

## LECTURE NOTES 8

### A More Sophisticated Treatment of EM Wave Propagation in Conducting Media

In the previous P436 Lecture Notes 7, we discussed the propagation of *EM* waves in conducting media (*e.g.* metals), taking into account Ohm's Law  $\vec{J}(\vec{r}, t) = \sigma_c \vec{E}(\vec{r}, t)$  and the continuity equation  $\vec{\nabla} \cdot \vec{J}(\vec{r}, t) = -\partial \rho_{free}(\vec{r}, t) / \partial t$  in the conductor.

However, this treatment of conducting media neglected certain *inertial effects* associated with the “free” / conduction electrons in the metal conductor – we need to consider more carefully the actual motion of the “free” / conduction electrons in the conductor, and their response *e.g.* to the application of a monochromatic plane *EM* wave of angular frequency  $\omega$ .

Since the “free” / conduction electrons in a metal are not bound to individual atoms in a conductor, there are no “elastic” restoring forces acting on the “free” / conduction electrons *{i.e.  $k_e = 0$ }*, as there was in the case of the polarization of bound atomic electrons, *e.g.* in a dispersive, linear non-conducting medium. Thus, the differential equation describing the motion of the “free” / conduction electrons in a metal is of the general form:

$$\vec{F}_e(\vec{r}, t) = m_e^* \vec{a}(\vec{r}, t) = m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) + 0 \vec{r}(t) = -e\vec{E}(\vec{r}, t)$$

where the driving force  $-e\vec{E}(\vec{r}, t)$  = the charge of the electron ( $-e$ ) times the electric field  $\vec{E}(\vec{r}, t)$  of the monochromatic plane *EM* wave propagating in the conducting metal.

Thus, we obtain a first-order inhomogeneous differential equation of the form:

$$m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) = -e\vec{E}(\vec{r}, t)$$

*n.b.* here again, we neglect the effects of the magnetic Lorentz force term,

$$\left| \vec{F}_{Lorentz}(\vec{r}, t) \right| = \left| -e\vec{v}(\vec{r}, t) \times \vec{B}(\vec{r}, t) \right| \ll \left| e\vec{E}(\vec{r}, t) \right| \quad \text{because: } v = |\vec{v}| \approx v_D \ll v_{prop}$$

Note also that  $m_e^*$  = mass of the “free” / conduction electron in the metal conductor which, depending on the microscopic/quantum mechanical details of the conductor is *{often}* not equal to the mass of a {truly free, isolated} electron,  $m_e = 9.1 \times 10^{-31}$  kg.

The quantity  $m_e^* \gamma$  is known as the frictional / Joule dissipation constant, associated with electron energy / momentum losses – *i.e.* “free” / conduction electron scattering losses in the conductor. The damping constant  $\gamma$  *{n.b. units of angular frequency (radians/sec)}* is associated with dissipative/ energy loss mechanism(s) of the motion of “free” / conduction electrons in the metal – “free” / conduction electrons scattering off of each other, off of atoms, lattice vibrations, crystal defects and impurities in real metals. Thus, we can also define a relaxation time constant  $\tau_{rlx} \equiv 1/\gamma$  associated with the “free” / conduction electrons present in the metal conductor.

The free current density in the metal conductor is  $\vec{J}_{free}(\vec{r}, t) = -n_e e \vec{v}(\vec{r}, t)$  {Amps/m<sup>2</sup>}, where:  
 $n_e$  = “free” / conduction electron number density in the metal conductor {#/unit volume = #/m<sup>3</sup>}.  
 $q = -e$  = the electric charge of “free” / conduction electrons {Coulombs}.  
 $\vec{v}(\vec{r}, t)$  = the {drift} velocity ( $\ll c$ ) of “free” / conduction electrons in metal at  $(\vec{r}, t)$  {m/s}.

Thus: 
$$m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{v}(\vec{r}, t) = -e \vec{E}(\vec{r}, t) \quad \Leftarrow \text{multiply through by } -n_e e:$$

$$\begin{aligned} & -n_e e m_e^* \frac{d\vec{v}(\vec{r}, t)}{dt} - n_e e m_e^* \gamma \vec{v}(\vec{r}, t) = +n_e e^2 \vec{E}(\vec{r}, t) \\ & = m_e^* \left( -n_e e \frac{d\vec{v}(\vec{r}, t)}{dt} \right) - m_e^* \gamma (n_e e \vec{v}(\vec{r}, t)) = +n_e e^2 \vec{E}(\vec{r}, t) \\ & = m_e^* \left( \frac{d}{dt} \underbrace{(-n_e e \vec{v}(\vec{r}, t))} \right) + m_e^* \gamma \underbrace{(-n_e e \vec{v}(\vec{r}, t))} = +n_e e^2 \vec{E}(\vec{r}, t) \end{aligned}$$

$$\therefore m_e^* \frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + m_e^* \gamma \vec{J}_{free}(\vec{r}, t) = n_e e^2 \vec{E}(\vec{r}, t) \quad \text{or:} \quad \frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + \gamma \vec{J}_{free}(\vec{r}, t) = \left( \frac{n_e e^2}{m_e^*} \right) \vec{E}(\vec{r}, t)$$

which is also a first-order linear, inhomogeneous differential equation.

Recall that the solution to an inhomogeneous differential equation is the sum of the solution to the corresponding homogeneous differential equation, plus a particular solution satisfying the inhomogeneous differential equation. Thus, we first need to obtain the solution to the homogeneous differential equation, *i.e.* when no EM waves are present in the conducting material, *vis.*  $\vec{E}(\vec{r}, t) = 0$ .

a.) If  $\vec{E}(\vec{r}, t) = 0$ , the general solution to the first-order linear, homogeneous differential equation:

$$\frac{d\vec{J}_{free}(\vec{r}, t)}{dt} + \gamma \vec{J}_{free}(\vec{r}, t) = 0$$

is of the form of a damped exponential (because  $\gamma > 0$ ):

$$\vec{J}_{free}(\vec{r}, t) = \vec{J}_{o, free}(\vec{r}) e^{-\gamma t} = J_o(\vec{r}) e^{-t/\tau_{rlx}} \quad \text{where:} \quad \tau_{rlx} \equiv 1/\gamma$$

→ Transient free current(s) will decay to  $1/e = e^{-1} = 0.3679$  of their initial  $t = 0$  value ( $\vec{J}_o(\vec{r})$ ) in a characteristic relaxation time  $t = \tau_{rlx} \equiv 1/\gamma$ .

Using the continuity equation  $\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\partial \rho_{free}(\vec{r}, t) / \partial t$  {expressing conservation of free electric charge} in the above homogeneous equation, with  $\vec{E}(\vec{r}, t) = 0$ :

$$\boxed{\frac{d}{dt} \left( \underbrace{\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t)} \right) + \gamma \left( \underbrace{\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t)} \right) = \left( \frac{n_e e^2}{m_e^*} \right) \left( \underbrace{\vec{\nabla} \cdot \vec{E}(\vec{r}, t)}_{=0} \right)}$$

$$= -\frac{\partial^2 \rho_{free}(\vec{r}, t)}{\partial t^2} - \gamma \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} = 0$$

thus:  $\boxed{\frac{\partial^2 \rho_{free}(\vec{r}, t)}{\partial t^2} + \gamma \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} = 0} \Rightarrow \boxed{\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} + \gamma \rho_{free}(\vec{r}, t) = 0}$

$$\Rightarrow \boxed{\rho_{free}(\vec{r}, t) = \rho_{o_{free}}(\vec{r}) e^{-\gamma t} = \rho_{o_{free}}(\vec{r}) e^{-t/\tau_{rx}}}$$

But from P436 Lecture Notes 7, p. 1 we also obtained the relation  $\rho_{free}(\vec{r}, t) = \rho_{o_{free}}(\vec{r}) e^{-\sigma_C t / \epsilon}$ . Thus comparing these two equations, we see that:  $\gamma = \sigma_C / \epsilon = 1 / \tau_{rx}$ .

b.) For a static non-zero electric field  $\vec{E}(\vec{r}) = \vec{E}_o$ , then this static problem can have no explicit time dependence, hence  $d\vec{J}_{free}(\vec{r}, t) / dt = 0$  and thus the first-order linear, inhomogeneous equation becomes:

$$\boxed{\gamma \vec{J}_{free}(\vec{r}) = \left( \frac{n_e e^2}{m_e^*} \right) \vec{E}(\vec{r}) = \left( \frac{n_e e^2}{m_e^*} \right) \vec{E}_o = \text{constant}} \quad \text{or:} \quad \boxed{\vec{J}_{free}(\vec{r}) = \left( \frac{n_e e^2}{m_e^* \gamma} \right) \vec{E}(\vec{r}) = \left( \frac{n_e e^2}{m_e^* \gamma} \right) \vec{E}_o = \text{constant}}$$

The static / DC continuity equation for free charge is:  $\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}) = -\frac{\partial \rho_{free}(\vec{r})}{\partial t} = 0$

with static solution(s):  $\vec{J}_{free}(\vec{r}) = \text{constant} = fcn(\vec{r}, \{\text{or } t\})$  and  $\rho_{free}(\vec{r}) \neq fcn(t)$ .

However, Ohm's Law (for DC / steady free electric currents) is:  $\vec{J}_{free}(\vec{r}) = \sigma_C \vec{E}(\vec{r})$  where:  $\sigma_C$  = the static conductivity of the metal (Siemens/m) =  $1 / \rho_C \leftarrow 1 / \text{static resistivity} (\Omega \cdot \text{m})$ .

Thus, we see here that:  $\boxed{\vec{J}_{free}(\vec{r}) = \left( \frac{n_e e^2}{m_e^* \gamma} \right) \vec{E}(\vec{r}) = \sigma_C \vec{E}(\vec{r})}$  i.e. the static conductivity  $\boxed{\sigma_C = \left( \frac{n_e e^2}{m_e^* \gamma} \right)}$

or equivalently, the static resistivity:  $\boxed{\rho_C \equiv \frac{1}{\sigma_C} = \left( \frac{m_e^* \gamma}{n_e e^2} \right)}$ . {n.b. both purely real quantities}

The static / DC resistance  $R$  of e.g. a long wire is thus:  $\boxed{R = \frac{\rho_C \ell}{A_{\perp}} = \left( \frac{m_e^* \gamma}{n_e e^2} \right) \frac{\ell}{A_{\perp}}}$  (Ohms)

{again, a purely real quantity}

c.) For a harmonically time-dependent {in general, complex} electric field (e.g. associated with a monochromatic plane EM wave (or an AC current, using Ohm's law), with angular frequency  $\omega = 2\pi f > 0$ :

$$\boxed{\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_o(\vec{r}) e^{-i\omega t}} \quad \text{and:} \quad \boxed{\frac{d\tilde{\vec{J}}_{free}(\vec{r}, t)}{dt} + \gamma \tilde{\vec{J}}_{free}(\vec{r}, t) = \left(\frac{n_e e^2}{m_e^*}\right) \tilde{\vec{E}}(\vec{r}, t)}$$

We assume that the {in general, complex} free current density solution to this linear inhomogeneous first-order differential equation is of the general form:  $\tilde{\vec{J}}_{free}(\vec{r}, t) = \tilde{\vec{J}}_{o, free}(\vec{r}) e^{-i\omega t}$ .

Then the solution to this linear, inhomogeneous first-order differential equation is:

$$\boxed{(-i\omega + \gamma) \tilde{\vec{J}}_{free}(\vec{r}, t) = \left(\frac{n_e e^2}{m_e^*}\right) \tilde{\vec{E}}(\vec{r}, t) = \gamma \left(\frac{n_e e^2}{m_e^* \gamma}\right) \tilde{\vec{E}}(\vec{r}, t) = \gamma \sigma_c \tilde{\vec{E}}(\vec{r}, t)} \quad \text{since:} \quad \boxed{\sigma_c = \left(\frac{n_e e^2}{m_e^* \gamma}\right)}$$

$$\Rightarrow \boxed{\tilde{\vec{J}}_{free}(\vec{r}, t) = \left(\frac{\gamma \sigma_c}{(-i\omega + \gamma)}\right) \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{\gamma \sigma_c}{(\gamma - i\omega)}\right) \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{\sigma_c}{(1 - i\omega/\gamma)}\right) \tilde{\vec{E}}(\vec{r}, t) \equiv \tilde{\sigma}_c(\omega) \tilde{\vec{E}}(\vec{r}, t)}$$

where:

the AC conductivity:  $\tilde{\sigma}_c(\omega) \equiv \left(\frac{\sigma_c}{(1 - i\omega/\gamma)}\right) = \left(\frac{(n_e e^2/m_e^* \gamma)}{(1 - i\omega/\gamma)}\right) \leftarrow n.b. \text{ complex quantity}$

and the AC resistivity:  $\tilde{\rho}_c(\omega) \equiv \frac{1}{\tilde{\sigma}_c(\omega)} = \rho_c (1 - i\omega/\gamma) = \left(\frac{m_e^* \gamma}{n_e e^2}\right) (1 - i\omega/\gamma) \leftarrow \text{complex quantity}$

Thus, we see that the {complex} AC resistance (aka impedance, Z) e.g. of a long wire is:

$$\boxed{\tilde{Z}(\omega) \equiv \tilde{R}_{AC}(\omega) = \frac{\tilde{\rho}_c(\omega) \ell}{A_{\perp}} = \left(\frac{\rho_c \ell}{A_{\perp}}\right) (1 - i\omega/\gamma) = \left(\frac{m_e^* \gamma}{n_e e^2}\right) \left(\frac{\ell}{A_{\perp}}\right) (1 - i\omega/\gamma) \leftarrow n.b. \text{ complex quantity}}$$

The AC continuity equation is:  $\boxed{\vec{\nabla} \cdot \tilde{\vec{J}}_{free}(\vec{r}, t) = -\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t}}$  with:  $\boxed{\vec{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t) = \frac{\tilde{\rho}_{free}(\vec{r}, t)}{\epsilon}}$  Gauss' Law

But:  $\boxed{\tilde{\vec{J}}_{free}(\vec{r}, t) = \tilde{\sigma}_c(\omega) \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{\sigma_c}{(1 - i\omega/\gamma)}\right) \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{(n_e e^2/m_e^* \gamma)}{(1 - i\omega/\gamma)}\right) \tilde{\vec{E}}(\vec{r}, t)}$

Thus:  $\boxed{\vec{\nabla} \cdot \tilde{\vec{J}}_{free}(\vec{r}, t) = \tilde{\sigma}_c(\omega) \vec{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{\sigma_c}{(1 - i\omega/\gamma)}\right) \vec{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t) = \left(\frac{(n_e e^2/m_e^* \gamma)}{(1 - i\omega/\gamma)}\right) \vec{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t)}$

$\therefore \boxed{-\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \left(\frac{\sigma_c}{(1 - i\omega/\gamma)}\right) \frac{\tilde{\rho}_{free}(\vec{r}, t)}{\epsilon}}$  or:  $\boxed{-\frac{\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \frac{\sigma_c/\epsilon}{(1 - i\omega/\gamma)} \tilde{\rho}_{free}(\vec{r}, t)}$

The solution to the above differential equation for the electric charge volume density is of the form:  $\tilde{\rho}_{free}(\vec{r}, t) = \tilde{\rho}_o(\vec{r})e^{-i\omega t}$  provided that:

$$\frac{-\partial \tilde{\rho}_{free}(\vec{r}, t)}{\partial t} = \frac{\sigma_c / \epsilon}{(1 - i\omega / \gamma)} \tilde{\rho}_{free}(\vec{r}, t) \Rightarrow i\omega = \frac{\sigma_c / \epsilon}{(1 - i\omega / \gamma)} \text{ n.b. implies } \epsilon \rightarrow \tilde{\epsilon}(\omega) \{\text{complex!}\}$$

$$\text{i.e. that: } \tilde{\epsilon}(\omega) = \frac{1}{i\omega} \frac{\sigma_c}{(1 - i\omega / \gamma)} \quad \text{but: } \tilde{\sigma}_c(\omega) \equiv \left( \frac{\sigma_c}{(1 - i\omega / \gamma)} \right) = \left( \frac{(n_e e^2 / m_e^* \gamma)}{(1 - i\omega / \gamma)} \right)$$

$$\text{Thus, we see that: } \tilde{\epsilon}(\omega) = \frac{\tilde{\sigma}_c(\omega)}{i\omega} = \frac{1}{i\omega} \frac{\sigma_c}{(1 - i\omega / \gamma)} \quad \Leftarrow \quad \exists \text{ a connection between complex AC electric permittivity } \tilde{\epsilon}(\omega) \text{ and complex AC conductivity } \tilde{\sigma}_c(\omega) \text{ in a metal / conductor.}$$

$$\text{Technically-speaking, this relation } \underline{\text{should}} \text{ be: } \tilde{\epsilon}(\omega) = \underbrace{\epsilon_o}_{\text{empty space}} + \underbrace{\frac{\tilde{\sigma}_c(\omega)}{i\omega}}_{\text{metal}}$$

because the vacuum pervades all space – however, the conducting medium and the vacuum coexist in the same region of space-time, invoking the superposition principle, the two electric permittivities (n.b. both scalar quantities) are additive.

However, it can be seen that because the {DC} conductivity of metals is so high {typically  $\sigma_c \sim 10^7$  Siemens/m}, that even for {angular} frequencies in the optical region { $\omega \sim 10^{16}$  radians/sec} the electric permittivity of free space,  $\epsilon_o$  is dwarfed by the second term, i.e.  $\epsilon_o = 8.85 \times 10^{-12} \ll |\sigma_c(\omega) / i\omega| = |\sigma_c / i\omega(1 - i\omega / \gamma)|$  (Farads/m), so often the  $\epsilon_o$  term is neglected/dropped in many textbooks... For technical correctness / completeness' sake, we will include it here. Thus, we see that even for metal conductors:

$$\frac{\tilde{\epsilon}(\omega)}{\epsilon_o} = 1 + \frac{\tilde{\sigma}_c(\omega) / \epsilon_o}{i\omega} = 1 + \tilde{\chi}_e(\omega).$$

For pure copper metal at low frequencies, i.e.  $\omega \rightarrow 0$ , the static conductivity is  $\sigma_c^{Cu} \approx 6 \times 10^7$  Siemens/m; the number density of free electrons in pure copper metal is  $n_e^{Cu} \approx 8.5 \times 10^{28} / \text{m}^3$  and using  $\sigma_c = (n_e e^2 / m_e^* \gamma) \Rightarrow \gamma_{Cu} = (n_e^{Cu} e^2 / m_e^* \sigma_c^{Cu}) = (n_e^{Cu} e^2 / m_e \sigma_c^{Cu}) \approx 4 \times 10^{13} \text{ sec}^{-1}$ , which corresponds to a relaxation time for “free” / conduction electrons in pure copper of  $\tau_{rlx}^{Cu} \approx 1 / \gamma_{Cu} \approx 2.5 \times 10^{-14} \text{ sec}$ .

**For Conducting Metals with “Free” Electrons:**  $\tilde{\mathbf{J}}_{free}(\vec{r}, t) = \tilde{\sigma}_c(\omega) \tilde{\mathbf{E}}(\vec{r}, t)$

Note that in the static limit ( $\omega \rightarrow 0$ ):

$$\left\{ \begin{array}{l} \tilde{\mathbf{E}}(\vec{r}, t) = \tilde{\mathbf{E}}_o(\vec{r}) e^{-i\omega t} \rightarrow \tilde{\mathbf{E}}_o(\vec{r}) \text{ static / constant} \\ \tilde{\mathbf{J}}(\vec{r}, t) = \tilde{\mathbf{J}}_o(\vec{r}) e^{-i\omega t} \rightarrow \tilde{\mathbf{J}}_o(\vec{r}) \text{ static / constant} \\ \tilde{\rho}(\vec{r}, t) = \tilde{\rho}_o(\vec{r}) e^{-i\omega t} \rightarrow \rho_o(\vec{r}) \text{ static / constant} \end{array} \right\} \begin{array}{l} \tilde{\mathbf{J}} = \sigma_c \tilde{\mathbf{E}} \\ \vec{\nabla} \cdot \tilde{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \end{array}$$

$$\tilde{\sigma}_c(\omega) = \left( \frac{\sigma_c}{(1 - i\omega/\gamma)} \right) \rightarrow \sigma_c = 1/\rho_c \text{ static / constant} \quad \sigma_c = \left( \frac{n_e e^2}{m_e^* \gamma} \right)$$

$$\tilde{\rho}_c(\omega) = \frac{1}{\tilde{\sigma}_c(\omega)} = \rho_c (1 - i\omega/\gamma) \rightarrow \rho_c \text{ static / constant}$$

$$\tilde{\epsilon}(\omega) = \epsilon_o + \frac{\tilde{\sigma}_c(\omega)}{i\omega} = \epsilon_o + \frac{\sigma_c}{i\omega(1 - i\omega/\gamma)} = \epsilon_o + \frac{\sigma_c}{\omega(\omega/\gamma + i)} \rightarrow \infty \text{ singular !!!}$$

In the previous P436 Lecture Notes 7.5 on dispersion in non-conducting, non-magnetic linear / homogeneous / isotropic media, we showed / derived the complex electric permittivity (due to bound atomic electrons) of the dielectric medium to be:

$$\tilde{\epsilon}(\omega) = \epsilon_o (1 + \tilde{\chi}_e(\omega)) = \epsilon_o \left( 1 + \left( \frac{n_e e^2}{\epsilon_o m_e} \right) \sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} \right) \text{ with: } \tilde{K}_e(\omega) = \frac{\tilde{\epsilon}(\omega)}{\epsilon_o} = 1 + \tilde{\chi}_e(\omega)$$

$$\text{and: } \tilde{n}(\omega) = \sqrt{\tilde{\epsilon}(\omega)/\epsilon_o} = \sqrt{\tilde{K}_e(\omega)} = \sqrt{1 + \tilde{\chi}_e(\omega)}$$

If only one / a single resonance exists, then this relation reduces to:

$$\tilde{\epsilon}_1(\omega) = \epsilon_o \left( 1 + \left( \frac{n_e e^2}{\epsilon_o m_e} \right) \frac{1}{[\omega_1^2 - \omega^2 - i\gamma\omega]} \right) \text{ with: } \omega_{1j} \equiv \sqrt{\omega_{0j}^2 - \left( \frac{n_e e^2}{3\epsilon_o m_e} \right)}$$

$$\tilde{\epsilon}_1(\omega) = \epsilon_o + \left( \frac{n_e e^2}{m_e} \right) \frac{1}{[\omega_1^2 - \omega^2 - i\gamma\omega]} \quad \omega_{0j} \equiv \sqrt{\frac{k_{ej}}{m_e}}$$

Comparing this expression to that for conducting media with “free” / conduction electrons:

$$\tilde{\epsilon}_{free}(\omega) = \epsilon_o + \frac{\sigma_c}{\omega(\omega/\gamma + i)} \text{ with static conductivity: } \sigma_c \equiv \left( \frac{n_e e^2}{m_e^* \gamma} \right)$$

$$\therefore \tilde{\epsilon}_{free}(\omega) = \epsilon_o + \left( \frac{n_e e^2}{m_e^*} \right) \left( \frac{1}{\gamma\omega} \right) \frac{1}{[\omega/\gamma + i]} = \epsilon_o + \left( \frac{n_e e^2}{m_e^*} \right) \frac{1}{[\omega^2 + i\gamma\omega]}$$

or:  $\tilde{\epsilon}_{free}(\omega) = \epsilon_o - \left( \frac{n_e e^2}{m_e^*} \right) \frac{1}{[\omega_0^2 - \omega^2 - i\gamma\omega]}$  with  $\omega_0 \equiv 0 \leftarrow i.e. \text{ a resonance at } \underline{\text{zero}} \text{ frequency!}$



We can (again) associate complex wavenumber  $\tilde{k}(\omega)$  with complex index of refraction  $\tilde{n}(\omega)$ :

$$\boxed{\tilde{k}(\omega) = \left(\frac{\omega}{c}\right) \tilde{n}(\omega)} \text{ or: } \boxed{\tilde{n}(\omega) = \left(\frac{c}{\omega}\right) \tilde{k}(\omega)} \text{ where the complex index of refraction:}$$

$$\boxed{\tilde{n}(\omega) \equiv n(\omega) + i\eta(\omega)} \text{ thus: } \boxed{n(\omega) = \Re e(\tilde{n}(\omega)) = \left(\frac{c}{\omega}\right) k(\omega)} \text{ and: } \boxed{\eta(\omega) = \Im m(\tilde{n}(\omega)) = \left(\frac{c}{\omega}\right) \kappa(\omega)}$$

Plugging in the expression  $\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$  for monochromatic plane *EM* waves propagating in a dispersive, non-magnetic conducting medium into the above wave equation, we obtain the so-called characteristic equation for this situation {Exercise - explicitly work this out yourselves!}:

$$\boxed{\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 + \frac{i\omega\mu_o\sigma_c}{(1-i\omega/\gamma)}} \text{ with: } \boxed{c^2 = \frac{1}{\epsilon_o\mu_o}} \text{ and: } \boxed{\sigma_c = \left(\frac{n_e e^2}{m_e^* \gamma}\right)} \text{ or: } \boxed{\gamma = \left(\frac{n_e e^2}{m_e^* \sigma_c}\right)}$$

$$\text{Then: } \boxed{\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega)} = (\text{Complex Index of Refraction})^2 \Rightarrow \boxed{\tilde{n}^2(\omega) = 1 + \frac{i\sigma_c/\epsilon_o}{\omega(1-i\omega/\gamma)}}$$

### The Low Frequency Behavior of a Dispersive, Non-Magnetic Conducting Medium:

$$\text{Definition of low frequency: } \boxed{(\omega/\gamma) \ll 1} \text{ where: } \boxed{\gamma = \left(\frac{n_e e^2}{m_e^* \sigma_c}\right)} \text{ and: } \boxed{\omega \ll (\sigma_c/\epsilon_o)}$$

In this regime, the characteristic/mean/average time between successive collisions experienced by the “free” / conduction electrons in the dispersive, non-magnetic conducting medium  $\langle \tau_{coll} \rangle$  is short compared to the oscillation period of *EM* waves,  $\tau = 1/f = 2\pi/\omega$ , i.e.  $\langle \tau_{coll} \rangle \ll \tau$ .

$$\text{For } \omega \approx 0 \text{ (but not } \equiv 0 \text{!): } \boxed{\tilde{k}^2(\omega) = \left(\frac{\omega}{c}\right)^2 + \frac{i\omega\mu_o\sigma_c}{(1-i\omega/\gamma)}} \Rightarrow \boxed{\tilde{k}^2(\omega) \approx \frac{i\omega\mu_o\sigma_c}{1} = i\omega\mu_o\sigma_c} \text{ keeping only terms linear in } \omega$$

$$\text{Now note that: } \boxed{\sqrt{i} = \sqrt{-1} = \left(\frac{1+i}{\sqrt{2}}\right)} \Leftarrow \{\text{Please work this out/derive this yourselves!}\}$$

$$\text{Thus for } \omega \approx 0 \text{ (i.e. } \omega/\gamma \ll 1 \text{): } \boxed{\tilde{k}^2(\omega) \approx i\omega\mu_o\sigma_c} \Rightarrow \boxed{\tilde{k}(\omega) = k(\omega) + i\kappa(\omega) \approx \sqrt{i\omega\mu_o\sigma_c} = \sqrt{\frac{\mu_o\omega\sigma_c}{2}} (1+i)}$$

From which we see immediately that for monochromatic plane *EM* waves propagating in a dispersive, non-magnetic conducting medium, that  $k(\omega) \approx \kappa(\omega) = \sqrt{\frac{\mu_o\omega\sigma_c}{2}}$  and thus here

$$\{\text{again}\} \text{ we see that the } \vec{B} \text{-field lags the } \vec{E} \text{-field by: } \boxed{\phi_k = \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa}{k}\right) \approx \tan^{-1}(1) = \frac{\pi}{4} = 45^\circ}$$

Similarly, for  $\omega \approx 0$  (i.e.  $\omega/\gamma \ll 1$ ) since:  $\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega)$

Then:  $\tilde{n}^2(\omega) = \left(\frac{c}{\omega}\right)^2 \tilde{k}^2(\omega) \approx i\omega\mu_o\sigma_c \left(\frac{c}{\omega}\right)^2 = i\cancel{\omega}\cancel{\mu_o}\sigma_c \left(\frac{1}{\cancel{\epsilon_o}\cancel{\omega^2}}\right) = +\frac{i\sigma_c}{\epsilon_o\omega}$ . Using:  $\sqrt{i} = \left(\frac{1+i}{\sqrt{2}}\right)$

$\Rightarrow \tilde{n}(\omega) = n(\omega) + i\eta(\omega) = \sqrt{\frac{i\sigma_c}{\epsilon_o\omega}} = \sqrt{\frac{\sigma_c}{2\epsilon_o\omega}}(1+i)$  which again implies that  $n(\omega) \approx \eta(\omega) = \sqrt{\frac{\sigma_c}{2\epsilon_o\omega}}$ .

Here again, we can define a low-frequency skin depth:  $\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\mu_o\omega\sigma_c}}$  same as before, but with  $\mu = \mu_o$

which is the characteristic distance scale that the  $\vec{E}$  and  $\vec{B}$  fields fall to  $1/e = e^{-1} = 0.3679$  of their original  $\{z = 0\}$  values.

We define a low-frequency absorption coefficient:  $\alpha(\omega) \equiv 2\kappa(\omega) = 2/\delta_{sc}(\omega) = \sqrt{2\mu_o\omega\sigma_c}$

and corresponding low-frequency extinction coefficient:  $\xi(\omega) \equiv 2\eta(\omega) \approx \sqrt{\frac{\sigma_c}{2\epsilon_o\omega}}$

The characteristic “distance” scales over which the {time-averaged} *EM* wave energy density  $\langle u(z) \rangle$  and intensity  $\langle I(z) \rangle$  fall to  $1/e = e^{-1} = 0.3679$  of their initial ( $z = 0$ ) values, respectively are:

$$\langle u_{EM}(z) \rangle = \langle u_{EM}(0) \rangle e^{-\alpha z} \quad \text{and} \quad \langle I(z) \rangle = \langle I(0) \rangle e^{-\xi(\frac{\omega}{c})z} \quad \left[ \text{Since both are proportional to } E^2 \right]$$

For pure copper metal, for which  $\gamma_{Cu} \approx 4.1 \times 10^{13} \text{ sec}^{-1}$  and  $\sigma_c^{Cu} \approx 6 \times 10^7 \text{ Siemens/m}$  and thus  $\sigma_c^{Cu}/\epsilon_o \approx 6.5 \times 10^{18} / \text{sec}$ , then for low frequencies, e.g.  $f = \omega/2\pi \leq 10^{12} \text{ Hz}$  we see that pure copper metal is in the low-frequency regime for vacuum wavelengths of:

$$\lambda_o = \frac{c}{f} \geq \frac{3 \times 10^8}{10^{12}} \approx 3 \times 10^{-4} \text{ m} = 300 \mu\text{m} \quad \text{cf w/} \quad \lambda_{Cu}(f) = \frac{2\pi}{k(f)} = 2\pi / \sqrt{\frac{\mu_o(2\pi f)\sigma_c^{Cu}}{2}} \geq 10^{-6} \text{ m} = 1 \mu\text{m}$$

In pure copper metal, the skin depth  $\delta_{sc}(\omega)$  associated with monochromatic *EM* waves of frequency  $f \sim 10^{12} \text{ Hz} = 1 \text{ THz} = 10^6 \text{ MHz}$  is:

$$\delta_{sc}^{Cu}(f = 1 \text{ THz}, \lambda_o = 300 \mu\text{m}) \approx \sqrt{\frac{2}{\mu_o\omega\sigma_c}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 2\pi \times 10^{12} \times 6 \times 10^7}} \approx 6.5 \times 10^{-8} \text{ m} = 65 \text{ nm}$$

Thus at “low” frequencies, we see that:

$$\left\{ \delta_{sc}(10^{12} \text{ Hz}) = 65 \text{ nm} \right\} \ll \left\{ \lambda_{Cu}(10^{12} \text{ Hz}) = 1 \mu\text{m} \right\} \ll \left\{ \lambda_o = 300 \mu\text{m} \right\}$$

In pure copper metal, the skin depth  $\delta_{sc}(\omega)$  associated with monochromatic  $EM$  waves of frequency  $f \sim 10^{10}$  Hz = 10 GHz =  $10^4$  MHz (e.g. cell phones) {which corresponds to a vacuum wavelength of  $\lambda_o = c/f = 3 \times 10^8 / 10^{10} = 3 \times 10^{-2} \text{ m} \approx 3 \text{ cm}$ } is:

$$\delta_{sc}^{Cu}(10^{10} \text{ Hz}) \equiv \frac{1}{\kappa(\omega)} \approx \sqrt{\frac{2}{\mu_o \omega \sigma_c}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 2\pi \times 10^{10} \times 6 \times 10^7}} \approx 6.5 \times 10^{-7} \text{ m} = 650 \text{ nm}$$

The wavelength of  $EM$  waves with frequency  $f = 10^{10}$  Hz propagating in the pure copper metal is:

$$\lambda_{Cu}(f) = \frac{2\pi}{k(f)} = 2\pi / \sqrt{\frac{\mu_o (2\pi f) \sigma_c^{Cu}}{2}} = 10^{-5} \text{ m} = 10 \mu\text{m}$$

Thus at “low” frequencies, we {again} see that:

$$\{\delta_{sc}(10^{10} \text{ Hz}) = 650 \text{ nm}\} \ll \{\lambda_{Cu}(10^{10} \text{ Hz}) = 10 \mu\text{m}\} \ll \{\lambda_o = 3.0 \text{ cm}\}$$

### The High Frequency Behavior of a Dispersive, Non-Magnetic Conducting Medium:

Definition of high frequency:  $(\omega/\gamma) \gg 1$  where:  $\gamma = \left( \frac{n_e e^2}{m_e^* \sigma_c} \right)$  and:  $\omega \gg (\sigma_c / \epsilon_o)$ .

The square of the complex index of refraction for a metal is:  $\tilde{n}^2(\omega) = 1 + \frac{i\sigma_c / \epsilon_o}{\omega(1 - i\omega/\gamma)}$

At high frequencies, the “free” / conduction electrons in a metal will undergo a great many oscillations before scattering – i.e. the period of oscillations  $\tau = 1/f = 2\pi/\omega$  is short compared to the mean time between scatterings, essentially the relaxation time,  $\tau_{relax} = 1/\gamma$ .

In the high-frequency regime, the volume charge densities of “free” electrons and positive ions in metal together can be thought of as a plasma – whose charge density oscillates longitudinally {i.e. in the direction of propagation of the  $EM$  wave} at the {natural} angular frequency of  $\omega_p$  known as the plasma frequency, defined as:

The plasma frequency in a metal/conductor:  $\omega_p \equiv \sqrt{\frac{n_e e^2}{\epsilon_o m_e^*}}$  (radians/sec) and  $f_p = \frac{\omega_p}{2\pi}$  (Hz).

For pure copper:  $\omega_p^{Cu} = 1.644 \times 10^{16}$  rad/sec  $\Rightarrow f_p^{Cu} = \frac{\omega_p^{Cu}}{2\pi} \approx 2.616 \times 10^{15}$  Hz

The corresponding plasma wavelength in copper is:  $\lambda_p^{Cu} = \frac{c}{f_p^{Cu}} = 1.147 \times 10^{-7} \text{ m} = 114.7 \text{ nm}$

$\Rightarrow$  Operative in the optical  $\rightarrow$  UV portion of the  $EM$  spectrum – typical for many metals !!!

We can rewrite the (complex index of refraction)<sup>2</sup> in terms of the plasma frequency  $\omega_p$ :

$$\tilde{n}^2(\omega) = 1 + \frac{i(\sigma_C/\epsilon_o)}{\omega(1-i\omega/\gamma)} = 1 + i \left( \frac{n_e e^2}{\epsilon_o m_e^*} \right) \left[ \frac{1}{\gamma\omega(1-i\omega/\gamma)} \right] = 1 - \underbrace{\left( \frac{n_e e^2}{\epsilon_o m_e^*} \right)}_{=\omega_p^2} \left[ \frac{1}{i\gamma\omega(1-i\omega/\gamma)} \right] \quad \begin{array}{l} \text{used:} \\ i \equiv \sqrt{-1} \\ i = -1/i \end{array}$$

Thus:  $\tilde{n}^2(\omega) = 1 - \frac{\omega_p^2}{(\omega^2 + i\gamma\omega)}$  where:  $\omega_p \equiv \sqrt{\frac{n_e e^2}{\epsilon_o m_e^*}}$  and:  $\gamma \equiv \left( \frac{n_e e^2}{m_e^* \sigma_C} \right) = \left( \frac{\epsilon_o}{\sigma_C} \right) \left( \frac{n_e e^2}{\epsilon_o m_e^*} \right) = \frac{\omega_p^2}{(\sigma_C/\epsilon_o)}$

Now:  $\tilde{n}^2 = (n + i\eta)(n + i\eta) = n^2 + 2in\eta + \eta^2$  with  $\text{Re}(\tilde{n}^2) = n^2 - \eta^2$  and  $\text{Im}(\tilde{n}^2) = 2n\eta$

Equating the real and imaginary portions of  $\tilde{n}^2 = (n^2 - \eta^2) + i(2n\eta)$  – use the following “trick”:

$$\frac{1}{\omega^2 + i\gamma\omega} = \frac{1}{(\omega^2 + i\gamma\omega)} \cdot \frac{\omega^2 - i\gamma\omega}{\omega^2 - i\gamma\omega} = \frac{\omega^2 - i\gamma\omega}{\omega^4 + \gamma^2\omega^2} = \frac{\omega^2 - \omega^2(i\gamma/\omega)}{\omega^2(\omega^2 + \gamma^2)} = \frac{\cancel{\omega^2}(1 - i\gamma/\omega)}{\cancel{\omega^2}(\omega^2 + \gamma^2)} = \frac{1 - i\gamma/\omega}{\omega^2 + \gamma^2}$$

$$\therefore \tilde{n}^2(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} = 1 - \frac{\omega_p^2(1 - i\gamma/\omega)}{\omega^2 + \gamma^2} = \underbrace{\left[ 1 - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right]}_{=(n^2 - \eta^2)} + i \underbrace{\left( \frac{(\gamma/\omega)\omega_p^2}{\omega^2 + \gamma^2} \right)}_{=2n\eta} = (n^2 - \eta^2) + i(2n\eta)$$

$$\therefore n^2(\omega) - \eta^2(\omega) = 1 - \frac{\omega_p^2}{(\omega^2 + \gamma^2)} \quad \text{and} \quad 2n(\omega)\eta(\omega) = \left( \frac{(\gamma/\omega)\omega_p^2}{\omega^2 + \gamma^2} \right)$$

Now define:  $\alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)}$  and  $\beta_x(\omega) \equiv (\gamma/\omega)$ .

Then:  $n^2(\omega) - \eta^2(\omega) = 1 - \alpha_x(\omega)$  and:  $2n(\omega)\eta(\omega) = (\gamma/\omega)\alpha_x(\omega) = \beta_x(\omega)\alpha_x(\omega)$

Thus:  $\eta = \frac{\beta_x \alpha_x}{2n} \Rightarrow n^2 - \eta^2 = 1 - \alpha_x \Rightarrow n^2 - \frac{\beta_x^2 \alpha_x^2}{4n^2} = (1 - \alpha_x)$  or:  $n^4 - (1 - \alpha_x)n^2 - \left( \frac{\beta_x \alpha_x}{2} \right)^2 = 0$

Let  $x \equiv n^2$ , then:  $x^2 - (1 - \alpha_x)x - \left( \frac{\beta_x \alpha_x}{2} \right)^2 = 0$  which is a quadratic equation of the form:

$ax^2 + bx + c = 0$  with:  $a = 1$ ,  $b = -(1 - \alpha_x)$  and  $c = -\left( \frac{\alpha_x \beta_x}{2} \right)^2$ .

The solution is:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(1 - \alpha_x) \pm \sqrt{(1 - \alpha_x)^2 + \alpha_x^2 \beta_x^2}}{2}$  must select +ve root on physical grounds!

The Complex Index of Refraction:  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$

$$n^2(\omega) = \frac{(1 - \alpha_x(\omega))}{2} \left[ 1 + \sqrt{1 + \left( \frac{\alpha_x(\omega)\beta_x(\omega)}{1 - \alpha_x(\omega)} \right)^2} \right] \quad \alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)}$$

Where: 
$$n(\omega) = \sqrt{\frac{(1 - \alpha_x(\omega))}{2} \left[ 1 + \sqrt{1 + \left( \frac{\alpha_x(\omega)\beta_x(\omega)}{1 - \alpha_x(\omega)} \right)^2} \right]}$$

And: 
$$\eta(\omega) = \frac{\alpha_x(\omega)\beta_x(\omega)}{2n(\omega)} \quad \text{where:} \quad \omega_p \equiv \sqrt{\frac{n_e e^2}{\epsilon_o m_e^*}} \quad \text{and:} \quad \gamma \equiv \left( \frac{n_e e^2}{m_e^* \sigma_c} \right) = \frac{\omega_p^2}{(\sigma_c / \epsilon_o)}$$

Typical plasma frequencies for metals are  $\omega_p \approx 10^{16}$  rad/sec.

Typical  $\gamma$ -factors for metals are  $\gamma = 1/\tau_{relax} \sim 10^{13} - 10^{14}$  rad/sec.

Hence, in a typical metal/conductor we see that:  $\omega_p/\gamma = \omega_p \tau_{relax} \sim 10^2 - 10^3 \gg 1$  i.e.  $\omega_p \gg \gamma$ .

Thus, at high frequencies ( $\omega \gg \omega_p \gg \gamma$ ):

Then: 
$$\alpha_x(\omega) \equiv \frac{\omega_p^2}{(\omega^2 + \gamma^2)} \approx \left( \frac{\omega_p}{\omega} \right)^2 \ll 1 \quad \text{and:} \quad \beta_x(\omega) \equiv \left( \frac{\gamma}{\omega} \right) \ll 1 \quad \text{at high frequencies.}$$

And: 
$$\frac{\alpha_x \beta_x}{1 - \alpha_x} \approx \frac{\left( \frac{\omega_p}{\omega} \right)^2 \left( \frac{\gamma}{\omega} \right)}{1 - \left( \frac{\omega_p}{\omega} \right)^2} = \left( \frac{\gamma}{\omega} \right) * \frac{\left( \frac{\omega_p}{\omega} \right)^2}{1 - \left( \frac{\omega_p}{\omega} \right)^2} \approx \left( \frac{\gamma}{\omega} \right) \left( \frac{\omega_p}{\omega} \right)^2 \ll 1$$

Then: 
$$n(\omega) \approx \sqrt{1 - \alpha_x(\omega)} \approx \sqrt{1 - \left( \frac{\omega_p}{\omega} \right)^2} \approx 1 - \frac{1}{2} \left( \frac{\omega_p}{\omega} \right)^2 \leq 1 \quad \{ \text{Using } \sqrt{1 - \epsilon^2} \approx 1 - \frac{1}{2} \epsilon^2 \text{ for } \epsilon \ll 1 \}$$

And: 
$$\eta(\omega) = \frac{\alpha_x(\omega)\beta_x(\omega)}{2n(\omega)} \approx \frac{\left( \frac{\gamma}{\omega} \right) \left( \frac{\omega_p}{\omega} \right)^2}{2 \left( 1 - \frac{1}{2} \left( \frac{\omega_p}{\omega} \right)^2 \right)} \approx \frac{1}{2} \left( \frac{\gamma}{\omega} \right) \left( \frac{\omega_p}{\omega} \right)^2 \ll 1 \quad \text{at high frequencies.}$$

$\Rightarrow$  At high frequencies, for  $\omega \gg \omega_p$  the complex index of refraction  $\tilde{n}(\omega)$  is predominantly real, i.e. the imaginary part  $\eta(\omega) \ll$  real part  $n(\omega)$  at high frequencies.

$\Rightarrow$  Metals are  $\approx$  transparent to  $\gamma$ -rays and x-rays for  $\omega \gg \omega_p$  !!!

Some {alkali} metals are transparent / begin to be transparent in the UV region of the *EM* spectrum!

Metal	$\lambda_p = \frac{c}{f_p} = \frac{2\pi c}{\omega_p}$	$\omega_p \equiv \sqrt{\frac{n_e e^2}{\epsilon_0 m_e^*}}$
${}^7_3\text{Li}$	155.0 nm	
${}^{23}_{11}\text{Na}$	209.0 nm	
${}^{39}_{19}\text{K}$	287.0 nm	
${}^{63}_{29}\text{Cu}$	114.7 nm	
Alkali Metals ${}^{85}_{37}\text{Rb}$	322.0 nm ← UV	

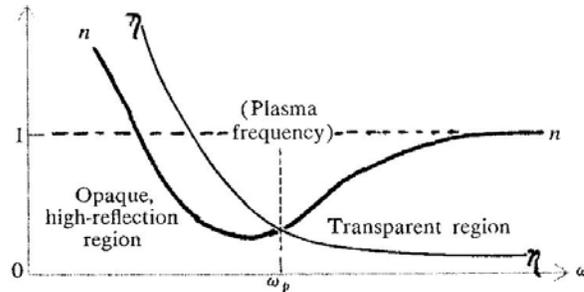


Figure 5.4. Index of refraction and extinction coefficient versus frequency for a metal.

$$\tilde{n} = n + i\eta$$

Above the plasma frequency ( $\omega > \omega_p$ ):  $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$  and  $\kappa(\omega) \ll k(\omega)$ , with:  $\tilde{k}(\omega) = \left(\frac{\omega}{c}\right)\tilde{n}(\omega)$  and thus:  $k(\omega) = \left(\frac{\omega}{c}\right)n(\omega)$  and:  $\kappa(\omega) = \left(\frac{\omega}{c}\right)\eta(\omega)$ .

Thus, since  $\kappa(\omega) \ll k(\omega)$  in the high-frequency ( $\omega \gg \omega_p$ ) regime, we see that:

$$\phi_k = \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \approx 0 \quad \text{for } \omega \gg \omega_p$$

and thus we see that  $\tilde{\vec{B}}$  is {nearly} in-phase with  $\tilde{\vec{E}}$  for *EM* waves propagating in metals/conductors with frequency above the plasma frequency ( $\omega > \omega_p$ ).

Above the plasma frequency, we also have  $\gamma \ll \omega$ , such that:

$$\tilde{\vec{F}}_e(\vec{r}, t) = m_e^* \tilde{\vec{a}}(\vec{r}, t) = m_e^* \frac{d\tilde{\vec{v}}(\vec{r}, t)}{dt} + m_e^* \gamma \tilde{\vec{v}}(\vec{r}, t) = -e\tilde{\vec{E}}(\vec{r}, t) \approx m_e^* \tilde{\vec{a}}(\vec{r}, t) = -e\tilde{\vec{E}}(\vec{r}, t)$$

⇒ Because  $\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ , and  $\tilde{\vec{a}}(\vec{r}, t) = \tilde{\vec{r}}(t) = \partial^2 \tilde{\vec{r}}(t) / \partial t^2$  we see that the “free” electrons in a metal oscillate in-phase with the driving  $\vec{E}$  of the monochromatic *EM* wave for  $\omega > \omega_p$ .

Above the plasma frequency  $\omega_p$ , a metal's complex index of refraction is  $\approx$  real (*i.e.* absorption is small), and the conductor becomes increasingly transparent as  $\omega$  is increased beyond  $\omega_p$ .

Note also that  $n(\omega) < 1$  for  $\omega > \omega_p$  {*i.e.* in the anomalous dispersion region}!!!

$$\boxed{n(\omega) \approx \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}} \text{ for } (\omega > \omega_p) \quad \text{and:} \quad \boxed{\eta(\omega) \approx \left(\frac{\gamma}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2} \text{ for } (\omega > \omega_p)$$

Since  $\tilde{n}^2(\omega) = \tilde{\epsilon}(\omega)/\epsilon_0$  for non-magnetic conductors,  $\Rightarrow \tilde{\epsilon}(\omega)$  is {also} predominantly real for non-magnetic conductors at high frequencies ( $\omega > \omega_p$ ), *i.e.*  $\tilde{\epsilon}(\omega)/\epsilon_0 = \tilde{K}_e(\omega) \approx 1$  for  $\omega > \omega_p$ ;

the skin depth for  $\omega > \omega_p$  is: 
$$\boxed{\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \left(\frac{c}{\omega}\right) \frac{1}{\eta(\omega)} = \left(\frac{c}{\gamma}\right) \left(\frac{\omega}{\omega_p}\right)^2}$$

Below the plasma frequency  $\omega_p$ , the index of refraction is complex, *i.e.*  $\eta(\omega)$  is large and therefore non-negligible; Metals are thus opaque for  $\omega < \omega_p$ .

Well below the plasma frequency  $\omega \ll \omega_p$ :  $k(\omega) \approx \kappa(\omega)$

and 
$$\boxed{\phi_k = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \approx \tan^{-1}(1) \approx \frac{\pi}{4} = 45^\circ} \Rightarrow \vec{B} \text{ lags } \vec{E} \text{ by } 45^\circ \text{ for } \omega \ll \omega_p$$

The “free” / conduction electrons in a dispersive conductor also lag  $\vec{E}$ -field by  $\approx$  same phase lag.

For  $\omega < \omega_p$ , since the incident *EM* wave is not transmitted, much of the *EM* wave is reflected.  
 $\Rightarrow$  Metals have high reflectance / high reflection coefficient  $R(\omega)$  for  $\omega < \omega_p$ .

In certain situations, such as *EM* waves propagating in the Earth's ionosphere, or *e.g.* in a tenuous electronic plasma in the laboratory, the electrons are {truly} free, hence damping is negligible ( $\gamma \approx 0$ ) in such situations.

Then for situations with negligible damping, *i.e.*  $\gamma \approx 0$ , the above relations simply somewhat:

$$\boxed{n^2(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} = K_e(\omega) = (1 + \chi_e(\omega)) = \Re e(\tilde{n}^2(\omega))} \Rightarrow \boxed{n^2(\omega) \approx 1 - \left(\frac{\omega_p}{\omega}\right)^2} \text{ with: } \boxed{\omega_p \equiv \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}}$$

The above relation holds {*i.e.* is valid} over a wide range of frequencies, including  $\omega < \omega_p$ .

For situations with  $\gamma \approx 0$ , the imaginary part of complex  $\tilde{n}(\omega)$ ,

$$\boxed{\eta(\omega) = \Im m(\tilde{n}(\omega)) \approx \left(\frac{\gamma}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2 \approx 0} \text{ because } \gamma \approx 0.$$

In this regime, where  $\gamma \approx 0$ , the wavenumber  $k$  is: 
$$\boxed{k \approx \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}$$

which is sometimes expressed as:  $\omega^2 = \omega_p^2 + ck^2$  ← dispersion relation for  $k(\omega)$

*n.b.* In dispersive conducting media,  $\gamma$  is not small, so the above formulae apply only for  $\omega \gg \omega_p$

In a tenuous plasma:  $k = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$  for  $\omega < \omega_p$  implies that  $k$  is purely imaginary!!!

⇒ *EM* waves with  $\omega < \omega_p$  incident on such a plasma are reflected for  $\omega < \omega_p$ , because the *EM* fields inside tenuous plasma are exponentially attenuated by the factor  $e^{-kz}$ .

It is precisely this fact that enables *e.g.* short-wave radio communication around the globe – the short-wave radio waves reflect off of the plasma in the earth’s ionosphere! {see handout...}

For tenuous electron plasmas in the laboratory:

$$n_e \approx 10^{18} - 10^{22} \text{ electrons/m}^3 \Rightarrow \omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}} \approx 6 \times 10^{10} - 6 \times 10^{12} \text{ radians/sec}$$

At  $\omega = 0$ , the (static) attenuation constant is:  $\kappa = \Im m(\tilde{k}) \approx \omega_p / c$  at  $\omega = 0$ .

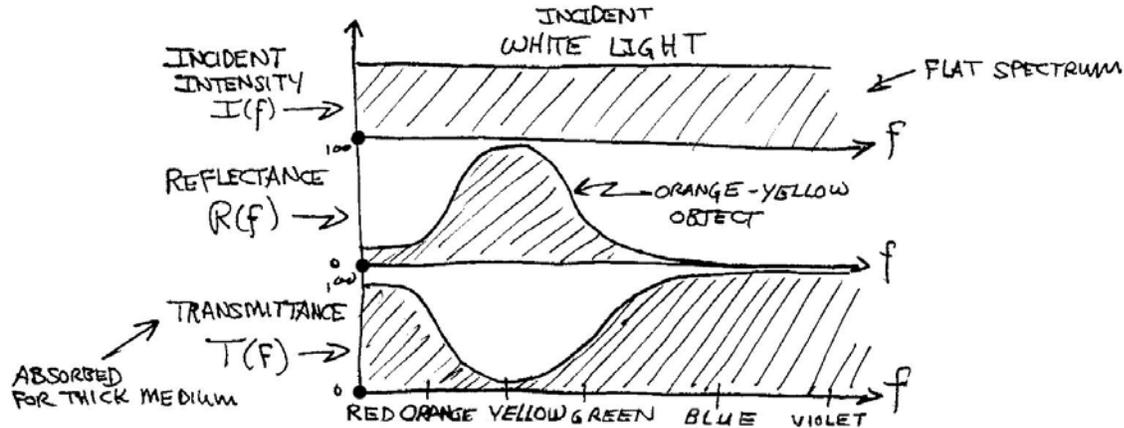
⇒ Skin depth in a metal (at  $\omega = 0$ ):

$$\delta_{sc}(\omega = 0) = \frac{1}{\kappa(\omega = 0)} = \frac{c}{\omega_p} \approx 0.5 \text{ cm} - 5 \times 10^{-3} \text{ cm} \text{ for static (or low-frequency) } EM \text{ fields}$$

Expulsion of *EM* fields within a plasma is well-known phenomena, *e.g.* in controlled thermonuclear processes, and can also be exploited *e.g.* in attempts at confinement of a hot plasma.

### Electromagnetic Wave Propagation in a Linear, Homogeneous, Isotropic Dispersive Conductive Medium

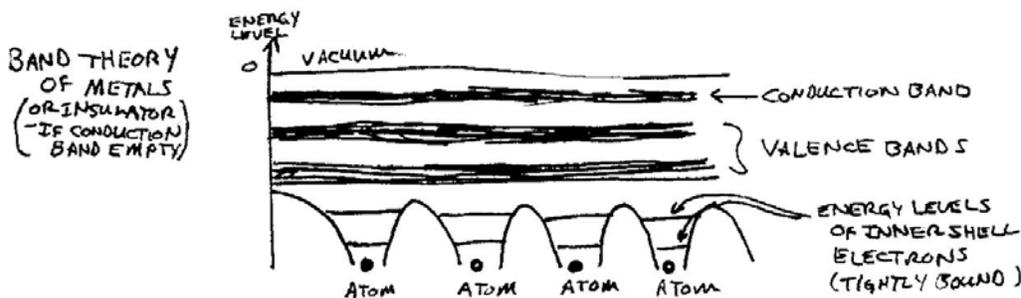
In a conducting medium, metals in particular, the optical / *EM* properties of many metals are dominated by the “free” electrons in the metal, resulting in high reflectance  $R \sim 85\text{-}95\%$ , nearly independent of the frequency / wavelength of the incident light, at least in the visible light region of the *EM* spectrum ( $\sim 350 \text{ nm} < \lambda < 780 \text{ nm}$ ). These “free” electrons in such metals as aluminum, tin, sodium, potassium, cesium, vanadium, niobium, gadolinium, holmium, yttrium, scandium, osmium, and even iron have a silvery-gray appearance {due to visible light photons scattering off of the “free” electrons in the metal} – essentially these metals are colorless because of their wavelength-independent reflectance, but remember / realize that the physical color of an object illuminated *e.g.* by white light (= flat distribution in frequency) is reflected light – hence the illuminated object is a poor absorber of light at that wavelength – indeed, the object preferentially absorbs light at other frequencies / wavelengths, and not so much at the frequency / wavelength associated with the color we perceive it to be, from reflected light – see figure below:



Some conducting metals, such as gold and copper, as well as various alloys – *e.g.* bronze and brass (zinc, tin and copper) – do not have a silvery-gray appearance, but have a yellow / orange / red “tinge” to them. This is because these metals preferentially absorb (*i.e.* transmit) in the green-blue region and thus reflect more-so in the yellow / orange / red region of the visible light spectrum, giving these metals their characteristic hues / colors.

Indeed, *e.g.* gold-metalized sun glasses – having a very thin layer of gold deposited on them – have transmittance  $T(f)$  which is peaked in the green portion of the visible spectrum. Goggles coated with a thin gold layer are used *e.g.* by people who work around high-temperature furnaces because while visible light from the furnace (which is peaked in the green) is transmitted, infrared light (*i.e.* heat) is strongly absorbed by gold!

Because “free” / conduction electrons in a metal have no resonances (except the one at  $\omega = 0$ ) because there are no restoring forces acting on them. In such metals as gold and copper, bound electrons *e.g.* in the so-called valence bands of the metal also play a non-negligible role *e.g.* in the optical properties of the metal – *i.e.* in the visible light portion of the *EM* spectrum!



Since (*i.e.* we assume) the response of the “free” electrons in the conduction band is separable / independent of the response of the bound electrons in the valence band(s) to incident *EM* waves (valid as long as the amplitudes of  $\vec{E} / \vec{B}$  are not too large). Thus we {again} use the principle of linear superposition *e.g.* for the complex electric permittivity:  $\tilde{\epsilon}_{tot}(\omega) = \tilde{\epsilon}_{bound}(\omega) + \tilde{\epsilon}_{free}(\omega)$

where: 
$$\tilde{\epsilon}_{bound}(\omega) = \epsilon_0 \left( 1 + \left( \frac{n_b^e e^2}{\epsilon_0 m_e^*} \right) \sum_{j=1}^n \frac{f_j^{bound}}{\omega_{1j}^2 - \omega^2 - i\gamma_j \omega} \right)$$
 with: 
$$\sum_{j=1}^n f_j^{bound} = 1$$

*n.b.*  $n_b^e = \#$  density ( $= \# / m^3$ ) of bound atomic electrons in the dispersive conducting medium

$$\text{and where: } \tilde{\epsilon}_{free}(\omega) = \epsilon_o \left( \frac{n_e^f e^2}{\epsilon_o m_e^*} \right) \frac{1}{[\omega_0^2 - \omega^2 - i\gamma_0 \omega]} = -\epsilon_o \frac{\omega_p^2}{[\omega^2 + i\gamma_0 \omega]} = -\epsilon_o \frac{\omega_p^2}{(\omega^2 + i\gamma_0 \omega)}$$

where:  $\omega_0 = 0$  and  $n_e^f = \#$  density ( $\#/m^3$ ) of “free” conduction-band electrons in the dispersive conducting medium, and the plasma frequency of the dispersive conducting medium is:

$$\omega_p \equiv \sqrt{\frac{n_e^f e^2}{\epsilon_o m_e^*}}$$

$$\text{Then: } \tilde{\epsilon}_{tot}(\omega) = \epsilon_o \left( 1 + \underbrace{\left( \frac{n_e^b e^2}{\epsilon_o m_e^*} \right) \sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} - \frac{\omega_p^2}{[\omega^2 + i\gamma_0 \omega]}}_{= \tilde{\chi}_e^{tot}(\omega)} \right) = \epsilon_o (1 + \tilde{\chi}_e^{tot}(\omega))$$

$$\tilde{\epsilon}_{tot}(\omega) = \epsilon_o (1 + \tilde{\chi}_e^{tot}(\omega))$$

If the dispersive conducting medium is non-magnetic ( $\mu \approx \mu_o$ ), the complex index of refraction  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$  is related to the complex permittivity  $\tilde{\epsilon}(\omega) = \epsilon(\omega) + i\zeta(\omega)$  by:

$$\tilde{n}^2(\omega) = \left( \frac{\tilde{\epsilon}(\omega)}{\epsilon_o} \right) = \left( \frac{\tilde{\epsilon}_{tot}(\omega)}{\epsilon_o} \right) = \left[ 1 + \left( \frac{n_e^b e^2}{\epsilon_o m_e^*} \right) \sum_{j=1}^n \frac{f_j^{osc}}{[\omega_{1j}^2 - \omega^2 - i\gamma_j \omega]} - \frac{\omega_p^2}{[\omega^2 + i\gamma_0 \omega]} \right] = (1 + \tilde{\chi}_e^{tot}(\omega))$$

$$\text{with: } \tilde{n}^2(\omega) = (n + i\eta)(n + i\eta) = (n^2 - \eta^2) + i(2n\eta)$$

Complex  $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$  is again related to complex  $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$  by:

$$\tilde{k}(\omega) = \left( \frac{\omega}{c} \right) \tilde{n}(\omega) \quad \text{or:} \quad \tilde{n}(\omega) = \left( \frac{c}{\omega} \right) \tilde{k}(\omega) \quad \text{with:} \quad n(\omega) = \left( \frac{c}{\omega} \right) k(\omega) \quad \text{and:} \quad \eta(\omega) = \left( \frac{c}{\omega} \right) \kappa(\omega)$$