

## 3 The effect of dielectric and magnetic media on electric and magnetic fields

### 3.1 Polarization

The terrestrial environment is characterized by dielectric media (*e.g.*, air, water) which are, for the most part, electrically neutral, since they are made up of neutral atoms and molecules. However, if these atoms and molecules are placed in an electric field they tend to *polarize*. Suppose that when a given neutral molecule is placed in an electric field  $\mathbf{E}$  the centre of charge of its constituent electrons (whose total charge is  $-q$ ) is displaced by a distance  $-\mathbf{r}$  with respect to the centre of charge of its constituent atomic nuclei. The *dipole moment* of the molecule is defined  $\mathbf{p} = q\mathbf{r}$ . If there are  $N$  such molecules per unit volume then the *electric polarization*  $\mathbf{P}$  (*i.e.*, the dipole moment per unit volume) is given by  $\mathbf{P} = N\mathbf{p}$ . More generally,

$$\mathbf{P}(\mathbf{r}) = \sum_i N_i \langle \mathbf{p}_i \rangle, \quad (3.1)$$

where  $\langle \mathbf{p}_i \rangle$  is the average dipole moment of the  $i$ th type of molecule in the vicinity of point  $\mathbf{r}$ , and  $N_i$  is the average number of such molecules per unit volume at  $\mathbf{r}$ .

It is easily demonstrated that any divergence of the polarization field  $\mathbf{P}(\mathbf{r})$  gives rise to an effective charge density  $\rho_b$  in the medium. In fact,

$$\rho_b = -\nabla \cdot \mathbf{P}. \quad (3.2)$$

This charge density is attributable to *bound charges* (*i.e.*, charges which arise from the polarization of neutral atoms), and is usually distinguished from the charge density  $\rho_f$  due to *free charges*, which represents a net surplus or deficit of electrons in the medium. Thus, the total charge density  $\rho$  in the medium is

$$\rho = \rho_f + \rho_b. \quad (3.3)$$

It must be emphasized that both terms in this equation represent real physical charge. Nevertheless, it is useful to make the distinction between bound and free charges, especially when it comes to working out the energy associated with electric fields in dielectric media.

Gauss' law takes the differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f + \rho_b}{\epsilon_0}. \quad (3.4)$$

This expression can be rearranged to give

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (3.5)$$

where

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (3.6)$$

is termed the *electric displacement*, and has the same dimensions as  $\mathbf{P}$  (dipole moment per unit volume). The divergence theorem tells us that

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f dV. \quad (3.7)$$

In other words, the flux of  $\mathbf{D}$  out of some closed surface  $S$  is equal to the total free charge enclosed within that surface. Unlike the electric field  $\mathbf{E}$  (which is the force acting on unit charge) or the polarization  $\mathbf{P}$  (the dipole moment per unit volume), the electric displacement  $\mathbf{D}$  has no clear physical meaning. The only reason for introducing it is that it enables us to calculate fields in the presence of dielectric materials without first having to know the distribution of polarized charges. However, this is only possible if we have a *constitutive relation* connecting  $\mathbf{E}$  and  $\mathbf{D}$ . It is conventional to assume that the induced polarization  $\mathbf{P}$  is directly proportional to the electric field  $\mathbf{E}$ , so that

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad (3.8)$$

where  $\chi_e$  is termed the *electric susceptibility* of the medium. It follows that

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E}, \quad (3.9)$$

where

$$\epsilon = 1 + \chi_e \quad (3.10)$$

is termed the *dielectric constant* or *relative permittivity* of the medium. (Likewise,  $\epsilon_0$  is termed the *permittivity of free space*.) It follows from Eqs. (3.5) and (3.9) that

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\epsilon_0 \epsilon}. \quad (3.11)$$

Thus, the electric fields produced by free charges in a dielectric medium are analogous to those produced by the same charges in a vacuum, except that they are reduced by a factor  $\epsilon$ . This reduction can be understood in terms of a polarization of the atoms or molecules of the dielectric medium that produces electric fields in opposition to that of given charge. One immediate consequence is that the capacitance of a capacitor is increased by a factor  $\epsilon$  if the empty space between the electrodes is filled with a dielectric medium of dielectric constant  $\epsilon$  (assuming that fringing fields can be neglected).

It must be understood that Eqs. (3.8)–(3.11) are just an *approximation* which is generally found to hold under terrestrial conditions (provided that the fields are not too large) for *isotropic* media. For anisotropic media (*e.g.*, crystals) Eq. (3.9) generalizes to

$$\mathbf{D} = \epsilon_0 \boldsymbol{\epsilon} \cdot \mathbf{E}, \quad (3.12)$$

where  $\boldsymbol{\epsilon}$  is a second rank tensor known as the *dielectric tensor*. For strong electric fields  $\mathbf{D}$  ceases to vary linearly with  $\mathbf{E}$ . Indeed, for sufficiently strong electric fields neutral molecules are disrupted and the whole concept of a dielectric medium becomes meaningless.

## 3.2 Boundary conditions for $\mathbf{E}$ and $\mathbf{D}$

When the space near a set of charges contains dielectric material of non-uniform dielectric constant then the electric field no longer has the same form as in vacuum. Suppose, for example, that the space is occupied by two dielectric media whose uniform dielectric constants are  $\epsilon_1$  and  $\epsilon_2$ . What are the matching conditions on  $\mathbf{E}$  and  $\mathbf{D}$  at the boundary between the two media?

Imagine a Gaussian pill-box enclosing part of the boundary surface between the two media. The thickness of the pill-box is allowed to tend towards zero, so that the only contribution to the outward flux of  $\mathbf{D}$  comes from its flat faces. These faces are parallel to the bounding surface and lie in each of the two media. Their outward normals are  $d\mathbf{S}_1$  (in medium 1) and  $d\mathbf{S}_2$ , where  $d\mathbf{S}_1 = -d\mathbf{S}_2$ . Assuming that there is no free charge inside the disk (which is reasonable in the

limit where the volume of the disk tends towards zero), then Eq. (3.7) yields

$$\mathbf{D}_1 \cdot d\mathbf{S}_1 + \mathbf{D}_2 \cdot d\mathbf{S}_2 = 0, \quad (3.13)$$

where  $\mathbf{D}_1$  is the electric displacement in medium 1 at the boundary with medium 2, *etc.* The above equation can be rewritten

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = 0, \quad (3.14)$$

where  $\mathbf{n}_{21}$  is the normal to the boundary surface, directed from medium 1 to medium 2. If the fields and charges are non time varying then Maxwell's equations yield  $\nabla \wedge \mathbf{E} = 0$ , which give the familiar boundary condition (obtained by integrating around a small loop which straddles the boundary surface)

$$(\mathbf{E}_2 - \mathbf{E}_1) \wedge \mathbf{n}_{21} = 0. \quad (3.15)$$

In other word, the normal component of the electric displacement and the tangential component of the electric field are both continuous across any boundary between two dielectric materials.

### 3.3 Boundary value problems with dielectrics - I

Consider a point charge  $q$  embedded in a semi-infinite dielectric  $\epsilon_1$  a distance  $d$  away from a plane interface which separates the first medium from another semi-infinite dielectric  $\epsilon_2$ . The interface is assumed to coincide with the plane  $z = 0$ . We need to find solutions to the equations

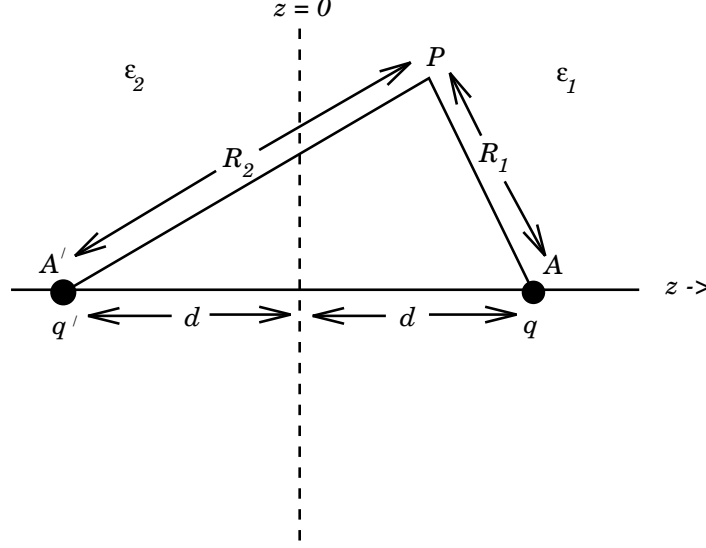
$$\epsilon_1 \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (3.16)$$

for  $z > 0$ ,

$$\epsilon_2 \nabla \cdot \mathbf{E} = 0 \quad (3.17)$$

for  $z < 0$ , and

$$\nabla \wedge \mathbf{E} = 0 \quad (3.18)$$



everywhere, subject to the boundary conditions at  $z = 0$  that

$$\epsilon_1 E_z(z = 0^+) = \epsilon_2 E_z(z = 0^-), \quad (3.19a)$$

$$E_x(z = 0^+) = E_x(z = 0^-), \quad (3.19b)$$

$$E_y(z = 0^+) = E_y(z = 0^-). \quad (3.19c)$$

In order to solve this problem we will employ a slightly modified form of the well known method of images. Since  $\nabla \wedge \mathbf{E} = 0$  everywhere, the electric field can be written in terms of a scalar potential. So,  $\mathbf{E} = -\nabla\phi$ . Consider the region  $z > 0$ . Let us assume that the scalar potential in this region is the same as that obtained when the whole of space is filled with the dielectric  $\epsilon_1$  and, in addition to the real charge  $q$  at position  $A$ , there is a second charge  $q'$  at the image position  $A'$  (see diagram). If this is the case then the potential at some point  $P$  in the region  $z > 0$  is given by

$$\phi(z > 0) = \frac{1}{4\pi\epsilon_0\epsilon_1} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right), \quad (3.20)$$

where  $R_1 = \sqrt{\rho^2 + (d - z)^2}$  and  $R_2 = \sqrt{\rho^2 + (d + z)^2}$ , when written in terms of cylindrical polar coordinates  $(\rho, \varphi, z)$ . Note that the potential (3.20) clearly is a

solution of Eq. (3.16) in the region  $z > 0$ . It gives  $\nabla \cdot \mathbf{E} = 0$ , with the appropriate singularity at the position of the point charge  $q$ .

Consider the region  $z < 0$ . Let us assume that the scalar potential in this region is the same as that obtained when the whole of space is filled with the dielectric  $\epsilon_2$  and a charge  $q''$  is located at the point  $A$ . If this is the case then the potential in this region is given by

$$\phi(z < 0) = \frac{1}{4\pi\epsilon_0\epsilon_2} \frac{q''}{R_1}. \quad (3.21)$$

The above potential is clearly a solution of Eq. (3.17) in the region  $z < 0$ . It gives  $\nabla \cdot \mathbf{E} = 0$ , with no singularities.

It now remains to choose  $q'$  and  $q''$  in such a manner that the boundary conditions (3.19) are satisfied. The boundary conditions (3.19b) and (3.19c) are obviously satisfied if the scalar potential is continuous at the interface between the two dielectric media:

$$\phi(z = 0^+) = \phi(z = 0^-). \quad (3.22)$$

The boundary condition (3.19a) implies a jump in the normal derivative of the scalar potential across the interface:

$$\epsilon_1 \frac{\partial \phi(z = 0^+)}{\partial z} = \epsilon_2 \frac{\partial \phi(z = 0^-)}{\partial z}. \quad (3.23)$$

The first matching condition yields

$$\frac{q + q'}{\epsilon_1} = \frac{q''}{\epsilon_2}, \quad (3.24)$$

whereas the second yields

$$q - q' = q''. \quad (3.25)$$

Here, use has been made of

$$\frac{\partial}{\partial z} \left( \frac{1}{R_1} \right)_{z=0} = -\frac{\partial}{\partial z} \left( \frac{1}{R_2} \right)_{z=0} = \frac{d}{(\rho^2 + d^2)^{3/2}}. \quad (3.26)$$

Equations (3.24) and (3.25) imply that

$$q' = - \left( \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \right) q, \quad (3.27a)$$

$$q'' = \left( \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} \right) q. \quad (3.27b)$$

The polarization charge density is given by  $\rho_b = -\nabla \cdot \mathbf{P}$ , However, inside either dielectric  $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ , so  $\nabla \cdot \mathbf{P} = \epsilon_0 \chi_e \nabla \cdot \mathbf{E} = 0$ , except at the point charge  $q$ . Thus, there is zero polarization charge density in either dielectric medium. At the interface  $\chi_e$  takes a discontinuous jump,

$$\Delta \chi_e = \epsilon_1 - \epsilon_2. \quad (3.28)$$

This implies that there is a polarization charge sheet on the interface between the two dielectric media. In fact,

$$\sigma_{\text{pol}} = -(\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n}_{21}, \quad (3.29)$$

where  $\mathbf{n}_{21}$  is a unit normal to the interface pointing from medium 1 to medium 2 (*i.e.*, along the positive  $z$ -axis). Since

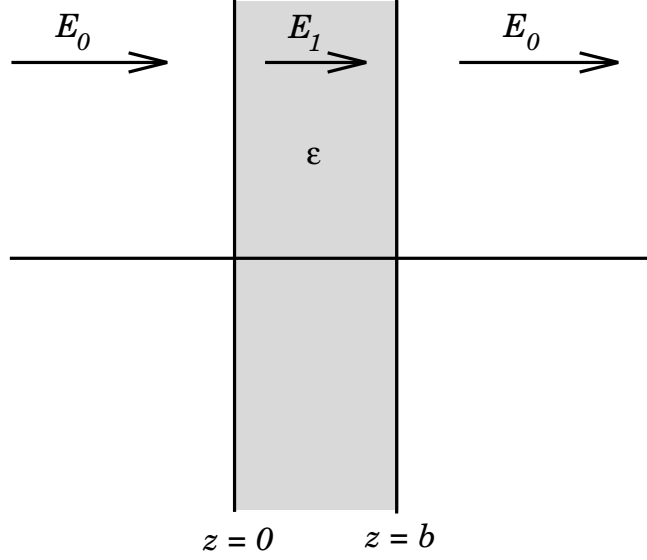
$$\mathbf{P}_i = \epsilon_0(\epsilon_i - 1)\mathbf{E} = -\epsilon_0(\epsilon_i - 1)\nabla\phi \quad (3.30)$$

in either medium, it is easy to demonstrate that

$$\sigma_{\text{pol}} = -\frac{q}{2\pi} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1(\epsilon_2 + \epsilon_1)} \frac{d}{(\rho^2 + d^2)^{3/2}}. \quad (3.31)$$

In the limit  $\epsilon_2 \gg \epsilon_1$ , the dielectric  $\epsilon_2$  behaves like a conducting medium (*i.e.*,  $\mathbf{E} \rightarrow 0$  in the region  $z < 0$ ), and the polarization surface charge density on the interface approaches that obtained in the case when the plane  $z = 0$  coincides with a conducting surface.

The above method can clearly be generalized to deal with problems involving many point charges in the presence of many different dielectric media whose interfaces form parallel planes.



### 3.4 Boundary value problems with dielectrics - II

Consider a plane slab of dielectric  $\epsilon$  lying between  $z = 0$  and  $z = b$ . Suppose that this slab is placed in a uniform  $z$ -directed electric field of strength  $E_0$ . What is the field strength  $E_1$  inside the slab?

Since there are no free charges and this is a one-dimensional problem, it is clear from Eq. (3.5) that the electric displacement  $D$  is the same in both the dielectric slab and the vacuum region which surrounds it. In the vacuum region  $D = \epsilon_0 E_0$ , whereas  $D = \epsilon_0 \epsilon E_1$  in the dielectric. It follows that

$$E_1 = \frac{E_0}{\epsilon}. \quad (3.32)$$

In other words, the electric field inside the slab is reduced by polarization charges. As before, there is zero polarization charge density inside the dielectric. However, there is a uniform polarization charge sheet on both surfaces of the slab. It is easily demonstrated that

$$\sigma_{\text{pol}}(z = b) = -\sigma_{\text{pol}}(z = 0) = \epsilon_0 E_0 \frac{\epsilon - 1}{\epsilon}. \quad (3.33)$$

In the limit  $\epsilon \gg 1$ , the slab acts like a conductor and  $E_1 \rightarrow 0$ .



Let us now generalize this result. Consider a dielectric medium whose dielectric constant  $\epsilon$  varies with  $z$ . The medium is assumed to be of finite extent and is surrounded by a vacuum, so that  $\epsilon(z) \rightarrow 1$  as  $|z| \rightarrow \infty$ . Suppose that this dielectric is placed in a uniform  $z$ -directed electric field  $E_0$ . What is the field  $E(z)$  inside the dielectric?

We know that the electric displacement inside the dielectric is given by  $D(z) = \epsilon_0 \epsilon(z) E(z)$ . We also know from Eq. (3.5) that, since there are no free charges and this is a one-dimensional problem,

$$\frac{dD(z)}{dz} = \epsilon_0 \frac{d[\epsilon(z)E(z)]}{dz} = 0. \quad (3.34)$$

Furthermore,  $E(z) \rightarrow E_0$  as  $|z| \rightarrow \infty$ . It follows that

$$E(z) = \frac{E_0}{\epsilon(z)}. \quad (3.35)$$

Thus, the electric field is inversely proportional to the dielectric constant inside the dielectric medium. The polarization charge density inside the dielectric is given by

$$\rho_b = \epsilon_0 \frac{dE(z)}{dz} = \epsilon_0 E_0 \frac{d}{dz} \left[ \frac{1}{\epsilon(z)} \right]. \quad (3.36)$$

### 3.5 Boundary value problems with dielectrics - III

Suppose that a dielectric sphere of radius  $a$  and dielectric constant  $\epsilon$  is placed in a  $z$ -directed electric field of strength  $E_0$  (in the absence of the sphere). What is the electric field inside and around the sphere?

Since this is a static problem we can write  $\mathbf{E} = -\nabla\phi$ . There are no free charges, so Eqs. (3.5) and (3.9) imply that

$$\nabla^2\phi = 0 \quad (3.37)$$

everywhere. The boundary conditions (3.14) and (3.15) reduce to

$$\epsilon \left. \frac{\partial\phi}{\partial r} \right|_{r=a^-} = \left. \frac{\partial\phi}{\partial r} \right|_{r=a^+}, \quad (3.38a)$$

$$\left. \frac{\partial \phi}{\partial \theta} \right|_{r=a^-} = \left. \frac{\partial \phi}{\partial \theta} \right|_{r=a^+}. \quad (3.38b)$$

Furthermore,

$$\phi(r, \theta, \varphi) \rightarrow -E_0 r \cos \theta \quad (3.39)$$

as  $r \rightarrow 0$ . Here,  $(r, \theta, \varphi)$  are spherical polar coordinates centred on the sphere.

Let us search for an axisymmetric solution,  $\phi = \phi(r, \theta)$ . Since the solutions to Poisson's equation are *unique*, we know that if we can find such a solution which satisfies all of the boundary conditions then we can be sure that this is the correct solution. Equation (3.37) reduces to

$$\frac{1}{r} \frac{\partial^2 (r\phi)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0. \quad (3.40)$$

Straightforward separation of the variables yields

$$\phi(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta), \quad (3.41)$$

where  $l$  is a non-negative integer, the  $A_l$  and  $B_l$  are arbitrary constants, and  $P_l(x)$  is a solution to Legendre's equation,

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP_l}{dx} \right] + l(l+1) P_l = 0, \quad (3.42)$$

which is single-valued, finite, and continuous in the interval  $-1 \leq x \leq +1$ . It can be demonstrated that Eq. (3.42) only possesses such solutions when  $l$  takes an integer value. The  $P_l(x)$  are known as *Legendre polynomials* (since they are polynomials of order  $l$  in  $x$ ), and are specified by *Rodrigues' formula*

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \quad (3.43)$$

Since Eq. (3.42) is a Sturm-Liouville type equation, and the Legendre polynomials satisfy Sturm-Liouville type boundary conditions at  $x = \pm 1$ , it immediately

follows that the  $P_l(\cos \theta)$  are orthogonal functions which form a *complete set* in  $\theta$ -space. The orthogonality relation can be written

$$\int_{-1}^1 P_{l'}(x) P_l(x) dx = \frac{2}{2l+1} \delta_{ll'}. \quad (3.44)$$

The Legendre polynomials form a complete set of angular functions, and it is easily demonstrated that the  $r^l$  and the  $r^{-(l+1)}$  form a complete set of radial functions. It follows that Eq. (3.41), with the  $A_l$  and  $B_l$  unspecified, represents a completely general axisymmetric solution of Eq. (3.37) which is well behaved in  $\theta$ -space. We now need to find values of the  $A_l$  and  $B_l$  which are consistent with the boundary conditions.

Let us divide space into the regions  $r \leq a$  and  $r > a$ . In the former region

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta), \quad (3.45)$$

where we have rejected the  $r^{-(l+1)}$  radial solutions because they diverge unphysically as  $r \rightarrow 0$ . In the latter region

$$\phi(r, \theta) = \sum_{l=0}^{\infty} (B_l r^l + C_l r^{-(l+1)}) P_l(\cos \theta). \quad (3.46)$$

However, it is clear from the boundary condition (3.39), and Eq. (3.43), that the only non-vanishing  $B_l$  is  $B_1 = -E_0$ . This follows since  $P_1(\cos \theta) = \cos \theta$ . The boundary condition (3.38b) (which integrates to give  $\phi(r = a^-) = \phi(r = a^+)$  for a potential which is well behaved in  $\theta$ -space) gives

$$A_1 = -E_0 + \frac{C_1}{a^3}, \quad (3.47)$$

and

$$A_l = \frac{C_l}{a^{2l+1}} \quad (3.48)$$

for  $l \neq 1$ . Note that it is appropriate to match the coefficients of the  $P_l(\cos \theta)$  since these functions are *orthogonal*. The boundary condition (3.38a) yields

$$\epsilon A_1 = -E_0 - 2 \frac{C_1}{a^3}, \quad (3.49)$$

and

$$\epsilon l A_l = -(l+1) \frac{C_l}{a^{2l+1}} \quad (3.50)$$

for  $l \neq 1$ . Equations (3.48) and (3.50) give  $A_l = C_l = 0$  for  $l \neq 1$ . Equations (3.47) and (3.49) reduce to

$$A_1 = - \left( \frac{3}{2 + \epsilon} \right) E_0, \quad (3.51a)$$

$$C_1 = \left( \frac{\epsilon - 1}{\epsilon + 2} \right) a^3 E_0. \quad (3.51b)$$

The solution is therefore

$$\phi = - \left( \frac{3}{2 + \epsilon} \right) E_0 r \cos \theta \quad (3.52)$$

for  $r \leq a$ , and

$$\phi = -E_0 r \cos \theta + \left( \frac{\epsilon - 1}{\epsilon + 2} \right) E_0 \frac{a^3}{r^2} \cos \theta \quad (3.53)$$

for  $r > a$ .

Equation (3.52) is the potential of a uniform  $z$ -directed electric field of strength

$$E_1 = \frac{3}{2 + \epsilon} E_0. \quad (3.54)$$

Note that  $E_1 < E_0$  provided that  $\epsilon > 1$ . Thus, the electric field strength is reduced inside the dielectric sphere due to partial shielding by polarization charges. Outside the sphere the potential is equivalent to that of the applied field  $E_0$  plus the field of a point electric dipole, located at the origin and pointing in the  $z$ -direction, whose dipole moment is

$$p = 4\pi\epsilon_0 \left( \frac{\epsilon - 1}{\epsilon + 2} \right) a^3 E_0. \quad (3.55)$$

This dipole moment can be interpreted as the volume integral of the polarization  $\mathbf{P}$  over the sphere. The polarization is

$$\mathbf{P} = \epsilon_0(\epsilon - 1) E_1 \hat{\mathbf{z}} = 3\epsilon_0 \left( \frac{\epsilon - 1}{\epsilon + 2} \right) E_0 \hat{\mathbf{z}}. \quad (3.56)$$

Since the polarization is uniform there is zero polarization charge density inside the sphere. However, there is a polarization charge sheet on the surface of the sphere whose density is given by  $\sigma_{\text{pol}} = \mathbf{P} \cdot \hat{\mathbf{r}}$  (see Eq. (3.29)). It follows that

$$\sigma_{\text{pol}} = 3\epsilon_0 \left( \frac{\epsilon - 1}{\epsilon + 2} \right) E_0 \cos \theta. \quad (3.57)$$

The problem of a dielectric cavity of radius  $a$  in a dielectric medium with dielectric constant  $\epsilon$  and with an applied electric field  $E_0$  parallel to the  $z$ -axis can be treated in much the same manner as that of a dielectric sphere. In fact, it is easily demonstrated that the results for the cavity can be obtained from those for the sphere by making the transformation  $\epsilon \rightarrow 1/\epsilon$ . Thus, the field inside the cavity is uniform, parallel to the  $z$ -axis, and of magnitude

$$E_1 = \frac{3\epsilon}{2\epsilon + 1} E_0. \quad (3.58)$$

Note that  $E_1 > E_0$  provided that  $\epsilon > 1$ . The field outside the cavity is the original field plus that of a  $z$ -directed dipole, located at the origin, whose dipole moment is

$$p = -4\pi\epsilon_0 \left( \frac{\epsilon - 1}{2\epsilon + 1} \right) a^3 E_0. \quad (3.59)$$

Here, the negative sign implies that the dipole points in the opposite direction to the external field.

### 3.6 The energy density within a dielectric medium

Consider a system of free charges embedded in a dielectric medium. The increase in the total energy when a small amount of free charge  $\delta\rho_f$  is added to the system is given by

$$\delta U = \int \phi \delta\rho_f d^3\mathbf{r}, \quad (3.60)$$

where the integral is taken over all space and  $\phi(\mathbf{r})$  is the electrostatic potential. Here, it is assumed that the original charges and the dielectric are held fixed, so

that no mechanical work is performed. It follows from Eq. (3.5) that

$$\delta U = \int \phi \nabla \cdot \delta \mathbf{D} d^3 \mathbf{r}, \quad (3.61)$$

where  $\delta \mathbf{D}$  is the change in the electric displacement associated with the charge increment. Now the above equation can also be written

$$\delta U = \int \nabla \cdot (\phi \delta \mathbf{D}) d^3 \mathbf{r} - \int \nabla \phi \cdot \delta \mathbf{D} d^3 \mathbf{r}, \quad (3.62)$$

giving

$$\delta U = \int \phi \delta \mathbf{D} \cdot d\mathbf{S} - \int \nabla \phi \cdot \delta \mathbf{D} d^3 \mathbf{r}, \quad (3.63)$$

where use has been made of Gauss's theorem. If the dielectric medium is of finite spatial extent then we can neglect the surface term to give

$$\delta U = - \int \nabla \phi \cdot \delta \mathbf{D} d^3 \mathbf{r} = \int \mathbf{E} \cdot \delta \mathbf{D} d^3 \mathbf{r}. \quad (3.64)$$

This energy increment cannot be integrated unless  $\mathbf{E}$  is a known function of  $\mathbf{D}$ . Let us adopt the conventional approach and assume that  $\mathbf{D} = \epsilon_0 \epsilon \mathbf{E}$ , where the dielectric constant  $\epsilon$  is independent of the electric field. The change in energy associated with taking the displacement field from zero to  $\mathbf{D}(\mathbf{r})$  at all points in space is given by

$$U = \int_0^{\mathbf{D}} \delta U = \int_0^{\mathbf{D}} \int \mathbf{E} \cdot \delta \mathbf{D} d^3 \mathbf{r}, \quad (3.65)$$

or

$$U = \int \int_0^{\mathbf{E}} \frac{\epsilon_0 \epsilon \delta(E^2)}{2} d^3 \mathbf{r} = \frac{1}{2} \int \epsilon_0 \epsilon E^2 d^3 \mathbf{r}, \quad (3.66)$$

which reduces to

$$U = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3 \mathbf{r}. \quad (3.67)$$

Thus, the electrostatic energy density inside a dielectric is given by

$$W = \frac{\mathbf{E} \cdot \mathbf{D}}{2}. \quad (3.68)$$

This is a standard result which is often quoted in textbooks. Nevertheless, it is important to realize that the above formula is only valid in dielectric media in which the electric displacement  $\mathbf{D}$  varies *linearly* with the electric field  $\mathbf{E}$ .

### 3.7 The force density within a dielectric medium

Equation (3.67) was derived by considering a virtual process in which true charges are added to a system of charges and dielectrics which are held fixed, so that no mechanical work is done against physical displacements. Let us now consider a different virtual process in which the physical coordinates of the charges and dielectric are given a virtual displacement  $\delta \mathbf{r}$  at each point in space, but no free charges are added to the system. Since we are dealing with a conservative system, the energy expression (3.67) can still be employed, despite the fact that it was derived in terms of another virtual process. The variation in the total electrostatic energy  $\delta U$  when the system undergoes a virtual displacement  $\delta \mathbf{r}$  is related to the electrostatic force density  $\mathbf{f}$  acting within the dielectric medium via

$$\delta U = - \int \mathbf{f} \cdot \delta \mathbf{r} d^3 \mathbf{r}. \quad (3.69)$$

If the medium is moving with a velocity field  $\mathbf{u}$  then the rate at which electrostatic energy is drained from the  $\mathbf{E}$  and  $\mathbf{D}$  fields is given by

$$\frac{dU}{dt} = - \int \mathbf{f} \cdot \mathbf{u} d^3 \mathbf{r}. \quad (3.70)$$

Let us now consider the energy increment due to both a change  $\delta \rho_f$  in the free charge distribution and a change  $\delta \epsilon$  in the dielectric constant, caused by the displacements. From Eq. (3.67)

$$\delta U = \frac{1}{2\epsilon_0} \int [D^2 \delta(1/\epsilon) + 2\mathbf{D} \cdot \delta \mathbf{D} / \epsilon] d^3 \mathbf{r}, \quad (3.71)$$

or

$$\delta U = -\frac{\epsilon_0}{2} \int E^2 \delta \epsilon d^3 \mathbf{r} + \int \mathbf{E} \cdot \delta \mathbf{D} d^3 \mathbf{r}. \quad (3.72)$$

Here, the first term represents the energy increment due to the change in dielectric constant associated with the virtual displacements, whereas the second term corresponds to the energy increment caused by displacements of the free charges. The second term can be written

$$\int \mathbf{E} \cdot \delta \mathbf{D} d^3 \mathbf{r} = - \int \nabla \phi \cdot \delta \mathbf{D} d^3 \mathbf{r} = \int \phi \nabla \cdot \delta \mathbf{D} d^3 \mathbf{r} = \int \phi \delta \rho_f d^3 \mathbf{r}, \quad (3.73)$$

where surface terms have been neglected. Thus, Eq. (3.72) implies that

$$\frac{dU}{dt} = \int \left( \phi \frac{\partial \rho_f}{\partial t} - \frac{\epsilon_0}{2} E^2 \frac{\partial \epsilon}{\partial t} \right) d^3 \mathbf{r}. \quad (3.74)$$

In order to arrive at an expression for the force density  $\mathbf{f}$  we need to express the time derivatives  $\partial \rho / \partial t$  and  $\partial \epsilon / \partial t$  in terms of the velocity field  $\mathbf{u}$ . This can be achieved by adopting a dielectric equation of state; *i.e.*, a relation which gives the dependence of the dielectric constant  $\epsilon$  on the mass density  $\rho_m$ . Let us assume that  $\epsilon(\rho_m)$  is a known function. It follows that

$$\frac{D\epsilon}{Dt} = \frac{d\epsilon}{d\rho_m} \frac{D\rho_m}{Dt}, \quad (3.75)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad (3.76)$$

is the total time derivative (*i.e.*, the time derivative in a frame of reference which is locally co-moving with the dielectric.) The hydrodynamic equation of continuity of the dielectric is

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0, \quad (3.77)$$

which implies that

$$\frac{D\rho_m}{Dt} = -\rho_m \nabla \cdot \mathbf{u}. \quad (3.78)$$

It follows that

$$\frac{\partial \epsilon}{\partial t} = -\frac{d\epsilon}{d\rho_m} \rho_m \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \epsilon. \quad (3.79)$$

The conservation equation for the free charges is written

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) = 0. \quad (3.80)$$

Thus, we can express Eq. (3.74) in the form

$$\frac{dU}{dt} = \int \left[ -\phi \nabla \cdot (\rho_f \mathbf{u}) + \frac{\epsilon_0}{2} E^2 \frac{d\epsilon}{d\rho_m} \rho_m \nabla \cdot \mathbf{u} + \left( \frac{\epsilon_0}{2} E^2 \nabla \epsilon \right) \cdot \mathbf{u} \right] d^3 \mathbf{r}. \quad (3.81)$$



Integrating the first term by parts and neglecting any surface contributions, we obtain

$$-\int \phi \nabla \cdot (\rho_f \mathbf{u}) d^3 \mathbf{r} = \int \rho_f \nabla \phi \cdot \mathbf{u} d^3 \mathbf{r}. \quad (3.82)$$

Likewise,

$$\int \frac{\epsilon_0}{2} E^2 \frac{d\epsilon}{d\rho_m} \rho_m \nabla \cdot \mathbf{u} d^3 \mathbf{r} = - \int \frac{\epsilon_0}{2} \nabla \left( E^2 \frac{d\epsilon}{d\rho_m} \rho_m \right) \cdot \mathbf{u} d^3 \mathbf{r}. \quad (3.83)$$

Thus, Eq. (3.81) becomes

$$\frac{dU}{dt} = \int \left[ -\rho_f \mathbf{E} + \frac{\epsilon_0}{2} E^2 \nabla \epsilon - \frac{\epsilon_0}{2} \nabla \left( E^2 \frac{d\epsilon}{d\rho_m} \rho_m \right) \right] \cdot \mathbf{u} d^3 \mathbf{r}. \quad (3.84)$$

Comparing with Eq. (3.70), we see that the force density inside the dielectric is given by

$$\mathbf{f} = \rho_f \mathbf{E} - \frac{\epsilon_0}{2} E^2 \nabla \epsilon + \frac{\epsilon_0}{2} \nabla \left( E^2 \frac{d\epsilon}{d\rho_m} \rho_m \right). \quad (3.85)$$

The first term in the above equation is the standard electrostatic force density. The second term represents a force which appears whenever an inhomogeneous dielectric is placed in an electric field. The last term, known as the *electrostriction* term, gives a force acting on a dielectric in an inhomogeneous electric field. Note that the magnitude of the electrostriction force depends explicitly on the dielectric equation of state of the material, through  $d\epsilon/d\rho_m$ . The electrostriction term gives zero net force acting on any finite region of dielectric if we can integrate over a large enough portion of the dielectric that its extremities lie in a field free region. For this reason the term is frequently omitted, since in the calculation of the total forces acting on dielectric bodies it usually does not contribute. Note, however, that if the electrostriction term is omitted an incorrect pressure variation within the dielectric is obtained, even though the total force is given correctly.

### 3.8 The Clausius-Mossotti relation

Let us now investigate what a dielectric equation of state actually looks like. Suppose that a dielectric medium is made up of identical molecules which develop

a dipole moment

$$\mathbf{p} = \alpha \epsilon_0 \mathbf{E} \quad (3.86)$$

when placed in an electric field  $\mathbf{E}$ . The constant  $\alpha$  is called the *molecular polarizability*. If  $N$  is the number density of such molecules then the polarization of the medium is

$$\mathbf{P} = N\mathbf{p} = N\alpha\epsilon_0\mathbf{E}, \quad (3.87)$$

or

$$\mathbf{P} = \frac{N_A \rho_m \alpha}{M} \epsilon_0 \mathbf{E}, \quad (3.88)$$

where  $\rho_m$  is the mass density,  $N_A$  is Avogadro's number, and  $M$  is the molecular weight. But, how does the electric field experienced by an individual molecule relate to the average electric field in the medium? This is not a trivial question since we expect the electric field to vary strongly (on atomic length-scales) inside the dielectric.

Suppose that the dielectric is polarized with a mean electric field  $\mathbf{E}_0$  which is uniform (on macroscopic length-scales) and directed along the  $z$ -axis. Consider one of the molecules which constitute the dielectric. Let us draw a sphere of radius  $a$  about this particular molecule. This is intended to represent the boundary between the microscopic and the macroscopic range of phenomena affecting the molecule. We shall treat the dielectric outside the sphere as a continuous medium and the dielectric inside the sphere as a collection of polarized molecules. According to Eq. (3.29) there is a polarization surface charge of magnitude

$$\sigma_{\text{pol}} = -P \cos \theta \quad (3.89)$$

on the inside of the sphere, where  $(r, \theta, \varphi)$  are spherical polar coordinates, and  $\mathbf{P} = P \hat{\mathbf{z}} = \epsilon_0(\epsilon - 1)E_0 \hat{\mathbf{z}}$  is the uniform polarization of the dielectric. The magnitude of  $E_z$  at the molecule due to the surface charge is

$$E_z = -\frac{1}{4\pi\epsilon_0} \int \frac{\sigma_{\text{pol}} \cos \theta}{a^2} dS, \quad (3.90)$$

where  $dS = 2\pi a^2 \sin \theta d\theta$  is a surface element of the sphere. It follows that

$$E_z = \frac{P}{2\epsilon_0} \int_0^\pi \cos^2 \theta \sin \theta d\theta = \frac{P}{3\epsilon_0}. \quad (3.91)$$

It is easily demonstrated that  $E_\theta = E_\varphi = 0$  at the molecule. Thus, the field at the molecule due to the surface charges on the sphere is

$$\mathbf{E} = \frac{\mathbf{P}}{3\epsilon_0}. \quad (3.92)$$

The field due to the individual molecules within the sphere is obtained by summing over the dipole fields of these molecules. The electric field at a distance  $\mathbf{r}$  from a dipole  $\mathbf{p}$  is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \left[ \frac{\mathbf{p}}{r^3} - \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} \right]. \quad (3.93)$$

It is assumed that the dipole moment of each molecule within the sphere is the same, and also that the molecules are evenly distributed throughout the sphere. This being the case, the value of  $E_z$  at the molecule due to all of the other molecules within in the sphere,

$$E_z = -\frac{1}{4\pi\epsilon_0} \sum_{\text{mols}} \left[ \frac{p_z}{r^3} - \frac{3(p_x xz + p_y yz + p_z z^2)}{r^5} \right], \quad (3.94)$$

is zero, since

$$\sum_{\text{mols}} x^2 = \sum_{\text{mols}} y^2 = \sum_{\text{mols}} z^2 = \frac{1}{3} \sum_{\text{mols}} r^2 \quad (3.95)$$

and

$$\sum_{\text{mols}} xy = \sum_{\text{mols}} yz = \sum_{\text{mols}} zx = 0. \quad (3.96)$$

It is easily seen that  $E_\theta = E_\varphi = 0$ . Hence, the electric field at the molecule due to the other molecules within the sphere vanishes.

It is clear that the net electric field seen by an individual molecule is

$$\mathbf{E} = \mathbf{E}_0 + \frac{\mathbf{P}}{3\epsilon_0}. \quad (3.97)$$

This is *larger* than the average electric field  $\mathbf{E}_0$  in the dielectric. The above analysis indicates that this effect is ascribable to the long range (rather than the

short range) interactions of the molecule with the other molecules in the medium. Making use of Eq. (3.88) and the definition  $\mathbf{P} = \epsilon_0(\epsilon - 1)\mathbf{E}_0$ , we obtain

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{N_A \rho_m \alpha}{3M}. \quad (3.98)$$

This is called the *Clausius-Mossotti* relation. This formula is found to work pretty well for a wide class of dielectric liquids and gases. The Clausius-Mossotti relation yields

$$\frac{d\epsilon}{d\rho_m} = \frac{(\epsilon - 1)(\epsilon + 2)}{3\rho_m}. \quad (3.99)$$

### 3.9 Dielectric liquids in electrostatic fields

Consider the behaviour of an uncharged dielectric liquid placed in an electrostatic field. If  $p$  is the pressure in the liquid when in equilibrium with the electrostatic force density  $\mathbf{f}$ , then force balance requires that

$$\nabla p = \mathbf{f}. \quad (3.100)$$

It follows from Eq. (3.85) that

$$\nabla p = -\frac{\epsilon_0}{2} E^2 \nabla \epsilon + \frac{\epsilon_0}{2} \nabla \left( E^2 \frac{d\epsilon}{d\rho_m} \rho_m \right) = \frac{\epsilon_0 \rho_m}{2} \nabla \left( E^2 \frac{d\epsilon}{d\rho_m} \right). \quad (3.101)$$

We can integrate this equation to give

$$\int_{p_1}^{p_2} \frac{dp}{\rho_m} = \frac{\epsilon_0}{2} \left( \left[ E^2 \frac{d\epsilon}{d\rho_m} \right]_2 - \left[ E^2 \frac{d\epsilon}{d\rho_m} \right]_1 \right), \quad (3.102)$$

where 1 and 2 refer to two general points in the liquid. Here, it is assumed that the liquid possesses an equation of state, so that  $p = p(\rho_m)$ . If the liquid is essentially incompressible ( $\rho_m \simeq \text{constant}$ ) then

$$p_2 - p_1 = \frac{\epsilon_0 \rho_m}{2} \left[ E^2 \frac{d\epsilon}{d\rho_m} \right]_1^2. \quad (3.103)$$

Finally, if the liquid obeys the Clausius-Mossotti relation then

$$p_2 - p_1 = \left[ \frac{\epsilon_0 E^2}{2} \frac{(\epsilon - 1)(\epsilon + 2)}{3} \right]_1^2. \quad (3.104)$$

According to Eqs. (3.54) and (3.104), if a sphere of dielectric liquid is placed in a uniform electric field  $\mathbf{E}_0$  then the pressure inside the liquid takes the constant value

$$p = \frac{3}{2} \epsilon_0 E_0^2 \frac{\epsilon - 1}{\epsilon + 2}. \quad (3.105)$$

It is clear that the electrostatic forces acting on the dielectric are all concentrated at the edge of the sphere and are directed radially inwards; *i.e.*, the dielectric is *compressed* by the external electric field. This is a somewhat surprising result since the electrostatic forces acting on a rigid conducting sphere are concentrated at the edge of the sphere but are directed radially outwards. We might expect these two cases to give the same result in the limit  $\epsilon \rightarrow \infty$ . The reason that this does not occur is because a dielectric liquid is slightly compressible and is, therefore, subject to an electrostriction force. There is no electrostriction force for the case of a completely rigid body. In fact, the force density inside a rigid dielectric (for which  $\nabla \cdot \mathbf{u} = 0$ ) is given by Eq. (3.85) with the third term (the electrostriction term) missing. It is easily seen that the force exerted by an electric field on a rigid dielectric is directed outwards and approaches that exerted on a rigid conductor in the limit  $\epsilon \rightarrow 0$ .

As is well known, when a pair of charged (parallel plane) capacitor plates are dipped into a dielectric liquid the liquid is drawn up between the plates to some extent. Let us examine this effect. We can, without loss of generality, assume that the transition from dielectric to vacuum takes place in a continuous manner. Consider the electrostatic pressure difference between a point  $A$  lying just above the surface of the liquid in between the plates and a point  $B$  lying just above the surface of the liquid well away from the capacitor where  $E = 0$ . The pressure difference is given by

$$p_A - p_B = - \int_A^B \mathbf{f} \cdot d\mathbf{l}. \quad (3.106)$$

Note, however, that the Clausius-Mossotti relation yields  $d\epsilon/d\rho_m = 0$  at both  $A$  and  $B$ , since  $\epsilon = 1$  in a vacuum (see Eq. (3.99)). Thus, it is clear from Eq. (3.85)

that the electrostriction term makes no contribution to the line integral (3.106). It follows that

$$p_A - p_B = \frac{\epsilon_0}{2} \int_A^B E^2 \nabla \epsilon \cdot d\mathbf{l}. \quad (3.107)$$

The only contribution to this integral comes from the vacuum/dielectric interface in the vicinity of point  $A$  (since  $\epsilon$  is constant inside the liquid, and  $E = 0$  in the vicinity of point  $B$ ). Suppose that the electric field at point  $A$  has normal and tangential (to the surface) components  $E_n$  and  $E_t$ , respectively. Making use of the boundary conditions that  $E_t$  and  $\epsilon E_n$  are constant across a vacuum/dielectric interface, we obtain

$$p_A - p_B = \frac{\epsilon_0}{2} \left[ E_t^2 (\epsilon - 1) + \epsilon^2 E_n^2 (\epsilon) \int_1^\epsilon \frac{d\epsilon}{\epsilon^2} \right], \quad (3.108)$$

giving

$$p_A - p_B = \frac{\epsilon_0 (\epsilon - 1)}{2} \left[ E_t^2 + \frac{E_n^2}{\epsilon} \right]. \quad (3.109)$$

This electrostatic pressure difference can be equated to the hydrostatic pressure difference  $\rho_m g h$  to determine the height  $h$  that the liquid rises between the plates. At first sight, the above analysis appears to suggest that the dielectric liquid is drawn upward by a surface force acting on the vacuum/dielectric interface in the region between the plates. In fact, this is far from being the case. A brief examination of Eq. (3.104) shows that this surface force is actually directed downwards. According to Eq. (3.85), the force which causes the liquid to rise between the plates is a volume force which develops in the region of non-uniform electric field at the base of the capacitor, where the field splays out between the plates. Thus, although we can determine the height to which the fluid rises between the plates without reference to the electrostriction force, it is, somewhat paradoxically, this force which is actually responsible for supporting the liquid against gravity.

Let us consider another paradox concerning the electrostatic forces exerted in a dielectric medium. Suppose that we have two charges embedded in a uniform dielectric  $\epsilon$ . The electric field generated by each charge is the same as that in vacuum, except that it is reduced by a factor  $\epsilon$ . Therefore, we expect that the force exerted by one charge on another is the same as that in vacuum, except

that it is also reduced by a factor  $\epsilon$ . Let us examine how this reduction in force comes about. Consider a simple example. Suppose that we take a parallel plate capacitor and insert a block of solid dielectric between the plates. Suppose, further, that there is a small vacuum gap between the faces of the block and each of the capacitor plates. Let  $\pm\sigma$  be the surface charge densities on each of the capacitor plates, and let  $\pm\sigma_p$  be the polarization charge densities which develop on the outer faces of the intervening dielectric block. The two layers of polarization charge produce equal and opposite electric fields on each plate, and their effects therefore cancel each other. Thus, from the point of view of electrical interaction alone there would appear to be no change in the force exerted by one capacitor plate on the other when a dielectric slab is placed between them (assuming that  $\sigma$  remains constant during this process). That is, the force per unit area (which is attractive) remains

$$f_s = \frac{\sigma^2}{2\epsilon_0}. \quad (3.110)$$

However, in experiments in which a capacitor is submerged in a dielectric liquid the force per unit area exerted by one plate on another is observed to decrease to

$$f_s = \frac{\sigma^2}{2\epsilon_0\epsilon}. \quad (3.111)$$

This apparent paradox can be explained by taking into account the difference in liquid pressure in the field filled space between the plates and the field free region outside the capacitor. This pressure difference is balanced by internal elastic forces in the case of the solid dielectric discussed earlier, but is transmitted to the plates in the case of the liquid. We can compute the pressure difference between a point  $A$  on the inside surface of one of the capacitor plates and a point  $B$  on the outside surface of the same plate using Eq. (3.107). If we neglect end effects then the electric field is normal to the plates in the region between the plates and is zero everywhere else. Thus, the only contribution to the line integral (3.107) comes from the plate/dielectric interface in the vicinity of point  $A$ . Using Eq. (3.109), we find that

$$p_A - p_B = \frac{\epsilon_0}{2} \left(1 - \frac{1}{\epsilon}\right) E^2 = \frac{\sigma^2}{2\epsilon_0} \left(1 - \frac{1}{\epsilon}\right), \quad (3.112)$$

where  $E$  is the normal field strength between the plates in the absence of dielectric. The sum of this pressure force and the purely electrical force (3.110) yields a net attractive force per unit area

$$f_s = \frac{\sigma^2}{2\epsilon_0\epsilon} \quad (3.113)$$

acting between the plates. Thus, any decrease in the forces exerted by charges on one another when they are immersed or embedded in some dielectric medium can only be understood in terms of mechanical forces transmitted between these charges by the medium itself.

### 3.10 Magnetization

All matter is built up out of atoms, and each atom consists of electrons in motion. The currents associated with this motion are termed *atomic currents*. Each atomic current is a tiny closed circuit of atomic dimensions, and may therefore be appropriately described as a magnetic dipole. If the atomic currents of a given atom all flow in the same plane then the atomic dipole moment is directed normal to the plane (in the sense given by the right-hand rule) and its magnitude is the product of the total circulating current and the area of the current loop. More generally, if  $\mathbf{j}(\mathbf{r})$  is the atomic current density at the point  $\mathbf{r}$  then the magnetic moment of the atom is

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \wedge \mathbf{j} d^3\mathbf{r}, \quad (3.114)$$

where the integral is over the volume of the atom. If there are  $N$  such atoms or molecules per unit volume then the *magnetization*  $\mathbf{M}$  (*i.e.*, the magnetic dipole moment per unit volume) is given by  $\mathbf{M} = N\mathbf{m}$ . More generally,

$$\mathbf{M}(\mathbf{r}) = \sum_i N_i \langle \mathbf{m}_i \rangle, \quad (3.115)$$

where  $\langle \mathbf{m}_i \rangle$  is the average magnetic dipole moment of the  $i$ th type of molecule in the vicinity of point  $\mathbf{r}$ , and  $N_i$  is the average number of such molecules per unit volume at  $\mathbf{r}$ .

Consider a general medium which is made up of molecules which are polarizable and possess a net magnetic moment. It is easily demonstrated that any



circulation in the magnetization field  $\mathbf{M}(\mathbf{r})$  gives rise to an effective current density  $\mathbf{j}_m$  in the medium. In fact,

$$\mathbf{j}_m = \nabla \wedge \mathbf{M}. \quad (3.116)$$

This current density is called the *magnetization current density*, and is usually distinguished from the *true current density*  $\mathbf{j}_t$ , which represents the convection of free charges in the medium. In fact, there is a third type of current called a *polarization current*, which is due to the apparent convection of bound charges. It is easily demonstrated that the polarization current density  $\mathbf{j}_p$  is given by

$$\mathbf{j}_p = \frac{\partial \mathbf{P}}{\partial t}. \quad (3.117)$$

Thus, the total current density  $\mathbf{j}$  in the medium is given by

$$\mathbf{j} = \mathbf{j}_t + \nabla \wedge \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \quad (3.118)$$

It must be emphasized that all terms on the right-hand side of this equation represent real physical currents, although only the first term is due to the motion of real charges (over more than atomic dimensions).

The Ampère-Maxwell equation takes the form

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (3.119)$$

which can also be written

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}_t + \mu_0 \nabla \wedge \mathbf{M} + \mu_0 \frac{\partial \mathbf{D}}{\partial t}, \quad (3.120)$$

where use has been made of the definition  $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ . The above expression can be rearranged to give

$$\nabla \wedge \mathbf{H} = \mathbf{j}_t + \frac{\partial \mathbf{D}}{\partial t}, \quad (3.121)$$

where

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (3.122)$$

is termed the *magnetic intensity*, and has the same dimensions as  $\mathbf{M}$  (*i.e.*, magnetic dipole moment per unit volume). In a steady-state situation, Stokes's theorem tell us that

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{j}_t \cdot d\mathbf{S}. \quad (3.123)$$

In other words, the line integral of  $\mathbf{H}$  around some closed curve is equal to the flux of true current through any surface attached to that curve. Unlike the magnetic field  $\mathbf{B}$  (which specifies the force  $e \mathbf{v} \wedge \mathbf{B}$  acting on a charge  $e$  moving with velocity  $\mathbf{v}$ ) or the magnetization  $\mathbf{M}$  (the magnetic dipole moment per unit volume), the magnetic intensity  $\mathbf{H}$  has no clear physical meaning. The only reason for introducing it is that it enables us to calculate fields in the presence of magnetic materials without first having to know the distribution of magnetization currents. However, this is only possible if we possess a constitutive relation connecting  $\mathbf{B}$  and  $\mathbf{H}$ .

### 3.11 Magnetic susceptibility and permeability

In a large class of materials there exists an approximately linear relationship between  $\mathbf{M}$  and  $\mathbf{H}$ . If the material is isotropic then

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (3.124)$$

where  $\chi_m$  is called the magnetic susceptibility. If  $\chi_m$  is positive the material is called *paramagnetic*, and the magnetic field is strengthened by the presence of the material. If  $\chi_m$  is negative then the material is *diamagnetic* and the magnetic field is weakened in the presence of the material. The magnetic susceptibilities of paramagnetic and diamagnetic materials are generally extremely small. A few sample values are given in Table 1.<sup>10</sup>

A linear relationship between  $\mathbf{M}$  and  $\mathbf{H}$  also implies a linear relationship between  $\mathbf{B}$  and  $\mathbf{H}$ . In fact, we can write

$$\mathbf{B} = \mu \mathbf{H}, \quad (3.125)$$

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<sup>10</sup>Data obtained from the *Handbook of Chemistry and Physics*, Chemical Rubber Company Press, Boca Raton, FL

Material	$\chi_m$
Aluminium	$2.3 \times 10^{-5}$
Copper	$-0.98 \times 10^{-5}$
Diamond	$-2.2 \times 10^{-5}$
Tungsten	$6.8 \times 10^{-5}$
Hydrogen (1 atm)	$-0.21 \times 10^{-8}$
Oxygen (1 atm)	$209.0 \times 10^{-8}$
Nitrogen (1 atm)	$-0.50 \times 10^{-8}$

Table 1: Magnetic susceptibilities of some paramagnetic and diamagnetic materials at room temperature

where

$$\mu = \mu_0(1 + \chi_m) \quad (3.126)$$

is termed the magnetic *permeability* of the material in question. (Likewise,  $\mu_0$  is termed the *permeability of free space*.) It is clear from Table 1 that the permeabilities of common diamagnetic and paramagnetic materials do not differ substantially from that of free space. In fact, to all intents and purposes the magnetic properties of such materials can be safely neglected (*i.e.*,  $\mu = \mu_0$ ).

### 3.12 Ferromagnetism

There is, however, a third class of magnetic materials called *ferromagnetic* materials. Such materials are characterized by a possible permanent magnetization, and generally have a profound effect on magnetic fields (*i.e.*,  $\mu/\mu_0 \gg 1$ ). Unfortunately, ferromagnetic materials do *not* exhibit a linear dependence between  $\mathbf{M}$  and  $\mathbf{H}$  or  $\mathbf{B}$  and  $\mathbf{H}$ , so that we cannot employ Eqs. (3.124) and (3.125) with constant values of  $\chi_m$  and  $\mu$ . It is still expedient to use Eq. (3.125) as the definition of  $\mu$ , with  $\mu = \mu(\mathbf{H})$ , however this practice can lead to difficulties under certain circumstances. The permeability of a ferromagnetic material, as defined by Eq. (3.125), can vary through the entire range of possible values from zero to infinity, and may be either positive or negative. The most sensible approach is to consider each problem involving ferromagnetic materials separately, try to

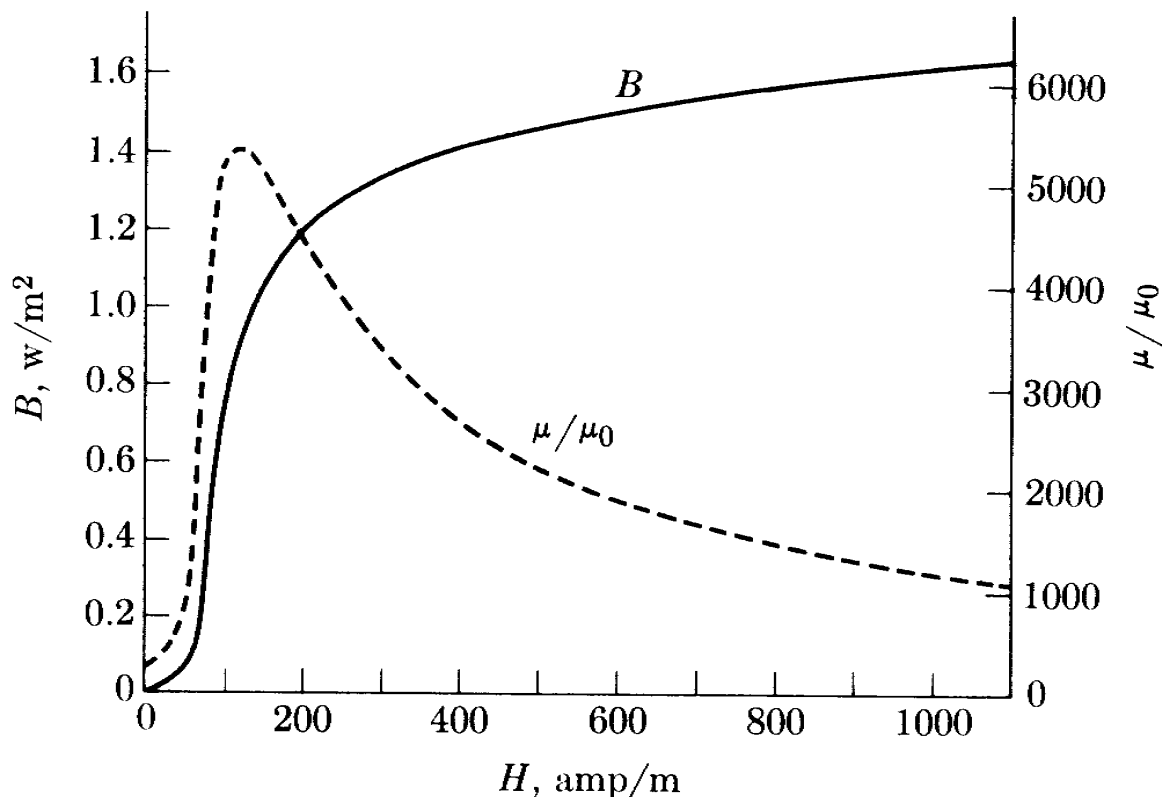


Figure 2: *Magnetization curve and relative permeability of commercial iron (annealed)*

determine which region of the  $\mathbf{B}$ - $\mathbf{H}$  diagram is important for the particular case in hand, and then make approximations appropriate to this region.

First, let us consider an unmagnetized sample of ferromagnetic material. If the magnetic intensity, which is initially zero, is increased *monotonically*, then the  $\mathbf{B}$ - $\mathbf{H}$  relationship traces out a curve such as that shown in Fig. 2. This is called a *magnetization curve*. It is evident that the permeabilities  $\mu$  derived from the curve (according to the rule  $\mu = B/H$ ) are always positive, and show a wide range of values. The maximum permeability occurs at the “knee” of the curve. In some materials this maximum permeability is as large as  $10^5 \mu_0$ . The reason for the knee in the curve is that the magnetization  $\mathbf{M}$  reaches a maximum value in the material, so that

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (3.127)$$

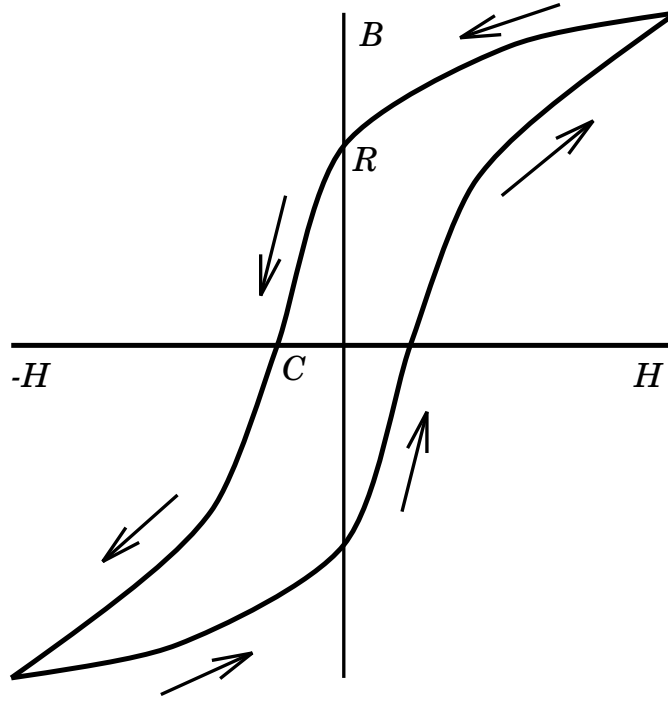


Figure 3: *Typical hysteresis loop of a ferromagnetic material*

continues to increase at large  $\mathbf{H}$  only because of the  $\mu_0 \mathbf{H}$  term. The maximum value of  $\mathbf{M}$  is called the *saturation magnetization* of the material.

Next, consider a ferromagnetic sample magnetized by the above procedure. If the magnetic intensity  $\mathbf{H}$  is decreased, the  $\mathbf{B}-\mathbf{H}$  relation does not follow back down the curve of Fig. 2, but instead moves along a new curve, shown in Fig. 3, to the point  $R$ . The magnetization, once established, does not disappear with the removal of  $\mathbf{H}$ . In fact, it takes a reversed magnetic intensity to reduce the magnetization to zero. If  $\mathbf{H}$  continues to build up in the reversed direction, then  $\mathbf{M}$  (and hence  $\mathbf{B}$ ) becomes increasingly negative. Finally, when  $\mathbf{H}$  increases again the operating point follows the lower curve of Fig. 3. Thus, the  $\mathbf{B}-\mathbf{H}$  curve for increasing  $\mathbf{H}$  is quite different to that for decreasing  $\mathbf{H}$ . This phenomenon is known as *hysteresis*.

The curve of Fig. 3 is called the hysteresis loop of the material in question. The value of  $\mathbf{B}$  at the point  $R$  is called the *retentivity* or *remanence*. The magnitude of  $\mathbf{H}$  at the point  $C$  is called the *coercivity*. It is evident that  $\mu$  is negative in the

second and fourth quadrants of the diagram and positive in the first and third quadrants. The shape of the hysteresis loop depends not only on the nature of the ferromagnetic material but also on the maximum value of  $\mathbf{H}$  to which the material is subjected. However, once this maximum value,  $\mathbf{H}_{\max}$ , becomes sufficient to produce saturation in the material the hysteresis loop does not change shape with any further increase in  $\mathbf{H}_{\max}$ .

Ferromagnetic materials are used either to channel magnetic flux (*e.g.*, around transformer circuits) or as sources of magnetic field (permanent magnets). For use as a permanent magnet, the material is first magnetized by placing it in a strong magnetic field. However, once the magnet is removed from the external field it is subject to a demagnetizing  $\mathbf{H}$ . Thus, it is vitally important that a permanent magnet should possess both a large remanence and a large coercivity. As will become clear later on, it is generally a good idea for the ferromagnetic materials used to channel magnetic flux around transformer circuits to possess small remanences and small coercivities.

### 3.13 Boundary conditions for $\mathbf{B}$ and $\mathbf{H}$

What are the matching conditions for  $\mathbf{B}$  and  $\mathbf{H}$  at the boundary between two media? The governing equations for a steady state situation are

$$\nabla \cdot \mathbf{B} = 0, \quad (3.128)$$

and

$$\nabla \wedge \mathbf{H} = \mathbf{j}_t. \quad (3.129)$$

Integrating Eq. (3.128) over a Gaussian pill-box enclosing part of the boundary surface between the two media gives

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n}_{21} = 0, \quad (3.130)$$

where  $\mathbf{n}_{21}$  is the unit normal to this surface directed from medium 1 to medium 2. Integrating Eq. (3.129) around a small loop which straddles the boundary surface yields

$$(\mathbf{H}_2 - \mathbf{H}_1) \wedge \mathbf{n}_{21} = 0, \quad (3.131)$$

assuming that there is no true current sheet flowing in this surface. In general, there is a magnetization current sheet flowing in the boundary surface whose density is given by

$$\mathbf{J}_m = \mathbf{n}_{21} \wedge (\mathbf{M}_2 - \mathbf{M}_1), \quad (3.132)$$

where  $\mathbf{M}_1$  is the magnetization in medium 1 at the boundary, *etc.* It is clear that the normal component of the magnetic field and the tangential component of the magnetic intensity are both continuous across any boundary between magnetic materials.

### 3.14 Permanent ferromagnets

Let us consider the magnetic field generated by a distribution of permanent ferromagnets. Let us suppose that the magnets in question are sufficiently “hard” that their magnetization is essentially independent of the applied field for moderate field strengths. Such magnets can be treated as if they contain a fixed, specified magnetization  $\mathbf{M}(\mathbf{r})$ .

Let us assume that there are no true currents in the problem, so that  $\mathbf{j}_t = 0$ . Let us also assume that we are dealing with a steady state situation. Under these circumstances Eq. (3.121) reduces to

$$\nabla \wedge \mathbf{H} = 0. \quad (3.133)$$

It follows that we can write

$$\mathbf{H} = -\nabla \phi_m, \quad (3.134)$$

where  $\phi_m$  is called the *magnetic scalar potential*. Now, we know that

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0. \quad (3.135)$$

Equations (3.134) and (3.135) combine to give

$$\nabla^2 \phi_m = -\rho_m, \quad (3.136)$$

where

$$\rho_m = -\nabla \cdot \mathbf{M}. \quad (3.137)$$

Thus, the *magnetostatic* field  $\mathbf{H}$  is determined by Poisson's equation. We can think of  $\rho_m$  as an *effective magnetic charge density*. Of course, this magnetic charge has no physical reality. We have only introduced it in order to make the problem of the steady magnetic field generated by a set of permanent magnets look formally the same as that of the steady electric field generated by a distribution of charges.

The unique solution of Poisson's equation, subject to sensible boundary conditions at infinity, is well known:

$$\phi_m(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\rho_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (3.138)$$

This yields

$$\phi_m(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (3.139)$$

If the magnetization field  $\mathbf{M}(\mathbf{r})$  is well behaved and localized we can integrate by parts to obtain

$$\phi_m(\mathbf{r}) = \frac{1}{4\pi} \int \mathbf{M}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}'. \quad (3.140)$$

Now

$$\nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right), \quad (3.141)$$

so our expression for the magnetic potential can be written

$$\phi_m(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (3.142)$$

Far from the region of non-vanishing magnetization the potential reduces to

$$\phi_m(\mathbf{r}) \simeq -\nabla \left( \frac{1}{4\pi r} \right) \cdot \int \mathbf{M}(\mathbf{r}') d^3\mathbf{r}' \simeq \frac{\mathbf{m} \cdot \mathbf{r}}{4\pi r^3}, \quad (3.143)$$

where  $\mathbf{m} = \int \mathbf{M} d^3\mathbf{r}$  is the total magnetic moment of the distribution. This is the scalar potential of a dipole. Thus, an arbitrary localized distribution of



magnetization asymptotically produces a dipole magnetic field whose strength is determined by the net magnetic moment of the distribution.

It is often a good approximation to treat the magnetization field  $\mathbf{M}(\mathbf{r})$  as a discontinuous quantity. In other words,  $\mathbf{M}(\mathbf{r})$  is specified throughout the “hard” ferromagnets in question, and suddenly falls to zero at the boundaries of these magnets. Integrating Eq. (3.137) over a Gaussian pill-box which straddles one of these boundaries leads to the conclusion that there is an *effective magnetic surface charge density*,

$$\sigma_m = \mathbf{n} \cdot \mathbf{M}, \quad (3.144)$$

on the surface of the ferromagnets, where  $\mathbf{M}$  is the surface magnetization, and  $\mathbf{n}$  is a unit outward directed normal to the surface. Under these circumstances Eq. (3.139) yields

$$\phi_m(\mathbf{r}) = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' + \frac{1}{4\pi} \int_S \frac{\mathbf{M}(\mathbf{r}') \cdot d\mathbf{S}'}{|\mathbf{r} - \mathbf{r}'|}, \quad (3.145)$$

where  $V$  represents the volume occupied by the magnets and  $S$  is the bounding surface to  $V$ . Here,  $d\mathbf{S}$  is an outward directed volume element to  $S$ . It is clear that Eq. (3.145) consists of a volume integral involving the volume magnetic charges  $\rho_m = -\nabla \cdot \mathbf{M}$  and a surface integral involving the surface magnetic charges  $\sigma_m = \mathbf{n} \cdot \mathbf{M}$ . If the magnetization is uniform throughout the volume  $V$  then the first term in the above expression vanishes and only the surface integral makes a contribution.

We can also write  $\mathbf{B} = \nabla \wedge \mathbf{A}$  in order to satisfy  $\nabla \cdot \mathbf{B} = 0$  automatically. It follows from Eqs. (3.121) and (3.122) that

$$\nabla \wedge \mathbf{H} = \nabla \wedge (\mathbf{B}/\mu_0 - \mathbf{M}) = 0, \quad (3.146)$$

which gives

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}_m, \quad (3.147)$$

since  $\mathbf{j}_m = \nabla \wedge \mathbf{M}$ . The unique solution to Eq. (3.147), subject to sensible boundary conditions at infinity, is very well known:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (3.148)$$

Thus,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \wedge \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (3.149)$$

If the magnetization field is discontinuous it is necessary to add a surface integral to the above expression. It is straightforward to show that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \wedge \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' + \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{M}(\mathbf{r}') \wedge d\mathbf{S}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (3.150)$$

It is clear that the above expression consists of a volume integral involving the volume magnetization currents  $\mathbf{j}_m = \nabla \wedge \mathbf{M}$  and a surface integral involving the surface magnetization currents  $\mathbf{J}_m = \mathbf{M} \wedge \mathbf{n}$  (see Eq. (3.132)). If the magnetization field is uniform throughout  $V$  then only the surface integral makes a contribution.

### 3.15 A uniformly magnetized sphere

Consider a sphere of radius  $a$ , with a uniform permanent magnetization  $\mathbf{M} = M_0 \hat{\mathbf{z}}$ , surrounded by a vacuum region. The simplest way of solving this problem is in terms of the scalar magnetic potential introduced in Eq. (3.134). From Eqs. (3.136) and (3.137), it is clear that  $\phi_m$  satisfies Laplace's equation,

$$\nabla^2 \phi_m = 0, \quad (3.151)$$

since there is zero volume magnetic charge density in a vacuum or a uniformly magnetized magnetic medium. However, according to Eq. (3.144), there is a magnetic surface charge density,

$$\sigma_m = \hat{\mathbf{r}} \cdot \mathbf{M} = M_0 \cos \theta, \quad (3.152)$$

on the surface of the sphere. One of the matching conditions at the surface of the sphere is that the tangential component of  $\mathbf{H}$  must be continuous. It follows from Eq. (3.134) that the scalar magnetic potential must be continuous at  $r = a$ , so that

$$\phi_m(r = a_+) = \phi_m(r = a_-). \quad (3.153)$$

Integrating Eq. (3.136) over a Gaussian pill-box straddling the surface of the sphere yields

$$\left[ \frac{\partial \phi_m}{\partial r} \right]_{r=a-}^{r=a+} = -\sigma_m = -M_0 \cos \theta. \quad (3.154)$$

In other words, the magnetic charge sheet on the surface of the sphere gives rise to a discontinuity in the radial gradient of the magnetic scalar potential at  $r = a$ .

The most general axisymmetric solution to Eq. (3.151) which satisfies physical boundary conditions at  $r = a$  and  $r = \infty$  is

$$\phi_m(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \quad (3.155)$$

for  $r < a$ , and

$$\phi_m(r, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos \theta) \quad (3.156)$$

for  $r \geq a$ . The boundary condition (3.153) yields

$$B_l = A_l a^{2l+1} \quad (3.157)$$

for all  $l$ . The boundary condition (3.154) gives

$$-\frac{(l+1) B_l}{a^{l+2}} - l A_l a^{l-1} = -M_0 \delta_{l1} \quad (3.158)$$

for all  $l$ , since  $P_l(\cos \theta) = \cos \theta$ . It follows that

$$A_l = B_l = 0 \quad (3.159)$$

for  $l \neq 1$ , and

$$A_1 = \frac{M_0}{3}, \quad (3.160a)$$

$$B_1 = \frac{M_0 a^3}{3}. \quad (3.160b)$$

Thus,

$$\phi_m(r, \theta) = \frac{M_0 a^2}{3} \frac{r}{a^2} \cos \theta \quad (3.161)$$

for  $r < a$ , and

$$\phi_m(r, \theta) = \frac{M_0 a^2}{3} \frac{a}{r^2} \cos \theta \quad (3.162)$$

for  $r \geq a$ . Since there is a uniqueness theorem associated with Poisson's equation, we can be sure that this axisymmetric potential is the only solution to the problem which satisfies physical boundary conditions at  $r = 0$  and infinity.

In the vacuum region outside the sphere

$$\mathbf{B} = \mu_0 \mathbf{H} = -\mu_0 \nabla \phi_m. \quad (3.163)$$

It is easily demonstrated from Eq. (3.162) that

$$\mathbf{B}(r > a) = \frac{\mu_0}{4\pi} \left[ -\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r}}{r^5} \right], \quad (3.164)$$

where

$$\mathbf{m} = \frac{4}{3} \pi a^3 \mathbf{M}. \quad (3.165)$$

This, of course, is the magnetic field of a magnetic dipole  $\mathbf{m}$ . Not surprisingly, the net dipole moment of the sphere is equal to the integral of the magnetization  $\mathbf{M}$  (which is the dipole moment per unit volume) over the volume of the sphere.

Inside the sphere we have  $\mathbf{H} = -\nabla \phi_m$  and  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ , giving

$$\mathbf{H} = -\frac{\mathbf{M}}{3}, \quad (3.166)$$

and

$$\mathbf{B} = \frac{2}{3} \mu_0 \mathbf{M}. \quad (3.167)$$

Thus, both the  $\mathbf{H}$  and  $\mathbf{B}$  fields are uniform inside the sphere. Note that the magnetic intensity is oppositely directed to the magnetization. In other words, the  $\mathbf{H}$  field acts to *demagnetize* the sphere. How successful it is at achieving

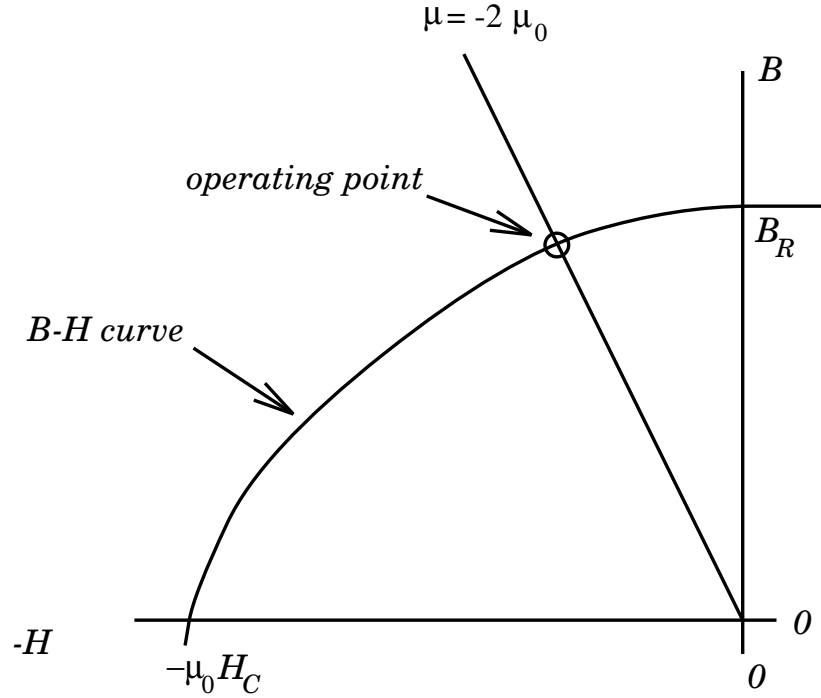


Figure 4: *Schematic demagnetization curve for a permanent magnet*

this depends on the shape of the hysteresis curve in the negative  $H$  and positive  $B$  quadrant. This curve is sometimes called the *demagnetization curve* of the magnetic material which makes up the sphere. Figure 4 shows a schematic demagnetization curve. The curve is characterized by two quantities: the retentivity  $B_R$  (*i.e.*, the residual magnetic field strength at zero magnetic intensity) and the coercivity  $\mu_0 H_c$  (*i.e.*, the negative magnetic intensity required to demagnetize the material: this quantity is conventionally multiplied by  $\mu_0$  to give it the units of magnetic field strength). The operating point (*i.e.*, the values of  $B$  and  $\mu_0 H$  inside the sphere) is obtained from the intersection of the demagnetization curve and the curve  $B = \mu H$ . It is clear from Eqs. (3.166) and (3.167) that

$$\mu = -2 \mu_0 \quad (3.168)$$

for a uniformly magnetized sphere in the absence of external fields. The magnetization inside the sphere is easily calculated once the operating point has been determined. In fact,  $M_0 = B - \mu_0 H$ . It is clear from Fig. 4 that for a magnetic material to be a good permanent magnet it must possess both a large retentivity *and* a large coercivity. A material with a large retentivity but a small coercivity

is unable to retain a significant magnetization in the absence of a strong external magnetizing field.

### 3.16 A soft iron sphere in a uniform magnetic field

The opposite extreme to a “hard” ferromagnetic material, which can maintain a large remnant magnetization in the absence of external fields, is a “soft” ferromagnetic material, for which the remnant magnetization is relatively small. Let us consider a somewhat idealized situation in which the remnant magnetization is negligible. In this situation there is no hysteresis, so the  $\mathbf{B}$ - $\mathbf{H}$  relation for the material reduces to

$$\mathbf{B} = \mu(B) \mathbf{H}, \quad (3.169)$$

where  $\mu(B)$  is a single valued function. The most commonly occurring “soft” ferromagnetic material is soft iron (*i.e.*, annealed, low impurity iron).

Consider a sphere of soft iron placed in an initially uniform external field  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ . The  $\mu_0 \mathbf{H}$  and  $\mathbf{B}$  fields inside the sphere are most easily obtained by taking the solutions (3.166) and (3.167) (which are still valid in this case), and superimposing on them the uniform field  $\mathbf{B}_0$ . We are justified in doing this because the equations which govern magnetostatic problems are *linear*. Thus, inside the sphere we have

$$\mu_0 \mathbf{H} = \mathbf{B}_0 - \frac{1}{3} \mu_0 \mathbf{M}, \quad (3.170a)$$

$$\mathbf{B} = \mathbf{B}_0 + \frac{2}{3} \mu_0 \mathbf{M}. \quad (3.170b)$$

Combining Eqs. (3.169) and (3.170) yields

$$\mu_0 \mathbf{M} = 3 \left( \frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \mathbf{B}_0, \quad (3.171)$$

with

$$\mathbf{B} = \left( \frac{3\mu}{\mu + 2\mu_0} \right) \mathbf{B}_0, \quad (3.172)$$

where, in general,  $\mu = \mu(B)$ . Clearly, for a highly permeable material (*i.e.*,  $\mu/\mu_0 \gg 1$ , which is certainly the case for soft iron) the magnetic field strength inside the sphere is approximately three times that of the externally applied field. In other words, the magnetic field is amplified inside the sphere.

The amplification of the magnetic field by a factor three in the high permeability limit is specific to a sphere. It can be shown that for elongated objects (*e.g.*, rods), aligned along the direction of the external field, the amplification factor can be considerably larger than this.

It is important to realize that the magnetization inside a ferromagnetic material cannot increase without limit. The maximum possible value of  $\mathbf{M}$  is called the saturation magnetization, and is usually denoted  $\mathbf{M}_s$ . Most ferromagnetic materials saturate when they are placed in external magnetic fields whose strengths are greater than, or of order, one tesla. Suppose that our soft iron sphere first attains the saturation magnetization when the unperturbed external magnetic field strength is  $B_s$ . It follows from (3.170b) and (3.171) (with  $\mu \gg \mu_0$ ) that

$$B = B_0 + 2B_s \quad (3.173)$$

inside the sphere, for  $B_0 > B_s$ . In this case, the field amplification factor is

$$\frac{B}{B_0} = 1 + 2 \frac{B_s}{B_0}. \quad (3.174)$$

Thus, for  $B_0 \gg B_s$  the amplification factor approaches unity. We conclude that if a ferromagnetic material is placed in an external field which greatly exceeds that required to cause saturation then the material effectively loses its magnetic properties, so that  $\mu \simeq \mu_0$ . Clearly, it is very important to avoid saturating the soft magnets used to channel magnetic flux around transformer circuits. This sets an upper limit on the magnetic field strengths which can occur in such circuits.

### 3.17 Magnetic shielding

There are many situations, particularly in experimental physics, where it is desirable to shield a certain region from magnetic fields. This can be achieved by

surrounding the region in question by a material of high permeability. It is vitally important that a material used as a magnetic shield does not develop a permanent magnetization in the presence of external fields, otherwise the material itself may become a source of magnetic fields. The most effective commercially available magnetic shielding material is called *Mumetal*, and is an alloy of 5% Copper, 2% Chromium, 77% Nickel, and 16% Iron. The maximum permeability of Mumetal is about  $10^5 \mu_0$ . This material also possesses a particularly low retentivity and coercivity. Unfortunately, Mumetal is *extremely* expensive. Let us investigate how much of this material is actually required to shield a given region from an external magnetic field.

Consider a spherical shell of magnetic shielding, made up of material of permeability  $\mu$ , placed in a formerly uniform magnetic field  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ . Suppose that the inner radius of the shell is  $a$  and the outer radius is  $b$ . Since there are no free currents in the problem, we can write  $\mathbf{H} = -\nabla\phi_m$ . Furthermore, since  $\mathbf{B} = \mu\mathbf{H}$  and  $\nabla \cdot \mathbf{B} = 0$ , it is clear that the magnetic scalar potential satisfies Laplace's equation,  $\nabla^2\phi_m = 0$ , throughout all space. The boundary conditions are that the potential must be well behaved at  $r = 0$  and  $r \rightarrow \infty$ , and also that the tangential and the normal components of  $\mathbf{H}$  and  $\mathbf{B}$ , respectively, must be continuous at  $r = a$  and  $r = b$ . The boundary conditions on  $\mathbf{H}$  merely imply that the scalar potential  $\phi_m$  must be continuous at  $r = a$  and  $r = b$ . The boundary conditions on  $\mathbf{B}$  yield

$$\mu_0 \frac{\partial\phi_m(r = a-)}{\partial r} = \mu \frac{\partial\phi_m(r = a+)}{\partial r}, \quad (3.175a)$$

$$\mu_0 \frac{\partial\phi_m(r = b+)}{\partial r} = \mu \frac{\partial\phi_m(r = b-)}{\partial r}. \quad (3.175b)$$

Let us try the following test solution for the magnetic potential:

$$\phi_m = -\frac{B_0}{\mu_0} r \cos \theta + \frac{\alpha}{r^2} \cos \theta \quad (3.176)$$

for  $r > b$ ,

$$\phi_m = \left( \beta r + \frac{\gamma}{r^2} \right) \cos \theta \quad (3.177)$$



for  $b \geq r \geq a$ , and

$$\phi_m = \delta r \cos \theta \quad (3.178)$$

for  $r < a$ . This potential is certainly a solution of Laplace's equation throughout space. It yields the uniform magnetic field  $\mathbf{B}_0$  as  $r \rightarrow \infty$ , and satisfies physical boundary conditions at  $r = 0$  and infinity. Since there is a uniqueness theorem associated with Poisson's equation, we can be certain that this potential is the correct solution to the problem provided that the arbitrary constants  $\alpha$ ,  $\beta$ , *etc.* can be adjusted in such a manner that the boundary conditions at  $r = a$  and  $r = b$  are also satisfied.

The continuity of  $\phi_m$  at  $r = a$  and  $r = b$  requires that

$$\beta a + \frac{\gamma}{a^2} = \delta a, \quad (3.179)$$

and

$$\beta b + \frac{\gamma}{b^2} = -\frac{B_0}{\mu_0} b + \frac{\alpha}{b^2}. \quad (3.180)$$

The boundary conditions (3.175) yield

$$\mu_0 \delta = \mu \left( \beta - \frac{2\gamma}{a^3} \right), \quad (3.181)$$

and

$$\mu_0 \left( -\frac{B_0}{\mu_0} - \frac{2\alpha}{b^3} \right) = \mu \left( \beta - \frac{2\gamma}{b^3} \right). \quad (3.182)$$

It follows that

$$\mu_0 \alpha = \left[ \frac{(2\mu + \mu_0)(\mu - \mu_0)}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2(a^3/b^3)(\mu - \mu_0)^2} \right] (b^3 - a^3) B_0, \quad (3.183a)$$

$$\mu_0 \beta = - \left[ \frac{3(2\mu + \mu_0)\mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2(a^3/b^3)(\mu - \mu_0)^2} \right] B_0, \quad (3.183b)$$

$$\mu_0 \gamma = - \left[ \frac{3(\mu - \mu_0)\mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2(a^3/b^3)(\mu - \mu_0)^2} \right] a^3 B_0, \quad (3.183c)$$

$$\mu_0 \delta = - \left[ \frac{9\mu\mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2(a^3/b^3)(\mu - \mu_0)^2} \right] B_0. \quad (3.183d)$$

Consider the limit of a thin, high permeability shell for which  $b = a + d$ ,  $d/a \ll 1$ , and  $\mu/\mu_0 \gg 1$ . In this limit, the field inside the shell is given by

$$\mathbf{B} \simeq \frac{3}{2} \frac{\mu_0}{\mu} \frac{a}{d} \mathbf{B}_0. \quad (3.184)$$

Thus, if  $\mu \simeq 10^5 \mu_0$  for Mumetal, then we can reduce the magnetic field strength inside the shell by almost a factor of 1000 using a shell whose thickness is only 1/100th of its radius. Clearly, a little Mumetal goes a long way! Note, however, that as the external field strength,  $B_0$ , is increased, the Mumetal shell eventually saturates, and  $\mu/\mu_0$  gradually falls to unity. Thus, extremely strong magnetic fields (typically,  $B_0 \gtrsim 1$  tesla) are hardly shielded at all by Mumetal, or similar magnetic materials.

### 3.18 Magnetic energy

Consider an electrical conductor. Suppose that a battery with an electromotive field  $\mathbf{E}'$  is feeding energy into this conductor. The energy is either dissipated as heat or is used to generate a magnetic field. Ohm's law inside the conductor gives

$$\mathbf{j}_t = \sigma(\mathbf{E} + \mathbf{E}'), \quad (3.185)$$

where  $\mathbf{j}_t$  is the true current density,  $\sigma$  is the conductivity, and  $\mathbf{E}$  is the inductive electric field. Taking the scalar product with  $\mathbf{j}_t$ , we obtain

$$\mathbf{E}' \cdot \mathbf{j}_t = \frac{j_t^2}{\sigma} - \mathbf{E} \cdot \mathbf{j}_t. \quad (3.186)$$

The left-hand side of this equation represents the rate at which the battery does work on the conductor. The first term on the right-hand side is the rate of Joule heating inside the conductor. We tentatively identify the remaining term with the rate at which energy is fed into the magnetic field. If all fields are quasi-stationary (*i.e.*, slowly varying) then the displacement current can be neglected, and the Ampère-Maxwell equation reduces to  $\nabla \wedge \mathbf{H} = \mathbf{j}_t$ . Substituting this

expression into Eq. (3.186) and integrating over all space, we get

$$\int \mathbf{E}' \cdot (\nabla \wedge \mathbf{H}) d^3\mathbf{r} = \int \frac{(\nabla \wedge \mathbf{H})^2}{\sigma} d^3\mathbf{r} - \int \mathbf{E} \cdot (\nabla \wedge \mathbf{H}) d^3\mathbf{r}. \quad (3.187)$$

The last term can be integrated by parts using the relation

$$\nabla \cdot (\mathbf{E} \wedge \mathbf{H}) = \mathbf{H} \cdot (\nabla \wedge \mathbf{E}) - \mathbf{E} \cdot (\nabla \wedge \mathbf{H}). \quad (3.188)$$

The divergence theorem plus the Faraday-Maxwell equation yield

$$\int \mathbf{E} \cdot (\nabla \wedge \mathbf{H}) d^3\mathbf{r} = - \int \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} d^3\mathbf{r} - \int (\mathbf{E} \wedge \mathbf{H}) \cdot d\mathbf{S}. \quad (3.189)$$

Since  $\mathbf{E} \wedge \mathbf{H}$  falls off at least as fast as  $1/r^5$  in electrostatic and quasi-stationary magnetic fields ( $1/r^2$  comes from electric monopole fields, and  $1/r^3$  from magnetic dipole fields), the surface integral in the above expression can be neglected. Of course, this is not the case for radiation fields, for which  $\mathbf{E}$  and  $\mathbf{H}$  fall off like  $1/r$ . Thus, the constraint of “quasi-stationarity” effectively means that the fields vary sufficiently slowly that any radiation fields can be neglected.

The total power expended by the battery can now be written

$$\int \mathbf{E}' \cdot (\nabla \wedge \mathbf{H}) d^3\mathbf{r} = \int \frac{(\nabla \wedge \mathbf{H})^2}{\sigma} d^3\mathbf{r} + \int \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} d^3\mathbf{r}. \quad (3.190)$$

The first term on the right-hand side has already been identified as the energy loss rate due to Joule heating. The last term is obviously the rate at which energy is fed into the magnetic field. The variation  $\delta U$  in the magnetic field energy can therefore be written

$$\delta U = \int \mathbf{H} \cdot \delta \mathbf{B} d^3\mathbf{r}. \quad (3.191)$$

This result is analogous to the result (3.64) for the variation in the energy of an electrostatic field.

In order to make Eq. (3.191) integrable, we must assume a functional relationship between  $\mathbf{H}$  and  $\mathbf{B}$ . For a medium which magnetizes linearly the integration can be carried out in much the same manner as Eq. (3.67), to give

$$U = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} d^3\mathbf{r}. \quad (3.192)$$

Thus, the magnetostatic energy density inside a linear magnetic material is given by

$$W = \frac{\mathbf{H} \cdot \mathbf{B}}{2}. \quad (3.193)$$

Unfortunately, most interesting magnetic materials, such as ferromagnets, exhibit a nonlinear relationship between  $\mathbf{H}$  and  $\mathbf{B}$ . For such materials, Eq. (3.191) can only be integrated between definite states, and the result, in general, depends on the past history of the sample. For ferromagnets, the integral of Eq. (3.191) has a finite, non-zero value when  $\mathbf{B}$  is integrated around a complete magnetization cycle. This cyclic energy loss is given by

$$\Delta U = \int \oint \mathbf{H} \cdot d\mathbf{B} d^3\mathbf{r}. \quad (3.194)$$

In other words, the energy expended per unit volume when a magnetic material is carried through a magnetization cycle is equal to the area of its hysteresis loop as plotted in a graph of  $B$  against  $H$ . Thus, it is particularly important to ensure that the magnetic materials used to form transformer cores possess hysteresis loops with comparatively small areas, otherwise the transformers are likely to be extremely inefficient.