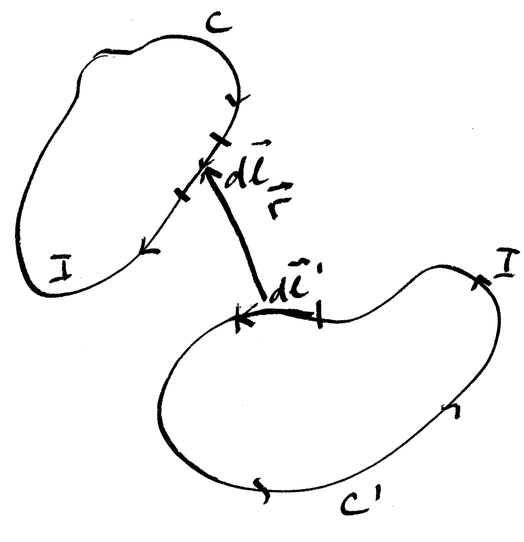
7. Magnetostatics

Our study of <u>magnetism</u> begins with the fields and forces created by <u>static currents</u>

Ampère observed that the (differential) force between two current elements $I\,d\vec{l}$ and $I'\,d\vec{l}'$ separated a distance $\vec{r}$ is

$$d^2 \vec{F}_{II'} = \frac{II'}{c^2} \frac{d\vec{l} \times (d\vec{l}' \times \vec{r})}{r^3}$$



(in gaussian units)

This is the magnetic analog of Coulomb's law.

It is useful to rewrite Ampère's law in a symmetric way: using

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \quad [\text{'bac-cab' rule}]$$

$$d\vec{l} \times (d\vec{l}' \times \vec{r}) = d\vec{l}'(d\vec{l}\cdot\vec{r}) - \vec{r}(d\vec{l}\cdot d\vec{l}')$$, so that

So that the force between the two loops becomes

$$\vec{F}_{cc'} = \frac{II'}{c^2}\left[\oint_{c'} d\vec{r}' \underbrace{\oint_{c} \frac{d\vec{r}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}_{-d\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)} - \oint_{c}\oint_{c'} \frac{(d\vec{r}\cdot d\vec{r}')\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right], \text{ or}$$

$$\vec{F}_{cc'} = -\frac{II'}{c^2}\oint_{c}\oint_{c'} \frac{(d\vec{r}\cdot d\vec{r}')\,(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}. \quad (N)$$

"Neumann's formula"

Exercise 18

Show that the force per unit length between two parallel wires of current I, I' is

$$\frac{\vec{F}_{II'}}{L} = -2\frac{II'}{c^2}\,\frac{\hat{d}}{d}.$$

Check the units in this equation ∎

7.3. The magnetic field

We attribute the force experienced by one of the current elements to the magnetic field $\vec{B}$ created by the other.

The field created by the loop c' is

$$\vec{B}(\vec{r}) = \frac{I'}{c} \oint \frac{d\vec{r}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \qquad \text{Biot-Savart Law}$$

The force experienced by the loop c is

$$\vec{F} = \frac{I}{c} \oint_c d\vec{r} \times \vec{B} \qquad \text{"Lorentz force"}$$

We can re-write these integral expressions in differential form:

$$d\vec{B} = \frac{I\, d\vec{l} \times \vec{r}}{c\, r^3} \quad \text{and} \quad d\vec{F} = \frac{I\, d\vec{r} \times \vec{B}}{c}.$$

Note that $I\, d\vec{r} = q\, \vec{v}$ is the charge crossing a surface $\perp d\vec{r}$ per unit time, and thus

$$\vec{F} = q\, \frac{\vec{v}}{c} \times \vec{B}$$ is the magnetic force acting on a charge q.

Adding the electric force:

$$\underline{\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)} \qquad \text{Lorentz - force.}$$

Note that the magnetic force does not do any work on the charge.

Just as we defined a <u>charge density ρ</u>, we define a current density $\vec{j}$. For a set of point charges q_i:

$$\vec{j} = \sum_i q_i \, \vec{v}_i \, \delta(\vec{r} - \vec{r}_i(t))$$

<u>Exercise 19</u>

Show that $\rho = \sum_i q_i \, \delta(\vec{r} - \vec{r}_i(t))$ and $\vec{j}$ obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

In magnetostatics $\rho(\vec{r}) = \text{const} \Rightarrow \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$. ■

Since $\vec{j}\, d^3r$ is the amount of charge that crosses a surface $\perp \vec{j}$ per unit time, we can replace $I\, d\vec{\ell} \to \vec{j}\, d^3r$ in Biot - Savart law:

$$B(\vec{r}) = \frac{1}{c} \int d^3r' \, \frac{j(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} .$$

7.7. Differential form of Magnetostatics

Taking the divergence of the last eq.:

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{1}{c} \int d^3r \; \vec{j}(\vec{r}\,') \, \vec{\nabla} \times \left(\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) = 0$$

since $\vec{\nabla} \times \vec{\nabla} f = 0$ and $\dfrac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = - \vec{\nabla} \left(\dfrac{1}{|\vec{r} - \vec{r}\,'|} \right)$.

We have also used $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = (\vec{\nabla} \times \vec{a}) \cdot \vec{b} - \vec{a} \cdot \vec{\nabla} \times \vec{b}$.

Similarly, one can show that

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} .$$

Therefore, the two basic eqs. of magnetostatics become

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} \end{cases}$$

Exercise 20

Show that $B(\vec{r}) = \frac{1}{c} \vec{\nabla} \times \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ and

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

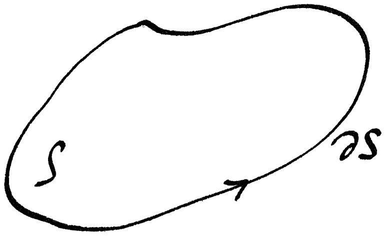
■

These eqs. are the analogues of

$$\vec{E} = -\vec{\nabla} \int d^3r \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ and

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho.$$

We can also rewrite $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$ into the integral equivalent of Gauss' law using Stokes' theorem:



$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \oint_{\partial S} \vec{B} \cdot d\vec{\ell} \quad , \quad \text{or}$$

$$\oint_{\partial S} \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} I_S \quad , \quad \text{where}$$

$I_S = \int_S \vec{j} \cdot d\vec{A}$ is the net current passing through the surface S. (Ampère's circuital law)

7.8. The vector potential

We know that if $\vec{B} = \vec{\nabla} \times \vec{A}$, then $\vec{\nabla} \cdot \vec{B} = 0$.

Is the converse true?

According to the Poincaré lemma, if $\vec{\nabla} \cdot \vec{B}$, $\vec{B} = \vec{\nabla} \times \vec{A}$ __locally__. If the 3-dim space in which $\vec{\nabla} \cdot \vec{B} = 0$ is __contractible__, then $\vec{B} = \vec{\nabla} \times \vec{A}$ __globally__ (A contractible space can be continuously deformed into a point).

with $\vec{B} = \vec{\nabla} \times \vec{A}$. $\vec{A}$ is know as the vector potential. Note that

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} \underset{\text{bac - cab}}{=} \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} .$$

Hence

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \frac{4\pi}{c} \vec{j} .$$

We can bring this eq. closer to Poisson's eq by noting that the <u>gauge transformation</u>

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi , \quad \text{for any scalar function } \chi$$

leaves the magnetic field invariant:

$$\vec{B} \to \vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\chi) = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}\chi}_{0} = \vec{B} \quad \checkmark$$

We can use this property to impose a <u>gauge condition</u> on $\vec{A}$. We can for instance impose Coulomb gauge : $\vec{\nabla} \cdot \vec{A} = 0$.

This is always possible: Assume $\vec{\nabla}\cdot\vec{A} \neq 0$. Then,

$\vec{A}' = \vec{A} + \vec{\nabla}\chi$ still satisfies $\vec{B} = \vec{\nabla}\times\vec{A}$.

Because $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}^2\chi$, if we solve

$$\vec{\nabla}^2\chi = -\vec{\nabla}\cdot\vec{A}, \quad \vec{\nabla}\cdot\vec{A}' = 0.$$

In Coulomb gauge, the eqs. of magnetostatics then reduce to

$$\vec{\nabla}^2\vec{A} = -\frac{4\pi}{c}\vec{j}.$$

Therefore, the solution for the vector potential is

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}.$$

Note that this solution is compatible with our gauge choice, since

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{c}\int d^3r' \, \vec{j}(\vec{r}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\frac{1}{c}\int d^3r' \, \vec{j}(\vec{r}')\,\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= \frac{1}{c}\int d^3r' \, \vec{\nabla}'\cdot\vec{j}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|} = 0, \text{ since } \vec{\nabla}'\cdot\vec{j}(\vec{r}') = 0.$$

Finally, using $\vec{B} = \vec{\nabla} \times \vec{A}$ we find

$$\vec{B} = \vec{\nabla} \times \left(\frac{1}{c} \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) =$$

$$= -\frac{1}{c} \int d^3r' \, \vec{J}(\vec{r}') \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \frac{1}{c} \int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3},$$

our previous Biot-Savart solution.