13. Electromagnetic Radiation and Scattering

We turn our attention now to the generation of electromagnetic radiation. We need to solve Maxwell's eqs. with time-dependent sources:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t \vec{B} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \end{cases}$$

where we have set (for simplicity) $\epsilon = \mu = 1$.

Introduce vector and scalar potentials:

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A} .$$

Homogeneous eqs. are then identically satisfied and remaining eqs. become

$$\begin{cases} \vec{\nabla}^2\phi + \frac{1}{c}\partial_t\vec{\nabla}\cdot\vec{A} = -4\pi\rho \\ \vec{\nabla}^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \frac{1}{c}\partial_t\vec{\nabla}\phi - \frac{1}{c^2}\partial_t^2\vec{A} = -\frac{4\pi}{c}\vec{j} \end{cases}$$

These equations satisfy by imposing the Lorenz gauge condition

$$\frac{1}{c}\partial_t \phi + \vec{\nabla}\cdot\vec{A} = 0 \qquad (\partial_\mu A^\mu = 0).$$

In Lorenz gauge, the Maxwell's eqs. simplify to:

$$\begin{cases} \vec{\nabla}^2\phi - \frac{1}{c^2}\partial_t^2\phi = -4\pi\rho \\ \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\partial_t^2\vec{A} = -\frac{4\pi}{c}\vec{j} \end{cases} \qquad (\Box A^\mu = -4\pi j^\mu)$$

Historical note: The "Lorenz gauge" is due to the danish physicist Ludvig Lorenz, although it is often (incorrectly) attributed to dutch physicist Hendrik Lorentz (author of the "Lorentz transformations".

Recall that $\vec{E}$ and $\vec{B}$ are invariant under the gauge transformations

$$\phi \to \phi' = \phi - \frac{1}{c}\partial_t \chi, \qquad \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$(A_\mu \to A_\mu + \partial_\mu \chi)$$

We can therefore impose Lorentz gauge by solving

$$\frac{1}{c}\partial_t \phi' + \vec{\nabla}\cdot\vec{A}' = \frac{1}{c}\partial_t\phi + \vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\partial_t^2\chi + \vec{\nabla}^2\chi \overset{!}{=} 0,$$

$$\text{or} \quad -\frac{1}{c^2}\partial_t^2\chi + \vec{\nabla}^2\chi = -\frac{1}{c}\partial_t\phi - \vec{\nabla}\cdot\vec{A}.$$

Note that Lorentz gauge does not uniquely specify ϕ, A. Any gauge transformation with

$$-\frac{1}{c^2}\partial_t^2\chi + \vec{\nabla}^2\chi = 0$$

preserves the Lorenz gauge condition

$\Rightarrow$ our solutions will contain "gauge modes".

13.3. Retarded solution

In order to solve the Lorentz gauge equations, it is convenient to work in Fourier space:

$$\vec{j} = e^{-i\omega t}\, e^{i\vec{k}\cdot\vec{r}}\, j_\omega(\vec{k})$$

(any other functional form can be constructed by superposition of $\vec{j}$'s of this form)

Then, the ansatz $\vec{A} = e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}} \vec{A}(\vec{k})$ gives

$$\left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right) \vec{A}(\vec{k}) = -\frac{4\pi}{c} \vec{j}_\omega(\vec{k}) \quad , \text{ or}$$

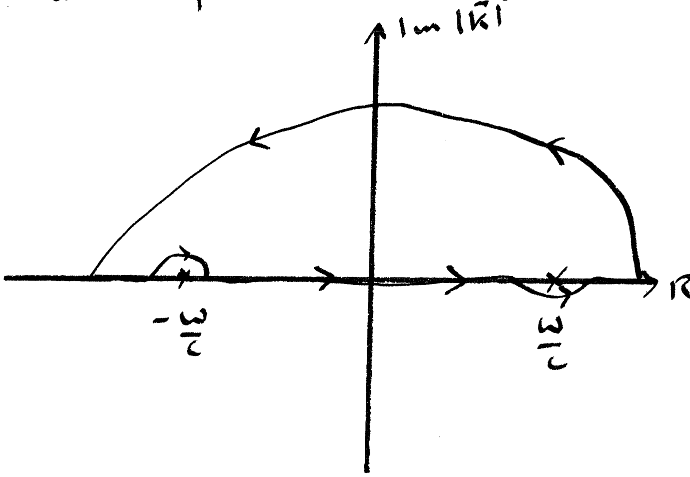
$$\vec{A}(\vec{k}) = \frac{4\pi}{c} \frac{\vec{j}_\omega(\vec{k})}{k^2 - \frac{\omega^2}{c^2}} .$$

Therefore, in real space, the solution is

$$\vec{A}(t,\vec{r}) = \frac{1}{(2\pi)^2}\int d\omega\, d^3k \, \frac{4\pi}{c} \frac{\vec{j}_\omega(\vec{k})}{k^2 - \frac{\omega^2}{c^2}} e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}} =$$

$$= \frac{1}{(2\pi)^4}\int d\omega\, d^3k\, dt'\, d^3k' \, \frac{4\pi}{c} \frac{\vec{j}(t',\vec{r}')}{k^2 - \frac{\omega^2}{c^2}} e^{-i\omega(t-t')} e^{i\vec{k}(\vec{r}-\vec{r}')}$$

Carry out d^3k integral. Use rotational invariance and proceed along deformed contour



$$\int d^3k \, \frac{e^{i\vec{k}(\vec{r}-\vec{r}')}}{k^2 - \frac{\omega^2}{c^2}}$$

$$= 2\pi \int d(\cos\theta)\, dk\, k^2 \, \frac{e^{ik|\vec{r}-\vec{r}'|\cos\theta}}{(k+\frac{\omega}{c})(k-\frac{\omega}{c})} = \frac{4\pi}{|\vec{r}-\vec{r}'|}\int_0^\infty dk\, k \, \frac{\sin(k|\vec{r}-\vec{r}'|)}{(k+\frac{\omega}{c})(k-\frac{\omega}{c})}$$

$$= \frac{4\pi}{i|\vec r-\vec r'|}\,\frac{1}{2}\int_{-\infty}^{\infty} dk\, k\, \frac{e^{ik|\vec r-\vec r'|}}{(k+\frac{\omega}{c})(k-\frac{\omega}{c})} = \frac{8\pi^2}{|\vec r-\vec r'|}\,\frac{1}{2}\cdot\frac{1}{2}\, e^{i\frac{\omega}{c}|\vec r-\vec r'|}$$

cos even,
sin odd

Cauchy

Therefore,

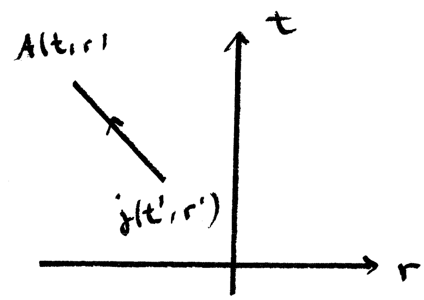
$$\vec A(t,\vec r) = \frac{1}{2}\frac{1}{(2\pi)^2}\,\frac{4\pi}{c}\int d\omega\, dt'\, d^3r'\, \frac{\vec j(t',\vec r')}{|\vec r-\vec r'|}\, e^{-i\omega(t-t')}\, e^{i\frac{\omega}{c}|\vec r-\vec r'|}$$

The integral over $d\omega$ gives $2\pi\, \underbrace{\delta(t-t'-|\vec r-\vec r'|/c)}_{\text{vanishes unless}}$,

$t' = t - \frac{|\vec r-\vec r'|}{c}$

and, hence,

$$\vec A(t,\vec r) = \frac{1}{c}\int d^3r'\, \frac{\vec j\left(t-\frac{|\vec r-\vec r'|}{c},\, \vec r'\right)}{|\vec r-\vec r'|}$$



$t' = t - \frac{|\vec r-\vec r'|}{c}$ is the retarded time

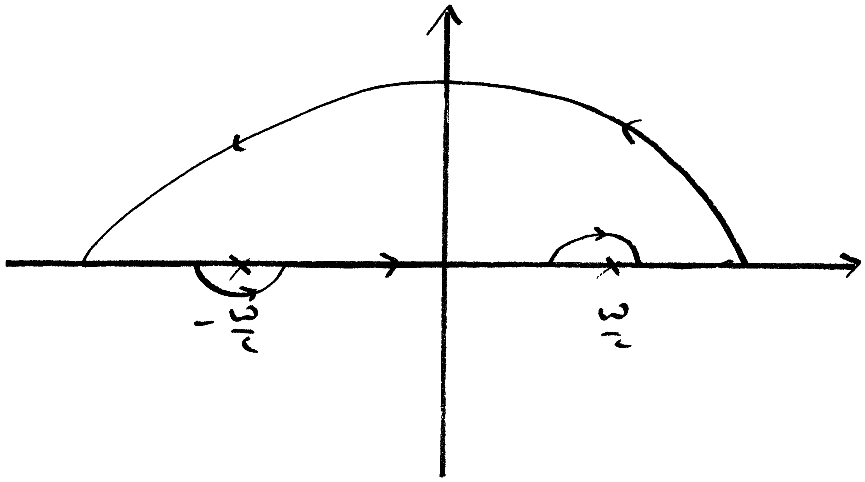
What we have shown is that

$$G^{(4)}(t,\vec x;t',\vec x') = \frac{\delta[t'-(t-\frac{|\vec r-\vec r'|}{c})]}{|\vec r-\vec r'|}$$

is a green's function, in the sense that

$$\left(-\frac{1}{c^2}\partial_t^2 + \bar{\nabla}^2\right) G(t,\bar{r};t',\bar{r}') = -4\pi\,\delta(t-t')\,\delta(\bar{r}-\bar{r}')$$

If we had chosen the contour



we would have gotten the advanced Green's fct

$$G^{(-)}(t,\bar{r};t',\bar{r}') = \frac{\delta\left[t' - \left(t + \frac{|\bar{r}-\bar{r}'|}{c}\right)\right]}{|\bar{r}-\bar{r}'|}.$$

- The choice of contour determines the b.c. satisfied by the Green's function.
- The latter are determined by physical considerations: causality.
- Feynman and Wheeler attempted to come up with a theory in which advanced and retarded potentials appear symmetrically.

Note that we have basically followed the procedure to construct Green's functions outlined in Lecture 6, Section 5.8.

Along the same lines, we find

$$\phi = \int d^3 r' \frac{\rho(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}')}{|\vec{r}-\vec{r}'|}.$$

13.4 Radiation Solution of the Wave Equation

Let us study the behavior of the solution for a given frequency of the source,

$$\vec{j}(t', \vec{r}') \propto e^{-i\omega t'},$$

$$\vec{A}(t, \vec{r}) = \frac{1}{c} \int d^3 r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \exp\left[-i\omega\left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right]$$

$$= \frac{e^{-i\omega t}}{c} \int d^3 r' \frac{\vec{j}(\vec{r}')\, e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}.$$

We are going to consider three regions:

with $\lambda = \frac{2\pi c}{\omega}$, d = dimensions of the source

i) Near (static) zone: $d \ll r \ll \lambda$

ii) Intermediate (induction) zone: $d \ll r \simeq \lambda$

iii) Far (radiation) zone: $d \ll \lambda \ll r$

- In the near zone, $e^{i\omega|\vec{r}-\vec{r}'|/c} = e^{i\frac{2\pi}{\lambda}|\vec{r}-\vec{r}'|} \simeq 1$

so the field is

$$\vec{A}(t,\vec{r}) \simeq \frac{e^{-i\omega t}}{c} \int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

as in magnetostatics, but with an oscillating field.

- In the far zone, we expand

$$|\vec{r}-\vec{r}'| = |\vec{r}| - \hat{r}\cdot\vec{r}' + \dots \quad \text{and get}$$

$$\vec{A}(\vec{r}) \simeq \frac{e^{-i\omega t}}{c} \int d^3r' \, \frac{\vec{j}(\vec{r}')\, e^{i\frac{\omega}{c}|\vec{r}|} \, e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'}}{|\vec{r}| - \hat{r}\cdot\vec{r}'} \simeq$$

$$\simeq e^{-i\omega t} \, \frac{e^{i\frac{\omega}{c}|\vec{r}|}}{c|\vec{r}|} \int d^3r' \, \vec{j}(\vec{r}') \left(1 + \frac{\hat{r}\cdot\vec{r}'}{|\vec{r}|} + \dots\right) e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'}$$

or $\quad \vec{A}(t,\vec{r}) = \dfrac{e^{i(\frac{\omega}{c}|\vec{r}| - \omega t)}}{c\,|\vec{r}|} \int d^3r' \; \vec{j}(\vec{r}') \, \exp(-i \frac{\omega}{c} \hat{r}\cdot\vec{r}')$.

We can expand the exponential to obtain the radiation field as an expansion of multipoles,

$$\exp(-i\frac{\omega}{c} \hat{r}\cdot\vec{r}') = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{\left(\frac{\omega}{c} \hat{r}\cdot\vec{r}'\right)^n}_{\sim \frac{d}{\lambda} \ll 1} .$$

The leading contribution stems from $n=1$, which gives

$$\int d^3r' \; \vec{j}(\vec{r}') \underset{\uparrow}{=} - \int d^3r' \; \vec{r}' \, (\vec{\nabla}'\cdot\vec{j}) \underset{\uparrow}{=} \frac{1}{c}\int d^3r' \; \vec{r}' \, \partial_t \rho = \frac{1}{c}\partial_t \vec{p}$$

$\int d^3r' \, j_i = \int d^3r' \, [\partial_j'(r_i' j^j) - r_i \partial_j j^i]$ $\qquad$ $c^{-1}\partial_t\rho + \vec{\nabla}\cdot\vec{j} = 0$

, where $\vec{p} = \int d^3r' \, r' \rho(\vec{r}')$ is the <u>dipole</u>

We can hence write

$$\vec{A}(t,\vec{r}) = \frac{e^{i\frac{\omega}{c}|\vec{r}|}}{c^2|\vec{r}|} \, \dot{\vec{p}} + \cdots$$

With $\vec{p} \propto e^{-i\omega t}$, this looks like an <u>outgoing spherical wave</u> of phase velocity $v_p = c$, with amplitude proportional to $\dot{\vec{p}}$ $\Rightarrow$ electromagnetic radiation requires a time-varying dipole.

There is <u>no monopole</u> radiation because electric charge is conserved:

We have for the scalar potential

$$\phi(t,\vec{r}) = \int d^3r' \frac{\rho(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}')}{|\vec{r}-\vec{r}'|} \simeq \int d^3r' \frac{\rho(t - \frac{|\vec{r}|}{c}, \vec{r}')}{|\vec{r}|}$$

$$= \left.\frac{Q_{tot}}{|\vec{r}|}\right|_{t' = t - \frac{|\vec{r}|}{c}}$$

Because charge is conserved, $\frac{dQ_{tot}}{dt} = 0$, so the monopole does not contribute to EM radiation.

In the radiation zone we find

$$\vec{B} = \vec{\nabla} \times \vec{A} \simeq i \frac{\omega}{c^2} \hat{r} \times \vec{A} = \frac{\omega^2}{c^2} \hat{r} \times \vec{p} \, \frac{e^{i\frac{\omega}{c}|\vec{r}|}}{|\vec{r}|}$$

transverse $(\perp \hat{r})$ and $\propto \frac{1}{|\vec{r}|}$

and from $\frac{1}{c}\partial_t \vec{E} = \vec{\nabla} \times \vec{B} - \cancel{\frac{4\pi}{c}\vec{j}}^{\,0}$ in the far zone $(\vec{j} = 0)$

$-i\frac{\omega}{c}\vec{E} = i\frac{\omega}{c}\hat{r} \times \vec{B}$, or $\vec{E} = -\hat{r} \times \vec{B}$ also transverse and $\propto \frac{1}{|\vec{r}|}$.