Remarks on units

Recall that we write Coulomb's law as

$$\vec{F} = \frac{q q'}{r^3} \vec{r} \qquad \text{(gaussian units)}$$

instead of the perhaps more familiar

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q q'}{r^3} \vec{r} \qquad \text{(Rationalized MKSA)}.$$

Here, $\frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$, where c is speed of light.

In gaussian units charged is measured in statcoulombs: 2 charges of 1 statcoulomb each experience a force of 1 dyne when separated by 1 cm.

In order to find the relation between Coulombs and statcoulombs, we use this definition:

$$\frac{1 \text{ dyne}}{1 \text{ Newton}} = \frac{(s/m)^2}{4\pi\epsilon_0} \left(\frac{1 \text{ statcoulomb}}{1 \text{ coulomb}} \right)^2 \frac{1}{\left(\frac{1\text{cm}}{1\text{m}}\right)^2}$$

using 1 Newton $= 10^5$ dyne, $1\text{m} = 10^2$ cm we find

$$1 \text{ Coulomb} = 10 \frac{c}{m/sec} \text{ statcoulomb} \simeq 3 \cdot 10^9 \text{ statcoulomb}$$

In our units, the potential created by a charge q is

$$\phi = \frac{q}{r} \qquad (\text{gaussian})$$

instead of the more familiar

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \qquad (\text{rationalized MKSA})$$

Therefore, in gaussian units the potential has units of $1 \text{ statvolt} \equiv \frac{\text{statcoulomb}}{\text{cm}}$

Alternatively, since $U = \frac{q q'}{r}$ (gaussian) is the potential energy of charge q' in the field created by q

$$1 \text{ statvolt} = \frac{1 \text{ erg}}{1 \text{ statcoulomb}} \qquad \text{(gaussian)}$$

Because

$$1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}}, \qquad \text{(rationalized MKSA)}$$

we find using $1 \text{ Joule} = 10^{7} \text{ ergs}$

$$1 \text{ statvolt} = \frac{10^{-7} \text{ Joule}}{10^{-1} \left(\frac{c}{m/s}\right)^{-1} \text{ Coulomb}} = \left(\frac{c}{m/s}\right) \cdot 10^{-6} \text{ Volt} \approx 300 \text{ Volt}.$$

In both systems of units the electric field is $\vec{E} = -\vec{\nabla}\phi$. Therefore, the units of <u>electric field</u> are

$$\frac{\text{statvolt}}{\text{cm}} \qquad \text{(gaussian)}$$

$$\frac{1 \text{ V}}{\text{m}} \qquad \text{(Rationalized MKSA)}$$

In a separation ansatz solutions the units typically show up in the expansion coeff. For instance,

in one case we saw

$$\phi(x,y,z) = \sum_{m,n} \alpha_{mn} \sin\left(\frac{m\pi x}{A}\right) \sin\left(\frac{n\pi y}{B}\right) \sinh(\gamma_{mn} z)$$

with $\gamma_{mn} = \pi \sqrt{\frac{m^2}{A^2} + \frac{n^2}{B^2}}$ and

$$\alpha_{mn} = \frac{1}{\sinh(\gamma_{mn} C)} \frac{4}{AB} \int_0^A dx \int_0^B dy \; V(x,y) \sin\left(\frac{m\pi x}{A}\right) \sin\left(\frac{n\pi y}{B}\right)$$

whereas in another instance

$$\phi = \sum_{lm} \frac{a_{lm}}{r^{l+1}} Y_{lm}(\theta,\varphi) \quad \text{with}$$

$$a_{lm} = R^{l+1} \int d\Omega \; V(\theta,\varphi) \, Y^*_{lm}(\theta,\varphi)$$

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going back to Poissons's equation:

we saw that a solution of $\nabla^2\phi = -4\pi\rho$ satisfies

$$\phi(\vec{r}) = \int d^3r' \; G(\vec{r},\vec{r}') \rho(\vec{r}') - $$

$$- \frac{1}{4\pi} \oint_{\partial V} [\phi(\vec{r}') \vec{\nabla}_{r'} G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}') \vec{\nabla}_{r'} \phi(\vec{r}')] d\vec{A},$$

where $G(\vec{r},\vec{r}')$ is a green's function:

$$\vec{\nabla}^2_{r'} \, G(\vec{r}, \vec{r}\,') = -4\pi \, \delta(\vec{r} - \vec{r}\,')$$

The Green's function is not unique: given a solution of Laplace's equation, $\vec{\nabla}^2 F(\vec{r}, \vec{r}\,') = 0$, we can construct an additional Green's function

$$G(\vec{r}, \vec{r}\,') \longrightarrow G(\vec{r}, \vec{r}\,') + F(\vec{r}, \vec{r}\,') .$$

We can use this freedom to simplify the contribution of the boundary term and incorporate boundary conditions:

i) For Dirichlet bc, choose

$$G(\vec{r}, \vec{r}\,') = 0 \quad \text{for } \vec{r}\,' \in \partial V .$$

Then,

$$\phi(\vec{r}) = \int_V d^3 r' \, G(\vec{r}, \vec{r}\,') \rho(\vec{r}\,') - \frac{1}{4\pi} \oint_{\partial V} \underbrace{\phi(\vec{r}\,')}_{\text{known.}} \vec{\nabla}_{r'} G(\vec{r}, \vec{r}\,') \cdot d\vec{A}$$

We won't consider Neumann bc here (see Jackson for how to deal with them)

Note that the Green's function is symmetric:

$$G(\vec{r}, \vec{r}\,') = G(\vec{r}\,', \vec{r}).$$

This follows by applying Green's first identity to $\phi = G(\vec{r}_1, \vec{r}\,')$ and $\psi = G(\vec{r}_2, \vec{r}\,')$.

5.7. Green's functions in specific cases

By construction, a Green's function is the potential created by a (unit) point charge, with appropriate b.c. Thus, the Green's function with the b.c. $\phi(r \to \infty) = 0$ is

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|}$$

If we are interested in the Green's function with Dirichlet b.c. on a sphere, we need the field of a point charge outside a grounded sphere. In this case, by the method of images:

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{R/r'}{|\vec{r} - R^2 \vec{r}\,'/r'^2|}$$

5.8. Constructing Green's functions

In some cases it is more convenient to "construct" the Green's function by using the eigensolutions of Laplace's equation:

Suppose that we have a set of functions $u_{\vec{k}}(\vec{x})$ such that

$$\vec{\nabla}^2 u_{\vec{k}}(\vec{x}) = \lambda_{\vec{k}}\, u_{\vec{k}}(\vec{x})$$

($\vec{k}$ is an array of labels)

Such eigenvectors can be taken to be an orthonormal set:

$$\sum_{\vec{k}} u^*_{\vec{k}}(\vec{x})\, u_{\vec{k}}(\vec{x}') = \delta(\vec{x}'-\vec{x})$$

Therefore,

$$G(\vec{x},\vec{x}') \equiv -4\pi \sum_{\vec{k}} \frac{u^*_{\vec{k}}(\vec{x})\, u_{\vec{k}}(\vec{x}')}{\lambda_{\vec{k}}}$$ is

a Green's function, as you can check by acting left and right with $\vec{\nabla}^2_{x'}$.

Note that the boundary conditions satisfied by G depend on the bc satisfied by the eigenfunctions $u_{\vec{k}}$.

Example

Suppose we demand that $u_{\vec{k}}(\vec{x})$ remain finite at $|\vec{x}| = \infty$. The solutions of

$$\vec{\nabla}^2 u_{\vec{k}} = \lambda_{\vec{k}} \, u_{\vec{k}} \quad \text{are then}$$

$$u_{\vec{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}, \quad \text{with} \quad \lambda_{\vec{k}} = -\vec{k}^2.$$

These solutions are "orthonormal":

$$\frac{1}{(2\pi)^3} \int d^3\vec{k} \; e^{i\vec{k}(\vec{x}-\vec{x}')} = \delta(\vec{x}-\vec{x}')$$

and form a basis in the space of real functions.

Furthermore,

$$-\frac{4\pi}{(2\pi)^3} \int d^3k \; \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{(-k^2)} = \frac{1}{|\vec{x}-\vec{x}'|}$$

Exercise 12

Show that

$$\frac{1}{2\pi^2}\int d^3k \; \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{k^2} = \frac{1}{|\vec{x}-\vec{x}'|}$$

■

Note: $\frac{1}{k^2}$ is basically the "propagator" of a <u>massless</u> particle in momentum space.

5.8.2 Reduction to a 1-dim. Green's function

Similar techniques can be applied to pieces of the $\vec{\nabla}^2$ operator. Consider for instance a Green's function in spherical words.

Suppose we make the ansatz

$$G(r,\theta,\varphi;\, r',\theta',\varphi') := -4\pi \sum_{lm} f_l(r,r')\, Y^*_{lm}(\theta,\varphi)\, Y_{lm}(\theta',\varphi').$$

Applying $\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{L^2}{r^2}$ we find

$$\vec{\nabla}_1^2 G = -4\pi \sum_{lm}\left\{\frac{1}{r'^2}\partial_{r'}\left[r'^2\partial_{r'} f_l(r,r')\right] - \frac{1}{r'^2}\, l(l+1) f_l(r,r')\right\} Y^*_{lm}(\theta,\varphi)\, Y_{lm}(\theta',\varphi')$$

$$\stackrel{!}{=} -4\pi\, \delta(r-r')\, \delta(\cos\theta-\cos\theta')\, \delta(\varphi-\varphi')$$

Because the Y_{lm} form a complete set on the sphere,

$$\sum_{lm} Y^*_{lm}(\theta,\varphi)\, Y_{lm}(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\,\delta(\varphi-\varphi')$$

we need to solve

$$\frac{1}{r'^2}\partial_{r'}\left[r'^2 \partial_{r'} f_l(r,r')\right] - l(l+1) f_l(r,r') = \delta(r-r'). \quad (*)$$

and we demand that $f_l(r, r'=R)=0$

(this would be the appropriate condition if we are interested in specifying ϕ at the surface of a sphere).

To solve $(*)$ we note that at $r\neq r'$, the lhs must vanish (it satisfies Laplace's equation). From the sols. in Lecture 5, we thus know that at $r\neq r'$,

$$f_l(r,r') = a(r)(r')^l + b(r)\cdot\frac{1}{(r')^{l+1}}$$