All of electromagnetism:

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu \qquad \text{inhomogeneous Maxwell's eqs.}$$

$$\partial_\mu F^{*\mu\nu} = 0 \qquad \text{Homogeneous " "}$$

$$\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu u^\nu \qquad \text{Lorentz force}$$

Charge conservation readily follows from the inhomogeneous eqs:

$$0 = \partial_\nu \partial_\mu F^{\mu\nu} = 4\pi \partial_\nu j^\nu$$

For many purposes (e.g. quantization) it is very important that these eqs. can be derived from appropriate action principles.

Consider the action

$$S_{EM}[A_\mu] = \int d^4x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right]$$

This action is manifestly Lorentz invariant:

$$S_{EM}[\Lambda_\mu{}^\nu A_\nu] = S_{EM}[A_\mu] \qquad \text{(Lorentz scalar)}$$

and it is also gauge invariant if the current is conserved:

$$S_{EM}[A_\mu + \partial_\mu \chi] = S_{EM}[A_\mu] - \int d^4x \, (\partial_\mu \chi) \, j^\mu =$$

$$= S_{EM}[A_\mu] + \int d^4x \; \chi \, \partial_\mu j^\mu \, ,$$

where we have used the divergence theorem in 4 dim.

<u>Exercise 48</u>

i) Derive Maxwell's eqs. (in covariant form) by extremizing S_{EM} wrt variations of $A_\mu(x)$.

ii) Derive the Lorentz force equation by extremizing the action

$$S_p = -\int d\lambda \left[m \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} + q \, A_\mu \frac{dx^\mu}{d\lambda} \right]$$

Hint: Recall Exercise 45

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Transformation of EM fields

Because $F_{\mu\nu}$ contains $\vec{E}$ and $\vec{B}$, and we know how $F_{\mu\nu}$ transforms under Lorentz transformations,

$$F'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\rho F_{\alpha\rho},$$

we can determine how $\vec{E}$ and $\vec{B}$ transform under Lorentz transformations:

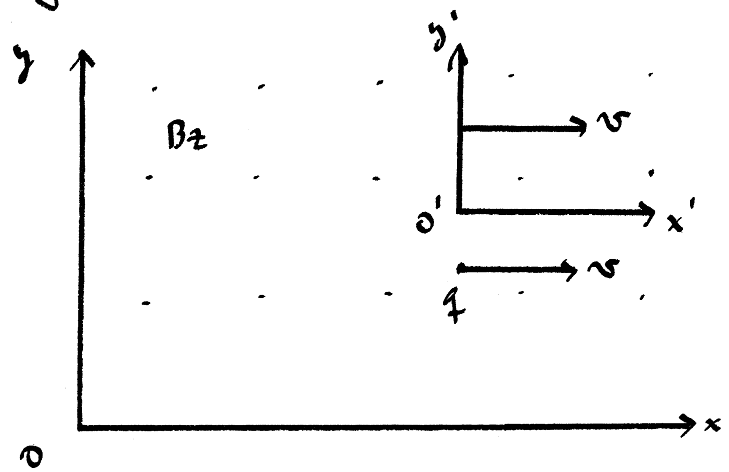
$$\begin{cases} \vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \dfrac{\gamma^2}{\gamma+1}\vec{\beta}\cdot(\vec{\beta}\cdot\vec{E}) \\ \vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \dfrac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}). \end{cases}$$

In particular, for a boost along the x direction:

$$\begin{cases} E_x' = E_x \\ E_y' = \gamma(E_y - \beta B_z) \\ E_z' = \gamma(E_z + \beta B_y) \end{cases} \qquad \begin{cases} B_x' = B_x \\ B_y' = \gamma(B_y + \beta E_z) \\ B_z' = \gamma(B_z - \beta E_y) \end{cases}$$

Example

Consider a particle moving at the same speed as an inertial frame O' in a magnetic field along the z-direction:



In reference frame O:

$$\frac{dp_y}{dt} = -q v B_z$$

In reference frame O':

$$\frac{dp'_y}{dt'} = \frac{dp'_y}{d\tau} = q E'_y = q \gamma(-\beta B_z) = -\gamma q v B_z$$

since $\gamma = \dfrac{dt}{d\tau}$, $\quad \dfrac{dp'_y}{dt} = -q v B_z$ ✓

because $p'_y = p_y$ for a boost along the x direction

The canonical stress tensor

In addition to Lorentz invariance and gauge invariance, the action of pure electromagnetism

$$S_{EM} = \int d^4x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right]$$

has additional symmetries: <u>spacetime translations</u>,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x^\mu + a^\mu) \quad , \text{ with}$$

a^μ a constant set of numbers.

According to a famous theorem by Emmy Noether, <u>global symmetries lead to conservation laws</u>.

In particular invariance under

time translations $\Rightarrow$ energy conservation

spatial " $\Rightarrow$ momentum "

We can use the invariance of S_{EM} under spacetime translations to calculate the energy and momentum of the EM fields

For a small (infinitesimal) translation, we can write

$$A_\mu'(x) = A_\mu(x^\nu + a^\nu) = A_\mu + a^\nu \partial_\nu A_\mu .$$

Consider now the change in S_{EM} under such transformation, and assume for later purposes that $a^\nu = a^\nu(x)$:

$$\delta S_{EM} = \int d^4x \left[-\frac{1}{16\pi} \delta(F_{\mu\nu} F^{\mu\nu}) \right] .$$

$$\delta(F_{\mu\nu} F^{\mu\nu}) = 2\, \delta F_{\mu\nu} F^{\mu\nu} = 2 \left[\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \right] F^{\mu\nu}$$

$$= 2 \left[\partial_\mu (a^\rho \partial_\rho A_\nu) - \partial_\nu (a^\rho \partial_\rho A_\mu) \right] F^{\mu\nu} =$$

$$= 2 \left[(\partial_\mu a^\rho) \partial_\rho A_\nu + a^\rho \partial_\rho \partial_\mu A_\nu - (\partial_\nu a^\rho) \partial_\rho A_\mu - a^\rho \partial_\rho \partial_\nu A_\mu \right] F^{\mu\nu}$$

$$= a^\rho \partial_\rho (F_{\mu\nu} F^{\mu\nu}) + 4 (\partial_\mu a^\rho)(\partial_\rho A_\nu) F^{\mu\nu} .$$

Therefore, after integration by parts:

$$\delta S_{EM} = \int d^4x \, (\partial_\mu a^\rho) \, T^\mu{}_\rho , \quad \text{where}$$

$$T^\mu{}_\rho = -\frac{1}{4\pi} F^{\mu\nu} \partial_\rho A_\nu + \frac{1}{16\pi} \delta^\mu{}_\rho F_{\alpha\beta} F^{\alpha\beta}$$

is the canonical <u>stress tensor</u>. Note that $T^\mu{}_\rho$ is <u>not</u> gauge invariant! $\rightsquigarrow$ Problems ...

Note that

- for constant a^ρ (global translations),

$\delta S_{EM} = 0$, invariant for all fields $A_\mu(x)$.

- for non-constant a^ρ, $\delta S_{EM} = 0$ if fields satisfy eqs. of motion, since then δS_{EM} for all variations. It follows that if EM fields satisfy eqs. of motion

$$\underline{\partial_\mu T^\mu{}_\rho = 0} \qquad \text{conservation laws}$$

We have a set of four conserved currents associated with each value of $\rho = 0, 1, 2, 3$:

$$\partial_\mu j^\mu(\rho) = 0, \quad \text{with} \quad j^\mu(\rho) = T^\mu{}_\rho .$$

$\rho = 0$ $[a^\mu = (\epsilon, 0, 0, 0)]$ is associated with time translations. The conserved "charge" is the energy

$$E = \int d^3x \, j^0(0) = \int d^3x \, T^0{}_0$$

Now,

$$T^0{}_0 = -\frac{1}{4\pi} F^{0i}\, \partial_0 A_i + \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$$

using that $F_{\alpha\beta}F^{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2)$ and $F_{0i} = E^i$,

$$T^0{}_0 = \frac{1}{4\pi} E^i (E^i + \partial_i A_0) - \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2)$$

discarding a total divergence $(\partial_i (A_0 E^i) = (\partial_i A_0) E_i)$

$$\underline{T^0{}_0 = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)}$$

is the <u>energy density</u> of the EM field.

Similarly, for $\rho = 1, 2, 3$ we get the conservation equations related to spatial translations. The conserved "charges" are the momenta

$$P_i = \int d^3x\; j^0{}_{(i)} = \int d^3x\; T^0{}_i .$$

Now,

$$T^0{}_i = -\frac{1}{4\pi} F^{0j}\, \partial_i A_j = \frac{1}{4\pi} E^j \big(\overbrace{\partial_i A_j - \partial_j A_i}^{F_{ij}} + \partial_j A_i \big)$$

and discarding again a divergence

$$T^0{}_i = \frac{1}{4\pi} E^j (-\epsilon_{ijk} B^k) = -\frac{1}{4\pi} (\vec{E} \times \vec{B})^i$$

It follows that the momentum of the EM field is

$$p^i = \frac{1}{4\pi} (\vec{E} \times \vec{B})^i \quad , \text{ or } \quad \underline{\vec{P} = \frac{1}{4\pi} (\vec{E} \times \vec{B})}$$

Finally, note that we can write energy conservation as

$$\partial_\mu j^\mu_{(0)} = 0 .$$

Therefore, $j^i_{(0)}$ is the energy current (energy flow):

$$\vec{S} = \vec{j}_{(0)} \qquad \text{Poynting vector.}$$

Hence, $S^i = j^i{}_0 = T^i{}_0 = -\frac{1}{4\pi} (F^{i0} \partial_0 A_0) - \frac{1}{4\pi} (F^{ij} \partial_0 A_j)$

$$= -\frac{1}{4\pi} (F^{i0} \partial_0 A_0) - \frac{1}{4\pi} F^{ij} (\partial_0 A_j - \partial_j A_0 + \partial_j A_0)$$

and discarding a spacetime divergence,

$$S^i = -\frac{1}{4\pi} F^{ij} F_{0j} = \frac{1}{4\pi} \epsilon^{ij}{}_k B^k E_j \quad , \text{ or}$$

$$\underline{\vec{S} = \frac{1}{4\pi} \vec{E} \times \vec{B}.} \qquad \text{Poynting vector.}$$