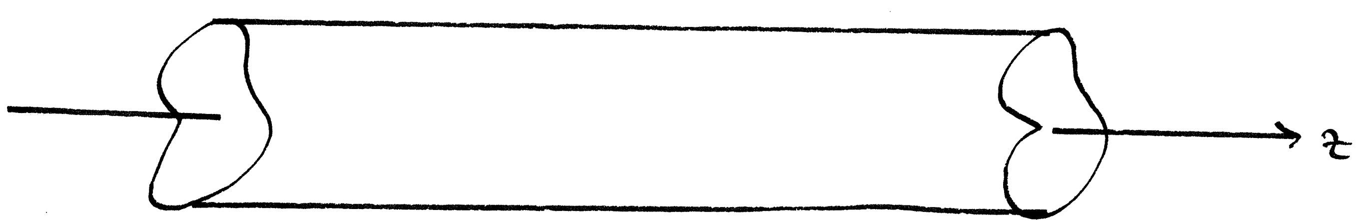
12. Wave guides (and cavities)

For practical purposes (transmission of electromagnetic signals), it is useful to study wave propagation in appropriately designed **wave guides**.

12.1. Cylindrical wave guides

A cylindrical wave guide is a (hollow) conducting cylinder of any cross-sectional shape:



A wave travelling along such a wave guide can be written, by symmetry, as

$$\vec{E}(t,\vec{r}) = \vec{E}(t,\vec{r}_T)\, e^{i(kz-\omega t)}, \text{ where } \vec{r}_T \cdot \hat{z} = 0.$$

The wave satisfies the wave equation

$$\vec{\nabla}^2 \vec{E} - \frac{\epsilon\mu}{c^2} \partial_t^2 \vec{E} = 0 \quad , \text{ which gives}$$

$$\left(\vec{\nabla}_T^2 - k^2 + \frac{\epsilon\mu\omega^2}{c^2} \right) \vec{E}(\vec{r}_T) \, \cancel{e^{i(kz-\omega t)}} = 0 .$$

cancel out

Here, $\vec{\nabla}_T^2$ is the Laplacian along the transverse directions.

Say, in

- cartesian coords. : $\vec{\nabla}_T^2 \equiv \partial_x^2 + \partial_y^2$
- cylindrical coords : $\vec{\nabla}_T^2 \equiv \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial y^2}$

Therefore, $\vec{E}(\vec{r}_T)$ satisfies the eigenvalue equation

$$\vec{\nabla}_T^2 \vec{E}(\vec{r}_T) = -\gamma^2 \vec{E}(\vec{r}_T) \quad , \text{ with}$$

$$\gamma^2 \equiv \frac{\epsilon\mu\omega^2}{c^2} - k^2$$

and appropriate boundary conditions.

Same considerations apply for $\vec{B}$:

$$\vec{\nabla}_T^2 \vec{B}(\vec{r}_T) = -\gamma^2 \vec{B}(\vec{r}_T) \quad \text{with appropriate bc.}$$

12.1.1. Phase and Group velocities

In order for the wave to propagate along z, k has to be real. Therefore, for a given value of γ, frequencies below

$$\omega_c \doteq \frac{\gamma c}{\sqrt{\epsilon\mu}}$$ cannot propagate.

With this definition we can write the dispersion relation as

$$k^2 = \frac{\epsilon\mu}{c^2}(\omega^2 - \omega_c^2).$$

Thus, the phase velocity is $v_p \doteq \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}} \frac{1}{\sqrt{1-\frac{\omega_c^2}{\omega^2}}}$

and the group velocity is

$$v_g = \frac{d\omega}{dk} \simeq \frac{c}{\sqrt{\epsilon\mu}}\sqrt{1-\frac{\omega_c^2}{\omega^2}}$$

(assuming ϵ, μ constants)

12.2. Eigenmodes in a wave guide

In order to determine the fields in the wave guide, we need to solve Maxwell's eqs.

Decomposing $\vec{\nabla} = \vec{\nabla}_T + \hat{k}\,\frac{\partial}{\partial z} = \vec{\nabla}_T + ik\hat{k}$, we find

$$\left.\begin{aligned}
&\vec{\nabla}\cdot\vec{D} = 0 \quad \Rightarrow \quad (\vec{\nabla}_T + ik\hat{k})\cdot\vec{E}(\vec{r}_T) = 0 \\
&\vec{\nabla}\times\vec{E} - \tfrac{1}{c}\partial_t\vec{B} = 0 \quad \Rightarrow \quad (\vec{\nabla}_T + ik\hat{k})\times\vec{E}(\vec{r}_T) = \frac{i\mu\omega}{c}\,\vec{H}(\vec{r}_T) \\
&\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad (\vec{\nabla}_T + ik\hat{k})\cdot\vec{H}(\vec{r}_T) = 0 \\
&\vec{\nabla}\times\vec{H} - \tfrac{1}{c}\partial_t\vec{D} = 0 \quad \Rightarrow \quad (\vec{\nabla}_T + ik\hat{k})\times\vec{H} = \frac{-i\epsilon\omega}{c}\,\vec{E}(\vec{r}_T)
\end{aligned}\right\}$$

We also decompose $\vec{E}$ and $\vec{H}$ in transverse and longitudinal parts:

$$\vec{E} = \vec{E}_T + \hat{k}\,E_z\,, \qquad \vec{H} = \vec{H}_T + \hat{k}\,H_z\,.$$

Then, the eqs. above become

$$\left.\begin{aligned}
&\vec{\nabla}_T\cdot\vec{E}_T = -ik\,E_z \;; \qquad \vec{\nabla}\cdot\vec{H}_T = -ik\,H_z \\
&\vec{\nabla}_T\times\vec{E}_T = \frac{i\mu\omega}{c}\,\hat{k}\,H_z \;; \qquad ik\hat{k}\times\vec{E}_T - \frac{i\mu\omega}{c}\vec{H}_T = \hat{k}\times\vec{\nabla}_T E_z \\
&\vec{\nabla}_T\times\vec{H}_T = \frac{-i\epsilon\omega}{c}\,\hat{k}\,E_z \;; \qquad ik\hat{k}\times\vec{H}_T + \frac{i\epsilon\omega}{c}\vec{E}_T = \hat{k}\times\vec{\nabla}_T H_z
\end{aligned}\right\}\text{(Max)}$$

For instance, the 2nd eq above becomes

$$(\vec{\nabla}_T + ik\hat{k}) \times (\vec{E}_T + \hat{k}\,E_z) = \frac{i\mu\omega}{c}(\vec{H}_T + \hat{k}\,H_z)$$

$$\parallel$$

$$\underbrace{\vec{\nabla}_T \times \vec{E}_T}_{\parallel \hat{k}} + \underbrace{\vec{\nabla}_T \times (\hat{k}E_z)}_{\perp \vec{k}} + \underbrace{ik\,\hat{k}\times\vec{E}_T}_{\perp \vec{k}} + ik\,\cancel{\hat{k}\times\hat{k}}\,E_z$$

12.2.1 TEM Waves

Let us look for solutions with $E_z = H_z = 0$.

These are transverse electric and magnetic waves (TEM).

From the eqs. above,

$$\vec{\nabla}_T \cdot \vec{E}_T = 0 \quad \text{and} \quad \vec{\nabla}_T \times \vec{E}_T = 0. \qquad (\text{TEM})$$

At this point we need to impose boundary conditions.

From $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t \vec{B}$ at the wave guide boundary

we get $\hat{n}\times\vec{E}_T = 0$ (for a perfect conductor)

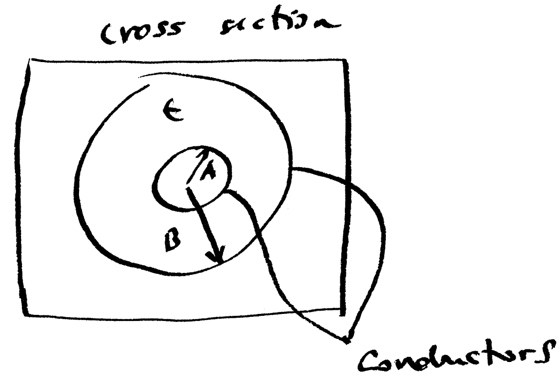
$$\left.\begin{array}{l} \hat{n}\times\vec{E}_T\big|_{\text{boundary}} \Rightarrow \text{boundary is equipotential} \\ \vec{\nabla}\cdot\vec{E}_T = 0 \;\&\; \vec{\nabla}_T\times\vec{E}_T = 0 \Rightarrow \text{electrostatics in empty space} \end{array}\right\} \Rightarrow$$

$\vec{E}_T$ must vanish inside the wave guide.

Therefore, in order to support a TEM wave, the wave guide requires more than a single conducting surface

Example:

- Coaxial cable



Exercise 32

Calculate $\vec{E}_T$ and $\vec{H}_T$ for a TEM mode inside a coaxial cable ∎

Note that for a TEM mode, from (TEM)

$$0 = \vec{\nabla}_T \times (\underbrace{\vec{\nabla}_T \times \vec{E}_T}_{0}) = \vec{\nabla}_T (\underbrace{\vec{\nabla}_T \cdot \vec{E}_T}_{0}) - \vec{\nabla}_T^2 \vec{E}_T = - \vec{\nabla}_T^2 \vec{E}_T$$

Therefore, $\gamma^2 = 0 \Rightarrow$ no cut-off freq. for TEM modes.

12.2.2 TM waves

Similarly, there exist solutions with

$H_z = 0 \Rightarrow$ Transverse Magnetic (TM)

$E_z = 0 \Rightarrow$ Transverse Electric (TE)

We focus on TM waves. The treatment of TE is analogous.

i) We begin by solving the eigenvalue problem

$\vec{\nabla}_T^2 E_z = -\gamma^2 E_z$ in the desired geometry, with

bc $E_z = 0 \Big|_{\text{boundary}}$ (from $\hat{n} \times \vec{E}\Big|_b = 0$)

ii) With E_z known and $H_z \equiv 0$ we get from (Mex)

$$i \vec{k} \times (\vec{k} \times \vec{H}_T) = -\frac{i\epsilon\omega}{c} \vec{k} \times \vec{E}_T \underset{\vec{k}\cdot\vec{H}_T = 0}{\overset{\text{bac-cab}}{\Rightarrow}} \underline{\vec{H}_T = \frac{\epsilon\omega}{ck} \hat{k} \times \vec{E}_T}$$

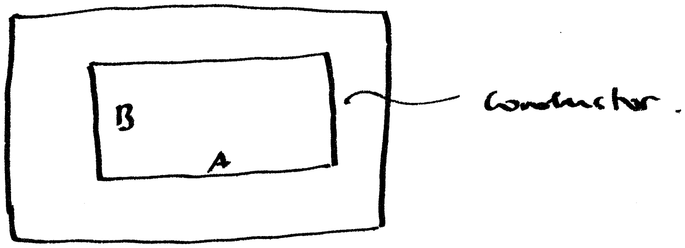
iii) It follows then

$$ik\hat{k} \times \vec{E}_T - \frac{i\mu\omega}{c} \frac{\epsilon\omega}{ck} \hat{k} \times \vec{E}_T = \hat{k} \times \vec{\nabla}_T E_z$$

$$\Rightarrow \underline{\vec{E}_T = \frac{ik}{\gamma^2} \vec{\nabla}_T E_z} .$$

Exercise 33

Calculate $\vec{E}$ and $\vec{H}$ in the TM mode of a rectangular wave guide

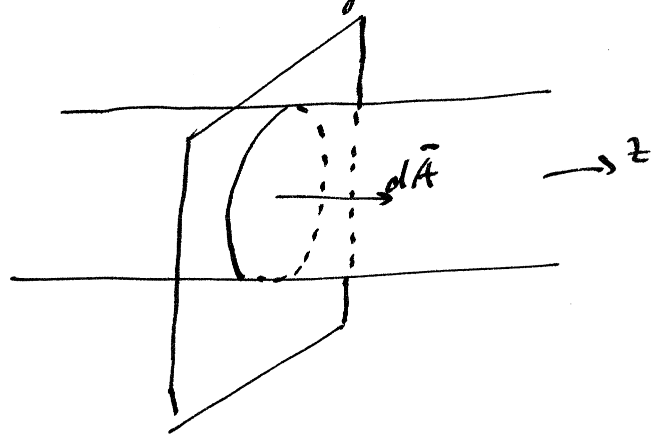


Cross section

12.3. Power Transmission and Attenuation

The power transmitted by the wave guide is

$P = \int \langle \vec{S} \rangle \cdot d\vec{A}$, where



$\langle \vec{S} \rangle$ is the average flux of electromagnetic energy (Poynting vector) $\hat{z}\cdot\langle\vec{S}\rangle = \frac{c}{16\pi}\hat{z}(\vec{E}_T \times \vec{H}_T^* + \vec{E}_T^* \times \vec{H}_T)$

using $\vec{H}_T = \frac{\epsilon\omega}{ck}\hat{k}\times\vec{E}_T$, P becomes

$$P = \frac{\epsilon\omega}{8\pi k}\int |\vec{E}_T|^2 dA = \frac{\epsilon\omega k}{8\pi\gamma^4}\int dA\, (\vec{\nabla}_T E_z^*)\cdot(\vec{\nabla}_T E_z)$$

Integration by parts then results in

$$\underline{P = \frac{\epsilon \omega k}{8\pi \gamma^2} \int |E_z|^2 \, dA .}$$

In real (as opposed to perfect) conductors, the fields penetrate the conductor (albeit with exponentially suppressed amplitudes). This leads to an energy flow $\perp$ to the wave guide boundary.

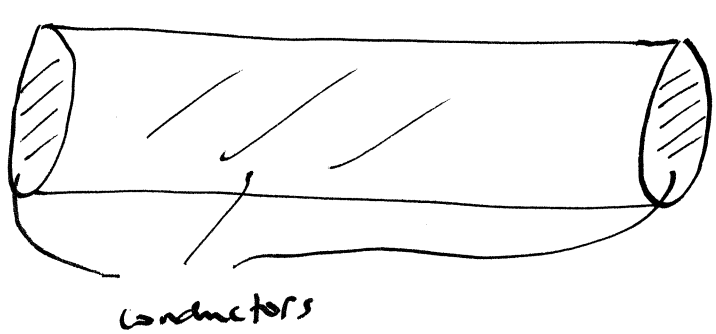
This energy loss leads to attenuation:

$$\frac{dP}{dz} = -\frac{P(z)}{L_{atten}} ,$$

where L_{atten} is the attenuation length.

12.4 Cavities

The same techniques can be applied to cavities, wave guides capped by conducting surfaces



Exercise 34

Find $\vec{E}$ and $\vec{H}$ inside a rectangular cavity (as in exercise 33) of length L for a TM mode. ∎