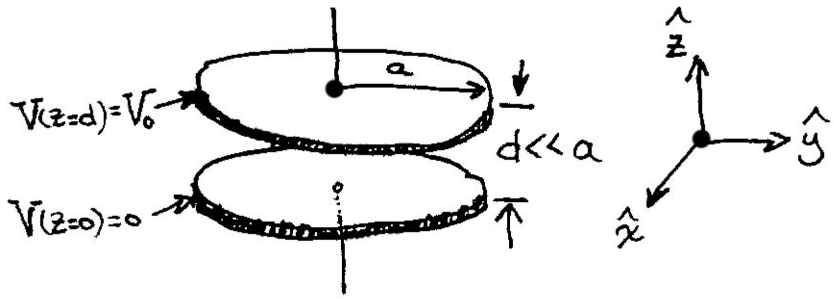
LECTURE NOTES 9

AC Electromagnetic Fields Associated with a Parallel-Plate Capacitor

Let's investigate the nature of AC electromagnetic fields associated with a parallel-plate capacitor, *e.g.* with circular plates of radius a separated by a small distance $d \ll a$ as shown in the figure below – we will neglect edge effects here:

3-D View:

$$\Delta V \equiv V(z=d) - V(z=0) = V_o$$



At DC ($f = 0$ Hz), we know the static solution to this problem, namely that the {free} charge Q_{free} on the capacitor is related to the potential difference ΔV across the capacitor's plates by: $Q_{free} = C\Delta V$ where the capacitance of the capacitor is: $C = \varepsilon_o A/d$ (Farads) for $d \ll a$; the area of one plate of the parallel plate capacitor is $A = \pi a^2$.

Since there is no free electric charge *between* the plates of the parallel plate capacitor, then for $d \ll a$, the solution to Laplace's Equation $\nabla^2 V(\vec{r}) = 0$ {derived from Gauss' Law $\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho_{free}(\vec{r})/\varepsilon_o = 0$, with $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$} yields:

$$\boxed{\Delta V \equiv V(z=d) - V(z=0) = -\int_{z=0}^{z=d} \vec{E}(\vec{r})\cdot d\vec{\ell}}$$

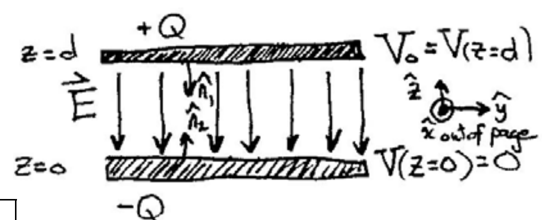
But: $\vec{E}(\vec{r}) = -E_o\hat{z}$ between the plates of the parallel plate capacitor for $d \ll a$

$$\therefore \quad \boxed{\Delta V \equiv V(z=d) - V(z=0) = (V_o - 0) = V_o = -E_o d}$$

$$\Rightarrow \quad \boxed{\vec{E}(\vec{r}) = -(V_o/d)\hat{z} = (\sigma_{free}/\varepsilon_o)\hat{n}_1}$$

where: $\boxed{\sigma_{free} = Q_{free}/A}$

Side View:



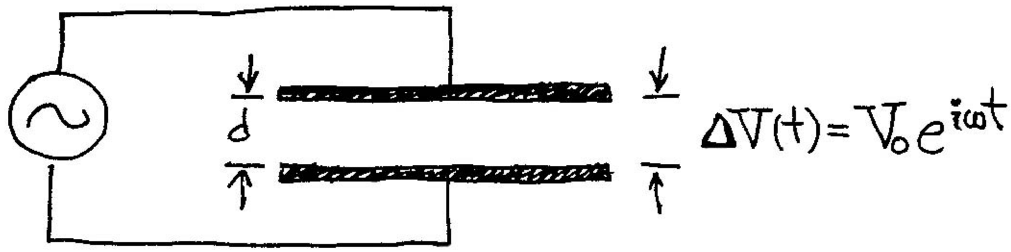
$$\Rightarrow \boxed{\sigma_{free}(z=d) = +\frac{Q_{free}}{A} = \frac{C\Delta V}{A} = \frac{\varepsilon_o \cancel{A} V_o}{\cancel{A} d} = \varepsilon_o \frac{V_o}{d} = \varepsilon_o E_o}$$

And: $$\boxed{\sigma_{free}(z=0) = -\frac{Q_{free}}{A} = -\frac{C\Delta V}{A} = -\frac{\varepsilon_o A V_o}{Ad} = -\varepsilon_o \frac{V_o}{d} = -\varepsilon_o E_o}$$

We ask:

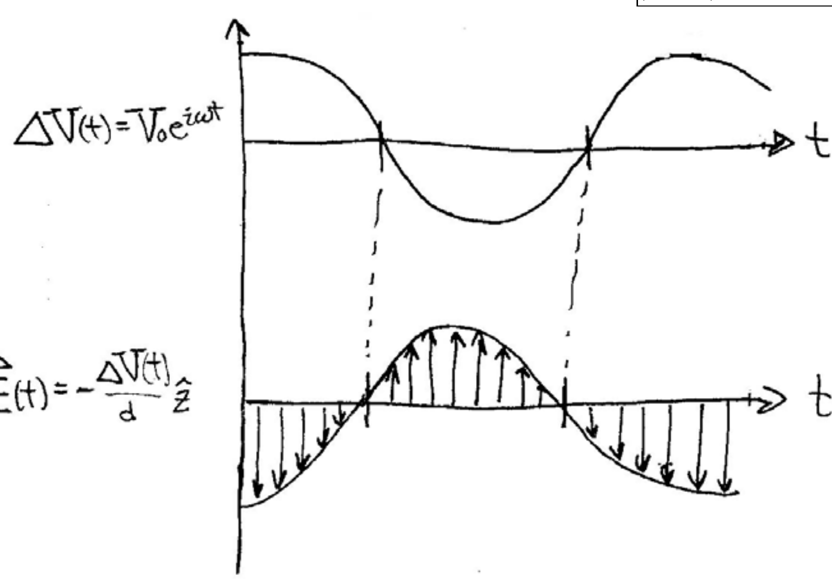
What happens when we *slowly* raise the frequency from $f = 0$ Hz (static E-field) to $f > 0$? *e.g.* Apply a sinusoidally time-varying potential difference across the plates of the capacitor of the form: $\Delta\tilde{V}(t) = V_o e^{i\omega t} = V_o[\cos\omega t + i\sin\omega t]$ $\Leftarrow$ single frequency, $f = \omega/2\pi$ *e.g.* using a sine-wave function generator , as shown in the figure below:

Function Generator



For $d \ll a$: $\boxed{\tilde{\vec{E}}(\vec{r},t) = -\frac{\Delta\hat{V}(t)}{d}\hat{z} = -\frac{V_o e^{i\omega t}}{d}\hat{z} = E_o e^{i\omega t}\hat{z}}$ with: $\boxed{E_o \equiv -V_o/d}$

The potential difference $\Delta\tilde{V}(t)$ and electric field $\tilde{\vec{E}}(t)$ *vs*. time *t*: $\boxed{\left|\tilde{\vec{E}}(t)\right| = \tilde{E}(t) = E_o e^{i\omega t}}$



Maxwell's Equations must be obeyed in the gap-region between the parallel plates of capacitor, where: $\tilde{\rho}_{free}(\vec{r},t) = \tilde{\rho}_{bound}(\vec{r},t) = 0$ and: $\tilde{\vec{J}}_{free}(\vec{r},t) = \tilde{\vec{J}}_{bound}(\vec{r},t) = 0$:

1) Coulomb's Law: $\boxed{\vec{\nabla}\cdot\tilde{\vec{E}}(\vec{r},t) = 0}$

2) No magnetic charges / monopoles: $\boxed{\vec{\nabla}\cdot\tilde{\vec{B}}(\vec{r},t) = 0}$

3) Faraday's Law: $\boxed{\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t) = -\partial\tilde{\vec{B}}(\vec{r},t)/\partial t}$ Maxwell's Displacement Current:

4) Ampere's Law: $\boxed{\vec{\nabla}\times\tilde{\vec{B}}(\vec{r},t) = \mu_o\varepsilon_o\,\partial\tilde{\vec{E}}(\vec{r},t)/\partial t = \mu_o\tilde{\vec{J}}_D(\vec{r},t)}$ where: $\boxed{\tilde{\vec{J}}_D(\vec{r},t) \equiv \varepsilon_o\,\partial\tilde{\vec{E}}(\vec{r},t)/\partial t}$

Ampere's Law (with Maxwell's Displacement Current) in integral form tells us that:

$$\boxed{\int_S \left(\vec{\nabla}\times\tilde{\vec{B}}(\vec{r},t)\right)\cdot d\vec{a} = \int_S \mu_o \tilde{\vec{J}}_D(\vec{r},t)\cdot d\vec{a}} \Leftarrow \textit{n.b.}\ \underline{\text{not}}\ \text{a}\ \underline{\textit{closed}}\ \text{surface!}$$

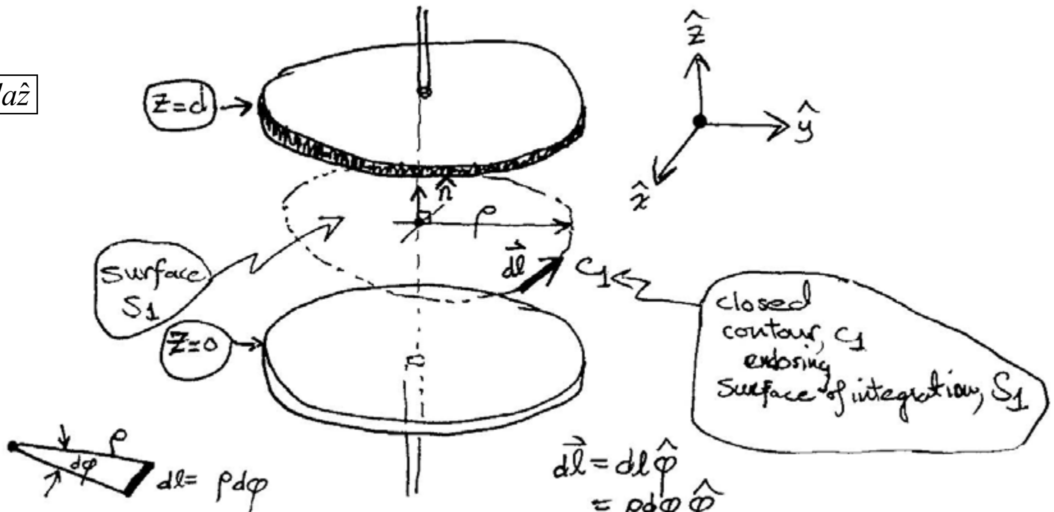
$$\boxed{\oint_C \tilde{\vec{B}}(\vec{r},t)\cdot d\vec{\ell} = \mu_o \int_S \tilde{\vec{J}}_D(\vec{r},t)\cdot d\vec{a} = \mu_o\varepsilon_o \int_S \partial\tilde{\vec{E}}(\vec{r},t)/\partial t\cdot d\vec{a}} \quad \boxed{\textbf{Using Stokes' Theorem}}$$

$$\therefore \boxed{\oint_C \tilde{\vec{B}}(\vec{r},t)\cdot d\vec{\ell} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_S \partial\tilde{\vec{E}}(\vec{r},t)/\partial t\right)\cdot d\vec{a}} \quad \text{where: } \boxed{c^2 = 1/\varepsilon_o\mu_o}$$

Let us consider a contour path of integration C_1 enclosing the surface S_1 as shown in the figure below:

$\boxed{d\vec{a} = da\hat{n} = +da\hat{z}}$

$\boxed{da = \rho\, d\rho d\varphi}$
(in cylindrical coordinates)



$\boxed{\tilde{\vec{E}}(\vec{r},t) = E_o e^{i\omega t}\hat{z}}$ and: $\boxed{\oint_{C_1} \tilde{\vec{B}}(\vec{r},t)\cdot d\vec{\ell} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_{S_1} \partial\tilde{\vec{E}}(\vec{r},t)/\partial t\right)\cdot d\vec{a}}$

<u>Note that</u>: $\tilde{\vec{B}}(\vec{r},t) = \tilde{B}(\rho,t)\hat{\varphi}$ due to the circular/azimuthal symmetry associated with this problem. $d\vec{\ell} = d\ell\hat{\varphi} = \rho\, d\varphi\hat{\varphi}$, and $d\vec{a} = da\hat{n} = +da\hat{z}$ by the right-hand rule, $d\vec{a} = \rho\, d\rho d\varphi\hat{z}$, $\boxed{da = \rho\, d\rho d\varphi}$ in cylindrical coordinates, thus $\vec{B} \parallel d\vec{\ell}$, and $\vec{E} \parallel d\vec{a}$, and:

$$\therefore \boxed{B(\rho,t)2\pi\rho = \frac{1}{c^2}\frac{\partial}{\partial t}E(t)\pi\rho^2} \Rightarrow \boxed{\tilde{\vec{B}}(\rho,t) = \frac{1}{c^2}\frac{\cancel{\pi}\rho^{\cancel{2}}}{2\cancel{\pi}\cancel{\rho}}\left(\frac{\partial\tilde{E}(t)}{\partial t}\right)\hat{\varphi} = \frac{\rho}{2c^2}\left(\frac{\partial\tilde{E}(t)}{\partial t}\right)\hat{\varphi}}$$

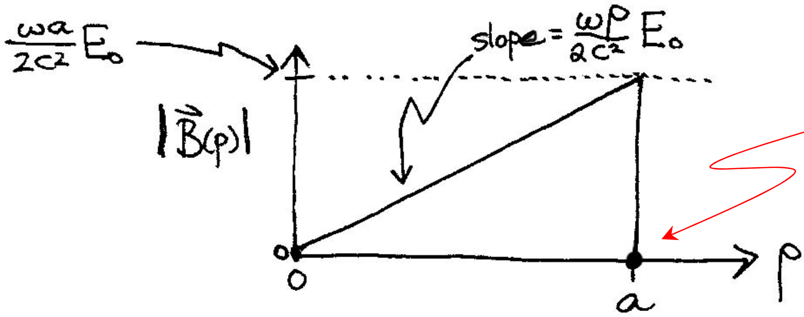
<u>But</u>: $\boxed{\tilde{E}(t) = E_o e^{i\omega t}} \Rightarrow \boxed{\frac{\partial\tilde{E}(t)}{\partial t} = i\omega E_o e^{i\omega t} = i\omega\tilde{E}(t)}$

$$\therefore \boxed{\tilde{\vec{B}}(\rho,t) = \frac{i\omega\rho}{2c^2}\tilde{E}(t)\hat{\varphi} = \underbrace{\frac{i\omega\rho}{2c^2}E_o}_{\equiv B_o(\rho)} e^{i\omega t}\hat{\varphi} = B_o(\rho)e^{i\omega t}\hat{\varphi}}$$

$$\Rightarrow \boxed{\tilde{\vec{B}}(\rho,t) = B_o(\rho)e^{i\omega t}\hat{\varphi} = \left[\frac{i\omega\rho}{2c^2}E_o\right]e^{i\omega t}\hat{\varphi} = i\left[\frac{\omega\rho}{2c^2}E_o\right]e^{i\omega t}\hat{\varphi}}$$

n.b. $\tilde{\vec{B}}(\rho,t)$ also oscillates sinusoidally like $\tilde{\vec{E}}(t)$ but is 90° out-of-phase with $\tilde{\vec{E}}(t)$.

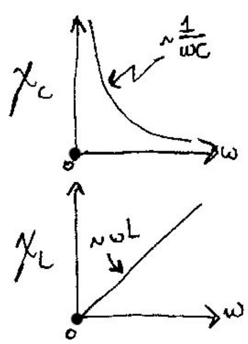
Note also that $\tilde{\vec{B}}(\rho,t)$ is *linearly* proportional to ρ (the radial distance from the axis of capacitor) and that $\tilde{\vec{B}}(\rho=0)=0$ at the center of the capacitor:



NOTE: $\tilde{\vec{B}}(\rho>a)=0$ because $\tilde{\vec{E}}(\rho>a)=0$ for $d \ll a$ in our model of this capacitor.

Thus, we see that for $\omega > 0$, $\exists$ (*i.e.* there exists) an azimuthal, ρ-dependent and time-varying magnetic field $\tilde{\vec{B}}(\rho,t)=\left[\frac{i\omega\rho E_o}{2c^2}\right]e^{i\omega t}\hat{\varphi}$ in the gap region of the parallel-plate capacitor, for $d \ll a$. Note also that the azimuthal magnetic field is also linearly proportional to $\omega = 2\pi f$, thus as the frequency increases, this magnetic field also increases in strength.
Note that for $\omega = 0$, $\tilde{\vec{B}}(\omega)=0$ as we obtained for the static limit case!

Furthermore, because the capacitor now has a non-zero magnetic field associated with it, for $\omega > 0$, the complex, frequency-dependent impedance $\tilde{Z}(\omega) \equiv R(\omega)+i\chi(\omega)$ (Ohms) {where $R(\omega)$ = AC resistance and $\chi(\omega)$ = AC reactance} of the parallel-plate capacitor is no longer just: $\tilde{Z}_C(\omega)=i\chi_C(\omega)=i(1/\omega C)$ (Ohms) where $\chi_C(\omega)=1/\omega C$ = the AC capacitive reactance of the capacitor (Ohms), with (complex) AC Ohm's Law: $\Delta\tilde{V}(\omega)=\tilde{I}(\omega)\bullet\tilde{Z}(\omega)$



Because of the existence of the magnetic field in gap-region of $\|$-plate capacitor, *EM* energy can also be/is stored in the magnetic field of $\|$-plate capacitor due to the inductance, L_C (Henrys) associated with the parallel-plate capacitor and hence it has an inductive reactance of $\chi_L(\omega)=\omega L$ and hence has an inductive complex impedance associated with it, of $Z_L(\omega)=i\chi_L(\omega)=i\omega L_C$ (Ohms). Since the inductance associated with this capacitor is in series with its capacitance, we add the two impedances:

$$\boxed{\tilde{Z}_C^{TOT}(\omega)=\tilde{Z}_C(\omega)+\tilde{Z}_L(\omega)=i\chi_C(\omega)+i\chi_L(\omega)=i\left(\frac{1}{\omega C}\right)+i\omega L_C}$$

$$\boxed{\tilde{Z}_C^{TOT}(\omega)=i\left(\frac{1}{\omega C}+\omega L_C\right)}$$

The {complex} form Ohm's Law {here} is thus: $\boxed{\Delta\tilde{V}(\omega)=\tilde{I}(\omega)\tilde{Z}_C^{TOT}(\omega)}$

Note that at low frequencies $(\omega \approx 0)$ for the parallel-plate capacitor with $d \ll a$, the capacitive reactance $\chi_C(\omega) = 1/\omega C \gg \chi_L(\omega) = \omega L_C$ and thus $\tilde{Z}_C^{TOT}(\omega \approx 0) \simeq \tilde{Z}_C(\omega \approx 0)$. However, at very high frequencies ($\omega \to \infty$), $\chi_C(\omega) \ll \chi_L(\omega) \Rightarrow \tilde{Z}_c^{TOT}(\omega \to \infty) \simeq \tilde{Z}_L(\omega \to \infty)$, *i.e.* in the very high frequency limit, this capacitor instead behaves like a pure inductor!!!

Note also that the electric, magnetic and total *EM* energy densities in the gap-region of the parallel plate capacitor, respectively are:

$$\boxed{u_E(\vec{r},t) = \frac{1}{2}\varepsilon_o\left|\tilde{\vec{E}}(\vec{r},t)\right|^2}, \quad \boxed{u_M(\vec{r},t) = \frac{1}{2\mu_o}\left|\tilde{\vec{B}}(\vec{r},t)\right|^2} \quad \text{and} \quad \boxed{u_{TOT}^{EM}(\vec{r},t) = u_E(\vec{r},t) + u_M(\vec{r},t)}$$

Now because the capacitor has a non-zero time-varying magnetic field: $\tilde{\vec{B}}(\rho,t) = \dfrac{i\omega\rho}{2c^2}E_o e^{i\omega t}\hat{\varphi}$

Faraday's Law $\vec{\nabla} \times \tilde{\vec{E}}(\vec{r},t) = -\partial\tilde{\vec{B}}(\vec{r},t)/\partial t$ tells us that there will be an *additional* {induced} electric field, because $\tilde{\vec{B}}(\vec{r},t)$ is *also* varying in time!!!

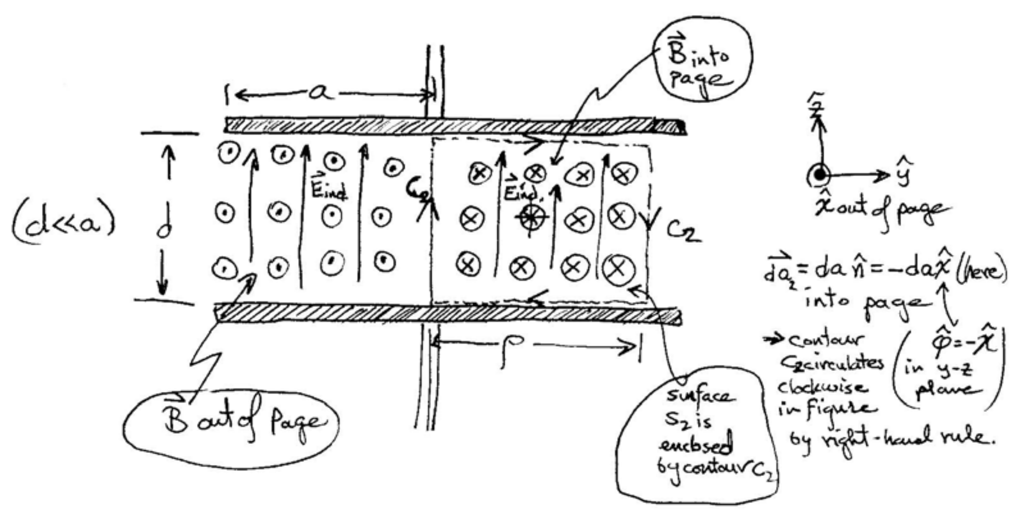
Faraday's Law in integral form is: $\boxed{\int_S\left(\vec{\nabla} \times \tilde{\vec{E}}(\vec{r},t)\right) \cdot d\vec{a} = -\dfrac{\partial}{\partial t}\left(\int_S \tilde{\vec{B}}(\vec{r},t) \cdot d\vec{a}\right) = -\dfrac{\partial\tilde{\Phi}_m(t)}{\partial t}}$

where $\tilde{\Phi}_m(t) \equiv \int_S \tilde{\vec{B}}(\vec{r},t) \cdot d\vec{a}$ is the magnetic flux (*Webers* = *Tesla*-m^2) enclosed by the surface S at time t. Applying Stokes' Theorem, we have: $\boxed{\oint_C \tilde{\vec{E}}(\vec{r},t) \cdot d\vec{\ell} = -\dfrac{\partial}{\partial t}\left(\int_S \tilde{\vec{B}}(\vec{r},t) \cdot d\vec{a}\right) = -\dfrac{\partial\tilde{\Phi}_m(t)}{\partial t}}$

where the contour C around a closed path of integration encloses the surface S through which magnetic flux $\tilde{\Phi}_m(t) \equiv \int_S \tilde{\vec{B}}(\vec{r},t) \cdot d\vec{a}$ passes.

Now $\tilde{\vec{B}} = \tilde{B}\hat{\varphi}$ (*i.e.* points in the $\hat{\varphi}$ {azimuthal} direction) and thus here we need $\tilde{\vec{B}} \parallel d\vec{a}$ hence $d\vec{a}_2 = da\hat{\varphi}$ also, and thus we take the closed contour C_2 line-integral path around the surface S_2 as shown in the side-view figure below:

Side-View:

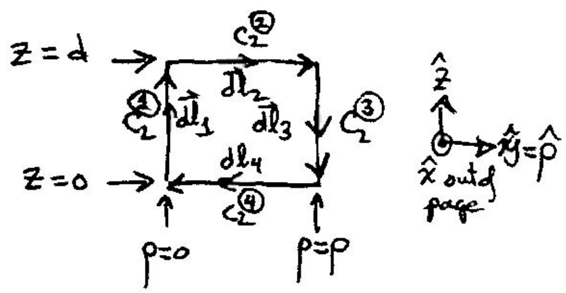


The <u>induced</u> electric field, as created by the time-varying magnetic field is:

$$\boxed{\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell} = -\frac{\partial}{\partial t}\left(\int_{S_2} \tilde{\vec{B}}(\vec{r},t)\cdot d\vec{a}_2\right) = -\frac{\partial \tilde{\Phi}_m(t)}{\partial t}}$$

where $\tilde{\Phi}_m(t) \equiv \int_{S_2} \tilde{\vec{B}}(\vec{r},t)\cdot d\vec{a}_2$ = magnetic flux enclosed by contour C_2 passing through surface S_2

<u>Then</u>: $\boxed{\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell} = \int_{(1)} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_1 + \int_{(2)} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_2 + \int_{(3)} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_3 + \int_{(4)} \tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_4}$



$d\vec{\ell}_1 = d\ell\hat{z}$

$d\vec{\ell}_2 = d\ell\hat{\rho}$ ($\hat{\rho} = \hat{y}$ here in $\hat{y}-\hat{z}$ plane)

$d\vec{\ell}_3 = d\ell(-\hat{z}) = -d\ell\hat{z}$

$d\vec{\ell}_4 = d\ell(-\hat{\rho}) = -d\ell\hat{\rho}$

Now $\boxed{\vec{\nabla}\times\tilde{\vec{E}}_{ind}(\vec{r},t) = -\partial\tilde{\vec{B}}(\vec{r},t)/\partial t}$ tells us that if $\vec{B} = B\hat{\varphi}$ direction, then in cylindrical coordinates:

$$\boxed{\vec{\nabla}\times\tilde{\vec{E}}_{ind}(\vec{r},t) = \underbrace{\left[\frac{1}{\rho}\cancelto{=0}{\frac{\partial\tilde{E}_z}{\partial\varphi}} - \cancelto{=0}{\frac{\partial\tilde{E}_\varphi}{\partial z}}\right]\hat{\rho}}_{=0} + \left[\cancelto{=0}{\frac{\partial\tilde{E}_\rho}{\partial z}} - \frac{\partial\tilde{E}_z}{\partial\rho}\right]\hat{\varphi} + \underbrace{\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\cancelto{=0}{\rho\tilde{E}_\varphi}\right) - \cancelto{=0}{\frac{\partial\tilde{E}_\rho}{\partial\varphi}}\right]\hat{z}}_{=0} = -\frac{\partial\tilde{E}_z(\vec{r},t)}{\partial\rho}\hat{\varphi}}$$

Thus, we see that $\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t) = \left|\vec{\nabla}\times\tilde{\vec{E}}(\vec{r},t)\right|\hat{\varphi}$ only, for all points (ρ,φ,z) in the gap region of $\|$-plate capacitor and for all times t. However, we see that due to the azimuthal / rotational symmetry associated with the cylindrical $\|$-plate capacitor, neither $\vec{E}_{ind}(\vec{r},t)$ nor $\vec{B}(\vec{r},t)$ can have any explicit φ-dependence, thus $\partial\tilde{E}_z/\partial\varphi = 0$ and $\partial\tilde{E}_\rho/\partial\varphi = 0$, which in turn respectively imply that $\partial\tilde{E}_\varphi/\partial z = 0$ and $\partial(\rho\tilde{E}_\varphi)/\partial\rho = 0$. Note further that Faraday's Law tells us that we must also have $\tilde{\vec{E}}_{ind}(\vec{r},t) \perp \tilde{\vec{B}}(\vec{r},t)$.

For $d \ll a$, the electric field in the gap region of the $\|$-plate capacitor <u>cannot</u> explicitly depend on z either. Thus, $\partial E_\rho/\partial z = 0 \Rightarrow \therefore$ the only surviving term in $\vec{\nabla}\times\tilde{\vec{E}}_{ind}(\vec{r},t)$ is:

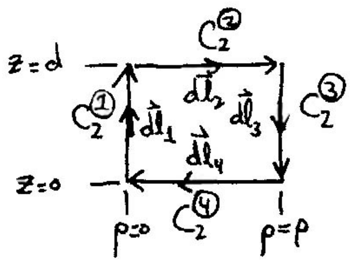
$$\boxed{\vec{\nabla}\times\tilde{\vec{E}}_{ind}(\vec{r},t) = -\frac{\partial\tilde{E}_z(\vec{r},t)}{\partial\rho}\hat{\varphi}}.$$

$\therefore\ \tilde{\vec{E}}_{ind}(\vec{r},t) = \tilde{E}_{ind}(\vec{r},t)\hat{z}$ *i.e.* the induced $\tilde{\vec{E}}$-field points in the $\hat{z}$ direction (must be $\perp$ $\tilde{\vec{B}} = \tilde{B}\hat{\varphi}$) which is satisfied because $\hat{z}\perp\hat{\varphi}$.

Thus, if the induced electric field $\tilde{\vec{E}}_{ind}(\vec{r},t)=\tilde{E}_{ind}(\vec{r},t)\hat{z}$ {only}, then we see that:

$$\oint_{C_2}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}=\int_{(1)}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_1+\int_{(2)}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_2+\int_{(3)}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_3+\int_{(4)}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}_4$$

$$=\int_{(1)}\tilde{E}_{ind}\hat{z}\cdot dz\hat{z}+\underbrace{\cancel{\int_{(2)}\tilde{E}_{ind}\hat{z}\cdot d\rho\hat{\rho}}}_{\substack{=0\\(\hat{z}\perp\hat{\rho})}}+\int_{(3)}\tilde{E}_{ind}\hat{z}\cdot(-dz\hat{z})+\underbrace{\cancel{\int_{(4)}\tilde{E}_{ind}\hat{z}(-d\rho\hat{\rho})}}_{\substack{=0\\(\hat{z}\perp\hat{\rho})}}$$



$$=\int_{(1)}\tilde{E}_{ind}\hat{z}\cdot dz\hat{z}+\int_{(3)}\tilde{E}_{ind}\hat{z}\cdot(-dz\hat{z})$$

$$=\int_{z=0}^{z=d}\tilde{E}_{ind}(\rho=0)\,dz-\int_{z=0}^{z=d}\tilde{E}_{ind}(\rho=\rho)\,dz$$

But $\tilde{\vec{E}}_{ind}(\vec{r},t)$ has no explicit z-dependence, thus:

$$\boxed{\oint_{C_2}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}=\tilde{E}_{ind}(\rho=0,t)\int_{z=o}^{z=d}dz\;-\;\tilde{E}_{nd}(\rho=\rho,t)\int_{z=0}^{z=d}dz=\tilde{E}_{ind}(\rho=0,t)*d+\tilde{E}_{ind}(\rho=\rho,t)*d}$$

<u>Or</u>: $\boxed{\oint_{C_2}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}=\left[\tilde{E}_{ind}(\rho=0,t)-\tilde{E}_{ind}(\rho=\rho,t)\right]d}$ where: $\boxed{\tilde{\vec{E}}_{ind}(\rho,t)=\tilde{E}_{ind}(\rho,t)\hat{z}}$

<u>But</u>: $\boxed{\oint_{C_2}\tilde{\vec{E}}_{ind}(\vec{r},t)\cdot d\vec{\ell}=-\frac{\partial}{\partial t}\int_{S_2}\tilde{\vec{B}}(\vec{r},t)\cdot d\vec{a}_2}$ where $S_2=$ surface enclosed by contour C_2

and $d\vec{a}_2=da\hat{n}_2=da\hat{\varphi}$ (*i.e.* S_2 lies in the *y-z* plane) $=-da\hat{x}$ {here} and $da=dydz=d\rho dz$

Now: $\boxed{\tilde{\vec{B}}(\rho,t)=B_o(\rho)e^{i\omega t}\hat{\varphi}=\left(\frac{i\omega\rho}{2c^2}\right)E_oe^{i\omega t}\hat{\varphi}}$ and $\hat{\varphi}=-\hat{x}$ (S_2 lies in the *y-z* plane)

$\therefore$ $\boxed{\int_{S_2}\tilde{\vec{B}}(\rho,t)\cdot d\vec{a}_2=\int_{\rho=0}^{\rho=\rho}\int_{z=0}^{z=d}\left(\frac{i\omega\rho}{2c^2}\right)E_0e^{i\omega t}\hat{\varphi}\cdot d\rho dz\hat{\varphi}}$ but: $\hat{\varphi}\cdot\hat{\varphi}=1$

$$\boxed{=\left(\frac{i\omega}{2c^2}\right)E_oe^{i\omega t}\int_{\rho=0}^{\rho=\rho}\int_{z=0}^{z=d}\rho d\rho dz=\left(\frac{i\omega d}{2c^2}\right)E_0e^{i\omega t}\int_{\rho=0}^{\rho=\rho}\rho d\rho}$$ and: $\int\rho d\rho=\frac{1}{2}\rho^2$

$\therefore$ $\boxed{\int_{S_2}\tilde{\vec{B}}(\rho,t)\cdot d\vec{a}_2=\left(\frac{i\omega\rho^2d}{4c^2}\right)E_0e^{i\omega t}}$

<u>Then</u>: $-\frac{\partial}{\partial t}\int_{S_2}\tilde{\vec{B}}(\vec{r},t)\cdot d\vec{a}_2=-\frac{\partial}{\partial t}\left[\left(\frac{i\omega\rho^2d}{4c^2}\right)E_0e^{i\omega t}\right]=-i\omega\left[\left(\frac{i\omega\rho^2d}{4c^2}\right)E_0e^{i\omega t}\right]$

<u>And</u>: $\boxed{(-i\omega)(i\omega)=(-i*i)\omega^2=+1\omega^2=\omega^2}$ {since $i=\sqrt{-1}$ and $-i=-\sqrt{-1}$ }

$$\therefore \quad \boxed{-\frac{\partial}{\partial t}\int_{S_2} \tilde{\vec{B}}(\vec{r},t)\bullet d\vec{a}_2 = \frac{\omega^2\rho^2 d}{4c^2}E_0 e^{i\omega t}}.$$

Then: $\boxed{\oint_{C_2} \tilde{\vec{E}}_{ind}(\vec{r},t)\bullet d\vec{\ell} = -\frac{\partial}{\partial t}\int_{S_2} \tilde{\vec{B}}(\vec{r},t)\bullet d\vec{a}_2}$ yields:

$$\boxed{\cancel{d}\left[\tilde{\vec{E}}_{ind}(\rho=0,t) - \tilde{\vec{E}}_{ind}(\rho=\rho,t)\right] = \frac{\omega^2\rho^2 \cancel{d}}{4c^2}E_0 e^{i\omega t}\hat{z}}$$

Note that the d's cancel on both sides of the above equation. Note also that because of the <u>explicit</u> ρ^2 dependence on the RHS of above equation, we see that $\tilde{\vec{E}}_{ind}(\rho=0,t)=0$.

Hence: $\boxed{\tilde{\vec{E}}_{ind}(\rho,t) = -\frac{\omega^2\rho^2}{4c^2}E_o e^{i\omega t}\hat{z} = -\left(\frac{\omega\rho}{2c}\right)^2 E_o e^{i\omega t}\hat{z}}$

Thus the <u>*total*</u> $\vec{E}$-field in the capacitor gap is:

$$\boxed{\tilde{\vec{E}}_{ToT}(\rho,t) = \tilde{\vec{E}}(t) + \tilde{\vec{E}}_{ind}(\rho,t) = E_o e^{i\omega t}\hat{z} - \left(\frac{\omega\rho}{2c}\right)^2 E_o e^{i\omega t}\hat{z} = \left(1-\left(\frac{\omega\rho}{2c}\right)^2\right)E_o e^{i\omega t}\hat{z}}$$

Thus, we see here that the <u>induced</u> electric field caused by the time-varying magnetic field points in the direction <u>opposite</u> to the initial/original $\tilde{\vec{E}}$-field, <u>reducing</u> the overall $\tilde{\vec{E}}$-field for $\rho > 0$, as we would expect from Lenz's Law.

However, note that we now also have an <u>additional</u> contribution to the $\tilde{\vec{B}}$-field inside the gap-region of the parallel plate capacitor, due to the presence of the <u>induced</u> $\tilde{\vec{E}}$-field contribution, $\tilde{\vec{E}}_{ind}(\rho,t)$.

Before we proceed further on this discussion, it would be best for us change our notation: Call our original time-dependent $\tilde{\vec{E}}$-field, $\tilde{\vec{E}}(\vec{r},t) = E_o e^{i\omega t} = \tilde{\vec{E}}_1(\vec{r},t)$.

This $\tilde{\vec{E}}$-field in turn creates a time-dependent $\tilde{\vec{B}}$-field by <u>Ampere's Law</u>:

$$\boxed{\vec{\nabla}\times\tilde{\vec{B}}_1(\vec{r},t) = \mu_o\varepsilon_o\frac{\partial\tilde{\vec{E}}_1(\vec{r},t)}{\partial t} = \frac{1}{c^2}\frac{\partial\tilde{\vec{E}}_1(\vec{r},t)}{\partial t}}.$$

However, because $\tilde{\vec{B}}_1(\vec{r},t)$ <u>*also*</u> varies in time, it turn creates <u>*another*</u> induced time-dependent electric field by <u>Faraday's Law</u>:

$$\boxed{\vec{\nabla}\times\tilde{\vec{E}}_2(\vec{r},t) = -\frac{\partial\tilde{\vec{B}}_1(\vec{r},t)}{\partial t}}.$$

But $\tilde{\vec{E}}_2(\vec{r},t)$ is <u>*also*</u> time-varying, and so it in turn produces another time-varying contribution to the magnetic field $\tilde{\vec{B}}_2(\vec{r},t)$.

But because $\tilde{\tilde{\vec{B}}}_2(\vec{r},t)$ is <u>*also*</u> time-varying, it in turn will induce <u>*another*</u> contribution to the electric field $\tilde{\tilde{\vec{E}}}_3(\vec{r},t)$ and so on… *i.e.*:

$$\boxed{\tilde{\tilde{\vec{E}}}_1(\vec{r},t)\xrightarrow{A.L.}\tilde{\tilde{\vec{B}}}_1(\vec{r},t)\xrightarrow{F.L.}\tilde{\tilde{\vec{E}}}_2(\vec{r},t)\xrightarrow{A.L.}\tilde{\tilde{\vec{B}}}_2(\vec{r},t)\xrightarrow{F.L.}\tilde{\tilde{\vec{E}}}_3(\vec{r},t)\xrightarrow{A.L.}\tilde{\tilde{\vec{B}}}_3(\vec{r},t)\xrightarrow{F.L.}\tilde{\tilde{\vec{E}}}_4(\vec{r},t)\xrightarrow{A.L.}\tilde{\tilde{\vec{B}}}_4(\vec{r},t)\xrightarrow{F.L.}\ldots}$$

<u>Then</u>: $$\boxed{\tilde{\tilde{\vec{E}}}_{TOT}(\vec{r},t)=\tilde{\tilde{\vec{E}}}_1(\vec{r},t)+\tilde{\tilde{\vec{E}}}_2(\vec{r},t)+\tilde{\tilde{\vec{E}}}_3(\vec{r},t)+\tilde{\tilde{\vec{E}}}_4(\vec{r},t)+\tilde{\tilde{\vec{E}}}_5(\vec{r},t)+\ldots=\sum_{n=1}^{\infty}\tilde{\tilde{\vec{E}}}_n(\vec{r},t)}$$

<u>And</u>: $$\boxed{\tilde{\tilde{\vec{B}}}_{TOT}(\vec{r},t)=\tilde{\tilde{\vec{B}}}_1(\vec{r},t)+\tilde{\tilde{\vec{B}}}_2(\vec{r},t)+\tilde{\tilde{\vec{B}}}_3(\vec{r},t)+\tilde{\tilde{\vec{B}}}_4(\vec{r},t)+\tilde{\tilde{\vec{B}}}_5(\vec{r},t)+\ldots=\sum_{n=1}^{\infty}\tilde{\tilde{\vec{B}}}_n(\vec{r},t)}$$

So thus we see that:

$$\oint_{C_1}\tilde{\tilde{\vec{B}}}_1(\vec{r},t)\cdot d\vec{\ell}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_{S_1}\tilde{\tilde{\vec{E}}}_1(\vec{r},t)\cdot d\vec{a}_1\right)$$
$$\oint_{C_2}\tilde{\tilde{\vec{E}}}_2(\vec{r},t)\cdot d\vec{\ell}=-\frac{\partial}{\partial t}\left(\int_{S_2}\tilde{\tilde{\vec{B}}}_1(\vec{r},t)\cdot d\vec{a}_2\right)$$
$$\oint_{C_1}\tilde{\tilde{\vec{B}}}_2(\vec{r},t)\cdot d\vec{\ell}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_{S_1}\tilde{\tilde{\vec{E}}}_2(\vec{r},t)\cdot d\vec{a}_1\right)$$
$$\oint_{C_2}\tilde{\tilde{\vec{E}}}_3(\vec{r},t)\cdot d\vec{\ell}=-\frac{\partial}{\partial t}\left(\int_{S_2}\tilde{\tilde{\vec{B}}}_2(\vec{r},t)\cdot d\vec{a}_2\right)$$
$$\oint_{C_1}\tilde{\tilde{\vec{B}}}_3(\vec{r},t)\cdot d\vec{\ell}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_{S_1}\tilde{\tilde{\vec{E}}}_3(\vec{r},t)\cdot d\vec{a}_1\right)$$
$$\oint_{C_2}\tilde{\tilde{\vec{E}}}_4(\vec{r},t)\cdot d\vec{\ell}=-\frac{\partial}{\partial t}\left(\int_{S_2}\tilde{\tilde{\vec{B}}}_3(\vec{r},t)\cdot d\vec{a}_2\right)$$

…. etc.

Algorithmically, this infinite sequence can be written as:

$$\boxed{n=1,\ \tilde{\tilde{\vec{E}}}_1(\vec{r},t)}$$

$$\boxed{\oint_{C_1}\tilde{\tilde{\vec{B}}}_n(\vec{r},t)\cdot d\vec{\ell}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\int_{S_1}\tilde{\tilde{\vec{E}}}_n(\vec{r},t)\cdot d\vec{a}_1\right)}\longrightarrow\boxed{\tilde{\tilde{\vec{B}}}_n(\vec{r},t)}$$

Infinite Loop

$$\boxed{\oint_{C_2}\tilde{\tilde{\vec{E}}}_{n+1}(\vec{r},t)\cdot d\vec{\ell}=-\frac{\partial}{\partial t}\left(\int_{S_2}\tilde{\tilde{\vec{B}}}_n(\vec{r},t)\cdot d\vec{a}_2\right)}\longrightarrow\boxed{\tilde{\tilde{\vec{E}}}_{n+1}(\vec{r},t)}$$

$$\boxed{n=n+1}$$

Where contour C_1 enclosing surface S_1 and area element $d\vec{a}_1$ are associated with the figure drawn on page 3 of these lecture notes, and where contour C_2 enclosing surface S_2 and area element $d\vec{a}_2$ are associated with the figure drawn on page 5 of these lecture notes.

It can thus be shown for the parallel-plate capacitor with $d \ll a$ that:

$$\boxed{\tilde{\vec{E}}_{TOT}(\rho,t) = \left[1 - \frac{1}{(1!)^2}\left(\frac{\omega\rho}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega\rho}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega\rho}{2c}\right)^6 + \ldots\right] E_o e^{i\omega t}\hat{z}} \quad \underline{\text{with}}: \boxed{E_o = -\left(\frac{V_o}{d}\right)}$$

$$\boxed{\tilde{\vec{B}}_{TOT}(\rho,t) = \left[1 - \frac{1}{(1!)^2}\left(\frac{\omega\rho}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega\rho}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega\rho}{2c}\right)^6 + \ldots\right] B_o e^{i\omega t}\hat{\varphi}} \quad \underline{\text{with}}: \boxed{B_o = \frac{i\omega\rho}{2c^2} E_o}$$

and where: $n! \equiv n\bullet(n-1)\bullet(n-2)\ldots 2\bullet 1$, $0!=1$, $1!=1$, $2!=2$, $3!=6$, $4!=24$, etc.

We also see that: $\boxed{B_n = \frac{i\omega\rho}{2c^2} E_n}$ and: $\boxed{E_{n+1} = \left(\frac{i\omega\rho}{2}\right) B_n = \left(\frac{i\omega\rho}{2}\right)\left(\frac{i\omega\rho}{2c^2}\right) E_n = -\left(\frac{\omega\rho}{2c}\right)^2 E_n}$

and: $\boxed{B_{n+1} = \left(\frac{i\omega\rho}{2c^2}\right) E_{n+1} = \left(\frac{i\omega\rho}{2c^2}\right)\left(\frac{i\omega\rho}{2}\right) B_n = -\left(\frac{\omega\rho}{2c}\right)^2 B_n}$

Due to the <u>cylindrical geometry</u> / azimuthal symmetry associated with this problem, it should not come as a surprise that:

<u>Defining</u>: $\boxed{x \equiv \frac{\omega\rho}{c} = k\rho}$ where $\boxed{k = \frac{\omega}{c}}$ = wavenumber

$\boxed{k = \frac{2\pi}{\lambda}}$ $\boxed{\lambda = \frac{c}{f}}$ $\boxed{f = \frac{\omega}{2\pi}}$

Then the quantity in square brackets on the previous page becomes:

$$\boxed{1 - \frac{1}{(1!)^2}\left(\frac{\omega\rho}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega\rho}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega\rho}{2c}\right)^6 + \ldots = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots}$$

The so-called "ordinary" Bessel function of the first kind, of order zero has a series expansion of the form:

$$\boxed{J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots}$$

$$\boxed{J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\bullet k!}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}}$$

$\boxed{k! \equiv k\bullet(k-1)\bullet(k-2)\ldots 3\bullet 2\bullet 1}$ where: $\boxed{\Gamma(k+1) = k!}$ (for k = integer)

In general, the series expansion of the ordinary Bessel functions of the first kind, of order n are:

$$\boxed{J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}}$$

Thus, for the cylindrical $\parallel$-plate capacitor with $d \ll a$ the electric and magnetic fields in the gap region are of the form:

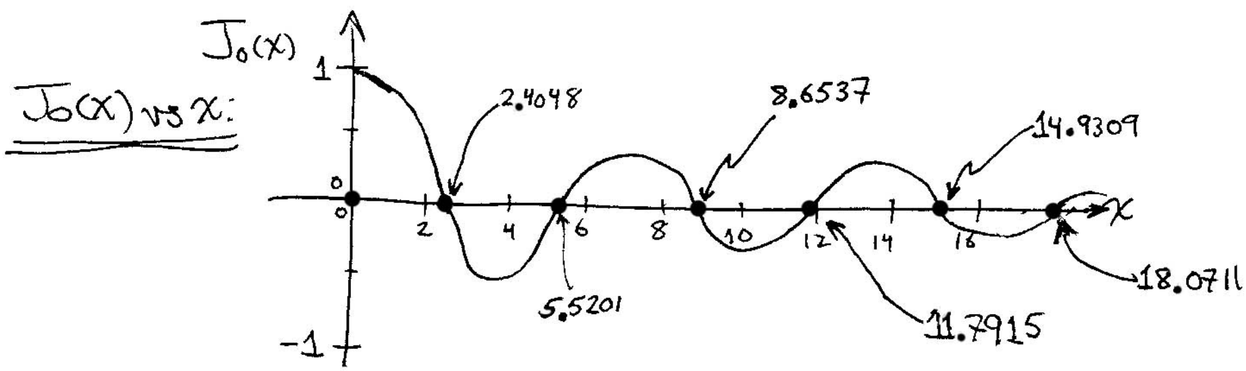
$$\boxed{\tilde{\vec{E}}(\rho,t) = J_0\left(\tfrac{\omega\rho}{c}\right)E_o e^{i\omega t}\hat{z} = J_0(k\rho)E_o e^{i\omega t}\hat{z}} \quad \text{with} \quad \boxed{k = \frac{\omega}{c}} \quad \text{and} \quad \boxed{E_o = -\frac{V_o}{d}}$$

$\vec{B}$ is 90° out-of-phase with $\vec{E}$ {here}:

$$\boxed{\tilde{\vec{B}}(\rho,t) = J_0\left(\tfrac{\omega\rho}{c}\right)B_o(\rho)e^{i\omega t}\hat{\varphi} = J_0(k\rho)B_o(\rho)e^{i\omega t}\hat{\varphi}} \quad \text{with} \quad \boxed{B_o(\rho) = \frac{i\omega\rho}{2c^2}E_o = i\left(\frac{\omega\rho}{2c^2}\right)E_o}$$

Note that for $\rho = 0$ that: $\boxed{\tilde{\vec{E}}(\rho=0,t) = E_o e^{i\omega t}}$ but: $\boxed{\tilde{\vec{B}}(\rho=0,t) = 0}$.

<u>The {Radial} Zeroes of $J_0(x)$:</u> $\boxed{x = k\rho = \left(\frac{\omega}{c}\right)\rho}$



n.b. the zeroes of $J_n(x)$ are <u>not</u> integer related!!

Since $\tilde{\vec{E}}(\rho,t) = J_0(k\rho)E_o e^{i\omega t}\hat{z}$ and $\tilde{\vec{B}}(\rho,t) = J_0(k\rho)B_o(\rho)e^{i\omega t}\hat{\varphi}$ we see that the zeroes x_n of $J_0(x)$ are physically where the electric and magnetic fields vanish (!!!), *i.e.* $\vec{E}(\rho,t) = 0$ and $\tilde{\vec{B}}(\rho,t) = 0$ when $\rho_n = x_n/k = cx_n/\omega$ with $x_1 = 2.4048$, $x_2 = 5.5201$, $x_3 = 8.6537$, etc.!!!

So let's now examine the frequency-dependence of the $\vec{E}$ and $\vec{B}$ fields of the $\parallel$-plate capacitor:

$$\boxed{\tilde{\vec{E}}(\rho,t) = J_0\left(\tfrac{\omega\rho}{c}\right)E_o e^{i\omega t}\hat{z} = J_0(k\rho)E_o e^{i\omega t}\hat{z}} \quad \text{with:} \quad \boxed{k = \frac{\omega}{c}} \quad \text{and:} \quad \boxed{E_o = -\frac{V_o}{d}}$$

$$\boxed{\tilde{\vec{B}}(\rho,t) = J_0\left(\tfrac{\omega\rho}{c}\right)B_o(\rho)e^{i\omega t}\hat{\varphi} = J_0(k\rho)B_o(\rho)e^{i\omega t}\hat{\varphi}} \quad \text{with:} \quad \boxed{B_o(\rho) = \frac{i\omega\rho}{2c^2}E_o = \frac{ik\rho}{2c}E_o}$$

a.) When: $\omega = 0$, $f = 0$ then: $k = \frac{\omega}{c} = 0 \Rightarrow \lambda = \infty$ (static case). Then: $x = k\rho = 0$ and $J_0(0) = 1$

$$\boxed{\tilde{\vec{E}}(\rho,t) = E_o\hat{z} = -\frac{V_o}{d}\hat{z}} \quad \text{with:} \quad \boxed{E_o = -\frac{V_o}{d}} \quad \text{and:} \quad \boxed{\tilde{\vec{B}}(\rho,t) = 0} \leftarrow$$ **n.b. Same result as original static calculation**

b.) When: $\omega \geq 0$, *e.g.* $f = 60\ Hz \rightarrow \omega = 2\pi f = 120\pi\ rad/sec$

Then: $k = \dfrac{2\pi}{\lambda} = \dfrac{\omega}{c} = \dfrac{120\pi\ rad\ /\ sec}{3\times10^{8} m\ /\ sec} = 1.257\times10^{-6}\ rad/meter$

Suppose the radius of the capacitor is $a = 1\ cm = 10^{-2}\ m$ (reasonable/typical diameter)

Then: $ka = 1.257\times10^{-8} = x$ (dimensionless)

And: $J_0(ka) = J_0\left(1.257\times10^{-8}\right) \simeq 1.0$ (*n.b.* see/refer to above graph of $J_0(x)$ *vs*. x)

Thus, we see that at $f = 60\ Hz$, the $\vec{E}$-field is ≈ that of the DC $\vec{E}$-field, and *e.g.* if $V_o = 10\ V$ and $d = 0.1\ mm \ll a = 1\ cm$, *i.e.* $d = 10^{-4}\ m \ll a = 10^{-2}\ m$, then: $E_o = -\dfrac{V_o}{d} = -\dfrac{10\ V}{10^{-4} m} = -10^{5}\ Volts/m$

and: $\left|B_o\left(\rho = a\right)\right| = \dfrac{\omega a}{2c^2} E_0 = \dfrac{ka}{2c} E_0 = \dfrac{1.257\times10^{-8}}{2\times3\times10^{8}}\times10^{5} = 2.1\times10^{-12}\ Tesla$ {*i.e.* is very small}.

<u>Another way to see this</u>: $c\left|B_o\left(\rho = a\right)\right| = 6.3\times10^{-4}\ Volts/m \ll \left|E_o\right| = 10^{5}\ Volts/m$

c.) Now suppose: $f = 1\ MHz = 10^{6}\ Hz$ and $\omega = 2\pi f = 2\pi\times10^{6} rads\ /\ sec$

Then: $k = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda} = \dfrac{2\pi\times10^{6}}{3\times10^{8}} \simeq 2.1\times10^{-2} = 0.021\ radians/m$ and if $a = 1\ cm = 0.01\ m$

Then: $(ka) = 0.021 \times 0.01 = 2.1 \times 10^{-4}$ and $J_0(ka) = J_0\ (2.1 \times 10^{-4}) \approx 1$ (still).

$\rightarrow E_o = -\dfrac{V_o}{d}$ (constant), and: $\left|B_o\left(\rho = a\right)\right| = \dfrac{ka}{2c} E_o = 3.5\times10^{-8} Tesla = 35\ nT$ (still very small)

for $E_o = -10^{5}$ Volts/meter and {still} $c\left|B_o\left(\rho = a\right)\right| = 10.5\ Volts/m \ll \left|E_o\right| = 10^{5}\ V/m$

for $V_o = 10$ Volts, $d = 0.1$ mm and $a = 1$ cm $= 10^{-2}$ m.

d.) Now suppose: $f = 100\ GHz = 10^{11}\ Hz$ and $\omega = 2\pi f = 6.3 \times 10^{11}\ rads/sec$

Then: $k = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda} = \dfrac{6.3\times10^{11}}{3\times10^{8}} = 2.1\times10^{3}$ radians/m

Then: $(ka) = 2.1 \times 10^{3} \times 10^{-2} = 21 \rightarrow J_0\ (k\rho)$ has 5 zeroes in it !!!

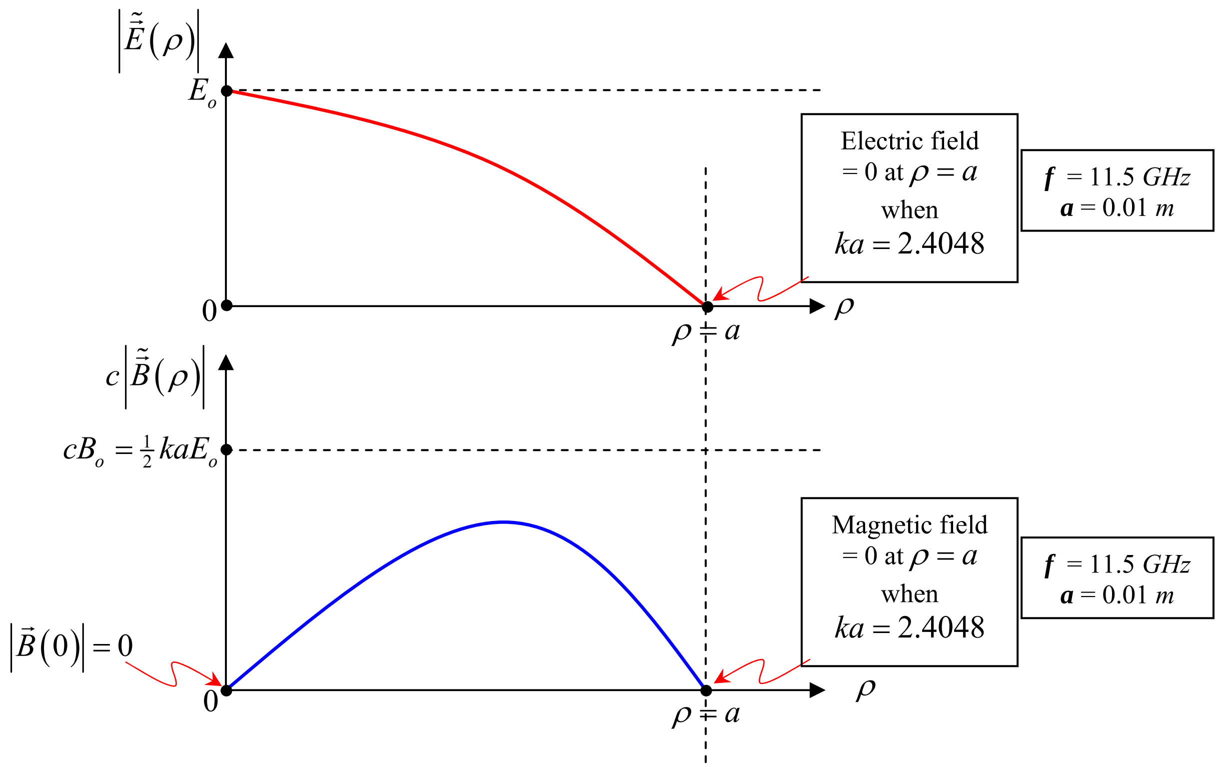
EEK!! → the $\vec{E}$-field points in the reverse direction depending on $0 \leq \rho \leq a$ value !!!

(see above graph of $J_0(x)$ *vs*. x on page 11 of these lecture notes)

<u>Suppose instead that we pick</u>: $ka = 2.4048 = x_1 = 1^{st}$ zero of $J_0(x) = J_0(k\rho)$ ($a = 10^{-2}\ m = 1\ cm$)

<u>Then</u>: $k = 240.48\ radians/m = \dfrac{\omega}{c} \rightarrow f = 1.15 \times 10^{10}\ Hz = 11.5\ GHz$ (in the microwave region)

<u>Then</u>: $\boxed{\tilde{\vec{E}}(\rho,t) = J_0(k\rho) E_o e^{i\omega t}\hat{z}}$ and $\boxed{\lambda = \dfrac{2\pi}{k} = 2.61\text{ cm}}$, $\boxed{\tilde{\vec{B}}(\rho,t) = J_0(k\rho) B_o e^{i\omega t}\hat{\varphi}}$, $\boxed{B_o = \dfrac{ik\rho}{2c} E_o}$



The Inductance of a Parallel-Plate Capacitor

Equate: $\boxed{W_m = \frac{1}{2}LI^2 = \frac{1}{2\mu_o}\int_v \left|\tilde{B}\right|^2 d\tau}$ $\boxed{\Delta\tilde{V} = \tilde{I}\tilde{Z}_{ToT}}$, $\boxed{\tilde{I} = \Delta\tilde{V}/\tilde{Z}_{ToT}}$ $\boxed{\Delta\tilde{V} = V_o e^{i\omega t}}$

Capacitance: $\boxed{C = \frac{\varepsilon_o A}{d}\left(\text{for } d \ll a\right)}$ $\boxed{\tilde{Z}_{ToT} = \tilde{Z}_C + \tilde{Z}_L = i\left(\frac{1}{\omega C} + \omega L\right)}$

$\boxed{\tilde{I} = \frac{V_o e^{i\omega t}}{i\left(\frac{1}{\omega C} + \omega L\right)}}$, $\boxed{\tilde{I}^* = \frac{V_o e^{-i\omega t}}{-i\left(\frac{1}{\omega C} + \omega L\right)}}$ then: $\boxed{|I|^2 = \tilde{I}\tilde{I}^* = \frac{V_o^2}{\left(\frac{1}{\omega C} + \omega L\right)^2}}$

Thus: $\boxed{W_m = \frac{1}{2}L|I|^2 = \frac{1}{2\mu_o}L\frac{V_o^2}{\left(\frac{1}{\omega C} + \omega L\right)^2} = \frac{1}{2\mu_o}\int_v \left|\tilde{B}\right|^2 d\tau = \int J_0^2(k\rho)\frac{k^2\rho^2}{4c^2}E_o^2 \rho\, d\rho\, d\varphi\, dz}$

$$= \frac{\cancel{1}}{\cancel{2}}L\frac{d^{\cancel{2}}\left(V_o^2/d^2\right)}{\left(\frac{1}{\omega C + \omega L}\right)^2} = \frac{\cancel{2}\pi\cancel{d}}{\cancel{2}\mu_o}\frac{k^2}{\underset{2}{\cancel{4}}c^2}E_o^2\int_0^a J_0^2(kp)\rho^3 d\rho$$

$$= L\frac{\cancel{E_o^2}\,d}{\left(\frac{1}{\omega C} + \omega L\right)^2} = \frac{\pi\varepsilon_o\cancel{\mu_o}k^2\cancel{E_o^2}}{2\cancel{\mu_o}}\int_o^a J_0^2(k\rho)\rho^3 d\rho \quad \text{with} \quad \frac{1}{c^2} = \varepsilon_o\mu_o,\ \omega = ck,\ k = \omega/c$$

$$\Rightarrow \boxed{\frac{L}{\left(\frac{1}{\omega C}+\omega L\right)^2} = \left[\frac{\pi\varepsilon_o\omega^2}{2c^2 d}\int_o^a J_0^2(kp)\rho^3 d\rho\right] \equiv \mathcal{A}}$$

$$\Rightarrow L = \mathcal{A}\left(\frac{1}{\omega C}+\omega L\right)^2 = \mathcal{A}\left(\frac{1}{\omega C}\right)^2 + 2\mathcal{A}\frac{\cancel{\omega} L}{\cancel{\omega} C} + \mathcal{A}\omega^2 L^2 = \mathcal{A}\left(\frac{1}{\omega C}\right)^2 + 2\mathcal{A}\left(\frac{L}{C}\right) + \mathcal{A}\omega^2 L^2$$

or: $\boxed{\underbrace{\mathcal{A}\omega^2}_{=a} L^2 + \underbrace{\left(\frac{2\mathcal{A}}{C}-1\right)}_{=b} L + \underbrace{\mathcal{A}\left(\frac{1}{\omega C}\right)^2}_{=c} = 0}$

$\Rightarrow$ Quadratic equation of the form: $\boxed{aL^2 + bL + c = 0}$, solve for L: $\boxed{L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

$$L = \frac{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right) \pm \sqrt{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right)^2 - {}^{4\mathcal{A}^2\cancel{\omega^2}}\!/\!_{\cancel{\omega^2}C^2}}}{2\mathcal{A}\omega^2} = \frac{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right) \pm \sqrt{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right)^2 - \left({}^{2\mathcal{A}}\!/\!_{C}\right)^2}}{2\mathcal{A}\omega^2}$$

$$L = \frac{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right) \pm \sqrt{1 - {}^{4\mathcal{A}}\!/\!_{C} + \cancel{{}^{4\mathcal{A}^2}\!/\!_{C^2}} - \cancel{{}^{4\mathcal{A}^2}\!/\!_{C^2}}}}{2\mathcal{A}\omega^2} = \frac{\left(1 - {}^{2\mathcal{A}}\!/\!_{C}\right) \pm \sqrt{1 - {}^{4\mathcal{A}}\!/\!_{C}}}{2\mathcal{A}\omega^2}$$

Physically, we want $\underline{L \to 0}$ when $\underline{\omega \to 0}$ $\therefore$ ***must*** choose – (negative) sign in above formula!

$\therefore \boxed{L = \frac{(1 - 2\mathcal{A}/C) - \sqrt{1 - 4\mathcal{A}/C}}{2\mathcal{A}\omega^2}}$ Now: $\boxed{\sqrt{1-\in} \simeq 1 - \frac{1}{2}\in + \frac{1}{8}\in^2 - \ldots}$ for $\boxed{\in \equiv \frac{4\mathcal{A}}{C} \ll 1}$, thus:

$$L \simeq \frac{\frac{1}{8}\left(\frac{4\mathcal{A}}{C}\right)^2}{2\mathcal{A}\omega^2} = \frac{\mathcal{A}}{\omega^2 C^2} = \frac{\frac{\pi \cancel{\varepsilon_o} \cancel{\omega^2}}{2c^2 \cancel{d}}\int_0^a J_0^2(k\rho)\rho^3 d\rho}{\cancel{\omega^2}\frac{\varepsilon_o^{\cancel{2}} A^2}{d^{\cancel{2}}}} = \frac{\pi d}{2c^2\varepsilon_o A^2}\int_0^a J_0^2(k\rho)\rho^3 d\rho = \frac{\mu_o \pi d}{2A^2}\int_0^a J_0^2(k\rho)\rho^3 d\rho$$

Where the capacitance and inductance of the parallel-plate capacitor, for $d \ll a$ are:

$\boxed{C = \frac{\varepsilon_o A}{d}}$ and $\boxed{L \simeq \frac{\mu_o \pi d}{2A^2}\int_0^a J_0^2(k\rho)\rho^3 d\rho}$ for $\boxed{\in \equiv \left(\frac{4\mathcal{A}}{C}\right) = \left[\frac{2\pi\omega^2}{Ac^2}\int_o^a J_0^2(kp)\rho^3 d\rho\right] \ll 1}$.

Note that for $ka =$

$$\left.\begin{array}{l} 2.4048 - 1^{\text{st}} \\ 5.5201 - 2^{\text{nd}} \\ 8.6537 - 3^{\text{rd}} \\ 11.7915 - 4^{\text{th}} \\ 14.9309 - 5^{\text{th}} \\ 18.0711 - 6^{\text{th}} \\ \quad\vdots \end{array}\right\} \text{zeroes of } J_0(ka)$$

The electric field $\tilde{\tilde{E}}(\rho = a) = 0$ for these values of ka, corresponding to wavelengths $\lambda = \frac{2\pi}{k}$

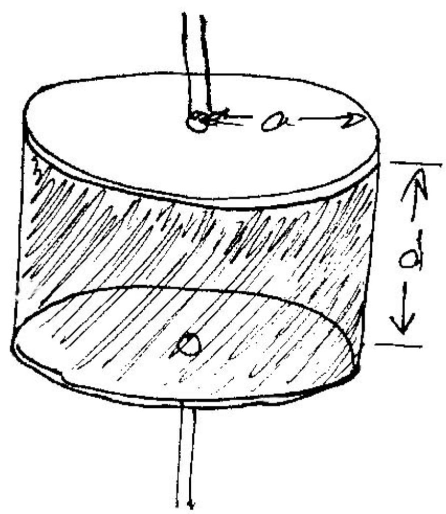
and frequencies $f = \frac{c}{\lambda} = \frac{\omega}{2\pi} = \frac{ck}{2\pi}$

Compare with radius $a = 1.0\ cm$ and diameter $D = 2a = 2.0\ cm$ of $\parallel$-plate cylindrical capacitor, as well as the gap dimension of $d = 0.1\ mm = 0.01\ cm$

$$\left\{\begin{array}{l}
\lambda_1 = 2.61\ cm \leftrightarrow f_1 = 1.15 \times 10^{10}\ Hz = 11.5\ GHz \\
\lambda_2 = 1.14\ cm \leftrightarrow f_2 = 2.64 \times 10^{10}\ Hz = 26.4\ GHz \\
\lambda_3 = 0.73\ cm \leftrightarrow f_3 = 4.13 \times 10^{10}\ Hz = 41.3\ GHz \\
\lambda_4 = 0.53\ cm \leftrightarrow f_4 = 5.63 \times 10^{10}\ Hz = 56.3\ GHz \\
\lambda_5 = 0.42\ cm \leftrightarrow f_5 = 7.13 \times 10^{10}\ Hz = 71.3\ GHz \\
\lambda_6 = 0.35\ cm \leftrightarrow f_6 = 8.63 \times 10^{10}\ Hz = 86.3\ GHz \\
\quad\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad \vdots
\end{array}\right.$$

Note that because the electric field $\tilde{\tilde{E}}(\rho = a) = 0$ for these specific frequencies (corresponding to the zeroes of $J_0(x)$=$J_0(ka)$), this means that physically, we could actually <u>short out</u> the capacitor at $\rho = a$ and it wouldn't make <u>*any*</u> difference to the behavior / physics of this "capacitor" at these specific frequencies f_1, f_2, f_3, . .!!!

For ka = zero of $J_0(ka)$ {*i.e.* $J_0(ka) = 0$), we can short out the capacitor by wrapping it *e.g.* with sheet metal at $\rho = a$, thus turning it into a <u>cylindrical, fully-enclosed can</u> with $d \ll a$!



$\rightarrow$ No change in physics for frequencies f_1, f_2, f_3, . . because $\vec{E}(\rho = a) = 0$ for these frequencies!

Thus, at these frequencies f_1, f_2, f_3, . . f_n corresponding to the zeroes of the Bessel Function $J_0(ka)$ (*i.e.* $J_0(ka) = 0$), a cylindrical conducting metal can of radius a and height $d \ll a$ is actually a <u>resonant</u> <u>cavity</u> with electric field:

$\tilde{\tilde{E}}(\rho, t) = J_0(k_n\rho) E_o e^{i\omega_n t}\hat{z}$ and magnetic field:

$\tilde{\tilde{B}}(\rho, t) = J_0(k_n\rho) B_0(\rho) e^{i\omega t}\hat{\varphi}$

subject to the <u>boundary conditions</u> that:

$\tilde{E}_{\parallel}(\rho = a, t) = \tilde{E}_z(\rho = a, t) = 0$ and also that:

$\tilde{B}_{\perp}(\rho = a, t) = \tilde{B}_\rho(\rho = a, t) = 0$

for: $k_n = \frac{\omega_n}{c}$, $\omega_n = 2\pi f_n$, $n = 1, 2, 3, \ldots$

with: $B_o = \frac{ik\rho}{2c} E_o$ and: $E_o = -V_o/d$

We will see shortly in the next set of P436 Lecture Notes (# 10) that the resonant frequencies of a resonant cavity and the allowed modes of *EM* wave propagation in wave guides can be derived directly from the wave equation for *EM* waves in these structures, as determined by the boundary conditions imposed on the *EM* waves by the conducting walls of these devices and also the allowed polarization states of these *EM* waves.

Here in these lecture notes, we obtained harmonic *EM* wave solutions for $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ in the gap region of a parallel plate capacitor (and cylindrical can capacitor, subject to boundary condition $E = 0$ at $\rho = a$) via a perturbative technique, analogous to what we did last semester in P435 for the $\vec{E}$-field associated with a dielectric sphere immersed in an initially uniform external $\tilde{\vec{E}}$-field and the $\vec{B}$-field associated with a magnetizable sphere immersed in an initially uniform external $\tilde{\vec{B}}$-field. (See/work Griffiths Problems 4.23 and 6.18).

"Homework" Exercises:

1.) Calculate the electric, magnetic and total energy densities $u_E(\rho,\varphi,z,t)$, $u_m(\rho,\varphi,z,t)$ and $u_{Tot}(\rho,\varphi,z,t)$ and their time averages; make *e.g.* plots of these *vs.* ρ. Investigate/plot their behavior for low frequencies $(\omega \approx 0)$ and at higher frequencies, when $\omega_n = ck_n = c(x_n/a)$ where $x_n = k_n a$ = zeroes of $J_0(x_n) = 0$.

2.) Calculate Poynting's vector $\tilde{\vec{S}}(\rho,\varphi,z,t) = \frac{1}{\mu_o}\tilde{\vec{E}}(\rho,\varphi,z,t) \times \tilde{\vec{B}}(\rho,\varphi,z,t)$ and its time average; make plots of $\left|\tilde{\vec{S}}(\rho)\right|$ *vs.* ρ, investigate/plot its behavior for low frequencies $(\omega \approx 0)$ and when $\omega_n = ck_n = c(x_n/a)$.

3.) Calculate the linear *EM* momentum density, $\tilde{\vec{\wp}}_{EM}(\rho,\varphi,z,t) = \varepsilon_o\mu_o\tilde{\vec{S}}(\rho,\varphi,z,t)$ and angular momentum density, $\tilde{\vec{\ell}}_{EM}(\rho,\varphi,z,t) = \vec{r} \times \tilde{\vec{\wp}}_{EM}(\rho,\varphi,z,t)$, and their time averages; make plots of $\tilde{\vec{\wp}}_{EM}(\rho)$ and $\tilde{\vec{\ell}}_{EM}(\rho)$ *vs.* ρ; investigate/plot their behavior for low frequencies $(\omega \approx 0)$ and when $\omega_n = ck_n = c(x_n/a)$.

