LECTURE NOTES 5

ELECTROMAGNETIC WAVES IN VACUUM

THE WAVE EQUATION FOR $\vec{E}$ AND $\vec{B}$

In regions of free space (*i.e.* the vacuum), where no electric charges, no electric currents and no matter of any kind are present, Maxwell's equations (in differential form) are:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

$\left(c^2=1/\varepsilon_o\mu_o\right)$

Set of coupled first-order partial differential equations

We can de-couple Maxwell's equations *e.g.* by applying the curl operator to equations 3) and 4):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\cancelto{=0}{\vec{\nabla}\cdot\vec{E}}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\cancelto{=0}{\vec{\nabla}\cdot\vec{B}}\right)-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$=-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

These are three-dimensional de-coupled <u>wave</u> equations for $\vec{E}$ and $\vec{B}$ - note that they have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.

Remember that each of the above equations is explicitly dependent on space and time, *i.e.* $\vec{E}=\vec{E}(\vec{r},t)$ and $\vec{B}=\vec{B}(\vec{r},t)$:

$$\boxed{\nabla^2\vec{E}(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \qquad \boxed{\nabla^2\vec{B}(\vec{r},t)=\frac{1}{c^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

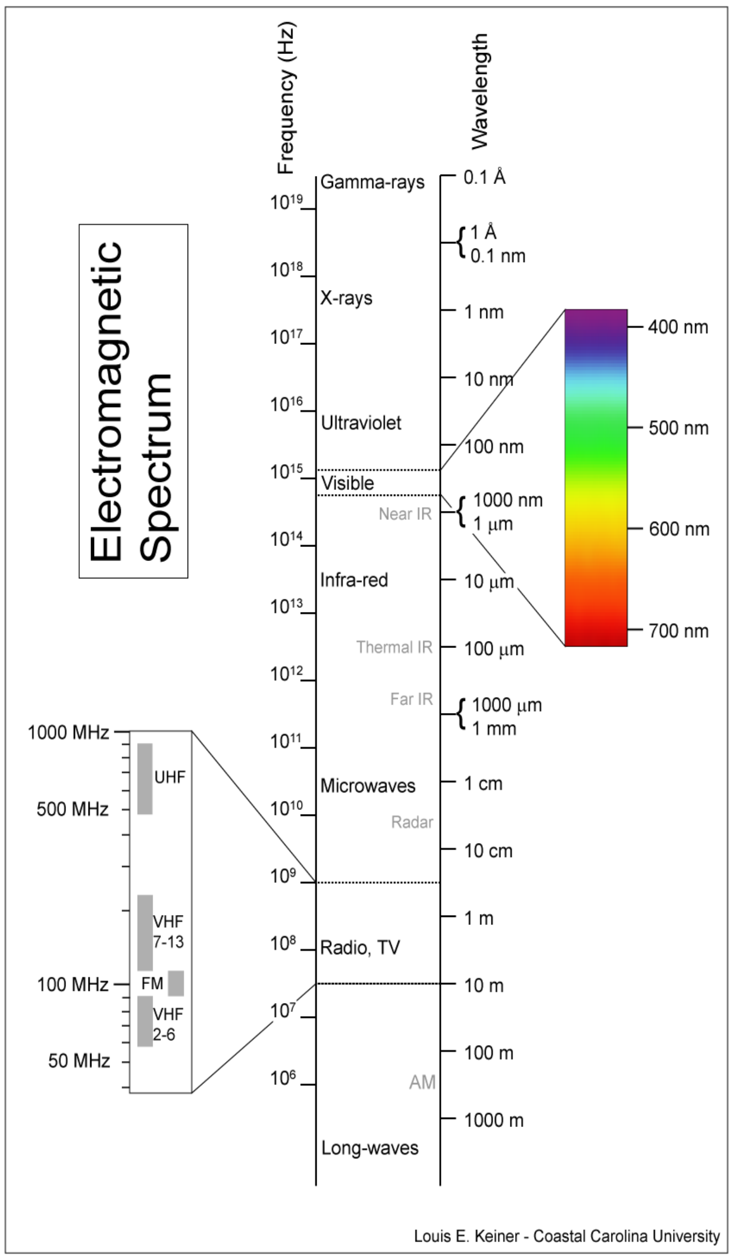
or:

$$\boxed{\nabla^2\vec{E}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0} \qquad \boxed{\nabla^2\vec{B}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0}$$

Thus, Maxwell's equations implies that empty space – the vacuum {which is <u>not</u> empty, at the microscopic scale} – supports the propagation of {<u>macroscopic</u>} electromagnetic waves, which propagate at the speed of light {in vacuum}: $\boxed{c=1/\sqrt{\varepsilon_o\mu_o}=3\times10^8\text{ m/s}}$.

EM waves have associated with them a frequency *f* and wavelength λ, related to each other via $c = f\lambda$. At the microscopic level, *EM* waves consist of large numbers of {massless} real photons, each carrying energy $\boxed{E = hf = hc/\lambda}$, linear momentum $|\vec{p}| = h/\lambda = hf/c = E/c$ and angular momentum $\boxed{|\vec{\ell}| = 1\hbar}$ where $h =$ Planck's constant $= 6.626\times10^{-34}$ Joule-sec and $\boxed{\hbar \equiv h/2\pi}$.

EM waves can have any frequency/any wavelength – the continuum of *EM* waves over the frequency region $0 < f < \infty$ (c.p.s. or Hertz {*aka* Hz}), or equivalently, over the wavelength region $0 < \lambda < \infty$ (m) is known as the electromagnetic spectrum, which has been divided up (for convenience) into eight bands as shown in the figure below (kindly provided by Prof. Louis E. Keiner, of Coastal Carolina University, Conway, SC):

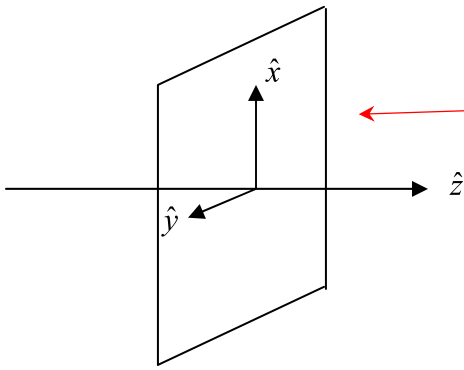


<u>Monochromatic *EM* Plane Waves</u>:

Monochromatic *EM* plane waves propagating in free space/the vacuum are sinusoidal *EM* plane waves consisting of a single frequency f, wavelength $\lambda = c/f$, angular frequency $\omega = 2\pi f$ and wavenumber $k = 2\pi/\lambda$. They propagate with speed $c = f\lambda = \omega/k$.

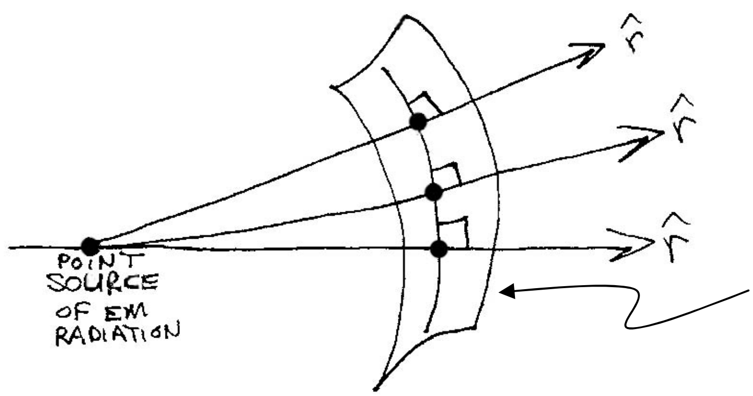
In the visible region of the *EM* spectrum {~380 nm (violet) $\leq \lambda \leq$ ~ 780 nm (red)}, *EM* light waves (consisting of real photons) of a given frequency / wavelength are perceived by the human eye as having a specific, single <u>color</u>. Hence we call such single-frequency, sinusoidal *EM* waves <u>mono-chromatic</u>.

EM waves that propagate *e.g.* in the $+\hat{z}$ direction but which <u>additionally</u> have no explicit *x*- or *y*-dependence are known as <u>plane</u> <u>waves</u>, because for a given time, t the <u>wave front(s)</u> of the *EM* wave lie in a <u>plane</u> which is $\perp$ to the $\hat{z}$ -axis, as shown in the figure below:



The planar wavefront associated with a plane *EM* wave propagating in the $+\hat{z}$ direction lies in the *x*-*y* plane.

Note that there also exist <u>spherical</u> *EM* waves – *e.g.* emitted from a <u>point</u> <u>source</u> (*e.g.* an atom) or a small antenna – the wavefronts associated with these *EM* waves are spherical, and thus do <u>*not*</u> lie in a <u>*plane*</u> $\perp$ to the direction of propagation of the *EM* wave:



Portion of a spherical wavefront associated with a spherical wave

n.b. If the point source is <u>infinitely</u> far away from observer, then a spherical wave → plane wave in this limit, (the radius of curvature → ∞); a spherical surface becomes planar as $R_C \to \infty$.

Criterion for a plane wave: $\boxed{\lambda \ll R_C}$

Monochromatic plane waves associated with $\vec{E}$ and $\vec{B}$:

$$\boxed{\tilde{\vec{E}}(z,t)=\tilde{\vec{E}}_o e^{\underbrace{i(kz-\omega t)}}} \qquad \boxed{\tilde{\vec{B}}(z,t)=\tilde{\vec{B}}_o e^{\underbrace{i(kz-\omega t)}}}$$

Propagating in $+\hat{z}$ direction

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n.b. complex vectors:

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e.g. $\boxed{\tilde{\vec{E}}_o=E_o e^{i\delta}\hat{x}}$

e.g. $\boxed{\tilde{\vec{B}}_o=B_o e^{i\delta}\hat{y}}$

n.b. The *real*, *physical* (instantaneous) fields are:

$$\left\{\begin{array}{l}\boxed{\vec{E}(\vec{r},t)\equiv\mathrm{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right)}\\ \boxed{\vec{B}(\vec{r},t)\equiv\mathrm{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right)}\end{array}\right\}$$

Very important to keep in mind!!

Note that Maxwell's equations for free space impose additional constraints on $\tilde{\vec{E}}_o$ and $\tilde{\vec{B}}_o$.

$\rightarrow$ Not just *any* $\tilde{\vec{E}}_o$ and/or $\tilde{\vec{B}}_o$ is acceptable / allowed !!!

$$\text{Since: } \vec{\nabla}\cdot\vec{E}=0 = \mathrm{Re}\left(\vec{\nabla}\cdot\tilde{\vec{E}}\right)=0 \qquad \text{and: } \vec{\nabla}\cdot\vec{B}=0 = \mathrm{Re}\left(\vec{\nabla}\cdot\tilde{\vec{B}}\right)=0$$

These two relations can only be satisfied $\forall(\vec{r},t)$ if $\vec{\nabla}\cdot\tilde{\vec{E}}=0\ \ \forall(\vec{r},t)$ and $\vec{\nabla}\cdot\tilde{\vec{B}}=0\ \ \forall(\vec{r},t)$.

In Cartesian coordinates: $\quad \vec{\nabla}=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}$

Thus: $\left(\vec{\nabla}\cdot\tilde{\vec{E}}\right)=0$ and $\left(\vec{\nabla}\cdot\tilde{\vec{B}}\right)=0$ become:

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{E}}_o e^{i(kz-\omega t)}\right)=0 \quad\text{and}\quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we *do* allow: $\quad \tilde{\vec{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in }\hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$

$$\tilde{\vec{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in }\hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then:

$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0}$$

$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0}$$

Or:

$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}=0}$$

$$\boxed{\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}=0}$$

Now: E_{ox}, E_{oy}, E_{oz} = Amplitudes (constants) of the electric field components in x, y, z directions respectively.

B_{ox}, B_{oy}, B_{oz} = Amplitudes (constants) of the magnetic field components in x, y, z directions respectively.

We see that: $\boxed{\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\leftarrow$ has no explicit x-dependence

And: $\boxed{\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\leftarrow$ has no explicit y-dependence

$\boxed{\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\leftarrow$ has no explicit x-dependence

And: $\boxed{\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\leftarrow$ has no explicit y-dependence

However: $\boxed{\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}}$

Thus: $\boxed{\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta}=ikE_{oz}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\Leftarrow$ true ***iff*** $\boxed{E_{oz}\equiv 0}$!!!

$\boxed{\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta}=ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta}=0}$ $\Leftarrow$ true ***iff*** $\boxed{B_{oz}\equiv 0}$!!!

- Thus, Maxwell's equations additionally tell us/impose the restriction that an electromagnetic plane wave *cannot* have *any* component of $\vec{E}$ or $\vec{B}$ $\parallel$ to (or anti-$\parallel$ to) the propagation direction (in this case here, the $\hat{z}$ -direction)

- Another way of stating this is that an *EM* wave cannot have any longitudinal components of $\vec{E}$ and $\vec{B}$ (*i.e.* components of $\vec{E}$ and $\vec{B}$ lying along the propagation direction).

- Thus, Maxwell's equations additionally tell us that an *EM* wave is a purely transverse wave (at least while it is propagating in free space) – *i.e.* the components of $\vec{E}$ and $\vec{B}$ must be $\perp$ to propagation direction.

- The plane of polarization of an *EM* wave is defined (by convention) to be parallel to $\vec{E}$.

Furthermore: Maxwell's equations impose yet another restriction on the allowed form of $\vec{E}$ and $\vec{B}$ for an *EM* wave:

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \quad \text{and/or:} \quad \boxed{\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

$$\underbrace{\boxed{= \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{E}}\right) = \mathrm{Re}\left(-\frac{\partial\tilde{\vec{B}}}{\partial t}\right)} \qquad \boxed{= \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}\right)}}_{\text{Can } only \text{ be satisfied } \forall\ (\vec{r},t)\ \textbf{iff}:}$$

$$\boxed{\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t}} \quad \text{and/or:} \quad \boxed{\vec{\nabla}\times\tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}}$$

Thus:

$$\boxed{\vec{\nabla}\times\tilde{\vec{E}} = \left(\overbrace{\cancel{\frac{\partial\tilde{E}_z}{\partial y}}}^{=0} - \frac{\partial\tilde{E}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial\tilde{E}_x}{\partial z} - \overbrace{\cancel{\frac{\partial\tilde{E}_y}{\partial x}}}^{=0}\right)\hat{y} + \left(\overbrace{\cancel{\frac{\partial\tilde{E}_y}{\partial x}}}^{=0} - \overbrace{\cancel{\frac{\partial\tilde{E}_x}{\partial y}}}^{=0}\right)\hat{z} = -\frac{\partial\tilde{B}_x}{\partial t}\hat{x} - \frac{\partial\tilde{B}_y}{\partial t}\hat{y} - \overbrace{\cancel{\frac{\partial\tilde{B}_z}{\partial t}}}^{=0}\hat{z}}$$

$$\boxed{\vec{\nabla}\times\tilde{\vec{B}} = \left(\overbrace{\cancel{\frac{\partial\tilde{B}_z}{\partial y}}}^{=0} - \frac{\partial\tilde{B}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial\tilde{B}_x}{\partial z} - \overbrace{\cancel{\frac{\partial\tilde{B}_y}{\partial x}}}^{=0}\right)\hat{y} + \left(\overbrace{\cancel{\frac{\partial\tilde{B}_y}{\partial x}}}^{=0} - \overbrace{\cancel{\frac{\partial\tilde{B}_x}{\partial y}}}^{=0}\right)\hat{z} = \frac{1}{c^2}\frac{\partial\tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial\tilde{E}_y}{\partial t}\hat{y} + \frac{1}{c^2}\overbrace{\cancel{\frac{\partial\tilde{E}_z}{\partial t}}}^{=0}\hat{z}}$$

With:

$$\boxed{\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \overbrace{\cancel{\tilde{E}_z}}^{=0}\hat{z} = \left(E_{ox}\hat{x} + E_{oy}\hat{y} + \overbrace{\cancel{E_{oz}}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}}$$

$$\boxed{\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \overbrace{\cancel{\tilde{B}_z}}^{=0}\hat{z} = \left(B_{ox}\hat{x} + B_{oy}\hat{y} + \overbrace{\cancel{B_{oz}}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}}$$

Thus:

$$\boxed{\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} = \left(E_{ox}\hat{x} + E_{oy}\hat{y}\right)e^{i(kz-\omega t)}e^{i\delta}}$$

$$\boxed{\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} = \left(B_{ox}\hat{x} + B_{oy}\hat{y}\right)e^{i(kz-\omega t)}e^{i\delta}}$$

$$\therefore \quad \boxed{\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{E}_y}{\partial z}\hat{x} + \frac{\partial\tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial\tilde{B}_x}{\partial t}\hat{x} - \frac{\partial\tilde{B}_y}{\partial t}\hat{y}}$$

$$\therefore \quad \boxed{\vec{\nabla}\times\tilde{\vec{B}} = -\frac{\partial\tilde{B}_y}{\partial z}\hat{x} + \frac{\partial\tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial\tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial\tilde{E}_y}{\partial t}\hat{y}}$$

Can only be satisfied / can only be true **iff** the $\hat{x}$ and $\hat{y}$ relations are separately / independently satisfied $\forall\ (\vec{r},t)$!

i.e. $\vec{\nabla}\times\tilde{\vec{E}}$: $\boxed{-\dfrac{\partial\tilde{E}_y}{\partial z}\hat{x} = -\dfrac{\partial\tilde{B}_x}{\partial t}\hat{x}} \Rightarrow \boxed{\dfrac{\partial\tilde{E}_y}{\partial z} = \dfrac{\partial\tilde{B}_x}{\partial t}} \Rightarrow \boxed{ikE_{oy} = -i\omega B_{ox}}$ (1)

$\boxed{+\dfrac{\partial\tilde{E}_x}{\partial z}\hat{y} = -\dfrac{\partial\tilde{B}_y}{\partial t}\hat{y}} \Rightarrow \boxed{\dfrac{\partial\tilde{E}_x}{\partial z} = -\dfrac{\partial\tilde{B}_y}{\partial t}} \Rightarrow \boxed{ikE_{ox} = +i\omega B_{oy}}$ (2)

$\vec{\nabla}\times\tilde{\vec{B}}$: $\boxed{-\dfrac{\partial\tilde{B}_y}{\partial z}\hat{x} = \dfrac{1}{c^2}\dfrac{\partial\tilde{E}_x}{\partial t}\hat{x}} \Rightarrow \boxed{-\dfrac{\partial\tilde{B}_y}{\partial z} = \dfrac{1}{c^2}\dfrac{\partial\tilde{E}_x}{\partial t}} \Rightarrow \boxed{-ikB_{oy} = -\dfrac{1}{c^2}i\omega E_{ox}}$ (3)

$\boxed{+\dfrac{\partial\tilde{B}_x}{\partial z}\hat{y} = \dfrac{1}{c^2}\dfrac{\partial\tilde{E}_y}{\partial t}\hat{y}} \Rightarrow \boxed{\dfrac{\partial\tilde{B}_x}{\partial z} = \dfrac{1}{c^2}\dfrac{\partial\tilde{E}_y}{\partial t}} \Rightarrow \boxed{ikB_{ox} = -\dfrac{1}{c^2}i\omega E_{oy}}$ (4)

From (1): $\boxed{ik\tilde{E}_{oy} = -i\omega B_{ox}} \Rightarrow \boxed{E_{oy} = -\left(\dfrac{\omega}{k}\right)B_{ox}}$ or: $\boxed{B_{ox} = -\left(\dfrac{k}{\omega}\right)E_{oy}}$

From (2): $\boxed{ik\tilde{E}_{ox} = +i\omega B_{oy}} \Rightarrow \boxed{E_{ox} = +\left(\dfrac{\omega}{k}\right)B_{oy}}$ or: $\boxed{B_{oy} = +\left(\dfrac{k}{\omega}\right)E_{ox}}$

From (3): $\boxed{-ikB_{oy} = -\dfrac{1}{c^2}i\omega E_{ox}} \Rightarrow \boxed{B_{oy} = +\dfrac{1}{c^2}\left(\dfrac{\omega}{k}\right)E_{ox}}$

From (4): $\boxed{ikB_{ox} = -\dfrac{1}{c^2}i\omega E_{oy}} \Rightarrow \boxed{B_{ox} = -\dfrac{1}{c^2}\left(\dfrac{\omega}{k}\right)E_{oy}}$

<u>Now</u>: $c = f\lambda = (2\pi f)\left(\dfrac{\lambda}{2\pi}\right) = \left(\dfrac{\omega}{k}\right)$ and $\dfrac{1}{c} = \left(k/\omega\right)$ $\left(k = 2\pi/\lambda\right)$

$\therefore$ $\underline{\vec{\nabla}\times\tilde{\vec{E}}}$: (1) $\boxed{B_{ox} = -\dfrac{1}{c}E_{oy}}$

(2) $\boxed{B_{oy} = +\dfrac{1}{c}E_{ox}}$

$\underline{\vec{\nabla}\times\tilde{\vec{B}}}$: (3) $\boxed{B_{oy} = +\dfrac{1}{c}E_{ox}}$

(4) $\boxed{B_{ox} = -\dfrac{1}{c}E_{oy}}$

Maxwell's Equations also have some redundancy encrypted into them!

So we really / actually <u>only</u> have <u>*two*</u> independent relations: $\boxed{B_{ox} = -\dfrac{1}{c}E_{oy}}$ and $\boxed{B_{oy} = +\dfrac{1}{c}E_{ox}}$

<u>But</u>: $\hat{z}\times\hat{y} = -\hat{x}$ $\hat{z}\times\hat{x} = +\hat{y}$

<u>Very Useful Table</u>:

$\hat{x}\times\hat{y} = \hat{z}$	$\hat{y}\times\hat{x} = -\hat{z}$
$\hat{y}\times\hat{z} = \hat{x}$	$\hat{z}\times\hat{y} = -\hat{x}$
$\hat{z}\times\hat{x} = \hat{y}$	$\hat{x}\times\hat{z} = -\hat{y}$

$\therefore$ We can write the above two relations succinctly/compactly with <u>one</u> relation: $\boxed{\tilde{\vec{B}}_o = \dfrac{1}{c}\left(\hat{z}\times\tilde{\vec{E}}_o\right)}$

Physically, the mathematical relation $\boxed{\vec{\tilde{B}}_o = \frac{1}{c}\left(\hat{z}\times\vec{\tilde{E}}_o\right)}$ states that $\vec{E}$ and $\vec{B}$ are:

a.) <u>in</u> <u>phase</u> with each other.

b.) <u>mutually</u> <u>perpendicular</u> to each other - *i.e.* $\left(\vec{E}\perp\vec{B}\right)\perp\hat{z}$ ($\hat{z}$ = propagation direction)

The $\vec{E}$ and $\vec{B}$ fields associated with this monochromatic plane *EM* wave are purely transverse { *n.b.* this is as also required by relativity at the <u>microscopic</u> level – for the extreme relativistic particles – the (massless) real photons traveling at the speed of light *c* that make up the macroscopic monochromatic plane *EM* wave.}

The <u>real</u> <u>*amplitudes*</u> of $\vec{E}$ and $\vec{B}$ are {also} related to each other by: $\boxed{B_o = \frac{1}{c}E_o}$

with $\boxed{B_o = \sqrt{B_{ox}^2 + B_{oy}^2}}$ and $\boxed{E_o = \sqrt{E_{ox}^2 + E_{oy}^2}}$

<u>Griffiths Example 9.2</u>:

A monochromatic (single-frequency) plane *EM* wave that is plane polarized/linearly polarized in the $+\hat{x}$ direction and is propagating in the $+\hat{z}$ direction, has:

$\boxed{\vec{E} = E\,\hat{x}}$ ⇐ definition of linearly polarized *EM* wave in the $+\hat{x}$ direction.

$$\therefore\quad \boxed{\vec{B} = \frac{1}{c}\left(\hat{z}\times\vec{E}\right) = \frac{1}{c}\left(\hat{z}\times E\hat{x}\right) = \frac{1}{c}E\underbrace{\left(\hat{z}\times\hat{x}\right)}_{=+\hat{y}\text{ by right-hand rule}} = \frac{1}{c}E\hat{y}}$$

<u>With</u>: $\boxed{\vec{B} = \frac{1}{c}\left(\hat{z}\times\vec{E}\right)}$, $\boxed{B = \frac{1}{c}E}$ and $\boxed{B_o = \frac{1}{c}E_o}$

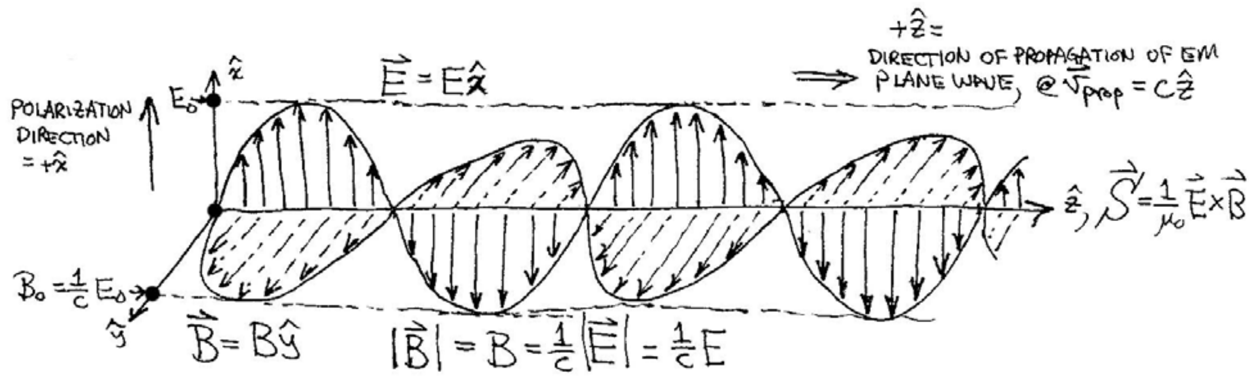
<u>Then</u>:
$$\boxed{\vec{\tilde{E}}(z,t) = \tilde{E}_o e^{i(kz-\omega t)}\hat{x} = E_o e^{i(kz-\omega t)}e^{i\delta}\hat{x} = E_o e^{i(kz-\omega t+\delta)}\hat{x}}$$
$$\boxed{\vec{\tilde{B}}(z,t) = \tilde{B}_o e^{i(kz-\omega t)}\hat{y} = B_o e^{i(kz-\omega t)}e^{i\delta}\hat{y} = B_o e^{i(kz-\omega t+\delta)}\hat{y}}$$
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

The *<u>physical</u>* (instantaneous) electric and magnetic fields are given by the following expressions:

$$\boxed{\vec{E}(z,t) = \text{Re}\left(\vec{\tilde{E}}(z,t)\right) = \text{Re}\left\{\overbrace{E_o\cos(kz-\omega t+\delta)\hat{x}}^{real} + i\,\overbrace{E_o\sin(kz-\omega t+\delta)\hat{x}}^{imaginary}\right\}}$$
$$\boxed{\vec{E}(z,t) = E_o\cos(kz-\omega t+\delta)\,\hat{x}}$$

$$\boxed{\vec{B}(z,t) = \text{Re}\left(\vec{\tilde{B}}(z,t)\right) = \text{Re}\left\{\overbrace{B_o\cos(kz-\omega t+\delta)\hat{y}}^{real} + i\,\overbrace{B_o\sin(kz-\omega t+\delta)\hat{y}}^{imaginary}\right\}}$$
$$\boxed{\vec{B}(z,t) = B_o\cos(kz-\omega t+\delta)\,\hat{y} = \frac{1}{c}E_o\cos(kz-\omega t+\delta)\,\hat{y}}$$

The *<u>physical</u>* (instantaneous) $\vec{E}$ and $\vec{B}$ fields are in-phase with each other for a linearly polarized *EM* wave



Note that: $\left(\vec{E} \perp \vec{B}\right) \perp \hat{z} \Rightarrow \vec{E} \perp \hat{z},\ \vec{B} \perp \hat{z}$ ($\hat{z}$ = direction of propagation of *EM* wave)

Instantaneous Poynting's Vector for a linearly polarized *EM* wave:

$$\boxed{\vec{S}(z,t) = \frac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}}$$

$$\boxed{\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}}$$

$$\boxed{\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\hat{z}} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$\Rightarrow$ *EM* Power flows in the direction of propagation of the *EM* wave (*here*, the $+\hat{z}$ direction)

Generalization for Propagation of Monochromatic Plane *EM* Waves in an Arbitrary Direction

Obviously, there is nothing special / profound with regard to plane *EM* waves propagating in a specific direction in free space / the vacuum. They can propagate in any direction. We can easily generalize the mathematical description for monochromatic plane *EM* waves traveling in an arbitrary direction as follows:

Introduce the notion / concept of a wave vector (or propagation vector) $\vec{k}$ which points in the direction of propagation, whose magnitude $|\vec{k}| = k$. Then the scalar product $\vec{k}\bullet\vec{r}$ is the appropriate generalization of kz:

If: $\vec{k} = k\hat{z}$ with $|\vec{k}| = k$ and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ with $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

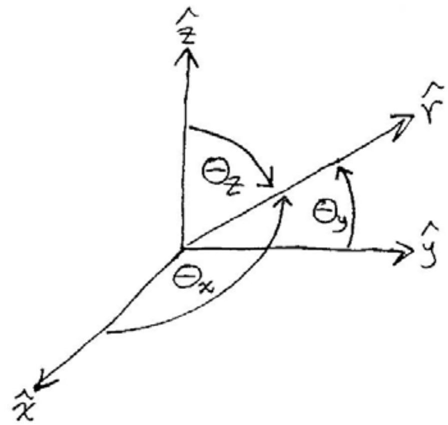
Then: $\left(\vec{k}\bullet\vec{r}\right) = k\hat{z}\bullet(x\hat{x}+y\hat{y}+z\hat{z}) = kz$

If: $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ with $|\vec{k}| = \sqrt{k_x^2+k_y^2+k_z^2}$ and $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$ with $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

Then: $\left(\vec{k}\bullet\vec{r}\right) = k_x x + k_y y + k_z z$

Now: $$\left.\begin{aligned} k_x &= k\cos\Theta_x \\ k_y &= k\cos\Theta_y \\ k_z &= k\cos\Theta_z \end{aligned}\right\}$$ where $\cos\Theta_x,\ \cos\Theta_y,\ \cos\Theta_z$ = direction cosines *w.r.t.* (with respect to) the $\hat{x},\ \hat{y},\ \hat{z}$-axes respectively

Direction Cosines: $$\left\{\begin{aligned} \cos\Theta_x &= \sin\theta\cos\varphi \\ \cos\Theta_y &= \sin\theta\sin\varphi \\ \cos\Theta_z &= \cos\theta \end{aligned}\right\}$$ in spherical-polar coordinates



Note: $$\boxed{\begin{aligned} &\sqrt{\cos^2\Theta_x+\cos^2\Theta_y+\cos^2\Theta_z} \\ &=\sqrt{\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta} \\ &=\sqrt{\sin^2\theta+\cos^2\theta}=1 \end{aligned}}$$

If *e.g.* $\vec{k}\,\|\,\vec{r}$ then: $\vec{k}\cdot\vec{r}=kr$. We explicitly demonstrate this in spherical polar coordinates:

For $\vec{k}\,\|\,\vec{r}$: $$\left\{\begin{aligned} k_x &= k\cos\Theta_x = k\sin\theta\cos\varphi \\ k_y &= k\cos\Theta_y = k\sin\theta\sin\varphi \\ k_z &= k\cos\Theta_z = k\cos\theta \end{aligned}\right\}$$ and: $$\left\{\begin{aligned} x &= r\cos\Theta_x = r\sin\theta\cos\varphi \\ y &= r\cos\Theta_y = r\sin\theta\sin\varphi \\ z &= r\cos\Theta_z = r\cos\theta \end{aligned}\right\}$$

Then: $$\begin{aligned} \left(\vec{k}\cdot\vec{r}\right) &= k_x x+k_y y+k_z z = kx\cos\Theta_x+ky\cos\Theta_y+kz\cos\Theta_z \\ &= kr\cos^2\Theta_x+kr\cos^2\Theta_y+kr\cos^2\Theta_z \\ &= kr\sin^2\theta\cos^2\varphi+kr\sin^2\theta\sin^2\varphi+kr\cos^2\theta \\ &= kr\left\{\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta\right\} = kr\left\{\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+\cos^2\theta\right\} \\ &= kr\left\{\sin^2\theta+\cos^2\theta\right\}=kr \end{aligned}$$

Thus, most generally, we can write the $\tilde{\vec{E}}(\vec{r},t)$ and $\tilde{\vec{B}}(\vec{r},t)$-fields as:

$$\boxed{\tilde{\vec{E}}(\vec{r},t)=\tilde{E}_o e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\hat{n}}$$

$$\boxed{\tilde{\vec{B}}(\vec{r},t)=\tilde{B}_o e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\left(\hat{k}\times\hat{n}\right)}$$

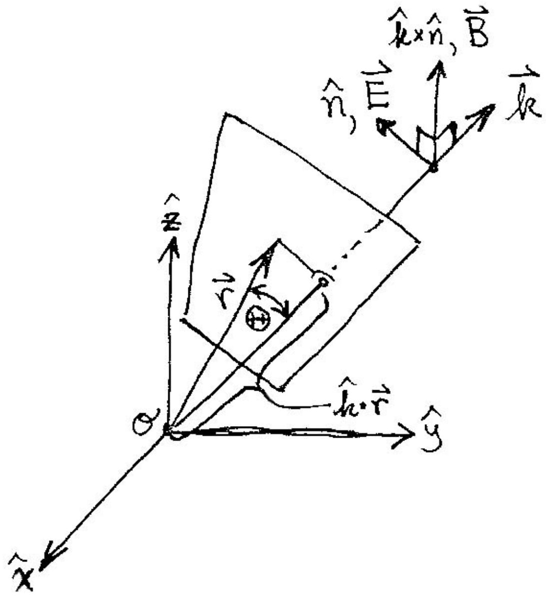
$$\boxed{\tilde{\vec{B}}=\frac{1}{c}\hat{k}\times\tilde{\vec{E}}}$$

where: $\hat{n}\equiv$ polarization vector $\boxed{\hat{n}\perp\hat{k}}$

i.e. $\boxed{\hat{n}\cdot\hat{k}=0}$ because $\tilde{\vec{E}}$ is transverse

We *must* have: $\boxed{\tilde{\vec{B}}(\vec{r},t)\perp\tilde{\vec{E}}(\vec{r},t)\perp\hat{k}}$ *i.e.* $\boxed{\tilde{\vec{E}}\cdot\tilde{\vec{B}}=0}$ and $\boxed{\tilde{\vec{E}}\cdot\hat{k}=0}$ and $\boxed{\tilde{\vec{B}}\cdot\hat{k}=0}$

The Direction of Propagation of a Monochromatic Plane *EM* Wave: $\hat{k}$



The Real/Physical (Instantaneous) *EM* Fields are:

$$\boxed{\vec{E}(\vec{r},t) = \text{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right) = E_o \cos\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right)\hat{n}}$$

where: $\hat{n} \equiv$ polarization vector $\left(\parallel \vec{E}\right)$

$$\boxed{\vec{B}(\vec{r},t) = \text{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right) = B_o \cos\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right)\left(\hat{k}\times\hat{n}\right)}$$

$\boxed{\left(B_o = \frac{1}{c}E_o\right)}$ in free space

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM* Waves

Instantaneous Energy Density Associated with an *EM* Wave:

$$\boxed{u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right) = u_{elect}(\vec{r},t) + u_{mag}(\vec{r},t)}$$

where: $\boxed{u_{elect}(\vec{r},t) = \frac{1}{2}\varepsilon_o E^2(\vec{r},t)}$ and $\boxed{u_{mag}(\vec{r},t) = \frac{1}{2\mu_o}B^2(\vec{r},t) = \frac{1}{2}\varepsilon_o E^2(\vec{r},t)}$

But: $\boxed{B^2 = \frac{1}{c^2}E^2}$ for *EM* waves in vacuum, and $\boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$

Thus: $u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \cancel{\mu_o}}{\cancel{\mu_o}}E^2(\vec{r},t)\right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t)\right)$

Or: $\boxed{u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right)}$ $\left(\frac{\text{Joules}}{\text{m}^3}\right)$

n.b. $\boxed{u_{elect}(\vec{r},t) = u_{mag}(\vec{r},t)}$ for *EM* waves propagating in the vacuum !!!!

Instantaneous Poynting's Vector Associated with an *EM* Wave:

$$\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)=\frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a linearly polarized monochromatic plane *EM* wave propagating in the vacuum, *e.g.*:

$$\boxed{\vec{E}(\vec{r},t)=E_o\cos(kz-\omega t+\delta)\,\hat{x}}\quad\text{and:}\quad\boxed{\vec{B}(\vec{r},t)=B_o\cos(kz-\omega t+\delta)\,\hat{y}}$$

Then: $\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\,\hat{z}}$ but: $\boxed{B_o=\frac{1}{c}E_o}$ for *EM* waves in vacuum.

Thus: $\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu_o c}E_o^2\cos^2(kz-\omega t+\delta)\,\hat{z}}\leftarrow$ multiply RHS by $1=\left(\frac{c}{c}\right)$

Hence: $\boxed{\vec{S}(\vec{r},t)=c\left(\frac{1}{\mu_o c^2}\right)E_o^2\cos^2(kz-\omega t+\delta)\,\hat{z}}$ but: $\boxed{\frac{1}{c^2}=\varepsilon_o\mu_o}$

Thus: $\boxed{\vec{S}(\vec{r},t)=c\left(\frac{\varepsilon_o\cancel{\mu_o}}{\cancel{\mu_o}}\right)E_o^2\cos^2(kz-\omega t+\delta)\,\hat{z}=c\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)\,\hat{z}}$

But: $\boxed{u_{EM}(\vec{r},t)=\varepsilon_oE^2(\vec{r},t)=\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)}$

$\therefore$ $\boxed{\vec{S}(\vec{r},t)=cu_{EM}(\vec{r},t)\,\hat{z}}$ *Here*, in ***this*** example, the propagation velocity of *energy*: $\boxed{\vec{v}_{prop}=c\hat{z}}$

$\Rightarrow$ Poynting's Vector = Energy Density * (Energy) Propagation Velocity: $\boxed{\vec{S}(\vec{r},t)=u_{EM}(\vec{r},t)\,\vec{v}_{prop}}$

Instantaneous Linear Momentum Density Associated with an *EM* Wave:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)}\quad\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For linearly polarized monochromatic plane *EM* waves propagating in the vacuum:

$$\boxed{\vec{\wp}_{EM}=\frac{1}{c^{\cancel{2}}}\cancel{c}\,\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)\,\hat{z}=\frac{1}{c}\underbrace{\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)}_{=u_{EM}}\hat{z}}$$

But: $\boxed{u_{EM}(\vec{r},t)=\varepsilon_oE^2(\vec{r},t)=\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)}$

$$\therefore\quad\boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon_o\mu_o\vec{S}(\vec{r},t)=\frac{1}{c^2}\vec{S}(\vec{r},t)=\frac{1}{c}u_{EM}(\vec{r},t)\,\hat{z}}\quad\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

<u>Instantaneous Angular Momentum Density Associated with an *EM* wave</u>:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t)} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

<u>But</u>: $$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c}u_{EM}(\vec{r},t)\hat{z}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$\therefore$ for an *EM* wave propagating in the $+\hat{z}$ direction:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \frac{1}{c^2}\vec{r} \times \vec{S}(\vec{r},t) = \frac{1}{c}u_{EM}(\vec{r},t)(\vec{r} \times \hat{z})} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

n.b. depends on the choice of origin

The <u>*instantaneous*</u> *EM* power flowing into/out of volume *v* with bounding surface *S* <u>enclosing</u> volume *v* (containing *EM* fields in the volume *v*) is:

$$\boxed{P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = \int_v \frac{\partial u_{EM}(\vec{r},t)}{\partial t} d\tau = -\oint_S \vec{S}(\vec{r},t) \cdot d\vec{a}} \qquad \text{(Watts)}$$

n.b. <u>closed</u> surface *S* enclosing volume *v*.

The <u>instantaneous</u> *EM* power crossing an (imaginary) surface (*e.g.* a 2-D plane – a window!) is:

$$\boxed{P_{EM}(t) = -\int_S \vec{S}(\vec{r},t) \cdot d\vec{a}_\perp}$$

The <u>instantaneous</u> total *EM* <u>energy</u> contained in volume *v* is: $\boxed{U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\, d\tau}$ (Joules)

The <u>instantaneous</u> total *EM* <u>linear momentum</u> contained in the volume *v* is:

$$\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t) d\tau} \qquad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The <u>instantaneous</u> total *EM* <u>angular momentum</u> contained in the volume *v* is:

$$\boxed{\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\, d\tau} \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

<u>Time-Averaged Quantities Associated with *EM* Waves</u>:

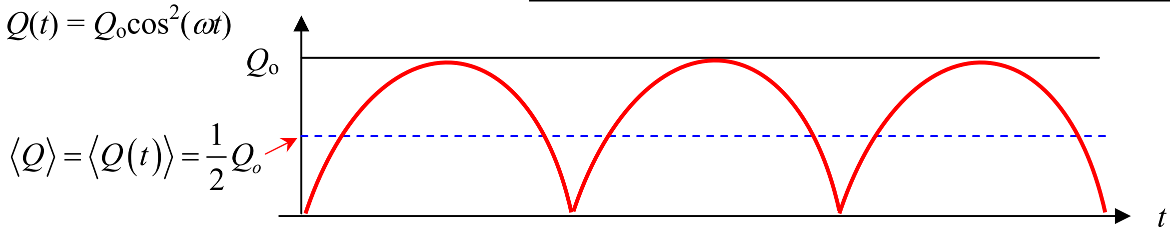
Frequently, we are <u>*not*</u> interested in knowing the instantaneous power P(*t*), energy / energy density, Poynting's vector, linear and angular momentum, *etc*.- *e.g.* simply because experimental measurements of these quantities are very often <u>averages</u> over <u>many</u> extremely fast cycles of oscillation…

(*e.g.* period of oscillation of <u>*light*</u> wave $\tau_{light} = 1/f_{light} \simeq \dfrac{1}{10^{15}\text{ cps}} = 10^{-15}\text{ sec/cycle} = 1\text{ femto-sec}$)

$\therefore\ \Rightarrow$ We want/need <u>time</u> <u>averaged</u> expressions for each of these quantities (*e.g.* in order to compare directly with experimental data) *e.g.* for monochromatic plane *EM* <u>*light*</u> waves:

If we have *e.g.* a "generic" instantaneous physical quantity of the form: $\boxed{Q(t) = Q_o\cos^2(\omega t)}$

The time-average of $Q(t)$ is defined as: $\boxed{\langle Q(t)\rangle \equiv \langle Q\rangle = \frac{1}{\tau}\int_{t=0}^{t=\tau} Q(t)\,dt = \frac{Q_o}{\tau}\int_{t=0}^{t=\tau}\cos^2(\omega t)\,dt}$



The time average of the $\cos^2(\omega t)$ function:

$$\boxed{\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2} + \frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau - 0) + \left(\frac{\sin 2\omega\tau}{2\omega} - 0\right)\right] = \frac{1}{2\tau}\left[\tau + \frac{\sin 2\omega\tau}{2\omega}\right]}$$

<u>But</u>: $\boxed{\omega\tau = 2\pi f\tau}$ and: $\boxed{f = 1/\tau}$ $\therefore$ $\boxed{\omega\tau = 2\pi(\tau/\tau) = 2\pi}$ $\therefore$ $\boxed{\sin(\omega\tau) = \sin(2\pi) = 0}$

$$\therefore\ \boxed{\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{2\cancel{\tau}}[\cancel{\tau}] = \frac{1}{2}}\ \therefore\ \boxed{\langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o}$$

Thus, the <u>time-averaged</u> quantities associated with an *EM* wave propagating in free space are:

EM Energy Density:	$\boxed{u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t)\rangle}$	Total *EM* Energy:	$\boxed{U_{EM}(t) \Rightarrow \langle U_{EM}(t)\rangle}$
Poynting's Vector:	$\boxed{\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t)\rangle}$	*EM* Power:	$\boxed{P_{EM}(t) \Rightarrow \langle P_{EM}(t)\rangle}$
Linear Momentum Density:	$\boxed{\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\wp}_{EM}(\vec{r},t)\rangle}$	Linear Momentum:	$\boxed{\vec{p}_{EM}(t) \Rightarrow \langle \vec{p}_{EM}(t)\rangle}$
Angular Momentum Density:	$\boxed{\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\ell}_{EM}(\vec{r},t)\rangle}$	Angular Momentum:	$\boxed{\vec{\mathcal{L}}_{EM}(t) \Rightarrow \langle \vec{\mathcal{L}}_{EM}(t)\rangle}$

For a monochromatic *EM* plane wave propagating in free space / vacuum in $\hat{z}$ direction:

Time – averaged quantities for *EM* plane wave propagating in the $+\hat{z}$ direction

$$\boxed{\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}\varepsilon_o E_o^2} \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\boxed{\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{2}c\varepsilon_o E_o^2\hat{z} = c\left\langle u_{EM}(\vec{r},t)\right\rangle\hat{z}} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\boxed{\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle = \frac{1}{2c}\varepsilon_o E_o^2\hat{z} = \frac{1}{c^2}\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle\hat{z}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\boxed{\left\langle \vec{\ell}_{EM}(\vec{r},t)\right\rangle = \left(\vec{r}\times\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\left\langle \vec{S}(\vec{r},t)\right\rangle\right) = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle\left(\hat{r}\times\hat{z}\right)} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

We define the *intensity* I associated with an *EM* wave as the <u>time average</u> of the <u>magnitude</u> of Poynting's vector:

Intensity of an *EM* wave: $$\boxed{I(\vec{r}) \equiv \left\langle S(\vec{r},t)\right\rangle = \left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle = c\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}c\varepsilon_o E_o^2} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The <u>intensity</u> of an *EM* wave is also known as the <u>*irradiance*</u> of the *EM* wave – it is the radiant power incident per unit area upon a surface.

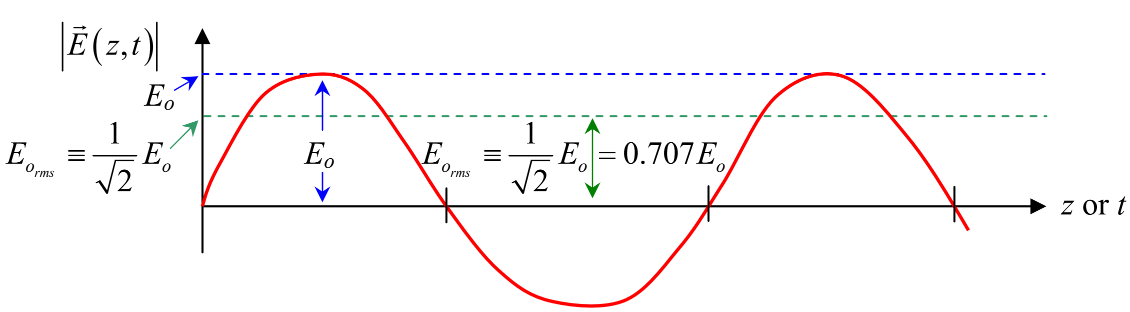
When working with <u>*time-averaged*</u> quantities such as $\left\langle u_{EM}(\vec{r},t)\right\rangle$, $\left\langle \vec{S}(\vec{r},t)\right\rangle$, $\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle$, $\left\langle \vec{\ell}_{EM}(\vec{r},t)\right\rangle$, *etc*. it is convenient/useful to define the so-called <u>root-mean-square</u> ($\equiv$ RMS) values of the $\vec{E}$ and $\vec{B}$ electric and magnetic field <u>*amplitudes*</u> (using the mathematical definition of RMS from probability and statistics):

For a <u>monochromatic</u> (*i.e.* <u>single</u> frequency, <u>sinusoidally-varying</u>) *EM* wave (<u>only</u>):

$$\boxed{\vec{E}_{rms} \equiv \frac{1}{\sqrt{2}}\vec{E}} \quad \Rightarrow \quad \boxed{E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o = 0.707E_o}$$

$$\boxed{\vec{B}_{rms} \equiv \frac{1}{\sqrt{2}}\vec{B}} \quad \Rightarrow \quad \boxed{B_{o_{rms}} \equiv \frac{1}{\sqrt{2}}B_o = 0.707B_o}$$

<u>Where</u>: E_o = <u>peak</u> (*i.e.* <u>max</u>) value of the $\vec{E}$-field = <u>amplitude</u> of the $\vec{E}$-field.
B_o = <u>peak</u> (*i.e.* <u>max</u>) value of the $\vec{B}$-field = <u>amplitude</u> of the $\vec{B}$-field.



Thus we see that:

$$\boxed{\vec{E}_{rms}\bullet\vec{E}_{rms}=\left(\frac{1}{\sqrt{2}}\vec{E}\right)\left(\frac{1}{\sqrt{2}}\vec{E}\right)=\frac{1}{2}\vec{E}\bullet\vec{E}} \text{ and } \boxed{\vec{B}_{rms}\bullet\vec{B}_{rms}=\left(\frac{1}{\sqrt{2}}\vec{B}\right)\left(\frac{1}{\sqrt{2}}\vec{B}\right)=\frac{1}{2}\vec{B}\bullet\vec{B}}$$

i.e. that:

$$\boxed{E_{rms}^2=\frac{1}{2}E^2=\frac{1}{2}E_{peak}^2}\Rightarrow\boxed{E_{o_{rms}}^2=\frac{1}{2}E_o^2} \text{ and } \boxed{B_{rms}^2=\frac{1}{2}B^2=\frac{1}{2}B_{peak}^2}\Rightarrow\boxed{B_{o_{rms}}^2=\frac{1}{2}B_o^2}$$

Then:

For mono-chromatic *EM* plane waves (only):

$$\boxed{\left\langle u_{EM}^{rms}(t)\right\rangle=\frac{1}{2}\left\langle u_{EM}(t)\right\rangle=\frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}=\frac{1}{4}\varepsilon_o E_o^2=\frac{1}{2}\varepsilon_o E_{o_{rms}}^2}\quad\left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\boxed{\left\langle \vec{S}_{rms}(t)\right\rangle=\frac{1}{2}\left\langle \vec{S}(t)\right\rangle=\frac{1}{2}c\left\langle u_{EM}(t)\right\rangle\hat{z}=c\left\langle u_{EM}^{rms}(t)\right\rangle\hat{z}}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\boxed{\left\langle \vec{\wp}_{EM}^{rms}(t)\right\rangle=\frac{1}{2c^2}\left\langle \vec{S}(t)\right\rangle=\frac{1}{2c}\left\langle u_{EM}(t)\right\rangle\hat{z}=\frac{1}{c^2}\left\langle \vec{S}_{rms}(t)\right\rangle=\frac{1}{c}\left\langle u_{EM}^{rms}(t)\right\rangle\hat{z}}\quad\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\boxed{\left\langle \vec{\ell}_{EM}^{rms}(t)\right\rangle=\frac{1}{2}\vec{r}\times\left\langle \vec{\wp}_{EM}(t)\right\rangle=\vec{r}\times\left\langle \vec{\wp}_{EM}^{rms}(t)\right\rangle=\frac{1}{c^2}\left(\vec{r}\times\left\langle \vec{S}_{rms}(t)\right\rangle\right)=\frac{1}{c}\left\langle u_{EM}^{rms}(t)\right\rangle\left(\vec{r}\times\hat{z}\right)}\quad\left(\frac{\text{kg}}{\text{m-sec}}\right)$$

$$\boxed{I_{rms}=\left|\left\langle \vec{S}_{rms}(t)\right\rangle\right|=\left\langle\left|\vec{S}_{rms}(t)\right|\right\rangle=\frac{1}{2}I=\frac{1}{2}\left\langle\left|\vec{S}(t)\right|\right\rangle=c\left\langle u_{EM}^{rms}(t)\right\rangle=\frac{1}{2}c\varepsilon_o E_{o_{rms}}^2}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

<u>Real world example</u>: 120 Vac/60 Hz "wall power" refers to the <u>RMS</u> AC voltage!
The <u>peak</u> voltage (*i.e.* voltage <u>amplitude</u>) is $V_{peak}=\sqrt{2}V_{rms}=\sqrt{2}\cdot 120=169.7\simeq 170.0$ Volts.

n.b. For *EM* waves ≠ sinusoidal waves, the root-mean-square (RMS) must be defined properly / mathematically – *e.g.* the RMS value of square or triangle wave amplitudes (from Fourier analysis these consist of linear combinations of infinite # of <u>harmonics</u>)

$$\boxed{E_{rms}^{\sqcap\sqcup\sqcap}\neq\frac{1}{\sqrt{2}}E^{\sqcap\sqcup\sqcap}}\qquad\boxed{E_{rms}^{\wedge\wedge}\neq\frac{1}{\sqrt{2}}E^{\wedge\wedge}}$$ (See/refer to probability & statistics reference books!!)

<u>Radiation Pressure</u>: $\mathrm{P}_{rad}\left(\dfrac{\text{Newtons}}{\text{m}^2}\right)$

When an *EM* wave impinges (*i.e.* is incident) on a ***perfect absorber*** (*e.g.* a totally ***black*** object with absorbance {*aka* absorption coefficient} $A = 1$, as "seen" at the frequency of the *EM* wave), all of the *EM* energy (<u>by definition</u>) is ***absorbed*** {ultimately winding up as heat…}.

By conservation of energy, linear momentum & angular momentum the object being irradiated by the incident *EM* wave acquires energy, linear momentum & angular momentum from the incident *EM* wave.

The *EM* Radiation Pressure acting on a ***perfect absorber*** for a ***normally incident*** *EM* wave is defined as:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \frac{\text{Time-Averaged}\left|\text{Force}\right|}{\perp\text{ Unit Area}} = \frac{\left\langle\left|\vec{F}_{EM}^{net}(t)\right|\right\rangle}{A_\perp}}\quad\left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

However, the *<u>time-averaged</u>* *EM* force is defined as:

$$\boxed{\left\langle\vec{F}_{EM}^{net}(t)\right\rangle \equiv \frac{d\left\langle\vec{p}_{EM}(t)\right\rangle}{dt} = \frac{\left\langle\Delta\vec{p}_{EM}(t)\right\rangle}{\Delta t}} = \boxed{\text{time rate of change of the time-averaged linear momentum}}$$

$\therefore$ the *EM* Radiation Pressure at ***normal incidence*** is: $\boxed{\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \dfrac{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle}{\Delta t}\dfrac{1}{A_\perp}}\ \left(\dfrac{\text{Newtons}}{\text{m}^2}\right)$

In a time interval $\Delta t \gg \tau = 1/f$, the *<u>time-averaged</u>* magnitude of the *EM* linear momentum *<u>transfer</u>* $\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle$ at ***normal incidence*** to a ***perfect absorber*** of *EM* radiation is:

$$\boxed{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle = \left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\,\Delta V}$$

EM Linear momentum density Volume of *EM* wave associated with time interval Δt

The volume associated with an *EM* wave propagating in <u>free space</u> over a time interval Δt is:

$\boxed{\Delta V = A_\perp\cdot(c\Delta t)}$ where $c\Delta t$ = distance traveled by the *EM* wave in the time interval Δt.

$$\therefore\quad\boxed{\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = \frac{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle}{\Delta t}\frac{1}{A_\perp} = \frac{\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\Delta V}{\Delta t}\frac{1}{A_\perp} = \frac{\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle\cancel{A_\perp}\,c\,\cancel{\Delta t}}{\cancel{\Delta t}\ \cancel{A_\perp}} = c\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle}$$

Thus, we see that for a monochromatic *EM* plane wave propagating in free space ***normally incident*** on a ***perfect absorber*** ($A = 1$):

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\begin{smallmatrix}\text{perfect}\\ \text{absorber}\end{smallmatrix}(A=1)\right\} = c\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle = \frac{1}{2}\varepsilon_o E_o^2 = \left\langle u_{EM}\right\rangle = I/c}\quad\left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

For a ***perfect*** ***reflector*** (*e.g.* a ***perfect*** mirror, with reflection coefficient (*aka* reflectance) $R = 1$ $\{A = 0\}$), note that:

$$\boxed{\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle^{\substack{perfect\\reflector}} = 2\times\left\langle\left|\Delta\vec{p}_{EM}(t)\right|\right\rangle^{\substack{perfect\\absorber}}}$$

Since $\Delta\vec{p}_{EM} \equiv \vec{p}_{EM}^{\,initial} - \vec{p}_{EM}^{\,final}$ and $\vec{p}_{EM}^{\,final} = -\vec{p}_{EM}^{\,initial}$ for an *EM* wave reflecting off of a ***perfect*** ***reflector***, then $\Delta\vec{p}_{EM} \equiv \vec{p}_{EM}^{\,initial} - \vec{p}_{EM}^{\,final} = \vec{p}_{EM}^{\,initial} + \vec{p}_{EM}^{\,initial} = 2\vec{p}_{EM}^{\,initial}$

i.e. an *EM* wave that reflects off of (*i.e.* "bounces" off of) a ***perfect*** ***reflector*** delivers ***twice*** (2×) the momentum kick (*i.e.* impulse) to the ***perfect*** ***reflector*** than the same *EM* wave that is absorbed by a ***perfect*** ***absorber***! Thus at ***normal*** ***incidence***:

$$\therefore\ \boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{reflector}}(R=1)\right\} = 2\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} = 2\left(I/c\right)}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

Note that for a ***partially*** reflecting surface, with reflection coefficient $R < 1$, since $R + A = 1$, the radiation pressure associated with an *EM* wave propagating in free space and reflecting off of a ***partially*** reflecting surface at ***normal*** ***incidence*** is given by:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = A\cdot\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} + 2R\cdot\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(R=1)\right\} = (A+2R)\left(I/c\right)}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

Since $A = 1 - R$, we can equivalently re-write this relation as:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = (A+2R)\left(I/c\right) = (1-R+2R)\left(I/c\right) = (1+R)\left(I/c\right)}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

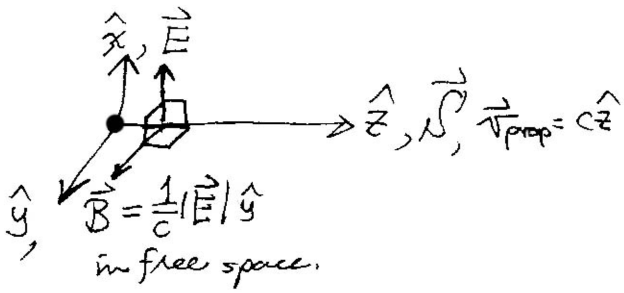
If the *EM* wave is *not* at normal incidence on the absorbing/reflecting surface, but instead makes a finite angle θ with respect to the unit normal of the surface, these relations need to be modified, due to the cosine θ factor $\langle\vec{S}\rangle\cdot\hat{n} = \langle|\vec{S}|\rangle\cos\theta = I\cos\theta$ associated with the flux of *EM* energy/momentum $\langle\vec{\wp}_{EM}(t)\rangle\cdot\hat{n} = \langle|\vec{\wp}_{EM}(t)|\rangle\cos\theta = \varepsilon_o\mu_o\langle|\vec{S}(t)|\rangle\cos\theta = \tfrac{1}{c^2}\langle|\vec{S}(t)|\rangle\cos\theta = \tfrac{1}{c^2}I\cos\theta$ crossing the surface area $A_\perp$ at a finite angle θ:

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{absorber}}(A=1)\right\} = \left(I/c\right)\cos\theta}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

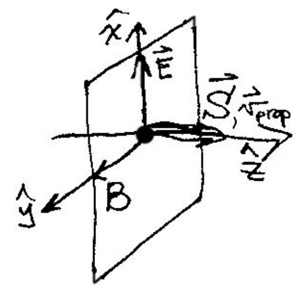
$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{perfect}\\\text{reflector}}(R=1)\right\} = 2\left(I/c\right)\cos\theta}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

$$\boxed{\mathrm{P}_{EM}^{Rad}\left\{\substack{\text{partial}\\\text{reflector}}(R+A=1)\right\} = (A+2R)\left(I/c\right)\cos\theta = (1+R)\left(I/c\right)\cos\theta}\ \left(\frac{\text{Newtons}}{\text{m}^2}\right)$$

Maxwell's equations (and relativity) for the macroscopic $\vec{E}$ and $\vec{B}$ fields associated with an *EM* wave propagating in free space mandate / require that $\vec{E} \perp \vec{B} \perp$ propagation direction (here = $\hat{z}$) $\{\vec{v}_{prop} = c\hat{z}\}$, as shown in the figure below:



Compare this microscopic picture to that of a classical / macroscopic *EM* plane wave, polarized in the x-hat direction:

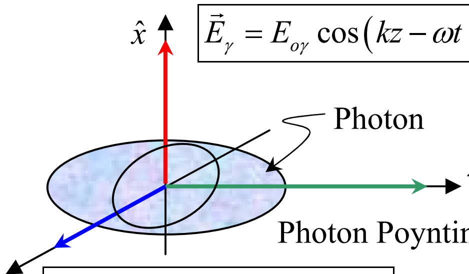


Macroscopic *EM* waves propagating in free space are purely transverse waves, *i.e.* $\vec{E} \perp \vec{B}$, and both of the $\vec{E}$ and $\vec{B}$ fields are also $\perp$ to the propagation direction of the *EM* wave, *e.g.* $\vec{v}_{prop} = c\hat{z}$. Thus, $\vec{E} \perp \vec{v}_{prop} = c\hat{z}$ and $\vec{B} \perp \vec{v}_{prop} = c\hat{z}$.

The behavior of the macroscopic $\vec{E}$ and $\vec{B}$ fields associated with *e.g.* a monochromatic *EM* plane wave propagating in free space, at the microscopic scale is simply the sum over (*i.e.* linear superposition of) the $\vec{E}$ and $\vec{B}$-field contributions from {large numbers of} individual real photons making up the *EM* field.

Each real photon has associated with it, its own $\vec{E}$ and $\vec{B}$ field – *e.g.* a linearly polarized real photon, polarized in $+\hat{x}$ direction:

$\boxed{\vec{E}_\gamma = E_{o\gamma}\cos(kz - \omega t + \delta)\,\hat{x}}$ ($+\hat{x}$ = polarization direction)



Real Photon Momentum: $\vec{p}_\gamma = h/\lambda\,\hat{z}$

Photon Poynting's vector: $\boxed{\vec{S}_\gamma = \tfrac{1}{\mu_o}\left(\vec{E}_\gamma \times \vec{B}_\gamma\right) \parallel +\hat{z}}$

$\boxed{\vec{B}_\gamma = B_{o\gamma}\cos(kz - \omega t + \delta)\,\hat{y}}$

$\boxed{\vec{B}_\gamma = \frac{1}{c}\hat{k} \times \vec{E}_\gamma}$ where the unit wavevector $\boxed{\hat{k} = +\hat{z}}$ {here} and $\boxed{B_{o\gamma} = \frac{1}{c}E_{o\gamma}}$ in vacuum.

Real photon energy: $E_\gamma = hf = p_\gamma c = |\vec{p}_\gamma|c$ (Total Relativistic Energy^2 = $E_\gamma^2 = p_\gamma^2 c^2 + \overset{=0}{\cancel{m_\gamma^2 c^4}}$)

Real photon momentum (deBroglie relation): $m_\gamma c^2 \equiv 0$ for real photon

$\boxed{p_\gamma = h/\lambda}$ and $c = f\lambda$ c = speed of light (in vacuum) = 3×10^8 m/sec

Question: How many real photons per second are emitted *e.g.* from a 10 mW laser?
(mW = milli-Watt = 10^{-3} Watt)

Answer: Depends on the color (*i.e.* wavelength λ, frequency f, photon energy) of the laser beam!
$E_\gamma = hf$

When we say 10 mW laser, what precisely does this refer to?

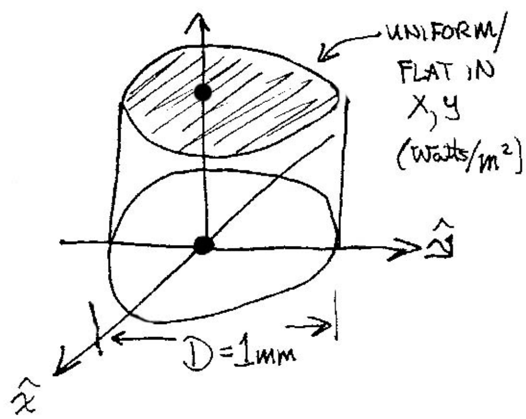
It refers to the *time-averaged EM* power:

$$\langle P_{laser}(t)\rangle = 10\text{ mW} = 10\times10^{-3}\text{ Watts} = 0.010\text{ Watts}\left(= 0.010\text{ Joules/sec}\right)$$

Let's assume that the laser beam points in the $+\hat{z}$ direction.

Also assume that the diameter of the laser beam is D = 1 mm = 0.001 m (typical).
Further assume (for simplicity's sake): Power flux density = intensity profile $I(x,y)$ is uniform in x and y over the diameter of the laser beam (not true in real life – laser beams have ~ Gaussian intensity profiles in x and y (*i.e.* $I(\rho) = I_o e^{-\rho^2/2\sigma^2}$); note that there also exist *e.g.* diffraction {beam-spreading} effects that should/need to be taken into account…)

$$I(x,y) = \left\langle \left|\vec{S}(x,y,t)\right| \right\rangle$$



In Δt = 1 second, the time-averaged energy associated with the 10 mW laser beam is:

$$\langle \Delta E_{laser}(t)\rangle = \langle P_{laser}(t)\rangle \Delta t$$
$$\langle \Delta E_{laser}(t)\rangle = 0.010\text{ Watts}*1\text{ sec}$$
$$\langle \Delta E_{laser}(t)\rangle = 0.010\ \frac{\text{Joules}}{\text{sec}}*1\text{ sec}$$
$$\langle \Delta E_{laser}(t)\rangle = 0.010\text{ Joules} = \text{Time-averaged energy of laser beam}$$

The {instantaneous} energy of the laser beam crosses an imaginary planar surface that is $\perp$ to the laser beam.

If the laser has red light, *e.g.* λ_{red} = 750 nm (*n.b.* 1 nm = 1 nano-meter = 10^{-9} meters) or if the laser has blue light, *e.g.* λ_{blue} = 400 nm

Since $f = c/\lambda$ the corresponding photon frequencies associated with red and blue laser light are:

$$f_{red}^{\gamma} = \frac{c}{\lambda_{red}^{\gamma}} = \frac{3\times10^{8}\,m/s}{750\times10^{-9}\,m} = 4.0\times10^{14}\ \text{cycles/sec (= Hertz, or Hz)}$$

$$f_{blue}^{\gamma} = \frac{c}{\lambda_{blue}^{\gamma}} = \frac{3\times10^{8}\,m/s}{400\times10^{-9}\,m} = 7.5\times10^{14}\ \text{cycles/sec (= Hertz, or Hz)}$$

The energy associated with a single, real photon is: $\boxed{E_{\gamma} = hf^{\gamma} = hc/\lambda^{\gamma}}$, where h = Planck's constant: h = 6.626 x 10^{-34} Joule-sec and c = 3 x 10^{8} m/sec (speed of light in vacuum). Thus, the corresponding photon energies associated with red and blue laser light are:

$$\boxed{E_{\gamma}^{red} = hf_{red}^{\gamma} = hc/\lambda_{red}^{\gamma}}\ \text{and:}\ \boxed{E_{\gamma}^{blue} = hf_{blue}^{\gamma} = hc/\lambda_{blue}^{\gamma}}\quad \text{since } f = c/\lambda$$

$$\boxed{E_{\gamma}^{red} = hf_{red}^{\gamma} = 6.626\times10^{-34}\,\text{Joule/sec}\ \times\ 4.0\times10^{14}\,/\text{sec} = 2.6504\times10^{-19}\,\text{Joules}}\ \text{(red light)}$$

$$\boxed{E_{\gamma}^{blue} = hf_{blue}^{\gamma} = 6.626\times10^{-34}\,\text{Joule/sec}\ \times\ 7.5\times10^{14}\,/\text{sec} = 4.9695\times10^{-19}\,\text{Joules}}\ \text{(blue light)}$$

In a time interval of $\Delta t = 1$ sec, the time-averaged energy $\boxed{\langle \Delta E_{laser}(t)\rangle = \langle \Delta N_{\gamma}(t)\rangle E_{\gamma}}$ where $\langle \Delta N_{\gamma}(t)\rangle$ is the {time-averaged} number of photons crossing a $\perp$ area in the time interval Δt.

Thus, the number of red (blue) photons emitted from a red (blue) laser in a $\Delta t = 1$ sec time interval is:

$$\text{\# red photons:}\quad \boxed{\langle \Delta N_{\gamma}^{red}(t)\rangle = \frac{\langle \Delta E_{laser}(t)\rangle}{E_{\gamma}^{red}} = \frac{0.010\ \text{Joules}}{2.6504\times10^{-19}\ \text{Joules/photon}} = 3.7730\times10^{16}}$$

$$\text{\# blue photons:}\quad \boxed{\langle \Delta N_{\gamma}^{blue}(t)\rangle = \frac{\langle \Delta E_{laser}(t)\rangle}{\mathrm{E}_{\gamma}^{blue}} = \frac{0.010\ \text{Joules}}{4.9695\times10^{-19}\ \text{Joules/photon}} = 2.0123\times10^{16}}$$

Thus, the {time-averaged} <u>*rate*</u> of emission of red (blue) photons from a red (blue) laser is:

$$\boxed{\langle R_{\gamma}^{red}(t)\rangle = \frac{\langle \Delta N_{\gamma}^{red}(t)\rangle}{\Delta t} = 3.7730\times10^{16}}\ \text{red photons/sec}$$

$$\boxed{\langle R_{\gamma}^{blue}(t)\rangle = \frac{\langle \Delta N_{\gamma}^{blue}(t)\rangle}{\Delta t} = 2.0123\times10^{16}}\ \text{blue photons/sec}$$

<u>Note</u>: In a time interval of $\Delta t = 1$ sec, photons (of any color / λ^{γ} / f^{γ} / E_{γ}) will travel a distance of $d = c\Delta t =\ 3\times10^{8}\,\text{m/s}\ \times\ 1\ \text{s}\ =\ 3\times10^{8}\,\text{meters}$

If the flux of photons is assumed (for simplicity) to be uniform across the $D = 1$ mm diameter laser beam, then the time-averaged flux of photons (#/m²/sec) is:

$$\left\langle \mathscr{F}_\gamma^{red}(t)\right\rangle = \frac{\left\langle R_\gamma^{red}(t)\right\rangle}{A_\perp^{laser}} = \frac{3.7730\times10^{16}\left(\gamma/\text{sec}\right)}{\pi\left(\frac{10^{-3}\,\text{m}}{2}\right)^2} = 4.8039\times10^{22}\left(\text{red }\gamma/\text{m}^2/\text{sec}\right)$$

$$\left\langle \mathscr{F}_\gamma^{blue}(t)\right\rangle = \frac{\left\langle R_\gamma^{blue}(t)\right\rangle}{A_\perp^{laser}} = \frac{2.0123\times10^{16}\left(\gamma/\text{sec}\right)}{\pi\left(\frac{10^{-3}\,\text{m}}{2}\right)^2} = 2.562\times10^{22}\left(\text{blue }\gamma/\text{m}^2/\text{sec}\right)$$

If each photon has E_γ Joules of energy, then power associated with red (blue) laser beam:

$$\left\langle P_\gamma^{red}(t)\right\rangle / A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = E_\gamma^{red}\bullet\left\langle \mathscr{F}_\gamma^{red}(t)\right\rangle = 2.6504\times10^{-19}\,\text{Joules}\times 4.8039\times10^{22}\left(\text{red }\gamma/\text{m}^2/\text{sec}\right)$$

$$\boxed{=1.2732\times10^4\ \text{Watts/m}^2}$$

$$\left\langle P_\gamma^{blue}(t)\right\rangle / A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = E_\gamma^{blue}\bullet\left\langle \mathscr{F}_\gamma^{blue}(t)\right\rangle = 4.9695\times10^{-19}\,\text{Joules}\times 2.5621\times10^{22}\left(\text{blue }\gamma/\text{m}^2/\text{sec}\right)$$

$$\boxed{=1.2732\times10^4\ \text{Watts/m}^2}$$

Thus we see that:

$$\boxed{\left\langle P_\gamma^{red}(t)\right\rangle / A_\perp^{laser} = \left\langle P_\gamma^{blue}(t)\right\rangle / A_\perp^{laser} = \left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = 1.2732\times10^4\ \text{Watts/m}^2} \leftarrow \text{10 mW laser}$$

n.b. This is precisely *why* you shouldn't look into a laser beam {with your ***one*** remaining eye}!!!

Time-averaged linear momentum density:

$$\left|\left\langle \vec{\wp}_\gamma^{red}(t)\right\rangle\right| = \varepsilon_o\mu_o\left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \frac{1}{c^2}\left|\left\langle \vec{S}_\gamma^{red}(t)\right\rangle\right| = \frac{1}{c}\left\langle u_\gamma^{red}(t)\right\rangle\hat{z} = 1.4147\times10^{-13}\ \text{kg/m}^2\text{-sec}$$

$$\left|\left\langle \vec{\wp}_\gamma^{blue}(t)\right\rangle\right| = \varepsilon_o\mu_o\left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = \frac{1}{c^2}\left|\left\langle \vec{S}_\gamma^{blue}(t)\right\rangle\right| = \frac{1}{c}\left\langle u_\gamma^{blue}(t)\right\rangle\hat{z} = 1.4147\times10^{-13}\ \text{kg/m}^2\text{-sec}$$

Thus: $\boxed{\left|\left\langle \vec{\wp}_\gamma^{red}\right\rangle\right| = \left|\left\langle \vec{\wp}_\gamma^{blue}\right\rangle\right| = 1.4147\times10^{-13}\ \text{kg/m}^2\text{-sec}}$

Momentum density, Poyntings vector, energy density are independent of frequency / wavelength / photon energy

The time-averaged linear momentum contained in $\Delta t = 1$ second's worth of laser beam:

Time averaged linear momentum: $\left\langle\left|\Delta\vec{p}_\gamma(t)\right|\right\rangle$ = momentum density $\left\langle\left|\vec{\wp}_\gamma(t)\right|\right\rangle$ x volume ΔV

Volume $\boxed{\Delta V = A_\perp^{laser} * (c\Delta t)}$ $\left(\text{m}^3\right)$

Distance light travels in Δt sec.

Red light momentum:

$$\boxed{\left\langle\left|\Delta\vec{p}_{\gamma}^{\,red}(t)\right|\right\rangle=\left\langle\left|\vec{\wp}_{\gamma}^{\,red}(t)\right|\right\rangle c\Delta t A_{\perp}=1.4147\times10^{-13}\times3\times10^{8}\times1\times\pi\times\left(\frac{0.001}{2}\right)^{2}}\left(\frac{\text{kg-m}}{\text{sec}}\right)$$
$$\boxed{=3.3333\times10^{-11}\ \text{kg-m/sec}}$$

Blue light momentum:

$$\boxed{\left\langle\left|\Delta\vec{p}_{\gamma}^{\,blue}(t)\right|\right\rangle=\left\langle\left|\vec{\wp}_{\gamma}^{\,blue}(t)\right|\right\rangle c\Delta t A_{\perp}=1.4147\times10^{-13}\times3\times10^{8}\times1\times\pi\times\left(\frac{0.001}{2}\right)^{2}}\left(\frac{\text{kg-m}}{\text{sec}}\right)$$
$$\boxed{=3.3333\times10^{-11}\ \text{kg-m/sec}}$$

Thus: $\boxed{\left\langle\left|\Delta\vec{p}_{\gamma}^{\,red}(t)\right|\right\rangle=\left\langle\left|\Delta\vec{p}_{\gamma}^{\,blue}(t)\right|\right\rangle=3.3333\times10^{-11}\ \text{kg-m/sec}}$

{"TRICK"}:

The time-averaged energy density $\langle u_{EM}(t)\rangle$ = time-averaged momentum density $\left\langle\left|\vec{\wp}_{EM}(t)\right|\right\rangle * c$ (Since photon energy, $E_{\gamma}=p_{\gamma}c$). Thus:

$$\boxed{\left\langle u_{\gamma}^{red}(t)\right\rangle=\left\langle\left|\vec{\wp}_{\gamma}^{\,red}(t)\right|\right\rangle c=1.4147\times10^{-13}\left(\frac{\text{kg}}{\text{m}^2/\text{sec}}\right)\times3\times10^{8}\,(\text{m/s})=4.2441\times10^{-5}\left(\text{Joules/m}^3\right)}$$
$$\boxed{\left\langle u_{\gamma}^{blue}(t)\right\rangle=\left\langle\left|\vec{\wp}_{\gamma}^{\,blue}(t)\right|\right\rangle c=1.4147\times10^{-13}\left(\frac{\text{kg}}{\text{m}^2/\text{sec}}\right)\times3\times10^{8}\,(\text{m/s})=4.2441\times10^{-5}\left(\text{Joules/m}^3\right)}$$

$$\boxed{\text{Joule}=\frac{\text{kg-m}^2}{\text{s}^2}}\quad\Rightarrow\quad\boxed{\frac{\text{Joule}}{\text{m}^2}=\frac{\text{kg}}{\text{m/s}^2}}$$

The time-averaged energy contained in $\Delta t=1$ second's worth of laser beam is:
The time-averaged energy $\langle U_{\gamma}(t)\rangle$ = time-averaged energy density $\langle u_{\gamma}(t)\rangle *$ volume ΔV

$$\boxed{\Delta V=A_{\perp}^{laser}*(c\Delta t)}$$

$$\therefore\ \boxed{\left\langle U_{\gamma}^{red}(t)\right\rangle=\left\langle u_{\gamma}^{red}(t)\right\rangle * A_{\perp}c\Delta t=4.2441\times10^{-5}\left(\frac{\text{Joules}}{\text{m}^3}\right)*\pi\times\left(\frac{0.001}{2}\right)^{2}*3\times10^{8}*1\left(\text{m}^3\right)}$$
$$\boxed{=0.010\,\text{Joules}=10\,\text{mJ}}$$

$$\boxed{\left\langle U_{\gamma}^{blue}(t)\right\rangle=\left\langle u_{\gamma}^{blue}(t)\right\rangle * A_{\perp}c\Delta t=4.2441\times10^{-5}\left(\frac{\text{Joules}}{\text{m}^3}\right)*\pi\times\left(\frac{0.001}{2}\right)^{2}*3\times10^{8}*1\left(\text{m}^3\right)}$$
$$\boxed{=0.010\,\text{Joules}=10\,\text{mJ}}$$

The time-averaged power in the laser beam: $\boxed{\left\langle P_{laser}^{red}(t)\right\rangle=\frac{\left\langle U_{laser}(t)\right\rangle}{\Delta t}=10\text{mW}=\left\langle \text{P}_{laser}^{blue}(t)\right\rangle}$

Time-averaged Power (Watts) = $\dfrac{d\langle U(t)\rangle}{dt}\left(\text{Joules}/\text{sec}\right)$ $\qquad\Delta t=1$ sec

Note: P_{laser} (laser power) is measured by the total time-averaged energy $\langle U(t)\rangle$ deposited in (a very accurately) known time interval Δt using an absolutely calibrated photodiode (*e.g.* by NIST).

A typical time interval $\Delta t = 10$ secs $\rightarrow$ $\Delta t \gg \tau$ (oscillation period) = $1/f$!!

$$\boxed{\tau_{red} = 1/f_{red} = 2.500\times10^{-15}\text{ sec} = 2.500\text{ femto-sec} = 2.500\text{ fs}}$$

$$\boxed{\tau_{blue} = 1/f_{blue} = 1.333\times10^{-15}\text{ sec} = 1.333\text{ femto-sec} = 1.333\text{ fs}}$$

$\rightarrow$ The laser power measured is time-averaged power, *i.e.* $\boxed{\langle P_{laser}(t)\rangle = \frac{1}{2}P_{laser}^{peak}(t)}$

Consider (the time-averaged) energy density associated with this 10 mW laser:

$$\boxed{\langle u_{EM}(t)\rangle = 4.2441\times10^{-5}}\ \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

Now: $\boxed{\langle u_{EM}(t)\rangle = \langle u_{elect}(t)\rangle + \langle u_{mag}(t)\rangle = \frac{1}{2}\varepsilon_o E_o^2 = \frac{1}{2}u_{EM}^{peak}(t)}$

And because: $\boxed{|\vec{B}(t)| = \frac{1}{c}|\vec{E}(t)|}$ for *EM* waves propagating in free space / vacuum ($\frac{1}{c^2} = \varepsilon_o\mu_o$)

We showed that: $\boxed{\langle u_{elect}(t)\rangle = \langle u_{mag}(t)\rangle}$

$$\boxed{\langle u_{elect}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}} \qquad \boxed{B_o^2 = \frac{1}{c^2}E_o^2 = \varepsilon_o\mu_o E_o^2}$$

$$\boxed{\langle u_{mag}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2\mu_o}B_o^2\right\} = \frac{1}{2}\left\{\frac{\varepsilon_o \cancel{\mu_o}}{2\cancel{\mu_o}}E_o^2\right\} = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\}}$$

Now: E_o = amplitude of the macroscopic electric field: $\boxed{\vec{E}(z,t) = E_o\cos(kz-\omega t+\delta)\hat{x}}$

B_o = amplitude of the macroscopic magnetic field: $\boxed{\vec{B}(z,t) = B_o\cos(kz-\omega t+\delta)\hat{y}}$

Define the RMS (Root-Mean-Square) amplitudes of the $\vec{E}$ and $\vec{B}$ fields:

$$\boxed{E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o} \Rightarrow \boxed{E_{o_{rms}}^2 = \frac{1}{2}E_o^2}$$

$$\boxed{B_{o_{rms}} \equiv \frac{1}{\sqrt{2}}B_o} \Rightarrow \boxed{B_{o_{rms}}^2 = \frac{1}{2}B_o^2 = \frac{1}{2c^2}E_o^2}\ \text{in free space / vacuum}$$

Then: $\boxed{\langle u_{elect}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2}\varepsilon_o E_o^2\right\} = \frac{1}{2}\varepsilon_o E_{o_{rms}}^2}$ (Joules/m^3)

$$\boxed{\langle u_{mag}(t)\rangle = \frac{1}{2}\left\{\frac{1}{2\mu_o}B_o^2\right\} = \frac{1}{2\mu_o}B_{o_{rms}}^2 = \frac{1}{2}\varepsilon_o E_{o_{rms}}^2}\ \text{in free space / vacuum}$$

So if: $\boxed{\langle u_{EM}(t)\rangle = \langle u_{elect}(t)\rangle + \langle u_{mag}(t)\rangle = 2\langle u_{elect}(t)\rangle}$ in free space / vacuum

$\boxed{= 2\varepsilon_o E^2_{o_{\text{rms}}} = 4.2441\times 10^{-5}}$ Joules/m^3

Then: $\boxed{E^2_{o_{\text{rms}}} = \frac{1}{2\varepsilon_o}\langle u_{EM}(t)\rangle}$ where $\varepsilon_o = 8.85\times 10^{-12}$ Farads/m = electric permittivity of free space

Thus: $\boxed{E^{laser}_{o_{\text{rms}}} = \sqrt{\frac{1}{2\varepsilon_o}\langle U_{EM}(t)\rangle} = \left[\frac{4.2441\times 10^{-5}\,\text{Joules/m}}{2\times 8.85\times 10^{-12}\,\text{Farads/m}}\right]^{1/2}}$ (Volts/m)

$\boxed{E^{laser}_{o\ \text{rms}} \simeq 1.5485\times 10^3\ \text{Volts/m} = 1548.5\ \text{Volts/m}}$ (*n.b.* same for red vs. blue laser light!)

Then: $\boxed{E^{laser}_{o} = E^{laser}_{peak} = \sqrt{2}\ E^{laser}_{o\ \text{rms}} \simeq 2190\ \text{Volts/m}}$

Then: $\boxed{B^{laser}_{o_{\text{rms}}} = \frac{1}{c}E^{laser}_{o_{\text{rms}}} \simeq 5.1616\times 10^{-6}\,\text{Tesla}\left(= 5.1616\times 10^{-2}\,\text{Gauss}\right)}$ 1 Tesla = 10^4 Gauss

SI (MKS) CGS Units

Thus: $\boxed{B^{laser}_{o} = \frac{1}{c}E^{laser}_{o} = \sqrt{2}\ B^{laser}_{o_{\text{rms}}} \simeq 7.2996\times 10^{-2}\ \text{Gauss}}$

Now earlier (above) we calculated the (time-averaged) number of photons present in the {red and blue} laser beams that were emitted in a time interval of $\Delta t = 1$ sec.

red photons emitted in $\Delta t = 1$ sec: $\boxed{\langle \Delta N^{red}_{\gamma}(t)\rangle = 3.7730\times 10^{16}}$ red photons

blue photons emitted in $\Delta t = 1$ sec: $\boxed{\langle \Delta N^{blue}_{\gamma}(t)\rangle = 2.0123\times 10^{16}}$ blue photons

The volume associated with a $D = 1$ mm diameter laser beam turned on for $\Delta t = 1$ sec is:

$$\boxed{\Delta V = A_{\perp} c\Delta t = \pi\left(\frac{D}{2}\right)^2 c\Delta t = \pi\left(\frac{0.001}{2}\right)^2 \bullet 3\times 10^8 \bullet 1 = 235.6194\ \text{m}^3}$$

The (time-averaged) number density $\boxed{\langle n_{\gamma}(t)\rangle = \frac{\langle \Delta N_{\gamma}(t)\rangle}{\Delta V}}$ of {red and blue} photons in the laser beam is:

$$\boxed{\langle n^{red}_{\gamma}(t)\rangle = \frac{\langle \Delta N^{red}_{\gamma}(t)\rangle}{\Delta V} = \frac{3.7730\times 10^{16}}{2.3562\times 10^{2}} = 1.6009\times 10^{14}}\ \text{red photons/m}^3$$

$$\boxed{\langle n^{blue}_{\gamma}(t)\rangle = \frac{\langle \Delta N^{blue}_{\gamma}(t)\rangle}{\Delta V} = \frac{2.0123\times 10^{16}}{2.3562\times 10^{2}} = 8.5405\times 10^{13}}\ \text{blue photons/m}^3$$

Then the (time-averaged) energy density $\langle u_{EM}(t)\rangle$ of the {red and blue} laser beam is:

Red photon energy: $\boxed{E^{red}_{\gamma} = hf^{red}_{\gamma} = 2.6504\times 10^{-19}\,\text{Joules}}$

Blue photon energy: $\boxed{E^{blue}_{\gamma} = hf^{blue}_{\gamma} = 4.9695\times 10^{-19}\,\text{Joules}}$

$$\boxed{\langle u_{EM}^{red}(t)\rangle = \langle n_\gamma^{red}(t)\rangle E_\gamma^{red} = 1.6009\times10^{14}\bullet 2.6504\times10^{-19} = 4.2442\times10^{-5}\ (\text{Joules/m}^3)}$$
$$\boxed{\langle u_{EM}^{blue}(t)\rangle = \langle n_\gamma^{blue}(t)\rangle E_\gamma^{blue} = 8.5405\times10^{13}\bullet 4.9695\times10^{-19} = 4.2442\times10^{-5}\ (\text{Joules/m}^3)}$$

The (time-averaged) energy $\langle U_{EM}(t)\rangle = \langle u_{EM}(t)\rangle * \Delta V$ of the {red and blue} laser beams is:

$$\boxed{\langle U_{EM}^{red}(t)\rangle = \langle u_{EM}^{red}(t)\rangle * \Delta V = 4.2442\times10^{-5} * 2.3562\times10^{2} = 0.010 \text{ Joules } = 10 \text{ mJoules}}$$
$$\boxed{\langle U_{EM}^{blue}(t)\rangle = \langle u_{EM}^{blue}(t)\rangle * \Delta V = 4.2442\times10^{-5} * 2.3562\times10^{2} = 0.010 \text{ Joules } = 10 \text{ mJoules}}$$

Now here is something quite interesting: Given that $\boxed{E_{o_{rms}} \equiv E_o/\sqrt{2}}$ for a monochromatic *EM* wave propagating in free space/the vacuum, with time-averaged *EM* energy density:

$$\boxed{\langle u_{EM}(t)\rangle = 2\varepsilon_o E_{o_{rms}}^2 = \varepsilon_o E_o^2} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

<u>But</u>: $\boxed{\langle u_{EM}(t)\rangle = \langle n_\gamma(t)\rangle E_\gamma} \left(\frac{\text{Joules}}{\text{m}^3}\right)$ $\begin{cases} \langle n_\gamma(t)\rangle = \text{photon number density (\#/m}^3\text{) in laser beam} \\ E_\gamma = hf_\gamma = hc/\lambda_\gamma = \text{energy/photon (Joules)} \end{cases}$

$\therefore$ $\boxed{2\varepsilon_o E_{o_{rms}}^2 = \langle n_\gamma(t)\rangle E_\gamma}$

<u>Or</u>: $\boxed{E_{o_{rms}}^2 = \dfrac{\langle n_\gamma(t)\rangle}{2\varepsilon_o} E_\gamma}$

This formula explicitly connects the <u>amplitudes</u> of the <u>macroscopic</u> $\vec{E}$ and $\vec{B}$ fields (since $B_o = E_o/c$) with the <u>microscopic</u> <u>constituents</u> of the $\vec{E}$ and $\vec{B}$ fields (i.e. the <u>photons</u>)!!!

n.b. This formula physically says that the <u>number</u> of {real} photons in the *EM* wave (each of photon energy E_γ) is proportional to E_o^2 = the <u>square</u> of the <u>macroscopic</u> electric field <u>amplitude</u>!

We can write this as: $\boxed{\langle n_\gamma(t)\rangle = 2\varepsilon_o E_{o_{rms}}^2 / E_\gamma}$ and note also that: $\boxed{E_{o_{rms}} = \sqrt{\dfrac{\langle n_\gamma(t)\rangle}{2\varepsilon_o} E_\gamma}}$!!!

Thus, we can now see that the {time-averaged} *EM* energy density:

$$\boxed{\langle u_{EM}(\vec{r},t)\rangle = 2\varepsilon_o E_{o_{rms}}^2(\vec{r}) = \langle n_\gamma(\vec{r},t)\rangle E_\gamma} \quad \underline{\text{with}}: \quad \boxed{\int_v \langle u_{EM}(\vec{r},t)\rangle d\tau = U_{EM}}$$

plays a role analogous to that of the <u>probability</u> <u>density</u> in <u>quantum</u> <u>mechanics</u>:

$$\boxed{\mathcal{P}(\vec{r},t) = \langle \psi(\vec{r},t) | \psi(\vec{r},t)\rangle = |\psi(\vec{r},t)|^2} \quad \underline{\text{with}}: \quad \boxed{\int_v \mathcal{P}(\vec{r},t)\, d\tau = 1}$$

Since: $\boxed{\langle n_\gamma(\vec{r},t)\rangle = \langle u_{EM}(\vec{r},t)\rangle / E_\gamma = 2\varepsilon_o E_{o_{rms}}^2(\vec{r})/E_\gamma}$ and: $\boxed{\int_v \langle n_\gamma(\vec{r},t)\rangle d\tau = \langle \Delta N_\gamma\rangle}$,

Then: $\boxed{\langle \mathcal{P}_\gamma(\vec{r},t)\rangle \equiv \langle n_\gamma(\vec{r},t)\rangle / \langle \Delta N_\gamma\rangle = \langle \psi_\gamma(\vec{r},t) | \psi_\gamma(\vec{r},t)\rangle = |\psi_\gamma(\vec{r},t)|^2}$!!!

Thus, we also see that the electric field $\vec{E}(\vec{r},t)$ plays a role analogous to that of the probability density <u>amplitude</u> $\psi(\vec{r},t)$ in quantum mechanics!!!

The (real) photon number density in the laser beam is: $\boxed{\langle n_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta V} = \dfrac{\langle \Delta N_\gamma(t)\rangle}{A_\perp c\Delta t}}$ $\left(\dfrac{\#}{\text{m}^3}\right)$

Then: $\boxed{2\varepsilon_o E_{o_{\text{rms}}}^2 A_\perp c\Delta t = \langle \Delta N_\gamma(t)\rangle E_\gamma}$ or: $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{\langle \Delta N_\gamma(t)\rangle}{2\varepsilon_o A_\perp c\Delta t} E_\gamma}$

But: $\boxed{\langle R_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta t}}$ = the time averaged rate of photons in laser beam (#/sec)

$\therefore$ $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{1}{2\varepsilon_o}\dfrac{\langle R_\gamma(t)\rangle}{A_\perp c} E_\gamma}$ and

$\boxed{\langle \mathscr{F}_\gamma(t)\rangle = \dfrac{\langle \Delta N_\gamma(t)\rangle}{\Delta t}\Big/ A_\perp = \langle R_\gamma(t)\rangle \Big/ A_\perp}$ $\left(\dfrac{\#}{\text{m}^2\text{-s}}\right)$ = flux of photons in the laser beam

$\therefore$ $\boxed{E_{o_{\text{rms}}}^2 = \dfrac{1}{2\varepsilon_o c}\langle \mathscr{F}_\gamma(t)\rangle E_\gamma}$ and $\boxed{\langle u_{EM}(t)\rangle = 2\varepsilon_o E_{o_{\text{rms}}}^2 = \dfrac{1}{c}\langle \mathscr{F}_\gamma(t)\rangle E_\gamma = \langle n_\gamma(\vec{r},t)\rangle E_\gamma}$ $\left(\dfrac{\text{Joules}}{\text{m}^3}\right)$

Thus, we see that the {real} photon flux: $\boxed{\langle \mathscr{F}_\gamma(t)\rangle = c\langle n_\gamma(\vec{r},t)\rangle}$ $\left(\dfrac{\#}{\text{m}^2\text{-s}}\right)$

Thus, the intensity {*aka* irradiance} of the laser beam is:

$$\boxed{I \equiv \left\langle \left|\vec{S}(t)\right| \right\rangle = c\langle u_{EM}(t)\rangle = 2\varepsilon_o E_{o_{\text{rms}}}^2 = \langle \mathscr{F}_\gamma(t)\rangle E_\gamma = c\langle n_\gamma(t)\rangle E_\gamma} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The {time-averaged} <longitudinal separation distance> between photons is defined as:

$$\boxed{\langle \Delta d_{\|}(t)\rangle \equiv \frac{c\Delta t}{\langle N_\gamma(t)\rangle}} \quad (\text{m})$$

For $\Delta t = 1$ sec: $\boxed{\langle \Delta d_{\|}^{red}(t)\rangle = \dfrac{3\times 10^8\,\text{m}}{3.773\times 10^{16}\,\gamma's} = 7.85\times 10^{-9}\,\text{m} \sim 8\times 10^{-9}\,\text{m} \simeq 8\text{ nm}}$ (1 nm = 10^{-9} m)

$$\boxed{\langle \Delta d_{\|}^{blue}(t)\rangle = \frac{3\times 10^8\,\text{m}}{2.0123\times 10^{16}\,\gamma's} = 1.49\times 10^{-8}\,\text{m} \sim 15\times 10^{-9}\,\text{m} \simeq 15\text{ nm}}$$

Recall that: $\lambda_\gamma^{red} = 750$ nm and $\lambda_\gamma^{blue} = 400$ nm

Thus: $\boxed{\lambda_\gamma \gg \langle \Delta d_{\|}(t)\rangle}$ for either red or blue laser light.

The {time-averaged} <transverse separation distance> between photons is defined as:

$$\boxed{\langle \Delta d_{\perp}(t)\rangle \equiv \frac{\sqrt{A_\perp}}{\langle N_\gamma(t)\rangle}}$$

Thus:

$$\boxed{\left\langle \Delta d_{\perp}^{red}(t) \right\rangle = \frac{\sqrt{\pi \left(\frac{0.001}{2}\right)^2}}{3.773\times 10^{16}} = 2.35\times 10^{-20}\,\mathrm{m}}$$

$$\boxed{\left\langle \Delta d_{\perp}^{blue}(t) \right\rangle = \frac{\sqrt{\pi \left(\frac{0.001}{2}\right)^2}}{2.0123\times 10^{16}} = 4.40\times 10^{-20}\,\mathrm{m}}$$

