LECTURE NOTES 18.75

The Relativistic Version of Maxwell's Stress Tensor $T^{\mu\nu}$

Despite the fact that we know that the *EM* energy density $\boxed{u_{EM} = \tfrac{1}{2}\varepsilon_o E^2 + \tfrac{1}{2\mu_o}B^2}$ and Poynting's Vector, $\boxed{\vec{S} = \tfrac{1}{\mu_o}\vec{E}\times\vec{B}}$ are not Lorentz invariant quantities, we ask: Is there a related entity, relativistic in nature, from which we *can* understand the transformation properties of u_{EM} and $\vec{S}$, in going from one IRF(S) to another IRF(S')?

The answer is *yes* – the 4-dimensional relativistic generalization of the 3-dimensional classical electrodynamics Maxwell's stress tensor: $\boxed{\overleftrightarrow{T}_{ij} \to T^{\mu\nu}}$!!!

Recall that the classical-electrodynamics 3-dimensional Maxwell stress tensor is $\overleftrightarrow{T}$, a 9-component, rank two 3 × 3 symmetric tensor (i.e. a matrix) whose elements are:

n.b. T_{ij} elements are symmetric: $T_{ij} = T_{ji} \Rightarrow \overleftrightarrow{T}$ is a symmetric rank-2 tensor.

$$\Rightarrow \boxed{T_{ij} \equiv \varepsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)} \quad \text{where: } \boxed{i,j = 1:3}$$

The Kronecker δ-function: $\boxed{\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}}$

Physically, $\overleftrightarrow{T}$ is the force per unit area (or stress) acting on a surface of interest.

$T_{ij} \equiv$ force per unit area in i^{th} direction acting on an element of surface in the j^{th} direction.

$\Rightarrow$ Thus: T_{xx}, T_{yy}, T_{zz}, physically represent pressures. (SI units: N/m^2)

$\Rightarrow$ And: T_{xy}, T_{xz}, T_{yx}, T_{yz}, T_{zx}, T_{zy} physically represent shears. (SI units: N/m^2)

n.b. SI units of $\overleftrightarrow{T}$ all same (= pressure): $\dfrac{N}{m^2} = \dfrac{kg\text{-}m/s^2}{m^2} = \dfrac{kg}{m\text{-}s^2}$

= Same SI units as energy density $\dfrac{J}{m^3} = \dfrac{N-m}{m^3} = \dfrac{N}{m^2}$

In classical electrodynamics, the force per unit volume (*aka* force density) is:

$$\boxed{\vec{f}(\vec{r},t) = \nabla\bullet\overleftrightarrow{T}(\vec{r},t) - \frac{1}{c^2}\frac{\partial\vec{S}(\vec{r},t)}{\partial t}} \quad \text{where: } \boxed{\vec{S}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)} = \text{Poynting's Vector}$$

SI units of force density: N/m^3
= energy flux (watts/m^2) through surface

The total force is therefore: $$\boxed{\vec{F}(t) = \int_v \vec{f}(\vec{r},t)\,d\tau = \int_v\left(\vec{\nabla}\bullet\overleftrightarrow{T}(\vec{r},t)\right)d\tau - \frac{1}{c^2}\int_v\frac{\partial\vec{S}(\vec{r},t)}{\partial t}d\tau}$$

SI units of force = $N = kg\text{-}m/s^2$

Use the divergence theorem on the 1st integral: $$\boxed{\vec{F}(t) = \oint_S\overleftrightarrow{T}(\vec{r},t)\bullet d\vec{a} - \frac{1}{c^2}\int_v\frac{\partial\vec{S}(\vec{r},t)}{\partial t}d\tau}$$

In going from the classical electrodynamics 3-D spatial version of Maxwell's stress tensor $\overleftrightarrow{T}$ a 3 × 3 = 9 element symmetric rank two tensor T_{ij}

$$\boxed{T_{ij} \equiv \varepsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) = \varepsilon_o c^2\left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_iB_j\right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2\right)\right]} \quad \boxed{c^2 = \frac{1}{\varepsilon_o\mu_o}}$$

$$\boxed{i, j = 1:3}$$

to the relativistic 4-dimensional space-time version of Maxwell's stress tensor $T^{\mu\nu}$ a 4 × 4 = 16 element symmetric tensor $(\mu,\nu = 0:3)$ we expect the "new"/additional temporal components of $T^{\mu\nu}$ i.e. a new top row (row # $\mu = 0$ column # $\nu = 0:3$) and a new LHS column (row # $\mu = 0:3$, column # $\nu = 0$) to:

a.) be symmetric, i.e. $T^{0\nu} = +T^{\nu 0}$
b.) have the same physical SI units (N/m^2 = pressure/energy density) as $\overleftrightarrow{T}_{ij}$
c.) have something to do with the temporal aspects of *EM* field energy flow
d.) be related to the *EM* field tensor, $F^{\mu\nu}$ (or equivalently, $G^{\mu\nu}$)

We define the relativistic version of Maxwell's stress tensor as:

$$\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2\left[\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}\right) - \frac{1}{4}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right]}$$ {n.b. implicit sum over repeated indices!}

Where the "flat" space-time metric tensor, $\delta^{\mu\nu}$ (analogous to δ_{ij}) is defined as:

$$\boxed{\delta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}} \quad \text{where: } \boxed{\delta^{\mu\nu} \equiv \begin{cases} -1 \text{ for } \mu = \nu = 0 \\ +1 \text{ for } \mu = \nu = 1,2,3 \\ \ \ 0 \text{ for } \mu \neq \nu \end{cases}}$$

Note that the differences in the physical appearance between definitions of 3-D $\overleftrightarrow{T}_{ij}$ and 4-D $T^{\mu\nu}$

$$\boxed{T_{ij} \equiv \varepsilon_o c^2\left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_iB_j\right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2\right)\right] \qquad \delta_{ij} \equiv \begin{pmatrix} +1 & & \\ & +1 & \\ & & +1 \end{pmatrix}}$$

$$\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2\left[\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}\right) - \frac{1}{4}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right] \qquad \delta^{\mu\nu} \equiv \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}}$$

n.b. symmetric tensors, SI units:
$$N/m^2 = \frac{kg}{m-s^2}$$

arise from our (& Griffiths') definition of the anti-symmetric *EM* field tensor, $F^{\mu\nu}$:

$$\boxed{F^{\mu\nu} \equiv \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}}$$

and: $$\boxed{\left(F_{\lambda\sigma}F^{\lambda\sigma}\right) = 2\left(B^2 - \frac{1}{c^2}E^2\right) = -2\left(\frac{E^2}{c^2} - B^2\right)}$$

(See Griffiths problem 12.50, p. 537 P436 HW #14)

For clarity's sake, here we present the results for $T^{\mu\nu}$ and place the (tedious!) calculational details in Appendix "A" of these lecture notes.

Recall that Poynting's Vector, $\boxed{\vec{S} \equiv \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)}$ SI units: $\boxed{\text{Watts}/m^2 = \frac{J/s}{m^2} = \frac{J}{m^2\text{-}s} = \frac{N\text{-}m}{m^2\text{-}s} = \frac{N}{m\text{-}s}}$

Recall that *EM* field linear momentum *density*: $\boxed{\vec{\wp}_{EM} \equiv \varepsilon_o\left(\vec{E}\times\vec{B}\right)}$

$$\boxed{\vec{\wp}_{EM} \equiv \varepsilon_o\left(\vec{E}\times\vec{B}\right) = \varepsilon_o\mu_o\left[\frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\right] = \varepsilon_o\mu_o\vec{S} = \frac{1}{c^2}\vec{S}} \Leftarrow \boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$$

$$\therefore \quad \boxed{\vec{\wp}_{EM}c = \frac{1}{c}\vec{S} = \frac{1}{c}\left[\frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\right] = \frac{1}{\mu_o}\left[\left(\frac{\vec{E}}{c}\times\vec{B}\right)\right] = \varepsilon_o c^2\left(\frac{\vec{E}}{c}\times\vec{B}\right)}$$

SI units of $\boxed{\vec{\wp}_{EM}c = \frac{1}{c}\vec{S}:}$ $\boxed{\frac{kg}{m^2-s}\cdot\frac{m}{s} = \frac{kg}{m-s^2}}$ but: $\boxed{1N = kg - m/s^2}$

$= \boxed{\frac{kg-m}{m^2-s^2} = \frac{N}{m^2}}$ = pressure = $\boxed{\frac{N-m}{m^3} = \frac{J}{m^3}}$ = energy density

Recall that the *EM* field energy *density* u_{EM} is defined as:

$$\boxed{u_{EM} \equiv \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 = \frac{1}{2}\varepsilon_o\left(\vec{E}\bullet\vec{E}\right) + \frac{1}{2\mu_o}\vec{B}\bullet\vec{B} = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2}\right) + \frac{1}{2}\varepsilon_o c^2 B^2 = \frac{1}{2}\varepsilon_o c^2\left[\left(\frac{E^2}{c^2}\right) + B^2\right]}$$

SI units: $\frac{J}{m^3} = \frac{N-m}{m^3} = \frac{N}{m^2}$ = energy *density* = pressure = force per unit area

= {linear} momentum flux density

Explicitly reminding the reader that the 3-D classical electrodynamics version of Maxwell's stress tensor $\overleftrightarrow{T}$ is defined as:

$$\overleftrightarrow{T} = T_{ij} = \begin{array}{cc} \begin{array}{ccc} \multicolumn{3}{c}{\text{Column } \# \, j = 1{:}3} \\ 1=x & 2=y & 3=z \end{array} & \text{Row } \# \, i = 1{:}3 \\ \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} & \begin{array}{c} 1 = x \\ 2 = y \\ 3 = z \end{array} \end{array} = \boxed{\varepsilon_o c^2 \left[\left(\frac{E_i}{c}\frac{E_j}{c} + B_i B_j \right) - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2 \right) \right]} \quad \boxed{\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}}$$

Row # (1:3) Column # (1:3)

$$\boxed{\overleftrightarrow{T} = \begin{pmatrix} \varepsilon_o c^2 \left[\left(\frac{E_x^2}{c^2} + B_x^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] & \varepsilon_o c^2 \left[\frac{E_x}{c}\frac{E_y}{c} + B_x B_y \right] & \varepsilon_o c^2 \left[\frac{E_x}{c}\frac{E_z}{c} + B_x B_z \right] \\ \varepsilon_o c^2 \left[\frac{E_y}{c}\frac{E_x}{c} + B_y B_x \right] & \varepsilon_o c^2 \left[\left(\frac{E_y^2}{c^2} + B_y^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] & \varepsilon_o c^2 \left[\frac{E_y}{c}\frac{E_z}{c} + B_y B_z \right] \\ \varepsilon_o c^2 \left[\frac{E_z}{c}\frac{E_x}{c} + B_z B_x \right] & \varepsilon_o c^2 \left[\frac{E_z}{c}\frac{E_y}{c} + B_z B_y \right] & \varepsilon_o c^2 \left[\left(\frac{E_z^2}{c^2} + B_z^2 \right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2 \right) \right] \end{pmatrix}}$$

$$\boxed{\overleftrightarrow{T} = \frac{1}{2}\varepsilon_o c^2 \begin{pmatrix} \left(\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2} \right) + \left(B_x^2 - B_y^2 - B_z^2 \right) & 2\left[\frac{E_x}{c}\frac{E_y}{c} + B_x B_y \right] & 2\left[\frac{E_x}{c}\frac{E_z}{c} + B_x B_z \right] \\ 2\left[\frac{E_y}{c}\frac{E_x}{c} + B_y B_x \right] & \left(\frac{E_y^2}{c^2} + B_y^2 \right) + \left(-B_x^2 + B_y^2 - B_z^2 \right) & 2\left[\frac{E_y}{c}\frac{E_z}{c} + B_y B_z \right] \\ 2\left[\frac{E_z}{c}\frac{E_x}{c} + B_z B_x \right] & 2\left[\frac{E_z}{c}\frac{E_y}{c} + B_z B_y \right] & \left(-\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2} \right) + \left(-B_x^2 - B_y^2 + B_z^2 \right) \end{pmatrix}}$$

$\Rightarrow$ Note that the T_{ij} elements <u>are</u> symmetric, i.e. $\boxed{T_{ji} = +T_{ij}}$

<u>Physically</u>:

$T_{ij}\,(i, j = 1:3)$ are pure <u>space-space</u> components.

$T_{ij} = i^{th}$ component of force across unit area perpendicular to j^{th} direction.

$T_{ii} = T_{xx}, T_{yy}, T_{zz}$ represent <u>pressures</u> on enclosing surfaces in the *x*, *y*, *z* directions, respectively.

$T_{ij} = (T_{xy} = T_{yx}), (T_{xz} = T_{zx}), (T_{yz} = T_{zy})$ represent <u>stresses</u> (or <u>shears</u>) on enclosing surfaces in the *x*, *y*, or *z* directions

<u>But note *also* that physically</u>:

T_{ij} = –*ve* of the rate of <u>flow</u> of the i^{th} component of *EM* field linear momentum $\vec{p}$ through unit area whose normal is in the j^{th} direction, i.e. $-T_{ij}$ is the i^{th} component of the <u>linear</u> <u>momentum</u> <u>flux</u> <u>*density*</u> transported in the j^{th} direction by the *EM* fields.

Then, the relativistic 4-D space-time version of Maxwell's stress tensor defined as:

$$\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(F^{\mu\sigma} F^{\nu}{}_{\sigma} \right) - \frac{1}{4} \delta^{\mu\nu} \left(F_{\lambda\sigma} F^{\lambda\sigma} \right) \right]}$$ gives:

$$\begin{array}{cc} \text{Row \#} & \text{Column \#} \\ 0{:}3 & 0{:}3 \end{array} \quad T^{\mu\nu} = \begin{array}{c} \text{Column \#} \\ \begin{array}{cccc} 0=t & 1=x & 2=y & 3=z \end{array} \\ \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \end{array} \begin{array}{c} \text{Row \#} \\ 0=t \\ 1=x \\ 2=y \\ 3=z \end{array} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix}$$

Now while we might <u>hope</u> that the $\{\mu,\nu = 1:3\}$ pure <u>space-space</u> elements of the relativistic 4-D space-time stress tensor $T^{\mu\nu}$ would be identical to that of the 3-D classical electrodynamics / Maxwell's stress tensor $\overleftrightarrow{T}_{ij}$, because of our definitions for $T^{\mu\nu}$ and $F^{\mu\nu}$ what we instead obtain is:

$$\boxed{T^{\mu\nu}\Big|_{\mu,\nu=1:3} = -\overleftrightarrow{T}_{ij}\Big|_{i,j=1:3}}$$

Thus, we see that the pure <u>space-space</u> $\{\mu,\nu = 1:3\}$ components of the relativistic stress tensor $T^{\mu\nu}$ <u>physically represent *EM* field linear momentum flux densities</u>, the <u>*negative*</u> of which physically corresponds to stresses/shears on bounding surfaces!

The <u>temporal</u> components of $T^{\mu\nu}$ are:

$$\boxed{\frac{1}{\mu_o} = \varepsilon_o c^2}$$

<u>$\nu = 0$ $\left(1^{\text{st}}\text{ column}\right)$:</u>

$$\boxed{T^{00} = u_{EM} = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2} + B^2\right)}$$

$$\boxed{T^{10} = \wp_x c = \varepsilon_o c^2\left(\frac{E_y}{c}B_z - \frac{E_z}{c}B_y\right) = \varepsilon_o c\left(E_y B_z - E_z B_y\right)}$$

$$\boxed{T^{20} = \wp_y c = \varepsilon_o c^2\left(\frac{E_z}{c}B_x - \frac{E_x}{c}B_z\right) = \varepsilon_o c\left(E_z B_x - E_x B_z\right)}$$

$$\boxed{T^{30} = \wp_z c = \varepsilon_o c^2\left(\frac{E_x}{c}B_y - \frac{E_y}{c}B_x\right) = \varepsilon_o c\left(E_x B_y - E_y B_x\right)}$$

Recalling that Poynting's Vector: $\boxed{\vec{S} \equiv \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)}$ $\therefore$ $\boxed{\frac{1}{c}\vec{S} = \frac{1}{\mu_o}\left(\frac{\vec{E}}{c}\times\vec{B}\right) = \varepsilon_o c^2\left(\frac{\vec{E}}{c}\times\vec{B}\right) = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)}$

<u>Thus:</u> $\boxed{\frac{1}{c}\vec{S} = \frac{1}{c}S_x\hat{x} + \frac{1}{c}S_y\hat{y} + \frac{1}{c}S_z\hat{z} = \varepsilon_o c\left(E_y B_z - E_z B_y\right)\hat{x} + \varepsilon_o c\left(E_z B_x - E_x B_z\right)\hat{y} + \varepsilon_o c\left(E_x B_y - E_y B_x\right)\hat{z}}$

$\underline{\mu = 0 \ \left(1^{\text{st}} \text{ row}\right):}$

$$\boxed{\begin{aligned} T^{01} &= \frac{1}{c}S_x = \varepsilon_o c\left(E_y B_z - E_z B_y\right) \\ T^{02} &= \frac{1}{c}S_y = \varepsilon_o c\left(E_z B_x - E_x B_z\right) \\ T^{03} &= \frac{1}{c}S_z = \varepsilon_o c\left(E_x B_y - E_y B_x\right) \end{aligned}}$$

Thus, we (<u>explicitly</u>) see that:

$$\left.\begin{aligned} \boxed{T^{10} = T^{01}} &\Rightarrow \boxed{\wp_x c = \frac{1}{c}S_x} \\ \boxed{T^{20} = T^{02}} &\Rightarrow \boxed{\wp_y c = \frac{1}{c}S_y} \\ \boxed{T^{30} = T^{03}} &\Rightarrow \boxed{\wp_z c = \frac{1}{c}S_z} \end{aligned}\right\} \text{ or: } \boxed{\vec{\wp} c = \frac{1}{c}\vec{S} = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)}$$

Thus the components of the relativistic 4-D/spacetime version of Maxwell's stress tensor $T^{\mu\nu}$ are $\{T^{\mu\nu} = T^{\nu\mu} =$ <u>symmetric</u> tensor$\}$: $\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2\left[\left(F^{\mu\sigma}F_{\sigma}^{\ \nu}\right) - \frac{1}{4}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right]}$

$$\boxed{T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & -T_{xx} & -T_{xy} & -T_{xz} \\ T^{yt} & -T_{yx} & -T_{yy} & -T_{yz} \\ T^{zt} & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}}$$

3-D space-space components of classical electrodynamics' Maxwell's stress tensor $\overleftrightarrow{T}_{ij}$

$$T^{\mu\nu} = \begin{pmatrix} u_{EM} = \frac{1}{2}\varepsilon_o c^2\left(\left(\frac{E}{c}\right)^2 + B^2\right) & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \wp_x c & -\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2} + B_x^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_x}{c}\frac{E_y}{c}\right) + \left(B_x B_y\right)\right] & -\varepsilon_0 c^2\left[\left(\frac{E_x}{c}\frac{E_z}{c}\right) + \left(B_x B_z\right)\right] \\ \wp_y c & -\varepsilon_o c^2\left[\left(\frac{E_y}{c}\frac{E_x}{c}\right) + \left(B_y B_x\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_y^2}{c^2} + B_y^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_y}{c}\frac{E_z}{c}\right) + \left(B_y B_z\right)\right] \\ \wp_z c & -\varepsilon_o c^2\left[\left(\frac{E_z}{c}\frac{E_x}{c}\right) + \left(B_z B_x\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_z}{c}\frac{E_y}{c}\right) + \left(B_z B_y\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_z^2}{c^2} + B_z^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2\right)\right] \end{pmatrix}$$

Physically:

$$
T^{\mu\nu} = \begin{pmatrix}
\begin{array}{l} EM \text{ field energy} \\ \text{density, } u_{EM} \end{array} & \begin{array}{l} x\text{-flow of } EM \\ \text{field energy, } S_x/c \end{array} & \begin{array}{l} y\text{-flow of } EM \\ \text{field energy, } S_y/c \end{array} & \begin{array}{l} z\text{-flow of } EM \\ \text{field energy, } S_y/c \end{array} \\[2ex]
\begin{array}{l} \text{flux of } EM \text{ field} \\ \text{momentum density, } \wp_x \end{array} & \begin{array}{l} x - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_x \end{array} & \begin{array}{l} y - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_x \end{array} & \begin{array}{l} z - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_x \end{array} \\[2ex]
\begin{array}{l} \text{flux of } EM \text{ field} \\ \text{momentum density, } \wp_y \end{array} & \begin{array}{l} x - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_y \end{array} & \begin{array}{l} y - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_y \end{array} & \begin{array}{l} z - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_y \end{array} \\[2ex]
\begin{array}{l} \text{flux of } EM \text{ field} \\ \text{momentum density, } \wp_z \end{array} & \begin{array}{l} x - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_z \end{array} & \begin{array}{l} y - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_z \end{array} & \begin{array}{l} z - \text{flow of} \\ EM \text{ field linear} \\ \text{momentum density, } \wp_z \end{array}
\end{pmatrix}
$$

SI units of $T^{\mu\nu}$: energy density = $J/m^2 = N\text{-}m/m^2 = N/m^2 =$ pressure

$$\boxed{T^{00} = u_{EM} = \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2} + B^2\right) = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2} \quad \underline{\text{using}}: \boxed{\left(c^2 = \frac{1}{\varepsilon_o \mu_o}\right)}$$

$$\boxed{T^{10} = \wp_x c = \varepsilon_o c^2\left(\frac{E_y}{c}B_z - \frac{E_z}{c}B_y\right) = \varepsilon_o c\left(E_y B_z - E_z B_y\right) = \frac{1}{c}S_x = T^{01}}$$

$$\boxed{T^{20} = \wp_x c = \varepsilon_o c^2\left(\frac{E_z}{c}B_x - \frac{E_x}{c}B_z\right) = \varepsilon_o c\left(E_z B_x - E_x B_z\right) = \frac{1}{c}S_y = T^{02}}$$

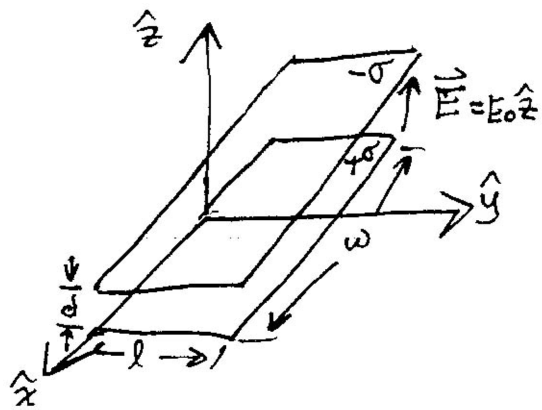
$$\boxed{T^{30} = \wp_z c = \varepsilon_o c^2\left(\frac{E_x}{c}B_y - \frac{E_y}{c}B_x\right) = \varepsilon_o c\left(E_x B_y - E_y B_x\right) = \frac{1}{c}S_z = T^{03}}$$

$$\boxed{T^{01} = \frac{1}{c}S_x = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_x = \varepsilon_o c\left(E_y B_z - E_z B_y\right) = \wp_x c = T^{10}}$$

$$\boxed{T^{02} = \frac{1}{c}S_y = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_y = \varepsilon_o c\left(E_z B_x - E_x B_z\right) = \wp_y c = T^{20}}$$

$$\boxed{T^{03} = \frac{1}{c}S_z = \varepsilon_o c\left(\vec{E}\times\vec{B}\right)_z = \varepsilon_o c\left(E_x B_y - E_y B_x\right) = \wp_z c = T^{30}}$$

As a simple example of the use of the 4-D/relativistic version of Maxwell's stress tensor $T^{\mu\nu}$, let us consider the purely electrostatic problem of a parallel-plate capacitor at rest in the lab frame IRF(S), with large area plates || to the x-y plane as shown in the figure below:



Inside the || plates $\{ d \ll \omega, \ell \}$:

$\vec{E} = E_o \hat{z} \quad \left(\vec{E} = 0 \text{ elsewhere}\right)$

$\vec{B} = 0$ everywhere

<u>Then</u>:

$$\boxed{T^{\mu\nu} = \begin{pmatrix} u_{EM} & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \wp_x c & -T_{xx} & -T_{xy} & -T_{xz} \\ \wp_y c & -T_{yx} & -T_{yy} & -T_{yz} \\ \wp_z c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\varepsilon_o E_o^2 & 0 & 0 & 0 \\ 0 & +\frac{1}{2}\varepsilon_o E_o^2 & 0 & 0 \\ 0 & 0 & +\frac{1}{2}\varepsilon_o E_o^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\varepsilon_o E_o^2 \end{pmatrix}}$$

$$\boxed{T^{\mu\nu} = \tfrac{1}{2}\varepsilon_o E_o^2 \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}} \Leftarrow \boxed{\text{Only the } \underline{\text{diagonal}} \text{ elements of } T^{\mu\nu} = T^{\mu\mu} \text{ are non-zero}}$$

$\boxed{T^{00} = u_{EM} = +\frac{1}{2}\varepsilon_o E_o^2} \Rightarrow$ +*ve* Energy Density (J/m^3)

$\boxed{T^{11} = -T_{xx} = +\frac{1}{2}\varepsilon_o E_o^2} \Rightarrow$ *+ve EM* <u>Pressure</u> = $\boxed{+\frac{1}{2}\varepsilon_o E_o^2}$ in $\hat{x}$-direction !!!

$\boxed{T^{22} = -T_{yy} = +\frac{1}{2}\varepsilon_o E_o^2} \Rightarrow$ *+ve EM* <u>Pressure</u> = $\boxed{+\frac{1}{2}\varepsilon_o E_o^2}$ in $\hat{y}$-direction !!!

$\boxed{T^{33} = -T_{zz} = -\frac{1}{2}\varepsilon_o E_o^2} \Rightarrow$ *−ve EM* <u>Pressure</u> = $\boxed{-\frac{1}{2}\varepsilon_o E_o^2}$ in $\hat{z}$-direction {n.b. $\vec{E} = E_o \hat{z}$ }!!!

$\Rightarrow$ Plates of capacitor attracted to each other – net attractive force acting on bottom/top plates:

<u>Tension</u>: $\boxed{\vec{F}_{bot} = +\sigma \ell w E_o \hat{z} = +Q E_o \hat{z}}$, $\boxed{\vec{F}_{top} = -\sigma \ell w E_o \hat{z} = -Q E_o \hat{z} = -\vec{F}_{bot}}$

Appendix A: Calculation of the Elements of the Relativistic Version of Maxwell's Stress Tensor

$$\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(F^{\mu\sigma} F^{\nu}{}_{\sigma} \right) - \frac{1}{4} \delta^{\mu\nu} \left(F_{\lambda\sigma} F^{\lambda\sigma} \right) \right]}$$

SI units of $T^{\mu\nu}$: energy density = $J/m^2 = N\text{-}m/m^2 = N/m^2$ = pressure

"flat" space-time metric tensor: $\boxed{\delta^{\mu\nu} \equiv \begin{cases} -1 \text{ for } \mu = \nu = 0 \\ \;\;0 \text{ for } \mu \neq \nu \\ +1 \text{ for } \mu = \nu = 1, 2 \text{ or } 3 \end{cases}}$ i.e. $\boxed{\delta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}}$

Anti-symmetric *EM* field tensor, $F^{\mu\nu}$: $\boxed{F^{\mu\nu} = -F^{\nu\mu}}$

Column # ν

Column #

Row #

$$\boxed{F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} \end{array}} \begin{array}{c} \text{Row \# } \mu \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

First, we need to calculate: $\boxed{F_{\lambda\sigma} F^{\lambda\sigma} = F^{\lambda\sigma} F_{\lambda\sigma}}$ where: $\boxed{F^{\mu\nu} = F^{\text{row \# column \#}}}$

$F^{\mu}{}_{\nu}$ means that we place a minus (−) sign in front of the $\underline{\underline{\nu = 0}}$ temporal components, i.e. elements in the first horizontal row ($\underline{\underline{\mu = 0}}$) of $F^{\mu\nu}$.

$F_{\mu}{}^{\nu}$ means that we place a minus (−) sign in front of the $\underline{\underline{\mu = 0}}$ temporal components, i.e. elements in the first vertical column ($\underline{\underline{\nu = 0}}$) of $F^{\mu\nu}$.

$F_{\mu\nu}$ means that we place a minus (−) sign in front of ***both*** of the $\underline{\underline{\nu = 0}}$ temporal components, elements in the first horizontal row ($\underline{\underline{\mu = 0}}$) .**AND**. the $\underline{\underline{\mu = 0}}$ temporal components, elements in the first vertical column ($\underline{\underline{\nu = 0}}$) of $F^{\mu\nu}$.

$$\therefore \quad \boxed{\begin{aligned} F_{\lambda\sigma} F^{\lambda\sigma} = &+F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} \\ &-F^{10}F^{10} + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} \\ &-F^{20}F^{20} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} \\ &-F^{30}F^{30} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \end{aligned}}$$

$$\begin{array}{l} F_{\lambda\sigma}F^{\lambda\sigma} = +0^2 \quad -\left(E_x/c\right)^2 - \left(E_y/c\right)^2 - \left(E_z/c\right)^2 \\ \quad -\left(-E_x/c\right)^2 \quad +0^2 \quad +\left(B_z\right)^2 \quad +\left(-B_y\right)^2 \\ \quad -\left(-E_y/c\right)^2 \quad +\left(B_z\right)^2 \quad +0^2 \quad +\left(B_x\right)^2 \\ \quad -\left(-E_z/c\right)^2 \quad +\left(B_y\right)^2 \quad +\left(-B_x\right)^2 \quad +0^2 \end{array} \Rightarrow \begin{array}{l} F_{\lambda\sigma}F^{\lambda\sigma} = -\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z}{c^2} \\ \quad -\frac{E_x^2}{c^2} + B_z^2 + B_y^2 \\ \quad -\frac{E_y^2}{c^2} + B_z^2 + B_x^2 \\ \quad -\frac{E_z^2}{c^2} + B_y^2 + B_x^2 \end{array}$$

$$F_{\lambda\sigma}F^{\lambda\sigma} = 2\left[-\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + \left(B_x^2 + B_y^2 + B_z^2\right)\right] = 2\left(-\frac{E^2}{c^2} + B^2\right) = 2\left(B^2 - \frac{E^2}{c^2}\right) = -2\left(\frac{E^2}{c^2} - B^2\right)$$

$$\therefore \quad \boxed{\left(F_{\lambda\sigma}F^{\lambda\sigma}\right) = \left(F^{\lambda\sigma}F_{\lambda\sigma}\right) = 2\left(B^2 - \frac{E^2}{c^2}\right) = -2\left(\frac{E^2}{c^2} - B^2\right)}$$

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Next, we need to calculate $\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}\right)$ which is a 4 × 4 = 16 element tensor $= S^{\mu\nu}$

n.b. repeated indices (σ) are summed over $\sigma = 0:3$ for <u>each</u> element of $S^{\mu\nu}$!!!

<u>Calculate the 16 elements of 4 × 4 tensor</u>: $\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma}}$

<u>$\nu = 0$ (1st column) of $S^{\mu\sigma}$ $(\mu = 0:3)$</u>:

$$S^{00} \equiv F^{0\sigma}F^{0}{}_{\sigma} = -F^{00}F^{00} + F^{01}F^{01} + F^{02}F^{02} + F^{03}F^{03} = -0^2 + \left(\frac{E_x}{c}\right)^2 + \left(\frac{E_y}{c}\right)^2 + \left(\frac{E_z}{c}\right)^2 = \frac{E^2}{c^2}$$

$$\begin{aligned} S^{10} &\equiv F^{1\sigma}F^{0}{}_{\sigma} = -F^{10}F^{00} + F^{11}F^{01} + F^{12}F^{02} + F^{13}F^{03} \\ &= -\left(-\frac{E_x}{c}\right)(0) + (0)\left(\frac{E_x}{c}\right) + B_z\left(\frac{E_y}{c}\right) + \left(-B_y\right)\left(\frac{E_z}{c}\right) = \frac{1}{c}\left(E_yB_z - E_zB_y\right) \end{aligned}$$

$$\begin{aligned} S^{20} &\equiv F^{2\sigma}F^{0}{}_{\sigma} = -F^{20}F^{00} + F^{21}F^{01} + F^{22}F^{02} + F^{23}F^{03} \\ &= -\left(-\frac{E_y}{c}\right)(0) + \left(-B_z\right)\left(\frac{E_x}{c}\right) + (0)\left(\frac{E_y}{c}\right) + \left(B_x\right)\left(\frac{E_z}{c}\right) = \frac{1}{c}\left(E_zB_x - E_xB_z\right) \end{aligned}$$

$$\begin{aligned} S^{30} &\equiv F^{3\sigma}F^{0}{}_{\sigma} = -F^{30}F^{00} + F^{31}F^{01} + F^{22}F^{02} + F^{23}F^{03} \\ &= -\left(-\frac{E_z}{c}\right)(0) + \left(B_y\right)\left(\frac{E_x}{c}\right) + \left(-B_x\right)\left(\frac{E_y}{c}\right) + (0)\left(\frac{E_z}{c}\right) = \frac{1}{c}\left(E_xB_y - E_yB_x\right) \end{aligned}$$

$\nu = 1$ (2nd column) of $S^{\mu\sigma}$ $(\mu = 0:3)$:

$$\boxed{\begin{aligned} S^{01} &\equiv F^{0\sigma}F^{1}{}_{\sigma} = -F^{00}F^{10} + F^{01}F^{11} + F^{02}F^{12} + F^{03}F^{13} \\ &= -(0)\left(-\frac{E_x}{c}\right) + \left(\frac{E_x}{c}\right)(0) + \left(\frac{E_y}{c}\right)(B_z) + \left(\frac{E_z}{c}\right)(-B_y) = \frac{1}{c}\left(E_y B_z - E_z B_y\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{11} &\equiv F^{1\sigma}F^{1}{}_{\sigma} = -F^{10}F^{10} + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} \\ &= -\left(-\frac{E_x}{c}\right)\left(-\frac{E_x}{c}\right) + (0)(0) + (B_z)(B_z) + (-B_y)(-B_y) = -\frac{E_x^2}{c^2} + B_y^2 + B_z^2 \end{aligned}}$$

$$\boxed{\begin{aligned} S^{21} &\equiv F^{2\sigma}F^{1}{}_{\sigma} = -F^{20}F^{10} + F^{21}F^{11} + F^{22}F^{12} + F^{23}F^{13} \\ &= -\left(-\frac{E_y}{c}\right)\left(-\frac{E_x}{c}\right) + (-B_z)(0) + (0)(B_z) + (B_x)(-B_y) = -\left(\frac{E_x E_y}{c^2} + B_x B_y\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{31} &\equiv F^{3\sigma}F^{1}{}_{\sigma} = -F^{30}F^{10} + F^{31}F^{11} + F^{32}F^{12} + F^{33}F^{13} \\ &= -\left(-\frac{E_z}{c}\right)\left(-\frac{E_x}{c}\right) + (B_y)(0) + (-B_x)(B_z) + (0)(-B_y) = -\left(\frac{E_x E_z}{c^2} + B_x B_z\right) \end{aligned}}$$

$\nu = 2$ (3rd column) of $S^{\mu\sigma}$ $(\mu = 0:3)$:

$$\boxed{\begin{aligned} S^{02} &\equiv F^{0\sigma}F^{2}{}_{\sigma} = -F^{00}F^{20} + F^{01}F^{21} + F^{02}F^{22} + F^{03}F^{23} \\ &= -(0)\left(-\frac{E_y}{c}\right) + \left(\frac{E_x}{c}\right)(-B_z) + \left(\frac{E_y}{c}\right)(0)\left(\frac{E_z}{c}\right)(B_x) = \frac{1}{c}\left(E_z B_x - E_x B_z\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{12} &\equiv F^{1\sigma}F^{2}{}_{\sigma} = -F^{10}F^{20} + F^{11}F^{21} + F^{12}F^{22} + F^{13}F^{23} \\ &= -\left(-\frac{E_x}{c}\right)\left(-\frac{E_y}{c}\right) + (0)(-B_z) + (B_z)(0) + (-B_y)(B_x) = -\left(\frac{E_x E_y}{c^2} + B_x B_y\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{22} &\equiv F^{2\sigma}F^{2}{}_{\sigma} = -F^{20}F^{20} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} \\ &= -\left(-\frac{E_y}{c}\right)\left(-\frac{E_y}{c}\right) + (-B_z)(-B_z) + (0)(0) + (B_x)(B_x) = -\frac{E_y^2}{c^2} + B_x^2 + B_z^2 \end{aligned}}$$

$$\boxed{\begin{aligned} S^{23} &\equiv F^{3\sigma}F^{2}{}_{\sigma} = -F^{30}F^{20} + F^{31}F^{21} + F^{32}F^{22} + F^{33}F^{23} \\ &= -\left(-\frac{E_z}{c}\right)\left(-\frac{E_y}{c}\right) + (B_y)(-B_z) + (-B_x)(0) + (0)(B_x) = -\left(\frac{E_y E_z}{c^2} + B_y B_z\right) \end{aligned}}$$

$\nu = 3$ (4th column) of $S^{\mu\sigma}$ $(\mu = 0:3)$:

$$\boxed{\begin{aligned} S^{03} &\equiv F^{0\sigma}F^{3}{}_{\sigma} = -F^{00}F^{30} + F^{01}F^{31} + F^{02}F^{32} + F^{03}F^{33} \\ &= -(0)\left(-\frac{E_z}{c}\right) + \left(\frac{E_x}{c}\right)(B_y) + \left(\frac{E_y}{c}\right)(-B_x) + \left(\frac{E_z}{c}\right)(0) = \frac{1}{c}\left(E_x B_y - E_y B_x\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{13} &\equiv F^{1\sigma}F^{3}{}_{\sigma} = -F^{10}F^{30} + F^{11}F^{31} + F^{12}F^{32} + F^{13}F^{33} \\ &= -\left(-\frac{E_x}{c}\right)\left(\frac{E_z}{c}\right) + (0)(B_y) + (B_z)(-B_x) + (-B_y)(0) = -\left(\frac{E_x E_z}{c^2} + B_x B_z\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{23} &\equiv F^{2\sigma}F^{3}{}_{\sigma} = -F^{20}F^{30} + F^{21}F^{31} + F^{22}F^{32} + F^{23}F^{33} \\ &= -\left(-\frac{E_y}{c}\right)\left(\frac{E_z}{c}\right) + (-B_z)(B_y) + (0)(-B_x) + (B_x)(0) = -\left(\frac{E_y E_z}{c^2} + B_y B_z\right) \end{aligned}}$$

$$\boxed{\begin{aligned} S^{33} &\equiv F^{3\sigma}F^{3}{}_{\sigma} = -F^{30}F^{30} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \\ &= -\left(-\frac{E_z}{c}\right)\left(-\frac{E_z}{c}\right) + (B_y)(B_y) + (-B_x)(-B_x) + (0)(0) = -\frac{E_z^2}{c^2} + B_x^2 + B_y^2 \end{aligned}}$$

Thus, collecting our results for the {intermediate} tensor $\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma}}$:

$$\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma} = \begin{pmatrix} S^{00} & S^{01} & S^{02} & S^{03} \\ S^{10} & S^{11} & S^{12} & S^{13} \\ S^{20} & S^{21} & S^{22} & S^{23} \\ S^{30} & S^{31} & S^{32} & S^{33} \end{pmatrix}}$$

$$\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma} = \begin{pmatrix} \frac{E^2}{c^2} & \frac{1}{c}\left(E_y B_z - E_z B_y\right) & \frac{1}{c}\left(E_z B_x - E_x B_z\right) & \frac{1}{c}\left(E_x B_y - E_y B_x\right) \\ \frac{1}{c}\left(E_y B_z - E_z B_y\right) & -\frac{E_x^2}{c^2} + B_y^2 + B_z^2 & -\left(\frac{E_x E_y}{c^2} + B_x B_y\right) & -\left(\frac{E_x E_z}{c^2} + B_x B_z\right) \\ \frac{1}{c}\left(E_z B_x - E_x B_z\right) & -\left(\frac{E_x E_y}{c^2} + B_x B_y\right) & -\frac{E_y^2}{c^2} + B_x^2 + B_z^2 & -\left(\frac{E_y E_z}{c^2} + B_y B_z\right) \\ \frac{1}{c}\left(E_x B_y - E_y B_x\right) & -\left(\frac{E_x E_z}{c^2} + B_x B_z\right) & -\left(\frac{E_y E_z}{c^2} + B_y B_z\right) & -\frac{E_z^2}{c^2} + B_x^2 + B_y^2 \end{pmatrix}}$$

Note that $S^{\mu\nu}$ is a symmetric matrix, i.e. $S^{\mu\nu} = S^{\nu\mu}$ {which is good, because $T^{\mu\nu}$ is symmetric!}

$\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma}}$ and: $\boxed{S^{\nu\mu} \equiv F^{\nu\sigma}F^{\mu}{}_{\sigma}}$

$\therefore$ $\boxed{F^{\mu\sigma}F^{\nu}{}_{\sigma} = F^{\nu\sigma}F^{\mu}{}_{\sigma}}$ and since each $\boxed{F^{\mu\sigma}F^{\nu}{}_{\sigma}}$, or equivalently $\boxed{F^{\nu\sigma}F^{\mu}{}_{\sigma}}$ is the sum of 4 terms $(\sigma = 0:3)$ then we may also write these as: $\boxed{F^{\mu\nu}F^{\nu}{}_{\sigma} = F^{\nu}{}_{\sigma}F^{\mu\sigma}}$ and: $\boxed{F^{\nu\sigma}F^{\mu}{}_{\sigma} = F^{\mu}{}_{\sigma}F^{\nu\sigma}}$.

$$\therefore \boxed{S^{\mu\nu} = S^{\nu\mu} = F^{\mu\sigma}F^{\nu}{}_{\sigma} = F^{\nu\sigma}F^{\mu}{}_{\sigma} = F^{\nu}{}_{\sigma}F^{\mu\sigma} = F^{\mu}{}_{\sigma}F^{\nu\sigma}}$$

Then:
$$\boxed{T^{\mu\nu} \equiv \varepsilon_0 c^2\left[\underbrace{\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}\right)}_{\equiv S^{\mu\nu}} - \frac{1}{4}\delta^{\mu\nu}\underbrace{\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)}_{= -2\left(\frac{E^2}{c^2} - B^2\right)}\right]}$$
where:
$$\boxed{\delta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}}$$

$$\boxed{S^{\mu\nu} \equiv F^{\mu\sigma}F^{\nu}{}_{\sigma} = \begin{pmatrix} \frac{E^2}{c^2} & \frac{1}{c}\left(E_yB_z - E_zB_y\right) & \frac{1}{c}\left(E_zB_x - E_xB_z\right) & \frac{1}{c}\left(E_xB_y - E_yB_x\right) \\ \frac{1}{c}\left(E_yB_z - E_zB_y\right) & -\frac{E_x^2}{c^2} + B_y^2 + B_z^2 & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) \\ \frac{1}{c}\left(E_zB_x - E_xB_z\right) & -\left(\frac{E_xE_y}{c^2} + B_xB_y\right) & -\frac{E_y^2}{c^2} + B_x^2 + B_z^2 & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) \\ \frac{1}{c}\left(E_xB_y - E_yB_x\right) & -\left(\frac{E_xE_z}{c^2} + B_xB_z\right) & -\left(\frac{E_yE_z}{c^2} + B_yB_z\right) & -\frac{E_z^2}{c^2} + B_x^2 + B_y^2 \end{pmatrix}}$$

Note that since $\delta^{\mu\nu}$ only contributes to the diagonal elements of $T^{\mu\nu}$, let us compute these:

$$\boxed{\begin{aligned} T^{00} &= \varepsilon_o c^2\left[S^{00} - \frac{1}{4}2\left(\frac{E^2}{c^2} - B^2\right)\right] = \varepsilon_o c^2\left[\frac{E^2}{c^2} - \frac{1}{2}\left(\frac{E^2}{c^2} - B^2\right)\right] \\ &= \varepsilon_o c^2\left[\frac{1}{2}\frac{E^2}{c^2} + \frac{1}{2}B^2\right] = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2}\varepsilon_o c^2 B^2 = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_o}B^2 = u_{EM} \end{aligned}}$$

$$\boxed{\begin{aligned} T^{11} &= \varepsilon_o c^2\left[S^{11} + \frac{1}{4}2\left(\frac{E^2}{c^2} - B^2\right)\right] = \varepsilon_o c^2\left[-\frac{E_x^2}{c^2} + B_y^2 + B_z^2 + \frac{1}{2}\frac{E^2}{c^2} - \frac{1}{2}B^2\right] \\ &= \varepsilon_o c^2\left[-\frac{E_x^2}{c^2} + \frac{1}{2}\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + B_y^2 + B_z^2 - \frac{1}{2}\left(B_x^2 + B_y^2 + B_z^2\right)\right] \\ &= \varepsilon_o c^2\left[\frac{1}{2}\left(-\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + \frac{1}{2}\left(-B_x^2 + B_y^2 + B_z^2\right)\right] = \frac{\varepsilon_o c^2}{2}\left[\left(-\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right)\right] \\ &= \frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + \left(-B_x^2 + B_y^2 + B_z^2\right)\right] \\ &= -\frac{1}{2}\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2}\right) + \left(B_x^2 - B_y^2 - B_z^2\right)\right] = -\varepsilon_o c^2\left[\left(\frac{E_x^2}{c^2} + B_x^2\right) - \frac{1}{2}\left(\frac{E^2}{c^2} + B^2\right)\right] \end{aligned}}$$

From the above elements S^{11}, S^{22}, S^{33} and <u>their cyclic permutation symmetries</u> we see that:

$$\boxed{T^{22} = -\frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}-\frac{E_z^2}{c^2}\right)+\left(-B_x^2+B_y^2-B_z^2\right)\right] = -\varepsilon_o\left[\left(\frac{E_y^2}{c^2}+B_y^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]}$$

<u>and</u>:

$$\boxed{T^{33} = -\frac{1}{2}\varepsilon_o c^2\left[\left(-\frac{E_x^2}{c^2}-\frac{E_y^2}{c^2}+\frac{E_z^2}{c^2}\right)+\left(-B_x^2-B_y^2+B_z^2\right)\right] = -\varepsilon_o\left[\left(\frac{E_z^2}{c^2}+B_z^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right]}$$

Then for <u>all other remaining</u> elements of the relativistic version of Maxwell's stress tensor, $T^{\mu\nu}\big|_{\mu\neq\nu}$, we see that since: $\boxed{\delta^{\mu\nu}\big|_{\mu\neq\nu}=0}$ <u>then</u>: $\boxed{T^{\mu\nu}\big|_{\mu\neq\nu} = \varepsilon_o c^2\left(F^{\mu\sigma}F_\sigma^{\ \nu}\right)\big|_{\mu\neq\nu} = \varepsilon_o c^2 S^{\mu\nu}\big|_{\mu\neq\nu}}$

<u>Thus</u>:

$$\boxed{T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\varepsilon_o c^2\left(\frac{E^2}{c^2}+B^2\right)=u_{EM} & \varepsilon_o c\left(E_yB_z-E_zB_y\right) & \varepsilon_o c\left(E_zB_x-E_xB_z\right) & \varepsilon_o c\left(E_xB_y-E_yB_x\right) \\ \varepsilon_o c\left(E_yB_z-E_zB_y\right) & -\varepsilon_o c\left[\left(\frac{E_x^2}{c^2}+B_x^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right] \\ \varepsilon_o c\left(E_zB_x-E_xB_z\right) & -\varepsilon_o c^2\left[\left(\frac{E_xE_y}{c^2}\right)+B_xB_y\right] & -\varepsilon_o c\left[\left(\frac{E_y^2}{c^2}+B_y^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right] \\ \varepsilon_o c\left(E_xB_y-E_yB_x\right) & -\varepsilon_o c^2\left[\left(\frac{E_xE_z}{c^2}\right)+B_xB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_yE_z}{c^2}\right)+B_yB_z\right] & -\varepsilon_o c^2\left[\left(\frac{E_z^2}{c^2}+B_z^2\right)-\frac{1}{2}\left(\frac{E^2}{c^2}+B^2\right)\right] \end{pmatrix}}$$

Now Poynting's vector:

$$\boxed{\begin{aligned}\vec{S} &= \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) = \varepsilon_o c^2\left(\vec{E}\times\vec{B}\right) = \varepsilon_o c^2\left[\left(\vec{E}\times\vec{B}\right)_x\hat{x}+\left(\vec{E}\times\vec{B}\right)_y\hat{y}+\left(\vec{E}\times\vec{B}\right)_z\hat{z}\right] \\ &= \varepsilon_o c^2\left[\left(E_yB_z-E_zB_y\right)\hat{x}+\left(E_zB_x-E_xB_z\right)\hat{y}+\left(E_xB_y-E_yB_x\right)\hat{z}\right]\end{aligned}}$$

<u>Thus</u>: $\boxed{\frac{1}{c}\vec{S} = \varepsilon_o c\left[\left(E_yB_z-E_zB_y\right)\hat{x}+\left(E_zB_x-E_xB_z\right)\hat{y}+\left(E_xB_y-E_yB_x\right)\hat{z}\right]}$

But relativistic linear momentum density $\boxed{\vec{\wp} = \frac{1}{c^2}\vec{S}}$ <u>or</u>: $\boxed{\vec{\wp}c = \frac{1}{c}\vec{S}}$. Thus, we see that:

$$\boxed{T^{00} = u_{EM} \quad T^{01} = \frac{1}{c}S_x = \wp_x c = T^{10} \quad T^{02} = \frac{1}{c}S_y = \wp_y c = T^{20} \quad T^{03} = \frac{1}{c}S_z = \wp_z c = T^{30}}$$

$$\boxed{\begin{aligned} T^{10} &= \wp_x c = \frac{1}{c}S_x = T^{01} \\ T^{20} &= \wp_y c = \frac{1}{c}S_y = T^{02} \\ T^{30} &= \wp_z c = \frac{1}{c}S_z = T^{03}\end{aligned}}$$

Thus: $\boxed{T^{\mu\nu} \equiv \varepsilon_o c^2 \left[\left(F^{\mu\sigma} F^{\nu}{}_{\sigma} \right) - \frac{1}{4} \delta^{\mu\nu} \left(F_{\lambda\sigma} F^{\lambda\sigma} \right) \right]}$ gives:

$$T^{\mu\nu} = \begin{pmatrix} u_{EM} & \frac{1}{c} S_x & \frac{1}{c} S_y & \frac{1}{c} S_z \\ \wp_x c & -\varepsilon_o c \left[\left(\frac{E_x^2}{c^2} + B_x^2 \right) - \frac{1}{2} \left(\frac{E^2}{c^2} + B^2 \right) \right] & -\varepsilon_o c^2 \left[\left(\frac{E_x E_y}{c^2} \right) + B_x B_y \right] & -\varepsilon_o c^2 \left[\left(\frac{E_x E_z}{c^2} \right) + B_x B_z \right] \\ \wp_y c & -\varepsilon_o c^2 \left[\left(\frac{E_x E_y}{c^2} \right) + B_x B_y \right] & -\varepsilon_o c \left[\left(\frac{E_y^2}{c^2} + B_y^2 \right) - \frac{1}{2} \left(\frac{E^2}{c^2} + B^2 \right) \right] & -\varepsilon_o c^2 \left[\left(\frac{E_y E_z}{c^2} \right) + B_y B_z \right] \\ \wp_z c & -\varepsilon_o c^2 \left[\left(\frac{E_x E_z}{c^2} \right) + B_x B_z \right] & -\varepsilon_o c^2 \left[\left(\frac{E_y E_z}{c^2} \right) + B_y B_z \right] & -\varepsilon_o c^2 \left[\left(\frac{E_z^2}{c^2} + B_z^2 \right) - \frac{1}{2} \left(\frac{E^2}{c^2} + B^2 \right) \right] \end{pmatrix}$$

The classical electrodynamics 3-D force *density* is: $\boxed{\vec{f} = \vec{\nabla} \cdot \vec{T} - \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} = -\frac{\partial \left(\frac{1}{c} \vec{S} \right)}{\partial (ct)} + \vec{\nabla} \cdot \vec{T}}$ SI units: N/m^3

where: $\boxed{\vec{f} = f_x \hat{x} + f_y \hat{y} + f_z \hat{z}}$

But: $\boxed{T^{\mu\nu}\big|_{\mu,\nu=1:3} = -\vec{T}_{ij}\big|_{i,j=1:3}}$ and: $-\frac{\partial \left(\frac{1}{c} \vec{S} \right)}{\partial (ct)} = -\frac{\partial \left(\frac{1}{c} S_x \hat{x} + \frac{1}{c} S_y \hat{y} + \frac{1}{c} S_z \hat{z} \right)}{\partial x^0} = -\frac{\partial \left(T^{01} \hat{x} + T^{02} \hat{y} + T^{03} \hat{z} \right)}{\partial x^0}$

The relativistic electrodynamics force *density* 4-vector is: $\boxed{f^{\nu} = \left(f^0, \vec{f} \right)}$ (SI units: N/m^3)

The zeroth/temporal/scalar component of the 4-vector f^{ν} is: $f^0 = -\frac{\partial \left(T^{00} \right)}{\partial x^0} = -\frac{\partial (u_{EM})}{\partial (ct)}$

Thus, the relativistic electrodynamics force *density* 4-vector is:

$$\boxed{f^{\nu} = -\partial_{\mu} T^{\mu\nu} = -\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = -\left(\frac{\partial T^{0\nu}}{\partial x^0} + \frac{\partial T^{1\nu}}{\partial x^1} + \frac{\partial T^{2\nu}}{\partial x^2} + \frac{\partial T^{3\nu}}{\partial x^3} \right)}$$

where: $\boxed{\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}}$ is the covariant 4-D ***gradient*** operator.

The *total* relativistic force 4-vector can likewise be obtained by noting that the 3-D spatial classical electrodynamics version of the 3-D total force vector is:

$$\boxed{\vec{F} = \int_v \vec{f} \, d\tau = \int_v \left(-\frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} + \vec{\nabla} \cdot \vec{T} \right) = \int_v \left(= \frac{\partial \left(\frac{1}{c} \vec{S} \right)}{\partial (ct)} + \vec{\nabla} \cdot \vec{T} \right) d\tau}$$

Thus: $\boxed{F^{\nu} = \int_{v_4} f^{\nu} \, d\tau_4 = -\int_{v_4} \partial_{\mu} T^{\mu\nu} \, d\tau_4 = -\int_{v_4} \left(\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} \right) d\tau_4}$ where: $\boxed{F^{\nu} = \left(F^0, \vec{F} \right)}$ and:

The 4-D space-time volume v_4 has volume element $d\tau_4 = cdt \; dxdydz$ $\boxed{\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}}$

If one wishes to transform these results in one IRF(S) to another IRF(S') moving with relative velocity $\vec{v}$ with respect to IRF(S), there are two equivalent methods to accomplish this task:

Method I:

First, Lorentz transform the $\vec{E}$ and $\vec{B}$ fields via:

$$\boxed{\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma\left(1-\left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) & \gamma\vec{\beta}\times \\ -\gamma\vec{\beta}\times & \gamma\left(1-\left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) \end{pmatrix}}_{\text{n.b. operator matrix}} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}} \text{ where: } \boxed{\begin{matrix}\gamma = \frac{1}{\sqrt{1-\beta^2}} \\ \vec{\beta} = \frac{\vec{v}}{c}\end{matrix}}$$

See/read P436 Lecture Notes 19 for further details of this method.

Then compute the new $F'^{\mu\nu}$, $S'^{\mu\nu} = F'^{\mu\sigma}F'^{\nu}_{\sigma}$ and $F'_{\lambda\sigma}F'^{\lambda\sigma}$ and thus compute the new $T'^{\mu\nu}$ in IRF(S'):

$$\boxed{T'^{\mu\nu} = \varepsilon_0 c^2\left[\left(F'^{\mu\sigma}F'^{\nu}_{\sigma}\right) - \frac{1}{2}\delta^{\mu\nu}\left(F'_{\lambda\sigma}F'^{\lambda\sigma}\right)\right]}, \quad \boxed{f'^{\nu} = -\frac{\partial T'^{\mu\nu}}{\partial x^{\mu}}}, \quad \boxed{F'^{\nu} = \int_{v_4} f'^{\nu} d\tau_4 = -\int_{v_4}\left(\frac{\partial T'^{\mu\nu}}{\partial x^{\mu}}\right)d\tau_4}$$

This method has the advantage that all quantities, e.g. $\vec{E}', \vec{B}', \vec{S}', \vec{\wp}'$, $F'^{\mu\nu}$, $S'^{\mu\nu} = F'^{\mu\sigma}F'^{\nu}_{\sigma}$, $T'^{\mu\nu}$, f'^{ν} and F'^{ν} are explicitly known/calculated in IRF(S').

Method II:

Lorentz transform $T^{\mu\nu}$ directly, since the Lorentz transformation of $F^{\mu\nu}$ in IRF(S) to $F'^{\mu\nu}$ in IRF(S') moving with velocity $\vec{v}$ relative to IRF(S) is given by: $\boxed{F'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\sigma}F^{\mu\nu}}$

Then: $\boxed{T'^{\mu\nu} = \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\sigma}T^{\mu\nu}}$

Note that since: $\boxed{T^{\mu\nu} = \varepsilon_o c^2\left[\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}\right) - \tfrac{1}{4}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right] = \varepsilon_o c^2\left[S^{\mu\nu} - \tfrac{1}{4}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right]}$

Then: $\boxed{T'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}T^{\mu\nu} = \varepsilon_o c^2\left[\Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}S^{\mu\nu} - \tfrac{1}{4}\Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}\delta^{\mu\nu}\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right]}$

Recall that: $\boxed{\left(F_{\lambda\sigma}F^{\lambda\sigma}\right) = \left(F^{\lambda\sigma}F_{\lambda\sigma}\right) = -2\left(\frac{E^2}{c^2} - B^2\right)}$ = relativistic invariant

i.e. $F_{\lambda\sigma}F^{\lambda\sigma} = F'_{\lambda\sigma}F'^{\lambda\sigma}$ = same value in any/all IRF'S!!!

Define: $\boxed{S'^{\mu\nu} \equiv \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}S^{\mu\nu}}$ and $\boxed{\delta'^{\mu\nu} \equiv \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}\delta^{\mu\nu}}$

Also note that ε_o and c^2, μ_o are relativistically invariant scalar quantities (same in all IRF's)

Then: $\boxed{T'^{\mu\nu} = \varepsilon_o c^2\left[S'^{\mu\nu} - \tfrac{1}{4}\delta'^{\mu\nu}\left(F'_{\lambda\sigma}F'^{\lambda\sigma}\right)\right]}$

Once $T'^{\mu\nu}$ is obtained, calculate $\boxed{f'^{\nu} = -\frac{\partial T'^{\mu\nu}}{\partial x^{\mu}}}$ and $\boxed{F'^{\nu} = \int_{v_4} f'^{\nu}\, d\tau_4 = -\int_{v_4}\left(\frac{\partial T'^{\mu\nu}}{\partial x^{\mu}}\right)d\tau_4}$