LECTURE NOTES 3

Conservation Laws (continued): Angular Momentum Associated with *EM* Fields

We have learned that the macroscopic *EM* fields have associated with them:

• *EM* Energy:

EM Energy Density: $$\boxed{u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)} \qquad \left(\text{Joules/m}^3\right)$$

EM Energy: $$\boxed{U_{EM}(t)\equiv\int_v u_{EM}(\vec{r},t)\,d\tau=\int_v\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)d\tau} \quad (\text{Joules})$$

• Poynting's Vector: $$\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)} \qquad \left(\frac{\text{Watts}}{\text{m}^2}=\frac{\text{Joules}}{\text{m}^2\text{-sec}}\right)$$

• *EM* Linear Momentum:

EM Linear Momentum Density:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t)\equiv\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)=\varepsilon_o\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)} \quad \left(\text{kg/m}^2\text{-sec}\right)$$

EM Linear Momentum:

$$\boxed{\vec{p}_{EM}(t)\equiv\int_v\vec{\wp}_{EM}(\vec{r},t)\,d\tau=\frac{1}{c^2}\int_v\vec{S}(\vec{r},t)\,d\tau=\varepsilon_o\int_v\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)d\tau} \quad \left(\text{kg-m/sec}\right)$$

The macroscopic *EM* fields can *additionally* have associated with them:

• *EM* Angular Momentum:

EM Angular Momentum Density:

$$\boxed{\begin{aligned}\vec{\ell}_{EM}(\vec{r},t)&\equiv\vec{r}\times\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{r}\times\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{r}\times\vec{S}(\vec{r},t)\\&=\varepsilon_o\left[\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\right]\end{aligned}} \qquad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

EM Angular Momentum:

$$\boxed{\begin{aligned}\vec{\mathcal{L}}_{EM}(t)&\equiv\int_v\vec{\ell}_{EM}(\vec{r},t)\,d\tau=\int_v\left(\vec{r}\times\vec{\wp}_{EM}(\vec{r},t)\right)d\tau=\frac{1}{c^2}\int_v\left(\vec{r}\times\vec{S}_{EM}(\vec{r},t)\right)d\tau\\&=\varepsilon_o\int_v\left[\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\right]d\tau\end{aligned}} \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

• Note that even STATIC $\vec{E}$ & $\vec{B}$ fields can carry net linear momentum $\vec{p}_{EM}\neq fcn(t)$ and net angular momentum $\vec{\mathcal{L}}_{EM}\neq fcn(t)$ as long as $\vec{E}\times\vec{B}$ is non-zero! Again, at the microscopic level, virtual photons associated with the macroscopic *EM* fields carry angular momentum $\vec{\mathcal{L}}$ as well as linear momentum $\vec{p}$ and (kinetic) energy *E*!

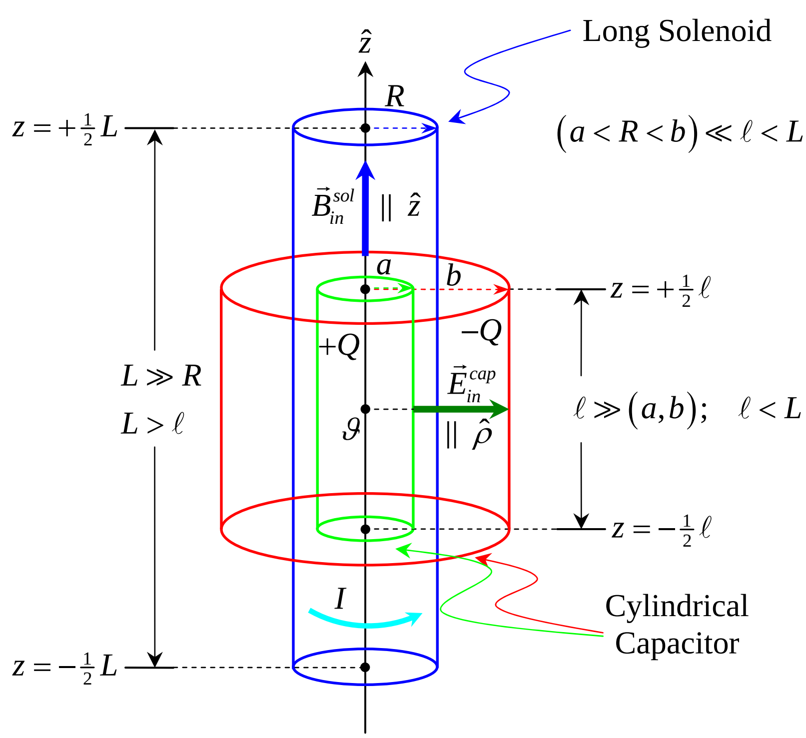
• Only when the *EM* field contributions are included for the total linear momentum $\vec{p}_{Tot}$ and the total angular momentum $\vec{\mathcal{L}}_{Tot}$, *i.e.* $\vec{p}_{Tot}=\vec{p}_{mech}+\vec{p}_{EM}$ and $\vec{\mathcal{L}}_{Tot}=\vec{\mathcal{L}}_{mech}+\vec{\mathcal{L}}_{EM}$ is conservation of linear momentum and conservation of angular momentum separately, independently satisfied.

Griffiths Example 8.4

EM Angular Momentum Associated with a Long Solenoid & Coaxial Cylindrical Capacitor.

- Consider a long solenoid of radius R and length $L \gg R$, with n turns per unit length $\left(n = N_{Tot}/L\right)$ carrying a steady/DC current of I Amperes.
- Coaxial with the long solenoid is a cylindrical (*i.e.* coaxial) capacitor consisting of two long cylindrical conducting tubes, one inside the solenoid, of radius $a < R$ and one outside the solenoid, of radius $b > R$. The cylinders are free to *rotate* about the $\hat{z}$ -axis.
- Both cylindrical conducting tubes have same length $\ell \le L$ with $\ell \gg a$ & $\ell \gg b$.
- The inner (outer) conducting cylindrical tube carries electric charge $+Q$ ($-Q$) uniformly distributed over their surfaces, respectively.

$\Rightarrow$ When the current I in the long solenoid is slowly/gradually reduced (see *e.g.* Griffiths Example 7.8, *p*. 306-7), the cylindrical conducting tubes begin to rotate - the inner (outer) conducting cylindrical tube rotating counter-clockwise (clockwise), respectively as viewed from above!!!



QUESTION: From where/how does the mechanical angular momentum originate?

ANSWER: The mechanical angular momentum imparted/transferred to the cylindrical conducting tubes was initially stored in the *EM* fields associated with this system:

The $\vec{B}$ -field associated with the long solenoid: $\boxed{\vec{B}_{inside}^{sol} = \mu_o n I \hat{z} \ \left(\rho \le R, |z| \le L/2\right)}$ and

The $\vec{E}$ -field associated with the cylindrical capacitor: $\boxed{\vec{E}_{inside}^{cap} = \frac{Q}{2\pi\varepsilon_o \ell}\left(\frac{1}{\rho}\hat{\rho}\right) \ \begin{pmatrix} a \le \rho \le b, \\ |z| \le \ell/2 \end{pmatrix}}$

n.b. $\vec{E}^{cap}_{inside}$ is non-zero for $\{\, a \le \rho \le b$ **.and.** $|z| \le \ell/2 \,\}$, $\vec{B}^{sol}_{inside}$ is non-zero for $\{\, \rho \le R \;\; (R < b)$ **.and.** $|z| \le L/2 \;\; (L > \ell) \,\}$. Hence, $\Rightarrow$ in the region $\{\, (a \le \rho \le R)$ & $|z| \le \ell/2 \,\}$ <u>only</u>:

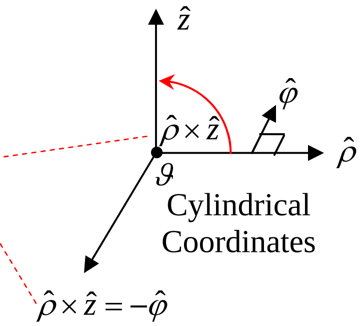
- <u>Poynting's Vector (Energy Flux Density)</u>:

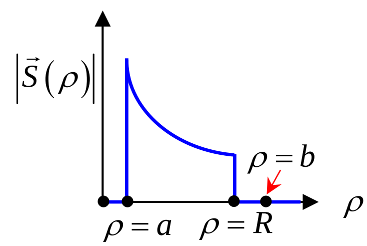
$$\boxed{\vec{S}(\vec{r}) = \left(\vec{E}(\vec{r}) \times \vec{B}(\vec{r})\right) = \frac{1}{\cancel{\mu_o}}\left(\frac{Q}{2\pi\varepsilon_o \ell}\left(\frac{1}{\rho}\hat{\rho}\right)\right) \times \left(\cancel{\mu_o} nI\hat{z}\right) = \frac{nQI}{2\pi\varepsilon_o \rho\ell}\underbrace{(\hat{\rho}\times\hat{z})}_{=-\hat{\varphi}}} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Very useful table:

$\hat{\rho}\times\hat{\varphi} = +\hat{z}$	$\hat{\varphi}\times\hat{\rho} = -\hat{z}$
$\hat{\varphi}\times\hat{z} = +\hat{\rho}$	$\hat{z}\times\hat{\varphi} = -\hat{\rho}$
$\hat{z}\times\hat{\rho} = +\hat{\varphi}$	$\hat{\rho}\times\hat{z} = -\hat{\varphi}$

$$\boxed{\vec{S}(\rho) = -\frac{nQI}{2\pi\varepsilon_o \rho\ell}\hat{\varphi}} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$





EM energy/*EM* power <u>circulates</u> in the $-\hat{\varphi}$ direction in the region $(a \le \rho \le R)$ & $|z| < \ell/2$:

- *EM* field linear momentum density: $\boxed{\vec{\wp}_{EM}(\vec{r}) = \varepsilon_o\mu_o\vec{S}(\vec{r}) = \varepsilon_o\left(\vec{E}(\vec{r})\times\vec{B}(\vec{r})\right)}$

$$\boxed{\vec{\wp}_{EM}(\vec{r}) = \cancel{\varepsilon_o}\mu_o\left(-\frac{nQI}{2\pi\cancel{\varepsilon_o}\rho\ell}\hat{\varphi}\right) = -\frac{\mu_o nQI}{2\pi\rho\ell}\hat{\varphi}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

- *EM* angular momentum density:

$$\boxed{\vec{\ell}_{EM}(\vec{r}) = \vec{r}\times\vec{\wp}_{EM}(\vec{r}) = \vec{\rho}\times\vec{\wp}_{EM}(\vec{r}) = \vec{\rho}\times\left[\varepsilon_o\left(\vec{E}(\vec{r})\times\vec{B}(\vec{r})\right)\right]} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

n.b. $\vec{\rho} = \rho\hat{\rho}$ in cylindrical coordinates, thus:

$$\boxed{\vec{\ell}_{EM}(\vec{r}) = \vec{\rho}\times\vec{\wp}_{EM}(\vec{r}) = \cancel{\rho}\hat{\rho}\times\left(-\frac{\mu_o nQI}{2\pi\cancel{\rho}\ell}\hat{\varphi}\right) = -\frac{\mu_o nQI}{2\pi\ell}\underbrace{(\hat{\rho}\times\hat{\varphi})}_{=+\hat{z}} = -\frac{\mu_o nQI}{2\pi\ell}\hat{z}} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

<u>Note that</u>: $\left|\vec{\ell}_{EM}(\vec{r})\right| = \dfrac{\mu_o nQI}{2\pi\ell} = \underline{\underline{\text{constant}}}$!!! $\vec{\ell}_{EM}(\vec{r})$ points in $-\hat{z}$ direction.

$\Rightarrow$ We then compute the *EM* angular momentum $\vec{\mathcal{L}}_{EM}$ by integrating $\vec{\ell}_{EM}(\vec{r})$ over the volume *v* corresponding to the region $(a \le \rho \le R)$ & $|z| < \ell/2$:

$$\vec{\mathcal{L}}_{EM} = \int_v \vec{\ell}_{EM}(\vec{r})\,d\tau = \int_{\rho=a}^{\rho=R}\int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\frac{1}{2}\ell}^{z=+\frac{1}{2}\ell}\underbrace{\left[-\frac{\mu_o nQI}{2\pi\ell}\hat{z}\right]}_{\text{constant vector!}}\rho\,d\rho\,d\varphi\,dz$$

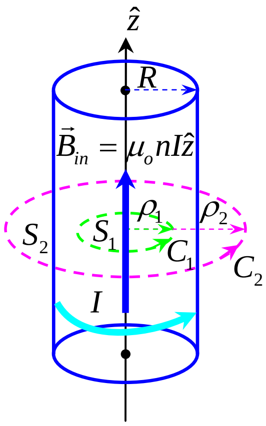
$$= \left[-\frac{\mu_o nQI}{2\pi\ell}\hat{z}\right]\underbrace{\int_{\rho=a}^{\rho=R}\int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\frac{1}{2}\ell}^{z=+\frac{1}{2}\ell}\underbrace{\rho\,d\rho\,d\varphi\,dz}_{d\tau}}_{}$$

= volume *v* of region $\{\, a \le \rho \le R$ **.and.** $|z| \le \ell/2 \,\}$

$$= \left[-\frac{\mu_o nQI}{2\cancel{\pi}\cancel{\ell}}\hat{z}\right]\overbrace{\cancel{\pi}\left(R^2 - a^2\right)\cancel{\ell}} = -\tfrac{1}{2}\mu_o nQI\left(R^2 - a^2\right)\hat{z}$$

Thus, the *EM* angular momentum is $\boxed{\vec{\mathcal{L}}_{EM} = -\tfrac{1}{2}\mu_o nQI\left(R^2-a^2\right)\hat{z}}$ $\left(\text{kg-m}^2/\text{sec}\right)$

When the current I in the long solenoid is slowly/gradually reduced, the changing magnetic field induces a changing circumferential electric field, by Faraday's Law: $\boxed{\oint_C \vec{E}\bullet d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\bullet d\vec{a}}$



Since: $\boxed{\vec{B}^{sol}_{inside} = \mu_o nI\hat{z}\ \ (\rho \le R)}$ then

for contour C_1 $(\rho \le R)$:

$$\boxed{\vec{E}^{IN}(\rho \le R) = -\frac{1}{2\cancel{\pi}\cancel{\rho}}\left(\mu_o n\frac{dI}{dt}\cancel{\pi}\rho^{\cancel{2}}\hat{\varphi}\right) = -\frac{1}{2}\mu_o n\frac{dI}{dt}\rho\hat{\varphi}}$$

for contour C_2 $(\rho \ge R)$:

$$\boxed{\vec{E}^{OUT}(\rho \ge R) = -\frac{1}{2\cancel{\pi}\rho}\left(\mu_o n\frac{dI}{dt}\cancel{\pi}R^2\hat{\varphi}\right) = -\frac{1}{2}\mu_o n\frac{dI}{dt}\left(\frac{R^2}{\rho}\right)\hat{\varphi}}$$

The {instantaneous} mechanical <u>torque</u> exerted on the <u>inner</u> conducting cylinder (of radius *a*) by the <u>tangential</u> $\vec{E}$-field $\vec{E}^{IN}(\rho \le R)$ is:

$$\boxed{\vec{N}^{mech}_a(\vec{r},t) = \vec{r}\times\vec{F}_E(\vec{r},t)\Big|_{\rho=a} = \vec{r}\times\left(+Q\vec{E}^{IN}(\rho,t)\right)\Big|_{\rho=a} = +a\hat{\rho}\times\left(-\frac{1}{2}\mu_o nQ\frac{dI(t)}{dt}a\hat{\varphi}\right)}$$

$$\boxed{\vec{N}^{mech}_a(\rho=a,t) = -\mu_o nQ\frac{dI(t)}{dt}a^2\underbrace{\hat{\rho}\times\hat{\varphi}}_{=+\hat{z}} = -\mu_o nQ\frac{dI(t)}{dt}a^2\hat{z}} \quad \left(\text{N-m}=\frac{\text{kg-m}^2}{\text{sec}^2}\right)$$

But torque (by its definition) = time rate of change of angular momentum, *i.e.* $\boxed{\vec{N}(t) \equiv \frac{d\vec{\mathcal{L}}(t)}{dt}}$

$\Rightarrow$ The corresponding {increase} in the <u>mechanical</u> angular momentum the inner cylinder acquires in the time it takes the current in the solenoid to decrease from $I(t=0)=I$ to $I(t=t_{final})=0$ is given by: $\boxed{\vec{N}^{mech}(t) \equiv \frac{d\vec{\mathcal{L}}^{mech}(t)}{dt}}$ $\Rightarrow$ $\boxed{d\vec{\mathcal{L}}^{mech}(t) = \vec{N}^{mech}(t)\,dt}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$ <u>and</u>:

$$\boxed{\Delta\vec{\mathcal{L}}^{mech}_a = \int_{\vec{\mathcal{L}}^{mech}_{init\,a}(t=0)=0}^{\vec{\mathcal{L}}^{mech}_{final\,a}(t=t_{final})} d\vec{\mathcal{L}}^{mech} = \vec{\mathcal{L}}^{mech}_{final\,a}(t=t_{final}) - \underbrace{\cancel{\vec{\mathcal{L}}^{mech}_{init\,a}(t=0)}}_{=0} = \vec{\mathcal{L}}^{mech}_{final\,a}(t=t_{final}) = \int_{t=0}^{t=t_{final}}\vec{N}^{mech}_a(t)\,dt}$$

$$\boxed{\begin{aligned}\Delta\vec{\mathcal{L}}^{mech}_a &= \vec{\mathcal{L}}^{mech}_{final\,a}(t=t_{final}) = \int_{t=0}^{t=t_{final}}\vec{N}^{mech}_a dt = \int_{t=0}^{t=t_{final}}\left(-\frac{1}{2}\mu_o nQ\frac{dI(t)}{dt}a^2\hat{z}\right)dt \\ &= -\frac{1}{2}\mu_o nQa^2\int_{t=0}^{t=t_{final}}\frac{dI(t)}{\cancel{dt}}\cancel{dt} = -\frac{1}{2}\mu_o nQa^2\int_{I(t)=I}^{I(t)=0}dI(t) = -\frac{1}{2}\mu_o nQa^2\hat{z}\left[0-I\right]\end{aligned}}$$

<u>Thus</u>: $\boxed{\vec{\mathcal{L}}^{mech}_{final\,a}(t=t_{final}) = \Delta\vec{\mathcal{L}}^{mech}_a = +\frac{1}{2}\mu_o nQa^2 I\hat{z}}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$ $\Leftarrow$ *n.b.* points in $+\hat{z}$ direction.

$\Rightarrow$ the inner conducting cylinder {viewed from above} rotates <u>counter-clockwise</u> $\left(@\,t=t_{final}\right)$!

Similarly, the {instantaneous} mechanical torque exerted on the <u>outer</u> conducting cylindrical tube (of radius *b*) by the <u>tangential</u> $\vec{E}$-field $\vec{E}^{OUT}(\rho \ge R)$ is:

$$\boxed{\vec{N}_b^{mech}(\vec{r},t) = \vec{r}\times\vec{F}_E(\vec{r},t)\Big|_{\rho=b} = \vec{r}\times\left(-Q\vec{E}^{OUT}(\rho,t)\right)\Big|_{\rho=b} = +b\hat{\rho}\times\left(+\frac{1}{2}\mu_0 nQ\frac{dI(t)}{dt}\left(\frac{R^2}{\rho}\right)\hat{\varphi}\right)}$$

$$\boxed{\vec{N}_b^{mech}(\rho=b,t) = +\frac{1}{2}\mu_o nQ\frac{dI(t)}{dt}R^2\underbrace{(\hat{\rho}\times\hat{\varphi})}_{=+\hat{z}} = +\frac{1}{2}\mu_o nQ\frac{dI(t)}{dt}R^2\hat{z}} \quad \left(\text{N-m}=\frac{\text{kg-m}^2}{\text{sec}^2}\right)$$

$\Rightarrow$ The corresponding {increase} in the <u>mechanical</u> angular momentum the outer cylinder acquired in the time it takes the current in the solenoid to decrease from $I(t=0)=I$ to $I(t=t_{final})=0$ is given by:

$$\boxed{\Delta\vec{\mathcal{L}}_b^{mech} = \int_{\vec{\mathcal{L}}_{init\,b}^{mech}(t=0)=0}^{\vec{\mathcal{L}}_{final\,b}^{mech}(t=t_{final})} d\vec{\mathcal{L}}_b^{mech} = \vec{\mathcal{L}}_{final\,b}^{mech}(t=t_{final}) - \underbrace{\cancel{\vec{\mathcal{L}}_{init\,b}^{mech}(t=0)}}_{=0} = \vec{\mathcal{L}}_{final\,b}^{mech}(t=t_{final}) = \int_{t=0}^{t=t_{final}}\vec{N}_b^{mech}(t)\,dt}$$

$$\boxed{\begin{aligned}\Delta\vec{\mathcal{L}}_b^{mech} &= \vec{\mathcal{L}}_{final\,b}^{mech}(t=t_{final}) = \int_{t=0}^{t=t_{final}}\left(+\frac{1}{2}\mu_o nQ\frac{dI(t)}{dt}R^2\hat{z}\right)dt = +\frac{1}{2}\mu_o nQR^2\hat{z}\int_{t=0}^{t=t_{final}}\frac{dI(t)}{\cancel{dt}}\cancel{dt}\\ &= +\frac{1}{2}\mu_o nQR^2\hat{z}\int_{I(t=0)=I}^{I(t=t_{final})=0}dI(t) = +\frac{1}{2}\mu_o nQR^2\hat{z}\left[0-I\right]\end{aligned}}$$

Thus: $\boxed{\vec{\mathcal{L}}_{final\,b}^{mech}(t=t_{final}) = \Delta\vec{\mathcal{L}}_b^{mech} = -\frac{1}{2}\mu_o nQIR^2\hat{z}}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$ $\Leftarrow$ *n.b.* points in $-\hat{z}$ direction.

$\Rightarrow$ outer conducting tube rotates <u>clockwise</u> at $\left(@\,t=t_{final}\right)$ viewed from above!

Now note that, for $t \ge t_{final}$: $\boxed{\vec{\mathcal{L}}_{final\,Tot}^{mech} \equiv \vec{\mathcal{L}}_{final\,a}^{mech} + \vec{\mathcal{L}}_{final\,b}^{mech} = \sum_{i=1}^{2}\vec{\mathcal{L}}_{final\,i}^{mech}}$

$$\boxed{\vec{\mathcal{L}}_{final\,Tot}^{mech} = +\frac{1}{2}\mu_o nQIa^2\hat{z} - \frac{1}{2}\mu_o nQIR^2\hat{z} = -\frac{1}{2}\mu_o nQI\left(R^2-a^2\right)\hat{z}}$$

But this is <u>precisely</u> the *EM* field angular momentum, for $t \le 0$: $\boxed{\vec{\mathcal{L}}_{EM} = -\frac{1}{2}\mu_o nQI\left(R^2-a^2\right)\hat{z}}$

i.e. $\boxed{\vec{\mathcal{L}}_{EM}(t\le 0) = \vec{\mathcal{L}}_{final\,Tot}^{mech}(t\ge t_{final}) = -\frac{1}{2}\mu_o nQI\left(R^2-a^2\right)\hat{z}}$ $\left(\text{kg-m}^2/\text{sec}\right)$

Thus, we explicitly see that angular momentum is conserved – angular momentum that was originally stored in the *EM* fields of this device is converted to mechanical angular momentum as the current in the long solenoid is slowly/steadily decreased!

Again, microscopically, angular momentum is carried by virtual photons associated with the macroscopic $\vec{E}\,\&\,\vec{B}$ fields in this region of space. The angular momentum, as initially carried by the *EM* fields in the region $\left(a \le \rho \le R \text{ and } |z| \le \ell/2\right)$ is transferred to the two charged conducting inner/outer cylindrical tubes as the current flowing conducting in the solenoid is slowly decreased from $I \to 0$, the cylinders acquiring non-zero mechanical angular momentum, the total of which = initial EM angular momentum! Note also the time-reversed situation (*I* increasing, *i.e.* from $0 \to I$) also does the time-reversed thing – because the *EM* force/interaction (microscopically & macroscopically) obeys <u>time-reversal invariance</u>!!!

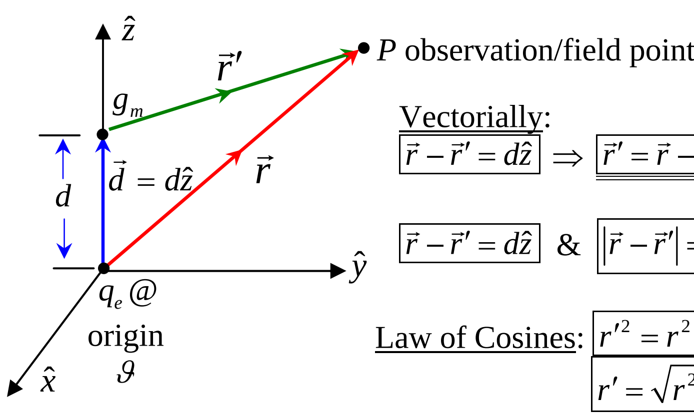
The *EM* Field Energy Density u_{EM}, Poynting's Vector $\vec{S}$, Linear Momentum Density $\vec{\wp}_{EM}$ and Angular Momentum Density $\vec{\ell}_{EM}$ Associated with a Point Electric Charge q_e and a Point Magnetic Monopole g_m

n.b. This is a static problem – *i.e.* it has no time dependence!

Point electric charge at origin: $\boxed{\vec{E}(\vec{r})=\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r^2}\hat{r}=\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r^3}\vec{r}}$ $\boxed{\vec{r}=r\hat{r}}$

Point magnetic monopole *e.g.* located at $\vec{d}=d\hat{z}$:

$$\boxed{\vec{B}(\vec{r})=\left(\frac{\mu_o}{4\pi}\right)\frac{g_m}{r'^2}\hat{r}'=\left(\frac{\mu_0}{4\pi\varepsilon_0}\right)\frac{g_m}{r'^3}\vec{r}'} \qquad \boxed{\vec{r}'=r'\hat{r}'}$$



Vectorially: $\boxed{\vec{r}-\vec{r}'=d\hat{z}} \Rightarrow \boxed{\vec{r}'=\vec{r}-d\hat{z}}$

$\boxed{\vec{r}-\vec{r}'=d\hat{z}}$ & $\boxed{|\vec{r}-\vec{r}'|=d}$

Law of Cosines: $\boxed{r'^2=r^2+d^2-2rd\cos\theta}$

$\boxed{r'=\sqrt{r^2+d^2-2rd\cos\theta}}$

$$\therefore \boxed{\vec{B}(\vec{r})=\left(\frac{\mu_o}{4\pi}\right)\frac{g_m}{r'^3}\vec{r}'=\left(\frac{\mu_o}{4\pi}\right)\frac{g_m(\vec{r}-d\hat{z})}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}}$$

EM field energy density:

$$\boxed{u_{EM}(\vec{r})=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r})+\frac{1}{\mu_o}B^2(\vec{r})\right)=\frac{1}{2}\left(\varepsilon_o\vec{E}(\vec{r})\bullet\vec{E}(\vec{r})+\frac{1}{\mu_o}\vec{B}(\vec{r})\bullet\vec{B}(\vec{r})\right)}$$

$$u_{EM}(\vec{r})=\frac{1}{2}\left\{\varepsilon_o\left[\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r^2}\right]^2+\frac{1}{\mu_o}\left[\left(\frac{\mu_o}{4\pi}\right)\frac{g_m}{\underbrace{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}_{=r'^3}}\right]^2\overbrace{\left(\vec{r}'\bullet\vec{r}'\right)}^{=r'^2}\right\}$$

$$u_{EM}(\vec{r})=\frac{1}{2}\left\{\varepsilon_o\left[\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r^2}\right]^2+\frac{1}{\mu_o}\left[\left(\frac{\mu_o}{4\pi}\right)\frac{g_m}{\left(r^2+d^2-2rd\cos\theta\right)}\right]^2\right\}$$

$$u_{EM}(\vec{r})=\frac{1}{2}\left\{\cancel{\varepsilon_o}\left(\frac{q_e^2}{16\pi^2\varepsilon_o^{\cancel{2}}r^4}\right)+\frac{1}{\cancel{\mu_o}}\left[\left(\frac{\mu_o^{\cancel{2}}}{16\pi^2}\right)\frac{g_m^2}{\left(r^2+d^2-2rd\cos\theta\right)^2}\right]\right\} \quad \text{but: } \varepsilon_o\mu_o=1/c^2$$

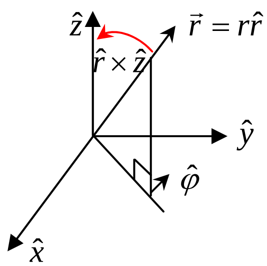
$$\boxed{u_{EM}(\vec{r}) = \frac{1}{32\pi^2\varepsilon_o}\left\{\frac{q_e^2}{r^4} + \frac{g_m^2/c^2}{\left(r^2+d^2-2rd\cos\theta\right)^2}\right\} = \frac{1}{32\pi^2\varepsilon_o}\left(\frac{1}{r^4}\right)\left\{q_e^2 + \frac{g_m^2/c^2}{\left(1+\left[\frac{d}{r}\right]^2 - 2\left[\frac{d}{r}\right]\cos\theta\right)^2}\right\}}$$

n.b. for $r \gg d$ (also true for $d = 0$): $\boxed{u_{EM}(\vec{r}) \simeq \frac{1}{32\pi^2\varepsilon_o}\left(\frac{1}{r^4}\right)\left\{q_e^2 + g_m^2/c^2\right\}}$ $\left(\frac{\text{Joules}}{\text{m}^3}\right)$

Poynting's Vector: $\boxed{\vec{S}(\vec{r}) = \frac{1}{\mu_o}\vec{E}(\vec{r})\times\vec{B}(\vec{r})}$ $\left(\frac{\text{Watts}}{\text{m}^2}\right)$

$$\vec{S}(\vec{r}) = \frac{1}{\cancel{\mu_o}}\left[\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r^3}\vec{r}\right]\times\left[\left(\frac{\cancel{\mu_o}}{4\pi}\right)\frac{g_m(\vec{r}-d\hat{z})}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right]$$

In spherical coordinates: $\vec{r}\times\vec{r}\equiv 0$, $\hat{z} = \left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)$, $\vec{r}\times\hat{z} = -r\sin\theta\hat{\varphi}$ and $\hat{r}\times\hat{\theta} = \hat{\varphi}$, $\vec{r} = r\hat{r}$.



$$\boxed{\begin{aligned}&n.b. \left(\hat{r}\times\hat{z}\right) = -\sin\theta\hat{\varphi}\\ &\text{is } \perp \text{ to } \hat{r} \ \left(\text{and} \perp \text{to } \hat{z}\right)\end{aligned}}$$

$$\therefore\ \vec{S}(\vec{r}) = \frac{1}{16\pi^2\varepsilon_o}\ \frac{q_e g_m d\vec{r}\times(-\hat{z})}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} = -\frac{d}{16\pi^2\varepsilon_o}\ \frac{q_e g_m}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\overbrace{\left(\vec{r}\times\hat{z}\right)}^{=-r\sin\theta\,\hat{\varphi}}$$

$$\boxed{\vec{S}(\vec{r}) = -\frac{d}{16\pi^2\varepsilon_o}\ \frac{q_e g_m}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\overbrace{\left(\vec{r}\times\hat{z}\right)}^{=-r\sin\theta\,\hat{\varphi}} = +\frac{d}{16\pi^2\varepsilon_o}\ \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\sin\theta\,\hat{\varphi}}$$

Note that Poynting's vector $\vec{S}(\vec{r})\,||\,\hat{\varphi}$ – *i.e. EM* energy is circulating in the $+\hat{\varphi}$ (azimuthal) direction in a ***static*** problem! Note also that $\vec{S}(\vec{r})$ vanishes when $d = 0$ (*i.e.* monopole g_m is on top of electric charge q_e) and also vanishes whenever $\hat{r}\,||$ (or anti-$||$) to $\hat{z}$ (then $\hat{r}\times\hat{z} = \sin\theta\hat{\varphi} = 0$)!

EM Field Linear momentum density: $\boxed{\vec{\wp}_{EM}(\vec{r}) = \varepsilon_o\mu_o\vec{S}(\vec{r}) = \vec{S}(\vec{r})/c^2 = \varepsilon_o\vec{E}(\vec{r})\times\vec{B}(\vec{r})}$ $\left(\frac{\text{kg}}{\text{m}^2\text{-s}}\right)$

$$\boxed{\vec{\wp}_{EM}(\vec{r}) = -\frac{\mu_o d}{16\pi^2}\frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\overbrace{\left(\vec{r}\times\hat{z}\right)}^{=-r\sin\theta\,\hat{\varphi}} = +\frac{\mu_o d}{16\pi^2}\ \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\sin\theta\,\hat{\varphi}}$$

Here again, note that $\vec{\wp}_{EM}(\vec{r})\,||\,\hat{\varphi}$ – *i.e. EM* linear momentum density is circulating in the $+\hat{\varphi}$ (azimuthal) direction in a static problem! Note also that $\vec{\wp}_{EM}(\vec{r})$ vanishes when $d = 0$ and also vanishes whenever $\hat{r}\,||$ (or anti-$||$) to $\hat{z}$ (then $\hat{r}\times\hat{z} = \sin\theta\hat{\varphi}$ vanishes)!

EM Field Angular momentum density: $\boxed{\ell_{EM}(\vec{r}) = \vec{r} \times \vec{\wp}_{EM}(\vec{r})}$ $\left(\frac{\text{kg}}{\text{m-s}}\right)$

$$\vec{\ell}_{EM}(\vec{r}) = -\frac{\mu_o d}{16\pi^2} \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \vec{r} \times (\hat{r} \times \hat{z}) \qquad \text{but: } \vec{r} = r\hat{r}$$

$$\therefore\ \vec{\ell}_{EM}(\vec{r}) = -\frac{\mu_o d}{16\pi^2} \frac{q_e g_m}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \hat{r} \times (\hat{r} \times \hat{z})$$

Now: $\hat{r} \times (\hat{r} \times \hat{z}) = \hat{r}(\hat{r}\bullet\hat{z}) - \hat{z}(\hat{r}\bullet\hat{r}) = \hat{r}(\hat{r}\bullet\hat{z}) - \hat{z} = \hat{r}\cos\theta - \hat{z} = \cancel{\hat{r}\cos\theta} - \cancel{\hat{r}\cos\theta} + \hat{\theta}\sin\theta = \hat{\theta}\sin\theta$

where: $(\hat{r}\bullet\hat{z}) = \hat{r}\bullet\left\{\hat{r}\cos\theta - \hat{\theta}\sin\theta\right\} = \cos\theta$ and: $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$\therefore\ \boxed{\vec{\ell}_{EM}(\vec{r}) = -\frac{\mu_o d}{16\pi^2} \frac{q_e g_m}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \sin\theta\hat{\theta}} \quad \left(\frac{\text{kg}}{\text{m-s}}\right)$$

EM Field Energy Density: $\boxed{u_{EM}(\vec{r}) = \frac{1}{32\pi^2\varepsilon_o}\left(\frac{1}{r^2}\right)\left\{\frac{q_e^2}{r^2} + \frac{g_m^2/c^2}{\left(r^2+d^2-2rd\cos\theta\right)^2}\right\}}$ $\left(\frac{\text{Joules}}{\text{m}^3}\right)$

Poynting's Vector: $\boxed{\vec{S}(\vec{r}) = +\frac{d}{16\pi^2\varepsilon_o} \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \sin\theta\,\hat{\varphi}}$ $\left(\frac{\text{Watts}}{\text{m}^2}\right)$

EM Linear Momentum Density: $\boxed{\vec{\wp}_{EM}(\vec{r}) = +\frac{\mu_o d}{16\pi^2} \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \sin\theta\,\hat{\varphi}}$ $\left(\frac{\text{kg}}{\text{m}^2\text{-s}}\right)$

EM Angular Momentum Density: $\boxed{\vec{\ell}_{EM}(\vec{r}) = -\frac{\mu_o d}{16\pi^2} \frac{q_e g_m}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \sin\theta\hat{\theta}}$ $\left(\frac{\text{kg}}{\text{m-s}}\right)$

The Total *EM* Field Energy: $U_{EM} = \int_v u_{EM}(\vec{r})\,d\tau = \infty$ (Joules) because $\vec{E}(\vec{r}=0)$ and $\vec{B}(\vec{r}=\vec{r}')$ both diverge/are both singular (at $\vec{r}=0$ and $\vec{r}=\vec{r}'$ respectively) – so this is not a surprise!!!

However, the *EM* Power flowing through/crossing the enclosing surface S is zero (!!!):

$$\boxed{P_{EM} = -\oint_S \vec{S}(\vec{r})\bullet d\vec{a} = -\frac{d}{16\pi^2\varepsilon_o}(q_e g_m)\oint_S \frac{1}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} \sin\theta\,\hat{\varphi}\bullet d\vec{a} = 0} \ (\text{Watts}) \ !!!$$

But: $\vec{S}(\vec{r}) \neq 0$!!! The *EM* Power $P_{EM} = 0$ because $d\vec{a} = da\hat{n} = da\hat{r}$ and $\hat{\varphi}\bullet\hat{r} = 0$, *i.e.* $\hat{\varphi}$ is always $\perp$ to $\hat{r}$!!! $\Rightarrow$ *EM* field energy associated with electric charge – magnetic monopole $(q_e - g_m)$ system circulates! (*i.e*. is fully contained within enclosing surface S !!!)

<u>Total *EM* Field Linear Momentum</u>:

$$\boxed{\vec{p}_{EM} = \int_v \vec{\wp}_{EM}(r)\,d\tau = +\frac{\mu_o d}{16\pi^2}\int_v \frac{q_e g_m}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\sin\theta\,\hat{\varphi}\;d\tau} \quad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

Note that: $\boxed{\oint_S \vec{\wp}_{EM}(\vec{r})\cdot d\vec{a} = 0}$ because $\{\vec{\wp}_{EM}(\vec{r}) \parallel \hat{\varphi}\} \perp \{d\hat{a} \parallel \hat{r}\} \Rightarrow$ *EM* Field Linear Momentum <u>circulates</u> (*i.e.* *EM* field linear momentum is also fully contained within the enclosing surface *S*)!

$$\boxed{\begin{aligned}\vec{p}_{EM} &= +\frac{\mu_o d}{16\pi^2}(q_e g_m)\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{\cancel{r^2}\,dr\sin^2\theta\,d\theta\,d\varphi}{\cancel{r^2}\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\,\hat{\varphi}\\ &= +\frac{\mu_o d}{16\pi^2}(q_e g_m)\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{dr\sin^2\theta\,d\theta\,d\varphi}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\,\hat{\varphi}\end{aligned}}$$

Let's do the φ-integral first (trivial – get 2π):

$$\vec{p}_{EM} = +\frac{\mu_o d}{8\pi}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{dr\sin^2\theta\,d\theta}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\,\hat{\varphi}$$

Next, let's do the *r*-integral:

$$\begin{aligned}\vec{p}_{EM} &= +\frac{\mu_o d}{8\pi}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\frac{2\left(2r-2d\cos\theta\right)\sin^2\theta\,d\theta}{\left(4d^2-\left(2d\cos\theta\right)^2\right)\sqrt{r^2+d^2-2rd\cos\theta}}\,\hat{\varphi}\\ &= +\frac{\mu_o d}{8\pi}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\left.\frac{\cancel{4}\left(r-d\cos\theta\right)\sin^2\theta\,d\theta}{\cancel{4}d^2\left(1-\cos\theta^2\right)\sqrt{r^2+d^2-2rd\cos\theta}}\right|_{r=0}^{r=\infty}\hat{\varphi}\\ &= +\frac{\mu_o \cancel{d}}{8\pi d^{\cancel{2}}}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\left.\frac{\left(r-d\cos\theta\right)\cancel{\sin^2\theta}\,d\theta}{\cancel{\sin^2\theta}\sqrt{r^2+d^2-2rd\cos\theta}}\right|_{r=0}^{r=\infty}\hat{\varphi}\\ &= +\frac{\mu_o}{8\pi d}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\left.\frac{\left(r-d\cos\theta\right)d\theta}{\sqrt{r^2+d^2-2rd\cos\theta}}\right|_{r=0}^{r=\infty}\hat{\varphi}\\ &= +\frac{\mu_o}{8\pi d}(q_e g_m)\int_{\theta=0}^{\theta=\pi}\left[1+\cos\theta\right]d\theta\,\hat{\varphi}\end{aligned}$$

Finally, we carry out the θ-integral:

$$\vec{p}_{EM} = +\frac{\mu_o}{8\pi d}(q_e g_m)\left[\theta+\sin\theta\right]_{\theta=0}^{\theta=\pi}\hat{\varphi} = +\frac{\mu_o}{8\pi d}(q_e g_m)\left[(\pi-0)+(0-0)\right]\hat{\varphi} = +\frac{\mu_o}{8\cancel{\pi}d}(q_e g_m)\cancel{\pi}\hat{\varphi}$$

<u>or</u>: $\boxed{\vec{p}_{EM} = +\frac{\mu_o q_e g_m}{8d}\hat{\varphi}} \quad \left(\frac{\text{kg-m}}{\text{sec}}\right)$

Note that the *EM* field linear momentum $\vec{p}_{EM}$ is finite as long as the electric charge-magnetic monopole separation distance $d > 0$. When the electric charge q_e is on top of the monopole g_m, then $d = 0$ and $\vec{p}_{EM}$ becomes ***infinite***.

Total *EM* Field Angular Momentum: $\boxed{\vec{\mathcal{L}}_{EM} = \int_v \vec{\ell}_{EM}(\vec{r})\,d\tau}$ $\left(\text{kg-m}^2/\text{sec}\right)$

$$\boxed{\vec{\mathcal{L}}_{EM} = -\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_v \frac{\overbrace{(\hat{r}\cos\theta - \hat{z})}^{=\sin\theta\hat{\theta}}}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\,d\tau = -\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{(\hat{r}\cos\theta-\hat{z})\,r^2\sin\theta d\theta d\varphi}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}}$$

Let's work this out in Cartesian coordinates: $\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$

$$\vec{\mathcal{L}}_{EM} = -\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{\left\{\left[\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z}\right]\cos\theta-\hat{z}\right\}r^2\sin\theta d\theta d\varphi dr}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}$$

$$\vec{\mathcal{L}}_{EM} = -\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{\left\{\sin\theta\cos\theta\cos\varphi\hat{x}+\sin\theta\cos\theta\sin\varphi\hat{y}+\left(\cos^2\theta-1\right)\hat{z}\right\}r^2\sin\theta d\theta d\varphi dr}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}$$

Now note that: $\int_{\varphi=0}^{\varphi=2\pi}\{\ \}\cos\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\{\ \}\sin\varphi d\varphi$

$$= \{\ \}\int_{\varphi=0}^{\varphi=2\pi}\cos\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\{\ \}\sin\varphi d\varphi = 0$$

$$= \{\ \}\left(-\sin\varphi\right)\Big|_{\varphi=0}^{\varphi=2\pi} = \{\ \}\left(\cos\varphi\right)\Big|_{\varphi=0}^{\varphi=2\pi} = 0$$

Thus, the integrals over the φ-variable for both the $\hat{x}$ and $\hat{y}$ terms explicitly vanish,

$\Rightarrow$ $\hat{x}$ - and $\hat{y}$ - components of $\vec{\mathcal{L}}_{EM}$ both vanish due to manifest axial/azimuthal symmetry (rotational invariance) of this problem about the $\hat{z}$ -axis; only the $\hat{z}$-term remains:

$$\vec{\mathcal{L}}_{EM} = -\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{\left(\cos^2\theta-1\right)\hat{z}}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}r^2 dr\sin\theta d\theta d\varphi$$

Now: $1 = \sin^2\theta + \cos^2\theta \ \Rightarrow\ -\sin^2\theta = \left(\cos^2\theta - 1\right)$

$$\therefore\ \boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o d\; q_e g_m}{16\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty}\frac{\sin^2\theta\hat{z}}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}r^2 dr\sin\theta d\theta d\varphi}$$

Let's do the φ-integral first - (trivial), since integrand has no explicit φ-dependence, get:

$$\boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o d\; q_e g_m}{8\pi}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty}\frac{\sin^2\theta\hat{z}}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}r^2 dr\sin\theta d\theta}$$

Next, let's do the r-integral – noting that:

$$\int_{r=0}^{r=\infty}\frac{r^2 dr}{r\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} = \int_{r=0}^{r=\infty}\frac{r dr}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} = \frac{\left(r\cos\theta-d\right)}{d\left(1-\cos^2\theta\right)\sqrt{r^2+d^2-2rd\cos\theta}}\Bigg|_{r=0}^{r=\infty}$$

$$= \left[\frac{\cos\theta}{d\left(1-\cos^2\theta\right)} + \frac{\cancel{d}}{\cancel{d}\left(1-\cos^2\theta\right)d}\right] = \frac{\left(1+\cos\theta\right)}{d\left(1-\cos^2\theta\right)} = \frac{\cancel{\left(1+\cos\theta\right)}}{d\cancel{\left(1+\cos\theta\right)}\left(1-\cos\theta\right)} = \frac{1}{d\left(1-\cos\theta\right)}$$

$$\therefore \boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o \cancel{d}\, q_e g_m}{8\pi}\int_{\theta=0}^{\theta=\pi}\frac{\sin^3\theta d\theta}{\cancel{d}\left(1-\cos\theta\right)}\hat{z} = +\frac{\mu_o q_e g_m}{8\pi}\int_{\theta=0}^{\theta-\pi}\frac{\sin^2\theta\sin\theta d\theta}{\left(1-\cos\theta\right)}\hat{z}}$$

Finally, let's do the θ-integral:

Let $u=\cos\theta$ and $du = d\cos\theta = -\sin\theta d\theta$, and $\sin^2\theta = 1-\cos^2\theta = 1-u^2$
For $\theta = 0 \Rightarrow u=+1$ and for $\theta=\pi \Rightarrow u=-1$

<u>Then</u>: $\boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o q_e g_m}{8\pi}\int_{u=-1}^{u=+1}\frac{\left(1-u^2\right)du}{\left(1-u\right)}\hat{z}}$ <u>but</u>: $\boxed{1-u^2 = \left(1-u\right)\left(1+u\right)}$

$$\boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o q_e g_m}{8\pi}\int_{u=-1}^{u=+1}\frac{\cancel{\left(1-u\right)}\left(1+u\right)}{\cancel{\left(1-u\right)}}du\hat{z} = \frac{\mu_o q_e g_m}{8\pi}\int_{u=-1}^{u=+1}\left(1+u\right)du\hat{z}}$$

$$\boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o q_e g_m}{8\pi}\left[u+\frac{1}{2}u^2\right]\hat{z}\Bigg|_{u=-1}^{u=+1} = \frac{\mu_o q_e g_m}{8\pi}\left[2\right]\hat{z} = \frac{\mu_o q_e g_m}{4\pi}\hat{z}}$$

$$\boxed{\vec{\mathcal{L}}_{EM} = +\left(\frac{\mu_o}{4\pi}\right)q_e g_m\hat{z}} \quad \left(\text{kg-m}^2/\text{sec}\right)$$

Note that the *EM* field angular momentum associated with the electric charge-magnetic monopole system is <u>independent</u> of the $q_e - g_m$ separation distance, d !!!

Quantum mechanically $\vec{\mathcal{L}}_{EM}$ is <u>quantized</u> in integer (or even half-integer) units of $\hbar \equiv h/2\pi$, where *h* = Planck's constant,

i.e. $\boxed{\left|\vec{\mathcal{L}}_{EM}\right| = \ell\hbar = \frac{\ell h}{2\pi} = \left(\frac{\mu_o}{4\pi}\right)q_e g_m} \Rightarrow \boxed{q_e\cdot\mu_o g_m = 2\ell h}$!!!

However, recall the Dirac Quantization Condition (P435 Lecture Notes 18) which arose from insisting on the single-valued nature of the electron's wavefunction circling/orbiting a {presumed} heavy magnetic monopole:

$e\cdot\mu_o g_m = \frac{e g_m}{\varepsilon_o c^2} = nh$ (SI units)	Dirac Quantization Condition

These two formulae agree if $2\ell = n$ or $\ell = n/2$, thus if *n* = 1,2,3,… then $\ell = \frac{1}{2},1,\frac{3}{2},2...$ and

$$\boxed{\left|\vec{\mathcal{L}}_{EM}\right| = \ell\hbar = \tfrac{1}{2}\hbar, 1\hbar, \tfrac{3}{2}\hbar, 2\hbar ... = \frac{\ell h}{2\pi} = \left(\frac{\mu_o}{4\pi}\right)q_e g_m} \quad \left(\text{kg-m}^2/\text{sec}\right)$$

The *EM* Field Energy Density u_{EM}, Poynting's Vector $\vec{S}$, Linear Momentum Density $\vec{\wp}_{EM}$ and Angular Momentum Density $\vec{\ell}_{EM}$ Associated with a Point Electric Charge q_e and a Point/Pure Magnetic Dipole Moment $\vec{m} = m\hat{z}$

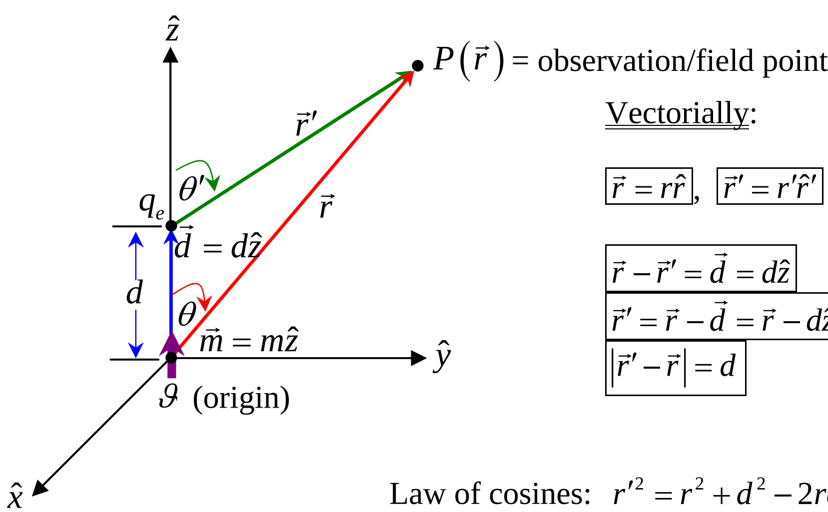
n.b. This is {again} a static problem – has no time dependence!

1) This time, we locate the point charge q_e at $\vec{d} = d\hat{z}$: $\boxed{\vec{E}(\vec{r}) = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r'^2}\hat{r}' = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r'^3}\vec{r}'}$

2) We locate the pure/point magnetic dipole moment $\vec{m} = m\hat{z}$ at the origin:

$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left\{\frac{m}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] - \left(\frac{8\pi}{3}\right)\vec{m}\,\delta^3(\vec{r})\right\}} \longleftarrow \text{ in coordinate-free form}$$

$$\boxed{\vec{B}(\vec{r}) = \left(\frac{\mu_o}{4\pi}\right)\left\{\frac{m}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) - \left(\frac{8\pi}{3}\right)\vec{m}\,\delta^3(\vec{r})\right\}} \longleftarrow \text{ in spherical coordinates}$$



Vectorially:

$\boxed{\vec{r} = r\hat{r}}$, $\boxed{\vec{r}' = r'\hat{r}'}$ and $\boxed{\vec{m} = m\hat{z}}$

$\boxed{\vec{r} - \vec{r}' = \vec{d} = d\hat{z}}$

$\boxed{\vec{r}' = \vec{r} - \vec{d} = \vec{r} - d\hat{z}}$

$\boxed{|\vec{r}' - \vec{r}| = d}$

Law of cosines: $r'^2 = r^2 + d^2 - 2rd\cos\theta$

$r' = \sqrt{r^2 + d^2 - 2rd\cos\theta} = |\vec{r}'|$

EM Energy Density:

$$\boxed{u_{EM}(\vec{r}) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r}) + \frac{1}{\mu_o}B^2(\vec{r})\right) = \frac{1}{2}\left(\varepsilon_o \vec{E}(\vec{r})\cdot\vec{E}(\vec{r}) + \frac{1}{\mu_o}\vec{B}(\vec{r})\cdot\vec{B}(\vec{r})\right)}$$

$$\boxed{\begin{aligned} u_{EM}(\vec{r}) = \frac{1}{2}\Bigg\{&\varepsilon_o\left(\frac{1}{4\pi\varepsilon_o}\right)^2\frac{q_e^2}{(r^2+d^2-2rd\cos\theta)^2} + \frac{1}{\mu_o}\left(\frac{\mu_o}{4\pi}\right)^2\Bigg[\frac{m^2}{r^6}\Big(\underbrace{4\cos^2\theta + \sin^2\theta}_{=(3\cos^2\theta+1)}\Big) \\ &-2\frac{m^2}{r^3}\left(\frac{8\pi}{3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\cdot\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\delta^3(\vec{r}) + \left(\frac{8\pi}{3}\right)^2 m^2\left(\delta^3(\vec{r})\right)^2\Bigg]\Bigg\}\end{aligned}}$$

$$\boxed{\begin{aligned} u_{EM}(\vec{r}) = \frac{1}{2}\Bigg\{ & \varepsilon_o \left(\frac{1}{4\pi\varepsilon_o}\right)^2 \frac{q_e^2}{\left(r^2+d^2-2rd\cos\theta\right)^2} + \frac{1}{\mu_o}\left(\frac{\mu_o}{4\pi}\right)^2 \Bigg[\frac{m^2}{r^6}\left(3\cos^2\theta+1\right) \\ & -\frac{16\pi}{3}\frac{m^2}{r^3}\underbrace{\left(2\cos^2\theta-\sin^2\theta\right)}_{=3\cos^2\theta-1}\delta^3(\vec{r}) + \left(\frac{8\pi}{3}\right)^2 m^2\left(\delta^3(\vec{r})\right)^2\Bigg]\Bigg\} \end{aligned}}$$

$$\boxed{\begin{aligned} u_{EM}(\vec{r}) = \frac{1}{2}\Bigg\{ & \varepsilon_o \left(\frac{1}{4\pi\varepsilon_o}\right)^2 \frac{q_e^2}{\left(r^2+d^2-2rd\cos\theta\right)^2} + \frac{1}{\mu_o}\left(\frac{\mu_o}{4\pi}\right)^2 \Bigg[\frac{m^2}{r^6}\left(3\cos^2\theta+1\right) \\ & -\frac{16\pi}{3}\frac{m^2}{r^3}\left(3\cos^2\theta-1\right)\delta^3(\vec{r}) + \left(\frac{8\pi}{3}\right)^2 m^2\left(\delta^3(\vec{r})\right)^2\Bigg]\Bigg\} \end{aligned}}$$

$$\boxed{\begin{aligned} u_{EM}(\vec{r}) = \frac{1}{32\pi^2}\Bigg\{ & \left(\frac{q_e^2}{\varepsilon_o}\right)\frac{1}{\left[r^2+d^2-2rd\cos\theta\right]^2} \\ & +\mu_o\left(\frac{m^2}{r^6}\right)\left[\left(3\cos^2\theta+1\right) - \frac{16\pi}{3}r^3\left(3\cos^2\theta-1\right)\delta^3(\vec{r}) + \left(\frac{8\pi}{3}\right)^2 r^6\left(\delta^3(\vec{r})\right)^2\right]\Bigg\} \end{aligned}}$$

n.b. If $r \gg d$ (or $d=0$) then for $r>0$: $\boxed{u_{EM}(\vec{r}) \simeq \frac{1}{32\pi r^4}\left\{\left(\frac{q_e^2}{\varepsilon_o}\right) + \mu_o\left(\frac{m^2}{r^2}\right)\left(3\cos^2\theta+1\right)\right\}}$

<u>Poynting's Vector</u>: $\boxed{S(\vec{r}) = \frac{1}{\mu_o}\left(E(\vec{r})\times B(\vec{r})\right)}$

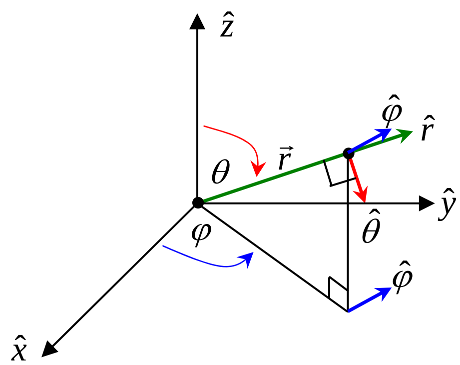
$$\boxed{\begin{aligned} S(\vec{r}) &= \frac{1}{\cancel{\mu_o}}\left[\left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_e}{r'^3}\vec{r}'\right]\times\left[\left(\frac{\cancel{\mu_o}}{4\pi}\right)\left(\frac{m}{r^3}\right)\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) - \left(\frac{8\pi}{3}\right)m\hat{z}\,\delta^3(\vec{r})\right] \\ &= \frac{1}{16\pi^2\varepsilon_o}\left(\frac{q_e m}{r'^3 r^3}\right)\left\{\vec{r}'\times\left[\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) - \left(\frac{8\pi}{3}\right)r^3\hat{z}\,\delta^3(\vec{r})\right]\right\} \end{aligned}}$$

<u>but</u>: $\boxed{\vec{r}' = \vec{r}-\vec{d} = \vec{r}-d\hat{z} = r\hat{r}-d\hat{z}}$ <u>and</u> $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ <u>then</u>:

$$\boxed{\begin{aligned} \vec{r}'\times\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) &= \left(r\hat{r}-d\hat{z}\right)\times\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) \\ &= 2r\cos\theta\underbrace{\left(\cancel{\hat{r}\times\hat{r}}\right)}_{=0} + r\sin\theta\underbrace{\left(\hat{r}\times\hat{\theta}\right)}_{=+\hat{\varphi}} - 2d\cos\theta\underbrace{\left(\hat{z}\times\hat{r}\right)}_{=+\sin\theta\hat{\varphi}} - d\sin\theta\underbrace{\left(\hat{z}\times\hat{\theta}\right)}_{=+\cos\theta\hat{\varphi}} \\ &= \left(r\sin\theta - 2d\sin\theta\cos\theta - d\sin\theta\cos\theta\right)\hat{\varphi} = \left(r\sin\theta - 3d\sin\theta\cos\theta\right)\hat{\varphi} \end{aligned}}$$

and:

$$\boxed{\begin{aligned} \vec{r}'\times\hat{z} &= \left(r\hat{r}-d\hat{z}\right)\times\hat{z} = r\left(\hat{r}\times\hat{z}\right) - d\left(\cancel{\hat{z}\times\hat{z}}\right) = r\left(\hat{r}\times\hat{z}\right) \\ &= r\hat{r}\times\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) = r\left(\cancel{\hat{r}\times\hat{r}}\right)\cos\theta - r\underbrace{\left(\hat{r}\times\hat{\theta}\right)}_{=+\hat{\varphi}}\sin\theta = -r\sin\theta\hat{\varphi} \end{aligned}}$$



In spherical coordinates:

$\boxed{\hat{r}\times\hat{\theta}=+\hat{\varphi}}$ and $\boxed{\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}}$

$$\begin{aligned}\hat{z}\times\hat{r}&=\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\times\hat{r}\\&=\cos\theta\underbrace{\left(\cancel{\hat{r}\times\hat{r}}\right)}_{=0}-\sin\theta\underbrace{\left(\hat{\theta}\times\hat{r}\right)}_{=-\varphi}=+\underline{\sin\theta\hat{\varphi}}\end{aligned}$$

$$\begin{aligned}\hat{z}\times\hat{\theta}&=\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\times\hat{\theta}\\&=\cos\theta\underbrace{\left(\hat{r}\times\hat{\theta}\right)}_{=+\hat{\varphi}}-\sin\theta\underbrace{\left(\cancel{\hat{\theta}\times\hat{\theta}}\right)}_{=0}=+\underline{\cos\theta\hat{\varphi}}\end{aligned}$$

$$\therefore\quad\boxed{\begin{aligned}\vec{S}(\vec{r})&=\frac{1}{16\pi^2\varepsilon_o}\left(\frac{q_e m}{r^3r'^3}\right)\left[\left(r\sin\theta-3d\sin\theta\cos\theta\right)+\left(\frac{8\pi}{3}\right)r^4\sin\theta\,\delta^3(\vec{r})\right]\hat{\varphi}\\&=\frac{1}{16\pi^2\varepsilon_o}\left(\frac{q_e m}{r^3r'^3}\right)\left[\left(r-3d\cos\theta\right)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]\sin\theta\hat{\varphi}\end{aligned}}$$

But: $r'=\sqrt{r^2+d^2-2rd\cos\theta}$

$$\therefore\quad\boxed{\vec{S}(\vec{r})=\frac{1}{16\pi^2\varepsilon_o}(q_e m)\left(\frac{\left[\left(r-3d\cos\theta\right)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\sin\theta\hat{\varphi}}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Note that:

1.) $\vec{S}(\vec{r})$ points in the $+\hat{\varphi}$-direction!!!

2.) $\vec{S}(\vec{r})$ vanishes (for $r > 0$) when: $\left(1-3\left(\tfrac{d}{r}\right)\cos\theta\right)=0$!!!

   i.e. when: $\cos\theta=\tfrac{1}{3}\left(\tfrac{d}{r}\right)=\tfrac{d}{3r}$ $\leftarrow$ equation for a line-curve (corresponds to a surface in φ !!)

3.) $\vec{S}(\vec{r})$ *also* vanishes (for $r > 0$) when: $\sin\theta=0$ *i.e.* at $\theta=0$ and $\theta=\pi$ *i.e.* @ *N/S* poles!

4.) Note also that {here} $\vec{S}(\vec{r})$ does not vanish when $d = 0$ (i.e. when point electric charge q_e and point magnetic dipole moment $\vec{m}=m\hat{z}$ are on top of/coincident with each other!!

5.) For $r\gg d$ (or $d=0$, with $r > 0$): $\boxed{\vec{S}(\vec{r})\simeq\frac{1}{16\pi^2\varepsilon_o}\left(\frac{q_e m}{r^5}\right)\sin\theta\hat{\varphi}}$ $\left(\frac{\text{Watts}}{\text{m}^2}\right)$

EM Field Linear Momentum Density: $\boxed{\vec{\wp}_{EM}(\vec{r})=\varepsilon_o\mu_o\vec{S}(\vec{r})}$

$$\boxed{\vec{\wp}_{EM}(\vec{r})=\varepsilon_o\mu_o\vec{S}(\vec{r})=\frac{\mu_o}{16\pi^2}(q_e m)\left(\frac{\left[\left(r-3d\cos\theta\right)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\sin\theta\hat{\varphi}}\quad\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Same comments made above for $\vec{S}(\vec{r})$ apply here for $\vec{\wp}_{EM}(\vec{r})$.

<u>*EM* Field Angular Momentum Density</u>: $\boxed{\vec{\ell}_{Em}(\vec{r}) = \vec{r}\times\vec{\wp}_{EM}(\vec{r})}$ where $\vec{r} = r\hat{r}$ and $\hat{r}\times\hat{\varphi} = -\hat{\theta}$

$$\boxed{\vec{\ell}_{EM}(\vec{r}) = -\frac{\mu_o}{16\pi^2}(q_e m)\left(\frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\sin\theta\hat{\theta}}$$

<u>Very Useful Table</u>:

$\hat{r}\times\hat{\theta} = +\hat{\varphi}$	$\hat{\theta}\times\hat{r} = -\hat{\varphi}$
$\hat{\theta}\times\hat{\varphi} = +\hat{r}$	$\hat{\varphi}\times\hat{\theta} = -\hat{r}$
$\hat{\varphi}\times\hat{r} = +\hat{\theta}$	$\hat{r}\times\hat{\varphi} = -\hat{\theta}$

Note that for $0 < r/3d < |\cos\theta|$ that $\vec{\ell}_{EM}(\vec{r})$ points in: $\begin{cases} -\hat{\theta} \text{ direction for } & 0 \le \theta < \pi/2 \\ +\hat{\theta} \text{ direction for } & \pi/2 \le \theta < \pi \end{cases}$

<u>Energy in *EM* Field</u>:

$$\boxed{\begin{aligned} U_{EM} = \int_v u_{EM}(\vec{r})\,d\tau = \frac{1}{32\pi^2}\int_v &\left\{\frac{q_e^2}{\varepsilon_o}\frac{1}{\left[r^2+d^2-2rd\cos\theta\right]^2}\right. \\ &\left.+\mu_o\left(\frac{m^2}{r^6}\right)\left[\left(3\cos^2\theta+1\right)-\frac{16\pi}{3}r^3\left(3\cos^2\theta-1\right)\delta^3(\vec{r})+\left(\frac{8\pi}{3}\right)^2 r^6\left(\delta^3(\vec{r})\right)^2\right]\right\}d\tau \end{aligned}}$$

$U_{EM} = \infty$ $\left(\vec{E} \text{ diverges at } \vec{r}=\vec{d},\ \vec{B} \text{ diverges at } \vec{r}=0\right)$ $\quad d\tau = r^2 dr\sin\theta d\theta d\varphi$

<u>Power in *EM* Field</u> crossing/passing through enclosing surface *S*: $\boxed{P_{EM} = -\oint_S \vec{S}(\vec{r})\cdot d\vec{a} = 0}$

because $\boxed{d\vec{a} = da\hat{n} = da\hat{r}}$ but $\vec{S}$ points in the $\hat{\varphi}$-direction. $\Rightarrow$ *EM* energy <u>circulates</u> within volume *v*, enclosed by surface *S* !!!

<u>Total Linear Momentum in *EM* Field</u>:

$$\boxed{\vec{p}_{EM} = \int_v \vec{\wp}_{EM}(\vec{r})\,d\tau = \frac{\mu_o}{16\pi^2}(q_e m)\int_v\left(\frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\sin\theta\hat{\varphi}\,d\tau}$$

Carrying out the 3-D volume integral is ~ tedious. We do not explicitly wade through this here. The contributions from each of the 3 terms associated with the numerator in the integrand are *a*.) finite, *b*.) logarithmically-divergent, and *c*.) zero respectively. Thus $\vec{p}_{EM} = \infty$ here, and also note that each of these 3 terms is proportional to $\frac{\mu_o}{d^2}(q_e m)$, which is strongly divergent as the electric charge q_e – point/pure magnetic dipole $\vec{m}$ separation distance $d \to 0$.

Note again that $\boxed{\oint_S \vec{\wp}_{EM}(\vec{r})\cdot d\vec{a} = 0}$ *i.e. EM* field linear momentum circulates in $\hat{\varphi}$-direction.

Total Angular Momentum in *EM* Field:

$$\boxed{\begin{aligned}\vec{\mathcal{L}}_{EM} &= \int_v \vec{\ell}_{EM}(\vec{r})\,d\tau = -\frac{\mu_o}{16\pi^2}(q_e m)\int_v \left(\frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{r^2\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\sin\theta\hat{\theta}\,d\tau \\ &= -\frac{\mu_o}{16\pi^2}(q_e m)\int_v \left(\frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{\cancel{r^2}\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\right)\cancel{r^2}dr\sin^2\theta d\theta d\varphi\,\hat{\theta}\end{aligned}}$$

$$\boxed{\vec{\mathcal{L}}_{EM} = -\frac{\mu_o}{16\pi^2}(q_e m)\int_v \frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}dr\sin^2\theta d\theta d\varphi\,\hat{\theta}} \quad \boxed{d\tau = r^2 dr\sin\theta d\theta d\varphi}$$

We again choose to work this out in Cartesian coordinates, so $\boxed{\sin\theta\hat{\theta} = \hat{r}\cos\theta - \hat{z}}$

$$\boxed{\vec{\mathcal{L}}_{EM} = -\frac{\mu_o}{16\pi^2}(q_e m)\int_v \frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}\left[\hat{r}\cos\theta-\hat{z}\right]dr\sin\theta d\theta d\varphi}$$

Then $\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$, thus:

$$\begin{aligned}\left[\hat{r}\cos\theta-\hat{z}\right]\sin\theta &= \left[\sin\theta\cos\theta\cos\varphi\hat{x}+\sin\theta\cos\theta\sin\varphi\hat{y}+\cos^2\theta\hat{z}-\hat{z}\right]\sin\theta \\ &= \sin^2\theta\cos\theta\cos\varphi\hat{x}+\sin^2\theta\cos\theta\sin\varphi\hat{y}+\sin\theta\cos^2\theta\hat{z}-\sin\theta\hat{z} \\ &= \sin^2\theta\cos\theta\cos\varphi\hat{x}+\sin^2\theta\cos\theta\sin\varphi\hat{y}-\sin\theta\left(1-\cos^2\theta\right)\hat{z} \\ &= \sin^2\theta\cos\theta\cos\varphi\hat{x}+\sin^2\theta\cos\theta\sin\varphi\hat{y}-\sin^3\theta\hat{z}\end{aligned}$$

Again, the integrals for the $\hat{x}$ and $\hat{y}$ components of $\vec{\mathcal{L}}_{EM}$ will contribute nothing when the integrals of $\int_{\varphi=0}^{\varphi=2\pi}\{...\}\cos\varphi d\varphi$ and $\int_{\varphi=0}^{\varphi=2\pi}\{...\}\sin\varphi d\varphi$ are carried out – only the $\hat{z}$ term survives the φ-integration:

$$\boxed{\vec{\mathcal{L}}_{EM} = +\frac{\mu_o}{8\pi}(q_e m)\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty}\frac{\left[(r-3d\cos\theta)+\left(\frac{8\pi}{3}\right)r^4\delta^3(\vec{r})\right]}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}dr\sin^3\theta d\theta\,\hat{z}}$$

Carrying out the remainder of the integration is ~ somewhat tedious, so we don't explicitly wade through this here, but interestingly enough, it yields a <u>finite</u> result (for $d > 0$):

$\boxed{\vec{\mathcal{L}}_{EM} = \frac{\mu_o}{8\pi d}(q_e m)\left[4-\pi\right]\hat{z}}$, which diverges as the electric charge q_e – point magnetic dipole moment $\vec{m}$ separation distance $d \to 0$, which coincides with that of a real/physical electron – *i.e.* a point electric charge –*e* with point magnetic dipole moment of magnitude $\boxed{\mu = \frac{e\hbar}{2m_e}}$.

The main purpose of the above example, aside from its instructional use as academic exercise to illustrate a simple static electromagnetic system in which energy, linear momentum and angular momentum are all involved, is also to emphasize/underscore the important point that real/physical electrons simultaneously have *both* a point electric charge *and* a point magnetic dipole moment – both of which are necessary ingredients in order to be able to transfer {apparently} arbitrarily large amounts of energy, linear and angular momentum to other such particles via the electromagnetic interaction. Without the simultaneous presence of both an electric charge and a magnetic dipole moment, transfer of linear & angular momentum could not occur!

It is not surprising that "classical" macroscopic electrodynamics "fails" here to correctly quantitatively explain the physics operative at the microscopic scale – the domain of quantum mechanics (and beyond – i.e. the structure of space-time itself at the smallest distance scales).

Despite more than 100 years of collective effort, since explicit discovery the electron by J.J. Thompson in 1897, and the discovery of electron spin and the electron's magnetic dipole moment by first observed experimentally by O. Stern & W. Gerlach in 1922 and subsequently explained theoretically by W. Pauli and S. Goudsmit and G. Uhlenbeck in 1925, today, we still have gained no fundamental insight as to what precisely electric charge is, nor do we understand the physics origins of intrinsic spin angular momentum (associated with either spin-½ fermions {and the accompanying Pauli exclusion "principle"} or integer spin bosons, such as the photon {and their accompanying "gregarious" nature at the quantum level – the opposite of that for fermions!}, nor any fundamental explanation of the existence of the intrinsic magnetic dipole moment(s) associated with all of the fundamental, point-like electrically-charged particles – three generations of integer-charged point-like leptons (e, μ, τ) and six point-like quarks $+2/3:(u, c, t)$ and $-1/3:(d, s, b)$. Note that the $W^{\pm}$ boson – the spin-1 electrically-charged mediator of the weak interactions also has a magnetic dipole moment, as well as an electric quadrupole moment. These same fundamental particles also interact via the weak interaction and thus have weak charges and weak magnetic moments {the $W^{\pm}$ boson also additionally has a weak quadrupole moment}. The spin-½ quarks additionally interact via the strong interactions, and hence have strong "chromo-electric" charges ("red", "green" & "blue") as well as strong "chromo-magnetic" dipole moments.

Thus, point "charge" and point magnetic dipole moments, etc. associated with the all of the fundamental particles we know and love transcends each of the individual forces, and in fact points to/hints at a single common explanation. We do know that intrinsic spin and the accompanying magnetic dipole moments of these particles are indeed manifestly fully-relativistic phenomena, and thus "hint" at an explanation operative only at the smallest conceivable distance scale, where the quantum behavior of space-time itself becomes manifest – i.e. the so-called Planck distance scale, also known as the Planck length: $L_P = \sqrt{\hbar G_N / c^3} = 1.61624 \times 10^{-35}$ meters, with corresponding Planck time $t_P = L_P / c = \sqrt{\hbar G_N / c^5} = 5.39056 \times 10^{-44}$ seconds! It may seem surprising that Newton's gravitational constant G_N enters here – however, Einstein's general theory of relativity tells us that there is an intrinsic link between the gravitational "force" as we understand it macroscopically in the every-day world and the curvature of space-time!

