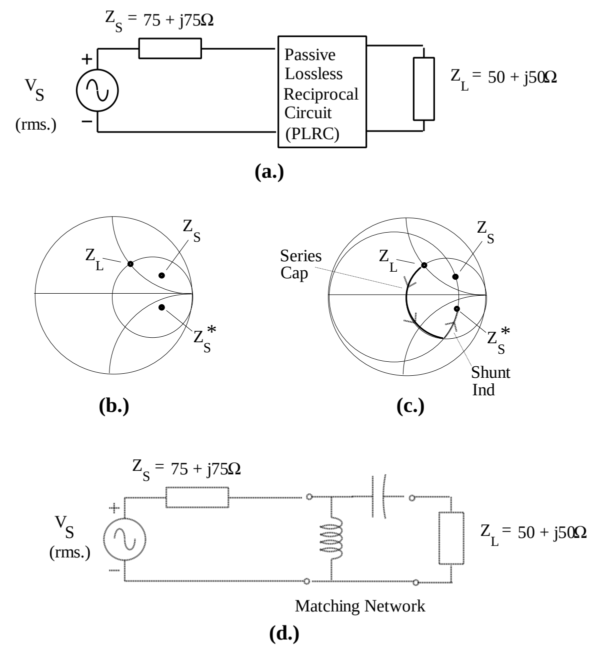

Figure 8.8 **(a.)** An inter-stage matching network inserted between a source and a load. **(b.)** impedances shown on the unit smith chart. **(c.)** path between Z_L and Z_S^*, **(d.)** lumped element realization of a matching network

The same matching network could have been created by looking at the source and backing again through the matching network until finally looking into the output side of the matching network. Similar paths on the Smith Chart are illustrated in figure 8.9.

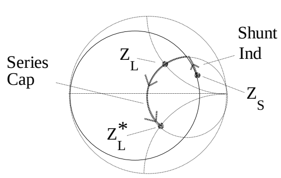

Figure 8.9 Realization of a matching network finding segments of constant resistance and conductance circles from the source to the conjugate of the load.

Input and output matching networks can also be realized using *pairs* of reactive, lumped element components. Eight different lumped element matching networks are possible using pairs of inductors and capacitors. Such networks are referred to as "L" networks because topology of the schematic representation resembles an "L" shape, i.e., there is always a series element and a shunt elements. In the case of an input or output matching network the circuit can be developed following constant resistance and conductance circle segments from a complex impedance (complex reflection coefficient) to the center of the Unit Smith Chart which represents Z_0 ($\Gamma_0 = 0$). The 8 networks are shown in the Smith Chart are illustrated in figure 8.10 and the shaded areas indicate impedances that can be matched to Z_0.

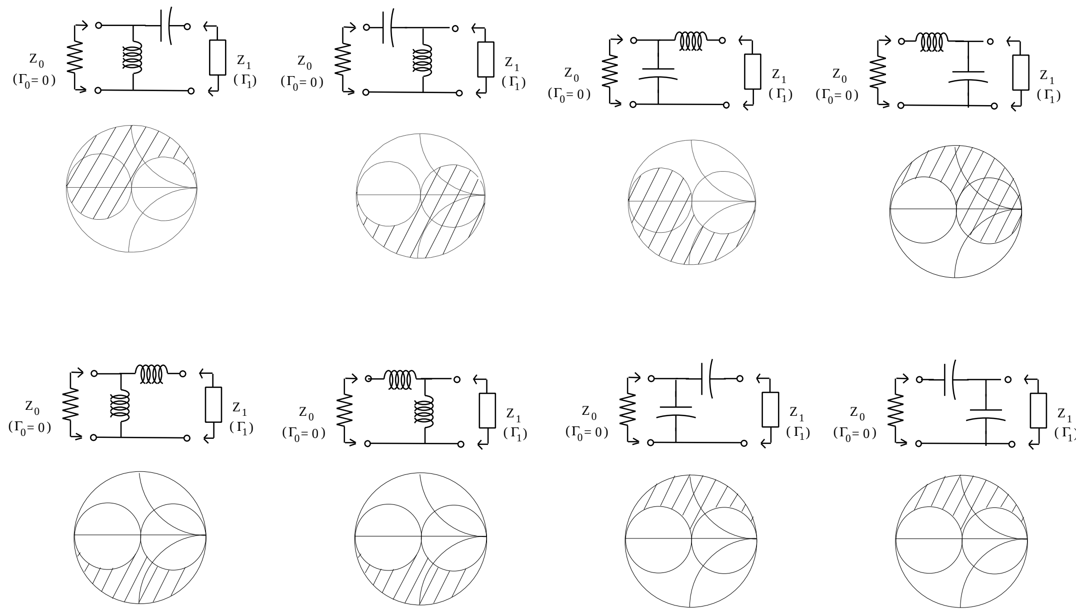

Figure 8.10 Realization of an input/output matching network finding segments of constant resistance and conductance circles from the source to the conjugate of the load.

An alternative two-component network is possible for input/output matching circuits if one employs a transformer as one of the components. The principle of operation consists of connecting a reactive element in series or in shunt with the complex impedance so that the combined impedance is purely resistive (or conductive). Then a transformer is connected to step the real impedance up or down to Z_0. This is illustrated in figure 8.11.

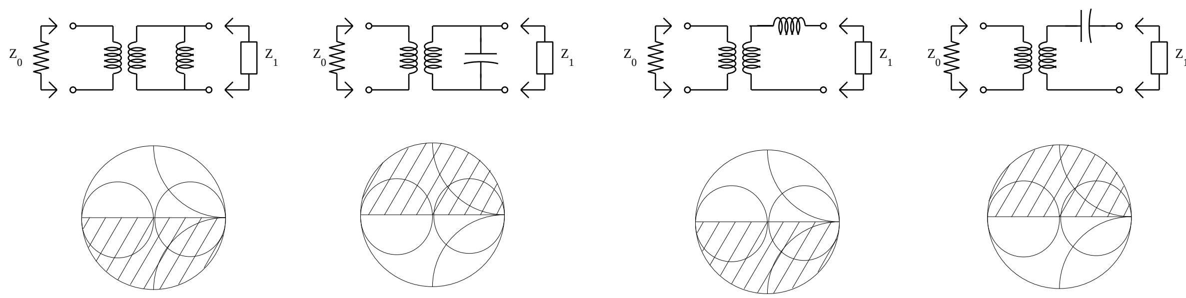

Figure 8.11 Realization of a input/output matching network using a transformer as one of two lumped-elements (shaded area shows permissible impedances for Z_1

8.3 DISTRIBUTED-ELEMENT MATCHING NETWORKS

Matching networks using distributed elements can follow the same pattern as those using lumped elements where an inductor can be replaced with a short circuited transmission line and a capacitor replaced by an open circuited transmission line, where in both cases the transmission lines are assumed to be less than $l/4$ in length. In this case the circuits of figure 8.10 become those illustrated in figure 8.12.

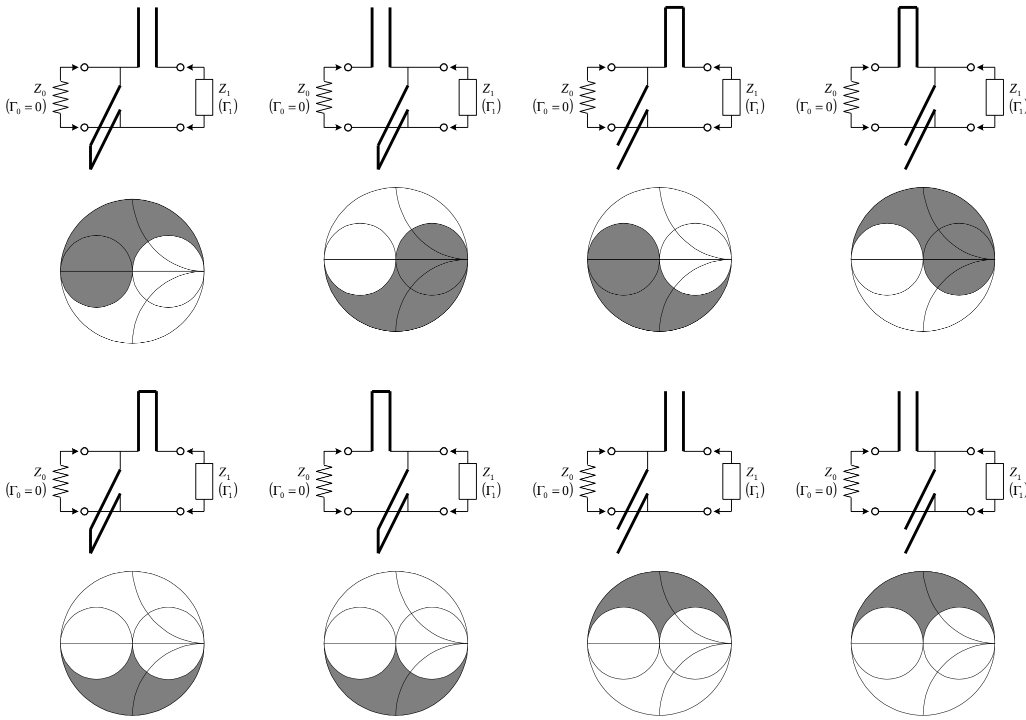

Figure 8.12 Realization of an input/output matching network using distributed element components. All transmission line elements are assumed to be less than $l/4$ in length. Shaded area equals permissible impedances Z_1.

However, if circuits are to be designed using microstrip technology then series stubs are not possible and matching networks must be designed which us only shunt stubs. This is a relatively easy task since one may design an input/output matching network by starting at the complex impedance and connecting a length of transmission line which rotates the impedance until it intersects the unity conductance circle. At this point an open or short-circuited stub can be attached in shunt with a length to cancel the reactance and move the impedance to the center of the Smith chart. This is illustrated in figure 8.13.

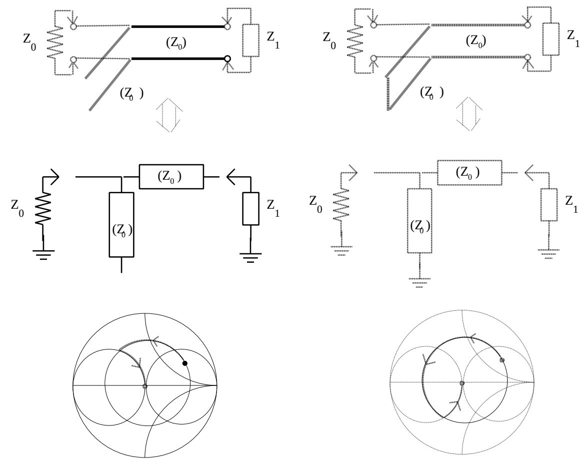

Figure 8.13 Realization of a distributed element, two component input/output matching network.

Another distributed element input/output matching network approach is to use a quarter-wave transformer. In this case an open or short circuit stub is placed in shunt with the complex impedance and adjusted so that the combined impedance is real. A quarter wave transmission line is then inserted with a characteristic impedance designed to transform the real impedance to Z_0. This is illustrated in figure 8.14.

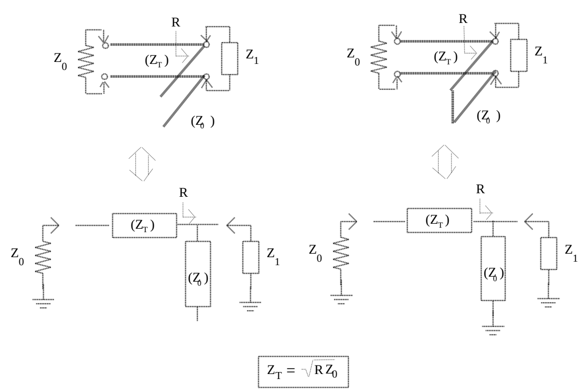

Figure 8.14 Realization of a distributed element, two component input/output matching network using a stub and a quarter wave transformer.

A variation of the last technique is to use λ/8 shorted or open stubs to cancel the reactive part of the complex impedance before transforming with the quarter wave transmission line. Recall that the impedance for a short-circuited transmission line is given by the expression $Z_{SC} = jZ_1 \tan q$, where $q = bL = 2pL / l$. If L = λ/8 then $q = p/4$ and $\tan q = 1$ and $Z_{SC} = jZ_1$. Therefore, the impedance for a short circuited stub of length λ/8 is determined by Z_1, which can be adjusted in microstrip by changing the width of the line. Thus, if the complex impedance has a negative imaginary part (i.e., its capacitive) the line width of a λ/8 shorted stub can be set to cancel the reactive part of the complex impedance. In a similar manner an open circuited stub has an input impedance given by $Z_{OC} = -jZ_1 \cot q$ and if the stub length is λ/8 then the input impedance is given by $Z_{OC} = -jZ_1$. In this case if the imaginary part of complex impedance is positive (i.e., it is inductive) then a $\lambda/8$ open circuited stub can be connected in shunt to cancel the reactance.

8.4 BANDWIDTH CONSIDERATIONS FOR MATCHING NETWORKS

The networks presented thus far are examples of narrow band matching circuits. However, for many applications they are sufficient to meet the task. When wider bandwidths are required one must first consider the nature of the complex impedances being matched. Fano has developed a set of integrals that show an important trade-off when considering the bandwidth of a matching network. He considered four types of complex loads, (1.) a resistor in parallel with a capacitor, (2.) an inductor in series with a resistance, (3.) an inductor in parallel with a resistor, and (4.) a capacitor in series with a resistor. These cases are shown in figure 8.15.

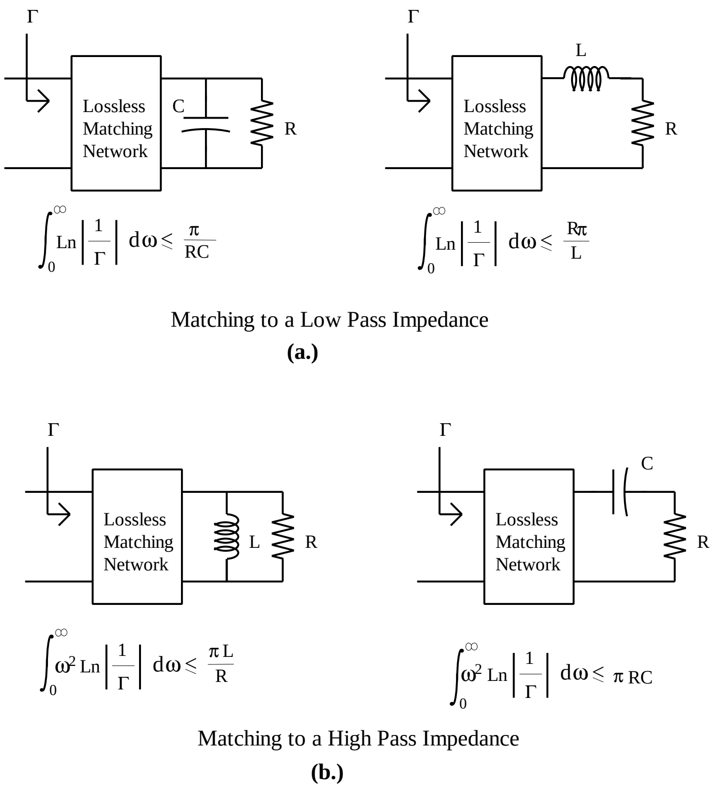

Figure 8.15 Fano's integral relationships for impedances consisting of series and parallel combinations of R + L and R + C.

Since perfect input/output matching means that $\Gamma = 0$ from the Fano relationships it is clear first that perfect matching can at most occur only at discrete frequency points and not over a continuous band. Secondly, the wider the band then for the same complex impedance the less perfect the match will be. An exception to this is the case where the impedance is consists of only a resistance. In the low pass case C = L = 0 which means that the bound on the integral is infinite and in that case the match can be perfect over a band. In the high pass case C=L=∞ and again the bound on the integral is infinite implying that a perfect match is possible over a band.

An example circuit, which can be used to match two real impedances, is cascading quarter-wave transmission lines of different impedance. In microstrip the lines would have different width and so such circuits are sometimes referred to as step impedance matching networks or stepped impedance transformers.

If wider bandwidth or improved performance is desired then additional quarter-wave sections can be added to the matching network. This is illustrated in figure 8.16.

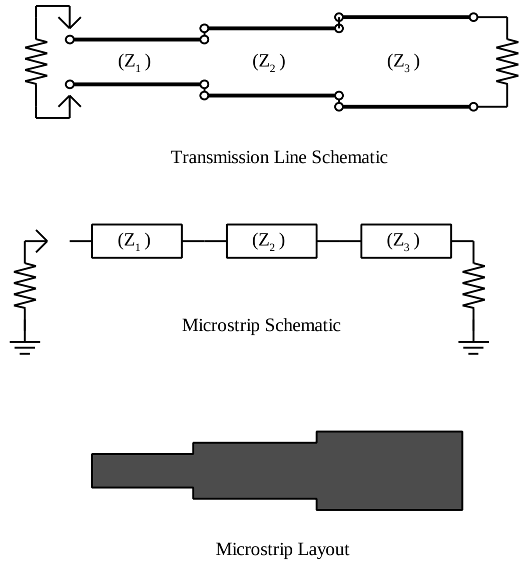

Figure 8.16 A stepped impedance matching network

In general increasing the bandwidth of a circuit requires added additional components or sections.