Classical, Non-Relativistic Theory of Scattering of Electromagnetic Radiation

We present here the theory of scattering of electromagnetic radiation from the classical, non-relativistic physics approach. A quantum-mechanical, fully-relativistic of this subject matter can also be obtained via the use of relativistic quantum electrodynamics {QED}, however, this more sophisticated level of treatment is simply beyond the scope of this course.

"Generic" Theoretical Definition of a Scattering Cross Section:

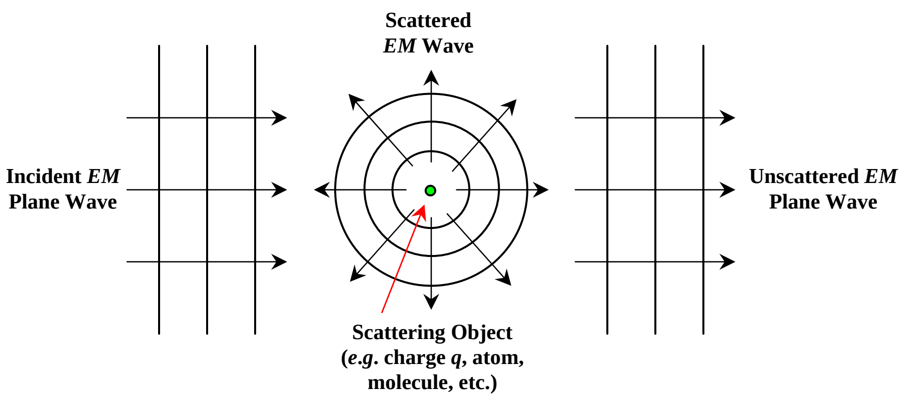
There are all sorts of scattering processes that occur in nature; generically all of them can be defined in terms of a scattering ***cross section*** σ_{scat}, which has SI units of area, *i.e.* m^2.

In classical, non-relativistic physics the **total** scattering cross section σ_{scat} for a given *EM* wave scattering process defined as the ratio of the total, time-averaged radiated power $\langle P_{rad}(t)\rangle$ (SI units: *Watts*) radiated by a "target" object (*e.g.* a charged particle, an atom or molecule, *etc*.) undergoing that scattering process, normalized to the incident intensity $I_{inc}(\vec{r}=0)=\langle|\vec{S}_{inc}(\vec{r}=0,t)|\rangle$ (SI units: $Watts/m^2$), evaluated at the location of the target/scattering object (usually at the origin, $\vartheta(\vec{r}=0)$):

Total Scattering Cross Section: $\boxed{\sigma_{scat}\equiv\dfrac{\langle P_{rad}(t)\rangle}{I_{inc}(\vec{r}=0)}=\dfrac{\langle P_{rad}(t)\rangle}{\langle|\vec{S}_{inc}(\vec{r}=0,t)|\rangle}}$ (SI units of area, *i.e.* m^2.)

This relation is known as the ***total*** scattering cross section – because it contains no ***angular*** (θ,φ) information about the nature of the scattering process – these have been integrated out.

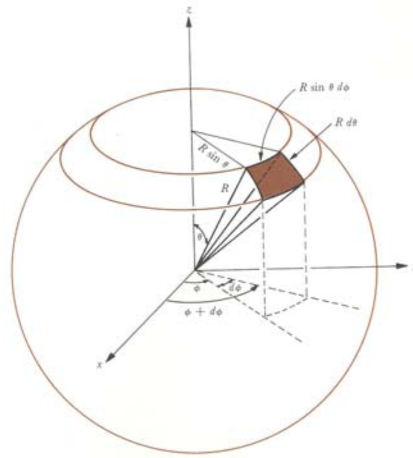
Physically speaking, the total scattering cross section can be thought of as an effective cross-sectional area (hence the name cross section) per scattering object of the incident wave front that is required to deliver the power that is scattered out of the incident wave front and into 4π steradians (*i.e.* into any/all angles θ,φ) as shown schematically in the figure below. The scattering object absorbs energy/power from the cross sectional area σ_{scat} in the incident *EM* wave and then re-radiates (*i.e.* scatters) this absorbed energy.



Note also that the scattering cross section is explicitly defined using time-averaged (rather than instantaneous) quantities in both the numerator and denominator in order to facilitate direct comparison between theoretical prediction(s) *vs*. experimental measurement(s).

The time-averaged power radiated into a differential/infinitesimal element of solid angle $\boxed{d\Omega = d\cos\theta d\varphi = \sin\theta d\theta d\varphi}$ (SI units: *steradians*; *aka sterad*, and/or *sr*) is:

$$\boxed{\frac{d\langle P_{rad}(\theta,\varphi,t)\rangle}{d\Omega} = \langle \vec{S}_{rad}(\vec{r},t)\rangle \cdot r^2\hat{r}} \text{ (SI units: } Watts/steradian = Watts/sr\text{).}$$



The infinitesimal vector area element $d\vec{a}_{sph} = da_{sph}\hat{r}$ associated with a sphere of radius r {centered on the scattering object - the "target" - located at the origin ϑ} is related to the differential solid angle element $d\Omega$ via $\boxed{d\vec{a}_{sph} = r^2 d\Omega\,\hat{r} = r^2 d\cos\theta d\varphi\,\hat{r} = r^2\sin\theta d\theta d\varphi\,\hat{r}}$. *EM* radiation scattered from the target object into $d\Omega$ passes through this area element $d\vec{a}_{sph}$. The figure on the right shows the relation between the differential solid angle element $\boxed{d\Omega = d\cos\theta d\varphi = \sin\theta d\theta d\varphi}$ and the infinitesimal area element $\boxed{da_{sph} = (rd\cos\theta)(rd\varphi) = (r\sin\theta d\theta)(rd\varphi) = r^2 d\Omega}$.

$\Rightarrow$ Note also that $\boxed{d\langle P_{rad}(\theta,\varphi,t)\rangle/d\Omega}$ has ***no*** r-dependence!

The ***differential*** angular scattering cross section is defined as:

$$\boxed{\frac{d\sigma_{scat}(\theta,\varphi)}{d\Omega} = \frac{d^2\sigma_{scat}(\theta,\varphi)}{d\cos\theta d\varphi} \equiv \frac{1}{\left\langle\left|\vec{S}_{inc}(\vec{r}=0,t)\right|\right\rangle}\frac{d\langle P_{rad}(\theta,\varphi,t)\rangle}{d\Omega} = \frac{1}{\left\langle\left|\vec{S}_{inc}(\vec{r}=0,t)\right|\right\rangle}\frac{d^2\langle P_{rad}(\theta,\varphi,t)\rangle}{d\cos\theta d\varphi}} \quad \text{SI units: } m^2/Sr$$

Note that the choice of {the usual} polar and azimuthal angles (θ,φ) to describe the two independent scattering angles means that the *EM* wave that is incident on the "target" object (free charge q, atom or molecule, *etc*.) is propagating in the $+\hat{z}$ direction.

It is also instructive to write out the differential scattering relation in more explicit detail:

$$\boxed{\frac{d\sigma_{scat}(\theta,\varphi)}{d\Omega} \equiv \frac{1}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle\Big|_{r=0}}\frac{d\langle P_{rad}(\theta,\varphi,t)\rangle}{d\Omega} = \frac{\langle\vec{S}_{rad}(\vec{r},t)\rangle\cdot r^2\hat{r}}{\langle\vec{S}_{inc}(\vec{r},t)\rangle\cdot\hat{z}\Big|_{r=0}} = \frac{\langle\vec{E}_{rad}(\vec{r},t)\times\vec{B}_{rad}(\vec{r},t)\rangle\cdot r^2\hat{r}}{\langle\vec{E}_{inc}(\vec{r},t)\times\vec{B}_{inc}(\vec{r},t)\rangle\cdot\hat{z}\Big|_{r=0}}} \quad \text{SI units: } m^2/Sr$$

As we derived in P436 Lecture Notes 14 (p. 1-5) from the Taylor series expansions of $\rho(\vec{r}',t_r)$ and $\vec{J}(\vec{r}',t_r)$ to first order in r', the only contribution to the *EM* power radiated is associated with a non-zero value of {some kind of/"generic"} time-varying electric dipole moment $\vec{p}(\vec{r}',t_r) = p(\vec{r}',t_r)\hat{z}$ {*n.b.* oriented parallel to the $\hat{z}$-axis}, where, in the "far-zone" limit {$\boxed{r'_{\max}/r \ll 1}$ and $\boxed{\omega r'_{\max}/c = kr'_{\max} = 2\pi r'_{\max}/\lambda \ll 1}$}, the instantaneous differential retarded *EM* power radiated by the time-varying E1 electric dipole, with retarded time $\boxed{t_r = t_o \equiv t - r/c}$ is:

$$\boxed{\frac{dP_r^{rad}(\theta,\varphi,t)}{d\Omega} = \vec{S}^{rad}(\vec{r},t)\cdot r^2\hat{r} \simeq \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2 c}\sin^2\theta} \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

The accompanying retarded electric and magnetic fields, *EM* energy density, Poynting's vector associated with the radiating E1 electric dipole, in the "far-zone" limit are:

$$\boxed{\vec{E}_{\mathrm{r}}(\vec{r},t) \simeq \frac{\mu_o \ddot{p}(t_o)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}}, \quad \boxed{\vec{B}_{\mathrm{r}}(\vec{r},t) \simeq \frac{\mu_o \ddot{p}(t_o)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\varphi}} \text{ with } \boxed{\vec{B}_{\mathrm{r}}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}_{\mathrm{r}}(\vec{r},t)}$$

$$i.e. \ \boxed{\vec{B}_{\mathrm{r}} \perp \vec{E}_{\mathrm{r}} \perp \hat{r} \ \{\| \hat{k}\}}$$

and: $$\boxed{u_{\mathrm{r}}^{rad}(\vec{r},t) \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)} \quad \text{and: } \boxed{\vec{S}_{\mathrm{r}}^{rad}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}_{\mathrm{r}}(\vec{r},t)\times\vec{B}_{\mathrm{r}}(\vec{r},t)}$$

$$\boxed{\vec{S}_{\mathrm{r}}^{rad}(\vec{r},t) \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\underbrace{(\hat{\theta}\times\hat{\varphi})}_{=\hat{r}} = \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} = \vec{c}\,u_{\mathrm{r}}^{rad}(\vec{r},t)} \text{ with } \boxed{\begin{array}{l}\vec{c} \equiv c\hat{r}\\ \hat{r} \,\|\, \hat{k}\end{array}}$$

The total <u>instantaneous</u> retarded *EM* power radiated by a time-varying E1 electric dipole into 4π steradians, with vector area element $\boxed{d\vec{a}_\perp = r^2\sin\theta d\theta d\varphi\hat{r} = r^2 d\Omega\hat{r}}$ in the "far-zone" limit is:

$$\boxed{P_{\mathrm{r}}^{rad}(t) = \int_S \vec{S}_{\mathrm{r}}^{rad}(\vec{r},t)\cdot d\vec{a}_\perp \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\sin\theta d\theta d\varphi = \frac{\mu_o \ddot{p}^2(t_o)}{6\pi^2 c}} \text{ (Watts)}$$

We can then take the necessary time-averages of the instantaneous flux of incident *EM* energy (*i.e*. Poynting's vector) and the instantaneous <u>differential</u> radiated *EM* power and/or <u>total</u> radiated *EM* power to use in the above cross section formulae.

<u>Scattering of *EM* Radiation from a *Free* Electric Point Charge: *Thomson Scattering*</u>

Suppose a <u>*free*</u> electric point charge *q* is located at the origin $\vartheta(x,y,z) = (0,0,0)$ with a plane *EM* wave propagating in free space in the $+\hat{z}$ direction and incident on the free charge *q*, which has mass *m*. Noting that $\vec{k}_{inc} = k\hat{k}_{inc}$ and that $\hat{k}_{inc} \,\|\, \hat{z}$ and using complex notation:

$$\boxed{\tilde{\vec{E}}_{inc}(\vec{r},t) = \tilde{\vec{E}}_o(\vec{r})e^{i(kz-\omega t)}}, \quad \boxed{\tilde{\vec{B}}_{inc}(\vec{r},t) = \tilde{\vec{B}}_o(\vec{r})e^{i(kz-\omega t)}} \text{ with } \boxed{\tilde{\vec{B}}_o(\vec{r}) = \frac{1}{c}\left(\hat{k}_{inc}\times\tilde{\vec{E}}_o(\vec{r})\right) = \frac{1}{c}\left(\hat{z}\times\tilde{\vec{E}}_o(\vec{r})\right)}$$

The <u>polarization</u> of the incident *EM* wave is (by convention) taken parallel to the $\tilde{\vec{E}}_o(\vec{r})$ direction. For definiteness' sake, let us assume the incident *EM* plane wave to be linearly polarized in the $\hat{x}$-direction, and for simplicity, let us also assume that $\boxed{\tilde{\vec{E}}_o(\vec{r}) = E_o\hat{x}}$, and therefore:

$$\boxed{\tilde{\vec{B}}_o(\vec{r}) = \frac{1}{c}\left(\hat{k}_{inc}\times\tilde{\vec{E}}_o(\vec{r})\right) = \frac{1}{c}\left(\hat{z}\times\tilde{\vec{E}}_o(\vec{r})\right) = \frac{E_o}{c}\underbrace{(\hat{z}\times\hat{x})}_{=-\hat{y}} = \frac{E_o}{c}(-\hat{y}) = -\frac{E_o}{c}\hat{y}}.$$

Poynting's vector for the incident *EM* plane wave is:

$$\boxed{\tilde{\vec{S}}_{inc}(\vec{r},t) = \frac{1}{\mu_o}\tilde{\vec{E}}_{inc}(\vec{r},t)\times\tilde{\vec{B}}_{inc}(\vec{r},t) = \frac{1}{\mu_o c}E_o^2 e^{2i(kz-\omega t)}\underbrace{(\hat{x}\times\hat{y})}_{=+\hat{z}} = +\varepsilon_o c E_o^2 e^{2i(kz-\omega t)}\hat{z}} \quad \text{(using } \varepsilon_o c = \frac{1}{\mu_o c}\text{)}$$

The *EM* plane wave incident on the free point electric charge q located at the origin ϑ causes the point charge q to accelerate/move, because two forces act on the point charge – an electric force and a magnetic/Lorentz force. Noting that the resulting time-dependent position of the point charge is at the source position $r' \simeq 0$ (*i.e.* in the neighborhood/vicinity of the origin ϑ):

$$\boxed{\tilde{\vec{F}}_{tot}(r' \simeq 0,t) = q\tilde{\vec{E}}_{inc}(r' \simeq 0,t) + q\vec{v}(r' \simeq 0,t)\times\tilde{\vec{B}}_{inc}(r' \simeq 0,t) = m\tilde{\vec{a}}(r' \simeq 0,t)}$$

The {transverse} electric field of the incident *EM* plane wave in the vicinity of $r' \simeq 0$ is:

$$\boxed{\tilde{\vec{E}}_{inc}(r' \simeq 0,t) = E_o e^{-i\omega t}\hat{x}}$$

The {transverse} magnetic field of the incident *EM* plane wave in the vicinity of $r' \simeq 0$ is:

$$\boxed{\tilde{\vec{B}}_{inc}(r' \simeq 0,t) = \frac{1}{c}\left(\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(r' \simeq 0,t)\right) = \frac{1}{c}\left(\hat{z}\times\tilde{\vec{E}}_{inc}(r' \simeq 0,t)\right) = \frac{1}{c}E_o e^{-i\omega t}\underbrace{(\hat{z}\times\hat{x})}_{=+\hat{y}} = +\frac{1}{c}E_o e^{-i\omega t}\hat{y}}$$

Thus: $$\boxed{\tilde{\vec{F}}_{tot}(\vec{r}' \simeq 0,t) = qE_o e^{-i\omega t}\hat{x} + qE_o e^{-i\omega t}\underbrace{\cancel{\frac{\vec{v}(\vec{r}' \simeq 0,t)\times\hat{y}}{c}}}_{(v/c)\ll 1} = m\tilde{\vec{a}}(\vec{r}' \simeq 0,t)}.$$

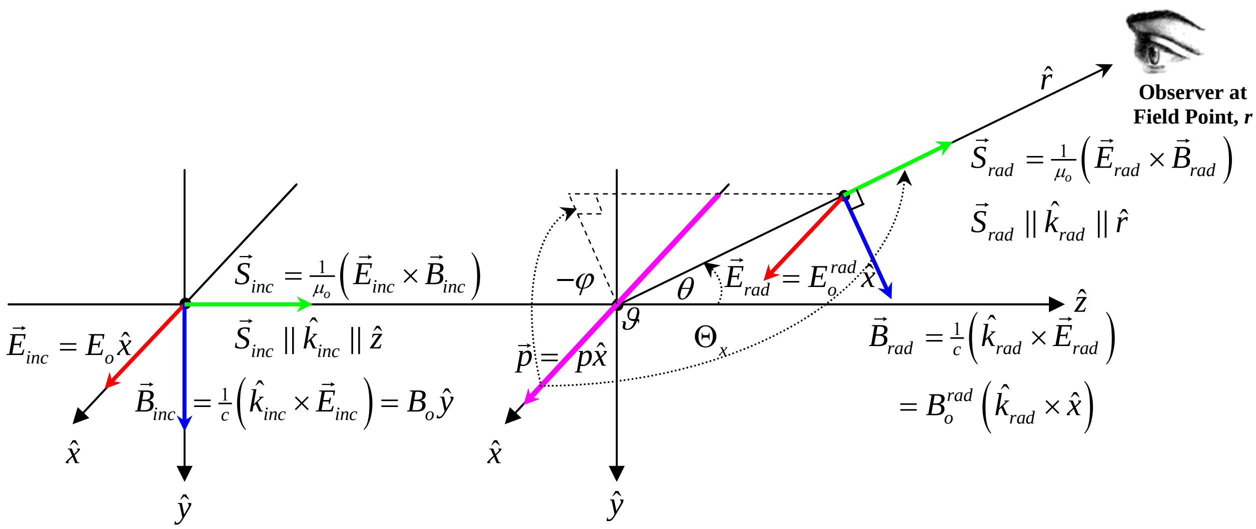
However, for ***non-relativistic*** scattering of an *EM* plane wave by a point electric charge q, where the motion of the charge is always such that $v \ll c$ $\left(\text{or: } \beta \equiv v/c \ll 1\right)$, the 2nd (magnetic) Lorentz force term will correspondingly always be small in comparison to the 1st (electric) force term, hence we will ***neglect*** the magnetic Lorentz force term in our treatment here.

Physically, the transverse electric field of the incident *EM* plane wave $\boxed{\tilde{\vec{E}}_{inc}(0,t) = E_o e^{-i\omega t}\hat{x}}$ exerts a time-dependent force $\boxed{\tilde{\vec{F}}_{tot}(0,t) = qE_o e^{-i\omega t}\hat{x} = m\tilde{\vec{a}}(0,t) = m\ddot{\tilde{\vec{x}}}(0,t) = -m\omega^2 x_o e^{-i\omega t}\hat{x}}$ on the free electric charge q, causing it to oscillate back and forth along the $\hat{x}$-axis in a time-dependent manner: $\boxed{\tilde{\vec{x}}(0,t) = x_o e^{-i\omega t}\hat{x}}$ with (real) amplitude x_o. Note that the wavelength of associated with the incident *EM* plane wave for ***non-relativistic*** scattering is such that the variation of the electric field strength in the vicinity of the free electric charge is negligible, *i.e.* that: $\boxed{x_o \ll \lambda = c/f}$.

Note also that the instantaneous acceleration of the charge q is $\boxed{\tilde{\vec{a}}(0,t) = \ddot{\tilde{\vec{x}}}(0,t) = -\omega^2 x_o e^{-i\omega t}\hat{x}}$.

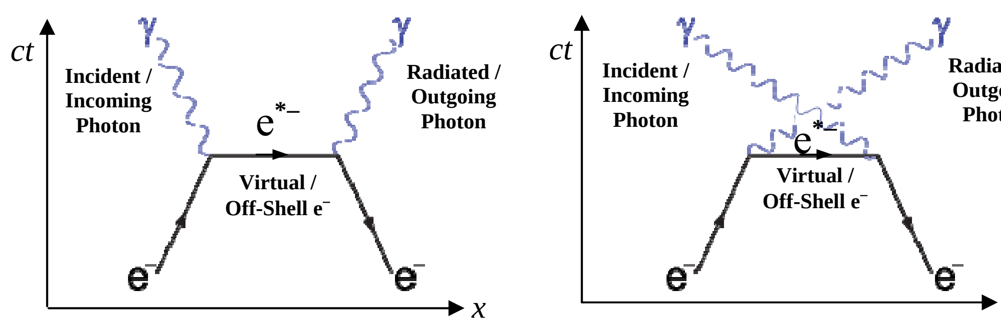
The *E*-field induced oscillatory motion of the point electric charge q thus creates an induced time-dependent electric dipole moment: $\boxed{\tilde{\vec{p}}(0,t) \equiv q\tilde{\vec{x}}(0,t) = qx_o e^{-i\omega t}\hat{x} = p_o e^{-i\omega t}\hat{x}}$ oriented parallel to the electric field $\boxed{\tilde{\vec{E}}_{inc}(0,t) = E_o e^{-i\omega t}\hat{x}}$ of the incident *EM* plane wave.

The induced oscillating electric dipole moment $\tilde{\vec{p}}(0,t)$ subsequently radiates electric dipole (E1) *EM* radiation. Energy from the incident *EM* wave is {temporarily} absorbed by the point charge q {this is necessary in order to get the charge q moving - *i.e.* to accelerate it}. The incident *EM* energy absorbed by the charge q is radiated a short time later. Thus, this overall process is one type of scattering of *EM* radiation! The geometrical setup for this situation is shown in the figure below:



Note also that the instantaneous mechanical power associated with the oscillating free charge q is $\boxed{\tilde{P}_{mech}(t) = \tilde{\vec{F}}_{tot}(0,t)\bullet\tilde{\vec{v}}(0,t)}$, arising from the absorption of energy from the incident EM wave by the free charge q. The velocity of the oscillating charge is $\boxed{\tilde{\vec{v}}(0,t) = d\tilde{\vec{x}}(0,t)/dt = -i\omega x_o e^{-i\omega t}\hat{x}}$ and hence $\boxed{\tilde{P}_{mech}(t) = \left(-m\omega^2 x_o e^{-i\omega t}\hat{x}\right)\bullet\left(-i\omega x_o e^{-i\omega t}\hat{x}\right) = im\omega^3 x_o e^{-2i\omega t}}$.

At the microscopic level, real photons of {angular} frequency ω associated with the incident EM wave (*e.g.* real photons in a laser beam that comprise the macroscopic EM wave output from the laser) are {temporarily} absorbed by the electric point charge q, and then re-emitted {a short time later} as {quantized} EM radiation {of the same frequency} – the emitted photons are associated with the outgoing, or *scattered* EM wave! Two space-time/Feynman diagrams showing examples of this QED scattering process for an electron are shown in the figure below:



The characteristic time interval Δt associated with the photon-free charge absorption-re-radiation process is governed by the Heisenberg uncertainty principle $\boxed{\Delta E \Delta t \le \frac{1}{2}\hbar}$ where $\boxed{\hbar \equiv h/2\pi}$ and h is Planck's constant, $\boxed{\hbar = 1.0546\times10^{-34}\ Joule/sec = 6.582\times10^{-16}\ eV/sec}$ {since $\boxed{1\,eV = 1\,electron\text{-}Volt = 1.602\times10^{-19}\ Joules}$}. For an incident photon *e.g.* with energy $\boxed{E_\gamma = hf_\gamma = hc/\lambda_\gamma = 1\,eV}$, since $\boxed{hc \simeq 1240\ eV\text{-}nm}$ this photon has a corresponding wavelength of $\boxed{\lambda_\gamma = 1240\ nm}$ {which is in the infra-red portion of the EM spectrum}. Then using $\boxed{\Delta E = E_\gamma = 1\,eV}$ in the Heisenberg uncertainty relation, we see that the corresponding time interval Δt is:

$$\boxed{\Delta t \le \tfrac{1}{2}\hbar/\Delta E = \tfrac{1}{2}\hbar/E_\gamma = \tfrac{1}{2}\left(6.582\times10^{-16}\, eV/sec/1\,eV\right) = 3.261\times10^{-16}\, sec = 0.3261\times10^{-15}\, sec = 0.326\, femto\text{-}sec}$$

Note that the period of oscillation τ associated with a $\boxed{E_\gamma = hf_\gamma = 1\,eV}$ photon is:

$$\boxed{\tau_\gamma = 1/f_\gamma = \lambda_\gamma/c = 1240\times10^{-9}\, m/3\times10^{8}\, m/s = 4.13\times10^{-15}\, sec = 4.13\, femto\text{-}sec}$$

Thus, we see that the period of oscillation τ associated with a 1 eV photon is ~ $10\times$ longer than the characteristic time interval Δt associated with the photon-free charge absorption-re-emission process.

Please also note that had we _not_ neglected the magnetic $\left(q\vec{v}\times\vec{B}\right)$ Lorentz force term in the original force equation, the *B*-field induced motion of the point charge q would have corresponded to the creation of an induced, time-dependent magnetic dipole moment at $r' \simeq 0$:

$$\boxed{\tilde{\vec{m}}(0,t) \equiv \tilde{\vec{I}}(0,t)\, A_\perp = I_o A_\perp e^{-i\omega t}\hat{y} = m_o e^{-i\omega t}\hat{y}}$$

which in turn also would have subsequently radiated magnetic dipole (M1) *EM* radiation.

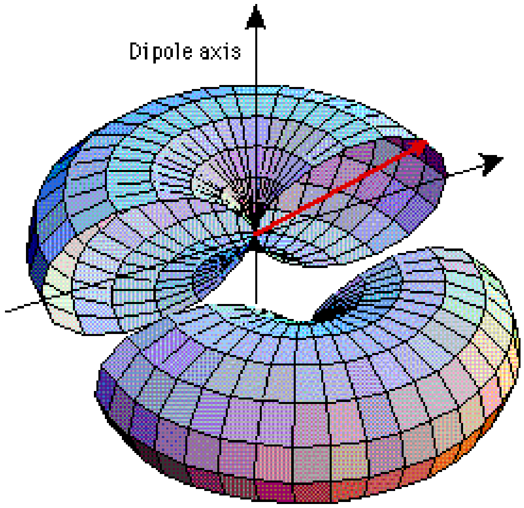
However, because $B_o = E_o/c \ll E_o$ we also see that $m_o \ll p_o c$, and thus the power radiated by this induced, time-dependent magnetic dipole would be $\ll$ than the power radiated by the induced, time-dependent electric dipole. As we have seen in P436 Lecture Notes 13.5 (p. 11), for "equal" strength dipoles $\left(\text{i.e. } m_o = p_o c\right)$, the amount of M1 radiation is far less than that for E1 radiation.

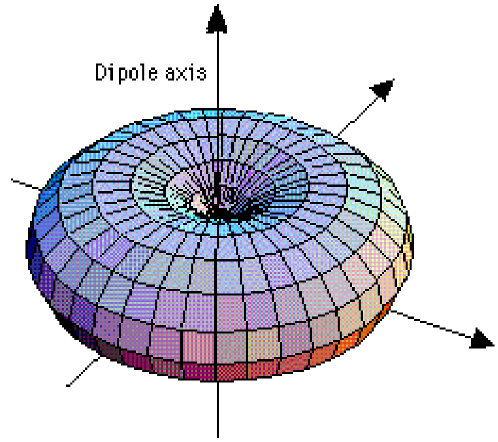
Note further that, formally mathematically speaking, M1 radiation appears at second-order in the Taylor series expansion of $\rho\left(\vec{r}',t_r\right)$ and $\vec{J}\left(\vec{r}',t_r\right)$, hence another reason why we neglected this term, since we initially stated that we were only working to first order in this expansion.

In the "far-zone" limit, the instantaneous differential retarded *EM* power radiated by an oscillating E1 electric dipole {situated at the origin, *n.b.* oriented along the $\hat{z}$-direction} into solid angle element $d\Omega$ is (see P436 Lecture Notes 14, *p*. 5):

$$\boxed{\frac{dP_r^{rad}(\theta,\varphi,t)}{d\Omega} = \vec{S}_{rad}(\vec{r},t)\cdot r^2\hat{r} \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\sin^2\theta}\left(\frac{\text{Watts}}{\text{steradian}}\right)$$

The angular distribution of the *EM* power radiated by an oscillating E1 electric dipole with time- dependent electric dipole moment $\vec{p}(t) = p_o\hat{a}e^{-i\omega t}$ oriented parallel to an arbitrary $\hat{a}$-axis (*e.g.* where $\hat{a} = \hat{x}, \hat{y}$ or $\hat{z}$ axis) is shown in the figure below. The intensity {*aka* ***irradiance***} $\boxed{I_{rad}(\vec{r}) = \left|\left\langle \vec{S}_{rad}(\vec{r},t)\right\rangle\right|}$ $\left(Watts/m^2\right)$ is proportional to the distance from the origin $\vartheta = (0,0,0)$ to an arbitrary point $\vec{r} = (r,\theta,\varphi)$ on the 3-D surface of the figure.





However, for the problem we have at hand, our electric dipole is oriented in the $\hat{x}$-direction. Thus we must carry out a rotation of the above result such that it is appropriate for our situation:

$$\boxed{\frac{dP_{\text{r}}^{rad}(\theta,\varphi,t)}{d\Omega}=\vec{S}^{rad}(r,\theta,\varphi,t)\bullet r^2\hat{r}\simeq\frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2 c}\sin^2\Theta_x}\left(\frac{\text{Watts}}{\text{steradian}}\right)$$

Here, Θ_x is the opening angle between the observation/field point unit vector $\hat{r}$ and the $\hat{x}$-axis. Thus, $\cos\Theta_x$ is the direction cosine between the observation/field point unit vector $\hat{r}$ and the $\hat{x}$-axis: *i.e.* $\boxed{\cos\Theta_x=\hat{r}\bullet\hat{x}=\sin\theta\cos\varphi}$, in terms of the usual polar (θ) and azimuthal (φ) angles.

Note also that since: $\boxed{|\vec{r}\times\vec{x}|=|\vec{r}||\vec{x}|\sin\Theta_x}$ or: $\boxed{\frac{|\vec{r}\times\vec{x}|}{|\vec{r}||\vec{x}|}=|\hat{r}\times\hat{x}|=\sin\Theta_x}$ then:

$$\boxed{\begin{aligned}\sin^2\Theta_x&=(\hat{r}\times\hat{x})\bullet(\hat{r}\times\hat{x})\\&=\left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{x}\right]\bullet\left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{x}\right]\\&=\sin^2\theta\sin^2\varphi+\cos^2\theta\end{aligned}}$$

which can also be obtained from:

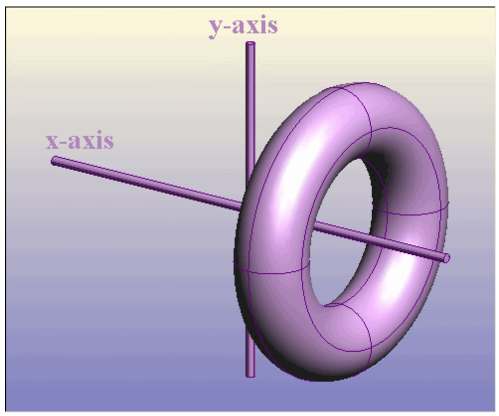
$$\boxed{\begin{aligned}\sin^2\Theta_x&=1-\cos^2\Theta_x=1-(\hat{r}\bullet\hat{x})^2=1-(\sin^2\theta\cos^2\varphi)=1-\sin^2\theta(1-\sin^2\varphi)\\&=1-\sin^2\theta+\sin^2\theta\sin^2\varphi=\cos^2\theta+\sin^2\theta\sin^2\varphi=\sin^2\theta\sin^2\varphi+\cos^2\theta\end{aligned}}$$

Thus, in terms of the usual polar (θ) and azimuthal (φ) angles, for the oscillating E1 electric dipole oriented along the $\hat{x}$-axis:

$$\boxed{\frac{dP_r^{rad}(\theta,\varphi,t)}{d\Omega} = \vec{S}^{rad}(r,\theta,\varphi,t)\bullet r^2\hat{r} \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\sin^2\Theta_x} \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

Becomes: $$\boxed{\frac{dP_r^{rad}(r,\theta,\varphi,t)}{d\Omega} = \vec{S}^{rad}(r,\theta,\varphi,t)\bullet r^2\hat{r} \simeq \frac{\mu_o \ddot{p}^2(t_o)}{16\pi^2 c}\left(\sin^2\theta\sin^2\varphi + \cos^2\theta\right)} \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

The angular distribution of the *EM* energy radiated from the oscillating E1 electric dipole oriented along the $\hat{x}$-axis is thus similar to that shown in the figure below:



In the above formula, the second time-derivative of the electric dipole moment $\ddot{\vec{p}}(t_o)$ is to be evaluated at the retarded time $\boxed{t_o \equiv t - r/c}$ and also computed from the local origin, $\vartheta\ \{\vec{r}\,' = 0\}$: $\boxed{\ddot{\tilde{\vec{p}}}(0,t_o) = \ddot{\tilde{\vec{p}}}(0,t-r/c)}$.

The <u>instantaneous</u> induced electric dipole moment is: $\boxed{\tilde{\vec{p}}(0,t_o) \equiv q\tilde{\vec{x}}(0,t_o) = qx_o e^{-i\omega t_o}\hat{x} = p_o e^{-i\omega t_o}\hat{x}}$.

Thus: $\boxed{\ddot{\tilde{\vec{p}}}(0,t_o) = -\omega^2 p_o e^{-i\omega t_o}\hat{x} = -\omega^2 q x_o e^{-i\omega t_o}\hat{x} = q\left(-\omega^2 x_o\right)e^{-i\omega t_o}\hat{x} = q\ddot{\tilde{\vec{x}}}(0,t_o) = q\tilde{\vec{a}}(0,t_o)}$.

From the above force equation, we see that: $\boxed{\tilde{\vec{a}}(0,t_o) = \frac{q}{m}\tilde{\vec{E}}_{inc}(0,t_o) = \frac{q}{m}E_o e^{-i\omega t_o}\hat{x}}$

but: $\boxed{\tilde{\vec{a}}(0,t_o) = \ddot{\tilde{\vec{x}}}(0,t_o) = -\omega^2 x_o e^{-i\omega t_o}\hat{x}}$ and thus we see that: $\boxed{-\omega^2 x_o = \frac{q}{m}E_o}$ or: $\boxed{x_o = -\left(\frac{q}{m\omega^2}E_o\right)}$

and therefore: $\boxed{\ddot{\tilde{\vec{p}}}(0,t_o) = q\tilde{\vec{a}}(0,t_o) = q\left(-\omega^2 x_o\right)e^{-i\omega t}\hat{x} = \left(\frac{q^2}{m}E_o\right)e^{-i\omega t_o}\hat{x}}$

n.b. <u>*here*</u>, $\ddot{\tilde{\vec{p}}}(0,t_o)$ is <u>***independent***</u> of frequency ω, because: $\boxed{p_o = qx_o = -q\left(\frac{q}{m\omega^2}E_o\right) = -\frac{q^2}{m\omega^2}E_o}$,

i.e. $\boxed{p_o = qx_o \propto \frac{1}{\omega^2}}$!!! Thus: $\boxed{\ddot{\tilde{\vec{p}}}^2(0,t_o) = \ddot{\tilde{\vec{p}}}(0,t_o)\bullet\ddot{\tilde{\vec{p}}}(0,t_o) = \left(\frac{q^2}{m}E_o\right)^2 e^{-2i\omega t_o}}$.

The time average of {the real part of !!!} this quantity, averaged over one complete cycle of oscillation $\boxed{\tau = 1/f = 2\pi/\omega}$ is simply half of this value: $\boxed{\left\langle \ddot{\vec{p}}^2(0,t_o)\right\rangle = \frac{1}{2}\ddot{\vec{p}}^2(0,t_o) = \frac{1}{2}\left(\frac{q^2}{m}E_o\right)^2}$

since $\boxed{\left\langle \text{Re}\left\{e^{-2i\omega t_o}\right\}\right\rangle = \frac{1}{\tau}\int_{one\ cycle} \text{Re}\left\{e^{-2i\omega t_o'}\right\} dt_o' = \frac{1}{\tau}\int_{one\ cycle} \cos^2 \omega t_o'\, dt_o' = \frac{1}{2}}$.

Note also that by carrying out the time-averaging process, this also helps us to completely side-step/avoid the difficulty associated with experimentally dealing with the retarded time $\boxed{t_o \equiv t - r/c}$!

Thus, the time-averaged differential power radiated by the oscillating E1 electric dipole is:

$$\boxed{\frac{d\left\langle P_{\text{r}}^{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} = \left\langle \vec{S}^{rad}(r,\theta,\varphi,t_o)\right\rangle \bullet r^2\hat{r} \simeq \frac{\mu_o\left\langle \ddot{p}^2(t_o)\right\rangle}{16\pi^2 c}\sin^2\Theta_x = \frac{1}{2}\cdot\frac{\mu_o}{16\pi^2 c}\left(\frac{q^2E_o}{m}\right)^2 \sin^2\Theta_x} \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

Or: $$\boxed{\frac{d\left\langle P_{\text{r}}^{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} = \left\langle \vec{S}^{rad}(r,\theta,\varphi,t_o)\right\rangle \bullet r^2\hat{r} \simeq \frac{1}{2}\cdot\frac{\mu_o}{16\pi^2 c}\left(\frac{q^2E_o}{m}\right)^2 \left(\sin^2\theta\sin^2\varphi + \cos^2\theta\right)} \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

Likewise, the time average of the {magnitude of} the instantaneous flux of *EM* energy incident on the free charge {located at the origin $\vartheta(\vec{r}=0)$}, $\boxed{\left|\vec{S}_{inc}(0,t_o)\right| = \varepsilon_o E_o^2 c e^{-2i\omega t_o}}$ is half of this value, *i.e.*:

$$\boxed{I_{inc}(0) \equiv \left\langle \left|\vec{S}_{inc}(0,t_o)\right|\right\rangle = \frac{1}{2}\left|\vec{S}_{inc}(0,t_o)\right| = \frac{1}{2}\varepsilon_o E_o^2 c} \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The scattering of *EM* plane waves by a free electric charge q is known as **Thomson scattering** in honor of J.J. Thomson – the discoverer of the electron {in 1897}.

Thus, the differential Thomson scattering cross section {per scattering object of charge q} for an incident plane *EM* wave propagating in the $+\hat{z}$-direction and linearly polarized in the $\hat{x}$-direction is:

$$\boxed{\begin{aligned}\frac{d\sigma_T^{LPx}(\theta,\varphi)}{d\Omega} &= \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{d\left\langle P_{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} \simeq \frac{1}{\cancel{\frac{1}{2}}\varepsilon_o \cancel{E_o^2} c}\,\frac{\mu_o \cancel{\frac{1}{2}}\left(\frac{q^4 \cancel{E_o^2}}{m^2}\right)}{16\pi^2 c}\sin^2\Theta_x \\ &= \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \sin^2\Theta_x \quad \left\{\text{using } \mu_o = 1/\varepsilon_o c^2\right\}\end{aligned}}$$

i.e.

$$\boxed{\frac{d\sigma_T^{LPx}(\theta,\varphi)}{d\Omega} \simeq \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \sin^2\Theta_x = \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \left(\sin^2\theta\sin^2\varphi + \cos^2\theta\right)} \left(m^2/sr\right)$$

There are *three* interesting things associated with this result:

The differential Thomson scattering cross section is independent of *both* the frequency of the incident *EM* radiation, and note also that it is independent of the strength (*i.e.* amplitude) of the incident electric field, E_o !

Note also that the differential Thomson scattering cross section varies as the *square* of the electric charge, *i.e.* the Thomson scattering cross section is the same for $+q$ vs. $-q$ charged particles, and note also that a scattering object with free electric charge $q = +2e$ (such as an α-particle) has a Thomson scattering cross section $4\times$ greater than that associated with a scattering object {of the same mass, m} that has free electric charge $q = +e$.

Then since:

$$\boxed{\left\langle P_{\mathrm{r}}^{rad}(t_o)\right\rangle = \int \frac{d\left\langle P_{\mathrm{r}}^{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} d\Omega \simeq \frac{\mu_o}{32\pi^2 c}\left(\frac{q^2 E_o}{m}\right)^2 \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)\sin\theta d\theta d\varphi}$$

Carrying out the azimuthal angle integrals first: $\boxed{\int_{\varphi=0}^{\varphi=2\pi}\sin^2\varphi\, d\varphi = \pi}$ and $\boxed{\int_{\varphi=0}^{\varphi=2\pi} d\varphi = 2\pi}$.

Then carrying out the polar angle integration:

$$\boxed{\int_{\theta=0}^{\theta=\pi}\sin^2\theta\sin\theta d\theta = \int_{\theta=0}^{\theta=\pi}\sin^3\theta d\theta = \left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]\Bigg|_{\theta=0}^{\theta=\pi} = \left[1-\frac{1}{3}\right]-\left[-1+\frac{1}{3}\right] = 2-\frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}}$$

And: $\boxed{\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta d\theta = -\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\theta=\pi} = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}}$

Thus: $\boxed{\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)\sin\theta d\theta d\varphi = \left(\pi\cdot\frac{4}{3}\right)+\left(2\pi\cdot\frac{2}{3}\right) = \frac{4\pi}{3}+\frac{4\pi}{3} = \frac{8\pi}{3}}$

Then: $\boxed{\left\langle P_{\mathrm{r}}^{rad}(t_o)\right\rangle \simeq \frac{\mu_o}{{}_4\cancel{32}\pi^{\cancel{2}}c}\left(\frac{q^2 E_o}{m}\right)^2 \frac{\cancel{8}\,\cancel{\pi}}{3} = \frac{\mu_o}{12\pi c}\left(\frac{q^2 E_o}{m}\right)^2}$

Thus, the total Thomson scattering cross section {per scattering object of charge q} for an incident plane *EM* wave propagating in the $+\hat{z}$-direction and linearly polarized in the $\hat{x}$-direction is:

$$\boxed{\sigma_T^{LPx} = \int \frac{d\sigma_T^{LPx}(\theta,\varphi)}{d\Omega} d\Omega \equiv \frac{\left\langle P_{rad}(t_o)\right\rangle}{\left\langle\left|\vec{S}_{inc}(t_o)\right|\right\rangle} \simeq \frac{\dfrac{\mu_o}{{}^{6}\cancel{12}\pi\cancel{c}}\left(\dfrac{q^4 \cancel{E_o^2}}{m^2}\right)}{\dfrac{1}{\cancel{2}}\varepsilon_o\cancel{E_o^2}\cancel{c}} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \quad \left\{\text{using } \mu_o = 1/\varepsilon_o c^2\right\}}$$

Thus: $\boxed{\sigma_T^{LPx} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2}$ (m^2)

If the electrically-charged scattering object *e.g.* is an electron, with electric charge $q = -e$, and rest mass m_e then the quantity: $\boxed{r_e \equiv \left(e^2/4\pi\varepsilon_o m_e c^2\right) \simeq 2.82\times10^{-15}\,m}$ is known as the so-called **classical electron radius**.

The **classical electron radius** r_e is defined as the radial distance from an electron where the {magnitude of the} potential energy $|U_e(r_e)| = |eV_e(r_e)|$ associated with a unit test charge $q = e$ equals the rest mass energy of the electron $E_{e^-}^{rest} = m_e c^2$, *i.e.*

$$\boxed{|U_e(r_e)| = |eV_e(r_e)| = \frac{e^2}{4\pi\varepsilon_o r_e} = m_e c^2} \text{ thus: } \boxed{r_e \equiv \frac{e^2}{4\pi\varepsilon_o m_e c^2} \simeq 2.82\times10^{-15}\,m}$$

The differential and the total Thomson scattering cross sections for free electron scattering of a linear polarized plane *EM* wave, written in terms of the classical electron radius r_e are:

$$\boxed{\frac{d\sigma_{T\,e^-}^{LPx}(\theta,\varphi)}{d\Omega} \simeq r_e^2 \sin^2\Theta_x = r_e^2\left(\sin^2\theta\sin^2\varphi + \cos^2\theta\right)} \left(m^2/sr \text{ per electron}\right)$$

and:

$$\boxed{\sigma_{T\,e^-}^{LPx} = \frac{8\pi}{3} r_e^2} \left(m^2 \text{ per electron}\right) \text{ where: } \boxed{r_e \equiv \frac{e^2}{4\pi\varepsilon_o m_e c^2} \simeq 2.82\times10^{-15}\,m}$$

Numerically, we see that:

$$\boxed{\sigma_{T\,e^-}^{LPx} \simeq \frac{8\pi}{3} r_e^2 = \frac{8\pi}{3}\left(2.82\times10^{-15}\right)^2 = 66.6\times10^{-30} \simeq 0.67\times10^{-28}} \left(m^2 \text{ per electron}\right)$$

Physicists get tired of writing down {astronomically} small numbers all the time, so we have defined a convenient unit of area for cross sections, known as a ***barn*** {originating from the phrase: "It's as big as a barn"}: $\boxed{1\text{ barn} \equiv 10^{-28}\,m^2 = \left(10^{-14}\right)^2 m^2 = \left(10\times10^{-15}\right)^2 m^2 = \left(10\,fm\right)^2 = 100\,fm^2}$ where 1 *fm* = 1 Fermi = 10^{-15} m (in honor of Enrico Fermi, nuclear physicist of mid-20th century).

Thus: $$\boxed{\sigma_{T\,e^-}^{LPx} \simeq \frac{8\pi}{3} r_e^2 = \frac{8\pi}{3}\left(2.82\times10^{-15}\right)^2 \simeq 0.67\times10^{-28}\,m^2 \text{ per electron} = 0.67\ \textit{barns} \text{ per electron}}$$

In order to obtain a more physically intuitive understanding of the magnitude of this (and various other) total scattering cross sections, note that the characteristic size of the nucleus of an atom is typically $\boxed{r_{nucleus} \sim (1-few)\ fm = (1-few)\times10^{-15}\,m}$ whereas the characteristic size of an atom is typically $\boxed{r_{atom} \sim (1-few)\text{ Ångstroms} = (1-few)\times10^{-10}\,m = 0.1(1-few)\,nm}$.

Thus, the geometrical cross-sectional area of a typical nucleus in an atom is $\boxed{A_{nucleus} = \pi r_{nucleus}^2 \sim (0.03-1)\times10^{-28}\,m = (0.03-1)\ barns}$, whereas the geometrical cross-sectional area of a typical atom is huge: $\boxed{A_{atom} = \pi r_{atom}^2 \sim (0.03-1)\times10^{-18}\,m = (0.03-1)\times10^{+10}\ barns\,!!!}$.

Hence, we see that the free electron Thomson scattering cross section is comparable to the geometrical cross-sectional area for a typical nucleus, and is very much smaller than the geometrical cross-sectional area for a typical atom, *i.e.*

$$\boxed{\left(\sigma_{T\,e^-}^{LPx} = 8\pi r_e^2/3 = 0.67\ \ barns\right) \sim \left(A_{nucleus} = (0.03-1)\ \ barns\right) \ll \left(A_{atom} = (0.03-1)\times 10^{+10}\ barns\ \right)}$$

Note that the Thomson scattering of a linearly-polarized *EM* plane wave propagating in the $\hat{z}$-direction has **rotational invariance** about the $\hat{z}$-axis. In the original problem above, we could have alternatively chosen the polarization of the *EM* plane wave to be parallel to *e.g.* the $-\hat{y}$-axis instead of the $\hat{x}$-axis, *i.e.* $\boxed{\tilde{\vec{E}}_{inc}(\vec{r}\simeq 0,t) = -E_o e^{-i\omega t}\hat{y}}$, then the differential Thomson scattering cross section for a free charge q would instead then be:

$$\boxed{\frac{d\sigma_T^{LPy}(\theta,\varphi)}{d\Omega} = \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{d\left\langle P_{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} \simeq \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t)\right|\right\rangle}\frac{\mu_o\left\langle \ddot{p}^2(t_o)\right\rangle}{16\pi^2 c}\sin^2\Theta_y}$$

where the direction cosine $\boxed{\cos\Theta_y = \hat{r}\bullet(-\hat{y}) = -\sin\theta\sin\varphi}$ and: $\boxed{\left|\hat{r}\times(-\hat{y})\right| = \sin\Theta_y}$ thus:

$$\boxed{\begin{aligned}\sin^2\Theta_y &= \left(\hat{r}\times(-\hat{y})\right)\bullet\left(\hat{r}\times(-\hat{y})\right) = (\hat{r}\times\hat{y})\bullet(\hat{r}\times\hat{y})\\ &= \left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{y}\right]\bullet\left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{y}\right]\\ &= \sin^2\theta\cos^2\varphi+\cos^2\theta\end{aligned}}$$

which can also be obtained from:

$$\boxed{\begin{aligned}\sin^2\Theta_y &= 1-\cos^2\Theta_y = 1-(\hat{r}\bullet\hat{y})^2 = 1-\left(\sin^2\theta\sin^2\varphi\right) = 1-\sin^2\theta\left(1-\cos^2\varphi\right)\\ &= 1-\sin^2\theta+\sin^2\theta\cos^2\varphi = \cos^2\theta+\sin^2\theta\cos^2\varphi = \sin^2\theta\cos^2\varphi+\cos^2\theta\end{aligned}}$$

Note that {importantly}, going through the same derivational steps as done originally, the induced dipole moment associated with an incident *EM* plane wave propagating in the $\hat{z}$-direction but linearly polarized in the $-\hat{y}$-direction is: $\boxed{\tilde{\vec{p}}(0,t) \equiv q\tilde{\vec{y}}(0,t) = -qy_o e^{-i\omega t}\hat{y} = -p_o e^{-i\omega t}\hat{y}}$, thus we see that the induced dipole moment simply follows/tracks the polarization of the incident *EM* plane wave, *i.e.* $\tilde{\vec{p}}(0,t)$ is parallel to the polarization/*E*-field vector of the incident *EM* plane wave!

The intensity maximum of the scattered radiation {here} would then be oriented perpendicular to the $\hat{y}$-axis instead of the perpendicular to the $\hat{x}$-axis {*n.b.* please see/refer to the 3-D figure on page 7 of these lecture notes}.

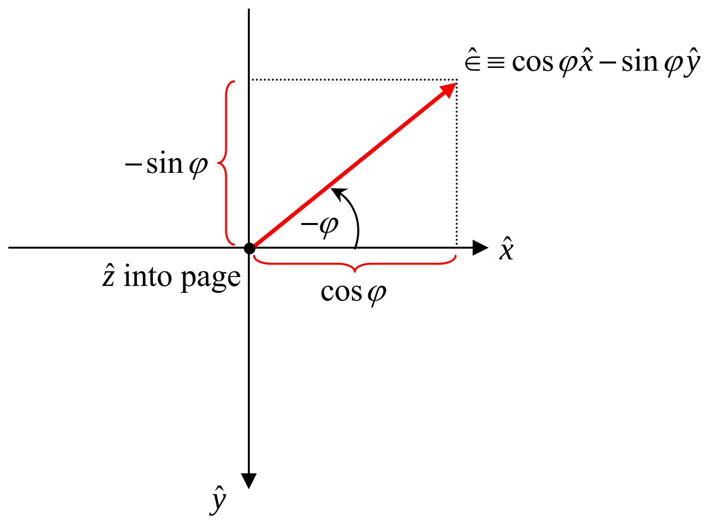
The differential Thomson scattering cross section for an incident *EM* plane wave propagating in the $\hat{z}$-direction but linearly polarized in the $-\hat{y}$-direction is:

$$\boxed{\frac{d\sigma_T^{LPy}(\theta,\varphi)}{d\Omega} \simeq \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\sin^2\Theta_y = \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)}\ \left(m^2/sr\right)$$

Integrating this expression over the polar and azimuthal angles (θ,φ), we obtain precisely the same result for the <u>total</u> Thomson scattering cross section as above for the original $\hat{x}$-polarization case:

$$\boxed{\sigma_T^{LPy} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \sigma_T^{LPx}} \left(m^2\right)$$

More generally, for an incident *EM* plane wave propagating in the $\hat{z}$-direction with ***<u>arbitrary</u>*** linear polarization $\boxed{\hat{\in} \equiv \cos\varphi\hat{x} - \sin\varphi\hat{y}}$ and $\boxed{\tilde{\vec{E}}_{inc}(\vec{r}\simeq 0,t) = E_o e^{-i\omega t}\,\hat{\in}}$ as shown in the figure below {*cf* with/see also the 3-D figure shown on p. 4 of these lecture notes}:



In this more general situation, the induced dipole moment is again parallel to/tracks the polarization vector of the incident *EM* plane wave:

$$\boxed{\tilde{\vec{p}}(0,t) \equiv q\,\tilde{\vec{\in}}(0,t) = qr_o e^{-i\omega t}\left(\cos\varphi\hat{x} - \sin\varphi\hat{y}\right) = p_o e^{-i\omega t}\left(\cos\varphi\hat{x} - \sin\varphi\hat{y}\right)}$$

Thus, here the differential Thomson scattering cross section for a free charge q is:

$$\boxed{\frac{d\sigma_T^{LP\in}(\theta,\varphi)}{d\Omega} = \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{d\left\langle P_{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} \simeq \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{\mu_o\left\langle \ddot{p}^2(t_o)\right\rangle}{16\pi^2 c}\sin^2\Theta_\in}$$

where {here} the <u>direction cosine</u> associated with an ***<u>arbitrary</u>*** linear polarization is:

$$\boxed{\cos\Theta_\in = \hat{r}\cdot\hat{\in} = \left(\sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}\right)\cdot\left(\cos\varphi\hat{x} - \sin\varphi\hat{y}\right) = \sin\theta\left(\cos^2\varphi - \sin^2\varphi\right) = \sin\theta\cos 2\varphi}$$

and: $\boxed{\left|\hat{r}\times\hat{\in}\right| = \sin\Theta_\in}$ thus:

$$\boxed{\begin{aligned}\sin^2\Theta_\in &= (\hat{r}\times\hat{\in})\cdot(\hat{r}\times\hat{\in}) \qquad\qquad \hat{\in}\equiv\cos\varphi\hat{x}-\sin\varphi\hat{y}\\ &= \left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{\in}\right]\cdot\left[(\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z})\times\hat{\in}\right]\\ &= 4\sin^2\theta\cos^2\varphi\sin^2\varphi+\cos^2\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}=\sin^2\theta\underbrace{(4\cos^2\varphi\sin^2\varphi)}_{=\sin^2 2\varphi}+\cos^2\theta\\ &= \sin^2\theta\sin^2 2\varphi+\cos^2\theta\end{aligned}}$$

which can also be obtained from:

$$\boxed{\begin{aligned}\sin^2\Theta_\in &= 1-\cos^2\Theta_\in = 1-(\hat{r}\cdot\hat{\in})^2 = 1-\sin^2\theta\cos^2 2\varphi = \sin^2\theta+\cos^2\theta-\sin^2\theta\cos^2 2\varphi\\ &= \sin^2\theta\underbrace{(1-\cos^2 2\varphi)}_{=\sin^2 2\varphi}+\cos^2\theta = \sin^2\theta\sin^2 2\varphi+\cos^2\theta\end{aligned}}$$

This relation can *also* be obtained via a 3rd method – noting that since:

$$\boxed{\cos\Theta_\in = \hat{r}\cdot\hat{\in} = \hat{r}\cdot(\cos\varphi\hat{x}-\sin\varphi\hat{y}) = \cos\varphi(\hat{r}\cdot\hat{x})+\sin\varphi(\hat{r}\cdot(-\hat{y})) = \cos\varphi\cos\Theta_x+\sin\varphi\cos\Theta_y}$$

where: $\boxed{\cos\Theta_x = \hat{r}\cdot\hat{x} = \sin\theta\cos\varphi}$ and: $\boxed{\cos\Theta_y = \hat{r}\cdot(-\hat{y}) = -\hat{r}\cdot\hat{y} = -\sin\theta\sin\varphi}$ then:

$$\boxed{\begin{aligned}\sin^2\Theta_\in &= 1-\cos^2\Theta_\in = 1-(\hat{r}\cdot\hat{\in})^2 = 1-\left[\hat{r}\cdot(\cos\varphi\hat{x}-\sin\varphi\hat{y})\right]^2 = 1-\left[\cos\varphi(\hat{r}\cdot\hat{x})+\sin\varphi(\hat{r}\cdot(-\hat{y}))\right]^2\\ &= 1-\left[\cos\varphi(\sin\theta\cos\varphi)+\sin\varphi(-\sin\theta\sin\varphi)\right]^2 = 1-\left[\sin\theta\cos^2\varphi-\sin\theta\sin^2\varphi\right]^2\\ &= 1-\sin^2\theta\underbrace{(\cos^2\varphi-\sin^2\varphi)^2}_{=\cos^2 2\varphi} = 1-\sin^2\theta\cos^2 2\varphi = \cos^2\theta+\sin^2\theta-\sin^2\theta\cos^2 2\varphi\\ &= \cos^2\theta+\sin^2\theta\underbrace{(1-\cos^2 2\varphi)}_{=\sin^2 2\varphi} = \cos^2\theta+\sin^2\theta\sin^2 2\varphi = \sin^2\theta\sin^2 2\varphi+\cos^2\theta\end{aligned}}$$

Thus, we see that:

$$\boxed{\cos^2\Theta_x = (\hat{r}\cdot\hat{x})^2 = \sin^2\theta\cos^2\varphi}$$

$$\boxed{\cos^2\Theta_y = (\hat{r}\cdot(-\hat{y}))^2 = (-\hat{r}\cdot\hat{y})^2 = (-\sin\theta\sin\varphi)^2 = \sin^2\theta\sin^2\varphi}$$

$$\boxed{\begin{aligned}\cos^2\Theta_\in &= (\hat{r}\cdot\hat{\in})^2 = 1-\sin^2\Theta_\in = 1-\cos^2\theta-\sin^2\theta\sin^2 2\varphi = \sin^2\theta-\sin^2\theta\sin^2 2\varphi\\ &= \sin^2\theta\underbrace{(1-\sin^2 2\varphi)}_{=\cos^2 2\varphi} = \sin^2\theta\cos^2 2\varphi\end{aligned}}$$

with $\boxed{\hat{\in}\equiv\cos\varphi\hat{x}-\sin\varphi\hat{y}}$,

Note that when $\varphi = 0$: $\hat{\in}=\hat{x}$ and $\cos^2\Theta_\in = \cos^2\Theta_x = \sin^2\theta$.

Note that when $\varphi = -\frac{\pi}{2}$: $\hat{\in}=-\hat{y}$ and $\cos^2\Theta_\in = \cos^2\Theta_y = \sin^2\theta$.

and:

$$\boxed{\sin^2\Theta_x = (\hat{r}\times\hat{x})\cdot(\hat{r}\times\hat{x}) = \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$\boxed{\sin^2\Theta_y = (\hat{r}\times(-\hat{y}))\cdot(\hat{r}\times(-\hat{y})) = (\hat{r}\times\hat{y})\cdot(\hat{r}\times\hat{y}) = \sin^2\theta\cos^2\varphi + \cos^2\theta}$$

$$\boxed{\sin^2\Theta_\in = (\hat{r}\times\hat{\in})\cdot(\hat{r}\times\hat{\in}) = \sin^2\theta\sin^2 2\varphi + \cos^2\theta}$$

Note that when $\varphi = 0$: $\hat{\in} = \hat{x}$ and $\sin^2\Theta_\in = \sin^2\Theta_x = \cos^2\theta$.

Note that when $\varphi = -\frac{\pi}{2}$: $\hat{\in} = -\hat{y}$ and $\sin^2\Theta_\in = \sin^2\Theta_y = \cos^2\theta$.

Thus, the differential Thomson scattering cross section for an incident *EM* plane wave propagating in the $\hat{z}$ -direction with ***arbitrary*** linear polarization in the $\boxed{\hat{\in} \equiv \cos\varphi\hat{x} - \sin\varphi\hat{y}}$-direction is:

$$\boxed{\frac{d\sigma_T^{LP\in}(\theta,\varphi)}{d\Omega} \simeq \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \sin^2\Theta_\in = \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \left(\sin^2\theta\sin^2 2\varphi + \cos^2\theta\right)} \ \left(m^2/sr\right)$$

Then since $\sigma_T = \int\left(\frac{d\sigma_T(\theta,\varphi)}{d\Omega}\right)d\Omega$, carrying out the angular integration over the polar (θ) and azimuthal (φ) angles: $\int_{\varphi=0}^{\varphi=\pi}\int_{\theta=0}^{\theta=\pi}\left(\sin^2\theta\sin^2 2\varphi + \cos^2\theta\right)\sin\theta d\theta d\varphi$, carrying out the φ -integrals:

$$\int_{\varphi=0}^{\varphi=2\pi}\sin^2 2\varphi d\varphi = \left[\frac{\varphi}{2} - \frac{\sin 4\varphi}{8}\right]\Bigg|_{\varphi=0}^{\varphi=2\pi} = \frac{2\pi}{2} = \pi \ \text{ and } \ \int_{\varphi=0}^{\varphi=\pi} d\varphi = 2\pi$$

Using $\sin^2\theta = 1 - \cos^2\theta$ and making the substitution $u = \cos\theta$, hence $du = d\cos\theta = -\sin\theta d\theta$; and when $\theta = 0$: $u = \cos 0 = 1$, when $\theta = \pi$: $u = \cos\pi = -1$, thus the θ -integrals are:

$$\int_{\theta=0}^{\theta=\pi}\sin^2\theta\sin\theta d\theta = \int_{u=-1}^{u=+1}\left(1-u^2\right)du = \left[u - \tfrac{1}{3}u^3\right]\Big|_{u=-1}^{u=+1} = \left[1-\tfrac{1}{3}\right] - \left[-1+\tfrac{1}{3}\right] = \left[1-\tfrac{1}{3}\right] + \left[1-\tfrac{1}{3}\right] = 2\left[1-\tfrac{1}{3}\right] = \tfrac{4}{3}$$

$$\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta d\theta = \int_{u=-1}^{u=+1}u^2 du = \tfrac{1}{3}u^3\Big|_{u=-1}^{u=+1} = \tfrac{1}{3} - \left(-\tfrac{1}{3}\right) = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

Putting all of these results together:

$$\int_{\varphi=0}^{\varphi=\pi}\int_{\theta=0}^{\theta=\pi}\left(\sin^2\theta\sin^2 2\varphi + \cos^2\theta\right)\sin\theta d\theta d\varphi = \frac{4}{3}\cdot\pi + \frac{2}{3}\cdot 2\pi = \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{8\pi}{3} \ \ !!!$$

$$\boxed{\sigma_T^{LP\in} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \sigma_T^{LPx} = \sigma_T^{LPy} = \sigma_T} \ \left(m^2\right)$$

Thus we have explicitly shown that we obtain precisely the same result for the total Thomson scattering cross section for an arbitrary polarization $\boxed{\hat{\in} \equiv \cos\varphi\hat{x} - \sin\varphi\hat{y}}$ of an incident *EM* plane wave as that obtained above for either the original $\hat{x}$ - and/or the $-\hat{y}$ - polarization cases.

Due to the manifest rotational invariance/symmetry of this problem about the $\hat{z}$-axis {the propagation direction of the incident *EM* plane wave}, intuitively we can understand why this must be true, since the orientation of the induced electric dipole moment $\tilde{\vec{p}}(\vec{r}' \simeq 0, t)$ is such that it is always parallel to/tracks the polarization vector $\hat{\in}$ of the incident *EM* plane wave.

Thus, we see that while the polarization vector $\hat{\in}$ of the incident *EM* plane wave certainly matters greatly for the differential Thomson scattering cross section $d\sigma_T(\theta,\varphi)/d\Omega$, the total Thomson scattering cross section σ_T is unaffected/does not depend on the polarization vector $\hat{\in}$ of the incident *EM* plane wave.

Since left- and right-circularly polarized *EM* plane waves are {complex, but} linear orthogonal combinations of linearly-polarized *EM* plane waves: $\boxed{\hat{\in}_{LCP} \equiv \frac{1}{\sqrt{2}}(\hat{x}+i\hat{y})}$ and $\boxed{\hat{\in}_{RCP} \equiv \frac{1}{\sqrt{2}}(\hat{x}-i\hat{y})}$ {note that we have normalized these polarization vectors such that $\boxed{\hat{\in}_{LCP} \bullet \hat{\in}^*_{LCP} = \hat{\in}_{RCP} \bullet \hat{\in}^*_{RCP} = 1}$}, then we can also see that the total Thomson scattering cross section for *LCP* or *RCP* incident *EM* plane waves is also:

$$\boxed{\sigma_T = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2} \quad (m^2)$$

We leave it to the interested reader to determine the analytic form of the differential Thomson scattering cross sections for *LCP* or *RCP* incident *EM* plane waves.

Using the same line of reasoning, we can additionally see that the total Thomson scattering cross section for ***unpolarized*** *EM* plane waves incident on a free charge q is also

$$\boxed{\sigma_T^{unpol} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \sigma_T} \quad (m^2)$$

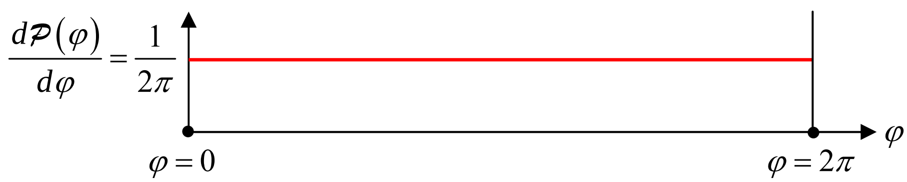
because an unpolarized *EM* plane wave is equivalent to a randomly-polarized *EM* plane wave, whose time-dependent polarization vector $\hat{\in}(t)$ changes randomly from one moment to the next within the azimuthal interval $0 \le \varphi < 2\pi$. For a randomly-distributed φ variable, note that the probability distribution function $dP(\varphi)/d\varphi$ is flat within the azimuthal interval $0 \le \varphi < 2\pi$.

The instantaneous orientation of the induced electric dipole moment $\tilde{\vec{p}}(\vec{r}' \simeq 0, t)$ is parallel to the polarization vector $\hat{\in}(t)$ of the randomly-polarized incident *EM* plane wave on a moment-to-moment basis; thus, while the differential angular distribution of the scattered radiation is changing on a moment-to-moment basis, the total Thomson scattering cross section σ_T is unchanged/time-independent for an unpolarized/randomly-polarized incident *EM* plane wave, because the (θ,φ) angular dependence has been integrated out. We explicitly prove this, below.

The probability is unity (*i.e.* 100%) for a randomly-polarized *EM* plane wave to have linear polarization $\hat{\in}(t)$ instantaneously oriented *somewhere* within the azimuthal interval $0 \le \varphi < 2\pi$.

Mathematically this means that the φ-integral of the probability density $\boxed{\int_{\varphi=0}^{\varphi=2\pi} \left(\frac{d\mathcal{P}(\varphi)}{d\varphi} \right) d\varphi = 1}$.

Since the azimuthal probability density $\left(\frac{d\mathcal{P}(\varphi)}{d\varphi} \right)$ is flat {*i.e.* is constant} for a randomly-distributed φ distribution, then we can take $\left(\frac{d\mathcal{P}(\varphi)}{d\varphi} \right)$ outside of this integral, and then since $\boxed{\int_{\varphi=0}^{\varphi=2\pi} d\varphi = 2\pi}$, we see that the probability density $\boxed{\frac{d\mathcal{P}(\varphi)}{d\varphi} = \frac{1}{2\pi}}$ for an unpolarized/randomly-polarized *EM* plane wave, as shown in the figure below:



If the polarization state $\hat{\in}(t)$ of the incident *EM* plane wave is changing randomly from moment-to-moment, one might worry that the process of averaging the instantaneous differential radiated power $d\langle P_{rad}(\theta,\varphi,t_o)\rangle / d\Omega$ over one period/one cycle of oscillation τ would be insufficient – *i.e.* it would be very "noisy" due to rapid fluctuations/random temporal changes in the polarization state $\hat{\in}(t)$ with time *t*.

At the microscopic level, an unpolarized macroscopic *EM* plane wave consists of real photons, each with a randomly oriented $\vec{E}$-field/randomly oriented polarization vector $\hat{\in}_\gamma$. Individual photons which Thomson scatter off of a free charge *q* are first absorbed by the charge *q* and then re-radiated a short time later, *e.g.* with characteristic time interval $\Delta t \le 0.326\, fs$ for a 1 *eV* photon, compared to the period of oscillation for a 1 *eV* photon of $\tau_\gamma = 4.13\, fs$, ~ $10\times$ longer than Δt.

Thus, for *EM* plane waves incident on a free charge *q* one might well worry that averaging over a single period of oscillation/single cycle τ would likely to yield a noisy result due to fluctuations. However, in actual/real-life scattering experiments, precisely because of such concerns, the averaging time interval Δt_{avg} is frequently orders of magnitude longer than either of these two time scales, typically Δt_{avg} is micro-seconds, to milli-seconds, seconds and/or even longer in order to significantly reduce the level of such {statistical} fluctuations.

Since $\boxed{\left|\hat{r}\times\hat{\in}(t)\right| = \sin\Theta_\in(t)}$, then with <u>no loss of generality</u>, we can very easily modify the time-averaging of the differential power/differential scattering cross section formulae simply by moving the $\sin^2\Theta_\in(t)$ factor {*n.b.* previously assumed to time-independent in the above examples} <u>inside</u> the time-averaging process, *i.e.*:

$$\boxed{\begin{aligned}\frac{d\sigma_T^{unpol}(\theta,\varphi)}{d\Omega} &= \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{d\left\langle P_{rad}(\theta,\varphi,t_o)\right\rangle}{d\Omega} \simeq \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{\mu_o\left\langle \ddot{p}^2(t_o)\sin^2\Theta_\in(t)\right\rangle}{16\pi^2 c} \\ &= \frac{1}{\left\langle\left|\vec{S}_{inc}(0,t_o)\right|\right\rangle}\frac{\mu_o\left\langle \ddot{p}^2(t_o)\left(\sin^2\theta\sin^2 2\varphi(t)+\cos^2\theta\right)\right\rangle}{16\pi^2 c}\end{aligned}}$$

Note that the polar angle θ is <u>fixed</u> by the observer being at the field/observation point *P*. The time-averaged value of the $\sin^2 2\varphi(t_o)$ factor is actually a <u>probability-weighted</u> integral over all possible φ-values that can/do occur during the time-averaging process over Δt_{avg}:

$$\boxed{\begin{aligned}\left\langle\sin^2 2\varphi(t_o)\right\rangle &= \int_{\varphi=0}^{\varphi=2\pi}\left(\frac{d\mathcal{P}(\varphi)}{d\varphi}\right)\sin^2 2\varphi(t_o)\,d\varphi = \int_{\varphi=0}^{\varphi=2\pi}\left(\frac{1}{2\pi}\right)\sin^2 2\varphi(t_o)\,d\varphi \\ &= \frac{1}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\sin^2 2\varphi(t_o)\,d\varphi = \frac{1}{2\pi}\left[\frac{\varphi}{2}-\frac{\sin 4\varphi}{8}\right]_{\varphi=0}^{\varphi=2\pi} = \frac{1}{\cancel{2\pi}}\cdot\frac{\cancel{2\pi}}{2} = \frac{1}{2}\end{aligned}}$$

Thus, the <u>differential</u> Thomson scattering cross section for an <u>unpolarized</u> macroscopic *EM* plane wave incident on a free electric charge *q*, averaged over a <u>long</u> time interval Δt_{avg} is:

$$\boxed{\begin{aligned}\frac{d\sigma_T^{unpol}(\theta,\varphi)}{d\Omega} &\simeq \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left\langle\sin^2\Theta_\in\right\rangle = \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\sin^2\theta\left\langle\sin^2 2\varphi(t_o)\right\rangle+\cos^2\theta\right) \\ &= \left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\frac{1}{2}\sin^2\theta+\cos^2\theta\right) = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\sin^2\theta+2\cos^2\theta\right) \\ &= \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\underbrace{\left\{\sin^2\theta+\cos^2\theta\right\}}_{=1}+\cos^2\theta\right) = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(1+\cos^2\theta\right)\end{aligned}}$$

$$\boxed{\frac{d\sigma_T^{unpol}(\theta,\varphi)}{d\Omega} \simeq \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(1+\cos^2\theta\right)}\ \left(m^2/sr\right) \Leftarrow$$ *n.b.* has <u>no</u> φ-dependence!

Hence, we see that for an <u>unpolarized</u> macroscopic *EM* plane wave incident on a free electric charge *q*, averaged over a <u>long</u> time interval Δt_{avg}, the <u>differential</u> Thomson scattering cross section has ***<u>no</u>*** φ-dependence, as we anticipated.

The total Thomson scattering cross section for an unpolarized macroscopic *EM* plane wave incident on a free electric charge *q*, averaged over a long time interval Δt_{avg} is:

$$\sigma_T^{unpol} = \int\left(\frac{d\sigma_T(\theta,\varphi)}{d\Omega}\right)d\Omega = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(1+\cos^2\theta\right)\sin\theta d\theta d\varphi$$

But:

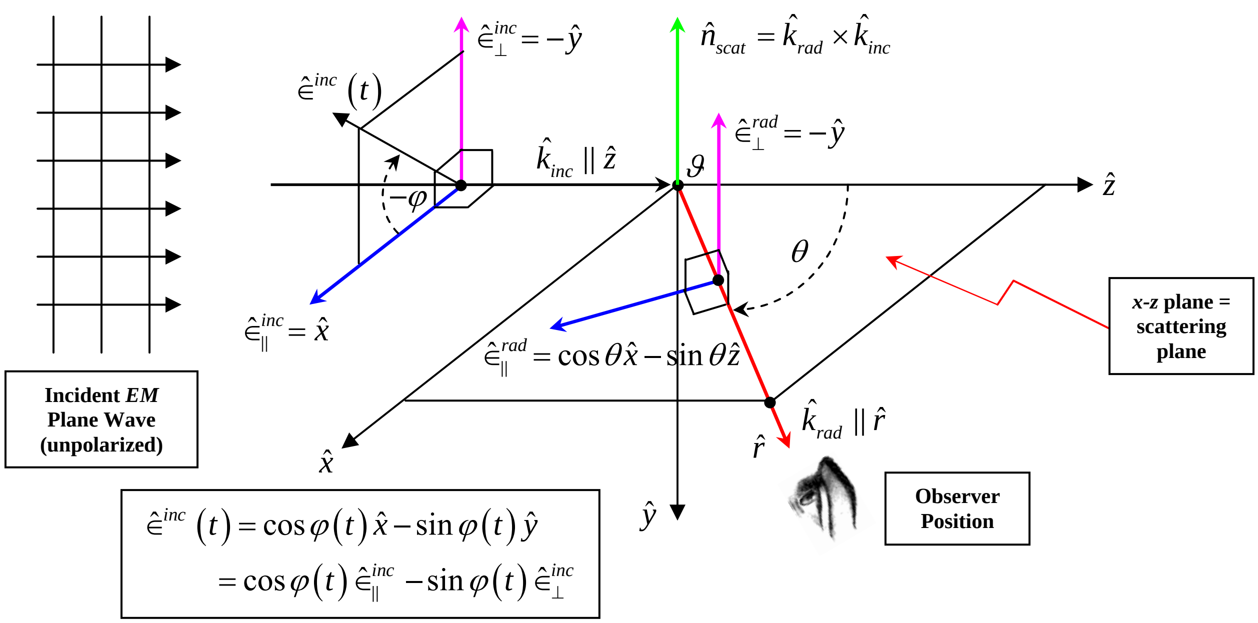
$$\frac{1}{2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(1+\cos^2\theta\right)\sin\theta d\theta d\varphi = \frac{\cancel{2}\pi}{\cancel{2}}\int_{\theta=0}^{\theta=\pi}\left(1+\cos^2\theta\right)\sin\theta d\theta$$
$$= \pi\int_{\theta=0}^{\theta=\pi}\sin\theta d\theta + \pi\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta d\theta = 2\pi + \pi\cdot\frac{2}{3} = \frac{6\pi}{3} + \frac{2\pi}{3} = \frac{8\pi}{3}$$

Thus, here again we see that the Thomson scattering cross section for an unpolarized / randomly-polarized macroscopic *EM* plane wave incident on a free electric charge *q*, averaged over a long time interval Δt_{avg} is:

$$\boxed{\sigma_T^{unpol} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \sigma_T}\ \left(m^2\right)$$

Even though an incident *EM* plane wave may be unpolarized, this does *not* mean that an observer at the field/observation point $P(\vec{r}) = P(r,\theta,\varphi)$ will observe unpolarized scattered radiation – quite the contrary! The reason for this is simple – for a specific field/observation point $P(r,\theta,\varphi)$ the *EM* radiation that is scattered into that specific (θ,φ) angular region depends sensitively on the incident polarization state! We can easily see this from the following:

Consider an observer located in the x-z plane at $P(r,\theta,\varphi=0)$ as shown in the figure below:



$$\hat{\in}^{inc}(t) = \cos\varphi(t)\hat{x} - \sin\varphi(t)\hat{y}$$
$$= \cos\varphi(t)\hat{\in}_\parallel^{inc} - \sin\varphi(t)\hat{\in}_\perp^{inc}$$

In the above figure, note that the *x*-*z* scattering plane is *defined* by the two wavevectors: $\hat{k}_{inc} \parallel \hat{z}$ and $\hat{k}_{rad} \parallel \hat{r}$, where {here, with $\varphi = 0$ } $\boxed{\hat{r} = \sin\theta\cos\varphi\hat{x} + \cos\theta\hat{z} = \sin\theta\hat{x} + \cos\theta\hat{z}}$.
The unit normal to the scattering plane $\hat{n}_{scat}$ is defined by: $\boxed{\hat{n}_{scat} = \hat{k}_{rad} \times \hat{k}_{inc}}$.

For an unpolarized/arbitrary/random polarization of the incident *EM* plane wave, from the above figure, it can also be seen that the {instantaneous} polarization unit vector $\hat{\in}^{inc}(t)$ associated with the incident unpolarized *EM* plane wave can be decomposed into a component parallel to/lying within the *x*-*z* scattering plane $\hat{\in}_{\parallel}^{inc}$ and a component perpendicular to the *x*-*z* scattering plane $\hat{\in}_{\perp}^{inc}$: $\boxed{\hat{\in}^{inc}(t) = \cos\varphi(t)\hat{x} - \sin\varphi(t)\hat{y} = \cos\varphi(t)\hat{\in}_{\parallel}^{inc} - \sin\varphi(t)\hat{\in}_{\perp}^{inc}}$.

Likewise, for the scattered *EM* radiation, whatever instantaneous polarization $\hat{\in}^{rad}(t)$ exists can be decomposed into a component parallel to/lying within the *x*-*z* scattering plane $\hat{\in}_{\parallel}^{rad}$ and a component perpendicular to the *x*-*z* scattering plane $\hat{\in}_{\perp}^{rad}$.

However, as we have seen above for Thomson scattering with $\hat{x}$ and $-\hat{y}$ linearly-polarized incident *EM* waves, the orientation of the incident vs. scattered polarization vectors is unchanged by the scattering process in the following sense:

In the above figure, for an incident *EM* plane wave with linear polarization $\hat{\in}_{\parallel}^{inc} = \hat{x}$ parallel to (*i.e.* lying in) the *x*-*z* scattering plane, the scattered polarization is $\hat{\in}_{\parallel}^{rad} = \cos\theta\hat{x} - \sin\theta\hat{z}$ which also lies in the *x*-*z* scattering plane – notice the two blue polarization vectors in the above figure.

For an incident *EM* plane wave with linear polarization $\hat{\in}_{\perp}^{inc} = -\hat{y}$ perpendicular to the *x*-*z* scattering plane, the scattered polarization is $\hat{\in}_{\perp}^{rad} = -\hat{y} = \hat{\in}_{\perp}^{inc}$ which is also perpendicular to the *x*-*z* scattering plane – notice the two magenta polarization vectors in the above figure.

Thus, we can decompose the differential Thomson scattering cross section into that in which the polarization vector of the scattered *EM* radiation lies in (*i.e.* parallel to) the *x*-*z* scattering plane and that in which the polarization vector of the scattered radiation is perpendicular to the *x*-*z* scattering plane, which we already have (!) from our above *LPx* and *LPy* results, namely:

$$\boxed{\begin{aligned}\frac{d\sigma_T^{unpol\parallel}(\theta,\varphi=0)}{d\Omega} &= \frac{1}{2}\frac{d\sigma_T^{LPx}(\theta,\varphi=0)}{d\Omega} \simeq \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \sin^2\Theta_x \qquad \left(m^2/sr\right)\\ &= \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\sin^2\theta\cancel{\sin^2\varphi} + \cos^2\theta\right) = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\cos^2\theta\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{d\sigma_T^{unpol\perp}(\theta,\varphi=0)}{d\Omega} &= \frac{1}{2}\frac{d\sigma_T^{LPy}(\theta,\varphi=0)}{d\Omega} \simeq \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \sin^2\Theta_y = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(\sin^2\theta\underbrace{\cos^2\varphi}_{=1} + \cos^2\theta\right)\\ &= \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{=1} = \frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 \qquad \left(m^2/sr\right)\end{aligned}}$$

n.b. The 1/2 factor in the above arises from statistically projecting $\hat{\in}^{inc}(t)$ onto $\hat{\in}_{\parallel}^{inc} = \hat{x}$ and $\hat{\in}_{\perp}^{inc} = -\hat{y}$.

Thus, for an observation point $P(r,\theta,\varphi=0)$ lying in the *x*-*z* scattering plane:

$$\boxed{\frac{d\sigma_T^{unpol\,\parallel}(\theta,\varphi=0)}{d\Omega}=\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\cos^2\theta}\quad\left(m^2/sr\right)$$

And:

$$\boxed{\frac{d\sigma_T^{unpol\,\perp}(\theta,\varphi=0)}{d\Omega}=\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2}\quad\left(m^2/sr\right)\quad\Leftarrow\ \textit{n.b.}\text{ has }\underline{\text{no}}\ \theta\text{-dependence!}$$

Note further that:

$$\boxed{\begin{aligned}\frac{d\sigma_T^{unpol}(\theta,\varphi=0)}{d\Omega}&=\frac{d\sigma_T^{unpol\,\parallel}(\theta,\varphi=0)}{d\Omega}+\frac{d\sigma_T^{unpol\,\perp}(\theta,\varphi=0)}{d\Omega}\\&=\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\cos^2\theta+\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\\&=\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(1+\cos^2\theta\right)\quad\left(m^2/sr\right)\end{aligned}}$$

Thus:

$$\boxed{\frac{d\sigma_T^{unpol}(\theta,\varphi=0)}{d\Omega}\simeq\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_o mc^2}\right)^2\left(1+\cos^2\theta\right)=\frac{d\sigma_T^{unpol}(\theta,\varphi)}{d\Omega}}\quad\left(m^2/sr\right)$$

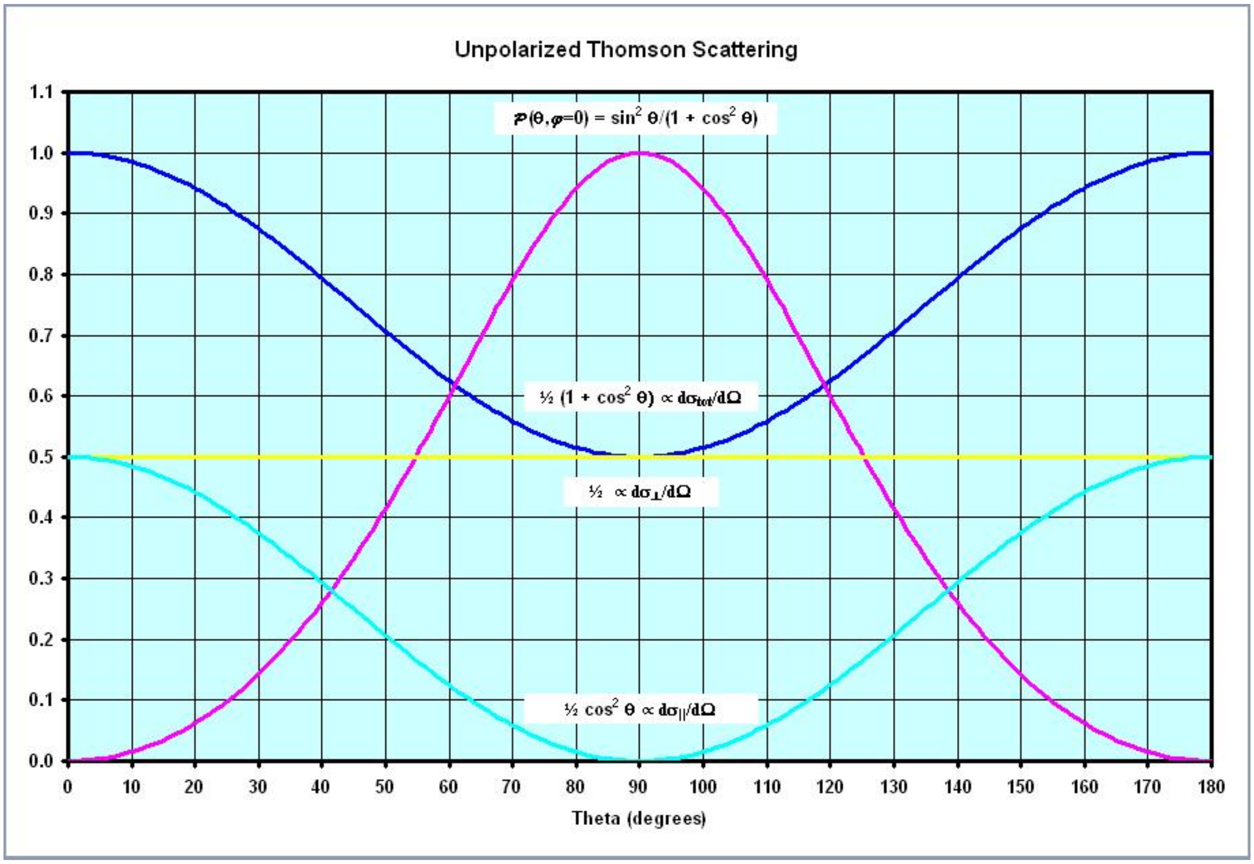
We now introduce a quantity known as "***the polarization***" $\mathcal{P}(\theta,\varphi=0)$ which is formally a specific type of ***asymmetry parameter*** $\mathcal{A}(x)$ – the normalized/fractional difference between two related variables $\boxed{\mathcal{A}(x)\equiv\frac{a(x)-b(x)}{a(x)+b(x)}}$. Thus, in general $\mathcal{A}(x)$ ranges between $\boxed{-1\le\mathcal{A}(x)\le+1}$. (*n.b.* Sometimes $\mathcal{A}(x)$ is expressed in terms of a *percentage*).

Here, for our current Thomson scattering physics situation, the polarization {asymmetry} $\mathcal{P}(\theta,\varphi=0)$ is defined as the normalized/fractional difference (*i.e.* asymmetry) between the perpendicular $(\perp)$ vs. parallel $(\parallel)$ differential Thomson scattering cross sections:

$$\boxed{\mathcal{P}_T^{unpol}(\theta,\varphi=0)\equiv\frac{\dfrac{d\sigma_T^{unpol\,\perp}(\theta,\varphi=0)}{d\Omega}-\dfrac{d\sigma_T^{unpol\,\parallel}(\theta,\varphi=0)}{d\Omega}}{\dfrac{d\sigma_T^{unpol\,\perp}(\theta,\varphi=0)}{d\Omega}+\dfrac{d\sigma_T^{unpol\,\parallel}(\theta,\varphi=0)}{d\Omega}}=\frac{1-\cos^2\theta}{1+\cos^2\theta}=\frac{\sin^2\theta}{1+\cos^2\theta}}$$

Here, we see that due to the physics associated with Thomson scattering of an unpolarized *EM* plane wave from a free electric charge, $\mathcal{P}_T^{unpol}(\theta,\varphi=0)$ ranges between $\boxed{0\le\mathcal{P}_T^{unpol}(\theta,\varphi=0)\le1}$. Due the geometrical constraints imposed in the overall scattering process, the instantaneous orientation of the induced electric dipole yields useful non-zero time-averaged information!

The angular dependence of the normalized differential Thomson scattering cross sections $\boxed{\left(\dfrac{d\sigma_T^{unpol}(\theta,\varphi=0)}{d\Omega}\right)\Bigg/\left(\dfrac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \dfrac{1}{2}\left(1+\cos^2\theta\right)}$, $\boxed{\left(\dfrac{d\sigma_T^{unpol\perp}(\theta,\varphi=0)}{d\Omega}\right)\Bigg/\left(\dfrac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \dfrac{1}{2}}$, $\boxed{\left(\dfrac{d\sigma_T^{unpol\parallel}(\theta,\varphi=0)}{d\Omega}\right)\Bigg/\left(\dfrac{q^2}{4\pi\varepsilon_o mc^2}\right)^2 = \dfrac{1}{2}\cos^2\theta}$ and the polarization $\boxed{\mathcal{P}_T^{unpol}(\theta,\varphi=0) = \dfrac{\sin^2\theta}{1+\cos^2\theta}}$ for unpolarized Thomson scattering as a function of θ are shown together in the figure below:



Various things can be seen/learned from the four curves on the above graph:

a.) As mentioned above, the $\perp$ differential Thomson scattering cross section is constant/flat, independent of the scattering angle θ. The || component is maximal at $\theta = 0^o$ and $\theta = 180^o$.

b.) At $\theta = 0^o$ (forward scattering) and at $\theta = 180^o$ (backward scattering) the $\perp$ vs. || differential Thomson scattering cross sections are equal to each other, and because of this, the polarization {asymmetry} vanishes, *i.e.* $\boxed{\mathcal{P}_T^{unpol}(\theta=0,\varphi=0) = \mathcal{P}_T^{unpol}(\theta=180^o,\varphi=0) = 0}$.

c.) At $\theta = 90^o$ the || differential Thomson scattering cross section vanishes, and because of this, the polarization {asymmetry} is maximal, *i.e.* $\boxed{\mathcal{P}_T^{unpol}(\theta=90^o,\varphi=0) = 1}$, thus at $\theta = 90^o$, the Thomson scattering of an unpolarized *EM* wave by a free charge *q* is purely/100% due to the perpendicular $(\perp)$ polarization component (only) of the incident *EM* wave!!!

Scattering of *EM* Radiation by Neutral Atoms and/or Molecules

As discussed previously in P436 Lecture Notes 7.5 (Dispersion Phenomena in Linear Dielectric Media) atoms and molecules are composite objects – consisting of {relatively light} electrons bound to {relatively massive} nuclei. When a monochromatic (*i.e.* single-frequency) *EM* plane wave is incident on a neutral atom (or molecule) – for simplicity's sake, assumed to be spherical in shape – the electric field of the incident *EM* plane wave $\tilde{\vec{E}}_{inc}(\vec{r},t)$ induces electric dipole moment(s) in the neutral atom/molecule {primarily} due to jiggling the light electrons at the angular frequency ω of the incident *EM* plane wave, arising from the driving force $-e\tilde{\vec{E}}_{inc}(\vec{r},t)$ acting on the bound electrons:

$$\boxed{m_e\ddot{\tilde{\vec{r}}}(t)+m_e\gamma\dot{\tilde{\vec{r}}}(t)+k_e\tilde{\vec{r}}(t)=-e\tilde{\vec{E}}_{inc}(\vec{r},t)} \leftarrow \text{inhomogeneous } 2^{\text{nd}}\text{-order differential eqn.}$$

$$\boxed{\underbrace{m_e\frac{\partial^2\tilde{\vec{r}}(t)}{\partial t^2}}_{m_e\vec{a}}+\underbrace{m_e\gamma\frac{\partial\tilde{\vec{r}}(t)}{\partial t}}+\underbrace{k_e\tilde{\vec{r}}(t)}=\underbrace{-e\tilde{\vec{E}}(\vec{r},t)}} \leftarrow$$

n.b. we have {again} neglected the $e\tilde{\vec{v}}\times\tilde{\vec{B}}\ \left(\ll e\tilde{\vec{E}}\right)$ term here...

Velocity-dependent damping term
$\gamma \equiv$ damping constant

Potential Force
(binding of atomic electrons to atom)

Driving Force

$\boxed{m_e = \text{electron mass } = 9.1\times10^{-31}\,kg}$

For a driving force term sinusoidally varying in time with angular frequency ω associated with a monochromatic *EM* plane wave with linear polarization in the $\hat{x}$-direction incident on the atom/molecule {located at the origin, $\vartheta(\vec{r}=0)$ }: $\boxed{-e\tilde{\vec{E}}_{inc}(\vec{r}=0,t)=-eE_o e^{-i\omega t}\hat{x}}$

the inhomogeneous force equation becomes: $\boxed{m_e\ddot{\tilde{\vec{x}}}+m_e\gamma\dot{\tilde{\vec{x}}}+k_e\tilde{\vec{x}}=-eE_o e^{-i\omega t}\hat{x}}$.

The solution of this inhomogeneous second order differential equation is: $\boxed{\tilde{\vec{x}}(\vec{r}=0,t)=\tilde{x}_o e^{-i\omega t}\hat{x}}$

where: $\boxed{\tilde{x}_o=\dfrac{eE_o/m_e}{\left[\left(\omega^2-\omega_0^2\right)+i\gamma\omega\right]}}$ = Atomic electron spatial displacement amplitude {n.b. complex!}

and: $\boxed{\omega_0 \equiv \sqrt{k_e/m_e}}$ (radians/sec) = characteristic/natural resonance {angular} frequency.

The corresponding {complex} induced electric dipole moment $\tilde{\vec{p}}(\vec{r}=0,t)$ is:

$$\boxed{\tilde{\vec{p}}(\vec{r}=0,t)=-e\tilde{\vec{x}}(t)=-e\tilde{x}_o e^{-i\omega t}\hat{x}=-\left(\frac{e^2E_o}{m_e}\right)\frac{1}{\left[\left(\omega^2-\omega_0^2\right)+i\gamma\omega\right]}e^{-i\omega t}\hat{x}}$$

Using the "standard trick" $\boxed{z=\dfrac{1}{x+iy}=\dfrac{1}{x+iy}\dfrac{x-iy}{x-iy}=\dfrac{x-iy}{x^2+y^2}}$ with $\boxed{x\equiv\left(\omega^2-\omega_0^2\right)}$ and $\boxed{y\equiv\gamma\omega}$

we can rationalize/rewrite this as:

$$\boxed{\tilde{\vec{p}}(\vec{r}=0,t)=-\left(\frac{e^2E_o}{m_e}\right)\left\{\frac{\left(\omega^2-\omega_0^2\right)-i\gamma\omega}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\right\}e^{-i\omega t}\hat{x}}$$

The above expression describes the induced electric dipole moment $\tilde{\vec{p}}(\vec{r}=0,t)$ of an atom / molecule associated with a *single* *QM* transition/resonance at angular frequency $\boxed{\omega_0 \equiv \sqrt{k_e/m_e}}$ and natural linewidth $\boxed{\Gamma \equiv \gamma/2\pi\,(FWHM)}$ sec^{-1}. However, real atoms/molecules are governed by quantum mechanics and in general have many possible quantum mechanical bound states of the atomic electrons, with transitions between them {resonances} with associated transition energies $\Delta E_j = \hbar\omega_{0_j}$ and linewidths Γ_j, as dictated by quantum mechanical selection rules. Thus, a more realistic model of the atom/molecule that takes into account the various *QM* transitions / resonances present in an atom/molecule, properly weighted for each such resonance, gives:

$$\boxed{\tilde{\vec{p}}_{tot}(\vec{r}=0,t) = -\left(\frac{e^2 E_o}{m_e}\right)\left\{\sum_{j=1}^{n} f_j^{osc}\frac{\left(\omega^2-\omega_{0_j}^2\right)-i\gamma_j\omega}{\left[\left(\omega^2-\omega_{0_j}^2\right)^2+\gamma_j^2\omega^2\right]}\right\}e^{-i\omega t}\hat{x}}$$

where the angular frequency and natural linewidth associated with the j^{th} resonance are: $\boxed{\omega_{0_j} \equiv \sqrt{k_{e_j}/m_e}}$ rad/sec and $\boxed{\Gamma_j \equiv \gamma_j/2\pi\,(FWHM)}$ sec^{-1}respectively, and the so-called "oscillator strength" f_j^{osc} associated with the j^{th} resonance is such that: $\boxed{\sum_{j=1}^{n} f_j^{osc} = 1}$.

In the above expression for $\tilde{\vec{p}}_{tot}(\vec{r}=0,t)$, note that the *n* individual contributions to the overall / total induced electric dipole moment of the atom/molecule are ***coherently*** added together, *i.e.* added together at the ***amplitude*** level – the tacit assumption that has been made here is that the wavelength λ associated with the incident monochromatic *EM* plane wave/outgoing scattered/radiated *EM* wave is much larger than the characteristic size of the atom/molecule, *i.e.* $\lambda \gg r_{atom}, r_{molecule}$ and thus variation of phase(s) $e^{i\varphi} \sim e^{ikr}$ associated with the incoming/outgoing *EM* waves, *e.g.* over the diameter of the atom/molecule are negligible and hence can be/are neglected. Note that this approximation is certainly valid *e.g.* for *EM* radiation in the optical portion of the *EM* spectrum (and below), since for visible light: $\lambda_{violet} \sim 400\,nm$, $\lambda_{red} \sim 650\,nm$ whereas typically $r_{atom}, r_{molecule} \sim (0.1-few)\,nm$.

Then evaluating $\ddot{\tilde{\vec{p}}}_{tot}(\vec{r}=0,t)$ at the retarded time t_o:

$$\boxed{\ddot{\tilde{\vec{p}}}_{tot}(\vec{r}=0,t_o) = +\omega^2\left(\frac{e^2 E_o}{m_e}\right)\left\{\sum_{j=1}^{n} f_j^{osc}\frac{\left(\omega^2-\omega_{0_j}^2\right)-i\gamma_j\omega}{\left[\left(\omega^2-\omega_{0_j}^2\right)^2+\gamma_j^2\omega^2\right]}\right\}e^{-i\omega t_o}\hat{x}}$$

And thus:

$$\boxed{\ddot{\tilde{p}}_{tot}^2(0,t_o) = \ddot{\tilde{\vec{p}}}_{tot}(0,t_o)\bullet\ddot{\tilde{\vec{p}}}_{tot}(0,t_o) = \omega^4\left(\frac{e^2 E_o}{m_e}\right)^2\left\{\sum_{j=1}^{n} f_j^{osc}\frac{\left(\omega^2-\omega_{0_j}^2\right)-i\gamma_j\omega}{\left[\left(\omega^2-\omega_{0_j}^2\right)^2+\gamma_j^2\omega^2\right]}\right\}^2 e^{-2i\omega t_o}}$$

From the discussion above (p. 3-9) for Thomson scattering of a monochromatic *EM* plane wave linearly polarized in the $\hat{x}$-direction incident on a free electric charge q located at the origin ϑ, the instantaneous differential power radiated by an induced atomic/molecular electric dipole moment, oriented along the $\hat{x}$-axis {due to the polarization of the incident *EM* radiation} is:

$$\boxed{\frac{dP_{\mathrm{r}}^{rad}\left(\theta,\varphi,t_o\right)}{d\Omega}=\vec{S}^{rad}\left(r,\theta,\varphi,t_o\right)\bullet r^2\hat{r}\simeq\frac{\mu_o\ddot{\vec{p}}_{tot}^{\,2}\left(0,t_o\right)}{16\pi^2c}\sin^2\Theta_x}\left(\frac{\text{Watts}}{\text{steradian}}\right)$$

where $\ddot{p}^2\left(0,t_o\right)$ is given above; and: $\boxed{\sin^2\Theta_x=\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)}$.

Thus: $$\boxed{\frac{dP_{\mathrm{r}}^{rad}\left(\theta,\varphi,t_o\right)}{d\Omega}=\vec{S}^{rad}\left(r,\theta,\varphi,t_o\right)\bullet r^2\hat{r}\simeq\frac{\mu_o\ddot{\vec{p}}_{tot}^{\,2}\left(0,t_o\right)}{16\pi^2c}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)}\left(\frac{\text{Watts}}{\text{steradian}}\right)$$

Again, carrying out the time-averaging process on this quantity, and taking only the real part of it for the physically-meaningful result for "far-zone" radiation associated with this scattering process, however, because the above expression for $\ddot{p}_{tot}^2\left(0,t_o\right)$ is extremely complicated, for the purpose of discussing the salient physics features/behavior, we will assume for simplicity's sake that only a single resonance exists (with $f_1^{osc}=1$), rather than n of them. The full-blown expression for *n* resonances can *e.g.* be coded up on a computer and results obtained numerically.

Thus, with the simplifying assumption of a single resonance in the atom/molecule:

$$\boxed{\begin{aligned}\frac{d\left\langle P_{\mathrm{r}}^{rad}\left(\theta,\varphi,t_o\right)\right\rangle}{d\Omega}&=\left\langle\vec{S}^{rad}\left(r,\theta,\varphi,t_o\right)\right\rangle\bullet r^2\hat{r}\simeq\frac{1}{2}\frac{\mu_o\operatorname{Re}\left\{\ddot{\vec{p}}^{\,2}\left(0,t_o\right)\right\}}{16\pi^2c}\sin^2\Theta_x\\&=\frac{1}{2}\cdot\frac{\mu_o}{16\pi^2c}\left(\frac{e^2E_o}{m_e}\right)^2\left\{\frac{\omega^4\cancel{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]^{\cancel{2}}}\right\}\sin^2\Theta_x\ \text{ and using: }\ \mu_o=\frac{1}{\varepsilon_oc^2}\\&=\frac{1}{2}\varepsilon_oE_o^2c\left(\frac{e^2}{4\pi\varepsilon_om_ec^2}\right)^2\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)\end{aligned}}\left(\frac{\text{Watts}}{\text{sr}}\right)$$

The time average of the {magnitude of} the instantaneous flux of *EM* energy incident on the electric dipole {located at the origin ϑ} is: $\boxed{I_{inc}\left(0\right)=\left\langle\left|\vec{S}_{inc}\left(0,t_o\right)\right|\right\rangle=\frac{1}{2}\left|\vec{S}_{inc}\left(0,t_o\right)\right|=\frac{1}{2}\varepsilon_oE_o^2c}\left(\frac{\text{Watts}}{\text{m}^2}\right)$ and thus:

$$\boxed{\begin{aligned}\frac{d\sigma_{atom}\left(\theta,\varphi\right)}{d\Omega}&\equiv\frac{1}{\left\langle\left|\vec{S}_{inc}\left(0,t_o\right)\right|\right\rangle}\frac{d\left\langle P_{rad}\left(\theta,\varphi,t_o\right)\right\rangle}{d\Omega}\simeq\frac{\cancel{\frac{1}{2}\varepsilon_oE_o^2c}}{\cancel{\frac{1}{2}\varepsilon_oE_o^2c}}\left(\frac{e^2}{4\pi\varepsilon_om_ec^2}\right)^2\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\sin^2\Theta_x\\&=\left(\frac{e^2}{4\pi\varepsilon_om_ec^2}\right)^2\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)\end{aligned}}\left(\frac{m^2}{sr}\right)$$

Since the classical electron radius is: $\boxed{r_e \equiv e^2/4\pi\varepsilon_o m_e c^2 \simeq 2.82\times10^{-15}\,m}$, we can write the <u>differential</u> scattering cross section {per atom/molecule} for an incident *EM* plane wave propagating in the $+\hat{z}$-direction and linearly polarized in the $\hat{x}$-direction in terms of r_e as:

$$\boxed{\frac{d\sigma_{atom}^{LPx}(\theta,\varphi)}{d\Omega} \simeq r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

Hence, we see that this result is very similar to that obtained for electron Thomson scattering for an incident *EM* plane wave propagating in the $+\hat{z}$-direction and linearly polarized in the $\hat{x}$-direction:

$$\boxed{\frac{d\sigma_{T\,e^-}^{LPx}(\theta,\varphi)}{d\Omega} \simeq r_e^2\sin^2\Theta_x = r_e^2\left(\sin^2\theta\sin^2\varphi+\cos^2\theta\right)} \quad \left(m^2/sr \text{ per electron}\right)$$

For the atomic/molecular <u>differential</u> scattering cross section, we simply have the additional dimensionless factor $\boxed{\omega^4\Big/\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}$ arising from the physics associated with the {quantum mechanical} internal structure of the atom/molecule!

Thus, since the <u>total</u> cross section for Thomson scattering of an *EM* plane wave incident on an electron is: $\boxed{\sigma_T = \frac{8\pi}{3}r_e^2}$ $\left(m^2 \text{ per electron}\right)$, the <u>total</u> cross section for atomic/molecular scattering of an *EM* plane wave is: $\boxed{\sigma_{atom} = \frac{8\pi}{3}r_e^2\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}}$ $\left(m^2 \text{ per atom/molecule}\right)$.

Similarly, for each of the other polarizations of the incident monochromatic *EM* plane wave discussed above for Thomson scattering, we obtain very similar results for atomic/molecular scattering:

$$\boxed{\frac{d\sigma_{atom}^{LPy}(\theta,\varphi)}{d\Omega} \simeq r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(\sin^2\theta\cos^2\varphi+\cos^2\theta\right)} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

$$\boxed{\frac{d\sigma_{atom}^{LP\in}(\theta,\varphi)}{d\Omega} \simeq r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(\sin^2\theta\sin^2 2\varphi+\cos^2\theta\right)} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

$$\boxed{\frac{d\sigma_{atom}^{unpol}(\theta,\varphi=0)}{d\Omega} \simeq \frac{1}{2}r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\left(1+\cos^2\theta\right)} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

with:

$$\boxed{\frac{d\sigma_{atom}^{unpol\perp}(\theta,\varphi=0)}{d\Omega} \simeq \frac{1}{2}r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

and:

$$\boxed{\frac{d\sigma_{atom}^{unpol\parallel}(\theta,\varphi=0)}{d\Omega} \simeq \frac{1}{2}r_e^2 \frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}\cos^2\theta} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

Thus, we also see that the ***polarization*** {asymmetry} for atomic/molecular scattering by an unpolarized *EM* plane wave is the same as that for Thomson scattering:

$$\boxed{\mathcal{P}_{atom}^{unpol}(\theta,\varphi=0) \equiv \frac{\dfrac{d\sigma_{atom}^{unpol\perp}(\theta,\varphi=0)}{d\Omega}-\dfrac{d\sigma_{atom}^{unpol\parallel}(\theta,\varphi=0)}{d\Omega}}{\dfrac{d\sigma_{atom}^{unpol\perp}(\theta,\varphi=0)}{d\Omega}+\dfrac{d\sigma_{atom}^{unpol\parallel}(\theta,\varphi=0)}{d\Omega}} = \frac{1-\cos^2\theta}{1+\cos^2\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}}$$

Hence the results on the graph shown on page 22 of these lecture notes for differential Thomson scattering are in fact *also* valid for atomic/molecular scattering by an unpolarized *EM* plane wave.

Whereas the Thomson/free electron scattering cross section results are independent of the frequency of the incident *EM* plane wave, from the above results, it is manifestly apparent that the atomic/molecular bound electron scattering cross section results depend very sensitively on the frequency of the incident *EM* plane wave.

Let us examine the frequency behavior of the resonance lineshape factor in the above atomic/molecular scattering cross section results. $\boxed{\dfrac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}}$

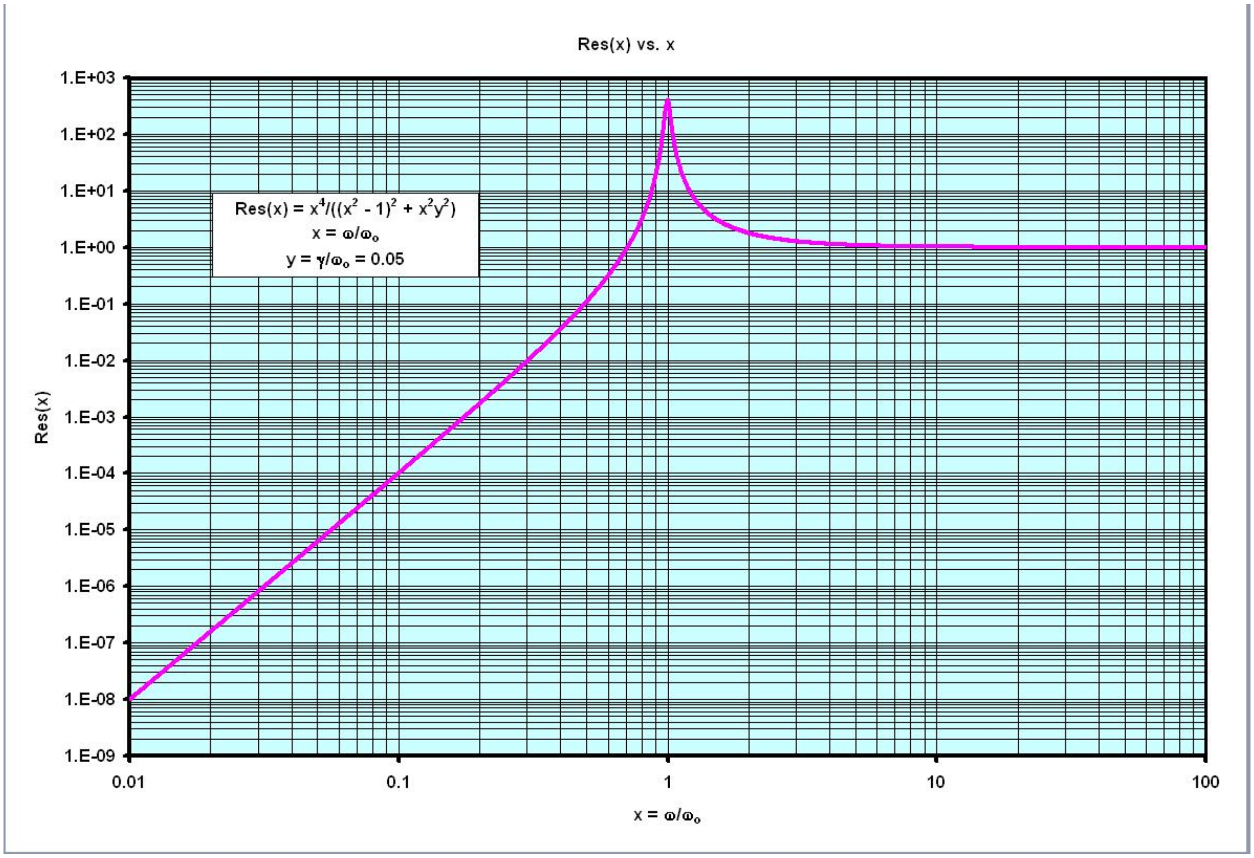
Noting that in atoms/molecules, the natural line widths associated with resonances/transitions between distinct quantum states {typically in the *UV* portion of the *EM* spectrum for atoms} are quite narrow, *i.e.* that:

$$\boxed{\left\{\Gamma \equiv \gamma/2\pi\,(FWHM)\right\} \ll \left\{\omega_0 \equiv \sqrt{k_e/m_e}\right\}}.$$

Defining $\boxed{x \equiv \omega/\omega_0}$ and $\boxed{y \equiv \gamma/\omega_0}$, a log-log plot of the resonance lineshape $\boxed{\dfrac{x^4}{\left[\left(x^2-1\right)^2+x^2y^2\right]}}$ vs. x is shown in the figure below.

Below the peak of the resonance $(\omega < \omega_0)$, note the linear 4 decade increase in the lineshape per 1 decade increase in $x \equiv \omega/\omega_0$ which is due to the ω^4 dependence of the lineshape.

Above the peak of the resonance $(\omega > \omega_0)$, note that the lineshape is flat with frequency, *i.e.* it is independent of frequency!



Referring to the above plot of the resonance lineshape, we see that there are <u>three</u> distinct frequency regions to consider:

1.) <u>Low frequencies</u>: $\omega \ll \omega_0$

When $\omega \ll \omega_0$, the factor $\left(\omega^2 - \omega_0^2\right)^2 \approx \omega_0^4$ in the resonance lineshape $\boxed{\dfrac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}}$ and additionally, since $\gamma \ll \omega_0$, then for $\omega \ll \omega_0$:

$$\boxed{\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]} \approx \frac{\omega^4}{\left[\omega_0^4+\gamma^2\omega^2\right]} \approx \frac{\omega^4}{\omega_0^4} = \left(\frac{\omega}{\omega_o}\right)^4 = \frac{1}{r_e^2}\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega}}.$$

Thus, at <u>low</u> frequencies $\left(\omega \ll \omega_0\right)$ the differential and total atomic/molecular scattering cross sections are <u>strongly</u> frequency-dependent, and behave as:

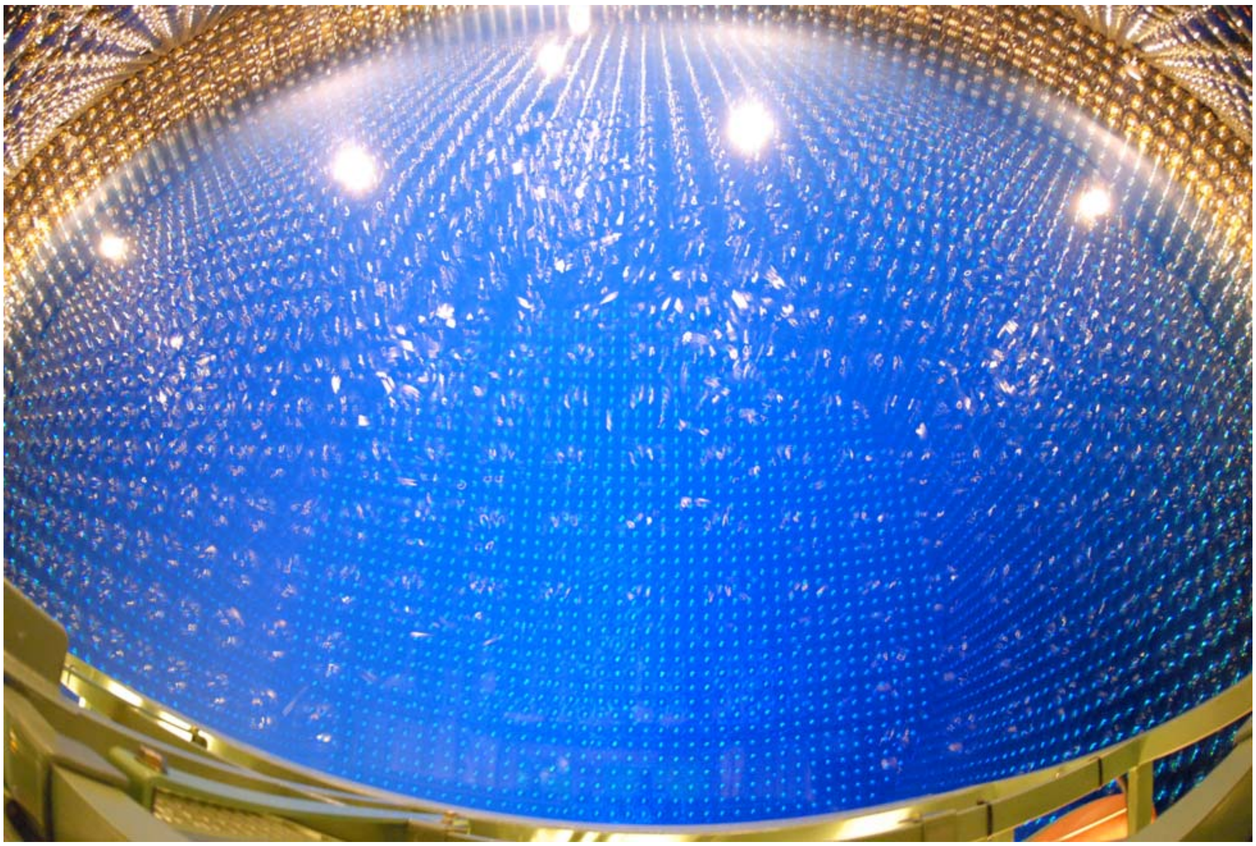
$$\boxed{\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega} \approx r_e^2\left(\frac{\omega}{\omega_o}\right)^4 \times \{\text{Angular Factor}\}} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

and:

$$\boxed{\sigma_{atom} \approx \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\omega_o}\right)^4} \quad \left(m^2 \text{ per atom/molecule}\right)$$

This is type of atomic/molecular scattering of *EM* plane waves at low frequencies $(\omega \ll \omega_0)$ is known as ***Rayleigh scattering***, in honor or Lord Rayleigh, who carried out early theoretical work associated with this topic in the latter part of the 19^{th} century.

The strong ω^4 frequency dependence of the Rayleigh scattering cross section explains why the sky and *e.g.* pure water (!) appears blue. Blue light is Rayleigh-scattered ~ 4-5× more than red light! The following picture shows the gorgeous blue color arising from Rayleigh scattering of light in the ultra-pure H2O tank of the Super-Kamiokande experiment (located in Japan):



The behavior of the polarization {asymmetry} $\boxed{\mathcal{P}_{atom}^{unpol}(\theta,\varphi=0)=\sin^2\theta\big/\left(1+\cos^2\theta\right)}$ for unpolarized incident *EM* plane waves in the visible portion of the *EM* spectrum also explains why the light from the sky is polarized, and especially so at $\left(\theta=90^o,\varphi=0\right)$! Please go back/ refer to/look at P436 Lecture Notes 13, *p*. 17-18 where we discussed this originally.

2.) Resonance: $\omega \approx \omega_0$

On (or near) a resonance $\omega \approx \omega_0$ (typically in the UV portion of the spectrum for atoms) the factor $\left(\omega^2-\omega_0^2\right)^2 \approx 0$ and thus for $\omega \approx \omega_0$:

$$\boxed{\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]} \approx \frac{\omega_0^4}{\gamma^2\omega_0^2} = \frac{\omega_0^2}{\gamma^2} = \left(\frac{\omega_0}{\gamma}\right)^2 = \frac{1}{r_e^2}\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega}}$$

Since $\gamma \ll (\omega_0 \approx \omega)$ then $(\omega_0/\gamma)^2 \gg 1$ and thus we see that on (or near) a resonance the differential and total atomic/molecular ***resonant*** scattering cross sections become extremely large – incident *EM* radiation is absorbed/re-emitted/scattered prolifically on/near a resonance:

$$\boxed{\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega} \approx r_e^2\left(\frac{\omega_o}{\gamma}\right)^2 \times \{\text{Angular Factor}\}} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

and:

$$\boxed{\sigma_{atom} \approx \frac{8\pi}{3}r_e^2\left(\frac{\omega_o}{\gamma}\right)^2} \quad \left(m^2 \text{ per atom/molecule}\right)$$

3.) High frequencies: $\omega \gg \omega_0$

For high frequencies $(\omega \gg \omega_0)$ the bound atomic/molecular electrons behave as if they are free – *i.e.* as in Thomson scattering of free electrons! When $\omega \gg \omega_0$ the behavior of the factor $\left(\omega^2-\omega_0^2\right)^2$ in $\boxed{\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]}}$ is $\left(\omega^2-\omega_0^2\right)^2 \approx \omega^4$ and additionally, since $\gamma \ll \omega_0$, then for $\omega \gg \omega_0$:

$$\boxed{\frac{\omega^4}{\left[\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2\right]} \approx \frac{\omega^4}{\left[\omega^4+\gamma^2\omega^2\right]} = \frac{\omega^4}{\omega^2\left[\omega^2+\gamma^2\right]} = \frac{\omega^2}{\left[\omega^2+\gamma^2\right]} \approx \frac{\omega^2}{\omega^2} = 1 = \frac{1}{r_e^2}\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega}}.$$

Thus, at high frequencies $(\omega \gg \omega_0)$ the differential and total atomic/molecular scattering cross sections are frequency-independent, and behave essentially identical to those associated with Thomson scattering of free electrons:

$$\boxed{\frac{d\sigma_{atom}(\theta,\varphi)}{d\Omega} \approx r_e^2 \times \{\text{Angular Factor}\}} \quad \left(m^2/sr \text{ per atom/molecule}\right)$$

and:

$$\boxed{\sigma_{atom} \approx \frac{8\pi}{3}r_e^2} \quad \left(m^2 \text{ per atom/molecule}\right)$$

4.) Extremely High frequencies: $\hbar\omega \approx m_e c^2$ (and beyond).

At extremely high frequencies, when $E_\gamma = \hbar\omega \approx m_e c^2$ the scattering of *EM* radiation by atoms/molecules is no longer in the non-relativistic regime, and *n.b.* *also* violates the second condition of our original Taylor series expansion for *EM* radiation in the "far-zone" limit $\{\boxed{r'_{\max}/r \ll 1}$ and $\boxed{\omega r'_{\max}/c = kr'_{\max} = 2\pi r'_{\max}/\lambda \ll 1}\}$ because for $E_\gamma = \hbar\omega \approx m_e c^2$ (and beyond), the wavelength λ is comparable to (and/or smaller than) the size of the atom/molecule $r'_{\max}$, thus the requirement $2\pi r'_{\max}/\lambda \ll 1$ is *not* satisfied. Thus use of (any) of the above formulae would be extremely precarious in this regime.

This is the regime of hard *x*- and γ-ray scattering by {essentially free} electrons bound to atoms/molecules, and is known as ***Compton scattering***, which is essentially "billiard-ball" *x*- and/or γ-ray photon-electron elastic scattering. In this high frequency/high energy regime, the scattered *EM* radiation has a different (*i.e.* lower) frequency than the incident radiation. The classical theory of the scattering of *EM* radiation is unable to explain, in terms of any kind of macroscopic *EM* wave phenomena.

Only the relativistic quantum mechanical theory of photons interacting with electrons {QED} succeeds in properly explaining Compton scattering of high-energy *x*- and γ-rays by electrons (and other charged particles).

For $\hbar\omega \ll m_e c^2$ the frequency shift is small, but not precisely zero. The classical theory works adequately well in this regime. We will discuss Compton scattering in more detail when we get to the subject of relativistic kinematics (P436 Lect. Notes 17).

Scattering of *EM* Radiation by a *Collection* of Free Charges, Neutral Atoms and/or Molecules

Thus far, we have discussed scattering of *EM* radiation by a single free charge *q* and/or a single neutral atom/molecule. What happens when a macroscopic *EM* plane wave scatters from a collection/ensemble of *many* such objects?

Consider what happens when an incident *EM* plane wave scatters from just *two* such objects. Suppose the first scattering object is located at the origin $\vartheta(\vec{r}_1 = 0)$ (as before), the other is located at an arbitrary position $\vec{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$.

Since the incident *EM* wave is a plane wave, the solutions of the two corresponding inhomogeneous force equations are such that, at the common retarded time t_o the magnitudes of the {complex} induced electric dipole moments are equal to each other: $\left|\tilde{\vec{p}}_1(\vec{r}_1, t_o)\right| = \left|\tilde{\vec{p}}_2(\vec{r}_2, t_o)\right|$, *i.e.* $\tilde{\vec{p}}_2(\vec{r}_2, t_o)$ can only differ from $\tilde{\vec{p}}_1(\vec{r}_1, t_o)$ by a relative phase

$$\boxed{\tilde{\vec{p}}_2(\vec{r}_2, t_o) = \tilde{\vec{p}}_1(\vec{r}_1, t_o)e^{i\delta_{21}} = \tilde{\vec{p}}_1(\vec{r}_1, t_o)e^{i\omega\Delta t_{21}} = \tilde{\vec{p}}_1(\vec{r}_1, t_o)e^{i\vec{k}_{inc}\cdot\vec{r}_2}}.$$

due to relative arrival time difference $\boxed{\Delta t_{21} = \left(\vec{k}_{inc}\cdot\vec{r}_2\right)/\omega}$ of the incident *EM* wave at the 2nd scattering object, at position $\vec{r}_2$ relative to the first, located at origin $\vartheta(\vec{r}_1 = 0)$.

However, this is only half of the story. Because $\tilde{\vec{p}}_1$ is located at position $\vec{r}_1 = 0$ and $\tilde{\vec{p}}_2$ is located at position $\vec{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$, the *EM* waves simultaneously radiated by each of the two electric dipoles at the common retarded time t_o will arrive at the observation/field point position $P(r,\theta,\varphi,t)$ such that the *EM* "far-zone" radiation fields associated with dipole # 2 have associated with them an additional {relative} phase shift of $e^{i\delta'_{21}} = e^{i\omega\Delta t'_{21}} = e^{-i\vec{k}_{rad}\cdot\vec{r}_2}$ due to relative arrival time difference $\boxed{\Delta t'_{21} = -\left(\vec{k}_{rad}\cdot\vec{r}_2\right)/\omega}$ of the scattered *EM* wave at the observation point $P(r,\theta,\varphi,t)$ from the 2nd scattering object, at position $\vec{r}_2$ relative to the first, located at origin $\vartheta(\vec{r}_1 = 0)$.

Thus, there is an overall phase shift for scatterer # 2 at position $\vec{r}_2$ relative to scatterer # 1 located at origin $\vartheta(\vec{r}_1 = 0)$ of: $\boxed{e^{i\delta_{21}}e^{i\delta'_{21}} = e^{i\omega\Delta t_{21}}e^{i\omega\Delta t'_{21}} = e^{i\vec{k}_{inc}\cdot\vec{r}_2}e^{-i\vec{k}_{rad}\cdot\vec{r}_2} = e^{i\left(\vec{k}_{inc}-\vec{k}_{rad}\right)\cdot\vec{r}_2} = e^{i\Delta\vec{k}\cdot\vec{r}_2}}$ where: $\boxed{\Delta\vec{k} \equiv \vec{k}_{inc} - \vec{k}_{rad}}$.

In general, the scattered *EM* wave(s) radiated from the two induced electric dipole moments $\tilde{\vec{p}}_1(\vec{r}_1,t)$ and $\tilde{\vec{p}}_2(\vec{r}_2,t)$, since the radiation electric fields also obey the superposition principle, will subsequently interfere with each other in the "far-zone" at the observation point $P(r,\theta,\varphi,t)$.

We can therefore generalize the above 2-scatterer result for the scattering of an incident *EM* wave to that for a collection of N identical scattering objects, each of which is located at position $\vec{r}_n = x_n\hat{x} + y_n\hat{y} + z_n\hat{z}$ for the n^{th} scattering object, $n = 1, 2, 3, \ldots N$. Via use of the principle of linear superposition, each such identical scattering object will contribute $\vec{E}^{scat}_{r_n}(\vec{r},t)$ to the overall "far-zone" *EM* radiation field at the observation point $P(r,\theta,\varphi,t)$. Since the differential and total scattering cross sections are both proportional to the modulus-squared of the overall/total *EM* radiation field $\left|\vec{E}^{scat}_{r_{tot}}(\vec{r},t)\right|^2$, where:

$$\boxed{\vec{E}^{scat}_{r_{tot}}(\vec{r},t) = \sum_{n=1}^{N}\vec{E}^{scat}_{r_n}(\vec{r},t) = \sum_{n=1}^{N}\vec{E}^{scat}_{r_1}(\vec{r},t)e^{i\Delta\vec{k}\cdot\vec{r}_n} = \vec{E}^{scat}_{r_1}(\vec{r},t)\sum_{n=1}^{N}e^{i\Delta\vec{k}\cdot\vec{r}_n}} \text{ where } \boxed{\vec{E}^{scat}_{r_n}(\vec{r},t) = \vec{E}^{scat}_{r_1}(\vec{r},t)e^{i\Delta\vec{k}\cdot\vec{r}_n}}$$

where $\vec{E}^{scat}_{r_1}(\vec{r},t)$ is the scattered "far-zone" radiation field at the observation point $P(r,\theta,\varphi,t)$ associated with a single scattering object located at the origin $\vartheta(\vec{r}_1 = 0)$.

Then we see that: $\boxed{\left|\vec{E}^{scat}_{r_{tot}}(\vec{r},t)\right|^2 = \left|\sum_{n=1}^{N}e^{i\Delta\vec{k}\cdot\vec{r}_n}\right|^2\left|\vec{E}^{scat}_{r_1}(\vec{r},t)\right|^2}$

Defining the so-called complex ***structure factor***: $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) \equiv \left|\sum_{n=1}^{N}e^{i\Delta\vec{k}(\theta,\varphi)\cdot\vec{r}_n}\right|^2}$

and neglecting ***multiple scattering*** effects[1], the differential scattering cross section associated with an incident macroscopic, monochromatic *EM* plane wave scattering off of a collection/ensemble of N identical scattering objects (*e.g.* free electrons, atoms/molecules, etc) can be written as:

[1] i.e. the mean free path for scattering is large compared to the overall spatial dimensions of the collection of scatterers.

$$\boxed{\frac{d\sigma_N(\theta,\varphi)}{d\Omega} = \mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right)\frac{d\sigma_1(\theta,\varphi)}{d\Omega} = \left|\sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_n}\right|^2 \frac{d\sigma_1(\theta,\varphi)}{d\Omega}}$$

where $d\sigma_1(\theta,\varphi)/d\Omega$ is the differential scattering cross section associated with a *single* scattering object located at the origin $\mathcal{G}(\vec{r}_1 = 0)$.

It can be seen that the numerical effect of the structure factor $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) \equiv \left|\sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_n}\right|^2}$ on the overall differential scattering cross section $d\sigma_N(\theta,\varphi)/d\Omega$ depends very sensitively on the exact/precise details of the spatial distribution of the N identical scattering objects.

First, let us consider again the case of only two scattering objects for forward scattering, when $\theta = 0$. Then for *two* identical scattering objects, the structure factor:

$$\boxed{\begin{aligned}\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) &\equiv \left|\sum_{n=1}^{2} e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_n}\right|^2 = \left(1+e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}\right)\left(1+e^{-i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}\right) = 1+e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}+e^{-i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}+1 \\ &= 2+\left(e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}+e^{-i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2}\right) = 2+2\cos\left(\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2\right) = 2\left[1+\cos\left(\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_2\right)\right]\end{aligned}}$$

However, for forward $(\theta = 0)$ scattering this means that: $\boxed{\vec{k}_{rad} = \vec{k}_{inc} = \hat{z}}$ *i.e.* $\boxed{\vec{k}_{rad} \parallel \vec{k}_{inc} \parallel \hat{z}}$ and thus: $\boxed{\Delta\vec{k}(\theta = 0,\varphi) = \vec{k}_{inc} - \vec{k}_{rad} = 0}$ and hence: $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta=0,\varphi)=0\right) = 2\left[1+\cos(0)\right] = 4}$.

Thus we see that two identical scattering objects ***always*** constructively interfere with each other for forward $(\theta = 0)$ scattering, independent of the location $\vec{r}_2$ of the 2nd scatterer relative to that of the first, located at the origin $\mathcal{G}(\vec{r}_1 = 0)$. For forward $(\theta = 0)$ scattering the {relative} time delay $\Delta t_2 = z_2/c$ in the arrival of the incident *EM* plane wave at the *z*-location of the 2nd scattering object is exactly compensated by the {relative} decrease in the arrival time $\Delta t_2 = -z_2/c$ of the scattered/radiated wave in the "far-zone" at the observer's position at $(r,\theta = 0,\varphi)$.

Noting that since: $\boxed{\Delta\vec{k}(\theta \equiv 0,\varphi) = \vec{k}_{inc} - \vec{k}_{rad} = 0}$ for ***all*** N identical scattering objects for forward $(\theta \equiv 0)$ scattering, we see that for forward $(\theta \equiv 0)$ scattering of an incident *EM* plane wave by *N* identical scattering objects, the structure factor $\mathcal{F}\left(\Delta\vec{k}(\theta \equiv 0,\varphi) = 0\right)$ becomes:

$$\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta\equiv 0,\varphi)=0\right) = \left|\sum_{n=1}^{2} e^{i\Delta\vec{k}(\theta=0)\bullet\vec{r}_n}\right|^2 = \left|\sum_{n=1}^{2} e^{i\vec{0}\bullet\vec{r}_n}\right|^2 = \left|e^{i\vec{0}\bullet\vec{r}_1}+e^{i\vec{0}\bullet\vec{r}_2}+\ldots+e^{i\vec{0}\bullet\vec{r}_N}\right|^2 = \underbrace{\left|1+1+\ldots+1\right|^2}_{=|N|^2} = N^2}$$

and thus we see that N identical scattering objects ***always*** constructively interfere with each other for forward $(\theta \equiv 0)$ scattering, independent of the *z*-location of the *n*th scatterer relative to that of the first {located at the origin $\mathcal{G}(\vec{r}_1 = 0)$ }.

Thus, the forward $(\theta \equiv 0)$ differential scattering cross section associated with N identical scatterers is:

$$\boxed{\frac{d\sigma_N(\theta \equiv 0,\varphi)}{d\Omega} = N^2 \frac{d\sigma_1(\theta \equiv 0,\varphi)}{d\Omega}}$$

It can also be seen that for backward $(\theta \equiv \pi)$ scattering the situation is ***not*** the same as for forward $(\theta \equiv 0)$ scattering. Again, we consider the two-scatterer case: when $\theta = \pi$ then: $\boxed{\vec{k}_{rad} = -\vec{k}_{inc} = -\hat{z}}$ and: $\boxed{\Delta\vec{k}(\theta \equiv \pi,\varphi) = \vec{k}_{inc} - \vec{k}_{rad} = 2\vec{k}_{inc} \neq 0}$ and: $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta \equiv \pi,\varphi)\right) = 2\left[1+\cos\left(2\vec{k}_{inc}\bullet\vec{r}_2\right)\right]}$.

When: $\boxed{\left(2\vec{k}_{inc}\bullet\vec{r}_2\right) = 2n\pi,\ n = 1,2,3...}$ {*i.e.* certain (angular) frequencies $\boxed{\omega_n = ck_n = n\pi c/z_2}$}

Then: $\boxed{\cos\left(2\vec{k}_{inc}\bullet\vec{r}_2\right) = \cos(2n\pi) = 1}$ and: $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta = \pi,\varphi)\right) = 4}$ resulting in constructive interference for backward $(\theta \equiv \pi)$ scattering.

When: $\boxed{2\left(\vec{k}_{inc}\bullet\vec{r}_2\right) = (2n-1)\pi,\ n = 1,2,3...}$ {*i.e.* certain other (angular) frequencies $\boxed{\omega_n' = ck_n' = \tfrac{1}{2}(2n-1)\pi c/z_2}$} then: $\boxed{\cos\left(2\vec{k}_{inc}\bullet\vec{r}_2\right) = \cos\left((2n-1)\pi\right) = -1}$ and: $\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta \equiv \pi,\varphi)\right) = 0}$ resulting in destructive interference for backward $(\theta = \pi)$ scattering!

Thus, for backward $(\theta \equiv \pi)$ scattering with N identical scatterers, ***only*** if the scatterers are arranged in some kind of highly-organized, regular array/3-D lattice (*e.g.* such as a crystal), will ***coherent*** backward scattering effects (*i.e.* constructive/destructive interference effects) be observable.

In general, if the N identical scatterers are randomly organized, such as in a plasma, a gas, a liquid or an amorphous solid, then it can be shown that the terms with $m \neq n$ in the structure factor:

$$\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) \equiv \left|\sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet\vec{r}_n}\right|^2 = \sum_{n=1}^{N}\sum_{m=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet(\vec{r}_n-\vec{r}_m)}}$$

contribute very little to the overall value of $\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right)$, due to ***stochastic*** cancellations {*n.b.* the same principle is used to balance turbine blade assemblies in constructing jet engines}.

Thus, for a random collection of N identical scatterers, when $m = n$ the double series associated with $\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right)$ becomes:

$$\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) = \sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet(\vec{r}_n-\vec{r}_n)} = \sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\bullet(0)} = \sum_{n=1}^{N} 1 = N}$$

and thus the differential scattering cross section associated with N randomly distributed identical scatterers (except for the precisely $\theta \equiv 0$ forward scattering) is:

$$\boxed{\frac{d\sigma_N(\theta > 0,\varphi)}{d\Omega} = N \frac{d\sigma_1(\theta > 0,\varphi)}{d\Omega}}$$

The overall scattering in this randomly-distributed situation is known as "***incoherent scattering***".

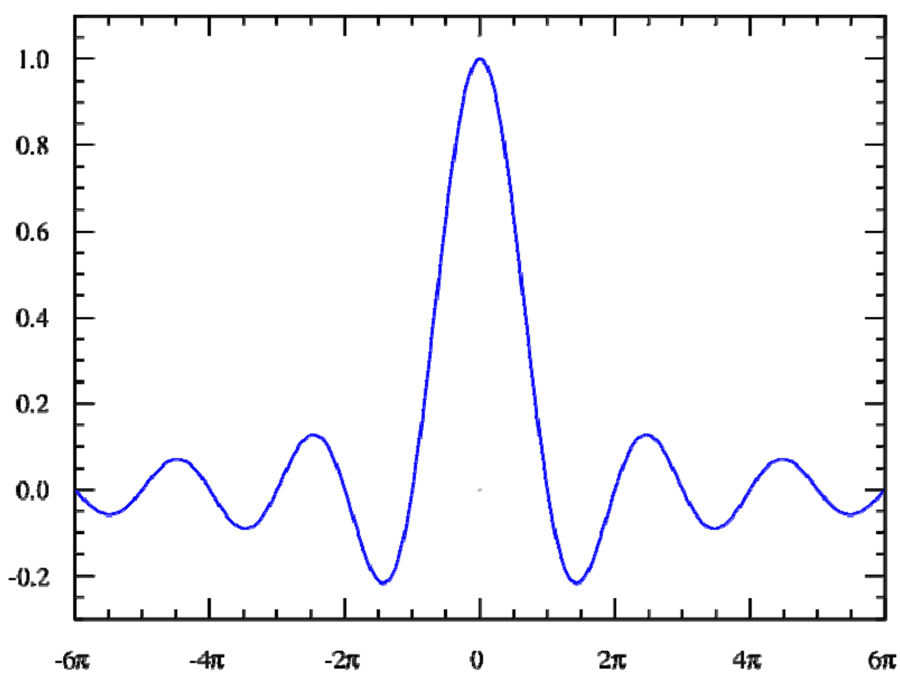
For the situation associated with the scattering of a macroscopic, monochromatic *EM* plane wave incident on a highly regular, cubical 3-D crystalline-type array/lattice consisting of N identical scattering objects (*e.g.* atoms or molecules) of dimensions $W \times H \times D = L_x \times L_y \times L_z$ and lattice constant / lattice spacing a and $N = N_x N_y N_z = (L_x/a)\cdot(L_y/a)\cdot(L_z/a)$ where the N_j are the # of lattice sites in the j^{th} direction, the structure factor for this highly regular array of N identical scatterers is:

$$\boxed{\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right) \equiv \left|\sum_{n=1}^{N} e^{i\Delta\vec{k}(\theta,\varphi)\cdot\vec{r}_n}\right|^2 = N^2\left\{\left(\frac{\sin^2\left(\tfrac{1}{2}N_x\Delta k_x a\right)}{N_x^2\sin^2\left(\tfrac{1}{2}\Delta k_x a\right)}\right)\cdot\left(\frac{\sin^2\left(\tfrac{1}{2}N_y\Delta k_y a\right)}{N_y^2\sin^2\left(\tfrac{1}{2}\Delta k_y a\right)}\right)\cdot\left(\frac{\sin^2\left(\tfrac{1}{2}N_z\Delta k_z a\right)}{N_z^2\sin^2\left(\tfrac{1}{2}\Delta k_z a\right)}\right)\right\}}$$

At low frequencies/long wavelengths $(\lambda \gg a)$, only the forward-scattering peak at $\Delta k_j a = 0$ contributes to $\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right)$ because {here} the maximum possible value of $\Delta k_j a$ is: $\boxed{2ka = 4\pi a/\lambda \ll 1}$. In this forward-scattering/small angle regime, using the small-angle Taylor series approximation for $\sin^2\left(\tfrac{1}{2}\Delta k_x a\right) \simeq \left(\tfrac{1}{2}\Delta k_x a\right)^2$, the structure factor becomes:

$$\boxed{\begin{aligned}\mathcal{F}\left(\Delta\vec{k}(\theta\approx 0,\varphi)\right) &\simeq N^2\left\{\left(\frac{\sin^2\left(\tfrac{1}{2}N_x\Delta k_x a\right)}{\left(\tfrac{1}{2}N_x\Delta k_x a\right)^2}\right)\cdot\left(\frac{\sin^2\left(\tfrac{1}{2}N_y\Delta k_y a\right)}{\left(\tfrac{1}{2}N_y\Delta k_y a\right)^2}\right)\cdot\left(\frac{\sin^2\left(\tfrac{1}{2}N_z\Delta k_z a\right)}{\left(\tfrac{1}{2}N_z\Delta k_z a\right)^2}\right)\right\}\\ &= N^2\text{sinc}^2\left(\tfrac{1}{2}N_x\Delta k_x a\right)\cdot\text{sinc}^2\left(\tfrac{1}{2}N_y\Delta k_y a\right)\cdot\text{sinc}^2\left(\tfrac{1}{2}N_z\Delta k_z a\right)\end{aligned}} \quad \text{where} \quad \boxed{\text{sinc}(x) \equiv \frac{\sin x}{x}}$$

A graph sinc(x) vs. x is shown in the figure below:



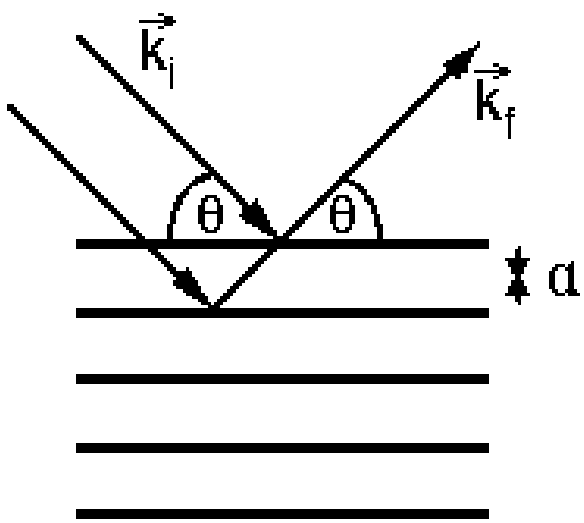
In the low-frequency/long-wavelength regime, from the above graph of sinc(x) vs. x it can be seen that the structure factor $\mathcal{F}\left(\Delta\vec{k}\left(\theta\approx 0,\varphi\right)\right)$ is only appreciable when $\boxed{\tfrac{1}{2}N_j\left|\Delta k_j\right|a\le\pi}$, which corresponds to an angular region of forward scattering $\boxed{\Delta\theta\approx\left|\Delta k_j\right|a\le 2\pi/N_j}$, which becomes exceedingly small as $N_j\to\infty$.

Note that for a typical lattice constant $\boxed{a\sim 5\text{Å}=0.5\,nm=5\times10^{-10}\,m}$ and a typical macroscopic sample size of $\boxed{L_j=1\,cm=0.01\,m}$ then: $\boxed{N_j=L_j/a\approx 10^{-2}/5\times10^{-10}=2\times10^{7}}$, corresponding to a forward scattering angular region of: $\boxed{\Delta\theta\le 2\pi/2\times10^{7}\approx\pi\times10^{-7}\,radians\sim 0.3\,\mu rad}$!!!

Thus, in regular crystalline arrays of scattering objects, *e.g.* single crystals of transparent minerals such as diamond, emerald, quartz, rock salt, *etc*. there is essentially <u>no</u> scattering at long wavelengths {except in the <u>extreme</u> forward direction}. The very small amount of large-angle scattering that does occur in such samples is caused by/due to transitory thermal vibrations of the atoms in the lattice away from the "perfect" configuration of the 3-D crystalline lattice.

At high frequencies/short wavelengths $\left(\lambda<2a\sim 1\,nm\right)$ *i.e.* when $\boxed{ka>\pi}$ – typically in the *x*-ray region of the *EM* spectrum – the structure factor $\mathcal{F}\left(\Delta\vec{k}\left(\theta,\varphi\right)\right)$ has <u>maxima</u> when the so-called <u>***Bragg scattering condition***</u> is satisfied: $\boxed{\Delta k_j a=2n\pi}$ or:

$\boxed{\Delta k_j=2k_{inc}\sin\theta_j=\frac{4\pi}{\lambda}\sin\theta_j=\frac{2n\pi}{a}}$, *i.e.* when $\boxed{n\lambda=2a\sin\theta_j}$.



Crystal planes spaced distance a apart

Typical X-ray diffraction pattern

Experimental Aspects, Applications and Uses of the Measurement of Differential and Total Scattering Cross Sections

At the beginning of these lecture notes, the theoretical/mathematical definitions of the differential and total scattering cross sections were given:

Differential Scattering Cross Section {SI units: m^2/sr per scattering object}:

$$\boxed{\frac{d\sigma_{scat}(\theta,\varphi)}{d\Omega} \equiv \frac{1}{I_{inc}(\vec{r}=0)}\frac{d\left\langle P_{rad}(\theta,\varphi,t)\right\rangle}{d\Omega} = \frac{1}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle\Big|_{r=0}}\frac{d\left\langle P_{rad}(\theta,\varphi,t)\right\rangle}{d\Omega} = \frac{\left\langle\vec{S}_{rad}(\vec{r},t)\right\rangle\bullet r^2\hat{r}}{\left\langle\vec{S}_{inc}(\vec{r},t)\right\rangle\bullet\hat{z}\Big|_{r=0}}}$$

The differential scattering cross section {SI units: m^2/sr} is the time-averaged differential/angular *EM* power {SI units: *Watts/sr*} radiated by a scattering object (or a collection/ensemble of scattering objects) into solid angle $\boxed{d\Omega(\theta,\varphi) = d\cos\theta d\varphi = \sin\theta d\theta d\varphi}$, normalized to the time-averaged ***flux*** of *EM* energy (= *EM* intensity I_{inc}, *aka* ***irradiance***) {SI units: *Watts*} incident on the scattering object/objects located at the origin $\vartheta(\vec{r}=0)$.

From the RHS of the above equation, we also see that:

$$\boxed{\frac{d\sigma_{scat}(\theta,\varphi)}{d\Omega} \equiv \frac{Scattered\ \langle Flux\rangle\ of\ EM\ Radiation/Unit\ Solid\ Angle}{Incident\ \langle Flux\rangle\ of\ EM\ Radiation/Unit\ Area}\left(\frac{Watts/sr}{Watts/m^2}\right) = \left(\frac{m^2}{sr}\right)}$$ **per scattering object**

Total Scattering Cross Section {SI units: Area, *i.e.* m^2 per scattering object}:

$$\boxed{\sigma_{scat} \equiv \frac{\left\langle P_{rad}(t)\right\rangle}{I_{inc}(\vec{r}=0)} = \frac{\left\langle P_{rad}(t)\right\rangle}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle\Big|_{r=0}} = \frac{\int d\left\langle P_{rad}(\theta,\varphi,t)\right\rangle/d\Omega}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle\Big|_{r=0}} = \frac{\int d\left\langle\vec{S}_{rad}(\vec{r},t)\right\rangle\bullet r^2\hat{r}/d\Omega}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle\Big|_{r=0}} = \int\frac{d\sigma_{scat}(\theta,\varphi)}{d\Omega}d\Omega}$$

The total scattering cross section{SI units: m^2} is the time-averaged total *EM* power {SI units: *Watts*} radiated into *all* angles {*i.e.* 4π steradians (*sr*)} by a scattering object (or a collection / ensemble of scattering objects) normalized to the time-averaged ***flux*** of *EM* energy (= *EM* intensity I_{inc}, *aka* ***irradiance***) {SI units: *Watts*} incident on the scattering object/objects located at the origin $\vartheta(\vec{r}=0)$.

From the RHS of the above equation, we also see that:

$$\boxed{\sigma_{scat} \equiv \frac{Total\ Scattered\ \langle Flux\rangle\ of\ EM\ Radiation\ into\ 4\pi\ sr}{Incident\ \langle Flux\rangle\ of\ EM\ Radiation/Unit\ Area}\left(\frac{Watts}{Watts/m^2}\right) = \left(m^2\right)}$$ **per scattering object**

In many common experimental situations associated with the scattering of a macroscopic *EM* plane wave incident on a collection/ensemble of N identical scattering objects – *e.g.* free electrons/ions in plasmas, or *e.g.* neutral atoms/molecules in solids, liquids and/or gases, the {instantaneous} 3-D positions of the scattering objects are randomly distributed. For a collection of N identical randomly distributed scattering objects, we have shown that the structure factor $\mathcal{F}\left(\Delta\vec{k}(\theta,\varphi)\right)=N$ {except for precisely $\theta\equiv 0$ forward scattering, where $\mathcal{F}\left(\Delta\vec{k}(\theta\equiv 0,\varphi)\right)=N^2$ } and thus the so-called ***incoherent*** differential scattering cross section associated with the collective scattering of an incident *EM* plane wave by N identical randomly distributed scatterers (except for precisely $\theta\equiv 0$ forward scattering) is:

$$\boxed{\frac{d\sigma_N(\theta>0,\varphi)}{d\Omega}=N\frac{d\sigma_1(\theta>0,\varphi)}{d\Omega}}$$

Next, we consider a monochromatic macroscopic *EM* plane wave propagating in the $+\hat{z}$ -direction normally incident on a target consisting of a slab of "generic" matter (plasma, gas, liquid or solid) of macroscopic volume $V=H\times W\times D\left(m^3\right)$ consisting of a total of N randomly distributed identical scattering objects, characterized by a number density $n=N/V\left(\#/m^3\right)$ and mass volume density $\rho=M/V\left(kg/m^3\right)$. The incident *EM* plane wave uniformly illuminates the cross sectional area $A_\perp=H\times W$ of the target slab of "generic" matter.

The macroscopic *EM* plane wave enters the front face of the target at normal incidence at $z=0$, and after propagating an infinitesimal longitudinal distance dz into the target, the $dN=N(dz/D)$ randomly-distributed scattering objects contained within this infinitesimal thickness dz of the target have collectively absorbed and then re-radiated into 4π steradians a time-averaged amount of power $\langle dP_o\rangle$ from the incident *EM* plane wave of {*n.b.* turning the total cross section (per scattering object!) relation around: $\langle P_{rad}(t)\rangle=\sigma_{scat}I_o\,(Watts)$ per scattering object}:

$$\boxed{\langle dP_o\rangle=dN\sigma I_o=N(dz/D)\sigma I_o=N(dz/A_\perp D)\sigma I_oA_\perp=(N/V)dz\sigma I_oA_\perp=n\sigma dzI_oA_\perp\,(Watts)}$$

which corresponds to a *decrease*/*reduction*/*loss* in the incident intensity I_o over the infinitesimal distance dz of:

$$\boxed{dI_o=\frac{\langle dP_o\rangle}{A_\perp}=-n\sigma I_odz\left(Watts/m^2\right)}\text{ with corresponding –ve slope: }\boxed{\frac{dI_o}{dz}=-n\sigma I_o\left(Watts/m^3\right)}$$

Then for propagation of the macroscopic *EM* wave a longitudinal distance z into the target, this latter relation becomes:

$$\boxed{\frac{dI(z)}{dz}=-n\sigma I(z)\left(Watts/m^3\right)}\text{ which can be rearranged as: }\boxed{\frac{dI(z)}{dz}+n\sigma I(z)=0}$$

This relation is a simple homogeneous first-order differential equation with boundary conditions: $\boxed{I(z=0)=I_o}$ and: $\boxed{I(z=\infty)=0}$. The specific solution to this differential equation is: $\boxed{I(z)=I_o e^{-n\sigma z}}$.

Thus we see that the incident *EM* plane wave is ***exponentially*** attenuated to $1/e=e^{-1}=\ 0.368$ of its initial intensity I_o in propagating a characteristic longitudinal distance known as the ***attenuation length*** $\boxed{\lambda_{atten}\equiv 1/n\sigma}$ (m) into the target. Then: $\boxed{I(z)=I_o e^{-z/\lambda_{atten}}=I_o e^{-n\sigma z}}$.

The ***reciprocal*** of the ***attenuation length*** λ_{atten} is known as the ***absorption coefficient***: $\boxed{\alpha\equiv 1/\lambda_{atten}=n\sigma}$ (m^{-1}). Then: $\boxed{I(z)=I_o e^{-\alpha z}=I_o e^{-z/\lambda_{atten}}=I_o e^{-n\sigma z}}$.

Using an isothermal model of the earth's atmosphere (*i.e.* the density $\rho_{\rm air}$ varies exponentially with altitude), $n_{air}\sim 2.7\times 10^{25}\ \text{molecules/m}^3$, typical value(s) of the attenuation length for Rayleigh scattering of visible light by N_2 and O_2 molecules in the earth's atmosphere are:

Red light ($\lambda = 650$ nm): $\lambda_{atten}\sim 188\ km$
Green light ($\lambda = 520$ nm): $\lambda_{atten}\sim\ 77\ km$
Violet light ($\lambda = 410$ nm): $\lambda_{atten}\sim\ 30\ km$

The % scattering of {direct} sunlight in the earth's atmosphere as a function of wavelength λ is shown in the figure below:

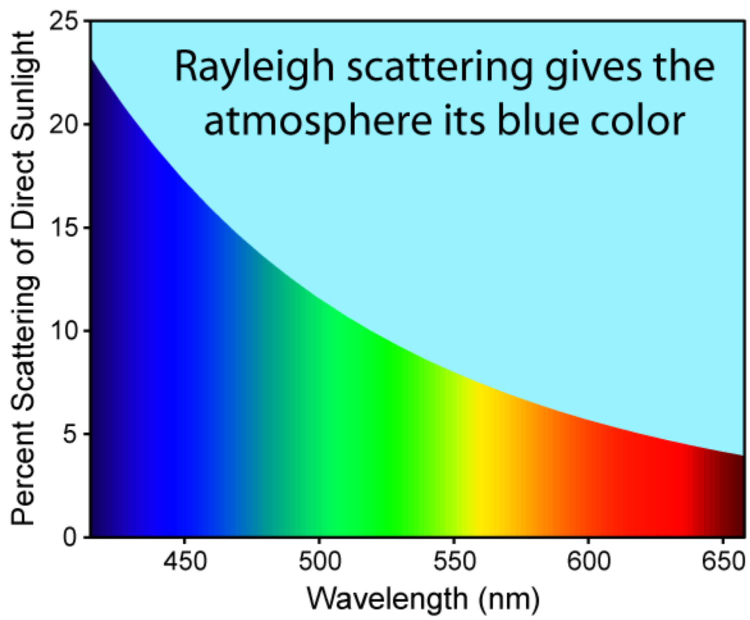
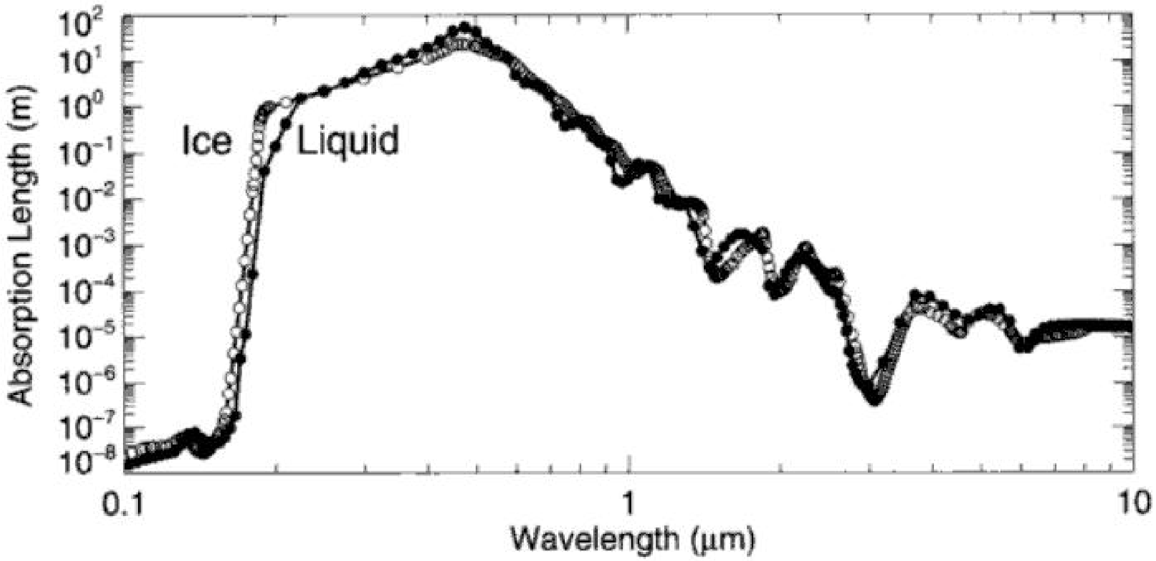
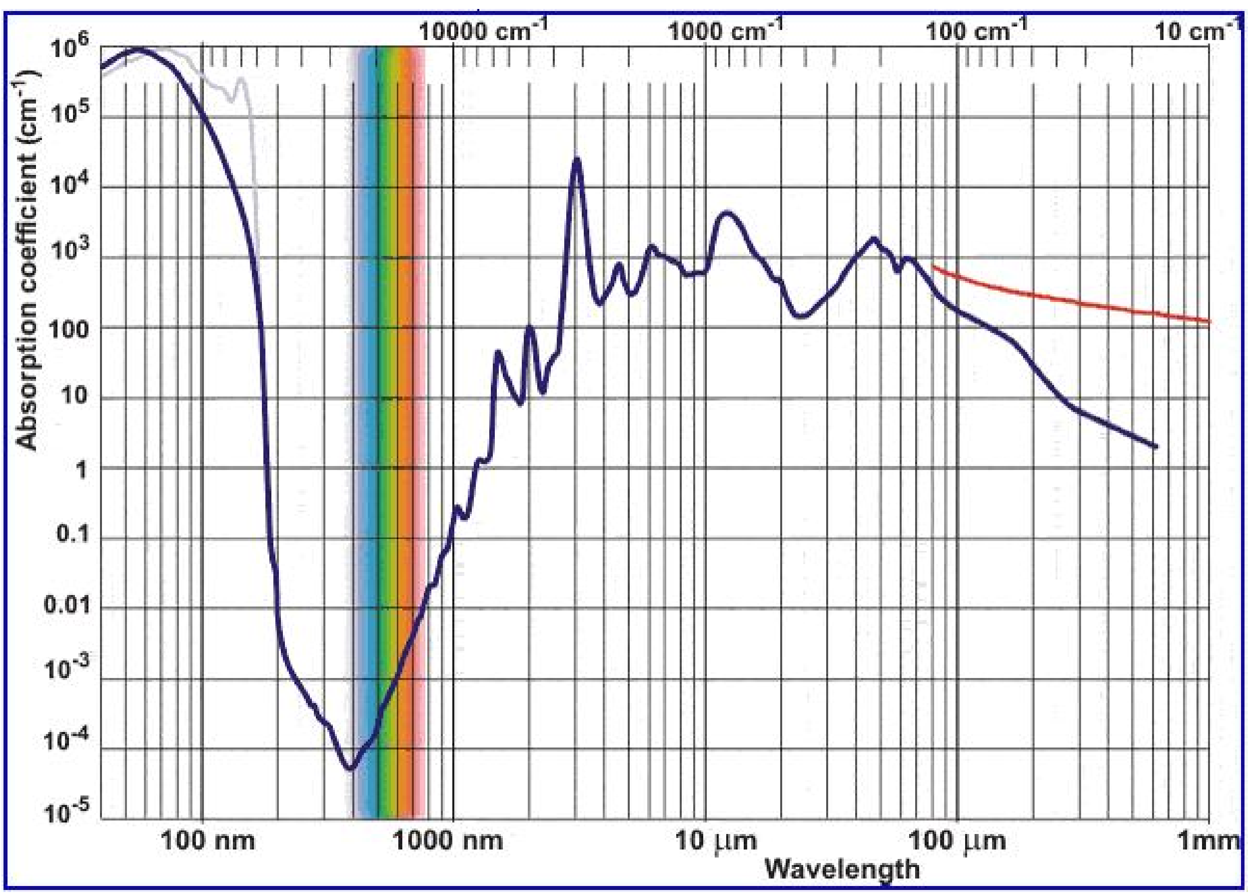


Image: Copyright Robert A. Rohde, 2007
http://en.wikipedia.org/wiki/Image:Rayleigh_sunlight_scattering.png

Note also that Rayleigh scattering of the sun's light by air molecules in the earth's atmosphere is also is responsible for giving the sky its apparent height above the ground.

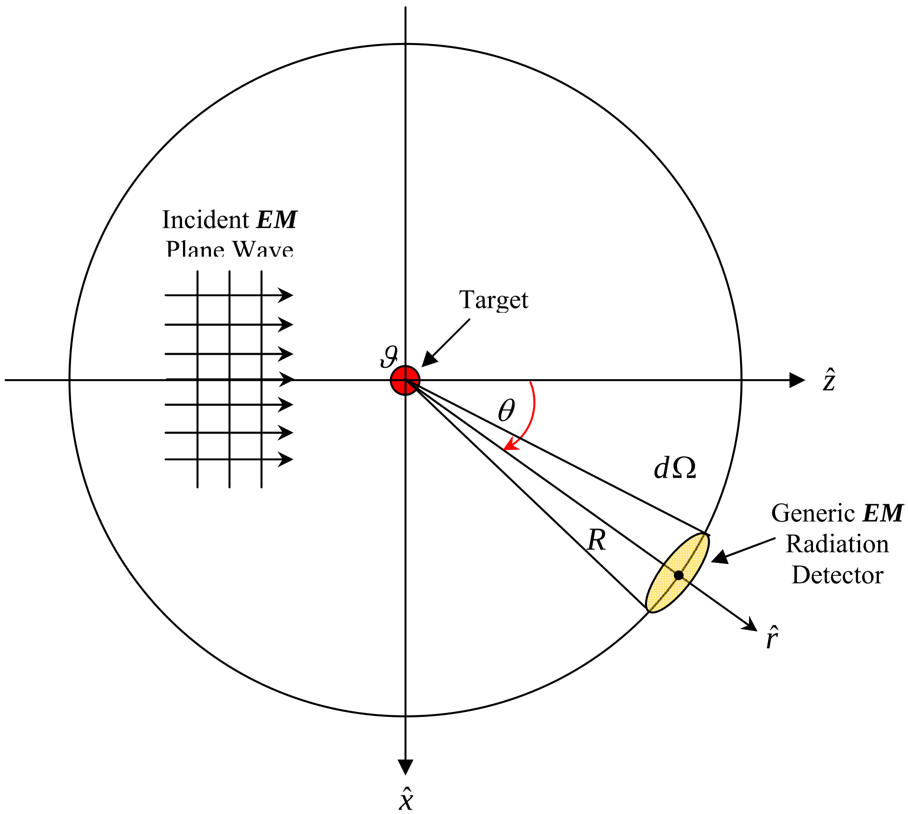
Plots of the attenuation/absorption length λ_{atten} and the absorption coefficient $\alpha \equiv 1/\lambda_{atten}$ for pure water (& ice) vs. wavelength are shown in the figures below:





Typical Experimental Apparatus to Measure a Differential Scattering Cross Section

A typical experimental setup used to measure a differential scattering cross section is shown in a plan view in the figure below, in the *x*-*z* scattering plane:

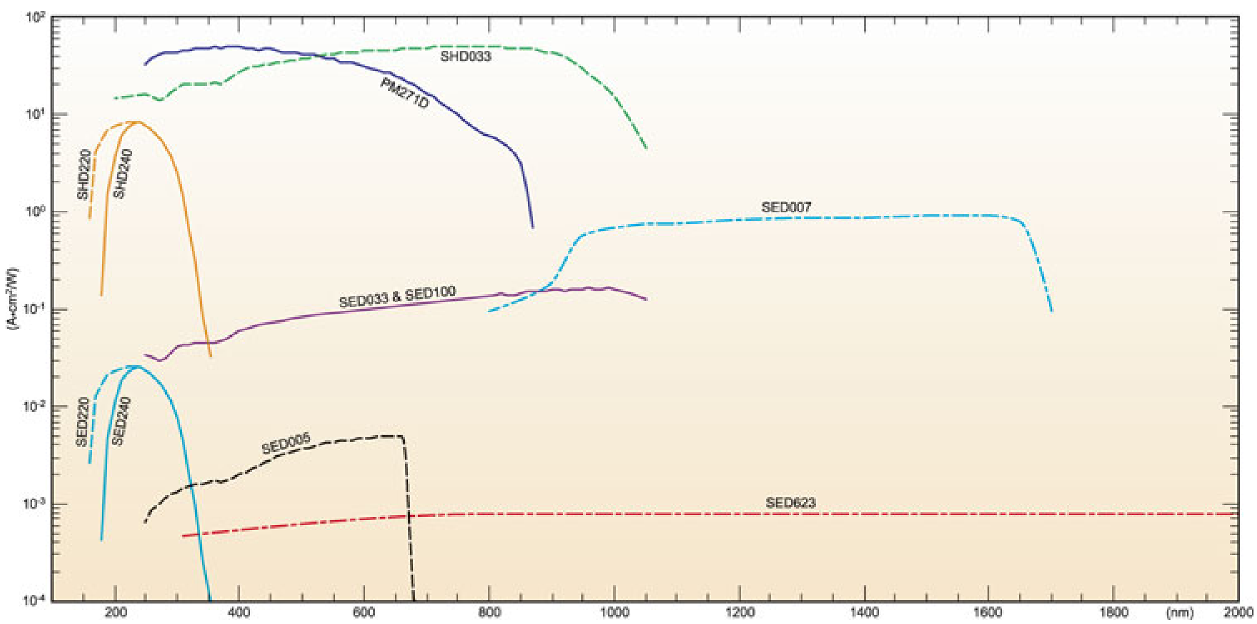


The "generic" *EM* radiation detector, located a distance *R* away from the center of the target at polar angle θ subtends a solid angle $d\Omega$ from a point target. If the target has finite spatial extent, then one obtains the solid angle subtended by the detector for a finite-volume target by integrating over the volume of the target to obtain the target-weighted solid angle. In principle, corrections also need to be made for a.) attenuation of the incident *EM* intensity in propagating through the target and b.) multiple scattering of the incident *EM* radiation and also of the scattered *EM* radiation in getting out the target; all of which become increasingly important for a thick target.

Depending on the physics associated with the scattering of incident *EM* radiation in the target, a specific choice must be made for the detector that is used for measuring the *EM* radiation scattered into solid angle $d\Omega(\theta,\varphi=0)$. For example, for if one is interested in measuring the differential scattering cross section associated with a continuous, high-intensity beam of incident *EM* radiation in the infra-red (IR), visible or ultra-violet (UV) light region, use of an absolutely-calibrated {NIST-traceable} spectro-photometer with associated readout electronics {*aka* radiometer} is frequently used, as shown in the figures below:

International Light IL1700 Research Radiometer + Photometer:

Absolute calibration of photometer response (*Ampere-cm*^{2}/*Incident Watt*) vs. wavelength λ:



← UV →| ←Visible → | ← IR → Wavelength, λ (nm)

The spectrophotometer {of active area A_{sp} } is frequently some kind of photodiode. The photocurrent {in Amperes} produced in the junction of the photodiode by the *EM* radiation incident on the spectrophotometer is accurately measured by the accompanying electronics of the radiometer. Note that there is great flexibility in the experimental setup – it could use incident EM radiation that is polarized in some manner (*e.g. LPx*, *LPy*, *RCP*, *LCP* etc) and/or for *e.g.* unpolarized *EM* radiation incident on the target, polarizing filters can be placed in front of the spectrophotometer to measure *e.g.* $d\sigma_{\parallel}^{unpol}(\theta,\varphi=0)/d\Omega$ and $d\sigma_{\perp}^{unpol}(\theta,\varphi=0)/d\Omega$ and the polarization {asymmetry} $\mathcal{P}_{atom}^{unpol}(\theta,\varphi=0)$, *etc*.

In some physics situations, the illuminated target is *e.g.* a biological sample of some kind and the detector of the scattered radiation is *e.g.* a microscope + CCD (or CMOS) camera, usually oriented at fixed scattering angle, *e.g.* $(\theta = \pi/2, \varphi = 0)$. Again the incident *EM* radiation could {additionally} be polarized in some manner, or if unpolarized EM radiation is incident, then polarizing filters placed upstream of the camera can be used to analyze the || vs. $\perp$ polarization states of the scattered *EM* radiation.

In other physics situations where the intensity of the incident *EM* radiation is extremely low, the photodetector of choice often is a low-noise photomultiplier tube (PMT) or avalanche photodiode (APD), often thermo-cooled {or even LN2 or LHe-cooled!} in order to reduce finite temperature/thermal "dark noise". These detectors then count single photons scattered from the target. The {wavelength-dependent} quantum efficiency (QE) of the photon detector used in such scattering experiments must therefore be accurately known; typical QE's are on the order of ~ 10-20%.

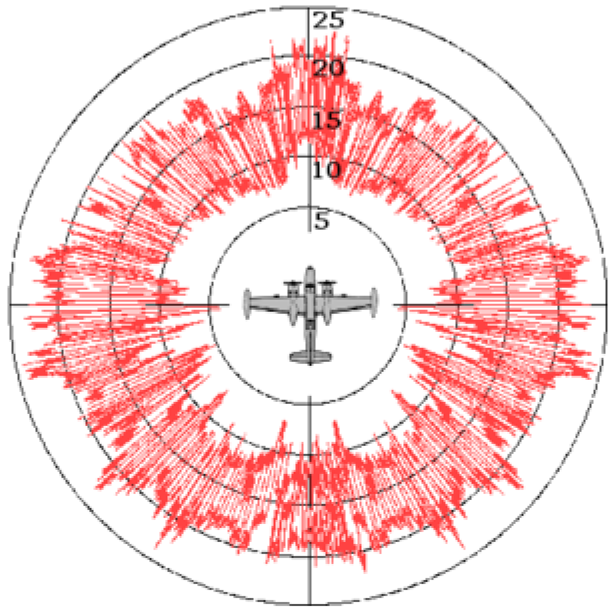
In P436 Lecture Notes 5 (*p*. 18-26) we discussed the connection between intensity I and the {time-averaged} number of photons $\langle n_\gamma(t)\rangle$ *e.g.* associated with a laser beam with photon energy $\boxed{E_\gamma = hf_\gamma = hc/\lambda_\gamma}$:

$$\boxed{I \equiv \left\langle \left|\vec{S}(t)\right| \right\rangle = c\left\langle u_{EM}(t)\right\rangle = 2\varepsilon_o E_{o_{\rm rms}}^2 = \left\langle \mathscr{F}_\gamma(t)\right\rangle E_\gamma = c\left\langle n_\gamma(t)\right\rangle E_\gamma} \quad \left(\frac{Watts}{m^2}\right)$$

Thus, counting photons {quanta of the *EM* field} within solid angle $d\Omega$ at fixed scattering angle $(\theta, \varphi = 0)$ is equivalent to measuring intensity I_{scat} within solid angle $d\Omega$ at fixed scattering angle $(\theta, \varphi = 0)$.

Sometimes an *EM* wave scattering experiment involves *e.g.* microwaves or radio waves – *e.g.* such as in RADAR applications. In engineering parlance, the type of scattering cross section(s) that we have been discussing here in these lecture notes are all known as ***bi-static*** cross section measurements, because the location of the source used to illuminate the target with *EM* radiation is distinct from the location of the detector of the scattered *EM* radiation. A ***mono-static*** cross section measurement is where the source and detector are co-located at the same point – *i.e.* the detector only measures back-scattered $(\theta = \pi)$ *EM* radiation. Additionally, engineers define the RADAR differential scattering cross sections (RCS) differently than physicists, as: $4\pi(d\sigma/d\Omega)$.

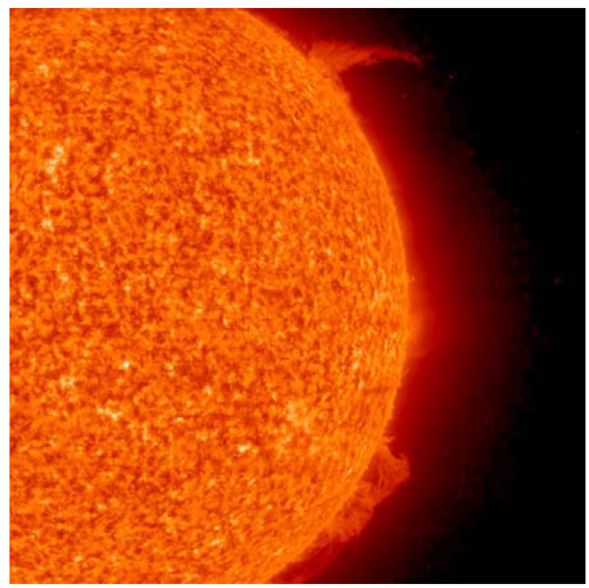
A typical differential RCS *e.g.* for a B-26 Invader is shown as a polar plot in the figure on the right.



All airplanes {unless "stealthified"} have a characteristic/distinct type of differential RCS that can be useful *e.g.* in identifying the type of plane.

In plasma physics – *e.g.* fusion/tokamak applications, the differential Thomson scattering cross section {at small scattering angles} is commonly used as a diagnostic tool *e.g.* using a {fairly high-powered} laser to monitor the number density and also the temperature of the plasma.

Thomson scattering of electrons in the tenuous plasma surrounding our sun can be seen *e.g.* during a total eclipse of the sun – this is the sun's corona! NASA's STEREO mission generates 3-D images of the sun by measuring the sun's so-called *K*-corona using two satellites, as shown in the RHS picture below:



The maximum possible *EM* luminosity of a star is determined by the balance between the inward-directed gravitational force $\boxed{F_{grav} = G_N M_{star} m_p / r^2}$ and the outward-directed force due to radiation pressure associated with Thomson scattering of electrons $\boxed{F_{rad} = \mathscr{L}_{star} \sigma_T^{e^-} / 4\pi c r^2}$ on the plasma surrounding the star (assumed here to be 100% ionized hydrogen). The condition that $\boxed{F_{grav} \geq F_{rad}}$ constrains the maximum luminosity of the star, known as the ***<u>Eddington limit</u>*** (in honor of Sir Arthur S. Eddington):

$$\boxed{\begin{aligned} \mathscr{L}_{star} &\leq 4\pi G_N M_{star} m_p c / \sigma_T^{e^-} \simeq 1.3\times 10^{31}\, Watts \\ &\leq 3.3\times 10^4 \left(M_{star} / M_\odot \right) \mathscr{L}_\odot \end{aligned}}$$

where the sun's mass: $M_\odot \simeq 2\times 10^{30}\, kg$ and our suns' solar luminosity: $\mathscr{L}_\odot \simeq 3.85\times 10^{26}\, Watts$

The cosmic microwave background (CMB) – a relic of the Big Bang – is partially linearly polarized due to Thomson scattering on electrons. CMB experiments such as WMAP and the future Planck mission measure/will measure the polarization of the CMB radiation.