LECTURE NOTES 10.5

EM Standing Waves in Resonant Cavities

- One can create a resonant cavity for *EM* waves by taking a waveguide (of arbitrary shape) and closing/capping off the two open ends of the waveguide.
- Standing *EM* waves exist in (excited) resonant cavity (= linear superposition of two counter-propagating traveling EM waves of same frequency).
- Analogous to standing acoustical/sound waves in an acoustical enclosure.
- Rectangular resonant cavity – use Cartesian coordinates
- Cylindrical resonant cavity – use cylindrical coordinates
- Spherical resonant cavity – use spherical coordinates

} to solve the *EM* wave eqn.

A.) Rectangular Resonant Cavity ($L \times W \times H = a \times b \times d$) with perfectly conducting walls (*i.e.* no dissipation/energy loss mechanisms present), with $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le d$.

n.b. Again, by convention: $a > b > d$.

Since we have rectangular symmetry, we use Cartesian coordinates - seek monochromatic *EM* wave solutions of the general form:

$$\boxed{\begin{aligned}\tilde{\vec{E}}(x,y,z,t) &= \tilde{\vec{E}}_o(x,y,z)e^{-i\omega t}\\ \tilde{\vec{B}}(x,y,z,t) &= \tilde{\vec{B}}_o(x,y,z)e^{-i\omega t}\end{aligned}}$$

Subject to the boundary conditions $E_{\parallel} = 0$ and $B_{\perp} = 0$ at all inner surfaces.

Maxwell's Equations (inside the rectangular resonant cavity – away from the walls):

(1) Gauss' Law:

$$\boxed{\vec{\nabla}\cdot\vec{E} = 0} \Rightarrow \boxed{\vec{\nabla}\cdot\tilde{\vec{E}}_o = 0}$$

(2) No Monopoles:

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \Rightarrow \boxed{\vec{\nabla}\cdot\tilde{\vec{B}}_o = 0}$$

(3) Faraday's Law:

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \Rightarrow \boxed{\vec{\nabla}\times\tilde{\vec{E}}_o = i\omega\tilde{\vec{B}}_o}$$

(4) Ampere's Law:

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}} \Rightarrow \boxed{\vec{\nabla}\times\tilde{\vec{B}}_o = -i\frac{\omega}{c^2}\tilde{\vec{E}}_o}$$

Take the curl of (3): = 0 {Gauss' Law}

$$\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\tilde{\vec{E}}_o\right) = i\omega\vec{\nabla}\times\tilde{\vec{B}}_o = \vec{\nabla}\times\left(\cancel{\vec{\nabla}\cdot\tilde{\vec{E}}_o}\right) - \nabla^2\tilde{\vec{E}}_o = i\omega\left(\vec{\nabla}\times\tilde{\vec{B}}_o\right) = \left(\frac{\omega}{c}\right)^2\tilde{\vec{E}}}$$ {using (4) Ampere's Law}

$$\Rightarrow \boxed{\begin{aligned}\nabla^2\tilde{E}_{ox} &= -\left(\frac{\omega}{c}\right)^2\tilde{E}_{ox}\\ \nabla^2\tilde{E}_{oy} &= -\left(\frac{\omega}{c}\right)^2\tilde{E}_{oy}\\ \nabla^2\tilde{E}_{oz} &= -\left(\frac{\omega}{c}\right)^2\tilde{E}_{oz}\end{aligned}}$$

$$\boxed{\left\{\begin{aligned}\tilde{E}_{ox} &= \tilde{E}_{ox}(x,y,z)\\ \tilde{E}_{oy} &= \tilde{E}_{oy}(x,y,z)\\ \tilde{E}_{oz} &= \tilde{E}_{oz}(x,y,z)\end{aligned}\right\}\text{ i.e. each is a fcn}(x,y,z)}$$

For each component $\{x, y, z\}$ of $\tilde{\vec{E}}_o(x, y, z)$ we try product solutions and then use the separation of variables technique:

$\boxed{\tilde{E}_{o_i}(x,y,z) \equiv X_i(x) Y_i(y) Z_i(z)}$ for $\boxed{\nabla^2 \tilde{E}_{o_i}(x,y,z) = -\left(\frac{\omega}{c}\right)^2 \tilde{E}_{o_i}}$ where subscript $\boxed{i = x, y, z}$.

$$\boxed{Y_i(y) Z_i(z) \frac{\partial^2 X_i(x)}{\partial x^2} + X_i(x) Z_i(z) \frac{\partial^2 Y_i(y)}{\partial^2 y} + X_i(x) Y_i(y) \frac{\partial^2 Z_i(z)}{\partial^2 z} = -\left(\frac{\omega c}{c}\right)^2 X_i(x) Y_i(y) Z_i(z)}$$

Divide both sides by $\boxed{X_i(x) Y_i(y) Z_i(z)}$:

The wave equation becomes: $$\boxed{\underbrace{\frac{1}{X_i(x)} \frac{\partial^2 X_i(x)}{\partial x^2}}_{fcn(x)\text{only}} + \underbrace{\frac{1}{Y_i(y)} \frac{\partial^2 Y_i(y)}{\partial y^2}}_{= fcn(y)\text{only}} + \underbrace{\frac{1}{Z_i(z)} \frac{\partial^2 Z_i(z)}{\partial z^2}}_{= fcn(z)\text{only}} = -\left(\frac{\omega}{c}\right)^2}$$

This equation must hold/be true for arbitrary (x, y, z) pts. interior to resonant cavity $\left\{\begin{matrix} 0 \le x \le a \\ 0 \le y \le b \\ 0 \le z \le d \end{matrix}\right\}$

This can only be true if:

$$\boxed{\frac{1}{X_i(x)} \frac{\partial^2 X_i(x)}{\partial x^2} = -k_x^2 = \text{constant}} \Rightarrow \boxed{\frac{\partial^2 X_i(x)}{\partial x^2} + k_x^2 X_i(x) = 0}$$

$$\boxed{\frac{1}{Y_i(y)} \frac{\partial^2 Y_i(y)}{\partial y^2} = -k_y^2 = \text{constant}} \Rightarrow \boxed{\frac{\partial^2 Y_i(y)}{\partial y^2} + k_y^2 Y_i(y) = 0}$$

$$\boxed{\frac{1}{Z_i(z)} \frac{\partial^2 Z_i(z)}{\partial z^2} = -k_z^2 = \text{constant}} \Rightarrow \boxed{\frac{\partial^2 Z_i(z)}{\partial z^2} + k_z^2 Z_i(z) = 0}$$

n.b. We want oscillatory (not damped) solutions !!!

with: $\boxed{k^2 \equiv k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2}$ $\Leftarrow$ characteristic equation

General solution(s) are of the form: $(i = x, y, z)$:

$$\boxed{\tilde{E}_{o_i}(x,y,z) = \left[A_i \cos(k_x x) + B_i \sin(k_x x)\right] \times \left[C_i \cos(k_y y) + D_i \sin(k_y y)\right] \times \left[E_i \cos(k_z z) + F_i \sin(k_z z)\right]}$$

n.b. In general, k_x, k_y and k_z should each have subscript $i = x, y, z,$ but we will shortly find out that k_{x_i} = same for all $i = x, y, z,$ k_{y_i} = same for all $i = x, y, z,$ and k_{z_i} = same for all $i = x, y, z$.

Boundary Conditions: $\boxed{E_{\parallel}=0}$ @ boundaries and $\boxed{B_{\perp}=0}$ @ boundaries:

$$\boxed{E_{ox}=0 \text{ at } \begin{Bmatrix} y=0, y=b \\ z=0, z=d \end{Bmatrix}} \Rightarrow \text{coefficients } \boxed{\begin{Bmatrix} \tilde{E}_x=0 \\ \tilde{C}_x=0 \end{Bmatrix}} \text{ and } \boxed{\begin{Bmatrix} k_y=n\pi/b,\ n=1,2,3,.... \\ k_z=\ell\pi/d,\ \ell=1,2,3,.... \end{Bmatrix}}$$

$$\boxed{E_{oy}=0 \text{ at } \begin{Bmatrix} x=0, x=a \\ z=0, z=d \end{Bmatrix}} \Rightarrow \text{coefficients } \boxed{\begin{Bmatrix} \tilde{A}_y=0 \\ \tilde{E}_y=0 \end{Bmatrix}} \text{ and } \boxed{\begin{Bmatrix} k_x=m\pi/b,\ m=1,2,3,.... \\ k_z=\ell\pi/d,\ \ell=1,2,3,.... \end{Bmatrix}}$$

$$\boxed{E_{oz}=0 \text{ at } \begin{Bmatrix} x=0, x=a \\ y=0, y=b \end{Bmatrix}} \Rightarrow \text{coefficients } \boxed{\begin{Bmatrix} \tilde{A}_z=0 \\ \tilde{C}_z=0 \end{Bmatrix}} \text{ and } \boxed{\begin{Bmatrix} k_x=m\pi/a,\ m=1,2,3,.... \\ k_y=n\pi/d,\ n=1,2,3,.... \end{Bmatrix}}$$

n.b. $\underline{m=0}$, and/or $\underline{n=0}$ and/or $\underline{\ell=0}$ are not allowed, otherwise $\boxed{\tilde{E}_{o_i}(x,y,z)\equiv 0}$ (trivial solution).

<u>Thus we have (absorbing constants/coefficients, & dropping *x,y,z* subscripts on coefficients)</u>:

$$\boxed{\begin{aligned} &\tilde{E}_{ox}(x,y,z)=\left[\tilde{A}\cos(k_x x)+\tilde{B}\sin(k_x x)\right]\sin(k_y y)\sin(k_z z) \\ &\tilde{E}_{oy}(x,y,z)=\sin(k_x x)\sin(k_y y)\left[\tilde{C}\cos(k_y y)+\tilde{D}\sin(k_y y)\right]\sin(k_z z) \\ &\tilde{E}_{ox}(x,y,z)=\sin(k_x x)\sin(k_y y)\left[\tilde{E}\cos(k_z z)+\tilde{F}\sin(k_z z)\right] \end{aligned}}$$

But (1) Gauss' Law: $\boxed{\vec{\nabla}\bullet\tilde{\vec{E}}=0} \Rightarrow \boxed{\dfrac{\partial\tilde{E}_{ox}}{\partial x}+\dfrac{\partial\tilde{E}_{oy}}{\partial y}+\dfrac{\partial\tilde{E}_{oz}}{\partial z}=0}$

<u>Thus</u>:

$$\boxed{\begin{aligned} &k_x\left[-\tilde{A}\sin(k_x x)+\tilde{B}\cos(k_x x)\right]\sin(k_y y)\sin(k_z z) \\ +\ &k_y\sin(k_x x)\left[-\tilde{C}\sin(k_y y)+\tilde{D}\cos(k_y y)\right]\sin(k_z z) \\ +\ &k_z\sin(k_x x)\sin(k_y y)\left[-\tilde{E}\sin(k_z z)+\tilde{F}\cos(k_z z)\right]=0 \end{aligned}}$$

This equation <u>must</u> be satisfied for any/<u>all</u> points inside rectangular cavity resonator.
In particular, it has to be satisfied at $(x,y,z)=\underline{\underline{(0,0,0)}}$.

We see that for the locus of points associated with ($x=0,y,z$) and ($x,y=0,z$) and ($x,y,z=0$), we <u>must</u> have $\boxed{\tilde{B}=\tilde{D}=\tilde{F}=0}$ in the above equation.

Note also that for the locus of points associated with $(x=m\pi/2k_x\,,y,z)$ and $(x,y=n\pi/2k_y\,,z)$ and $(x,y,z=\ell\pi/2k_z)$ where $m,n,\ell=$ <u>odd</u> integers (1, 3, 5, 7, *etc*. …) we must have: $\boxed{\tilde{A}k_x+\tilde{C}k_y+\tilde{E}k_z=0}$. Note further that this relation is <u>*automatically*</u> satisfied for $m,n,\ell=$ <u>even</u> integers (2, 4, 6, 8, *etc*. …).

Thus:
$$\boxed{\left.\begin{array}{l}\tilde{E}_{ox}(x,y,z)=\tilde{A}\cos(k_x x)\sin(k_y y)\sin(k_z z)\\ \tilde{E}_{oy}(x,y,z)=\tilde{C}\sin(k_x x)\cos(k_y y)\sin(k_z z)\\ \tilde{E}_{oz}(x,y,z)=\tilde{E}\sin(k_x x)\sin(k_y y)\sin(k_z z)\end{array}\right\}\left\{\begin{array}{ll}k_x=\left(\frac{m\pi}{a}\right) & m=1,2,3,4,\ldots\\ k_y=\left(\frac{n\pi}{b}\right) & n=1,2,3,4,\ldots\\ k_z=\left(\frac{\ell\pi}{d}\right) & \ell=1,2,3,4,\ldots\end{array}\right.}$$

With: $\tilde{E}_o(x,y,z)=\tilde{E}_{ox}\hat{x}+\tilde{E}_{oy}\hat{y}+\tilde{E}_{oz}\hat{z}$ *n.b.*: $m=n=\ell=0$ simultaneously is not allowed!

Now use Faraday's Law to determine $\vec{B}$:

$$\boxed{\vec{B}=-\frac{i}{\omega}\left(\vec{\nabla}\times\vec{E}\right)}$$

$$\boxed{\tilde{B}_{ox}=-\frac{i}{\omega}\left(\frac{\partial\tilde{E}_{oz}}{\partial y}-\frac{\partial\tilde{E}_{oy}}{\partial z}\right)=-\frac{i}{\omega}\left[\tilde{E}k_y\sin(k_x x)\cos(k_y y)\cos(k_z z)-\tilde{C}k_z\sin(k_x x)\cos(k_y y)\cos(k_z z)\right]}$$

$$\boxed{\tilde{B}_{oy}=-\frac{i}{\omega}\left(\frac{\partial\tilde{E}_{ox}}{\partial z}-\frac{\partial\tilde{E}_{oz}}{\partial x}\right)=-\frac{i}{\omega}\left[\tilde{A}k_z\cos(k_x x)\sin(k_y y)\cos(k_z z)-\tilde{E}k_x\cos(k_x x)\sin(k_y y)\cos(k_z z)\right]}$$

$$\boxed{\tilde{B}_{oz}=-\frac{i}{\omega}\left(\frac{\partial\tilde{E}_{oy}}{\partial x}-\frac{\partial\tilde{E}_{ox}}{\partial y}\right)=-\frac{i}{\omega}\left[\tilde{C}k_x\cos(k_x x)\cos(k_y y)\sin(k_z z)-\tilde{A}k_y\cos(k_x x)\cos(k_y y)\sin(k_z z)\right]}$$

OR:
$$\boxed{\begin{aligned}\tilde{\vec{B}}_o(x,y,z)&=\tilde{B}_{ox}\hat{x}+\tilde{B}_{oy}\hat{y}+\tilde{B}_{oz}\hat{z}\\ &=-\frac{i}{\omega}\Big\{\left(\tilde{E}k_y-\tilde{C}k_z\right)\sin(k_x x)\cos(k_y y)\cos(k_z z)\,\hat{x}\\ &\qquad+\left(\tilde{A}k_z-\tilde{E}k_x\right)\cos(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{y}\\ &\qquad+\left(\tilde{C}k_x-\tilde{A}k_y\right)\cos(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{y}\Big\}\end{aligned}}$$

This expression for $\tilde{\vec{B}}_o(x,y,z)$ (already) automatically satisfies boundary condition (2) $\boxed{B_\perp=0}$:

$B_{ox}=0$ at $x=0, x=a$	$B_{oy}=0$ at $y=0, y=b$	$B_{oz}=0$ at $z=0, z=d$
with $k_x\equiv\left(\frac{m\pi}{a}\right)$	with $k_y\equiv\left(\frac{n\pi}{b}\right)$	with $k_z\equiv\left(\frac{\ell\pi}{d}\right)$
$m=0,1,2,\ldots$	$n=0,1,2,\ldots$	$\ell=0,1,2,\ldots$

Does $\boxed{\vec{\nabla}\bullet\tilde{\vec{B}}_o(x,y,z)=0}$???

$$\vec{\nabla}\bullet\tilde{\vec{B}}_o(x,y,z)=\frac{\partial\tilde{B}_{ox}}{\partial x}+\frac{\partial\tilde{B}_{oy}}{\partial y}+\frac{\partial\tilde{B}_{oz}}{\partial z}$$

$$=-\frac{i}{\omega}\Big\{k_x\left(\tilde{E}k_y-\tilde{C}k_z\right)\cos(k_xx)\cos(k_yy)\cos(k_zz)$$
$$+k_y\left(\tilde{A}k_z-\tilde{E}k_x\right)\cos(k_xx)\cos(k_yy)\cos(k_zz)$$
$$+k_z\left(\tilde{C}k_x-\tilde{A}k_y\right)\cos(k_xx)\cos(k_yy)\cos(k_zz)\Big\}$$

$$=-\frac{i}{\omega}\Big\{\cancel{k_xk_y\tilde{E}}-\cancel{k_xk_z\tilde{C}}+\cancel{k_yk_z\tilde{A}}-\cancel{k_xk_y\tilde{E}}+\cancel{k_xk_z\tilde{C}}-\cancel{k_yk_z\tilde{A}}\Big\}\times\left[\cos(k_xx)\cos(k_yy)\cos(k_zz)\right]$$

$\therefore\ \boxed{\vec{\nabla}\bullet\tilde{\vec{B}}_o(x,y,z)=0}$ YES!!!

For TE modes:

$\boxed{\vec{E}_z=0}\Rightarrow$ coefficient $\boxed{\tilde{E}=0}$. Then $\boxed{\tilde{A}k_x+\tilde{C}k_y+\tilde{E}k_z=0}$ tells us that: $\boxed{\tilde{A}k_x+\tilde{C}k_y=0}$ or: $\boxed{\tilde{C}=-\tilde{A}\left(\frac{k_x}{k_y}\right)}$

$$\tilde{E}_{ox}(x,y,z)=\tilde{A}\cos(k_xx)\sin(k_yy)\sin(k_zz)$$
$$\tilde{E}_{oy}(x,y,z)=-\tilde{A}\left(\frac{k_x}{k_y}\right)\sin(k_xx)\cos(k_yy)\sin(k_zz)$$
$$\tilde{E}_{oz}(x,y,z)=0$$

$$k_x=\left(\frac{m\pi}{a}\right),\ m=1,2,\ldots$$
$$k_y=\left(\frac{n\pi}{b}\right),\ n=1,2,\ldots$$
$$k_z=\left(\frac{\ell\pi}{d}\right),\ \ell=1,2,\ldots$$

($n=0$ is NOT allowed for TE modes!!!)

$$\tilde{B}_{ox}(x,y,z)=-\frac{i}{\omega}\tilde{A}\left(\frac{k_x}{k_y}\right)k_z\sin(k_xx)\cos(k_yy)\cos(k_zz)$$
$$\tilde{B}_{oy}(x,y,z)=-\frac{i}{\omega}\tilde{A}\cos(k_xx)\sin(k_yy)\cos(k_zz)$$
$$\tilde{B}_{oz}(x,y,z)=+\frac{i}{\omega}\tilde{A}\left[\left(\frac{k_x}{k_y}\right)k_x+k_y\right]\cos(k_xx)\cos(k_yy)\sin(k_zz)$$

The lowest $TE_{m,n,\ell}$ mode $(a>b>d)$ is: TE_{111}

For TM modes:

$\boxed{\vec{B}_z = 0} \Rightarrow \boxed{\left(\tilde{C}k_x - \tilde{A}k_y\right) = 0}$ or: $\boxed{\tilde{C} = +\tilde{A}\left(\frac{k_y}{k_x}\right)}$

$$\boxed{\tilde{E}_{ox}(x,y,z) = \tilde{A}\cos(k_x x)\sin(k_y y)\sin(k_z z)} \quad \boxed{k_x = \left(\frac{m\pi}{a}\right), m = 1,2,\ldots}$$

$$\boxed{\tilde{E}_{oy}(x,y,z) = \tilde{A}\left(\frac{k_y}{k_x}\right)\sin(k_x x)\cos(k_y y)\cos(k_z z)} \quad \boxed{k_y = \left(\frac{n\pi}{b}\right), n = 1,2,\ldots}$$

($m = 0$ is NOT allowed for TM modes!!!) $\quad \boxed{k_z = \left(\frac{\ell\pi}{d}\right), \ell = 1,2,\ldots}$

$$\boxed{\tilde{E}_{oz}(x,y,z) = -\tilde{A}\left[\left(\frac{k_x}{k_z}\right) + \left(\frac{k_y}{k_x}\right)\left(\frac{k_y}{k_z}\right)\right]\sin(k_x x)\sin(k_y y)\cos(k_z z)}$$

$$\boxed{\tilde{B}_{ox}(x,y,z) = +\frac{i}{\omega}\left\{\tilde{A}\left[k_x + \left(\frac{k_y}{k_x}\right)k_y\right]\left(\frac{k_y}{k_z}\right) + \tilde{A}\left(\frac{k_y}{k_x}\right)k_z\right\}\sin(k_x x)\cos(k_y y)\cos(k_z z)}$$

$$\boxed{\tilde{B}_{oy}(x,y,z) = -\frac{i}{\omega}\left\{\tilde{A}k_z + \tilde{A}\left[k_x + \left(\frac{k_y}{k_x}\right)k_y\right]\left(\frac{k_x}{k_x}\right)\right\}\cos(k_x x)\sin(k_y y)\cos(k_z z)}$$

$$\boxed{\tilde{B}_{oz}(x,y,z) = 0}$$

The lowest $TM_{m,n,\ell}$ mode $(a > b > d)$ is: TM_{111}

For either TE or TM modes: $\boxed{k^2 \equiv k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2}$ with:

$$\boxed{k_x = \left(\frac{m\pi}{a}\right), m = 1,2,\ldots} \quad \boxed{k_y = \left(\frac{n\pi}{b}\right), n = 1,2,\ldots} \quad \boxed{k_z = \left(\frac{\ell\pi}{d}\right), \ell = 1,2,\ldots}$$

The angular cutoff frequency for m,n,ℓ^{th} mode is the same for TE/TM modes in a rectangular cavity:

$$\boxed{\omega_{mn\ell} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}}$$ and: $\boxed{v_{prop} = \frac{\omega}{k} = c = v_{phase}}$ no dispersion.

B.) The Spherical Resonant Cavity

The general problem of *EM* modes in a spherical cavity is mathematically considerably more involved (*e.g.* than for the rectangular cavity) due to the vectorial nature of the $\vec{E}$ and $\vec{B}$-fields. $\Rightarrow$ For simplicity's sake, it is conceptually easier to consider the scalar wave equation, with a scalar field $\psi(\vec{r},t)$ satisfying the free-source wave equation $\boxed{\nabla^2\psi(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t}=0}$

which can be Fourier-analyzed in the time domain $\boxed{\psi(\vec{r},t)=\int_{-\infty}^{\infty}\psi(\vec{r},\omega)\,e^{-i\omega t}d\omega}$ with each Fourier component $\psi(\vec{r},\omega)$ satisfying the Helmholtz Wave Equation: $\boxed{(\nabla^2+k^2)\psi(\vec{r},\omega)=0}$ with $\boxed{k^2=(\omega/c)^2}$ *i.e.* no dispersion.

In spherical coordinates the Laplacian operator is:

$$\boxed{\nabla^2\psi(\vec{r},\omega)=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\psi(\vec{r},\omega)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi(\vec{r},\omega)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi(\vec{r},\omega)}{\partial\varphi^2}}$$

To solve this scalar wave equation – we again try a product solution of the form:

$$\boxed{\psi(\vec{r},\omega)=\frac{R(r)}{r}P(\theta)Q(\varphi)e^{i\omega t}}\Rightarrow\boxed{\sum_{\ell,m}f_{\ell m}(r)\underbrace{Y_{\ell m}(\theta,\varphi)}_{\substack{\text{spherical}\\\text{harmonics}}}}$$

The $Y_{\ell m}(\theta,\varphi)$ satisfy the angular portion of scalar wave equation…

Plug this $\psi(\vec{r},\omega)$ into the above scalar wave equation, use the separation of variables technique:

Get radial equation: $\boxed{\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{\ell(\ell+1)}{r^2}\right]f_\ell(r)=0}$ where $\ell=0,1,2,\ldots$

Let $\boxed{f_\ell(r)=\frac{1}{\sqrt{r}}u_\ell(r)}$. Then we obtain Bessel's equation with index $\boxed{\nu=\ell+\tfrac{1}{2}}$:

$$\boxed{\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(\ell+\frac{1}{2})^2}{r^2}\right]u_\ell(r)=0}$$

Solutions of the (radial) Bessel's equation are of the form: $\boxed{f_{\ell m}(r)=\frac{A_{\ell m}}{\sqrt{r}}\underbrace{J_{\ell+\frac{1}{2}}(kr)}_{\substack{\text{Bessel fcn of 1st}\\\text{kind of order }\ell+\frac{1}{2}}}+\frac{B_{\ell m}}{\sqrt{r}}\underbrace{N_{\ell+\frac{1}{2}}(kr)}_{\substack{\text{Bessel fcn of 2nd}\\\text{kind of order }\ell+\frac{1}{2}}}}$

It is customary to define so-called spherical Bessel functions and spherical Hankel functions:

$$\boxed{j_\ell(x)\equiv\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}J_{\ell+\frac{1}{2}}(x)}\quad\text{where:}\quad\boxed{x=kr}$$

$$\boxed{n_\ell(x)\equiv\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}N_{\ell+\frac{1}{2}}(x)}$$

and: $$h_\ell^{(1,2)}(x) \equiv \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \left[J_{\ell+\frac{1}{2}}(x) \pm i N_{\ell+\frac{1}{2}}(x) \right] = j_\ell(x) \pm i\, n_\ell(x)$$

n.b. If $x = kr$ is real, then $h_\ell^{(2)}(x) = h_\ell^{*(1)}(x)$

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\sin(x)}{x}\right)$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\cos(x)}{x}\right)$$

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0^{(1)}(x) = \frac{e^{ix}}{ix}$$

$$n_0(x) = -\frac{\cos(x)}{x} \qquad h_0^{(2)}(x) = \frac{e^{-ix}}{-ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right)$$

$$n_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x} \qquad h_1^{(2)}(x) = -\frac{e^{-ix}}{x}\left(1 - \frac{i}{x}\right)$$

For $x \ll 1,\ \ell$:

$$j_\ell(x) \approx \frac{x^\ell}{(2\ell+1)!!}\left(1 - \frac{x^2}{2(2\ell+3)} + \dots\right)$$

where: $(2\ell+1)!! = (2\ell+1)(2\ell-1)(2\ell-3)\dots\times 5\times 3\times 1$

$$n_\ell(x) \approx -\frac{(2\ell-1)!!}{x^{\ell+1}}\left(1 - \frac{x^2}{2(1-2\ell)} + \dots\right)$$

For $x \gg 1,\ \ell$:

$$j_\ell(x) \approx \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$$

$$n_\ell(x) \approx -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)$$

The general solution to Helmholtz's equation in spherical coordinates can be written as:

$$\psi(\vec{r},t) = \sum_{\ell,m}\left[\underbrace{A_{\ell m}^{(1)}} h_\ell^{(1)}(kr) + \underbrace{A_{\ell m}^{(2)}} h_\ell^{(2)}(kr) \right] Y_{\ell m}(\theta,\varphi)$$

Coefficients are determined by boundary conditions.

For the case of *EM* waves in a spherical resonant cavity we will (here) only consider TM modes, which for spherical geometry means that the radial component of $\vec{B}$, $B_r = 0$. We further assume (for simplicity's sake) that the $\vec{E}$ and $\vec{B}$-fields do not have any explicit φ-dependence.

Hence: $$\boxed{Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1(\ell-m)!}{4\pi(\ell+m)!}}\,\underbrace{P_\ell^m(\cos\theta)}\,e^{im\phi}}$$

Will have some restrictions imposed on it Associated Legendré Polynomial

If $\boxed{B_r=0}$ and $\vec{B}\neq$ explicit function of φ, then: $\boxed{\vec{\nabla}\bullet\vec{B}=0}\Rightarrow\boxed{B_\varphi\neq0}$ {necessarily}

But: $\boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}}$ requires: $\boxed{E_\varphi=0}$

$\rightarrow$ TM modes with no explicit φ-dependence involve only E_r, E_θ and B_φ

Combining $\boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}}$ and $\boxed{\vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$ with harmonic time dependence $e^{-i\omega t}$ of solutions,

We obtain: $\boxed{\left(\frac{\omega}{c}\right)^2\vec{B}-\vec{\nabla}\times\vec{\nabla}\times\vec{B}=0}$

The φ-component of this equation is:

$$\boxed{\left(\frac{\omega}{c}\right)^2(rB_\varphi)+\frac{\partial^2}{\partial r^2}(rB_\varphi)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta rB_\varphi)\right]=0}$$

But: $$\boxed{\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta rB_\varphi)\right]=\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(rB_\varphi)}{\partial\theta}\right)-\frac{rB_\varphi}{\sin^2\theta}}_{\sim\text{Legendré equation with } m=\pm1}}$$

$\Rightarrow$ Try product solutions of the form: $\boxed{B_\varphi(r,\theta)=\frac{u_\ell(r)}{r}P_\ell^1(\cos\theta)}$

Substituting this into the above equation gives a differential equation for $u_\ell(r)$ of the form of:

Bessel's equation: $\boxed{\frac{d^2u_\ell(r)}{dr^2}+\left[\left(\frac{\omega}{c}\right)^2-\frac{\ell(\ell+1)}{r^2}\right]u_\ell(r)\equiv0}$ with $\ell=0,1,2,3,\ldots$ defining the angular dependence of the TM modes.

Let us consider a resonant spherical cavity as two concentric, perfectly conducting spheres of inner radius *a* and outer radius *b*.

If $\boxed{B_\varphi(r,\theta)=\frac{u_\ell(r)}{r}P_\ell^1(\cos\theta)}$, the radial and tangential electric fields (using Ampere's Law) are:

$$\boxed{E_r(r,\theta)=\frac{ic^2}{\omega r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\varphi)=-\frac{ic^2}{\omega r}\ell(\ell+1)\frac{u_\ell(r)}{r}P_\ell(\cos\theta)}$$

$$\boxed{E_\theta(r,\theta)=-\frac{ic^2}{\omega r}\frac{\partial}{\partial r}(rB_\varphi)=-\frac{ic^2}{\omega r}\frac{\partial u_\ell(r)}{\partial r}P_\ell^1(\cos\theta)}$$

But $\boxed{E_\theta = E_\parallel}$ which <u>must</u> vanish at $r = a$ and $r = b$ $\Rightarrow$ $\boxed{\left.\frac{\partial u_\ell(r)}{\partial r}\right|_{r=a} = \left.\frac{\partial u_\ell(r)}{\partial r}\right|_{r=b} = 0}$

The solutions of the radial Bessel equation are spherical Bessel functions (or spherical Hankel functions).

The above radial boundary conditions on $\left.\frac{\partial u_\ell(r)}{\partial r}\right|_{\substack{r=a\\r=b}} = 0$ lead to <u>transcendental</u> equations for the characteristic frequencies, ω_ℓ {eeeEEK}!!!

However {<u>don't</u> panic!}, if: $(b - a) = h$ is such that $h \ll a$ <u>then</u>: $\boxed{\frac{\ell(\ell+1)}{r^2} \approx \frac{\ell(\ell+1)}{a^2} = \text{constant!!!}}$

And thus in this situation, the solutions of Bessel's equation:

$$\boxed{\frac{d^2u_\ell(r)}{dr^2} + \left[\left(\frac{\omega}{c}\right)^2 - \frac{\ell(\ell+1)}{a^2}\right]u_\ell(r) = 0} \Rightarrow \boxed{\frac{d^2u_\ell(r)}{dr^2} + k^2u_\ell(r) = 0} \text{ where: } \boxed{k^2 = \left(\frac{\omega}{c}\right)^2 - \frac{\ell(\ell+1)}{a^2}}$$

are simply sin (kr) and cos (kr) !!! *i.e.* $\boxed{u_\ell(r) = A\cos(kr) + B\sin(kr)}$

<u>Then</u>: $\boxed{\left.\frac{\partial u_\ell}{\partial r}\right|_{r=a} = -kA\sin(ka) + kB\cos(ka) = 0}$ and $\boxed{\left.\frac{\partial u_\ell}{\partial r}\right|_{r=b} = -kA\sin(kb) + kB\cos(kb) = 0}$

For $\boxed{(b-a) = h \ll a}$ an approximate solution is: $\boxed{u_\ell(r) \simeq A\cos[kr - ka]}$

with: $\boxed{kh = k(b-a) = n\pi}$, $n = 0, 1, 2, \ldots$

<u>Thus</u>: $\boxed{k_{n\ell}^2 = \left(\frac{\omega}{c}\right)^2 - \frac{\ell(\ell+1)}{a^2} = \left(\frac{n\pi}{h}\right)^2}$, $n = 0, 1, 2, 3, \ldots$ and $\ell = 0, 1, 2, 3, \ldots$

The corresponding <u>angular cutoff frequency</u> is:

$$\boxed{\omega_{n\ell} \simeq c\sqrt{k_n^2 + \frac{\ell(\ell+1)}{a^2}} \simeq c\sqrt{\left(\frac{n\pi}{h}\right)^2 + \frac{\ell(\ell+1)}{a^2}}} \text{ for } \boxed{h \ll a}, \ n = 0, 1, 2, 3, \ldots \text{ and } \ell = 0, 1, 2, 3, \ldots$$

Because $\boxed{h \ll a}$, we see that the modes with $n = 1, 2, 3, \ldots$ turn out to have relatively high frequencies $\boxed{\omega_{n\ell} \approx c\left(\frac{n\pi}{h}\right)}$ for $n \ge 1$. However, the $n = 0$ modes have relatively low frequencies:

$$\boxed{\omega_{0\ell} \simeq c\sqrt{\frac{\ell(\ell+1)}{a^2}} \simeq \frac{c}{a}\sqrt{\ell(\ell+1)}} \text{ for } \boxed{h \ll a}.$$

An <u>exact</u> solution (correct to first order in (h/a) expansion) for $n = 0$ is: $\boxed{\omega_{0\ell} = \frac{c}{\left(a + \frac{1}{2}h\right)}\sqrt{\ell(\ell+1)}}$

These eigen-mode frequencies are known as <u>Schumann</u> <u>resonance</u> <u>frequencies</u>. $\ell = 1, 2, 3, \ldots$
(W.O. Schumann – Z. Naturforsch. <u>72</u>, 149, 250 (1952))

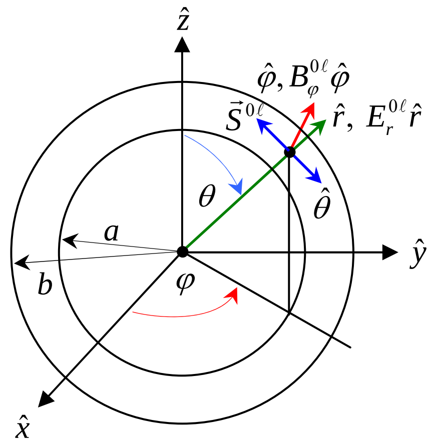
For $n = 0$, the *EM* fields are: $\boxed{E_\theta^{0\ell} = 0}$, $\boxed{E_r^{0\ell} \sim \frac{1}{r^2} P_\ell(\cos\theta)}$ and $\boxed{B_\varphi^{0\ell} \sim \frac{1}{r} P_\ell^1(\cos\theta)}$

Very Useful Table:

$\hat{r} \times \hat{\theta} = \hat{\varphi}$ $\qquad \hat{\theta} \times \hat{r} = -\hat{\varphi}$

$\hat{\theta} \times \hat{\varphi} = \hat{r}$ $\qquad \hat{\varphi} \times \hat{\theta} = -\hat{r}$

$\hat{\varphi} \times \hat{r} = \hat{\theta}$ $\qquad \hat{r} \times \hat{\varphi} = -\hat{\theta}$



Poynting's vector:

$$\boxed{\vec{S}_{0\ell} = \frac{1}{\mu_0}\left(\vec{E}_{0\ell} \times \vec{B}_{0\ell}\right) \sim (\hat{r} \times \hat{\varphi})\frac{1}{r^3} P_\ell(\cos\theta) P_\ell^1(\cos\theta) \sim \frac{1}{r^3} P_\ell(\cos\theta) P_\ell^1(\cos\theta)(-\hat{\theta})} \Leftarrow \boxed{\text{Circumpolar } \textbf{\textit{N-S}} \text{ waves!}}$$

The Earth's surface and the Earth's ionosphere behave as a spherical resonant cavity (!!!) with the Earth's surface {approximately} as the inner spherical surface: $a \simeq r_{\mathrm{E}} \equiv r_\oplus = 6378\,\mathrm{km}$ $= 6.378 \times 10^6\,\mathrm{m}$ (= Earth's mean equatorial radius), the height h (above the surface of the Earth) of the ionosphere is: $h \simeq 100\ \mathrm{km} = 10^5\,\mathrm{m}\ (\ll a) \rightarrow\ b = a + h\ \simeq 6.478 \text{ x } 10^6$ m.

For the $n = 0$ Schumann resonances: $\boxed{\omega_{0\ell} = \frac{c}{\left(a + \frac{1}{2}h\right)}\sqrt{\ell(\ell+1)}}$ for $\boxed{h \ll a}$.

$\underline{\ell = 1:}$ $\boxed{\omega_{01} \simeq \frac{c\sqrt{2}}{\left(a+\frac{1}{2}h\right)}}$ $\Rightarrow$ $\boxed{f_{01} = \frac{\omega_{01}}{2\pi} = 10.5\text{ Hz}}$

$\underline{\ell = 2:}$ $\boxed{\omega_{02} \simeq \frac{c\sqrt{6}}{\left(a+\frac{1}{2}h\right)}}$ $\Rightarrow$ $\boxed{f_{02} = \frac{\omega_{02}}{2\pi} = 18.3\text{ Hz}}$

$\underline{\ell = 3:}$ $\boxed{\omega_{03} \simeq \frac{c\sqrt{12}}{\left(a+\frac{1}{2}h\right)}}$ $\Rightarrow$ $\boxed{f_{03} = \frac{\omega_{03}}{2\pi} = 25.7\text{ Hz}}$

$\underline{\ell = 4:}$ $\boxed{\omega_{04} \simeq \frac{c\sqrt{20}}{\left(a+\frac{1}{2}h\right)}}$ $\Rightarrow$ $\boxed{f_{04} = \frac{\omega_{04}}{2\pi} = 33.2\text{ Hz}}$

$\underline{\ell = 5:}$ $\boxed{\omega_{05} \simeq \frac{c\sqrt{30}}{\left(a+\frac{1}{2}h\right)}}$ $\Rightarrow$ $\boxed{f_{05} = \frac{\omega_{05}}{2\pi} = 46.7\text{ Hz}}$ (. . . etc.)

n.b. For the $n = 1$ Schumann resonances: $f_{1\ell} \approx 1.5\ KHz$

The $n = 0$ Schumann resonances in the Earth-ionosphere cavity manifest themselves as peaks in the noise power spectrum in the VLF (Very Low Frequency) portion of the *EM* spectrum → VLF *EM* standing waves in the spherical cavity of the Earth-ionosphere system!!!

Schumann resonances in the Earth-ionosphere cavity are excited by the radial $\vec{E}$-field component of lightning discharges (the frequency component of *EM* waves produced by lightning at these Schumann resonance frequencies).

Lightning discharges (anywhere on Earth) contain a wide spectrum of frequencies of *EM* radiation – the frequency components $f_{01}, f_{02}, f_{03}, f_{04}, \ldots$ excite these resonant modes – the Earth literally "rings like a bell" at these frequencies!!! The $n = 0$ Schumann resonances are the lowest-lying normal modes of the Earth-ionosphere cavity.

Schumann resonances were first definitively observed in 1960. (M. Balser and C.A. Wagner, Nature 188, 638 (1960)).

→ Nikola Tesla *may* have observed them before 1900!!! (Before the ionosphere was known to even exist!!!) He also estimated the lowest modal frequency to be $f_{01} \sim 6$ Hz!!!

$n = 0$ Schumann Resonances:

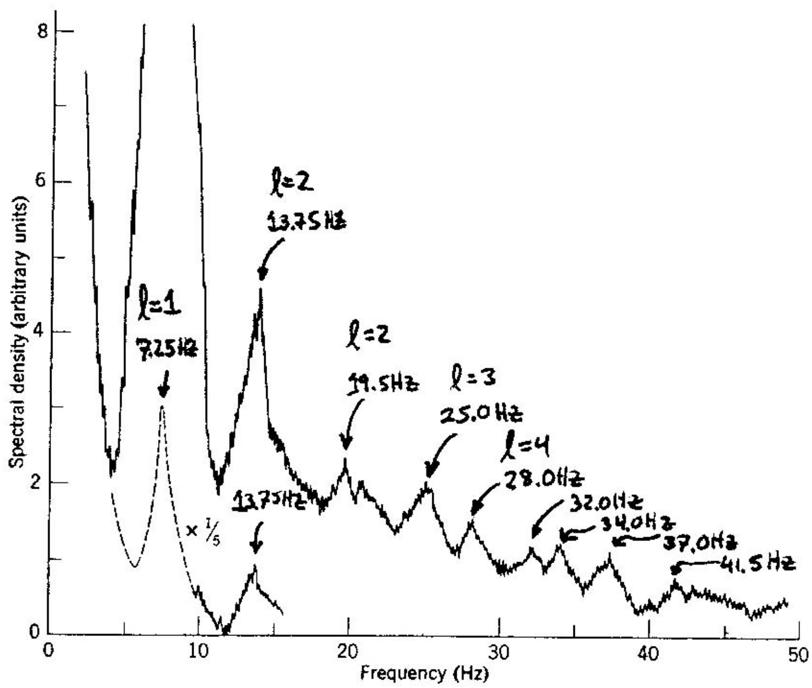
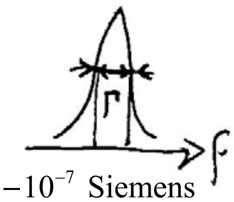


Figure 8.9 Typical noise power spectrum at low frequencies (integrated over 30 s), observed at Lavangsdalen, Norway on June 19, 1965. The prominent Schumann resonances at 8, 14, 20, and 26 Hz, plus peaks at 32, 37, and 43 Hz as well as smaller structure are visible. [After A. Egeland and T. R. Larsen, *Phys. Norv.* **2**, 85 (1967).]

The observed Schumann resonance frequencies are systematically <u>lower</u> than predicted, (primarily) due to <u>damping</u> <u>effects</u>: $\boxed{\omega^2 \approx \omega_0^2\left[1-\frac{1+i}{Q}\right]}$ where Q = Quality factor $\equiv \frac{\omega_0}{\Gamma}$ = "Q" of resonance, and Γ = width at half maximum of power spectrum:



The Earth's surface is also not perfectly conducting.
Seawater conductivity $\sigma_C \simeq 0.1$ Siemens!!
Neither is the ionosphere! $\rightarrow$ Ionosphere's conductivity $\sigma_C \simeq 10^{-4} - 10^{-7}$ Siemens

- On July 9, 1962, a nuclear explosion (EMP) detonated at <u>high altitude</u> (400 km) over Johnson Island in the Pacific {<u>Test Shot</u>: Starfish Prime, Operation Dominic I}.
  - <u>Measurably</u> affected the Earth's ionosphere and radiation belts on a world-wide scale!
  - Sudden <u>decrease</u> of ~ 3 – 5% in Schumann frequencies – increase in height of ionosphere!
  - Change in height of ionosphere: $\boxed{\Delta h = h' - h \simeq 2\cdot(0.03-0.05)R_\oplus \approx 400-600\,km}$!!!
  - Height changes decayed away after ~ several hours.
  - Artificial radiation belts lasted several years!

- Note that # of lightning strikes, (*e.g.* in tropics) is <u>strongly</u> correlated to average temperature. $\Rightarrow$ Scientists have used Schumann resonances & monthly mean magnetic field strengths to monitor lightning rates and thus monitor monthly temperatures – they all correlate very well!!!
- Monitoring Schumann Resonances → Global Thermometer → useful for Global Warming studies!!

<u>Earth Coordinate System</u>

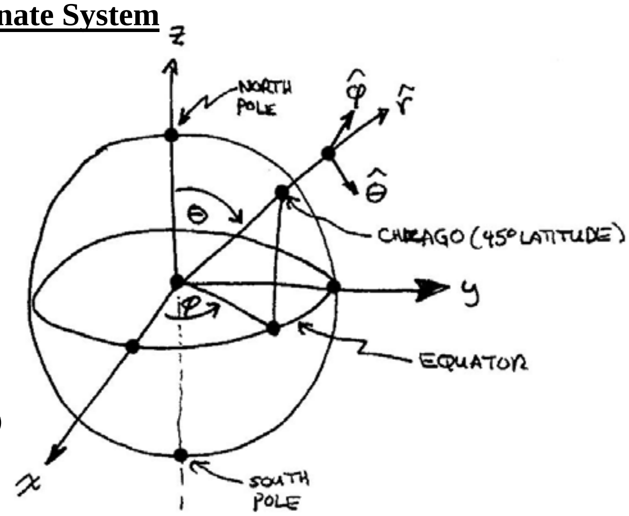
$\boxed{f_{0\ell} \simeq \frac{c}{2\pi\left(a+\frac{1}{2}h\right)}\sqrt{\ell(\ell+1)}}$

$\boxed{E_\theta^{0\ell} = 0}$ (north – south)

$\boxed{E_r^{0\ell} \sim \frac{1}{r^2}P_\ell(\cos\theta)}$ (up – down)

$\boxed{B_\varphi^{0\ell} \sim \frac{1}{r}P_\ell^1(\cos\theta)}$ (east – west)

$\boxed{\vec{S}_{0\ell} \sim \frac{1}{r^3}P_\ell(\cos\theta)P_\ell^1(\cos\theta)(-\hat{\theta})}$ (north – south)

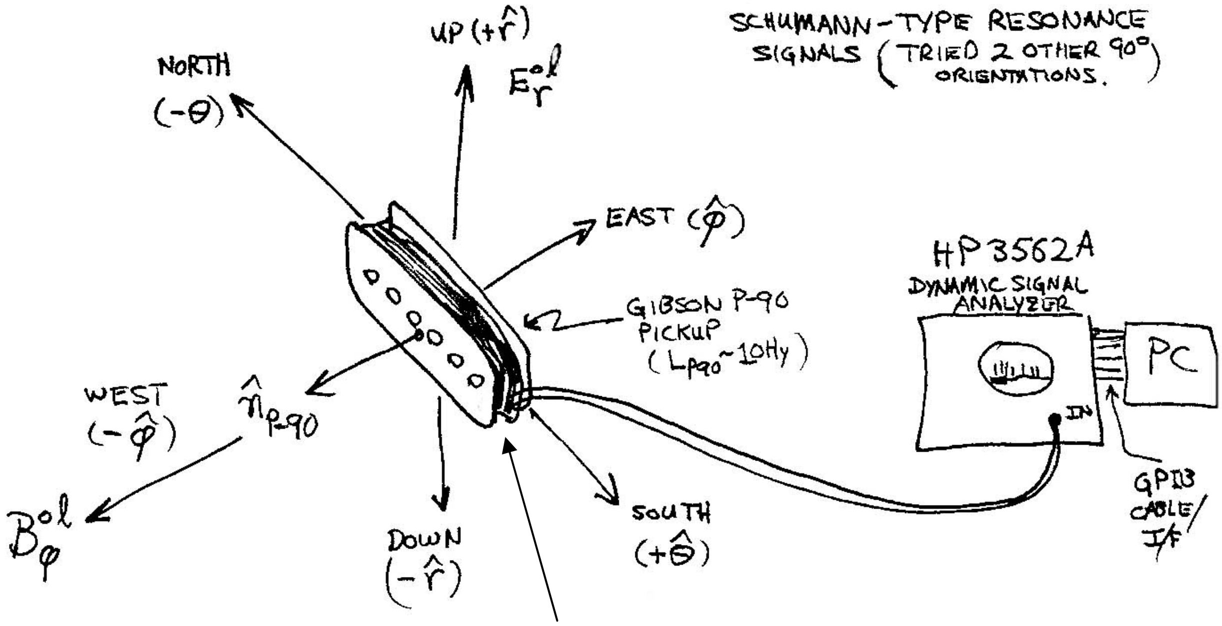


<u>For the n = 0 modes of Schumann Resonances</u>:
$\boxed{\vec{E} \parallel \hat{r}}$ (up – down) $\boxed{\vec{B} \parallel \hat{\varphi}}$ (east – west) $\boxed{\vec{S} \parallel -\hat{\theta}}$ (north – south)

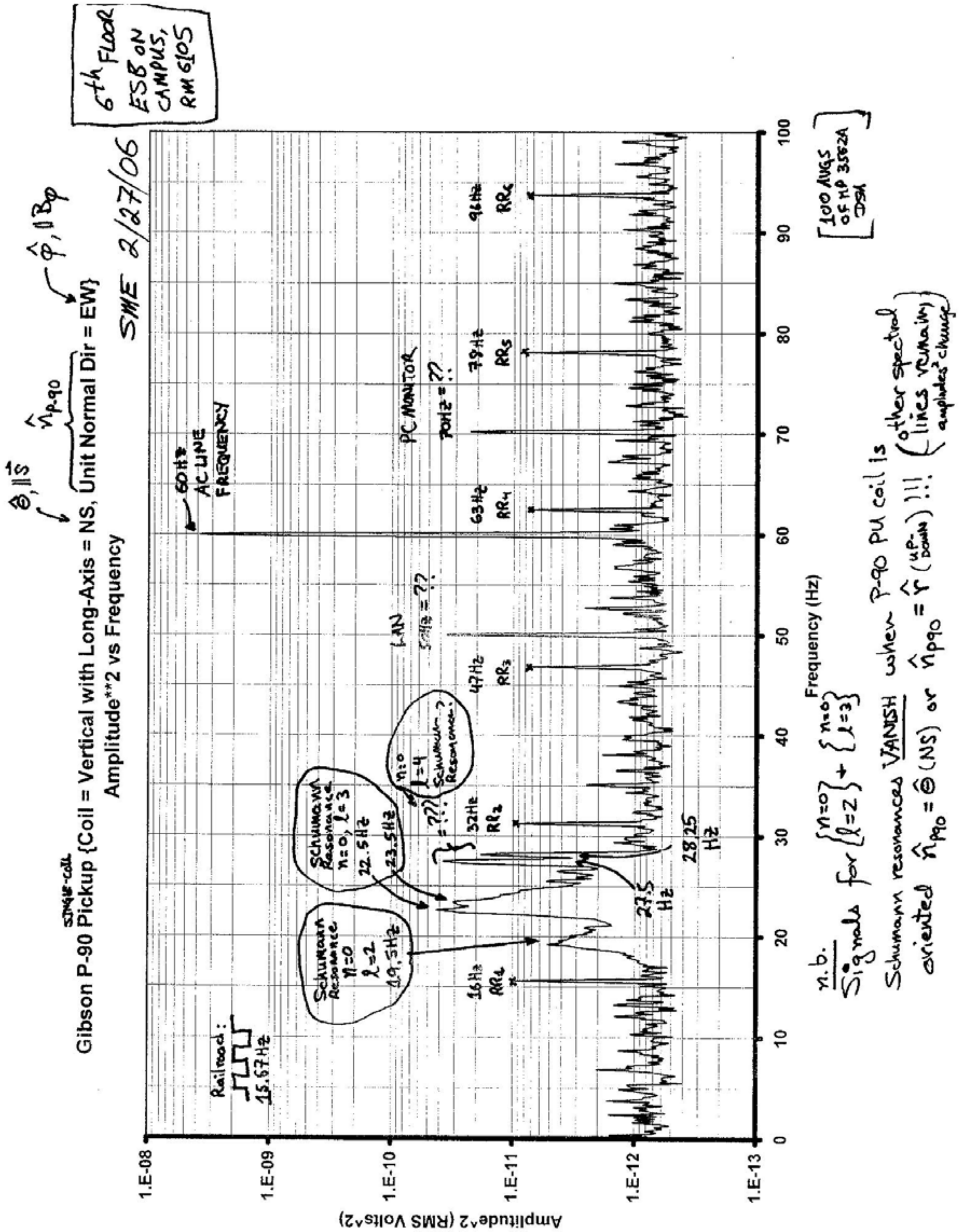
We can observe Schumann resonances right here in town / @ UIUC!! Use *e.g.* Gibson P-90 single-coil electric guitar pickup $\left(L_{P-90} \simeq 10 \text{ Henrys, } \sim 10\text{K turns \#42AWG copper wire}\right)$ for detector of Schumann waves and a spectrum analyzer (*e.g.* HP 3562A Dynamic Signal Analyzer) – read out the HP 3562A into PC via GPIB.

$\rightarrow$ Orientation/alignment of Gibson P-90 electric guitar pickup is important – want axis of pickup aligned $\boxed{\parallel \vec{B} \parallel \hat{\varphi}}$ (*i.e.* oriented east – west) as shown in figure below. *n.b.* only this orientation yielded Schumann-type resonance signals {also tried 2 other 90° orientations {up-down} and {north-south} but observed no signal(s) for Schumann resonances for these.}

Electric guitar PU's are very sensitive – *e.g.* they can easily detect car / bus traffic on street below from 6105 ESB (6th Floor Lab) – can easily see car/bus signal from PU on a 'scope!!!



n.b. PU housed in 4π closed, grounded aluminum sheet-metal box to suppress electric noise.


Gibson P-90 single-coil Pickup {Coil = Vertical with Long-Axis = NS, Unit Normal Dir = EW}
Amplitude**2 vs Frequency
Amplitude^2 (RMS Volts^2)
Frequency (Hz)
1.E-08
1.E-09
1.E-10
1.E-11
1.E-12
1.E-13
0
10
20
30
40
50
60
70
80
90
100
SME 2/27/06
6th FLOOR ESB ON CAMPUS, RM 6105
100 AVGS OF HP 3562A DSA
Railroad: 15.67 Hz
60 Hz AC LINE FREQUENCY
PC MONITOR 70 Hz = ??
LAN 50 Hz = ??
Schumann Resonance n=0 ℓ=2 19.5 Hz
Schumann Resonance n=0, ℓ=3 22.5 Hz 23.5 Hz
n=0 ℓ=4 Schumann Resonance?
16 Hz RR1
32 Hz RR2
47 Hz RR3
63 Hz RR4
78 Hz RR5
96 Hz RR6
27.5 Hz
28.25 Hz
n.b. Signals for {n=0, ℓ=2} + {n=0, ℓ=3} Schumann resonances VANISH when P-90 PU coil is oriented n̂P90 = θ̂ (NS) or n̂P90 = r̂ (UP-DOWN) !!!
(other spectral lines remain) (amplitudes change)