

LECTURE NOTES 7.5

Dispersion: The Frequency-Dependence of the Electric Permittivity $\epsilon = \epsilon(\omega)$ and the Electric Susceptibility $\chi_e(\omega)$:

Over the entire *EM* frequency interval $\{0 \leq f \leq \infty \text{ Hz}\}$, the speed of propagation v_{prop} of monochromatic (*i.e.* single-frequency) *EM* waves in matter is often not constant, not independent of frequency: $v_{prop} \neq \text{constant}$; $v_{prop} = fcn(\text{frequency}, f) = v_{prop}(f)$, because matter is composite.

At the microscopic level, matter is comprised of atoms/molecules – these have resonances in energy/energy levels that are governed by the laws of quantum mechanics that are operative at the atomic/molecular scale...

If: $v_{prop} = fcn(\text{frequency}, f) = v_{prop}(f)$ but: $v_{prop}(f) = f\lambda \Rightarrow \lambda = fcn(\text{frequency}, f)$

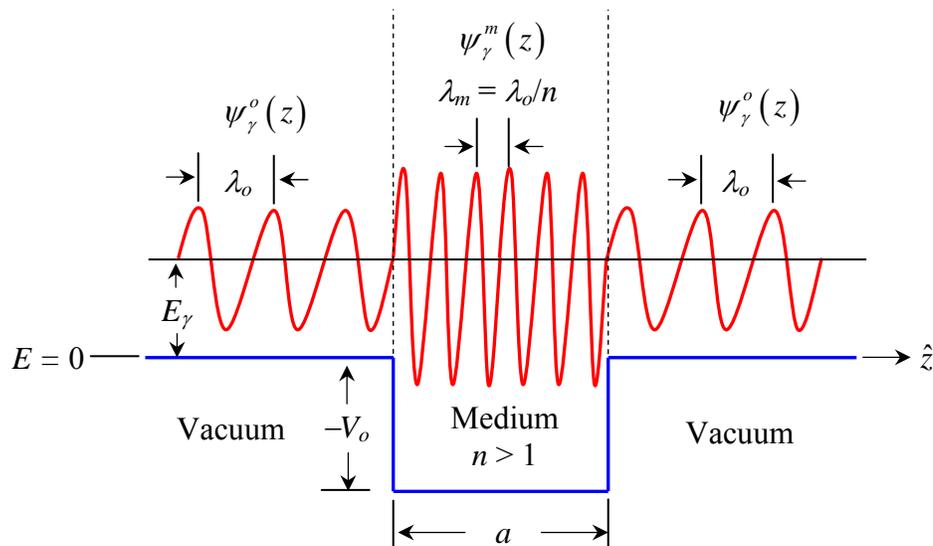
Thus: $v_{prop}(f) = f\lambda(f)$ or, since: $\omega \equiv 2\pi f$ and: $k \equiv 2\pi/\lambda \Rightarrow k(\omega) \equiv 2\pi/\lambda(\omega)$

Thus: $v_{prop}(\omega) = f\lambda(\omega) = 2\pi f \cdot \lambda(\omega) / 2\pi = \omega/k(\omega)$

For a monochromatic / single-frequency *EM* wave $v_{prop}(\omega) = \omega/k(\omega)$ ← *EM* wave velocity
a.k.a. phase-velocity

The microscopic reason for this is that in a macroscopic medium, a (*real*) photon's energy $E_\gamma = hf_\gamma = \text{constant}$ / is unchanged / same as that in the vacuum, however the photon's momentum $p_\gamma = h/\lambda_\gamma$ (De Broglie Relation) does change, relative to the {real} photon's momentum in the vacuum. In a macroscopic medium, if $\lambda = \lambda(\omega)$, then $p_\gamma(\omega) = h/\lambda_\gamma(\omega)$.

{Real} Photons Propagating Through A Finite-Thickness Macroscopic Linear Medium: {Optical Potential Model}



Macroscopically, the frequency-dependence of the wavelength $\lambda = \lambda(\omega)$, or wavenumber $k(\omega) = 2\pi/\lambda(\omega)$, and linear momentum $p(\omega)$ associated with macroscopic *EM* waves propagating in a macroscopic, linear/homogeneous/isotropic medium arises from the frequency-dependence of the macroscopic electric permittivity $\varepsilon(\omega)$ (or equivalently the electric susceptibility $\chi_e(\omega)$) since:

$$\boxed{\varepsilon(\omega) = \varepsilon_o (1 + \chi_e(\omega))}$$

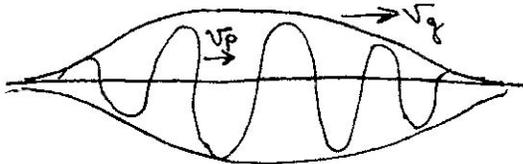
The frequency dependence of the macroscopic electric permittivity $\varepsilon(\omega)$ is known as dispersion; a medium that has $\varepsilon \neq \text{constant}$ with varying frequency is known as a dispersive medium.

For non-magnetic/non-conducting linear/homogeneous/isotropic media, the index of refraction $n = \sqrt{\varepsilon/\varepsilon_o}$. Thus if $\varepsilon = \varepsilon(\omega) = \varepsilon_o (1 + \chi_e(\omega))$ then $n(\omega) = \sqrt{\varepsilon(\omega)/\varepsilon_o}$.

For a wave packet (= a group {= superposition/linear combination} of waves of many frequencies – as explained by Mssr. Fourier), the envelope of the wave packet travels with (in general, frequency-dependent) group velocity:

$$\boxed{v_{group}(\omega) \equiv \frac{d\omega}{dk(\omega)} = \frac{1}{dk(\omega)/d\omega} = \left[\frac{dk(\omega)}{d\omega} \right]^{-1}}$$

A propagating wave packet:



$$\boxed{v_p(\omega) = v_{phase}(\omega) = \omega/k(\omega)}$$

$$\boxed{v_g(\omega) = v_{group}(\omega) \equiv \left[dk(\omega)/d\omega \right]^{-1}}$$

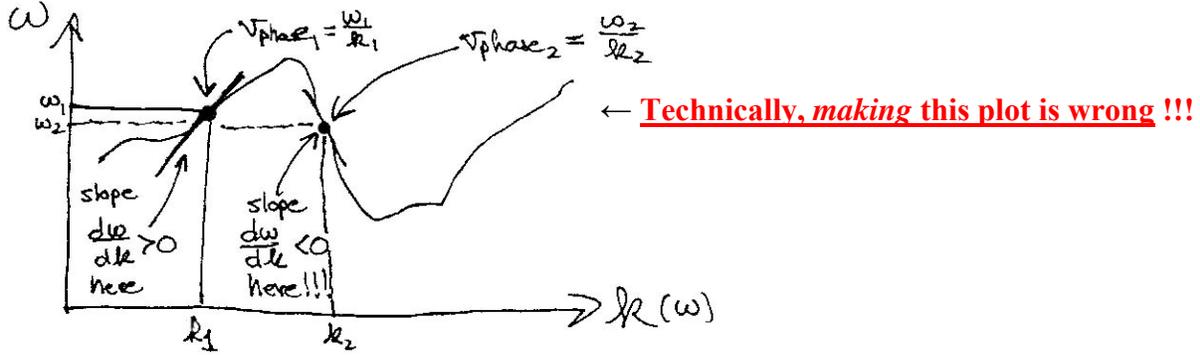
If the phase velocity $v_p(\omega) = \omega/k(\omega)$ is frequency dependent, and $v_{group}(\omega) = \left[dk(\omega)/d\omega \right]^{-1} \neq v_{phase}(\omega)$ {e.g. as in the case for surface waves on water, where $v_p = 2v_g$ } the exact relationship between v_{phase} and v_{group} depends on the detailed nature of the medium (as we shall soon see. . .).

Note that energy carried by a wave packet travels at the group velocity, v_{group} and not at the phase velocity v_{phase} . Note that in certain circumstances, v_{phase} can exceed c {= speed of light in the vacuum} but in these situations, no energy (and/or information) is transmitted – these are transmitted at $v_{group} < c$ always, by causality... A physical/mechanical example: calculate the phase velocity of the intersection point of the two halves of a scissors as the blades of the scissors are closed. {Ans: $v_p^{scissors} \rightarrow \infty$!!!}



An Important Detail:

Physically, the phase velocity $v_{phase}(\omega) \equiv \omega/k(\omega)$ = a single point on the ω vs. $k(\omega)$ curve, whereas the group velocity $v_{group}(\omega) \equiv "d\omega/dk(\omega)"$ = the slope of the ω vs. $k(\omega)$ curve:

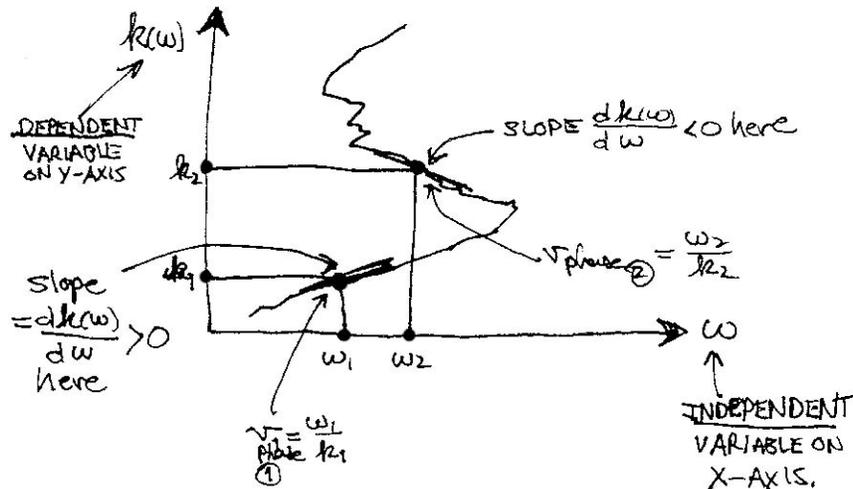


Why is this plot technically wrong ???

Because ω is the **independent** variable {always plotted on the axis of abscissas (i.e. the x-axis)} and $k(\omega)$ is the **dependent** variable {always plotted on the axis of ordinates (i.e. the y-axis)}.

Thus, the **technically correct** way is to plot $k(\omega)$ vs. ω : {i.e. k depends on ω , **not** vice-versa!}

Then:
$$v_{group}(\omega) \equiv 1 / \left(\frac{dk(\omega)}{d\omega} \right) = \left[\frac{dk(\omega)}{d\omega} \right]^{-1} \quad \text{i.e.} \quad v_{group}(\omega) \equiv 1 / \text{slope of } \{k(\omega) \text{ vs. } \omega\} \text{ graph :}$$



Dispersion Phenomena in Linear Dielectrics

In a non-conducting, linear, homogeneous, isotropic medium there are no *free* electrons (i.e. $\rho_{free}(\vec{r}) = 0$). Atomic electrons are permanently bound to nuclei of atoms comprising the medium. $\Rightarrow \exists$ no preferential direction / no preferential directions in such an {isotropic} medium.

Suppose each atomic electron (charge $-e$) in a dielectric is *displaced* by a small distance \vec{r} from its *equilibrium position*, e.g. by application of a *static* electric field $\vec{E}(\vec{r})$.

The resulting *macroscopic* electric polarization (aka electric dipole moment per unit volume) is:

$$\boxed{\vec{P}(\vec{r}) = n_e \vec{p}(\vec{r})} \quad \text{where: } n_e = (\text{atomic}) \text{ electron number density } (\#e/m^3)$$

and the {induced} atomic/molecular electric dipole moment is: $\vec{p}(\vec{r}) = -e\vec{r}$ {here}, where \vec{r} is the {vector} displacement of the atomic electron from its equilibrium { $\vec{E} = 0$ } position.

Thus: $\boxed{\vec{P}(\vec{r}) = n_e \vec{p}(\vec{r}) = -n_e e \vec{r}}$

The atomic electrons are each elastically bound to their equilibrium positions with a force constant k_e . The force equation for *each* atomic electron is thus: $\vec{F}_e(\vec{r}) = -e\vec{E}(\vec{r}) = k_e \vec{r}$

The *static* polarization is therefore given by: $\boxed{\vec{P}(\vec{r}) = n_e \vec{p}(\vec{r}) = -n_e e \vec{r} = -n_e e \left(\frac{-e\vec{E}(\vec{r})}{k_e} \right) = + \frac{n_e e^2}{k_e} \vec{E}(\vec{r})}$

However, if the \vec{E} -field *varies* with time, i.e. $\vec{E} = \vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(kz - \omega t)}$ e.g. due to a monochromatic EM plane wave incident on an atom, the above relation is incorrect !

A more correct {“semi-classical”} way to treat this situation is to consider the bound atomic electrons as classical, damped, forced harmonic oscillators, as mathematically described by the following differential equation:

$$m_e \ddot{\vec{r}} + m_e \gamma \dot{\vec{r}} + k_e \vec{r} = -e\vec{E}(\vec{r}) \quad \leftarrow \text{inhomogeneous } 2^{\text{nd}}\text{-order differential equation}$$

$$m_e \frac{\partial^2 \vec{r}(t)}{\partial t^2} + m_e \gamma \frac{\partial \vec{r}(t)}{\partial t} + k_e \vec{r}(t) = -e\vec{E}(\vec{r}, t)$$

n.b. we have neglected the $e\vec{v} \times \vec{B}$ ($\ll e\vec{E}$) term here...

Velocity-dependent damping term
 $\gamma \equiv$ damping constant

Potential Force (binding of atomic electrons to atom)

Driving Force

$m_e = \text{electron mass} = 9.1 \times 10^{-31} \text{ kg}$

Suppose the driving / forcing term varies sinusoidally/is harmonic/periodic in time with angular frequency ω , i.e. $\vec{F}_e(\vec{r}, t) = -e\vec{E}(\vec{r}, t) = -e\vec{E}_0 e^{-i\omega t} \hat{r}$ because $\vec{E}(\vec{r}, t) = \vec{E}_0 e^{-i\omega t} \hat{r}$.

n.b. The electric field \vec{E} is now complex \vec{E} .

Then the inhomogeneous force equation becomes: $m_e \ddot{\tilde{r}} + m_e \gamma \dot{\tilde{r}} + k_e \tilde{r} = -e \tilde{E}_o e^{-i\omega t} \hat{r}$ with $\tilde{r}(t) = \tilde{r}(\omega) \hat{r}$.

In the steady state, we have:

$$m_e \ddot{\tilde{r}} + m_e \gamma \dot{\tilde{r}} + k_e \tilde{r} = -e \tilde{E}_o e^{-i\omega t} \hat{r}$$

Since \tilde{r} physically represents the {vector} spatial displacement of each atomic electron from its equilibrium $\{\vec{E} = 0\}$ position, then: $\tilde{r}(t) = \tilde{r}_o(\omega) e^{-i\omega t} \hat{r}$ {n.b. $\tilde{r}(t)$ is now complex}.

Thus:

$$m_e \ddot{\tilde{r}} + m_e \gamma \dot{\tilde{r}} + k_e \tilde{r} = -e \tilde{E}_o e^{-i\omega t} \hat{r}$$

$$\begin{aligned} m_e \frac{\partial^2 \tilde{r}(t)}{\partial t^2} + m_e \gamma \frac{\partial \tilde{r}(t)}{\partial t} + k_e \tilde{r}(t) &= -e \tilde{E}(\tilde{r}, t) \\ -m_e \omega^2 \tilde{r}_o e^{-i\omega t} - i\omega m_e \gamma \tilde{r}_o e^{-i\omega t} + k_e \tilde{r}_o e^{-i\omega t} &= -e \tilde{E}_o e^{-i\omega t} \\ (m_e \omega^2 - k_e + i\omega m_e \gamma) \tilde{r}_o &= e \tilde{E}_o \end{aligned}$$

Divide this equation through by m_e :

$$\left(\omega^2 - \left(\frac{k_e}{m_e} \right) + i\omega \gamma \right) \tilde{r}_o = \frac{e}{m_e} \tilde{E}_o$$

Define: $\omega_0^2 \equiv \left(\frac{k_e}{m_e} \right)$ or: $\omega_0 \equiv \sqrt{\frac{k_e}{m_e}}$ = characteristic/natural resonance {angular} frequency.

Then: $(\omega^2 - \omega_0^2 + i\gamma\omega) \tilde{r}_o = \left(\frac{e}{m_e} \right) \tilde{E}_o$ or: $\tilde{r}_o(\omega) = \frac{\left(\frac{e}{m_e} \right) \tilde{E}_o}{\left[\omega^2 - \omega_0^2 + i\gamma\omega \right]}$ = Atomic electron spatial displacement amplitude {n.b. complex!}

Now: $\tilde{P}(\tilde{r}, t) = -n_e e \tilde{r}(t) = -n_e e \tilde{r}_o(\omega) e^{-i\omega t} \hat{r}$ {n.b. \tilde{P} is now complex and frequency-dependent!!!}

Thus: $\tilde{P}(\tilde{r}, t) = \frac{-n_e \left(\frac{e^2}{m_e} \right) (\tilde{E}_o e^{-i\omega t} \hat{r})}{\left[\omega^2 - \omega_0^2 + i\gamma\omega \right]} = + \frac{n_e \left(\frac{e^2}{m_e} \right)}{\left[\omega_0^2 - \omega^2 - i\gamma\omega \right]} \tilde{E}(\tilde{r}, t)$ ← n.b. Note the re-ordering of terms in the denominator

For $\omega = 0$: $\tilde{P}(\omega = 0) = \frac{n_e \left(\frac{e^2}{m_e} \right)}{\omega_0^2} \tilde{E}_o \hat{r} = \frac{n_e \left(\frac{e^2}{m_e} \right)}{\left(\frac{k_e}{m_e} \right)} \tilde{E}_o \hat{r} = \left(\frac{n_e e^2}{k_e} \right) \tilde{E}_o \hat{r}$ ← Static polarization $\tilde{P}(\omega = 0)$ is in-phase with \tilde{E}

Note also that the phase of {complex} $\tilde{P}(\omega)$ depends on the {angular} frequency ω - i.e. $\tilde{P}(\omega)$ lags behind $\tilde{E}(\omega)$ by a phase angle of:

$$\phi_p(\omega) = \tan^{-1} \left[\frac{\Im m(\tilde{P}(\tilde{r}, t))}{\Re e(\tilde{P}(\tilde{r}, t))} \right] = \tan^{-1} \left[\frac{\gamma\omega}{(\omega_0^2 - \omega^2)} \right] \leftarrow \text{n.b. The damping constant } \gamma \text{ has the same units as } \omega: \text{ radians/sec}$$

When: $\omega < \omega_0 = \sqrt{\frac{k_e}{m_e}}$, $\phi_p > 0 \Rightarrow \tilde{\mathbf{P}}$ lags $\tilde{\mathbf{E}}$. When: $\omega > \omega_0 = \sqrt{\frac{k_e}{m_e}}$, $\phi_p < 0 \Rightarrow \tilde{\mathbf{P}}$ leads $\tilde{\mathbf{E}}$.

From the above formula, note that if the damping constant, $\gamma = 0$ then $\phi_p = 0$, the electric polarization $\tilde{\mathbf{P}}(\omega)$ is then always in-phase with $\tilde{\mathbf{E}}(\omega)$ because if $\gamma = 0$, then $\Im m(\tilde{\mathbf{P}}(\vec{r}, t)) = 0$, *i.e.* the electric polarization $\tilde{\mathbf{P}}(\omega)$ is purely real! Physically, a damping constant of $\gamma = 0$ means that the width $\Gamma = \gamma/2\pi$ of the atomic/molecular resonance is infinitely narrow, and thus there are no dissipative processes (*i.e.* energy loss mechanisms) present at the microscopic atomic/molecular level in this macroscopic medium! Note also that γ has physical/SI units of radians/second.

Note further that $\tilde{\mathbf{E}}$ in the above expression is actually $\tilde{\mathbf{E}}_{int}$ – the internal macroscopic electric field of the dielectric: $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_{int} = \tilde{\mathbf{E}}_{ext} + \tilde{\mathbf{E}}_p$, the sum of the macroscopic external applied electric field and the macroscopic electric field due to the polarization of the dielectric medium.

The electric field due to polarization of the medium is: $\tilde{\mathbf{E}}_p = -\frac{1}{3\epsilon_o} \tilde{\mathbf{P}}$ See P435 Lect. Notes 10, p. 1-6, see also P435 Lect. Notes 9, p. 26

Thus: $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_{int} = \tilde{\mathbf{E}}_{ext} - \frac{1}{3\epsilon_o} \tilde{\mathbf{P}}$ Therefore: $\tilde{\mathbf{P}} = \frac{n_e \left(\frac{e^2}{m_e} \right)}{\left[\omega_o^2 - \omega^2 - i\gamma\omega \right]} \left[\tilde{\mathbf{E}}_{ext} - \frac{1}{3\epsilon_o} \tilde{\mathbf{P}} \right]$ where: $\omega_o \equiv \sqrt{\frac{k_e}{m_e}}$

Now solve for $\tilde{\mathbf{P}}$: Skipping writing out some of the {tedious} complex algebra, we obtain:

$$\tilde{\mathbf{P}} = \frac{n_e \left(\frac{e^2}{m_e} \right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega \right]} \tilde{\mathbf{E}}_{ext} \quad \text{where:} \quad \omega_1 \equiv \sqrt{\omega_o^2 - \left(\frac{n_e e^2}{3\epsilon_o m_e} \right)} = \text{effective angular resonance frequency of bound atomic electrons}$$

This formula is essentially identical *e.g.* to the {complex} displacement amplitude formula for a driven harmonic oscillator, and/or that for the {complex} AC voltage amplitude in an LCR circuit, and for many other physical systems exhibiting a {damped} resonance-type behavior.

Now if $\tilde{\mathbf{E}}_{ext} = \vec{E}$ -field associated with a monochromatic plane *EM* wave propagating in a dielectric medium: $\tilde{\mathbf{E}}_{ext}(z, t) = \tilde{E}_o e^{i(kz - \omega t)}$, then because of the linear relationship between the polarization $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}_{ext}(z, t) = \underbrace{\tilde{E}_o e^{i(kz - \omega t)}}_{\text{for EM plane wave}} \hat{x}$, Gauss' Law becomes (since $\rho_{free}(\vec{r}) = 0$):

$$\vec{\nabla} \cdot \tilde{\mathbf{E}}_{ext} = -\frac{1}{\epsilon_o} \vec{\nabla} \cdot \tilde{\mathbf{P}} = \tilde{\rho}_{bound} = 0$$

The wave equation {for a dielectric medium with $\tilde{\rho}_{free}(\vec{r}) = 0$ and $\tilde{\mathbf{J}}_{free} = 0$ } becomes:

$$\nabla^2 \tilde{\mathbf{E}}_{ext} - \frac{1}{c^2} \frac{\partial^2 \tilde{\mathbf{E}}_{ext}}{\partial t^2} = \mu_o \frac{\partial^2 \tilde{\mathbf{P}}}{\partial t^2} = \frac{\mu_o n_e \left(\frac{e^2}{m_e} \right)}{\left[\omega_1^2 - \omega^2 - i\gamma\omega \right]} \frac{\partial^2 \tilde{\mathbf{E}}_{ext}}{\partial t^2} \quad \text{with:} \quad \frac{1}{c^2} = \epsilon_o \mu_o$$

$$\text{Or: } \boxed{\nabla^2 \tilde{E}_{ext} = \frac{1}{c^2} \left[1 + \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \frac{1}{\omega_1^2 - \omega^2 - i\gamma\omega} \right] \frac{\partial^2 \tilde{E}_{ext}}{\partial t^2}} \quad \text{with: } \boxed{\omega_1 \equiv \sqrt{\omega_0^2 - \frac{n_e e^2}{3\epsilon_o m}}}$$

The general solution to this dispersive wave equation is of the form:

$$\boxed{\tilde{E}_{ext}(z, t) = \tilde{E}_o e^{i(\tilde{k}z - \omega t)}} \quad \text{with complex } \boxed{\tilde{k} = k + i\kappa} \quad \text{and} \quad \boxed{\tilde{k}^2 = \frac{\omega^2}{c^2} \left[1 + \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \frac{1}{\omega_1^2 - \omega^2 - i\gamma\omega} \right]}$$

Thus, we also see here that the complex wavenumber $\tilde{k} = k + i\kappa$ is explicitly dependent on the angular frequency ω , *i.e.* $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$.

We further see that monochromatic plane *EM* waves propagating in a dispersive dielectric medium are exponentially attenuated, because $\tilde{E}_{ext}(z, t) = \tilde{E}_o e^{i(\tilde{k}z - \omega t)} = \tilde{E}_o e^{-\kappa z} e^{i(kz - \omega t)}$, *i.e.* the $\kappa(\omega) = \Im m(\tilde{k}(\omega))$ term corresponds to absorption/dissipation in the macroscopic dielectric, and is physically related to, and thus proportional to the damping constant γ .

Note that we can also write: $\tilde{P}(z, t, \omega) \equiv \epsilon_o \tilde{\chi}_e(\omega) \tilde{E}_{ext}(z, t, \omega)$ and thus the macroscopic electric susceptibility $\tilde{\chi}_e(\omega)$ {here} is also {now} complex and is also frequency-dependent, *i.e.* $\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$. The $\zeta_e(\omega) = \Im m(\tilde{\chi}_e(\omega))$ term corresponds to absorption/dissipation in the macroscopic dielectric, and is physically related to, and thus proportional to the damping constant γ . The corresponding dissipative energy losses at the microscopic, atomic/molecular level in the macroscopic dielectric ultimately wind up as heat!

$$\text{Since: } \boxed{\tilde{P}(z, t) = \frac{n_e \left(\frac{e^2}{m_e} \right)}{\omega_1^2 - \omega^2 - i\gamma\omega} \tilde{E}_{ext}(z, t) = \left(\frac{\epsilon_o}{\epsilon_o} \right) \frac{n_e \left(\frac{e^2}{m_e} \right)}{\omega_1^2 - \omega^2 - i\gamma\omega} \tilde{E}_{ext}(z, t)} \quad \text{with } \boxed{\omega_1 = \sqrt{\omega_0^2 - \frac{n_e e^2}{3\epsilon_o m_e}}}$$

$$\therefore \text{ The complex electric susceptibility is: } \boxed{\tilde{\chi}_e(\omega) = \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \frac{1}{\omega_1^2 - \omega^2 - i\gamma\omega} \equiv \chi_e(\omega) + i\zeta_e(\omega)}$$

Now before we go much further with this, we need to discuss another aspect of our model – namely that in most linear dielectric materials, the atoms comprising the material are multi-electron atoms, and consequently there are many different binding energies – the outer shell atomic electrons are weakly bound, hence have small k_e , and thus small $\omega_0 = \sqrt{k_e/m_e}$, whereas the inner-shell electrons are much more tightly bound, hence have larger k_e , larger $\omega_0 = \sqrt{k_e/m_e}$.

Furthermore, in complex media, *i.e.* dielectrics with more than one kind of atom, electrons can be shared between atoms – *i.e.* they are bound to molecules *e.g.* the π -electrons in benzene ring / aromatic hydrocarbon-type compounds, which can be weakly bound in some molecules.

Thus, there can be also be {molecular} resonances *e.g.* in the microwave and infra-red regions of the *EM* spectrum – atomic resonances are typically in the optical and UV regions {for the outer-most shell electrons}, as well as in the far UV and x-ray regions {for the inner-shell electrons}!

Allowing for all such resonances, we can write the {complex} electric polarization $\tilde{\mathbf{P}}$ as a summation over all of the resonances present in the linear dielectric as follows:

$$\tilde{\mathbf{P}}(z, t) = \frac{n_e e^2}{m_e} \left(\sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2 - \omega^2 - i\gamma_j \omega} \right) \tilde{\mathbf{E}}_{ext}(z, t) \quad \text{where:} \quad \omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \frac{n_e e^2}{3\epsilon_o m_e}} \quad \text{and:} \quad \omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$$

and where: $f_j^{osc} \equiv$ oscillator strength of j^{th} resonance, defined such that $\sum_{j=1}^n f_j^{osc} = 1$

Physically: $f_j^{osc} =$ fractional strength of the j^{th} resonance and $\gamma_j = 2\pi \times$ width of the j^{th} resonance.

Thus, we see that the complex electric susceptibility $\tilde{\chi}_e(\omega) \equiv \chi_e(\omega) + i\zeta_e(\omega)$ is:

$$\tilde{\chi}_e(\omega) = \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left(\sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2 - \omega^2 - i\gamma_j \omega} \right) \equiv \chi_e(\omega) + i\zeta_e(\omega)$$

The complex electric permittivity $\tilde{\epsilon}(\omega) = \epsilon_o (1 + \tilde{\chi}_e(\omega)) \equiv \epsilon(\omega) + i\zeta(\omega)$ of a dispersive, linear dielectric medium is:

$$\tilde{\epsilon}(\omega) = \epsilon_o (1 + \tilde{\chi}_e(\omega)) = \epsilon_o \left(1 + \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left(\sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2 - \omega^2 - i\gamma_j \omega} \right) \right) \equiv \epsilon(\omega) + i\zeta(\omega)$$

with the relations: $\epsilon(\omega) = \Re e(\tilde{\epsilon}(\omega)) = \epsilon_o (1 + \chi_e(\omega))$ and $\zeta(\omega) = \Im m(\tilde{\epsilon}(\omega)) = \epsilon_o \zeta_e(\omega)$.

Thus, monochromatic plane *EM* wave solutions to the dispersive wave equation are of the form:

$$\tilde{\mathbf{E}}(z, t, \omega) = \tilde{\mathbf{E}}_o e^{i(\tilde{k}z - \omega t)} \quad \text{with \{complex\} wavenumber } \tilde{k}(\omega) = k(\omega) + i\kappa(\omega) \equiv \sqrt{\tilde{\epsilon}(\omega) \mu_o} \omega$$

Thus:
$$\tilde{\mathbf{E}}_{ext}(z, t, \omega) = \tilde{\mathbf{E}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\mathbf{E}}_o \underbrace{e^{-\kappa z}}_{\substack{\text{exponential} \\ \text{damping} \\ \text{of EM wave}}} e^{i(kz - \omega t)}$$

Introducing a {frequency-dependent} complex wavenumber $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$ is equivalent to introducing a {frequency-dependent} complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$.

For a linear, dispersive dielectric, the complex index of refraction and complex wavenumber are

{simply} related to each other by: $\tilde{k}(\omega) = \left(\frac{\omega}{c} \right) \tilde{n}(\omega)$

$$\therefore \boxed{(k(\omega) + i\kappa(\omega)) = \left(\frac{\omega}{c}\right)(n(\omega) + i\eta(\omega)) = \left(\frac{\omega}{c}\right)n(\omega) + i\left(\frac{\omega}{c}\right)\eta(\omega)}$$

$$\Rightarrow \boxed{k(\omega) = \left(\frac{\omega}{c}\right)n(\omega)} \quad \text{and} \quad \boxed{\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega)}$$

The complex index of refraction $\tilde{n}(\omega)$ is related to the complex electric permittivity $\tilde{\epsilon}(\omega) = 1 + \tilde{\chi}_e(\omega)$ and thus the complex electric susceptibility $\tilde{\chi}_e(\omega)$ via the relation $\tilde{n}(\omega) = \sqrt{\tilde{\epsilon}(\omega)/\epsilon_0} = \sqrt{1 + \tilde{\chi}_e(\omega)}$,

$$\text{Squaring both sides: } \boxed{\tilde{n}^2(\omega) = \frac{\tilde{\epsilon}(\omega)}{\epsilon_0} = 1 + \tilde{\chi}_e(\omega) = 1 + \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \left(\sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} \right)}$$

But:

$$\boxed{\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 \left(1 + \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} \right] \right) = (k(\omega) + i\kappa(\omega))^2 = k^2(\omega) + 2ik(\omega)\kappa(\omega) + \kappa^2(\omega)}$$

$$\text{Since: } \boxed{\tilde{n}(\omega) = \left(\frac{c}{\omega}\right)\tilde{k}(\omega)} \text{ then: } \boxed{\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega) = 1 + \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} \right]}$$

$$\boxed{= (n(\omega) + i\eta(\omega))^2 = n^2(\omega) + 2in(\omega)\eta(\omega) - \eta^2(\omega)}$$

$$\text{Using the "standard" trick: } \boxed{\tilde{z} = \frac{1}{x - iy} = \frac{1}{x - iy} \frac{x + iy}{x + iy} = \frac{x + iy}{x^2 + y^2}}, \quad \boxed{\Re(\tilde{z}) = \frac{x}{x^2 + y^2}} \quad \text{and} \quad \boxed{\Im(\tilde{z}) = \frac{y}{x^2 + y^2}}$$

Then equating the real and imaginary parts of the LHS & RHS of the above equation, we obtain:

$$\boxed{n^2(\omega) - \eta^2(\omega) = 1 + \frac{n_e e^2}{\epsilon_0 m_e} \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]}$$

$$\boxed{2n(\omega)\eta(\omega) = \frac{n_e e^2}{\epsilon_0 m_e} \left[\sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]}$$

2 equations and 2 unknowns:
 $\{n(\omega) \& \eta(\omega)\}$
 \rightarrow solve for $n(\omega) \& \eta(\omega)$

$$\text{First define: } \boxed{\alpha_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]}$$

$$\boxed{\beta_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\gamma_j \omega)}{[(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2]} \right]} \quad (\text{n.b. } \beta_x(\omega) > 0, \text{ is always positive})$$

$$\text{Then: } \boxed{n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)} \quad \text{and} \quad \boxed{2n(\omega)\eta(\omega) = \beta_x(\omega)} \quad \Rightarrow \quad \boxed{\eta(\omega) = \beta_x(\omega)/2n(\omega)}$$

Thus: $n^2(\omega) - \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)^2 = (1 + \alpha_x(\omega)) \Leftrightarrow$ multiply equation through by $n^2(\omega)$

$$n^4(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = (1 + \alpha_x(\omega))n^2(\omega)$$

Or: $n^4(\omega) - (1 + \alpha_x)n^2(\omega) - \left(\frac{\beta_x(\omega)}{2}\right)^2 = 0 \Leftrightarrow$ *n.b.* This may look like a **quartic** equation, but it is actually a **quadratic** equation !!!

Define: $x \equiv n^2(\omega)$. Temporarily we suppress the (ω) -dependence in the following:

Then: $x^2 - (1 + \alpha_x)x - \left(\frac{\beta_x}{2}\right)^2 = 0 \rightarrow ax^2 + bx + c = 0$ with $a=1$, $b=-(1+\alpha)$, $c=-\left(\frac{\beta}{2}\right)^2$

Solutions / roots of this quadratic equation are of the general form: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow x = \frac{+(1 + \alpha_x) \pm \sqrt{(1 + \alpha_x)^2 + 4\left(\frac{\beta_x}{2}\right)^2}}{2} = \frac{1}{2} \left[(1 + \alpha_x) \pm \sqrt{(1 + \alpha_x)^2 + \beta_x^2} \right]$$

i.e. $x = \frac{1}{2}(1 + \alpha_x) \left[1 \pm \sqrt{1 + \left(\frac{\beta_x}{(1 + \alpha_x)}\right)^2} \right]$ *n.b.* the term: $\left(\frac{\beta_x}{(1 + \alpha_x)}\right)^2 > 0$

\rightarrow Must select +ve root on physical grounds, since $x \equiv n^2 > 0$.

$$\therefore x = n^2 = \frac{1}{2}(1 + \alpha) \left[1 + \sqrt{1 + \left(\frac{\beta_x}{(1 + \alpha_x)}\right)^2} \right]$$

Finally, we obtain:

$$n(\omega) \equiv \Re(\tilde{n}(\omega)) = \sqrt{\left(\frac{1 + \alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1 + \alpha_x(\omega))}\right)^2} \right]}$$

Complex index of refraction:
 $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$

$$\eta(\omega) \equiv \Im(\tilde{n}(\omega)) = \frac{\beta_x(\omega)}{2n(\omega)} = \frac{\beta_x(\omega)/2}{\sqrt{\left(\frac{1 + \alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{(1 + \alpha_x(\omega))}\right)^2} \right]}}$$

Where:

$$\alpha_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right]$$

$$\beta_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\gamma_j \omega)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right]$$

Obviously, *explicitly* writing out the full mathematical formulae for $n(\omega)$ and $\eta(\omega)$ is quite tedious – but these can be reasonably-easily coded up {i.e. a computer program} and plots of $n(\omega)$ vs. ω and $\eta(\omega)$ vs. ω can be obtained. We can also then obtain the following:

The complex relations: $\tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega)$ and $\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \left(\frac{\omega}{c} \right) \tilde{n}(\omega)$

and thus: $k(\omega) = \left(\frac{\omega}{c} \right) n(\omega)$ and $\kappa(\omega) = \left(\frac{\omega}{c} \right) \eta(\omega)$.

The {frequency-dependent} intensity/irradiance $I(z, \omega) = \langle |\vec{S}(z, t, \omega)| \rangle$ associated with a monochromatic plane *EM* wave propagating in a linear, dispersive dielectric is also exponentially decreased by a factor of $1/e = e^{-1}$ of its original value in going a characteristic distance of: $z = 1/\alpha(\omega) = 1/2\kappa(\omega) \equiv \ell_{atten}(\omega)$ i.e. defining: $\ell_{atten}(\omega) \equiv 1/\alpha(\omega) = 1/2\kappa(\omega) =$ intensity attenuation length – which is \sim analogous to the skin depth, $\delta_{sc} \equiv 1/\kappa$ for metals / conductors. However, note that $\delta_{sc} \equiv 1/\kappa$ is associated with the attenuation of the \vec{E} and \vec{B} -fields, whereas attenuation effects in intensity/irradiance, I varies as the square of the \vec{E} -field:

$$I(z) = \langle |\vec{S}(z, t)| \rangle \propto \langle E_{ext}^2(z, t) \rangle, \text{ hence: } I(z) \propto E_o^2 e^{-2\kappa(\omega)z} = E_o^2 e^{-\alpha(\omega)z}$$

In the exponential z -dependent term $e^{-2\kappa(\omega)z}$, since the energy densit(ies) $\langle u_{E,M}(z, t) \rangle$ and intensity $I(z) = \langle |\vec{S}(z, t)| \rangle$ are both proportional to E^2 i.e. both proportional to $e^{-2\kappa(\omega)z}$, we define the {frequency-dependent} absorption coefficient $\alpha(\omega) \equiv 2\kappa(\omega) = 1/\ell_{atten}(\omega)$.

Similarly, for the {frequency-dependent} complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ we can also define the {frequency-dependent} extinction coefficient: $\xi(\omega) \equiv 2\eta(\omega)$.

Since: $\kappa(\omega) = \left(\frac{\omega}{c} \right) \eta(\omega) \Rightarrow 2\kappa(\omega) = \left(\frac{\omega}{c} \right) 2\eta(\omega)$ thus: $\alpha(\omega) = \left(\frac{\omega}{c} \right) \xi(\omega) = 2 \left(\frac{\omega}{c} \right) \eta(\omega)$.

The absorption coefficient: $\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right) 2\eta(\omega) = \left(\frac{\omega}{c}\right) \xi(\omega) = 1/\ell_{\text{atten}}(\omega)$ and

The extinction coefficient: $\xi(\omega) \equiv 2\eta(\omega)$.

Typical values of the (real) index of refraction $n(\omega)$ for solids and liquids are $n(\omega) \approx 1.3 - 1.7$ in the visible light region of *EM* spectrum, e.g. $n_{\text{glass}}(\omega) \approx 1.5$, $n_{\text{H}_2\text{O}}(\omega) \approx 1.3$, $n_{\text{plastic}}(\omega) \approx 1.7$.

Then if: $n(\omega) = \sqrt{\left(\frac{1+\alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right]} = 1.5 \Leftarrow$ index of refraction of glass in the visible light region

Then: $n^2(\omega) = \left(\frac{1+\alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right] = (1.5)^2 = 2.25$

Thus: $(1+\alpha_x(\omega)) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right] = 4.50 \Leftarrow$ One equation & two unknowns: $\alpha_x(\omega)$ and $\beta_x(\omega)$

→ Need another relation / independent constraint!!

Note that glass doesn't have significant absorption in the visible light region, but typical such solid / liquid materials have (measured) absorption coefficients for visible light in the range of:

$\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right) \eta(\omega) \approx 10^{-2} - 10^{-1} \text{ m}^{-1} \Leftarrow$ i.e. intensity I falls off to $1/e = e^{-1} = 0.3679$ of initial ($z = 0$) value after light travels a distance $\sim 10 - 100$ m

So suppose: $\alpha(\omega) \equiv 2\kappa(\omega) = \left(\frac{\omega}{c}\right) \eta(\omega) \approx 10^{-1} \text{ m}^{-1}$ in glass for visible light, $\omega_{\text{vis}} \approx 10^{16}$ radians/sec

$$\rightarrow \eta(\omega) = \left(\frac{c}{\omega}\right) \alpha(\omega) \approx \left(\frac{3 \times 10^8}{10^{16}}\right) 10^{-1} = 3 \times 10^{-9} \ll 1$$

Now: $\eta(\omega) = \left(\frac{\beta_x(\omega)}{2n(\omega)}\right)$ and $n(\omega) \approx 1.5$ for glass in visible light range of *EM* spectrum.

$$\rightarrow \eta(\omega) = \frac{1}{3} \beta_x(\omega) \text{ or: } \beta_x(\omega) = 3\eta(\omega) \approx 9 \times 10^{-9} \ll 1 \text{ in the visible light range for glass}$$

Then: $(1+\alpha_x(\omega)) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1+\alpha_x(\omega)}\right)^2}\right] = 4.50$ and: $\beta_x(\omega) \approx 9 \times 10^{-9}$.

Now solve for $\alpha_x(\omega)$:

$$1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2} = 4.50 / (1 + \alpha_x(\omega)) \Rightarrow 1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2 = \left[\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right]^2$$

$$\Rightarrow \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right) = \sqrt{\left[\left(\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right)^2\right]} - 1 \Rightarrow \beta_x(\omega) = (1 + \alpha_x(\omega)) \sqrt{\left[\left(\frac{4.50}{(1 + \alpha_x(\omega))} - 1\right)^2\right]} - 1$$

This has a solution when: $\alpha_x(\omega) \approx 1.25$ for: $\beta_x(\omega) = 9 \times 10^{-9} \ll \alpha_x(\omega) \leftarrow$ Obtained via numerical methods using a computer

Thus, for $n(\omega) = 1.5$ for glass in the visible light region of the *EM* spectrum with $\alpha_x(\omega) \approx 1.25$ and $\beta_x(\omega) = 9 \times 10^{-9}$ (i.e. $\beta_x(\omega)/(1 + \alpha_x(\omega)) \ll 1$), as an explicit check, we see that:

$$n(\omega) = \sqrt{\left(\frac{1 + \alpha_x(\omega)}{2}\right) \left[1 + \sqrt{1 + \left(\frac{\beta_x(\omega)}{1 + \alpha_x(\omega)}\right)^2}\right]} = \sqrt{\left(\frac{1 + 1.25}{2}\right) \left[1 + \sqrt{1 + \left(\frac{9 \times 10^{-9}}{1 + 1.25}\right)^2}\right]} \approx \sqrt{\left(\frac{1 + 1.25}{2}\right) [1 + \sqrt{1}]}$$

$$= \sqrt{\left(\frac{1 + 1.25}{2}\right) [2]} = \sqrt{2.25} = 1.5$$

Thus we also see that: $\alpha_x(\omega) \approx n^2(\omega) - 1 \equiv \left(\frac{n_e e^2}{\epsilon_o m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_{1j}^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2}\right]$ i.e. $n^2(\omega) = 1 + \alpha_x(\omega)$

for typical materials – glass, water, plastic – in the visible light region of the *EM* spectrum, $\omega \approx 10^{16}$ radians / sec.

Whereas: $\beta_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_o m_e}\right) \left[\sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2}\right] \ll 1$

for these same materials – glass, water, plastic – in the visible light region of the *EM* spectrum, $\omega \approx 10^{16}$ radians / sec.

Our original equations were: $n^2(\omega) - \eta^2(\omega) = 1 + \alpha_x(\omega)$ and $2n(\omega)\eta(\omega) = \beta_x(\omega)$ or: $\eta(\omega) = \beta_x(\omega)/2n(\omega)$ with: $\alpha_x(\omega) \approx 1.25$ and $\beta_x(\omega) \approx 9 \times 10^{-9}$ for $n(\omega) = 1.5$ (for glass) with visible light and: $\eta(\omega) = \beta_x(\omega)/2n(\omega) = 3 \times 10^{-9}$.

We now see more clearly that: $\eta(\omega) \ll n(\omega)$ in the visible light region of the *EM* spectrum for glass, *i.e.* the complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega) \approx 1.25 + 9 \times 10^{-9}i$ for glass is predominantly real in the visible light region of the *EM* spectrum.

Thus, for glass in the visible light region of the *EM* spectrum:

$$n^2(\omega) - \eta^2(\omega) \approx n^2(\omega) = 1 + \alpha_x(\omega) = 1 + \left(\frac{n_e e^2}{\epsilon_0 m} \right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx (1.5)^2 = 2.25$$

and:

$$2n(\omega)\eta(\omega) = \beta(\omega)_x \equiv \left(\frac{n_e e^2}{\epsilon_0 m} \right) \left[\sum_{j=1}^n \frac{f_j^{osc} \gamma_j \omega}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx 9 \times 10^{-9}$$

$$\alpha_x(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_0 m} \right) \left[\sum_{j=1}^n \frac{f_j^{osc} (\omega_j^2 - \omega^2)}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right] \approx 1.25$$

Note that these results that we just obtained for glass in the visible light region of the *EM* spectrum do not hold true for all frequencies of *EM* waves {visible light region is in fact only narrow portion of the *EM* spectrum}!!! In particular, these results do not hold at {or near} an atomic (or molecular) resonance!

Let us consider a “simplified” atomic/molecular system, that of having only a single resonance frequency (*i.e.* a single bound-state quantum energy level), then:

$$\omega_1 \equiv \sqrt{\omega_0^2 - \left(\frac{n_e e^2}{3\epsilon_0 m_e} \right)} = 2\pi f_1 \quad \text{with:} \quad \omega_0 \equiv \sqrt{\frac{k_e}{m_e}}$$

or:
$$f_1 = \frac{1}{2\pi} \omega_1 = \frac{1}{2\pi} \sqrt{\omega_0^2 - \left(\frac{n_e e^2}{3\epsilon_0 m_e} \right)}$$

n.b. Oscillator strength $f_1^{osc} = 1$ {here} because only have a single resonance!

Then:
$$n(\omega) = \sqrt{\left(\frac{1 + \alpha_x^1(\omega)}{2} \right) \left[1 + \sqrt{1 + \left(\frac{\beta_x^1(\omega)}{1 + \alpha_x^1(\omega)} \right)^2} \right]}$$

$$\alpha_x^1(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_0 m_e} \right) \left[\frac{f_1^{osc} (\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + \gamma_1^2 \omega_1^2} \right]$$

$$\eta(\omega) = \frac{\beta_x^1(\omega)}{2n(\omega)} = \frac{(\beta_x^1(\omega)/2)}{\sqrt{\left(\frac{1 + \alpha_x^1(\omega)}{2} \right) \left[1 + \sqrt{1 + \left(\frac{\beta_x^1(\omega)}{1 + \alpha_x^1(\omega)} \right)^2} \right]}}$$

$$\beta_x^1(\omega) \equiv \left(\frac{n_e e^2}{\epsilon_0 m_e} \right) \left[\frac{f_1^{osc} \gamma_1 \omega}{(\omega_1^2 - \omega^2)^2 + \gamma_1^2 \omega_1^2} \right]$$

The figure on the left (immediately below) shows the behavior of the real and imaginary parts of the complex index of refraction of a dispersive, linear medium, *i.e.* $\{n(\omega) \text{ vs. } \omega\}$ and $\{\eta(\omega) \text{ vs. } \omega\}$ for a single atomic resonance. The figure on the right (immediately below) shows the behavior of $\{n(\omega) - 1 \text{ vs. } \omega\}$ and the absorption coefficient $\{\alpha(\omega) \equiv 2\kappa(\omega) \text{ vs. } \omega\}$ for a single atomic resonance.

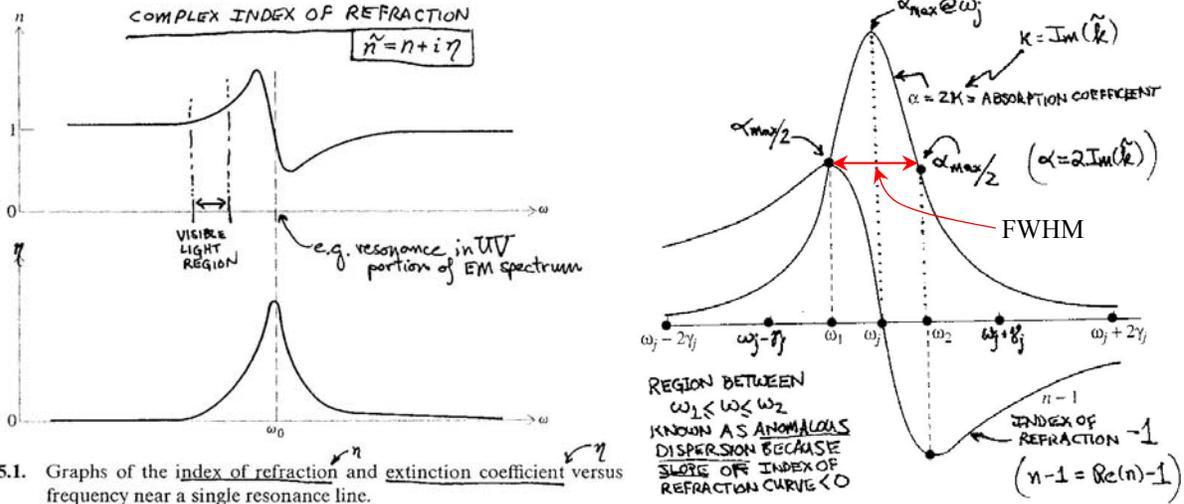
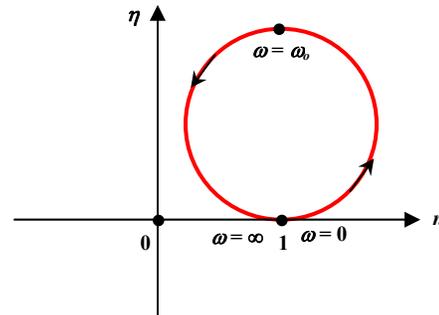
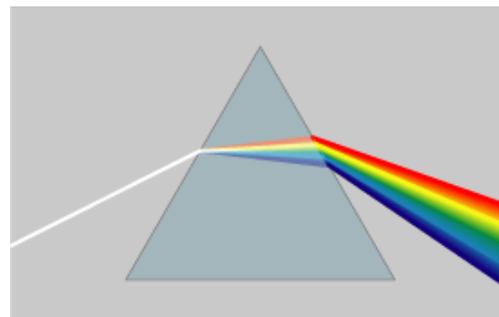
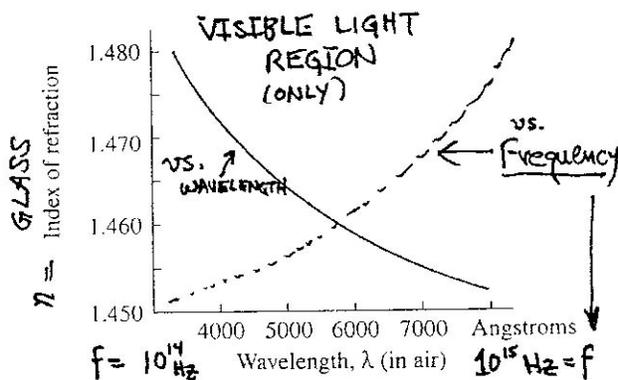


Figure 5.1. Graphs of the index of refraction and extinction coefficient versus frequency near a single resonance line.

n.b. The above curves are “classic” features of a complex resonance – with center / resonance frequency ω_j and damping constant $\gamma_j \equiv \omega_2 - \omega_1 = 2\pi\Gamma = 2\pi^*$ width (FWHM) of the resonance. In the complex plane:



In the visible light region of the *EM* spectrum, the graph below shows both the frequency and wavelength behavior of the {real} index of refraction of glass, *i.e.* $n(f) \text{ vs. } f$ {dotted line} and $n(\lambda) \text{ vs. } \lambda$ {solid line}. Note that since $v_{prop} = f\lambda \Rightarrow f = v_{prop}/\lambda$ or $\lambda = v_{prop}/f$

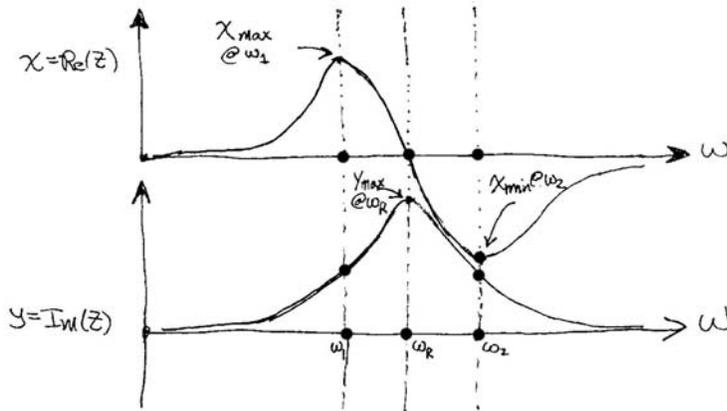


{1 Ångstrom = $10^{-10} \text{ m} = 0.1 \text{ nm}$ }

n.b. Media which are very transparent *e.g.* in the visible light region are often almost (or are) opaque in the so-called anomalous dispersion region of a resonance, $\omega_1 < \omega_R < \omega_2$ – *i.e.* the FWHM region of the atomic resonance, where the extinction coefficient $\eta(\omega)$ becomes very large – *EM* waves near the resonance frequency ω_R are very rapidly exponentially attenuated!

The General Behavior of “Classic” Complex Resonance:

$$\tilde{z}(\omega) = x(\omega) + iy(\omega)$$

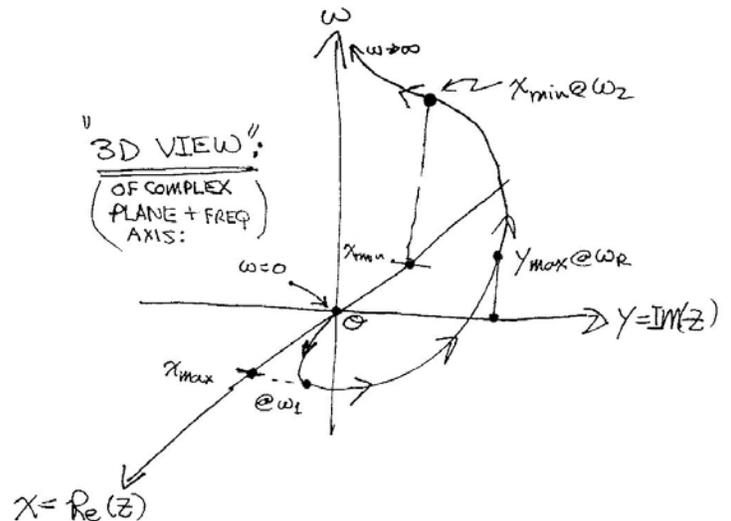
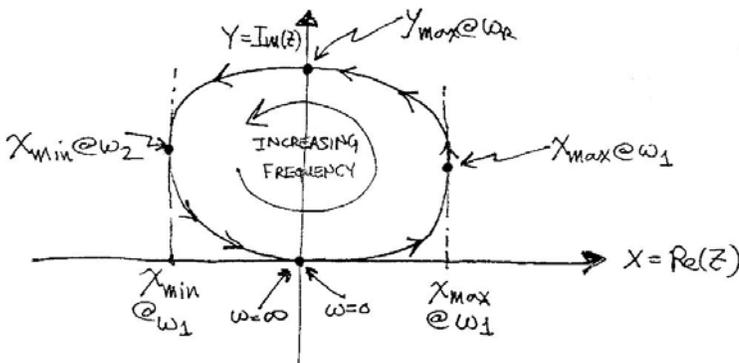


n.b. in some complex systems *e.g.* the resonance of a *LCR* circuit, $\Re\{\tilde{z}\}$ & $\Im\{\tilde{z}\}$ are interchanged from what is drawn here!

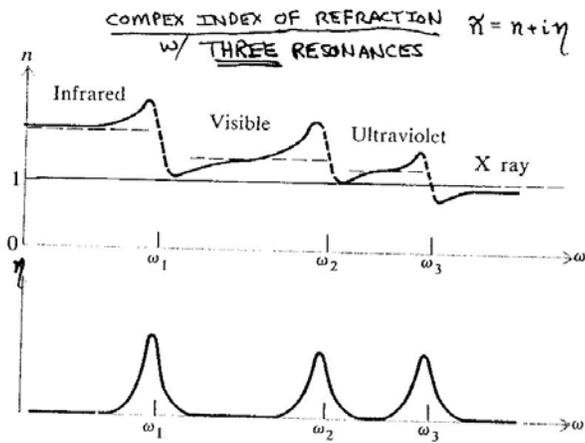
i.e. $\Re\{\tilde{z}\} \rightleftharpoons \Im\{\tilde{z}\}$

Note that the shape of the curve for the magnitude of \tilde{z} , $|\tilde{z}(\omega)| = \sqrt{\tilde{z}(\omega) \cdot \tilde{z}^*(\omega)} = \sqrt{x^2(\omega) + y^2(\omega)}$, is very similar to shape of the $\Im(z(\omega))$ curve {as shown here}.

Trajectory of $\tilde{z}(\omega)$ in the complex plane:



A more realistic microscopic picture of an atomic system – with many electrons with many quantum bound states → **many** resonances in a dispersive, linear macroscopic dielectric!!!



Index of refraction and extinction coefficient for a hypothetical substance with absorption bands in the infrared, visible, and ultraviolet regions of the spectrum.

Exercise(s): Draw out the corresponding trajectories of complex $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ and $\tilde{\epsilon}(\omega) = \epsilon(\omega) + i\xi(\omega)$ for the above triple-resonance cases in the complex plane!

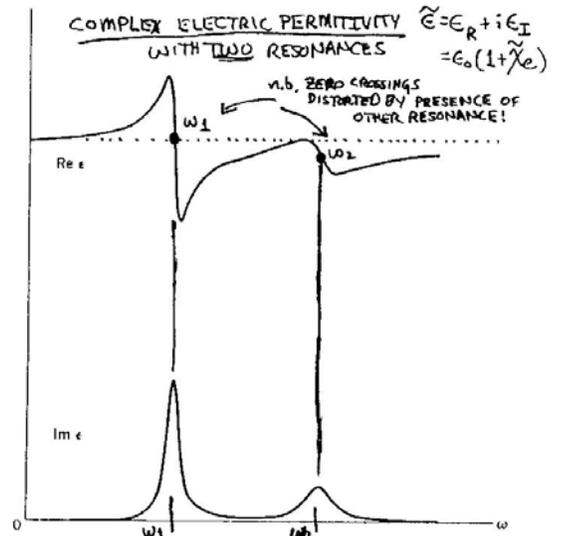


Figure 7.8 Real and imaginary parts of the dielectric constant $\epsilon(\omega)/\epsilon_0$ in the neighborhood of two resonances. The region of anomalous dispersion is also the frequency interval where absorption occurs.

In the high-frequency region, above the highest resonant frequency (typically in UV or x-ray region), the index of refraction is predicted to be $n(\omega) < 1.0$ (i.e. actually less than that of the vacuum).

Indeed, this phenomenon has been explicitly observed / measured e.g. in quartz (SiO_2) using x-rays:

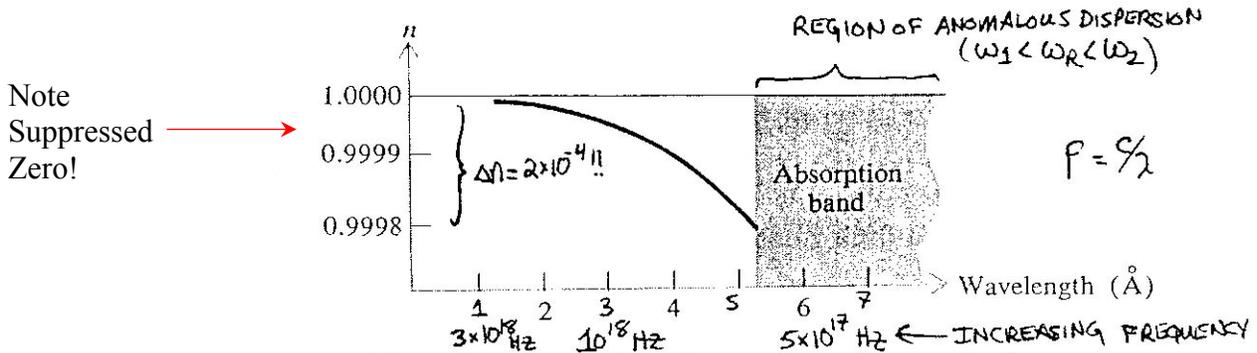


Figure 5.3. Measured index of refraction of quartz in the x-ray region.

Note that physically the damping constant γ_j = width of the j^{th} resonance is inversely related to the lifetime τ_j associated with the corresponding excited state of the constituent atoms/molecules of the dispersive, linear dielectric, since at the microscopic level, the {real} photons associated with the monochromatic plane EM wave have energy $E_\gamma = hf$ and {assuming the atoms/molecules of the dispersive, linear dielectric to all be in their ground state, with ground state energy E_1 }, then if the monochromatic plane EM wave has {angular} frequency $\omega = \omega_R = 2\pi f_R$ = the resonance frequency of the bound atomic electrons, then we see that $\Delta E_{j1} = E_j - E_1 = E_\gamma = hf_R$ at resonance!

At a resonance, *e.g.* when $\omega = \omega_{1j}$, the {real} photons in the monochromatic plane *EM* wave easily stimulate the atomic electrons, causing them to resonate – the {real} photons are absorbed, enabling the atomic electron to make a transition from the ground state {with energy E_1 } to the j^{th} excited state {with energy E_j } via an electric dipole transition, if so allowed by quantum-mechanical selection rules. The j^{th} excited atomic state has {mean} lifetime τ_j associated with it, thus the atomic electron de-excites back to the ground state by emitting a {real} photon of this same frequency. The miracle of all of this is that {real} photons associated with the *EM* field are actually interacting simultaneously with all of the atoms in the dispersive linear dielectric (within the coherence length of the photon) at any given instant in time, thus the resultant “scattered” photon that is {ultimately} emitted, actually must be summed over the response of the ensemble of many atoms – the miraculous result of which is forward scattering of the photons associated with the macroscopic *EM* wave, but with a {frequency-dependent} phase shift, which is related to the resonance lineshape and the finite lifetime τ_j of the excited state of the atom!

At a resonance, *e.g.* when $\omega = \omega_{1k}$, a large, transitory/transient {complex and frequency-dependent} electric dipole moment $\tilde{\vec{p}}(\vec{r}, \omega) = -e\tilde{\vec{r}}(\omega) = -e\tilde{r}(\omega)\hat{r}$ is induced in the atom, where:

$$\tilde{\vec{r}}(\omega) = r_o \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left(\sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} \right) \equiv r(\omega) + i\rho(\omega)$$

Note that here we can also make a direct connection with quantum mechanics – the electric dipole moment operator $\tilde{\vec{p}}(\vec{r}, \omega)$ and position operator $\tilde{\vec{r}}(\omega)$ operating *e.g.* on the ground state wavefunction of the atom/molecule $|\psi_1(\vec{r})\rangle$, *i.e.* $\tilde{\vec{p}}(\vec{r}, \omega)|\psi_1(\vec{r})\rangle$ and $\tilde{\vec{r}}(\omega)|\psi_1(\vec{r})\rangle$.

We can *e.g.* compute the expectation value of the modulus squared of the electric dipole moment $\langle \psi_1(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega)^2 | \psi_1(\vec{r}) \rangle$ of the atom/molecule. Inserting a complete set of states $\sum_{j=1}^{\infty} \langle \psi_j(\vec{r}) | \psi_j(\vec{r}) \rangle = \sum_{j=1}^{\infty} |\psi_j(\vec{r})\rangle \langle \psi_j(\vec{r})| = 1$ into this expression, we can then obtain quantum mechanical predictions for the {squares} the oscillator strengths f_j^{osc} :

$$\sum_{j=1}^{\infty} \langle \psi_1(\vec{r}) | \tilde{\vec{p}}^*(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle \langle \psi_j(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_1(\vec{r}) \rangle = \sum_{j=1}^{\infty} \left| \langle \psi_1(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle \right|^2$$

The transition rate $\Gamma_{\vec{1j}}$ (= # atoms/molecules per second) from the ground state to the j^{th} excited state {via an electric dipole transition, as allowed by quantum mechanical selection rules} is proportional to $\langle \psi_j(\vec{r}) | \tilde{\vec{p}}(\vec{r}, \omega) | \psi_1(\vec{r}) \rangle$,

whereas the transition rate $\Gamma_{\bar{j}1}$ (= # atoms/molecules per second) from the j^{th} excited state to the ground state {via an electric dipole transition, as allowed by quantum mechanical selection rules} is proportional to $\langle \psi_1(\vec{r}) | \tilde{p}^*(\vec{r}, \omega) | \psi_j(\vec{r}) \rangle$.

Note that by the {microscopic} manifest time-reversal invariance of the electromagnetic interaction, the transition rates are identical, *i.e.* $\Gamma_{\bar{i}j} \equiv \Gamma_{\bar{j}i} = \gamma_j / 2\pi = \text{“damping constant”}$ in our semi-classical model!

Note further that the lifetimes τ_j of the excited states of atoms are {inversely} related to the widths γ_j of the j^{th} resonances/widths of the j^{th} excited states by the Heisenberg uncertainty principle: $\Delta E \Delta t \geq \hbar$, where $\hbar \equiv h/2\pi$ and $h = \text{Planck's constant}$. If we set this relation to its minimum, *i.e.* $\Delta E \Delta t = \hbar$ then:

$$\boxed{(\hbar \gamma_j) * \tau_j = \hbar} \Rightarrow \boxed{\hbar \gamma_j = \hbar / \tau_j} \text{ or: } \boxed{1/\tau_j = \Gamma_{\bar{i}j} \equiv \Gamma_{\bar{j}i} = \gamma_j / 2\pi}$$

If one stays well away / far from the resonance frequencies of bound-state atomic electrons, then far from a resonance, the resonance factor:

$$\boxed{\frac{1}{(\omega_{1j}^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \approx \frac{1}{(\omega_{1j}^2 - \omega^2)^2}} \text{ i.e. far from a resonance: } \boxed{(\omega_{1j}^2 - \omega^2)^2 \gg \gamma_j^2 \omega^2}$$

Thus, far from a resonance / resonances, relatively little absorption/dissipation occurs {*i.e.* $\eta^2(\omega) \ll n^2(\omega)$, such that $\tilde{n}(\omega) = n(\omega) + \tilde{\eta}(\omega) \simeq n(\omega)$ is predominantly real} and hence:

$$\boxed{n^2(\omega) - \eta^2(\omega) \simeq n^2(\omega) \simeq 1 + \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n \frac{f_j^{\text{osc}}}{\omega_{1j}^2 - \omega^2} \right]}$$

Now:
$$\boxed{\frac{1}{\omega_{1j}^2 - \omega^2} = \frac{1/\omega_{1j}^2}{\left[1 - (\omega^2/\omega_{1j}^2) \right]} \approx \frac{1}{\omega_{1j}^2} \left(1 + \frac{\omega^2}{\omega_{1j}^2} \right) = \frac{1}{\omega_{1j}^2} + \frac{\omega^2}{\omega_{1j}^4}}$$

Then:
$$\boxed{n^2(\omega) - \eta^2(\omega) \simeq n^2(\omega) \simeq 1 + \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n \left(\frac{f_j^{\text{osc}}}{\omega_{1j}^2} \right) + \omega^2 \sum_{j=1}^n \left(\frac{f_j^{\text{osc}}}{\omega_{1j}^4} \right) \right]}$$

If $\boxed{n^2 = 1 + \epsilon}$ and $\boxed{\epsilon \ll 1} \Rightarrow \boxed{n = \sqrt{1 + \epsilon} \simeq 1 + \frac{1}{2}\epsilon}$

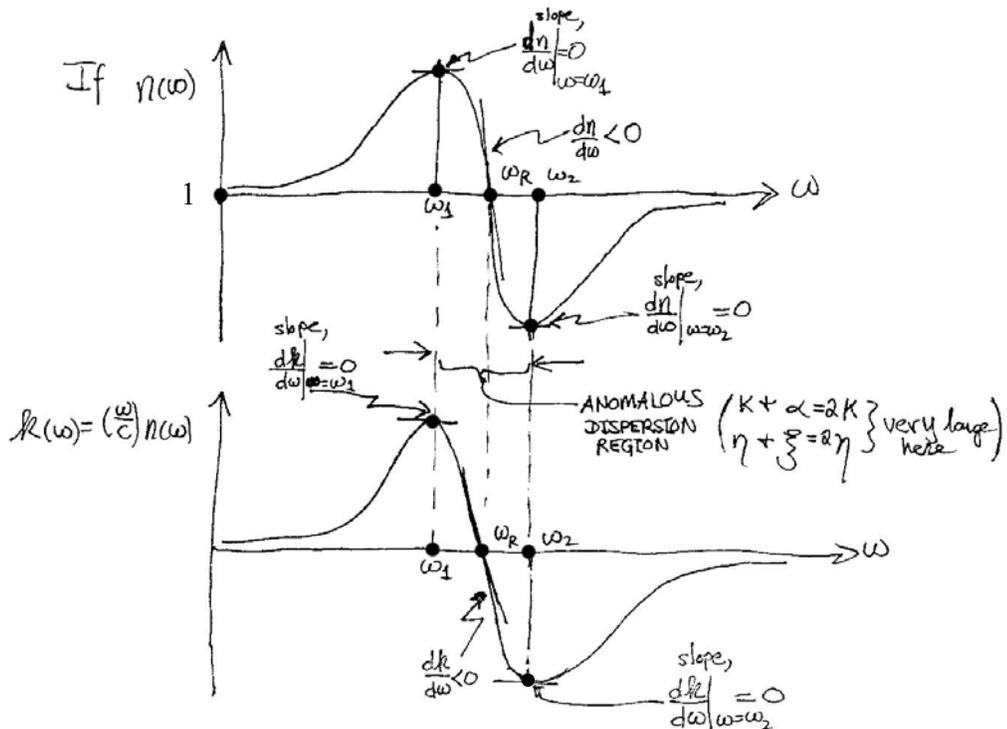
Thus, far from a resonance/resonances:
$$\boxed{n(\omega) \simeq 1 + \frac{1}{2} \left(\frac{n_e e^2}{\epsilon_o m_e} \right) \left[\sum_{j=1}^n \left(\frac{f_j^{\text{osc}}}{\omega_{1j}^2} \right) + \omega^2 \sum_{j=1}^n \left(\frac{f_j^{\text{osc}}}{\omega_{1j}^4} \right) \right]}$$

But: $\lambda_o = \frac{2\pi}{k_o} = \frac{2\pi c}{\omega}$ = vacuum wavelength \Rightarrow Cauchy's Formula: $n(\lambda_o) \approx 1 + A \left(1 + \frac{B}{\lambda_o^2} \right)$

Where: A = Coefficient of *Refraction* and B = Coefficient of *Dispersion*.

Since: $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$ and/or: $\tilde{k}(\omega) = k(\omega) + ik(\omega)$ and: $\tilde{k}(\omega) = \left(\frac{\omega}{c} \right) \tilde{n}(\omega)$

Then: $k(\omega) = \left(\frac{\omega}{c} \right) n(\omega)$ and: $\kappa(\omega) = \left(\frac{\omega}{c} \right) \eta(\omega)$, thus:



The phase velocity: $v_{phase} = \frac{\omega}{k} = \frac{c}{n(\omega)} > c$ if $n(\omega) < 1$

The group velocity: $v_{group} = \left(\frac{d\omega}{dk} \right) = 1 / \left(\frac{dk(\omega)}{d\omega} \right) = \left[\frac{dk(\omega)}{d\omega} \right]^{-1}$

Note that at the “turning points” of either the $\{n(\omega) \text{ vs. } \omega\}$ or $\{k(\omega) \text{ vs. } \omega\}$ graphs, *i.e.* at {angular} frequencies $\omega = \omega_1$ and/or $\omega = \omega_2$ where the slope $dk(\omega)/d\omega = 0 \Rightarrow v_{group} = \infty$!!!

Note further that in the {angular} frequency region $\omega_1 < \omega < \omega_2$ {the “anomalous dispersion” region}, since the slope $dk(\omega)/d\omega < 0$ then the group velocity $v_{group}(\omega) = 1/(dk(\omega)/d\omega) < 0$!!! {Hence the name anomalous dispersion...}

This phenomenon has been experimentally verified (see *e.g.* C.G.B. Garrett & D.E. McCumber, Phys. Rev. A, 1, p. 305 (1970)). If the medium is not too thick, a Gaussian pulse with a central frequency near an absorption line (*i.e.* near a resonance, ω_R) and with pulse width $\Delta t \gg \tau_R = 1/\gamma_R$ propagates with appreciable absorption, but (more or less) retains its shape.

The peak of the Gaussian pulse propagates at v_{group} even when the group velocity is negative!!!

- Useful for pulse re-shaping applications - leading edge is less attenuated than trailing edge.
- Can actually have the peak of a greatly attenuated pulse emerge from the absorber before the peak of the incident pulse enters the absorber (= definition of negative group velocity)!!!
 {*i.e.* microscopically, if the absorber is not too thick, then some photons can make it all the way through the absorber w/o interacting at all – this probability is exponentially suppressed.
- Has applications/uses *e.g.* in optical mammography/breast cancer screening for women...}
- See J.D. Jackson's Electrodynamics, 3rd Edition, pages 325-26 for more details!

Finally, if we set $\omega \equiv 0$, then we obtain the static (*i.e.* zero-frequency) limit of {all of} these quantities; note also that they become purely real in this limit:

Static Polarization:
$$\vec{P}(0) = \left(\frac{ne^2}{m} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2}$$
 where
$$\omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \left(\frac{ne^2}{3\epsilon_o m_e} \right)}$$
 and since $\vec{P} = \epsilon_o \chi_e \vec{E}_{ext}$

Static Electricity Susceptibility:
$$\chi_e(0) = \left(\frac{ne^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2}$$
 and
$$\omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$$

Static Index of Refraction:
$$n(0) = \sqrt{1 + \alpha_x(0)} = \sqrt{1 + \chi_e(0)} = \sqrt{K_e(0)}$$

But:
$$K_e(\omega) = \epsilon(\omega)/\epsilon_o = (1 + \chi_e(\omega)) \Rightarrow \epsilon(0) = \epsilon_o (1 + \chi_e(0))$$
 and thus:

Static Dielectric Constant:
$$K_e(0) = \epsilon(0)/\epsilon_o = (1 + \chi_e(0)) = 1 + \alpha_x(0) = n^2(0)$$

But:
$$\alpha_x(0) = \left(\frac{ne^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} = \chi_e(0)$$

$$\therefore K_e(0) = 1 + \left(\frac{ne^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2} = n^2(0)$$
 and
$$\chi_e(0) = K_e(0) - 1 = \left(\frac{ne^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{\omega_{1j}^2}$$

⇒ Note that the static dielectric constant {as measured at $f = 0$ Hz/DC} is $K_e(0) > 1.0$ because it contains information about all of the {quantum mechanical} resonances/excited states $\omega_{1j} > 0$ present in the dispersive, linear medium, even into the x-ray region at $\omega_{1j} \approx 10^{18-19}$ Hz and beyond !!!

⇒ Equivalently, armed now with this knowledge of the microscopic behavior of a dispersive, linear medium, an electric susceptibility $\chi_e(0) > 0$ {or equivalently, a dielectric constant $K_e(0) > 1$ } *instantly* tells us that there are indeed {quantum mechanical} resonances/excited states present in the {composite} atoms/molecules that make up the *macroscopic* material of the dispersive, linear medium!!!

⇒ A wonderful macroscopic example of dispersion is the rainbow. However, at the *microscopic* level, the wavelength-dependence of the index of refraction of light arises as a consequence of the resonant behavior of quantum mechanical bound states of electrons in atoms of the water molecule (H₂O) responding to *EM* light waves {= visible light photons} coming from our sun. If *no* such composite behavior existed at the *microscopic* level, there would be *no* rainbows to enjoy in the *macroscopic* everyday world!

