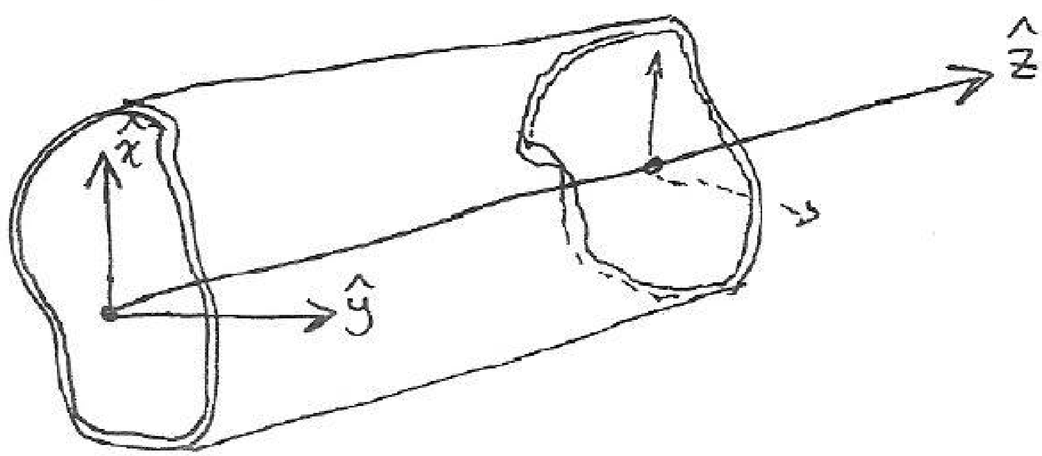
LECTURE NOTES 10

WAVE GUIDES and GUIDED *EM* WAVES

We consider/investigate the conditions under which *EM* waves can propagate when confined to the interior of some kind of "hollow" pipe – also know as a wave guide. In the real world wave guides consisting of *e.g.* rectangular, cylindrical, or arbitrarily cross-section shaped conducting and/or superconducting hollow metal pipes can be used to transport *EM* waves and *EM* energy in the radio and microwave region of the *EM* spectrum, whereas, *e.g.* glass or plastic optical fibers act as wave guides in the infrared, visible and even the UV portions of the *EM* spectrum.

We consider first the simplest type of a wave guide – a *perfect* conductor $(\sigma_C = \infty,\ \rho_C = 1/\sigma_C = 0)$ such that inside the walls of the perfect conductor: $\tilde{\vec{E}} = 0$ & $\tilde{\vec{B}} = 0$.

n.b. in a perfect conductor $\tilde{\vec{E}}(\vec{r},t) = 0$ and by Faraday's Law, if $\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t) = 0 \Rightarrow \partial\tilde{\vec{B}}(\vec{r},t)/\partial t = 0$. So if $\tilde{\vec{B}}(\vec{r},t=0) = 0$, it will remain $= 0\ \forall\, t$. A superconductor is a perfect conductor with $\tilde{\vec{B}}(\vec{r},t) = 0$ inside it (magnetic flux is expelled from a SC material – known as the Meissner effect).



The boundary conditions at the inner walls of a perfect conductor are:

$\boxed{\oint_C \tilde{\vec{E}}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \tilde{\vec{B}}\cdot d\vec{a} = 0}$ $\Rightarrow$ (1) Tangential $\vec{E}$ continuous: $\tilde{E}_{\parallel} = 0$ (since $\tilde{E}_{\parallel}^{inside} = 0$)

$\boxed{\oint_S \tilde{\vec{B}}\cdot d\vec{a} = 0}$ $\Rightarrow$ (2) Normal $\tilde{\vec{B}}$ continuous: $\tilde{B}_{\perp} = 0$ (since $\tilde{B}_{\perp}^{inside} = 0$)

Note that free surface charges σ_{free} and free surface currents $\vec{K}_{free}$ will be induced on the inner surfaces of this perfectly-conducting wave guide so as to "enforce" these boundary conditions.

We are interested in/seek monochromatic/single-frequency plane traveling wave-type solutions - that propagate down the inside of the wave guide, *e.g.* in the $+\hat{z}$ direction of the above figure. Generically, these must be of the form:

$$\boxed{\vec{\tilde{E}}(x,y,z,t)=\vec{\tilde{E}}_o(x,y)e^{i(k_z z-\omega t)}}$$
$$\boxed{\vec{\tilde{B}}(x,y,z,t)=\vec{\tilde{B}}_o(x,y)e^{i(k_z z-\omega t)}}$$

> ***n.b.*** for the cases of interest to us, the wave number k_z will turn out to be real.

In the interior region of the wave guide, away from (*i.e.* not inside) the walls, Maxwell's equations must be satisfied, which, for empty space or *e.g.* air with $\varepsilon_{air}\simeq\varepsilon_o$ and $\mu_{air}\simeq\mu_o$ are:

(1) Gauss' Law: $\boxed{\vec{\nabla}\cdot\vec{\tilde{E}}=0}$ (2) No magnetic charges/monopoles: $\boxed{\vec{\nabla}\cdot\vec{\tilde{B}}=0}$

(3) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{\tilde{E}}=-\partial\vec{\tilde{B}}/\partial t}$ (4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{\tilde{B}}=\varepsilon_o\mu_o\,\partial\vec{\tilde{E}}/\partial t=\left(1/c^2\right)\partial\vec{\tilde{E}}/\partial t}$

The question then is, what restrictions arising from the boundary conditions (1) $\tilde{E}^{\parallel}=0$ and (2) $\tilde{B}^{\perp}=0$ are imposed on $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ in satisfying Maxwell's equations (1) – (4) above?

Note also that confined *EM* waves (*e.g.* for propagation inside of wave guides) are not (in general) purely transverse waves!

The boundary conditions (1) $\tilde{E}^{\parallel}=0$ and 2) $\tilde{B}^{\perp}=0$ will (in general, for confined waves) require longitudinal components: $\boxed{\tilde{E}_{o_z}(x,y) \text{ and } \tilde{B}_{o_z}(x,y)}$. Generically, our $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ - fields interior to the wave guide will thus be of the form(s):

$$\boxed{\vec{\tilde{E}}(x,y,z,t)=\vec{\tilde{E}}_o(x,y)e^{i(k_z z-\omega t)}}\quad \text{with: } \boxed{\vec{\tilde{E}}_o(x,y)=\tilde{E}_{o_x}(x,y)\hat{x}+\tilde{E}_{o_y}(x,y)\hat{y}+\tilde{E}_{o_z}(x,y)\hat{z}}$$

$$\text{and: } \boxed{\vec{\tilde{B}}(x,y,z,t)=\vec{\tilde{B}}_o(x,y)e^{i(k_z z-\omega t)}}\quad \text{with: } \boxed{\vec{\tilde{B}}_o(x,y)=\tilde{B}_{o_x}(x,y)\hat{x}+\tilde{B}_{o_y}(x,y)\hat{y}+\tilde{B}_{o_z}(x,y)\hat{z}}$$

If these expressions are inserted into (3) Faraday's Law and (4) Ampere's Law (above) we obtain:

(3) Faraday's Law:

(i) $$\boxed{\frac{\partial\tilde{E}_{o_y}}{\partial x}-\frac{\partial\tilde{E}_{o_x}}{\partial y}=i\omega\tilde{B}_{o_z}}$$

(ii) $$\boxed{\frac{\partial\tilde{E}_{o_z}}{\partial y}-\frac{\partial\tilde{E}_{o_y}}{\partial z}=i\omega\tilde{B}_{o_x}}$$
$$\boxed{\frac{\partial\tilde{E}_{o_z}}{\partial y}-ik_z\tilde{E}_{o_y}=i\omega\tilde{B}_{o_x}}$$

(iii) $$\boxed{\frac{\partial\tilde{E}_{o_x}}{\partial z}-\frac{\partial\tilde{E}_{o_z}}{\partial x}=i\omega\tilde{B}_{o_y}}$$
$$\boxed{ik_z\tilde{E}_{o_x}-\frac{\partial\tilde{E}_{o_z}}{\partial x}=i\omega\tilde{B}_{o_y}}$$

(4) Ampere's Law:

(iv) $$\boxed{\frac{\partial\tilde{B}_{o_y}}{\partial x}-\frac{\partial\tilde{B}_{o_x}}{\partial y}=-\frac{i\omega}{c^2}\tilde{E}_{o_z}}$$

(v) $$\boxed{\frac{\partial\tilde{B}_{o_z}}{\partial y}-\frac{\partial\tilde{B}_{o_y}}{\partial z}=-\frac{i\omega}{c^2}\tilde{E}_{o_x}}$$
$$\boxed{\frac{\partial\tilde{B}_{o_z}}{\partial y}-ik_z\tilde{B}_{o_y}=-\frac{i\omega}{c^2}\tilde{E}_{o_x}}$$

(vi) $$\boxed{\frac{\partial\tilde{B}_{o_x}}{\partial z}-\frac{\partial\tilde{B}_{o_z}}{\partial x}=-i\frac{\omega}{c^2}\tilde{E}_{o_y}}$$
$$\boxed{ik_z\tilde{B}_{o_x}-\frac{\partial\tilde{B}_{o_z}}{\partial x}=-i\frac{\omega}{c^2}\tilde{E}_{o_y}}$$

> Note the cyclic permutations in x, y, z for (i)-(iii) and (iv)-(vi).

We can use the four equations (ii), (iii), (v), and (vi) to solve for $\tilde{E}_{o_x}, \tilde{E}_{o_y}, \tilde{B}_{o_x}$ and $\tilde{B}_{o_y}$ in terms of $\tilde{E}_{o_z}$ and $\tilde{B}_{o_z}$, which, after some algebra yield:

(a) $$\boxed{\tilde{E}_{o_x} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{E}_{o_z}}{\partial x} + \omega \frac{\partial \tilde{B}_{o_z}}{\partial y}\right)}$$

(b) $$\boxed{\tilde{E}_{o_y} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{E}_{o_z}}{\partial y} - \omega \frac{\partial \tilde{B}_{o_z}}{\partial x}\right)}$$

(c) $$\boxed{\tilde{B}_{o_x} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{B}_{o_z}}{\partial x} - \omega \frac{\partial \tilde{E}_{o_z}}{\partial y}\right)}$$

(d) $$\boxed{\tilde{B}_{o_y} = \frac{i}{(\omega/c)^2 - k_z^2}\left(k_z \frac{\partial \tilde{B}_{o_z}}{\partial y} + \omega \frac{\partial \tilde{E}_{o_z}}{\partial x}\right)}$$

We now insert (a) – (d) above into the other two Maxwell's equations:
(1) Gauss' Law $\vec{\nabla} \cdot \tilde{\vec{E}} = 0$ and (2) No magnetic charges $\vec{\nabla} \cdot \tilde{\vec{B}} = 0$:

$$\boxed{\frac{\partial \tilde{E}_{o_x}}{\partial x} + \frac{\partial \tilde{E}_{o_y}}{\partial y} + \frac{\partial \tilde{E}_{o_z}}{\partial z} = 0} \quad \text{and:} \quad \boxed{\frac{\partial \tilde{B}_{o_x}}{\partial x} + \frac{\partial \tilde{B}_{o_y}}{\partial y} + \frac{\partial \tilde{B}_{o_z}}{\partial z} = 0}$$

We obtain (after some more algebra): two <u>decoupled</u> wave equations for $\tilde{E}_{o_z}$ and $\tilde{B}_{o_z}$:

(α) $$\boxed{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k_z^2\right]\tilde{E}_{o_z} = 0}$$

(β) $$\boxed{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k_z^2\right]\tilde{B}_{o_z} = 0}$$

For monochromatic plane *EM* traveling waves propagating in the $+\hat{z}$ direction:

Longitudinal component of $\tilde{\vec{E}}$
If: $\boxed{\tilde{E}_{o_z} = 0}$, these *EM* waves correspond to **<u>TE</u>** (Transverse Electric) waves.

Longitudinal component of $\tilde{\vec{B}}$
If: $\boxed{\tilde{B}_{o_z} = 0}$, these *EM* waves correspond to **<u>TM</u>** (Transverse Magnetic) waves.

If ***<u>both</u>*** $\boxed{\tilde{E}_{o_z} = \tilde{B}_{o_z} = 0}$, these *EM* waves correspond to **<u>TEM</u>** (Transverse Electric & Magnetic) waves.

n.b. <u>TEM</u> waves *<u>cannot</u>* propagate in *<u>hollow</u>* wave guides.
{they *<u>can</u>* propagate *e.g.* in a *<u>coaxial</u>* waveguide structure with a *<u>center</u>* conductor}.

If $\boxed{\tilde{E}_{o_z} = 0}$ {TE waves}, then Gauss' Law ($\vec{\nabla}\bullet\tilde{\vec{E}} = 0$) becomes: $\boxed{\dfrac{\partial \tilde{E}_{o_x}}{\partial x} + \dfrac{\partial \tilde{E}_{o_y}}{\partial y} = 0}$

If $\boxed{\tilde{B}_{o_z} = 0}$ {TM waves}, then Faraday's Law $\left(\vec{\nabla}\times\tilde{\vec{E}}\right)_z = -\dfrac{\partial \tilde{B}_z}{\partial t}$ becomes: $\boxed{\dfrac{\partial \tilde{E}_{o_y}}{\partial x} - \dfrac{\partial \tilde{E}_{o_x}}{\partial y} = 0}$

If ***both*** $\boxed{\tilde{E}_{o_z} = \tilde{B}_{o_z} = 0}$ {TEM waves}, from (α) and (β) above, we see that $k = \omega/c$.
$\Rightarrow$ we must go back and fully solve equations (i) – (vi) on page 2 (above).

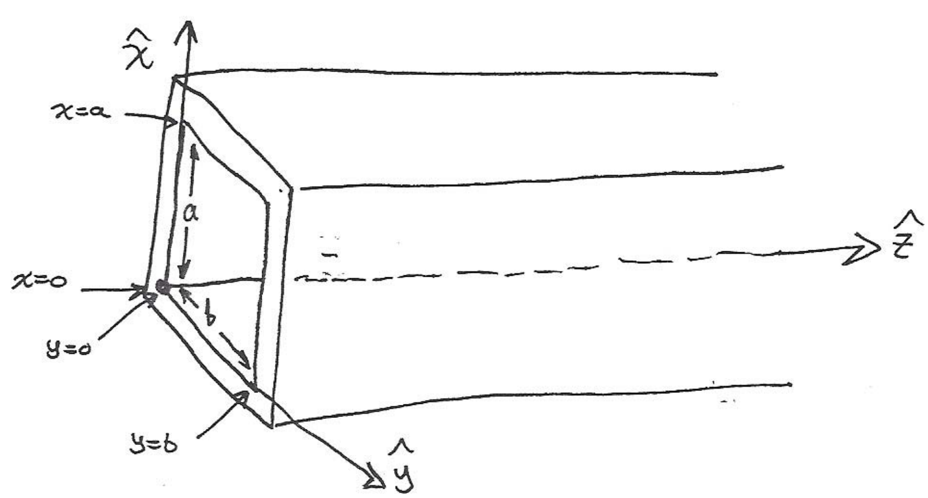
Note that $\tilde{\vec{E}}_o$ *does* satisfy $\vec{\nabla}\bullet\tilde{\vec{E}} = 0$ and $\vec{\nabla}\times\tilde{\vec{E}} = 0$ (*i.e.* $\tilde{\vec{E}}_o$ has zero divergence and zero curl)
$\Rightarrow \tilde{\vec{E}}_o = -\vec{\nabla}\tilde{V}_{scalar}$ $\Rightarrow$ then $\tilde{V}_{scalar}$ satisfies Laplaces' equation: $\vec{\nabla}\bullet\left(-\vec{\nabla}\tilde{V}\right) = -\nabla^2\tilde{V} = 0$

But boundary condition (1): $\boxed{\tilde{E}^{\parallel} = 0}$ at the inner surface of waveguide $\Rightarrow$ the inner surface of the waveguide is an equipotential, *i.e.* $\tilde{V} =$ constant on the inner surface of the wave guide.

If the inside of the waveguide is completely hollow, since Laplace's equation does not allow local maxima or minima anywhere except on the surfaces, then {here} the potential $\tilde{V}$ interior to this wave guide ***must*** be a constant everywhere $\Rightarrow \tilde{\vec{E}}_o = -\vec{\nabla}\tilde{V} = 0$ everywhere inside the waveguide. $\Rightarrow$ No TEM wave propagation can occur in hollow wave guides {*unless the wavelength $\lambda \ll$ cross-sectional dimensions *a*, *b* of the waveguide – then TEM waves are a special / limiting case of TE waves… *e.g. EM* light waves in an optical fiber = waveguide!!!}.

Propagation of TE Waves in a Perfectly Conducting Hollow Rectangular Waveguide ($\sigma_C = \infty$):

Consider a perfectly conducting, hollow rectangular waveguide of (inner) height *a* and width *b* as shown in the figure below {*n.b.* important: $a \geq b$ by convention!!!}:



For TE waves: $\boxed{\tilde{E}_{o_z}(x,y)=0}$, then:

$$(\alpha)\quad \boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{E}_{o_z}=0} \Rightarrow \underline{0=0}\ \ (\textit{i.e.}\ \underline{\text{no}}\ \text{information}).$$

$$(\beta)\quad \boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{B}_{o_z}=0} \Rightarrow \boxed{\tilde{B}_{o_z}(x,y)=0}\ \text{for TE waves.}$$

The boundary condition for $\tilde{\vec{B}}_o(x,y)$ is $\boxed{\tilde{B}^{\perp}=0}$ on the inner walls of waveguide.

But: $\boxed{\tilde{\vec{B}}_o(x,y)=\tilde{B}_{o_x}(x,y)\hat{x}+B_{o_y}(x,y)\hat{y}+B_{o_z}(x,y)\hat{z}}$. Then, referring to the above figure:

$\boxed{B^{\perp}=0}$ in the $\hat{x}$-direction: $\boxed{\tilde{B}_{o_x}(x=0,y)=\tilde{B}_{o_x}(x=a,y)=0}$

$\boxed{B^{\perp}=0}$ in the $\hat{y}$-direction: $\boxed{\tilde{B}_{o_y}(x,y=0)=\tilde{B}_{o_y}(x,y=b)=0}$

But from equations (c) and (d) above:

$$\text{(c)}\quad \boxed{\tilde{B}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial x}-\omega\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial y}}\right)}$$

then: $\boxed{\tilde{B}_{o_x}(x=0,y)=\tilde{B}_{o_x}(x=a,y)=0} \Rightarrow \boxed{\frac{\partial\tilde{B}_{o_z}(x=0,y)}{\partial x}=\frac{\partial\tilde{B}_{o_z}(x=a,y)}{\partial x}=0}$

n.b. These terms = 0 because $E_{oz}(x,y)=0$ for TE waves.

$$\text{(d)}\quad \boxed{\tilde{B}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial\tilde{B}_{o_z}(x,y)}{\partial y}+\omega\cancel{\frac{\partial\tilde{E}_{o_z}(x,y)}{\partial x}}\right)}$$

then: $\boxed{\tilde{B}_{o_y}(x,y=0)=\tilde{B}_{o_y}(x,y=b)=0} \Rightarrow \boxed{\frac{\partial\tilde{B}_{o_z}(x,y=0)}{\partial y}=\frac{\partial\tilde{B}_{o_z}(x,y=b)}{\partial y}=0}$

Now, to solve the wave equation for $\tilde{B}_{o_z}(x,y)$:

Namely (β) $\boxed{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{B}_{o_z}(x,y)=0}$

Use the ***separation of variables technique*** – try a ***product solution*** of the form: $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{X}(x)\tilde{Y}(y)}$

Inserting this into the above equation (β): $\boxed{\tilde{Y}(y)\frac{\partial^2\tilde{X}(x)}{\partial x^2}+\tilde{X}(x)\frac{\partial^2\tilde{Y}}{\partial y^2}+\left[\left(\frac{\omega}{c}\right)^2-k_z^2\right]\tilde{X}(x)\tilde{Y}(y)=0}$

Divide through by $\boxed{\tilde{X}((x)\tilde{Y}(y)}$:

$$\boxed{\underbrace{\frac{1}{\tilde{X}(x)}\frac{\partial^2\tilde{X}(x)}{\partial x^2}}_{fcn\ \text{of}\ x\ \text{only}}+\underbrace{\frac{1}{\tilde{Y}(y)}\frac{\partial^2\tilde{Y}(y)}{\partial y^2}}_{fcn\ \text{of}\ y\ \text{only}}=-\left[\left(\frac{\omega}{c}\right)^2-k_z^2\right]=\text{constant}}$$

The above relation can be true for arbitrary (x,y) points ***iff*** (if and only if):

$$(\gamma)\quad \boxed{\left(\frac{1}{\tilde{X}(x)}\frac{\partial^2\tilde{X}(x)}{\partial x^2}\right)=-k_x^2=\text{constant}}$$

$$(\delta)\quad \boxed{\left(\frac{1}{\tilde{Y}(y)}\frac{\partial^2 Y(y)}{\partial y^2}\right)=-k_y^2=\text{constant}'\left(\neq -k_x^2\right)}$$

Then the so-called *characteristic* equation becomes:

$$\boxed{-k_x^2-k_y^2=-\left[\left(\frac{\omega}{c}\right)^2-k_z^2\right]=\text{constant}''} \quad \underline{\text{or}}:\ \boxed{k_z^2(\omega)=\left(\frac{\omega}{c}\right)^2-k_x^2-k_y^2} \Leftarrow$$

n.b. $k_z(\omega)$ is frequency dependent!

We can rewrite the *characteristic* equation as: $\boxed{\left(\frac{\omega}{c}\right)^2=k_x^2+k_y^2+k_z^2(\omega)=k^2=\left|\vec{k}\right|^2=\vec{k}\bullet\vec{k}}$

The general solutions of the equations $\boxed{\frac{\partial^2\tilde{X}(x)}{\partial x^2}+k_x^2\tilde{X}(x)=0}$ and: $\boxed{\frac{\partial^2 Y(y)}{\partial y^2}+k_y^2\tilde{Y}(y)=0}$

are of the form: $\boxed{\tilde{X}(x)=\tilde{A}_x\cos(k_x x)+\tilde{B}_x\sin(k_x x)}$ and: $\boxed{\tilde{Y}(y)=\tilde{A}_y\cos(k_y y)+\tilde{B}_y\sin(k_y y)}$

Now the boundary condition $\boxed{\tilde{B}^{\perp}=0}$ requires <u>not</u> <u>only</u>:

(c): $\boxed{\tilde{B}_{o_x}(x=0,y)=\tilde{B}_{o_x}(x=a,y)=0}$ <u>but also</u>: $\boxed{\frac{\partial\tilde{B}_{o_z}(x=0,y)}{\partial x}=\frac{\partial\tilde{B}_{o_z}(x=a,y)}{\partial x}=0}$

(d): $\boxed{\tilde{B}_{o_y}(x,y=0)=\tilde{B}_{o_y}(x,y=b)=0}$ <u>but also</u>: $\boxed{\frac{\partial\tilde{B}_{o_z}(x,y=0)}{\partial y}=\frac{\partial\tilde{B}_{o_z}(x,y=b)}{\partial y}=0}$

Since $\tilde{B}_{ox}(x,y)=\tilde{X}(x)\tilde{Y}(y)$ these LATTER boundary conditions require:

$$\boxed{\frac{\partial\tilde{X}(x=0)}{\partial x}=\frac{\partial\tilde{X}(x=a)}{\partial x}=0} \text{ and } \boxed{\frac{\partial\tilde{Y}(y=0)}{\partial y}=\frac{\partial\tilde{Y}(y=a)}{\partial y}=0}$$

So <u>if</u>: $\boxed{\tilde{X}(x)=\tilde{A}_x\cos(k_x x)+\tilde{B}_x\sin(k_x x)}$ and $\boxed{\tilde{Y}(y)=\tilde{A}_y\cos(k_y y)+\tilde{B}_y\sin(k_y y)}$

<u>Then</u>: $\boxed{\frac{\partial\tilde{X}(x)}{\partial x}=-k_x\tilde{A}_x\sin(k_x x)+k_x\tilde{B}_x\cos(k_x x)}$ and $\boxed{\frac{\partial\tilde{Y}(y)}{\partial y}=-k_y\tilde{A}_y\sin(k_y y)+k_y\tilde{B}_y\cos(k_x x)}$

<u>Thus</u>: $\boxed{\frac{\partial\tilde{X}(x=0)}{\partial x}=0}$ <u>requires</u>: $\boxed{\tilde{B}_x=0}$ and $\boxed{\frac{\partial\tilde{Y}(y=0)}{\partial y}=0}$ <u>requires</u>: $\boxed{\tilde{B}_y=0}$

Likewise:

$$\boxed{\frac{\partial \tilde{X}(x=a)}{\partial x}=0} \text{ requires: } \boxed{k_x a = m\pi,\ m=0,1,2,3,\ldots} \text{ or: } \boxed{k_x=\left(\frac{m\pi}{a}\right), m=0,1,2,3,\ldots}$$

and: $$\boxed{\frac{\partial Y(y=b)}{\partial y}=0} \text{ requires: } \boxed{k_y b = n\pi,\ n=0,1,2,3,\ldots} \text{ or: } \boxed{k_y=\left(\frac{n\pi}{b}\right),\ n=0,1,2,3,\ldots}$$

Then $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{X}(x)\tilde{Y}(y)}$ becomes, after absorbing the coefficients $\tilde{A}_x$ & $\tilde{A}_y$ into a single coefficient $\tilde{B}_o$:

$$\boxed{\tilde{B}_{o_z}(x,y)=\tilde{B}_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$$

The full (x, y, z, t) dependence is: $\boxed{\tilde{B}_z(x,y,z,t)=\tilde{B}_{o_z}(x,y)e^{i(k_z z-\omega t)}}$

The characteristic equation then becomes:

$$\boxed{k_z^2(\omega)=\left(\frac{\omega}{c}\right)^2-k_x^2-k_y^2=\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2 \qquad \begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$$

Thus, having found/determined $\tilde{B}_{o_z}(x,y)$ and, since for the TE mode $\boxed{\tilde{E}_{o_z}(x,y)\equiv 0}$, we can now determine $\tilde{E}_{o_x}, \tilde{E}_{o_y}, \tilde{B}_{o_x}$, and $\tilde{B}_{o_y}$ using equations (a) – (d) above:

(a) $$\boxed{\tilde{E}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\cancel{\frac{\partial \tilde{E}_{o_z}(x,y)}{\partial x}}+\omega\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial y}\right)=\frac{i\omega}{(\omega/c)^2-k_z^2}\left(\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial y}\right)}$$

(b) $$\boxed{\tilde{E}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\cancel{\frac{\partial \tilde{E}_{o_z}(x,y)}{\partial y}}-\omega\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial x}\right)=\frac{-i\omega}{(\omega/c)^2-k_z^2}\left(\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial x}\right)}$$

(c) $$\boxed{\tilde{B}_{o_x}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial x}-\omega\cancel{\frac{\partial \tilde{E}_{o_z}(x,y)}{\partial y}}\right)=\frac{ik}{(\omega/c)^2-k_z^2}\left(\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial x}\right)}$$

(d) $$\boxed{\tilde{B}_{o_y}(x,y)=\frac{i}{(\omega/c)^2-k_z^2}\left(k_z\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial y}+\omega\cancel{\frac{\partial \tilde{E}_{o_z}(x,y)}{\partial x}}\right)=\frac{ik}{(\omega/c)^2-k_z^2}\left(\frac{\partial \tilde{B}_{o_z}(x,y)}{\partial y}\right)}$$

But: $\boxed{\tilde{B}_{o_z}(x,y)=\tilde{B}_o\cos(k_x x)\cos(k_y y)}$ with $\boxed{k_x=\left(\frac{m\pi}{a}\right)}$, $\boxed{k_y=\left(\frac{n\pi}{b}\right)}$ and $\boxed{\begin{matrix} m=0,1,2,3,\ldots \\ n=0,1,2,3,\ldots \end{matrix}}$

Explicitly carrying out the spatial differentiation in (a)-(d) above, then for TE wave propagation:

(a) $\boxed{\tilde{E}_{o_x}(x,y) = \dfrac{-i\omega k_y}{(\omega/c)^2 - k_z^2}\tilde{B}_o \cos(k_x x)\sin(k_y y)}$ with $\boxed{k_x = \left(\dfrac{m\pi}{a}\right)}$, $\boxed{k_y = \left(\dfrac{n\pi}{b}\right)}$, $\boxed{\begin{array}{l} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \end{array}}$

(b) $\boxed{\tilde{E}_{o_y}(x,y) = \dfrac{+i\omega k_x}{(\omega/c)^2 - k_z^2}\tilde{B}_o \sin(k_x x)\cos(k_y y)}$ and: $\boxed{k_z^2 = \left(\dfrac{\omega}{c}\right)^2 - k_x^2 - k_y^2}$ and:

(c) $\boxed{\tilde{E}_{o_z}(x,y) \equiv 0}$

(d) $\boxed{\tilde{B}_{o_x}(x,y) = \dfrac{-ik_z k_x}{(\omega/c)^2 - k_z^2}\tilde{B}_o \sin(k_x x)\cos(k_y y)}$

(e) $\boxed{\tilde{B}_{o_y}(x,y) = \dfrac{-ik_z k_y}{(\omega/c)^2 - k_z^2}\tilde{B}_o \cos(k_x x)\sin(k_y y)}$ and:

(f) $\boxed{\tilde{B}_{o_z}(x,y) = \tilde{B}_o \cos(k_x x)\cos(k_y y)}$

The full (x, y, z, t) – dependence is:

$$\boxed{\begin{array}{l}
\tilde{\vec{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \tilde{E}_z\hat{z}\left\{\begin{array}{l}
\text{(a) } \tilde{E}_x(x,y,z,t) = \tilde{E}_{o_x}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-i\omega k_y}{(\omega/c)^2 - k_z^2}B_o \cos(k_x x)\sin(k_y y)e^{i(k_z z - \omega t)} \\
\text{(b) } \tilde{E}_x(x,y,z,t) = \tilde{E}_{o_y}(x,y)e^{i(k_z z - \omega t)} = \dfrac{+i\omega k_x}{(\omega/c)^2 - k_z^2}B_o \sin(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \\
\text{(c) } \tilde{E}_z(x,y,z,t) = \tilde{E}_{o_z}(x,y)e^{i(k_z z - \omega t)} = 0
\end{array}\right. \\
\tilde{\vec{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \tilde{B}_z\hat{z}\left\{\begin{array}{l}
\text{(d) } \tilde{B}_x(x,y,z,t) = \tilde{B}_{o_x}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-ik_z k_x}{(\omega/c)^2 - k_z^2}B_o \sin(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \\
\text{(e) } \tilde{B}_y(x,y,z,t) = \tilde{B}_{o_y}(x,y)e^{i(k_z z - \omega t)} = \dfrac{-ik_z k_y}{(\omega/c)^2 - k_z^2}B_o \cos(k_y x)\sin(k_y y)e^{i(k_z z - \omega t)} \\
\text{(f) } \tilde{B}_z(x,y,z,t) = \tilde{B}_{o_z}(x,y)e^{i(k_z z - \omega t)} = B_o \cos(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)}
\end{array}\right.
\end{array}}$$

Note that for the TE mode(s) of propagation of *EM* waves in a rectangular waveguide, the $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ -fields are <u>in</u> <u>phase</u> with each other – the *x*, *y* and *z*-components of $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ all have the common phase factor $e^{i(k_z z - \omega t)}$.

The wave number $\boxed{k_z(\omega)=\sqrt{\left(\frac{\omega}{c}\right)^2-k_x^2-k_y^2}=\frac{2\pi}{\lambda_z}}$ with $\boxed{k_x=\left(\frac{m\pi}{a}\right)}$, $\boxed{k_y=\left(\frac{n\pi}{b}\right)}$ and $\boxed{\begin{array}{l}m=0,1,2,3,\ldots\\ n=0,1,2,3,\ldots\end{array}}$

Thus: $$\boxed{k_z(\omega)=\frac{2\pi}{\lambda_z(\omega)}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left\lfloor k_x^2+k_y^2\right\rfloor}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}}$$

We can define a so-called {angular} cutoff frequency for the $(m,n)^{th}$ TE mode as:

$$\boxed{\omega_{m,n}\equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}}$$

Thus, we can rewrite the above relation as:

$$\boxed{k_z^{m,n}(\omega)=\sqrt{\left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{\omega_{m,n}}{c}\right)^2}=\frac{1}{c}\sqrt{\omega^2-\omega_{m,n}^2}}$$

Note that for {angular} frequencies *below* the cutoff frequency: $\boxed{\omega<\omega_{m,n}\equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}}$

Then: $\left(\omega^2-\omega_{m,n}^2\right)<0$ and: $k_z^{m,n}(\omega)=\frac{1}{c}\sqrt{\omega^2-\omega_{m,n}^2}$ becomes *imaginary*, hence: $e^{i(k_z z)}\rightarrow e^{-k_z z}$ which means that when $\omega<\omega_{m,n}$, the *EM* wave for the $(m,n)^{th}$ mode is *exponentially* *damped*.

Note also that $m=n=0$ corresponds to $k_x=k_y=0$ with $k_z^{0,0}(\omega)=\omega/c$. But then, from the above $\tilde{\vec{E}}$- and $\tilde{\vec{B}}$-field relations on the previous page, we see that for *this* kind of TE wave, that:

$\boxed{\tilde{E}_x=\tilde{E}_y=\tilde{E}_z=0}$ and $\boxed{\tilde{B}_x=\tilde{B}_y=0}$ with $\boxed{\tilde{B}_z=B_o e^{i(kz-\omega t)}=0}$.

$\Rightarrow$ This is ***not*** a proper kind of propagating *EM* wave, because then $\tilde{\vec{E}}=\tilde{\vec{B}}=0$ everywhere!!!

Thus, the ***lowest*** *non-trivial* propagating TE-type *EM* wave is the TE_{10} mode, where the notation TE_{mn} designates the $(m,n)^{th}$ mode of propagation. Note again, that by convention, the index associated with the *largest* transverse dimension (here a) with corresponding integer index m is given *first*.

Thus, for the ***lowest*** TE mode, $\text{TE}_{1,0}$: $\boxed{k_z^{1,0}(\omega)=\frac{1}{c}\sqrt{\omega^2-\omega_{1,0}^2}=\frac{1}{c}\sqrt{\omega^2-\left(\frac{\pi c}{a}\right)^2}}$

and we see that $k_{1,0}\geq 0$ {*i.e.* is a purely *real* quantity} when: $\boxed{\omega^2-\omega_{1,0}^2=\omega^2-(\pi c/a)^2>0}$
i.e. when: $\boxed{\omega>\omega_{1,0}\equiv(\pi c/a)(radians/sec)}$, or : $\boxed{f>f_{1,0}=(\omega_{1,0}/2\pi)=c/2a\ \ (Hz)}$.

A Numerical Example:

Suppose the rectangular wave guide's transverse internal dimensions are $a = 2\,cm$ and $b = 1\,cm$

Then: $\boxed{\omega_{1,0} = \pi c/a = 3\pi\times10^8\ m/s/0.02m = 1.5\pi\times10^{10}\ radians/sec = 4.71\times10^{10}\ radians/sec}$

This corresponds to a cutoff frequency of: $\boxed{f_{1,0} = \omega_{1,0}/2\pi = \tfrac{3}{4}\times10^{10}\ /s = 7.5\,GHz}$
which is in the microwave portion of the *EM* spectrum, and corresponds to a wavelength of:

$$\boxed{\lambda_z^{1,0} = c/f_{1,0} = 3\times10^8\ m/s/\tfrac{3}{4}\times10^{10}\ /s = 4\times10^{-2}m = 4.0\,cm} \quad i.e.\ \boxed{\lambda_z^{1,0} = 2a}\ !!!$$

Thus, we see that if $\lambda_z > \lambda_z^{1,0} = 2a$, we ***cannot*** propagate $TE_{1,0}$ waves because: $\boxed{f < f_{1,0} = \frac{c}{\lambda_z^{1,0}} = \frac{c}{2a}}$.

We also see that if $\lambda_z < \lambda_z^{1,0} = 2a$, then we ***can*** propagate $TE_{1,0}$ wave because: $\boxed{f > f_{1,0} = \frac{c}{\lambda_z^{1,0}} = \frac{c}{2a}}$.

Precisely *at* the angular cutoff frequency for the $TE_{1,0}$ mode, *i.e.* $\omega = \omega_{1,0} = \pi c/a$, we see that the wavenumber $k_z^{1,0}(\omega) = \frac{1}{c}\sqrt{\omega^2 - \omega_{1,0}^2} = 0 = 2\pi/\lambda_z^{1,0}(\omega)$ and thus $\lambda_z^{1,0}(\omega) = \infty$ for $\omega = \omega_{1,0}$,
where $\lambda_z^{1,0}(\omega)$ = the wavelength of the *EM* wave *in* the waveguide for the $TE_{1,0}$ mode.

Now suppose that $\omega > \omega_{1,0}$ then: $\boxed{k_z^{m,n}(\omega) = \frac{2\pi}{\lambda_z^{m,n}(\omega)} = \frac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}$

The higher the angular frequency ω is, it then becomes possible to propagate $TE_{m,n}$ waves in *more* than just one mode. There exists an angular cutoff frequency for *each* $TE_{m,n}$ mode:

Angular cutoff frequency for each TE_{mn} mode: $\boxed{\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$

Another Numerical Example:

A rectangular wave guide's transverse internal dimensions are (again) $a = 2\,cm$ and $b = 1\,cm$. Suppose that: $f = 20\,GHz = 2\times10^{10}\,Hz$, thus: $\omega = 2\pi f = 4\pi\times10^{10} = 12.56\times10^{10}\,radians/sec$ with corresponding vacuum wavelength $\lambda_o = c/f = 1.5\,cm$.

Which TE$_{m,n}$ modes are ***accessible*** ? $\boxed{\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$

$$\omega_{1,0} = c\sqrt{\left(\frac{1\pi}{a}\right)^2} = \left(\frac{\pi c}{a}\right) = 4.71\times10^{10}\,radians/sec$$

$$\omega_{0,1} = c\sqrt{\left(\frac{1\pi}{b}\right)^2} = \left(\frac{\pi c}{b}\right) = 9.42\times10^{10}\,radians/sec$$

$$\omega_{1,1} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \pi c\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 10.53\times10^{10}\,radians/sec$$

$$\omega_{2,0} = c\sqrt{\left(\frac{2\pi}{a}\right)^2} = \frac{2\pi c}{a} = 9.42\times10^{10}\,radians/sec$$

$$\omega_{3,0} = c\sqrt{\left(\frac{3\pi}{a}\right)^2} = \frac{3\pi c}{a} = 14.14\times10^{10}\,radians/sec$$

TOO HIGH !!!

$$\omega_{2,1} = c\sqrt{\left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \pi c\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \pi c\sqrt{2\left(\frac{1}{b}\right)^2} = 13.33\times10^{10}\,radians/sec$$

Thus, for $f = 20\,GHz \Rightarrow \omega = 12.56\times10^{10}$ radians/sec we <u>can</u> access/<u>can</u> propagate TE$_{m,n}$ waves in the following 4 modes:

$TE_{1,0}$:	$\omega_{1,0} =$	$4.71\times10^{10}\,radians/sec$
$TE_{0,1}$:	$\omega_{0,1} =$	$9.42\times10^{10}\,radians/sec$
$TE_{2,0}$:	$\omega_{2,0} =$	$9.42\times10^{10}\,radians/sec$
$TE_{1,1}$:	$\omega_{1,1} =$	$10.53\times10^{10}\,radians/sec$

n.b. Degenerate, <u>because</u> ***a*** = 2***b*** !!!

<u>TE$_{m,n}$ Wavenumbers and Wavelengths Inside the Waveguide</u>: $\boxed{a = 2\,cm \text{ and } b = 1\,cm}$

$f = 20\,GHz$
$= 2\times10^{10}\,Hz$

$\omega = 12.56\times10^{10}$ $radians/sec$

$$\begin{cases} TE_{1,0}:\ k_z^{1,0}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} = 388.31\,m^{-1},\ \lambda_z^{1,0}(\omega) = \frac{2\pi}{k_{1,0}(\omega)} = 1.620\,cm \\ TE_{0,1}:\ k_z^{0,1}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2} = 277.06\,m^{-1},\ \lambda_z^{0,1}(\omega) = \frac{\pi}{k_{0,1}(\omega)} = 2.268\,cm \\ TE_{2,0}:\ k_z^{2,0}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{2\pi}{a}\right)^2} = 277.06\,m^{-1},\ \lambda_z^{2,0}(\omega) = \frac{\pi}{k_{2,0}(\omega)} = 2.268\,cm \\ TE_{1,1}:\ k_z^{1,1}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]} = 228.23\,m^{-1},\ \lambda_z^{1,1}(\omega) = \frac{\pi}{k_{1,1}(\omega)} = 2.750\,cm \end{cases}$$

Degenerate !!!

Compare these to vacuum wavenumber $k_o = \frac{2\pi}{\lambda_o} =$ $418.88\,m^{-1}$ and vacuum wavelength $\lambda_o = 1.5\,cm$.

Note that the wavenumbers and wavelengths inside the wave guide will change when the frequency f (or $\omega = 2\pi f$) changes, because $\boxed{k_z^{m,n}(\omega) = \frac{2\pi}{\lambda_z^{m,n}(\omega)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}$

Physically, the <u>*phase*</u> velocity (also known as the <u>*wave*</u> velocity) $v_{\phi_z}^{m,n}(\omega)$ is the speed of propagation of <u>*planes*</u> of <u>*constant*</u> <u>*phase*</u> $\varphi_{m,n}(\omega) \equiv \left[k_z^{m,n}(\omega) z - \omega t\right] = constant$ and is associated with the $e^{i(k_z z - \omega t)}$ phase-factor of the EM wave for each individual TE$_{m,n}$ mode.

Since $\varphi_{m,n}(\omega) \equiv \left[k_z^{m,n}(\omega) z - \omega t\right] = constant$, then: $\partial\varphi_{m,n}(\omega)/\partial t = 0$ which means that: $\boxed{\frac{\partial\varphi_{m,n}(\omega)}{\partial t} = \frac{\partial}{\partial t}\left[k_z^{m,n}(\omega) z(t) - \omega t\right] = k_z^{m,n}(\omega)\frac{\partial z(t)}{\partial t} - \omega = 0}$, or that: $\boxed{k_z^{m,n}(\omega)\frac{\partial z(t)}{\partial t} = \omega}$, or: $\boxed{\frac{\partial z(t)}{\partial t} = \frac{\omega}{k_z^{m,n}(\omega)}}$. The phase velocity $\boxed{v_{\phi_z}^{m,n}(\omega) \equiv \frac{\partial z(t)}{\partial t} = \frac{\omega}{k_z^{m,n}(\omega)}}$

Thus, the <u>*phase*</u> velocity of a TE$_{m,n}$ wave for the $(m,n)^{th}$ mode is:

$$\boxed{v_{\varphi_z}^{m,n}(\omega) \equiv \frac{\omega}{k_z^{m,n}(\omega)} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}}$$

Since:

$\boxed{\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$ then we see that the <u>*phase*</u> velocity of a TE$_{m,n}$ wave is:

$\boxed{v_{\varphi_z}^{m,n}(\omega) \equiv \frac{\omega}{k_z^{m,n}(\omega)} = \frac{c}{\sqrt{1-\left(\omega_{m,n}/\omega\right)^2}} > c}$ for the $(m,n)^{th}$ allowed TE$_{m,n}$ mode!!!

For the $(m,n)^{th}$ $\text{TE}_{m,n}$ mode, *EM energy* in the waveguide propagates at the *group velocity*:

$$\boxed{v_{g_z}^{m,n}(\omega) \equiv 1\Big/\left(\frac{dk_z^{m,n}(\omega)}{d\omega}\right) = \left(\frac{dk_z^{m,n}(\omega)}{d\omega}\right)^{-1}} \Leftarrow$$

See P436 HW #7, Griffiths problem 9.29, page 411

Let's calculate $v_{g_z}^{m,n}(\omega)$:

$$\boxed{\frac{dk_z^{m,n}(\omega)}{d\omega} = \frac{d}{d\omega}\left\{\frac{1}{c}\sqrt{\omega^2-\omega_{m,n}^2}\right\} = \frac{2\omega}{2c}\bullet\frac{1}{\sqrt{\omega^2-\omega_{m,n}^2}} = \frac{\omega/c}{\sqrt{\omega^2-\omega_{m,n}^2}}}$$

Thus: $$\boxed{v_{g_z}^{m,n}(\omega) = \frac{1}{\dfrac{dk_{m,n}(\omega)}{d\omega}} = \frac{\sqrt{\omega^2-\omega_{m,n}^2}}{\omega/c} = c\sqrt{1-\left(\frac{\omega_{m,n}}{\omega}\right)^2}}$$ where: $$\boxed{\omega_{m,n} \equiv c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}}$$

It can be seen from the above relation that $v_g^{m,n}(\omega) < c$ {always!}, as required by causality…

Note further that: $$\boxed{v_{\varphi_z}^{m,n}(\omega)\cdot v_{g_z}^{m,n}(\omega) = \frac{c}{\sqrt{1-(\omega_{m,n}/\omega)^2}}\cdot c\sqrt{1-(\omega_{m,n}/\omega)^2} = c^2}$$

The instantaneous surface charge and current densities induced on the *inner* surfaces of the {perfectly conducting} waveguide due to the *EM* fields within the waveguide can be obtained from:

$$\boxed{\sigma_{surf}^{ind}(x,y,z,t) = \varepsilon_o\vec{E}_{surf}(x,y,z,t)\bullet\hat{n}_{surf}(x,y,z)}$$

and:

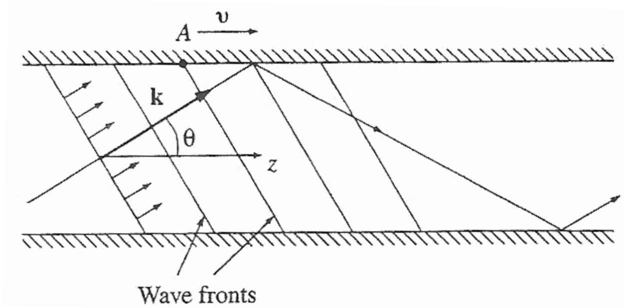
$$\boxed{\vec{K}_{surf}^{ind}(x,y,z,t) = \tfrac{1}{\mu_o}\hat{n}_{surf}(x,y,z)\times\vec{B}_{surf}(x,y,z,t)}$$

where $\hat{n}_{surf}(x,y,z)$ is the local {inward-pointing} unit normal at (x,y,z) associated with a given inner surface of the waveguide, and $\vec{E}_{surf}(x,y,z,t)$, $\vec{B}_{surf}(x,y,z,t)$ are the instantaneous electric, magnetic fields evaluated at (x,y,z,t) on that surface.

Note that $\vec{E}_{surf}(x,y,z,t)\bullet\hat{n}_{surf}(x,y,z)$ is the instantaneous local *normal* (*i.e.* $\perp$) component of the electric field at (x,y,z,t) on that surface, whereas $\hat{n}_{surf}(x,y,z)\times\vec{B}_{surf}(x,y,z,t)$ is the instantaneous local *tangential* (*i.e.* $||$) component of the magnetic field at (x,y,z,t) on that surface.

The Physical Picture of *EM* Waves Propagating Inside a Wave Guide.

Consider an *ordinary* monochromatic *EM* plane wave *initially* propagating at speed $c = \omega/|\vec{k}|$ in the $\hat{k}$-direction, making an angle θ with respect to the $\hat{z}$-axis, as shown in the figure below:



Because the inner walls of the wave guide are perfectly conducting, they are lossless, *i.e.* perfectly reflecting. The *EM* waves are thus *multiply-reflected* {*n.b.* with π phase shift at *each* reflection} as they "bounce" down the wave guide – interfering with each other in such a way as to form *standing wave* patterns of wavelength $\lambda_x = 2a/m$ in the $\hat{x}$-direction and $\lambda_y = 2b/n$ in the $\hat{y}$-direction!!!

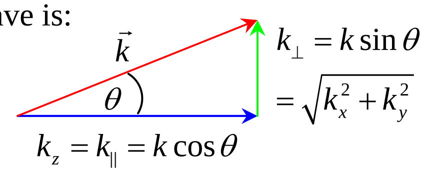
The *x*, *y* wavelengths respectively correspond to the *x*, *y*-wavenumbers $k_x = 2\pi/\lambda_x = m\pi/a$ in the $\hat{x}$-direction, and $k_y = 2\pi/\lambda_y = n\pi/b$ in the $\hat{y}$-direction. In the $\hat{z}$-direction, the ensemble (*i.e.* *group*) of reflected waves results in a *traveling wave*, with *z*-wavenumber:

$$\boxed{k_z^{m,n}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(k_x^2 + k_y^2\right)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} = \frac{1}{c}\sqrt{\omega^2 - \omega_{m,n}^2}}$$

where: $\boxed{\omega_{m,n} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}}$

The propagation wave**vector** associated with the *initial* plane wave is:

$$\boxed{\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = \left(\frac{m\pi}{a}\right)\hat{x} + \left(\frac{n\pi}{b}\right)\hat{y} + k_z^{m,n}(\omega)\hat{z}}$$



Thus, because *m*, *n* = 0, 1, 2, 3,… (*n.b.* both *m* = *n* = 0 simultaneously is ***not*** allowed), only ***certain*** angles $\theta_{m,n}$ will lead to one of the allowed standing wave patterns in *x* and *y*:

$$\boxed{\cos\theta_{m,n} = \frac{k_z^{m,n}(\omega)}{|\vec{k}|} = \frac{\sqrt{\omega^2 - \omega_{m,n}^2}/c}{\omega/c} = \sqrt{1 - (\omega_{m,n}/\omega)^2}}$$ where: $\boxed{\omega_{m,n} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}}$

This "original" plane *EM* wave, traveling at angle $\theta_{m,n}$ with respect to the $\hat{z}$-axis travels at speed $c = \omega/|\vec{k}|$ (*i.e.* we assume that the medium (*e.g.* air, or vacuum) inside the wave guide has $\varepsilon = \varepsilon_o$ and $\mu = \mu_o$).

But because this plane *EM* wave makes an angle $\theta_{m,n}$ with respect to the $\hat{z}$-axis, the component of the initial wave's speed *projected* *along* the $\hat{z}$-axis is *less* than c :

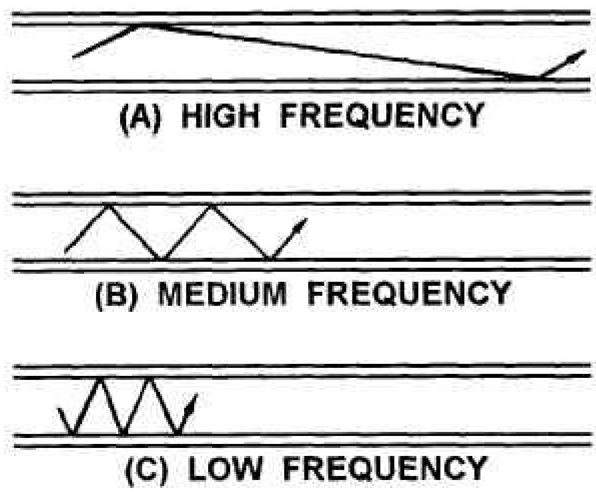
$$\boxed{v_z(\omega) = c\cos\theta_{m,n}(\omega) = c\sqrt{1-\left(\omega_{m,n}/\omega\right)^2} = v_{g_z}^{m,n}(\omega) = \text{group velocity!}}$$

The *phase* velocity (*aka* wave velocity) is the speed at which wavefronts (planes of constant phase) (*e.g*. point *A* in the figure on the previous page) propagate down the wave guide – these *can* move *much* faster than *c*, because:

$$\boxed{v_{\varphi_z}^{m,n}(\omega) = \frac{c}{\cos\theta_{m,n}(\omega)} = \frac{c}{\sqrt{1-\left(\omega_{m,n}/\omega\right)^2}}}$$

Note that if $\theta_{m,n} = 90^\circ$ (*i.e.* $\cos\theta_{m,n} = 0$), {*i.e*. when $\omega = \omega_{m,n}$}, for which $v_g^{m,n}(\omega) = 0$ and $v_{\varphi_z}^{m,n}(\omega) = \infty$!!!

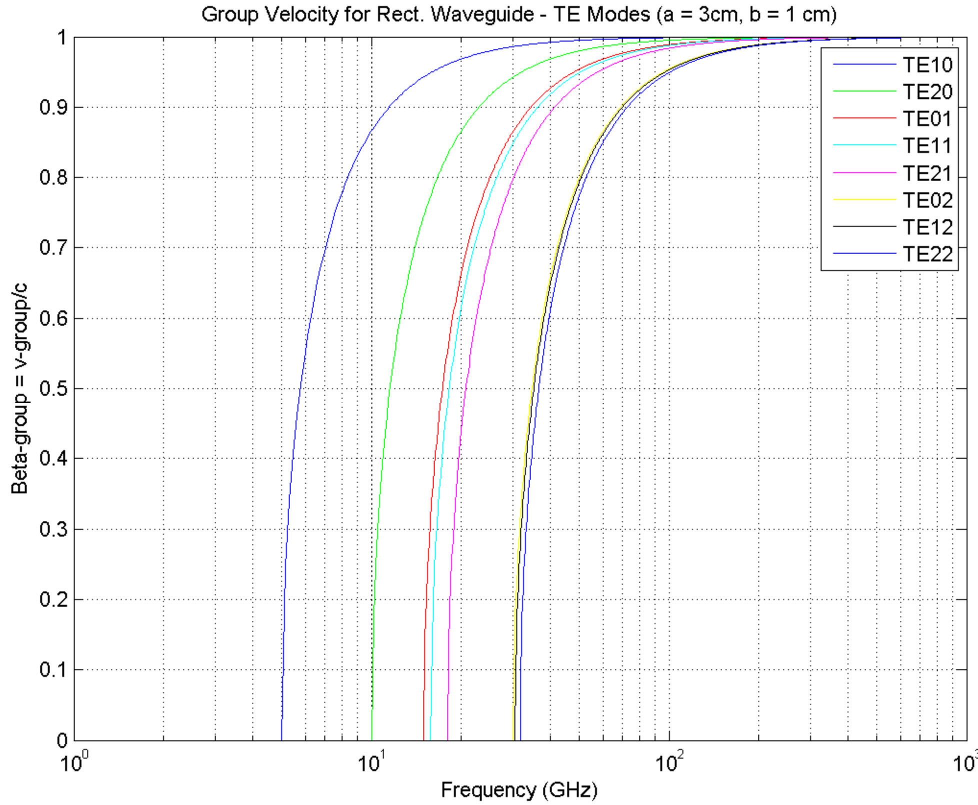
Physically, this corresponds to *standing* waves in (*x*,*y*), *i.e*. NO propagation along $\hat{z}$-direction $\Rightarrow$ *i.e*. a 2-D resonant cavity!!!

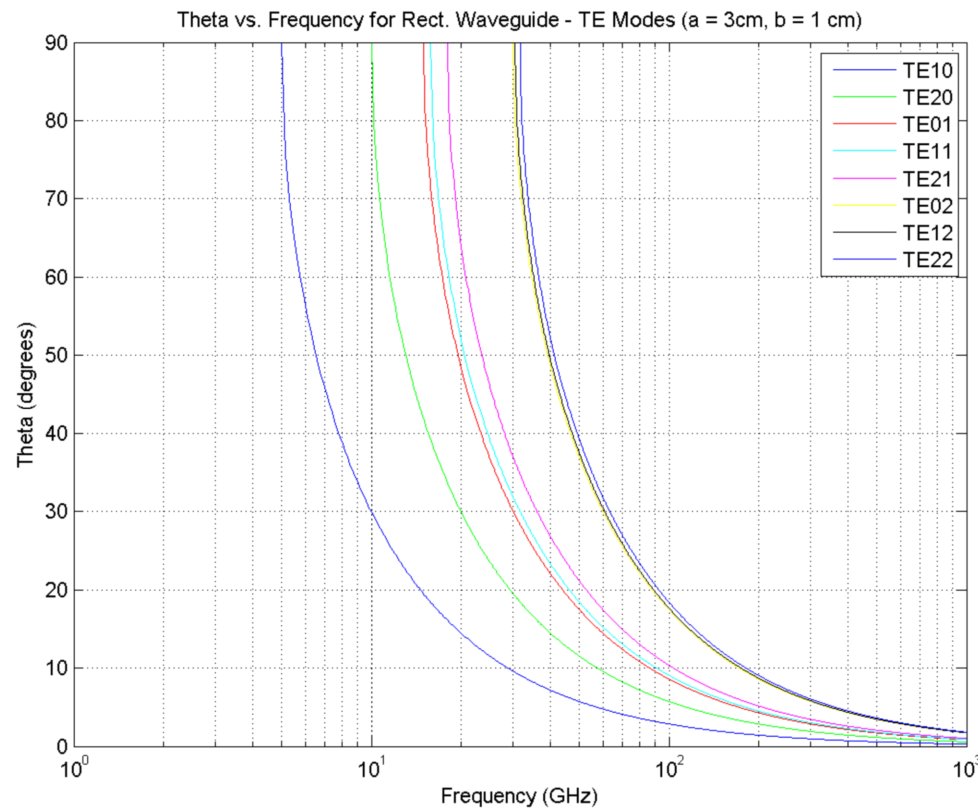


Thus, the allowed solution(s) that we obtained on *p*. 8 above for the *x*, *y* and *z* components of the electric and magnetic fields for *TE* mode propagation of electromagnetic waves down a waveguide actually/physically represent the *steady*-*state* ensemble (*i.e*. *group*) wave solution associated with the *collective* effect(s) of the multiply-reflected waves interfering with each other as they propagate down the waveguide!

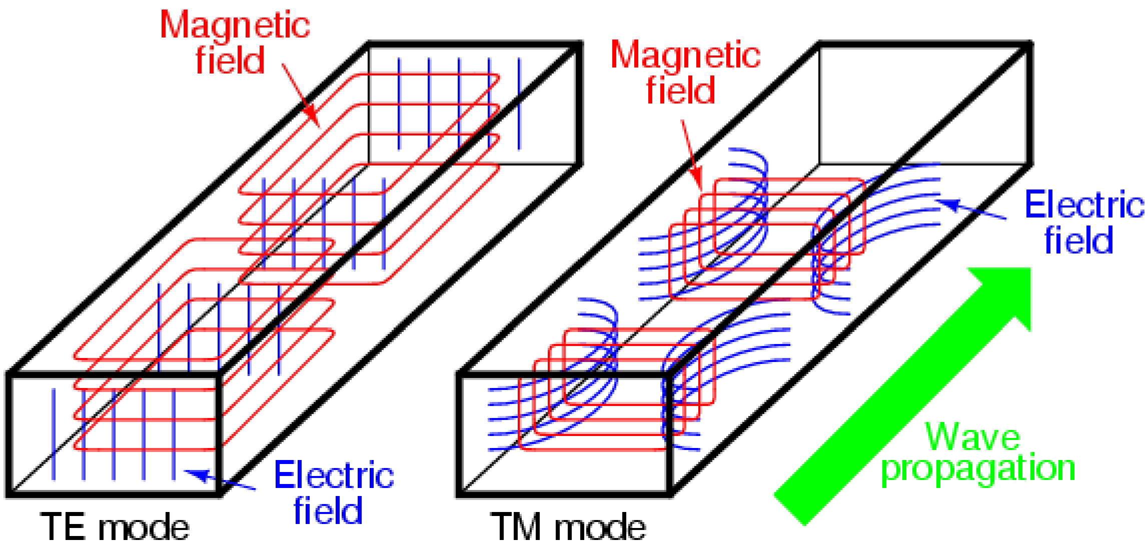
This *group* of waves for the (m,n^{th}) *TE* (and/or TE) mode propagates down the waveguide at the *group* velocity $v_{g_z}^{m,n}(\omega) = c\sqrt{1-\left(\omega_{m,n}/\omega\right)^2} = c\cos\theta_{m,n}(\omega)$ (hence the origin of its name!).

In the two figures below, we show plots of $\beta_{g_z}^{m,n}(f) \equiv v_{g_z}^{m,n}(\omega)/c = \sqrt{1-\left(f_{m,n}/f\right)^2}$ *vs*. *f* and $\theta_{m,n} = \cos^{-1}\left(\beta_{g_z}^{m,n}(f) \equiv v_{g_z}^{m,n}(\omega)/c = \sqrt{1-\left(f_{m,n}/f\right)^2}\right)$ *vs*. *f*:


Group Velocity for Rect. Waveguide - TE Modes (a = 3cm, b = 1 cm)
Beta-group = v-group/c
Frequency (GHz)
TE10
TE20
TE01
TE11
TE21
TE02
TE12
TE22


Theta vs. Frequency for Rect. Waveguide - TE Modes (a = 3cm, b = 1 cm)
Theta (degrees)
Frequency (GHz)
TE10
TE20
TE01
TE11
TE21
TE02
TE12
TE22

3-D Picture of $\vec{E}$ and $\vec{B}$-fields in Rectangular Wave Guide for $TE_{1,0}$ Mode:



Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

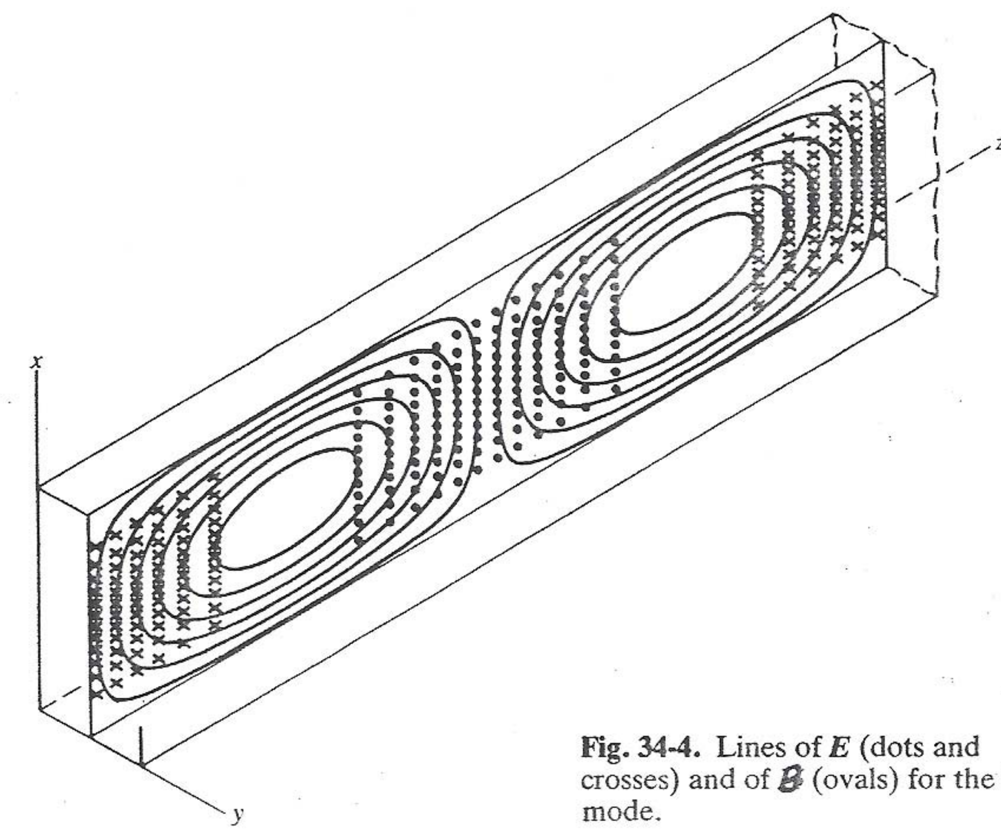
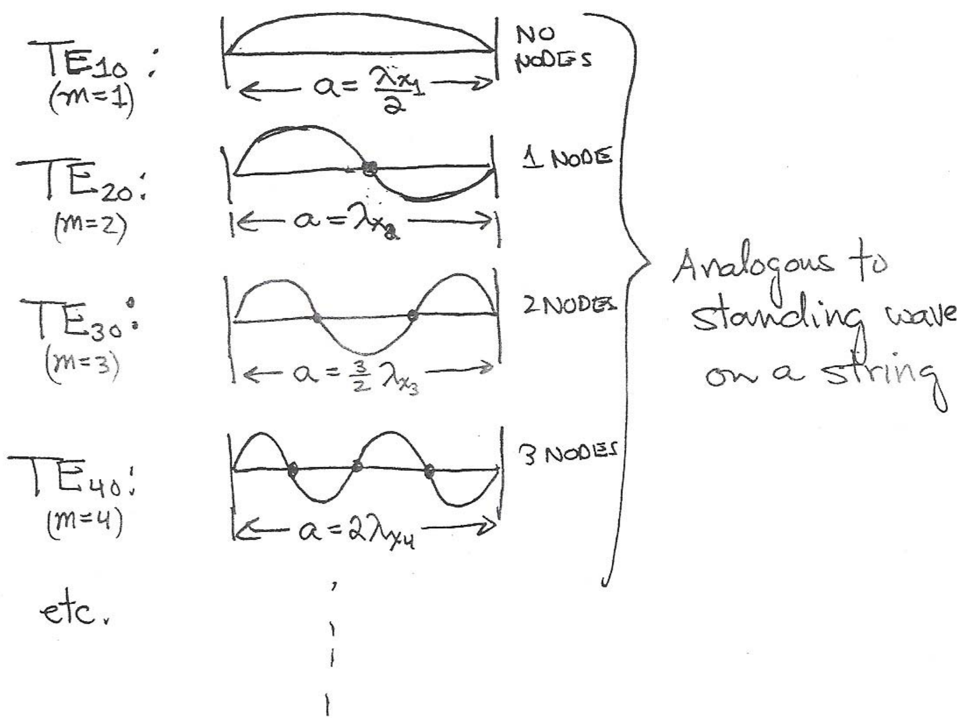


Fig. 34-4. Lines of $\boldsymbol{E}$ (dots and crosses) and of $\boldsymbol{B}$ (ovals) for the TE_{10} mode.

For $TE_{0,1}$ mode, rotate above pix by 90°

For $\text{TE}_{m,0}$ modes - $\exists$ nodes at the mid-plane:



Time-Averaged Power Transmitted Down a Rectangular Wave Guide in $\text{TE}_{m,n}$ Modes:

In order to calculate the time-averaged power transmitted down a rectangular wave guide {of cross-sectional area $A_\perp = ab\,(= h \times w)$ } we integrate the time-averaged Poynting vector, $\left\langle \vec{S}(\vec{r},t)\right\rangle_t$ over the cross-sectional area of the waveguide:

$\hat{n} = +\hat{z}$ direction (here)

$$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \int_{A_\perp}\left\langle \vec{S}_{m,n}(x,y,z,t)\right\rangle \cdot d\vec{a}_\perp = \int_{y=o}^{y=b}\int_{x=o}^{x=a}\left\langle \vec{S}_{m,n}(x,y,z,t)\right\rangle \cdot \hat{n}dxdy} \quad \boxed{d\vec{a}_\perp = \hat{n}\ dxdy = \hat{z}\ dxdy}$$

From Griffiths Problem 9.11 (p.382): $\boxed{\left\langle \vec{S}_{m,n}(x,y,z,t)\right\rangle = \frac{1}{2\mu_o}\Re e\left(\tilde{\vec{E}}_{m,n}(x,y,z,t) \times \tilde{\vec{B}}_{m,n}^{*}(x,y,z,t)\right)}$

{Because $\boxed{\langle f\ g\rangle = \tfrac{1}{2}\text{Re}\left(\tilde{f}\ \tilde{g}^*\right)}$, where * denotes complex conjugation}

For the TE$_{m,n}$ modes in a rectangular wave guide:

$$\boxed{\tilde{\vec{E}}_{m,n}(x,y,z,t)=\tilde{\vec{E}}_{o_{m,n}}(x,y)e^{i(k_z z-\omega t)}} \text{ and: } \boxed{\tilde{\vec{B}}^*_{m,n}(x,y,z,t)=\tilde{\vec{B}}^*_{o_{m,n}}(x,y)e^{-i(k_z z-\omega t)}} \quad \boxed{m,n=0,1,2,3,\ldots}$$

with: $\boxed{k_z=k_z^{m,n}\equiv\sqrt{\left(\frac{\omega}{c}\right)^2-\left[k_{x_m}^2+k_{y_n}^2\right]}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}}$, $\boxed{k_{x_m}\equiv\left(\frac{m\pi}{a}\right)}$, $\boxed{k_{y_n}\equiv\left(\frac{n\pi}{b}\right)}$

and with: $\boxed{\tilde{\vec{E}}_{o_{mn}}(x,y)=\tilde{E}_{ox_{mn}}\hat{x}+\tilde{E}_{oy_{mn}}\hat{y}+\tilde{E}_{oz_{mn}}\hat{z}}$ and $\boxed{\tilde{\vec{B}}_{o_{mn}}(x,y)=\tilde{B}_{ox_{mn}}\hat{x}+\tilde{B}_{oy_{mn}}\hat{y}+\tilde{B}_{oz_{mn}}\hat{z}}$

$$\tilde{E}_{ox_{mn}}(x,y)=\frac{i\omega\left(-k_{y_n}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\cos\left(k_{x_m}x\right)\sin\left(k_{y_n}y\right)=\frac{i\omega}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-n\pi}{b}\right)B_o\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\tilde{E}_{oy_{mn}}(x,y)=\frac{-i\omega\left(-k_{x_m}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\sin\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=\frac{-i\omega}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-m\pi}{a}\right)B_o\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\tilde{E}_{oz_{mn}}(x,y)=0$$

$$\tilde{B}^*_{ox_{mn}}(x,y)=\frac{-ik_{mn}\left(-k_{x_m}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\sin\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=\frac{ik_{mn}}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-m\pi}{a}\right)B_o\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\tilde{B}^*_{oy_{mn}}(x,y)=\frac{-ik_{mn}\left(-k_{y_n}\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}B_o\cos\left(k_{x_m}x\right)\sin\left(k_{y_n}y\right)=\frac{ik_{mn}}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{-n\pi}{b}\right)B_o\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\tilde{B}^*_{oz_{mn}}(x,y)=B_o\cos\left(k_{x_m}x\right)\cos\left(k_{y_n}y\right)=B_o\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

<u>Then</u>: $\boxed{\left\langle\vec{S}(x,y,z,t)\right\rangle=\frac{1}{2\mu_o}\Re e\left(\tilde{\vec{E}}(x,y,z,t)\times\tilde{\vec{B}}^*(x,y,z,t)\right)}$ $\Leftarrow$ <u>Note</u>: All time dependence vanishes { $e^{i(k_z z-\omega t)}$ factor}

<u>Very Useful Table</u>:

$\hat{x}\times\hat{y}=\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$	$\hat{x}\times\hat{x}=0$
$\hat{y}\times\hat{z}=\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$	$\hat{y}\times\hat{y}=0$
$\hat{z}\times\hat{x}=\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$	$\hat{z}\times\hat{z}=0$

<u>Then</u>: $\left(\tilde{\vec{E}}=\tilde{E}_x\hat{x}+\tilde{E}_y\hat{y}+\tilde{E}_z\hat{z}\right)\times\left(\tilde{\vec{B}}^*=B_x^*\hat{x}+B_y^*\hat{y}+B_x^*\hat{z}\right)$

$$\begin{aligned}
&=\tilde{E}_x\tilde{B}_y^*(\hat{x}\times\hat{y})+\tilde{E}_x\tilde{B}_z^*(\hat{x}\times\hat{z})=\tilde{E}_x\tilde{B}_y^*\hat{z}-\tilde{E}_xB_z^*\hat{y}\\
&+\tilde{E}_y\tilde{B}_x^*(\hat{y}\times\hat{x})+\tilde{E}_y\tilde{B}_z^*(\hat{y}\times\hat{z})-\tilde{E}_y\tilde{B}_x^*\hat{z}+\tilde{E}_yB_z^*\hat{x}\\
&+\tilde{E}_z\tilde{B}_x^*(\hat{z}\times\hat{x})+\tilde{E}_z\tilde{B}_y^*(\hat{z}\times\hat{y})-\tilde{E}_z\tilde{B}_x^*\hat{y}-\tilde{E}_zB_y^*\hat{x}\\
&=\left(\tilde{E}_y\tilde{B}_z^*-\tilde{E}_z\tilde{B}_y^*\right)\hat{x}+\left(\tilde{E}_z\tilde{B}_x^*-\tilde{E}_x\tilde{B}_z^*\right)\hat{y}+\left(\tilde{E}_x\tilde{B}_y^*-\tilde{E}_y\tilde{B}_x^*\right)\hat{z}
\end{aligned}$$

But $\boxed{E_{z_{mn}} = 0}$ for TE$_{m,n}$ modes, and skipping (<u>much</u>) algebra:

<u>Then</u>:
$$\boxed{\begin{aligned}\frac{1}{2\mu_o}\left(\tilde{\vec{E}}_{m,n}\times\tilde{\vec{B}}_{m,n}\right) = \frac{1}{2\mu_o}\Bigg\{ &\frac{i\pi\omega B_o^2}{(\omega/c)^2 - k_{z_{mn}}^2}\left(\frac{m}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\hat{x} \\ &+\frac{i\pi\omega B_o^2}{(\omega/c)^2 - k_{z_{mn}}^2}\left(\frac{n}{b}\right)\cos^2\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{y} \\ +\frac{\pi^2\omega k_{z_{m,n}}B_o^2}{\left[(\omega/c)^2 - k_{z_{mn}}^2\right]^2}&\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{z}\Bigg\}\end{aligned}}$$

<u>Then</u>: $\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle = \frac{1}{2\mu_o}\Re e\left(\tilde{\vec{E}}_{m,n}(x,y,z,t)\times\tilde{\vec{B}}_{m,n}^*(x,y,z,t)\right)}$

$$\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle = \frac{\pi^2\omega k_{z_{m,n}}B_o^2}{2\mu_o\left[(\omega/c)^2 - k_{z_{mn}}^2\right]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]}$$

<u>Note that</u>: $\boxed{\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle = \left\langle\left|\vec{S}_{m,n}(x,y,z,t)\right|\right\rangle\hat{z}}$ $\Leftarrow$ points in $+\hat{z}$ direction, as it should!!

<u>Then</u>: $\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \int_{A_\perp}\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle\bullet d\vec{a}_\perp = \int_{y=o}^{y=b}\int_{x=o}^{x=a}\left\langle\vec{S}_{m,n}(x,y,z,t)\right\rangle\bullet\hat{n}dxdy}$ $\boxed{d\vec{a}_\perp = \hat{n}\ dxdy = \hat{z}\ dxdy}$

$$= \frac{\pi^2\omega k_{z_{m,n}}B_o^2}{2\mu_o\left[(\omega/c)^2 - k_{z_{mn}}^2\right]^2}\left[\left(\frac{n}{b}\right)^2\int_{y=o}^{y=b}\int_{x=o}^{x=a}\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dxdy + \left(\frac{m}{a}\right)^2\int_{y=o}^{y=b}\int_{x=o}^{x=a}\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)dxdy\right]$$

<u>But</u>: $\boxed{\int_o^a\sin^2\left(\frac{m\pi x}{a}\right)dx = \int_o^a\cos^2\left(\frac{m\pi x}{a}\right)dx = \left(\frac{a}{2}\right)}$ and: $\boxed{\int_o^b\sin^2\left(\frac{n\pi y}{b}\right)dy = \int_o^b\cos^2\left(\frac{n\pi x}{b}\right)dx = \left(\frac{b}{2}\right)}$

$\therefore \boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \frac{\pi^2\omega k_{z_{m,n}}B_o^2 ab}{8\mu_o\left[(\omega/c)^2 - k_{z_{mn}}^2\right]^2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$ (Watts) with: $\boxed{\begin{array}{l} m = 0,1,2,\ldots\ (m,n \text{ not both} = 0 \\ n\ = 0,1,2,\ldots\ \text{ simutaneously!)}\end{array}}$

<u>But</u>: $\boxed{k_z(\omega) = k_{z_{m,n}}(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} = \left(\frac{\omega}{c}\right)\sqrt{1 - \left(\frac{c}{\omega}\right)^2\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}$

<u>Now</u>: $\boxed{\left(\frac{\omega}{c}\right) = k_o = \frac{2\pi}{\lambda_o}}$ <u>where</u>: $k_o =$ <u>vacuum</u> wavenumber $\boxed{k_{x_m} = \left(\frac{m\pi}{a}\right)}$, $\boxed{k_{y_n} = \left(\frac{n\pi}{b}\right)}$

$\Rightarrow$ $\boxed{\lambda_o = 2\pi\left(\frac{c}{\omega}\right)}$ <u>where</u>: $\lambda_o =$ <u>vacuum</u> wavelength

Thus:

$$\boxed{\left(\frac{c}{\omega}\right)^2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]=\left[2\pi\left(\frac{c}{\omega}\right)\right]^2\left[\left(\frac{m}{2a}\right)^2+\left(\frac{n}{2b}\right)^2\right]=\lambda_o^2\left[\left(\frac{m}{2a}\right)^2+\left(\frac{n}{2b}\right)^2\right]=\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}$$

Thus: $$\boxed{k_{z_{m,n}}(\omega)=\sqrt{k_o^2-k_{x_m}^2-k_{y_n}^2}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}=k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}}$$

$\therefore$ $$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{\omega B_o^2 ab}{8\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}}$$

or: $$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{1}{2\mu_o\omega}\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]k_o\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2b}\right)^2\right]}}$$

But: $$\boxed{\int_{A_\perp}\left\langle\left|\tilde{\vec{E}}_x(x,y,z,t)\right|^2\right\rangle dxdy=\frac{\omega^2 k_{y_n}^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}=\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left(\frac{n\pi}{b}\right)^2}$$ with: $\boxed{k_{y_n}\equiv\left(\frac{n\pi}{b}\right)}$

$$\boxed{\int_{A_\perp}\left\langle\left|\tilde{\vec{E}}_y(x,y,z,t)\right|^2\right\rangle dxdy=\frac{\omega^2 k_{x_m}^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}=\frac{\omega^2 B_o^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left(\frac{m\pi}{a}\right)^2}$$ with: $\boxed{k_{x_m}\equiv\left(\frac{m\pi}{a}\right)}$

Defining:

$$\left.\begin{aligned}&\boxed{\left|\tilde{E}_{o_x}^{m,n}\right|\equiv\frac{\omega k_{y_n}B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}=\frac{\omega B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{n\pi}{b}\right)}\\&\boxed{\left|\tilde{E}_{o_y}^{m,n}\right|\equiv\frac{\omega k_{x_m}B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}=\frac{\omega B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\left(\frac{m\pi}{a}\right)}\end{aligned}\right\}$$

Magnitudes of $\hat{x},\hat{y}$ electric field amplitudes

Then: $$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{k_o}{2\mu_o\omega}\left(\left|\tilde{E}_{o_x}^{mn}\right|^2+\left|\tilde{E}_{o_y}^{mn}\right|^2\right)\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\sqrt{1-\left[\left(\frac{m\lambda_o}{2a}\right)^2+\left(\frac{n\lambda_o}{2a}\right)^2\right]}}$$

But: $\boxed{\frac{k_o}{\omega}=\frac{1}{c}}\Rightarrow\boxed{\frac{k_o}{\mu_o\omega}=\frac{1}{\mu_o}\left(\frac{k_o}{\omega}\right)=\frac{1}{\mu_o c}}$ but: $\boxed{c=\sqrt{\frac{1}{\varepsilon_o\mu_o}}}$ $\therefore$ $\boxed{\frac{k_o}{\mu_o\omega}=\frac{1}{\mu_o c}=\frac{\sqrt{\varepsilon_o\mu_o}}{\mu_o}=\sqrt{\frac{\varepsilon_o}{\mu_o}}}$

$\therefore$ $$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle=\frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2+\frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)ab\sqrt{1-\left[\left(\frac{m\lambda_o}{a}\right)^2+\left(\frac{n\lambda_o}{b}\right)^2\right]}}$$

$\lambda_o\equiv$ vacuum wave length $=c/f$

$\Rightarrow$ The time-averaged power transported down the hollow rectangular waveguide for the $\text{TE}_{mn}{}^{th}$ mode is proportional to the square of the *E*-field amplitudes in the $\hat{x}$ and $\hat{y}$ direction!!

We introduce the parameter $\boxed{Z_o(\vec{r}) \equiv \left|\tilde{\vec{E}}_\perp(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_\perp(\vec{r})\big/\mu_o\right| = \mu_o c = \sqrt{\mu_o/\varepsilon_o} = 120\pi\ \Omega \simeq 377\Omega}$

= the *EM* wave impedance of *free space* (*n.b.* Z_o is a purely real, scalar quantity because there is no dissipation!). For TE$_{mn}$ modes of *EM* wave propagation in a waveguide that has perfectly conducting walls (*i.e.* no dissipation/no losses); the *EM* wave impedance of the waveguide is also purely real:

$$\boxed{Z_{TE}^{m,n}(\omega) \equiv \left|\tilde{\vec{E}}_{TE}^{\perp}(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_{TE}^{\perp}(\vec{r})\big/\mu_o\right| = Z_o\left(\lambda_z^{m,n}(\omega)\big/\lambda_o\right) = Z_o\left(k_o\big/k_{z_{m,n}}(\omega)\right)}.$$

Then since:

$$\boxed{k_{z_{m,n}}(\omega) = \frac{2\pi}{\lambda_z^{m,n}(\omega)} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = k_o\sqrt{1 - \left[\left(\frac{m\lambda_o}{2a}\right)^2 + \left(\frac{n\lambda_o}{2b}\right)^2\right]} = \sqrt{k_o^2 - k_{x_m}^2 - k_{y_n}^2} < k_o}$$

or equivalently $\boxed{\lambda_z^{m,n}(\omega) > \lambda_o}$ we see that: $\boxed{Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)\big/\lambda_o\right) = Z_o\left(k_o\big/k_{z_{m,n}}(\omega)\right) > 377\Omega}$ for TE$_{m,n}$ modes of *EM* wave propagation in a waveguide.

We can thus write the *EM* power transmitted down the waveguide for TE$_{mn}$ modes as:

$$\boxed{\left\langle P_{m,n}^{trans}(z,t)\right\rangle = \frac{1}{2Z_o}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2 + \frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)A_\perp\left(\frac{\lambda_o}{\lambda_z^{m,n}(\omega)}\right) = \frac{1}{2}\left(\frac{1}{4}\left|\tilde{E}_{o_x}^{m,n}\right|^2 + \frac{1}{4}\left|\tilde{E}_{o_y}^{m,n}\right|^2\right)A_\perp \Big/ Z_{TE}^{m,n}(\omega)}$$

Where $A_\perp = ab$ = cross-sectional area of the rectangular waveguide.
Note that this expression is analogous to $\langle P\rangle = \frac{1}{2}V_{peak}^2\big/R$, since $E^2 A_\perp \sim (Volts/m)^2 * m^2 = Volts^2$.

The Energy Density $\langle u_{m,n}\rangle$ Stored in a Rectangular Waveguide - TE$_{m,n}$ Mode

Again, from Griffiths Problem 9.11 (*p*. 382) since $\boxed{\langle fg\rangle = \tfrac{1}{2}\mathrm{Re}\left(\tilde{f}\,\tilde{g}^*\right)}$

Then: $\boxed{\langle u\rangle = \frac{1}{4}\Re e\left\{\varepsilon_o \tilde{\vec{E}}\bullet\tilde{\vec{E}}^* + \frac{1}{\mu_o}\tilde{\vec{B}}\bullet\tilde{\vec{B}}^*\right\}}$

Then in the (m,n^{th}) TE mode: $\boxed{\langle u_{m,n}\rangle = \frac{1}{4}\Re e\left\{\varepsilon_o \tilde{\vec{E}}_{m,n}\bullet\tilde{\vec{E}}^*_{m,n} + \frac{1}{\mu_o}\tilde{B}_{m,n}\bullet\tilde{\vec{B}}^*_{m,n}\right\}}$

Where: $\boxed{\tilde{\vec{E}}_{m,n} = \tilde{E}_{x_{m,n}}\hat{x} + \tilde{E}_{y_{m,n}}\hat{y} + \overset{\text{0 for TE modes}}{\cancel{\tilde{E}_{z_{m,n}}}}\hat{z}}$ *n.b.* $\boxed{|A|^2 \equiv A\cdot A^*}$

And: $\boxed{\tilde{\vec{B}}_{m,n} = \tilde{B}_{x_{m,n}}\hat{x} + \tilde{B}_{y_{m,n}}\hat{y} + \tilde{B}_{z_{m,n}}\hat{z}}$

Then: $\boxed{\langle u_{m,n}\rangle = \frac{1}{4}\mathrm{Re}\left\{\varepsilon_o\left[\left|E_{x_{m,n}}\right|^2 + \left|E_{y_{m,n}}\right|^2 + \overset{\text{0 for TE modes}}{\cancel{\left|E_{z_{m,n}}\right|^2}}\right] + \frac{1}{\mu_o}\left[\left|B_{x_{m,n}}\right|^2 + \left|B_{y_{m,n}}\right|^2 + \left|B_{z_{m,n}}\right|^2\right]\right\}}$

$$\boxed{\begin{aligned}\langle u_{m,n}\rangle = \frac{\varepsilon_o}{4}&\left(\frac{\omega\pi B_o}{\left[(\omega/c)^2 - k^2_{z_{mn}}\right]}\right)^2\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\ &+\frac{1}{4\mu_o}\left\{\left(\frac{k_{m,n}\pi B_o}{\left[(\omega/c)^2 - k^2_{z_{mn}}\right]}\right)^2\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right.\\ &\qquad\left.+B_o^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right\}\end{aligned}}$$

The time-averaged energy <u>per unit length</u> (*Joules*/*m*) in the waveguide for the (m,n^{th}) TE mode is:

$\boxed{\langle E_{m,n}\rangle / L \equiv \int_{A_\perp}\langle u_{m,n}\rangle da_\perp = \int_{y=o}^{y=b}\int_{x=o}^{x=a}\langle u_{m,n}\rangle\, dxdy}$ where L (meters) is the length of the waveguide.

$$\boxed{\begin{aligned}\langle E_{m,n}\rangle/L \equiv \int_{A_\perp}\langle u_{m,n}\rangle da_\perp = \frac{\varepsilon_o}{4}&\left(\frac{\pi\omega B_o}{\left[(\omega/c)^2 - k^2_{m,n}\right]}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right]\\ &+\frac{1}{4\mu_o}\left\{\left(\frac{\pi k_{m,n}B_o}{\left[(\omega/c)^2 - k^2_{z_{mn}}\right]}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)B_o^2\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\langle E_{m,n}\rangle/L \equiv \int_{A_\perp}\langle u_{m,n}\rangle da_\perp = \frac{\varepsilon_o}{4}&\left(\frac{\pi\omega B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\right)^2\left(\frac{ab}{4}\right)\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]\\ &+\frac{1}{4\mu_o}\left\{\left(\frac{\pi k_{m,n}B_o}{\left[(\omega/c)^2-k_{z_{mn}}^2\right]}\right)^2\left(\frac{ab}{4}\right)\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\left(\frac{ab}{4}\right)B_o^2\right\}\end{aligned}}$$

Now: $\boxed{\langle P_{m,n}^{trans}(z,t)\rangle = \int_{A_\perp}\langle \vec{S}_{m,n}\rangle\bullet d\vec{a}_\perp = \frac{\omega k_{m,n}B_o^2 ab}{8\mu_o\left[(\omega/c)^2-k_{z_{mn}}^2\right]^2}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}$

and: $\boxed{k_{z_{mn}}^2 = \left(\frac{\omega}{c}\right)^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]} \Rightarrow \boxed{\left[\left(\frac{\omega}{c}\right)^2-k_{z_{mn}}^2\right]=\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\equiv\left(\frac{\omega_{m,n}^2}{c^2}\right)}$

Thus: $\boxed{\langle P_{m,n}^{trans}\rangle = \int_{A_\perp}\langle \vec{S}_{m,n}\rangle\bullet d\vec{a}_\perp = \frac{\omega k_{z_{mn}}ab}{8\mu_o\omega_{mn}^2}c^2B_o^2}$

And: $\boxed{\frac{\langle E_{m,n}\rangle}{L} = \int_{A_\perp}\langle u_{m,n}\rangle da_\perp = \frac{\omega^2 ab}{8\mu_o\omega_{mn}^2}B_o^2}$

Note that the ratio of: $\boxed{\frac{\langle P_{m,n}^{trans}\rangle}{\langle E_{m,n}\rangle/L} = \frac{Watts}{Joules/m} = \frac{Joules/sec}{Joules/m} = \frac{m}{sec}}$ (*i.e.* speed)

$$\boxed{\frac{\langle P_{m,n}^{trans}\rangle}{\langle E_{m,n}\rangle/L} = \frac{\frac{\omega k_{z_{mn}}ab}{8\mu_o\omega_{mn}^2}c^2B_o^2}{\frac{\omega^2 ab}{8\mu_o\omega_{mn}^2}B_o^2} = \frac{k_{m,n}c^2}{\omega} = \left(\frac{c}{\omega}\right)(k_{m,n}c)}$$

But: $\boxed{k_{z_{mn}}^2c^2 = \omega^2-\omega_{mn}^2}$ or: $\boxed{k_{z_{mn}}c = \sqrt{\omega^2-\omega_{mn}^2}}$

$$\therefore \boxed{\frac{\langle P_{m,n}^{trans}\rangle}{\langle E_{m,n}\rangle/L} = \left(\frac{c}{\omega}\right)\sqrt{\omega^2-\omega_{mn}^2} = c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} = v_{g_z}^{mn}(\omega)}\ !!!$$

or: $\boxed{v_{g_z}^{mn}(\omega) = c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} = \frac{\langle P_{m,n}^{trans}\rangle}{\langle E_{m,n}\rangle/L}}$ Thus we see that: $\boxed{\langle P_{m,n}^{trans}\rangle = v_{g_z}^{m,n}(\omega)\cdot\langle E_{m,n}\rangle/L}$

Propagation of TM Waves in a Hollow Rectangular Waveguide

For propagation TM waves in a hollow waveguide, $\tilde{E}_z \neq 0$, but $\underline{\underline{\tilde{B}_z = 0}}$.

$\therefore$ We need to solve 3-D wave equation: $\boxed{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k_z^2\right] E_z = 0}$

subject to boundary conditions on the inner walls of rectangular waveguide: $\boxed{\tilde{E}_{\parallel} = 0 \text{ and } \tilde{B}_{\perp} = 0}$

Following the procedure we developed for the TE mode case, let: $\boxed{\tilde{\tilde{E}}_z(x,y) = \tilde{X}(x)\tilde{Y}(y)}$

$$\boxed{\tilde{X}(x) = \tilde{A}_x \cos k_x x + \tilde{B}_x \sin k_x x = 0 \ \{\text{at } x = 0 \text{ and } x = a\} \Rightarrow \tilde{A}_x = 0} \Rightarrow \boxed{\tilde{X}(x) = \tilde{B}_x \sin k_x x}$$
$$\boxed{\tilde{Y}(y) = \tilde{A}_y \cos k_y y + \tilde{B}_y \sin k_y y = 0 \ \{\text{at } y = 0 \text{ and } y = b\} \Rightarrow \tilde{A}_y = 0} \Rightarrow \boxed{\tilde{Y}(y) = \tilde{B}_y \sin k_y y}$$

Because $m, n = 0, 1, 2, 3\ldots$ the lowest *non-trivial* TM$_{mn}$ mode is TM$_{11}$

$$\boxed{\begin{cases} k_x \equiv \left(\frac{m\pi}{a}\right), \ m = 1,2,3,\ldots & n.b.\ m = 0 \text{ is NOT allowed here!!! } (\Rightarrow X(x) \equiv 0 \text{ everywhere!!!}) \\ k_y \equiv \left(\frac{n\pi}{a}\right), \ n = 1,2,3,\ldots & n.b.\ n = 0 \text{ is NOT allowed here!!! } (\Rightarrow Y(y) \equiv 0 \text{ everywhere!!!}) \end{cases}}$$

Then: $\boxed{\tilde{\tilde{E}}_z(x,y) = \tilde{E}_o \sin(k_x x)\sin(k_y y) = \tilde{E}_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)}$ with $\boxed{m, n = 1, 2, 3, \ldots}$

All the rest is the {nearly} the same as for TE waves – $\left\langle P_{m,n}^{trans}\right\rangle, \left\langle u_{m,n}\right\rangle$, *etc.*

Then: $\boxed{k_{z_{mn}} = \sqrt{(\omega/c)^2 - (m\pi/a)^2 - (n\pi/b)^2}}$ (as before)

The cutoff angular frequency: $\boxed{\omega_{mn} \equiv c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}} \Rightarrow \boxed{k_{z_{mn}} = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}}$

Phase (*aka* wave) velocity: $\boxed{v_{\phi_z}^{m,n} = c\frac{1}{\sqrt{1 - (\omega_{m,n}/\omega)^2}}}$ Group velocity: $\boxed{v_{g_z}^{m,n}(\omega) = c\sqrt{1 - (\omega_{m,n}/\omega)^2}}$

One *difference* for TM modes *vs*. TE modes is that the wave impedance is:

$$\boxed{Z_{TM}^{m,n}(\omega) = Z_o\left(\lambda_o/\lambda_z^{m,n}(\omega)\right) = Z_o\left(k_{z_{mn}}(\omega)/k_o\right)} \ vs. \ \boxed{Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) = Z_o\left(k_o/k_{z_{mn}}(\omega)\right)}.$$

Since $\lambda_z^{mn}(\omega) > \lambda_o$ and $Z_o \equiv \sqrt{\mu_o/\varepsilon_o} = 120\pi\,\Omega \simeq 377\Omega$ then we see that:

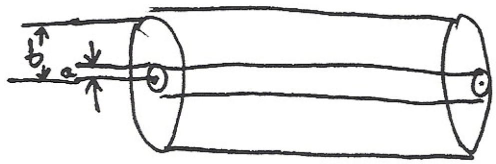
$$\boxed{Z_{TM}^{m,n}(\omega) = Z_o\left(\lambda_o/\lambda_z^{m,n}(\omega)\right) < 377\Omega} \ \underline{\text{whereas}}: \ \boxed{Z_{TE}^{m,n}(\omega) = Z_o\left(\lambda_z^{m,n}(\omega)/\lambda_o\right) > 377\Omega}$$

The ratio of the lowest TM mode to the lowest TE mode is:

$$\boxed{\left(\omega_{m,n}^{TM}/\omega_{m,n}^{TE}\right) = \left(\omega_{11}^{TM}/\omega_{10}^{TE}\right) = \sqrt{(1/a)^2 + (1/b)^2}\Big/\sqrt{(1/a)^2} = \sqrt{1 + (a/b)^2}}$$

Propagation of TEM Waves in a Coaxial Transmission Line

- We have previously shown that a hollow waveguide ***cannot*** support TEM waves $\{\tilde{E}_z = \tilde{B}_z = 0\}$
- However, a coaxial transmission line, consisting of an inner, long straight wire of radius *a*, surrounded by a cylindrical conducting sheath of radius $b > a$ ***does*** support the propagation of TEM waves:



For TEM waves: $k = \omega/c$. TEM waves travel at speed of light $c \Rightarrow$ non-dispersive!

For TEM waves, Maxwell's equations give:

(1) Gauss' Law: $\vec{\nabla}\cdot\tilde{\vec{E}} = 0$

$$\frac{\partial \tilde{E}_{ox}}{\partial x} + \frac{\partial \tilde{E}_{oy}}{\partial_y} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{E}_{ox}}{\partial x} = -\frac{\partial \tilde{E}_{oy}}{\partial y}}$$

(2) No monopoles: $\vec{\nabla}\cdot\tilde{\vec{B}} = 0$

$$\frac{\partial \tilde{B}_{ox}}{\partial x} + \frac{\partial \tilde{B}_{oy}}{\partial y} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{B}_{ox}}{\partial x} = -\frac{\partial \tilde{B}_{oy}}{\partial y}}$$

(3) Faraday's Law: $\left(\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial \tilde{\vec{B}}}{\partial t}\right)$

(4) Ampere's Law: $\left(\vec{\nabla}\times\tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial \tilde{\vec{E}}}{\partial t}\right)$

(i) $\boxed{\frac{\partial \tilde{E}_{oy}}{\partial x} - \frac{\partial \tilde{E}_{ox}}{\partial y} = i\omega \overbrace{\cancel{\tilde{B}_{oz}}}^{=0} = 0}$

(ii) $\boxed{\overbrace{\cancel{\frac{\partial \tilde{E}_{oz}}{\partial y}}}^{=0} - ik\tilde{E}_{oy} = i\omega \tilde{B}_{ox}}$

(iii) $\boxed{ik\tilde{E}_{ox} - \overbrace{\cancel{\frac{\partial \tilde{E}_{oz}}{\partial x}}}^{=0} = i\omega \tilde{B}_{oy}}$

(iv) $\boxed{\frac{\partial \tilde{B}_{oy}}{\partial x} - \frac{\partial \tilde{B}_{ox}}{\partial y} = -\frac{i\omega}{c^2} \overbrace{\cancel{\tilde{E}_{oz}}}^{=0} = 0}$

(v) $\boxed{\overbrace{\cancel{\frac{\partial \tilde{B}_{oz}}{\partial y}}}^{=0} = -\frac{i\omega}{c^2}\tilde{E}_{ox}}$

(vi) $\boxed{ikB_{ox} - \overbrace{\cancel{\frac{\partial \tilde{B}_{oz}}{\partial x}}}^{=0} = -\frac{i\omega}{c^2}\tilde{E}_{oy}}$

which can be rewritten:

(i) $\boxed{\frac{\partial \tilde{E}_{oy}}{\partial x} = \frac{\partial \tilde{E}_{ox}}{\partial y}}$

(ii) $\boxed{\tilde{B}_{ox} = -\frac{k}{\omega}\tilde{E}_{oy} = -\frac{1}{c}\tilde{E}_{oy}}$

(iii) $\boxed{\tilde{B}_{oy} = \frac{k}{\omega}\tilde{E}_{ox} = +\frac{1}{c}\tilde{E}_{ox}}$

(iv) $\boxed{\frac{\partial \tilde{B}_{oy}}{\partial x} = \frac{\partial \tilde{B}_{ox}}{\partial y}}$

(v) $\boxed{\tilde{B}_{oy} = \frac{\omega}{c^2 k}\tilde{E}_{ox} = \frac{1}{c}\tilde{E}_{ox}}$

(vi) $\boxed{\tilde{B}_{ox} = -\frac{\omega}{c^2 k}\tilde{E}_{oy} = -\frac{1}{c}\tilde{E}_{oy}}$

Note that equations (iii) and (v) above give the same relation $\boxed{\tilde{B}_{oy} = \frac{1}{c}\tilde{E}_{ox}}$

as do equations (ii) and (vi), $\boxed{\tilde{B}_{ox} = -\frac{1}{c}\tilde{E}_{oy}}$.

The following six relations:

$$\boxed{\tilde{B}_{oy} = \frac{1}{c}\tilde{E}_{ox}} \qquad \boxed{\tilde{B}_{ox} = -\frac{1}{c}\tilde{E}_{oy}}$$

$$\boxed{\frac{\partial \tilde{E}_{ox}}{\partial y} = \frac{\partial \tilde{E}_{oy}}{\partial x}} \qquad \boxed{\frac{\partial \tilde{B}_{ox}}{\partial y} = \frac{\partial \tilde{B}_{oy}}{\partial x}}$$

$$\boxed{\frac{\partial \tilde{E}_{ox}}{\partial x} = -\frac{\partial \tilde{E}_{oy}}{\partial y}} \qquad \boxed{\frac{\partial \tilde{B}_{ox}}{\partial x} = -\frac{\partial \tilde{B}_{oy}}{\partial y}}$$

Are precisely the same equations of <u>electrostatics</u> and <u>magnetostatics</u> for empty space (*i.e.* the <u>vacuum</u>) in <u>*two*</u> dimensions.

Since a coaxial cable has <u>*cylindrical*</u> geometry/<u>*cylindrical*</u> symmetry, the TEM electric field (as in case of the infinite line charge) <u>*must*</u> be of the form:

$$\boxed{\tilde{\vec{E}}_o(\rho,\varphi) = \frac{\tilde{A}}{\rho}\hat{\rho}} \quad \text{where } \tilde{A} = \text{constant.}$$

Similarly, the TEM magnetic field (as in the case of infinite line current) <u>*must*</u> be of the form:

$$\boxed{\tilde{\vec{B}}_o(\rho,\varphi) = \frac{\tilde{A}}{\rho c}\hat{\varphi}}$$

Then for TEM wave propagation in a coaxial transmission line:

$$\boxed{\tilde{\vec{E}}(\rho,\varphi,z,t) = \tilde{\vec{E}}_o(\rho,\varphi)e^{i(kz-\omega t)} = \frac{\tilde{A}}{\rho}e^{i(kz-\omega t)}\hat{\rho}}$$

$$\boxed{\tilde{\vec{B}}(\rho,\varphi,z,t) = \tilde{B}_o(\rho,\varphi)e^{i(kz-\omega t)} = \frac{\tilde{A}}{\rho c}e^{i(kz-\omega t)}\hat{\varphi}}$$

$$\boxed{k = \frac{\omega}{c}} \qquad \boxed{v_{group} = v_{phase} = c} \qquad \Rightarrow \ \underline{\textbf{\textit{no}}} \text{ dispersion!}$$

Note that there are <u>***no***</u> restrictions on the value of k for TEM waves in a coaxial cable.

For TEM *EM* wave propagation in a coaxial transmission line that has <u>perfectly</u> <u>conducting</u> <u>walls</u> (*i.e.* <u>no</u> dissipation/<u>no</u> losses), the *EM* wave impedance is (again) purely real:

$$\boxed{Z_{TEM}^{coax}(\omega) \equiv \left|\tilde{\vec{E}}_{TEM}^{\perp}(\vec{r})\right| \Big/ \left|\tilde{\vec{B}}_{TEM}^{\perp}(\vec{r})\big/\mu_o\right| = \sqrt{\mu_o/\varepsilon_o} = Z_o = 120\pi\,\Omega = 377\Omega}.$$