CHAPTER

4

CIRCUIT TECHNOLOGY

4.1 LOSSY TRANSMISSION LINE

Since all physical transmission media have some loss one way or the other, it is for the practical reason that one should look into the steady state characteristics of lossy transmission lines. Now, consider a small segment of a lossy transmission line distributed circuit model as shown in Figure 4.1. The losses in the transmission media are depicted by the series and the shunt resistors. These resistors represent the finite conductivity of the conductors and the dielectric insulator between the conductors respectively. The constants *R*, *G*, *L*, and *C* are defined as per unit length circuit parameters. The total length of the transmission line segment is Δx.

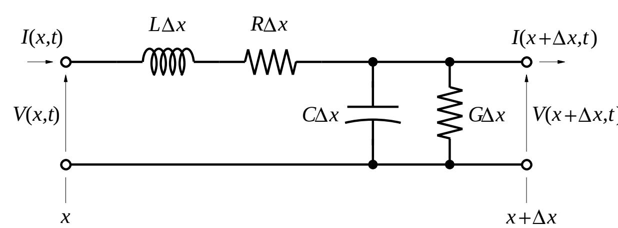


Figure 4.1 Distributed circuit model of a lossy transmission line.

Applying the Kirchoff's law to the series and shunt elements respectively, one gets

$$V(x,t)-V(x+\Delta x,t)=R\Delta x I(x,t)+L\Delta x\frac{\partial I(x,t)}{\partial t} \tag{4.1}$$

$$I(x,t)-I(x+\Delta x,t)=G\Delta x V(x+\Delta x,t)+C\Delta x\frac{\partial V(x+\Delta x,t)}{\partial t} \tag{4.2}$$

Dividing both sides of the above equations by Δx and taking the limit of both equations for $\Delta x \to 0$, the resulting equations become

$$-\frac{\P V(x,t)}{\P x}=RI(x,t)+L\frac{\P I(x,t)}{\P t} \tag{4.3}$$

$$-\frac{\P I(x,t)}{\P x}=GV(x,t)+C\frac{\P V(x,t)}{\P t} \tag{4.4}$$

Since we are mainly interested in sinusoidal steady state solution
s of the voltage and current, and there are no reason why the dependence of the variables x and t are not separable, we can assume the solutions to be $V(x,t)=V(x)e^{jwt}$ and $I(x,t)=I(x)e^{jwt}$. Substituting these for the voltage and current in (4.3) and (4.4) yields

$$\frac{dV(x)}{dx}=-(R+jwL)I(x) \tag{4.5}$$

$$\frac{dI(x)}{dx}=-(G+jwC)V(x) \tag{4.6}$$

At this point, the time dependence is removed from the differential equations and one is dealing with only the voltage and current phasors. Further derivation of (4.5) and (4.6) resulted in two second order linear differential equations.

$$\frac{d^2V(x)}{dx^2}=(R+jwL)(G+jwC)V(x) \tag{4.7}$$

$$\frac{d^2I(x)}{dx^2}=(R+jwL)(G+jwC)I(x) \tag{4.8}$$

Since equations (4.7) and (4.8) are really the same, their solutions must be linearly dependent to each other. But this is what was expected anyway, since the ratio of the voltage and current is the characteristic impedance of the transmission line. Working with (4.7), the solution must bear the form of $e^{\pm gx}$, where g can be found by substituting for $V(x)$, which yields

$$g=\sqrt{(R+jwL)(G+jwC)} \tag{4.9}$$

The general solution of the voltage phasor is then

$$V(x)=V^+e^{-gx}+V^-e^{gx} \tag{4.10}$$

where the + and - superscripts are chosen to indicate the propagation direction of the voltage wave. Current phasor can be derived from (4.10) and (4.5), i.e.

$$I(x)=I^+e^{-gx}-I^-e^{gx}=\frac{g}{R+jwL}\left(V^+e^{-gx}-V^-e^{gx}\right) \tag{4.11}$$

The characteristic impedance of a lossy transmission line is therefore

$$Z_c=\frac{R+jwL}{g}=\sqrt{\frac{R+jwL}{G+jwC}} \tag{4.12}$$

Example 4.1.1 *For a* ***low loss transmission line****, it is generally assumed that* $R << wL$ *and* $G << wC$*. With these assumptions, how can one simplify the characteristic impedance* Z_c *and the propagation constant* g *of a low loss transmission line? What is the phase velocity of the signal traveling in the low loss transmission line?*

Applying the assumptions to equation (4.12), the characteristic impedance of a low loss line can be readily derived, which is

$$Z_c \cong \sqrt{L/C} \tag{4.13}$$

Therefore, the characteristic impedance of a low loss line is the same as that of a lossless line. For the propagation constant, one should proceed as follows. From (4.9),

$$g = \sqrt{RG - w^2 LC + jw\left(RC + LG\right)} \tag{4.14}$$

With the low loss assumption, we can dismissed the RG term in the equation above since it is too small compared to other terms. Thus,

$$g \cong \sqrt{-w^2 LC + jw\left(RC + LG\right)} \tag{4.15}$$

Applying the binomial expansion, i.e. $(a - x)^n = a^n + na^{n-1}x + \dfrac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$, to (4.15).

$$g \cong \left(-w^2 LC\right)^{\frac{1}{2}} + \frac{1}{2}\left(-w^2 LC\right)^{-\frac{1}{2}} jw\left(RC + LG\right) + \frac{1}{8}\left(-w^2 LC\right)^{-\frac{3}{2}} w^2 \left(RC + LG\right)^2 + \cdots$$

or

$$g \cong jw\sqrt{LC} + \frac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) + \frac{j}{8}w\sqrt{LC}\left(\frac{R}{wL} + \frac{G}{wC}\right)^2 + \cdots \tag{4.16}$$

Apparently, any term after the second in the above equation is negligible based on the small loss assumptions. And, finally,

$$g \cong \boldsymbol{a} + j\boldsymbol{b} = \frac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) + jw\sqrt{LC} \tag{4.17}$$

The phase constant $\boldsymbol{b}$ is apparently the same as the wave number defined in a lossless transmission line. Since the wave is attenuated by the factor of e^{-ax} as it traveled in the $+x$ direction, $\boldsymbol{a}$ is called the attenuation constant. One can readily see that

$$\boldsymbol{a} = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}\left(RY_c + GZ_c\right) \tag{4.18}$$

The complete solution of the voltage wave that travels in the $+x$ direction is $V^+ e^{-gx + jwt}$. The phase velocity is derived from the time displacement of the constant phase point, i.e. assuming the exponential term to be a constant, $-gx + jwt = u$, and taking the time derivative of it. This yields

$$v_p = \frac{dx}{dt} = \frac{jw}{g} \cong \frac{jw}{\frac{1}{2}\sqrt{LC}\left(\frac{R}{L}+\frac{G}{C}\right) + jw\sqrt{LC}} = \frac{1}{\frac{1}{2j}\sqrt{LC}\left(\frac{R}{wL}+\frac{G}{wC}\right)+\sqrt{LC}} \cong \frac{1}{\sqrt{LC}} \tag{4.19}$$

The phase velocity of the signal in a low loss line is the same as that in a lossless line. It is customary to expressed the characteristic impedance in terms of the phase velocity and one of the circuit parameter as below.

$$Z_c = \sqrt{\frac{L}{C}} = v_p L = \frac{1}{v_p C} \tag{4.20}$$

At high frequencies, the electromagnetic field can penetrate only a small distance into a conductor. In fact, the amplitude of the fields decay exponentially from its value at the surface of the conductor according to e^{-u/d_s}. Here, u is the normal distance into the conductor and d_s is the skin depth given below.

$$d_s = \sqrt{\frac{2}{w m_m s_m}} \tag{4.21}$$

where w is the angular frequency of the signal, m_m is the permeability of the conductor, and s_m is the conductivity of the conductor. At one skin depth, i.e. $u = d_s$, the field strength is 37% of the surface value. At three skin depth, i.e. $u = 3d_s$, the field strength is down to only 5% of its surface value. For copper, a commonly used conductor in microwave transmission media, the skin depth at 100 MHz is 6.6 micron or 0.26 mils. At 10 GHz, the skin depth is 10 times smaller than that. Therefore, when considering the finite conductance of a lossy transmission line, the series resistance *R* in Figure 4.1 is usually expressed in terms of the surface resistance R_m, which is defined as

$$R_m = \frac{1}{s_m d_s} \tag{4.22}$$

The exact formula for the series resistance of a particular transmission media also depends on the geometry of the media itself.

In Figure 4.1, the admittance *Y* of the shunt components can be expressed as

$$Y = jwC + G = jwC\left(1 - j\frac{G}{wC}\right)$$

The term G/wC, which causes the admittance *Y* to deviate from a pure imaginary value with non-zero dielectric conductivity, is called the *loss tangent* and conventionally denoted as "tan d". For a particular transmission media, the geometric factor for the conductance and capacitance of the dielectric is the same and can be eliminated from the loss tangent formula. In other words, $G = ks_d$ and $C = ke_d$, where *k* is the geometric factor, s_d and e_d is the conductivity and the permittivity of the dielectric respectively. Therefore,

$$\tan d = \frac{s_d}{w e_d} \tag{4.23}$$

and the dielectric conductance G in terms of the loss tangent is

$$G = w C \tan d \tag{4.24}$$

The mathematical derivations in this section are based on the circuit model in Figure 4.1. In relating this model to physical transmission lines, one should bear in mind that it only applies to transmission lines with two conductors and also the current must only flow in the same direction as the propagation of signal waves.

4.2 COAXIAL LINE

The most commonly seen RF and microwave transmission media is probably the coaxial line (or coaxial cable). A diagram showing the construction of a coaxial line is shown in Figure 4.2. Because of the small skin depth at RF and microwave frequencies, the electromagnetic field is practically confined in between the conductors of the coaxial line making it a non-radiating transmission media.

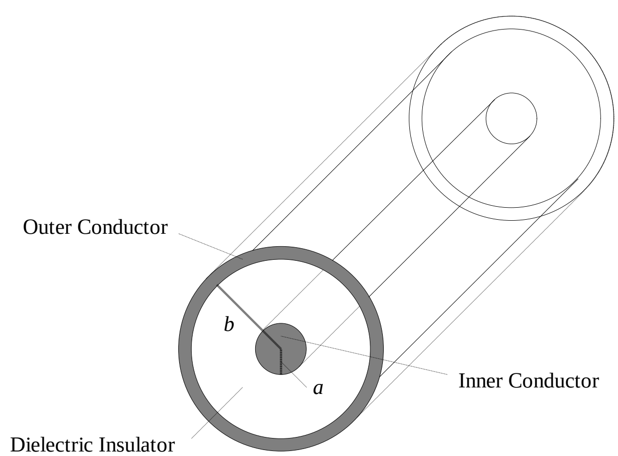
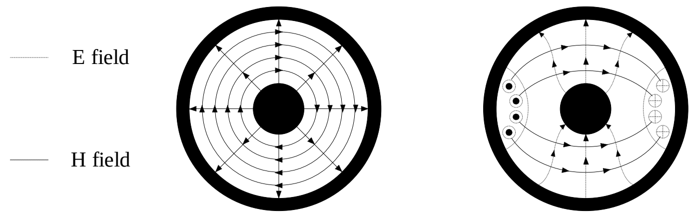


Figure 4.2 A coaxial line segment.

The electromagnetic field can propagate down the coaxial line in different fashion which is called modes. The primary mode of propagation in the coaxial line is called TEM (Transverse ElectroMagnetic wave) mode. In this mode all EM fields are transverse to the propagation direction as seen in Figure 4.3(a). TEM mode of propagation is possible at all frequencies. Higher order modes of propagation happens above a certain cutoff frequency f_c for each particular mode. Below the cutoff frequency, the fields of that mode decay exponentially with the propagation distance and thus termed evanescent mode. Figure 4.3(b) shows the electromagnetic field pattern of the TE_{11} mode.



(a) **(b)**

Figure 4.3 The electromagnetic field pattern for the **(a)** TEM mode and **(b)** TE_{11} mode.

Single mode operation is of essential importance in a transmission media. This is because the phase velocities of different propagating modes are not the same and interference may occur some distance down the transmission line if multiple modes of propagation are allowed to exist at the same time. Since TE_{11} mode has the lowest cutoff frequency of all higher order modes in coaxial line, it sets the upper frequency limit for the single mode, i.e. TEM, operation. The approximated formula for the TE_{11} mode cutoff frequency, which are accurate to within 5 percent for $b/a < 7$, is

$$f_{cutoff,\mathrm{TE}_{11}} \cong \frac{1}{p(a+b)\sqrt{m_d e_d}} \tag{4.25}$$

Since there are no transverse current in the TEM mode propagation, the formulae derived in the previous section are readily applicable to the coaxial line. The per unit length capacitance and inductance parameters, shown below, can be easily found by solving for the static EM field in the coaxial structure.

$$C = \frac{2pe_d}{\ln(b/a)} \tag{4.26}$$

$$L = \frac{m_d}{2p}\ln\frac{b}{a} \tag{4.27}$$

From (4.19), the phase velocity are

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{m_d e_d}} \tag{4.28}$$

which is the same as the phase velocity of EM waves in an unbounded dielectric of the same material. This turns out to be true for TEM mode in any transmission media that support it. The characteristic impedance of the coaxial line are

$$Z_c = \sqrt{\frac{L}{C}} = \frac{1}{2p}\sqrt{\frac{m_d}{e_d}}\ln\frac{b}{a} \tag{4.29}$$

The per unit length series resistance, contributed by both the inside and outside conductors, is

$$R = \frac{R_m}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \tag{4.30}$$

The per unit length shunt conductance come directly from equations (4.24) and (4.26) to be

$$G = \omega C \tan\delta = \omega\,\frac{2\pi\varepsilon_d}{\ln(b/a)}\tan\delta \tag{4.31}$$

Example 4.2.1 *An industrial standard 50 ohms Teflon filled 0.086 inches (indicating the outer conductor outside diameter) semi-rigid cable has a .020 inches diameter inside conductor. What is the inside diameter of the outer conductor assuming the relative permittivity of Teflon is 2.1 and the relative permeability is 1? What is the maximum frequency that this cable can operate under single TEM mode? What is the per foot loss in dB at 10 GHz assuming the conductors are made of copper and the loss tangent of Teflon is .00015?*

From (4.29),

$$50 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{2.1\times\varepsilon_0}}\ln\frac{b}{.02} \Rightarrow b = 0.067\text{inches}$$

where $\mu_0 = 4\pi\times10^{-7}\,\text{henry/m}$ and $\varepsilon_0 = 10^{-9}/36\pi\,\text{farad/m}$ are the permeability and permittivity of vacuum respectively.

Since $b/a = 3.35 < 7$, from (4.25)

$$f_{cutoff,\mathrm{TE}_{11}} \cong \frac{3\times10^8}{\pi 0.0254(.02+.067)\sqrt{2.1}} = 29.8\text{GHz}$$

However, it is generally a good idea to set the maximum frequency for single TEM mode operation to be 5% lower than the cutoff frequency of the TE_{11} mode. Therefore, the maximum frequency for single TEM mode operation is approximately 28.3 GHz.

The conductivity of copper is 5.8×10^7 mhos / m. From (4.21), the skin depth of copper at 10 GHz is 0.66 micron. From (4.22) and (4.30), the per foot series resistance is

$$R = \frac{1}{2\pi\sigma_m\delta_s}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{12}{2\pi\times5.8\times10^7\times0.66\times10^{-6}}\left(\frac{1}{0.02}+\frac{1}{0.067}\right) = 3.24\text{ohms/ft}$$

From (4.31), the per foot shunt conductance is

$$G = 2\pi\ \ f\,\frac{2\pi\varepsilon_d}{\ln(b/a)}\tan\delta = 12\times0.0254\times2\pi\times10^{10}\times\frac{2\pi\times2.1\times10^{-9}/36\pi}{\ln(0.067/0.02)}\times0.00015 = 277\times10^{-6}\text{mhos/ft}$$

From (4.18) and section 2.4,

$$\frac{\Delta P_{dB}}{\Delta d} = 8.686 \times a = 8.686 \times \frac{1}{2}\left(\frac{R}{50} + G \times 50\right) = 0.342\text{dB/ft}$$

4.3 MICROSTRIP LINE

Microstrip lines are widely used in the microwave integrated circuits from hybrids to monolithics. Figure 4.4 shows an example of a microstrip transmission line.

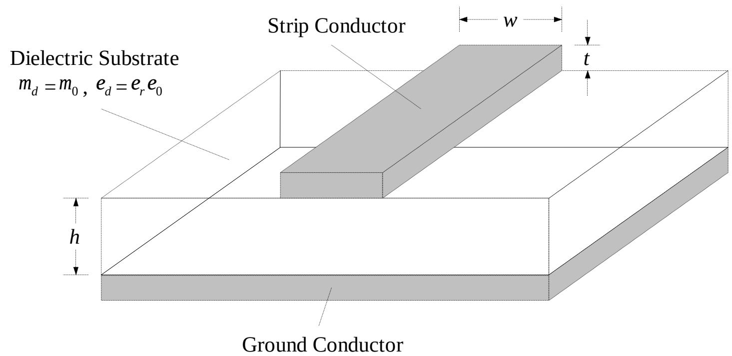


Figure 4.4 A microstrip line segment.

The advantages of using the microstrip line is that it is well suited for the printed circuit fabrication process, and because the strip conductor is exposed on the top side, component mounting is relatively easy. However, since the dielectric around the conductor is inhomogeneous, EM fields can not propagate in TEM mode. This is because a TEM wave must have a velocity c in the air above the strip and one of $c/\sqrt{e_r}$ in the substrate below. Because of the different velocities needed for TEM propagation, it is not possible to form a single TEM mode. Consequently, small amount of transverse fields must exist to equalize the propagation velocities in different dielectrics. This hybrid presence of TE and TM modes form the principle mode of propagation in a microstrip line which is named quasi-TEM mode. At low frequencies, up to a few GHz, static analysis and TEM mode formulae can be used to approximate the quasi-TEM transmission line characteristics. At frequencies above that, dynamic analysis must be used to solve for all fields components in order to characterize the microstrip line. For the purpose of understanding the basic circuit parameters for microstrip lines, static analysis will be discussed here. Dynamic analysis is beyond the scope of this text.

For the microstrip line in Figure 4.4, the propagation velocity and characteristic impedance can be approximated at low frequencies by applying the TEM mode equations (4.19) and (4.20) which are restated below.

$$v_p = \frac{1}{\sqrt{LC}} \tag{4.32}$$

$$Z_c = \sqrt{\frac{L}{C}} = v_p L = \frac{1}{v_p C} \tag{4.33}$$

Now, if we remove the dielectric substrate under the strip and fill it with air as shown in Figure 4.5(a), the transmission line is surrounded by homogeneous dielectric and true TEM mode can propagate on the line.

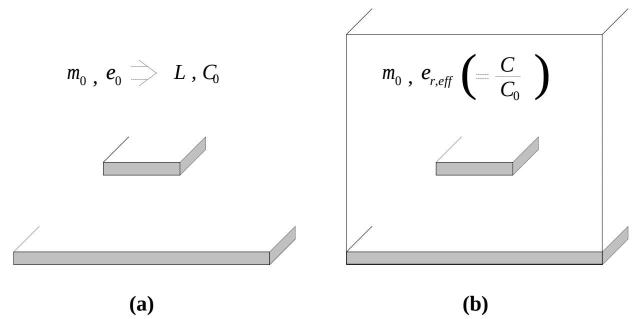


Figure 4.5 **(a)** The static analysis of quasi-TEM line using parameters from a transmission line with homogeneous air dielectric with otherwise the same dimensions, and **(b)** The equivalent TEM transmission line with homogeneous dielectric having effective relative permittivity $e_{r,\text{eff}}$.

Applying equations (4.19) and (4.20) again,

$$c = \frac{1}{\sqrt{LC_0}} \tag{4.34}$$

$$Z_0 = \sqrt{\frac{L}{C_0}} = cL = \frac{1}{cC_0} \tag{4.35}$$

where c is the speed of light in air. Since we assumed the dielectric substrate in the microstrip line is non-magnetic, i.e. $m_r = 1$, the per unit length inductance L remains the same. By substituting for L in (4.32) and (4.33) with that in (4.34) and (4.35) respectively, one gets

$$v_p = \frac{c}{\sqrt{C/C_0}} = \frac{c}{\sqrt{e_{r,\text{eff}}}} \tag{4.36}$$

$$Z_c = \frac{1}{c\sqrt{CC_0}} = \frac{Z_0}{\sqrt{C/C_0}} = \frac{Z_0}{\sqrt{e_{r,\text{eff}}}} \tag{4.37}$$

The term C/C_0 is called the effective relative permittivity of the microstrip line in Figure 4.4, i.e.

$$e_{r,\text{eff}} = \frac{C}{C_0} \tag{4.38}$$

Both C and C_0 can be found by solving the static field equations for the structures in Figure 4.4 and figure 4.5(a). This is usually done by numerical methods and conformal mapping which will not be discussed here. However, once $e_{r,\mathrm{eff}}$ is calculated from the static capacitances, the microstrip line in Figure 4.4 can be approximated by a TEM strip line surrounded homogeneously with dielectric whose relative permittivity is $e_{r,\mathrm{eff}}$ as shown in Figure 4.5(b). In dynamic analysis, although static capacitances are not defined, $e_{r,\mathrm{eff}}$ is still being used by solving for phase velocity in the quasi-TEM mode and modifying the definition using (4.36), i.e.

$$e_{r,eff} = (c/v_p)^2 \tag{4.39}$$

If the width of the microstrip line is changed smaller, both C and C_0 decrease. Consequently, from (4.37), Z_c becomes higher. Likewise, widening the line width reduces its characteristic impedance. The effective relative permittivity also changes with line width but not as obvious from (4.38) because of the ratio. Consider a very wide line. In this cas, almost all electric field lines are packed between the strip and the ground plane with or without the substrate. Therefore, the ratio C/C_0 is very close to the relative permittivity of the substrate e_r, but can never quite become that. As the width of the line shrinks, propotionally more and more field lines reach the air space above the substrate. As a result, the ratio C/C_0 drops, but it can never reach 1. So, the effective relative permittivity of the microstrip line increases as the width of the line increases, decreases as the width of the line decreases. However, its value will always be in between the relative permittivity of the substrate and the air, i.e. $1 < e_{r,\mathrm{eff}} < e_r$.

When a cover is added or lowered towards the microstrip line, both static capacitance C and C_0 increases. From (4.37), Z_c decreases. Also, proportionally, since more electric field is above the microstrip in the air space than before, C increases slower than C_0, therefore $e_{r,\mathrm{eff}}$ also decreases. Similar effect can also be produced by adding or closing in a side wall to the microstrip line.

With the presence of transverse field components in the quasi-TEM transmission lines, the characteristic impedance is not uniquely defined as of the TEM case. This is because the voltage between the conductors depends on the integration path chosen, and the longitudinal current flowing in the strip and the ground plane are different. Although different definitions of characteristic impedance for microstrip have been used in research articles, however, conventionally the voltage is defined along the center line of the strip and the current is defined as the longitudinal current in the strip itself. Approximate formulas for the characteristic impedance have been derived by various methods in the articles with about 1% accuracy which is more than sufficient in practice. The equations by Hammerstad and Jensen are shown below as an example.

Hammerstad and Jensen equations:

For the air filled structure of the microstrip line as shown in Figure 4.5(a), The characteristic impedance Z_0 with zero thickness microstrip, i.e. $t = 0$, is

$$Z_0(w/h, t = 0, e_r = 1) = \frac{\left(\sqrt{m_0/e_0}\right)}{(w/h) + 1.98 \cdot (w/h)^{0.172}} \tag{4.40}$$

This is a simplified equation of Hammerstad and Jensen for $w/h > 0.06$ and up to 0.3% error in impedance calculation. The characteristic impedance of the microstrip line can be obtained with (4.37) and the formula for $e_{r,\text{eff}}$ below.

$$e_{r,\text{eff}}\left(w/h, e_r, t=0\right) = \frac{e_r+1}{2} + \frac{e_r-1}{2}\cdot\left(1+\frac{10}{(w/h)}\right) - a\cdot b \tag{4.41}$$

where

$$a = 1+\frac{1}{49}\cdot\ln\left(\frac{(w/h)^4+3.7\times10^{-4}\cdot(w/h)^2}{(w/h)^4+0.432}\right)+\frac{1}{18.7}\cdot\ln\left(1+1.69\times10^{-4}\cdot(w/h)^3\right) \tag{4.42}$$

$$b = 0.564\cdot\left(\frac{e_r-0.9}{e_r+3}\right)^{0.053} \tag{4.43}$$

When the thickness of the microstrip is to be taken into account, the microstrip line is usually modeled as an equivalent microstrip line with zero strip thickness but slightly larger width. In other words, $w_{eq} = w + \Delta w_t$. A level of complication is added here because the modeled incremental width is different for the microstrip line and its air filled counter part. For the latter one, notation $w_{eq,0} = w + \Delta w_{t,0}$ is used where 0 stands for homogeneous air dielectric. Hammerstad and Jensen gave empirical equations from a functional approximation of the numerical results for finite strip thickness case as follows.

$$\Delta w_{t,0} = \frac{t}{p}\cdot\ln\left(1+\frac{4e}{(t/h)\cdot\coth^2\left(\sqrt{6.517\cdot(w/h)}\right)}\right) \tag{4.44}$$

$$\Delta w_t = \frac{\Delta w_{t,0}}{2}\cdot\left(1+\frac{1}{\cosh\sqrt{e_r-1}}\right) \tag{4.45}$$

To find the characteristic impedance of finite thickness line, the following formula must be carefully used.

$$Z_c\left(w/h, t/h, e_r\right) = Z_c\left(w_{eq}/h, t=0, e_r\right) = \frac{Z_0\left(w_{eq}/h, t=0\right)}{\sqrt{e_{r,\text{eff}}\left(w_{eq}/h, t=0, e_r\right)}} \tag{4.46}$$

$$e_{r,\text{eff}}\left(w/h, t/h, e_r\right) = \left(\frac{Z_0\left(w_{eq,0}/h, t=0\right)}{Z_c\left(w_{eq}/h, t=0, e_r\right)}\right)^2 = e_{r,\text{eff}}\left(w_{eq}/h, t=0, e_r\right)\cdot\left(\frac{Z_0\left(w_{eq,0}/h, t=0\right)}{Z_0\left(w_{eq}/h, t=0\right)}\right)^2 \tag{4.47}$$

Higher order mode of propagation can occur in a microstrip line with the lowest cutoff frequency belonging to HE_1 mode where H stands for *Hybrid*. Because increasing conductor width decreases f_{cutoff,HE_1}, it is important to determine the cutoff frequency of the widest stripline in a circuit.

Attenuation in microstrip line resulted from conductor and dielectric loss as before. Though the formula will not be discussed here, basic parameters such as s_m and $\tan d$ are used in the calculations. One other source of loss often discussed is the conductor roughness at the strip and substrate contact. Substrate manufacturer provide the surface roughness figure for calculation of this loss.