CHAPTER
14

GAIN AND STABILITY

14.1 INTRODUCTION

Two port circuits are now considered where $|S_{21}| > 1$ which means that more power exits port 2 than enters port 1. The extra power comes from sources internal to the two port circuit. These extra sources are usually DC power sources which provide a bias for active elements. The active elements are able to transfer DC power into AC or microwave power so that increased power can be delivered to a load. This chapter will consider two fundamental properties of such circuit. The first property is *gain* which has several different definitions appropriate to various applications. Stability, meaning the likelihood that a circuit will oscillate, is important when considering active circuits since a designer is always faced with a trade-off between gain and stability.

It is important to also understand that active circuits can have a reflection coefficieint that exceeds unity. This means that more power power bounces back from a port than was incident upon it. If the reflection coefficient is greater than one, $|\Gamma| > 1$, then $|(Z - Z_o)/(Z + Z_o)| > 1$ and $|(Z - Z_o)/(Z + Z_o)|^2 > 1$ which implies $[(Z - Z_o)/(Z + Z_o)][(Z - Z_o)/(Z + Z_o)]^* > 1$. Cross multiplication results in $(Z - Z_o)(Z^* - Z_o) > (Z + Z_o)(Z^* + Z_o)$ which implies that $0 > Z + Z^*$ or equivaltently, $0 > 2\,\mathrm{Re}\{Z\}$, i.e., the resistance is negative. Therefore, *the reflection coefficient magnitude is greater than unity if and only if the associated impedance has a negative real part.* This is consistent with the Smith chart representation of reflection coefficients. Recall that constant resistance curve for negative values are circle which are outside of the circles associated with positive values. This is shown below in figure 14.1 where the Unit smith Chart is shaded to show the region of positive resistance. The magnitude of the reflection coeficient is less that unity, and hence the name.

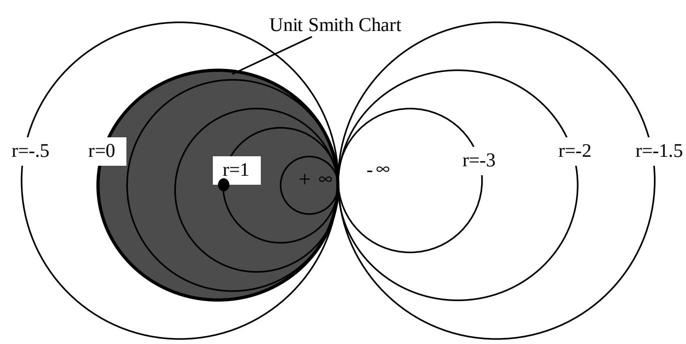


Figure 14.1. Smith Chart circles showing curves of constant resistance for positive and negative resistance.

The Smith chart can also be represented using the stereographic projection of the complex plane onto a sphere having unity diameter. The projection consists of the mapping shown in the figure below where straight line segments connect the north pole of the shere with points on the plane

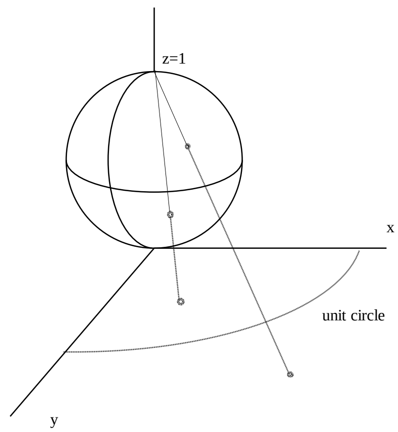


Figure 14.2. Illustration of the stereographic projection of the complex plane.

Note using the stereographic projection one sees that straight lines on the plane map into circles on the sphere which pass through the north pole. From the vantage point of the sphere representation one sees that the points $+\infty$, $-\infty$, $+j\infty$, and $-j\infty$ are the same point, i.e., the north pole. Also it can be shown that circles on the sphere map to circles in the plane. Note that when viewed in terms of the sphere straight line as a special case of a circle. That is *a straight line is a circle which goes throught the point infinity. The set of circles and straight lines is sometimes called generalized circles*. The Unit Smith Chart on the sphere is the southern hmeisphere. Thus a circle in the northern hemisphere maps to a circle outside of the Unit Smith Chart in the plane.

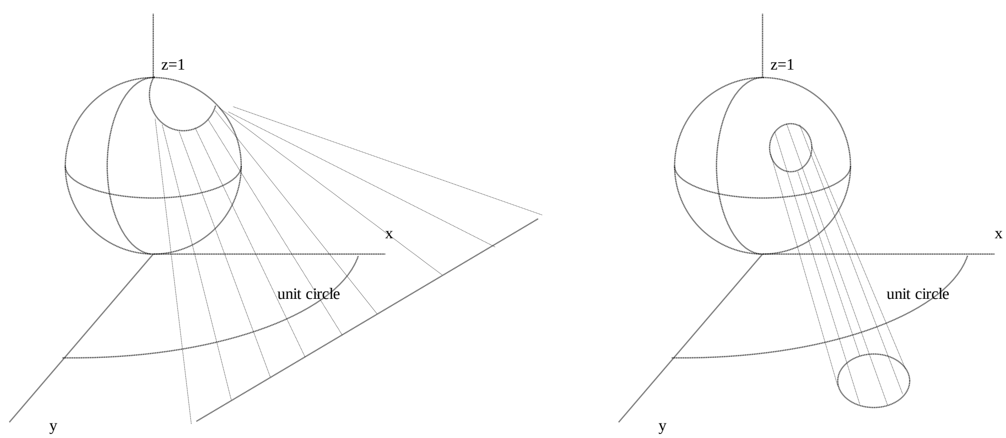


Figure 14.3. Line and circle in the plane map to circles (only) on the sphere..

A bilinear of linear fractional transfromation is a complex function of the form $w = (az+b)/(cz+d) with ad - bc \neq 0$, where $w = u + jv$.

Theorem: *A line fractional transformation maps generalized circles into generalized circles*.

The smith Chart mapping $\Gamma = (Z - Z_o)/(Z + Z_o)$ is a bilinear transformation and illustrates this property as shown in the figures below.

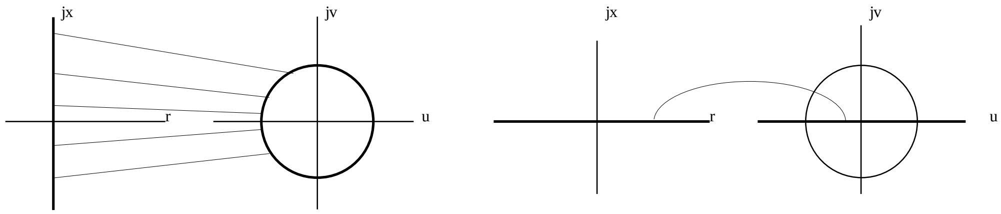


Figure 14.4. The Smith chart mapping illustrates that generalized circles map into generalized circles..

TWO PORT STABILITY AND DEFINITION OF m

New issues arise with active circuits two port circuits. The impedance on one side of the circuit can effect the reflection coeficient on the other side. The load impedance is described in terms of a load reflection coeficient, Γ_L and the source impedance in terms of a source reflection coeficient, Γ_S. The input reflection coeficient, Γ_{IN} is the reflection coeficient seen at the input of a two port circuit with the load connected. The output reflection coeficient, Γ_{OUT}, is the reflection coeficient at the coutput of a two port circuit with the source impedance connected. This is illustrated below:

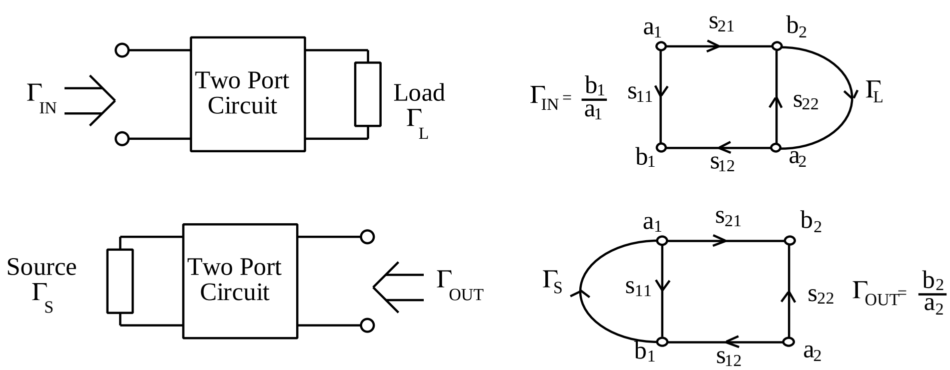


Figure 14.5. Illustration of the input and output reflection coefficient.

$$\Gamma_{IN} = f(\Gamma_L) = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}$$

$$\Gamma_L = f^{-1}(\Gamma_{IN}) = \frac{S_{11} - \Gamma_{IN}}{\Delta - S_{22}\Gamma_{IN}}$$

$$\Gamma_{OUT} = g(\Gamma_S) = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S}$$

$$\Gamma_S = g^{-1}(\Gamma_{OUT}) = \frac{S_{22} - \Gamma_{OUT}}{\Delta - S_{11}\Gamma_{OUT}}$$

where $\Delta = S_{11}S_{22} - S_{12}S_{21}$

Note that the expressions above are all Bilinear or Linear Fractional Transformations. An important question is whether a load (or source) can cause the otherside to have a negative resistance. If one asks what loads result in the input having a passive reflection coefficient then $\left|\Gamma_{IN}\left(\Gamma_L\right)\right| < 1$. Equivalently, what loads make the input look stable. An insight is obtained by asking the question what loads cause the input to be on the border of stable/unstable, i.e., $\left|\Gamma_{IN}\left(\Gamma_L\right)\right| = 1$. This is a circle and therefore the values of Γ_L must be a circle from the property of bilinear transformations. One can think of the problem as a complex transformation from the Γ_{IN} plane to the Γ_L plane. This is illustrated as

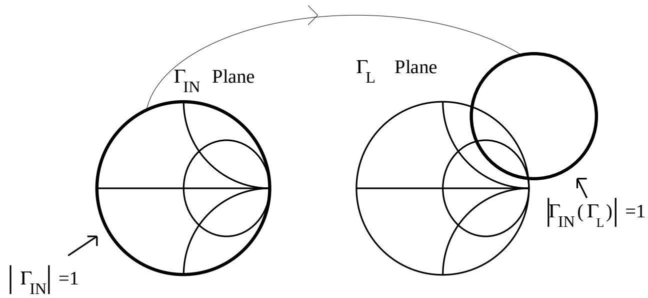


Figure 14.6. Illustration of a possible mapping of the circle $\left|\Gamma_{IN}\right| = 1$ Γ_{L}-plane.

The load plane is divided into two regions by a circle (generalized). However, it is not clear, though, how the unit disk in the Γ_{IN} plane is mapped in the Γ_L plane. The unit disk may have mapped to the inside of the circle in the Γ_L plane or to the out side. The region outside of the circle is called a disk complement. These two possiblities illustrated below:

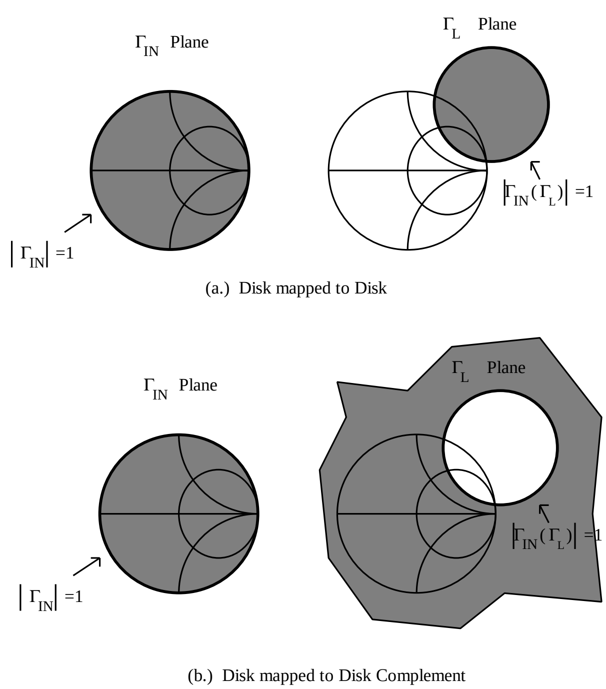
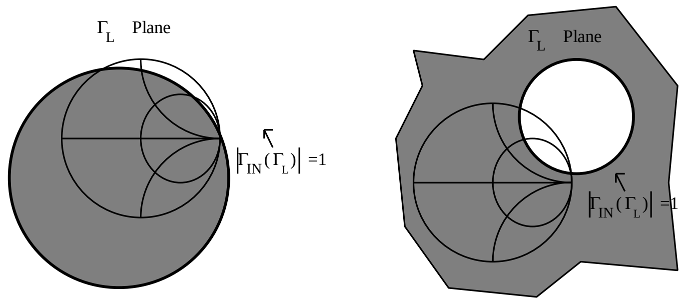


Figure 14.7. **(a.)** Illustration of the unit disk mapped to a disk. **(b.)** Illustration of the unit disk maped to a disk complement.

The circle in the load plane is referred to as the *load stability circle* since it show the boundary of possible load which result in $\Gamma_{IN} = 1$. The circle in the load stability circle depends only upon the S-parameters of the two port. Based on this it is clear that care must be taken when connecting a load to a two port circuit since a passive load may cause the input impedance to have a negative resistance implying that the circuit may be vulnerable to oscillations. In general the load stability circle and stable region for transistor look like one of the following cases:



(a.) **(b.)**

Figure 14.8. **(a.)** Example showing stable region as disk in the load plane. **(b.)** Example showing stable region as a disk complement in the load plane.

Assuming that the center of the Smith chart is stable then a nautural measure of stability is defined by determining the distance from the center of the Smith chart to the nearest unstable point. **This length is called** m

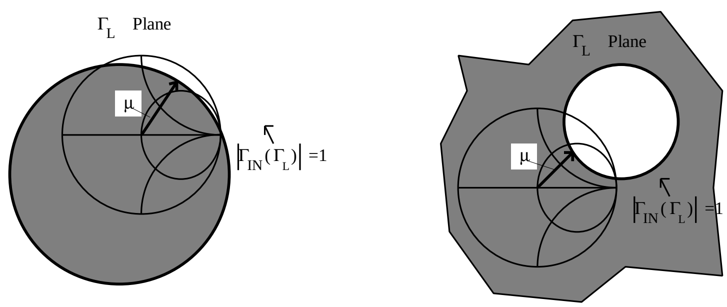


Figure 14.9. Illustration of the definition of the parameter m .

An important conclusion follows from the above:

If m>1 then any passive load is ok.

In this case the two port circuit is said to be unconditionally stable. Otherwise the two port is said to be potentially unstable, or only conditionally stable.

A convention that is very useful in drawing stability circles is to indicate the stable region using tick marks. The above cases would look like

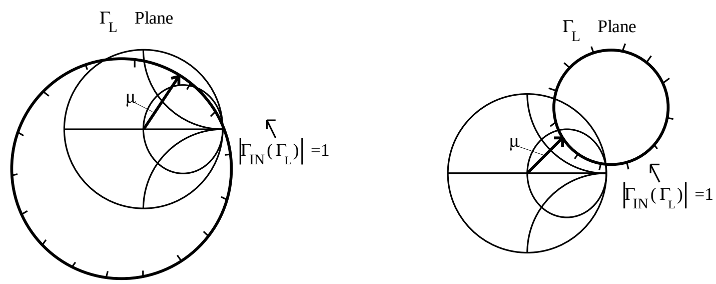


Figure 14.10. Illustration of the convention of using "tick marks"to designate the stable region determned by the stability circle.

μ is found to be

$$\boxed{m = \frac{1-|S_{11}|^2}{|S_{22} - S_{11}^{*}\Delta| + |S_{12}S_{21}|}}$$

To facilitate the theoretical development, the following algebraic combinations of *S*-parameters are defined.

$$B_1 = D_1 + E_1$$
$$B_2 = D_2 + E_2$$

$$C_1 = S_{11} - \Delta S_{22}^{*}$$
$$C_2 = S_{22} - \Delta S_{11}^{*}$$

$$D_1 = |S_{11}|^2 - |\Delta|^2$$
$$D_2 = |S_{22}|^2 - |\Delta|^2$$

$$E_1 = 1 - |S_{22}|^2$$
$$E_2 = 1 - |S_{11}|^2$$

$$k = \frac{E_1 - D_1}{2|S_{12}S_{21}|} = \frac{E_2 - D_2}{2|S_{12}S_{21}|}$$

Additionally, the following easily verified relationship turns out to be useful.

$$|C_1|^2 = |S_{12}S_{21}|^2 + D_1E_1$$
$$|C_2|^2 = |S_{12}S_{21}|^2 + D_2E_2$$

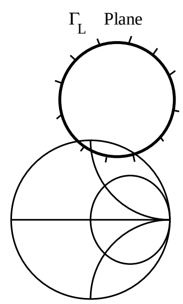
The stable region in the source plane, whose boundary is the load stability circle, is also referred to as the *input stable region in the load plane* since it defines the Γ_L values that result in $|\Gamma_{IN}(\Gamma_L)| < 1$. Using (3), this results in region which is either a disk or disk complement. In either case the boundary is the circle with radius, r_L, and center, C_L, given by

$$r_L = \left|\frac{S_{12}S_{21}}{D_2}\right|$$

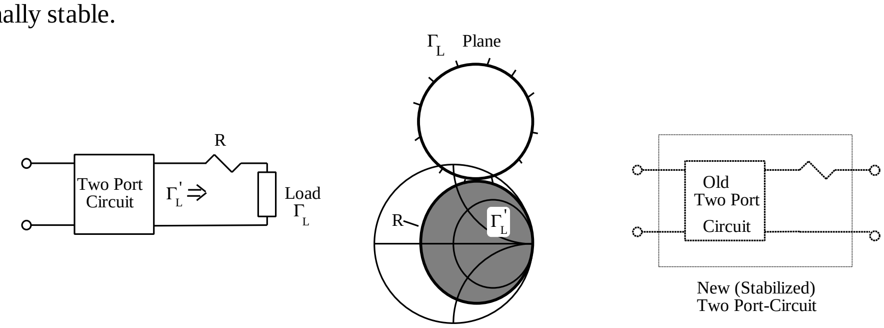
$$C_L = \frac{C_2^*}{D_2}$$

If $D_2 < 0$ then the stable region is the "disk", $\left|\Gamma_L - C_L\right| < r_L$, whereas, if $D_2 > 0$ then the stable region is the "disk complement", $\left|\Gamma_L - C_L\right| > r_L$. The subscript "2" is often used to designate the right side of the circuit and "1" the left side.

Note that if a two port circuit is potentially unstable then it can be stabilized by adding resistors to the circuit. For example suppose a two port circuit produces the following load stability circle

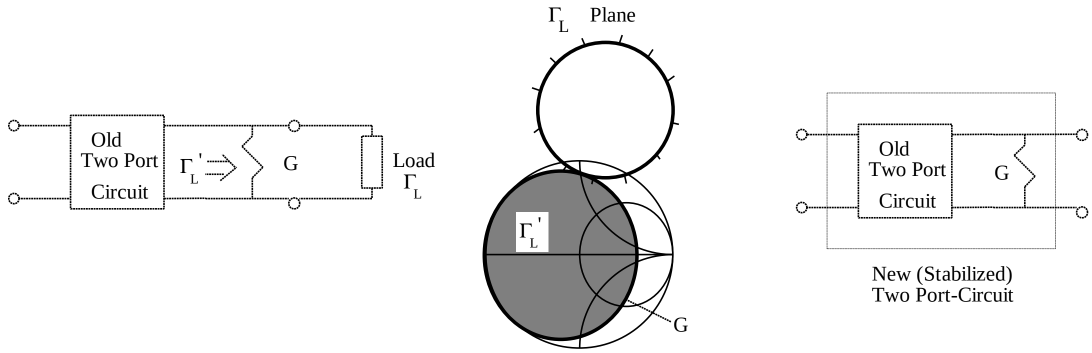


If a series resistor is insert with the load as shown and if the reistance is chosen to equal to or greater than that of the constant resistance circle shown then the new two port which includes the series resistor will be unconditionally stable.



(a.) **(b.)** **(c.)**

If μ is dtermined for the new two-port circuit then it will be found that μ>1, i.e., the new circuit is uncondtionally stable. Note that a shunt resistor could have been used also to stablize the circuit. In this case it value would have been determined by choosing a conductance which is equal to exceeds that of the constant conductance circle in the Smith chart as shown. The circuit would look like that shown in the figure below. Again μ>1 for the new two port circuit.



In all of the previous discussion the role of the source can be interchanged with the load if the input is interchanged with the output. Therefore one can examine what happens to the output for different source impedances. Source stability circles are similarly defined and the distance to the nearest unstable point is denoted as μ'. The circuit can be similarly stabilized by using series or shunt resistors whose values can be determined analogously in the source plane. In this case

$$m' = \frac{1 - |S_{22}|^2}{|S_{11} - S_{22}^* \Delta| + |S_{12} S_{21}|}$$

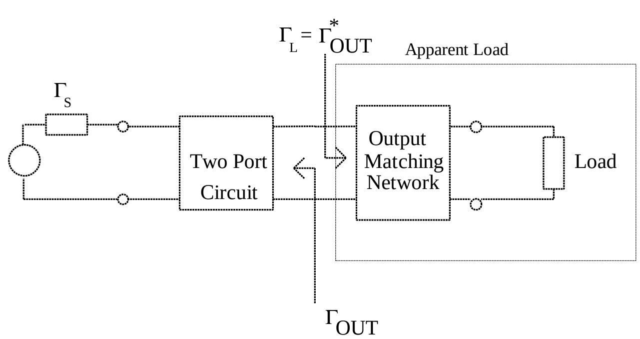
Important Theorem:

m>1 if and only if m'>1

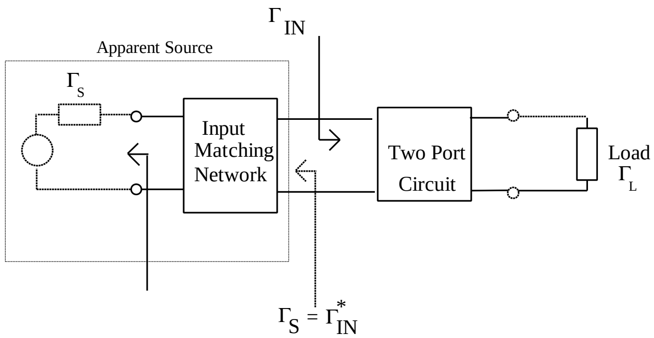
In genral a two port circuit can be stablized at the source or load side (or even a combination)

MAXIMUM POWR TRANSFER AND CONJUGATE MATCH

Given a voltage source with a source impedance Z_S then maximum power is transfered or deliver to the load when it has an impedance equal to $Z_L = Z_S^*$. In this case the load and source are said to conjugately matched. In terms of reflection coefficients and and the **conjugate match condition** is $\Gamma_L = \Gamma_S^*$. If a two port circuit is connected to a source then the load can be chosen so that $\Gamma_L = \Gamma_{OUT}^*$ and we know that the maximum power from the combined source and two-port is delivered to the load. In the case where the load is pre-specified and thereby constrained and different from Γ_{OUT}^* then one can create a lossless matching network so that the load appears to be conjugately matched. This is illustrated below



In a similar manner if a load is connected to two-port circuit then Γ_{IN} can be examined to see if it is conjuagetly matched to the source. If not then a matching network could be inserted so that the matching condition is met. This is illustrated by



PROBLEM: In general you can not do both of the above matching procedures since the input and load are related and the output and the source are related. The following theorems lead to an understanding of when the input and output of a two port circuit can be simultaneously matched.

Theorem: *m>1 (unconditionally stable) if and only if k>1 and* $B_1 > 0$.

Theorem: *m>1 (unconditionally stable) if and only if k>1 and* $B_2 > 0$

Theorem: *m>1 (unconditionally stable) if and only if k>1 and* $|\Delta| < 1$

Theorem: *m>1 (unconditionally stable) if and only if k>1 and* $1-|S_{11}|^2 > |S_{12}S_{21}|$

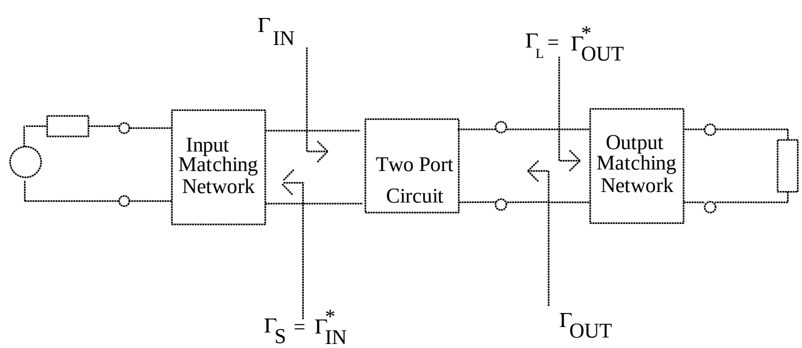
Theorem: *m>1 (unconditionally stable) if and only if k>1 and* $1-|S_{22}|^2 > |S_{12}S_{21}|$

Since μ'>1 when μ>1 then all of the above could have been stated in terms of μ'>1.

Theorem: (Simultaneous Conjugate Match)

If m>1 (two port uncondtionally stable) then it is always possible to find a source and load combination such that the input and output are simultaneously matched

In that case you can design an amplifier whose input and output are matched. The canonical form is



To find the source and load reflection coefficient values for a simultaneous conjugate match the following equations must be solved

$$\Gamma_S = \Gamma_{IN}^*$$
$$\Gamma_L = \Gamma_{OUT}^*$$

$$\Gamma_S^* = \Gamma_{IN}$$
$$\Gamma_L^* = \Gamma_{OUT}$$

$$\Gamma_S^* = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L} = \frac{S_{11}-\Delta\Gamma_L}{1-S_{22}\Gamma_L}$$
$$\Gamma_L^* = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1-S_{11}\Gamma_S} = \frac{S_{22}-\Delta\Gamma_S}{1-S_{11}\Gamma_S}$$

Clearing the denominator

$$(1-S_{22}\Gamma_L)\Gamma_S^* = S_{11} - \Delta\Gamma_L$$

$$\Gamma_L = \frac{S_{22}^* - \Delta^*\Gamma_S^*}{1-S_{11}^*\Gamma_S^*}$$

$$\Gamma_S^* - S_{22}\Gamma_S^*\left(\frac{S_{22}^* - \Delta^*\Gamma_S^*}{1-S_{11}^*\Gamma_S^*}\right) = S_{11} - \Delta\left(\frac{S_{22}^* - \Delta^*\Gamma_S^*}{1-S_{11}^*\Gamma_S^*}\right)$$

$$\Gamma_S^*\left(1-S_{11}^*\Gamma_S^*\right) - S_{22}\Gamma_S^*\left(S_{22}^* - \Delta^*\Gamma_S^*\right) = S_{11}\left(1-S_{11}^*\Gamma_S^*\right) - \Delta\left(S_{22}^* - \Delta^*\Gamma_S^*\right)$$

$$\Gamma_S\left(1-S_{11}\Gamma_S\right) - S_{22}^*\Gamma_S\left(S_{22} - \Delta\Gamma_S\right) = S_{11}^*\left(1-S_{11}\Gamma_S\right) - \Delta^*\left(S_{22} - \Delta\Gamma_S\right)$$

$$\Gamma_S - S_{11}\Gamma_S^2 - |S_{22}|^2\Gamma_S + S_{22}^*\Delta\Gamma_S^2 = S_{11}^* - |S_{11}|^2\Gamma_S - \Delta^*S_{22} + |\Delta|^2\Gamma_S$$

$$-S_{11}\Gamma_S^2 + S_{22}^*\Delta\Gamma_S^2 + \Gamma_S - |S_{22}|^2\Gamma_S + |S_{11}|^2\Gamma_S - |\Delta|^2\Gamma_S - S_{11}^* + \Delta^*S_{22} = 0$$

$$\left(-S_{11} + S_{22}^*\Delta\right)\Gamma_S^2 + \left(1 - |S_{22}|^2 + |S_{11}|^2 - |\Delta|^2\right)\Gamma_S - \left(S_{11}^* - \Delta^*S_{22}\right) = 0$$

$$-C_1\Gamma_S^2 + (E_1 + D_1)\Gamma_S - C_1^* = 0$$

$$-C_1\Gamma_S^2 + B_1\Gamma_S - C_1^* = 0$$

$$\Gamma_S = \frac{B_1 \pm \sqrt{B_1^2 - 4|C_1|^2}}{2C_1}$$

Similarly,

$$\Gamma_L = \frac{B_2 \pm \sqrt{B_2^2 - 4|C_2|^2}}{2C_2}$$

$$\Gamma_S^+ = \frac{B_1 + \sqrt{B_1^2 - 4|C_1|^2}}{2C_1}$$

$$\Gamma_S^- = \frac{B_1 - \sqrt{B_1^2 - 4|C_1|^2}}{2C_1}$$

Note that

$$|\Gamma_S^+||\Gamma_S^-| = 1$$

One solution is seen to be inside the USC and one outside

Substitution shows that

$$B_1^2 - 4|C_1|^2 = B_2^2 - 4|C_2|^2 = 4|S_{21}S_{21}|^2(k^2 - 1)$$

Therefore, μ>1 implies k>1 and

$$B_1^2 - 4|C_1|^2 > 0$$
$$B_2^2 - 4|C_2|^2 > 0$$

$$|\Gamma_S^+|^2 - |\Gamma_S^-|^2 = \frac{B_1\sqrt{B_1^2 - 4|C_1|^2}}{|C_1|^2} > 0$$

since μ>1 also implies $B_1 > 0$. Therefore

$$|\Gamma_S^+|^2 > |\Gamma_S^-|^2$$

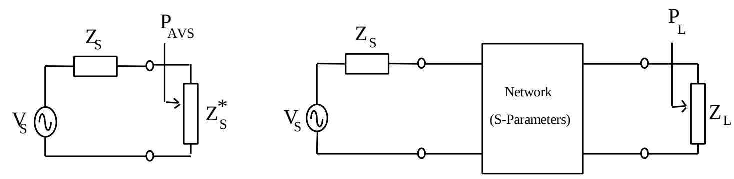
and Γ_S^- is the desired match. A similar analysis shows that the desired match for the load is Γ_L^-. The simultaneous match reflection coefficients is therefore

$$\boxed{\begin{aligned} \Gamma_{MS} &= \frac{B_1 - \sqrt{B_1^2 - 4|C_1|^2}}{2C_1} \\ \Gamma_{ML} &= \frac{B_2 - \sqrt{B_2^2 - 4|C_2|^2}}{2C_2} \end{aligned}}$$

TRANSDUCER, AVAILABLE, AND OPERATING POWER GAINS

The following gains are defined

Transducer gain = Power delivered to the load / Available power from source



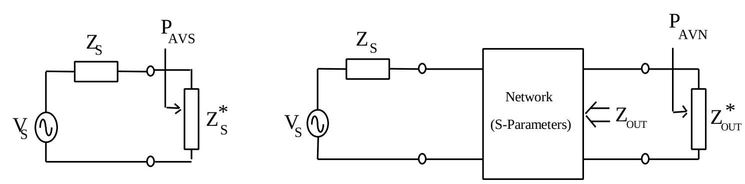
$$G_T = \frac{P_L}{P_{AVS}}$$

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left|(1-S_{11}\Gamma_S)(1-S_{22}\Gamma_L)-S_{12}S_{21}\Gamma_S\Gamma_L\right|^2}$$

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{|(1-S_{11}\Gamma_S)|^2|1-\Gamma_{OUT}\Gamma_L|^2}$$

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{|(1-\Gamma_{IN}\Gamma_S)|^2|1-S_{22}\Gamma_L|^2}$$

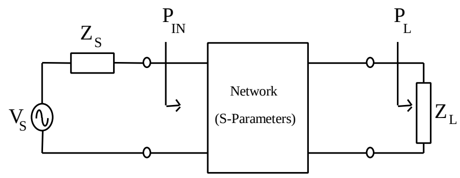
Available gain = power available from network / power available from source



$$G_A = \frac{P_{AVN}}{P_{AVS}}$$

$$G_A = \frac{\left(1-|\Gamma_S|^2\right)|S_{21}|^2}{|(1-S_{11}\Gamma_S)|^2\left(1-|\Gamma_{OUT}|^2\right)}$$

Operating Power gain = Power delivered to load / Power into the network



$$G_P = \frac{P_L}{P_{IN}}$$

$$G_P = \frac{|S_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left(1-|\Gamma_{IN}|^2\right)|1-S_{22}\Gamma_L|^2}$$

Note:

$$G_T = G_T([S], \Gamma_S, \Gamma_L)$$

$$G_A = G_A([S], \Gamma_S) = G_T\left([S], \Gamma_S, \Gamma_{OUT}^*\right)$$

$$G_P = G_P([S], \Gamma_L) = G_T\left([S], \Gamma_{IN}^*, \Gamma_L\right)$$

$$G_A = \frac{P_L}{P_{AVS}} \cdot \frac{P_{AVN}}{P_L} = G_T\left(\frac{P_{AVN}}{P_L}\right) \ge G_T$$

$$G_P = \frac{P_L}{P_{AVN}} \cdot \frac{P_{AVN}}{P_{IN}} = G_T\left(\frac{P_{AVN}}{P_{IN}}\right) \ge G_T$$

DERIVATION OF GAIN FORMULAS

In general the power delivered by a source to a load is

$$a_1 = b_s + \Gamma_s b_1$$
$$b_2 = a_1$$
$$b_1 = a_2$$
$$a_2 = \Gamma_L b_2$$

$$(b_2) = b_s + \Gamma_s(\Gamma_L b_2)$$

$$b_2 = \frac{b_s}{1-\Gamma_s\Gamma_L}$$

$$P_L = |b_2|^2\left(1-|\Gamma_L|^2\right)$$

$$P_L = \frac{|b_S|^2\left(1-|\Gamma_L|^2\right)}{|1-\Gamma_S\Gamma_L|^2}$$

If the load is conjugately matched to the source then $P_L = P_{AVS}$ and

$$P_{AVS} = \frac{|b_S|^2}{1-|\Gamma_S|^2}$$

(First) Transducer Gain Formula

$$a_1 = b_s + \Gamma_s b_1$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$a_2 = \Gamma_L b_2$$

Solve for b_2

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} b_s + \Gamma_s b_1 \\ \Gamma_L b_2 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \left[\begin{pmatrix} b_s \\ 0 \end{pmatrix} + \begin{pmatrix} \Gamma_s b_1 \\ \Gamma_L b_2 \end{pmatrix} \right]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \Gamma_s b_1 \\ \Gamma_L b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} b_s \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} S_{11}\Gamma_s & S_{12}\Gamma_L \\ S_{21}\Gamma_s & S_{22}\Gamma_L \end{pmatrix} \right] \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} b_s \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-S_{11}\Gamma_s & -S_{12}\Gamma_L \\ -S_{21}\Gamma_s & 1-S_{22}\Gamma_L \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_s \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}$$

$$inv\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{b_s \begin{pmatrix} 1-S_{22}\Gamma_L & S_{12}\Gamma_L \\ S_{21}\Gamma_s & 1-S_{11}\Gamma_s \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}}{\det\begin{pmatrix} 1-S_{11}\Gamma_s & -S_{12}\Gamma_L \\ -S_{21}\Gamma_s & 1-S_{22}\Gamma_L \end{pmatrix}}$$

$$b_2 = \frac{b_s\left(S_{11}S_{21}\Gamma_s + (1-S_{11}\Gamma_s)S_{21}\right)}{(1-S_{11}\Gamma_s)(1-S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_s\Gamma_L}$$

$$b_2 = \frac{b_s S_{21}}{(1-S_{11}\Gamma_s)(1-S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_s\Gamma_L}$$

Equivalent expression

$$b_2 = \frac{bsS_{21}}{\left[(1 - S_{11}\Gamma_s) - \frac{S_{21}S_{12}\Gamma_s\Gamma_L}{1 - S_{22}\Gamma_L}\right](1 - S_{22}\Gamma_L)}$$

$$b_2 = \frac{bsS_{21}}{\left[1 - \left(S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1 - S_{22}\Gamma_L}\right)\Gamma_s\right](1 - S_{22}\Gamma_L)}$$

$$b_2 = \frac{bsS_{21}}{(1 - \Gamma_{IN}\Gamma_s)(1 - S_{22}\Gamma_L)}$$

Similarly,

$$b_2 = \frac{bsS_{21}}{(1 - S_{11}\Gamma_s)(1 - \Gamma_{OUT}\Gamma_L)}$$

Each of the expressions for b_2 can be substituted into the following to obtain an expression for the power delivered to the load, i.e.,

$$P_L = |b_2|^2\left(1 - |\Gamma_L|^2\right)$$

Therefore,

$$P_L = \frac{|b_S|^2|S_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_S\Gamma_L|^2}$$

Dividing by the power available from the source gives

$$G_T = \frac{\left(1 - |\Gamma_S|^2\right)|S_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_S\Gamma_L|^2} \qquad \text{Q.E.D.}$$

(Second) Transducer Gain Formula

The second transducer gain formula can be derived by factoring the term $(1 - S_{11}\Gamma_S)$ from the denominator of the above formula substituting the output reflection coefficent expression

$$\Gamma_{OUT} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S}$$

to get

$$\mathrm{G_T} = \frac{\left(1 - |\Gamma_S|^2\right)|S_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|(1 - S_{11}\Gamma_S)|^2|1 - \Gamma_{OUT}\Gamma_L|^2}$$

This expression can also be obtained from the following. The over all circuit can be viewed as a load connected to a Thevenin Equivalent source which consists of the original source cascaded with the two port. This new source would have a reflection coefficient and independent source parameter b'_S. The new parameters can be obtained as follows

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$a_1 = b_s + \Gamma_s b_1$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} b_s + \Gamma_s b_1 \\ a_2 \end{pmatrix}$$

$$b_1 = S_{11}(b_s + \Gamma_s b_1) + S_{12} a_2$$
$$b_2 = S_{21}(b_s + \Gamma_s b_1) + S_{22} a_2$$

$$b_1(1 - S_{11}\Gamma_s) = S_{11} b_s + S_{12} a_2$$

$$b_1 = \frac{S_{11} b_s + S_{12} a_2}{1 - S_{11}\Gamma_s}$$

$$b_2 = S_{21}\left[b_s + \Gamma_s \left(\frac{S_{11} b_s + S_{12} a_2}{1 - S_{11}\Gamma_s} \right) \right] + S_{22} a_2$$

$$b_2 = S_{21} b_s + \frac{\Gamma_s S_{11}}{1 - S_{11}\Gamma_s} b_s + \frac{S_{12} S_{21} a_2}{1 - S_{11}\Gamma_s} + S_{22} a_2$$

$$b_2 = \frac{S_{21}}{1 - S_{11}\Gamma_s} b_s + \left(S_{22} + \frac{S_{12} S_{21}}{1 - S_{11}\Gamma_s} \right) a_2$$

$$b_2 = b'_s + \Gamma_{OUT} a_2$$

$$P_L = \frac{|b'_S|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - \Gamma_{OUT}\Gamma_L|^2}$$

$$b'_S = \frac{b_S S_{21}}{1 - S_{11}\Gamma_S}$$

and

$$\Gamma_{OUT} = S_{22} + \frac{S_{12} S_{21} \Gamma_S}{11 - S_{11}\Gamma_S}$$

$$P_L = \frac{|b'_S|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - \Gamma_{OUT}\Gamma_L|^2}$$

$$P_L = \frac{|b_S|^2 |S_{21}|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - S_{11}\Gamma_S|^2 |1 - \Gamma_{OUT}\Gamma_L|^2}$$

$$P_{AVS} = \frac{|b_S|^2}{1 - |\Gamma_S|^2}$$

$$G_T = \frac{P_L}{P_{AVS}} = \frac{\left(1 - |\Gamma_S|^2\right)|S_{21}|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - S_{11}\Gamma_S|^2 |1 - \Gamma_{OUT}\Gamma_L|^2}$$

Note also that the available power from the network is

$$P_{AVN} = \frac{|b'_S|^2}{\left(1 - |\Gamma_{OUT}|^2\right)} = \frac{b_S S_{21}}{\left(1 - |\Gamma_{OUT}|^2\right)\left(1 - S_{11}\Gamma_S\right)}$$

(Third) Formula for Transducer Gain

Alternatively, the power delivered to the load can be found by examining the linear two port circuit from the input. First one finds the in signal parameter a_1 in terms of b_S by recognizing that the effects of the remainder of the circuit+load are represented by the input reflection coefficient. Therefore,

$$a_1 = \frac{b_S}{1 - \Gamma_S \Gamma_{IN}}$$

The next step is to find the relationship between b_2 and a_1 which can be found from the transfer ratio analysis

$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$b_2 = S_{21}a_1 + S_{22}\Gamma_L b_2$$

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L} a_1$$

$$P_L = |b_2|^2 \left(1 - |\Gamma_L|^2\right)$$

$$P_L = \frac{|S_{21}|^2 |a_1|^2 \left(1-|\Gamma_L|^2\right)}{|1-S_{22}\Gamma_L|^2} = \frac{|b_S|^2 |S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{|1-S_{22}\Gamma_L|^2 |1-\Gamma_S\Gamma_{IN}|^2}$$

$$G_T = \frac{P_L}{P_{AVS}} = \frac{\left(1-|\Gamma_S|^2\right)|S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{|1-S_{22}\Gamma_L|^2 |1-\Gamma_S\Gamma_{IN}|^2} \qquad \text{Q.E.D.}$$

Note that

$$P_{IN} = \frac{|b_S|^2 \left(1-|\Gamma_{IN}|^2\right)}{|1-\Gamma_S\Gamma_{IN}|^2}$$

Available Gain Formula

$$G_A = \frac{P_{AVN}}{P_{AVS}}$$

Substitution from above

$$P_{AVS} = \frac{|b_S|^2}{1-|\Gamma_S|^2}$$

$$P_{AVN} = \frac{|b_S|^2 |S_{21}|^2}{\left(1-|\Gamma_{OUT}|^2\right)\left(1-S_{11}\Gamma_S\right)}$$

$$\mathrm{G}_{\mathrm{A}} = \frac{P_{AVN}}{P_{AVS}} = \frac{\left(1-|\Gamma_{\mathrm{S}}|^2\right)|S_{21}|^2}{|(1-S_{11}\Gamma_S)|^2 \left(1-|\Gamma_{OUT}|^2\right)} \qquad \text{Q.E.D.}$$

Power Gain Formula

$$P_{IN} = \frac{|b_S|^2 \left(1-|\Gamma_{IN}|^2\right)}{|1-\Gamma_S\Gamma_{IN}|^2}$$

$$P_L = \frac{|b_S|^2 |S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{|1-S_{22}\Gamma_L|^2 |1-\Gamma_S\Gamma_{IN}|^2}$$

$$G_P = \frac{|S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{\left(1-|\Gamma_{IN}|^2\right)|1-S_{22}\Gamma_L|^2} \qquad \text{Q.E.D.}$$

Maximum Transducer Gain for (m>1) Unconditionally stable two-port

For an unconditionally stable device the maximum transducer gain will occur for a source and load whose impedance equates to the simultaneous conjugate match conditions, i.e.,

$$\Gamma_S = \Gamma_{MS}$$
$$\Gamma_L = \Gamma_{ML}$$

$$\mathrm{G}_{\mathrm{T,MAX}} = \frac{\left(1-|\Gamma_{\mathrm{MS}}|^2\right)|S_{21}|^2\left(1-|\Gamma_{ML}|^2\right)}{\left|(1-S_{11}\Gamma_{MS})(1-S_{22}\Gamma_{ML})-S_{12}S_{21}\Gamma_{MS}\Gamma_{ML}\right|^2}$$

$$\mathrm{G}_{\mathrm{T,MAX}} = \frac{|S_{21}|}{|S_{12}|}\left(k-\sqrt{k^2-1}\right)$$

Theorem: μ=1 implies k=1.

Therefore for an amplifier that is just barely stable the gain equals

$$\mathrm{G}_{\mathrm{T,MAX}} = \frac{|S_{21}|}{|S_{12}|}\left(k-\sqrt{k^2-1}\right) with k = 1$$

$$\mathrm{G}_{\mathrm{MSG}} = \frac{|S_{21}|}{|S_{12}|}$$

Note

$$\mathrm{G}_{\mathrm{MSG}} \geq G_{T,MAX}$$

Gain Circles (Available Gain)

$$\mathrm{G}_{\mathrm{A}} = \frac{\left(1-|\Gamma_{\mathrm{S}}|^2\right)|S_{21}|^2}{|(1-S_{11}\Gamma_S)|^2\left(1-|\Gamma_{OUT}|^2\right)}$$

$$\mathrm{G}_{\mathrm{A}} = \frac{\left(1-|\Gamma_{\mathrm{S}}|^2\right)|S_{21}|^2}{|1-S_{11}\Gamma_S|^2-|S_{22}-\Delta\Gamma_S|^2}$$

$$\mathrm{G}_{\mathrm{A}} = g_a|S_{21}|^2$$

$$\mathrm{g}_a = \frac{\left(1-|\Gamma_{\mathrm{S}}|^2\right)}{|1-S_{11}\Gamma_S|^2-|S_{22}-\Delta\Gamma_S|^2}$$

$$|1-S_{11}\Gamma_S|^2-|S_{22}-\Delta\Gamma_S|^2\mathrm{g}_a = 1-|\Gamma_{\mathrm{S}}|^2$$

$$(1-S_{11}\Gamma_S)(1-S_{11}\Gamma_S)^*\mathrm{g}_a-(S_{22}-\Delta\Gamma_S)(S_{22}-\Delta\Gamma_S)^*\mathrm{g}_a = 1-|\Gamma_{\mathrm{S}}|^2$$

$$1 - S_{11}\Gamma_S - S_{11}^*\Gamma_S^* + |S_{11}|^2|\Gamma_S|^2 - |S_{22}|^2 + S_{22}\Delta^*\Gamma_S^* + S_{22}^*\Delta\Gamma_S - |\Delta|^2|\Gamma_S|^2 = \frac{1-|\Gamma_S|^2}{g_a}$$

$$\left(|S_{11}|^2 - |\Delta|^2 + \frac{1}{g_a}\right)|\Gamma_S|^2 - \left(S_{11} - S_{22}^*\Delta\right)\Gamma_S - \left(S_{11} - S_{22}^*\Delta\right)^*\Gamma_S^* + 1 - |S_{22}|^2 - \frac{1}{g_a} = 0$$

$$\left(D_1 + \frac{1}{g_a}\right)|\Gamma_S|^2 - C_1\Gamma_S - C_1\Gamma_S^* = \frac{1}{g_a} - E_1$$

$$C_{ga} = \frac{C_1^*}{D_1 + \frac{1}{g_a}}$$

$$r_{ga} = \left|\frac{\left[1 - 2k|S_{12}S_{21}|g_a + |S_{12}S_{21}|^2 g_a^2\right]^{1/2}}{1 + g_a D_1}\right|$$

Nested circles

Peak gain occurs when

$$r_{ga} = 0$$

Occurs when

$$1 - 2k|S_{12}S_{21}|g_a + |S_{12}S_{21}|^2 g_a^2 = 0$$

$$g_a = \frac{1}{|S_{12}S_{21}|}\left(k - \sqrt{k^2 - 1}\right)$$

$$G_{A,MAX} = \frac{|S_{21}|}{|S_{12}|}\left(k - \sqrt{k^2 - 1}\right)$$