3. Einstein's two postulates

After Michelson and Morley's failure to detect our velocity wrt the aether, Einstein followed a rather different alternative.

He maintained the

1. Principle of Special Relativity

- The law of physics are the same in all inertial frames

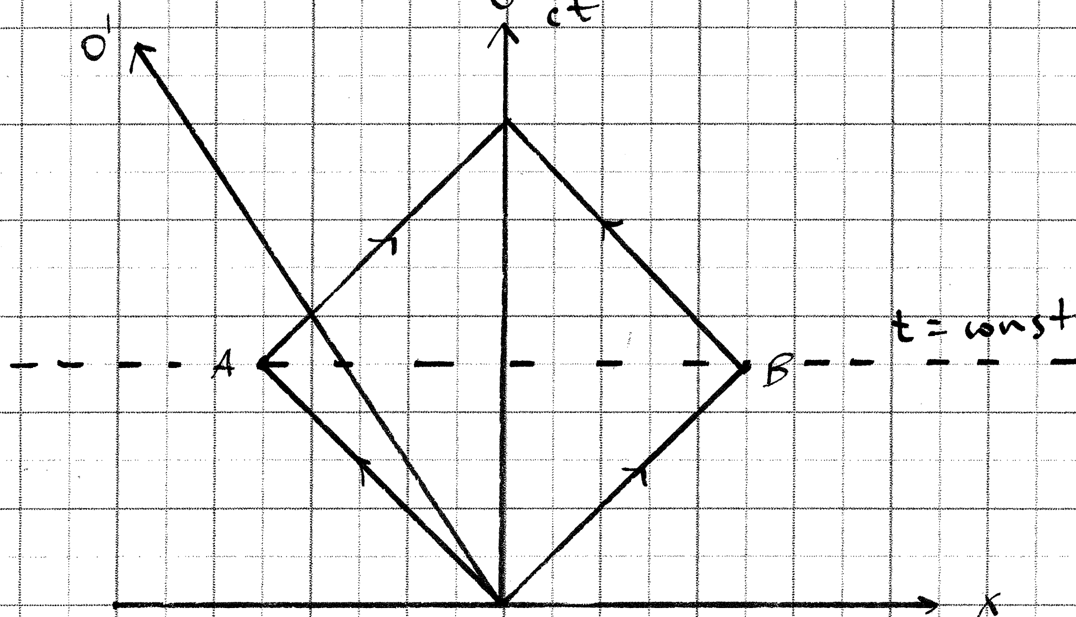
and also kept the predictions of Maxwell's eqs.:

2. Principle of the constancy of the speed of light

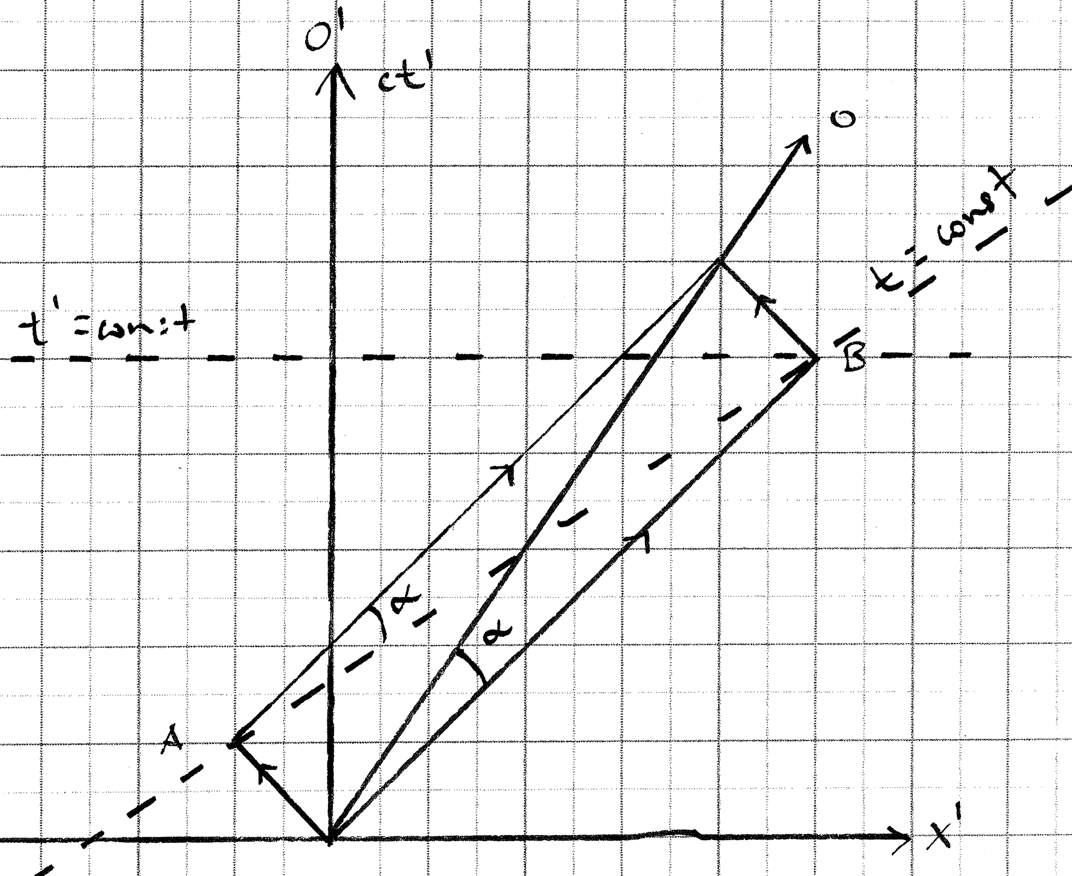
- The speed of light is a constant c, no matter what the state of motion of the source is.

Several striking consequences follow from the 2 principles:

- **Relativity of simultaneity:**



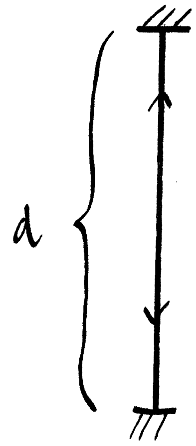
Events A & B are simultaneous for O



Events A & B are not simultaneous in O'

• Time between events is frame-dependent:

Consider a moving "light clock"

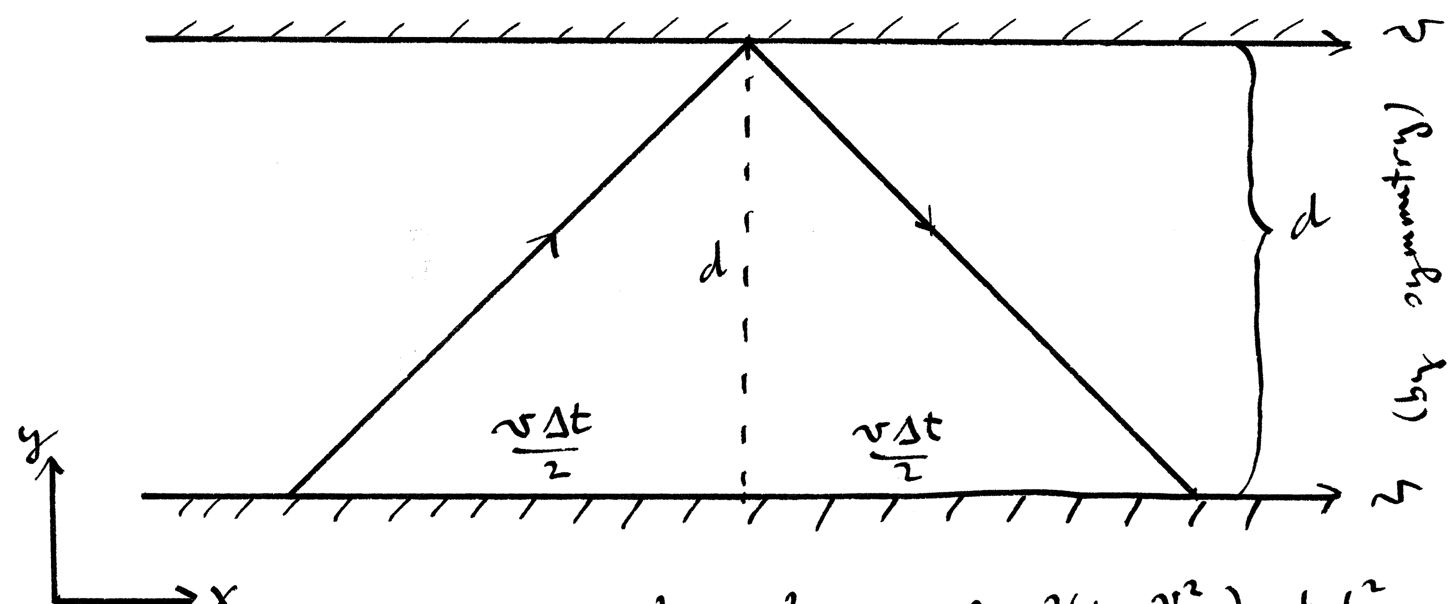


In a frame in which the clock is at rest, the clocks "ticks" every

$$\Delta \tau = \frac{2d}{c}$$

• In a frame in which the clock is moving at constant speed v (along the x direction), the clock ticks every Δt, where

$$c\,\Delta t = 2 \cdot \sqrt{\frac{v^2 \Delta t^2}{4} + d^2}$$



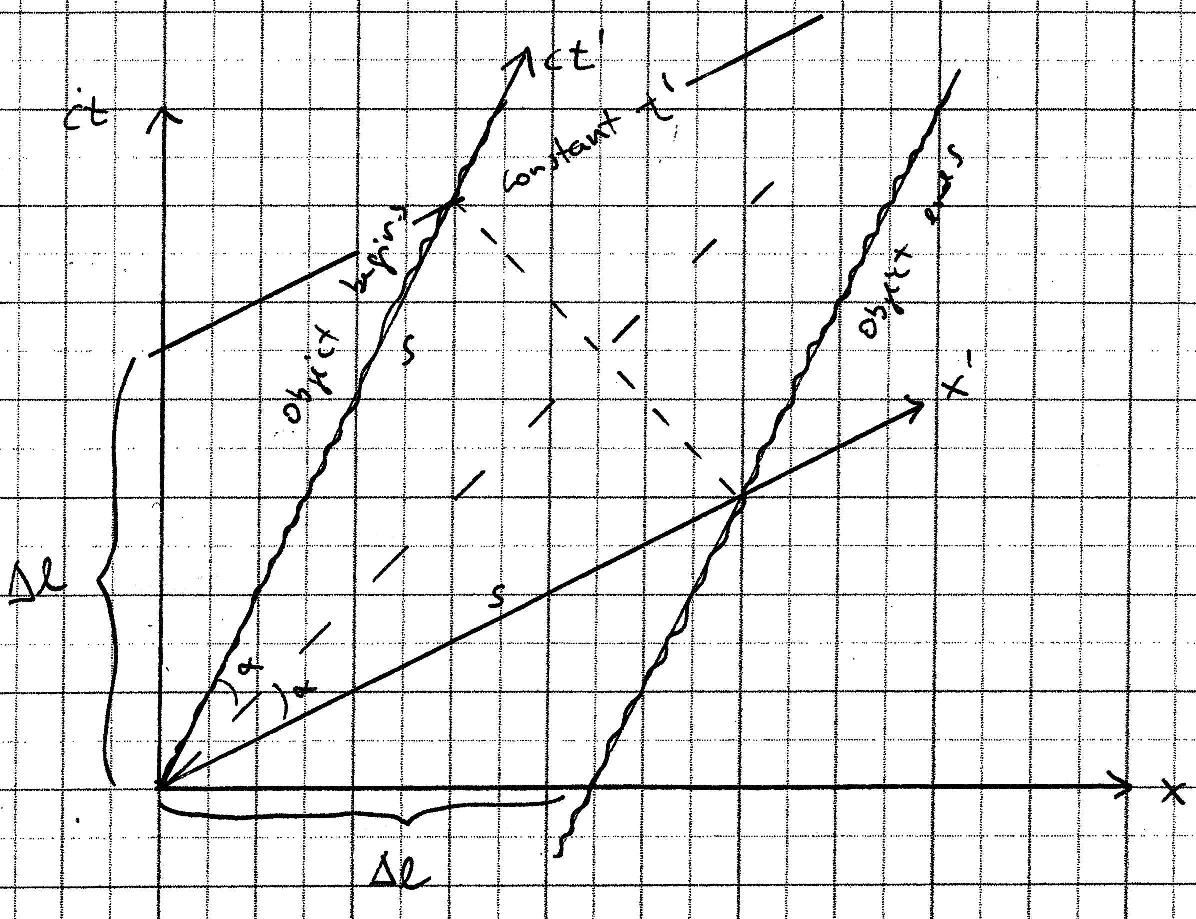
$$c^2 \Delta t^2 = v^2 \Delta t^2 + 4d^2, \quad c^2 \Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = 4d^2$$

It follows that $\boxed{\Delta t = \dfrac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}}$

Example: Muons

Time dilation: Moving clocks tick slower.

- The length of an object is frame dependent:



From time dilation above:

$$\frac{s}{c} = \frac{\Delta l / c}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \boxed{\Delta l = s \cdot \sqrt{1 - \frac{v^2}{c^2}}}$$

Length contraction: Moving objects appear to be shorter.

4. Lorentz transformations

As we saw, galilean transformations do not preserve Postulate 2 (constant c), so we have to find the appropriate set of coordinate transformations between inertial frames. Look for transformations that

i) Preserve the speed of light – postulate 2.

ii) Are linear in t and $\vec{x}$ –

no preferred point in space or time

iii) Map an object at constant spatial coords. to an object moving at constant speed v – inertial frames move at constant speed wrt each other

It can be shown that the corresponding coordinate transformations include a <u>Lorentz transformation</u>

$$(T)\quad \begin{cases} ct' = \gamma(ct - \vec{\beta}\cdot\vec{x}) \\ \vec{x}' = \vec{x} + \frac{\gamma-1}{\beta^2}(\vec{\beta}\cdot\vec{x})\vec{\beta} - \gamma\vec{\beta}ct \end{cases}$$

where $\vec{\beta} = \frac{\vec{v}}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta^2 = \frac{\vec{v}^2}{c^2}$.

For motion along the x-axis, $\vec{v} = v\,\vec{e}_x$,

this simplifies to

$$(\tilde{T}) \quad \begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(-\beta ct + x) \\ y' = y \\ z' = z \end{cases}$$

Note that in the limit of non-relativistic velocities ($c \to \infty$), the Lorentz transformation reduces to a Galilean transformation.

<u>Exercise 17</u>

Show that (T) satisfies properties i) – iii) above

Inspection of these equations shows that they simplify if we replace $ct \to t$ (i.e. if we measure time in units of the distance travelled by light during that time.) Effectively this amounts to working in units with <u>$c = 1$</u> , as in what follows.

Exercise 38

Derive time-dilation and length contraction from the Lorentz transformation ∎

5. Spacetime geometry

The Lorentz transformation $(\tilde{T})$ can be cast in a form that reveals the geometric origin of the transformation.

Set $\cosh\theta \equiv \gamma$ and $\gamma\beta \equiv \sinh\theta$.

This is possible because $\gamma^2 = \dfrac{1}{1-\beta^2}$.

In this form, $(\tilde{T})$ reads

$$(\tilde{T})\begin{cases} t' = \cosh\theta \cdot t - \sinh\theta \cdot t \\ x' = -\sinh\theta \cdot t + \cosh\theta \cdot x \\ y' = y \\ z' = z \end{cases}$$

This is reminiscent of a rotation on a plane:

$$\begin{cases} x' = \cos\varphi \, x + \sin\varphi \cdot y \\ y' = -\sin\varphi \cdot x + \cos\varphi \cdot y \\ z' = z \end{cases}$$

We can therefore interpret a Lorentz transformation as a "rotation in spacetime." When we change inertial frames we merely are contemplating spacetime from a different "orientation."

This analogy goes much further:

- On 3-dimensional (flat) space, there is a special class of coordinate systems in which the distance between two points satisfies

$$dl^2 = dx^2 + dy^2 + dz^2 \quad - \quad \text{Cartesian coordinates.}$$

The set of all rotations $R \in SO(3)$ is a subgroup of those coordinate transformations that preserve this line element: rotations and translations.

Similarly

- In (flat) 4-dimensional spacetime, there is a special class of coordinate systems in which the spacetime interval between two events is

$$(S) \qquad d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \qquad \text{— inertial frames.}$$

Lorentz transformations are a subgroup of all the transformations that preserve (S). The latter define the Poincaré group, which consists of spacetime translations, rotations and boosts (T) (note that the Poincaré group is a 10-parameter group).

Exercise 39

Show that spacetime translations, rotations and boosts preserve (S) ∎

Exercise 40 (Relativistic Addition of Velocities)

A particle moves at speed $\bar{v}'$ in inertial frame O', which moves at speed v in inertial frame O.

i) What is the speed of the particle in O?

Suppose now that $\vec{v}' \parallel \vec{v} \parallel \hat{e}_x$. The particle defines an inertial frame O''. ii) How is the rapidity θ'' of the LT between O and O' related to those belonging to the LT between O and O' and O' and O''?