5 Radiation and scattering

5.1 Basic antenna theory

It possible to solve exactly for the radiation pattern emitted by a linear antenna fed with a sinusoidal current pattern. Assuming that all fields and currents vary in time like $\mathrm{e}^{-\mathrm{i}\,\omega t}$, and adopting the Lorentz gauge, it is easily demonstrated that the vector potential obeys the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\boldsymbol{A} = -\mu_0\,\boldsymbol{j}, \tag{5.1}$$

where $k = \omega/c$. The Green's function for this equation, subject to the Sommerfeld radiation condition (which ensures that sources radiate waves instead of absorbing them), is given by Eq. (2.123). Thus, we can invert Eq. (5.1) to obtain

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{e}^{\,\mathrm{i}\,k|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3\boldsymbol{r}'. \tag{5.2}$$

The electric field in the source free region follows from the Ampère-Maxwell equation and $\boldsymbol{B} = \nabla\wedge\boldsymbol{A}$,

$$\boldsymbol{E} = \frac{\mathrm{i}}{k}\nabla\wedge c\boldsymbol{B}. \tag{5.3}$$

Now

$$|\boldsymbol{r}-\boldsymbol{r}'| = r\sqrt{1 - 2\,\boldsymbol{n}\cdot\boldsymbol{r}'/r + r'^2/r^2}, \tag{5.4}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. Assuming that $r' \ll r$, this expression can be expanded binomially to give

$$|\boldsymbol{r}-\boldsymbol{r}'| = r\left[1 - \frac{\boldsymbol{n}\cdot\boldsymbol{r}'}{r} + \frac{r'^2}{2r^2} - \frac{1}{8}\left(\frac{2\boldsymbol{n}\cdot\boldsymbol{r}'}{r}\right)^2 + \cdots\right], \tag{5.5}$$

where we have retained all terms up to order $(r'/r)^2$. This expansion occurs in the complex exponential of Eq. (5.2); *i.e.*, it determines the oscillation phase of each element of the antenna. The quadratic terms in the expansion can be neglected provided they can be shown to contribute a phase shift which is significantly less than 2π. Since the maximum possible value of r' is $d/2$, for a linear antenna which extends along the z-axis from $z = -d/2$ to $z = d/2$, the phase shift associated with the quadratic terms is insignificant as long as

$$r \gg \frac{k d^2}{16\pi} = \frac{d^2}{8\lambda}, \tag{5.6}$$

where $\lambda = 2\pi/k$ is the wavelength of the radiation. This constraint is known as the *Fraunhofer limit.*

In the Fraunhofer limit we can approximate the phase variation of the complex exponential in Eq. (5.2) by a linear function of r':

$$|\boldsymbol{r} - \boldsymbol{r}'| \rightarrow r - \boldsymbol{n}\cdot\boldsymbol{r}'. \tag{5.7}$$

The denominator $|\boldsymbol{r} - \boldsymbol{r}'|$ in the integrand of Eq. (5.2) can be approximated as r provided that the distance from the antenna is much greater than its length; *i.e.*, provided that

$$r \gg d. \tag{5.8}$$

Thus, Eq. (5.2) reduces to

$$\boldsymbol{A}(\boldsymbol{r}) \simeq \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{r}\int \boldsymbol{j}(\boldsymbol{r}')\,\mathrm{e}^{-\mathrm{i}\,k\boldsymbol{n}\cdot\boldsymbol{r}'}\,d^3\boldsymbol{r}' \tag{5.9}$$

when the constraints (5.6) and (5.8) are satisfied. If the additional constraint

$$kr \gg 1 \tag{5.10}$$

is also satisfied, then the electromagnetic fields associated with Eq. (5.9) take the form

$$\boldsymbol{B}(\boldsymbol{r}) \simeq \mathrm{i}\,k\,\boldsymbol{n}\wedge\boldsymbol{A} = \mathrm{i}\,k\,\frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{r}\int \boldsymbol{n}\wedge\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{e}^{-\mathrm{i}\,k\boldsymbol{n}\cdot\boldsymbol{r}'}\,d^3\boldsymbol{r}', \tag{5.11a}$$

$$\boldsymbol{E}(\boldsymbol{r}) \simeq c\boldsymbol{B}\wedge\boldsymbol{n} = \mathrm{i}\,ck\,(\boldsymbol{n}\wedge\boldsymbol{A})\wedge\boldsymbol{n}. \tag{5.11b}$$

These are clearly radiation fields, since they are mutually orthogonal, transverse to the radius vector, and satisfy $E = cB \propto r^{-1}$. The three constraints (5.6), (5.8), and (5.10), can be summed up in a single inequality:

$$d \ll \sqrt{\lambda r} \ll r. \tag{5.12}$$

The current density associated with a linear, sinusoidal, centre-fed antenna is

$$\boldsymbol{j}(\boldsymbol{r}) = I\sin(kd/2 - k|z|)\,\delta(x)\,\delta(y)\,\hat{\boldsymbol{z}} \tag{5.13}$$

for $|z| < d/2$, with $\boldsymbol{j}(\boldsymbol{r}) = 0$ for $|z| \geq d/2$. In this case, Eq. (5.9) yields

$$\boldsymbol{A}(\boldsymbol{r}) = \hat{\boldsymbol{z}}\,\frac{\mu_0\,I}{4\pi}\frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{r}\int_{-d/2}^{d/2}\sin(kd/2 - k|z|)\,\mathrm{e}^{-\mathrm{i}\,kz\cos\theta}\,dz, \tag{5.14}$$

where $\cos\theta = \boldsymbol{n}\cdot\hat{\boldsymbol{z}}$. The result of this straightforward integration is

$$\boldsymbol{A}(\boldsymbol{r}) = \hat{\boldsymbol{z}}\,\frac{\mu_0\,I}{4\pi}\frac{2\,\mathrm{e}^{\,\mathrm{i}\,kr}}{kr}\left[\frac{\cos(kd\,\cos\theta/2) - \cos(kd/2)}{\sin^2\theta}\right]. \tag{5.15}$$

Note from Eqs. (5.11) that the electric field lies in the plane containing the antenna and the radius vector to the observation point. The time-averaged power radiated by the antenna per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{\mathrm{Re}\,(\boldsymbol{n}\cdot\boldsymbol{E}\wedge\boldsymbol{B}^*)\,r^2}{2\mu_0} = \frac{ck^2\,\sin^2\theta\,|A|^2\,r^2}{2\mu_0}. \tag{5.16}$$

Thus,

$$\frac{dP}{d\Omega} = \frac{\mu_0 c\,I^2}{8\pi^2}\left|\frac{\cos(kd\,\cos\theta/2) - \cos(kd/2)}{\sin\theta}\right|^2. \tag{5.17}$$

The angular distribution of power depends on the value of kd. In the long wavelength limit $kd \ll 1$ the distribution reduces to

$$\frac{dP}{d\Omega} = \frac{\mu_0 c\,I_0^{\,2}}{128\pi^2}\,(kd)^2\sin^2\theta, \tag{5.18}$$

where $I_0 = I\,kd/2$ is the peak current in the antenna. It is easily shown from Eq. (5.13) that the current distribution in the antenna is linear:

$$I(z) = I_0(1 - 2|z|/d) \tag{5.19}$$

for $|z| < d/2$. This type of antenna corresponds to a short (compared to the wavelength) oscillating electric dipole, and is generally known as a *Hertzian oscillating dipole*. The total power radiated is

$$P = \frac{\mu_0 c\,I_0^{\,2}\,(kd)^2}{48\pi}. \tag{5.20}$$

In order to maintain the radiation, power must be supplied continuously to the oscillating dipole from some generator. By analogy with the heating power produced in a resistor,

$$\langle P\rangle_{\rm heat} = \langle I^2\rangle R = \frac{{I_0}^2\,R}{2}, \tag{5.21}$$

we can define the factor which multiplies ${I_0}^2/2$ in Eq. (5.20) as the *radiation resistance* of the dipole antenna:

$$R_{\rm rad} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{(kd)^2}{24\pi} = 197\left(\frac{d}{\lambda}\right)^2 \text{ ohms}. \tag{5.22}$$

Since we have assumed that $\lambda \gg d$, this radiation resistance is necessarily very small. Typically, in devices of this sort the radiated power is swamped by the ohmic losses appearing as heat. Thus, a "short" dipole is a very inefficient radiator. Practical antennas have dimensions which are comparable with the wavelength of the emitted radiation.

Probably the most common practical antennas are the half-wave antenna ($kd = \pi$) and the full-wave antenna ($kd = 2\pi$). In the former case, Eq. (5.17) reduces to

$$\frac{dP}{d\Omega} = \frac{\mu_0 c\,I^2}{8\pi^2}\,\frac{\cos^2(\pi\cos\theta/2)}{\sin^2\theta}. \tag{5.23}$$

In the latter case, Eq. (5.17) yields

$$\frac{dP}{d\Omega} = \frac{\mu_0 c\,I^2}{2\pi^2}\,\frac{\cos^4(\pi\cos\theta/2)}{\sin^2\theta}. \tag{5.24}$$

The half-wave antenna radiation pattern is very similar to the characteristic $\sin^2\theta$ pattern of a Hertzian dipole. However, the full-wave antenna radiation pattern is considerably sharper (*i.e.*, it is more concentrated in the transverse directions $\theta = \pm\pi/2$).

The total power radiated by a half-wave antenna is

$$P = \frac{\mu_0 c\,I^2}{4\pi}\int_0^\pi \frac{\cos^2(\pi\cos\theta/2)}{\sin\theta}\,d\theta. \tag{5.25}$$

The integral can be evaluated numerically to give 1.2188. Thus,

$$P = 1.2188\,\frac{\mu_0 c\, I^2}{4\pi}. \tag{5.26}$$

Note from Eq. (5.13) that I is equivalent to the peak current flowing in the antenna. Thus, the radiation resistance of a half-wave antenna is given by $P/(I^2/2)$, or

$$R_{\rm rad} = \frac{0.6094}{\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} = 73 \text{ ohms}. \tag{5.27}$$

This resistance is substantially larger than that for a Hertzian dipole (see Eq. (5.22)). In other words, a half-wave antenna is a far more efficient radiator of electromagnetic radiation than a Hertzian dipole. According to standard transmission line theory, if a transmission line is terminated by a resistor whose resistance matches the characteristic impedance of the line, then all of the power transmitted down the line is dissipated in the resistor. On the other hand, if the resistance does not match the impedance of the line then some of the power is reflected and returned to the generator. We can think of a half-wave antenna, centre-fed by a transmission line, as a 73 ohm resistor terminating the line. The only difference is that the power absorbed from the line is radiated rather than dissipated as heat. Thus, in order to avoid problems with reflected power the impedance of a transmission line feeding a half-wave antenna must be 73 ohms. Not surprisingly, 73 ohm impedance is one of the standard ratings for the co-axial cables used in amateur radio.

5.2 Antenna directivity and effective area

We have seen that standard antennas emit more radiation in some directions than in others. Indeed, it is topologically impossible for an antenna to emit *transverse* waves uniformly in all directions (for the same reason that it is impossible to comb the hair on a sphere in such a manner that there is no parting). One of the aims of antenna engineering is to design antennas which transmit most of their radiation in a particular direction. By a reciprocity argument, such an antenna, when used as a receiver, is preferentially sensitive to radiation incident from the same direction.

The *directivity* or *gain* of an antenna is defined as the ratio of the *maximum* value of the power radiated per unit solid angle, to the average power radiated per unit solid angle:

$$G = \frac{(dP/d\Omega)_{\rm max}}{P/4\pi}. \tag{5.28}$$

Thus, the directivity measures how much more intensely the antenna radiates in its preferred direction than a mythical "isotropic radiator" would when fed with the same total power. For a Hertzian dipole the gain is 3/2. For a half-wave antenna the gain is 1.64. To achieve a directivity which is significantly greater than unity, the antenna size needs to be much larger than the wavelength. This is usually achieved using a phased array of half-wave or full-wave antennas.

Antennas can be used to receive, as well as emit, electromagnetic radiation. The incoming wave induces a voltage in the antenna which can be detected in an electrical circuit connected to the antenna. In fact, this process is equivalent to the emission of electromagnetic waves by the antenna viewed in reverse. In the theory of electrical circuits, a receiving antenna is represented as an e.m.f connected in series with a resistor. The e.m.f., $V_0 \cos\omega t$, represents the voltage induced in the antenna by the incoming wave. The resistor, $R_{\rm rad}$, represents the power re-radiated by the antenna (here, the real resistance of the antenna is neglected). Let us represent the detector circuit as a single load resistor $R_{\rm load}$ connected in series with the antenna. The question is: how can we choose $R_{\rm load}$ so that the maximum power is extracted from the wave and transmitted to the load resistor? According to Ohm's law:

$$V_0 \cos\omega t = I_0 \cos\omega t\,(R_{\rm rad} + R_{\rm load}), \tag{5.29}$$

where $I = I_0 \cos\omega t$ is the current induced in the circuit.

The power input to the circuit is

$$P_{\rm in} = \langle V I\rangle = \frac{{V_0}^{\,2}}{2(R_{\rm rad} + R_{\rm load})}. \tag{5.30}$$

The power transferred to the load is

$$P_{\rm load} = \langle I^2 R_{\rm load}\rangle = \frac{R_{\rm load}\,{V_0}^{\,2}}{2(R_{\rm rad} + R_{\rm load})^2}. \tag{5.31}$$

The power re-radiated by the antenna is

$$P_{\rm rad} = \langle I^2 R_{\rm rad}\rangle = \frac{R_{\rm rad}\, {V_0}^2}{2(R_{\rm rad}+R_{\rm load})^2}. \tag{5.32}$$

Note that $P_{\rm in} = P_{\rm load} + P_{\rm rad}$. The maximum power transfer to the load occurs when

$$\frac{\partial P_{\rm load}}{\partial R_{\rm load}} = \frac{{V_0}^2}{2}\left[\frac{R_{\rm load}-R_{\rm rad}}{(R_{\rm rad}+R_{\rm load})^3}\right] = 0. \tag{5.33}$$

Thus, the maximum transfer rate corresponds to

$$R_{\rm load} = R_{\rm res}. \tag{5.34}$$

In other words, the resistance of the load circuit must match the radiation resistance of the antenna. For this optimum case,

$$P_{\rm load} = P_{\rm rad} = \frac{{V_0}^2}{8R_{\rm rad}} = \frac{P_{\rm in}}{2}. \tag{5.35}$$

So, even in the optimum case one *half* of the power absorbed by the antenna is immediately re-radiated. If $R_{\rm load} \neq R_{\rm res}$ then more than one half of the absorbed power is re-radiated. Clearly, an antenna which is receiving electromagnetic radiation is also emitting it. This is how the BBC catch people who do not pay their television license fee in England. They have vans which can detect the radiation emitted by a TV aerial whilst it is in use (they can even tell which channel you are watching!).

For a Hertzian dipole antenna interacting with an incoming wave whose electric field has an amplitude E_0 we expect

$$V_0 = E_0\, d/2. \tag{5.36}$$

Here, we have used the fact that the wavelength of the radiation is much longer than the length of the antenna, and that the relevant e.m.f. develops between the two ends and the centre of the antenna. We have also assumed that the antenna is properly aligned (*i.e.*, the radiation is incident perpendicular to the axis of the antenna). The Poynting flux of the incoming wave is

$$\langle u_{\rm in}\rangle = \frac{\epsilon_0 c\, {E_0}^2}{2}, \tag{5.37}$$

whereas the power transferred to a properly matched detector circuit is

$$P_{\rm load} = \frac{{E_0}^2 d^2}{32 R_{\rm rad}}. \tag{5.38}$$

Consider an idealized antenna in which all incoming radiation incident on some area $A_{\rm eff}$ is absorbed and then magically transferred to the detector circuit with no re-radiation. Suppose that the power absorbed from the idealized antenna matches that absorbed from the real antenna. This implies that

$$P_{\rm load} = \langle u_{\rm in}\rangle A_{\rm eff}. \tag{5.39}$$

The quantity $A_{\rm eff}$ is called the *effective area* of the antenna; it is the area of the idealized antenna which absorbs as much net power from the incoming wave as the actual antenna. Alternatively, $A_{\rm eff}$ is the area of the incoming wavefront which is captured by the receiving antenna and fed to its load circuit. Thus,

$$P_{\rm load} = \frac{{E_0}^2 d^2}{32 R_{\rm rad}} = \frac{\epsilon_0 c\, {E_0}^2}{2}\, A_{\rm eff}, \tag{5.40}$$

giving

$$A_{\rm eff} = \frac{d^2}{16\,\epsilon_0 c\, R_{\rm rad}} = \frac{3}{8\pi}\,\lambda^2. \tag{5.41}$$

It is clear that the effective area of a Hertzian dipole antenna is of order the wavelength squared of the incoming radiation.

We can generalize from this analysis of a special case. The directivity of a Hertzian dipole is 3/2. Thus, the effective area of the *isotropic radiator* (the mythical reference antenna against which directivities are measured) is

$$A_0 = \frac{2}{3} A_{\rm Hertzian\ dipole} = \frac{\lambda^2}{4\pi}, \tag{5.42}$$

or

$$A_0 = \pi \lambda\!\!\!^{-2}, \tag{5.43}$$

where $\lambda\!\!\!^{-} = \lambda/2\pi$. Here, we have used the formal definition of the effective area of an antenna: $A_{\rm eff}$ is that area which, when multiplied by the time-averaged Poynting flux of the incoming wave, equals the maximum power received by the antenna (when its orientation is optimal). Clearly, the effective area of an isotropic radiator is the same as the area of a circle whose radius is the reduced wavelength $\lambda\!\!\!^{-}$.

We can take yet one more step and conclude that the effective area of any antenna of directivity G is

$$A_{\rm eff} = G\,\pi\lambda\!\!\!^{-2}. \tag{5.44}$$

Of course, to realize this full capture area the antenna must be orientated properly.

Let us calculated the coupling or *insertion loss* of an antenna-to-antenna communications link. Suppose that a generator delivers the power $P_{\rm in}$ to a transmitting antenna, which is aimed at a receiving antenna a distance r away. The receiving antenna (properly aimed) then captures and delivers the power $P_{\rm out}$ to its load circuit. From the definition of directivity, the transmitting antenna produces the time-averaged Poynting flux

$$\langle u\rangle = G_t\,\frac{P_{\rm in}}{4\pi r^2} \tag{5.45}$$

at the receiving antenna. The received power is

$$P_{\rm out} = \langle u\rangle\,G_r\,A_0. \tag{5.46}$$

Here, G_t is the gain of the transmitting antenna, and G_r is the gain of the receiving antenna. Thus,

$$\frac{P_{\rm out}}{P_{\rm in}} = G_t G_r\left(\frac{\lambda}{4\pi r}\right)^2 = \frac{A_t A_r}{\lambda^2 r^2}, \tag{5.47}$$

where A_t and A_r are the effective areas of the transmitting and receiving antennas, respectively. This result is known as the *Friis transmission formula*. Note that it depends on the product of the gains of the two antennas. Thus, a properly aligned communications link has the same insertion loss operating in either direction.

A thin wire linear antenna might appear to be essentially one dimensional. However, the concept of an effective area shows that it possesses a second dimension determined by the wavelength. For instance, for a half-wave antenna, the gain of which is 1.64, the effective area is

$$A_{\rm eff} = 1.64\, A_0 = \frac{\lambda}{2}\, (0.26\, \lambda). \tag{5.48}$$

Thus, we can visualize the capture area as a rectangle which is the physical length of the antenna in one direction, and approximately one quarter of the wavelength in the other.

5.3 Antenna arrays

Consider a linear array of N half-wave antennas arranged along the x-axis with a uniform spacing Δ. Suppose that each antenna is aligned along the z-axis, and also that all antennas are driven *in phase*. Let one end of the array coincide with the origin. The field produced in the radiation zone by the end-most antenna is given by (see Eq. (5.15))

$$\boldsymbol{A}(\boldsymbol{r}) = \hat{\boldsymbol{z}}\, \frac{\mu_0\, I}{4\pi}\, \frac{2}{kr}\, \frac{\cos(\pi \cos\theta/2)}{\sin^2\theta}\, {\rm e}^{\,{\rm i}\,(kr-\omega t)}, \tag{5.49}$$

where I is the peak current flowing in each antenna. The fields produced at a given point in the radiation zone by successive elements of the array differ in phase by an amount $\alpha = k\Delta\, \sin\theta\, \cos\varphi$. Here, r, θ, φ are conventional spherical polar coordinates. Thus, the total field is given by

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \hat{\boldsymbol{z}}\, \frac{\mu_0\, I}{4\pi}\, \frac{2}{kr}\, \frac{\cos(\pi \cos\theta/2)}{\sin^2\theta} \\ &\quad \times \left[1 + {\rm e}^{\,{\rm i}\,\alpha} + {\rm e}^{\,2{\rm i}\,\alpha} + \cdots + {\rm e}^{\,(N-1){\rm i}\,\alpha}\right] {\rm e}^{\,{\rm i}\,(kr-\omega t)}. \end{aligned} \tag{5.50}$$

The series in square brackets is a geometric progression in $\beta = \exp(\,{\rm i}\,\alpha)$, the sum of which is

$$1 + \beta + \beta^2 + \cdots \beta^{N-1} = \frac{\beta^N - 1}{\beta - 1}. \tag{5.51}$$

Thus, the term in square brackets becomes

$$\frac{{\rm e}^{\,{\rm i}\,N\alpha} - 1}{{\rm e}^{\,{\rm i}\,\alpha} - 1} = {\rm e}^{\,{\rm i}\,(N-1)\alpha/2}\, \frac{\sin(N\alpha/2)}{\sin(\alpha/2)}. \tag{5.52}$$

It follows from Eq. (5.16) that the radiation pattern due to the array takes the form

$$\frac{dP}{d\Omega} = \left(\frac{\mu_0 c I^2}{8\pi^2} \frac{\cos^2(\pi \cos\theta/2)}{\sin^2\theta}\right) \left(\frac{\sin^2(N\alpha/2)}{\sin^2(\alpha/2)}\right). \tag{5.53}$$

We can think of this formula as the product of the two factors in large parentheses. The first is just the standard radiation pattern of a half-wave antenna. The second arises from the linear array of N elements. If we retained the same array, but replaced the elements by something other than half-wave antennas, then the first factor would change, but not the second. If we changed the array, but not the elements, then the second factor would change but the first would remain the same. Thus, we can think of the radiation pattern as the product of two independent factors, the *element function* and the *array function*. This independence follows from the Fraunhofer approximation (5.6), which justifies the linear phase shifts of Eq. (5.7).

The array function in this case is

$$f(\alpha) = \frac{\sin^2(N\alpha/2)}{\sin^2(\alpha/2)}, \tag{5.54}$$

where

$$\alpha = k\Delta \sin\theta \cos\varphi. \tag{5.55}$$

The function $f(\alpha)$ has nulls whenever the numerator vanishes; that is, whenever

$$\pm\alpha = \frac{2\pi}{N}, \frac{4\pi}{N}, \cdots \frac{(N-1)\,2\pi}{N}; \frac{(N+1)\,2\pi}{N} \cdots. \tag{5.56}$$

However, when $\pm\alpha = 0, 2\pi, \cdots$, the denominator also vanishes, and the l'Hôpital limit is easily seen to be $f(0, 2\pi, \cdots) \to N^2$. These limits are known as the *principle maxima* of the function. Secondary maxima occur approximately at the maxima of the numerator; that is, at

$$\pm\alpha = \frac{3\pi}{N}, \frac{5\pi}{N}, \cdots \frac{(2N-3)\,2\pi}{N}; \frac{(2N+3)\,2\pi}{N} \cdots. \tag{5.57}$$

There are $(N-2)$ secondary maxima between successive principal maxima.

Now, the maximum possible value of α is $k\varDelta = 2\pi\varDelta/\lambda$. Thus, when the element spacing $\varDelta$ is less than the wavelength there is only one principle maximum (at $\alpha = 0$), directed perpendicular to the array (*i.e.*, at $\varphi = \pm\pi/2$). Such a system is called a *broadside array.* The secondary maxima of the radiation pattern are called *side lobes.* In the direction perpendicular to the array, all elements contribute in phase, and the intensity is proportional to the square of the sum of the individual amplitudes. Thus, the peak intensity for an N element array is N^2 times the intensity of a single antenna. The angular half-width of the principle maximum (in φ) is approximately $\varDelta\varphi \simeq \lambda/N\varDelta$. Although the principle lobe clearly gets narrower in the azimuthal angle φ as N increases, the lobe width in the polar angle θ is mainly controlled by the element function, and is thus little affected by the number of elements. A radiation pattern which is narrow in one angular dimension, but broad in the other, is called a *fan beam.*

Arranging a set of antennas in a regular array has the effect of taking the azimuthally symmetric radiation pattern of an individual antenna and concentrating it into some narrow region of azimuthal angle of extent $\varDelta\varphi \simeq \lambda/N\varDelta$. The net result is that the gain of the array is larger than that of an individual antenna by a factor of order

$$\frac{2\pi\, N\varDelta}{\lambda}. \tag{5.58}$$

It is clear that the boost factor is of order the linear extent of the array divided by the wavelength of the emitted radiation. Thus, it is possible to construct a very high gain antenna by arranging a large number of low gain antennas in a regular pattern and driving them in phase. The optimum spacing between successive elements of the array is of order the wavelength of the radiation.

A linear array of antenna elements which are spaced $\varDelta = \lambda/2$ apart and driven with alternating phases has its principle radiation maximum along $\varphi = 0$ and π, since the field amplitudes now add in phase in the plane of the array. Such a system is called an *end-fire array.* The direction of the principle maximum can be changed at will by introducing the appropriate phase shift between successive elements of the array. In fact, it is possible to produce a radar beam which sweeps around the horizon, without any mechanical motion of the array, by varying the phase difference between successive elements of the array electronically.

5.4 Thomson scattering

When an electromagnetic wave is incident on a charged particle, the electric and magnetic components of the wave exert a Lorentz force on the particle, setting it into motion. Since the wave is periodic in time, so is the motion of the particle. Thus, the particle is accelerated and, consequently, emits radiation. More exactly, energy is absorbed from the incident wave by the particle and re-emitted as electromagnetic radiation. Such a process is clearly equivalent to the scattering of the electromagnetic wave by the particle.

Consider a linearly polarized, monochromatic, plane wave incident on a particle carrying a charge q. The electric component of the wave can be written

$$\boldsymbol{E} = \boldsymbol{e}\, E_0\, \mathrm{e}^{\,\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{5.59}$$

where E_0 is the peak amplitude of the electric field, $\boldsymbol{e}$ is the polarization vector, and $\boldsymbol{k}$ is the wave vector (of course, $\boldsymbol{e}\cdot\boldsymbol{k} = 0$). The particle is assumed to undergo small amplitude oscillations about an equilibrium position which coincides with the origin of the coordinate system. Furthermore, the particle's velocity is assumed to remain sub-relativistic, which enables us to neglect the magnetic component of the Lorentz force. The equation of motion of the charged particle is approximately

$$\boldsymbol{f} = q\boldsymbol{E} = m\ddot{\boldsymbol{s}}, \tag{5.60}$$

where m is the mass of the particle, $\boldsymbol{s}$ is its displacement from the origin, and $\dot{}$ denotes $\partial/\partial t$. According to Eq. (2.321), the time-averaged power radiated per unit solid angle by an accelerating, non-relativistic, charged particle is given by

$$\frac{dP}{d\Omega} = \frac{q^2\langle\ddot{s}^2\rangle}{16\pi^2\epsilon_0 c^3}\,\sin^2\theta, \tag{5.61}$$

where $\langle\cdots\rangle$ denotes a time average. However,

$$\langle\ddot{s}^2\rangle = \frac{q^2}{m^2}\langle E^2\rangle = \frac{q^2 {E_0}^2}{2m^2}. \tag{5.62}$$

Hence, the scattered power per unit solid angle is given by

$$\frac{dP}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0\, mc^2}\right)^2 \frac{\epsilon_0 c\, {E_0}^2}{2}\sin^2\theta. \tag{5.63}$$

The time-averaged Poynting flux of the incident wave is

$$\langle u \rangle = \frac{\epsilon_0 c\, {E_0}^2}{2}. \tag{5.64}$$

It is convenient to define the *scattering cross section* as the equivalent area of the incident wavefront which delivers the same power as that re-radiated by the particle:

$$\sigma = \frac{\text{total re-radiated power}}{\langle u \rangle}. \tag{5.65}$$

By analogy, the *differential scattering cross section* is defined

$$\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\langle u \rangle}. \tag{5.66}$$

It follows from Eqs. (5.63), (5.64), and (5.66) that

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0\, mc^2}\right)^2 \sin^2\theta. \tag{5.67}$$

The total scattering cross section is then

$$\sigma = \int_0^\pi \frac{d\sigma}{d\Omega}\, 2\pi\, \sin\theta\, d\theta = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0\, mc^2}\right)^2. \tag{5.68}$$

The quantity θ appearing in Eq. (5.67) is the angle subtended between the direction of acceleration of the particle and the direction of the outgoing radiation (which is parallel to the unit vector $\boldsymbol{n}$). In the present case, the acceleration is due to the electric field, so it is parallel to the polarization vector $\boldsymbol{e}$. Thus, $\cos\theta = \boldsymbol{e}\cdot\boldsymbol{n}$.

Up to now, we have only considered the scattering of linearly polarized radiation by a charged particle. Let us now calculate the angular distribution of scattered radiation for the commonly occurring case of randomly polarized incident radiation. It is helpful to set up a right-handed coordinate system based on the three mutually orthogonal unit vectors $\boldsymbol{e}$, $\boldsymbol{e}\wedge\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{k}}$. In terms of these unit vectors, we can write

$$\boldsymbol{n} = \sin\varphi\cos\psi\, \boldsymbol{e} + \sin\varphi\sin\psi\, \boldsymbol{e}\wedge\hat{\boldsymbol{k}} + \cos\varphi\, \hat{\boldsymbol{k}}, \tag{5.69}$$

where φ is the angle subtended between the direction of the incident radiation and that of the scattered radiation, and ψ is an angle which specifies the orientation of the polarization vector in the plane perpendicular to $\boldsymbol{k}$ (assuming that $\boldsymbol{n}$ is known). It is easily seen that

$$\cos\theta = \boldsymbol{e}\cdot\boldsymbol{n} = \cos\psi\,\sin\varphi, \tag{5.70}$$

so

$$\sin^2\theta = 1 - \cos^2\psi\,\sin^2\varphi. \tag{5.71}$$

Averaging this result over all possible polarizations of the incident wave (*i.e.*, over all possible values of the polarization angle ψ), we obtain

$$\overline{\sin^2\theta} = 1 - \overline{\cos^2\psi}\,\sin^2\varphi = 1 - (\sin^2\varphi)/2 = \frac{1+\cos^2\varphi}{2}. \tag{5.72}$$

Thus, the differential scattering cross section for unpolarized incident radiation (obtained by substituting $\overline{\sin^2\theta}$ for $\sin^2\theta$ in Eq. (5.67)) is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = \left(\frac{q^2}{4\pi\epsilon_0\,mc^2}\right)^2\frac{1+\cos^2\varphi}{2}. \tag{5.73}$$

It is clear that the differential scattering cross section is independent of the frequency of the incident wave, and is also symmetric with respect to forward and backward scattering. The frequency of the scattered radiation is the same as that of the incident radiation. The total scattering cross section is obtained by integrating over the entire solid angle of the polar angle φ and the azimuthal angle ψ. Not surprisingly, the result is exactly the same as Eq. (5.68).

The classical scattering cross section (5.73) is modified by quantum effects when the energy of the incident photons, $\hbar\omega$, becomes comparable with the rest mass of the scattering particle, mc^2. The scattering of a photon by a charged particle is called *Compton scattering*, and the quantum mechanical version of the Compton scattering cross section is known as the *Klein-Nishina formula.* As the photon energy increases, and eventually becomes comparable with the rest mass energy of the particle, the Klein-Nishina formula predicts that forward scattering of photons becomes increasingly favored with respect to backward scattering. The Klein-Nishina cross section *does*, in general, depend on the frequency of the incident photons. Furthermore, energy and momentum conservation demand a shift in the frequency of scattered photons with respect to that of the incident photons.

If the charged particle in question is an electron then Eq. (5.68) reduces to the well known *Thomson scattering cross section*

$$\sigma_{\rm Thomson} = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0\, m_e c^2}\right)^2 = 6.65\times 10^{-29}\,{\rm m}^2. \tag{5.74}$$

The quantity $e^2/(4\pi\epsilon_0\, m_e c^2) = 2.8\times 10^{-15}$ m is called the *classical electron radius* (it is the radius of spherical shell of total charge e whose electrostatic energy equals the rest mass energy of the electron). Thus, as a scatterer the electron acts rather like a solid sphere whose radius is of order the classical electron radius. Since this radius is extremely small, it is clear that scattering of radiation by a single electron (or any other charged particle) is a very weak process.

5.5 Rayleigh scattering

Let us now consider the scattering of electromagnetic radiation by a harmonically bound electron; *e.g.*, an electron orbiting an atomic nucleus. We have seen in Section 4.2 that such an electron satisfies an equation of motion of the form

$$\ddot{\boldsymbol{s}} + \gamma_0\,\dot{\boldsymbol{s}} + {\omega_0}^2\boldsymbol{s} = -\frac{e}{m_e}\,\boldsymbol{E}, \tag{5.75}$$

where ω_0 is the characteristic oscillation frequency of the electron, and $\gamma_0 \ll \omega_0$ is the damping rate of such oscillations. Assuming an ${\rm e}^{-{\rm i}\,\omega t}$ time dependence of both $\boldsymbol{s}$ and $\boldsymbol{E}$, we find that

$$\ddot{\boldsymbol{s}} = \frac{\omega^2}{{\omega_0}^2 - \omega^2 - {\rm i}\,\gamma_0\,\omega}\frac{e}{m_e}\,\boldsymbol{E}. \tag{5.76}$$

It follows, by analogy with the analysis in the previous section, that the total scattering cross section is given by

$$\sigma = \sigma_{\rm Thomson}\,\frac{\omega^4}{({\omega_0}^2 - \omega^2)^2 + (\gamma_0\,\omega)^2}. \tag{5.77}$$

The angular distribution of the radiation is the same as that in the case of a free electron.

The maximum value of the cross section (5.77) is obtained when $\omega \simeq \omega_0$; *i.e.*, for resonant scattering. In this case, the scattering cross section can become very large. In fact,

$$\sigma \simeq \sigma_{\rm Thomson} \left(\frac{\omega_0}{\gamma_0}\right)^2, \tag{5.78}$$

which is generally far greater than the Thomson scattering cross section.

For strong binding, $\omega \ll \omega_0$, Eq. (5.77) reduces to

$$\sigma \simeq \sigma_{\rm Thomson} \left(\frac{\omega}{\omega_0}\right)^4, \tag{5.79}$$

giving a scattering cross section which depends on the inverse fourth power of the wavelength of the incident radiation. Equation (5.79) is known as the *Rayleigh scattering cross section*, and is appropriate to the scattering of visible radiation by gas molecules. This is Rayleigh's famous explanation of the blue sky: the air molecules of the atmosphere preferentially scatter the shorter wavelength components out of "white" sunlight which grazes the atmosphere. Conversely, sunlight viewed directly through the long atmospheric path at sunset appears reddened. The Rayleigh scattering cross section is much less than the Thompson scattering cross section (for $\omega \ll \omega_0$). However, this effect is offset to some extent by the fact that the density of neutral molecules in a gas (*e.g.*, the atmosphere) is much larger than the density of free electrons typically encountered in a plasma.