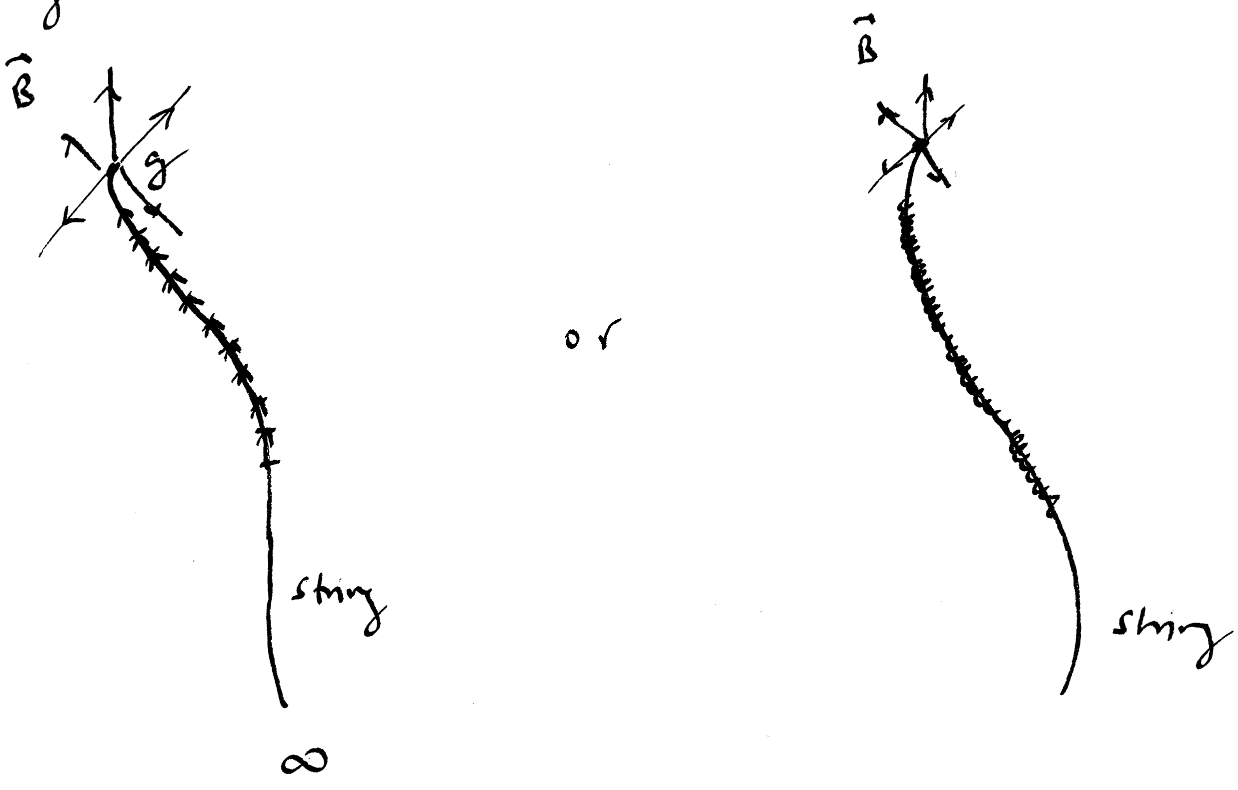
In 1931 Dirac showed that the existence of a single magnetic monopole (of charge $\rho_e = 0$; $\rho_m = g$) would imply the quantization of electric charge.

Dirac modeled his monopole (in the context of conventional electrodynamics) as a "string" of aligned dipoles extending to infinity, or a tightly wound solenoid along the string:

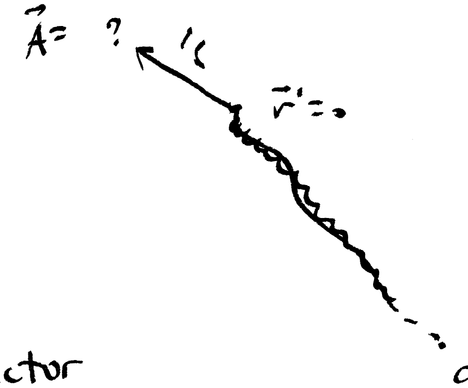


From Lecture 12, the vector potential (in Coulomb gauge) created by a dipole $d\vec{\mu}$ located at $\vec{r}\,'$ is

$$\vec{A}(\vec{r}) = d\vec{\mu}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \qquad \left[\text{we had } \vec{A}(\vec{r}) = \vec{\mu} \times \frac{\vec{r}}{|\vec{r}|^3} \text{ for } \vec{r}'=0\right]$$

$$= -d\vec{\mu} \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

with $d\vec{\mu} = g\, d\vec{r}'$, $\updownarrow d\vec{r}'$ (g at the top, $-g$ at the bottom), the potential becomes



$$\vec{A}(\vec{r}) = -g \int_{\text{string}} d\vec{r}' \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

To calculate $\vec{B}$, we use the vector identity $\vec{\nabla}\times(\vec{a}\times\vec{b}) = \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a}) + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b}$:

$$\vec{B}(\vec{r}) \equiv \vec{\nabla}\times\vec{A} = -g \int_{\vec{r}'=\infty}^{\vec{r}'=0} \Big[d\vec{r}' \underbrace{\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}_{-4\pi\,\delta(\vec{r}-\vec{r}')} + \underbrace{(d\vec{r}'\cdot\vec{\nabla}')}_{\text{derivative along string}} \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\Big]$$

Therefore, away from the string,

$$\vec{B}(\vec{r}) = -g\,\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = g\,\frac{\vec{r}}{|\vec{r}|^2},$$

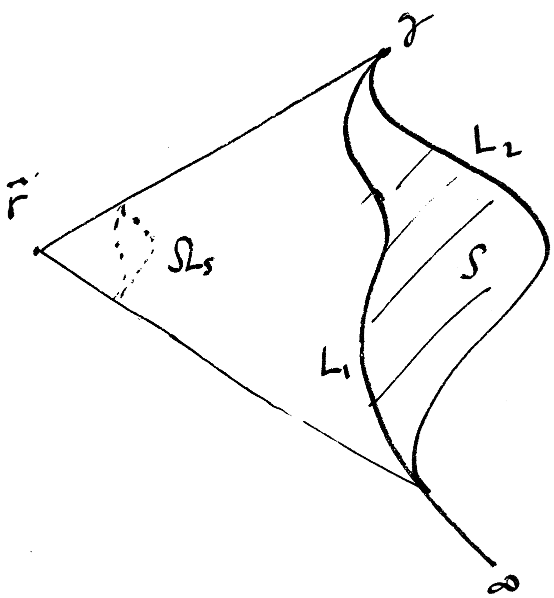
the field of a magnetic monopole of charge g!

[Note that

$$-g\int d\vec{r}'\,(-4\pi)\,\delta(\vec{r}-\vec{r}') = 4\pi g \int d\lambda\, \frac{d\vec{r}'}{d\lambda}\, \delta(\vec{r}-\vec{r}'(\lambda)) \quad]$$

In order for Dirac's monopole to be physically sensible, the location of the string should be irrelevant. One can indeed show that the location of the string can be altered by a gauge transformation.

Consider two different strings L_1 and L_2.



Then, the vector potentials of the two strings satisfy

$$\vec{A}_2(\vec{r}) = \vec{A}_1(\vec{r}) + g\,\vec{\nabla}\Omega_s(\vec{r}),$$

where Ω_s is the solid angle subtended by S at $\vec{r}$.

If we now consider the Hamiltonian of an electron in an electromagnetic field,

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi,$$

it is easy to check that if $i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$, then the solution of $i\hbar\frac{\partial|\psi'\rangle}{\partial t} = H'|\psi'\rangle$, with

$$H' = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}'\right)^2 + e\phi'$$

and

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \quad , \quad \phi' = \phi - \frac{1}{c}\partial_t \chi \qquad \text{is}$$

$$\langle \vec{r}|\psi'\rangle = \exp\left[ie\chi/\hbar c\right]\langle \vec{r}|\psi\rangle .$$

Therefore, the wave functions of an electron in the fields of two different strings are related by

$$\langle \vec{r}|\psi_2\rangle = \exp\left[ieg\,\Omega_S(\vec{r})/\hbar c\right]\langle \vec{r}|\psi_1\rangle$$

But Ω_S changes by 2π when $\vec{r}$ moves from slightly above S to slightly below

we avoid the ambiguity in the wave function if

$$\frac{ieg\cdot 4\pi}{\hbar c} = i\,n\cdot(2\pi) \, , \quad n\in\mathbb{Z} .$$

Hence, $$\frac{eg}{\hbar c} \overset{!}{=} \frac{n}{2} \, , \quad n\in\mathbb{Z}$$

charge quantization.

The magnetic field of a monopole,

$$\vec{B}(\vec{r}) = g \, \frac{\vec{r}}{|\vec{r}|^3}$$ helps to illustrate

Poincaré's lemma:

From electrostatics we know that $\vec{\nabla} \cdot \vec{B} = 0$, except at the origin, where $\vec{B}$ is singular.

Consider hence the space

$$\underline{P} = \mathbb{R}^3 \setminus \{ \vec{0} \} .$$

Then, $\vec{\nabla} \cdot \vec{B} = 0$ in $\underline{P}$, but we <u>cannot</u> write $\vec{B} = \vec{\nabla} \times \vec{A}$ with a globally defined $\vec{A}$.

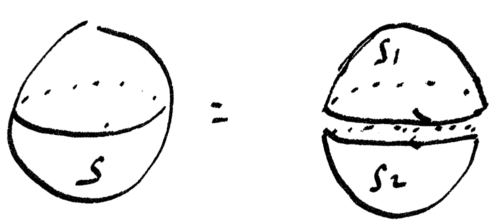
If we could, we would have

$$\int_{S} \vec{B} \cdot d\vec{S} = \int_{S_1} \vec{B} \, d\vec{S} + \int_{S_2} \vec{B} \cdot d\vec{S} = \int_{S_1} \vec{\nabla} \times \vec{A} \, d\vec{S} + \int \vec{\nabla} \times \vec{A} \cdot d\vec{S}$$

$$\parallel$$

$$4\pi g$$

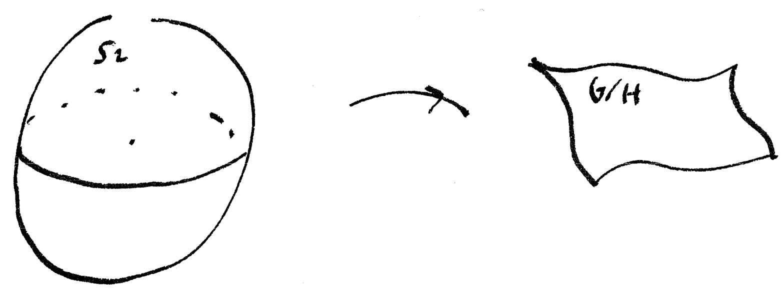
$$= \int_{\partial S_1} \vec{A} \cdot d\vec{l} + \int_{\partial S_2} \vec{A} \, d\vec{l} = 0 + 0 . \;\lightning$$



The "problem" is that $P = \mathbb{R}^3 \setminus \{\vec{0}\}$ is <u>not</u> a contractible space.

In "spontaneously broken" gauge theories there exist regular monopole solutions (e.g. the 't Hooft-Polyakov monopole).

The existence of monopoles in these theories is related to the topological properties of maps between S_2 (the two-sphere) and the vacuum manifold of the theory, $\pi_2(G/H)$ (homotopy group).



10. Electromagnetic Plane Waves

Let us investigate now the full Maxwell's eqs. in the absence of charges and currents (in a medium or in vacuum):

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} &= 0 \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{H} - \frac{1}{c}\partial_t\vec{D} &= 0 \end{aligned}\right\}$$

with $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$.

Then,

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{1}{c}\partial_t\vec{B} \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{B} &= \left(\frac{\epsilon\mu}{c}\right)\partial_t\vec{E} \end{aligned}\right\}$$

Taking say $\vec{\nabla}\times$ and using $\vec{\nabla}\times\vec{\nabla}\times\vec{a} = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}$:

$$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{0} - \vec{\nabla}^2\vec{E} + \frac{1}{c}\partial_t\underbrace{\vec{\nabla}\times\vec{B}}_{\frac{\epsilon\mu}{c}\partial_t\vec{E}} = 0 \quad , \text{ or}$$

$$\underline{\frac{\epsilon\mu}{c^2}\partial_t^2\vec{E} - \vec{\nabla}^2\vec{E} = 0}$$ a wave equation!

A convenient set of solutions is the *plane wave* $\vec{E} = \vec{E}_0 \cdot \exp[i\vec{k}\vec{x} - i\omega t]$, where

$$\frac{\epsilon\mu}{c^2}\omega^2 - \vec{k}^2 = 0, \quad \text{or} \quad \omega = \frac{c}{\sqrt{\epsilon\mu}}|\vec{k}|$$

(Dispersion relation)

The *phase velocity* of the wave is [from $ikx - \omega t = \text{const}$]

$$v \equiv \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}} = \frac{c}{n}, \quad \text{where}$$

$n \equiv \sqrt{\epsilon\mu}$ is the medium's index of refraction

One can similarly show that $\vec{B}$ obeys the same wave equation, with the same set of solutions,

$$\vec{B} = \vec{B}_0 \exp[i\vec{k}\cdot\vec{x} - i\omega t], \quad \omega = \frac{c|\vec{k}|}{\sqrt{\epsilon\mu}}.$$

For these plane waves,

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0 \end{array}\right\} \text{ wave is transverse } (\vec{E}, \vec{B} \perp \vec{k})$$

Since $\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0$, $i\vec{k}\times\vec{E}_0 = i\frac{\omega}{c}\vec{B}_0$, or

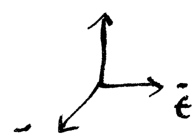
$$\vec{B}_0 = n\,\hat{k}\times\vec{E}_0$$

Of course, $\vec{E}$ and $\vec{B}$ are real, so the understanding is that we are supposed to take the real parts of $\vec{E}$ and $\vec{B}$.

10.3. Polarization

To study polarization, assume without loss of generality that $\vec{k} \parallel \hat{z}$. Define then

$$\vec{\epsilon}_1 \equiv \hat{x} \; ; \quad \vec{\epsilon}_2 \equiv \hat{y} \; , \quad \vec{\epsilon}_3 = \hat{z}$$



Then, only such plane wave can be written as the linear compination

$$\vec{E} = (E_x \hat{\epsilon}_1 + E_y \hat{\epsilon}_2)\, e^{i\vec{k}\cdot\vec{r} - i\omega t} \; ,$$

for complex numbers E_x and E_y.

If $E_x = |E_x| e^{i\gamma_x}$ and $E_y = |E_y| e^{i\gamma_y}$ have the same phase $(\gamma_x = \gamma_y)$, this describes <u>linearly polarized</u> light along the direction

$$\vec{E}_0 = |E_x| \hat{\epsilon}_1 + |E_y| \hat{\epsilon}_2 .$$

If $\delta_x \neq \delta_y$, light is elliptically polarize

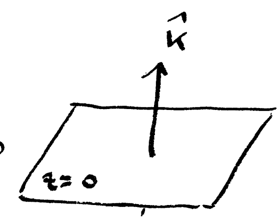
A convenient basis to discuss elliptically polarized light is provided by

$$\hat{\epsilon}_\pm \equiv \frac{1}{\sqrt{2}} \left(\hat{\epsilon}_1 \pm i \, \hat{\epsilon}_2 \right),$$

which satisfy $\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_\mp = 0$, $\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_\pm = 1$

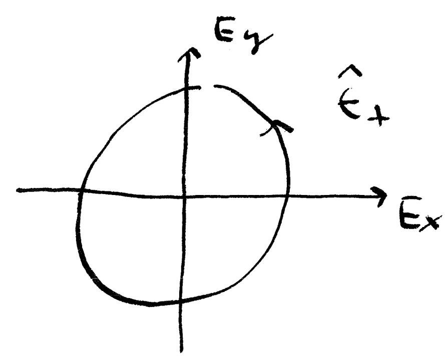
and $\hat{\epsilon}^*_\pm \cdot \hat{\epsilon}_3 = 0$.

With $\vec{E} = \hat{\epsilon}_\pm \, e^{i\vec{k}\cdot\vec{x} - i\omega t}$, at $x = y = z = 0$

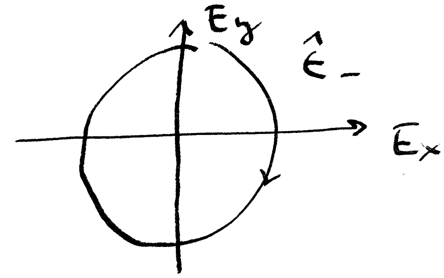


$$E_x = \cos(\omega t) \qquad (= \mathrm{Re}\, \vec{E}_x)$$

$$E_y = \pm \sin(\omega t) \qquad (= \mathrm{Re}\, \vec{E}_y)$$



$\hat{\epsilon}_+$ describes right circularly polarized light.



$\hat{\epsilon}_-$ describes left circularly polarized light.

Exercise 27

Show that if $\vec{E} = E_+ (\hat{\epsilon}_+ + r e^{i\alpha} \hat{\epsilon}_-) e^{-i\omega t}$,

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{E_+}{\sqrt{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} (1+r)\cos(\omega t - \frac{\alpha}{2}) \\ (1-r)\sin(\omega t - \frac{\alpha}{2}) \end{pmatrix}.$$

Interpret this result geometrically. ∎