## 5.8.2. Reduction to a 1-dim Green's function

Our goal was to construct a Green's function

$$\vec{\nabla}^2_{r'} \, G(\vec{r}, \vec{r}\,') = -4\pi \, \delta(\vec{r} - \vec{r}\,')$$

in spherical coordinates.

<u>Ansatz</u>: $G(r, \theta, \varphi; r', \theta', \varphi') = -4\pi \sum_{lm} f_l(r, r') \, Y^*_{lm}(\theta, \varphi) \, Y_{lm}(\theta', \varphi')$

We need to solve [Note that $\delta(\vec{r} - \vec{r}\,') = \frac{1}{r^2} \delta(r - r') \, \delta(\varphi - \varphi') \, \delta(\cos\theta - \cos\theta')$]

$$\frac{1}{r'^2} \partial_{r'} \left[ r'^2 \partial_{r'} f_l(r, r') \right] - l(l+1) f_l(r, r') = \frac{1}{r^2} \delta(r - r') \qquad (*)$$

and we demand $f_l(r, r' = R) = 0$, as appropriate for a Dirichlet problem with a sphere of radius R.

Assume we are interested in the <u>interior</u> of the sphere, $r, r' \leq R$.

To solve $(*)$ we note that at $r \neq r'$, the lhs must vanish (it satisfies Laplace's equation)

From Lecture 5, we know that the sols. of Laplace's eq. are

$$f_\ell^{(1)}(r') = (r')^\ell \quad \text{and} \quad f_\ell^{(2)}(r') = \frac{1}{(r')^{\ell+1}}$$

Consider now the two regions

I : $0 \le r' < r < R$

II : $0 \le r < r' \le R$

Because the potential should be regular at $r' = 0$ in Region I,

$$f_\ell^{(I)}(r,r') = a(r)\,(r')^\ell.$$

On the other hand, the potential should vanish at $r' = R$ (Dirichlet):

$$f_\ell^{(II)}(r,r') = b(r)\cdot(r')^\ell + c(r)\,\frac{r'}{(r')^{\ell+1}},$$

$$f_\ell^{(II)}(r,r'=R) = b(r)\,(R)^\ell + c(r)\,\frac{1}{R^{\ell+1}} \stackrel{!}{=} 0 \Rightarrow c(r) = -R^{2\ell+1}\,b(r)$$

$$f_\ell^{(II)}(r,r') = b(r)\left[(r')^\ell - \frac{R^{2\ell+1}}{(r')^{\ell+1}}\right]$$

In order to determine $b(r)$ and $c(r)$ we recall that $G(\vec{r},\vec{r}\,') = G(\vec{r}\,',\vec{r})$. In particular, $f_\ell(r,r')$ needs to be continuous at $r=r'$ (otherwise its derivative does not exist)

$\Rightarrow$

$$f_\ell^{(I)} = c(r)\cdot\left[r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right](r')^\ell \qquad 0\le r'\le r < R$$

$$f_\ell^{(II)} = c(r)\quad r^\ell\cdot\left[(r')^\ell - \frac{R^{2\ell+1}}{(r')^{\ell+1}}\right] \qquad 0\le r< r'\le R.$$

Finally, to determine the unknown function $c(r)$, we note by integrating $(*)$: $\int_{r-\epsilon}^{r+\epsilon} dr'$

$$r^2\,\partial_{r'} f_\ell(r,r')\Big|_{r'=r-\epsilon}^{r'=r+\epsilon} = 1 \quad , \text{ which leads to}$$

$$c(r)\,r^\ell\left[\ell\; r^{\ell-1} + (\ell+1)\,\frac{R^{2\ell+1}}{r^{\ell+2}}\right] - c(r)\left[r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right]\ell\; r^{\ell-1} \overset{!}{=} \frac{1}{r^2}$$

or

$$c(r)\,(\ell+1)\,\frac{R^{2\ell+1}}{r^2} + c(r)\,\ell\,\frac{R^{2\ell+1}}{r^2} \overset{!}{=} 1 \Rightarrow c(r)\,(2\ell+1)\,\frac{R^{2\ell+1}}{r^2} = \frac{1}{r^2}$$

so finally

$$f_\ell^{(I)} = \frac{1}{2\ell+1}\left[\frac{r^\ell}{R^{2\ell+1}} - \frac{1}{r^{\ell+1}}\right](r')^\ell \,, \qquad 0 \le r' < r < R$$

$$f_\ell^{(II)} = \frac{r^\ell}{2\ell+1}\left[\frac{(r')^\ell}{R^{2\ell+1}} - \frac{1}{(r')^{\ell+1}}\right] \,, \qquad 0 \le r < r' < R.$$

Introducing $r_< = \min(r, r')$, $r_> = \max(r, r')$

$$f_\ell(r, r') = \frac{r_<^\ell}{2\ell+1}\left[\frac{r_>^\ell}{R^{2\ell+1}} - \frac{1}{r_>^{\ell+1}}\right].$$

If we are interested in a problem where the bc demand that $\phi = 0$ at infinity, we let $R \to \infty$. The Green's function becomes

$$G(r, \theta, \varphi \,;\, r', \theta', \varphi') = 4\pi \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta, \varphi)\, Y_{\ell m}(\theta', \varphi').$$

<u>Exercise 13</u>

Calculate the Green's function in the <u>exterior</u> of the sphere: $r, r' \geq R$.

Note that the last equation implies that

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma),$$

where $\gamma$ is the angle between $\vec{r}$ and $\vec{r}'$.

## 4.2.4. Multipole expansion

Suppose we are interested in the potential created by a charge distribution $\rho(\vec{r}')$.

Let us assume that the charge is confined within a sphere of radius $R$ : $\rho(\vec{r}')=0$, $|\vec{r}'|>R$.

Because the potential outside the sphere satisfies $\vec{\nabla}^2\phi=0$, we know from Lecture 5

$$\phi(\vec{r}) = \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}},$$

since we know that $\phi(\vec{r})=0$ at $|\vec{r}|=\infty$.

In fact, since $G(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|}$ is the appropriate Green's function, we know that

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad , \quad \text{or}$$

$$\phi(\vec{r}) = 4\pi \sum_{\ell m} \frac{1}{2\ell+1}\left(\int d^3r' \, (r')^\ell \rho(r',\theta',\varphi') \, Y^*_{\ell m}(\theta',\varphi')\right) \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}$$

It follows immediately that

$$\underline{q_{\ell m} = \int d^3r' \; Y^*_{\ell m}(\theta',\varphi')(r')^\ell \rho(\vec{r}')}$$

These coefficients are the charge multipoles.

<u>Examples</u>

- For $\ell = m = 0$: With $Y_{00} = \frac{1}{\sqrt{4\pi}}$

$q_{00} = \int d^3r' \, \rho(\vec{r}')$ is just the total charge.

In the limit $|\vec{r}| \to \infty$, the charge gives the dominant contribution to the potential.

- For $\ell = 1$: with

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\varphi} \; ; \; Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \; ; \; Y_{1-1} = +\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\varphi}$$

and $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$
\begin{cases}
q_{11} = -\sqrt{\dfrac{3}{8\pi}} \displaystyle\int d^3r' \; (x' - iy')\, \rho(\vec{r}\,') \\[2ex]
q_{10} = \sqrt{\dfrac{3}{4\pi}} \displaystyle\int d^3r' \; z' \rho(\vec{r}\,') \\[2ex]
q_{1-1} = +\sqrt{\dfrac{3}{8\pi}} \displaystyle\int d^3r' \; (x' + iy')\, \rho(\vec{r}\,')
\end{cases}
$$

Recall now from quantum mechanics that a given <u>vector</u> $\vec{v} = (v_x, v_y, v_z)$ (which transforms in a 3-dim. rep. of $SO(3)$) has "spherical components"

$$v_{11} = -\frac{v_x + i v_y}{\sqrt{2}} \;; \quad v_{10} = v_z \;; \quad v_{1-1} = \frac{v_x - i v_y}{\sqrt{2}} .$$

Therefore, the $l=1$ multipoles are just the spherical components of the <u>dipole</u>

$$\vec{p} = \int d^3r' \; \vec{r}\,' \rho(\vec{r}\,') .$$

(up to a constant and complex conjugation)

- For $l=2$, with

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\varphi} \; ; \quad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2\theta - \frac{1}{2} \right)$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3r' \, (x'-iy')^2 \rho(\vec{r}\,')$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int d^3r' \; z' \, (x'-iy') \rho(\vec{r}\,')$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3r' \, (3z'^2 - r'^2) \rho(\vec{r}\,')$$

These are just the components of the <u>quadrupole</u> tensor

$$Q_{ij} = \int d^3r' \, (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{r}\,') , \qquad (i,j=1,2,3).$$

which is <u>symmetric</u> and <u>traceless</u>, and thus transforms in the spin-2 representation of $SO(3)$:

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}} \left( Q_{11} - 2i Q_{12} - Q_{22} \right) \; ; \quad q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} \left( Q_{13} - i Q_{23} \right)$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \, Q_{33} .$$

## Exercise 14

i) Show that

$$\phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2} Q_{ij} \frac{r^i r^j}{r^5} + \cdots$$

by expanding $G(\vec{r},\vec{r}\,') = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$ in $\phi(\vec{r}) = \int d^3r' \dfrac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

as a Taylor series around the origin $\vec{r}\,' = 0$.

ii) Show that $\phi(\vec{r}) = \dfrac{\vec{p}\cdot\vec{r}}{r^3}$ is the

field created by two charges $q$ and $-q$

separated by $\vec{L}$ as $\vec{L} \to 0$, with

$$\vec{p} \equiv q\vec{L} \overset{!}{=} \text{const}.$$

iii) Calculate the electric field created by such a dipole. ∎

### 2.3.2. Forces and Torques

We can use these expansions to calculate potentials, forces and torques of charge distributions in external fields:

Consider for instance the potential:

$$U = \int d^3r \; \rho(\vec{r}) \, \phi(\vec{r}) .$$

If charge localized around $\vec{r}=0$, expand

$$\phi(\vec{r}) = \phi(0) + \vec{r} \cdot \vec{\nabla}\phi(0) + \frac{1}{2} r^i r^j \frac{\partial^2 \phi}{\partial r^i \partial r^j}(0) + \dots$$

Therefore,

$$U = Q\,\phi(0) - \vec{p} \cdot \vec{E}(0) + \dots$$

The force acting on a charge distr. is

$$\vec{F} = \int d^3r \; \rho(\vec{r}) \, \vec{E}(\vec{r}).$$

As before, we find

$$\vec{F} = Q\,\vec{E}(0) + (\vec{p}\cdot\vec{\nabla})\,\vec{E}\,\Big|_0 + \dots$$

The torque acting on a charge distr. is

$$\vec{\tau} = \int d^3r \; \vec{r} \times (\rho\, \vec{E}(\vec{r}))$$

Expanding $\vec{E} = \vec{E}(0) + \dots$

$$\vec{\tau} = \vec{p} \times \vec{E}(0) + \dots$$