

LECTURE NOTES 16

THE STRUCTURE OF SPACE-TIME

Lorentz Transformations Using Four-Vectors:

Space-time {as we all know...} has four dimensions:

1 time dimension & 3 {orthogonal} space dimensions: $(t, \vec{r} = x\hat{x} + y\hat{y} + z\hat{z})$.

Einstein's Theory of (Special) Relativity:

1-D time and 3-D space are placed on an equal/symmetrical footing with each other.

We use 4-vector/tensor notation for relativistic kinematics and relativistic electrodynamics because the mathematical description of the physics takes on a simpler, and more elegant appearance; the principles and physical consequences of the physics are also made clearer / more profound!

Lorentz Transformations Expressed in 4-Vector Notation:

We define any 4-vector: $x^\mu \equiv (x^0, x^1, x^2, x^3)$ Note the {contravariant} superscripts, here!

Where, by convention, the 0th component of the 4-vector, $x^0 =$ is the temporal (time-like), {i.e. scalar} component of the 4-vector x^μ , and (x^1, x^2, x^3) are the (x, y, z) spatial / space-like {i.e. 3-vector} components of the 4-vector x^μ , respectively.

n.b. Obviously, the physical SI units of the 4-vector components must all be the same!!!

For space-time 4-vectors, we define {contravariant/superscript} x^μ as:

$$\left. \begin{array}{l} x^0 \equiv ct \\ x^1 \equiv x \\ x^2 \equiv y \\ x^3 \equiv z \end{array} \right\} x^\mu \equiv (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

Then the Lorentz transformation of space-time quantities in IRF(S) to IRF(S'), the latter of which is moving e.g. with velocity $\vec{v} = +v\hat{x}$ relative to IRF(S) is given by:

$$\left. \begin{array}{l} \text{Original} \\ \text{4-vector} \\ \text{Notation:} \end{array} \right\} \left\{ \begin{array}{l} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x'^0 = \gamma(x^0 - \beta x^1) \\ x'^1 = \gamma(x^1 - \beta x^0) \\ x'^2 = x^2 \\ x'^3 = x^3 \end{array} \right\} \left. \begin{array}{l} \text{New/Tensor} \\ \text{4-vector} \\ \text{Notation} \end{array} \right\}$$

Where: $\beta \equiv v/c$ and: $\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$

We can also write these four equations (either version) in matrix form:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Each of the four above equations of the RHS representation can also be written compactly and elegantly in tensor notation as:

$$\boxed{x'^{\mu} = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} x^{\nu}} \quad \text{where: } \mu = 0, 1, 2, 3 \quad \text{and: } \Lambda \equiv \text{Lorentz Transformation Matrix}$$

$$\Lambda \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 & \Lambda_3^0 \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \\ \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 \end{pmatrix} \quad \begin{array}{l} \leftarrow \text{row index} \\ \downarrow \text{column index} \end{array}$$

where: $\Lambda_{\nu}^{\mu} = \Lambda_{\text{column}}^{\text{row}} = (\mu - \nu)^{\text{th}}$ element of Λ
 Superscript, $\mu = 0, 1, 2, 3 =$ row index
 Subscript, $\nu = 0, 1, 2, 3 =$ column index

We explicitly write out each of the four equations associated with $x'^{\mu} = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} x^{\nu}$ for $\mu = 0, 1, 2, 3$:

$x'^0 = \sum_{\nu=0}^3 \Lambda_{\nu}^0 x^{\nu} = \Lambda_0^0 x^0 + \Lambda_1^0 x^1 + \Lambda_2^0 x^2 + \Lambda_3^0 x^3 = \gamma(x^0 - \beta x^1)$
$x'^1 = \sum_{\nu=0}^3 \Lambda_{\nu}^1 x^{\nu} = \Lambda_0^1 x^0 + \Lambda_1^1 x^1 + \Lambda_2^1 x^2 + \Lambda_3^1 x^3 = \gamma(x^1 - \beta x^0)$
$x'^2 = \sum_{\nu=0}^3 \Lambda_{\nu}^2 x^{\nu} = \Lambda_0^2 x^0 + \Lambda_1^2 x^1 + \Lambda_2^2 x^2 + \Lambda_3^2 x^3 = x^2$
$x'^3 = \sum_{\nu=0}^3 \Lambda_{\nu}^3 x^{\nu} = \Lambda_0^3 x^0 + \Lambda_1^3 x^1 + \Lambda_2^3 x^2 + \Lambda_3^3 x^3 = x^3$

We can write this relation even more compactly using the Einstein Summation Convention:
 Repeated indices are always summed over:

$$\boxed{x'^{\mu} = \Lambda_{\nu}^{\mu} x^{\nu} = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} x^{\nu}}$$

The RHS of this equation has repeated index ν , which implicitly means we are to sum over it, i.e.

Thus, $\boxed{x'^{\mu} = \Lambda_{\nu}^{\mu} x^{\nu}}$ is simply shorthand notation for: $\boxed{x'^{\mu} = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} x^{\nu}}$

People (including Einstein) got / get tired of explicitly writing all of the summation symbols

$$\sum_{\nu=0}^3 \text{ all the time / everywhere....}$$

The nature / composition of the Lorentz transformation matrix Λ (a rank-two, $4 \times 4 = 16$ component tensor) defines the space-time structure of our universe, i.e. specifies the rules for transforming from one IRF to another IRF.

Generally speaking {broadband mathematically}, one can define a 4-vector a^μ to be anything one wants, however for Special Relativity and for Lorentz transformations between one IRF and another, our 4-vectors are only those which transform from one IRF to another IRF as:

$$a'^{\mu} = \sum_{\nu=0}^3 \Lambda_{\nu}^{\mu} a^{\nu} \Rightarrow a'^{\mu} = \Lambda_{\nu}^{\mu} a^{\nu}$$

This compact relation mathematically defines the space-time nature, or structure of our universe!

For a Lorentz transformation along the $\hat{1} = \hat{x}$ axis, with: $\vec{v} = +v\hat{x}$ and thus $\vec{\beta} = \beta\hat{x}$, $\vec{\beta} = \vec{v}/c$, for a 4-vector $a^\mu = (a^0, a^1, a^2, a^3)$, where a^0 is the temporal/scalar component and $\vec{a} = (a^1, a^2, a^3) = (a_x, a_y, a_z)$ are the $(\hat{x}, \hat{y}, \hat{z})$ spatial/3-vector components of the 4-vector a^μ , then $a'^{\mu} = \Lambda_{\nu}^{\mu} a^{\nu}$ written out in matrix form is:

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \gamma(a^0 - \beta a^1) \\ \gamma(a^1 - \beta a^0) \\ a^2 \\ a^3 \end{pmatrix}$$

Dot Products with 4-Vectors:

In “standard” 3-D space-type vector algebra, we have the familiar dot product:

$$\vec{a} \cdot \vec{b} \equiv (a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) = a_x b_x + a_y b_y + a_z b_z = \text{scalar quantity (i.e. = pure \#)}$$

∃ A relativistic 4-vector analog of this, but it is NOT simply the sum of like components.

Instead, the zeroth component product of a relativistic 4-vector dot product has a minus sign:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \Leftarrow \text{four-dimensional scalar product / dot product (= pure \#)}$$

Just as an ordinary / “normal” 3-D vector product $\vec{a} \cdot \vec{b}$ is invariant (i.e. unchanged) under 3-D space rotations ($\vec{a} \cdot \vec{b}$ is the length of vector \vec{b} projected onto \vec{a} {and/or vice versa} – a length does not change under a 3-D space rotation), the four-dimensional scalar product between two relativistic 4-vectors is invariant (i.e. unchanged) under any/all Lorentz transformations, from one IRF(S) to another IRF(S').

i.e. The scalar product/dot product of any two relativistic 4-vectors is a Lorentz invariant quantity.
 \Rightarrow The scalar product/dot product of any two relativistic 4-vectors has the same numerical value in any/all IRFs !!!

Thus: $\underbrace{-a'^0b'^0 + a'^1b'^1 + a'^2b'^2 + a'^3b'^3}_{\text{In IRF}(S')} = \underbrace{-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3}_{\text{In IRF}(S)} = \text{pure \#}$

However, in order to keep track of the minus sign associated with the temporal component of a 4-vector, especially when taking (i.e. computing) a scalar/dot product, we introduce the notion of contravariant and covariant 4-vectors.

What we have been using thus far in these lecture notes are contravariant 4-vectors a^μ , denoted by the superscript μ :

$$\boxed{a^\mu = (a^0, a^1, a^2, a^3)} = \text{Contravariant 4-vector: } a^\mu = a^{\text{row}} = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

A covariant 4-vector a_μ is denoted by its subscript μ :

$$\boxed{a_\mu = (a_0, a_1, a_2, a_3)} = \text{Covariant 4-vector: } a_\mu = a_{\text{column}} = (a_0 \ a_1 \ a_2 \ a_3)$$

The temporal/zeroth component {only} of covariant a_μ differs from that of contravariant a^μ by a minus sign:

$$\begin{array}{l} \boxed{a_0 = -a^0} \\ \boxed{a_1 = +a^1} \\ \boxed{a_2 = +a^2} \\ \boxed{a_3 = +a^3} \end{array} \left. \begin{array}{l} \leftarrow \text{Note the - sign difference !!!} \\ \\ \end{array} \right\} \text{Same sign for both contravariant and covariant 4-vectors}$$

Thus, raising {or lowering} the index μ of a 4-vector, e.g. $\boxed{a_\mu \rightarrow a^\mu}$ or $\boxed{a^\mu \rightarrow a_\mu}$ changes the sign of the zeroth (i.e. temporal/scalar) component of the 4-vector {only}.

⇒ That's why we have to pay {very} close attention to subscripts vs. superscripts here !!!

Thus, a 4-vector scalar/dot product (= a Lorentz invariant quantity) may be written using contravariant and covariant 4-vectors as:

$$\boxed{\sum_{\mu=0}^3 a_\mu b^\mu} = \boxed{\sum_{\mu=0}^3 a^\mu b_\mu} = \boxed{-a_0b^0 + a_1b^1 + a_2b^2 + a_3b^3} = \boxed{-a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3} = \text{pure \#}$$

This {again} can be written more compactly / elegantly / succinctly using the Einstein summation convention (i.e. summing over repeated indices) as:

$$\boxed{a_\mu b^\mu = a^\mu b_\mu} = \boxed{-a_0b^0 + a_1b^1 + a_2b^2 + a_3b^3} = \boxed{-a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3} = \text{pure \#}$$

There is an interesting parallel between relativistic Lorentz transformations (to/from different IRF's in space-time) and spatial rotations in 3-D space:

A spatial rotation in 3-dimensional Euclidean space (e.g. for a rotation about the \hat{z} -axis) can be written in matrix form as:

$$\underbrace{\begin{pmatrix} a'_x \\ a'_y \\ a'_z \end{pmatrix}}_{\vec{a}' \text{ or } a'^{\mu}} = \underbrace{\begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\vec{R} \text{ or } R^{\mu}} \underbrace{\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}}_{\vec{a} \text{ or } a^{\mu}} = \begin{pmatrix} a_x \cos \varphi + a_y \sin \varphi \\ -a_x \sin \varphi + a_y \cos \varphi \\ a_z \end{pmatrix}$$

$= 2^{\text{nd}} \text{ rank, } 3 \times 3 = 9 \text{ component tensor}$

$$\Rightarrow \boxed{\vec{a}' = \vec{R} \cdot \vec{a}} \text{ (in 3-D vector notation) or: } \boxed{a'^{\mu} = R^{\mu}_{\nu} a^{\nu}} \text{ (in tensor notation, } \mu, \nu = 1:3)$$

Compare this to the Lorentz transformation (e.g. along the \hat{x} -axis) for 4-vectors in space-time:

$$\underbrace{\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix}}_{a'^{\mu}} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^{\mu}_{\nu}} \underbrace{\begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}}_{a^{\mu}} = \begin{pmatrix} \gamma a^0 - \gamma\beta a^1 \\ -\gamma\beta a^0 + \gamma a^1 \\ a^2 \\ a^3 \end{pmatrix} \Rightarrow \boxed{a'^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}}$$

$\mu, \nu = 0:3$

Comparing the matrix for \vec{R} or R^{μ} with that of Λ^{μ}_{ν} , we can see that a Lorentz transformation from one IRF to another is analogous to/has similarities to a physical rotation in 3-D Euclidean space – i.e. a Lorentz transformation is a certain kind of rotation in space-time – where the rotation is between the longitudinal space dimension (= the direction of the Lorentz boost, *a.k.a.* the “boost axis”) and time!

In order to make this parallel somewhat sharper, we introduce a new kinematic variable, known as the rapidity (ζ), which is defined as:

$$\boxed{\zeta \equiv \tanh^{-1} \beta = \tanh^{-1}(v/c)} \text{ or: } \boxed{\tanh \zeta \equiv \beta = v/c} \text{ where: } \boxed{-1 \leq \beta (= v/c) \leq +1} \text{ thus: } \boxed{-\infty \leq \zeta \leq +\infty}$$

$$\text{Since: } \boxed{\tanh \zeta = \frac{\sinh \zeta}{\cosh \zeta}} \text{ and: } \boxed{\cosh^2 \zeta - \sinh^2 \zeta = 1}$$

$$\text{Then: } \boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2 \zeta}} = \frac{\cosh \zeta}{\sqrt{\cosh^2 \zeta - \sinh^2 \zeta}} = \cosh \zeta} \text{ i.e. } \boxed{\gamma = \cosh \zeta}$$

$$\therefore \boxed{\gamma\beta = \cosh \zeta \tanh \zeta = \cancel{\cosh \zeta} \frac{\sinh \zeta}{\cancel{\cosh \zeta}} = \sinh \zeta} \text{ i.e. } \boxed{\gamma\beta = \sinh \zeta} \text{ and: } \boxed{\beta = \tanh \zeta}$$

Then the Lorentz transformation (along the \hat{x} -axis) $a'^{\mu} = \Lambda_{\nu}^{\mu} a^{\nu}$ of a 4-vector a^{μ} can be written {using $\beta = \tanh \zeta$, $\gamma = \cosh \zeta$ and $\gamma\beta = \sinh \zeta$ } as:

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} a^0 \cosh \zeta - a^1 \sinh \zeta \\ -a^0 \sinh \zeta + a^1 \cosh \zeta \\ a^2 \\ a^3 \end{pmatrix}$$

Again, compare this with the 3-D space rotation of a 3-D space-vector \vec{a} about the \hat{z} -axis:

$$\begin{pmatrix} a'_x \\ a'_y \\ a'_z \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} a_x \cos \varphi + a_y \sin \varphi \\ -a_x \sin \varphi + a_y \cos \varphi \\ a_z \end{pmatrix}$$

We see that the above Lorentz transformation is similar (but not identical) to the expression for the 3-D Euclidean geometry spatial rotation!

However because of the $\sinh \zeta$ and $\cosh \zeta$ nature associated with the Lorentz transformation we see that the Lorentz transformation is in fact a hyperbolic rotation in space-time – i.e. the transformation of the longitudinal space dimension associated with the axis parallel to the Lorentz boost direction and time is that of a hyperbolic-type rotation!!!

The use of the rapidity variable, ζ has benefits e.g. for the Einstein Velocity Addition Rule:

If $\vec{u} = d\vec{x}/dt$ = the velocity of a particle as seen by a {stationary} observer in IRF(S) and $\vec{u}' = d\vec{x}'/dt'$ = the velocity of the particle as seen by a {stationary} observer in IRF(S') and $\vec{v} = v\hat{x}$ = the {relative} velocity between IRF(S) and IRF(S'), then u' is related to u by:

$$u' = \frac{u - v}{1 - uv/c^2} \leftarrow \text{Einstein Velocity Addition Rule (1-D Case)}$$

We can re-write this as: $u'/c = \frac{u/c - v/c}{1 - uv/c^2} \Rightarrow \beta'_{u'} = \frac{\beta_u - \beta}{1 - \beta_u \beta}$

Then since: $\beta \equiv \tanh \zeta$ we can similarly define: $\beta_u \equiv \tanh \zeta_u$ and $\beta'_{u'} \equiv \tanh \zeta'_{u'}$.

Then: $\beta'_{u'} = \frac{\beta_u - \beta}{1 - \beta_u \beta} \Rightarrow \tanh \zeta'_{u'} = \frac{\tanh \zeta_u - \tanh \zeta}{1 - \tanh \zeta_u \tanh \zeta} \equiv \tanh(\zeta_u - \zeta) \leftarrow \begin{matrix} \text{See e.g. CRC Handbook} \\ \text{r.e. trigonometric identities} \\ \text{for hyperbolic functions!} \end{matrix}$

Thus: $\tanh \zeta'_{u'} = \tanh(\zeta_u - \zeta)$

Or: $\zeta'_{u'} = \zeta_u - \zeta \leftarrow \text{Rapidity form of the Einstein Velocity Addition Rule (1-D Case)}$

\Rightarrow Rapidities $\zeta \equiv \tanh^{-1} \beta$ are additive quantities in going from one IRF to another IRF !!!

Rapidity Addition Law: $\zeta' = \zeta_u - \zeta$

$$\zeta' = \tanh^{-1}\left(\frac{u'}{c}\right)$$

$$\zeta_u = \tanh^{-1}\left(\frac{u}{c}\right)$$

$$\zeta = \tanh^{-1}\left(\frac{v}{c}\right) \equiv \tanh^{-1}(\beta)$$

Velocities (certainly) are **not** additive in going from one IRF to another.

However: $\zeta = \tanh^{-1}(\beta)$ rapidities **are** additive in this regard.

We explicitly show that the “dot products” $x_\mu x^\mu$ and $x'_\mu x'^\mu$ are Lorentz invariant quantities:

$$x_\mu = (x_0 \ x_1 \ x_2 \ x_3) = (-ct \ x \ y \ z)$$

$$x'_\mu = (x'_0 \ x'_1 \ x'_2 \ x'_3) = (-ct' \ x' \ y' \ z')$$

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{aligned} x_\mu x^\mu &= (x_0 \ x_1 \ x_2 \ x_3) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = (-ct \ x \ y \ z) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \\ &= x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 = -(ct)^2 + x^2 + y^2 + z^2 = x^\mu x_\mu \end{aligned}$$

$$\begin{aligned} x'_\mu x'^\mu &= (x'_0 \ x'_1 \ x'_2 \ x'_3) \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = (-ct' \ x' \ y' \ z') \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \\ &= x'_0 x'^0 + x'_1 x'^1 + x'_2 x'^2 + x'_3 x'^3 = -(ct')^2 + x'^2 + y'^2 + z'^2 = x'^\mu x'_\mu \end{aligned}$$

But: $x'^\mu = \Lambda^\mu_\nu x^\nu$ and: $x'_\mu = \Lambda^\nu_\mu x_\nu$

For Lorentz Transform (a.k.a. Lorentz “Boost”) along \hat{x} direction:

$$x'^\mu = \Lambda^\mu_\nu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta x) \\ \gamma(x - \beta ct) \\ y \\ z \end{pmatrix}$$

And: $x'_\mu = \Lambda_\mu^\nu x_\nu = (-ct \ x \ y \ z) \begin{pmatrix} +\gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (-\gamma(ct - \beta x) \ \gamma(x - \beta ct) \ y \ z)$

Thus:

$$\begin{aligned} x'_\mu x'^\mu &= (-\gamma(ct - \beta x) \ \gamma(x - \beta ct) \ y \ z) \begin{pmatrix} \gamma(ct - \beta x) \\ \gamma(x - \beta ct) \\ y \\ z \end{pmatrix} \\ &= -\gamma^2 (ct - \beta x)^2 + \gamma^2 (x - \beta ct)^2 + y^2 + z^2 \\ &= -\gamma^2 [(ct)^2 - 2\beta xct + \beta^2 x^2] + \gamma^2 [x^2 - 2\beta xct + \beta^2 (ct)^2] + y^2 + z^2 \\ &= -\gamma^2 (ct)^2 + \cancel{2\gamma^2\beta xct} - \gamma^2 \beta^2 x^2 + \gamma^2 x^2 - \cancel{2\gamma^2\beta xct} + \gamma^2 \beta^2 (ct)^2 + y^2 + z^2 \\ &= -\gamma^2 (ct)^2 + \gamma^2 \beta^2 (ct)^2 + \gamma^2 x^2 - \gamma^2 \beta^2 x^2 + y^2 + z^2 \\ &= -\gamma^2 (1 - \beta^2) (ct)^2 + \gamma^2 (1 - \beta^2) x^2 + y^2 + z^2 \end{aligned}$$

But: $\gamma^2 \equiv 1/1 - \beta^2$

$\therefore x'^\mu x'_\mu = -\left(\frac{1 - \beta^2}{1 - \beta^2}\right) (ct)^2 + \left(\frac{1 - \beta^2}{1 - \beta^2}\right) x^2 + y^2 + z^2 = -(ct)^2 + x^2 + y^2 + z^2$

i.e. $x'^\mu x'_\mu = x'_\mu x'^\mu = -(ct')^2 + x'^2 + y'^2 + z'^2 = -(ct)^2 + x^2 + y^2 + z^2 = x^\mu x_\mu = x_\mu x^\mu$

$\therefore x'_\mu x'^\mu = x'^\mu x'_\mu = x_\mu x^\mu = x^\mu x_\mu$ are indeed Lorentz invariant quantities!

Various Forms of Lorentz Transformations from IRF(S) to IRF(S'):

1.) 1-D Lorentz Transform / "Boost" along the \hat{x} direction:

$$\beta_x \equiv \frac{v_x}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta_x^2}}$$

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_\nu^\mu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta_x x) \\ \gamma(x - \beta_x ct) \\ y \\ z \end{pmatrix}$$

2.) 1-D Lorentz Transform / “Boost” along the \hat{y} direction:

$$\beta_y \equiv \frac{v_y}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta_y^2}}$$

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_v^{\mu} x^{\nu} = \begin{pmatrix} \gamma & 0 & -\gamma\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta_y & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta_y y) \\ x \\ \gamma(y - \beta_y ct) \\ z \end{pmatrix}$$

 3.) 1-D Lorentz Transform / “Boost” along the \hat{z} direction:

$$\beta_z \equiv \frac{v_z}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta_z^2}}$$

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_v^{\mu} x^{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta_z z) \\ x \\ y \\ \gamma(z - \beta_z ct) \end{pmatrix}$$

 4.) 3-D Lorentz Transform / “Boost” along arbitrary \hat{r} direction:

$$\vec{\beta} \equiv \frac{\vec{v}}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

First, we define:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

In IRF(S)

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\vec{\beta} \equiv \frac{\vec{v}}{c}$$

$$\vec{\beta} = \beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z}$$

$$\beta \equiv |\vec{\beta}| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$$

Then:

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_v^{\mu} x^{\nu} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

{n.b. By inspection of this 3-D Λ -matrix for 1-D motion (i.e. only along \hat{x} , \hat{y} , or \hat{z}) it is easy to show that this expression reduces to the appropriate 1-D Lorentz transformation 1.) – 3.) above.}

Or:

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_{\vec{v}}^{\mu} x^{\nu} = \begin{pmatrix} \gamma(ct - \vec{\beta} \cdot \vec{r}) \\ x + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}) \beta_x - \gamma \beta_x ct \\ y + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}) \beta_y - \gamma \beta_y ct \\ z + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}) \beta_z - \gamma \beta_z ct \end{pmatrix} \quad \text{with: } \begin{cases} \vec{\beta} = \beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z} \\ \beta \equiv |\vec{\beta}| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} \\ \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \end{cases}$$

n.b. The $x'^0 = ct'$ equation follows trivially from $x'^0 = ct'$ in 1.) through 3.) above.

The 3-D spatial part may be written vectorially as: $\vec{r}' = \vec{r} + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}) \vec{\beta} - \gamma \vec{\beta} ct$

which may appear to be a more complicated expression, but it's really only sorting out components of \vec{r} and \vec{r}' that are \perp and \parallel to \vec{v} for separate treatment.

See J.D. Jackson's "Electrodynamics", 3rd Edition, p. 525 & p. 547 for more information.

Various Forms of Inverse Lorentz Transformations from IRF(S') to IRF(S):

1.) 1-D Lorentz Transform / "Boost" along the \hat{x}' direction: $\beta'_x \equiv \frac{v'_x}{c} = -\frac{v_x}{c} = -\beta_x$ $\gamma' = \frac{1}{\sqrt{1-\beta_x'^2}} = \gamma$

$$x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda_{\vec{v}}^{\prime\mu} x'^{\nu} = \begin{pmatrix} \gamma & +\gamma\beta_x & 0 & 0 \\ +\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct' + \beta_x x') \\ \gamma(x' + \beta_x ct') \\ y' \\ z' \end{pmatrix}$$

2.) 1-D Lorentz Transform / "Boost" along the \hat{y}' direction: $\beta'_y \equiv \frac{v'_y}{c} = -\frac{v_y}{c} = -\beta_y$ $\gamma' = \frac{1}{\sqrt{1-\beta_y'^2}} = \gamma$

$$x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda_{\vec{v}}^{\prime\mu} x'^{\nu} = \begin{pmatrix} \gamma & 0 & +\gamma\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ +\gamma\beta_y & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct' + \beta_y y') \\ x' \\ \gamma(y' + \beta_y ct') \\ z' \end{pmatrix}$$

3.) 1-D Lorentz Transform / “Boost” along the \hat{z}' direction: $\beta'_z \equiv \frac{v'_z}{c} = -\frac{v_z}{c} = -\beta_z$ $\gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \gamma$

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda_{\nu'}^{\mu} x'^{\nu} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct' + \beta_z z') \\ x' \\ y' \\ \gamma(z' + \beta_z ct') \end{pmatrix}$$

4.) 3-D Lorentz Transform / “Boost” along arbitrary \hat{r} direction: $\vec{\beta}' \equiv \frac{\vec{v}'}{c} = -\frac{\vec{v}}{c} = -\vec{\beta}$ $\gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \gamma$

First, we define:

$$\left. \begin{aligned} \vec{r}' &= x'\hat{x}' + y'\hat{y}' + z'\hat{z}' \\ \vec{v}' &= v'_x\hat{x}' + v'_y\hat{y}' + v'_z\hat{z}' = -\vec{v} \\ \vec{\beta}' &= \beta'_x\hat{x}' + \beta'_y\hat{y}' + \beta'_z\hat{z}' \\ &= -\vec{\beta} = -\beta_x\hat{x} - \beta_y\hat{y} - \beta_z\hat{z} \end{aligned} \right\} \text{In IRF}(S')$$

$$\begin{aligned} r' &= |\vec{r}'| = \sqrt{x'^2 + y'^2 + z'^2} \\ v' &= |\vec{v}'| = \sqrt{v_x'^2 + v_y'^2 + v_z'^2} = v \\ \beta' &\equiv |\vec{\beta}'| = \sqrt{\beta_x'^2 + \beta_y'^2 + \beta_z'^2} \\ &= \beta = |\vec{\beta}| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} \end{aligned}$$

Then:

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda_{\nu'}^{\mu} x'^{\nu} = \begin{pmatrix} \gamma & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & 1 + \frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ +\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ +\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

{n.b. By inspection of this 3-D Λ' -matrix for 1-D motion (i.e. only along \hat{x}' , \hat{y}' , or \hat{z}') it is easy to show that this expression reduces to the appropriate 1-D Lorentz transformation 1.) – 3.) above.}

Or:

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda_{\nu'}^{\mu} x'^{\nu} = \begin{pmatrix} \gamma(ct' + \vec{\beta} \cdot \vec{r}') \\ x' + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}') \beta_x + \gamma\beta_x ct' \\ y' + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}') \beta_y + \gamma\beta_y ct' \\ z' + \frac{(\gamma-1)}{\beta^2} (\vec{\beta} \cdot \vec{r}') \beta_z + \gamma\beta_z ct' \end{pmatrix} \quad \text{with:} \quad \begin{aligned} \vec{\beta}' &= \beta'_x\hat{x}' + \beta'_y\hat{y}' + \beta'_z\hat{z}' = -\vec{\beta} \\ \beta' &\equiv |\vec{\beta}'| = \sqrt{\beta_x'^2 + \beta_y'^2 + \beta_z'^2} = \beta \\ \gamma' &\equiv \frac{1}{\sqrt{1-\beta'^2}} = \gamma \end{aligned}$$

n.b. The $x^0 = ct$ equation follows trivially from $x^0 = ct$ in 1') through 3') above.

The 3-D spatial part may be written vectorially as: $\vec{r} = \vec{r}' + \frac{(\gamma - 1)}{\beta^2} (\vec{\beta} \cdot \vec{r}') \vec{\beta} + \gamma \vec{\beta} ct$

which may appear to be a more complicated expression, but it's really only sorting out components of \vec{r}' and \vec{r} that are \perp and \parallel to \vec{v}' for separate treatment.

Note also that $x^\mu = \Lambda_\nu^\mu x'^\nu$ but $x'^\nu = \Lambda_\tau^\nu x^\tau$. $\therefore x^\mu = \Lambda_\nu^\mu x'^\nu = \Lambda_\nu^\mu \Lambda_\tau^\nu x^\tau$

The quantity: $\Lambda_\nu^\mu \Lambda_\tau^\nu = 1_\tau^\mu$ = identity (i.e. unit) 4x4 matrix = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ← n.b. Lorentz-invariant quantity!!!

Thus: $x^\mu = \Lambda_\nu^\mu x'^\nu = \Lambda_\nu^\mu \Lambda_\tau^\nu x^\tau = 1_\tau^\mu x^\tau = x^\mu$ i.e. $\Lambda_\nu^\mu \Lambda_\tau^\nu = \Lambda_\tau^\nu \Lambda_\nu^\mu = 1_\tau^\mu$

We define the relativistic space-time interval between two “events” as the

Space-time difference: $\Delta x^\mu \equiv x_A^\mu - x_B^\mu$ ← known as the space-time displacement 4-vector

Event A occurs at space-time coordinates: $x_A^\mu = \begin{pmatrix} x_A^0 \\ x_A^1 \\ x_A^2 \\ x_A^3 \end{pmatrix} = \begin{pmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{pmatrix}$

Event B occurs at space-time coordinates: $x_B^\mu = \begin{pmatrix} x_B^0 \\ x_B^1 \\ x_B^2 \\ x_B^3 \end{pmatrix} = \begin{pmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{pmatrix}$

The scalar 4-vector product of $\Delta x_\mu \Delta x^\mu = \Delta x^\mu \Delta x_\mu$ is a Lorentz invariant quantity, = same numerical value in any IRF, known as the interval I between two events:

Lorentz-Invariant Interval: $I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu$ = same numerical value in all IRF's.

$$I = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 \underbrace{(t_A - t_B)^2}_{\Delta t_{AB}^2} + \underbrace{(x_A - x_B)^2}_{\Delta x_{AB}^2} + \underbrace{(y_A - y_B)^2}_{\Delta y_{AB}^2} + \underbrace{(z_A - z_B)^2}_{\Delta z_{AB}^2}$$

$$= -c^2 \Delta t_{AB}^2 + \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2$$

Define the usual 3-D spatial distance: $d_{AB} \equiv \sqrt{\Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2}$

$\therefore I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t_{AB}^2 + \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2 = -c^2 \Delta t_{AB}^2 + d_{AB}^2$ ← Lorentz invariant quantity, same numerical value in all IRF's

Thus, when Lorentz transform from one IRF(S) to another IRF(S'):

$$\text{In IRF}(S): \quad I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t_{AB}^2 + d_{AB}^2$$

$$\text{In IRF}(S'): \quad I' \equiv \Delta x'^\mu \Delta x'_\mu = \Delta x'_\mu \Delta x'^\mu = -c^2 \Delta t'_{AB}{}^2 + d'_{AB}{}^2$$

Because the interval I is a Lorentz-invariant quantity, then:

$$I = \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu = I' = \Delta x'^\mu \Delta x'_\mu = \Delta x'_\mu \Delta x'^\mu$$

$$\text{Or:} \quad -c^2 \Delta t_{AB}^2 + d_{AB}^2 = -c^2 \Delta t'_{AB}{}^2 + d'_{AB}{}^2$$

Work this out / prove to yourselves that it is true \rightarrow follow procedure / same as on pages 7-8 of these lecture notes.

$$\text{Note that:} \quad \begin{matrix} \Delta t_{AB} \neq \Delta t'_{AB} \\ [IRF(S)] \quad [IRF(S')] \end{matrix} \text{ and } \begin{matrix} d_{AB} \neq d'_{AB} \\ [IRF(S)] \quad [IRF(S')] \end{matrix}$$

Time dilation in IRF(S') relative to IRF(S) is exactly compensated by spatial Lorentz contraction in IRF(S') relative to IRF(S), keeping the interval I the same (i.e. Lorentz invariant) in all IRF's !

\Rightarrow Profound aspect / nature of space-time!

Depending on the details of the two events (A & B), the interval

$$I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t_{AB}^2 + d_{AB}^2 \text{ can be positive, negative, or zero:}$$

$$I < 0: \text{ Interval } I \text{ is time-like: } c^2 \Delta t_{AB}^2 > d_{AB}^2$$

e.g. If two events A & B occur at same spatial location, then $\vec{r}_A = \vec{r}_B \rightarrow d_{AB} = 0$,

\Rightarrow The two events A & B occur at different times, thus: $\Delta t_{AB} \neq 0$.

$$I > 0: \text{ Interval } I \text{ is space-like: } c^2 \Delta t_{AB}^2 < d_{AB}^2$$

e.g. If two events A & B occur simultaneously / at same time, then $t_A = t_B \rightarrow \Delta t_{AB} = 0$,

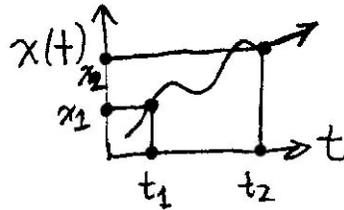
\Rightarrow The two events A & B occur at different spatial locations, thus $d_{AB} \neq 0$.

$$I = 0: \text{ Interval } I \text{ is light-like: } c^2 \Delta t_{AB}^2 = d_{AB}^2$$

e.g. Two events A & B are connected by a signal traveling at the speed of light (in vacuum).

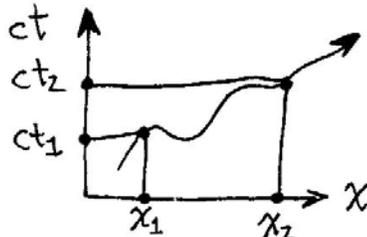
Space-time Diagrams \equiv Minkowski Diagrams

On a “normal”/Galilean space-time diagram, we plot $x(t)$ vs. t :



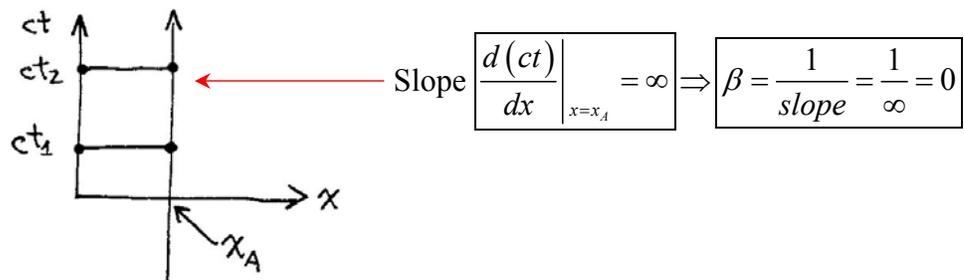
Velocity, $v(t) = \text{slope} \left(\left. \frac{dx(t)}{dt} \right|_t \right)$ of $x(t)$ vs. t graph at time t .

In relativity, we {instead} plot ct vs. x (danged theorists!!!) for the space-time diagram:
(*a.k.a.* Minkowski Diagram)



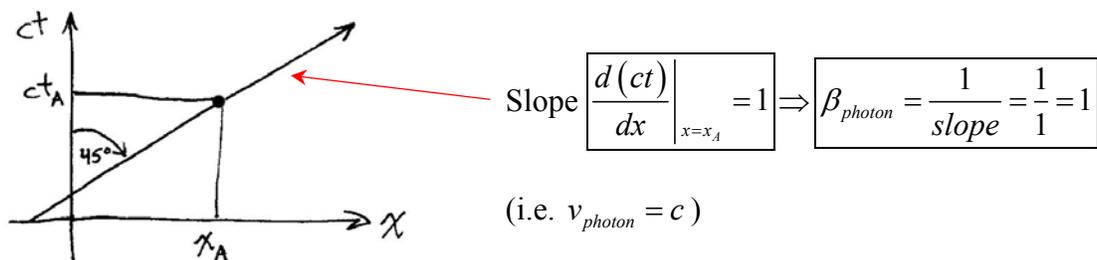
Dimensionless velocity β : $\left(\frac{v}{c} \right) = \beta = \frac{1}{\text{slope}} = \frac{1}{\left(\left. \frac{d(ct)}{dx} \right|_x \right)}$ of ct vs. x graph at point x .

A particle at rest in an IRF is represented by a vertical line on the relativistic space-time diagram:

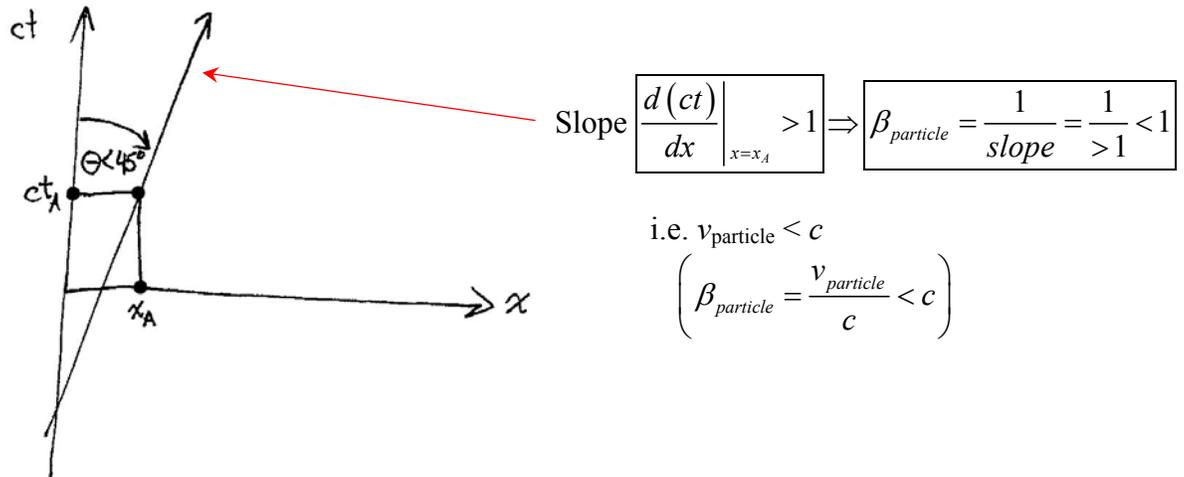


The “trajectory” of a particle in the space-time diagram makes an angle $\theta = 0^\circ$ with respect to vertical (ct) axis.

A photon traveling at $v = c$ is represented by a straight line at 45° with respect to the vertical (ct) axis:

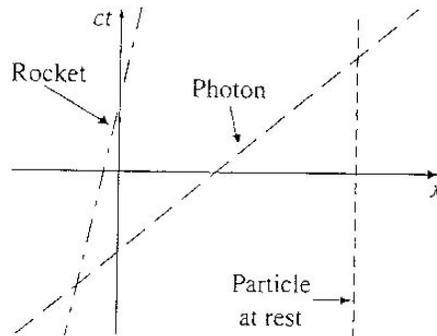


A particle traveling at constant speed $v < c$ ($\beta < 1$) is represented by a straight line making an angle $\theta < 45^\circ$ with respect to the vertical (ct) axis:



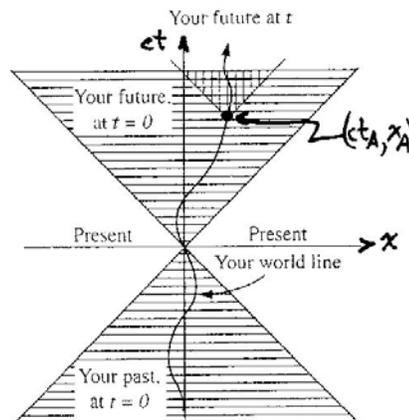
The trajectory of a particle on a relativistic space-time/Minkowski diagram is known as the world line of the particle.

All three situations superimposed together on the Minkowski/space-time diagram:



Suppose you set out from $t = 0$ at the origin of your own Minkowski diagram. Because your velocity can never exceed c ($v \leq c$, i.e. $\beta \leq 1$), your trajectory (your world line) in the ct vs. x space-time diagram can never have $|slope| = |dct/dx| < 1$, anywhere along it.

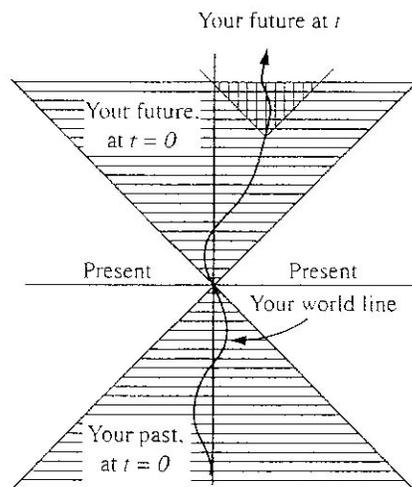
\Rightarrow Your “motion” in the Minkowski diagram is restricted to the wedge-shaped region bounded by the two 45° lines (with respect to vertical (ct) axis) as shown in the figure below:



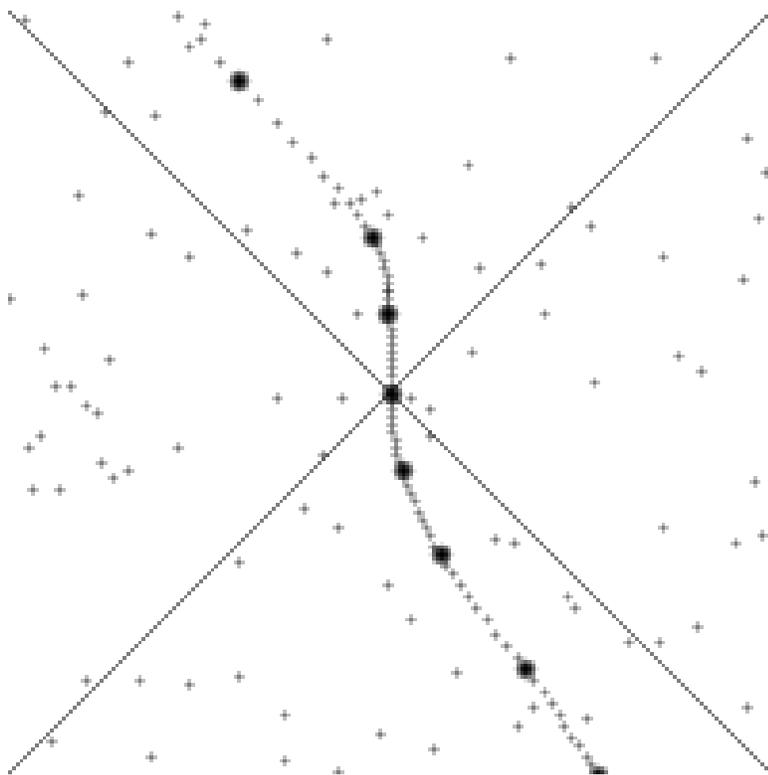
- The $\pm 45^\circ$ wedge-shaped region above the horizontal x -axis ($ct > 0$) is your future at $t = 0$ = locus of all space-time points potentially accessible to you.
- Of course, as time goes on, as you do progress along your world line, your “options” progressively narrow – your future at any moment $t > 0$ is the $\pm 45^\circ$ wedge constructed from / at whatever space-time point (ct_A, x_A) you are at, at that point in space (x_A) at the time t_A .
- The backward $\pm 45^\circ$ wedge below the horizontal x -axis ($ct < 0$) is your past at $t = 0$ = locus of all points potentially accessible to you.
- The space-time regions outside the present and past $\pm 45^\circ$ wedges in the Minkowski diagram are inaccessible to you, because you would have to travel faster than speed of light c to be in such regions!
- A space-time diagram with one time dimension (vertical axis) and 3 space dimensions (3 horizontal axes: x , y and z) is a 4-dimensional diagram – can’t draw it on 2-D paper!
- In a 4-D Minkowski Diagram, $\pm 45^\circ$ wedges become 4-D “hypercones”.
 “future” = contained within the forward light cone.
 “past” = contained within the backward light cone.

The slope of the world line / the trajectory connecting two events on a space-time diagram tells you at a glance whether the invariant interval $I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu$ is:

- Time-like (Slope $\frac{d(ct)}{dx} > 1$) (all points in your future and your past are time-like)
- Space-like (Slope $\frac{d(ct)}{dx} < 1$) (all points in your present are space-like)
- Light-like (Slope $\frac{d(ct)}{dx} \equiv 1$) (all points on your light cone(s) are light-like)



Changing Views of Relativistic Space-time Along the World line of a Rapidly Accelerating Observer

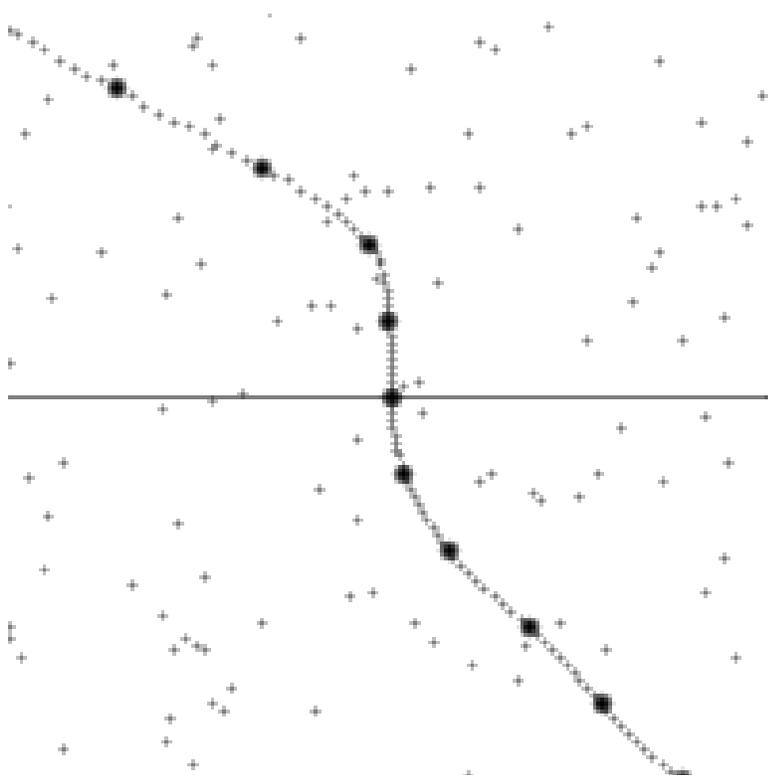


For relativistic space-time, the vertical axis is $c \times \text{time}$, the horizontal axis is distance; the dashed line is the space-time trajectory ("world line") of the observer. The small dots are arbitrary events in space-time.

The lower quarter of the diagram (within the light cone) shows events (dots) in the past that are visible to the user, the upper quarter (within the light cone) shows events (dots) in the future that will be able to see the observer.

The slope of the world line (deviation from vertical) gives the relative velocity to the observer. Note how the view of relativistic space-time changes when the observer accelerates {see relativistic animation}.

Changing Views of Galilean Space-time Along the World Line of a Slowly Accelerating Observer



In non-relativistic Galilean/ Euclidean space, the vertical axis is $c \times \text{time}$, the horizontal axis is distance; the dashed line is the space-time trajectory ("world line") of the observer. The small dots are arbitrary events in space-time.

The lower half of the diagram shows (past) events that are "earlier" than the observer, the upper half shows (future) events that are "later" than the observer.

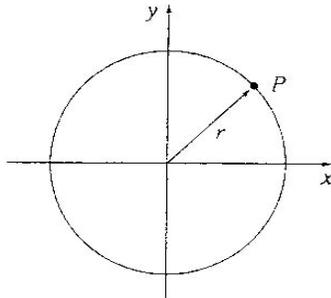
The slope of the world line (deviation from vertical) gives the relative velocity to the observer. Note how the view of Galilean / Euclidean space-time changes when the observer accelerates {see Galilean animation}.

Note that time in space-time is not “just another coordinate” (like x, y, z) – its “mark of distinction” is the minus sign in the invariant interval:

$$I \equiv \Delta x^\mu \Delta x_\mu = \Delta x_\mu \Delta x^\mu = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

The minus sign in the invariant interval (arising from / associated with time dimension) imparts a rich structure to $\sinh, \cosh, \tanh \dots$ the hyperbolic geometry of relativistic space-time versus the circular geometry of Euclidean 3-dimensional space.

In Euclidean 3-D space, a rotation {e.g. about the \hat{z} -axis} of a point P in the x - y plane describes a circle – the locus of all points at a fixed distance $r = \sqrt{x^2 + y^2}$ from the origin:



$r = \text{constant}$ (i.e. is invariant) under a rotation in Euclidean / 3-D space.

For a Lorentz transformation in relativistic space-time, the interval

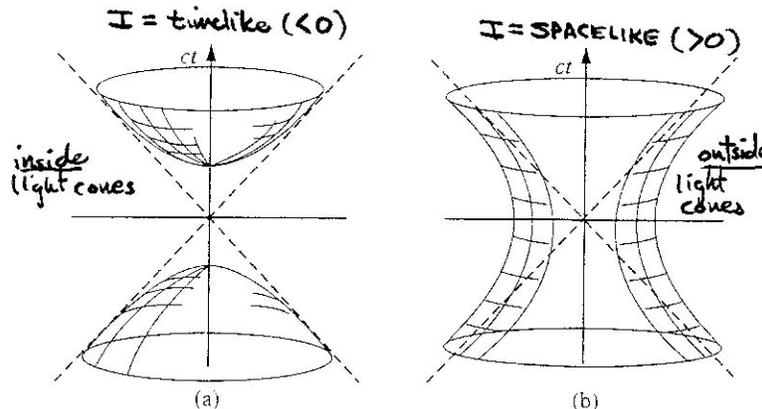
$$I \equiv x^\mu x_\mu = x_\mu x^\mu = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

is a Lorentz invariant quantity (i.e. is preserved under any/all Lorentz transformations from one IRF to another).

The locus of all points in space-time with a given / specific value of I is a hyperbola (for ct and Δx (i.e. 1 space dimension) only):

$$I = -(c\Delta t)^2 + \Delta x^2$$

If we include e.g. the \hat{y} -axis, the locus of all points in space-time with a given / specific value of $I = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2$ is a hyperboloid of revolution:



When the invariant interval I is time-like ($I < 0$) \rightarrow surface is a hyperboloid of two sheets.

When the invariant interval I is space-like ($I > 0$) \rightarrow surface is a hyperboloid of one sheet.

- When carrying out a Lorentz transformation from IRF(S) to IRF(S') (where IRF(S') is moving with respect to IRF(S) with velocity \vec{v}) the space-time coordinates (x, ct) of a given event will change (via appropriate Lorentz transformation) to (x', ct') .
- The new coordinates (x', ct') will lie on the same hyperbola as (x, ct) !!!
- By appropriate combinations of Lorentz transformations and rotations, a single space-time point (x, ct) can generate the entire surface of a given hyperboloid (i.e. only the one that the original space-time point (x, ct) is on).
- \exists no Lorentz transformations from the upper \rightarrow lower sheet of the time-like ($I < 0$) hyperboloid of two sheets (and vice versa).
- \exists no Lorentz transformations from the upper or lower sheet of the time-like ($I < 0$) hyperboloid of two sheets to the space-like ($I > 0$) hyperboloid of one sheet (and vice versa).
- In discussion of the simultaneity of events, reversing of the time-ordering of events is, in general not always possible.

\rightarrow If the invariant interval $I = -(c\Delta t)^2 + d^2 < 0$ (i.e. is time-like) the time-ordering is absolute (i.e. the time-ordering cannot be changed).

\rightarrow If the invariant interval $I = -(c\Delta t)^2 + d^2 > 0$ (i.e. is space-like) the time-ordering of events depends on the IRF in which they are observed.

In terms of the space-time/Minkowski diagram for time-like invariant intervals $I = -(c\Delta t)^2 + d^2 < 0$:

- An event on the upper sheet of a time-like hyperboloid (n.b. lies inside of light cone) definitely occurred after time $t = 0$.
- An event on lower sheet of a time-like hyperboloid (n.b. also lies inside of light cone) definitely occurred before time $t = 0$.
- For an event occurring on a space-like hyperboloid (invariant interval $I = -(c\Delta t)^2 + d^2 > 0$, the space-like hyperboloid lies outside of the light cone) the event can occur either at positive or negative time t – it depends on the IRF from which the event is viewed!
- This rescues the notion of causality!
To an observer in one IRF: “Event A caused event B”
To another “observer” (outside of light cone, in another IRF) could say: “B preceded A”.
- If two events are time-like separated (within the light cone) \rightarrow they obey causality.
- If the invariant interval $I = x_\mu x^\mu = x^\mu x_\mu = -(c\Delta t)^2 + d^2 < 0$ (i.e. is time-like) between two events (i.e. they lie within the light cone) then the temporal ordering is same \forall (for all) observers – i.e. causality is obeyed.

- Causality is IRF-dependent for the space-like invariant interval

$I = x_\mu x^\mu = x^\mu x_\mu = -(c\Delta t)^2 + d^2 > 0$ between two events (i.e. they lie outside the light cone). Temporal-ordering is IRF-dependent / not the same for all observers.

- We don't live outside the light cone (outside the light cone $\rightarrow \beta > 1$).

Another Perspective on the Structure of Space-Time:



Mathematician Herman Minkowski (1864-1909) in 1907 introduced the notion of 4-D space-time (not just space and time separately). In his mathematical approach to special relativity and inertial reference frames, space and time Lorentz transform (e.g. along the \hat{x} direction) as given above, however in his scheme the contravariant x^μ and covariant x_μ 4-vectors that he advocated using were:

$$x^\mu = \begin{pmatrix} ict \\ x \\ y \\ z \end{pmatrix} \text{ and } x_\mu = (ict \ x \ y \ z)$$

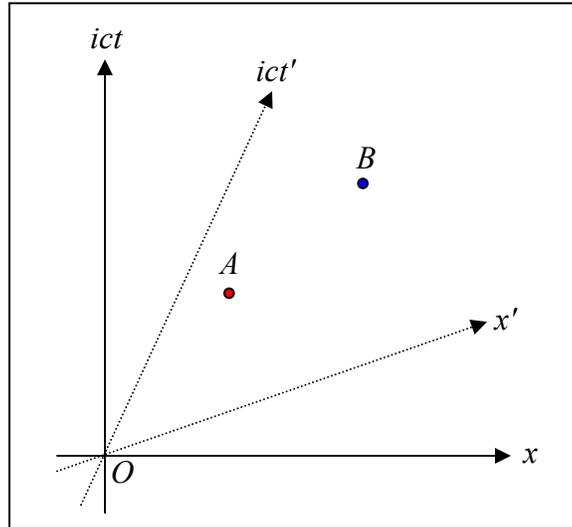
It can be readily seen that the Lorentz invariant quantity $x_\mu x^\mu = -(ct)^2 + x^2 + y^2 + z^2$ is the same as always, but here the -ve sign in the temporal (0) index is generated by $i*i = -1$.

Thus, in Minkowski's notation $x'^\mu = \Lambda_\nu^\mu x^\nu$ for a 1-D Lorentz transform along the \hat{x} -direction is:

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ict' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_\nu^\mu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ict \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ict - \beta_x x) \\ \gamma(x - \beta_x ict) \\ y \\ z \end{pmatrix} \text{ with: } \begin{cases} \beta_x \equiv \frac{v_x}{c} \\ \gamma = \frac{1}{\sqrt{1 - \beta_x^2}} \end{cases}$$

The physical interpretation of the “ ict ” temporal component vs. the x, y, z spatial components of the four-vectors x^μ and x_μ is that there exists a complex, 90° phase relation between space and time in special relativity – i.e. flat space-time. We've seen this before, e.g. for {zero-frequency} virtual photons, where the relation for the relativistic total energy associated with a virtual photon is $E_{\gamma^*}^2 = p_{\gamma^*}^2 c^2 + m_{\gamma^*}^2 c^4 = hf_{\gamma^*} = 0 \Rightarrow p_{\gamma^*} c = \pm i m_{\gamma^*} c^2$.

In the flat space-time of special relativity, graphically this means that Lorentz transformations from one IRF to another are related to each other e.g. via the {flat} space-time diagram as shown in the figure below:



This formalism works fine in flat space-time/special relativity, but in curved space-time / general relativity, it is cumbersome to work with – the complex phase relation between time and space is no longer 90° , it depends on the {local} curvature of space-time!

Imagine taking the above flat space-time 2-D surface and curving it e.g. into potato-chip shape!!! Then imagine taking the 4-D flat space-time and curving/warping it per the curved 4-D space-time in proximity to a supermassive black hole or a neutron star!!!

Thus, for people working in general relativity, the use of the modern 4-vector notation e.g. for contravariant x^μ and covariant x_μ is strongly preferred, e.g.

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \text{ and } x_\mu = (-ct \ x \ y \ z)$$

In flat space-time/special relativity, the modern mathematical notation works equally well and then also facilitates people learning the mathematics of curved space-time/general relativity.

Using the rule for the temporal (0) component of covariant x_μ that $x_0 = -x^0$, then Lorentz invariant quantities such as $x_\mu x^\mu = -(ct)^2 + x^2 + y^2 + z^2$ are “automatically” calculated properly.

However, the physical interpretation of the complex phase relation between time and space (and the temporal-spatial components of {all} other 4-vectors) often gets lost in the process... which is why we explicitly mention this here...