Clarification

Last time we saw that the electric field $\vec{E}$ is <u>conservative</u>: $\oint \vec{E}\cdot d\vec{r} = 0$. This allows us to define an electric potential $\phi(\vec{r})$ such that

$$W = q\int_{r_A}^{r_B} \vec{E}\cdot d\vec{r} = -q(\phi(\vec{r}_B) - \phi(\vec{r}_A))$$

Since $W = \Delta K$ (change in kinetic energy), we find $K_B - K_A = -q(\phi(\vec{r}_B) - \phi(\vec{r}_A))$, or

$$K_B + q\phi(\vec{r}_B) = K_A + q\phi(\vec{r}_A).$$

Therefore $E = K + q\phi$ is <u>conserved</u>.

$U(\vec{r}) = q\,\phi(\vec{r})$ is the electrostatic (potential) energy.

1. 4. Gauss' Law

states that $\oint_{\partial V} \vec{E}\cdot d\vec{A} = 4\pi Q_{int}$.

Often, it is simpler to use Gauss' law (instead of Coulomb's) to calculate the electric field: (Jump to example on page 5)

Exercise 1

Calculate the electric field created by

i) A uniformly charged wire

ii) A uniformly charged sheet

1.5 The variation of $\vec{E}$

We have seen that $\vec{E} = -\vec{\nabla}\phi$, and $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$. Putting both together we find

$$\vec{\nabla}\cdot\vec{\nabla}\phi \equiv \nabla^2\phi \equiv \Delta\phi = -4\pi\rho$$

$$\Delta\phi = -4\pi\rho$$

is Poisson's equation. In vacuum, $\Delta\phi = 0$.

Note that in a cartesian coordinate system,

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 .$$

Curl

given an arbitrary vector field $\vec{E}$, we define its curl, $\vec{\nabla} \times \vec{E}$ ($\mathrm{rot}\,\vec{E}$, $\mathrm{curl}\,\vec{E}$) by the equation

$$(\vec{\nabla} \times \vec{E})_i = \epsilon_{ijk}\, \partial_j E_k$$

where ϵ_{ijk} is totally antisymmetric, and $\epsilon_{123} = +1$.

We use Einstein's summation convention throughout.

The curl of a gradient is zero,

$$\vec{\nabla} \times \vec{\nabla} \phi = 0$$

Exercise 2

Show that $\vec{\nabla} \times \vec{\nabla} \phi = 0$ and $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

Hint: use the ϵ symbol!

Stokes' theorem states that for an arbitrary (smooth) vector field $\vec{E}$,

$$\int_A \vec{\nabla}\times\vec{E}\cdot d\vec{A} = \oint_{\partial A} \vec{E}\cdot d\vec{r}$$

$\uparrow$ surface integral $\qquad\qquad$ $\uparrow$ path integral.

In particular, since $\vec{\nabla}\times\vec{E} = 0$ (because $\vec{E} = -\vec{\nabla}\phi$)

$$\oint_{\partial A} \vec{E}\cdot d\vec{r} = 0$$

, as we previously saw.

To summarize:

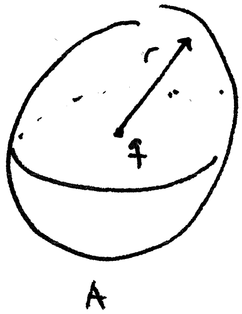
$$\begin{tikzcd}[column sep=large]
\underset{\text{conservative}}{\oint \vec{E}\cdot dr = 0} \arrow[rr, shift left] \arrow[dr, shift left] & & \underset{\text{derivable from a potential}}{\vec{E} = -\vec{\nabla}\phi} \arrow[ll, shift left] \arrow[dl, shift right] \\
& \underset{\text{irrotational}}{\vec{\nabla}\times\vec{E} = 0} \arrow[ul, shift left] \arrow[ur, shift right, "\text{locally}"', "\text{(Poincaré lemma)}"] &
\end{tikzcd}$$

Example

Consider the field of a point charge q,

$$\rho = q\,\delta(\vec{r}).$$

Choosing a spherical surface centered at q:



$$\oint \vec{E}\cdot d\vec{A} = 4\pi r^2 E \overset{!}{=} 4\pi q$$

gauss' law.

Hence, $E = \dfrac{q}{r^2}$.

On the other hand, we know that the electric field created by a point charge is

$$\vec{E} = \frac{q}{|\vec{r}-\vec{r}'|}(\vec{r}-\vec{r}')$$

(Coulomb). Since $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ it must be that

$$\vec{\nabla}_r\cdot\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right) = 4\pi\,\delta(\vec{r}-\vec{r}'),$$

as we previously claimed

Exercise 2

Calculate the electric field created by a uniformly charged ring of radius R.

■

Exercise 3

Find the electric field created by a uniformly charged solid sphere

■

Exercise 4

Find the charge distribution that produces the potential $\phi(r) = q \dfrac{e^{-\mu r}}{r}$

(this is known as the Yukawa potential)

■

2. Developments in electrostatics

2.1. Conductors

By definition, conductors contain charges that are able to move freely (typically e^-).

As a consequence, if an electric field is applied to a conductor, charges will rearrange themselves until $\vec{E} = 0$. Therefore,

1. $\vec{E} = 0$ inside a conductor

$$\downarrow \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

2. There are no charges inside a conductor. Excess charges must be on the surface.

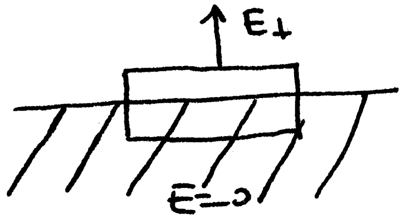
$$\downarrow \quad \vec{E} = 0 \text{ inside}$$

3. The entire conductor has the same electric potential

$$\downarrow$$

4. $\vec{E}$ is normal to the surface of the conductor

Gauss' law



$$\oint \vec{E}\cdot d\vec{A} = 4\pi Q_{int}$$

$$\parallel$$

$$E_\perp \cdot A = 4\pi\sigma A$$

5. Just outside the conductor $E_\perp = 4\pi\sigma$

σ is the (surface) charge density.

Similarly, one can show that for cavities inside a conductor:

6. $\vec{E} = 0$ inside an empty cavity inside a conductor

7. The surface density σ vanishes on the surface of an empty cavity inside a conductor

If the cavity is not empty, but contains charges

8. The total charge on the surface of a cavity inside a conductor is the negative of the total charge within the cavity.

9. The electric field outside a conductor is not affected by the motion of the charges within the cavity.

Exercise 5

Prove statement 6. above

Applications

- Faraday ice bucket experiment
- Van de graaf generator
- Faraday cage

2.2. Electrostatic energy

Recall that we have identified $U = q\phi$, as the potential energy of a charge q in a given electric field.

Imagine we want to calculate how much work is needed to create a given field configuration with charges q_i, $i = 1, \dots n$.

We start with a single charge at $\vec{r}_1$,

$$U_1 = 0, \qquad \phi_1 = \frac{q_1}{|\vec{r} - \vec{r}_1|}$$

When we add a second charge q_2 by moving it from ∞ to $\vec{r}_2$, we need to perform a work

$$U_2 = q_2 \phi_1 = \frac{q_1 q_2}{|r_2 - r_1|}$$

$$\phi_2 = \frac{q_1}{|\vec{r} - r_1|} + \frac{q_2}{|\vec{r} - \vec{r}_2|}$$

In general, to add the charge q_i to the configuration of charges $q_1, \dots q_{i-1}$, we need to perform work

$$\Delta U_i = q_i \phi_{i-1}, \quad \text{where}$$

$$\phi_{i-1} = \sum_{n=1}^{i-1} \frac{q_n}{|\vec{r} - \vec{r}_n|}$$ is the potential created by the charges $q_1, \dots q_{i-1}$

Thus, the amount of energy needed is

$$U_N = \sum_{i=2}^{N} \Delta U_i = \sum_{i=2}^{N} q_i \phi_{i-1} = \sum_{i=2}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|}$$

$$= \sum_{j<i}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_{i \neq j}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

For a continuous charge distribution, this becomes

$$U = \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{1}{2} \int d^3r \, \rho(\vec{r}) \phi(\vec{r}).$$

since $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$