10. Electromagnetic Plane Waves

Let us look now at the full Maxwell's eqs. in the absence of charges and currents (in a medium or in vacuum):

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{D} = 0 \\ &\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0 \\ &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{H} - \frac{1}{c}\partial_t\vec{D} = 0 \end{aligned}\right\} \quad \text{with} \quad \vec{D} = \epsilon\vec{E} \quad \text{and} \quad \vec{B} = \mu\vec{H}.$$

If ϵ and μ constant:

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{B} = \left(\frac{\epsilon\mu}{c}\right)\partial_t\vec{E}.$$

using $\vec{\nabla}\times\vec{\nabla}\times\vec{a} = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}$:

$$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{0} - \nabla^2\vec{E} + \frac{1}{c}\partial_t\underbrace{\vec{\nabla}\times\vec{B}}_{\frac{\epsilon\mu}{c}\partial_t\vec{E}} = 0 .$$

Hence,

$$\boxed{\frac{\epsilon\mu}{c^2}\partial_t^2\vec{E} - \vec{\nabla}^2\vec{E} = 0}$$

a wave equation!

A convenient set of solutions is given by the plane waves $\vec{E} = \vec{E}_0 \exp[i\vec{k}\cdot\vec{x} - i\omega t]$,

where $\frac{\epsilon\mu}{c^2}\omega^2 - \vec{k}^2 = 0$, or $\omega = \frac{c}{\sqrt{\epsilon\mu}} |\vec{k}|$

Dispersion relation.

The phase velocity of the wave is

(from $i\vec{k}\cdot\vec{x} - i\omega t$ = const)

$$v \equiv \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}} \equiv \frac{c}{n}$$, where

$n = \sqrt{\epsilon\mu}$ is the medium's index of refraction.

$\vec{B}$ obeys the same wave equation, with the same set of solutions,

$$\vec{B} = \vec{B}_0 \exp[i\vec{k}\cdot\vec{x} - i\omega t], \quad \omega = \frac{c|\vec{k}|}{n}.$$

Of course, $\vec{E}$ and $\vec{B}$ are real, so the understanding is that we are supposed to take the real parts of $\vec{E}$ and $\vec{B}$.

For these plane-wave solutions,

$$\vec{\nabla}\cdot\vec{E}=0 \;\Rightarrow\; \vec{k}\cdot\vec{E}=0$$
$$\vec{\nabla}\cdot\vec{B}=0 \;\Rightarrow\; \vec{k}\cdot\vec{B}=0$$
$\Big\}$ wave is <u>transverse</u>

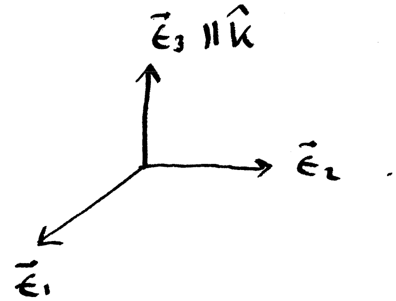
Because $\vec{\nabla}\times\vec{E}+\frac{1}{c}\partial_t\vec{B}=0$, $i\vec{k}\times\vec{E}_0=i\frac{\omega}{c}\vec{B}_0$, or

$$\vec{B}_0=n\,\hat{k}\times\vec{E}_0 .$$

10.3. Polarization

To study polarization, assume without loss of generality that $\vec{k}\parallel\hat{z}$. Define then

$$\vec{\epsilon}_1\equiv\hat{x}\quad;\quad\vec{\epsilon}_2=\hat{y}\quad;\quad\vec{\epsilon}_3=\hat{z}$$



Then, any such plane wave can be written as the linear combination

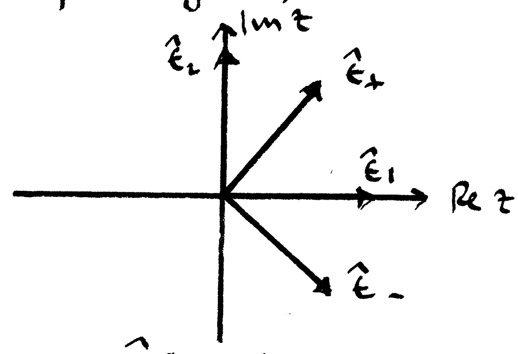
$$\vec{E}=(E_x\,\hat{\epsilon}_1+E_y\,\hat{\epsilon}_2)\,e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

with complex E_x, E_y. If $E_x=|E_x|e^{i\varphi_x}$ and $E_y=|E_y|e^{i\varphi_y}$ have the same phase ($\varphi_x=\varphi_y$), this describes linearly polarized light along the direction

$$|E_x|\,\hat{\epsilon}_1 + |E_y|\,\hat{\epsilon}_2$$

If $\delta_x \neq \delta_y$ light is elliptically polarized.

A convenient basis to discuss elliptically polarized light is provided by

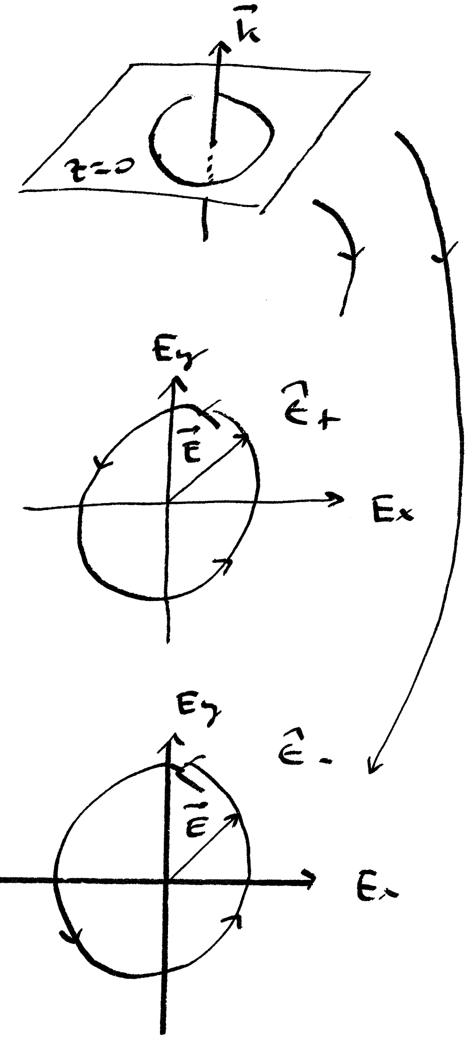


$$\hat{\epsilon}_\pm \equiv \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_1 \pm i\,\hat{\epsilon}_2\right).$$

which satisfy $\hat{\epsilon}_\pm^* \cdot \hat{\epsilon}_\mp = 0$, $\hat{\epsilon}_\pm^* \cdot \hat{\epsilon}_\pm = 1$, $\hat{\epsilon}_\pm^* \cdot \hat{\epsilon}_3 = 0$.

With $\vec{E} = \hat{\epsilon}_\pm\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$, at

$$\begin{cases} E_x = \cos(\omega t) \\ E_y = \pm\sin(\omega t) \end{cases}$$



$\hat{\epsilon}_+$ describes right circularly polarized light

$\hat{\epsilon}_-$ describes left circularly polarized light

Exercise 27

Show that if $\vec{E} = E_+ (\hat{\epsilon}_+ + r e^{i\alpha} \hat{\epsilon}_-) e^{-i\omega t}$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{E_+}{\sqrt{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} (1+r)\cos(\omega t - \alpha/2) \\ (1-r)\sin(\omega t - \alpha/2) \end{pmatrix}$$

Interpret this result geometrically

In general, light is not fully polarized (or not polarized at all). To describe this situation it is convenient to adopt a "polarization matrix" analogous to the density matrix of QM.

Suppose that a fraction p_i of the photons have a polarization vector $\vec{\epsilon}_i = \alpha_{ix} \hat{\epsilon}_1 + \alpha_{iy} \hat{\epsilon}_2$.

Then define the matrix (linear operator)

$$\rho = \sum_i p_i \vec{\epsilon}_i \vec{\epsilon}_i^{\,+}$$

By definition

i) $\rho^+ = \rho$ hermitian

ii) $\vec{E}^+ \rho \vec{E} \geq 0 \quad \forall \vec{E}$ $\quad \rho$ positive definite

iii) $\underline{P} \cdot \bar{k} = \bar{k}^+ \underline{P} = 0$, $\underline{P}$ transverse.

The polarization matrix can be described in terms of the four <u>Stokes parameters</u>

$$\begin{cases} I = \hat{\epsilon}_1^+ \underline{P} \, \hat{\epsilon}_1 + \hat{\epsilon}_2^+ \underline{P} \, \hat{\epsilon}_2 \\ Q = \hat{\epsilon}_1^+ \underline{P} \, \hat{\epsilon}_1 - \hat{\epsilon}_2^+ \underline{P} \, \hat{\epsilon}_2 \\ U = 2 \, \mathrm{Re} \, [\hat{\epsilon}_2^+ \underline{P} \, \hat{\epsilon}_1] \\ V = 2 \, \mathrm{Im} \, [\hat{\epsilon}_2^+ \underline{P} \, \hat{\epsilon}_1] \end{cases}$$

We shall later see that the averaged flow of electromagnetic energy for a plane wave with polarization vector $\bar{E}_0$ is

$$\langle \vec{S} \rangle = \frac{c \hat{k}}{8\pi} \sqrt{\frac{\epsilon}{\mu}} \, |\bar{E}_0|^2 .$$

Thus, I represents the intensity of radiation.

<u>Exercise 28</u>

Calculate $\{I, Q, U, V\}$ for light

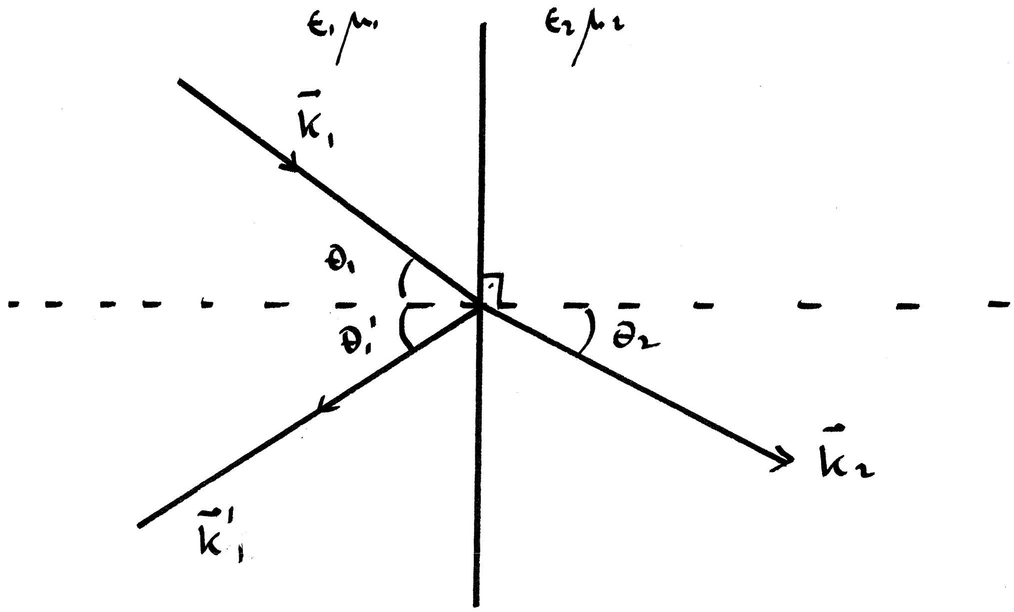
i) Polarized $\parallel$ x direction ii) Polarized y direction

iii) 45° degrees wrt. x axis iv) left polarized v) right polarized.

10.4. Reflection and Refraction at a planar interface

Consider a wave approaching a planar interface between two dielectric media.

We expect a reflected and a refracted wave:



As before, from the Maxwell's eqs. we find that the normal components of $\vec{D}$ and $\vec{B}$, and the tangential components of $\vec{E}$ and $\vec{H}$ have to be continuous (careful with $\vec{E}$ and $\vec{H}$!):

$$\left\{\begin{aligned}
&\hat{n}\cdot\left(\epsilon_1 \vec{E}_1 e^{i\vec{k}_1\cdot\vec{r}} + \epsilon_1 \vec{E}_1' e^{i\vec{k}_1'\cdot\vec{r}}\right) \overset{!}{=} \hat{n}\cdot \epsilon_2 \vec{E}_2 e^{i\vec{k}_2\cdot\vec{r}} \\
&\hat{n}\cdot\left(\vec{B}_1 e^{i\vec{k}\cdot\vec{r}} + \vec{B}_1' e^{i\vec{k}_1'\cdot\vec{r}}\right) \overset{!}{=} \hat{n}\, \vec{B}_2 e^{i\vec{k}_2\cdot\vec{r}} \\
&\hat{n}\times\left(\vec{E}_1 e^{i\vec{k}_1\cdot\vec{r}} + \vec{E}_1' e^{i\vec{k}_1'\cdot\vec{r}}\right) \overset{!}{=} \hat{n}\times \vec{E}_2 e^{i\vec{k}_2\cdot\vec{r}} \\
&\hat{n}\times\left(\frac{\vec{B}_1}{\mu_1} e^{i\vec{k}_1\cdot\vec{r}} + \frac{\vec{B}_1'}{\mu_1} e^{i\vec{k}_1'\cdot\vec{r}}\right) \overset{!}{=} \hat{n}\times \frac{\vec{B}_2}{\mu_2} e^{i\vec{k}_2\cdot\vec{r}}
\end{aligned}\right.$$

These equations demand that

$$\vec{k}_1 \cdot \vec{r} = \vec{k}_1' \cdot \vec{r} = \vec{k}_2 \cdot \vec{r} \quad \text{at the interface.}$$

Therefore, the projections of $\vec{k}_1$, $\vec{k}_1'$ and $\vec{k}_2$ onto the interface are all equal:

$$\vec{k}_1^{\parallel} = \vec{k}_1'^{\parallel} = \vec{k}_2^{\parallel}$$

$\Rightarrow$ $\vec{k}$, $\vec{k}'$ and $\vec{k}_2$ all lie on a plane &

$$k_1 \sin\theta_1 = k_1' \sin\theta_1' = k_2 \sin\theta_2$$

Using $k = \frac{n}{c}\omega$ it follows that

$$\underline{n_1 \sin\theta_1 = n_2 \sin\theta_2} \quad \text{Snell's law.}$$

The electric and magnetic fields are then determined by the conditions

$$\left\{ \begin{array}{l} \epsilon_1 \, \hat{n} \cdot (\vec{E}_1 + \vec{E}_1') = \epsilon_2 \, \hat{n} \cdot \vec{E}_2 \\ \sqrt{\epsilon_1 \mu_1} \; \hat{n} \cdot (\hat{k}_1 \times \vec{E}_1 + \hat{k}_1' \times \vec{E}_1') = \sqrt{\epsilon_2 \mu_2} \; \hat{n} \cdot (\hat{k}_2 \times \vec{E}_2) \\ \hat{n} \times (\vec{E}_1 + \vec{E}_1') = \hat{n} \times \vec{E}_2 \\ \sqrt{\dfrac{\epsilon_1}{\mu_1}} \; \hat{n} \times (\hat{k}_1 \times \vec{E}_1 + \hat{k}_1' \times \vec{E}_1') = \sqrt{\dfrac{\epsilon_2}{\mu_2}} \; \hat{n} \times (\hat{k}_2 \times \vec{E}_2) \end{array} \right.$$

Note that if $n_2 < n_1$ the equation

$$\sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1$$

does not have a (real) solution for incident angles $\theta > \theta_c = \arcsin \frac{n_2}{n_1}$.

This leads to <u>total internal reflection</u>.

The amounts of reflected and transmitted radiation can be described in terms of the reflection and transfission coefficients

$$R = \frac{\hat{n} \cdot \langle \vec{S}_1' \rangle}{\hat{n} \cdot \langle \vec{S}_1 \rangle} \quad ; \quad T = \frac{\hat{n} \cdot \langle \vec{S}_2 \rangle}{\hat{n} \cdot \langle \vec{S}_1 \rangle}$$

which satisfy $R + T = 1$ (conservation of energy)

<u>Exercise 29</u>

Calculate T for light polarized

i) $\perp$ to plane of incidence

ii) $\parallel$ " " "

This implies in particular that unpolarized incident light typically yields (partially) polarized reflected and transmitted light.

For example for a wave polarized $\parallel$ to the incident plane $R = 0$ at the Brewster angle

$$\theta_B \simeq \arctan \frac{n_2}{n_1}$$

Thus, unpolarized light incident at an angle θ_B produces fully polarized reflected light, with polarization $\perp$ to plane of incidence.