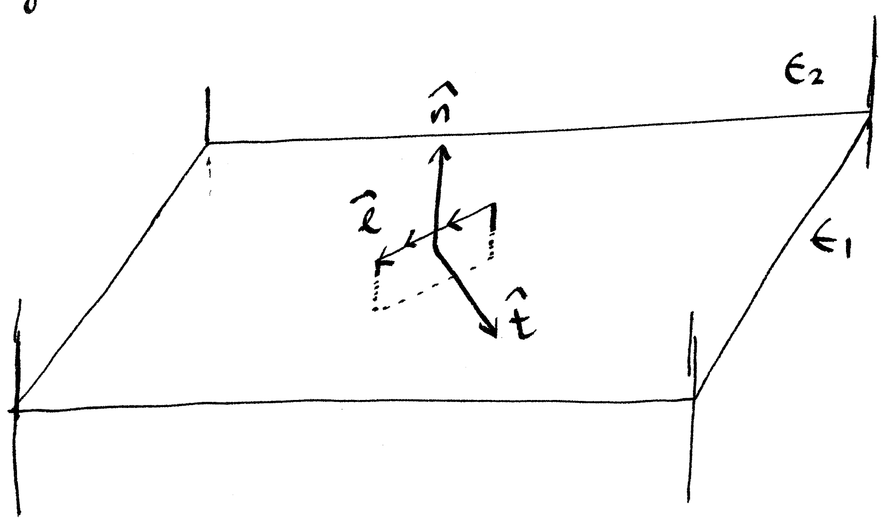
6.4.1 Boundary conditions on $\vec{D}$, $\vec{E}$.

Recall that $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 4\pi\sigma$.

Thus, if $\sigma = 0$, continuous at boundary

Apply now $\vec{\nabla} \times \vec{E} = 0$ along the contour



$$(\hat{t} \times \hat{n}) \cdot (\vec{E}_2 - \vec{E}_1) \cdot \ell = 0.$$

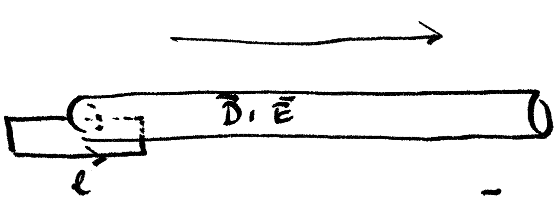
since $(\hat{t} \times \hat{n}) \cdot (\vec{E}_2 - \vec{E}_1) = [\hat{n} \times (\vec{E}_2 - \vec{E}_1)] \cdot \hat{t} \overset{!}{=} 0 \quad \forall \hat{t}$

it follows that $\underline{\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0}$

Tangential component $E_{\parallel}$ continuous at boundary.

Examples

i) A needle aligned $\parallel$ $\vec{E}_0$:

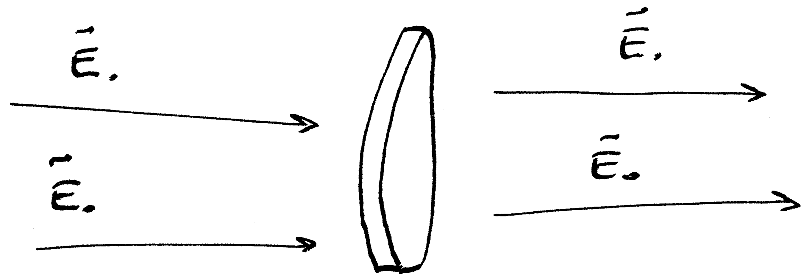


Consider the contour in the figure:

$$\vec{E}_0 \cdot \vec{l} - \vec{E}\,\frac{\vec{l}}{2} - \vec{E}_0 \cdot \frac{\vec{l}}{2} \stackrel{!}{=} 0 \quad \Rightarrow \quad \vec{E} = \vec{E}_0$$

Therefore, $\vec{D} = \epsilon \vec{E} = \epsilon \vec{E}_0$

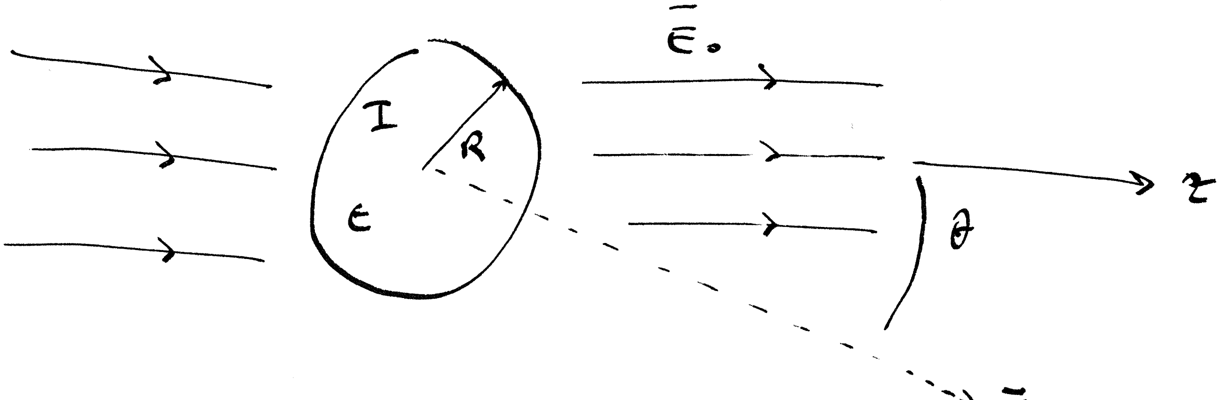
ii) A very thin lamina



With $\sigma = 0$, using $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 4\pi\sigma$,

$\vec{D} = \vec{D}_0$, or $\epsilon \vec{E} = \vec{E}_0$

iii) A dielectric sphere



Solve Laplace's eqs. in regions I & II (no macroscopic charges). Use azimuthal symmetry:

$$\text{I} : \quad \phi_I = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) \qquad (m=0)$$

$$\text{II} : \quad \phi_{II} = \sum_{l=0}^{\infty} \left(\frac{b_l}{r^{l+1}} + c_l r^l \right) P_l(\cos\theta).$$

Boundary conditions demand that

$$\phi_{II} \to -E_0 z = -E_0 r \cos\theta$$

$$\Rightarrow \quad c_l = \delta_{l1} \cdot (-E_0) \quad , \text{ since } \quad P_1(x) = x.$$

Impose now junction conditions at the interface, $r = R$:

i) $\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) \overset{!}{=} 0 \quad \Rightarrow$

$$\Rightarrow \quad -\left.\frac{\partial \phi_{II}}{\partial r}\right|_R = -\epsilon \left.\frac{\partial \phi_I}{\partial r}\right|_R$$

ii) $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \quad \Rightarrow$

$$\Rightarrow \quad -\frac{1}{R}\left.\frac{\partial \phi_{II}}{\partial \theta}\right|_R = -\frac{1}{R}\left.\frac{\partial \phi_I}{\partial \theta}\right|_R$$

Condition ii) gives

$$a_1 R = \frac{b_1}{R^2} + (-E_0) R \quad , \text{ or } \quad a_1 = -E_0 + \frac{b_1}{R^3} \qquad (\ell=1)$$

$$a_\ell R^\ell = \frac{b_\ell}{R^{\ell+1}} \quad , \text{ or } \quad a_\ell = \frac{b_\ell}{R^{2\ell+1}} \qquad (\ell \neq 1) .$$

Condition i) gives

$$\epsilon a_1 = -E_0 - 2 \frac{b_1}{R^3} \qquad (\ell=1)$$

$$\epsilon \ell a_\ell = -(\ell+1) \frac{b_\ell}{R^{2\ell+1}} \qquad (\ell \neq 1)$$

Recall that the Legendre polynomials are complete on the unit interval $[-1,1]$, so conditions i) & ii) require each individual ℓ term to vanish.

The $\ell \neq 1$ conditions imply $a_\ell = b_\ell = 0 \quad (\ell \neq 1)$.

Therefore, the potential becomes

$$\phi_I = -\left(\frac{3}{\epsilon+2}\right) E_0 \, r \cos\theta$$

$$\phi_{II} = -E_0 \, r \cos\theta + \left(\frac{\epsilon-1}{\epsilon+2}\right) E_0 \frac{R^3}{r^2} \cos\theta$$

The electric field inside the sphere is uniform and $\parallel$ to $\vec{E}_0$.

The electric field outside the sphere is a superposition of the constant field $\vec{E}_0$ plus the contribution of a dipole

$$\vec{p} = \frac{\epsilon-1}{\epsilon+2}\,\vec{E}_0\,R^3.$$

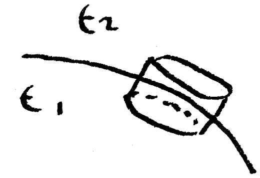
(Recall the expansion $\phi = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \cdots$)

using $\vec{\underline{P}} = \chi_e\,\vec{E}$ and $\epsilon = 1 + 4\pi\chi_e$ we can calculate the polarization inside the sphere:

$$\vec{\underline{P}} = \frac{\epsilon-1}{4\pi}\,\vec{E} = 3\,\frac{\epsilon-1}{\epsilon+2}\,\frac{1}{4\pi}\,\vec{E}_0 = \text{const}.$$

The volume $\int_I d^3V\ \vec{\underline{P}} = \vec{p}$. Recall that the polarization is the macroscopic average of the dipole density, and that $-\vec{\nabla}\cdot\vec{\underline{P}}$ contributes to the averaged charge density: $\rho_{pol} = -\vec{\nabla}\cdot\vec{\underline{P}}$

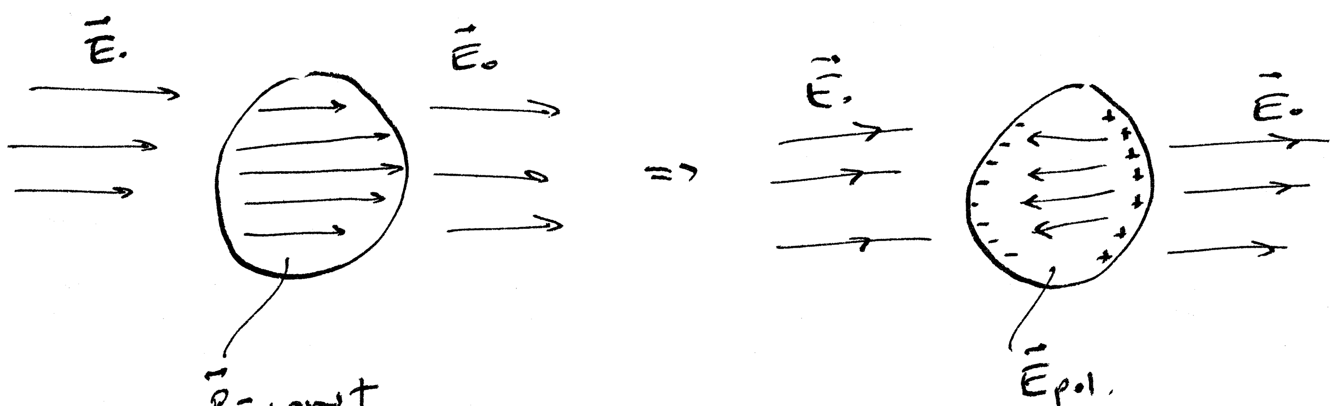
Applying the divergence theorem to a pillbox at the interface we thus find



$$\sigma_{pol} = -(\vec{P}_2 - \vec{P}_1)\cdot\hat{n}$$

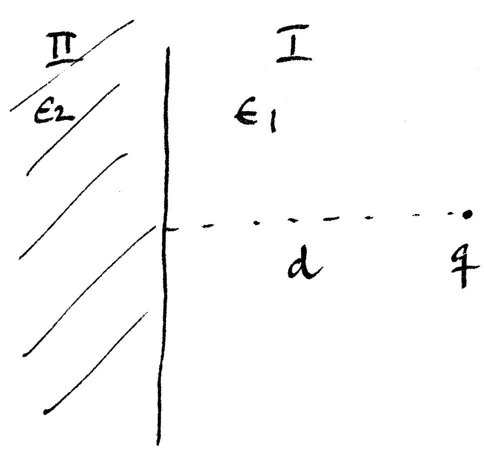
Using the previous results and $\vec{P}_2 = 0$ (outside)

$$\sigma = \frac{3}{4\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)\vec{E}_0\cdot\hat{n} = \frac{3}{4\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)E_0\cos\theta$$



Exercise 16

i) Find the potential in region I of the following configuration:



ii) Calculate the polarization surface charge density σ_{pol} at the interface.

iii) What happens in the limit $\epsilon_2 \gg \epsilon_1$? ∎

6.7. Electrostatic energy in dielectrics

In lecture 3 we proved that the potential energy in a given charge distribution $\rho(\vec{r})$ is

$$U = \frac{1}{2} \int d^3r \; \rho(\vec{r}) \, \phi(\vec{r})$$

This remains true if the medium is linear, that is, if $\vec{D}$ is proportional to $\vec{E}$.

using $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$ and $\vec{E} = -\vec{\nabla}\phi$

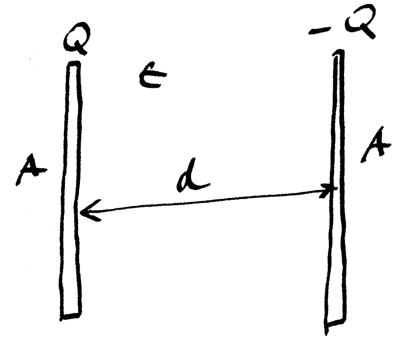
the divergence theorem gives (integrate by parts)

$$U = \frac{1}{8\pi} \int d^3r \; \vec{E} \cdot \vec{D}.$$

If the field is non-linear the energy may depend on the history of the system (hysteresis).

Exercise 17

Calculate the energy stored in a parallel plate capacitor of area A and charge Q



i) Under the assumption that the medium is linear, $\vec{D} = \epsilon \vec{E}$

ii) Under the assumption that the medium is non linear, $\vec{D} = \kappa E \, \vec{E}$.

iii) Calculate the force between the two plates in case i). $\blacksquare$

In general, the energy of a system depends on one (or several) coordinate ξ. Then, the force experienced by a dielectric the system in the ξ direction is

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)\Big|_Q .$$

The derivative is taken with the charges that create the fields kept constant, because it takes work to move those charges.

But if we want to calculate the force experienced by a dielectric in a system kept at constant potential, the force is

$$F_z = + \left(\frac{\partial W}{\partial z} \right) \Big|_V$$

In general, the force on the dielectric acts on the direction of stronger electric field (a dielectric is pulled into a capacitor)