Last time

Solution of Laplace's equation in spherical coordinates:

$$\phi = \sum_{\ell m} \frac{a_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) \quad \text{with}$$

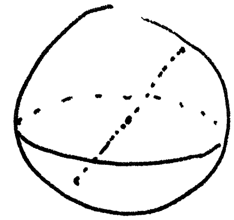
$$a_{\ell m} = R^{\ell+1} \int d\Omega \; V(\theta,\varphi) \, Y^*_{\ell m}(\theta,\varphi) \, .$$

In problems with azimuthal symmetry ($V = V(\theta)$) only harmonics with $m=0$ enter the expansion. In this case, the spherical harmonics reduce to the Legendre polynomials:

$$Y_{\ell m=0} = \frac{1}{\sqrt{4\pi}} P_\ell(\cos\theta) \, .$$

4.2.5 Spherical harmonics satisfy a number of useful relations. Among others:

- Under parity (spatial inversion)



$$Y_{lm}(\pi-\theta,\ \phi+\pi) = (-1)^l\, Y_{lm}(\theta,\varphi)$$

- under complex conjugation

$$Y_{lm}^*(\theta,\varphi) = (-1)^m\, Y_{l-m}(\theta,\varphi) .$$

- Addition theorem:

i) Let (θ_1, φ_1) and (θ_1', φ_1') be the coordinates of two points on the sphere.

ii) Let (θ_2, φ_2) and (θ_2', φ_2') be the coordinates of the same points after a (passive) rotation.

Then, $\displaystyle\sum_m Y_{lm}(\theta_1,\varphi_1)\, Y_{lm}^*(\theta_1',\varphi_1') = \sum_m Y_{lm}(\theta_2,\varphi_2)\, Y_{lm}^*(\theta_2',\varphi_2')$

This just reflects that

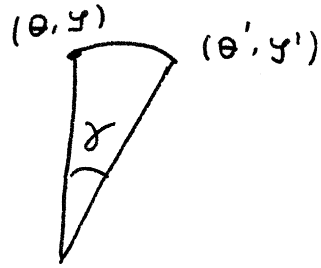
$\displaystyle\sum_m |lm\rangle\langle lm|$ is a <u>scalar</u> under rotations.

If we choose $\theta_2 = 0$, then $Y_{lm}(\theta_2=0, \varphi_2) = 0$

and the addition theorem reduces to

$$\sum_m Y_{lm}(\theta,\varphi)\, Y^*_{lm}(\theta',\varphi') = \frac{2l+1}{4\pi} P_l(\cos\gamma),$$

where γ is the angle



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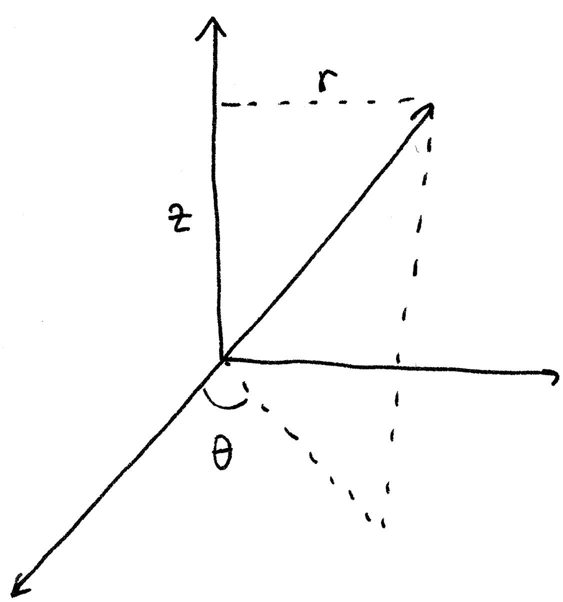
4.3. Example: Cylindrical coordinates

In problems with the geometry of a cylinder, cylindrical coordinates are useful:



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z.$$

In cylindrical coordinates, the Laplace operator is

$$\nabla^2\phi = \frac{1}{r}\left[\partial_r(r\,\partial_r\phi)\right] + \frac{1}{r^2}\partial_\theta^2\phi + \partial_z^2\phi$$

To solve Laplace's equation, we use the ansatz $\phi = R(r)\,\Theta(\theta)\,Z(z)$, (separation of variables) which leads to

$$\frac{\nabla^2 \phi}{\phi} = \frac{r\,(r R')'}{R} + \frac{\Theta''}{\Theta} + \frac{r^2 Z''}{Z} = 0 .$$

If the boundary conditions we are dealing with do not have a z-dependence, we set $Z \equiv 0$. We then need to solve

$$\Theta'' = \lambda_\theta \Theta .$$

Because $\Theta(\theta) \overset{!}{=} \Theta(\theta + 2\pi)$, the solutions are

$$\Theta = \alpha \sin(n\theta) + \beta \cos(n\theta) \qquad n = 0, 1, 2, \ldots$$

with $\lambda_\theta = -n$ we then need to solve

$$r(rR')' = n^2 R$$

For $n=0$ solutions are : $R(r) = 1$, $R = \log r$

$n \neq 0$ $R = r^n$, $R = r^{-n}$.

The general solution therefore is

$$\phi(r,\theta) = a_0 + b_0 \log r + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$$

Exercise 9

Suppose that the surface of the cylinder is kept at a potential $\phi(R,\theta) = V_0$, $0^\circ < \theta < 180^\circ$ and $\phi(R,\theta) = -V_0$, $180^\circ < \theta < 360^\circ$.

i) Find the potential inside and outside the cylinder.

ii) Find the surface charge distribution on the cylinder.

If the boundary conditions do not display translation symmetry along z-direction, we need to solve

$$\frac{(rR')'}{rR} - \frac{m^2}{r^2} = -\frac{Z''}{Z} ,$$

since

$$\Theta(\theta) = \alpha_m \cos(m\phi) + \beta_m \sin(m\phi)$$

Assuming $Z'' = \lambda_z Z$, with $\lambda_z = k^2$, we need to solve

$$(rR')' - \frac{m^2}{r} R = -k^2 rR.$$

Change variables: $x \equiv kr$, $y(x) \equiv R(r)$. Then

$$x[xy]' + (x^2 - m^2) y = 0$$

This is <u>Bessel's equation</u>. Its solutions are

the Bessel functions, $y_1 = J_m(x)$.

Neumann functions, $y_2 = N_m(x)$

(for m integer, these are linearly independent)

Note: All these differential operators are particular examples of a Sturm-Liouville problem:

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(y)\,y = \lambda\, w(x)\, y$$

For well-behaved $p(x)$, $q(x)$, $w(x)$, the corresponding differential operator is <u>hermitian</u>.

For appropriate boundary conditions on an interval $[0, L]$:

$$\begin{cases} \alpha_1 \, y(0) + \alpha_2 \, y'(0) = 0 \\ \beta_1 \, y(L) + \beta_2 \, y'(L) = 0 \end{cases}$$

Sturm's theorem applies. In particular the solutions of Sturm's equation form an <u>orthonormal basis</u> of the corresponding Hilbert space, with

$$\int_0^L dx \; y_n(x) \, y_m(x) \, w(x) = \delta_{mn} \, .$$

For a relatively complete list of the properties of the Bessel functions (and many others), see Abramowitz and Stegun, <u>Handbook of Mathematical Functions</u> (available on-line).

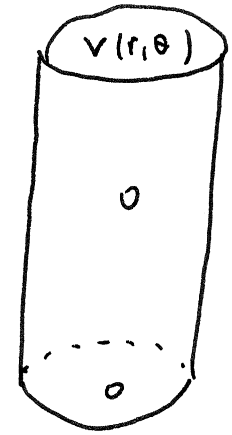
Exercise 10

Calculate the potential inside a hollow cylinder kept at a potential

$$\phi(r, \theta, z=0) = 0$$

$$\phi(R, \theta, z=0) = 0$$

$$\phi(r, \theta, L) = V(r, \theta)$$



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5. Green's functions

Thus far, we have solved the equations of electrostatics in the vacuum, $\rho = 0$.

Consider now these equations in the presence of charges:

$$\nabla^2 \phi = -4\pi\rho \qquad \text{(Poisson's equation).}$$

Suppose that we know a Green's function of the equation

$$\vec{\nabla}^2 G(\vec{r}, \vec{r}\,') = -4\pi\, \delta^{(3)}(\vec{r} - \vec{r}\,') .$$

Exercise 11

Prove Green's first identity

$$\int_V [\phi\, \nabla^2 \psi - \psi \nabla^2 \phi]\, dV = \oint_{\partial V} [\phi\, \vec{\nabla}\psi - \psi \vec{\nabla}\phi]\, d\vec{A}.$$

Applying this identity with $\phi = \phi(\vec{r}\,')$,

$\psi = G(\vec{r},\vec{r}\,')$

we get

$$\int_V [\phi(\vec{r}\,') \underbrace{\nabla^2_{r'} G(\vec{r},\vec{r}\,')}_{-4\pi\,\delta(\vec{r}-\vec{r}\,')} - G(\vec{r},\vec{r}\,') \underbrace{\nabla^2_{r'} \phi(\vec{r}\,')}_{-4\pi\,\rho(\vec{r}\,')}]\, d^3r'$$

$$= \oint_{\partial V} [\phi(\vec{r}\,') \vec{\nabla}_{r'} G(\vec{r},\vec{r}\,') - G(\vec{r},\vec{r}\,') \vec{\nabla}_{r'} \phi(\vec{r}\,')]\, d\vec{A}\,'$$

It follows that the solution to Poisson's eq. is

$$\phi(\vec{r}) = \int d^3r'\, G(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') -$$

$$- \frac{1}{4\pi} \oint_{\partial V} [\phi(\vec{r}\,') \vec{\nabla}_{r'} G(\vec{r},\vec{r}\,') - G(\vec{r},\vec{r}\,') \vec{\nabla}_{r'} \phi(\vec{r}\,')]\, d\vec{A}\,'.$$

Note that the Green's function is not unique: If $G_1(\vec{r},\vec{r}')$ is a solution of

$$\nabla^2_{r'} G_1(\vec{r},\vec{r}') = -4\pi \rho(\vec{r}) \quad , \text{ so is}$$

$$G_2(\vec{r},\vec{r}') = G_1(\vec{r},\vec{r}') + F(\vec{r},\vec{r}') \quad \text{for any } F \text{ with}$$

$$\nabla^2_{r'} F(\vec{r},\vec{r}') = 0 .$$

We can use this freedom to simplify the contribution of the boundary term:

i) For <u>Dirichlet</u> bc, choose

$$G(\vec{r},\vec{r}') = 0 \qquad \text{for } \vec{r}' \in \partial V$$

Then,

$$\phi(\vec{r}) = \int_V d^3r' \; G(\vec{r},\vec{r}')\,\rho(\vec{r}') - \frac{1}{4\pi}\oint_{\partial V} \phi(\vec{r}')\,\vec{\nabla}_{r'} G(\vec{r},\vec{r}')\, d\vec{A} .$$

ii) For <u>Neumann</u> boundary conditions, the natural choice would be $\vec{\nabla}_{r'} G(\vec{r},\vec{r}') = 0$ for $\vec{r}' \in \partial V$, but this is not possible because by

Gauss' law

$$\int_V \vec{\nabla}^2_{r'}\, G(\vec{r},\vec{r}\,')\, d^3r' = \int_{\partial V} \vec{\nabla}_{r'}\, G(\vec{r},\vec{r}\,')\, d\vec{A}\,'$$

$$\shortparallel$$

$$-4\pi \int_V d^3r'\; \delta(\vec{r}-\vec{r}\,') = -4\pi \qquad (\text{for } \vec{r} \text{ in } V).$$

We shall not consider Neumann bc in this case (see Jackson for how to deal with them.)