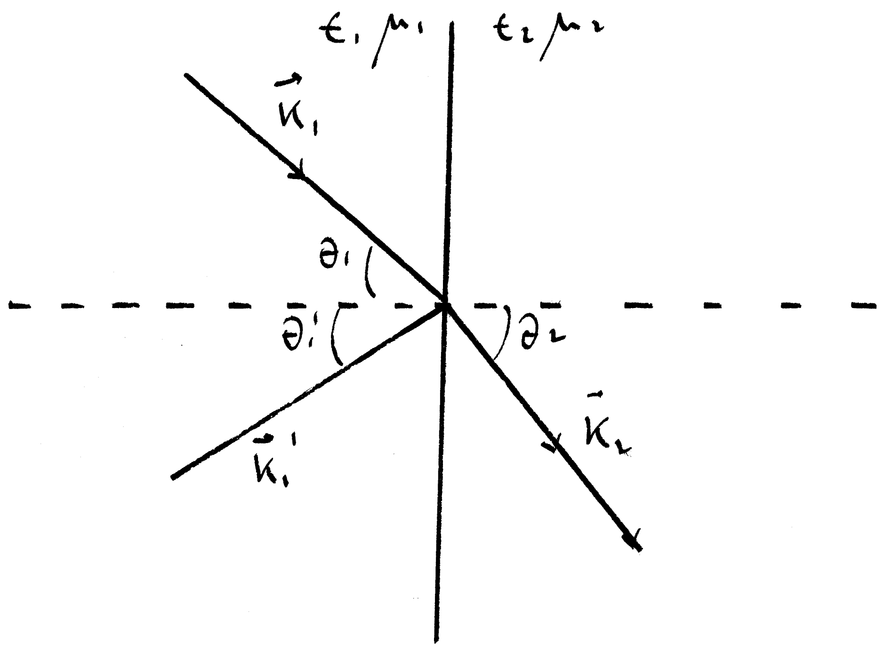
10.4. Reflection & Refraction (continued)



Maxwell's eqs $\Rightarrow$ $\underline{n_1 \sin\theta_1 = n_2 \sin\theta_2}$ Snell's law

Note that if $n_2 < n_1$ the equation

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$$

does not have a (real) solution for incident angles $\theta > \theta_c = \arcsin\frac{n_2}{n_1}$

This leads to <u>total internal reflection</u>

(applications: optical fibers, diving in a pool, etc.)

See Jackson or Franklin to see what happens to the "transmitted" wave

The electric and magnetic fields of the reflected and refracted components are determined by the junction conditions

$$\left\{\begin{aligned} &\epsilon_1\, \hat{n}\cdot(\vec{E}_1+\vec{E}_1') = \epsilon_2\, \hat{n}\cdot\vec{E}_2 \\ &\sqrt{\epsilon_1\mu_1}\;\hat{n}\cdot(\hat{k}_1\times\vec{E}_1+\hat{k}_1'\times\vec{E}_1') = \sqrt{\epsilon_2\mu_2}\;\hat{n}\cdot(\hat{k}_2\times\vec{E}_2) \\ &\hat{n}\times(\vec{E}_1+\vec{E}_1') = \hat{n}\times\vec{E}_2 \\ &\sqrt{\frac{\epsilon_1}{\mu_1}}\;\hat{n}\times(\hat{k}_1\times\vec{E}_1+\hat{k}_1'\times\vec{E}_1') = \sqrt{\frac{\epsilon_2}{\mu_2}}\;\hat{n}\times(\hat{k}_2\times\vec{E}_2) \end{aligned}\right.$$

The amount of reflected and transmitted radiation is described by the reflection and transmission coefficients

$$R \equiv \frac{\hat{n}\cdot\langle\vec{S}_1'\rangle}{\hat{n}\cdot\langle\vec{S}_1\rangle} \quad ; \quad T \equiv \frac{\hat{n}\cdot\langle\vec{S}_2\rangle}{\hat{n}\cdot\langle\vec{S}_1\rangle} .$$

where $\langle\vec{S}\rangle = \frac{c\hat{k}}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\,|\vec{E}|^2$ is averaged electromagnetic energy flow.

$R+T=1$ follows from Maxwell's eqs. and expresses conservation of energy.

Exercise 29

Calculate T for light polarized

i) $\perp$ to plane of incidence

ii) $\parallel$ " " "

You'll see that in general $T_{\parallel} \neq T_{\perp}$.

Therefore, typically, reflection and refraction produce partially polarized light, even if the incident light is not polarized.

For example: For a wave polarized $\parallel$ to incident plane, $R_{\parallel} = 0$ at the Brewster angle

$$\theta_B \simeq \arctan \frac{n_2}{n_1}.$$

$\Rightarrow$ The reflected light is fully polarized, with $\vec{E} \perp$ to incident plane

Application: Polarized glasses, CMB, ...

II. Electromagnetic waves in matter

Thus far we have made simplifying assumptions about the nature of the medium in which waves propagate. We relax in the following some of those assumptions:

II.1 Waves in a conducting medium

Consider the propagation of a wave in a conductor. By Ohm's law

$$\vec{j} = \sigma \vec{E}$$

the electric field generates a current, so

$$\vec{\nabla} \times \vec{H} - \frac{1}{c} \partial_t \vec{D} = \frac{4\pi}{c} \vec{j} \longrightarrow \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \mu \sigma \vec{E} + \frac{\mu \epsilon}{c} \partial_t \vec{E}$$

for a plane wave, this becomes

$$i \vec{k} \times \vec{B} = \frac{4\pi}{c} \mu \sigma \vec{E} - \frac{i \mu \epsilon}{c} \omega \vec{E} = - \frac{i \mu \epsilon \omega}{c} \left(1 + i \frac{4\pi\sigma}{\epsilon \omega} \right) \vec{E}$$

Recalling that

$$\cancel{i} \vec{k} \times \vec{E} = \cancel{i} \frac{\omega}{c} \vec{B}$$

we find

$$\frac{c}{\omega}\,\vec{k}\times(\vec{k}\times\vec{E}) = \frac{c}{\omega}\Big(\vec{k}\,\underbrace{(\vec{k}\cdot\vec{E})}_{0} - \vec{E}\,(\vec{k}\cdot\vec{k})\Big) = -\frac{\mu\epsilon\omega}{c}\left(1+i\,\frac{4\pi\sigma}{\epsilon\omega}\right)$$

bac - cab

or

$$|\vec{k}| = \underbrace{\frac{n\omega}{c}}\left(1+\frac{i4\pi\sigma}{\epsilon\omega}\right)^{1/2} \in \mathbb{C}$$

wave number in a dielectric.

writing $|\vec{k}| = \beta + i\,\frac{\alpha}{2}$, the electric field becomes

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} = \vec{E}_0 \exp\left[i\,\hat{k}\,(\beta+i\tfrac{\alpha}{2})\,\vec{x} - i\omega t\right] =$$

$$= \vec{E}_0 \exp\left(-\frac{\alpha}{2}\,\hat{k}\,\vec{x}\right) \exp\left[i\beta\,\hat{k}\,\vec{x} - i\omega t\right]$$

Therefore, the wave is exponentially attenuated.
The intensity $|\vec{E}|^2 \propto e^{-\alpha\hat{k}\vec{x}}$. Therefore, α^{-1} is the <u>attenuation length</u>.

Analogous considerations show that a plane wave can only penetrate a conductor up to a finite attenuation length.

11.3. Frequency dependence of permittivity

It turns out that our previous assumption that ϵ and μ do not depend on the frequency of the wave are not always justified.

Consider for instance an e^- bound to a molecule by a damped harmonic force. Then

$$\underset{\substack{\uparrow\\ \text{electric}\\ \text{force}}}{m\ddot{\vec{x}} = -e\vec{E}} - \underset{\substack{\uparrow\\ \text{damping}\\ \text{force}}}{m\gamma\dot{\vec{x}}} - \underset{\substack{\uparrow\\ \text{harmonic (restoring)}\\ \text{force.}}}{m\omega_0^2\vec{x}}$$

For static fields, the stationary solution is

$$\vec{x} = \frac{-e\vec{E}}{m\omega_0^2},$$ which leads to a

molecular dipole $\vec{p} = -e\vec{x} = \dfrac{e^2}{m\omega_0^2}\vec{E}$.

This would lead to a polarization (dipole density)

$$\vec{P} = N\frac{e^2}{m\omega_0^2}\vec{E},$$

where N is the density of molecules.

We have thus derived that $\vec{P} = \chi_e \vec{E}$, as we assumed previously.

If, on the other hand, $\vec{E} \propto e^{-i\omega t}$, the ansatz $\vec{x} \propto e^{-i\omega t}$ leads to

$$-m\omega^2 \vec{x} = -e\vec{E} - m\gamma(-i\omega)\vec{x} - m\omega_0^2 \vec{x}, \quad \text{or}$$

$$\vec{x} = \frac{-e\vec{E}}{m\left[\omega_0^2 - \omega^2 - i\omega\gamma\right]}, \quad \text{which yields}$$

the dipole moment and Polarization

$$\vec{p} = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\vec{E}; \qquad \vec{P} = \underbrace{\frac{e^2}{m}\frac{N}{\omega_0^2 - \omega^2 - i\omega\gamma}}_{\chi_e}\vec{E}.$$

Therefore $\epsilon = 1 + 4\pi\chi_e$ is
i) frequency-dependent
ii) complex!

In dielectrics we can typically set $\mu \simeq 1$, so that

$n = \sqrt{\epsilon\mu}$ is complex and frequency-dep. too!

with $|\vec{k}| = \frac{n}{c}\omega$, and $n = n_R(\omega) + i n_I(\omega)$

$|\vec{k}|$ becomes complex again:

$$|\vec{k}| = \frac{n_R(\omega)}{c}\omega + i\,\frac{n_I(\omega)}{c}\omega .$$

- The imaginary part of $|\vec{k}|$ leads to attenuation as before $\Rightarrow$ <u>absorption</u>. Absorption is strongest around resonances, $\omega = \omega_0$, where χ_e becomes purely imaginary.
- The real part of $|\vec{k}|$ (if frequency dependent) leads to <u>dispersion</u>: different frequency components travel with different phase velocities $v = \frac{c}{n_R(\omega)}$.

Thus, the shape of a pulse of radiation deforms with time. This is the familiar phenomenon we saw in QM for a free particle, where

$$\langle x|\psi(t)\rangle \propto e^{-iE(p)t/\hbar}\, e^{ipx/\hbar}, \text{ with } E = \frac{p^2}{2m}$$

A wave packet centered at wave number k_0 then travels at speed (group velocity)

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \neq v_p = \frac{\omega}{k} \quad \text{for}$$

non-linear dispersion relations.

In our context it is more appropriate to regard $k = |\vec{k}|$ as a function of ω. Thus

$$v_g = \frac{1}{\frac{dk}{d\omega}} = \frac{1}{\frac{dn_R}{d\omega}\cdot\frac{1}{c} + \frac{n_R}{c}\cdot 1} = \frac{c}{n_R + \frac{dn_R}{d\omega}}$$

Typically $n_R > 1$ and $\frac{dn_R}{d\omega} > 0$, so $v_g < c$, though in some materials $v_g > c$.

11.4 Causal relation between $\vec{D}$ and $\vec{E}$

With $\epsilon = \epsilon(\omega)$, the relation between $\vec{D}$ and $\vec{E}$ changes. By definition, in Fourier space we have

$$\begin{aligned} \vec{D}(\omega) &= \epsilon(\omega)\,\vec{E}(\omega). \\ &= [1 + \epsilon(\omega) - 1]\,\vec{E}(\omega) \end{aligned}$$

Therefore, in real space,

$$\vec{D}(t) = \vec{E}(t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, (\epsilon(\omega)-1)\, \vec{E}(\omega)\, e^{-i\omega t}.$$

Since $\vec{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tilde{t} \; \vec{E}(\tilde{t})\, e^{i\omega \tilde{t}}$, we find

$$\vec{D}(t) = \vec{E}(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, d\tilde{t} \;\; [\epsilon - 1](\omega)\; \vec{E}(\tilde{t})\; e^{i\omega(\tilde{t}-t)}$$

$$= \vec{E}(t) + \int_{-\infty}^{\infty} d\tilde{t} \;\; g(t-\tilde{t})\; \vec{E}(\tilde{t}),$$

where $g(t-\tilde{t}) = \int_{-\infty}^{\infty} d\omega \;\; [\epsilon(\omega)-1]\, e^{-i\omega(t-\tilde{t})}$ is

the convolution kernel.

<u>Causality</u> demands that $\vec{D}(t)$ be determined in terms of $\vec{E}(\tilde{t})$ for $\tilde{t} < t$, but, certainly, $\vec{D}(t)$ should not depend on $\vec{E}(\tilde{t})$ with $\tilde{t} > t$.

Therefore, causality demands

$$\underline{g(\Delta t) \overset{!}{=} 0 \qquad \text{for} \qquad \Delta t < 0.}$$