5 Electromagnetic Waves

5.1 General Form for Electromagnetic Waves.

In free space, Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{5.1.1}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{5.1.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.1.3}$$

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \tag{5.1.4}$$

In section 4.3 we derived wave equations for the scalar and vector potentials. Here we derive wave equations for $\vec{E}$ and $\vec{B}$ directly from Maxwell's equations.

Taking the curl of (5.1.2) gives:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = 0 \tag{5.1.5}$$

Substituting (5.1.4) this becomes:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0 \frac{\partial \vec{J}}{\partial t} \tag{5.1.6}$$

Now:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = \vec{\nabla} \frac{\rho}{\epsilon_0} - \vec{\nabla}^2 \vec{E} \tag{5.1.7}$$

So:

$$\vec{\nabla}^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t} + \frac{1}{\epsilon_0} \vec{\nabla} \rho \tag{5.1.8}$$

Similarly it can be shown that:

$$\vec{\nabla}^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = -\mu_0 \vec{\nabla} \times \vec{J} \tag{5.1.9}$$

These two equations have the form of the *nonhomogeneous wave eqautions.*

Away from sources, ie $\rho = 0$ and $\vec{J} = 0$, we have:

$$\vec{\nabla}^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{5.1.10}$$

Similarly it can be shown that:

$$\vec{\nabla}^2\vec{B} - \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0 \tag{5.1.11}$$

These wave equations have as solutions, waves travelling with a speed c $= \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$. For waves in a homogeneous, isotropic, linear and stationary (HILS) medium, we obtain:

$$\vec{\nabla}^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial\vec{J}}{\partial t} + \frac{1}{\epsilon}\vec{\nabla}\rho \tag{5.1.12}$$

$$\vec{\nabla}^2\vec{B} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} = -\mu\vec{\nabla}\times\vec{J} \tag{5.1.13}$$

Again, if $\rho\ = 0$ and $\vec{J}\ = 0$, these gives waves travelling with speed v $= \dfrac{1}{\sqrt{\mu\epsilon}}$. We can make another approximation. If the conductivity, σ, is constant, then

$$\vec{J} = \sigma\vec{E} \tag{5.1.14}$$

so:

$$\vec{\nabla}^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \sigma\mu\frac{\partial\vec{E}}{\partial t} = \frac{1}{\epsilon}\vec{\nabla}\rho \tag{5.1.15}$$

$$\vec{\nabla}^2\vec{B} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} - \sigma\mu\frac{\partial\vec{B}}{\partial t} = 0 \tag{5.1.16}$$

Note that although we have separate wave equations,$\vec{E}$ and $\vec{B}$ are still linked through Maxwell's equations. Therefore, purely magnetic or purely electric waves cannot exist. In some waves however, the energy density can be mostly magnetic or mostly electric.

5.2 Uniform Planes waves in a General Medium

A *wavefront* is a surface of unifrom phase. The wavefronts of a *plane* wave are planar. Uniform plane waves in unbounded media possess many properties which are independent of whether they travel in free space or in matter. Consider a general HILS medium characterized by ϵ_r, μ_r and σ. If we further assume the wave is sinusoidal, travelling in the positive Z-direction (ie $\vec{k} = k\hat{z}$) and that the $\vec{E}$ vectors are all parallel to a given direction, then the wave is said to be *linearly polarized.* If the wave is not linearly polarized, then it can be written as the sum of linearly waves.

$\vec{E}$ is of the form:

$$\vec{E} = \vec{E}_0\exp(i(\omega t - kz)), \tag{5.2.1}$$

where $\vec{E_0}$ is independent of time and spatial coordinates. The *wavenumber*, k, is real if there is *no attenuation*. Under these circumstances

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda} \tag{5.2.2}$$

where v is the *phase velocity* and λ the wavelength.

To find the relative orientations of $\vec{E}$ and $\vec{k}$ (the vector points in the direction of propagation of the wave ie the z-direction), we apply Maxwell's equations with the assumptions that, $\rho_f = 0, \vec{J_f} = \sigma\vec{E}$, so:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{5.2.3}$$

then since $\vec{E}$ depend only on z we get:

$$\frac{\partial}{\partial z}\hat{z} \cdot \vec{E} = 0 \tag{5.2.4}$$

So $\vec{E}$ is perpendicular to $\vec{z}$ and hence $\vec{k}$. This being the case, lets assign $\vec{E}$ to be parallel to the $\hat{x}$ direction and find the equation for $\vec{B}$. So we now have:

$$\vec{E} = E_0\hat{x}\exp(i(\omega t - kz)), \tag{5.2.5}$$

Applying:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{5.2.6}$$

Now the curl of E is:

$$\vec{\nabla} \times \vec{E} = \hat{x}(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}) + \hat{y}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \hat{z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \tag{5.2.7}$$

The only non zero term of which is $\hat{y}\frac{\partial E_x}{\partial z}$:

$$\Rightarrow \quad \frac{\partial\vec{B}}{\partial t} = -\frac{\partial E_x}{\partial z}\hat{y} \tag{5.2.8}$$

$$\Rightarrow \quad \frac{\partial\vec{B}}{\partial t} = \hat{y}ikE_0\exp(i(\omega t - kz)) \tag{5.2.9}$$

$$\Rightarrow \quad \vec{B} = \hat{y}(E_0\frac{k}{\omega}\exp(i(\omega t - kz)) \tag{5.2.10}$$

$$\Rightarrow \quad \vec{B} = \hat{y}(E_0\sqrt{\mu\epsilon}\exp(i(\omega t - kz)) \tag{5.2.11}$$

So $\vec{B}$, $\vec{E}$ and $\vec{k}$ are mutually perpendicular. ie. $\vec{E}$ and $\vec{B}$ are perpendicular to one another and to the direction of propagation. In fact $\vec{E} \times \vec{B}$ points in the direction of propagation.

To find the effect of a non-zero conductivity on the wave equations, we investigate:

$$\vec{\nabla} \times \vec{B} - \mu\epsilon \frac{\partial \vec{E}}{\partial t} = -\mu\sigma\vec{E} \tag{5.2.12}$$

Taking the time differential

$$\Rightarrow \quad \vec{\nabla} \times \vec{B} = \hat{x}E_0(i\omega\mu\epsilon - \mu\sigma)\exp(i(\omega t - kz)) \tag{5.2.13}$$

$$\Rightarrow -\frac{\partial B_y}{\partial z} = \mu E_0(i\omega\epsilon - \sigma)\exp(i(\omega t - kz)) \tag{5.2.14}$$

$$\Rightarrow \quad \vec{B} = \hat{y}E_0\frac{\omega\mu\epsilon - i\mu\sigma}{k}\exp(i(\omega t - kz)) \tag{5.2.15}$$

Equating the two expressions for $\vec{B}$ gives:

$$\frac{k}{\omega} = \frac{\mu(\omega\epsilon - i\sigma)}{k} \tag{5.2.16}$$

$$\Rightarrow \quad k^2 = \omega\mu(\omega\epsilon - i\sigma) \tag{5.2.17}$$

So k becomes complex if the medium has non zero conductivity. This in turn implies that the medium attentuates the wave. This makes sense in terms of Ohm's Law. There must be a conversion of wave energy into resistive heating.

Remember: $\vec{H} = \frac{1}{\mu}\vec{B}$, the ratio $\frac{E}{H}$ is the *characteritic impedence*, Z, of the medium in which the wave is propagating. Z is given by:

$$Z = \frac{E}{H} = \frac{k}{\omega\epsilon - i\sigma} \tag{5.2.18}$$

for non-zero conductivity

$$Z = \frac{\omega\mu}{k} = v\mu = \frac{\mu}{\sqrt{\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \tag{5.2.19}$$

for zero conductivity

5.3 Poynting Vector

Define:

$$\mathcal{P} = \vec{E} \times \vec{H} \tag{5.3.1}$$

as the Poynting Vector. In the previous section we saw that $\vec{E} \times \vec{B}$ was parallel to the direction of propagation of the wave. Consider now:

$$\begin{aligned} \vec{\nabla} \cdot \mathcal{P} &= \vec{\nabla} \cdot (\vec{E} \times \vec{H}) && (5.3.2) \\ &= \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) && (5.3.3) \end{aligned}$$

In a HILS medium, this becomes

$$\vec{\nabla}\cdot\mathcal{P} = -\vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t} - \vec{E}\cdot(\epsilon\frac{\partial\vec{E}}{\partial t} + \vec{J}_f) \tag{5.3.4}$$

$$= -\frac{\partial}{\partial t}(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}) - \vec{E}\cdot\vec{J}_f \tag{5.3.5}$$

$$\tag{5.3.6}$$

If we now integrate over a finite volume V with surface A and apply the divergence theorem, we obtain,

$$-\int_A \mathcal{P}\cdot d\vec{A} = -\int_A (\vec{E}\times\vec{H}\cdot d\vec{A} = \frac{d}{dt}\int_V (\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2})dV + \int_V \vec{E}\cdot\vec{J}_f dV \tag{5.3.7}$$

This is the *Poynting Theorem.* The LHS represents the rate at which electromagnetic energy flows into the volume V. The RHS consists of two terms. $\frac{d}{dt}\int_V (\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2})dV$ represents the rate of energy change in V associated with the electric and magnetic fields. The term $\int_V \vec{E}\cdot\vec{J}_f dV$, represents the energy lost through resistive heating. ie current times voltage drop = Power.

This equation is an expression of conservation of energy or if you like, a description of the Physical interpretation of the Poynting Vector $\mathcal{P}$. It is the flow of electromagnetic energy density, so if we integrate it over a closed volution we obtain the flux of energy in or out of the volume.

5.4 Group and Phase velocity

Consider a pulse. Because of Fourier's Theorem it can be expressed as superposition of plane waves.

$$u(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\exp(i(\omega(k)t - kz))dk \tag{5.4.1}$$

If it's frequency spectrum is peaked about some value of k, as is required to give a pulse, then we can do a Taylor series expansion about the central value, $k_0, \omega(k_0) = \omega_0$.

$$\omega(k) = \omega_0 + (\frac{d\omega}{dk})_{k_0}(k - k_0) + ... \tag{5.4.2}$$

So

$$u(k,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\exp(i((\omega_0 + (\frac{d\omega}{dk})_{k_0}(k-k_0))t - kz))dk \tag{5.4.3}$$

$$= \frac{1}{\sqrt{2\pi}}\exp(it(\omega_0 - k_0(\frac{d\omega}{dk})_{k_0}))\int_{-\infty}^{\infty} A(k)\exp(ik(\frac{d\omega}{dk}_{k_0} t - z))dk \tag{5.4.4}$$

$$= \exp(it(\omega_0 - k_0(\frac{d\omega}{dk})_{k_0}))f(z-(\frac{d\omega}{dk})_{k_0}t) \qquad (5.4.5)$$

$$= \exp(it(\omega_0 - k_0(\frac{d\omega}{dk})_{k_0}))f(z-v_g t) \qquad (5.4.6)$$

Where we subsumed $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\exp(ik(\frac{d\omega}{dk}_{k_0}t - z))dv$ into the function $f(z-(\frac{d\omega}{dk})_{k_0}t)$. Note that f has exactly the form required for the propagation of a wave. We explicitly recognize this by identifying:

$$v_g = (\frac{d\omega}{dk})_{k_0} \qquad (5.4.7)$$

as the *group velocity* of the pulse. v_g is speed with which the pulse as a whole moves through the medium and is the speed of energy transportation. The group velocity is in principle different from the *phase velocity* v_p,

$$v_p = \frac{\omega(k)}{k} \qquad (5.4.8)$$

For light, moving through a medium with a refractive index n,

$$\omega(k) = \frac{ck}{n(k)} \quad \Rightarrow \quad v_p(k_0) = \frac{c}{n(k_0)} \qquad (5.4.9)$$

whereas:

$$v_g(k_0) = (\frac{d\omega}{dk})_{k_0} = \frac{c}{n(k_0)}(1 - \frac{k_0}{n(k_0)}(\frac{dn}{dk})_{k_0}) \qquad (5.4.10)$$

Rewriting in terms of the frequencies gives:

$$v_g(\omega_0) = \frac{c}{n(\omega_0) + \omega_0(\frac{dn}{d\omega})_{\omega_0}} \qquad (5.4.11)$$

If the dispersion term $\omega\frac{dn}{d\omega}$ is small then the group velocity is compariable to the phase velocity.

5.5 Uniform Plane Waves in Free Space

In free space, we have $\epsilon_r = 1, \mu_r = 1, \sigma = 0$. There is no attentuation so from:

$$k = \omega\sqrt{\epsilon_0\mu_0} \qquad (5.5.1)$$

The speed of electromagnetic waves is, therefore,

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.99792458 \times 10^8 ms^{-1} \qquad (5.5.2)$$

Also, the *characteristic impedence of free space* is:

$$Z_0 = \frac{E}{H} = \frac{k}{\omega\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 3.767303 \times 10^2 = 377\Omega \qquad (5.5.3)$$

Since $\vec{B} = \mu_0 \vec{H}$ in free space

$$\frac{E}{B} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \quad or \quad | \vec{E} |= c | \vec{B} | \tag{5.5.4}$$

Then because the characteristic impedence of free space is <u>real</u>, the $\vec{E}$ and $\vec{H}$ vectors are *in phase.* Recall that the electric energy density is:

$$\varepsilon_E = \frac{\epsilon_0 E^2}{2} \tag{5.5.5}$$

and the magnetic energy density is:

$$\varepsilon_M = \frac{B^2}{2\mu_0} \tag{5.5.6}$$

The ratio of the electic and magnetic energy densities are equal since:

$$\frac{\varepsilon_E}{\varepsilon_M} = \frac{\epsilon_0 \mu_0 E^2}{B^2} = \frac{\epsilon_0 \mu_0}{\sqrt{\epsilon_0 \mu_0}^2} = 1 \tag{5.5.7}$$

The time averaged value of the total energy density at any point is:

$$\varepsilon' = \epsilon_0 \frac{E_{rms}^2}{2} + \frac{B_{rms}^2}{2\mu_0} = \epsilon_0 E_{rms}^2 = \frac{B_{rms}^2}{\mu_0} \tag{5.5.8}$$

the time-averaged Poynting vector is:

$$\mathcal{P}_{av} = \frac{1}{2} \mathcal{R}e(\vec{E} \times \vec{H}^*) \tag{5.5.9}$$

and for a uniform plane wave in space

$$\mathcal{P}_{av} = \frac{1}{2} \mathcal{R}e(EH^*)\hat{z} = c\epsilon_0 E_{rms}^2 \hat{z} = \frac{E_{rms}^2}{Z_0}\hat{z} = Z_0 H_{rms}^2 \hat{z} \tag{5.5.10}$$

5.6 Uniform Plane Waves in Nonconductors

The propagation of waves in dielectrics is basically the same as that in free space with (for HILS media) ϵ and μ replacing ϵ_0 and μ_0. The phase velocity is now

$$v_p = \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{\sqrt{\epsilon_r \mu_r}} = \frac{c}{n} \tag{5.6.1}$$

where n is the *index of refraction.* Note that the group velocity is equal the phase velocity for a plane wave of single k.

$$n = \sqrt{\epsilon_r \mu_r} \tag{5.6.2}$$

The speed of propagation of the wave, v, is less than in free space since both ϵ_r and μ_r are larger than unity. In nonmagnetic media,

$$n = \sqrt{\epsilon_r} \tag{5.6.3}$$

Generally, ϵ_r is a function of frequency (dispersion). The characteristic impedence of the medium is:

$$Z = \frac{E}{H} = \frac{\mu}{\epsilon} \approx 377\sqrt{\mu_r \epsilon_r}\Omega \tag{5.6.4}$$

Note that the electric and magnetic fields are still in phase, the electric and magnetic energy desnities are again equal and the time averaged energy density is:

$$\varepsilon'_{av} = \frac{\epsilon E_{rms}^2}{2} = \frac{H_{rms}}{2\mu} = \epsilon E^2 = \frac{B^2}{\mu} \tag{5.6.5}$$

The Poynting vector $\vec{E} \times \vec{H}$ again points in the direction of propagation:

$$\mathcal{P}_{av} = \frac{1}{2}\mathcal{R}e(EH^*)\hat{z} = \sqrt{\frac{\epsilon}{\mu}} E_{rms}^2 \hat{z} = v\epsilon E_{rms}^2 \hat{z} \tag{5.6.6}$$

The time averaged Poynting vector is again equal to the phase velocity multiplied by the time-averaged energy density.

5.7 Uniform Plane Waves in Conductors

The main differences between waves in conductors and those in vacuo and lossless dielectrics is that the *wavenumber*, k, is now *complex*. Thus the wave amplitude decreases expontially due to resistive heating fom electric current flows in the medium.

$$k^2 = k_0^2 \epsilon_r \mu_r (1 - i\frac{\sigma}{\omega\epsilon}) \tag{5.7.1}$$

where k_0 is the wavenumber in free space corresponding to waves with frequency ω. If we set $k = \beta - i\alpha$, then

$$\vec{E} = \vec{E}_0 \exp(-\alpha Z) \exp(i(\omega t - \beta z)) \tag{5.7.2}$$

where both α and β are positive.
The quantity $\frac{1}{\alpha}$ has units of *length* and is the *attenuation distance* or *skin depth* δ over which the amplitude decreases by a factor of e.

$$\delta = \frac{1}{\alpha} \tag{5.7.3}$$

In this section we will determine expressions for α and β in terms of ϵ_r, μ_r, σ and k_0.

The phase velocity v_p is:

$$v_p = \frac{\omega}{\beta} \tag{5.7.4}$$

Define $\mathcal{D}$, as:

$$\mathcal{D} = \frac{\sigma}{\omega\epsilon} = \mid \frac{\sigma E}{\epsilon\frac{\partial E}{\partial t}} \mid = \mid \frac{\sigma E}{\frac{\partial D}{\partial t}} \mid \tag{5.7.5}$$

Physically this is the magnitude of conduction current density divided by the Displacement current density.

So if D $\ll$ 1 the medium is a poor conductor (good dielectric) and most of the electric energy is carried by the Displacement current (ie the Electric field). If D $\gg$ 1 the medium is a good conductor (poor dielectric) and most energy is carried by the conduction electrons. Now our expression for k^2 can be written:

$$k^2 = (\beta - i\alpha)^2 = k_0^2\epsilon_r\mu_r(1 - i\frac{\sigma}{\omega\epsilon}) = k_0^2\epsilon_r\mu_r(1 - i\mathcal{D}) \tag{5.7.6}$$

where $k_0 = k(\omega)$ in free space.
Solving for α, β and k we get:

$$\alpha = k_0\sqrt{\frac{\epsilon_r\mu_r}{2}}\sqrt{\sqrt{1+\mathcal{D}^2} - 1} \tag{5.7.7}$$

$$\beta = k_0\sqrt{\frac{\epsilon_r\mu_r}{2}}\sqrt{\sqrt{1+\mathcal{D}^2} + 1} \tag{5.7.8}$$

$$k = k_0\sqrt{\epsilon_r\mu_r}(1+\mathcal{D}^2)^{\frac{1}{4}}\exp(-iTan^{-1}(\frac{\alpha}{\beta})) \tag{5.7.9}$$

For a *low-loss dielectric* $\sigma \ll \omega\epsilon, \mathcal{D} \ll 1$ so

$$\alpha \approx \frac{\sqrt{\epsilon_r\mu_r}\mathcal{D}k_0}{2} = \sqrt{\frac{mu_r}{\epsilon_r}}\frac{\sigma c\mu_0}{2} = \frac{k_0 N\mathcal{D}}{2} \tag{5.7.10}$$

$$\beta \approx k_0\sqrt{\epsilon_r\mu_r} = k_0 N, \qquad N = \sqrt{\mu_r\epsilon_r} \tag{5.7.11}$$

So

$$v_p = \frac{\omega}{\beta} \approx \frac{c}{\sqrt{\epsilon_r\mu_r}} \tag{5.7.12}$$

We see that the phase velocity is unaffected by the conductivity, whose effect is to attentuate the wave.

For a *good conductor* $\mathcal{D} \gg 1, (\sigma \gg \omega\epsilon_0)$ and

$$k^2 = -i\mathcal{D}\epsilon_r\mu_r k_0^2 = -i\sigma\mu\omega \tag{5.7.13}$$

$$\Rightarrow \quad k = \sqrt{\frac{\sigma\mu\omega}{2}}(1-i) \tag{5.7.14}$$

$$\Rightarrow \quad \alpha = \beta = \sqrt{\frac{\sigma\mu\omega}{2}} \tag{5.7.15}$$

So the *skin depth*

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\sigma\mu\omega}} = \frac{1}{\beta} \tag{5.7.16}$$

Note that since $k = \beta - i\alpha$, the above relations imply that the wave is attentuated by a factor e after propagating just one wavelength within the medium. Consequently high frequency E/M waves are excluded from conductors except for a very thin layer on the surface. This is what is meant by the term "*skin depth*".

The impedence Z is given by

$$Z = \sqrt{\frac{\mu}{\epsilon}} \frac{\exp\left(iTan^{-1}(\frac{\alpha}{\beta})\right)}{(1+\mathcal{D}^2)^{\frac{1}{4}}} \quad \Omega \tag{5.7.17}$$

Since Z is complex, $\vec{E}$ and $\vec{H}$ are not in phase. $\vec{E}$ leads $\vec{H}$ by the angle:

$$\theta = Tan^{-1}(\frac{\alpha}{\beta}) \tag{5.7.18}$$

So

$$\vec{E} = \vec{E}_0 \exp(-\alpha z) \exp(i(\omega t - \beta z)) \tag{5.7.19}$$

$$\vec{H} = \vec{H}_0 \exp(-\alpha z) \exp(i(\omega t - \beta z - \theta)) \tag{5.7.20}$$

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu}{\epsilon}} \frac{1}{(1+\mathcal{D}^2)^{\frac{1}{4}}} \tag{5.7.21}$$

The ratio of the electric energy to the magnetic energy densities is

$$\frac{\varepsilon'_e}{\varepsilon'_m} = \frac{1}{\sqrt{1+\mathcal{D}^2}} \tag{5.7.22}$$

the time-averaged Poynting vector is:

$$\mathcal{P}_{av} = \frac{1}{2} E_0 H_0 \cos\theta \exp(-2\alpha z) = \sqrt{\frac{\epsilon}{\mu}} (1+\mathcal{D}^2)^{\frac{1}{4}} E_{rms}^2 \cos(\theta) \exp(-2\alpha z) \tag{5.7.23}$$

Note that if $\vec{E}$ and $\vec{H}$ are 90° out of phase then no energy is carried by the wave. For a good conductor, $\alpha = \beta$ so $\theta = \frac{\pi}{4}$, ie $\vec{E}$ leads $\vec{H}$ by 45°.

The index of refraction of a *good conductor*

$$n = \frac{c}{\frac{\omega}{\beta}} = \frac{c\beta}{\omega} = c\sqrt{\frac{\sigma\mu}{2\omega}} \tag{5.7.24}$$

so that there is a strong dependence of the speed of the wave on the frequency. In addition n can become very large for a good conductor, since $\sigma \approx 10^7$ for metals. (Copper has n = 1.1×10^8 at 1 megahertz.)
The phase velocity is

$$v_p = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\sigma\mu}} \tag{5.7.25}$$

and the group velocity is

$$v_g = \frac{1}{\frac{d\beta}{d\omega}} = 2\sqrt{\frac{2\omega}{\sigma\mu}} = 2v_p \tag{5.7.26}$$

provided σ and μ have no dependence on ω.

Finally if $\mathcal{D}$ is very large the ratio of the densities is

$$\frac{\varepsilon'_e}{\varepsilon'_m} = \frac{1}{D} \ll \frac{1}{50} \tag{5.7.27}$$

The energy is thus mostly magnetic since σ is large. This implies that $\frac{E}{J_f}$ is small. E is weak, but J_F and hence, H is large.

Summary:
Differences between waves in good conductors and lossless dielectrics

- In conductors there is exponential attenuation of the fields.
- $\vec{E}$ and $\vec{H}$ are not in phase in a good conductor.
- In a good conductor the group velocity is approximately twice the phase velocity.
- E/M waves are attentuated by a factor e after propagating just one wavelength within a good conductor. This is why the penetration depth of a wave is refered to as the "*skin depth*".