# LECTURE NOTES 19

## LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS (SLIGHT RETURN)

Before continuing on with our onslaught of the development of relativistic electrodynamics via tensor analysis, I want to briefly discuss an equivalent, simpler method of Lorentz transforming the *EM* fields $\vec{E}$ and $\vec{B}$ from one IRF(*S*) to another IRF(*S′*), which also sheds some light (by contrast) on how the *EM* fields Lorentz transform *vs*. "normal" 4-vectors.

In P436 Lecture Notes 18.5 {p.18-22} we discussed the tensor algebra method for Lorentz transformation of the electromagnetic field e.g. in the lab frame IRF(*S*), represented by the *EM* field tensor $F^{\mu\nu}$ to another frame IRF(*S′*), represented by the *EM* field tensor $F'^{\mu\nu}$ via the relation:

$$\boxed{F'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}F^{\lambda\sigma}}.$$

Analytically carrying out this tensor calculation by hand can be tedious and time-consuming. If such calculations are to be carried out repeatedly/frequently, we encourage people to code this up and simply let the computer do the repetitive work, which it excels at.

For 1-dimensional Lorentz transformations (only) there is a simpler, less complicated, perhaps somewhat more intuitive method. Starting with the algebraic rules for Lorentz-transforming { $\vec{E}$ and $\vec{B}$ } in one IRF(*S*) to { $E'$ and $\vec{B}'$ } in another IRF(*S′*) e.g. moving with relative velocity $\boxed{\vec{v} = +v\hat{x}}$ with respect to IRF(*S*):

|| component(s): $\boxed{E'_x = E_x}$ $\boxed{B'_x = B_x}$ $\boxed{\gamma \equiv 1\big/\sqrt{1-\beta^2}}$

$\perp$ components: $\boxed{E'_y = \gamma\left(E_y - \beta c B_z\right)}$ $\boxed{B'_y = \gamma\left(B_y + \beta E_z/c\right)}$ $\boxed{\beta \equiv v/c}$

$\boxed{E'_z = \gamma\left(E_z + \beta c B_y\right)}$ $\boxed{B'_z = \gamma\left(B_z + \beta E_y/c\right)}$

We can write these relations more compactly and elegantly by resolving them into their || and $\perp$ components relative to the boost direction: here, || is along $\boxed{\vec{v} = +v\hat{x}}$ and $\perp$ is perpendicular to $\vec{v}$, defined as follows {n.b <u>in general</u>, $\vec{v}$ could be || e.g. to $\hat{x}, \hat{y}, \hat{z}$ or $\hat{r}$ }:

$$\boxed{E'^{\parallel} = E^{\parallel}} \qquad \boxed{\vec{\beta} \equiv \vec{v}/c}$$

$$\boxed{E'^{\perp} = \gamma\left(E^{\perp} + \vec{v}\times B^{\perp}\right) = \gamma\left(E^{\perp} + \vec{\beta}c\times B^{\perp}\right)} \qquad \boxed{\gamma \equiv 1\big/\sqrt{1-\beta^2}}$$

$$\boxed{B'^{\parallel} = B^{\parallel}}$$

$$\boxed{B'^{\perp} = \gamma\left(B^{\perp} - \frac{1}{c^2}\vec{v}\times E^{\perp}\right) = \gamma\left(B^{\perp} - \frac{1}{c}\vec{\beta}\times E^{\perp}\right)}$$

Now since $\boxed{\vec{v} = +v\hat{x}}$ {here} then $\boxed{E^{\parallel} \equiv E_x}$, $\boxed{B^{\parallel} \equiv B_x}$ and $\boxed{B^{\perp} \equiv B_y\hat{y} + B_z\hat{z}}$, $\boxed{E^{\perp} \equiv E_y\hat{y} + E_z\hat{z}}$ {and similarly for corresponding quantities in IRF(*S′*)}.

Since $\boxed{\vec{v} = +v\hat{x}}$ and $\boxed{E^{\parallel} \equiv E_x}$ then: $\boxed{\vec{v} \times E^{\perp} = \vec{v} \times E_x \hat{x} = 0}$

And likewise, since $\boxed{B^{\parallel} \equiv B_x}$ then: $\boxed{\vec{v} \times B^{\parallel} = \vec{v} \times B_x \hat{x} = 0}$.

Thus, we can {<u>*safely*</u>} write: $\boxed{\vec{v} \times E^{\perp} = \vec{v} \times \vec{E}}$ and $\boxed{\vec{v} \times B^{\perp} = \vec{v} \times \vec{B}}$, as long as $\vec{v}$ is <u>*always*</u> || to <u>*one*</u> of the components of $\vec{E}$ and $\vec{B}$ - e.g. $\hat{x}$ or $\hat{y}$ or $\hat{z}$ .

Then we can write the Lorentz transformation of *EM* fields as:

$$\boxed{E'^{\parallel} = E^{\parallel}} \qquad \boxed{\beta \equiv v/c}$$

$$\boxed{E'^{\perp} = \gamma\left(E^{\perp} + \vec{v} \times \vec{B}\right) = \gamma\left(E^{\perp} + \vec{\beta} c \times \vec{B}\right)} \qquad \boxed{\vec{\beta} \equiv \vec{v}/c}$$

$$\boxed{B'^{\parallel} = B^{\parallel}}$$

$$\boxed{B'^{\perp} = \gamma\left(B^{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}\right) = \gamma\left(B^{\perp} - \frac{1}{c}\vec{\beta} \times \vec{E}\right)} \qquad \boxed{\gamma = 1\Big/\sqrt{1-\beta^2}}$$

This can be written more compactly in 2-D matrix form as:

<u>*EM* Fields</u>: "<u>Normal" 4-Vector</u>:

$$\boxed{\begin{pmatrix} E'^{\parallel} \\ cB'^{\parallel} \end{pmatrix} = \begin{pmatrix} E^{\parallel} \\ cB^{\parallel} \end{pmatrix}} \quad \rightarrow \quad \boxed{E'^{\parallel} = E^{\parallel} \text{ and } B'^{\parallel} = B^{\parallel}} \quad \leftarrow \text{compare to} \rightarrow \quad \boxed{x'^{\perp} = x^{\perp}}$$

$$\boxed{\begin{pmatrix} E'^{\parallel} \\ cB'^{\parallel} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Unit Matrix}} \begin{pmatrix} E^{\parallel} \\ cB^{\parallel} \end{pmatrix}} \quad \boxed{\begin{pmatrix} E'^{\perp} \\ cB'^{\perp} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & +\gamma\vec{\beta}\times \\ -\gamma\vec{\beta}\times & \gamma \end{pmatrix}}_{\text{Operator Matrix}} \begin{pmatrix} E^{\perp} \\ cB^{\perp} \end{pmatrix}} \leftrightarrow \boxed{\begin{pmatrix} x'^{\parallel} \\ ct' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}}_{\text{Scalar Matrix}} \begin{pmatrix} x^{\parallel} \\ ct \end{pmatrix}}$$

Thus, we see that for the *EM* fields *vs*. the 3-D space-part of a "normal" 4-vector, the || *vs*. $\perp$ components are <u>switched</u>, $\vec{B}$ transforms "sort of" like time *t*, but 2 × 2 Lorentz boost matrices for ( $\vec{E}$ and $\vec{B}$ ) vs. 4-vectors are <u>*not*</u> the same (they are *similar*, but they are <u>not</u> identical).

We can also write compact <u>inverse</u> Lorentz transformations (e.g. from IRF(*S′*) rest frame → IRF(*S*) lab frame):

<u>*EM* Fields</u>: "<u>Normal" 4-Vector</u>:

$$\boxed{\begin{pmatrix} E^{\parallel} \\ cB^{\parallel} \end{pmatrix} = \begin{pmatrix} E'^{\parallel} \\ cB'^{\parallel} \end{pmatrix}} \quad \rightarrow \quad \boxed{E^{\parallel} = E'^{\parallel} \text{ and } B^{\parallel} = B'^{\parallel}} \quad \leftarrow \text{compare to} \rightarrow \quad \boxed{x^{\perp} = x'^{\perp}}$$

$$\boxed{\begin{pmatrix} E'^{\parallel} \\ cB'^{\parallel} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Unit Matrix}} \begin{pmatrix} E^{\parallel} \\ cB^{\parallel} \end{pmatrix}} \quad \boxed{\begin{pmatrix} E^{\perp} \\ cB^{\perp} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\vec{\beta}\times \\ +\gamma\vec{\beta}\times & \gamma \end{pmatrix}}_{\text{Operator Matrix}} \begin{pmatrix} E'^{\perp} \\ cB'^{\perp} \end{pmatrix}} \leftrightarrow \boxed{\begin{pmatrix} x^{\parallel} \\ ct \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix}}_{\text{Scalar Matrix}} \begin{pmatrix} x'^{\parallel} \\ ct' \end{pmatrix}}$$

For a <u>general</u> Lorentz transformation (i.e. *<u>no</u>* restriction on the orientation of $\vec{v}$ {arbitrary}):

A.) Lorentz transformation from IRF($S$) → IRF($S'$):

$$\boxed{\vec{E}' = \gamma\left(\vec{E} + \vec{\beta}c \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\bullet\vec{E}\right)}$$

$$\boxed{\vec{B}' = \gamma\left(\vec{B} - \frac{1}{c}\vec{\beta} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\bullet\vec{B}\right)}$$

$\boxed{\vec{\beta} = \vec{v}/c}$ $\boxed{\beta = v/c}$

$\boxed{\gamma = 1\big/\sqrt{1-\beta^2}}$

<u>Or</u>:

$$\boxed{\begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma\left(1 - \left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) & +\gamma\vec{\beta}\times \\ -\gamma\vec{\beta}\times & \gamma\left(1 - \left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) \end{pmatrix}}_{\text{operator matrix}} \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix}}$$

B.) Inverse Lorentz transformation from IRF($S'$) → IRF($S$):

$$\boxed{\vec{E} = \gamma\left(\vec{E}' - \vec{\beta}c \times \vec{B}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\bullet\vec{E}'\right)}$$

$$\boxed{\vec{B} = \gamma\left(\vec{B}' + \frac{1}{c}\vec{\beta} \times \vec{E}'\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\bullet\vec{B}'\right)}$$

Switch $\boxed{\vec{\beta} \to -\vec{\beta}}$, $\boxed{\vec{E} \to \vec{E}'}$ and $\boxed{\vec{B} \to \vec{B}'}$ in above relations

<u>Or</u>:

$$\boxed{\begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} = \times \underbrace{\begin{pmatrix} \gamma\left(1 - \left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) & -\gamma\vec{\beta}\times \\ +\gamma\vec{\beta}x & \gamma\left(1 - \left(\frac{\gamma}{\gamma+1}\right)\vec{\beta}\vec{\beta}\bullet\right) \end{pmatrix}}_{\text{operator matrix}} \begin{pmatrix} \vec{E}' \\ c\vec{B}' \end{pmatrix}}$$

---

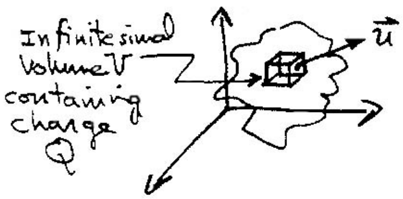
## <u>Electrodynamics in Tensor Notation</u>

So now that we know how to represent the *EM* field in relativistic tensor notation (as $F^{\mu\nu}$ or $G^{\mu\nu}$), we can also reformulate *<u>all</u>* <u>laws of electrodynamics</u> (e.g. Maxwell's equations, the Lorentz force law, the continuity equation {expressing electric charge conservation}, etc. . . ) in the mathematical language of tensors.

In order to begin this task, we must first determine how the sources of the *EM* fields – the electric charge density $\rho$ (a scalar quantity) and the electric current density $\vec{J}$ (a vector quantity) Lorentz transform.

The electric charge density $\boxed{\rho = Q/V}$ = charge per unit volume (Coulombs/m$^3$)

Imagine a cloud of electric charge drifting by. Concentrate on an infinitesimal volume $V$ containing charge $Q$ moving at (ordinary) velocity $\vec{u}$ :




Then: $\boxed{\rho = Q/V}$ = charge density (Coulombs/m$^3$)

And: $\boxed{\vec{J} = \rho\vec{u}}$ = current density (Amps/m$^2$).

A subtle, but important detail:

If there is only one species (i.e. kind / type) of charge carrier contained within the infinitesimal volume $V$, then they all travel at the same (average / mean) speed $\vec{u}$ .

However, if there are multiple species (kinds / types) of charge carriers (e.g. with different masses) or different signs of charge carriers contained within the the infinitesimal volume $V$ (e.g. electrons $e^-$ with rest masses $m_e c^2$ and protons $p$ with rest masses $m_P c^2$) then the different constituents / species *must* be treated separately in the following:

If $\exists$ $N$ species:

Current density $\boxed{\vec{J}_i = \rho_i \vec{u}_i}$ for the $i^{\text{th}}$ species ($i = 1, \ldots, N$), the electric charge density $\boxed{\rho_i = Q_i/V}$

And: $\boxed{\vec{J} = \sum_{i=1}^{N} \vec{J}_i = \sum_{i=1}^{N} \rho_i \vec{u}_i}$

We also need to express $\rho$ and $\vec{J}$ in terms of the proper charge density $\rho^0$ = volume charge density defined in the rest frame of the charge $Q$, IRF($S_0$).

The infinitesimal rest volume / proper volume = $V_0$ {defined in rest/proper frame IRF($S_0$)}

The proper charge density: $\boxed{\rho_0 = Q/V_0}$ $\leftarrow$ Recall that electric charge $Q$ (like $c$) is a Lorentz invariant scalar quantity

Because the longitudinal direction of motion undergoes Lorentz contraction from the rest frame IRF($S_0$) in the Lorentz transformation $\rightarrow$ another reference frame, e.g. lab frame IRF($S$)

Then: $\boxed{V = \frac{1}{\gamma_u} V_0}$ where: $\boxed{V_0 = \ell_0 w_0 d_0}$ and: $\boxed{V = \ell w d}$, where: $\boxed{\gamma_u = \frac{1}{\sqrt{1-\beta_u^2}}}$ and: $\boxed{\beta_u = \frac{u}{c}}$

If the Lorentz transformation is along (i.e. $\parallel$ to) the length $\ell, \ell_0$ of the infinitesimal volumes

Then: $\boxed{\ell = \frac{1}{\gamma_u} \ell_0}$ and the $\perp$ components of the volumes are unchanged: $\boxed{w_0 = w}$, $\boxed{d_0 = d}$.

Then if: $\boxed{V = \frac{1}{\gamma_u} V_0}$ $\rightarrow$ $\boxed{\rho = \frac{Q}{V} = \gamma_u \left(\frac{Q}{V_0}\right) = \gamma_u \rho_0}$ $\therefore$ $\boxed{\vec{J} = \rho\vec{u} = \gamma_u \rho_0 \vec{u} = \rho_0 \left(\gamma_u \vec{u}\right)}$

Recall that the 3-D vector associated with the proper velocity is: $\boxed{\vec{\eta} = \gamma_u \vec{u} \quad \left(\equiv \frac{d\vec{\ell}}{d\tau}\right)}$ $\therefore$ $\boxed{\vec{J} = \rho_0 \vec{\eta}}$

The zeroth (i.e. temporal/scalar) component of the proper 4-velocity is: $\boxed{\eta_0 \equiv \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \gamma_u c}$

The corresponding zeroth (i.e. temporal/scalar) component of the current density 4-vector $J^\mu$ is:

$$\boxed{J^0 \equiv \rho_0\eta^0 = \rho_0\gamma_u c = (\gamma_u\rho_0)c = \rho c = c\rho}$$

The current density 4-vector is: $\boxed{J^\mu = (c\rho, \vec{J}) = (c\rho, J_x, J_y, J_z)}$ (SI units: Amps/m$^2$)

Then: $\boxed{J^\mu = \rho_0\eta^\mu}$ where: $\boxed{\eta^\mu = (\gamma_u c, \gamma_u\vec{u}) = \gamma_u(c, \vec{u}) = \gamma_u(c, u_x, u_y, u_z)}$

constant / scalar quantity

$\therefore J^\mu$ is a proper four vector, i.e. $J^\mu$ = proper current density 4-vector.

Thus: $\boxed{J^\mu J_\mu = J_\mu J^\mu}$ is a Lorentz invariant quantity. Is it ???

$$\boxed{J^\mu J_\mu = J_\mu J^\mu = \rho_0^2\eta^\mu\eta_\mu = \rho_0^2\gamma_u^2\left(-c^2 + \underbrace{u_x^2 + u_y^2 + u_z^2}_{=u^2}\right) = \rho_0^2\left(\frac{-c^2+u^2}{1-u^2/c^2}\right) = -(\rho_0 c)^2\left(\frac{1-u^2/c^2}{1-u^2/c^2}\right) = -(\rho_0 c)^2}$$

$\boxed{J^\mu J_\mu = J_\mu J^\mu = -(\rho_0 c)^2 = \rho_0^2\eta^\mu\eta_\mu = \rho_0^2\eta_\mu\eta^\mu}$ {we also know that: $\boxed{\eta^\mu\eta_\mu = \eta_\mu\eta^\mu = -c^2}$}

Yes, $\boxed{J^\mu J_\mu = J_\mu J^\mu}$ *is* a Lorentz invariant quantity!

The 3-D continuity equation mathematically expresses local conservation of electric charge (using differential vector calculus):

$\boxed{\vec{\nabla}\bullet\vec{J}(\vec{r},t) = -\frac{\partial\rho(\vec{r},t)}{\partial t}}$ $\rho(\vec{r},t)$ = scalar point function, $\vec{J} = \vec{J}(\vec{r},t)$ = 3-D vector point function

$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}}$ (in Cartesian coordinates)

We can also express the continuity equation in 4-vector tensor notation:

$\boxed{\vec{\nabla}\bullet\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3}\frac{\partial J^i}{\partial x^i}}$ And: $\boxed{\frac{\partial\rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}}$ ($J^0 = c\rho$)

**n.b. Repeated indices implies summation!**

Then: $\boxed{\vec{\nabla}\bullet\vec{J} = -\frac{\partial\rho}{\partial t}} \Rightarrow \boxed{\vec{\nabla}\bullet\vec{J} + \frac{\partial\rho}{\partial t} = 0} \Rightarrow \boxed{\sum_{i=1}^{3}\frac{\partial J^i}{\partial x^i} + \frac{\partial J^0}{\partial x^0} = \frac{\partial J^0}{\partial x^0} + \sum_{i=1}^{3}\frac{\partial J^i}{\partial x^i} = 0} = \boxed{\sum_{i=1}^{3}\frac{\partial J^i}{\partial x^i} = \frac{\partial J^\mu}{\partial x^\mu} = 0}$

Thus: $\boxed{\vec{\nabla}\bullet\vec{J} = -\frac{\partial\rho}{\partial t}}$ or $\boxed{\vec{\nabla}\bullet\vec{J} + \frac{\partial\rho}{\partial t} = 0} \Rightarrow \boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$ Continuity Equation (local charge conservation)

Physically, note that $\dfrac{\partial J^\mu}{\partial x^\mu}$ is the 4-dimensional space-time <u>divergence</u> of the current density 4-vector $J^\mu = \left(c\rho, \vec{J}\right)$. The 4-current density $J^\mu = \left(c\rho, \vec{J}\right)$ is divergenceless because $\boxed{\dfrac{\partial J^\mu}{\partial x^\mu} = 0}$

The 4-vector operator $\dfrac{\partial}{\partial x^\mu}$ is sometimes called the 4-D gradient operator, it is also sometimes called the quad operator $\square^\mu$ (or "quad" for short).

The <u>contravariant</u> quad 4-vector operator: $\boxed{\square^\mu \equiv \dfrac{\partial}{\partial x^\mu} = \left(+\dfrac{1}{c}\dfrac{\partial}{\partial t}, \vec{\nabla}\right)}$

The <u>covariant</u> quad 4-vector operator: $\boxed{\square_\mu \equiv \dfrac{\partial}{\partial x_\mu} = \left(-\dfrac{1}{c}\dfrac{\partial}{\partial t}, \vec{\nabla}\right)}$

<u>Then</u>: $\boxed{\square^\mu \square_\mu = \square_\mu \square^\mu = \dfrac{\partial}{\partial x^\mu}\dfrac{\partial}{\partial x_\mu} = \dfrac{\partial}{\partial x_\mu}\dfrac{\partial}{\partial x^\mu} = \dfrac{\partial^2}{\partial x_\mu \partial x^\mu} = -\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} + \nabla^2 \equiv \square^2}$

= D'Alembertian 4-vector operator = 4-D Laplacian operator = Lorentz invariant quantity!

So we could equivalently write the relativistic 4-D continuity equation as:

$$\boxed{\square^\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = 0} \text{ i.e. "}\boxed{\vec{\nabla}_4 \bullet \vec{J}_4 = 0}\text{"}$$

Since the 4-vector product of <u>*any*</u> two (bona-fide) relativistic 4-vectors is a Lorentz invariant quantity (i.e. the same value in any/all IRF's):

$$\therefore \quad \boxed{\square^\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} = 0}$$ is also a Lorentz invariant quantity !!!

$\rightarrow$ Electric charge is (locally) conserved in any/all IRF's (as it must be!!!)

However, because the 4-D gradient operator $\dfrac{\partial}{\partial x_\mu}$ functions like a <u>covariant</u> 4-vector, e.g. when it operates on <u>contravariant</u> $J^\mu$ (or any other <u>contravariant</u> 4-vectors), it is often given the shorthand notation $\boxed{\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}}$ and because the 4-D gradient operator $\dfrac{\partial}{\partial x^\mu}$ functions like a <u>contravariant</u> 4-vector, e.g. when it operates on <u>covariant</u> $J_\mu$ (or any other <u>covariant</u> 4-vectors), it is given the shorthand notation $\boxed{\partial^\mu \equiv \dfrac{\partial}{\partial x_\mu}}$. See/work thru Griffiths Problem 12.55 {p. 543) for more details.

Thus we see {again} that: $\boxed{\partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \square^\mu \square_\mu = \square_\mu \square^\mu = \square^2}$ is a Lorentz invariant quantity, and $\boxed{\partial_\mu J^\mu = \square^\mu J^\mu = \dfrac{\partial J^\mu}{\partial x^\mu} = 0}$ is also a Lorentz invariant quantity.

## Maxwell's Equations in Tensor Notation

Maxwell's Equations:

1) Gauss' Law: $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_o}\rho}$

2) No Magnetic Monopoles: $\boxed{\nabla\cdot\vec{B} = 0}$

3) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$ with Maxwell's Displacement Current

Can be written as 4-derivatives of the relativistic *EM* field tensors $F^{\mu\nu}$ and $G^{\mu\nu}$:

$$\underbrace{\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu} \quad \text{and:} \quad \boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}}_{\text{Summation over } \nu = 0:3 \text{ implied}}$$

1) Gauss' Law $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_o}\rho}$. If $\mu = 0$ in $\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu}$, i.e. $\boxed{\frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_o J^0}$, $\nu = 0:3$

Physically, $\mu = 0$ is the temporal/scalar component of any space-time 4-vector.

Then: $\boxed{\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}}$ first row of $F^{\mu\nu}$

Row #

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}}$$

Column #

$$\therefore \boxed{\frac{\partial F^{0\nu}}{\partial x^0} = 0 + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}\vec{\nabla}\cdot\vec{E}} \quad \text{and:} \quad \boxed{\mu_o J^0 = \mu_o(c\rho)}$$

$$\therefore \boxed{\frac{1}{c}\vec{\nabla}\cdot\vec{E} = \mu_o c\rho} \quad \text{or:} \quad \boxed{\vec{\nabla}\cdot\vec{E} = \mu_o c^2\rho} \quad \text{but:} \quad \boxed{\mu_o c^2 = \frac{1}{\varepsilon_o}} \quad \therefore \boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_o}\rho}$$

Gauss' Law arises from the $\mu = 0$ (scalar / temporal) component of the 4-vector relation:

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu}$$

4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B}=\mu_o\vec{J}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$ If $\mu=1$ in $\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu}=\mu_o J^\mu}$

Then: $\boxed{\frac{\partial F^{1\nu}}{\partial x^\nu}=\frac{\partial F^{10}}{\partial x^0}+\frac{\partial F^{11}}{\partial x^1}+\frac{\partial F^{12}}{\partial x^2}+\frac{\partial F^{13}}{\partial x^3}}$ second row of $F^{\mu\nu}$

$$\boxed{\frac{\partial F^{1\nu}}{\partial x^\nu}=-\frac{1}{c^2}\frac{\partial \mathrm{E}_x}{\partial t}+\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=\left(-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}\right)_x}$$ and: $\boxed{\mu_0 J^1=\mu_o J_x}$

$$\therefore \quad \boxed{\frac{\partial F^{1\nu}}{\partial x^\nu}=\left(-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}\right)_x=\mu_o J_x}$$

Then for $\mu=2$ and $\mu=3$ (third and fourth rows of $F^{\mu\nu}$), likewise we find that:

$$\boxed{\frac{\partial F^{2\nu}}{\partial x^\nu}=\left(-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}\right)_y=\mu_o J_y} \quad \underline{\text{and}}: \quad \boxed{\frac{\partial F^{3\nu}}{\partial x^\nu}=\left(-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}\right)_z=\mu_o J_z}$$

$$\therefore \quad \boxed{\left.\frac{\partial F^{\mu\nu}}{\partial x^\nu}\right|_{\mu=1:3}=\mu_o J^\mu\Big|_{\mu=1:3}} \Rightarrow \boxed{\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}=\mu_o\vec{J}}$$

3-D spatial components of 4-vector $J^\mu$

Or: $\boxed{\vec{\nabla}\times\vec{B}=\mu_o\vec{J}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$ Ampere's Law with Maxwell's Displacement Current term !!!

→ Ampere's Law arises from the $\mu=1:3$ (3-D spatial / vector component) of 4-vector relation:

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu}=\mu_0 J^\mu} \rightarrow \boxed{\vec{\nabla}\bullet\vec{E}=\frac{1}{\varepsilon_o}\rho} \quad (\mu=0 \text{ temporal / scalar component})$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu}=\mu_0 J^\mu} \rightarrow \boxed{\vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\mu_o\vec{J}} \quad (\mu=1:3 \text{ 3-D spatial / vector component})$$

Thus, Gauss' Law and Ampere's Law form a 4-vector:

$$\boxed{\mu_o J^\mu=\left(\underbrace{\frac{1}{c}\vec{\nabla}\bullet\vec{E}=\overbrace{\frac{1}{c\varepsilon_o}\rho}^{=\mu_o J^0}}_{},\ \underbrace{\vec{\nabla}\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\mu_o\vec{J}}_{}\right)=\left(\frac{\partial F^{\mu\nu}}{\partial x^\nu}=\mu_o J^\mu\right)}$$

Gauss' Law temporal / scalar component of $\mu_o J^\mu$

Ampere's Law 3-D spatial / vector component of component of $\mu_o J^\mu$

And: $\boxed{J^\mu J_\mu=J_\mu J^\mu=-\rho_0^2c^2}$ = Lorentz invariant quantity {from above, page 5}.

2) $\boxed{\vec{\nabla}\bullet\vec{B}=0}$ no magnetic monopoles / no magnetic charges. If $\mu=0$ in $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}=0}$

$\mu=0$ is the temporal (scalar) component of space-time "null" 4-vector $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}=0^\mu=(0,\vec{0})=0}$

Then: $\boxed{\dfrac{\partial G^{0\nu}}{\partial x^\nu}=\dfrac{\partial G^{00}}{\partial x^0}+\dfrac{\partial G^{01}}{\partial x^1}+\dfrac{\partial G^{02}}{\partial x^2}+\dfrac{\partial G^{03}}{\partial x^3}=0}$ ← First row of $G^{\mu\nu}$

Row #

$$\boxed{G^{\mu\nu}=\begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}}$$

Column #

$\therefore$ $\boxed{\dfrac{\partial G^{0\nu}}{\partial x^\nu}=0+\dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}=0=\vec{\nabla}\bullet\vec{B}=0}$ $\therefore$ $\boxed{\vec{\nabla}\bullet\vec{B}=0\rightleftharpoons\dfrac{\partial G^{0\nu}}{\partial x^\nu}=0}$

3) Faraday's Law: $\boxed{\vec{\nabla}\bullet\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}}$ If $\mu=1$ in $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}=0}$

Then: $\boxed{\dfrac{\partial G^{1\nu}}{\partial x^\nu}=\dfrac{\partial G^{10}}{\partial x^0}+\dfrac{\partial G^{11}}{\partial x^1}+\dfrac{\partial G^{12}}{\partial x^2}+\dfrac{\partial G^{13}}{\partial x^3}=0}$ ← Second row of $G^{\mu\nu}$

$$\boxed{\frac{\partial G^{1\nu}}{\partial x^\nu}=-\frac{1}{c}\frac{\partial B_x}{\partial t}-\frac{1}{c}\frac{\partial E_z}{\partial y}+\frac{1}{c}\frac{\partial E_y}{\partial z}=-\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)_x=0}$$

$\therefore$ $\boxed{\dfrac{\partial G^{1\nu}}{\partial x^\nu}=0}$ gives $\boxed{\left(\dfrac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)_x=0}$

Likewise, for $\mu=2$ and $\mu=3$ (third and fourth rows of $G^{\mu\nu}$)

$\boxed{\dfrac{\partial G^{2\nu}}{\partial x^\nu}=0}$ gives $\boxed{\left(\dfrac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)_y=0}$ and: $\boxed{\dfrac{\partial G^{3\nu}}{\partial x^\nu}=0}$ gives $\boxed{\left(\dfrac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)_z=0}$

$\therefore$ $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}\Big|_{\mu=1:3}=0}$ gives $\boxed{\dfrac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}=0}$ or: $\boxed{\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}}$

Thus: $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}=0}$

→ $\boxed{\vec{\nabla}\bullet\vec{B}=0}$ ($\mu=0$ temporal / scalar component)

→ $\boxed{\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}}$ ($\mu=1:3$ 3-D spatial / vector component)

Arise from temporal ($\mu=0$) and spatial ($\mu=1:3$) component of the "null" 4-vector $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu}=0}$

Thus, in relativistic 4-vector / tensor notation, Maxwell's 4 equations {written in language of 3-D differential vector calculus}:

Maxwell's Equations:

1) Gauss' Law: $\boxed{\vec{\nabla}\bullet\vec{E} = \frac{1}{\varepsilon_o}\rho}$

2) No Magnetic Monopoles: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$

3) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$ with Maxwell's Displacement Current

are elegantly represented by two simple 4-vector equations:

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu}$$

$\mu = 0$ temporal/scalar component: $\boxed{\vec{\nabla}\bullet\vec{E} = \frac{1}{\varepsilon_o}\rho}$ 1) Gauss' Law

$\mu = 1:3$ spatial/vector component: $\boxed{\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_o\vec{J}}$ 4) Ampere's Law

**With Maxwell's Displacement Current**

$$\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

$\mu = 0$ temporal/scalar component: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$ 2) No Magnetic Charges

$\mu = 1:3$ spatial/vector component: $\boxed{\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t} = 0}$ 3) Faraday's Law

---

**<u>Griffiths Problem 12.53</u>:**

We can show that Maxwell's two equations $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$ and $\boxed{\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t} = 0}$ that are contained in $\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$ can also be obtained from (the more cumbersome / inelegant relation):

$$\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0} \rightleftharpoons \boxed{\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\nu}}{\partial x^\nu} = 0}$$

Since there are 3 indices in the latter equation $\mu = 0:3$, $\nu = 0:3$, $\lambda = 0:3$, there are actually 64 $(= 4^3)$ separate equations!!! However many of these 64 equations are either trivial or redundant.

Suppose two indices are the same (e.g. $\mu = \nu$)

Then: $\boxed{\frac{\partial F_{\mu\mu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\mu} = 0}$. But the *EM* field tensor $F_{\mu\nu}$ (like $F^{\mu\nu}$) is anti-symmetric.

$\therefore$ $\boxed{F_{\mu\mu} = 0}$ and: $\boxed{F_{\mu\lambda} = -F_{\lambda\mu}}$. Thus, $\geq$ 2 indices the same gives the trivial relation $0 = 0$.

Thus, in order to obtain a / any non-trivial result, $\mu$, $\nu$, and $\lambda$ must *all* be different from each other.

1) The indices $\mu$, $\nu$, and $\lambda$ could all be spatial indices, such as: $\mu = 1$ ($x$), $\nu = 2$ ($y$), $\lambda = 3$ ($z$) (or permutations thereof).

Or:

2) One index could be temporal, and two indices could be spatial, such as: $\mu = 0$, $\nu = 1$, $\lambda = 2$ (or permutations thereof), or: $\mu = 0$, $\nu = 1$, $\lambda = 3$

1) For the case(s) of all spatial indices, e.g. $\mu = 1$, $\nu = 2$, $\lambda = 3$:

$$\boxed{\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0} = \boxed{\frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0} = \boxed{\vec{\nabla} \bullet \vec{B} = 0}$$

All other permutations involving the all-spatial indices $\{1, 2, 3\}$ yield the same relation $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$ or minus it: i.e. $\boxed{-\vec{\nabla} \bullet \vec{B} = 0}$.

2) For the case of one temporal and two spatial indices, e.g. $\mu = 0$, $\nu = 1$, $\lambda = 2$:

$$\boxed{\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0} = \boxed{-\frac{1}{c}\frac{\partial E_x}{\partial y} + \frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{1}{c}\frac{\partial E_y}{\partial x} = 0} = \boxed{\frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{1}{c}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = 0}$$

$$= \boxed{\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E}\right)_z = 0}$$

Other Permutations:

For $\nu = 0$, $\mu$ & $\lambda = 1{:}3$ and $\lambda = 0$, $\mu$ & $\nu = 1{:}3$ get redundant results (same as above).

If $\mu = 0$, $\nu = 1$, $\lambda = 3$ get $y$ – component of above relation!
If $\mu = 0$, $\nu = 2$, $\lambda = 3$ get $x$ – component of above relation!

$\therefore$ $\boxed{\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0}$ contains: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$ and: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

**Duality Transformation of the Relativistic *EM* Field Tensors $F^{\mu\nu}$ and $G^{\mu\nu}$:**

The duality transformation for the specific case of space-time "rotating" $\vec{E} \to c\vec{B}$ and $\boxed{c\vec{B} \to -\vec{E}}$ ($\varphi_{\text{duality}} = 90^{\text{o}}$) takes $\boxed{F^{\mu\nu} \to G^{\mu\nu}}$, and can be mathematically represented in tensor notation as:

$$\boxed{G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}}$$

where: $F_{\lambda\sigma}$ is the {doubly} covariant form of the contravariant tensor $F^{\lambda\sigma}$.

and: $\varepsilon^{\mu\nu\lambda\sigma}$ is the totally anti-symmetric rank-four tensor.

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 \text{ for all even permutations of } \mu = 0,\ \nu = 1,\ \lambda = 2,\ \sigma = 3 \text{ (and)} \\ \ \ 0 \text{ if any two indices are equal/identical/the same.} \\ -1 \text{ for all odd permutations of } \mu = 0,\ \nu = 1,\ \lambda = 2,\ \sigma = 3 \end{cases}$$

Since $\varepsilon^{\mu\nu\lambda\sigma}$ is a rank-four tensor (= 4-dimensional "matrix") we can't write it down on 2-D paper all at once! $\varepsilon^{\mu\nu\lambda\sigma}$ has ($\mu$, $\nu$, $\lambda$, $\sigma$ = 0:3) → $4^4$ elements = 256 elements!!!

We could write out 16 {4×4} matrices – e.g. one $\mu$-$\nu$ matrix for each unique combination of $\lambda$ and $\sigma$:

$$\boxed{\varepsilon^{\mu\nu\lambda\sigma} = \left(\begin{array}{c:c:c:c}
\begin{matrix}\lambda{=}0 & \nu\to\mu \\ \sigma{=}0 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}0 & \nu\to\mu \\ \sigma{=}1 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}0 & \nu\to\mu \\ \sigma{=}2 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}0 & \nu\to\mu \\ \sigma{=}3 & (4{\times}4)\downarrow\end{matrix} \\ \hdashline
\begin{matrix}\lambda{=}1 & \nu\to\mu \\ \sigma{=}0 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}1 & \nu\to\mu \\ \sigma{=}1 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}1 & \nu\to\mu \\ \sigma{=}2 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}1 & \nu\to\mu \\ \sigma{=}3 & (4{\times}4)\downarrow\end{matrix} \\ \hdashline
\begin{matrix}\lambda{=}2 & \nu\to\mu \\ \sigma{=}0 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}2 & \nu\to\mu \\ \sigma{=}1 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}2 & \nu\to\mu \\ \sigma{=}2 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}2 & \nu\to\mu \\ \sigma{=}3 & (4{\times}4)\downarrow\end{matrix} \\ \hdashline
\begin{matrix}\lambda{=}3 & \nu\to\mu \\ \sigma{=}0 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}3 & \nu\to\mu \\ \sigma{=}1 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}3 & \nu\to\mu \\ \sigma{=}2 & (4{\times}4)\downarrow\end{matrix} & \begin{matrix}\lambda{=}3 & \nu\to\mu \\ \sigma{=}3 & (4{\times}4)\downarrow\end{matrix}
\end{array}\right)}$$

Define $\varepsilon^{\lambda\sigma}$ = totally anti-symmetric rank-two tensor: $\boxed{\varepsilon^{\lambda\sigma} = \begin{pmatrix} 0 & +1 & +1 & +1 \\ -1 & 0 & +1 & +1 \\ -1 & -1 & 0 & +1 \\ -1 & -1 & -1 & 0 \end{pmatrix}}$

Thus, we can define $\varepsilon^{\mu\nu\lambda\sigma}$ in terms of product of two $\varepsilon^{\lambda\sigma}$'s: $\boxed{\varepsilon^{\mu\nu\lambda\sigma} = \varepsilon^{\lambda\mu}\varepsilon^{\lambda\sigma}}$

---

## The Minkowski / Proper Force on a Point Electric Charge

The Minkowski force (*a.k.a.* proper force) $K^\mu$ acting on a point electric charge $q$ can be written in 4-vector / tensor notation in terms of the *EM* field tensor $F^{\mu\nu}$ and the proper 4-velocity $\eta^\mu$. Recall that:

$$\boxed{K^\mu \equiv \frac{dp^\mu}{d\tau} = \gamma_u \frac{dp^\mu}{dt} = \gamma_u F^\mu}$$ where the ordinary force: $\boxed{F^\mu = \frac{dp^\mu}{dt}}$ and: $\boxed{\gamma_u \equiv 1\Big/\sqrt{1-\beta_u^2}}$

However, we can *equivalently* write the Minkowski/proper force as: $\boxed{K^\mu = q\eta_\nu F^{\mu\nu}}$
where $\eta_\nu$ is the covariant form of the contravariant proper 4-velocity $\eta^\nu$.
i.e. we *contract* the *EM* field tensor $F^{\mu\nu}$ with the covariant proper 4-velocity $\eta_\nu$.

Since: $\boxed{K^\mu = \gamma_u F^\mu}$ and: $\boxed{\eta_\nu = \gamma_u u_\nu}$, where: $\boxed{\gamma_u \equiv 1\Big/\sqrt{1-\beta_u^2}}$ and: $\boxed{\beta_u = u/c}$

$\therefore$ $\boxed{K^\mu = q\eta_\nu F^{\mu\nu}} \Rightarrow \boxed{\gamma_u F^\mu = q\gamma_u u_\nu F^{\mu\nu}}$ or: $\boxed{F^\mu = q u_\nu F^{\mu\nu}}$

If $\mu = 1$ (i.e. row #1): $\boxed{K^1 = q\eta_\nu F^{1\nu} = q\left(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}\right)}$

$\boxed{u_\nu \equiv (c,\vec{u})} \leftrightarrow \boxed{\eta_\nu \equiv (\gamma_u c, \gamma_u \vec{u}) = \gamma_u u_\nu}$ where: $\boxed{\gamma_u = 1\Big/\sqrt{1-\beta_u^2}}$ and: $\boxed{\beta = u/c}$

Row #

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Column #

$$\therefore \boxed{K^1 = q\gamma_u\left[-c(-E_x/c) + u_x \cdot 0 + u_y B_z + u_z(-B_y)\right] = q\gamma_u\left[E_x + u_y B_z - u_z B_y\right] = q\gamma_u\left(\vec{E} + \vec{u}\times\vec{B}\right)_x}$$

$$\boxed{K^1 = q\gamma_u\left(\vec{E} + \vec{u}\times\vec{B}\right)_x}$$

Similarly, for $\mu = 2$, $\mu = 3$:

$$\boxed{K^2 = q\gamma_u\left(\vec{E} + \vec{u}\times\vec{B}\right)_y}$$

$$\boxed{K^3 = q\gamma_u\left(\vec{E} + \vec{u}\times\vec{B}\right)_z}$$

$\boxed{\vec{K} = q\gamma_u\left(\vec{E} + \vec{u}\times\vec{B}\right)}$ ← Minkowski 3-D Force Law

But: $\boxed{\vec{K} = \gamma_u \vec{F}}$

$\therefore$ $\boxed{\vec{F} = q\left(\vec{E} + \vec{u}\times\vec{B}\right)}$ ← Lorentz 3-D Force Law

For $\mu = 0$ (the temporal / scalar component) {see/work Griffiths Problem 12.54, page 541}:

$$\boxed{\begin{aligned} K^0 &= q\gamma_u\left(-c\bullet 0 + u_x\left(E_x/c\right) + u_y\left(E_x/c\right) + u_z\left(E_x/c\right)\right) = q\gamma_u \vec{u}\bullet\vec{E}/c \\ &= \quad -\eta^0 F^{00} + \quad \eta^1 F^{01} \quad + \quad \eta^2 F^{02} \quad + \quad \eta^3 F^{03} = \eta_\nu F^{0\nu} \end{aligned}}$$

n.b. this relation explicitly shows that $E_x$, $E_y$, $E_z$ are <u>temporal-spatial</u> (or <u>spatial-temporal</u>) components of $F^{\mu\nu}$, whereas $B_x$, $B_y$, $B_z$ are pure <u>spatial-spatial</u> components of $F^{\mu\nu}$ !!!

We also know that: $\boxed{K^0 \equiv \dfrac{dp^0}{d\tau} = \dfrac{1}{c}\dfrac{dE}{d\tau} = \dfrac{1}{c}\dfrac{dt}{d\tau}\dfrac{dE}{dt} = \dfrac{1}{c}\gamma_u\dfrac{dE}{dt}}$ $\therefore$ $\boxed{K^0 = q\gamma_u\vec{u}\bullet\vec{E}/c = \dfrac{1}{c}\gamma_u\dfrac{dE}{dt}}$

or: $\boxed{P_q = \dfrac{dE}{dt} = q\left(\vec{u}\bullet\vec{E}\right)}$ = {ordinary} relativistic power delivered to point electric charged particle (> 0)

$\boxed{P_q = \dfrac{dE}{dt} = q\left(\vec{u}\bullet\vec{E}\right) = \left(q\vec{E}\right)\bullet\vec{u} = \vec{F}\bullet\vec{u} = \dfrac{dW}{dt}}$ = Time rate of change of work done on changed particle by *EM* field

<u>Note:</u> The {ordinary} Lorentz force $\boxed{\vec{F} = q\vec{E} + q\left(\vec{u}\times\vec{B}\right)}$

$$\therefore \quad \boxed{\vec{F}\bullet\vec{u} = q\left(\vec{u}\bullet\vec{E}\right) + q\left[\vec{u}\bullet\left(\vec{u}\times\vec{B}\right)\right] = q\left(\vec{u}\bullet\vec{E}\right)}$$

<u>But</u>: $\boxed{\left(\vec{u}\times\vec{B}\right) \perp \vec{u}}$ $\therefore$ $\boxed{\vec{u}\bullet\left(\vec{u}\times\vec{B}\right) \equiv 0}$ $\Rightarrow$ <u>Magnetic Forces do no work</u> !!!

Thus we have the relations: $\boxed{K^\mu = \gamma_u F^\mu} = \boxed{K^\mu = q\eta_\nu F^{\mu\nu}}$ and also: $\boxed{F^\mu = qu_\nu F^{\mu\nu}}$ with $\boxed{\eta_\nu = \gamma_u u_\nu}$.

---

## <u>**The Relativistic 4-Vector Potential**</u> $A^\mu$

We know that the electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential $V$ and a vector potential $\vec{A}$ as:

$$\boxed{\vec{E}\left(\vec{r},t\right) = -\vec{\nabla}V\left(\vec{r},t\right) - \frac{\partial\vec{A}\left(\vec{r},t\right)}{\partial t}} \quad \underline{\text{and}}: \quad \boxed{\vec{B}\left(\vec{r},t\right) = \vec{\nabla}\times\vec{A}\left(\vec{r},t\right)}$$

Thus, it should not be surprising to realize that the scalar potential $V$ and the vector potential $\vec{A}$ form the temporal and spatial components (respectively) of the relativistic 4-vector potential $A^\mu$:

The 4-Vector Potential: $\boxed{A^\mu \equiv \left(V/c, \vec{A}\right) = \left(V/c, A_x, A_y, A_z\right)}$ SI Units: Newtons/Amp = "*p*/*q*" {momentum per Coulomb!}

n.b. SI units of $V$: $\boxed{\text{Volts} = \dfrac{\text{Newton-meters}}{\text{Coulomb}}}$ <u>then</u>: $\boxed{\dfrac{V}{c} = \dfrac{\text{N-m}}{\text{Coul}}\Big/\dfrac{\text{m}}{\text{sec}} = \dfrac{\text{N-sec}}{\text{Coul}} = \dfrac{\text{Newtons}}{\text{Amp}}}$

The *EM* field tensor $F^{\mu\nu}$ can be written in terms of covariant space-time derivatives of the 4-vector potential field $A^{\mu}$ as:

$$\boxed{F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}} \Leftarrow \text{ n.b. covariant differentiation here!!}$$

For the covariant derivatives $\boxed{\frac{\partial}{\partial x_{\mu}} \text{ and } \frac{\partial}{\partial x_{\nu}}}$ we need to change the sign of the temporal / scalar component relative to the contravariant derivatives $\boxed{\frac{\partial}{\partial x^{\mu}} \text{ and } \frac{\partial}{\partial x^{\nu}}}$.

Explicitly evaluate a few terms: $\boxed{A^{\mu} = \left(V/c, \vec{A}\right)}$

For $\mu = 0$ and $\nu = 1$:

$$\boxed{F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial(ct)} - \frac{\partial(V/c)}{\partial x} = -\frac{1}{c}\left(\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} V\right)_x = \frac{1}{c}\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right)_x = \frac{E_x}{c}}$$

Likewise, for ($\mu = 0$, $\nu = 2$) and ($\mu = 0$, $\nu = 3$) we obtain:

$$\boxed{F^{02} = \frac{1}{c}\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right)_y = \frac{E_y}{c}} \quad \text{and:} \quad \boxed{F^{03} = \frac{1}{c}\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right)_z = \frac{E_z}{c}}$$

For $\mu = 1$ and $\nu = 2$:

$$\boxed{F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \left(\vec{\nabla} \times \vec{A}\right)_z = B_z}$$

Likewise, for ($\mu = 1$, $\nu = 3$) and ($\mu = 2$, $\nu = 3$) we obtain:

$$\boxed{F^{13} = \left(\vec{\nabla} \times \vec{A}\right)_y = B_y} \quad \text{and:} \quad \boxed{F^{23} = \left(\vec{\nabla} \times \vec{A}\right)_x = B_x}$$

Note that the relativistic 4-potential formulation *automatically* takes care of the homogeneous Maxwell equation $\boxed{\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0}$ {it gives $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$ and: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$} because $\boxed{\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0}$ is equivalent to $\boxed{\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} = 0}$.

{See/read pages 10-11 of these lecture notes – also see/work Griffiths Problem 12.53, page 541}.

And since: $\boxed{F^{\mu\nu} = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}\right)} \Rightarrow \boxed{F_{\mu\nu} = \left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}\right)}$

Thus: $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\dfrac{\partial F_{\nu\lambda}}{\partial x^{\mu}}+\dfrac{\partial F_{\lambda\mu}}{\partial x^{\nu}}=0}$

$$=\boxed{\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial A_{\nu}}{\partial x^{\mu}}-\frac{\partial A_{\mu}}{\partial x^{\nu}}\right)+\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial A_{\lambda}}{\partial x^{\nu}}-\frac{\partial A_{\nu}}{\partial x^{\lambda}}\right)+\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial A_{\mu}}{\partial x^{\lambda}}-\frac{\partial A_{\lambda}}{\partial x^{\mu}}\right)=0}$$

$$=\boxed{\frac{\partial}{\partial x^{\lambda}}\frac{\partial A_{\nu}}{\partial x^{\mu}}-\frac{\partial}{\partial x^{\lambda}}\frac{\partial A_{\mu}}{\partial x^{\nu}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial A_{\lambda}}{\partial x^{\nu}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial A_{\nu}}{\partial x^{\lambda}}+\frac{\partial}{\partial x^{\nu}}\frac{\partial A_{\mu}}{\partial x^{\lambda}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial A_{\lambda}}{\partial x^{\mu}}=0}$$

$$=\boxed{\left(\frac{\partial}{\partial x^{\lambda}}\frac{\partial A_{\nu}}{\partial x^{\mu}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial A_{\nu}}{\partial x^{\lambda}}\right)+\left(\frac{\partial}{\partial x^{\nu}}\frac{\partial A_{\mu}}{\partial x^{\lambda}}-\frac{\partial}{\partial x^{\lambda}}\frac{\partial A_{\mu}}{\partial x^{\nu}}\right)+\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial A_{\lambda}}{\partial x^{\nu}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial A_{\lambda}}{\partial x^{\mu}}\right)=0}$$

$$=\boxed{\underbrace{\left(\frac{\partial}{\partial x^{\lambda}}\frac{\partial}{\partial x^{\mu}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\lambda}}\right)}_{=0}A_{\nu}+\underbrace{\left(\frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\lambda}}-\frac{\partial}{\partial x^{\lambda}}\frac{\partial}{\partial x^{\nu}}\right)}_{=0}A_{\mu}+\underbrace{\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial}{\partial x^{\mu}}\right)}_{=0}A_{\lambda}=0}$$

But: $\boxed{\dfrac{\partial}{\partial x^{\lambda}}\dfrac{\partial}{\partial x^{\mu}}=\dfrac{\partial}{\partial x^{\mu}}\dfrac{\partial}{\partial x^{\lambda}}}$ i.e. can change the order of differentiation – no effect!

$=\boxed{\dfrac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}=\dfrac{\partial^2}{\partial x^{\mu}\partial x^{\lambda}}}$ $\therefore$ $\boxed{0=0}$

$\therefore$ The relativistic 4-potential formulation $\boxed{F^{\mu\nu}=\left(\dfrac{\partial A^{\nu}}{\partial x_{\mu}}-\dfrac{\partial A^{\mu}}{\partial x_{\nu}}\right)}$ does indeed automatically satisfy $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^{\nu}}=0}$ because $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\dfrac{\partial F_{\nu\lambda}}{\partial x^{\mu}}+\dfrac{\partial F_{\lambda\mu}}{\partial x^{\nu}}=0}$ (shown to be equivalent to $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^{\nu}}=0}$) is satisfied / obeyed for $\boxed{F^{\mu\nu}=\left(\dfrac{\partial A^{\nu}}{\partial x_{\mu}}-\dfrac{\partial A^{\mu}}{\partial x_{\nu}}\right)}$.

Does the relativistic 4-potential formulation $\boxed{F^{\mu\nu}=\left(\dfrac{\partial A^{\nu}}{\partial x_{\mu}}-\dfrac{\partial A^{\mu}}{\partial x_{\nu}}\right)}$ satisfy the inhomogeneous Maxwell relation $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x^{\nu}}=\mu_o J^{\mu}}$ ???

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x^{\nu}}=\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial A^{\nu}}{\partial x_{\mu}}-\frac{\partial A^{\mu}}{\partial x_{\nu}}\right)=\frac{\partial^2 A^{\nu}}{\partial x^{\nu}\partial x_{\mu}}-\frac{\partial^2 A^{\mu}}{\partial x^{\nu}\partial x_{\nu}}=\mu_o J^{\mu}}$$

Switching the order of derivatives:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x^{\nu}}=\underbrace{\frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right)-\frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right)}=\mu_o J^{\mu}}$$

This is an intractable equation, as it stands now…

However, from our formulation of $F^{\mu\nu}$ in terms of (differences) in space-time derivatives of the 4-vector potential $A^{\mu}$: $\boxed{F^{\mu\nu} = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}\right)}$ it is clear that we can add to the 4-vector potential $A^{\mu}$ the space-time gradient of *any* scalar function $\lambda$: $\boxed{A^{\mu} \to A^{*\mu} = A^{\mu} + \frac{\partial \lambda}{\partial x_{\mu}}}$ !!!

The scalar and vector potentials $V$ and $\vec{A}$ are *not* uniquely determined by the *EM* fields $\vec{E}$ and $\vec{B}$.

Thus:

$$F^{*\mu\nu} = \left(\frac{\partial A^{*\nu}}{\partial x_{\mu}} - \frac{\partial A^{*\mu}}{\partial x_{\nu}}\right) = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} + \frac{\partial}{\partial x_{\mu}}\frac{\partial \lambda}{\partial x_{\nu}}\right) - \left(\frac{\partial A^{\mu}}{\partial x_{\nu}} + \frac{\partial}{\partial x_{\nu}}\frac{\partial \lambda}{\partial x_{\mu}}\right) = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}\right) + \underbrace{\left(\cancel{\frac{\partial^2 \lambda}{\partial x_{\mu}\partial x_{\nu}}} - \cancel{\frac{\partial^2 \lambda}{\partial x_{\nu}\partial x_{\mu}}}\right)}_{=0}$$

$$= \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}\right) = F^{\mu\nu} \text{ !!!}$$

$\therefore \boxed{F^{*\mu\nu} = F^{\mu\nu}}$ by $\boxed{A^{\mu} \to A^{*\mu} = A^{\mu} + \frac{\partial \lambda}{\partial x_{\mu}}}$ is <u>gauge</u> <u>invariance</u> associated with the *EM* field $F^{\mu\nu}$ !!!

We can *<u>exploit</u>* the gauge invariant properties of $F^{\mu\nu}$ to simplify the seemingly intractable relation:

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right) - \frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right) = \mu_o J^{\mu}}$$

Using the <u>Lorenz gauge condition</u>: $\boxed{\vec{\nabla}\bullet\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}} \Rightarrow \boxed{\vec{\nabla}\bullet\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0} \Rightarrow \boxed{\frac{\partial A^{\mu}}{\partial x^{\mu}} = 0}$

<u>We see that</u>: $\boxed{\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \frac{\partial}{\partial x_{\mu}}\overbrace{\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right)}^{=0!!} - \frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right) = \mu_o J^{\mu}} = \boxed{-\frac{\partial^2 A^{\mu}}{\partial x_{\nu}\partial x^{\nu}} = \mu_o J^{\mu}}$ <u>or</u>: $\boxed{\frac{\partial^2 A^{\mu}}{\partial x_{\nu}\partial x^{\nu}} = -\mu_o J^{\mu}}$

<u>But</u>: $\boxed{\Box^{\nu} = \frac{\partial}{\partial x^{\nu}}, \quad \Box_{\nu} = \frac{\partial}{\partial x_{\nu}}}$ <u>and</u>: $\boxed{\Box^2 \equiv \Box^{\nu}\Box_{\nu} = \Box_{\nu}\Box^{\nu} = \frac{\partial^2}{\partial x_{\nu}\partial x^{\nu}} = \frac{\partial^2}{\partial x^{\nu}\partial x_{\nu}} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}$

D'Alembertian operator (4-dimensional Laplacian operator)

$\therefore \boxed{\Box^2 A^{\mu} = -\mu_o J_{\mu}} = \boxed{\frac{\partial^2 A^{\mu}}{\partial x_{\nu}\partial x^{\nu}} = -\mu_o J^{\mu}} \Leftarrow$ Single 4-vector equation!

= ***<u>The</u>*** most elegant and simple formulation of Maxwell's equations – it contains <u>all four</u> of Maxwell's equations!!

Taken together with the continuity equation (charge conservation): $\boxed{\Box^{\mu} J^{\mu} = 0}$, these two relations compactly describe virtually all of {non-matter/free space} *EM* phenomena!!!

Note that the choice of the (instantaneous) Coulomb gauge $\boxed{\vec{\nabla}\bullet\vec{A}=0}$ is a bad one for use in relativistic electrodynamics, because $\underline{\vec{\nabla}\bullet\vec{A}}$ {alone} is ***not*** a Lorentz invariant quantity!

However: $\boxed{\vec{\nabla}\bullet\vec{A}(\vec{r},t)+\frac{1}{c^2}\frac{\partial V(\vec{r},t)}{\partial t}=\frac{\partial A^\mu(\vec{r},t)}{\partial x^\mu}=0}$ *is* a Lorentz invariant quantity because it is the product of two relativistic 4-vectors: $\frac{\partial}{\partial x^\mu}$ and $A^\mu$.

n.b. $\boxed{\vec{\nabla}\bullet\vec{A}=0}$ is "destroyed" by *any* Lorentz transformation from one IRF($S$) to another IRF($S'$) !!!

$\Rightarrow$ In order to *restore* $\boxed{\vec{\nabla}\bullet\vec{A}=0}$, one must perform an appropriate gauge transformation for *each* new inertial system entered, in *addition* to carrying out the Lorentz transformation itself !!!

In the Coulomb gauge, $A^\mu$ is *not* a "true" relativistic 4-vector, because $\boxed{A^\mu A_\mu = A_\mu A^\mu}$ is *not* a Lorentz invariant quantity in the Coulomb gauge !!!

n.b. The Coulomb gauge $\boxed{\vec{\nabla}\bullet\vec{A}=0}$ *is* useful when $\underline{v \ll c}$, i.e. for non-relativistic problems.