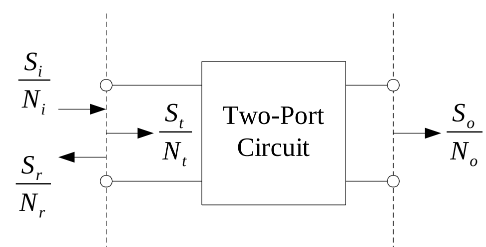
CHAPTER
16

NOISE

16.1 NOISE FIGURE AND NOISE TEMPERATURE

The noise factor is a measure of the signal to noise degradation caused by a circuit. Often one is interested in increasing the power associated with a desired signal and uses a circuit with gain to achieve this. The input noise is also amplified. Unfortunately, for any real circuit additional noise is added which results in a reduction in the signal to noise ratio at the output. The noise factor is defined such that an ideal *noiseless* system would have a noise factor of 1 (input SNR = output SNR). To further motivate the definition consider the illustration in Figure 16.1 below:



S_i = input signal power, S_o = output signal power
N_i = input noise power, N_o = output noise power

Figure 16.1 Input and output signal to noise ratio (SNR) of a two-port circuit.

Since that $S_r = S_i|\Gamma_{in}|^2$, $N_r = N_i|\Gamma_{in}|^2$, $S_t = S_i\left(1-|\Gamma_{in}|^2\right)$, and $N_t = N_i\left(1-|\Gamma_{in}|^2\right)$, therefore

$$\frac{S_i}{N_i} = \frac{S_r}{N_r} = \frac{S_t}{N_t} \tag{16.1}$$

The system designer is interested the degradation in signal to noise ratio caused by his circuit. The input/output SNR performance ratio, designed as g is

$$g = \frac{S_i / N_i}{S_o / N_o} \tag{16.2}$$

Noise is particularly important when very small signals are being amplified prior to some form of processing. In this case, one is interested in getting the maximum signal to noise ratio out of the device. Changing the output impedance will have the same effect on signal as the noise and therefore, the output is normally conjugately matched so that the maximum signal is delivered to the load. An input matching (or mismatching) network does not affect the input signal to noise ratio, but it can affect how much and where the internal noise sources deliver their power. For this reason the relevant amplifier gain is the available gain, G_A. Mismatching the input affects the gain, but the effect is the same for both the incoming signal and noise. However, the input mismatch can affect internal sources (of noise) and influence how much of their available power is delivered to the load. By choosing the input matching network appropriately the effectiveness of the internal noise sources to deliver power to the load can be minimized.

The output signal power is related to the input signal power by the gain, i.e.,

$$S_o = G_A \cdot S_i \tag{16.3}$$

and the performance ratio becomes,

$$g = \frac{N_o}{G_A N_i} \tag{16.4}$$

The noise contribution added by the amplifier can be represented by and equivalent noise source, ΔN, added to the input. The output noise is then $N_o = G_A \cdot (N_i + \Delta N)$ as illustrated by Figure 16.2 below.

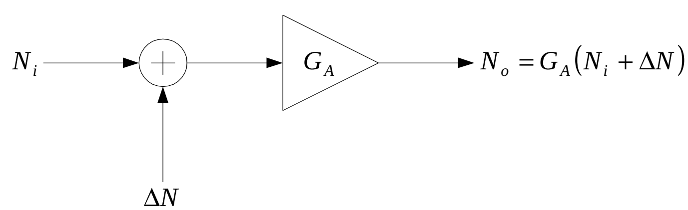


Figure 16.2 Contribution of the equivalent noise source ΔN of the amplifier.

The SNR performance ratio becomes

$$g = \frac{G_A(N_i + \Delta N)}{G_A N_i} = \frac{N_i + \Delta N}{N_i} = 1 + \frac{\Delta N}{N_i} \tag{16.5}$$

To evaluate the performance ratio one needs to know the equivalent input noise contributed by the circuit. A standard for measuring ΔN, called the *noise factor*, is defined using performance ratio, g, when the input noise, N_i, is that of a matched resistor at room temperature, i.e. $T_r = 290°K$, i.e. The noise factor is

$$F = 1 + \frac{\Delta N}{N_r} \tag{16.6}$$

where $N_r = KT_r B$. The *noise figure* is the logarithmic, i.e. dB, representation of the noise factor, or

$$F_{dB} = 10 \cdot \log(F) \tag{16.7}$$

When the noise factor is known, the equivalent input added noise contribution of the circuit, ΔN, can be found from

$$\Delta N = (F-1)N_r = (F-1)KT_r B \tag{16.8}$$

Another way of representing the equivalent added input noise contribution is to define ΔN in terms of the equivalent temperature that a matched resistor requires to generate it. Then

$$\Delta N = KT_{ckt} B \tag{16.9}$$

where T_{ckt} is called the noise temperature of the circuit and is a measure of the noise that the circuit its self adds. Equating the last two equations yields,

$$T_{ckt} = T_r(F-1) \tag{16.10}$$

A noise factor of 1 means that the circuit noise temperature is zero or that it contributes no additional noise of its own. A noise factor of 2 (or a noise figure of 3 dB) means that the circuit noise temperature is T_r, or 290°*K*, and the circuit itself is contributing added noise equivalent to a room temperature matched resistor added at the input.

Cascading amplifier is normally required and can be easily examined by whenever inter-stage matching networks exist to insure a conjugate match between gain block or that the gain blocks have previously been matched to 50 ohms. In either case the equivalent input noise ΔN_{eq} can be found by examining the following diagram.

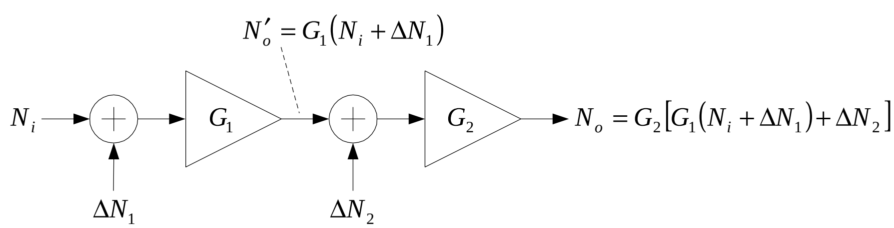


Figure 16.3 Contribution of the equivalent noise sources ΔN_1 and ΔN_2 of cascaded amplifiers.

Consequently,

$$N_o = G_2[G_1(N_i + \Delta N_1) + \Delta N_2] = G_1 G_2 N_i + G_1 G_2 \left[\Delta N_1 + \frac{\Delta N_2}{G_1}\right] \tag{16.11}$$

From the above equation, it is observed that the cascaded circuits can be represented as a single circuit with gain G_1G_2 and equivalent noise source ΔN_{eq} as shown below.

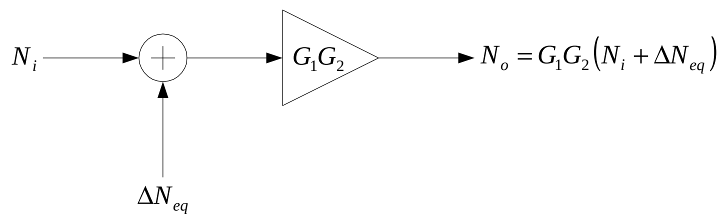


Figure 16.4 Equivalent noise source ΔN_{eq} of the cascaded amplifiers and its contribution.

Therefore, the equivalent added input noise for the cascaded pair, ΔN_{eq}, is

$$\Delta N_{eq} = \Delta N_1 + \frac{\Delta N_2}{G_1} \tag{16.12}$$

This equation illustrates how important it is for the first stage of cascaded amplifies to be a low noise circuit. The gain of the first stage effectively reduces the contribution from the following stage. Rewriting the above in terms of noise temperature yields

$$T_{eq} = T_1 + \frac{T_2}{G_1} \tag{16.13}$$

This expression can be converted easily to noise factor by substituting $T_{eq} = T_r\left(F_{eq} - 1\right)$, $T_1 = T_r\left(F_1 - 1\right)$, and $T_2 = T_r\left(F_2 - 1\right)$, to get

$$F_{eq} = F_1 + \frac{\left(F_2 - 1\right)}{G_1} \tag{16.14}$$

16.2 RANDOM VARIABLES AND STOCHASTIC PROSSES

The usual variables encountered in calculus are called deterministic meaning that for a physically well posed problems that a unique number can be assigned to the variable. If the problem is repeated, the exact same value for the variable will be the result. On the other hand, random variable is one in which a unique value is not assignable in general. Multiple executions of the same problem can result in different value assignments for each of the random variables. A random variable can be thought of has a hat full of paper tags with numbers on them. Each time a number is to be assigned to the random variable, a tag is withdrawn and its number assigned to the variable. Thus, the composition of the tags determines whether a certain value will be favored over others. A *random variable* x represents the collection of possible values that it can take on.

If one value appears on more tags, then it has a greater likelihood of being drawn. Or, equivalently, if the problem is performed many times then that value will occur more often. The weighting associated with a particular value for a random variable, x, is represented by a *probability density function*, $p(x)$. This function describes the relative likelihood that the assigned value for x will be between x and $x + dx$. The integral of $p(x)$ between two limits a and b is a measures of the fraction of experiments resulting in assigned values between a and b. Consequently, the probability density function is always normalize such that $\int_{-\infty}^{+\infty} p(x)dx = 1$, i.e., a hundred percent of the values occur between $-\infty$ and $+\infty$. The random variable can be pictured assuming values along a vertical number line according to an associated probability density functions drawn along side.

The mean value of a random variable is designated m (or m_x when it is necessary to distinguish between different random variables). It represents the average value of the values that would be assigned to the random variable. It depends, of course, on the specific random variable and its particular probability density function, $p(x)$. The mean value is given by the first moment integral

$$m = \int_{-\infty}^{+\infty} x\, p(x)dx \tag{16.15}$$

This first moment, or average, is also called the *expected value* and represented by a bracket symbol as shown

$$m = \langle x \rangle \tag{16.16}$$

Instantaneous voltage or current samples of noise can naturally be thought of as a random variable. One sample may result in a positive value while the next may be negative. However, we expect the average value of the noise to be zero since otherwise we could tap the noise as an energy source and produce work which would be a violation of the laws of thermodynamics. Hence, attention will be directed to random variables with a zero mean and such will be assumed unless otherwise stated.

The power associated with a noise voltage (or current) is represented by the average of square of the voltage (current) samples. The power is therefore represented by the second moment integral called the *variance*, $\mathrm{Var}(x)$, that equals

$$\mathrm{Var}(x) = \int_{-\infty}^{+\infty} x^2 p(x)dx \tag{16.17}$$

The *standard deviation*, s (or s_x) is the square root of the variance, i.e., s is the *rms noise voltage or rms noise current.*

For reasons which will be shortly motivated, a common random variable is one whose probability density function is normally distributed (a gaussian function), i.e.,

$$p(x) = \frac{1}{\sqrt{2p}\, s} \exp\left(-(x-m)^2 / 2 s^2\right) \tag{16.18}$$

where m is the mean and s is the standard deviation. For a zero mean gaussian function distributed random variable, the density function is given by

$$p(x) = \frac{1}{\sqrt{2p}\, s} \exp\left(-x^2 / 2 s^2\right) \tag{16.19}$$

The combination of two or more random variables are possible but their possible values depend upon a joint probability density function, $p(x, y)$. This function describes the relative likelihood of getting a value from the first random variable between x and $x + dx$ and a value for the second between y and $y + dy$. If the two random variables are independent, in that the outcome for one variable does not depend on the outcome of the other, then the joint probability density function can be expressed in the form as the multiple of the individual probability density function as below.

$$p(x, y) = p_x(x) \cdot p_y(y) \tag{16.20}$$

This means that the probability density function of the ordered pair of random variables (x, y) is $p(x, y)$.

New random variable can be created from existing random variables using algebraic manipulations. For example, random variables x and y can be added to form a new random variable z. The probability density function for z, such that $z = x + y$, depends upon the joint probability density function $p(x, y)$. If x and y are independent, the probability density function for z is given by

$$p_z(z) = \int_{-\infty}^{+\infty} p_x(x) p_y(z - x) dx = \int_{-\infty}^{+\infty} p_y(y) p_x(z - y) dy \tag{16.21}$$

The probability density function for a sum of two independent random variables is the convolution of the two probability density functions.

The central limit theorem says that *if a random variable consists of a sum of independent random variable components, then the probability density function for the sum approaches a gaussian function as the number of components increases.* The convergence to a guassian like distribution is quite fast as can be illustrated in Figure 16.5 by choosing two clearly non-gaussian distributions and computing the convolution to determine the probability density function for the sum. A uniform distribution convolved with a triangular one results in a quadratic one. If the sum of the random variables is added to a third one, then the quadratic probability density function is now being convolved with a third probability function, e.g. a trapezoidal one, the results are seen to already taking on the qualitative appearance of a gaussian function.

Electronic noise arises because of non-uniform current flow and because of conduction charges variations resulting from thermal excitations of electrons from lower bands in the atoms. These represents a large number of independent random effects which combine together to produce a noise voltage or current. As a results, it is reasonable to expect that a Gaussian distribution will describe noise voltage or current samples.

A complex random variable has a real and imaginary part, which are random variables, i.e.,

$$z = x + jy \tag{16.22}$$

For the purpose of analyzing noise, x and y may be assumed to be independent. In such case, the mean and variance of the random variable z are defined as

$$m = \langle z \rangle = \langle x \rangle + \langle jy \rangle = \langle x \rangle + j\langle y \rangle = m_x + jm_y \tag{16.23}$$

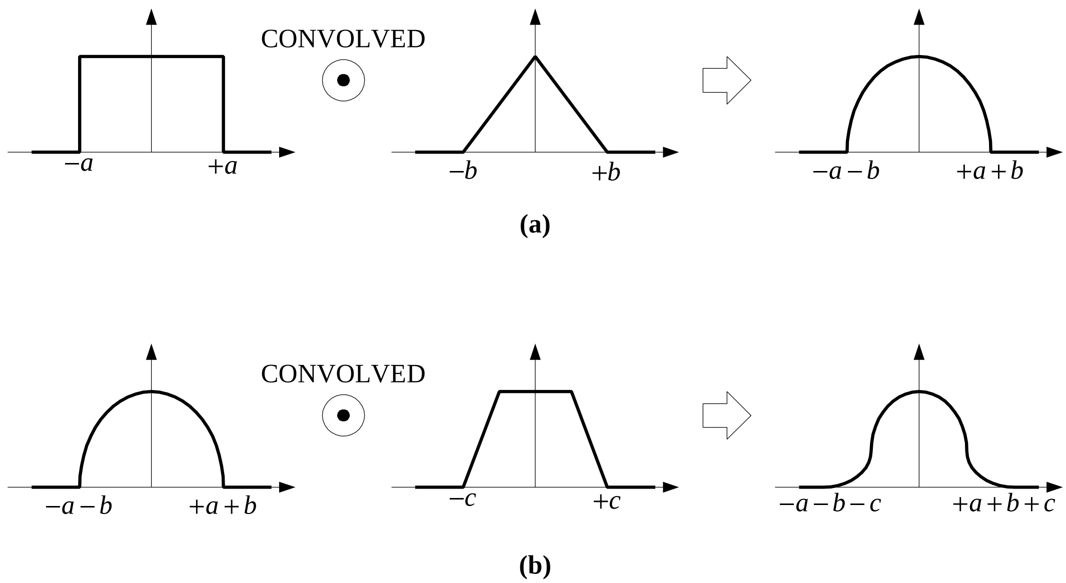


Figure 16.5 Illustration of the central limit theorem by **(a)** convolving a rectangular and a triangular shaped probability density function, and **(b)** convolving the resulting quadratic shaped probability density function with a third on which is trapezoidal.

$$\mathrm{Var}(z) = \left\langle |z|^2 \right\rangle = \left\langle zz^* \right\rangle = \left\langle x^2 + y^2 \right\rangle = \left\langle x^2 \right\rangle + \left\langle y^2 \right\rangle = \mathrm{Var}(x) + \mathrm{Var}(y) = s_x^2 + s_y^2 \qquad (16.24)$$

A measure of the correlation between two random variables can be found by looking at the average (or expected value) of the conjugate product. Two gaussian random variables are independent if and only if the average of their conjugate product equals zero. Suppose z is the sum of two independent complex random variables m and n (zero mean assumed), i.e., $z = m + n$, then the variance of z is the sum of the variances of m and n as shown in the derivation below.

$$\begin{aligned} \mathrm{Var}(z) &= \left\langle |z|^2 \right\rangle = \left\langle (m+n)(m+n)^* \right\rangle = \left\langle (m+n)(m^* + n^*) \right\rangle \\ &= \left\langle mm^* + mn^* + nm^* + nn^* \right\rangle \\ &= \left\langle |m|^2 \right\rangle + \left\langle mn^* \right\rangle + \left\langle nm^* \right\rangle + \left\langle |n|^2 \right\rangle \end{aligned}$$

The two middle terms of the last expression in the above derivation equal zero since the random variables m and n are independent, then

$$\mathrm{Var}(z) = \left\langle |m|^2 \right\rangle + \left\langle |m|^2 \right\rangle = \mathrm{Var}(m) + \mathrm{Var}(m) = s_m^2 + s_n^2 \qquad (16.25)$$

A stochastic process is function in a random variable, which is a function of a deterministic variable. Noise is a random variable that is a function of time and therefore we will discuss stochastic process in terms of time. A stochastic process can be thought of in an analogous way to that of a random variable. Imagine a hat with tags, but instead of discrete values the tags have functions of time listed on them. Sample functions are obtained by selecting the tags. If multiple tags have the same function then the multiple sampling of the stochastic process will favor that particular function. The collection of function is called an ensemble. The mean function can be calculated by averaging the functions in the ensemble.

Likewise a variance function can be visualize by averaging the square of the functions in the ensemble. This type of averaging is known as *ensemble averaging* and the mean and standard deviation are a function of time, $m(t)$ and $s(t)$. A stationary random process is one in which statistical quantities such as mean and variance are a constant, i.e., independent of time.

Given a particular sample function, $s(t)$, from the ensemble it is possible to determine a mean value by averaging over time. This mean value, designated as m, is given by

$$m = \frac{1}{2T}\int_{-T}^{+T} s(t)dt \tag{16.26}$$

Time averages are designated by a bar over the function, i.e., $m = \overline{s(t)}$. Likewise the variance can be calculated by averaging $[s(t)]^2$ over time, i.e., $\overline{[s(t)]^2}$. When the statistics over time are the same as the statistics over the ensemble then the process is said to be *ergodic*. Noise is an example of a stationary ergodic process and thus averages can be made on one sample over time or by computing them from multiple samples. A gaussian random process is a stationary ergodic process in which the statistics are described by a Gaussian distribution function.

Given a value for the random process at a time t_0, the value of the function at time t_1, in general, may not be independent random variables. The correlation of the random variables for separate times is described by the auto-correlation function or its Fourier transform, which is known as the power spectral density or *psd*. If the psd has a very low bandwidth then the values of the random process can not change fast implying that future value of the random process are most likely to be near the current value. However, if the psd is a wide band function then future value of the random process can jump quickly and bigger changes are equally as likely. A random process that has a flat psd is said to be white (all frequencies present) and it means that the random values between any points are uncorrelated or independent. Electronic noise is an example if a white gaussian stochastic (random) process.

Noise can be viewed in the time domain or in the frequency domain. It is easiest to think of discrete time samples of the noise as a time domain member of the ensemble. Data can be taken over multiple time windows resulting in a collection of such samples, which begins to comprise the ensemble. Likewise an FFT operation can be performed on each of the time samples resulting in a complex random variable associated with discrete frequencies. Since the FFT process is a linear function then the FFT of a gaussian random process must result in a complex gaussian variable for each frequency. If the process is white then the real and imaginary parts are independent.

$$S(w) = FFT[s(t)] \tag{16.27}$$

$$S(w) = x + jy \tag{16.28}$$

where x and y are independent gaussian random variables. In the frequency domain the random variable can be thought of as a phasor (containing amplitude and phase information) for that frequency of the noise. The amplitude is given by $\sqrt{x^2 + y^2}$ and the phase is given by $\tan^{-1}(y/x)$. Since the average power has to be the same whether viewed in the time or frequency domain then $\int_{-\infty}^{+\infty} |S(w)|^2 dw = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{+T} s(t)^2 dt = \mathrm{Var}(s) = s_s =$ mean square voltage (current). For this reason $|S(w)|^2$ is given the name power spectral density and has units of power per Hertz.

16.3 TWO PORT EQUIVALENT CIRCUITS

Equivalent circuits are an important circuit analysis technique. Circuits are equivalent if their current and voltage characteristics at their terminals are the same, i.e., from terminal point of view the circuits are the same. The conventional Thevenin equivalent circuit applies to a one port circuit. Any linear one port circuit can be replaced by a voltage source and a series impedance and exhibit the same terminal current/voltage behavior. We note that the one port may have embedded in it any number of dependent and independent sources. Also, the Thevenin impedance is not normally a simple resistance plus an inductor or capacitor but is a complicated reactance expression. Since the terminals must be the same, the equivalent circuit values can be obtained by simply applying the same voltages across the terminals of both circuits and adjusting the Thevenin components until the resulting current is the same. Applying this procedure with two different voltages result in a unique determination of the Thevenin circuit. A Norton Equivalent circuit consisting of a shunt current source and admittance can be similarly obtained.

A linear two port circuit with no independent sources can be represented by an impedance matrix as shown

$$\begin{bmatrix} V_2 \\ V_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_2 \\ I_1 \end{bmatrix} \tag{16.29}$$

If independent sources are included in the circuit, then the matrix representation has to change to accommodate the fact that a voltage can appear at a terminal when no current is flowing. However, the Z-matrix formulation can be easily amended by adding a constant voltage vector as shown

$$\begin{bmatrix} V_2 \\ V_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_2 \\ I_1 \end{bmatrix} + \begin{bmatrix} V_{20} \\ V_{10} \end{bmatrix} \tag{16.30}$$

Therefore, a circuit with independent sources can be equivalently represented by a two-port circuit without independent sources and two voltage sources in series with each output respectively as shown in Figure 16.6(a). The independent sources could be also accounted for by including shunt current sources at the terminals as shown in Figure 16.6(b). It is permissible to have a mixture of sources as illustrated in Figure 16.6(c). Also, the sources can be at either end of the circuit if a combination of current and voltage sources is used as shown in Figure 16.6(d). Likewise, the sources can both be on the output as illustrated in Figure 16.6(e). The order of the series voltage source or shunt current source can be interchanged (with different values), but the current source must be shunt and the voltage source must be in series.

The appropriate equivalent circuit to use depends upon the specific analysis objectives. Each of the equivalent circuits produces the same terminal characteristics as the original circuit.

16.4 THERMAL NOISE (KTB)

In any non-superconductive material at a temperature greater than absolute zero (0° Kelvin), electrons are constantly being excited between bound atomic states and conducted states. The number of electrons affected increases as the temperature increases. This variation of available electrons represent a variation in mobile charge and results in a random voltage appearing across the material. The average value of the random voltage is zero since the charge distribution at any instant of time is equally likely to result in a voltage in any direction. The voltage can be assumed to be gaussian random process since the material consists of many atoms whose influence can be expected to extend only over a few atomic spacings. Hence, the process can be expected to satisfy the central limit theorem and therefore result in a gaussian random voltage.

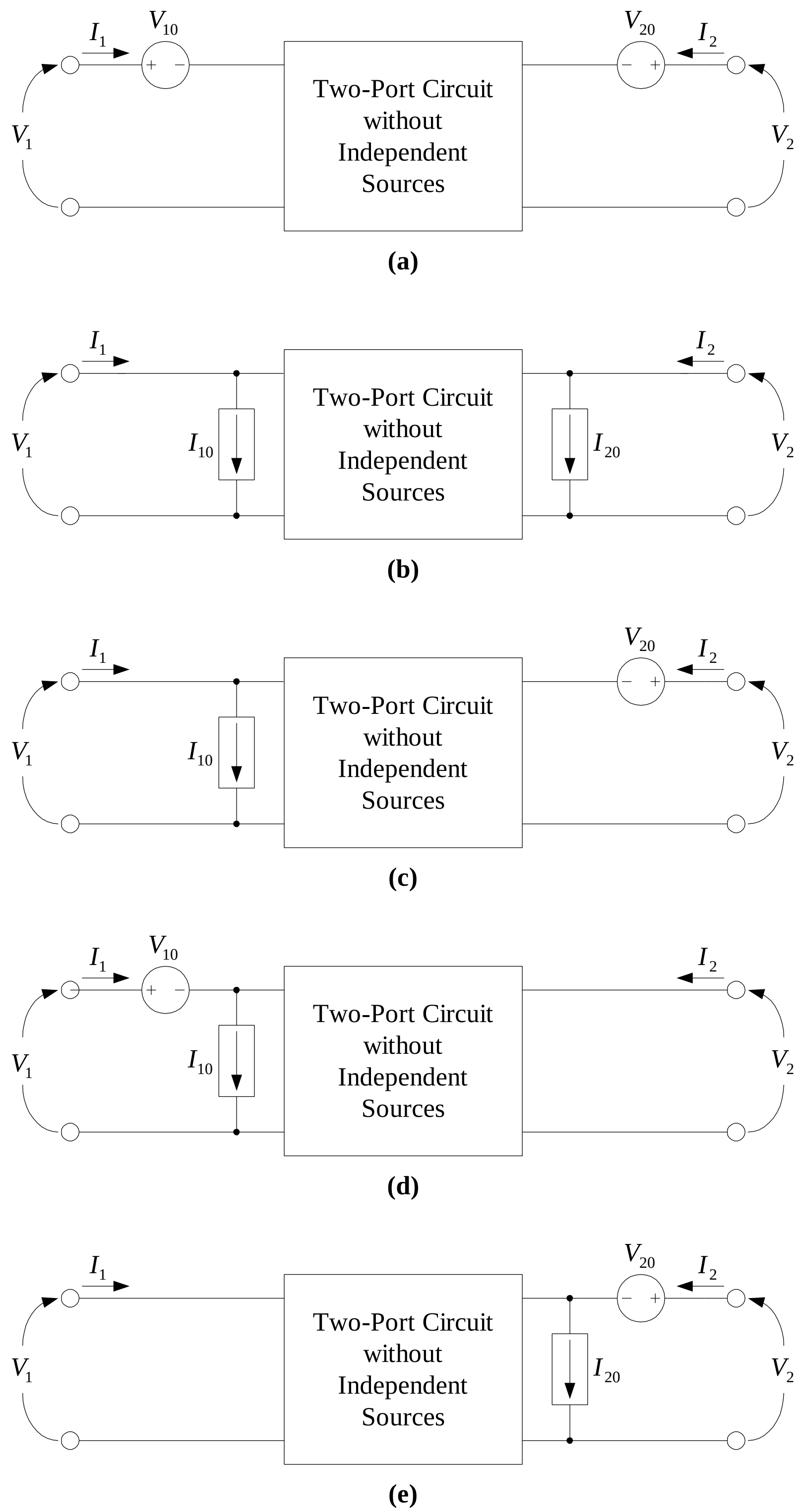


Figure 16.6 **(a)** Two-port Thevenin equivalent circuit, **(b)** two-port Norton equivalent circuit, **(c)** mixed use of the equivalent sources are permissible, **(d)** equivalent circuit with both sources at the input, **(e)** equivalent circuit with both sources at the output.

The variance of the voltage across a resistor has been measured by J.B. Johnson in an experiment [Thermal Agitation of Electricity in Conductors, *Phys*. Rev., vol 32 July, 1928, pp. 97-109] and was shown to be equal to

$$\mathrm{Var}(v) = s_v^2 = 4KTBR \tag{16.31}$$

where

K = Boltzmans Constant
T = Temperature in Kelvin
B = Bandwidth
R = Resistance in Ohms

For this reason thermal noise is sometimes referred to as Johnson noise and can be represented as an ideal (noiseless) resistor in series with a zero mean gaussian random noise voltage source.

The thermal noise voltage waveform consists of independent gaussian distributed voltage values. As a result there is no correlation between the voltages at one time and any other time. Thus, the noise has a power spectral distribution, which is uniform, i.e., it is white noise. The magnitude of each power spectral component equals $4KTR$ since its integral over the band must equal the total power of $4KTR \cdot B$. If $v(t)$ is the voltage waveform and $S(w) = \mathrm{FFT}[v(t)] = X(w) + jY(w)$, then

$$4KTBR = \left\langle v(t)^2 \right\rangle = \sum |S(w)|^2 \Delta w = \sum \left[X(w)^2 + Y(w)^2 \right] \Delta w = \sum P(w) \Delta w$$

Therefore,

$$P(w) = 4KTR \tag{16.32}$$

16.5 CIRCUIT COMBINATIONS OF THERMAL NOISE

A real resistor can be viewed as an ideal noiseless resistor, R, with a series voltage source, $V(w)$, whose value is defined as a complex gaussian zero-mean random variable with a white power spectral density which equals $4KTR$ as shown below.

Suppose two resistors are connected in series as depicted below.

Examining the problem from the frequency domain, $V_1(w)$ and $V_2(w)$ are complex gaussian stochastic processes with zero mean. The total voltage $V(w)$ equals the sum of the individual voltages

$$V(w) = V_1(w) + V_2(w)$$

The variance of V is

$$\left\langle |V(w)|^2 \right\rangle = \left\langle V(w)V(w)^* \right\rangle = \left\langle [V_1(w) + V_2(w)][V_1(w)^* + V_2(w)^*] \right\rangle$$

$$= \left\langle V_1(w)V_1(w)^* + V_1(w)V_2(w)^* + V_2(w)V_1(w)^* + V_2(w)V_2(w)^* \right\rangle$$

$$= \left\langle |V_1(w)|^2 \right\rangle + \left\langle V_1(w)V_2(w)^* \right\rangle + \left\langle V_2(w)V_1(w)^* \right\rangle + \left\langle |V_2(w)|^2 \right\rangle$$

Since the voltages are uncorrelated then the cross moments equal zero, i.e.,

$$\left\langle V_1(w)V_2(w)^* \right\rangle = \left\langle V_2(w)V_1(w)^* \right\rangle = 0 \tag{16.33}$$

and

$$\left\langle |V(w)|^2 \right\rangle = \left\langle |V_1(w)|^2 \right\rangle + \left\langle |V_2(w)|^2 \right\rangle \tag{16.34}$$

or

$$s_T^2 = s_1^2 + s_2^2 \tag{16.35}$$

Thus, for series connections of resistors the variances add, i.e., the power densities add,

$$s_T^2 = 4KTR_1 + 4KTR_2$$

$$= 4KT(R_1 + R_2)$$

$$= 4KTR_{SERIES}$$

$$\left\langle |V(w)|^2 \right\rangle = 4KTR_{SERIES} \tag{16.36}$$

Next consider two resistors connected in parallel as shown

The Thevenin Equivalent circuit consists of

$$R_{PARALLEL} = \frac{R_1 R_2}{(R_1 + R_2)} \text{ and } V(w) = \left[\frac{V_1(w) - V_2(w)}{R_1 + R_2}\right] R_2 + V_2(w)$$

The voltage becomes

$$V(w) = \frac{V_1(w)R_2 + V_2(w)R_1}{R_1 + R_2}$$

The variance of V equals

$$\left\langle |V(w)|^2 \right\rangle = \left\langle V(w)V(w)^* \right\rangle = \frac{1}{(R_1 + R_2)^2} \left\langle [V_1(w)R_2 + V_2(w)R_1][V_1(w)^* R_2 + V_2(w)^* R_1] \right\rangle$$

$$= \frac{1}{(R_1 + R_2)^2} \left\langle [V_1(w)V_1(w)^* R_2^2 + V_1(w)V_2(w)^* R_1 R_2 + V_2(w)V_1(w)^* R_1 R_2 + V_2(w)V_2(w)^* R_1^2] \right\rangle$$

$$= \frac{1}{(R_1 + R_2)^2} \left[R_2^2 \left\langle |V_1(w)|^2 \right\rangle + R_1 R_2 \left\langle V_1(w)V_2(w)^* \right\rangle + R_1 R_2 \left\langle V_2(w)V_1(w)^* \right\rangle + R_1^2 \left\langle |V_2(w)|^2 \right\rangle \right]$$

The cross moments are zero, and

$$\left\langle |V_1(w)|^2 \right\rangle = 4KTR_1$$

$$\left\langle |V_2(w)|^2 \right\rangle = 4KTR_2$$

resulting in

$$\left\langle |V(w)|^2 \right\rangle = \frac{1}{(R_1 + R_2)^2} \left[R_2^2 (4KTR_1) + R_1^2 (4KTR_2) \right]$$

$$= \frac{R_1 R_2}{(R_1 + R_2)^2} 4KT[R_2 + R_1] = 4KT\left[\frac{R_1 R_2}{R_1 + R_2}\right] = 4KTR_{PARALLEL}$$

Therefore,

$$\left\langle |V(w)|^2 \right\rangle = 4KTR_{PARALLEL} \tag{16.37}$$

The Thevenin voltage equals the thermal noise voltage associated with a resistor equal to parallel combination of the original two resistors, which is the Thevenin resistance.

Consider a circuit involving a resistor and a reactance as shown

There is no noise source associated with the reactance component since it represents inductance and capacitance effects and not conductance effects. The Thevenin equivalent impedance and source are

$$Z_T = \frac{jXR}{jX + R} = \frac{RX^2 + jXR^2}{X^2 + R^2} = \frac{RX^2}{X^2 + R^2} + j\frac{XR^2}{X^2 + R^2} = R_T + jX_T$$

$$V_T = \frac{jX}{jX + R} V(w)$$

The variance of V_T is

$$\left\langle \left| V_T \right|^2 \right\rangle = \left\langle V_T V_T^{\,*} \right\rangle = \left\langle \frac{jX}{jX + R} V(w) \frac{-jX}{-jX + R} V(w)^* \right\rangle$$

$$= \frac{X^2}{X^2 + R^2} \left\langle V(w) V(w)^* \right\rangle$$

$$= \frac{X^2}{X^2 + R^2} \left\langle \left| V(w) \right|^2 \right\rangle = 4KT\left[\frac{RX^2}{X^2 + R^2}\right] = 4KTR_T$$

Therefore,

$$\left\langle \left| V_T(w) \right|^2 \right\rangle = 4KTR_T \qquad (16.38)$$

where R_T is the resistor component of the Thevenin impedance. Hence, *the equivalent noise source equals the thermal noise associated with the resistive component of the Thevenin impedance*.

The noise power delivered from one resistor to another can be examined by the following circuit configuration

The noise power delivered by resistor 1 to resistor 2, designated P_{21}, can be found by ignoring source 2, i.e., replace it with a short circuit, and find the power dissipated in R_2 from the source V_1. In this case,

$$p_{21} = \left| \frac{V_1}{R_1 + R_2} \right|^2 R_2$$

$$P_{21} = \langle p_{21} \rangle = \frac{R_2}{(R_1 + R_2)^2} \left\langle |V_1|^2 \right\rangle = \frac{4KTR_1R_2}{(R_1 + R_2)^2} \tag{16.39}$$

The power delivered to R_1 from R_2, P_{12}, is given by

$$P_{12} = \frac{4KTR_1R_2}{(R_1 + R_2)^2} \tag{16.40}$$

The power delivered by R_1 to R_2 is the same as that delivered to R_1 by R_2 if they are at the same temperature. This has to be true since otherwise we would violate the laws of thermodynamics.

The maximum power is delivered from one resistor to the other when $R_1 = R_2$, i.e., the resistors are matched. The power so transferred is called the *available noise power* and equals KT for the power spectral density or KTB for the total noise power when integrated over the bandwidth B.

16.6 MINIMUM NOISE FIGURE

Need to compare (in other words, find the ratio of) the total noise at output to noise at output if circuit were truly noise free, i.e., when E and I were zero. Since the network is linear we can compare noise contribution at port $1-1'$ instead. Note that the noise temperature at input must be $290°K$.

$$\text{Total input noise}\, a \left\langle \left|I_{sc}\right|^2 \right\rangle = \left\langle \left|I_s\right|^2 \right\rangle + \left\langle \left|I + Y_s E\right|^2 \right\rangle$$

This is seen since

Using superposition

$$I_{sc} = I_s - EY_s - I$$

$$\left\langle \left|I_{sc}\right|^2 \right\rangle = \left\langle I_{sc} I_{sc}^* \right\rangle = \left\langle \left[I_s - \left(EY_s + I\right)\right]\left[I_s^* - \left(EY_s + I\right)^*\right] \right\rangle$$

$$= \left\langle I_s I_s^* \right\rangle - \left\langle I_s \left(EY_s + I\right)^* \right\rangle - \left\langle I_s^* \left(EY_s + I\right) \right\rangle + \left\langle \left(EY_s + I\right)\left(EY_s + I\right)^* \right\rangle$$

$$= \left\langle \left|I_s\right|^2 \right\rangle + \left\langle \left|EY_s + I\right|^2 \right\rangle$$

since I_s is independent of E and I, i.e., they are uncorrelated.

If the MESFET was truly noise free, then the input noise power would be proportional to $\left\langle \left|I_s\right|^2 \right\rangle$, i.e., E and I would be zero. Therefore, the noise factor F is

$$F = \frac{\left\langle \left|I_s\right|^2 \right\rangle + \left\langle \left|I + Y_s E\right|^2 \right\rangle}{\left\langle \left|I_s\right|^2 \right\rangle}$$

$$= 1 + \frac{\left\langle \left|I + Y_s E\right|^2 \right\rangle}{\left\langle \left|I_s\right|^2 \right\rangle}$$

E and I are not necessarily uncorrelated. Assume I consists of a component I_u which is uncorrelated to E and $\left(I - I_u\right)$ which is perfectly correlated to E. The correlated part of I can be related to E by a complex constant Y_c such that

$$I = I_u + Y_c E \tag{16.41}$$

The noise power defines real or equivalent circuit resistances

$$\left\langle |E|^2 \right\rangle = 4KT_r R_n \tag{16.42}$$

$$\left\langle |I_u|^2 \right\rangle = 4KT_r G_u \tag{16.43}$$

$$\left\langle |I_s|^2 \right\rangle = 4KT_r G_s \tag{16.44}$$

The numerator term in the noise factor expression can be evaluated as follows:

$$\left\langle |I + Y_s E|^2 \right\rangle = \left\langle \, [I_u + (Y_c + Y_s)E][I_u + (Y_c + Y_s)E]^* \right\rangle$$

$$= \left\langle |I_u|^2 \right\rangle + |Y_c + Y_s|^2 \left\langle |E|^2 \right\rangle$$

$$= 4KT_r G_u + |Y_c + Y_s|^2 \cdot 4KT_r R_n$$

F then equals

$$F = 1 + \frac{4KT_r G_u + |Y_c + Y_s|^2 \cdot 4KT_r R_n}{4KT_r G_s}$$

$$= 1 + \frac{G_u + R_n |Y_c + Y_s|^2}{G_s}$$

$$F = 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}\left[(G_s + G_c)^2 + (B_s + B_c)^2\right] \tag{16.45}$$

where

$$Y_S = G_S + jB_S \tag{16.46}$$

and

$$Y_c = G_c + jB_c \tag{16.47}$$

Note that we can reduce noise factor F if

$$B_s = -B_c \tag{16.48}$$

in that case

$$F = 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}(G_s + G_c)^2 \tag{16.49}$$

If we plot F as a function of G_s we see

To find $F_{\min}$, let $\dfrac{dF}{dG_s} = 0$ and solve for G_S. The solution found is the source conductance which, along with the source admittance set at $-B_c$, produces the minimum noise factor. This source conductance is denoted as G_m, and

$$G_m = \left(G_c^2 + \frac{G_u}{R_n} \right)^{1/2} \tag{16.50}$$

and

$$F_{\min} = 1 + 2R_n \left[G_c + \left(G_c^2 + \frac{G_u}{R_n} \right)^{1/2} \right] \tag{16.51}$$

In general F equals

$$F = F_{\min} + \frac{R_n}{G_s} \left[(G_s - G_m)^2 + (B_s - B_m)^2 \right] \tag{16.52}$$

16.7 NOISE CIRCLE

Let $\Gamma_s = \dfrac{Y_o - Y_s}{Y_o + Y_s}$ and $\Gamma_m = \dfrac{Y_o - Y_m}{Y_o + Y_m}$,then

$$F = F_{\min} + 4\frac{R_n}{Z_o} \cdot \frac{|\Gamma_s - \Gamma_m|^2}{|1 + \Gamma_m|^2 \left(1 - |\Gamma_s|^2\right)} \tag{16.53}$$

F completely specified by four parameters: $F_{\min}$, r_n, $|\Gamma_m|$, and $\angle\Gamma_m$, where $r_n = \dfrac{R_n}{Z_o}$, i.e., normalized resistance.

For a specific noise factor $F = F_i$, one can define

$$N_i = \frac{|\Gamma_s - \Gamma_m|^2}{1 - |\Gamma_s|^2} = \frac{F_i - F_{\min}}{4r_n} |1 + \Gamma_m|^2 \tag{16.54}$$

and

$$\left|\Gamma_s - \frac{\Gamma_m}{1+N_i}\right|^2 = \frac{N_i^2 + N_i\left(1-\left|\Gamma_m\right|^2\right)}{\left(1+N_i\right)^2} \tag{16.55}$$

Locust of constant noise factor (or in dB scale, noise figure) is a circle.