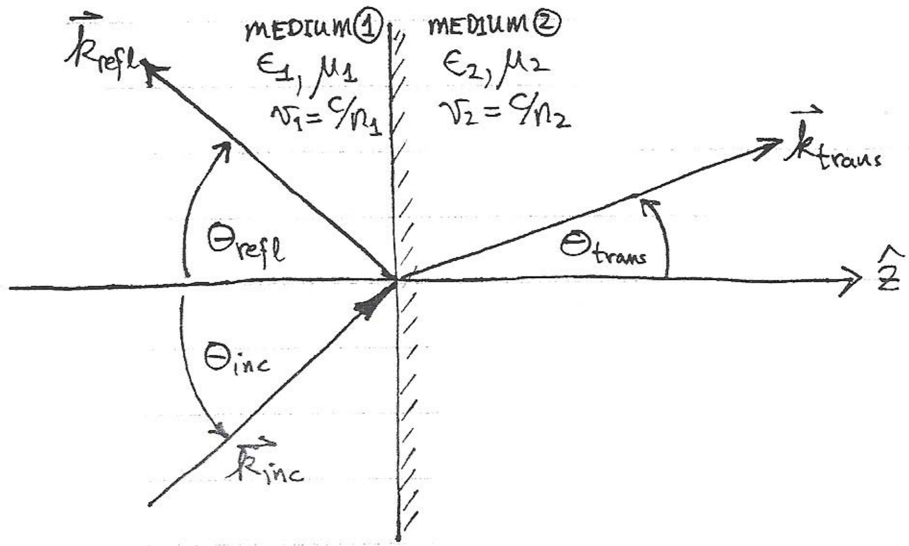
LECTURE NOTES 6.5

Reflection & Transmission of Monochromatic Plane *EM* Waves at Oblique Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

Suppose we have a monochromatic plane *EM* wave incident at an oblique angle θ_{inc} on a boundary between two linear/homogeneous/isotropic media, defined with respect to the normal to the interface, as shown in the figure below:



The incident *EM* wave is: $\boxed{\tilde{\vec{E}}_{inc}(\vec{r},t) = \tilde{\vec{E}}_{o_{inc}} e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}}$ and $\boxed{\tilde{\vec{B}}_{inc}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(\vec{r},t)}$

The reflected *EM* wave is: $\boxed{\tilde{\vec{E}}_{refl}(\vec{r},t) = \tilde{\vec{E}}_{o_{refl}} e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)}}$ and $\boxed{\tilde{\vec{B}}_{refl}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}(\vec{r},t)}$

The transmitted *EM* wave is: $\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}}$ and $\boxed{\tilde{\vec{B}}_{trans}(\vec{r},t) = \frac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}(\vec{r},t)}$

Note that all three *EM* waves have the same *frequency*, $f = \omega/2\pi$

This is due to the fact that at the microscopic level, the energy of real photon does *not* change in a medium, *i.e.* $E_\gamma^{vac} = E_\gamma^{med} = E_\gamma$, and since $E_\gamma = hf_\gamma$ for real photons, then $hf_\gamma^{vac} = hf_\gamma^{med} = hf_\gamma$. Thus the frequency of the photon does *not* change in a medium, *i.e.* $f_\gamma^{vac} = f_\gamma^{med} = f_\gamma$ {*n.b.* An experimental fact: *colors* of objects do *not* change when placed & viewed *e.g.* underwater}.

However, the momentum of a real photon *does* change in a medium! This is because the momentum of the real photon in a medium depends on index of refraction of that medium n_{med} via the relation $p_\gamma^{med} = n_{med}\, p_\gamma^{vac}$ where $n_{med} = c/v_{prop}^{med}$. Thus the photon momentum depends {inversely} on the speed of propagation in the medium!

From the DeBroglie relation between momentum and wavelength of the real photon $p_\gamma = h/\lambda_\gamma$ we see that $p_\gamma^{med} = n_{med} p_\gamma^{vac} = n_{med}\left(h/\lambda_\gamma^{vac}\right) = h\left(n_{med}/\lambda_\gamma^{vac}\right) = h/\lambda_\gamma^{med}$ and hence $\lambda_\gamma^{med} = \lambda_\gamma^{vac}/n_{med}$.

Thus, for <u>macroscopic</u> *EM* waves propagating in the two linear/homogeneous/isotropic media (1) and (2), we have $f_1 = f_2 = f$, and since $\omega = 2\pi f$ then $\omega_1 = \omega_2 = \omega$.

<u>But since</u>: $\omega = kv$ <u>then</u>: $\omega_1 = \omega_2 = \omega \;\Rightarrow\; k_1 v_1 = k_2 v_2$ <u>thus</u>: $\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$

<u>Now</u>: $\boxed{k_{inc} = \left|\vec{k}_{inc}\right| = 2\pi/\lambda_1}$; $\boxed{k_{refl} = \left|\vec{k}_{refl}\right| = 2\pi/\lambda_1}$; $\boxed{k_{trans} = \left|\vec{k}_{trans}\right| = 2\pi/\lambda_2}$

<u>And</u>: $\boxed{\omega = \omega_1 = \omega_2 = 2\pi\left(v_1/\lambda_1\right) = 2\pi\left(v_2/\lambda_2\right)}$

<u>Then</u>: $\boxed{\omega = 2\pi \underset{f_{inc}}{f_1} = 2\pi \underset{f_{refl}}{f_1} = 2\pi \underset{f_{trans}}{f_2}} \;\Rightarrow\; \boxed{f_1 = f_2 = f_{inc} = f_{refl} = f_{trans}}$

<u>Then</u>: $\boxed{\lambda_1 = \lambda_o/n_1}$ $\boxed{\lambda_2 = \lambda_o/n_2}$ <u>where</u>: $\boxed{\lambda_o = \text{ vacuum length } = c/f}$

<u>And</u>: $\boxed{v_1 = c/n_1}$ $\boxed{v_2 = c/n_2}$ <u>and</u>: $\boxed{k_o = \text{ vacuum wavenumber } = 2\pi/\lambda_o = \omega/c}$

<u>Thus</u>: $\boxed{k_1 = n_1 k_o}$ $\boxed{k_2 = n_2 k_o}$

<u>From</u>: $\boxed{\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2}$

We see that: $\boxed{k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right)k_{trans} = \left(\frac{v_2}{v_1}\right)k_2 = \left(\frac{n_1}{n_2}\right)k_{trans} = \left(\frac{n_1}{n_2}\right)k_2}$ Since $\boxed{v_i = c/n_i \quad i = 1,2}$

The total (*i.e.* combined) *EM* fields in medium 1):

$$\boxed{\tilde{\vec{E}}_{Tot_1}\left(\vec{r},t\right) = \tilde{\vec{E}}_{inc}\left(\vec{r},t\right) + \tilde{\vec{E}}_{refl}\left(\vec{r},t\right)} \text{ and } \boxed{\tilde{\vec{B}}_{Tot_1}\left(\vec{r},t\right) = \tilde{\vec{B}}_{inc}\left(\vec{r},t\right) + \tilde{\vec{B}}_{refl}\left(\vec{r},t\right)}$$

must be matched (*i.e.* joined smoothly) to the total *EM* fields in medium 2):

$$\boxed{\tilde{\vec{E}}_{Tot_2}\left(\vec{r},t\right) = \tilde{\vec{E}}_{trans}\left(\vec{r},t\right)} \text{ and } \boxed{\tilde{\vec{B}}_{Tot_2}\left(\vec{r},t\right) = \tilde{\vec{B}}_{trans}\left(\vec{r},t\right)}$$

using the boundary conditions BC1) → BC4) at $z = 0$ (in the *x*-*y* plane).

At $z = 0$, these four boundary conditions generically are of the form:

$$\boxed{\left(\frown\right) e^{i\left(\vec{k}_{inc}\cdot\vec{r} - \omega t\right)} + \left(\frown\right) e^{i\left(\vec{k}_{refl}\cdot\vec{r} - \omega t\right)} = \left(\frown\right) e^{i\left(\vec{k}_{trans}\cdot\vec{r} - \omega t\right)}}$$

These boundary conditions must hold for all (x,y) on the interface at $z = 0$, and also must hold for arbitrary times/any/all times, t. The above relation is already satisfied for arbitrary time, t, since the factor $e^{-i\omega t}$ is common to all terms.

Thus, the following generic relation must hold for any/all (x,y) on interface at at $\underline{\underline{z=0}}$:

$$\boxed{(\sim\!\sim)\, e^{i(\vec{k}_{inc}\cdot\vec{r})} + (\sim\!\sim)\, e^{i(\vec{k}_{refl}\cdot\vec{r})} = (\sim\!\sim)\, e^{i(\vec{k}_{trans}\cdot\vec{r})}}$$

When $\underline{\underline{z=0}}$ (*i.e.* at the interface in the x-y plane) we must have: $\vec{k}_{inc}\bullet\vec{r} = \vec{k}_{refl}\bullet\vec{r} = \vec{k}_{trans}\bullet\vec{r}$

More explicitly: $k_{inc_x}x + k_{inc_y}y + \underbrace{\cancel{k_{inc_z}z}}_{z=0} = k_{refl_x}x + k_{refl_y}y + \underbrace{\cancel{k_{refl_z}z}}_{z=0} = k_{trans_x}x + k_{trans_y}y + \underbrace{\cancel{k_{trans_z}z}}_{z=0}$

$\underline{\underline{\text{or}}}$: $k_{inc_x}x + k_{inc_y}y = k_{refl_x}x + k_{refl_y}y = k_{trans_x}x + k_{trans_y}y$ @ $z = 0$ in the x-y plane.

The above relation can *only* hold for arbitrary $(x, y, z = 0)$ ***iff*** (= if and only if):

$k_{inc_x}x = k_{refl_x}x = k_{trans_x}x \quad\Rightarrow\quad k_{inc_x} = k_{refl_x} = k_{trans_x}$

and: $k_{inc_y}y = k_{refl_y}y = k_{trans_y}y \quad\Rightarrow\quad k_{inc_y} = k_{refl_y} = k_{trans_y}$

Since this problem has rotational invariance (*i.e.* rotational symmetry) about the $\hat{z}$-axis, (see above pix on *p*. 1), without any loss of generality we can *e.g.* choose $\vec{k}_{inc}$ to lie entirely within the $\underline{\underline{x\text{-}z}}$ plane, as shown in the figure below…

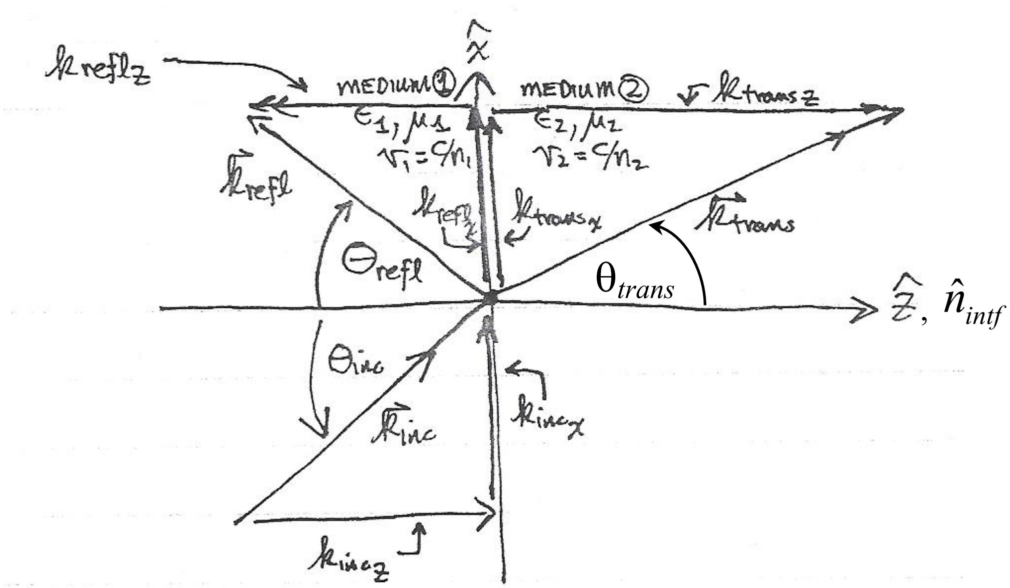
Then: $k_{inc_y} = k_{refl_y} = k_{trans_y} = 0$ and thus: $k_{inc_x} = k_{refl_x} = k_{trans_x}$.

i.e. the transverse components of $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ are all equal and point in the {same} $+\hat{x}$ direction.

The First Law of Geometrical Optics (All wavevectors k lie in a common plane):

The above result tells us that the three wave vectors $\vec{k}_{inc}, \vec{k}_{refl}$ and $\vec{k}_{trans}$ ALL LIE IN A PLANE known as the **plane of incidence** (here, the x-z plane) as shown in the figure below. Note that the plane of incidence also includes the unit normal to the interface, {*here*} $\hat{n}_{intf} = +\hat{z}$-axis.

The *x-z* Plane of Incidence:



The Second Law of Geometrical Optics (Law of Reflection):

From the above figure, we see that:

$$\boxed{k_{inc_x} = k_{inc}\sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl}\sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans}\sin\theta_{trans}}$$

But: $\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$

$\Rightarrow$ Angle of Incidence = Angle of Reflection $\boxed{\theta_{inc} = \theta_{refl}}$ Law of Reflection!

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle, θ_{trans} we see that: $\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}}$

In medium 1): $\boxed{k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o}$

where $\boxed{k_o = \text{ vacuum wave number } = 2\pi/\lambda_o}$ and $\boxed{\lambda_o = \text{ vacuum wave length}}$

In medium 2): $\boxed{k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o}$

Thus: $\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}} \Rightarrow \boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}}$

But since: $\boxed{k_{inc} = k_1 = n_1 k_o}$ and $\boxed{k_{trans} = k_2 = n_2 k_o}$

Then: $\boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}} \Rightarrow \boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}}$ Law of Refraction (Snell's Law)

Which can also be written as: $\boxed{\dfrac{\sin\theta_{trans}}{\sin\theta_{inc}} = \dfrac{n_1}{n_2}}$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1) we can also write Snell's Law as:

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \quad \text{or:} \quad \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}}$$

(incident) (transmitted)

Because of the above three laws of geometrical optics, we see that:

$\boxed{\vec{k}_{inc}\cdot\vec{r}\big|_{z=0} = \vec{k}_{refl}\cdot\vec{r}\big|_{z=0} = \vec{k}_{trans}\cdot\vec{r}\big|_{z=0}}$ everywhere on the interface at $z = 0$ {in the x-y plane}

Thus we see that: $\boxed{e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)}\big|_{z=0} = e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)}\big|_{z=0} = e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}\big|_{z=0}}$ everywhere on the interface at $z = 0$ {in the x-y plane}, valid also for arbitrary/any/all time(s) t, since ω is the same in either medium (1 or 2).

Thus, the boundary conditions BC 1) → BC 4) for a monochromatic plane *EM* wave incident on an interface at an oblique angle θ_{inc} between two linear/homogeneous/isotropic media become:

BC 1): Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges):

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}}} \quad \left\{\text{using } \vec{D} = \varepsilon\vec{E}\right\}$$

BC 2): Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_{x,y}}} + \tilde{E}_{o_{refl_{x,y}}}\right) = \tilde{E}_{o_{trans_{x,y}}}}$$

BC 3): Normal (*i.e.* *z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 4): Tangential (*i.e.* *x*-, *y*-) components of $\vec{H}$ continuous at $z = 0$ (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}}$$

Note that in each of the above, we also have the relation $\boxed{\vec{\tilde{B}}_o = \frac{1}{v}\hat{k}\times\vec{\tilde{E}}_o}$

For a monochromatic plane *EM* wave incident on a boundary between two linear / homogeneous / isotropic media at an oblique angle of incidence, there are ***three*** possible polarization cases to consider:

Case I): $\vec{E}_{inc} \perp$ plane of incidence – known as Transverse Electric (TE) Polarization
{ $\vec{B}_{inc} \parallel$ plane of incidence}

Case II): $\vec{E}_{inc} \parallel$ plane of incidence – known as Transverse Magnetic (TM) Polarization
{ $\vec{B}_{inc} \perp$ plane of incidence}

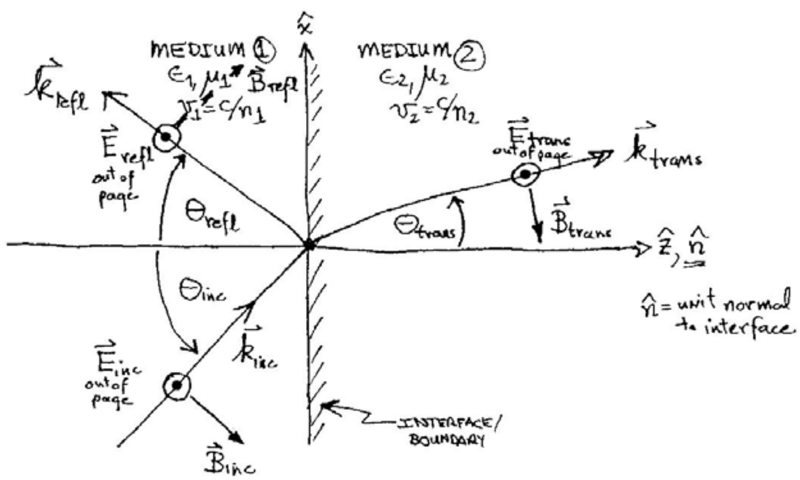
Case III): The most general case: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the plane of incidence.
{$\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the plane of incidence}
i.e. Case III is a linear vector combination of Cases I) and II) above!

Polarization for general case: $\boxed{\hat{n}_{inc} = \cos\varphi\hat{x} + \sin\varphi\hat{y} = \cos\varphi\,\hat{\epsilon}_\perp + \sin\varphi\,\hat{\epsilon}_\parallel}$

$\Rightarrow$ Simply decompose the linear polarization components of the general-case *EM* plane wave into its $\hat{x} = \hat{\epsilon}_\perp$ and $\hat{y} = \hat{\epsilon}_\parallel$ vector components – *i.e.* the *E*-field components perpendicular to and parallel to the plane of incidence, TE polarization and TM polarization respectively. Solve separately, then combine vectorially…

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE*) Polarization

A monochromatic plane *EM* wave is incident {from the left} on a boundary located at $z = 0$ in the *x*-*y* plane between two linear / homogeneous / isotropic media at an oblique angle of incidence. The polarization of the incident *EM* wave (*i.e.* the orientation of $\vec{E}_{inc}$ is *transverse* (*i.e.* $\perp$) to the plane of incidence {= the *x*-*z* plane containing the three wavevectors $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ *and* the unit normal to the boundary/interface, $\hat{n} = +\hat{z}$ }), as shown in the figure below:



Note that all three $\vec{E}$-field vectors are $\parallel \hat{y}$ {*i.e.* point out of the page} and thus all three $\vec{E}$-field vectors are $\parallel$ to the boundary/interface at $z = 0$, which lies in the *x*-*y* plane.

Since the three $\vec{B}$-field vectors are related to their respective $\vec{E}$-field vectors by the right-hand rule cross-product relation $\vec{B} = \frac{1}{v}\hat{k} \times \vec{E}$ then we see that all three $\vec{B}$-field vectors lie in the *x*-*z* plane {the plane of incidence}, as shown in the figure above.

The four boundary conditions on the {complex} $\vec{E}$- and $\vec{B}$-fields on the boundary at $z = 0$ are:

BC 1) Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges)

$$\boxed{\varepsilon_1\left(\overset{=0}{\cancel{\tilde{E}_{o_{inc_z}}}} + \overset{=0}{\cancel{\tilde{E}_{o_{refl_z}}}}\right) = \varepsilon_2 \overset{=0}{\cancel{\tilde{E}_{o_{trans_z}}}}} \Rightarrow \boxed{0 + 0 = 0} \quad \text{\{see/refer to above figure\}}$$

BC 2) Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_y}} + \tilde{E}_{o_{refl_y}}\right) = \tilde{E}_{o_{trans_y}}} \Rightarrow \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}} \quad \{n.b. \text{ All } E_x\text{'s} = 0 \text{ for } TE \text{ Polarization}\}$$

BC 3) Normal (*i.e.* *z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 3) {continued}: *n.b.* Since only the <u>*z*-components</u> of $\vec{B}$'s on either side of interface are involved here, and all unit wavevectors $\hat{k}_{inc}$, $\hat{k}_{refl}$ and $\hat{k}_{trans}$ lie in the plane of incidence (*x*-*y* plane) and all $\vec{E}$-field vectors are || to the $+\hat{y}$ direction for *TE* polarization, then because of the cross-product nature of $\tilde{\vec{B}} = \frac{1}{v}\hat{k}\times\tilde{\vec{E}}$, we only need the <u>*x*-components</u> of the unit wavevectors, *i.e.*:

$$\left.\begin{array}{|l|}\hline \hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z} \\ \hline \hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z} \\ \hline \hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z} \\ \hline \end{array}\right\} \boxed{\text{See/refer to above figure}}$$

$$\boxed{\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z}} = \boxed{\frac{1}{v_1}\left(\hat{k}_{inc_x}\times\tilde{E}_{o_{inc_y}}\hat{y} + \hat{k}_{refl_x}\times\tilde{E}_{o_{refl_y}}\hat{y}\right) = \frac{1}{v_2}\left(\hat{k}_{trans_x}\times\tilde{E}_{o_{trans_y}}\hat{y}\right)} \quad \boxed{\{\hat{x}\times\hat{y} = +\hat{z}\}}$$

$$= \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc}\{\hat{x}\times\hat{y}\} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\{\hat{x}\times\hat{y}\}\right) = \frac{1}{v_2}\left(\tilde{E}_{o_{trans}}\sin\theta_{trans}\{\hat{x}\times\hat{y}\}\right)}$$

$$= \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}}$$

BC 4) Tangential (*i.e.* *x*-, *y*-) components of $\vec{H}$ continuous at $z = 0$ (no free surface currents):

n.b. Same reasoning as in BC3 above, but here we only need the <u>*z*-components</u> of the unit wavevectors, *i.e.*:

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x} + \tilde{B}_{o_{refl_x}}\hat{x}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}} \quad \{n.b.\text{ All } B_y^{'s} = 0 \text{ for } TE \text{ Polarization – see above pix}\}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\hat{k}_{inc_z}\times\tilde{E}_{o_{inc_y}}\hat{y} + \hat{k}_{refl_z}\times\tilde{E}_{o_{refl_y}}\hat{y}\right) = \frac{1}{\mu_2 v_2}\left(\hat{k}_{trans_z}\times\tilde{E}_{o_{trans_y}}\hat{y}\right)} \quad \boxed{\{\hat{z}\times\hat{y} = -\hat{x}\}}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}\cos\theta_{inc}\{\hat{z}\times\hat{y}\} + \tilde{E}_{o_{refl}}\cos\theta_{refl}\{-\hat{z}\times\hat{y}\}\right) = \frac{1}{\mu_2 v_2}\left(\tilde{E}_{o_{trans}}\cos\theta_{trans}\{\hat{z}\times\hat{y}\}\right)}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}(-\cos\theta_{inc}) + \tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}(-\cos\theta_{trans})\hat{x}}$$

<u>Thus we obtain</u>: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$ (from BC 2))

Using the Law of Reflection $\theta_{inc} = \theta_{refl}$ on the BC 3) result: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Using Snell's Law / Law of Refraction:

$$\boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

<u>or</u>: $\boxed{v_2\sin\theta_{inc} = v_1\sin\theta_{trans}}$ <u>or</u>: $\boxed{\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

$\therefore$ $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\dfrac{v_1}{v_2} \cdot \dfrac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \tilde{E}_{o_{trans}}}$ *i.e.* BC 3) gives the same info as BC 1) !

From the BC 4) result: $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2 v_2} \cdot \dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Thus, {again} from BC 1) → BC 4) we actually have *only* two independent relations for the case of transverse electric (*TE*) polarization:

1) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

2) $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2 v_2} \cdot \dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Now: $\boxed{\beta \equiv \left(\dfrac{\mu_1 v_1}{\mu_2 v_2}\right)}$ and we define: $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ {*n.b.* Both α and β > 0}

Then eqn. 2) above becomes: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\ \tilde{E}_{o_{trans}}}$ and eqn. 1) is: $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Add eqn's 1) + 2) to get: $\boxed{2\tilde{E}_{o_{inc}} = (1+\alpha\beta)\tilde{E}_{o_{trans}}}$ $\Rightarrow$ $\boxed{\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ eqn. (1+2)

Subtract eqn's 2) – 1) to get: $\boxed{2\tilde{E}_{o_{refl}} = (1-\alpha\beta)\tilde{E}_{o_{trans}}}$ $\Rightarrow$ $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\alpha\beta}{2}\right)\tilde{E}_{o_{trans}}}$ eqn. (2–1)

Plug eqn. (2+1) into eqn. (2–1) to obtain: $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\alpha\beta}{2}\right)\left(\dfrac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$

Thus: $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ and $\boxed{\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$ or: $\boxed{\dfrac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)}$ and $\boxed{\dfrac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\dfrac{2}{1+\alpha\beta}\right)}$

n.b. since α and β > 0, then $\left(\dfrac{2}{1+\alpha\beta}\right) > 0$ and hence the transmitted wave is *always* in-phase with the incident wave for *TE* polarization.

The real / physical electric field amplitudes for transverse electric (*TE*) polarization are thus:

The Fresnel Equations for $\vec{E} \parallel$ to Interface
= $\vec{E} \perp$ Plane of Incidence = Transverse Electric (*TE*) Polarization

$\boxed{E^{TE}_{o_{refl}} = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}}$ and $\boxed{E^{TE}_{o_{trans}} = \left(\dfrac{2}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}}$ with $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ and $\boxed{\beta \equiv \left(\dfrac{\mu_1 v_1}{\mu_2 v_2}\right)}$

Now because the incident monochromatic plane *EM* wave strikes the interface (lying in the *x*-*y* plane) at an oblique angle θ_{inc}, the instantaneous power per unit area striking the interface is *not*

$$\boxed{\left|\vec{S}_{inc}\right| = \frac{1}{\mu_1}\left|\left(\vec{E}_{inc}\times\vec{B}_{inc}\right)\right|}$$ but instead is actually: $$\boxed{\left|\vec{S}_{inc}\bullet\hat{z}\right| = \left|\vec{S}_{inc}\right|\cos\theta_{inc}}$$ $\hat{z}$ = unit normal to the interface

Thus, the *time-averaged* incident intensity (*aka* irradiance) for an oblique angle of incidence is:

$$\boxed{I_{inc} \equiv \left|\left\langle\vec{S}_{inc}(t)\right\rangle\bullet\hat{z}\right| = \left|\left\langle\vec{S}_{inc}(t)\right\rangle\cos\theta_{inc}\right| = \left|\left\langle\vec{S}_{inc}(t)\right\rangle\right|\cos\theta_{inc}}$$

Note also that because the incident *EM* wave is now propagating in a physical linear / homogeneous / isotropic medium that Poynting's vector becomes:

$$\boxed{\vec{S}_{inc} = \frac{1}{\mu_1}\left(\vec{E}_{inc}\times\vec{B}_{inc}\right) = \frac{1}{\mu_1}\left(E_{o_{inc}}\hat{y}\times\left\{\frac{1}{v_1}\left(\hat{k}_{inc}\times E_{o_{inc}}\hat{y}\right)\right\}\right) = \frac{1}{v_1\mu_1}E^2_{o_{inc}}\left\{\hat{y}\times\left(\hat{k}_{inc}\times\hat{y}\right)\right\} = \frac{1}{v_1\mu_1}E^2_{o_{inc}}\hat{k}_{inc}}$$

But: $$\boxed{\left\{\hat{y}\times\left(\hat{k}_{inc}\times\hat{y}\right)\right\} = \hat{k}_{inc}\left(\hat{y}\bullet\hat{y}\right) - \hat{y}\underbrace{\left(\cancel{\hat{y}\bullet\hat{k}_{inc}}\right)}_{=0}}$$ and since: $\boxed{v_1^2 = 1/\varepsilon_1\mu_1}$

$$\therefore \boxed{\vec{S}_{inc} = \frac{1}{v_1\mu_1}E^2_{o_{inc}}\hat{k}_{inc} = \frac{\varepsilon_1}{v_1\left(\varepsilon_1\mu_1\right)}E^2_{o_{inc}}\hat{k}_{inc} = \frac{v_1^{\cancel{2}}\varepsilon_1}{\cancel{v_1}}E^2_{o_{inc}}\hat{k}_{inc} = v_1\varepsilon_1E^2_{o_{inc}}\hat{k}_{inc}}$$ and $$\boxed{\left\langle\vec{S}_{inc}(\vec{r},t)\right\rangle = \frac{1}{2}\vec{S}_{inc}(\vec{r},t)}$$

Thus, for *TE* polarization:

$$\boxed{I^{TE}_{inc} = \left|\left\langle\vec{S}^{TE}_{inc}(t)\right\rangle\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E^{TE}_{o_{inc}}\right)^2\right)\left|\hat{k}_{inc}\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E^{TE}_{o_{inc}}\right)^2\right)\cos\theta_{inc} = \frac{1}{2}\varepsilon_1 v_1\left(E^{TE}_{o_{inc}}\right)^2\cos\theta_{inc}}$$

Likewise, the reflected intensity is: $\boxed{I^{TE}_{refl} \equiv \left|\left\langle\vec{S}^{TE}_{refl}(t)\right\rangle\bullet\hat{z}\right|}$ $\theta_{refl} = \theta_{inc}$ by the Law of Reflection

Thus, for *TE* polarization: $$\boxed{I^{TE}_{refl} = \left|\left\langle\vec{S}^{TE}_{refl}(t)\right\rangle\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E^{TE}_{o_{refl}}\right)^2\right)\cos\theta_{refl} = \frac{1}{2}\varepsilon_1 v_1\left(E^{TE}_{o_{refl}}\right)^2\cos\theta_{inc}}$$

Likewise, the transmitted intensity is: $\boxed{I^{TE}_{trans} = \left|\left\langle\vec{S}^{TE}_{trans}(t)\right\rangle\bullet\hat{z}\right|}$ and using: $\boxed{v_2^2 = 1/\varepsilon_2\mu_2}$

Thus, for *TE* polarization: $$\boxed{I^{TE}_{trans} = \left|\left\langle\vec{S}^{TE}_{trans}(t)\right\rangle\bullet\hat{z}\right| = \left(\frac{1}{2}v_2\varepsilon_2\left(E^{TE}_{o_{trans}}\right)^2\right)\cos\theta_{trans} = \frac{1}{2}\varepsilon_2 v_2\left(E^{TE}_{o_{trans}}\right)^2\cos\theta_{trans}}$$

Thus the reflection and transmission coefficients for transverse electric (*TE*) polarization (with all $\vec{E}$-field vectors oriented $\perp$ to the plane of incidence) are:

$$\boxed{R_{TE} \equiv \frac{I^{TE}_{refl}}{I^{TE}_{inc}} = \frac{\frac{1}{2}\varepsilon_1 v_1\left(E^{TE}_{o_{refl}}\right)^2\cos\overbrace{\theta_{inc}}^{=\theta_{refl}}}{\frac{1}{2}\varepsilon_1 v_1\left(E^{TE}_{o_{inc}}\right)^2\cos\theta_{inc}} = \left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right)^2}$$ $$\boxed{T_{TE} \equiv \frac{I^{TE}_{trans}}{I^{TE}_{inc}} = \frac{\frac{1}{2}\varepsilon_2 v_2\left(E^{TE}_{o_{trans}}\right)^2\cos\theta_{trans}}{\frac{1}{2}\varepsilon_1 v_1\left(E^{TE}_{o_{inc}}\right)^2\cos\theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right)^2}$$

But: $\boxed{\beta \equiv \left(\dfrac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$ and: $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ $\therefore$ $\boxed{T_{TE} = \alpha\beta\left(\dfrac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$

And from above (*p*. 8): $\boxed{\left(\dfrac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)}$ and $\boxed{\left(\dfrac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\dfrac{2}{1+\alpha\beta}\right)}$

Thus: $\boxed{R_{TE} = \left(\dfrac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$ and: $\boxed{T_{TE} = \alpha\beta\left(\dfrac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \dfrac{4\alpha\beta}{(1+\alpha\beta)^2}}$

Explicit Check: Does $R_{TE} + T_{TE} = 1$? (*i.e.* is *EM* wave energy conserved?)

$$\frac{(1-\alpha\beta)^2}{(1+\alpha\beta)^2} + \frac{4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1-2\alpha\beta+\alpha^2\beta^2+4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1+2\alpha\beta+\alpha^2\beta^2}{(1+\alpha\beta)^2} = \frac{\cancel{(1+\alpha\beta)^2}}{\cancel{(1+\alpha\beta)^2}} = 1 \quad \text{Yes!!!}$$

Note that at normal incidence: $\boxed{\theta_{inc} = 0}$ $\Rightarrow$ $\boxed{\theta_{refl} = 0}$ and $\boxed{\theta_{trans} = 0}$ {See/refer to above figure}

Then: $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) = \dfrac{\cos 0}{\cos 0} = 1}$ $\Rightarrow$ $\boxed{\alpha = 1}$

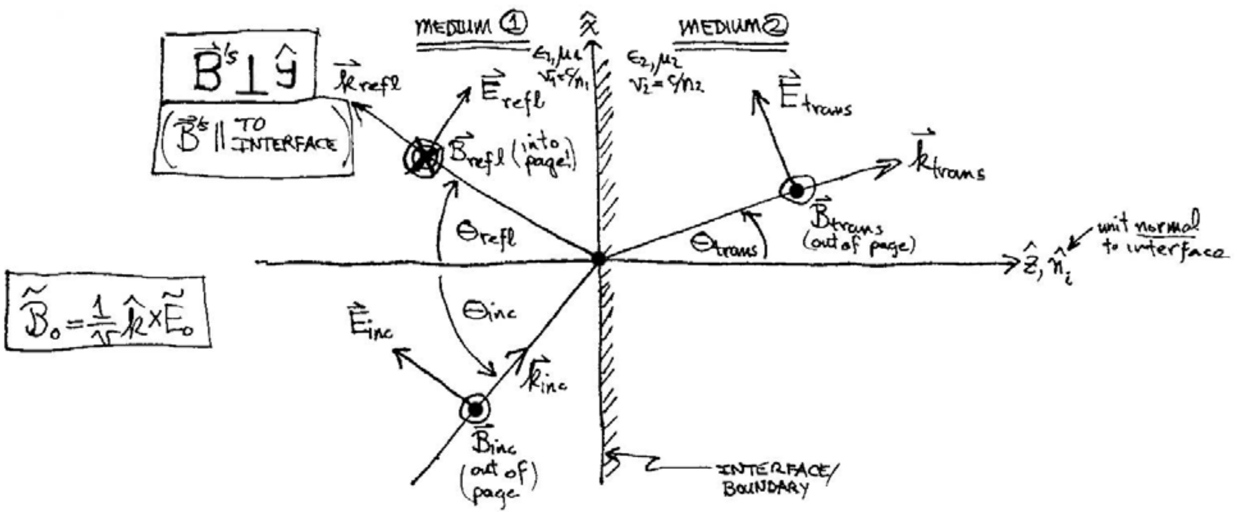
Thus at normal incidence: $\boxed{R_{TE}\Big|_{\theta_{inc}=0} = \left(\dfrac{1-\beta}{1+\beta}\right)^2}$ and $\boxed{T_{TE}\Big|_{\theta_{inc}=0} = \dfrac{4\beta}{(1+\beta)^2}}$

Note that these results for $R_{TE}\Big|_{\theta_{inc}=0}$ and $T_{TE}\Big|_{\theta_{inc}=0}$ are the same/identical to those we obtained previously for a monochromatic plane *EM* wave at normal incidence on interface!!!

In the special/limiting-case situation of normal incidence, where $\boxed{\theta_{inc} = \theta_{refl} = \theta_{trans} = 0}$, the plane of incidence *collapses* into a line (the $\hat{z}$ axis), the problem then has rotational invariance about the $\hat{z}$ axis, and thus for normal incidence the polarization direction associated with the spatial orientation of $\vec{E}_{inc}$ no longer has any physical consequence(s).

Case II): Electric Field Vectors Parallel to the Plane of Incidence: Transverse Magnetic (*TM*) Polarization

A monochromatic plane *EM* wave is incident {from the left} on a boundary located at $z = 0$ in the *x-y* plane between two linear / homogeneous / isotropic media at an oblique angle of incidence. The polarization of the incident *EM* wave (*i.e.* the orientation of $\vec{E}_{inc}$ is now *parallel* (*i.e.* $\parallel$) to the plane of incidence {= the *x-z* plane containing the three wavevectors $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ *and* the unit normal to the boundary/interface, $\hat{n} = +\hat{z}$ }), as shown in the figure below:



In this situation, all three $\vec{E}$-field vectors lie in the plane of incidence.

Since the three $\vec{B}$-field vectors are related to their respective $\vec{E}$-field vectors by the right-hand rule cross-product relation $\vec{B} = \frac{1}{v}\hat{k}\times\vec{E}$ then we see that all three $\vec{B}$-field vectors are $\parallel \hat{y}$ {*i.e.* either point out of or into the page} and thus are $\perp$ to the plane of incidence {hence the origin of the name transverse magnetic polarization}; note that all three $\vec{B}$-field vectors are also $\parallel$ to the boundary/interface at $z = 0$, which lies in the *x-y* plane as shown in the figure above.

The four boundary conditions on the {complex} $\vec{E}$- and $\vec{B}$-fields on the boundary at $z = 0$ are:

BC 1) Normal (*i.e.* *z*-) component of $\vec{D}$ continuous at $z = 0$ (no free surface charges)

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2\tilde{E}_{o_{trans_z}}}$$

$$\boxed{\varepsilon_1\left(-\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right) = \varepsilon_2\left(-\tilde{E}_{o_{trans}}\sin\theta_{trans}\right)}$$ {*n.b.* see/refer to above figure}

BC 2) Tangential (*i.e.* *x*-, *y*-) components of $\vec{E}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc_x}} + \tilde{E}_{o_{refl_x}}\right) = \tilde{E}_{o_{trans_x}}}$$

$$\boxed{\left(\tilde{E}_{o_{inc}}\cos\theta_{inc} + \tilde{E}_{o_{refl}}\cos\theta_{refl}\right) = \tilde{E}_{o_{trans}}\cos\theta_{trans}}$$ {*n.b.* see/refer to above figure}

BC 3) Normal (*i.e.* *z*-) component of $\vec{B}$ continuous at $z = 0$:

$$\boxed{\left(\cancelto{=0}{\tilde{B}_{o_{inc_z}}} + \cancelto{=0}{\tilde{B}_{o_{refl_z}}}\right) = \cancelto{=0}{\tilde{B}_{o_{trans_z}}}} \Rightarrow \boxed{0+0=0} \quad \{n.b. \text{ see/refer to above figure}\}$$

BC 4) Tangential (*i.e.* *x*-, *y*-) components of $\vec{H}$ continuous at $z = 0$ (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \quad \{n.b. \text{ All } B_x\text{'s} = 0 \text{ for } TM \text{ Polarization}\}$$

$$\therefore \quad \boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}}\hat{y} + \tilde{B}_{o_{refl_y}}\hat{y}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\hat{y}\right)} \quad n.b. \text{ Can use full cross-product(s) } \tilde{\vec{B}} = \tfrac{1}{v}\hat{k}\times\tilde{\vec{E}} \text{ here!}$$

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\hat{k}_{inc}\times\tilde{\vec{E}}_{o_{inc}} + \hat{k}_{refl}\times\tilde{\vec{E}}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\left(\hat{k}_{trans}\times\tilde{\vec{E}}_{o_{trans}}\right)}$$

Use right-hand rule for all cross-products

$$= \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}\hat{y} - \tilde{E}_{o_{refl}}\hat{y}\right) = \frac{1}{\mu_2 v_2}\left(\tilde{E}_{o_{trans}}\hat{y}\right)} \quad \{n.b. \text{ see/refer to above figure}\}$$

$$\therefore \quad \boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \Rightarrow \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$$

Redundant info – both BC's give same relation

From BC 1) at $z = 0$:

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} - \tilde{E}_{o_{refl}}\sin\theta_{refl}\right) = \varepsilon_2\left(\tilde{E}_{o_{trans}}\sin\theta_{trans}\right)}$$

<u>But</u>: $\boxed{\theta_{inc} = \theta_{refl}}$ (Law of Reflection) <u>and</u>: $\boxed{n_1 = \frac{c}{v_1}}$, $\boxed{n_2 = \frac{c}{v_2}}$

<u>And</u>: $\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \Rightarrow \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{\sin\theta_{trans}}{\sin\theta_{inc}} = \frac{n_1}{n_2}(\text{Snell's Law}) = \frac{v_2}{v_1}}$

$$\therefore \quad \boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2}\right)\tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\tilde{E}_{o_{trans}} = \beta\,\tilde{E}_{o_{trans}}}$$

From BC 4) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\tilde{E}_{o_{trans}} = \beta\tilde{E}_{o_{trans}}} \quad \text{where: } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$$

From BC 2) at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc}}\cos\theta_{inc} + \tilde{E}_{o_{refl}}\cos\theta_{refl}\right) = \tilde{E}_{o_{trans}}\cos\theta_{trans}} \quad \text{but: } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\therefore \quad \boxed{\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha\tilde{E}_{o_{trans}}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\tilde{E}_{o_{trans}}} \text{ and } \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha\tilde{E}_{o_{trans}}} \text{ with } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)} \text{ and } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

Solving these two above equations simultaneously, we obtain:

$$\boxed{2\tilde{E}_{o_{inc}} = (\alpha+\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}}$$

and: $$\boxed{2\tilde{E}_{o_{refl}} = (\alpha-\beta)\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}}}$$

$$\Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}}$$

The real / physical electric field amplitudes for transverse magnetic (*TM*) polarization are thus:

The Fresnel Equations for $\vec{B} \parallel$ to Interface

= $\vec{B} \perp$ Plane of Incidence = Transverse Magnetic (*TM*) Polarization

$$\boxed{E^{TM}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)E^{TM}_{o_{inc}}} \text{ and } \boxed{E^{TM}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)E^{TM}_{o_{inc}}} \text{ with } \boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}} \text{ and } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$$

Or: $$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)} \text{ and } \boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{2}{\alpha+\beta}\right)}$$

Note that the Fresnel relations for *TM* polarization are *not* identical to Fresnel relations for *TE* polarization:

$$\boxed{E^{TE}_{o_{refl}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}} \text{ and } \boxed{E^{TE}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)E^{TE}_{o_{inc}}} \text{ with } \boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)} \text{ and } \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

Or: $$\boxed{\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \text{ and: } \boxed{\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{2}{1+\alpha\beta}\right)}$$

We define the incident, reflected & transmitted intensities at oblique incidence for the *TM* case as we did for the *TE* case:

$$\boxed{I^{TM}_{inc} = v_1\left|\left\langle \vec{S}^{TM}_{inc}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E^{TM}_{o_{inc}}\right)^2\right)\cos\theta_{inc} = \frac{1}{2}\varepsilon_1 v_1\left(E^{TM}_{o_{inc}}\right)^2\cos\theta_{inc}}$$

$$\boxed{I^{TM}_{refl} = v_1\left|\left\langle \vec{S}^{TM}_{refl}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E^{TM}_{o_{refl}}\right)^2\right)\cos\theta_{refl} = \frac{1}{2}\varepsilon_1 v_1\left(E^{TM}_{o_{refl}}\right)^2\cos\theta_{inc}}$$

$$\boxed{I^{TM}_{trans} = v_2\left|\left\langle \vec{S}^{TM}_{trans}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_2\varepsilon_2\left(E^{TM}_{o_{trans}}\right)^2\right)\cos\theta_{trans} = \frac{1}{2}\varepsilon_2 v_2\left(E^{TM}_{o_{trans}}\right)^2\cos\theta_{trans}}$$

Thus, the reflection and transmission coefficients for transverse magnetic (*TM*) polarization (with all $\vec{B}$ -field vectors oriented $\perp$ to the plane of incidence) are:

$$\boxed{R_{TM} \equiv \frac{I_{refl}^{TM}}{I_{inc}^{TM}} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2} \quad \boxed{T_{TM} \equiv \frac{I_{trans}^{TM}}{I_{inc}^{TM}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}}$$

i.e. $\boxed{R_{TM} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2}$ <u>and</u>: $\boxed{T_{TM} = \alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}}$

Again, note that the reflection and transmission coefficients for transverse magnetic (TM) polarization are <u>*not*</u> identical/the same as those for the transverse electric case:

$\boxed{R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$ <u>and</u>: $\boxed{T_{TE} = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}}$

<u>Explicit Check</u>: Does $R_{TM} + T_{TM} = 1$? (*i.e.* is *EM* wave energy conserved?)

$$R_{TM} + T_{TM} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 + \frac{4\alpha\beta}{(\alpha+\beta)^2} = \frac{\alpha^2 - 2\alpha\beta + \beta^2 + 4\alpha\beta}{(\alpha+\beta)^2} = \frac{\alpha^2 + 2\alpha\beta + \beta^2}{(\alpha+\beta)^2} = \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2} = 1 \text{ Yes !!!}$$

Note again at <u>normal</u> incidence: $\theta_{inc} = 0 \;\Rightarrow\; \theta_{refl} = 0$ and $\theta_{trans} = 0$ {See/refer to above figure}

<u>Then</u>: $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) = \frac{\cos 0}{\cos 0} = 1} \Rightarrow \boxed{\alpha = 1}$

Thus at <u>normal</u> incidence: $\boxed{R_{TM}\Big|_{\theta_{inc}=0} = \left(\frac{1-\beta}{1+\beta}\right)^2}$ and $\boxed{T_{TM}\Big|_{\theta_{inc}=0} = \frac{4\beta}{(1+\beta)^2}}$

These <u>*are*</u> identical to those for the *TE* case at normal incidence, as expected – due to rotational invariance / symmetry about the $\hat{z}$ axis:

At <u>normal</u> incidence: $\boxed{R_{TE}\Big|_{\theta_{inc}=0} = \left(\frac{1-\beta}{1+\beta}\right)^2}$ and $\boxed{T_{TE}\Big|_{\theta_{inc}=0} = \frac{4\beta}{(1+\beta)^2}}$

The Fresnel Equations

TE Polarization

$$\boxed{\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right)=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}$$

$$\boxed{\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right)=\frac{2}{(1+\alpha\beta)}}$$

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$$

TM Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right)=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)}$$

$$\boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right)=\frac{2}{(\alpha+\beta)}}$$

$$\boxed{v_1=c/n_1=1/\sqrt{\varepsilon_1\mu_1}}$$

$$\boxed{v_2=c/n_2=1/\sqrt{\varepsilon_2\mu_2}}$$

Reflection and Transmission Coefficients *R* & *T*
R + *T* = 1

TE Polarization

$$\boxed{R_{TE}\equiv\frac{I^{TE}_{refl}}{I^{TE}_{inc}}=\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right)^2=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2}$$

$$\boxed{T_{TE}\equiv\left(\frac{I^{TE}_{trans}}{I^{TE}_{inc}}\right)=\alpha\beta\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right)^2=\frac{4\alpha\beta}{(1+\alpha\beta)^2}}$$

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$$

TM Polarization

$$\boxed{R_{TM}\equiv\frac{I^{TM}_{refl}}{I^{TM}_{inc}}=\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right)^2=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2}$$

$$\boxed{T_{TM}\equiv\left(\frac{I^{TM}_{trans}}{I^{TM}_{inc}}\right)=\alpha\beta\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right)^2=\frac{4\alpha\beta}{(\alpha+\beta)^2}}$$

$$\boxed{v_1=c/n_1=1/\sqrt{\varepsilon_1\mu_1}}$$

$$\boxed{v_2=c/n_2=1/\sqrt{\varepsilon_2\mu_2}}$$

Note that since $E^2_{o_{1,2}}=\langle n_{\gamma_{1,2}}(t)\rangle E_\gamma/\varepsilon_{1,2}$, the reflection coefficient/reflectance R can thus be seen as the statistical/ensemble average probability that at the microscopic scale, individual photons will be reflected at the interface: $R=\left(E_{o_{refl}}/E_{o_{inc}}\right)^2=\langle n_{\gamma_{refl}}(t)\rangle/\langle n_{\gamma_{inc}}(t)\rangle=P_{refl}$, and since $R+T=1$ then $T=1-R=1-P_{refl}=P_{trans}$, since we must have $P_{refl}+P_{trans}=1$!!!

Now we want to explore / investigate the physics associated with the Fresnel Equations and the reflection and transmission coefficients – comparing results for *TE* vs. *TM* polarization for the cases of external reflection ($n_1 < n_2$) and internal reflection $n_1 > n_2$)

Just as β can be written several different but equivalent ways (see above), so can the Fresnel Equations, as well as the expressions for *R* & *T* using various relations including Snell's Law.

Starting with the Fresnel Relations as given above, explicitly writing these out alternate versions:

Fresnel Equations

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} - \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} - \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}}$$

If we now neglect / ignore the magnetic properties of the two media – *e.g.* if paramagnetic / diamagnetic such that $|\chi_m| \ll 1$ then $\mu_1 \simeq \mu_2 \simeq \mu_o$ the Fresnel Relations then become:

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{n_1\cos\theta_{inc} - n_2\cos\theta_{trans}}{n_1\cos\theta_{inc} + n_2\cos\theta_{trans}}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{2n_1\cos\theta_{inc}}{n_1\cos\theta_{inc} + n_2\cos\theta_{trans}}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{-n_2\cos\theta_{inc} + n_1\cos\theta_{trans}}{n_2\cos\theta_{inc} + n_1\cos\theta_{trans}}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{2n_1\cos\theta_{inc}}{n_2\cos\theta_{inc} + n_1\cos\theta_{trans}}}$$

Using Snell's Law $n_1\sin\theta_1 = n_2\sin\theta_2 \;\Rightarrow\; n_{inc}\sin\theta_{inc} = n_{trans}\sin\theta_{trans}$ and various trigonometric identities, the above relations can also equivalently be written as:

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq -\frac{\sin(\theta_{inc} - \theta_{trans})}{\sin(\theta_{inc} + \theta_{trans})}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc} + \theta_{trans})}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq -\frac{\tan(\theta_{inc} - \theta_{trans})}{\tan(\theta_{inc} + \theta_{trans})}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{trans}}^{TM}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc} + \theta_{trans})\cos(\theta_{inc} - \theta_{trans})}}$$

n.b. the signs correlate to the *TE* & *TM* $\vec{E}$-field vectors as shown in the above figures!

We now use Snell's Law $n_{inc}\sin\theta_{inc} = n_{trans}\sin\theta_{trans}$ to eliminate θ_{trans} :

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

$$\boxed{\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{2\cos\theta_{inc}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

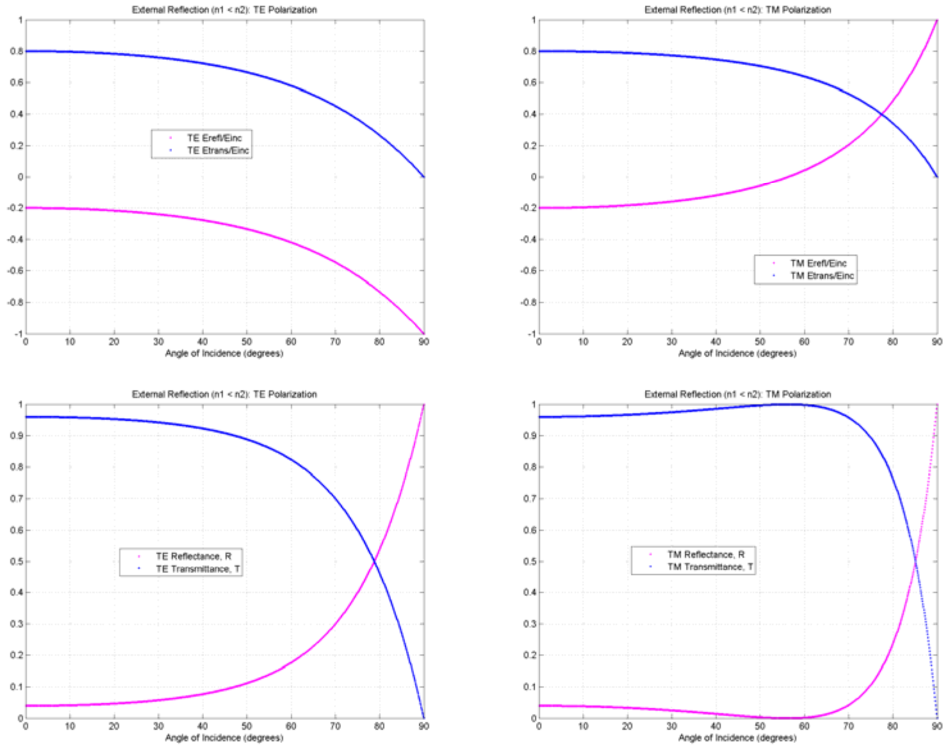
TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

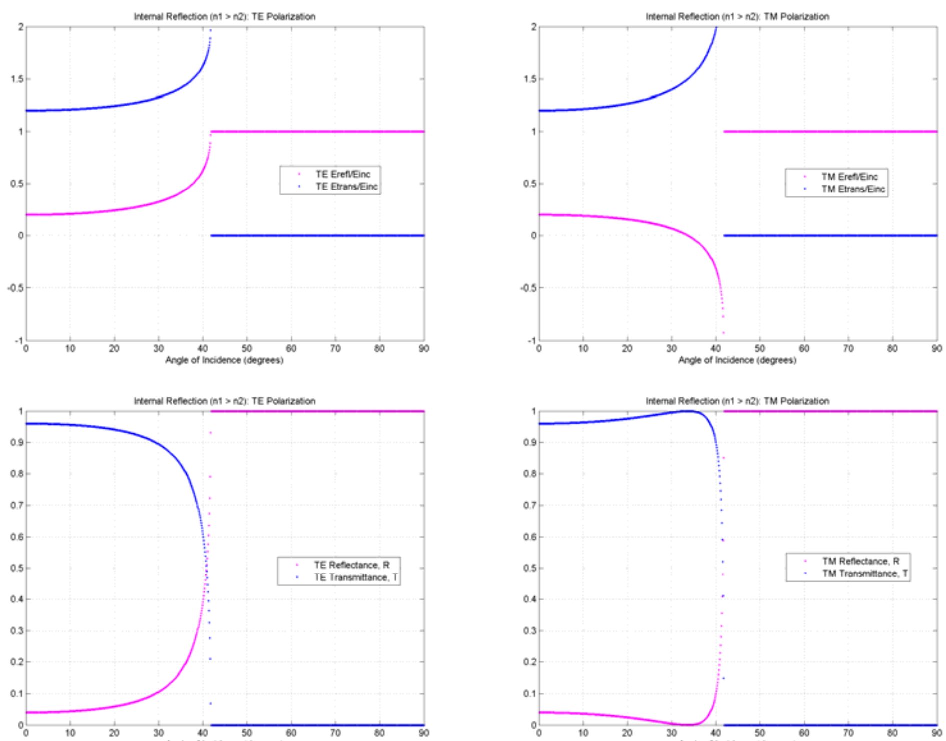
$$\boxed{\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{2\left(\frac{n_2}{n_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

The variation of $\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)$, $\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)$ and the reflection coefficient (*aka* the *reflectance*) $R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2$ and transmission coefficient (*aka* the *transmittance*) $T = \alpha\beta\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \frac{\sqrt{(n_2/n_1)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc}}\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2$ as a function of the angle of incidence θ_{inc} for *external* reflection ($n_1 < n_2$) and *internal* reflection ($n_1 > n_2$) for *TE* & *TM* polarization are shown in the figures below:

External Reflection ($n_1 = 1.0 < n_2 = 1.5$):



Internal Reflection ($n_1 = 1.5 > n_2 = 1.0$):



Comment 1):

When $\left(E_{refl}/E_{inc}\right)<0$, $E_{o_{refl}}$ is 180° out-of-phase with $E_{o_{inc}}$ since the ***numerators*** of the original Fresnel Equations for *TE* & *TM* polarization are $(1-\alpha\beta)$ and $(\alpha-\beta)$ respectively.

Comment 2):

For *TM* Polarization (only), there exists an angle of incidence where $\left(E_{refl}/E_{inc}\right)=0$, *i.e.* no reflected wave occurs at this angle for *TM* polarization! This angle is known as Brewster's angle θ_B (also known as the polarizing angle θ_P - because an incident wave which is a linear combination of *TE and TM* polarizations will have a reflected wave which is 100% pure-*TE* polarized for an incidence angle $\theta_{inc}=\theta_B=\theta_P$!!). * *n.b.* Brewster's angle θ_B exists for *both* external ($n_1 < n_2$) & internal reflection ($n_1 > n_2$) for *TM* polarization (only). *

Brewster's Angle θ_B / the Polarizing Angle θ_P for Transverse Magnetic (TM) Polarization

From the numerator of $\left(E_{o_{refl}}^{TM}/E_{o_{inc}}^{TM}\right)=\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)$ of the originally-derived expression for *TM* polarization, when this ratio = 0 at Brewster's angle θ_B = polarizing angle θ_P, we see that this occurs when $(\alpha-\beta)=0$, *i.e.* when $\alpha=\beta$.

But: $\boxed{\alpha\equiv\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}}$ and $\boxed{\beta\equiv\dfrac{\mu_1 n_2}{\mu_2 n_1}\simeq\dfrac{n_2}{n_1}}$ for $\boxed{\mu_1\simeq\mu_2\simeq\mu_o}$

Now: $\cos\theta_{trans}=\sqrt{1-\sin^2\theta_{trans}}$ and Snell's Law: $n_1\sin\theta_{inc}=n_2\sin\theta_{trans}\;\Rightarrow\;\sin\theta_{trans}=\left(\dfrac{n_1}{n_2}\right)\sin\theta_{inc}$

$\therefore$ at Brewster's angle $\theta_{inc}=\theta_B$ = polarizing angle θ_P where $\alpha=\beta$, this relation becomes:

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}=\beta\equiv\frac{\mu_1 n_2}{\mu_2 n_1}\simeq\frac{n_2}{n_1}}\text{ for }\boxed{\mu_1\simeq\mu_2\simeq\mu_o}\;\Rightarrow\;\boxed{\alpha=\frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}}}{\cos\theta_{inc}}\simeq\left(\frac{n_2}{n_1}\right)=\beta}$$

or: $$\boxed{1-\frac{1}{\beta^2}\sin^2\theta_{inc}=\beta^2\cos^2\theta_{inc}=\beta^2\left(1-\sin^2\theta_{inc}\right)}\leftarrow\text{Solve for }\sin^2\theta_{inc}$$

$$\boxed{1-\beta^2=\left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc}}\;\Rightarrow\;\boxed{\sin^2\theta_{inc}=\frac{1-\beta^2}{1/\beta^2-\beta^2}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}}$$

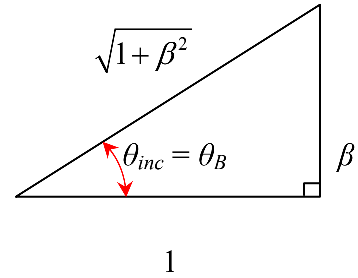
But: $\boxed{1-\beta^4=\left(1-\beta^2\right)\left(1+\beta^2\right)}$

$$\therefore\quad\boxed{\sin^2\theta_{inc}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)}=\frac{\beta^2}{1+\beta^2}}\;\Rightarrow\;\boxed{\sin\theta_{inc}=\frac{\beta}{\sqrt{1+\beta^2}}}$$

Geometrically: $\boxed{\sin\theta_{inc} = \frac{\beta}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{opp. side}}{\text{hypotenuse}}}$

$\boxed{\cos\theta_{inc} = \frac{1}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{adjacent}}{\text{hypotenuse}}}$

$\boxed{\tan\theta_{inc} = \beta} = \boxed{\frac{\text{opp. side}}{\text{adjacent}}} \simeq \left(\frac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where *__no__* *__reflected__* wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \dfrac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \dfrac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

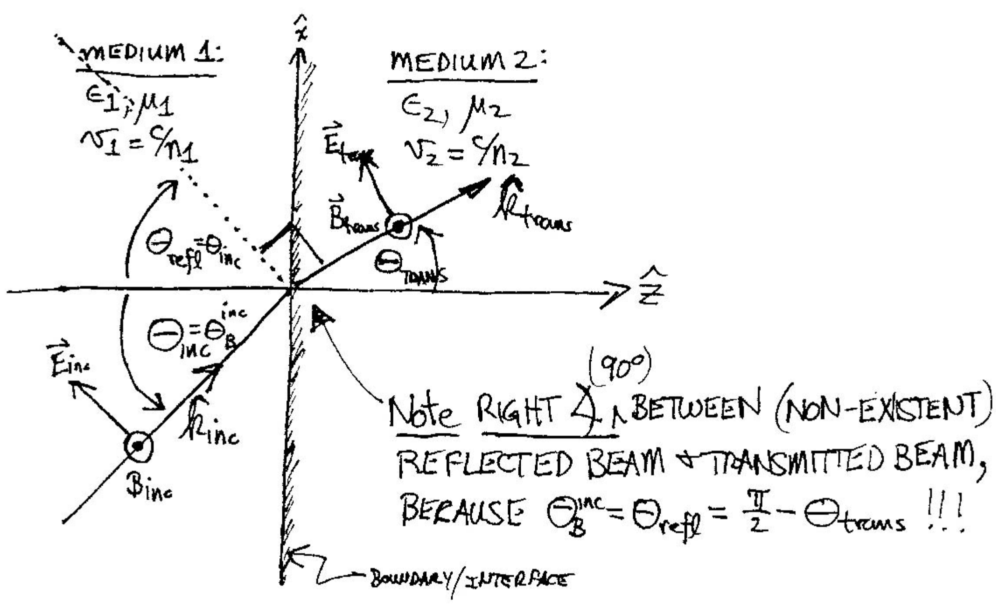
So what's so interesting about this???

__Well__: $\boxed{\cos\theta_B^{inc} = \sin\left(\tfrac{\pi}{2} - \theta_B^{inc}\right) = \sin\left(\tfrac{\pi}{2}\right)\cos\theta_B^{inc} - \cancelto{=0}{\cos\left(\tfrac{\pi}{2}\right)} \sin\theta_B^{inc} = \sin\theta_{trans}}$ *i.e.* $\boxed{\sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\theta_{trans}}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

__Thus__: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_P^{inc}$ and θ_{trans} are *__complimentary__* angles !!!

__*TM* Polarized *EM* Wave Incident at Brewster's Angle__ θ_B^{inc}:

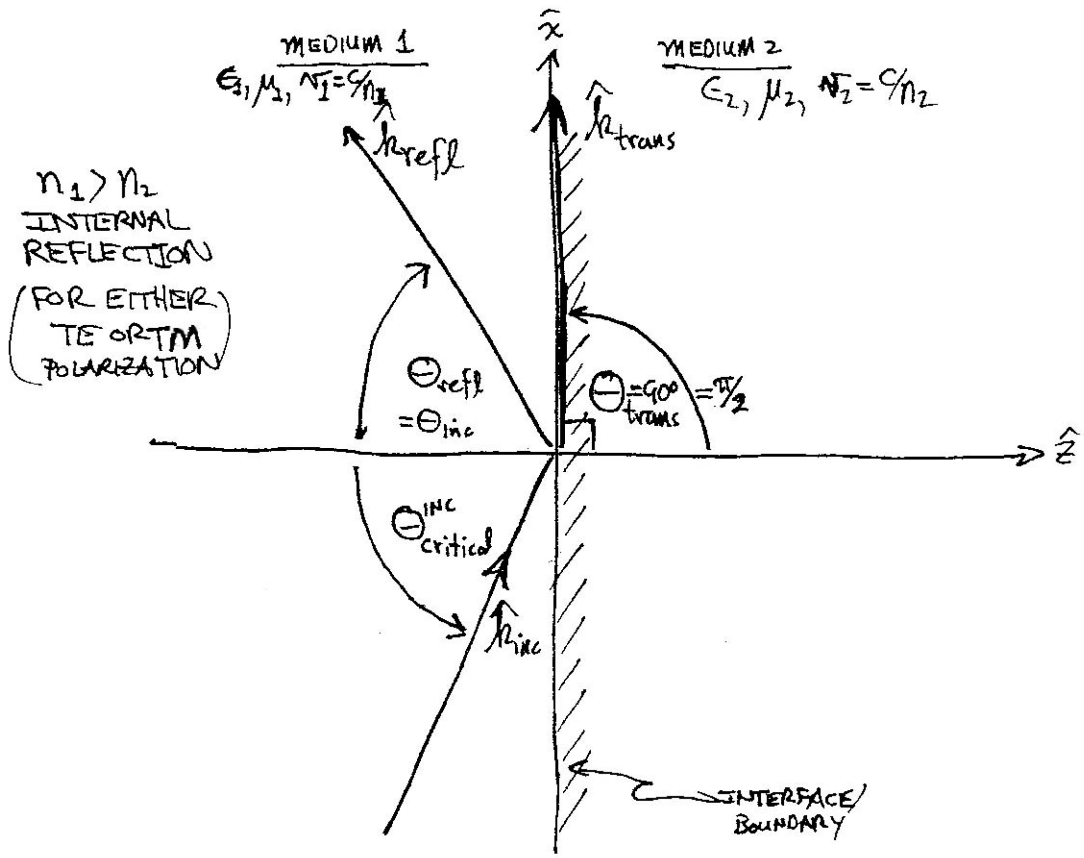


Thus, *e.g.* if an unpolarized *EM* wave (*i.e.* one which contains all polarizations/random polarizations) or an *EM* wave which is a linear combination of *TE* and *TM* polarization is incident on the interface between two linear/homogeneous/isotropic media at Brewster's angle $\theta_B^{inc} \equiv \theta_P^{inc}$, the reflected beam will be 100% pure *TE* polarization!! Hence this is why Brewster's angle θ_β is also known as the polarizing angle θ_P.

Comment 3):

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence $\theta_{critical}^{inc}$ past which *no* transmitted beam exists for either *TE* or *TM* polarization. The critical angle does *not* depend on polarization – it is actually dictated / defined by Snell's Law:

$$\boxed{n_1 \sin\theta_{critical}^{inc} = n_2 \sin\theta_{trans}^{\max} = n_2 \sin\left(\frac{\pi}{2}\right) = n_2} \quad \text{or:} \quad \boxed{\sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right)} \quad \text{or:} \quad \boxed{\theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

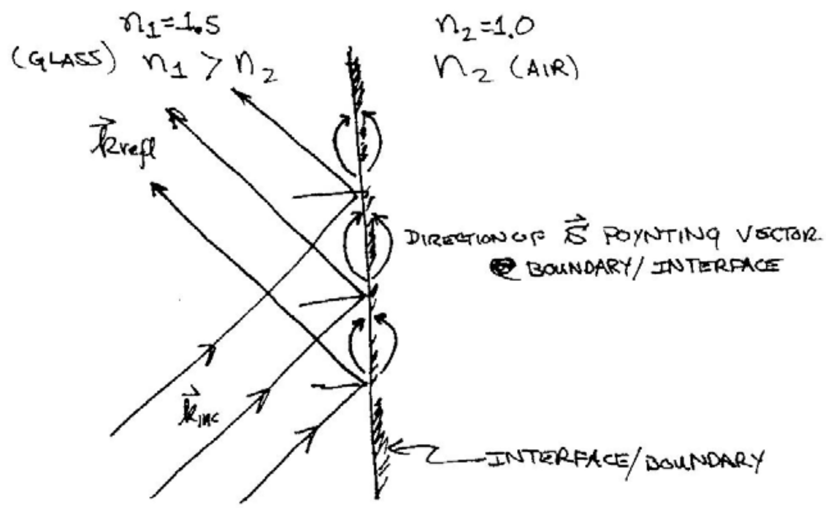


For $\theta_{inc} \geq \theta_{critical}^{inc}$, *no* transmitted beam exists → incident beam is totally internally reflected.

For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually exponentially damped – becomes a so-called:

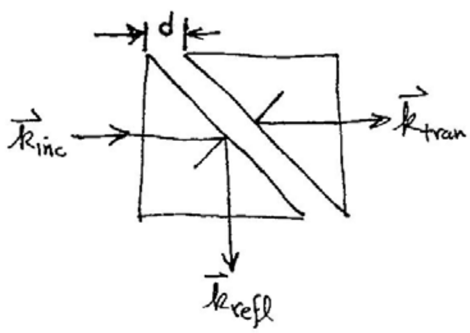
Evanescent Wave: $\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} \underbrace{e^{-\alpha z}}_{\text{Exp. damping in } z} \underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right) - \omega t\right)}}_{\text{Oscillatory along interface in } x\text{-direction}}}$ $\boxed{\alpha = k_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}}$

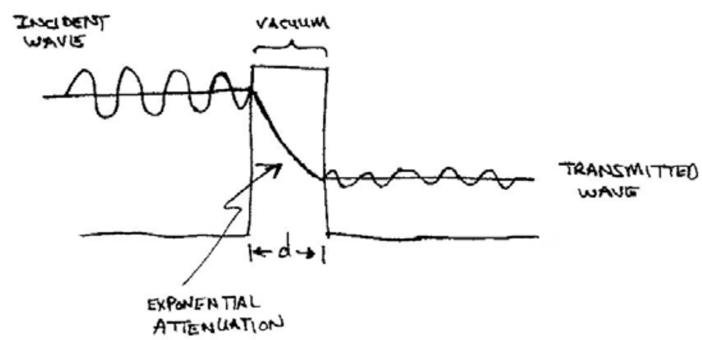
For Total Internal Reflection $\theta_{inc} \ge \theta_{critical}^{inc}$ **:**



Experimental demonstration that transmitted *EM* wave for $\theta_{inc} \ge \theta_{critical}^{inc}$ is exponentially damped $\Rightarrow$ Microscopically, this *is* an example of quantum mechanical barrier penetration / quantum mechanical tunneling phenomenon (using real photons)!!!

Use two 45° prisms – (*e.g.* glass {for light}, or $\underbrace{\text{paraffin \{for microwaves\}}}_{\text{UIUC Physics 401 experiment !!!}}$)





Phase shifts occur in reflected wave when $\theta_{inc} \ge \theta_{critical}^{inc}$ for total internal reflection ($n_1 > n_2$). Using the (last) version of Fresnel Equations (*p.* 17 of these lecture notes):

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{-\left(\frac{n_2}{n_1}\right)^2 \cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2 \cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}}$$

When $\theta_{inc} \geq \theta_{critical}^{inc}$, Snell's Law is: $\sin\theta_{critical}^{inc} = (n_2/n_1)$ {since $\sin\theta_{trans} = \sin 90^o = 1$}

The above ratios of E-field amplitudes become *complex* for internal reflection, because for $(n_2/n_1) < 1$ when $\sin^2\theta_{inc} > (n_2/n_1) < 1$, then $\sqrt{(n_2/n_1)^2 - \sin^2\theta_{inc}}$ becomes *imaginary*.

Thus for $\theta_{inc} \geq \theta_{critical}^{inc} = \sin^{-1}(n_2/n_1)$ for $n_1 > n_2$ (internal reflection), we can re-write the above $\vec{E}$-field ratios as:

TE Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\cos\theta_{inc} - i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}}$$

TM Polarization

$$\boxed{\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + i\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}}$$

It is easy to verify that these ratios lie on the unit circle in the complex plane – simply multiply them by their complex conjugates to show $AA^* = 1$, as they must for total internal reflection.

These formulae imply a phase change of the reflected wave (*relative* to incident wave) that depends on the angle of incidence $\theta_{inc} \geq \theta_{critical}^{inc} = \sin^{-1}(n_2/n_1)$ for total internal reflection.

We set: $\boxed{-\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right) = e^{-i\delta} = \frac{ae^{-i\alpha}}{ae^{+i\alpha}}} \Rightarrow \boxed{\delta = 2\alpha}$ and $\boxed{\tan(\delta/2) = \tan(\alpha)}$

Where δ = phase change (in radians) of the reflected wave relative to the incident wave.

Thus we see that (from the numerators of the above formulae) that:

$$\boxed{\tan\left(\frac{\delta_{TE}}{2}\right) = \frac{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_{inc}}} \quad \text{and:} \quad \boxed{\tan\left(\frac{\delta_{TM}}{2}\right) = \frac{\sqrt{\sin^2\theta_{inc} - \left(\frac{n_2}{n_1}\right)^2}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc}}}$$

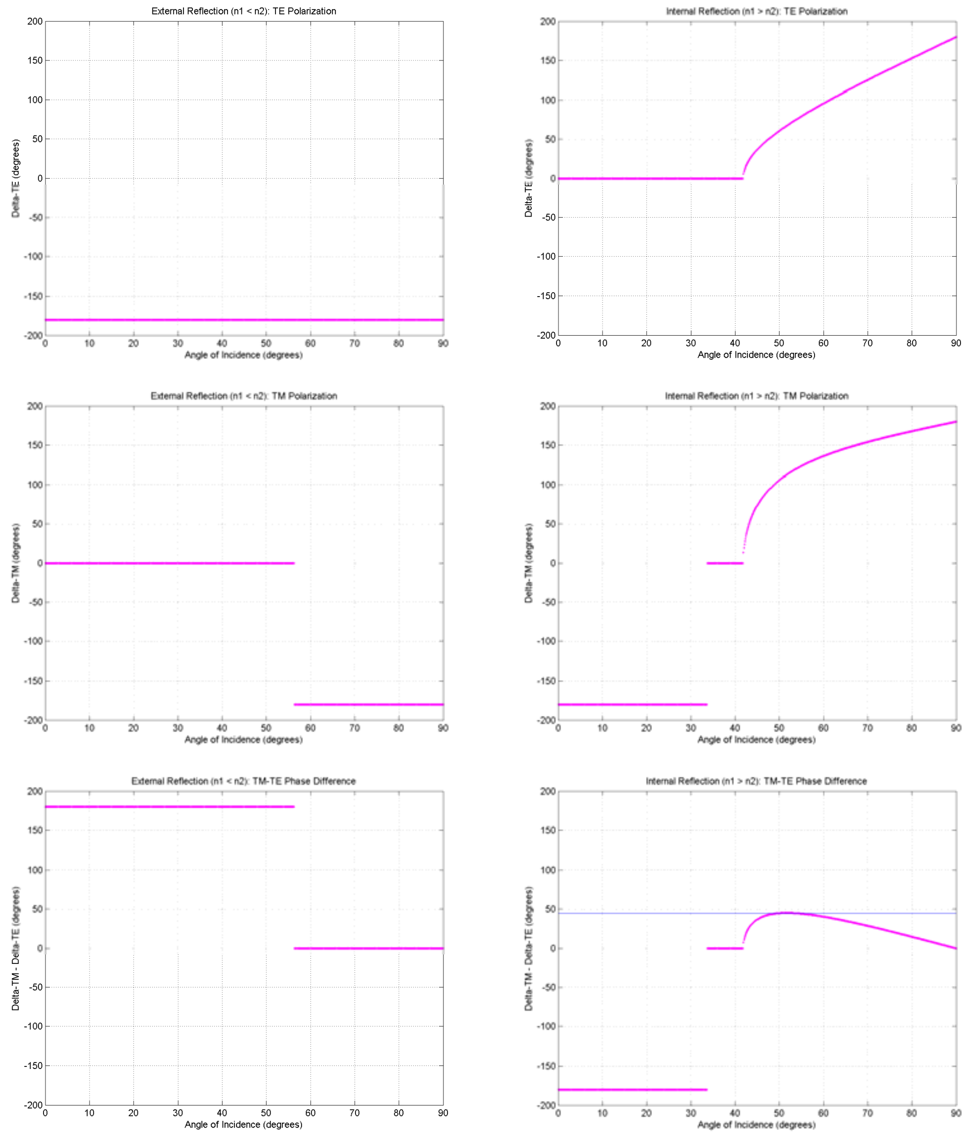
Then the relative phase *difference* $\Delta \equiv \delta_{TM} - \delta_{TE}$ between total internally-reflected TM vs. TE polarized waves can also be calculated:

$$\boxed{\tan\left(\frac{\Delta}{2}\right) = \tan\left(\frac{\delta_{TM} - \delta_{TE}}{2}\right) = \frac{\cos\theta_{inc}\sqrt{\sin^2\theta_{inc} - (n_2/n_1)^2}}{\sin^2\theta_{inc}}}$$

Phase shifts of the reflected wave ***relative*** to the incident wave for external, internal reflection and for TE, TM polarization are shown in the following graphs:

Phase Shifts Upon Reflection:

External Reflection ($n_1 = 1.0 < n_2 = 1.5$): Internal Reflection ($n_1 = 1.5 > n_2 = 1.0$):



Note that a phase shift of -180^{o} is equivalent to a phase shift of $+180^{o}$.

An Example of the {Clever} Use of Internal Reflection Phase Shifts - The Fresnel Rhomb:

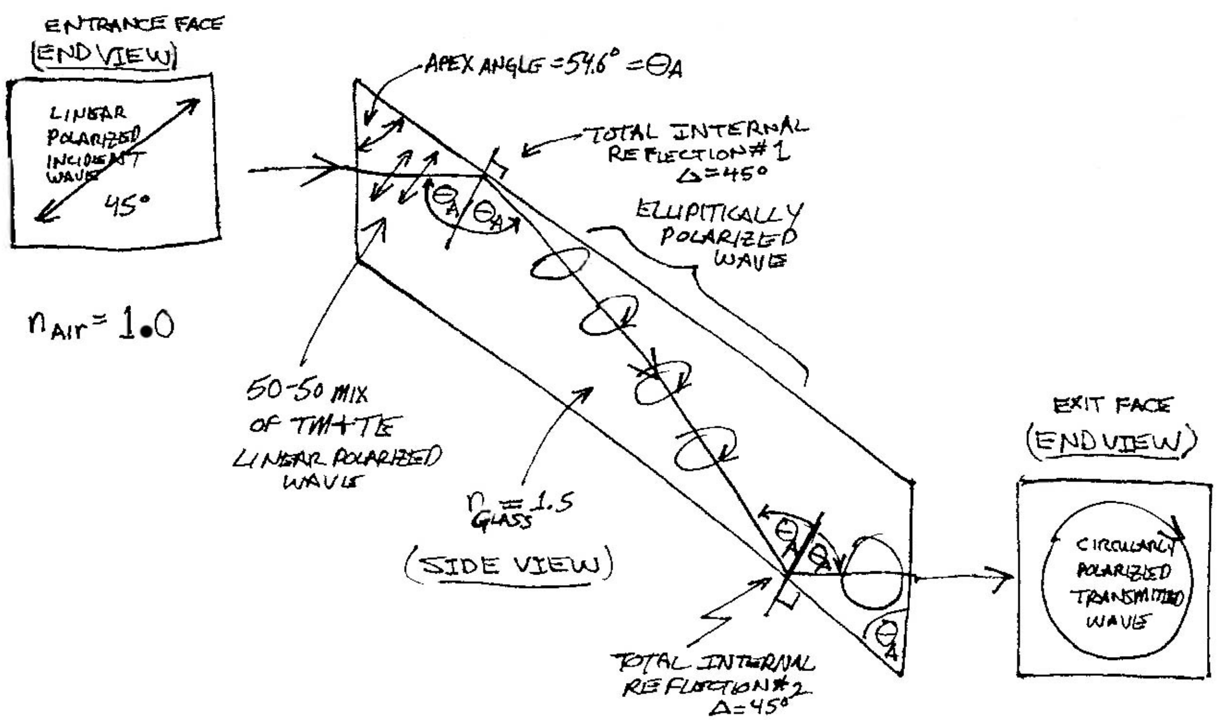
From last graph of the internal reflection phase shifts (above), we see that the relative *difference* in *TM* vs. *TE* phase shifts for total internal reflection at a glass-air interface ($n_1 = 1.5$ {glass}, $n_2 = 1.0$ {air}) is $\Delta \equiv \delta_{TM} - \delta_{TE} = \pi/4 = 45^o$ when $\theta_{inc} = 54.6^o$

Fresnel used this *TM* vs. *TE* relative phase-shift fact associated with total internal reflection and developed / designed a glass rhomb-shaped prism that converted linearly polarized light to circularly polarized light, as shown in the figure below.

He used light incident on the glass rhomb-shaped prism with polarization angle at 45^{o} with respect to face-edge of the glass rhomb (thus the incident light was a 50-50 mix of *TE* and *TM* polarization). Note that the transmitted wave actually undergoes *two* total internal reflections before emerging from rhomb at the exit face, with a -45^{o} relative phase *TM-TE* phase shift occurring at *each* total internal reflection. Thus, the first total internal reflection converts a linearly polarized wave into an elliptically polarized wave, the second total internal reflection converts the elliptically polarized wave into a circularly polarized wave!!!

The total phase shift (for 2 internal reflections): $\Delta_{tot} = 2\Delta = 2(\delta_{TM} - \delta_{TE}) = \pi/2 = 90^o$

(for rhomb apex angle $\theta_A = 54.6^{\text{o}}$, $n_{\text{air}} = 1.0$ and $n_{\text{glass}} = 1.5$)



NOTE: By time-reversal invariance of the *EM* interaction, we can also can see from the above that Fresnel's rhomb can also be used to convert circularly-polarized incident light into linearly polarized light!!!

Thus, the Fresnel relations for *TE* / *TM* polarization for internal / external reflection are valid / useful for any type of *EM* wave – linear, elliptic or circular polarization.

The general case is an *EM* wave which is a linear combination (of some kind, depending on nature and type of *EM* wave polarization state) of *TE* and *TM* polarization… more complicated!

Finally, again for the special / limiting case of <u>normal</u> incidence (where the plane of incidence collapses) the reflectance / reflection coefficient for both *TE* and *TM* polarization at $\theta_{inc} = 0$.

$$\boxed{R = \left(\frac{1-\left(\frac{n_2}{n_1}\right)^2}{1+\left(\frac{n_2}{n_1}\right)}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \approx 4\% \ \text{ for} \begin{Bmatrix} n_1 = 1.0 \text{ (air)} \\ n_2 = 1.5 \text{ (glass)} \end{Bmatrix}}$$

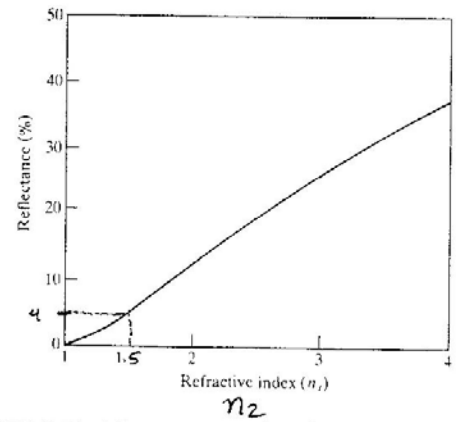


FIGURE 4.49 Reflectance at normal incidence in air (n_i = 1.0) at a single interface. $n_1 = n_{AIR} = 1.0$

<u>Can There be a Brewster's Angle θ_B^{inc} for Transverse Electric (*TE*) Polarization Reflection / Refraction at an Interface</u>?

<u>The Fresnel Equations</u>:

<u>TE Polarization</u> **<u>TM Polarization</u>**

$$\boxed{r_\perp \equiv \left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \qquad \boxed{r_\parallel \equiv \left(\frac{\mathrm{E}^{TE}_{0_{refl}}}{\mathrm{E}^{TE}_{0_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)}$$

<u>with</u>: $\boxed{\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$ <u>and</u>: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$

$$\boxed{v_1 = c/n_1 = 1\big/\sqrt{\varepsilon_1\mu_1}} \quad \boxed{v_2 = c/n_2 = 1\big/\sqrt{\varepsilon_2\mu_2}} \quad \boxed{n_1 = \sqrt{\varepsilon_1\mu_1/\varepsilon_0\mu_0}} \quad \boxed{n_2 = \sqrt{\varepsilon_2\mu_2/\varepsilon_0\mu_0}}$$

We saw P436 lecture notes above (*p*. 19-21) that for *TM* polarized *EM* Waves (where $\vec{B} \perp$ plane of incidence $\{$*i.e.* $\vec{B} \parallel$ to plane of the interface$\}$, with unit normal to the plane of incidence defined as $\hat{n}_{inc} \equiv \hat{k}_{inc} \times \hat{k}_{refl}$) that when $\theta_{inc} = \theta_B^{inc}$ = Brewster's angle (*a.k.a.* = θ_P^{inc} = Polarizing angle), that $E^{TM}_{o_{refl}} = 0$ because the <u>numerator</u> of $r_\parallel$, , $(\alpha - \beta) = 0$ *i.e.* $\alpha = \beta$ when $\theta_{inc} = \theta_B^{inc} = \theta_P^{inc}$ Thus, for a incident *TM* polarized monochromatic plane *EM* wave, when:

$$\boxed{\alpha = \beta\Big|_{\theta_{inc}=\theta_B^{inc}}} \Rightarrow \boxed{\left(\frac{\cos\theta^{TM}_{trans}}{\cos\theta^{TM}_{inc}}\right)_{TM} = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2} = \beta}$$

<u>or</u>: $$\boxed{\beta = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1}{\mu_2}\sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{\varepsilon_2}{\varepsilon_1}\sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$$

For non-magnetic media: $|\chi_m| \ll 1$ *i.e.* $\mu_1 \simeq \mu_2 \simeq \mu_0$ then: $\boxed{\left(\frac{\cos\theta_{trans}^{TM}}{\cos\theta_{inc}^{TM}}\right) \simeq \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}}$

We also derived the Brewster angle relation for *TM* polarization: $\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \frac{n_2}{n_1}}$

For the case of *TE* polarization, we see that: $E_{o_{refl}}^{TE} = 0$ when the numerator of $r_\perp$, $(1-\alpha\beta) = 0$ *i.e.* when: $\alpha\beta = 1$ or: $\beta = 1/\alpha$. What does this mean physically??

For *TE* Polarization: $\beta = 1/\alpha \Rightarrow$ $\boxed{\sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = 1\bigg/\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) = \left(\frac{\cos\theta_{inc}}{\cos\theta_{trans}}\right)}$

For non-magnetic media where: $|\chi_m| \ll 1$ *i.e.* $\mu_1 \simeq \mu_2 \simeq \mu_o$ then: $\boxed{\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) \simeq \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}}$

Thus: $\boxed{\left(\frac{n_2}{n_1}\right)^2 = \frac{\cos^2\theta_{inc}}{\cos^2\theta_{trans}} = \frac{\cos^2\theta_{inc}}{1-\sin^2\theta_{trans}} = \frac{1-\sin^2\theta_{inc}}{1-\sin^2\theta_{trans}}}$

From Snell's Law: $\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$ or: $\boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}}$

$\boxed{\sin\theta_{trans} = \left(\frac{n_1}{n_2}\right)\sin\theta_{inc}}$ or: $\boxed{\sin^2\theta_{trans} = \left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}}$

Then: $\boxed{\left(\frac{n_2}{n_1}\right)^2 = \frac{\left(1-\sin^2\theta_{inc}\right)}{\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}\right)}}$ or: $\boxed{\left(\frac{n_2}{n_1}\right)^2 - \cancel{\sin^2\theta_{inc}} = 1 - \cancel{\sin^2\theta_{inc}}}$

$\Rightarrow$ $\boxed{\left(\frac{n_2}{n_1}\right)^2 = 1}$ or: $\boxed{n_1 = n_2}$ $\Rightarrow$ can get $\theta_B^{inc} = \theta_P^{inc}$ for *TE* Polarization ***only*** when $n_1 = n_2$

i.e. ***no*** interface boundary, for non-magnetic material(s), where $|\chi_m| \ll 1$ and $\mu_1 \simeq \mu_2 \simeq \mu_0$.

Is there a possibility of a Brewster's angle for incident *TE* polarization for *magnetic* materials???

For incident *TE* polarization, we still need to satisfy the condition $\beta = 1/\alpha$.

i.e. $\boxed{\sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}} = \frac{\cos\theta_{inc}^B}{\cos\theta_{trans}}}$ or: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) = \frac{\cos^2\theta_{inc}^B}{\cos^2\theta_{trans}} = \frac{1-\sin^2\theta_{inc}^B}{1-\sin^2\theta_{trans}} = \frac{1-\sin^2\theta_{trans}^B}{1-\left(\frac{n_1}{n_2}\right)\sin^2\theta_{inc}^B}}$

but: $\boxed{\left(\frac{n_1}{n_2}\right)^2 = \left(\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\right)}$ thus: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right) = \frac{1-\sin^2\theta_{inc}^B}{1-\left(\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\right)\sin^2\theta_{inc}^B}}$

thus: $\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right)-\left(\frac{\cancel{\varepsilon_2}\mu_1}{\cancel{\varepsilon_1}\mu_2}\right)\left(\frac{\cancel{\varepsilon_1}\mu_1}{\cancel{\varepsilon_2}\mu_2}\right)\sin^2\theta_{inc}^B=1-\sin^2\theta_{inc}^B}$

$\boxed{\left(\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}\right)-\left(\frac{\mu_1}{\mu_2}\right)^2\sin^2\theta_{inc}^B=\left(1-\sin^2\theta_{inc}^B\right)}$ multiply both sides of this eqn. by $\left(\frac{\mu_2}{\mu_1}\right)$

$\boxed{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_1}{\mu_2}\right)\sin^2\theta_{inc}^B=\left(\frac{\mu_2}{\mu_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)\sin^2\theta_{inc}^B}$

$\boxed{\left[\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)\right]=\left[\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)\right]\sin^2\theta_{inc}^B}$

$\boxed{\theta_{inc}^B=0^o}$ $\boxed{\theta_{inc}^B=90^o}$

$\Rightarrow \boxed{\sin^2\theta_{inc}^B=\left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}\right]}$ Note: $\boxed{0\le\sin^2\theta_{inc}^B\le 1}$

Define: $\boxed{\sin\theta_{inc}^B=\sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}}\equiv\sqrt{A}}$ *i.e.* $\boxed{A\equiv\left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}\right]}$

Brewster's angle for *TE* polarization: $\boxed{\theta_{\substack{inc\\TE}}^B=\sin^{-1}\sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}}=\sin^{-1}\sqrt{A}}$

Let us assume that ε_1 and ε_2 are fixed {i.e. electric properties of medium 1) and 2) are fixed} but that we can engineer/design/manipulate the magnetic properties of medium 1) and 2) in such a way as to obtain a ratio $\left(\mu_1/\mu_2\right)\neq 1$ to give $0\le A\le 1$!!!

Then if $\theta_{\substack{inc\\TE}}^B=\sin^{-1}\sqrt{A}$ can be achieved, it might also be possible to engineer the magnetic properties $\left(\mu_1/\mu_2\right)$ such that $A<0$ – *i.e.* θ_{inc}^B becomes imaginary!!!

Note also that in the above formula that $\left(\mu_1/\mu_2\right)=1$ does *not* mean $\sin\theta_{inc}=\infty$ because the original formula for $\left(\mu_1/\mu_2\right)=1$ was:

$$\boxed{\left(\frac{n_2}{n_1}\right)^2\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_{inc}\right)=\left(1-\sin^2\theta_{inc}\right)}$$

which is perfectly mathematically fine/OK for $\left(n_2/n_1\right)=1$.

