6 Resonant cavities and wave guides

6.1 Introduction

Let us investigate the solution of the homogeneous wave equation in regions containing various geometric boundaries, particularly in regions bounded by conductors. The boundary value problem is of great theoretical significance and also has many practical electromagnetic applications, particularly in the microwave region of the spectrum.

6.2 Boundary conditions

Let us review the general boundary conditions on the field vectors at a surface between medium 1 and medium 2:

$$\boldsymbol{n}\cdot(\boldsymbol{D}_1 - \boldsymbol{D}_2) = \tau, \tag{6.1a}$$

$$\boldsymbol{n}\wedge(\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0, \tag{6.1b}$$

$$\boldsymbol{n}\cdot(\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \tag{6.1c}$$

$$\boldsymbol{n}\wedge(\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K}, \tag{6.1d}$$

where τ is used for the surface change density (to avoid confusion with the conductivity), and $\boldsymbol{K}$ is the surface current density. Here, $\boldsymbol{n}$ is a unit vector normal to the surface, directed from medium 2 to medium 1. We have seen in Section 4.4 that for normal incidence an electromagnetic wave falls off very rapidly inside the surface of a good conductor. Equation (4.35) implies that in the limit of perfect conductivity ($\sigma \to \infty$) the tangential component of $\boldsymbol{E}$ vanishes, whereas that of $\boldsymbol{H}$ may remain finite. Let us examine the behaviour of the normal components.

Let medium 1 be a good conductor for which $\sigma/\epsilon\epsilon_0\omega \gg 1$, whilst medium 2 is a perfect insulator. The surface change density is related to the currents flowing inside the conductor. In fact, the conservation of charge requires that

$$\boldsymbol{n}\cdot\boldsymbol{j} = \frac{\partial\tau}{\partial t} = -\mathrm{i}\,\omega\,\tau. \tag{6.2}$$

However, $\boldsymbol{n}\cdot\boldsymbol{j} = \boldsymbol{n}\cdot\sigma\boldsymbol{E}_1$, so it follows from Eq. (6.1)(a) that

$$\left(1 + \frac{\mathrm{i}\,\omega\epsilon_0\epsilon_1}{\sigma}\right)\boldsymbol{n}\cdot\boldsymbol{E}_1 = \frac{\mathrm{i}\,\omega\epsilon_0\epsilon_2}{\sigma}\,\boldsymbol{n}\cdot\boldsymbol{E}_2. \tag{6.3}$$

It is clear that the normal component of $\boldsymbol{E}$ within the conductor also becomes vanishingly small as the conductivity approaches infinity.

If $\boldsymbol{E}$ vanishes inside a perfect conductor then the curl of $\boldsymbol{E}$ also vanishes, and the time rate of change of $\boldsymbol{B}$ is correspondingly zero. This implies that there are no oscillatory fields whatever inside such a conductor, and that the boundary values of the fields outside are given by

$$\boldsymbol{n}\cdot\boldsymbol{D} = -\tau, \tag{6.4a}$$

$$\boldsymbol{n}\wedge\boldsymbol{E} = 0, \tag{6.4b}$$

$$\boldsymbol{n}\cdot\boldsymbol{B} = 0, \tag{6.4c}$$

$$\boldsymbol{n}\wedge\boldsymbol{H} = -\boldsymbol{K}. \tag{6.4d}$$

Here, $\boldsymbol{n}$ is a unit normal at the surface of the conductor pointing *into* the conductor. Thus, the electric field is normal and the magnetic field tangential at the surface of a perfect conductor. For good conductors these boundary conditions yield excellent representations of the geometrical configurations of external fields, but they lead to the neglect of some important features of real fields, such as losses in cavities and signal attenuation in wave guides.

In order to estimate such losses it is useful to see how the tangential and normal fields compare when σ is large but finite. Equations (4.5) and (4.34) yield

$$\boldsymbol{H} = \frac{1+\mathrm{i}}{\sqrt{2}}\sqrt{\frac{\sigma}{\mu_0\omega}}\,\boldsymbol{n}\wedge\boldsymbol{E} \tag{6.5}$$

at the surface of a conductor (provided that the wave propagates into the conductor). Let us assume, without obtaining a complete solution, that a wave with $\boldsymbol{H}$ very nearly tangential and $\boldsymbol{E}$ very nearly normal is propagated along the surface of the metal. According to the Faraday-Maxwell equation

$$|H_{\|}| \simeq \frac{k}{\mu_0\omega}\,|E_\perp| \tag{6.6}$$

just outside the surface, where k is the component of the propagation vector along the surface. However, Eq. (6.5) implies that a tangential component of $\boldsymbol{H}$ is accompanied by a small tangential component of $\boldsymbol{E}$. By comparing these two expressions, we obtain

$$\frac{|E_\parallel|}{|E_\perp|} \simeq k\sqrt{\frac{2}{\mu_0\omega\sigma}} = \frac{d}{\lambda\!\!\!^{-}}, \tag{6.7}$$

where d is the skin depth (see Eq. (4.36)) and $\lambda\!\!\!^{-} \equiv 1/k$. It is clear that the ratio of the tangential component of $\boldsymbol{E}$ to its normal component is of order the skin depth divided by the wavelength. It is readily demonstrated that the ratio of the normal component of $\boldsymbol{H}$ to its tangential component is of this same magnitude. Thus, we can see that in the limit of high conductivity, which means vanishing skin depth, no fields penetrate the conductor, and the boundary conditions are those given by Eqs. (6.4). Let us investigate the solution of the homogeneous wave equation subject to such boundary conditions.

6.3 Cavities with rectangular boundaries

Consider a vacuum region totally enclosed by rectangular conducting walls. In this case, all of the field components satisfy the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{6.8}$$

where ψ represents any component of $\boldsymbol{E}$ or $\boldsymbol{H}$. The boundary conditions (6.4) require that the electric field is normal to the walls at the boundary whereas the magnetic field is tangential. If a, b, and c are the dimensions of the cavity, then it is readily verified that the electric field components are

$$E_x = E_1\cos(k_1 x)\sin(k_2 y)\sin(k_3 z)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \tag{6.9a}$$

$$E_y = E_2\sin(k_1 x)\cos(k_2 y)\sin(k_3 z)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \tag{6.9b}$$

$$E_z = E_3\sin(k_1 x)\sin(k_2 y)\cos(k_3 z)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \tag{6.9c}$$

where

$$k_1 = \frac{l\,\pi}{a}, \tag{6.10a}$$

$$
\begin{aligned}
k_2 &= \frac{m\,\pi}{b}, && (6.10b)\\
k_3 &= \frac{n\,\pi}{c}, && (6.10c)
\end{aligned}
$$

with l, m, n integers. The allowed frequencies are given by

$$\frac{\omega^2}{c^2} = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right). \tag{6.11}$$

It is clear from Eq. (6.9) that at least two of the integers l, m, n must be different from zero in order to have non-vanishing fields. The magnetic fields obtained by the use of $\nabla\wedge\boldsymbol{E} = \mathrm{i}\,\omega\boldsymbol{B}$ automatically satisfy the appropriate boundary conditions, and are in phase quadrature with the electric fields. Thus, the sum of the total electric and magnetic energies within the cavity is constant, although the two terms oscillate separately.

The amplitudes of the electric field components are not independent, but are related by the divergence condition $\nabla\cdot\boldsymbol{E} = 0$, which yields

$$k_1\,E_1 + k_2\,E_2 + k_3\,E_3 = 0. \tag{6.12}$$

There are, in general, two linearly independent vectors $\boldsymbol{E}$ that satisfy this condition, corresponding to two polarizations. (The exception is the case that one of the integers l, m, n is zero, in which case $\boldsymbol{E}$ is fixed in direction.) Each vector is accompanied by a magnetic field at right angles. The fields corresponding to a given set of integers l, m, and n constitute a particular mode of vibration of the cavity. It is evident from standard Fourier theory that the different modes are *orthogonal* (*i.e.*, they are normal modes) and that they form a *complete set*. In other words, any general electric and magnetic fields which satisfy the boundary conditions (6.4) can be unambiguously decomposed into some linear combination of all of the various possible normal modes of the cavity. Since each normal mode oscillates at a specific frequency it is clear that if we are given the electric and magnetic fields inside the cavity at time $t = 0$ then the subsequent behaviour of the fields is *uniquely* determined for all time.

The conducting walls gradually absorb energy from the cavity, due to their finite resistivity, at a rate which can easily be calculated. For finite σ the small

tangential component of $\boldsymbol{E}$ at the walls can be estimated using Eq. (6.5):

$$\boldsymbol{E}_\parallel = \frac{1-\mathrm{i}}{\sqrt{2}}\sqrt{\frac{\mu_0\omega}{\sigma}}\,\boldsymbol{H}_\parallel \wedge \boldsymbol{n}. \tag{6.13}$$

Now, the tangential component of $\boldsymbol{H}$ at the walls is slightly different from that given by the ideal solution. However, this is a small effect and can be neglected to leading order in σ^{-1}. The time averaged energy flux into the walls is given by

$$\overline{\boldsymbol{N}} = \frac{1}{2}\,\mathrm{Re}\,(\boldsymbol{E}_\parallel \wedge \boldsymbol{H}_\parallel) = \frac{1}{2}\sqrt{\frac{\mu_0\omega}{2\sigma}}\,H_{\parallel 0}^{\,2}\,\boldsymbol{n} = \frac{H_{\parallel 0}^{\,2}}{2\sigma d}\,\boldsymbol{n}, \tag{6.14}$$

where $H_{\parallel 0}$ is the peak value of the tangential magnetic field at the walls predicted by the ideal solution. According to the boundary condition (6.4)(d), $H_{\parallel 0}$ is equal to the peak value of the surface current density K_0. It is helpful to define a surface resistance,

$$\overline{\boldsymbol{N}} = \overline{K^2} R_s\,\boldsymbol{n}, \tag{6.15}$$

where

$$R_s = \frac{1}{\sigma d}. \tag{6.16}$$

This approach makes it clear that the dissipation of energy is due to ohmic heating in a thin layer, whose thickness is of order the skin depth, on the surface of the conducting walls.

6.4 The quality factor of a resonant cavity

The quality factor Q of a resonant cavity is defined

$$Q = 2\pi\,\frac{\text{energy stored in cavity}}{\text{energy lost per cycle to walls}}. \tag{6.17}$$

For a specific normal mode of the cavity this quantity is independent of the mode amplitude. By conservation of energy the power dissipated in ohmic losses is minus the rate of change of the stored energy U. We can write a differential equation for the behaviour of U as a function of time:

$$\frac{dU}{dt} = -\frac{\omega_0}{Q}\,U, \tag{6.18}$$

where ω_0 is the oscillation frequency of the normal mode in question. The solution to the above equation is

$$U(t) = U(0)\, \mathrm{e}^{-\omega_0 t/Q}. \tag{6.19}$$

This time dependence of the stored energy suggests that the oscillations of the fields in the cavity are damped as follows:

$$E(t) = E_0\, \mathrm{e}^{-\omega_0 t/2Q}\, \mathrm{e}^{-\mathrm{i}\,(\omega_0+\Delta\omega) t}, \tag{6.20}$$

where we have allowed for a shift $\Delta\omega$ of the resonant frequency as well as the damping. A damped oscillation such as this does not consist of a pure frequency. Instead, it is made up of a superposition of frequencies around $\omega = \omega_0 + \Delta\omega$. Standard Fourier analysis yields

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(\omega)\, \mathrm{e}^{-\mathrm{i}\,\omega t}\, d\omega, \tag{6.21}$$

where

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} E_0\, \mathrm{e}^{-\omega_0 t/2Q}\, \mathrm{e}^{\mathrm{i}\,(\omega-\omega_0-\Delta\omega) t}\, dt. \tag{6.22}$$

It follows that

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0-\Delta\omega)^2 + (\omega_0/2Q)^2}. \tag{6.23}$$

The resonance shape has a full width Γ at half-maximum equal to ω_0/Q. For a constant input voltage, the energy of oscillation in the cavity as a function of frequency follows the resonance curve in the neighbourhood of a particular resonant frequency. It can be seen that the ohmic losses, which determine Q for a particular mode, also determine the maximum amplitude of the oscillation when the resonance condition is exactly satisfied, as well as the width of the resonance (*i.e.*, how far off the resonant frequency the system can be driven and still yield a significant oscillation amplitude).

6.5 Axially symmetric cavities

The rectangular cavity which we have just considered has many features in common with axially symmetric cavities of arbitrary cross section. In every cavity

the allowed values of the wave vector $\boldsymbol{k}$, and thus the allowed frequencies, are determined by the geometry of the cavity. We have seen that for each set of k_1, k_2, k_3 in a rectangular cavity there are, in general, two linearly independent modes; *i.e.*, the polarization remains arbitrary. We can take advantage of this fact to classify modes into two kinds according to the orientation of the field vectors. Let us choose one type of mode such that the electric field vector lies in the cross-sectional plane, and the other so that the magnetic field vector lies in this plane. This classification into transverse electric (TE) and transverse magnetic (TM) modes turns out to be possible for all axially symmetric cavities, although the rectangular cavity is unique in having one mode of each kind corresponding to each allowed frequency.

Suppose that the direction of symmetry is along the z-axis, and that the length of the cavity in this direction is L. The boundary conditions at $z = 0$ and $z = L$ demand that the z dependence of wave quantities be either $\sin k_3 z$ or $\cos k_3 z$, where $k_3 = n\pi/L$. In other words, every field component satisfies

$$\left(\frac{\partial^2}{\partial z^2} + {k_3}^2\right)\psi = 0, \tag{6.24}$$

as well as

$$(\nabla^2 + k^2)\psi = 0, \tag{6.25}$$

where ψ stands for any component of $\boldsymbol{E}$ or $\boldsymbol{H}$. The field equations

$$\nabla \wedge \boldsymbol{E} = \mathrm{i}\,\omega\mu_0\,\boldsymbol{H}, \tag{6.26a}$$

$$\nabla \wedge \boldsymbol{H} = -\mathrm{i}\,\omega\epsilon_0\,\boldsymbol{E} \tag{6.26b}$$

must also be satisfied.

Let us write each vector and each operator in the above equations as the sum of a transverse part, designated by the subscript s, and a component along z. We find that for the transverse fields

$$\mathrm{i}\,\omega\mu_0\,\boldsymbol{H}_s = \nabla_s \wedge \boldsymbol{E}_z + \nabla_z \wedge \boldsymbol{E}_s, \tag{6.27a}$$

$$-\mathrm{i}\,\omega\epsilon_0\,\boldsymbol{E}_s = \nabla_s \wedge \boldsymbol{H}_z + \nabla_z \wedge \boldsymbol{H}_s. \tag{6.27b}$$

When one side of Eqs. (6.27) is substituted for the transverse field on the right-hand side of the other, and use is made of Eq. (6.24), we obtain

$$\boldsymbol{E}_s = \frac{\nabla_s(\partial E_z/\partial z)}{k^2 - {k_3}^2} + \frac{\mathrm{i}\,\omega\mu_0}{k^2 - {k_3}^2}\,\nabla_s \wedge \boldsymbol{H}_z, \tag{6.28a}$$

$$\boldsymbol{H}_s = \frac{\nabla_s(\partial H_z/\partial z)}{k^2 - {k_3}^2} - \frac{\mathrm{i}\,\omega\epsilon_0}{k^2 - {k_3}^2}\,\nabla_s \wedge \boldsymbol{E}_z. \tag{6.28b}$$

Thus, all transverse fields can be expressed in terms of the z components of the fields, each of which satisfies the differential equation

$$\left[\nabla_s^2 + (k^2 - {k_3}^2)\right] A_z = 0, \tag{6.29}$$

where A_z stands for either E_z or H_z, and ∇_s^2 is the two-dimensional Laplacian operator.

The conditions on E_z and H_z at the boundary (in the transverse plane) are quite different: E_z must vanish on the boundary, whereas the normal derivative of H_z must vanish so that $\boldsymbol{H}_s$ in Eq. (6.28)(b) satisfies the appropriate boundary condition. When the cross section is a rectangle, these two conditions lead to the same eigenvalues of $(k^2 - {k_3}^2) = {k_s}^2 = {k_1}^2 + {k_2}^2$, as we have seen. Otherwise, they correspond to two *different* frequencies, one for which E_z is permitted but $H_z = 0$, and the other where the opposite is true. In every case, it is possible to classify the modes as transverse magnetic or transverse electric. Thus, the field components E_z and H_z play the role of independent potentials, from which the other field components of the TE and TM modes, respectively, can be derived using Eqs. (6.28).

The mode frequencies are determined by the eigenvalues of Eqs. (6.24) and (6.29). If we denote the functional dependence of E_z or H_z on the plane cross section coordinates by $f(x, y)$, then we can write Eq. (6.29) as

$$\nabla_s^2 f = -{k_s}^2 f. \tag{6.30}$$

Let us first show that ${k_s}^2 > 0$, and hence that $k > k_3$. Now,

$$f\,\nabla_s^2 f = \nabla_s\cdot(f\,\nabla_s f) - (\nabla_s f)^2. \tag{6.31}$$

It follows that

$$-{k_s}^2 \int f^2\, dV + \int (\nabla_s f)^2\, dV = \int f \nabla f \cdot d\boldsymbol{S}, \tag{6.32}$$

where the integration is over the transverse cross section. If either f or its normal derivative is to vanish on S, the conducting surface, then

$${k_s}^2 = \frac{\int (\nabla_s f)^2\, dV}{\int f^2\, dV} > 0. \tag{6.33}$$

We have already seen that $k_3 = n\pi/L$. The allowed values of k_s depend both on the geometry of the cross section and the nature of the mode.

For TM modes $H_z = 0$, and the z dependence of E_z is given by $\cos(n\pi\, z/L)$. Equation (6.30) must be solved subject to the condition that f vanish on the boundaries of the plane cross section, thus completing the determination of E_z and k. The transverse fields are special cases of Eqs. (6.28):

$$\boldsymbol{E}_s = \frac{1}{{k_s}^2} \nabla_s \frac{\partial E_z}{\partial z}, \tag{6.34a}$$

$$\boldsymbol{H} = \frac{\mathrm{i}\,\omega\epsilon_0}{{k_s}^2}\, \hat{\boldsymbol{z}} \wedge \nabla_s E_z. \tag{6.34b}$$

For TE modes, in which $E_z = 0$, the condition that H_z vanish at the ends of the cylinder demands the use of $\sin(n\pi\, z/L)$, and k_s must be such that the normal derivative of H_z is zero at the walls. Equations (6.28), giving the transverse fields, then become

$$\boldsymbol{H}_s = \frac{1}{{k_s}^2} \nabla_s \frac{\partial H_z}{\partial z}, \tag{6.35a}$$

$$\boldsymbol{E} = -\frac{\mathrm{i}\,\omega\mu_0}{{k_s}^2}\, \hat{\boldsymbol{z}} \wedge \nabla_s H_z, \tag{6.35b}$$

and the mode determination is completed.

6.6 Cylindrical cavities

Let us apply the methods of the previous section to the TM modes of a right circular cylinder of radius a. We can write

$$E_z = A f(r,\varphi)\cos(k_3 z)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \tag{6.36}$$

where $f(r,\varphi)$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + {k_s}^2 f = 0, \tag{6.37}$$

and (r,φ,z) are cylindrical polar coordinates. Let

$$f(r,\varphi) = g(r)\,\mathrm{e}^{\mathrm{i}\,m\varphi}. \tag{6.38}$$

It follows that

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dg}{dr}\right) + \left({k_s}^2 - \frac{m^2}{r^2}\right) g = 0, \tag{6.39}$$

or

$$z^2\,\frac{d^2 g}{dz^2} + z\,\frac{dg}{dz} + (z^2 - m^2)\,g = 0, \tag{6.40}$$

where $z = k_s r$. The above equation is known as *Bessel's equation.* The two linearly independent solutions of Bessel's equation are denoted $J_m(z)$ and $Y_m(z)$. In the limit $|z| \ll 1$ these solutions behave as z^m and z^{-m}, respectively, to lowest order . More exactly[16]

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!(m+k)!}, \tag{6.41a}$$

$$\begin{aligned} Y_m(z) &= -\frac{(z/2)^{-m}}{\pi}\sum_{k=0}^{m-1}\frac{(m-k-1)!\,(z^2/4)^k}{k!} + \frac{2}{\pi}\,\ln(z/2)\,J_m(z) \\ &\quad -\frac{(z/2)^m}{\pi}\sum_{k=0}^{\infty}[\psi(k+1)+\psi(m+k+1)]\,\frac{(-z^2/4)^k}{k!(m+k)!} \end{aligned} \tag{6.41b}$$

[16]M. Abramowitz, and I.A. Stegun, *Handbook of mathematical functions*, (Dover, New York, 1965), Cha. 9

for $|z| \ll 1$, where

$$\psi(1) = -\gamma, \tag{6.42a}$$

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}, \tag{6.42b}$$

and $\gamma = \sum_{k=1}^{\infty} k^{-1} = 0.57722$ is Euler's constant. Clearly, the J_m are well behaved in the limit $|z| \to 0$, whereas the Y_m are badly behaved.

The asymptotic behaviour of both solutions at large $|z|$ is

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos(z - m\,\pi/2 - \pi/4) + O(1/z), \tag{6.43a}$$

$$Y_m(z) = \sqrt{\frac{2}{\pi z}} \sin(z - m\,\pi/2 - \pi/4) + O(1/z). \tag{6.43b}$$

Thus, for $|z| \gg 1$ the solutions take the form of gradually decaying oscillations which are in phase quadrature. The behaviour of $J_0(z)$ and $Y_0(z)$ is shown in Fig. 21.

Since the axis $r = 0$ is included in the cavity the radial eigenfunction must be regular at the origin. This immediately rules out the $Y_m(k_s r)$ solutions. Thus, the most general solution for a TM mode is

$$E_z = A\, J_m(k_l r)\, \mathrm{e}^{\mathrm{i}\, m\varphi} \cos(k_3 z)\, \mathrm{e}^{-\mathrm{i}\,\omega t}. \tag{6.44}$$

The k_l are the eigenvalues of k_s, and are determined by the solutions of

$$J_m(k_l a) = 0. \tag{6.45}$$

The above constraint ensures that the tangential electric field is zero on the conducting walls surrounding the cavity ($r = a$).

The most general solution for a TE mode is

$$H_z = A\, J_m(k_l r)\, \mathrm{e}^{\mathrm{i}\, m\varphi} \sin(k_3 z)\, \mathrm{e}^{-\mathrm{i}\,\omega t}. \tag{6.46}$$

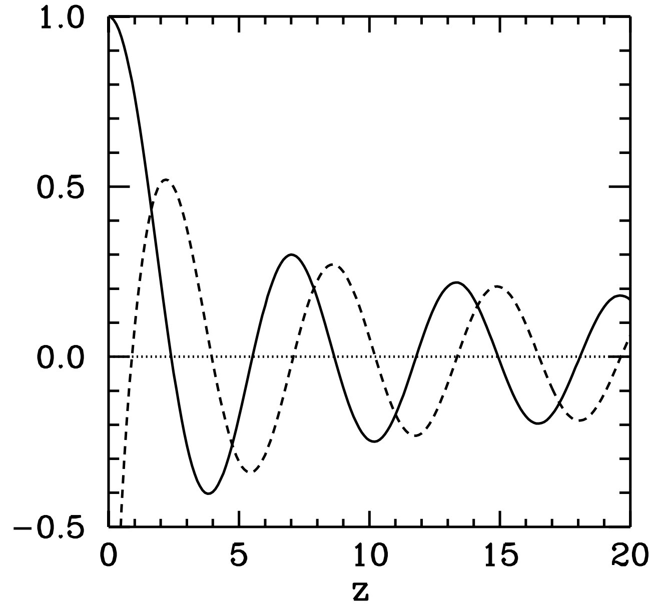

Figure 21: The Bessel functions $J_0(z)$ (solid line) and $Y_0(z)$ (dotted line)

In this case, the k_l are determined by the solution of

$$J_m'(k_l a) = 0, \tag{6.47}$$

where $'$ denotes differentiation with respect to the argument. The above constraint ensures that the normal magnetic field is zero on the conducting walls surrounding the cavity. The oscillation frequency of both the TM and TE modes is given by

$$\frac{\omega^2}{c^2} = k^2 = {k_l}^2 + \frac{n^2\pi^2}{L^2}. \tag{6.48}$$

If l is the ordinal number of a zero of a particular Bessel function of order m (l increases with increasing values of the argument), then each mode is characterized by three integers, l, m, n, as in the rectangular case. The lth zero of J_m is conventionally denoted $j_{m,l}$ [so, $J_m(j_{m,l}) = 0$]. Likewise, the lth zero of J_m' is denoted $j_{m,l}'$. Table 2 shows the first few zeros of J_0, J_0', J_1, and J_1'. It is clear that for fixed n and m the lowest frequency mode (i.e., the mode with the lowest value of k_l) is a TE mode. The mode with the next highest frequency is also a TE mode. The next highest frequency mode is a TM mode, and so on.

l	$j_{0,l}$	$j_{1,l}$	$j'_{0,l}$	$j'_{1,l}$
1	2.4048	3.8317	0.0000	1.8412
2	5.5201	7.0156	3.8317	5.3314
3	8.6537	10.173	7.0156	8.5363
4	11.792	13.324	10.173	11.706

Table 2: The first few values of $j_{0,l}$, $j_{1,l}$, $j'_{0,l}$, and $j'_{1,l}$

6.7 Wave guides

Let us consider the transmission of electromagnetic waves along the axis of a wave guide, which is simply a long, axially symmetric, hollow conductor with open ends. In order to represent a wave propagating along the z-direction, we can write the dependence of the fields on the coordinate variables and the time as

$$f(x, y)\, \mathrm{e}^{\,\mathrm{i}\,(k_g z - \omega t)}. \tag{6.49}$$

The *guide propagation constant*, k_g, is just the k_3 of previous sections, except that it is no longer restricted by the boundary conditions to take discrete values. The general considerations of Section 6.5 still apply, so that we can treat TM and TE modes separately. The solutions for f are identical to those for axially symmetric cavities already discussed. Although k_g is not restricted in magnitude, we note that for every eigenvalue of the two-dimensional equation, k_s, there is a lowest value of k, namely $k = k_s$ (often designated k_c for wave guides), for which k_g is real. This corresponds to the *cutoff frequency* below which waves are not transmitted by that mode, and the fields fall off exponentially with increasing z. In fact, the wave guide dispersion relation for a particular mode can easily be shown to take the form

$$k_g = \frac{\sqrt{\omega^2 - {\omega_c}^2}}{c}, \tag{6.50}$$

where

$$\omega_c = k_c\, c \equiv k_s\, c \tag{6.51}$$

is the cutoff frequency. There is an absolute cutoff frequency associated with the mode of lowest frequency; *i.e.*, the mode with the lowest value of k_c.

For real k_g (*i.e.*, $\omega > \omega_c$) it is clear from Eq. (6.50) that the wave is propagated along the guide with a phase velocity

$$u_p = \frac{\omega}{k_g} = \frac{c}{\sqrt{1 - {\omega_c}^2/\omega^2}}. \tag{6.52}$$

It is evident that the phase velocity is greater than that of electromagnetic waves in free space. This velocity is not constant, however, but depends on the frequency. The wave guide thus behaves as a dispersive medium. The group velocity of a wave pulse propagated along the guide is given by

$$u_g = \frac{d\omega}{dk_g} = c\sqrt{1 - {\omega_c}^2/\omega^2}. \tag{6.53}$$

It can be seen that u_g is always smaller than c, and also that

$$u_p\, u_g = c^2. \tag{6.54}$$

For a TM mode ($H_z = 0$) Eqs. (6.34) yield

$$\boldsymbol{E}_s = \frac{\mathrm{i}\, k_g}{{k_s}^2}\, \nabla_s E_z, \tag{6.55a}$$

$$\boldsymbol{H}_s = \frac{\omega\epsilon_0}{k_g}\, \hat{\boldsymbol{z}} \wedge \boldsymbol{E}_s, \tag{6.55b}$$

where use has been made of $\partial/\partial z = \mathrm{i}\, k_g$. For TE modes ($E_z = 0$) Eqs. (6.35) give

$$\boldsymbol{H}_s = \frac{\mathrm{i}\, k_g}{{k_s}^2}\, \nabla_s H_z, \tag{6.56a}$$

$$\boldsymbol{E}_s = -\frac{\omega\mu_0}{k_g}\, \hat{\boldsymbol{z}} \wedge \boldsymbol{H}_s. \tag{6.56b}$$

The time-average z component of the Poynting vector $\boldsymbol{N}$ is given by

$$\overline{N_z} = \frac{|\boldsymbol{E}_s \wedge {\boldsymbol{H}_s}^*|}{2}. \tag{6.57}$$

It follows that

$$\overline{N_z} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\sqrt{1-{\omega_c}^2/\omega^2}} \frac{H_{s0}^{\,2}}{2} \tag{6.58}$$

for TE modes, and

$$\overline{N_z} = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{1-{\omega_c}^2/\omega^2} \, \frac{H_{s0}^{\,2}}{2} \tag{6.59}$$

for TM modes. The subscript 0 denotes the peak value of a wave quantity.

Wave guide losses can be estimated by integrating Eq. (6.14) over the wall of the guide for any given mode. The energy flow of a propagating wave attenuates as e^{-Kz}, where

$$K = \frac{\text{power loss per unit length of guide}}{\text{power transmitted through guide}}. \tag{6.60}$$

Thus,

$$K = \frac{1}{2\sigma d} \int \left({H_s}^2 + {H_z}^2\right) dS \bigg/ \int \overline{N_z}\, dS\ , \tag{6.61}$$

where the numerator is integrated over unit length of the wall and the denominator is integrated over the transverse cross section of the guide. It is customary to define a *guide impedance* Z_g by writing

$$\int \overline{N_z}\, dS = \frac{Z_g}{2} \int H_{s0}^{\,2}\, dS. \tag{6.62}$$

It follows from Eqs. (6.58) and (6.59) that

$$Z_g = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\sqrt{1-{\omega_c}^2/\omega^2}} \tag{6.63}$$

for TE modes, and

$$Z_g = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{1-{\omega_c}^2/\omega^2} \tag{6.64}$$

for TM modes. For both types of mode $\boldsymbol{H}_s = (1/Z_g)\, \hat{\boldsymbol{z}} \wedge \boldsymbol{E}_s$.

6.8 Dielectric wave guides

We have seen that it is possible to propagate electromagnetic waves down a hollow conductor. However, other types of guiding structures are also possible. The general requirement for a guide of electromagnetic waves is that there be a flow of energy along the axis of the guiding structure but not perpendicular to it. This implies that the electromagnetic fields are appreciable only in the immediate neighbourhood of the guiding structure.

Consider an axisymmetric tube of arbitrary cross section made of some dielectric material and surrounded by a vacuum. This structure can serve as a wave guide provided that the dielectric constant of the material is sufficiently large. Note, however, that the boundary conditions satisfied by the electromagnetic fields are significantly different to those of a conventional wave guide. The transverse fields are governed by two equations; one for the region inside the dielectric, and the other for the vacuum region. Inside the dielectric we have

$$\left[\nabla_s^2 + \left(\epsilon_1 \frac{\omega^2}{c^2} - {k_g}^2\right)\right]\psi = 0. \tag{6.65}$$

In the vacuum region we have

$$\left[\nabla_s^2 + \left(\frac{\omega^2}{c^2} - {k_g}^2\right)\right]\psi = 0. \tag{6.66}$$

Here, $\psi(x,y)\,\mathrm{e}^{\,\mathrm{i}\,k_g z}$ stands for either E_z or H_z, ϵ_1 is the relative permittivity of the dielectric material, and k_g is the guide propagation constant. The guide propagation constant must be the same both inside and outside the dielectric in order to satisfy the electromagnetic boundary conditions at all points on the surface of the tube.

Inside the dielectric the transverse Laplacian must be negative, so that the constant

$${k_s}^2 = \epsilon_1 \frac{\omega^2}{c^2} - {k_g}^2 \tag{6.67}$$

is positive. Outside the cylinder the requirement of no transverse flow of energy can only be satisfied if the fields fall off exponentially (instead of oscillating).

Thus,

$$k_t{}^2 = k_g{}^2 - \frac{\omega^2}{c^2} \tag{6.68}$$

must be positive.

The oscillatory solutions (inside) must be matched to the exponentiating solutions (outside). The boundary conditions are the continuity of normal $\boldsymbol{B}$ and $\boldsymbol{D}$ and tangential $\boldsymbol{E}$ and $\boldsymbol{H}$ on the surface of the tube. These boundary conditions are far more complicated than those in a conventional wave guide. For this reason, the normal modes cannot usually be classified as either pure TE or pure TM modes. In general, the normal modes possess both electric and magnetic field components in the transverse plane. However, for the special case of a cylindrical tube of dielectric material the normal modes can have either pure TE or pure TM characteristics. Let us examine this case in detail.

Consider a dielectric cylinder of radius a and dielectric constant ϵ_1. For the sake of simplicity, let us only search for normal modes whose electromagnetic fields have no azimuthal variation. Equations (6.65) and (6.67) yield

$$\left(r^2\,\frac{d^2}{dr^2} + r\,\frac{d}{dr} + r^2 k_s{}^2\right)\psi = 0 \tag{6.69}$$

for $r < a$. The general solution to this equation is some linear combination of the Bessel functions $J_0(k_s r)$ and $Y_0(k_s r)$. However, since $Y_0(k_s r)$ is badly behaved at the origin ($r = 0$) the physical solution is $\psi \propto J_0(k_s r)$.

Equations (6.66) and (6.68) yield

$$\left(r^2\,\frac{d^2}{dr^2} + r\,\frac{d}{dr} - r^2 k_t{}^2\right)\psi = 0. \tag{6.70}$$

This can be rewritten

$$\left(z^2\,\frac{d^2}{dz^2} + z\,\frac{d}{dz} - z^2\right)\psi = 0, \tag{6.71}$$

where $z = k_t r$. This is type of *modified Bessel's equation*, whose most general form is

$$\left[z^2\,\frac{d^2}{dz^2} + z\,\frac{d}{dz} - (z^2 + m^2)\right]\psi = 0. \tag{6.72}$$

The two linearly independent solutions of the above equation are denoted $I_m(z)$ and $K_m(z)$. The asymptotic behaviour of these solutions at small $|z|$ is as follows:

$$I_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(k+m)!}, \tag{6.73a}$$

$$\begin{aligned} K_m(z) &= \frac{1}{2}\left(\frac{z}{2}\right)^{-m} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!}(-z^2/4)^k + (-1)^{m+1}\ln(z/2)I_m(z) \\ &\quad +(-1)^m \frac{1}{2}\left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} [\psi(k+1)+\psi(m+k+1)]\,\frac{(z^2/4)^m}{k!(m+k)!}. \end{aligned} \tag{6.73b}$$

Hence, I_m is well behaved in the limit $|z| \to 0$, whereas K_m is badly behaved. The asymptotic behaviour at large $|z|$ is

$$I_m(z) \simeq \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left[1 + O\left(\frac{1}{z}\right)\right], \tag{6.74a}$$

$$K_m(z) \simeq \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left[1 + O\left(\frac{1}{z}\right)\right]. \tag{6.74b}$$

Hence, I_m is badly behaved in the limit $|z| \to \infty$, whereas K_m is well behaved. The behaviour of $I_0(z)$ and $K_0(z)$ is shown in Fig. 22. It is clear that the physical solution to Eq. (6.70) (*i.e.*, the one which decays as $|r| \to \infty$) is $\psi \propto K_0(k_t r)$.

The physical solution is

$$\psi = J_0(k_s r) \tag{6.75}$$

for $r \leq a$, and

$$\psi = A\,K_0(k_t r) \tag{6.76}$$

for $r > a$. Here, A is an arbitrary constant, and $\psi(r)\,\mathrm{e}^{\,\mathrm{i}\,k_g z}$ stands for either E_z or H_z. It follows from Eqs. (6.28) (using $\partial/\partial\theta = 0$) that

$$H_r = \mathrm{i}\,\frac{k_g}{{k_s}^2}\frac{\partial H_z}{\partial r}, \tag{6.77a}$$

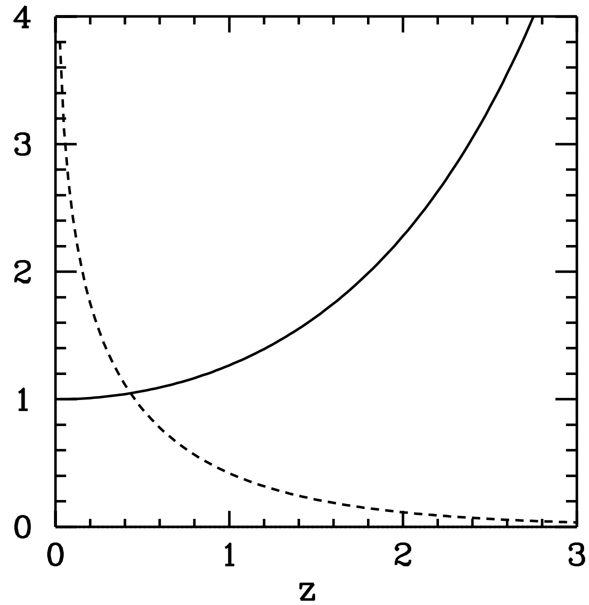

Figure 22: The Bessel functions $I_0(z)$ (solid line) and $K_0(z)$ (dotted line)

$$E_\theta = -\frac{\omega\mu_0}{k_g} H_r, \tag{6.77b}$$

$$H_\theta = \mathrm{i}\,\frac{\omega\epsilon_0\epsilon_1}{{k_s}^2}\frac{\partial E_z}{\partial r}, \tag{6.77c}$$

$$E_r = \frac{k_g}{\omega\epsilon_0\epsilon_1} H_\theta \tag{6.77d}$$

for $r \leq a$. There are an analogous set of relationships for $r > a$. The fact that the field components form two groups; (H_r, E_θ), which depend on H_z, and (H_θ, E_r), which depend on E_z; means that the normal modes take the form of either pure TE modes or pure TM modes.

For a TE mode ($E_z = 0$) we find that

$$H_z = J_0(k_s r), \tag{6.78a}$$

$$H_r = -\mathrm{i}\,\frac{k_g}{k_s} J_1(k_s r), \tag{6.78b}$$

$$E_\theta = \mathrm{i}\,\frac{\omega\mu_0}{k_s} J_1(k_s r) \tag{6.78c}$$

for $r \leq a$, and

$$H_z = A\,K_0(k_t r), \tag{6.79a}$$

$$H_r = \mathrm{i}\,A\,\frac{k_g}{k_t}\,K_1(k_t r), \tag{6.79b}$$

$$E_\theta = -\mathrm{i}\,A\,\frac{\omega\mu_0}{k_t}\,K_1(k_t r) \tag{6.79c}$$

for $r > a$. Here we have used

$$J_0'(z) = -J_1(z), \tag{6.80a}$$

$$K_0'(z) = -K_1(z), \tag{6.80b}$$

where $'$ denotes differentiation with respect to z. The boundary conditions require H_z, H_r, and E_θ to be continuous across $r = a$. Thus, it follows that

$$A\,K_0(k_t a) = J_0(k_s a), \tag{6.81a}$$

$$-A\,\frac{K_1(k_t r)}{k_t} = \frac{J_1(k_s a)}{k_s}. \tag{6.81b}$$

Eliminating the arbitrary constant A between the above two equations yields the dispersion relation

$$\frac{J_1(k_s a)}{k_s\,J_0(k_s a)} + \frac{K_1(k_t a)}{k_t\,K_0(k_t a)} = 0, \tag{6.82}$$

where

$$k_t^{\,2} + k_s^{\,2} = (\epsilon_1 - 1)\,\frac{\omega^2}{c^2}. \tag{6.82}$$

Figure 23 shows a graphical solution of the above dispersion relation. The roots correspond to the crossing points of the two curves; $-J_1(k_s a)/k_s J_0(k_s a)$ and $K_1(k_t a)/k_t K_0(k_t a)$. The vertical asymptotes of the first curve are given by the roots of $J_0(k_s a) = 0$. The vertical asymptote of the second curve occurs when $k_t = 0$; *i.e.*, when $k_s^{\,2} a^2 = (\epsilon_1 - 1)\,\omega^2 a^2/c^2$. Note from Eq. (6.82) that k_t decreases as k_s increases. In Fig. 23 there are two crossing points, corresponding to two distinct propagating modes of the system. It is evident that if the point $k_t = 0$

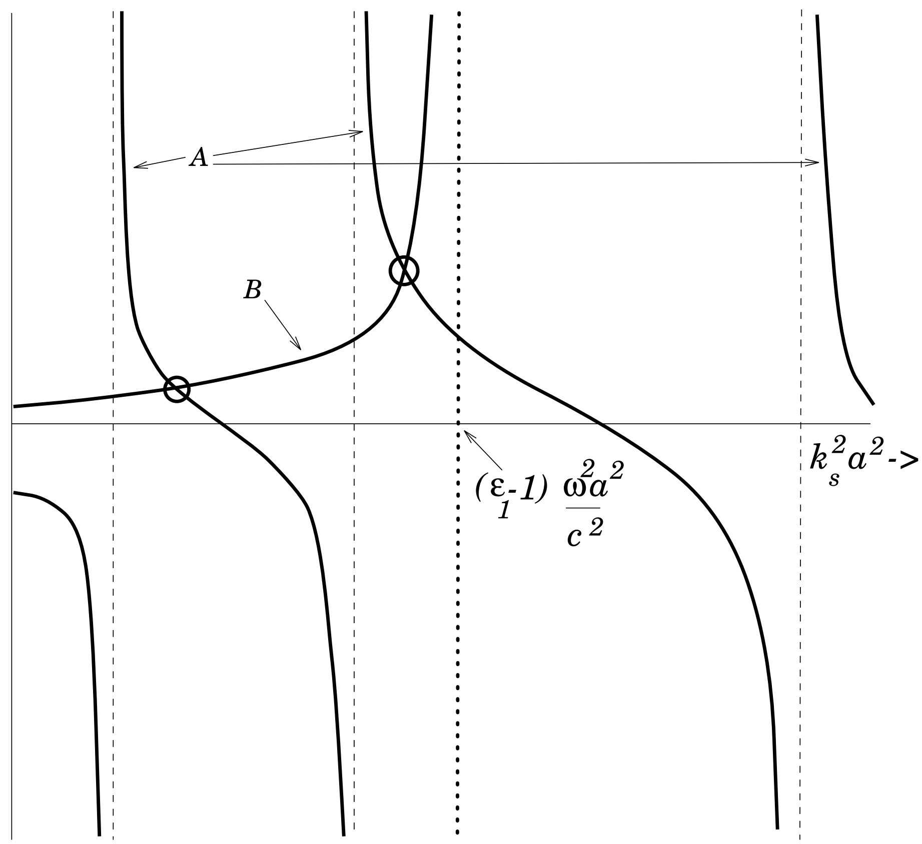

Figure 23: Graphical solution of the dispersion relation (6.82). The curve A represents $-J_1(k_s/a)/k_s J_0(k_s a)$. The curve B represents $K_1(k_t a)/k_t K_0(k_t a)$.

corresponds to a value of $k_s a$ which is less than the first root of $J_0(k_s a) = 0$, then there is no crossing of the two curves, and, hence, there are no propagating modes. Since the first root of $J_0(z) = 0$ occurs at $z = 2.4048$ (see Table 2) the condition for the existence of propagating modes can be written

$$\omega > \omega_{01} = \frac{2.4048\, c}{\sqrt{\epsilon_1 - 1}\, a}. \tag{6.83}$$

In other words, the mode frequency must lie above the cutoff frequency ω_{01} for the TE_{01} mode (here, the 0 corresponds to the number of nodes in the azimuthal direction, and the 1 refers to the 1st root of $J_0(z) = 0$). It is also evident that as the mode frequency is gradually increased the point $k_t = 0$ eventually crosses the second vertical asymptote of $-J_1(k_s/a)/k_s J_0(k_s a)$, at which point the TE_{02} mode can propagate. As ω is further increased more and more TE modes can propagate. The cutoff frequency for the TE_{0l} mode is given by

$$\omega_{0l} = \frac{j_{0l}\, c}{\sqrt{\epsilon_1 - 1}\, a}, \tag{6.84}$$

where j_{0l} is lth root of $J_0(z) = 0$ (in order of increasing z).

At the cutoff frequency for a particular mode $k_t = 0$, which implies from Eq. (6.68) that $k_g = \omega/c$. In other words, the mode propagates along the guide at the velocity of light in vacuum. Immediately below this cutoff frequency the system no longer acts as a guide but as an antenna, with energy being radiated radially. For frequencies well above the cutoff, k_t and k_g are of the same order of magnitude, and are large compared to k_s. This implies that the fields do not extend appreciably outside the dielectric cylinder.

For a TM mode ($H_z = 0$) we find that

$$E_z = J_0(k_s r), \tag{6.85a}$$

$$H_\theta = -\mathrm{i}\, \frac{\omega \epsilon_0 \epsilon_1}{k_s}\, J_1(k_s r), \tag{6.85b}$$

$$E_r = -\mathrm{i}\, \frac{k_g}{k_s}\, J_1(k_s r) \tag{6.85c}$$

for $r \leq a$, and

$$E_z = A\, K_0(k_t r), \tag{6.86a}$$

$$H_\theta = \mathrm{i}\, A\, \frac{\omega \epsilon_0}{k_t}\, K_1(k_t r), \tag{6.86b}$$

$$E_r = \mathrm{i}\, A\, \frac{k_g}{k_t}\, K_1(k_t r) \tag{6.86c}$$

for $r > a$. The boundary conditions require E_z, H_θ, and D_r to be continuous across $r = a$. Thus, it follows that

$$A\, K_0(k_t a) = J_0(k_s a), \tag{6.87a}$$

$$-A\, \frac{K_1(k_t r)}{k_t} = \epsilon_1 \frac{J_1(k_s a)}{k_s}. \tag{6.87b}$$

Eliminating the arbitrary constant A between the above two equations yields the dispersion relation

$$\epsilon_1 \frac{J_1(k_s a)}{k_s\, J_0(k_s a)} + \frac{K_1(k_t a)}{k_t\, K_0(k_t a)} = 0. \tag{6.88}$$

It is clear from this dispersion relation that the cutoff frequency for the TM_{0l} mode is exactly the same as that for the TE_{0l} mode. It is also clear that in the limit $\epsilon_1 \gg 1$ the propagation constants are determined by the roots of $J_1(k_s a) \simeq 0$. However, this is exactly the same as the determining equation for TE modes in a metallic wave guide of circular cross section (filled with dielectric of relative permittivity ϵ_1).

Modes with azimuthal dependence (*i.e.*, $m > 0$) have longitudinal components of both $\boldsymbol{E}$ and $\boldsymbol{H}$. This makes the mathematics somewhat more complicated. However, the basic results are the same as for $m = 0$ modes: for frequencies well above the cutoff frequency the modes are localized in the immediate vicinity of the cylinder.