15.4. Electromagnetic Fields of a Moving Charge

In lectures 20 and 21 we studied the radiation emitted by time-varying charge distributions. We proceed to study the radiation emitted by a single charge:

Recall that in Lorentz gauge ($\partial_\mu A^\mu = 0$), the inhomogeneous Maxwell's eqs. are

$$\Box A^\mu = 4\pi j^\mu, \quad \text{where} \quad \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$$

Recall that for a single charge

$$j^\mu = q \int d\tau \; u^\mu(\tau)\, \delta^{(4)}[x^\nu - x^\nu(\tau)].$$

To find A^μ we are going to look for an appropriate Green's function:

$$\underline{\Box_x D(x, x') = \delta^{(4)}(x^\nu - x'^\nu).}$$

Note that, as before,

$$\Box \exp(-i k_\mu x^\mu) = \eta^{\mu\nu} \partial_\mu \partial_\nu \exp(-i k_\mu x^\mu) = -\eta^{\mu\nu} k_\mu k_\nu \exp(-i k_\mu x^\mu)$$
$$\equiv - k^2 e^{-ik\cdot x}.$$

Therefore, we can immediately write the Green's function as

$k_\mu (x^\mu - x'^\mu)$

$$D(x, x') = -\frac{1}{(2\pi)^4} \int d^4k \, \frac{e^{-ik\cdot(x-x')}}{k^2}$$

$k_\nu k^\nu$.

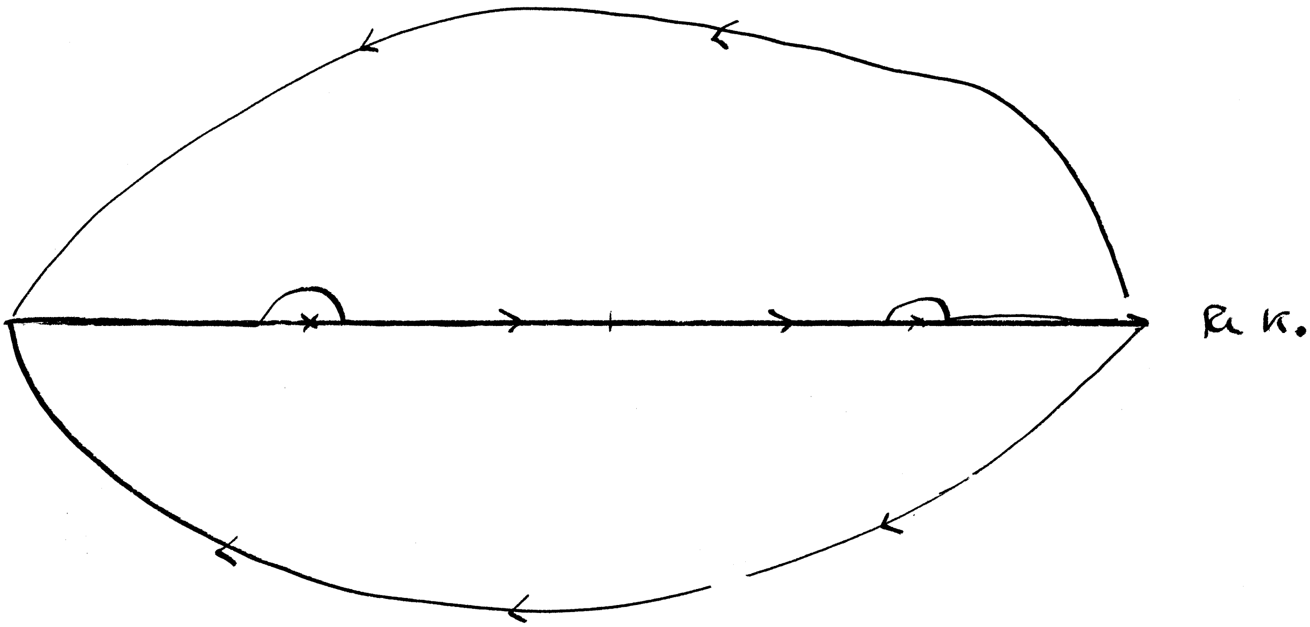
This can be checked by acting left and right with $\Box_x$.

To evaluate D, we integrate over k_0 first:

$$D(z) = -\frac{1}{(2\pi)^4} \int d^3k \, e^{i\vec{k}\cdot\vec{z}} \int dk_0 \, \frac{e^{-ik_0 z^0}}{k_0^2 - \vec{k}^2}$$

We encounter poles along integration contour at $k_0 = \pm |\vec{k}|$. Prescription to avoid poles determines the b.c. satisfied by the Green's fct.

We want to find the retarded Green's fct:



• For $z^0 = t^0 - t_0' < 0$ close the contour in upper half plane $\Rightarrow$

$$\int dk_0 \, \frac{e^{-ik_0 z^0}}{k_0^2 - |\vec{k}|^2} = 0 \quad \Rightarrow \quad \underline{D(z) = 0.}$$

Retarded.

• For $z^0 = t^0 - t_0' > 0$ close the contour in lower half plane.

$$\int dk_0 \, \frac{e^{-ik_0 z^0}}{(k_0 + |\vec{k}|)(k_0 - |\vec{k}|)} = -2\pi i \left[-\frac{e^{i|\vec{k}|z^0}}{2|\vec{k}|} + \frac{e^{-i|\vec{k}|z^0}}{2|\vec{k}|} \right] =$$

$$= -\frac{2\pi}{|\vec{k}|} \sin(|\vec{k}| z^0).$$

Consider hence

$$D(z) = \frac{\theta(z^0)}{(2\pi)^3} \int d^3k \; \frac{e^{i\vec{k}\cdot\vec{z}} \sin(|\vec{k}| z^0)}{|\vec{k}|} =$$

$$= \frac{\theta(z^0)}{(2\pi)^3} \int dk \cdot k^2 \; 2\pi \; d(\cos\theta) \; \frac{e^{i|\vec{k}||\vec{z}|\cos\theta} \sin(|\vec{k}| z^0)}{|\vec{k}|} =$$

$$= \frac{\theta(z^0)}{2\pi^2 |\vec{z}|} \int_0^\infty dk \; \sin(|\vec{k}|\cdot|\vec{z}|) \sin(|\vec{k}| z^0)$$

Because $\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$ and

cos is even,

$$= \frac{\theta(z^0)}{2\pi^2 |\vec{z}|} \cdot \frac{1}{4} \int_{-\infty}^{\infty} dk \left[e^{i|\vec{k}|(|\vec{z}| - z^0)} - e^{i|\vec{k}|(|\vec{z}| + z^0)} \right]$$

$$\Rightarrow \qquad \underline{D(x, x') = \frac{\theta(x^0 - x'^0)}{4\pi |\vec{x} - \vec{x}'|} \delta(x^0 - x'^0 - |\vec{x} - \vec{x}'|)}$$

This is the same retarded Green's fct of Lecture 20.

We can write it in a manifestly invariant form

by noting that

$$\delta(z^2) \equiv \delta(z^{0\,2} - |\vec{z}|^2) = \delta[(z^0 + |\vec{z}|)(z^0 - |\vec{z}|)]$$

$$= \frac{1}{2|\vec{z}|}\left[\delta(z_0 - |\vec{z}|) + \delta(z_0 + |\vec{z}|)\right],$$ where we

have used the identity

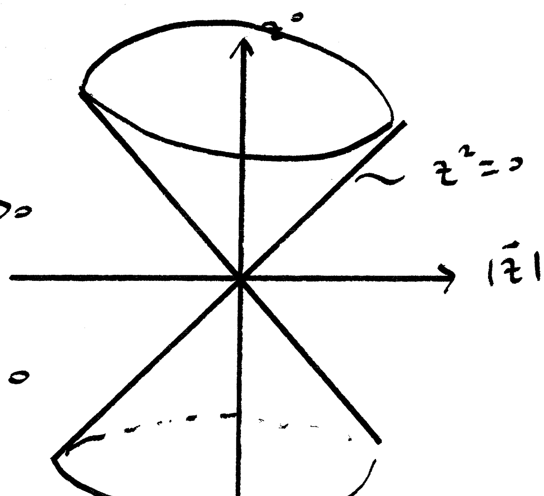
$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x - x_i) \quad \text{and} \quad f(x_i) = 0.$$

Therefore,

$$\boxed{D_{ret}(z) = \frac{1}{2\pi}\,\theta(z^0)\,\delta(z^2)}$$

The retarded Green's fct. has support in

the <u>future light cone</u>



We are finally ready to calculate the field of

a moving charge:

$$A^\mu(x) = 4\pi \int d^4x'\, D_{ret}(x, x')\, j^\mu(x'), \quad \text{or}$$

$$A^\mu(x) = \frac{4\pi}{2\pi}\int d^4x' \, \theta(x^0 - x'^0) \, \delta[(x-x')^2] \cdot q\int d\tau \, u^\mu(\tau)\, \delta(x' - x(\tau))$$

$$= 2q \int d\tau \, \theta(x^0 - x^0(\tau)) \, \delta[(x - x(\tau))^2] \, u^\mu(\tau).$$

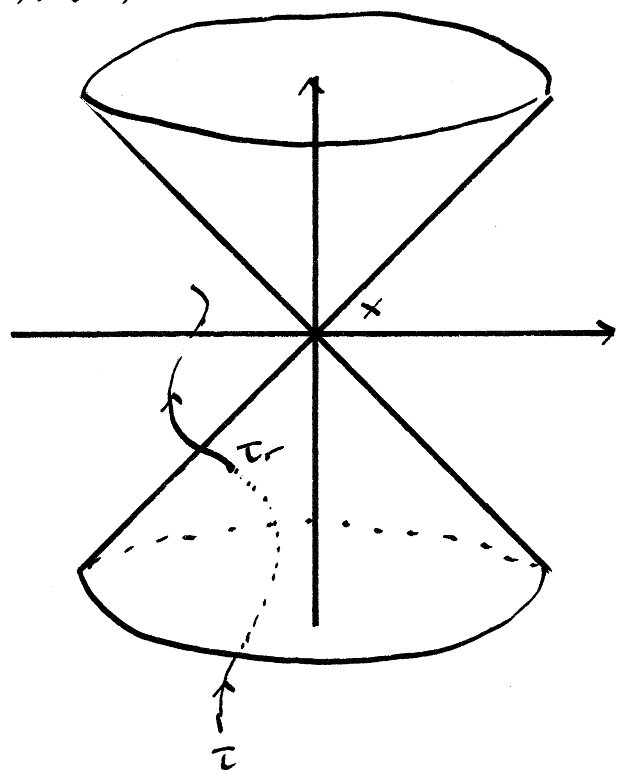
Using again that

$$\delta[(x - x(\tau))^2] = \frac{\delta(\tau - \tau_r)}{\left| 2(x - x(\tau))_\mu (-1) \frac{dx^\mu}{d\tau} \right|}$$

where τ_r is the solution of

$(x - x(\tau))^2 = 0$ with $x^0 \geq x^0(\tau)$

$$A^\mu(x) = \frac{q\, u^\mu(\tau_r)}{u^\mu(x_\mu - x_\mu(\tau_r))}$$



This is known as the Lienard-Wiechert potential.

Note that

$$u^\mu(x_\mu - x_\mu(\tau_r)) = u^0 (t - t(\tau_r)) - u^i (x_i - x_i(\tau_r)) =$$

$$= \gamma \cdot (t - t(\tau_r)) - \gamma v^i (x_i - x_i(\tau_r))$$

and because $t - t(\tau_r) = |\vec{x} - \vec{x}(\tau_r)| \equiv r$,

$$u^\mu (x_\mu - x_\mu(\tau)) = \gamma r (1 - \hat{r} \cdot \vec{v}).$$

Therefore, the potentials become

$$\phi(t, \vec{x}) = \left. \frac{q}{(r - \vec{r} \cdot \vec{v})} \right|_{ret} \quad \text{and}$$

$$\vec{A} = \left. \frac{q \vec{v}}{(r - \vec{r} \cdot \vec{v})} \right|_{ret}, \quad \text{where the label ret}$$

means that the expression is evaluated at the retarded time.

To calculate $\vec{E}$ and $\vec{B}$, we just use that

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}.$$ But we have

to be careful because $\vec{v}$ and $\vec{r} = \vec{x} - \vec{x}(\tau_r)$ depend on $\vec{x}$ through the retarded time.

The final expressions are:

$$\begin{cases} \vec{E}(t,\vec{x}) = \left.\dfrac{q\,(\hat{r}-\vec{v})}{r^2\gamma^2(1-\hat{r}\cdot\vec{v})^3}\right|_{ret} + \left.\dfrac{q\,\hat{r}\times[(\hat{r}-\vec{v})\times\dot{\vec{v}}]}{r(1-\hat{r}\cdot\vec{v})^3}\right|_{ret} \\[2em] \vec{B}(t,\vec{x}) = \hat{r}\times\vec{E}\Big|_{ret} \end{cases}$$

Two types of fields:

- velocity fields: Do not depend on $\dot{\vec{v}} = \dfrac{d\vec{v}}{dt}$
- acceleration fields: Depend on $\dfrac{d\dot{\vec{v}}}{dt}$.

Note that the velocity fields are proportional to $\dfrac{1}{r^2}$ $\longrightarrow$ Coulomb's law. For instance, for $\vec{v}\equiv 0$,

$$\vec{E} = \frac{q\,\hat{r}}{r^2}$$

The <u>acceleration</u> fields are proportional to $\dfrac{1}{r}$, and they are transverse to $\hat{r}$. They describe <u>outgoing radiation</u>. Electromagnetic radiation is produced by accelerated charges.

To calculate the radiated power we use again the Poynting vector:

$$\frac{d^2W}{dt\,d\Omega} = \left[r^2 \cdot (\hat{r} \cdot \vec{S}) \right] \quad , \quad \text{where} \quad \vec{S} = \frac{1}{4\pi} (\vec{E} \times \vec{B})$$

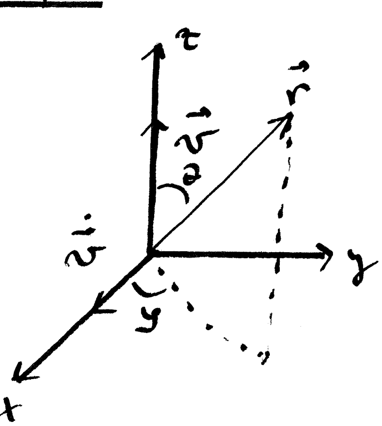
Because $\vec{E}$ and $\vec{B}$ arise from expressions evaluated at the retarded time, the energy emitted is captured by

$$\frac{d^2W}{dt_r\,d\Omega} = \frac{d^2W}{dt\,d\Omega} \frac{dt}{dt_r} = \frac{d^2W}{dt\,d\Omega} (1 - \hat{r} \cdot \vec{v}) \Big|_{ret} .$$

Therefore, using the acceleration fields:

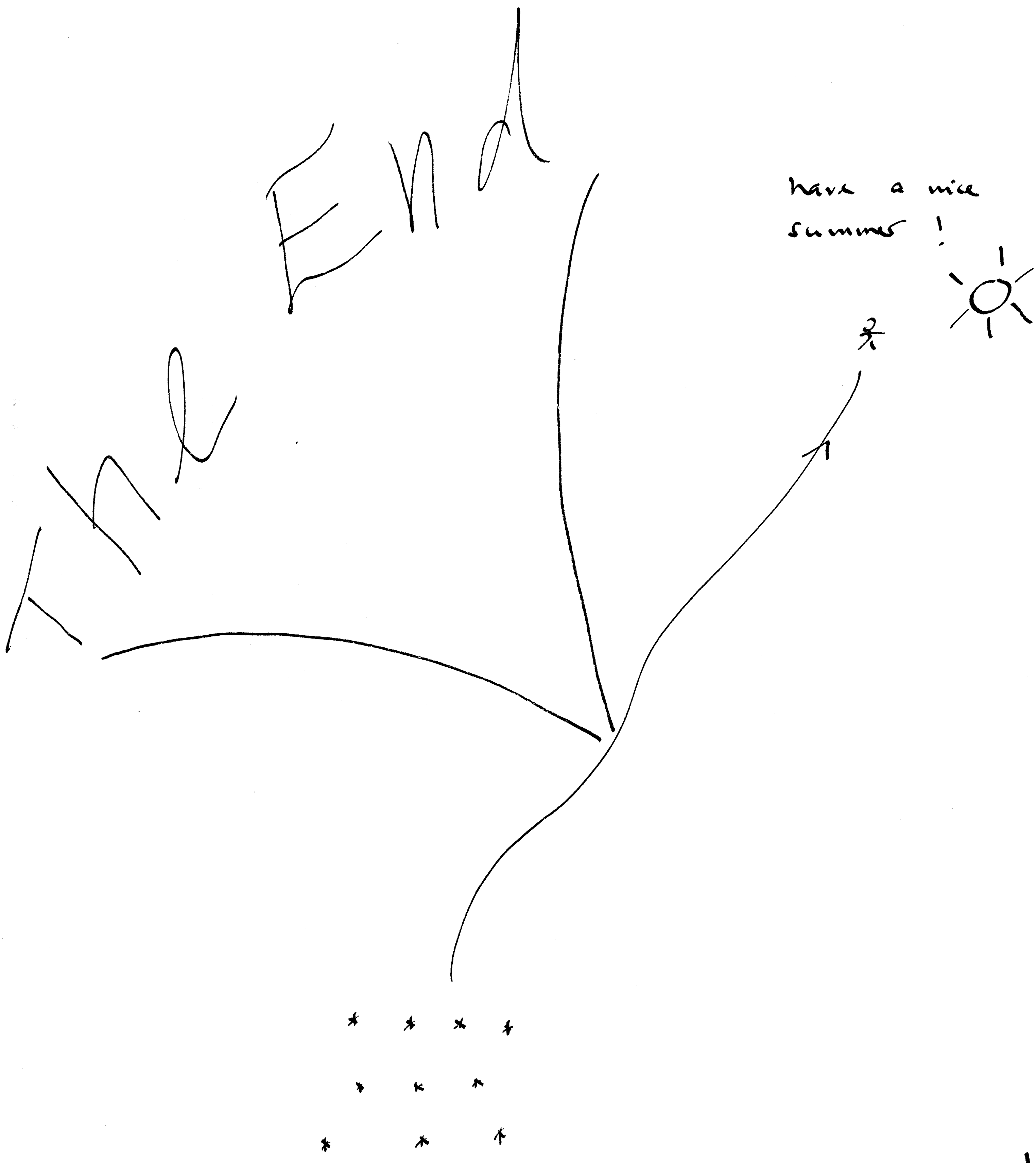
$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi} \frac{\left| \hat{r} \times [(\hat{r} - \vec{v}) \times \dot{\vec{v}}] \right|^2}{(1 - \hat{r} \cdot \vec{v})^5}$$

<u>Example</u>: For $\vec{v} \perp \dot{\vec{v}}$ (e.g. synchrotron radiation)



$$\frac{dP}{d\Omega} = \frac{q^2 |\dot{\vec{v}}|^2}{4\pi (1 - v\cos\theta)^3} \left[1 - \frac{\sin^2\theta \cos^2 y}{\gamma^2 (1 - v\cos\theta)^2} \right]$$

Peaks along the forward direction $\theta = 0$ because of the $1 - v\cos\theta$ factor in denominator.


The End
have a nice summer !