MP465: Electromagnetism

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1. Summary of Maxwell's Equations

From the introductory course you should be familiar with Maxwell's equations for an electric field, $\mathbf{E}(\mathbf{r},t)$, and a magnetic field, $\mathbf{B}(\mathbf{r},t)$, that, in general, can depend on both position $\mathbf{r}$ and time t.

$$\boxed{\begin{array}{ll} \nabla\times\mathbf{E}+\dfrac{\partial\mathbf{B}}{\partial t}=0 & \qquad \nabla.\mathbf{B}=0 \\ \nabla\times\mathbf{B}-\dfrac{1}{c^2}\dfrac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J} & \qquad \nabla.\mathbf{E}=\dfrac{\rho}{\epsilon_0} \end{array}}$$

The differential form of Gauss' law

$$\nabla.\mathbf{E}=\frac{\rho}{\epsilon_0} \tag{1}$$

states that the divergence of the electric field is proportional to the density of the electric charge: *i.e.* the electric field diverges away from the point where a source of electric charge is situated. The constant of proportionality, ϵ_0, is called *the electric permittivity of the vacuum.* The value of ϵ_0 depends on the system of units chosen: if charge is measured in Coulombs, C (the charge on an electron is: $e=-1.6\times 10^{-19}\,C$), then

$$\epsilon_0=8.854\times 10^{-12}\,C^2\,s^2\,kg^{-1}\,m^{-3}.$$

The numerical value of ϵ_0 is a constant of Nature: if ϵ_0 were larger than it is then the electric field due to a fixed charge would be correspondingly weaker, in the sense that a given charge would produce a smaller electric field; if ϵ_0 were smaller then the resulting electric field would be stronger.

The equation

$$\nabla.\mathbf{B}=0,$$

states that there are *no* free magnetic charges. No one has ever seen an isolated magnetic charge in the laboratory: the simplest source for the magnetic field is a magnetic dipole, which can be viewed as a pair of magnetic charges close to one another and of opposite sign, so the total magnetic charge is zero. Modern ideas that attempt to unify the forces of Nature, called *Grand Unified Theories*, or *GUT*'s for short, combine electromagnetism and the two nuclear forces (the **strong** and the **weak** nuclear forces) into a single mathematical formalism, and many such theories predict the existence of magnetic monopoles with extremely large masses, about 10^{16} times the mass of a proton, which would explain why they have never been produced in the laboratory. But it is not yet known if these theories are a correct description of Nature: for the purposes of this course we shall assume there are no magnetic monopoles.

The differential form of Farady's law of electromagnetic induction,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

dictates the form of the electric field generated by a time-varying magnetic field. The minus sign on the right-hand side here is the mathematical expression of *Lenz's Law*: any electric currents generated by $\mathbf{E}$ will always be in a direction such as to oppose the change in $\mathbf{B}$.

The fourth equation,

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J},$$

was Maxwell's *tour de force*. It describes how an electric current density, $\mathbf{J}$, acts as a source for the magnetic field and also a time varying electric field. Here μ_0 is *the magnetic permeability of the vacuum*, again for historical reasons. Just as for ϵ_0 the value of μ_0 depends on the units chosen, but the convention nowadays is to *define*

$$\mu_0 := 4\pi \times 10^{-7}\, kg\, m\, C^{-2}$$

and use this value to set the units. The speed of light also appears in Maxwell's fourth equation: c from the Latin *celeritas*, meaning speed. Originally c was measured, as metres per second, but now the metre itself is defined by setting c to be exactly

$$c := 299{,}792{,}458\, m\, s^{-1}.$$

In deriving his equations Maxwell discovered that c, ϵ_0 and μ_0 are not independent constants of Nature, but are related by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

a remarkable achievement.

Maxwell's equations are differential equations whose solutions determine $\mathbf{E}$ and $\mathbf{B}$ in terms of ρ and $\mathbf{J}$. They are a set of coupled, first order, partial differential equations. A very important and useful aspect of Maxwell's equations is that they are *linear*, and as a consequence once we have found some solutions we can just add them to get more solutions.

In addition to Maxwell's four equations above we need some other concepts from the earlier course.

1) The energy density, with units of $\frac{energy}{volume}$, stored in an electromagnetic field

$$w(\mathbf{r}, t) = \frac{1}{2}\left(\epsilon_0 \mathbf{E}.\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}.\mathbf{B}\right).$$

2) Electro-magnetic waves can transport energy through empty space. The energy flux carried by an electro-magnetic field, *i.e.* the energy crossing a unit cross-sectional area in unit time, $\frac{energy}{time \times area}$, usually called the *Poynting vector*, is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

3) Electro-magnetic fields can also carry momentum and exert a pressure. The momentum density, $\frac{momentum}{volume}$, is related to the Poynting vector through the speed of light. The momentum density is

$$\frac{1}{c^2} \mathbf{S} = \epsilon_0 \mathbf{E} \times \mathbf{B}.$$

4) Finally the Lorentz force, the force experienced by a charge moving with velocity $\mathbf{v}$ and carrying electric charge e in an electro-magnetic field, is

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Hence if the charge is stationary

$$\mathbf{F} = e\mathbf{E}$$

and the magnetic field has no effect. In a region of space where the electric field vanishes

$$\mathbf{F} = e\, \mathbf{v} \times \mathbf{B}$$

and the force is always at right-angles to both the particle's motion and the magnetic field: in particular, when $\mathbf{E} = 0$, $\mathbf{v}.\mathbf{F} = 0$ and the Lorentz force does no work on the particle.

2. Electrostatics

When there is no time dependence in the fields or charge distributions and $\mathbf{B} = 0$, which requires $\mathbf{J} = 0$, Maxwell's equations reduce to

$$\nabla.\mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}, \qquad \nabla \times \mathbf{E}(\mathbf{r}) = 0.$$

Consider first the electric field produced at a point $\mathbf{r}$ by a static point charge Q situated at $\mathbf{r}'$ which is given by Coulomb's law

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

(as an exercise, you should convince yourself that this indeed satisfies $\nabla \times \mathbf{E} = 0$). The point $\mathbf{r}$ here is called the *field point* and $\mathbf{r}'$ the *source point.* For a collection of N charges $Q_{(j)}$ situated at $\mathbf{r}_j$, $j = 1, \ldots N$, we can use linearity to obtain the total electric field at the field point $\mathbf{r}$ simply by adding the individual contributions from each charge:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} Q_{(j)} \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3}.$$

For a very large number of discrete charges it is convenient to approximate them by a continuous distribution with a charge density $\rho(\mathbf{r}')$, defined by taking a small volume $\delta V'$, surrounding $\mathbf{r}'$ containing charge $\delta Q'$, and taking the limit

$$\rho(\mathbf{r}') := \lim_{\delta V' \to 0} \frac{\delta Q'}{\delta V'},$$

assuming it exists. Then the total charge in a macroscopic volume V is

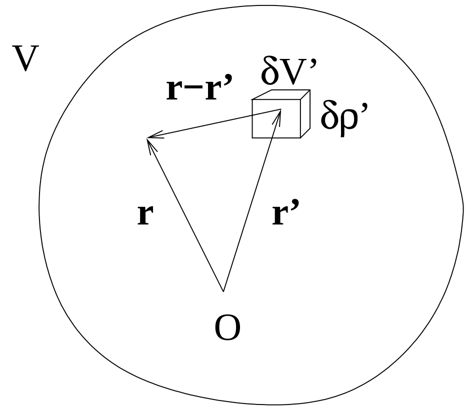
$$Q = \int_V \rho(\mathbf{r}') dV'$$

and the electric field at a point $\mathbf{r}$ due to Q is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV', \tag{2}$$

provided there is no more charge around other than that in V.*

* At first sight it may seem that we must keep $\mathbf{r}$ outside of V, in order to avoid a singularity when $\mathbf{r} = \mathbf{r}'$, but we shall see later that this is not necessary, provided ρ is finite at $\mathbf{r}$, (one can sometimes do integrals with an integrand which is infinite at isolated points and still get a finite answer).



Writing the electric field in this form allows some vector calculus manipulations which result in a simplification of the problem. We use the identity

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \qquad \text{for} \quad \mathbf{r} \neq \mathbf{r}'$$

(again when $\mathbf{r} = \mathbf{r}'$ the gradient is infinite, but we shall ignore this problem for the moment and see what happens) to derive

$$\nabla\left(\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'\right) = \int_V \rho(\mathbf{r}')\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' = -\int_V \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dV'.$$

It is important to realise that the differential operator ∇ here only operates on the field point $\mathbf{r}$ and not on the source point $\mathbf{r}'$, as far as ∇ is concerned $\mathbf{r}'$ is a constant which is why we can take ∇ inside the integral and pull it through $\rho(\mathbf{r}')$ with impunity.

It is therefore natural to define a scalar function

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'} \tag{3}$$

from which we can derive the electric field at $\mathbf{r}$ as minus the gradient of $\Phi(\mathbf{r})$,

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}).$$

The function $\Phi(\mathbf{r})$ is called the *electrostatic potential* at the point $\mathbf{r}$. Gauss' Law, equation (1), now states that

$$\nabla.\mathbf{E}(\mathbf{r}) = -\nabla^2\Phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

Thus Gauss' Law can be written as

$$\boxed{-\nabla^2\Phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}.} \tag{4}$$

With the definition $\mathbf{E} = -\nabla\Phi$ the other Maxwell equation for electrostatics, $\nabla \times \mathbf{E} = 0$, is automatic — the curl of a gradient is zero for any twice differentiable function.

The advantage of writing things this way is that, for a given charge distribution ρ, we have reduced the a set of four, first order, coupled differential equations

$$\nabla.\mathbf{E} = \rho/\epsilon_0 \qquad \nabla \times \mathbf{E} = 0$$

for three functions (the three components of $\mathbf{E}$) to a single partial differential equation for Φ, albeit a second order differential equation. In a region of space where there are no charges $\nabla^2\Phi = 0$ and we are dealing with Laplace's equation, which was studied in the Mathematical Methods course. Equation (4) is an inhomogeneous version of Laplace's equation, called *Poisson's equation.*

The Method of Images

In many practical situations there is a complication in that $\rho(\mathbf{r})$ is not always known explicitly. For example suppose we have a single charge Q near a large flat grounded conducting plate. 'Grounded' means that that the plate is earthed and is in effect in electrical contact with an infinite reservoir of charge which ensures that, wherever we place Q, the plate remains at the same potential, which we shall choose to be zero, the same potential as the Earth or the ground. A 2-dimensional surface which is constrained to have the same potential at all points is called an *equipotential surface.* Now positioning Q a distance a away from the plate will cause electric charge to be distributed on the plate which will be arranged in such a way as to ensure that the plate is at zero potential. There will be a surface charge density induced on the plate and we do not know what it is until we have solved the problem, but we cannot solve the problem without knowing the charge distribution on the plate.

We can break this impasse by using symmetry. Take the plate to be infinite in extent and co-incident with the $x = 0$ plane, in Cartesian co-ordinates, with the origin being the nearest point on the plate to Q and Q positioned at $x = a$, $y = z = 0$. The problem now is to find $\Phi(x, y, z)$ in the region $x \geq 0$. The method of images works by trying to find a charge distribution in the region $x < 0$ that forces $\Phi = 0$ in the $x = 0$ plane — these charges are called *image charges.* We then forget about the plate and calculate the potential in the region $x \geq 0$ due to Q and the image charges. By construction $\Phi(0, y, z) = 0$ and we shall get the correct Φ for all $x > 0$ too, that is the same Φ as would result from the metal plate and Q. It should be intuitively clear that placing a charge of the same magnitude as Q but opposite sign at $x = -a$, $y = z = 0$ produces a potential that exactly cancels that of Q in the $x = 0$ plane giving $\Phi = 0$ in the plane,

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r} - \boldsymbol{a}|} - \frac{1}{|\mathbf{r} + \boldsymbol{a}|}\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}}\right),$$

where $\boldsymbol{a} = a\hat{\mathbf{x}}$ ($\hat{\mathbf{x}}$ is a unit vector in the x-direction). $\Phi(x, y, z)$ vanishes when $x = 0$.

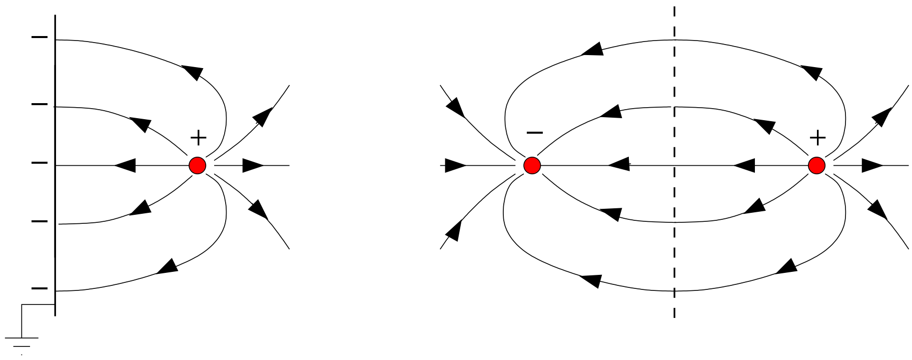
The electric field is

$$\mathbf{E} = -\nabla.\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{\mathbf{r} - \boldsymbol{a}}{|\mathbf{r} - \boldsymbol{a}|^3} - \frac{\mathbf{r} + \boldsymbol{a}}{|\mathbf{r} + \boldsymbol{a}|^3}\right).$$

At the surface of plate, $x = 0$, the electric field is

$$\mathbf{E} = -\frac{Q}{2\pi\epsilon_0}\left(\frac{1}{(a^2+y^2+z^2)^{3/2}}\right)\boldsymbol{a}, \tag{5}$$

which is perpendicular to the plate, into the plate if $Q > 0$ and out of the plate if $Q < 0$. This is a general feature of conductors. In a static situation the electric field is always normal to the surface of the conductor, any tangential component would result in a force on the charge carriers within the conductor making them move around until they have redistributed themselves so as to cancel the tangential component. For the same reason a static electric field is always zero inside a conductor.



Note that for $x < 0$ the electric field due to the image charge and Q does *not* co-incide with that due to the plate and Q, we only get the right answer for $x \geq 0$. For $x < 0$ the electric field is zero – the plate completely screens the negative x-region form Q.

We can use Gauss' Law to calculate the surface charge density, $\sigma(y,z)$, induced on the plate by Q. Consider a disc-shaped volume of small but finite thickness, like a coin, with the plate slicing through the middle of the coin so that it has one flat surface ('heads') in the region $x > 0$ and the other ('tails') in the region $x < 0$. Let $\mathbf{E}_+$ be the electric field (5) at the surface of the conductor on the positive-x side. Gauss' Law states that the total flux of electric field through the surface of the coin is equal to the total charge contained within the coin divided by ϵ_0 — if the area of the coin is δA then

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{1}{\epsilon_0}\int_{\delta A}\sigma dA.$$

If the area of the coin is small enough we can assume, with negligible error, that σ is constant throughout the whole of δA so

$$\int_{\delta A}\sigma dA = \sigma(y,z)\delta A, \tag{6}$$

when the coin is centred at (y,z) (at the end of the calculation we can send $\delta A \to 0$ so any approximations become exact). Also the electric field vanishes on the $x < 0$ side of the plate and the circular band round the edge of the coin does not contribute to the surface integral, if the coin is thin enough, so the surface integral is

$$\int_S \mathbf{E}.d\mathbf{S} = -\frac{Q}{2\pi\epsilon_0}\delta A\,\frac{\boldsymbol{a}.\hat{\mathbf{x}}}{(a^2+y^2+z^2)^{3/2}} = -\frac{Qa}{2\pi\epsilon_0}\frac{\delta A}{(a^2+y^2+z^2)^{3/2}} \tag{7}$$

where $\hat{\mathbf{x}}$ is the unit normal to the 'heads' ($x > 0$) surface of the coin. Equating (6) and (7) gives

$$\sigma(y,z) = -\frac{1}{2\pi}\frac{Qa}{(a^2+y^2+z^2)^{3/2}}.$$

The total charge induced on the plate is

$$Q_{induced} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(y,z)dydz = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Qa}{(a^2+y^2+z^2)^{3/2}}dydz.$$

The integral is most easily evaluated by using 2-dimensional polar co-ordinates with $y = r\sin\theta$ and $z = r\cos\theta$ ($0 \le r < \infty$, $0 \le \theta < 2\pi$), giving

$$Q_{induced} = -\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{Qa}{(a^2+r^2)^{3/2}}rdrd\theta = -\frac{Qa}{2}\int_{a^2}^{\infty}\frac{dv}{v^{3/2}} = -Q,$$

where we have used the change of variables $v = a^2+r^2$, $dv = 2rdr$. Hence the total charge induced on the plate is equal to the image charge.

Now consider a slightly different problem, that of a charge Q placed outside a grounded conducting sphere of radius R. We shall use spherical polar co-ordinates with the origin at the centre of the sphere, placing Q a distance a from the centre with $a > R$. Denote the position vector for Q by $\boldsymbol{a}$ and try placing an image charge $\tilde{Q}$ *inside* the sphere at a point $\tilde{\boldsymbol{a}}$, a distance $\tilde{a}$ from the centre with $\tilde{a} < R$. From symmetry we expect the full potential to be rotationally symmetric about the axis defined by $\boldsymbol{a}$, there is nothing in the configuration that can destroy this symmetry even after Q has induced a surface charge $\sigma(\theta,\phi)$ on the sphere: the surface charge should have this axial symmetry. It is therefore reasonable to guess that, if we can cook up the correct potential for the original problem using just a single image charge, then $\tilde{\boldsymbol{a}}$ should be co-linear with $\boldsymbol{a}$, so either $\tilde{\boldsymbol{a}} = \tilde{a}\hat{\boldsymbol{a}}$ or $\tilde{\boldsymbol{a}} = -\tilde{a}\hat{\boldsymbol{a}}$, with $\hat{\boldsymbol{a}} = \boldsymbol{a}/a$ a unit vector in the $\boldsymbol{a}$ direction, depending on whether $\tilde{Q}$ is on the same or the opposite side of the origin from Q. Now forget about the metal sphere and write the total potential due to Q and $\tilde{Q}$ at a field point $\mathbf{r}$ as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\mathbf{r}-\boldsymbol{a}|}+\frac{\tilde{Q}}{|\mathbf{r}-\tilde{\boldsymbol{a}}|}\right).$$

The potential at a point on the surface of the sphere, $\mathbf{r} = R\hat{\mathbf{r}}$ with $\hat{\mathbf{r}}$ the unit normal pointing out of the sphere, is now

$$\begin{aligned}\Phi(R\hat{\mathbf{r}}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|R\hat{\mathbf{r}}-\boldsymbol{a}|}+\frac{\tilde{Q}}{|R\hat{\mathbf{r}}-\tilde{\boldsymbol{a}}|}\right)\\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{R^2+a^2-2Ra\cos\theta}}+\frac{\tilde{Q}}{\sqrt{R^2+\tilde{a}^2\mp 2R\tilde{a}\cos\theta}}\right),\end{aligned}$$

where θ is the angle between the unit normal $\hat{\mathbf{r}}$ and $\boldsymbol{a}$. Our task now is to choose $\tilde{Q}$ and $\tilde{a}$ so that this vanishes at every point on the sphere, *i.e.* $\forall\, \theta$. This requires

$$\begin{aligned} \frac{Q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} &= -\frac{\tilde{Q}}{\sqrt{R^2 + \tilde{a}^2 \mp 2R\tilde{a}\cos\theta}} \\ \Leftrightarrow \qquad \frac{Q}{a\sqrt{\frac{R^2}{a^2} + 1 - 2\frac{R}{a}\cos\theta}} &= -\frac{\tilde{Q}}{R}\frac{1}{\sqrt{1 + \frac{\tilde{a}^2}{R^2} \mp 2\frac{\tilde{a}}{R}\cos\theta}} \end{aligned}$$

which can be achieved by setting

$$\frac{Q}{a} = -\frac{\tilde{Q}}{R}, \qquad \frac{R}{a} = \frac{\tilde{a}}{R}$$

and choosing the upper sign. So we have found a solution

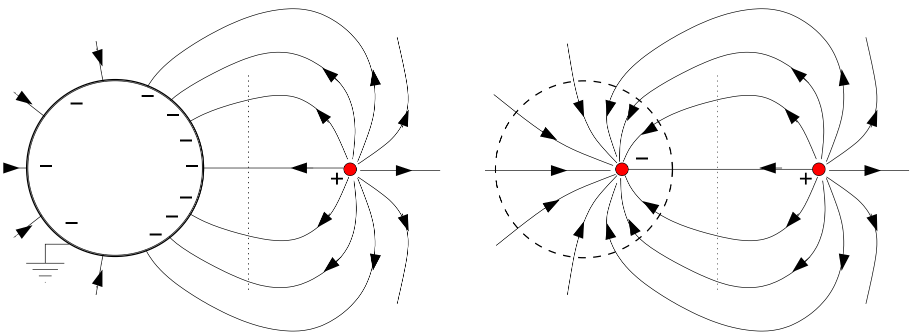
$$\tilde{Q} = -\frac{R}{a}Q, \qquad \tilde{a} = \frac{R^2}{a}, \qquad \tilde{\boldsymbol{a}} = \frac{R^2}{a^2}\boldsymbol{a}.$$

Note that $\tilde{a} < R$ and $|\tilde{Q}| < |Q|$ since $a > R$. Thus the full solution to our problem, $\forall\, r > R$, is

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r} - \boldsymbol{a}|} - \frac{R}{a}\frac{1}{|\mathbf{r} - \frac{R^2}{a^2}\boldsymbol{a}|}\right), \tag{8}$$

this vanishes at all points on the surface of the sphere by construction. Note however that it does not give the correct potential for the original problem at a field point *inside* the sphere: (8) is non-zero for $r < R$ but in the original problem $\Phi = 0$ inside the sphere, since the sphere is an equipotential surface and there are no physical charges inside the metal sphere so $\Phi = 0$ everywhere inside.

The electric field and surface charge density can now be derived from (8). In the limit $R \to \infty$ the sphere becomes a flat plane and the answer reduces to the previous case, once due account is taken for the shift in the position of the origin.



Green functions

The method of images that we have been describing is an example of the Green function technique that you have learned about in the mathematical methods course.

Recall that a Green function, $G(x,x')$, for a linear differential operator $\mathcal{L}$ is essentially an inverse of $\mathcal{L}$ in the sense that

$$\mathcal{L}G(x,x') = \delta(x-x')$$

where $\delta(x-x')$ is the Dirac δ-function. The Dirac δ-function has the property that, for any interval $[a,b]$ on the real line,

$$\int_a^b f(x')\delta(x-x')dx' = \begin{cases} f(x) & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

for any function $f(x)$. In particular for the simple case when $f(x)=1$

$$\int_a^b \delta(x-x')dx' = \begin{cases} 1 & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases},$$

which, if true for all intervals $[a,b]$, is sufficient to define $\delta(x-x')$.

The Green function is not unique, different boundary conditions lead to different Green functions. The differential operator $-\nabla^2$ is an example of a linear operator (usually called the *Laplacian*) and we will show that

$$-\nabla^2\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = \delta(\mathbf{r}-\mathbf{r}'),$$

where $\delta(\mathbf{r}-\mathbf{r}')$ is a 3-dimensional δ-function, in Cartesian co-ordinates

$$\delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z'),$$

so

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

is a Green function for the Laplacian.

To prove this first observe that

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3},$$

from which

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \nabla.\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = 0,$$

provided $\mathbf{r} \neq \mathbf{r}'$. This means that

$$\int_V \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' = 0,$$

if $\mathbf{r} \notin V$. For $\mathbf{r} \in V$ decompose V into a small spherical ball, B_ε, of radius ε centred on $\mathbf{r}$, and the rest of the volume, which we denote by $\bar{V}$, so that $V = \bar{V} + B_\varepsilon$. Now

$$\begin{aligned}\int_V \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' &= \int_{\bar{V}} \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' + \int_{B_\varepsilon} \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' \\ &= \int_{B_\varepsilon} \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV'.\end{aligned}$$

Now we observe that $\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = (\nabla')^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$, where ∇' involves differentiating with respect to $\mathbf{r}'$, and use the divergence theorem to write

$$\int_{B_\varepsilon} (\nabla')^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' = \int_{B_\varepsilon} \nabla'.\nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' = \int_{S_\varepsilon} \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) .\mathbf{n}' dS'$$

where S_ε is the surface of the ball, a sphere of radius ε with $\mathbf{r}' - \mathbf{r} = \varepsilon \mathbf{n}'$ and $\mathbf{n}'$ is a unit normal outward from the surface. Now, on the surface,

$$\nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{\varepsilon \mathbf{n}'}{\varepsilon^3} = -\frac{\mathbf{n}'}{\varepsilon^2}$$

and $dS' = \varepsilon^2 \sin\theta' d\theta' d\phi'$, so

$$\int_V \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' = \int_{S_\varepsilon} \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) .\mathbf{n}' dS' = -\int_{S_\varepsilon} \mathbf{n}'.\mathbf{n}' \sin\theta' d\theta' d\phi' = -4\pi.$$

Thus we have proven that

$$-\nabla^2 \int_V \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) dV' = \begin{cases} 1 & \mathbf{r} \in V \\ 0 & \mathbf{r} \notin V \end{cases},$$

for any volume V, which is equivalent to the statement that $-\nabla^2 \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = \delta(\mathbf{r}-\mathbf{r}')$. Hence we have proven that $\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ is a Green function for the Laplacian $-\nabla^2$.

We can now give a formal proof that the definition of Φ in (3) is compatible with Gauss' Law (4),

$$\begin{aligned}\nabla.\mathbf{E}(\mathbf{r}) = -\nabla^2\Phi(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0}\nabla^2 \left(\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV'\right) = -\frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}')\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dV' \\ &= \frac{1}{\epsilon_0} \int \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') dV' = \frac{\rho(\mathbf{r})}{\epsilon_0}.\end{aligned}$$

The Green function is not unique, different boundary conditions lead to different Green functions. We can add to $G(\mathbf{r}, \mathbf{r}')$ any function $F(\mathbf{r}, \mathbf{r}')$ for which $\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0$, for all $\mathbf{r}$ and $\mathbf{r}'$ both in V, to form a different Green function

$$G_F(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}, \mathbf{r}') + F(\mathbf{r}, \mathbf{r}')$$

and we still have

$$-\nabla^2 G_F(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$

$\forall\,\mathbf{r},\mathbf{r}' \in V$.

For example, if the region of interest V is the exterior of a grounded conducting sphere of radius R centred on the origin, the choice

$$F(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{R}{r'(|\mathbf{r}-\frac{R^2}{r'^2}\mathbf{r}'|)}$$

satisfies

$$\nabla^2 F(\mathbf{r},\mathbf{r}') = 0,$$

provided $r > R$, $r' > R$, and

$$G_F(\mathbf{r},\mathbf{r}') = G(\mathbf{r},\mathbf{r}') + F(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi}\frac{R}{r'\big(|\mathbf{r}-\frac{R^2}{r'^2}\mathbf{r}'|\big)}$$

satisfies

$$-\nabla^2 G_F(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}').$$

Physically $\frac{1}{\epsilon_0}G_F(\mathbf{r},\mathbf{r}')$ is the potential at the field point $\mathbf{r}$ due to a unit charge placed at the source point $\mathbf{r}'$, in the presence of the conducting sphere, provided both $r > R$ and $r' > R$.

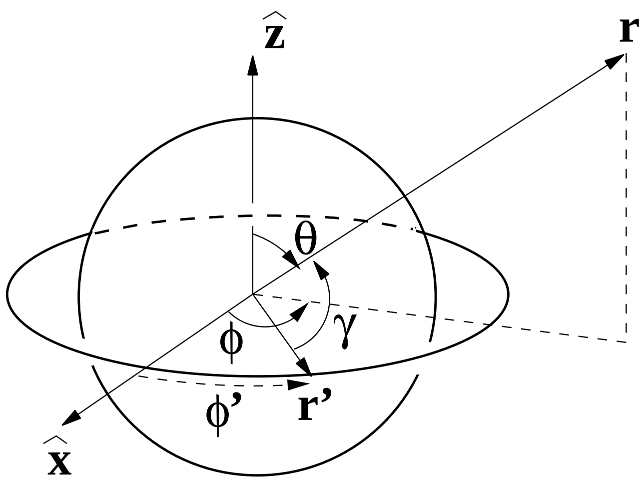
The power of the Green function method lies in the fact that we can immediately write down the potential at a field point $\mathbf{r}$ outside the sphere ($r > R$) for *any* charge distribution $\rho(\mathbf{r}')$ in the region $r' > R$ (which we denote by V) as

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')G_F(\mathbf{r}.\mathbf{r}')dV' = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R}{r'(|\mathbf{r}-\frac{R^2}{r'^2}\mathbf{r}'|)}\right)dV'.$$

This is guaranteed to vanish on the surface of the sphere $r = R$ since then $G_F(\mathbf{r},\mathbf{r}') = 0$ for all $\mathbf{r}'$.

Example

As an example of the Green function technique, consider a ring of charge, with total charge Q and radius a, encircling a grounded conducting sphere of radius R, with $a > R$.



Label the points on the ring by the azimuthal co-ordinate $0 \leq \phi' < 2\pi$ and denote the charge per unit length at ϕ' by $f(\phi')$, then the total charge on the ring is

$$Q = a \int_0^{2\pi} f(\phi')d\phi'$$

(we are using 3-dimensional polar co-ordinates (r', θ', ϕ') to label the source points, with the ring in the plane $\theta' = \pi/2$ and centred on the origin). The charge density due to this ring can be written using Dirac δ-functions as

$$\rho(\mathbf{r}') = \frac{1}{a}\delta(r' - a)\delta(\cos\theta')f(\phi'),$$

the prefactor $1/a$ being chosen so that

$$Q = \int_0^\infty \int_0^\pi \int_0^{2\pi} \rho(\mathbf{r}')r'^{\,2} \sin\theta' dr' d\theta d\phi'$$

and the δ-functions ensuring that $\rho(\mathbf{r}') = 0$ unless $r' = a$ and $\theta' = \pi/2$.

The potential at a field point $\mathbf{r}$ outside the sphere can immediately be written down as

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_R^\infty \int_0^\pi \int_0^{2\pi} \rho(\mathbf{r}') \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{R}{r'(|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'|)} \right) r'^{\,2} \sin\theta' dr' d\theta d\phi' \\ &= \frac{1}{4\pi a\epsilon_0} \int_R^\infty \int_0^\pi \int_0^{2\pi} f(\phi')\delta(r' - a)\delta(\cos\theta') \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{R}{r'(|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'|)} \right) r'^{\,2} \sin\theta' dr' d\theta d\phi'.\end{aligned}$$

Now $\mathbf{r}.\mathbf{r}' = rr'\cos\gamma$, where γ is the angle between $\mathbf{r}$ and $\mathbf{r}$,* so

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}}$$

and

$$\frac{R}{r'(|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'|)} = \frac{R}{r'\sqrt{r^2 + (R^4/r'^2) - 2r(R^2/r')\cos\gamma}} = \frac{1}{\sqrt{(r^2r'^2/R^2) + R^2 - 2rr'\cos\gamma}}.$$

Using these in the multiple integral above the δ-functions make the r' and θ' integrals trivial, they just set $r' = a$ and $\theta' = \pi/2$ in the integrand, giving

$$\Phi(\mathbf{r}) = \frac{a}{4\pi\epsilon_0} \int_0^{2\pi} f(\phi') \left(\frac{1}{\sqrt{r^2 - a^2 - 2ra\cos\gamma}} - \frac{1}{\sqrt{(ra/R)^2 + R^2 - 2ra\cos\gamma}} \right) d\phi',$$

* It is important to realise that γ depends on both the field point and the source point: in particular it is a function of θ' and ϕ', $\cos(\gamma(\theta, \phi, \theta'\phi')) = \sin\theta\sin\theta'\cos(\phi' - \phi) + \cos\theta\cos\theta'$, so it affects the integrals over dV'.

where it must be stressed that γ depends on ϕ', $\cos\gamma = \sin\theta\cos(\phi'-\phi)$, which its value when $\theta' = \pi/2$.

In the simplest case of a uniform ring of charge, $f(\phi') = Q/2\pi a$ is a constant, the charge per unit length, and

$$\Phi(\mathbf{r}) = \frac{Q}{8\pi^2\epsilon_0}\int_0^{2\pi}\left(\frac{1}{\sqrt{r^2-a^2-2ra\sin\theta\cos(\phi'-\phi)}}\right.$$
$$\left. - \frac{1}{\sqrt{(ra/R)^2+R^2-2ra\sin\theta\cos(\phi'-\phi)}}\right)d\phi'.$$

For example the potential at point $\mathbf{r} = \pm r\hat{\mathbf{z}}$ on an axis through the origin and perpendicular to the ring, so $\theta = 0$ or π, is

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2-a^2}} - \frac{R}{\sqrt{(ra)^2+R^4}}\right),$$

the first term being generated by the charge density on the ring itself and the second term being due to the surface charge density induced on the sphere by the ring. The second term is identical to the contribution of an ‘image’ ring of radius R^2/a and total charge $-RQ/a$ sitting inside the sphere.

Multipole expansions

A second example of a mathematical technique for solving problems in electrostatics is a method that provides an approximate solution to the problem in terms of an infinite series whose higher order terms are less and less important in certain situations. Provided the circumstances are right we can truncate the infinite series at a finite order and get an answer as close as we wish to the correct answer. In this class of problems we assume that $\rho(\mathbf{r}')$ is known, and there are no conducting surfaces around to confuse the issue, but the integral in (3) is too hard to perform analytically so we resort to an approximation technique, called a *multipole expansion*. A multipole expansion is really nothing more than a Taylor expansion of $1/|\mathbf{r}-\mathbf{r}'|$.

We assume that the charge distribution $\rho(\mathbf{r}')$ is confined to a volume V whose largest dimension is L, say. Then, choosing the origin to lie inside V, we can be confident that there is a region of space outside of V for which $r'/r < L/r << 1$, and we restrict ourselves to field points far enough away from V so that $L/r << 1$. Now use the Taylor expansion formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-3)}{3!}x^3 + \ldots \tag{9}$$

This expansion converges provided $|x| < 1$. We can now expand $1/|\mathbf{r}-\mathbf{r}'|$, by setting $n = -1/2$ and $x = -2(\mathbf{r}.\mathbf{r}'/r^2) + (r'/r)^2$ in (9),

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2-r'^2-2\mathbf{r}.\mathbf{r}'}} = \frac{1}{r}\left(1-2(\mathbf{r}.\mathbf{r}'/r^2)+(r'/r)^2\right)^{-1/2}$$

$$
\begin{aligned}
&= \frac{1}{r}\left(1 - \frac{1}{2}\left[-2(\mathbf{r}.\mathbf{r}'/r^2) + (r'/r)^2\right] + \frac{3}{8}\left[-2(\mathbf{r}.\mathbf{r}'/r^2) + (r'/r)^2\right]^2\right. \qquad (10)\\
&\qquad\qquad \left. - \frac{5}{16}\left[-2(\mathbf{r}.\mathbf{r}'/r^2) + (r'/r)^2\right]^3 + \cdots\right)\\
&= \frac{1}{r}\left(1 + \frac{\mathbf{r}.\mathbf{r}'}{r^2} + \frac{1}{2}\left(3\frac{(\mathbf{r}.\mathbf{r}')^2}{r^4} - \frac{r'^2}{r^2}\right) + \frac{1}{2}\left(5\frac{(\mathbf{r}.\mathbf{r}')^3}{r^6} - 3\frac{(\mathbf{r}.\mathbf{r}')r'^2}{r^4}\right) + o\left(\frac{r'}{r}\right)^4\right).
\end{aligned}
$$

Substituting this expansion in the definition of the scalar potential (3) gives

$$
\begin{aligned}
\Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_V \rho(\mathbf{r}')\left[1 + \frac{\mathbf{r}.\mathbf{r}'}{r^2} + \frac{1}{2r^2}\left(3\left(\frac{\mathbf{r}.\mathbf{r}'}{r}\right)^2 - r'^{\,2}\right)\right.\\
&\qquad\left. + \frac{\mathbf{r}.\mathbf{r}'}{2r^4}\left(5\left(\frac{\mathbf{r}.\mathbf{r}'}{r}\right)^2 - 3r'^{\,2}\right) + o\left(\frac{r'}{r}\right)^4\right] dV'\\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{1}{r^3}\sum_{i=1}^{3} Q_i x_i + \frac{1}{r^5}\sum_{i,j=1}^{3} Q_{ij}x_i x_j + \frac{1}{r^7}\sum_{i,j,k=1}^{3} Q_{ijk}x_i x_j x_k + o\left(\frac{L^4}{r^5}\right)\right),
\end{aligned}
$$

where x_i, $i = 1, 2, 3$ are Cartesian co-ordinates,

$$
Q = \int_V \rho(\mathbf{r}')dV'
$$

is the total charge contained in V, Q_i are the components of a vector,

$$
Q_i := \int_V \rho(\mathbf{r}')x'_i dV',
$$

called the *dipole moment* of the charge distribution,

$$
Q_{ij} := \frac{1}{2}\int_V \rho(\mathbf{r}')(3x'_i x'_j - \delta_{ij}r'^{\,2})dV'
$$

is called the *quadrupole moment* — it is a symmetric, traceless matrix, $\sum_{i=1}^{3} Q_{ii} = 0$ and

$$
Q_{ijk} = \frac{1}{2}\int_V \rho(\mathbf{r}')\left(5x'_i x'_j x'_k - \left(\delta_{ij}x'_k r'^{\,2} + \delta_{jk}x'_i r'^{\,2} + \delta_{ki}x'_j r'^{\,2}\right)\right) dV'
$$

(known as the *octopole moment*).

The expansion

$$
\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{1}{r^3}\sum_{i=1}^{3} Q_i x_i + \frac{1}{r^5}\sum_{i,j=1}^{3} Q_{ij}x_i x_j + \cdots\right)
$$

is called a *multipole expansion*, and the sum converges for $L/r < 1$. Note that the first term goes like $\sim 1/r$, the dipole term like $\sim 1/r^2$, the quadrupole term like $\sim 1/r^3$, and so on. For $L/R << 1$ the first few terms give a very good approximation to the correct potential at $\mathbf{r}$, provided they do not vanish. Indeed the most important contribution to the electric field comes from the first non-vanishing component, which can actually be proven to be independent of the choice of origin.

For example a very good approximation to the electric field surrounding a non-spherical neutral molecule, such as a water molecule H_2O, can be obtained from the potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{Q}.\mathbf{r}}{r^3}, \tag{11}$$

the first term vanishing since $Q = 0$ for a neutral molecule. A simple model for a charge distribution giving rise to such a potential is to take two charges of equal magnitude but opposite sign, q and $-q$ and place them a distance a apart. Placing the charges on the x-axis, symmetrically placed about the origin at $\mathbf{r}' = (a/2)\hat{\mathbf{x}}$ and $\mathbf{r}' = -(a/2)\hat{\mathbf{x}}$ the charge distribution can be represented by

$$\rho(\mathbf{r}') = q\big(\delta(x' - a/2) - \delta(x' + a/2)\big)\delta(y')\delta(z'),$$

where $x' = x'_1$, $y' = x'_2$ and $z' = x'_3$, so

$$Q = \int_{\mathbf{R}^3} \rho(\mathbf{r})dx'dy'dz' = q\int_{-\infty}^{\infty} ((\delta(x' - a/2) - \delta(x' + a/2)))dx' = q - q = 0,$$

obviously, while

$$\begin{aligned} Q_i &= \int_{\mathbf{R}^3} \rho(\mathbf{r})x'_i dx'dy'dz' \\ &= \begin{cases} q\left(\int_{-\infty}^{\infty}\big(\delta(x' - a/2) - \delta(x' + a/2)\big)x'dx'\right)\left(\int_{-\infty}^{\infty}\delta(y')dy'\right)\left(\int_{-\infty}^{\infty}\delta(z')dz'\right) = qa, & i = 1 \\ q\left(\int_{-\infty}^{\infty}\big(\delta(x' - a/2) - \delta(x' + a/2)\big)dx'\right)\left(\int_{-\infty}^{\infty}\delta(y')y'dy'\right)\left(\int_{-\infty}^{\infty}\delta(z')dz'\right) = 0, & i = 2 \\ q\left(\int_{-\infty}^{\infty}\big(\delta(x' - a/2) - \delta(x' + a/2)\big)dx'\right)\left(\int_{-\infty}^{\infty}\delta(y')dy'\right)\left(\int_{-\infty}^{\infty}\delta(z')z'dz'\right) = 0, & i = 3, \end{cases} \end{aligned}$$

giving the vector $\mathbf{Q} = qa\hat{\mathbf{x}}$. The next moment is

$$\begin{aligned} Q_{ij} &= \frac{1}{2}\int_{\mathbf{R}^3} \rho(\mathbf{r})(3x'_i x'_j - \delta_{ij}r'^{\,2})dx'dy'dz' \\ &= \frac{q}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(\delta(x' - a/2) - \delta(x' + a/2)\big)\delta(y')\delta(z')(3x'_i x'_j - \delta_{ij}r'^{\,2})dx'dy'dz', \end{aligned}$$

which actually vanishes for all i and j. The Q_{ijk} are non-zero but we do not need their explicit form here, all we need to know is that they are proportional to qa^3, because this allows us to write

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{ax}{r^3} + o\left(\frac{a^3}{r^4}\right)\right).$$

If we send $q \to \infty$ and $a \to 0$, keeping $p = qa$ finite, then in this limit

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{xp}{r^3}$$

is exact. This is the potential of a dipole, a pair of opposite sign but equal magnitude charges sitting close to one another, so that the total charge is zero. The potential falls like $\sim 1/r^2$ and the electric field like $\sim 1/r^3$, faster than than the field of a single point charge (which would be called an *electric monopole* in the language of multipole expansions).

An example of a charge distribution for which $Q = 0$ and $Q_i = 0$, but $Q_{ij} \neq 0$, so that the first multipole contribution to the potential is at the quadrupole level, is to take four charges at the edges of a square with a total charge summing to zero and two opposite edges being anti-parallel dipoles. For example take the square to lie in the x-y plane with sides of length a: with two charges $+q$ sitting at the two vertices $(x, y) = (a/2, a/2)$ and $(x, y) = (-a/2, -a/2)$ and two charges $-q$ sitting at the remaining two vertices $(x, y) = (a/2, -a/2)$ and $(x, y) = (-a/2, a/2)$. Then the charge distribution is

$$\begin{aligned}\rho(\mathbf{r}') = q\,[&\delta(x'-a/2)\delta(y'-a/2) + \delta(x'+a/2)\delta(y'+a/2) \\ &-\delta(x'-a/2)\delta(y'+a/2) - \delta(x'+a/2)\delta(y'-a/2)]\,\delta(z').\end{aligned}$$

Using the δ-functions to do the integrals in

$$Q_{ij} = \frac{1}{2}\int_{\mathbf{R}^3} \rho(\mathbf{r}) \left(3x_i'x_j' - \delta_{ij} r'^{\,2}\right) dx'dy'dz'$$

the δ_{ij} term vanishes and, in matrix form,

$$Q_{ij} = \frac{3qa^2}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(qa^4)$$

leading to the potential

$$\Phi(\mathbf{r}) = \frac{qa^2}{4\pi\epsilon_0} \left(\frac{3xy}{r^5} + o\left(\frac{a^4}{r^5}\right) \right).$$

Now sending $q \to \infty$ and $a \to 0$, this time keeping $p = qa^2$ finite, gives

$$\Phi(\mathbf{r}) = \frac{3p}{4\pi\epsilon_0} \left(\frac{xy}{r^5}\right),$$

which is a quadrupole potential. It falls off like $\sim 1/r^3$, giving an electric field whose magnitude falls off like $\sim 1/r^4$.

The multipole expansion in equation (10) is also an expansion in Legendre polynomials:

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}\left(1 + \frac{\mathbf{r}.\mathbf{r}'}{r^2} + \frac{1}{2}\left(3\frac{(\mathbf{r}.\mathbf{r}')^2}{r^4} - \frac{r'^2}{r^2}\right) + \frac{1}{2}\left(5\frac{(\mathbf{r}.\mathbf{r}')^3}{r^6} - 3\frac{(\mathbf{r}.\mathbf{r}')r'^2}{r^4}\right) + \cdots\right) \\ &= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\gamma + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left(3\cos^2\gamma - 1\right) + \frac{1}{2}\left(\frac{r'}{r}\right)^3\left(5\cos^3\gamma - 3\cos\gamma\right) + \cdots\right),\end{aligned}$$

where γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, $\mathbf{r}.\mathbf{r}' = rr'\cos\gamma$. The functions of γ appearing here are Legendre polynomials

$$\begin{aligned}
P_0(\gamma) &= 1 \\
P_1(\gamma) &= \cos\gamma \\
P_2(\gamma) &= \frac{1}{2}(3\cos^2\gamma - 1) \\
P_3(\gamma) &= \frac{1}{2}(5\cos^3\gamma - 3\cos\gamma).
\end{aligned}$$

Indeed one way to define the Legendre polynomials is from the expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma),$$

which converges for $r'/r < 1$. Thus the multipole expansion of the potential can be expressed as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{1}{r}\right)^{n+1}\Phi_n(\hat{\mathbf{r}}),$$

where the multipole moments, $\Phi_n(\hat{\mathbf{r}})$, are

$$\Phi_n(\hat{\mathbf{r}}) = \int_V \rho(\mathbf{r}')(r')^n P_n(\cos\gamma)dV'.$$

Remember that the angle γ appearing in these integrals depends on both the source point $\mathbf{r}'$ and the field point $\mathbf{r}$, because it is the angle between $\mathbf{r}$ and $\mathbf{r}'$, so $\Phi_n(\hat{\mathbf{r}})$ depends on the direction of the field point $\hat{\mathbf{r}} = \mathbf{r}/r$ for $n \geq 1$.

Electric dipoles are very important in understanding the properties of electrically neutral matter, such as a medium like water whose molecules behave like little electric dipoles. Consider an electric dipole $\mathbf{p} = aq\mathbf{n} = q\boldsymbol{a}$ of magnitude $p = aq$, consisting of two charges $\pm q$ a distance a apart in a line determined by the unit vector $\mathbf{n}$, at a point $\mathbf{r}$ in an external electric field $\mathbf{E}(\mathbf{r})$. We take $\mathbf{r}$ to be the mid-point of the line segment between q and $-q$ and $\mathbf{n} = \boldsymbol{a}/a$ to point from q to $-q$, so q is at $\mathbf{r} + (\boldsymbol{a}/2)\mathbf{n}$ and $-q$ at $\mathbf{r} - (\boldsymbol{a}/2)\mathbf{n}$. The total force on $\mathbf{p}$ due to the field $\mathbf{E}$ is the sum of the force $\mathbf{F}_q$ on q and the force $\mathbf{F}_{-q}$ on $-q$

$$\mathbf{F} = \mathbf{F}_q + \mathbf{F}_{-q} = q\mathbf{E}(\mathbf{r} + \boldsymbol{a}/2) - q\mathbf{E}(\mathbf{r} - \boldsymbol{a}/2) = q(\boldsymbol{a}.\nabla)\mathbf{E}(\mathbf{r}) + o(qa^2).$$

In the limit $q \to \infty$, $a \to 0$, keeping $p = qa$ finite,

$$\mathbf{F} = (\mathbf{p}.\nabla)\mathbf{E} = \nabla(\mathbf{p}.\mathbf{E}), \tag{12}$$

since

$$(\mathbf{p}.\nabla)E_i = \sum_{j=1}^{3} p_j\partial_j E_i = -\sum_{j=1}^{3} p_j\partial_j\partial_i\Phi = -\partial_i\sum_{j=1}^{3} p_j\partial_j\Phi = \partial_i(\mathbf{p}.\mathbf{E})$$

(the dipole moment $\mathbf{p}$ is independent of the position $\mathbf{r}$). This force can be derived from a potential energy $U(\mathbf{r}) = -\mathbf{p}.\mathbf{E}(\mathbf{r})$ associated with the dipole in the electric field

$$\mathbf{F} = -\nabla U(\mathbf{r}).$$

Note that for a constant electric field the force vanishes, but if the electric field depends on position the dipole tends to get pulled in towards regions of stronger field.

Although the net force vanishes in a constant field there is still a torque on $\mathbf{p}$: around the point $\mathbf{r}$ the total torque is (in the limit $\boldsymbol{a} \to 0$ with p finite)

$$\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{a} \times \mathbf{F}_q - \frac{1}{2}\boldsymbol{a} \times \mathbf{F}_{-q} = q\,\boldsymbol{a} \times \mathbf{E}(\mathbf{r}) + O(qa^2) \quad \longrightarrow \quad \mathbf{p} \times \mathbf{E}(\mathbf{r}). \tag{13}$$

This torque has the effect of twisting $\mathbf{p}$ to bring it parallel with $\mathbf{E}$: dipoles like to line up with an applied electric field.

The nature of the force on a system of static charges, due to an externally applied field $\mathbf{E}(\mathbf{r})$, is severely constrained by the fact that, in a region of space where $\nabla.\mathbf{E}(\mathbf{r}) = 0$, *i.e* away from the charges that generate the external field, a system of static charges cannot be in stable equilibrium — a result known as *Earnshaw's Theorem.* This is unfortunate from a practical point of view, because it means that we cannot use static electric fields to hold charged objects in any one place — if we could do so it might be possible to levitate objects against the force of gravity, for example. Earnshaw's theorem states that this is not possible.

To prove this, suppose we have a single static test charge q at a point $\mathbf{r}$ with a potential energy $U(\mathbf{r}) = q\Phi(\mathbf{r})$ in the external field $\mathbf{E}(\mathbf{r})$ (a more general system of charges can always be considered to be a linear superposition of individual point charges). Then the force on q is

$$\mathbf{F} = -\nabla U(\mathbf{r}) = -q\nabla\Phi(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$$

and demanding static equilibrium requires that the charge is not being accelerated, so $\mathbf{F} = 0$ and the potential energy $U = q\Phi$ must must satisfy

$$\nabla U(\mathbf{r}) = 0$$

at the point $\mathbf{r}$. For stable equilibrium it is necessary that

$$\nabla^2 U(\mathbf{r}) > 0 \quad \Rightarrow \quad \nabla^2\Phi(\mathbf{r}) \neq 0.$$

But we have assumed that

$$\nabla.\mathbf{E}(\mathbf{r}) = -\nabla^2\Phi(\mathbf{r}) = 0$$

giving a contradiction. Hence q cannot be in stable equilibrium at $\mathbf{r}$ if $\nabla.\mathbf{E}(\mathbf{r}) = 0$. In fact even $\nabla^2 U(\mathbf{r}) > 0$ is not sufficient for stable equilibrium, for example in Cartesians

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

and some of the partial derivative could be positive and one negative to give an overall positive quantity, but there is still an instability in the direction in which the partial derivative is negative. Stable equilibrium requires all three terms on the right hand side to be positive, but in only needs one of them to be negative for unstable equilibrium.

3. Magnetostatics

When there is no time dependence in the fields or current distributions and $\mathbf{E} = 0$, which requires $\rho = 0$, Maxwell's equations reduce to*

$$\nabla.\mathbf{B}(\mathbf{r}) = 0, \qquad \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}).$$

In a region of space where $\mathbf{J} = 0$ these equations are identical to those of electrostatics with $\rho(\mathbf{r}) = 0$, with $\mathbf{E}$ replaced by $\mathbf{B}$, and we can use similar mathematical techniques to solve them — define a magnetostatic potential $\Psi(\mathbf{r})$, the gradient of which is the magnetic field,

$$\mathbf{B}(\mathbf{r}) = -\nabla\Psi(\mathbf{r}).$$

The details however are often different because the physics of magnetic fields is different to that of electric fields leading to different kinds of boundary conditions: for example in electrostatics $\mathbf{E}$ is always normal to a conducting surface, which is therefore an equipotential surface, but we shall see later that $\mathbf{B}$ is always *tangential* to a conducting surface.

We shall pursue a different direction here. Our starting point is the Biot-Savart law for the magnetic field generated by a current density $\mathbf{J}$ in a volume V

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \tag{14}$$

For example if the current is carried in an infinitely long straight wire of constant cross-sectional area, ΔA, in the direction of the unit vector $\mathbf{n}$ then the current in the wire is $I = J\Delta A$ where $\mathbf{J} = J\mathbf{n}$ is the current density. The current is associated with a specific direction, so it is really a vector too, $\mathbf{I} = I\mathbf{n}$. Then the magnetic field generated by the current in the wire is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \frac{\mu_0}{4\pi} \int_L \frac{(\mathbf{J}(\mathbf{r}')\Delta A) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dl' = \frac{\mu_0}{4\pi} \int_L \frac{\mathbf{I} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dl',$$

where $\int_L \cdots dl'$ represents the integral along the length of the wire and we have assumed that $\mathbf{J}(\mathbf{r}')$ is constant across the cross-section of the wire.

* Note that, even though there is no time dependence in the equations here, and they represent a static situation, there must be moving electric charges to generate a current. These charges might even be accelerating, for example they could be moving in circles. For magnetostatics the important criterion is that the currents are independent of time and the *total* charge density is zero.

The Biot-Savart law is equivalent to

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}).$$

This follows from

$$\nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{15}$$

so

$$\begin{aligned}\nabla \times \mathbf{B}(\mathbf{r}) &= \nabla \times \left(\frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dV' \right) = -\frac{\mu_0}{4\pi} \int_V \nabla \times \left(\mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \right) dV' \\ &= -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dV' = \mu_0 \int_V \mathbf{J}(\mathbf{r}') \delta(\mathbf{r}-\mathbf{r}') dV' = \mu_0 \mathbf{J}(\mathbf{r}).\end{aligned}$$

Using the same vector identity (15) we can also write (14) as

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \times \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dV' = \frac{\mu_0}{4\pi} \int_V \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \times \mathbf{J}(\mathbf{r}') dV' \\ &= \nabla \times \left(\frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' \right).\end{aligned}$$

It is therefore convenient to define a *magnetic vector potential*, $\mathbf{A}(\mathbf{r})$, as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' \tag{16}$$

and then

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

is always true, even in the presence of non-zero currents. The vector potential is thus more general than the concept of a scalar potential for the magnetic field as the latter is only defined in regions of space where $\mathbf{J} = 0$.

In practical calculations the magnetic vector potential is perhaps not as useful as the electric scalar potential, involving as it does three triple integrations rather than one as in the scalar case. Nevertheless it is very important conceptually and we shall examine some of its properties. First we shall use conservation of charge to show that (16) has zero divergence,

$$\nabla . \mathbf{A}(\mathbf{r}) = 0.$$

To show this we need a mathematical expression of the fact that charge is conserved — electric charge can neither be created nor destroyed. Consider a volume of space V containing charge Q. For the purposes of this discussion we relax the condition that ρ and $\mathbf{J}$ should be time independent so we allow Q to vary with time but, if it does, charge must flow either into or out of V to compensate: if Q increases some charge must flow into V

to compensate, if Q decreases some charge must flow out of V. In either case there must be a flux of current through the surface, S, of V and

$$\frac{dQ}{dt} = -\int_S \mathbf{J}(\mathbf{r}').d\mathbf{S}',$$

the minus sign being due to the convention that $d\mathbf{S}'$ is an infinitesimal vector pointing *out* of V. Writing $Q(t) = \int_V \rho(\mathbf{r}', t)dV'$ we use the divergence theorem to express

$$\frac{dQ}{dt} = \frac{d\left(\int_V \rho(\mathbf{r}', t)dV'\right)}{dt} = \int_V \frac{\partial\rho(\mathbf{r}', t)}{\partial t}dV' = -\int_S \mathbf{J}(\mathbf{r}').d\mathbf{S}' = -\int_V \nabla'.\mathbf{J}(\mathbf{r}', t)dV'$$

(the prime on ∇' here indicates that it acts on the source point $\mathbf{r}'$). Hence in *any* volume of space

$$\int_V \frac{\partial\rho(\mathbf{r}', t)}{\partial t}dV' + \int_V \nabla'.\mathbf{J}(\mathbf{r}', t)dV' = 0$$

which can only be true if

$$\frac{\partial\rho(\mathbf{r}, t)}{\partial t} + \nabla.\mathbf{J}(\mathbf{r}, t) = 0 \tag{17}$$

at every point $\mathbf{r}$ and at all times t. This equation is a differential form of the statement that electric charge is conserved.

Returning now to statics, if ρ and $\mathbf{J}$ are independent of time we have

$$\nabla.\mathbf{J}(\mathbf{r}) = 0.$$

Now consider the divergence of $\mathbf{A}$ in (16),

$$\begin{aligned}\nabla.\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\nabla\left(\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'\right) = \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}').\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' \\ &= -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}').\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV',\end{aligned}$$

where in the last equation we have use the fact that, for any function $f(x)$,

$$\frac{\partial f(x-x')}{\partial x} = -\frac{\partial f(x-x')}{\partial x'}.$$

Now use the divergence theorem to write

$$\begin{aligned}-\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}').\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' &= \frac{\mu_0}{4\pi}\int_V \nabla'.\mathbf{J}(\mathbf{r}')\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' - \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}(\mathbf{r}').d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|} \\ &= -\frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}(\mathbf{r}').d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|},\end{aligned}$$

since $\nabla'.\mathbf{J}(\mathbf{r}') = 0$. Provided we chose V large enough to contain all the currents, so $\mathbf{J}(\mathbf{r}') = 0$ on the surface of V (or at least has no normal component on S), then

$$\int_S \frac{\mathbf{J}(\mathbf{r}').d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|} = 0$$

and we have proven that

$$\nabla.\mathbf{A}(\mathbf{r}) = 0$$

is a consequence of charge conservation.

For illustrative purposes we shall now derive the magnetic vector potential due to a finite segment of straight wire of length L carrying a current I. We choose co-ordinates so that the wire is aligned along the z-direction, centred on the origin so that it extends to $z = \pm L/2$. If the wire has constant cross-sectional area ΔA then, assuming $\mathbf{J}$ is constant, $\mathbf{J} = \mathbf{I}/\Delta A = \frac{I}{\Delta A}\hat{\mathbf{z}}$. From (16)

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' = \frac{\mu_0}{4\pi}\int_V \frac{I\hat{\mathbf{z}}}{\Delta A|\mathbf{r}-\mathbf{r}'|}dV' \\
&= \frac{\mu_0 I\hat{\mathbf{z}}}{4\pi}\int_V \frac{1}{\Delta A|\mathbf{r}-\mathbf{r}'|}dV' = \frac{\mu_0 I\hat{\mathbf{z}}}{4\pi}\int_{-L/2}^{L/2} \frac{1}{|\mathbf{r}-\mathbf{r}'|}dz'
\end{aligned}$$

with $\mathbf{r}' = z'\hat{\mathbf{z}}$. Now

$$\begin{aligned}
\int_{-L/2}^{L/2} \frac{1}{|\mathbf{r}-\mathbf{r}'|}dz' &= \int_{-L/2}^{L/2} \frac{1}{\sqrt{x^2+y^2+(z-z')^2}}dz' \\
&= \int_{\frac{-L/2-z}{\sqrt{x^2+y^2}}}^{\frac{L/2-z}{\sqrt{x^2+y^2}}} \frac{du}{\sqrt{1+u^2}} = \sinh^{-1}\left(\frac{L/2-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L/2+z}{\sqrt{x^2+y^2}}\right),
\end{aligned}$$

where we have used the substitution $u = (z'-z)/\sqrt{x^2+y^2}$. Hence

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\sinh^{-1}\left(\frac{L/2-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L/2+z}{\sqrt{x^2+y^2}}\right)\right]\hat{\mathbf{z}}.$$

Of course it is not possible to have an isolated segment of wire carrying a current which appears out of nothing at one end and disappears into nothing at the other, that would violate the principle of conservation of charge. We could connect four such segments into a square to make a continuous circuit or we could consider the limit of an infinitely long wire, $L \to \infty$. We shall analyse the closed loop later, in a more general setting, for the moment consider the case of $L \to \infty$. For large L

$$\sinh^{-1}\left(\frac{L}{2\sqrt{x^2+y^2}}\right) = -\ln\left(\frac{\sqrt{x^2+y^2}}{L}\right) + O\left(\frac{x^2+y^2}{L^2}\right)$$

so the vector potential behaves as

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{L^2}\right)\hat{\mathbf{z}} + O\left(\frac{1}{L^2}\right).$$

Hence,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{2\pi}\frac{(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})}{x^2+y^2} + O\left(\frac{1}{L^2}\right) \underset{L\to\infty}{\longrightarrow} \frac{\mu_0 I}{2\pi}\frac{(\hat{\mathbf{z}}\times\mathbf{r})}{x^2+y^2} = \frac{\mu_0 I}{2\pi}\frac{(\hat{\mathbf{z}}\times\hat{\mathbf{r}})}{\sqrt{x^2+y^2}}.$$

Thus $\mathbf{B}$ encircles the wire in a direction determined by a right-handed screw in the direction of the current and falls off inversely as the distance from the wire $1/\sqrt{x^2+y^2}$ when L is infinite.

Magnetic Field of a Localised Current Distribution — Multipole Expansions

Consider a closed loop, C, around which a current I is flowing (to be concrete one can consider a small wire, but this is not always a good picture). If $d\mathbf{r}'$ is an infinitesimal tangent vector to the loop at a source point $\mathbf{r}'$ then, using $\mathbf{J}(\mathbf{r}')dV' = Id\mathbf{r}'$ with I a constant, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},$$

where V is any volume containing C (the symbol $\oint$ reminds us that the integral is around a closed loop).

For any shape of loop and any current I we can approximate $\mathbf{A}(\mathbf{r}')$ using a multipole expansion, just as we did in electrostatics,

$$|\mathbf{r}-\mathbf{r}'|^{-1} = \left(r^2 + r'^2 - 2\mathbf{r}.\mathbf{r}'\right)^{-1/2} = \frac{1}{r}\left(1 - 2\frac{\mathbf{r}.\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1 + \frac{\mathbf{r}.\mathbf{r}'}{r^2} + o\left(\frac{r'}{r}\right)^2\right),$$

for $r' > r$ leading to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r}\left(\oint_C d\mathbf{r}' + \frac{1}{r^2}\oint_C (\mathbf{r}.\mathbf{r}')d\mathbf{r}' + \cdots\right),$$

where the dots represent higher order corrections — if L is the width of C at is widest point and the origin is taken near the centre of C then these terms are guaranteed to be less than L^3/r^3.

Now

$$\oint_C d\mathbf{r}' = \int_{\mathbf{r}_0}^{\mathbf{r}_0} d\mathbf{r}' = \Big[\mathbf{r}'\Big]_{\mathbf{r}_0}^{\mathbf{r}_0} = \mathbf{r}_0 - \mathbf{r}_0 = 0,$$

where $\mathbf{r}_0$ is any point on the curve C.* To evaluate the second integral, $\oint_C (\mathbf{r}.\mathbf{r}')d\mathbf{r}'$, use the vector identity

$$(\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} = -\mathbf{r}'(\mathbf{r}.d\mathbf{r}') + (\mathbf{r}.\mathbf{r}')d\mathbf{r}'$$

and Leibnitz rule, varying $\mathbf{r}'$ keeping $\mathbf{r}$ fixed,

$$d[\mathbf{r}'(\mathbf{r}'.\mathbf{r})] = \mathbf{r}'(\mathbf{r}.d\mathbf{r}') + (\mathbf{r}.\mathbf{r}')d\mathbf{r}'.$$

Adding these gives

$$(\mathbf{r}.\mathbf{r}')d\mathbf{r}' = \frac{1}{2}\left\{(\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} + d[\mathbf{r}'(\mathbf{r}'.\mathbf{r})]\right\},$$

so

$$\oint_C (\mathbf{r}.\mathbf{r}')d\mathbf{r}' = \frac{1}{2}\left(\oint_C \{(\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r}\} + \oint_C d[\mathbf{r}'(\mathbf{r}'.\mathbf{r})]\right) = \frac{1}{2}\left(\oint_C \mathbf{r}' \times d\mathbf{r}'\right)\times\mathbf{r} + \frac{1}{2}\left(\int_{\mathbf{r}_0}^{\mathbf{r}_0} d[\mathbf{r}'(\mathbf{r}'.\mathbf{r})]\right),$$

and again the second term vanishes because

$$\int_{\mathbf{r}_0}^{\mathbf{r}_0} d[\mathbf{r}'(\mathbf{r}'.\mathbf{r})] = \Big[\mathbf{r}'(\mathbf{r}'.\mathbf{r})\Big]_{\mathbf{r}_0}^{\mathbf{r}_0} = \mathbf{r}_0(\mathbf{r}_0.\mathbf{r}) - \mathbf{r}_0(\mathbf{r}_0.\mathbf{r}) = 0.$$

Thus

$$A(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^3}\left\{\frac{1}{2}\oint_C (\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r}\right\} + \cdots,$$

where the dots represent terms that are less and less important as $\mathbf{r}$ is taken further and further away from the loop C — these extra terms are of order $\frac{IL^3}{r^3}$ or less.

Define the *magnetic dipole moment* of the current distribution due to the loop C to be the vector

$$\mathbf{m} := \frac{I}{2}\oint_C (\mathbf{r}' \times d\mathbf{r}').$$

For a planar loop of area $\mathcal{A}$, *i.e.* a loop that lies in one 2-dimensional plane without bending into the third dimension,

$$\oint_C (\mathbf{r}' \times d\mathbf{r}') = \mathcal{A}\mathbf{n},$$

---

* As an exercise you may wish to convince yourself of this for the particular case of a circular loop of radius a lying in the x-y plane, taking $\mathbf{r}' = a(\cos\phi'\hat{\mathbf{x}} + \sin\phi'\hat{\mathbf{y}})$ in 2-dimensional polar co-ordinates with $0 \le \phi' < 2\pi$. For such a loop $d\mathbf{r}' = a(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}})d\phi'$ and

$$\oint_C d\mathbf{r}' = a\int_0^{2\pi}(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}})d\phi' = a\left(-\hat{\mathbf{x}}\int_0^{2\pi}\sin\phi' d\phi' + \hat{\mathbf{y}}\int_0^{2\pi}\cos\phi' d\phi'\right) = 0.$$

where $\mathbf{n}$ is a unit normal to the plane in which the loop lies.* Hence, at least for planar loops,

$$\mathbf{m} = I\mathcal{A}\mathbf{n}$$

is normal to the plane and proportional to the area of the loop.

In any case the expansion becomes more and more accurate as we take smaller and smaller loops $L \to 0$ and $I \to \infty$, keeping $IL^2 \approx I\mathcal{A}$, and thus $\mathbf{m}$, finite. In this limit the expression

$$A(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3} \tag{18}$$

becomes exact.

The magnetic field generated by this loop of current can now be calculated

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{3\frac{(\mathbf{m}.\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right\}.$$

Note that, as r increases, $\mathbf{A}$ falls off like $1/r^2$ and $\mathbf{B}$ falls off like $1/r^3$, just like the scalar potential and electric field for an electric dipole. In fact the geometrical form of $\mathbf{B}$ for a magnetic dipole field is identical that of $\mathbf{E}$ for an electric dipole field — it is the familiar dipole field produced by a bar magnet.

Atoms and molecules sometimes have magnetic dipole moments associated with them, or can develop magnetic dipole moments in the presence of an externally applied magnetic field. The underlying mechanism for this is fundamentally quantum mechanical in nature, but an heuristic understanding can be obtained by using the following classical arguments. Consider a charged particle, such as an electron, in a circular orbit around an atomic nucleus in an atom or a molecule. Denote the orbital angular momentum of the electron by $\mathbf{L}$ where

$$\mathbf{L} = M(\mathbf{r} \times \mathbf{v}),$$

with M is the electron mass, $\mathbf{r}$ its position and $\mathbf{v}$ its velocity. For a circular orbit $\mathbf{L}$ has magnitude $L = Mvr$. Since a charge is moving there is a current generated,

$$I = q\frac{v}{2\pi r}$$

which is the charge per unit time passing any point on the orbit. This current generates a magnetic dipole moment

$$m_L = I\pi r^2 = \frac{q}{2}vr = \frac{qL}{2M}.$$

Denoting the charge on the electron by $q = -e$ we have

$$m_L = -\frac{eL}{2M}.$$

Although we have derived this equation using classical physics, remarkably, it is also true in a full quantum mechanical treatment of the electron's dynamics. However this is not

* Again you should convince yourself that this is true for a circular loop.

the whole story since, in the theory of quantum mechanics, electrons have an intrinsic angular momentum of their own, which we denote by $\mathbf{S}$, which is not associated with any orbital momentum but arises from the intrinsic spin of the electron. This also generates a magnetic dipole moment

$$m_S = -\frac{eS}{M},$$

note there is no factor of 1/2 here — a fundamentally quantum mechanical result which we shall not derive here but merely state. The total angular momentum of the electron is then the sum of its orbital and intrinsic angular momentum, $\mathbf{L}+\mathbf{S}$, and the electron's motion generates a dipole moment for the atom of the form

$$\mathbf{m} = -\left(\frac{e}{2M}\right)(\mathbf{L}+2\mathbf{S}) = \frac{\mu_B}{\hbar}(\mathbf{L}+2\mathbf{S}), \tag{19}$$

where $\mu_B = -\frac{e\hbar}{2M}$ is called the *Bohr magneton.* If it were not for the strange factor of 2 in front of $\mathbf{S}$ the Bohr magneton would just be the ratio of the dipole moment to the total angular momentum of the electron, when the latter is measured in units of $\hbar$ — as it is the relation is a little is a more involved than that, but not much more involved.

Of course atoms and molecules usually have more than one electron and the true dipole moment will be a combination of all the dipole moments of the constituent electrons, but a full treatment would take us too far into the theory of addition of angular momenta in quantum mechanics. For the purpose of this course it is sufficient to observe that atomic and molecular magnetic dipole moments are related to the constituent electrons' angular momenta.

When a material made up of atoms or molecules with permanent non-zero magnetic moments is placed in an externally applied magnetic field $\mathbf{B}$ the dipole moments $\mathbf{m}$ like to line up with $\mathbf{B}$, just like electric dipole moments in and external electric field $\mathbf{E}$. One way of seeing this is to model a typical magnetic dipole by a small square loop of area a^2 carrying current I and placed in a constant external magnetic field, $\mathbf{B}$, so that the unit normal to the loop, $\mathbf{n}$,* is at an angle θ to the direction of $\mathbf{B}$. The magnetic dipole moment due to the loop is then

$$\mathbf{m} = Ia^2\mathbf{n}.$$

Choose axes so that $\hat{\mathbf{x}}$ is in the direction of $\mathbf{B}$, $\mathbf{B} = B\hat{\mathbf{x}}$. For simplicity we shall consider the case where $\mathbf{n}$ lies in the x-z plane, with $\mathbf{n} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}}$, so that two opposite sides of the square are in the y-direction and the other two opposite sides are in the x-z plane. There will be forces on the four sides of the loop due to the Lorentz force on the charges carriers, which cause the current, moving in the field $\mathbf{B}$,

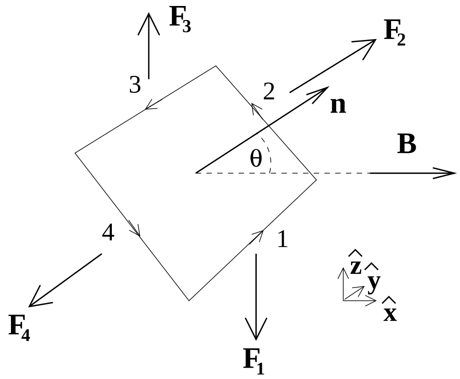
$$\mathbf{F} = e(\mathbf{v}\times\mathbf{B}).$$

On any infinitesimally small segment of the square loop of length dl in the direction $d\mathbf{l}$ the force will be

$$d\mathbf{F} = e(\mathbf{v}\times\mathbf{B})dN = I(d\mathbf{l}\times\mathbf{B}).$$

* Defined relative to the direction of I using the right-hand rule.

where dN is the number of charge carriers in the segment dl.



Label the sides 1 to 4, ascending in the direction of the current flow, starting with the side for which $\mathbf{I} = I\hat{\mathbf{y}}$. Denoting the force on side one by $\mathbf{F}_1$, on side two by $\mathbf{F}_2$, etc., we have: on side one

$$d\mathbf{l} = dl\hat{\mathbf{y}} \quad \Rightarrow \quad \mathbf{F}_1 = aIB(\hat{\mathbf{y}} \times \hat{\mathbf{x}}) = -aIB\hat{\mathbf{z}};$$

on side two

$$d\mathbf{l} = dl(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) \quad \Rightarrow \quad \mathbf{F}_2 = aIB(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) \times \hat{\mathbf{x}} = aIB\cos\theta\,\hat{\mathbf{y}};$$

on side three

$$d\mathbf{l} = -dl\hat{\mathbf{y}} \quad \Rightarrow \quad \mathbf{F}_3 = -aIB(\hat{\mathbf{y}} \times \hat{\mathbf{x}}) = aIB\hat{\mathbf{z}} = -\mathbf{F}_1;$$

while on the fourth side

$$d\mathbf{l} = dl(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{z}}) \quad \Rightarrow \quad \mathbf{F}_4 = aIB(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{z}}) \times \hat{\mathbf{x}} = -aIB\cos\theta\,\hat{\mathbf{y}} = -\mathbf{F}_2.$$

Thus the total force on the loop vanishes

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0.$$

There is however a net torque. The torque generated by $\mathbf{F}_s$ on side s of the square ($s = 1, 2, 3, 4$) is the same as it would be if $\mathbf{F}_s$ were applied to the mid-point of the relevant side. Denote the mid-points of the four edges by $\mathbf{r}_s$ then

$$\begin{aligned}
\mathbf{r}_1 &= \frac{a}{2}(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{z}}),\\
\mathbf{r}_2 &= \frac{a}{2}\hat{\mathbf{y}},\\
\mathbf{r}_3 &= \frac{a}{2}(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) = -\mathbf{r}_1,\\
\mathbf{r}_4 &= -\frac{a}{2}\hat{\mathbf{y}} = -\mathbf{r}_2.
\end{aligned}$$

$\mathbf{r}_2$ and $\mathbf{r}_4$ are parallel to $\mathbf{F}_2$ and $\mathbf{F}_4$ respectively, so they give no torque, leaving

$$\begin{aligned}
\boldsymbol{\tau} &= (\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_3 \times \mathbf{F}_3) = 2(\mathbf{r}_1 \times \mathbf{F}_1) = 2\left(\frac{a}{2}\right)(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{z}}) \times (-aIB\hat{\mathbf{z}})\\
&= a^2 IB\sin\theta\,\hat{\mathbf{y}} = mB\sin\theta\,\hat{\mathbf{y}} = \mathbf{m} \times \mathbf{B},
\end{aligned}$$

where $\mathbf{m} = Ia^2\mathbf{n}$

Although the torque has been derived for a special geometry, a square loop with two sides perpendicular to $\mathbf{B}$, the result is general: a magnetic dipole $\mathbf{m}$ in an external field $\mathbf{B}$ experiences a torque

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}.$$

This is exactly the same result (13) as for an electric dipole in an external electric field, the torque tends to twists the dipole so that it lines up with the field and the torque is zero when the dipole is parallel to the field: if the dipole points in the same direction as the field then it is in stable equilibrium, if the dipole points in the opposite direction to the field then it is in unstable equilibrium. This effect can be expressed in terms of potential energy, there is a potential energy U associated with the dipole,

$$U = -\mathbf{m}.\mathbf{B} = -mB\cos\theta,$$

that is minimised (stable equilibrium) when $\theta = 0$.

As for electrostatics, this expression for the potential energy of the dipole is also valid when $\mathbf{B}(\mathbf{r})$ depends on position, giving rise to a force on $\mathbf{m}$ when $\mathbf{B}(\mathbf{r})$ is not constant

$$\mathbf{F} = -\nabla U(\mathbf{r}) = \nabla\big(\mathbf{m}.\mathbf{B}(\mathbf{r})\big).$$

In a region of space where the current density generating the external field vanishes, so $\nabla \times \mathbf{B}(\mathbf{r}) = 0$, we can derive $\mathbf{B}$ from a magnetic scalar potential, $\mathbf{B}(\mathbf{r}) = -\nabla\Psi(\mathbf{r})$, and*

$$F_i = -\partial_i \left(\sum_{j=1}^{3} m_j \partial_j \Psi \right) = -\sum_{j=1}^{3} m_j \partial_i \partial_j \Psi = -\sum_{j=1}^{3} m_j \partial_j (\partial_i \Psi) = (\mathbf{m}.\nabla) B_i(\mathbf{r}),$$

so

$$\mathbf{F} = \nabla(\mathbf{m}.\mathbf{B}) = (\mathbf{m}.\nabla)\mathbf{B} \tag{20}$$

and, as in electrostatics (12), a dipole is attracted to regions of stronger $\mathbf{B}$.

There is a version of Earnshaw's theorem for magnetostatics, but things are more subtle when quantum effects are taken into account. For example consider a magnetic dipole $\mathbf{m}$ in an external field $\mathbf{B}$, in a region of space where $\nabla \times \mathbf{B} = 0$ so we can define a magnetic scalar potential $\mathbf{B}(\mathbf{r}) = -\nabla\Psi(\mathbf{r})$. The argument now exactly parallels that of the discussion of Earnshaw's theorem in electrostatics, but with $U(\mathbf{r}) = -\mathbf{m}.\mathbf{B}(\mathbf{r}) = \mathbf{m}.\nabla\Psi(\mathbf{r})$. Equilibrium requires the force on $\mathbf{m}$ to vanish, so $\mathbf{F} = -\nabla U = -(\mathbf{m}.\nabla)\nabla\Psi = 0$ Thus

$$\mathbf{F} = 0 \qquad \Leftrightarrow \qquad (\mathbf{m}.\nabla)\mathbf{B} = 0$$

so $\mathbf{B}$ is constant in the direction of $\mathbf{m}$. Stable equilibrium further requires that $\nabla^2 U(\mathbf{r})$ so

$$(\mathbf{m}.\nabla)\nabla^2\Psi > 0$$

* Remember $\mathbf{m}$ is constant, independent of $\mathbf{r}$.

but $\nabla.\mathbf{B} = 0 \Rightarrow -\nabla^2\Psi = 0$ so it is impossible to satisfy the condition for stable equilibrium in a region of space where $\nabla \times \mathbf{B} = 0$.

Unlike electrostatics, however, there is a way out of Earnshaw's theorem in magnetostatics, which relies on the quantum properties of magnetic dipoles. This is because magnetic dipoles are associated with angular momentum and, in the theory of quantum mechanics, angular momentum is quantised in units of $\hbar/2$, so we might expect magnetic dipoles to be quantised too and indeed they are. In classical physics the potential energy of a dipole in an external field, $U = -\mathbf{m}.\mathbf{B}$, can have any value between its maximum mB, when $\mathbf{m}$ is parallel to $\mathbf{B}$, and $-mB$, when $\mathbf{m}$ is anti-parallel to $\mathbf{B}$. In the theory of quantum mechanics magnetic dipoles are quantised in the same way as angular momentum — relative to a reference direction, which we take to be that of the external field, $\mathbf{n} = \mathbf{B}/B$. The dipole moment can only have a discrete set of values

$$\mathbf{m} = sg\mu_B\mathbf{n}$$

where s can take discrete values, either integral or half-integral ($\hbar s$ is like an angular momentum), μ_B is the Bohr magneton and g is a number, called the *Landé g-factor*, which can be calculated using the theory of addition of angular momentum in quantum mechanics. For example, from (19), $g = 1$ if $\mathbf{m}$ is due solely to the orbital angular momentum of a constituent electron while $g = 2$ if $\mathbf{m}$ is due solely to the intrinsic angular momentum of a constituent electron.* In general g is neither 1 nor 2, it can even be negative.

In any case the potential energy of such a quantised dipole in an external field $\mathbf{B} = B\mathbf{n}$ is

$$U = -\mathbf{m}.\mathbf{B} = -sg\mu_B B$$

and stable equilibrium requires both

$$\nabla U = 0 \quad \Rightarrow \quad \nabla B = 0 \quad \text{and} \quad \nabla^2 U > 0 \quad \Rightarrow \quad \begin{cases} \nabla^2 B > 0 & \text{if } sg < 0 \\ \nabla^2 B < 0 & \text{if } sg > 0. \end{cases}$$

If $gs < 0$, U is minimised at places where B is a minimum (*weak* field seeking states); if $gs > 0$, U is minimised at places where B is a maximum (*strong* field seeking states). In particular the dipole can be in stable equilibrium if we can cook up a magnetic field for which the magnitude of $\mathbf{B}$ has a minimum somewhere. A configuration like this is called a *magnetic trap*, because it is capable of trapping and holding individual molecules or atoms at fixed points in space, provided they have a suitable magnetic dipole moment. There is no analogue of this phenomenon for static electric fields because electric dipoles have nothing to do with angular momentum and so are not quantised.

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* For a free electron g is actually not quite 2. It can be calculated using the quantum theory of electrodynamics, *Quantum Electro-Dynamics* or QED. The deviation from 2 is known as the anomalous magnetic moment of the electron and it depends on the fine structure constant, $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx 1/137$. The best current measurement is $g - 2 = 0.0023193043718$. In fact comparison of the experimental value with the theoretical prediction of QED gives the most accurate current estimate of α.

Note that while $\nabla^2 B > 0$ is a *necessary* condition for weak-field seeking states to be in stable equilibrium it is not *sufficient*. Let $H_{ij} = \frac{\partial^2 B}{\partial x_i \partial x_j}$, with $x_1 = x$, $x_2 = y$ and $x_3 = z$, then a necessary and sufficient condition for weak-field seeking states to be in stable equilibrium is that all three eigenvalues of the symmetric matrix H_{ij} must be positive, not just its trace.

For example suppose that

$$\mathbf{B} = x\hat{\mathbf{x}} - y\hat{\mathbf{y}}$$

(you should convince yourself that this satisfies $\nabla.\mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$). Then $B = \sqrt{x^2 + y^2}$ clearly has a minimum at $x = y = 0$. B not is actually differentiable there, but clearly minimising B^2 is equivalent to minimising B, and $B^2 = x^2 + y^2$ is differentiable with

$$\nabla(B^2) = 2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$$

vanishing iff $x = y = 0$ while

$$\frac{\partial^2(B^2)}{\partial x_i \partial x_j} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A configuration like this is called a *linear trap*. Actually it only traps dipoles in two directions, the x and y directions — the dipoles will be in neutral equilibrium in the z-direction. A configuration that traps in 3-dimensions is

$$\mathbf{B} = \mathbf{B}_0 + a(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}),$$

where $\mathbf{B}_0$ is a constant vector and a is a constant (again check that this satisfies $\nabla.\mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$). In this case $B^2 = (B_{0,x} + ax)^2 + (B_{0,y} + ay)^2 + (B_{0,z} - 2az)^2$ and

$$\nabla(B^2) = 2a\big((B_{0,x} + ax)\hat{\mathbf{x}} + (B_{0,y} + ay)\hat{\mathbf{y}} - 2(B_{0,z} - 2az)\hat{\mathbf{z}}\big)$$

vanishes at

$$x = -\frac{B_{0,x}}{a}, \qquad y = -\frac{B_{0,y}}{a}, \qquad z = \frac{2B_{0,z}}{a},$$

while

$$\frac{\partial^2(B^2)}{\partial x_i \partial x_j} = \begin{pmatrix} 2a^2 & 0 & 0 \\ 0 & 2a^2 & 0 \\ 0 & 0 & 4a^2 \end{pmatrix},$$

and all three eigenvalues are positive.

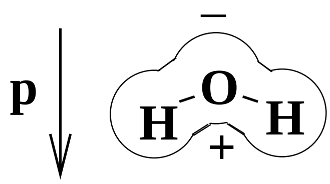
4. Maxwell's Equations in the Presence of Matter

Maxwell's equations are

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & \qquad \nabla.\mathbf{B} &= 0 \\ \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J} & \qquad \nabla.\mathbf{E} &= \frac{\rho}{\epsilon_0} \end{aligned}$$

where ρ accounts for all the electric charge and $\mathbf{J}$ for all the current density. Unfortunately we do not always know where all possible charges that can contribute to ρ and $\mathbf{J}$ might be.

For example, because the distribution of electric charge within a water molecule is not symmetric the molecule has a small electric dipole moment.



If there is no external electric field the dipole moments of each molecule point in random directions and they add up to zero, but if we place an electric charge in water the electric field of the charge will tend to align the electric dipole moments of the water molecules so that they are no longer random but instead add up to give a significant contribution to the total electric field in the water. A similar phenomenon can occur even in a medium whose individual molecules do not have a permanent dipole moment, such as molecular oxygen, O_2. This is a symmetric molecule and has no permanent electric dipole moment but, if an oxygen molecule is placed in an external electric field then the field tends to displace the negatively charged electrons in the oxygen relative to the positively charged oxygen atomic nuclei and the molecule develops a dipole moment, which would go away again if the external field were turned off.

Water and oxygen are examples of *polarisable media*, because they can develop a significant polarisation in response to an external field, even though they are not polarised in the absence of such a field. In both cases the problem of calculating the total electric field due to an electric charged placed in the medium (water or oxygen) becomes difficult.

Consider a small volume δV of a polarisable medium containing δN particles each with the same electric dipole moment $\mathbf{p}$. It the dipoles were perfectly aligned the total electric dipole moment of δV would be $\mathbf{p}\delta N$ (in practice the dipoles will never be perfectly aligned, due to their thermal motion, but we shall make this assumption here for simplicity). We define a *polarisation*, $\mathbf{P}$, by

$$\mathbf{P}\delta V = \mathbf{p}\delta N.$$

Now $\mathbf{P}$, the dipole moment in a unit volume of material, will contribute to the electric field. If δV is positioned at a source point $\mathbf{r}'$ then its contribution to the electrostatic potential at a field point $\mathbf{r}$ will be, from (11),

$$\delta\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}').(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\delta V'.$$

If the material of the medium is contained in a volume V then the contribution of the medium to $\Phi(\mathbf{r})$ is then $\Phi^{(\mathbf{P})}$ where

$$\Phi^{(\mathbf{P})}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\mathbf{P}(\mathbf{r}').(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV',$$

where the superscript $^{(\mathbf{P})}$ is to remind us that this is not the total electrostatic potential, but only the contribution coming from the dipoles in the medium. This contribution can be re-expressed as

$$\Phi^{(\mathbf{P})}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\mathbf{P}(\mathbf{r}').(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV' = \frac{1}{4\pi\epsilon_0}\int_V \mathbf{P}(\mathbf{r}').\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV',$$

where it is important to remember that ∇' acts on $\mathbf{r}'$ and not on $\mathbf{r}$. Integrating by parts

$$\int_V \mathbf{P}(\mathbf{r}').\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' = -\int_V \frac{\nabla'.\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' + \int_S \frac{\mathbf{P}(\mathbf{r}').d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|},$$

where S is the 2-dimensional surface bounding V. The electric field due to the polarisation of the medium is now

$$\mathbf{E}^{(\mathbf{P})}(\mathbf{r}) = -\nabla\Phi^{(\mathbf{P})}(\mathbf{r})$$

so

$$\nabla.\mathbf{E}^{(\mathbf{P})}(\mathbf{r}) = -\nabla^2\Phi^{(\mathbf{P})}(\mathbf{r}) = -\frac{1}{\epsilon_0}\nabla.\mathbf{P}(\mathbf{r}) + \frac{1}{\epsilon_0}\int_S \delta(\mathbf{r}-\mathbf{r}')\mathbf{P}(\mathbf{r}').d\mathbf{S}',$$

where we have used $-\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = 4\pi\delta(\mathbf{r}-\mathbf{r}')$. The last term on the right hand side is called the *surface polarisation*, it can be non-zero only when the field point $\mathbf{r}$ is taken to lie on the surface S. When $\mathbf{r} \notin S$

$$\nabla.\mathbf{E}^{(\mathbf{P})}(\mathbf{r}) = -\frac{1}{\epsilon_0}\nabla.\mathbf{P}(\mathbf{r}).$$

The total electric field is the sum of the electric field due to the charge density $\rho(\mathbf{r})$ introduced into the medium (we shall call these the *free* charges and denote the resulting electric field by $\mathbf{E}^{Free}(\mathbf{r})$) and the electric field induced by the polarisation, $\mathbf{E}^{(\mathbf{P})}$. $\mathbf{E}^{Free}(\mathbf{r})$ must satisfy

$$\nabla.\mathbf{E}^{Free}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

The total electric field $\mathbf{E}(\mathbf{r}) = \mathbf{E}^{Free}(\mathbf{r}) + \mathbf{E}^{(\mathbf{P})}(\mathbf{r})$ must then satisfy

$$\begin{aligned}\nabla.\mathbf{E}(\mathbf{r}) = \nabla.\mathbf{E}^{Free}(\mathbf{r}) + \nabla.\mathbf{E}^{(\mathbf{P})}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\epsilon_0} - \frac{1}{\epsilon_0}\nabla.\mathbf{P}(\mathbf{r}) \\ \Rightarrow \quad \nabla.(\epsilon_0\mathbf{E} + \mathbf{P}) &= \rho.\end{aligned}$$

In effect $-\nabla.\mathbf{P}$ gives an extra contribution to the charge density. It is now convenient to define a quantity called the *electric displacement vector*

$$\mathbf{D}(\mathbf{r}) = \epsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})$$

which satisfies the simple equation

$$\nabla.\mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}).$$

If we know $\rho(\mathbf{r})$ we can use all of the techniques that we have already studied to solve this equation for $\mathbf{D}$ — the net effect of the presence of the medium is simply to replace $\epsilon_0\mathbf{E}$ with $\mathbf{D}$. However we still cannot calculate $\mathbf{E}$, because the polarisation $\mathbf{P}$ depends on the total electric field $\mathbf{E}$, it is a function $\mathbf{P}(\mathbf{E})$, but $\mathbf{P}$ also contributes to $\mathbf{E}$ so we cannot calculate $\mathbf{E}$ without knowing $\mathbf{P}$ and we cannot calculate $\mathbf{P}$ without knowing $\mathbf{E}$. To make progress we need another assumption and in many practical situations it is sufficient to Taylor expand $\mathbf{P}(\mathbf{E})$, that is consider $\mathbf{P}(\mathbf{E})$ to be a function of the three component of the $\mathbf{E}$ and Taylor expand in these three variables. This is an acceptable procedure provided $\mathbf{E}$ is not too strong and, in practice, it is often only necessary to retain the first non-zero term in the expansion to get a good description of the physics. If there is a constant term in the Taylor expansion $\mathbf{P}(0) = \mathbf{P}_0$ then there will be a non-zero electric polarisation even when the electric field vanishes. Materials which sustain such a polarisation are called *ferro-electrics*, but this is not common. When there is no constant term in a Taylor expansion, as for most materials, the first non-zero term starts at the linear level. In fact for most media it is sufficient to take $\mathbf{P}$ to be a linear function of $\mathbf{E}$ and ignore quadratic terms. In a fluid, a liquid (like water) or a gas (like O_2), it seems reasonable that $\mathbf{P}$ will be parallel to, an in the same direction as, $\mathbf{E}$, though this is not necessarily true in a solid, such as a crystal. So for a fluid we write

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \tag{21}$$

where χ_e is a positive constant known as the *electric susceptibility* of the medium (it is a measure of how susceptible the medium is to being polarised when it is placed in an external electric field). A medium whose polarisation vector satisfies (21) is called a *linear medium.* For such a medium

$$\mathbf{D} = \epsilon_0(1+\chi_e)\mathbf{E} = \epsilon\mathbf{E}$$

where $\epsilon = \epsilon_0(1+\chi_e)$ is called the electric permittivity of the medium (in a vacuum $\chi_e = 0$ so $\epsilon = \epsilon_0$ is the same as the electric permittivity of the vacuum). In a linear medium

$$\nabla.\mathbf{D} = \epsilon\nabla.\mathbf{E} = \rho \quad \Rightarrow \quad \nabla.\mathbf{E} = \frac{\rho}{\epsilon}$$

and we can calculated $\mathbf{E}$ using all our earlier techniques, just replace ϵ_0 with ϵ everywhere.

Note that, since χ_e is positive in a polarisable medium, $\epsilon > \epsilon_0$, so the electric field produced by a charge density ρ is of exactly the same form as, but of a smaller magnitude than, the field produced by the same charge distribution in a vacuum. The polarisation of the medium is said to *screen* the free charges so they behave as though they have a smaller magnitude. Another name for a polarisable medium is a *dielectric* and ϵ is also called the *dielectric constant* of the medium. For example, in pure water at $20°C$, $\epsilon/\epsilon_0 = 1+\chi_e = 80$ and most solid dielectrics have χ_e in the range $1-20$. In the limit of $\chi_e \to \infty$ the electric field vanishes — this is what happens in a conductor, the electric field inside a conducting medium always vanishes even when an external field is applied.

There are also interesting effects when matter is put in an external magnetic field. Many materials, such as again water, consist of atoms or molecules that have small magnetic moments, or at least can develop one when an external magnetic field $\mathbf{B}$ is applied. These will tend to line up with $\mathbf{B}$ and generate a magnetic dipole moment which then modifies $\mathbf{B}$. Making the same assumptions as before, suppose a small volume δV of material contains δN magnetic dipoles, each with magnetic dipole moment $\mathbf{m}$, then the total dipole moment in δV will be given by

$$\mathbf{M}\delta V = \mathbf{m}\delta N,$$

where $\mathbf{M}$ is called the *magnetisation*. The material in δV situated at $\mathbf{r}'$ then contributes, from (18),

$$\delta\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)\delta V$$

to the magnetic vector potential, so the total contribution of all the material in a volume V is then

$$\mathbf{A}^{(\mathbf{M})}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV'.$$

Using $\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$ we can integrate by parts

$$\begin{aligned}\int_V \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dV' &= \int_V \mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV' = -\int_V \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{M}(\mathbf{r}')dV' \\ &= \int_V \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' + \int_S \frac{\mathbf{M}(\mathbf{r}')\times d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|},\end{aligned}$$

so we have

$$\mathbf{A}^{(\mathbf{M})}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{M}(\mathbf{r}')\times d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|}.$$

The magnetisation then contributes a term $\mathbf{B}^{(\mathbf{M})}(\mathbf{r}) = \nabla\times\mathbf{A}^{(\mathbf{M})}(\mathbf{r})$ to the total magnetic field. Now, using $\nabla\times\big(\nabla\times\mathbf{A}^{(\mathbf{M})}\big) = \nabla\big(\nabla.\mathbf{A}^{(\mathbf{M})}\big) - \nabla^2\mathbf{A}^{(\mathbf{M})}$ and $\nabla.\mathbf{A}^{(\mathbf{M})}(\mathbf{r}) = 0$,

$$\nabla\times\mathbf{B}^{(\mathbf{M})}(\mathbf{r}) = -\nabla^2\mathbf{A}^{(\mathbf{M})}(\mathbf{r}) = \mu_0\big(\nabla\times M(\mathbf{r})\big) + \frac{\mu_0}{4\pi}\int_S \delta(\mathbf{r}-\mathbf{r}')\big(\mathbf{M}(\mathbf{r}')\times d\mathbf{S}'\big).$$

The last term on the right hand side represents a current on the surface of V and vanishes if $\mathbf{r}$ is not on S.

When $\mathbf{r}\notin S$ the total magnetic field is $\mathbf{B} = \mathbf{B}^{(\mathbf{M})}(\mathbf{r}) + \mathbf{B}^{Free}$, where $\mathbf{B}^{Free}$ is the field produced by any currents $\mathbf{J}$ that are not associated with the internal structure of the atoms or molecules, *i.e.* currents that we force through the medium ourselves. These will give a contribution $\mathbf{B}^{Free}(\mathbf{r})$ satisfying $\nabla \times \mathbf{B}^{Free}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$. So the total field satisfies

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) + \mu_0 \big(\nabla \times \mathbf{M}(\mathbf{r})\big) \quad \Rightarrow \quad \nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \mathbf{J}.$$

In effect $\nabla \times \mathbf{M}$ acts like an extra contribution to the current. It is then convenient to define a quantity called the *magnetic intensity*

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

so that

$$\nabla \times \mathbf{H} = \mathbf{J}$$

and the net effect of the presence of the medium is to replace $\mathbf{B}$ with $\mu_0 \mathbf{H}$ in the original equation. We cannot however calculate $\mathbf{B}$ itself yet, just as in the discussion on electric polarisation $\mathbf{B}$ depends on $\mathbf{M}$ but $\mathbf{M}$ depends on $\mathbf{B}$ and we need some assumptions about $\mathbf{M}(\mathbf{B})$ to go any further. We shall Taylor expand in $\mathbf{B}$, which is a reasonable thing to do so as long as $\mathbf{B}$ is not too strong. Magnetic materials which have $\mathbf{M} \neq 0$ even when $\mathbf{B} = 0$ are called *ferromagnets* (examples are iron, nickel and cobalt). An ordinary bar magnet is such a material, for example — it has a non-zero magnetic dipole moment even in the absence of any external field. For ferromagnetic materials it is usually a very good approximation to keep only the first non-zero term in a Taylor expansion of $\mathbf{M}(\mathbf{B})$ and we can take $\mathbf{M}(\mathbf{B}) = \mathbf{M}_0$, a constant, so

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}_0.$$

In particular if there are no currents and $\mathbf{H} = 0$ then $\mathbf{B} = \mu_0 \mathbf{M}$ is non-zero even when $\mathbf{J} = 0$, this is the situation in a permanent magnet like a bar magnet.

For materials that do not have a permanent magnetisation, the Taylor expansion will start with a linear term and for fluids (but not necessarily for solids) we expect $\mathbf{M}$ to be parallel to $\mathbf{B}$.

$$\mathbf{M} = \frac{\chi_m}{\mu_0}\mathbf{B}$$

where χ_m is called the *magnetic susceptibility* of the medium.* This leads to

$$\mathbf{B} = \mu \mathbf{H},$$

where

$$\mu = \frac{\mu_0}{1 - \chi_m},$$

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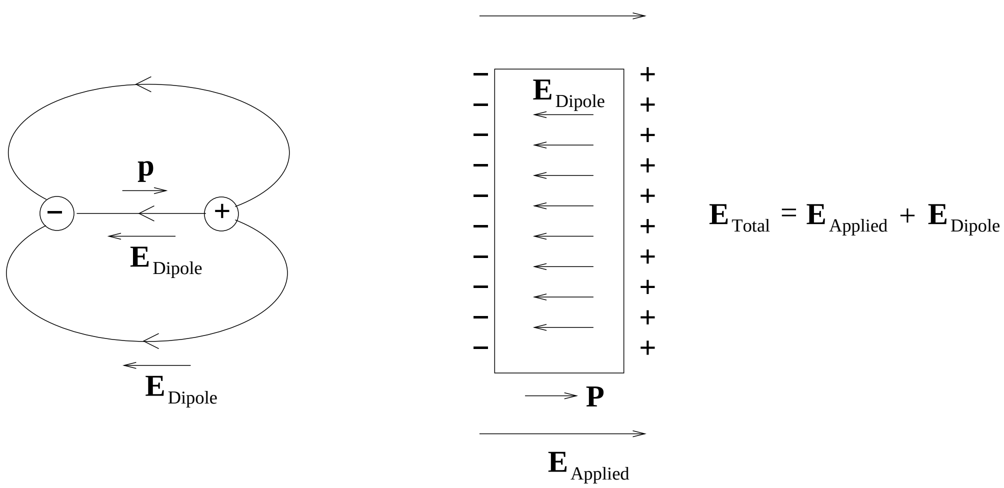
* The $1/\mu_0$ is conventional. Some textbooks define $\mathbf{M} = \chi_m \mathbf{H} \Rightarrow \mathbf{M} = \frac{1}{\mu_0}\frac{\chi_m}{(1+\chi_m)}\mathbf{B}$, but in most substances $|\chi_m| << 1$ so there is no practical difference (superconductors are an exception to this, they have a very large magnetic susceptibility).

and hence

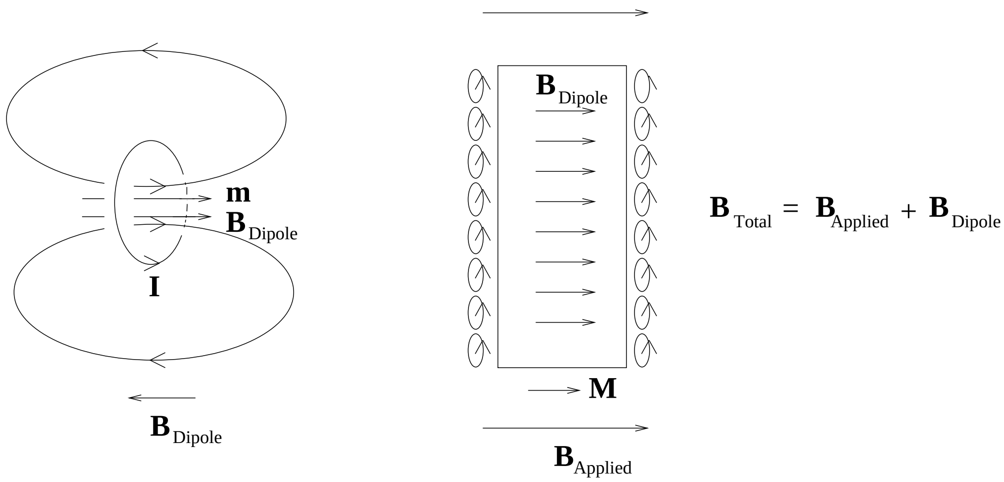
$$\nabla \times \mathbf{B} = \mu \mathbf{J}.$$

The net result is to replace μ_0 with μ in Maxwell's equations. μ is called the *magnetic permeability* of the medium and $\mu = \mu_0$ in a vacuum.

If $1 > \chi_m > 0$ then $\mu > \mu_0$ and the magnetic field generated by a given $\mathbf{J}$ is stronger than it would be in a vacuum — the atomic or molecular dipole moments $\mathbf{m}$ line up the external field and enhance it. A material with $\chi_m > 0$ is called *paramagnetic*, for example Aluminium and Magnesium are paramagnetic. At first sight it may seem strange that the electric field generated by a given ρ is *weaker* in a dielectric than in a vacuum while in a paramagnet the magnetic field generated by a given $\mathbf{J}$ is *stronger* than it would be in a vacuum. This can be understood intuitively by considering a slab of dielectric material in a constant electric field as being like an electric dipole itself, with a dipole moment sustained by a surface charge induced by the external field. The electric field inside a dipole aligned with an external field points in the *opposite* direction to the external field.



On the other hand for a magnetic dipole, thought of as a loop of current, aligned with an external magnetic field the magnetic field inside the dipole points in the *same* direction as the external magnetic field. A slab of paramagnetic material in an external magnetic field can be viewed as itself being a magnetic dipole with a dipole moment sustained by surface currents induced by the external field.



However magnetic effects are subtle and χ_m, unlike χ_e, can be negative. Materials with $\chi_m < 0$ are called *diamagnetic.* Put simply, if the atoms or molecules have non-zero angular momentum (arising, for example, from the spin of unpaired electrons) they will have a permanent magnetic dipole moment and will tend to be paramagnetic while if they have no unpaired electrons they will have no permanent dipole moment, but an externally applied magnetic field may generate one in which case they will be diamagnetic. This can be understood in terms of Lenz's law — the currents induced by the external field are in such a direction as to reduce the field. A diamagnet, like a dielectric, tends to reduce the applied field. Diamagnetic response to an external field is very common but diamagnetism is usually very weak and when either paramagnetism or ferromagnetism are present they usually dominate.

There are thus three types of magnetic materials: ferromagnetic, paramagnetic and diamagnetic. Some materials can have different magnetic properties under different conditions, for example ferromagnetic materials tend to lose there magnetism at high temperature, due to thermal motion of the atoms knocking their dipole moments out of alignment, and become paramagnetic.

Typically magnetic effects are small: $\chi_m \sim 10^{-2} - 10^{-5}$ for paramagnets and the most diamagnetic substance known, at room temperature, is a form of Carbon called pyrolytic graphite, with $\chi_m = -4.0 \times 10^{-4}$. The most diamagnetic metal is Bismuth with $\chi_m = -1.8 \times 10^{-4}$. Water is also diamagnetic, with $\chi_m \approx -5 \times 10^{-6}$. But magnetic effects are not always small: superconductors are materials in which $\chi_m \to -\infty$ at very low temperatures. Superconductors are perfect diamagnets with $\mu = 0$ and $\mathbf{B} = 0$ inside a superconductor, even when $\mathbf{J} \neq 0$. When a material becomes superconducting magnetic field lines are expelled to the exterior of the material — a phenomenon known as the *Meissner effect.* For example the element Mercury becomes superconducting below 4.2 K.

To summarise, the net result of the presence of matter in static situations is

$$\begin{array}{ccccccc}
\mathbf{E} & \to & \dfrac{\mathbf{D}}{\epsilon_0} & \qquad & \nabla.\mathbf{E} = \dfrac{1}{\epsilon_0}\rho & \to & \nabla.\mathbf{D} = \rho \\
\mathbf{B} & \to & \mu_0 \mathbf{H} & \qquad & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} & \to & \nabla \times \mathbf{H} = \mathbf{J}.
\end{array}$$

The equations $\nabla.\mathbf{B} = 0$ and $\nabla \times \mathbf{E} = 0$ are unchanged: since they are unaffected by the presence of sources then matter does not affect them. The same argument implies that, even for time varying fields, the equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla . \mathbf{B} = 0$$

are unaffected by the presence of matter. Using the substitutions above

$$\nabla \times \mathbf{B} - \epsilon_0 \mu_0 \left(\frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \mathbf{J} \quad \rightarrow \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J},$$

and we arrive at Maxwell's equations in the presence of matter

$$\boxed{\begin{array}{ll} \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 & \qquad \nabla . \mathbf{B} = 0 \\ \nabla \times \mathbf{H} - \dfrac{\partial \mathbf{D}}{\partial t} = \mathbf{J} & \qquad \nabla . \mathbf{D} = \rho \end{array}}$$

The same substitutions result in modifications of the energy density and the Poynting vector,

$$\begin{aligned} w &= \frac{1}{2}(\epsilon_0 \mathbf{E}.\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}.\mathbf{B}) \quad &\rightarrow \quad w &= \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H}) \\ \mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \quad &\rightarrow \quad \mathbf{S} &= (\mathbf{E} \times \mathbf{H}). \end{aligned}$$

From now on we shall restrict our attention to a linear medium, in which $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$. In this case

$$\begin{array}{ll} \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 & \qquad \nabla . \mathbf{B} = 0 \\ \nabla \times \mathbf{B} - \epsilon\mu \dfrac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J} & \qquad \nabla . \mathbf{E} = \dfrac{\rho}{\epsilon}, \end{array} \tag{22}$$

and

$$w = \frac{1}{2}(\epsilon \mathbf{E}.\mathbf{E} + \frac{1}{\mu}\mathbf{B}.\mathbf{B}) \qquad \mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B}).$$

5. Plane Waves and Radiation from Simple Systems

Differentiating Maxwell's equations (22) gives

$$0 = \nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = \nabla(\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{1}{\epsilon}\nabla\rho - \nabla^2 \mathbf{E} + \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \frac{\partial \mathbf{J}}{\partial t}$$

$$\Rightarrow \qquad \nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon}\nabla\rho + \mu \frac{\partial \mathbf{J}}{\partial t}$$

and

$$\mu(\nabla\times\mathbf{J}) = \nabla\times\left(\nabla\times\mathbf{B} - \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}\right) = \nabla(\nabla.\mathbf{B}) - \nabla^2\mathbf{B} - \mu\epsilon\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{B} + \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\Rightarrow \qquad \nabla^2\mathbf{B} - \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu(\nabla\times\mathbf{J}).$$

In a charge and current free region of space, $\rho = 0$ and $\mathbf{J} = 0$, Maxwell's equations imply (but are *not* equivalent to) a set of coupled, linear, homogeneous differential equations for $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{aligned}\nabla^2\mathbf{E} - \epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} &= 0\\ \nabla^2\mathbf{B} - \epsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} &= 0.\end{aligned} \tag{23}$$

These equations have wave-like solutions that move with speed $v = 1/\sqrt{\mu\epsilon}$, electromagnetic waves. To investigate this we shall adopt a complex notation and define oscillating complex electric and magnetic fields,

$$\underline{\mathcal{E}}(\mathbf{x},t) = \underline{\mathcal{E}}_0 \mathrm{e}^{i(\mathbf{k}.\mathbf{x}-\omega t)} \qquad \underline{\mathcal{B}}(\mathbf{x},t) = \underline{\mathcal{B}}_0 \mathrm{e}^{i(\mathbf{k}.\mathbf{x}-\omega t)} \tag{24}$$

where $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{B}}_0$ are constant complex vectors, $\mathbf{k}$ is a real vector (the wave-vector) and $\omega > 0$ (an angular frequency). This notation is a mathematical convenience, the true physical fields are just the real part of these, $\mathbf{E} = \Re(\underline{\mathcal{E}})$ and $\mathbf{B} = \Re(\underline{\mathcal{B}})$. For example if $\underline{\mathcal{E}}_0 = \mathbf{E}_0$ and $\underline{\mathcal{B}}_0 = \mathbf{B}_0$ are real vectors then

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = \mathbf{E}_0\cos(\mathbf{k}.\mathbf{x}-\omega t) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \mathbf{B}_0\cos(\mathbf{k}.\mathbf{x}-\omega t),$$

while, if $\underline{\mathcal{E}}_0 = \mathbf{E}_0\mathrm{e}^{i\delta}$ and $\underline{\mathcal{B}}_0 = \mathbf{B}_0 e^{i\delta}$ with $\mathbf{E}_0$ and $\mathbf{B}_0$ real vectors and δ a constant phase, then

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = \mathbf{E}_0\cos(\mathbf{k}.\mathbf{x}-\omega t+\delta) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \mathbf{E}_0\cos(\mathbf{k}.\mathbf{x}-\omega t+\delta).$$

As long as we only deal with expressions that are linear in $\mathbf{E}$ and $\mathbf{B}$ with real co-efficients, such as Maxwell's equations, then we can use this complex notation and just extract the real part at the end of the calculation.

In this notation equations (23) give

$$\begin{aligned}\nabla^2\underline{\mathcal{E}} - \epsilon\mu\frac{\partial^2\underline{\mathcal{E}}}{\partial t^2} &= (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2)\underline{\mathcal{E}} = 0\\ \nabla^2\underline{\mathcal{B}} - \epsilon\mu\frac{\partial^2\underline{\mathcal{B}}}{\partial t^2} &= (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2)\underline{\mathcal{B}} = 0\\ \Rightarrow \quad (-\mathbf{k}.\mathbf{k} + \epsilon\mu\omega^2) = 0 \quad &\Rightarrow \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}.\end{aligned}$$

These configurations correspond to waves of oscillating electric and magnetic fields with wave-length $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$ moving in the direction of the unit vector $\mathbf{n} = \mathbf{k}/k$ at speed $v = \omega/k = 1/\sqrt{\mu\epsilon}$.* Thus we can relate the speed of light in a medium, such as water or glass, to ϵ and μ. For most materials $\mu \approx \mu_0$

$$\frac{v}{c} = \sqrt{\frac{\epsilon_0}{\epsilon}} = \frac{1}{\sqrt{1+\chi_e}} < 1$$

so the refractive index is

$$n = \sqrt{1+\chi_e}$$

and the speed of light in the medium is related to the electric susceptibility.†

However this is not the whole story, equations (23) follow from, but do not imply, Maxwell's equations — information was thrown away in deriving them from (22) — to get the full picture we should substitute (24) into (22):

$$\nabla \times \underline{\mathcal{E}} + \frac{\partial \underline{\mathcal{B}}}{\partial t} = 0 \quad \Rightarrow \quad i(\mathbf{k} \times \underline{\mathcal{E}}_0) = i\omega \underline{\mathcal{B}}_0 \quad \Rightarrow \quad \underline{\mathcal{B}}_0 = \frac{1}{v}(\mathbf{n} \times \underline{\mathcal{E}}_0), \tag{25}$$

$$\nabla \times \underline{\mathcal{B}} - \mu\epsilon \frac{\partial \underline{\mathcal{E}}}{\partial t} = 0 \quad \Rightarrow \quad i(\mathbf{k} \times \underline{\mathcal{B}}_0) = -i\mu\epsilon\omega \underline{\mathcal{E}}_0 \quad \Rightarrow \quad \underline{\mathcal{E}}_0 = -v(\mathbf{n} \times \underline{\mathcal{B}}_0), \tag{26}$$

$$\nabla . \underline{\mathcal{E}} = 0 \quad \Rightarrow \quad \mathbf{k}.\underline{\mathcal{E}}_0 = 0 \quad \Rightarrow \quad \mathbf{n}.\underline{\mathcal{E}}_0 = 0, \tag{27}$$

$$\nabla . \underline{\mathcal{B}} = 0 \quad \Rightarrow \quad \mathbf{k}.\underline{\mathcal{B}}_0 = 0 \quad \Rightarrow \quad \mathbf{n}.\underline{\mathcal{B}}_0 = 0. \tag{28}$$

Thus $\mathbf{n}$, $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{B}}_0$ are mutually perpendicular. Let $\mathbf{n} = \mathbf{e}_3$ and introduce a right-handed orthonormal triple, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and $\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}$. Then there are two linearly independent possibilities: either $\underline{\mathcal{E}}_0$ is proportional to $\mathbf{e}_1$,

$$\underline{\mathcal{E}}_0 = \mathcal{E}_0 \mathbf{e}_1,$$

where $\mathcal{E}_0$ is a complex number, in which case (25) to (28) require

$$\underline{\mathcal{B}}_0 = \mathcal{B}_0 \mathbf{e}_2 = \frac{1}{v}\mathcal{E}_0 \mathbf{e}_2,$$

or $\underline{\mathcal{E}}_0$ is proportional to $\mathbf{e}_2$,

$$\underline{\mathcal{E}}_0 = \mathcal{E}'_0 \mathbf{e}_2,$$

---

* What we have described here is a *monochromatic* electro-magnetic wave traveling through a medium — we focused on a single frequency ω. In general a wave will consist of a superposition of many frequencies, perhaps centred around a maximum intensity of a given colour, but this can be described by adding different frequencies of different intensities — again the linearity of Maxwell's equation allows us to add solutions to get more solutions.

† The electric susceptibility can be a function of frequency: in water, for example, $\chi_e \approx 80$ for static fields but this is reduced to $\chi_e \approx 0.8$ at optical frequencies giving a refractive index of $n = 1.3$.

where $\mathcal{E}_0'$ is a complex number, in which case (25) to (28) require

$$\underline{\mathcal{B}}_0 = -\mathcal{B}_0\mathbf{e}_1 = -\frac{1}{v}\mathcal{E}_0'\mathbf{e}_1.$$

The most general wave-like solution of Maxwell's equations is a linear combination of these two possibilities,

$$\underline{\mathcal{E}}(\mathbf{x},t) = \big(\mathcal{E}_0\mathbf{e}_1 + \mathcal{E}_0'\mathbf{e}_2)\big)\mathrm{e}^{ik(\mathbf{x}.\mathbf{n}-vt)}, \qquad \underline{\mathcal{B}}(\mathbf{x},t) = \frac{1}{v}\big(-\mathcal{E}_0'\mathbf{e}_1 + \mathcal{E}_0\mathbf{e}_2\big)\mathrm{e}^{ik(\mathbf{x}.\mathbf{n}-vt)}.$$

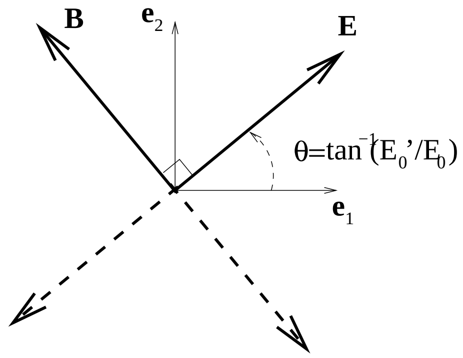
These two linearly independent possibilities are associated with the polarisation of light. If the two complex constants $\mathcal{E}_0$ and $\mathcal{E}_0'$ have the same complex phase δ, so $\mathcal{E}_0 = E_0\mathrm{e}^{i\delta}$ and $\mathcal{E}_0' = E_0'\mathrm{e}^{i\delta}$ with E_0 and E_0' real constants, and $\mathbf{n}$ is in the z-direction then $\mathbf{k}.\mathbf{n} = kz$ and the physical fields are

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = \big(E_0\mathbf{e}_1 + E_0'\mathbf{e}_2\big)\cos(kz - \omega t + \delta)$$

and

$$\mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \frac{1}{v}\big(E_0\mathbf{e}_2 - E_0'\mathbf{e}_1\big)\cos\big(kz - \omega t + \delta\big).$$

The electric and magnetic fields therefore keep a fixed orientation in space and are at right-angles to each other, and to the direction of motion $\mathbf{n}$ of the wave, but oscillate in magnitude. This is called a *plane polarised* wave.



Plane Polarised Wave

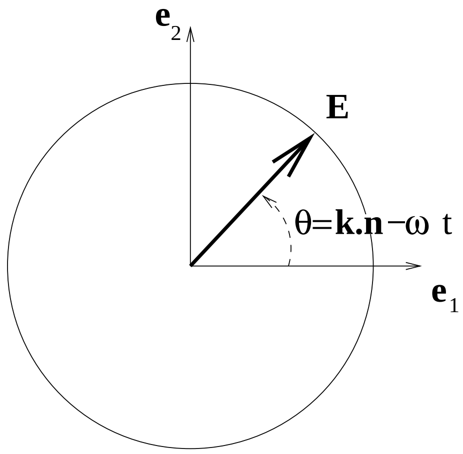
Other geometries are possible if $\underline{\mathcal{E}}_0$ and $\underline{\mathcal{E}}_0'$ have different complex phases, *e.g.* suppose $\mathcal{E}_0 = E_0$ and $\mathcal{E}_0' = iE_0'$ with E_0 and E_0' real. Then, again with $\mathbf{n}$ in the z-direction,

$$\mathbf{E}(\mathbf{x},t) = \Re\big(\underline{\mathcal{E}}(\mathbf{x},t)\big) = E_0\cos(kz-\omega t)\mathbf{e}_1 - E_0'\sin(kz-\omega t)\mathbf{e}_2$$

and

$$\mathbf{B}(\mathbf{x},t) = \Re\big(\underline{\mathcal{B}}(\mathbf{x},t)\big) = \frac{1}{v}\left\{E_0\cos(kz-\omega t)\mathbf{e}_2 + E_0'\sin(kz-\omega t)\mathbf{e}_1\right\},$$

and again $\mathbf{E}$ and $\mathbf{B}$ are always at right-angles to each other, and to $\mathbf{n}$, but this time they rotate both describing an ellipse: the wave is said to be *elliptically polarised.* If $E_0 = E_0'$ they describe a circle and the wave is *circularly polarised.* If $\underline{\mathcal{E}}_0' = -iE_0'$ the rotation is in the opposite direction (the two possible rotation directions for a circularly polarised electro-magnetic wave are called different *helicities*).



Circularly Polarised Wave

Electro-magnetic waves carry energy and we calculate the energy flux using the Poynting vector. The Poynting vector will depend on time and its average value over a cycle of oscillation is the more relevant quantity. First we must think a little about the meaning of our complex notation for quantities that are quadratic in the fields, in fact the complex notation is tailored towards calculating time-averages of quadratic quantities. To show this we shall prove a little lemma:

If $f(t) = f_0 \mathrm{e}^{-i\omega t}$ and $g(t) = g_0 \mathrm{e}^{-i\omega t}$, where f_0 and g_0 are independent of time t, then the time average of $\Re(f)\Re(g)$ over a complete cycle, $T = 2\pi/\omega$, is

$$\overline{fg} := \frac{1}{T}\int_0^T \Re\big(f(t)\big)\Re\big(g(t)\big)dt = \frac{1}{2}\Re(f_0^* g_0) \tag{29}$$

where f_0^* is the complex conjugate of f_0.

To prove this let

$$f_0 = u + iv \qquad \text{and} \qquad g_0 = \zeta + i\eta$$

where u, v, ζ and η are real and independent of t. Then

$$\begin{aligned}\Re\big(f(t)\big)\Re\big(g(t)\big) &= \big(u\cos(\omega t) + v\sin(\omega t)\big)\big(\zeta\cos(\omega t) + \eta\sin(\omega t)\big)\\ &= u\zeta\cos^2(\omega t) + v\eta\cos^2(\omega t) + (u\eta + v\zeta)\cos(\omega t)\sin(\omega t)\end{aligned}$$

so

$$\int_0^{\frac{2\pi}{\omega}} \big(\Re(f)\big)\big(\Re(g)\big)dt = u\zeta\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)dt + v\eta\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t)dt = \frac{\pi}{\omega}(u\zeta + v\eta),$$

since $\int_0^{\frac{2\pi}{\omega}}\cos(\omega t)\sin(\omega t)dt = 0$ and $\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)dt = \int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t)dt = \frac{\pi}{\omega}$. Hence the time average

$$\overline{fg} = \frac{1}{2}(u\zeta + v\eta).$$

But

$$\Re(f^*g) = \Re(f_0^* g_0) = u\zeta + v\eta,$$

which proves (29).

We can now apply this to calculate the time-average of the energy flux at a point $\mathbf{x}$ from the Poynting vector $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu$,

$$\overline{\mathbf{S}}(\mathbf{x}) = \frac{\omega}{2\pi\mu} \int_0^{\frac{2\pi}{\omega}} \big(\mathbf{E}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t)\big) dt = \frac{1}{2\mu} \Re(\underline{\mathcal{E}}_0^* \times \underline{\mathcal{B}}_0) = \frac{1}{2v\mu} \underline{\mathcal{E}}_0^* . \underline{\mathcal{E}}_0 \mathbf{n},$$

independent of $\mathbf{x}$ (equations (25) and (28) have been used in the last step above). This is related to the time-average of the energy density in the wave

$$\begin{aligned} \overline{w} &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \frac{1}{2} \left(\epsilon \mathbf{E}(\mathbf{x},t).\mathbf{E}(\mathbf{x},t) + \frac{1}{\mu} \mathbf{B}(\mathbf{x},t).\mathbf{B}(\mathbf{x},t) \right) dt \\ &= \frac{1}{4} \left(\epsilon \underline{\mathcal{E}}_0^* . \underline{\mathcal{E}}_0 + \frac{1}{\mu} \underline{\mathcal{B}}_0^* . \underline{\mathcal{B}}_0 \right) = \frac{\epsilon}{2} \underline{\mathcal{E}}_0^* . \underline{\mathcal{E}}_0 . \end{aligned}$$

So, since $v = 1/\sqrt{\epsilon\mu}$,

$$\overline{\mathbf{S}} = v\, \overline{w}\, \mathbf{n},$$

a very natural result stating that the time-averaged energy-flux is in the direction $\mathbf{n}$ of the wave and has a magnitude which is just the time-averaged energy times the speed of the wave.

Electro-magnetic waves are produced by oscillating charge and current distributions and in order to describe this we shall use the potentials rather than the fields.

Vector and Scalar Potentials

Since $\nabla.\mathbf{B} = 0$ we always have $\mathbf{B} = \nabla \times \mathbf{A}$, even in the presence of matter, so

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Hence $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ can be expressed as a gradient, $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi$, so

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

Notice that for any twice differentiable function $\Lambda(\mathbf{r},t)$

$$\begin{aligned} \mathbf{A}' &:= \mathbf{A} + \nabla\Lambda &\Rightarrow\qquad & \mathbf{B} = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} \\ \Phi' &:= \Phi - \frac{\partial \Lambda}{\partial t} &\Rightarrow\qquad & \mathbf{E} = -\nabla\Phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}. \end{aligned}$$

Thus Φ' and $\mathbf{A}'$ give rise to the same $\mathbf{E}$ and $\mathbf{B}$ fields as Φ and $\mathbf{A}$. The potentials Φ and $\mathbf{A}$ for an electro-magnetic field configuration are not unique, there is an ambiguity in their definition. The change

$$\begin{aligned} \mathbf{A}' &\rightarrow \mathbf{A} + \nabla\Lambda \\ \Phi' &\rightarrow \Phi - \frac{\partial\Lambda}{\partial t} \end{aligned} \tag{30}$$

is called a *gauge transformation*.* In the magnetostatics section we showed that $\mathbf{A}(\mathbf{r})$ arising from a given $J(\mathbf{r})$ satisfied $\nabla.\mathbf{A} = 0$, but we see now that this is not essential, if $\nabla.\mathbf{A} = 0$ then $\nabla.\mathbf{A}' \neq 0$ unless $\nabla^2\Lambda = 0$ which need not always be the case. Different choices of Λ lead to different gauges and a choice which gives $\nabla.\mathbf{A} = 0$ is called the *Coulomb gauge*, which is useful for problems in statics. For time varying fields the condition

$$\nabla.\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

is often convenient, this is called the *Lorentz gauge* (obviously the Lorentz gauge reduces to the Coulomb gauge when Φ is independent of t). For any potentials $(\Phi, \mathbf{A})$ it is always possible to find a Λ so that $(\Phi', \mathbf{A}')$ satisfy the Lorentz gauge condition, since

$$\begin{aligned} \nabla.\mathbf{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t} &= \nabla.\mathbf{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0 \\ \Rightarrow \quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} &= -\nabla.\mathbf{A} - \frac{1}{c^2}\frac{\partial\Phi}{\partial t}. \end{aligned}$$

The last equation here is just the inhomogeneous wave-equation for Λ, with a source $f(\mathbf{r}, t) := -\nabla.\mathbf{A} - \frac{1}{c^2}\frac{\partial\Phi}{\partial t}$, and this equation can always be solved to find Λ so that $(\Phi', \mathbf{A}')$ satisfy the Lorentz gauge condition.

However, even the Lorentz gauge condition does not completely remove the ambiguity in $(\Phi, \mathbf{A})$, for example if $(\Phi, \mathbf{A})$ satisfy the Lorentz gauge condition then

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\lambda, \qquad \Phi \rightarrow \Phi - \frac{\partial\lambda}{\partial t}$$

do too, provided λ satisfies the wave equation, $-\nabla^2\lambda + \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0$. This residual ambiguity in Φ and $\mathbf{A}$ does not affect any of the following analysis.

In terms of Φ and $\mathbf{A}$ two of Maxwell's equations are automatic,

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} \quad &\Rightarrow \quad \nabla.\mathbf{B} = 0 \\ \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \quad &\Rightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \end{aligned}$$

* The name is historical and, from a modern perspective, is rather inappropriate, but nevertheless it has stuck.

so we only need worry about the equations that involve sources ρ and $\mathbf{J}$. In the vacuum, with $\epsilon = \epsilon_0$ and $\mu = \mu_0$,[†]

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon} \quad \Rightarrow \quad -\nabla^2\Phi - \frac{\partial(\nabla.\mathbf{A})}{\partial t} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 J \quad \Rightarrow \quad \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial(\nabla\Phi)}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}.$$

In the Lorentz gauge $\frac{1}{c^2}\frac{\partial\Phi}{\partial t} = -\nabla.\mathbf{A}$ these reduce to the inhomogeneous wave-equations

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}, \qquad -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}.$$

In particular in a source free region of space, where $\rho = 0$ and $\mathbf{J} = 0$, the potentials satisfy the wave equation

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0, \qquad -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

and there will be wave-like solutions.

Radiation from Simple Systems

We shall now study the electromagnetic radiation produced by an oscillating distribution of charges and currents, using the method of Greens function. For simplicity we shall work in a vacuum and $\epsilon = \epsilon_0$ and $\mu = \mu_0$. In statics we solved

$$\nabla.\mathbf{E} = -\nabla^2\Phi = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{B} = -\nabla^2\mathbf{A} = \mu_0\mathbf{J} \qquad \text{(in the Coulomb gauge, } \nabla.\mathbf{A} = 0\text{)}$$

in a volume V using Green functions which satisfy $-\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$. For example, if V is unbounded space, $G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ gives

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')dV' = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dV'$$
$$\mathbf{A}(\mathbf{r}) = \mu_0\int_V \mathbf{J}(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')dV' = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}dV'.$$

In a dynamical situation, using the Lorentz gauge, we must solve

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$
$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J} \qquad (31).$$

[†] This whole analysis works equally well in a linear medium with $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$ and $c \to v$ everywhere in the equations.

Our strategy will again be to find suitable Green functions, but first we eliminate the time derivatives by using Fourier transforms. Define Fourier amplitudes

$$\widetilde{\Phi}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \Phi(\mathbf{r},t)e^{i\omega t}dt$$
$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r},t)e^{i\omega t}dt,$$

assuming the integrals exist. Given $\widetilde{\Phi}(\mathbf{r},\omega)$ and $\widetilde{\mathbf{J}}(\mathbf{r},\omega)$ the original charge and current densities can be re-constructed using the inverse transforms

$$\Phi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}d\omega \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{\mathbf{A}}(\mathbf{r},\omega)e^{-i\omega t}d\omega.$$

Multiplying (31) by $e^{i\omega t}$, integrating over all t and equating the integrands gives

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{\Phi} = \frac{\widetilde{\rho}}{\epsilon_0}, \qquad -\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{\mathbf{A}} = \mu_0\widetilde{\mathbf{J}},$$

where

$$\widetilde{\rho}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \rho(\mathbf{r},t)e^{i\omega t}dt, \qquad \widetilde{\mathbf{J}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r},t)e^{i\omega t}dt,$$

are the Fourier transforms of the charge and current densities. The problem is now reduced to finding Green functions $G_k(\mathbf{r},\mathbf{r}')$ for the operator $-(\nabla^2+k^2)$, called the *Helmholtz operator*,

$$-(\nabla^2+k^2)G_k(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

where $k=\omega/c$.

If V is unbounded space we can expect, from translational invariance, that $G_k(\mathbf{r},\mathbf{r}')$ should depend only on the difference $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, $G_k(\mathbf{R})$. Similarly rotational invariance implies that $G_k(\mathbf{R})$ should depend only on $R=|\mathbf{R}|$ and not on its direction, so there will be no angular dependence. Expressing ∇^2 in 3-dimensional polar co-ordinates, with the origin taken to be $\mathbf{R}=0$, we therefore have

$$\nabla^2 G_k(R) = \frac{1}{R}\left(\frac{d^2(RG_k)}{dR^2}\right) \quad\Rightarrow\quad \frac{1}{R}\left(\frac{d^2(RG_k)}{dR^2} + k^2(RG_k)\right) = \delta(R).$$

If $R \neq 0$

$$\left(\frac{d^2}{dR^2}+k^2\right)(RG_k) = 0$$

which has two linearly independent solutions which we denote by $G_k^{\pm}$,

$$RG_k^{\pm} = C_{\pm}e^{\pm ikR},$$

where $C_\pm$ are constants. When $k = 0$ the Helmholtz operator reduces to the Laplace operator, that we studied in the electrostatics section, with Green function $1/4\pi R$, so we can fix the normalisation

$$G_k(R) \quad \underset{k\to 0}{\longrightarrow} \quad G_0(R) = \frac{C_\pm}{R} = \frac{1}{4\pi R}.$$

So we choose $C_\pm = 1/4\pi$ and set*

$$G_k^\pm(R) = \frac{e^{\pm ikR}}{4\pi R} = \frac{e^{\pm i\frac{\omega}{c}R}}{4\pi R}.$$

The method of Green functions therefore leads to two linearly independent solutions in unbounded space for any given $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$,

$$\widetilde{\Phi}^\pm(\mathbf{r},\omega) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r},\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}^\pm(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r},\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'.$$

These reduce to the static result when $\omega = 0$. The inverse Fourier transforms give

$$\begin{aligned}
\Phi^\pm(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_V \frac{\widetilde{\rho}(\mathbf{r},\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'\right)e^{-i\omega t}d\omega \\
&= \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}(\mathbf{r}',\omega)e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|-i\omega t}d\omega\right)dV' \\
&= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}',t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV', \\
\mathbf{A}^\pm(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}dV'\right)e^{-i\omega t}d\omega \\
&= \frac{\mu_0}{4\pi}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\mathbf{J}}(\mathbf{r}',\omega)e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|-i\omega t}d\omega\right)dV' \\
&= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}',t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV',
\end{aligned}$$

where $t^\pm := t \mp \frac{1}{c}|\mathbf{r}-\mathbf{r}'|$. To summarise

$$\boxed{\begin{aligned}
\Phi^\pm(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}',t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV', \\
\mathbf{A}^\pm(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}',t^\pm)}{|\mathbf{r}-\mathbf{r}'|}dV'.
\end{aligned}}$$

---

* More generally we can take any linear combination

$$G_k(R) = \frac{1}{R}\left(C_+e^{ikR} + C_-e^{-ikR}\right)$$

as a Green function, provided $C_+ + C_- = 1/4\pi$. As an exercise check, given that $\nabla^2(1/R) = -4\pi\delta(\mathbf{R})$, that $(\nabla^2 + k^2)(e^{\pm ikR}/R) = -4\pi\delta(\mathbf{R})$.

These formulae have a very simple physical interpretation: $\Phi^+(\mathbf{r},t)$ at the field point $\mathbf{r}$ depends on $\rho(\mathbf{r}',t^+)$ at the source point $\mathbf{r}'$ not as it is at time t but as it was at time $t^+ = t - |\mathbf{r}-\mathbf{r}'|/c$, because it takes a finite time $|\mathbf{r}-\mathbf{r}'|/c$ for information, moving at the speed of light, about the charge distribution at $\mathbf{r}'$ to reach the point $\mathbf{r}$. Φ^+ and $\mathbf{A}^+$ are called *retarded* potentials, because of this time-lag. The second set of solutions, Φ^- and $\mathbf{J}^-$, correspond to the fields at $\mathbf{r}$ being influenced by what the charge and current distributions *will be* at the time $t^- = t + |\mathbf{r}-\mathbf{r}'|/c$ in the *future*, Φ^- and $\mathbf{J}^-$ are called *advanced* potentials. We shall restrict our attention to retarded potentials from now on.*

Multipole expansions

In principle the retarded potentials can be obtained by doing the integrals†

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}',\omega)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV',$$

for given ρ and $\mathbf{J}$ but, as in statics, this is often not possible analytically so we resort to a multipole approximation. We shall concentrate on a single frequency ω and consider, in complex notation, a charge and current distribution

$$\rho(\mathbf{r},t) = \widetilde{\rho}(\mathbf{r})e^{-i\omega t}, \qquad \mathbf{J}(\mathbf{r},t) = \widetilde{\mathbf{J}}(\mathbf{r})e^{-i\omega t}, \tag{32}$$

where $\widetilde{\rho}(\mathbf{r})$ and $\widetilde{\mathbf{J}}(\mathbf{r})$ are a static, possibly complex, charge and current density.‡ Their Fourier transforms are

$$\begin{aligned}\widetilde{\rho}(\mathbf{r},\omega') &= \int_{-\infty}^{\infty}\rho(\mathbf{r},t)e^{i\omega' t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}(\mathbf{r})e^{i(\omega'-\omega)t}dt = \widetilde{\rho}(\mathbf{r})\delta(\omega-\omega'),\\ \widetilde{\mathbf{J}}(\mathbf{r},\omega') &= \int_{-\infty}^{\infty}\mathbf{J}(\mathbf{r},t)e^{i\omega' t}dt == \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\mathbf{J}}(\mathbf{r})e^{i(\omega'-\omega)t}dt = \widetilde{\mathbf{J}}(\mathbf{r})\delta(\omega-\omega'),\end{aligned}$$

and for simplicity we shall omit the δ-functions and just use $\widetilde{\rho}(\mathbf{r})$ and $\widetilde{\mathbf{J}}(\mathbf{r})$ where it is understood that the angular frequency is ω. Similarly $\widetilde{\Phi}(\mathbf{r})$ and $\widetilde{\mathbf{A}}(\mathbf{r})$ are defined by

$$\widetilde{\Phi}(\mathbf{r},\omega') = \widetilde{\Phi}(\mathbf{r})\delta(\omega-\omega') \qquad \text{and} \qquad \widetilde{\mathbf{A}}(\mathbf{r},\omega') = \widetilde{\mathbf{A}}(\mathbf{r})\delta(\omega-\omega')$$

so

$$\widetilde{\Phi}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV', \qquad \widetilde{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\widetilde{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV'. \tag{33}$$

---

* Advanced potentials are important in the theory of relativistic quantum mechanics, where they are related to the existence of anti-particles.

† From now on we shall only consider retarded potentials and omit the superscript $^+$.

‡ As before the physical charge and current densities are the real parts of these.

We shall now show that, when $\omega \neq 0$, $\widetilde{\Phi}(\mathbf{r})$ and $\widetilde{\mathbf{A}}(\mathbf{r})$ are not independent — we can derive $\widetilde{\Phi}(\mathbf{r})$ from $\widetilde{\mathbf{A}}(\mathbf{r})$ using conservation of charge (17), which also follows from Maxwell's equations

$$\begin{array}{r} \nabla \times \mathbf{B} - \frac{1}{c^2}\dot{\mathbf{E}} = \mu_0 \mathbf{J} \\ \nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \end{array} \qquad \Rightarrow \qquad \nabla.\mathbf{J} = -\frac{1}{\mu_0 c^2}\nabla.\dot{\mathbf{E}} = -\dot{\rho},$$

since $\nabla.(\nabla \times \mathbf{B}) = 0$. The time dependence in (32) gives

$$\nabla.\widetilde{\mathbf{J}} = i\omega\widetilde{\rho} \qquad \Rightarrow \qquad \widetilde{\rho} = -\frac{i}{\omega}\nabla.\widetilde{\mathbf{J}}.$$

Hence

$$\begin{aligned}
\widetilde{\Phi}(\mathbf{r},\omega) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\widetilde{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV' = -\frac{i}{4\pi\epsilon_0\omega}\int_V \frac{\left(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\right)}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}dV' \\
&= \frac{i}{4\pi\epsilon_0\omega}\int_V \widetilde{\mathbf{J}}(\mathbf{r}').\nabla'\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)dV' = -\frac{i}{4\pi\epsilon_0\omega}\nabla.\left(\int_V \widetilde{\mathbf{J}}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dV'\right) \\
&= -\frac{ic^2}{\omega}\nabla.\mathbf{A}(\mathbf{r})
\end{aligned}$$

where we have integrated by parts and assumed that there is no flux of current through the bounding surface of V. This is in fact just the Lorentz gauge condition again

$$\widetilde{\Phi}(\mathbf{r}) = -\frac{ic^2}{\omega}\nabla.\widetilde{\mathbf{A}}(\mathbf{r}) \;\Rightarrow\; \nabla.\widetilde{\mathbf{A}}(\mathbf{r}) - \frac{i\omega}{c^2}\widetilde{\Phi}(\mathbf{r}) = 0 \;\Rightarrow\; \nabla.\mathbf{A}(\mathbf{r},t) + \frac{1}{c^2}\dot{\Phi}(\mathbf{r},t) = 0.$$

The multipole expansion follows from a Taylor expansion: in Cartesian co-ordinates, x_i, expanding around $\mathbf{r}' = 0$ and using the fact that $\frac{\partial}{\partial x_i'} = -\frac{\partial}{\partial x_i}$ when acting on a function of $|\mathbf{r}-\mathbf{r}'|$,

$$\begin{aligned}
\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} &= \frac{e^{ikr}}{r} + \sum_{i=1}^{3} x_i'\left[\frac{\partial}{\partial x_i'}\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)\right]_{\mathbf{r}'=0} + \frac{1}{2}\sum_{i,j=1}^{3} x_i'x_j'\left[\frac{\partial^2}{\partial x_i'\partial x_j'}\left(\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right)\right]_{\mathbf{r}'=0} + \dots \\
&= \frac{e^{ikr}}{r} - \sum_{i=1}^{3} x_i'\frac{\partial}{\partial x_i}\left(\frac{e^{ikr}}{r}\right) + \frac{1}{2}\sum_{i,j=1}^{3} x_i'x_j'\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{e^{ikr}}{r}\right) + \dots \\
&= \frac{e^{ikr}}{r} - e^{ikr}\sum_{i=1}^{3} x_i'\left(ik\frac{x_i}{r^2} - \frac{x_i}{r^3}\right) \\
&\quad + \frac{e^{ikr}}{2}\sum_{i,j=1}^{3} x_i'x_j'\left[\frac{ikx_j}{r}\left(ik\frac{x_i}{r^2} - \frac{x_i}{r^3}\right) + \delta_{ij}\left(\frac{ik}{r^2} - \frac{1}{r^3}\right) + x_i\left(\frac{3x_j}{r^5} - 2ik\frac{x_j}{r^4}\right)\right] + \dots \\
&= \frac{e^{ikr}}{r} + \frac{e^{ikr}}{r^3}(1-ikr)\sum_{i=1}^{3} x_i'x_i \\
&\qquad + \frac{e^{ikr}}{2r^5}\sum_{i,j=1}^{3}\left[x_ix_j\{3(1-ikr) - k^2r^2\} - \delta_{ij}r^2(1-ikr)\right]x_i'x_j' + \dots
\end{aligned}$$

Using this expansion in (33) gives

$$\widetilde{\Phi}(\mathbf{r}) = \frac{e^{ikr}}{4\pi\epsilon_0}\left\{\frac{\widetilde{Q}}{r} + \frac{(1-ikr)}{r^3}(\widetilde{\mathbf{Q}}.\mathbf{r}) + \sum_{i,j=1}^{3}\frac{x_i x_j}{2r^5}\left[(1-ikr)\big(3\widetilde{q}_{ij} - \delta_{ij}Tr(\widetilde{q})\big) - k^2r^2\widetilde{q}_{ij}\right]\right\} + \ldots$$

$$\widetilde{A}_i(\mathbf{r}) = \frac{\mu_0 e^{ikr}}{4\pi}\left\{\frac{1}{r}\int_V \widetilde{J}_i(\mathbf{r}')dV' + \frac{(1-ikr)}{r^3}\sum_{j=1}^{3}x_j\left[\int_V x'_j\widetilde{J}_i(\mathbf{r}')dV'\right] + \ldots\right\},$$

where

$$\widetilde{Q} = \int_V \widetilde{\rho}(\mathbf{r}')dV', \qquad \widetilde{\mathbf{Q}} = \int_V \mathbf{r}'\widetilde{\rho}(\mathbf{r}')dV' \qquad \text{and} \qquad \widetilde{q}_{ij} = \int_V x'_i x'_j\widetilde{\rho}(\mathbf{r}')dV'$$

are the multipole moments and $Tr(\widetilde{q}) = \sum_{i=1}^{3}\widetilde{q}_{ii}$.* In fact conservation of charge forces $\widetilde{Q} = 0$ since

$$\widetilde{Q} = \int_V \widetilde{\rho}(\mathbf{r}')dV' = -\frac{i}{\omega}\int_V \nabla'.\mathbf{J}(\mathbf{r}')dV' = -\frac{i}{\omega}\int_S \widetilde{\mathbf{J}}(\mathbf{r}').d\mathbf{S}' = 0$$

if there is no flux of current through the surface S bounding V.

As mentioned earlier $\widetilde{\Phi}$ and $\widetilde{\mathbf{A}}$ are not independent. From charge conservation, $\nabla.\widetilde{\mathbf{J}} = i\omega\widetilde{\rho}$, we have

$$\begin{aligned}\int_V \widetilde{J}_i(\mathbf{r}')dV' &= \sum_{j=1}^{3}\int_V \frac{\partial}{\partial x'_j}\big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dV' - \int_V x'_i\big(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\big)dV' \\ &= -i\omega\int_V x'_i\widetilde{\rho}(\mathbf{r}')dV' = -i\omega Q_i\end{aligned}$$

where again it has been assumed that there is no flux of current through the surface bounding V, so

$$\sum_{j=1}^{3}\int_V \frac{\partial}{\partial x'_j}\big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dV' = \sum_{j=1}^{3}\int_S \big(x'_i\widetilde{J}_j(\mathbf{r}')\big)dS'_j = 0$$

from the divergence theorem.

Also

$$\begin{aligned}\int_V \widetilde{x}'_j J_i(\mathbf{r}')dV' =& \sum_{k=1}^{3}\int_V \frac{\partial}{\partial x'_k}\big(x'_i x'_j\widetilde{J}_k(\mathbf{r}')\big)dV' - \int_V x'_i J_j(\mathbf{r}')dV' - \int_V x'_i x'_j\big(\nabla'.\widetilde{\mathbf{J}}(\mathbf{r}')\big)dV' \\ =& -\int_V x'_i J_j(\mathbf{r}')dV' - i\omega\int_V x'_i x'_j\widetilde{\rho}(\mathbf{r}')dV' = -\int_V x'_i J_j(\mathbf{r}')dV' - i\omega\widetilde{q}_{ij} \\ \Leftrightarrow \quad \int_V \widetilde{x}'_j J_i(\mathbf{r}')dV' =& \frac{1}{2}\int_V \big(\widetilde{x}'_j J_i(\mathbf{r}') - x'_i J_j(\mathbf{r}')\big)dV' - \frac{i\omega}{2}\widetilde{q}_{ij}.\end{aligned}$$

---

* Small $\widetilde{q}_{ij}$ is used here for the quadrupole moment because a capital Q_{ij} was used in the electrostatics section to denote the traceless part of the quadrupole moment, $\widetilde{Q}_{ij} = \frac{1}{2}\int_V(3x'_i x'_j - \delta_{ij}r'^2)\widetilde{\rho}(\mathbf{r}')dV'$.

The first term on the right hand side is anti-symmetric under interchange of the indices i and j and is called the magnetic dipole moment, it is equivalent to the vector

$$\widetilde{\mathbf{m}} = \frac{1}{2}\int_V \mathbf{r}' \times \widetilde{\mathbf{J}}(\mathbf{r}')dV',$$

while the second term is the electric quadrupole moment and is symmetric under interchange of i and j. Using these expressions

$$\widetilde{\Phi}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}e^{ikr}\left[\frac{(1-ikr)}{r^3}(\widetilde{\mathbf{Q}}.\mathbf{r}) + \frac{(1-ikr)}{r^5}\sum_{i,j=1}^{3} x_i x_j \left(\widetilde{Q}_{ij} - \frac{1}{2}k^2 r^2 \tilde{q}_{ij}\right) + \dots\right]$$

$$\widetilde{A}_i(\mathbf{r}) = \frac{\mu_0}{4\pi}e^{ikr}\left[-\frac{i\omega}{r}\widetilde{Q}_i + \frac{(1-ikr)}{r^3}\left\{(\widetilde{\mathbf{m}}\times\mathbf{r})_i - \frac{i\omega}{2}\sum_{j=1}^{3} x_j \tilde{q}_{ij}\right\} + \dots\right].$$

The three terms that are explicit on the right hand side of $\widetilde{\mathbf{A}}_i$ here are referred to respectively as the electric dipole term, $\widetilde{Q}_i$, the magnetic dipole term, $\widetilde{m}_i$ and the electric quadrupole term, $\widetilde{Q}_{ij}$. In a time independent situation, $\omega = 0$, $k = 0$, the electric dipole and quadrupole terms vanish leaving the familiar magnetic dipole term from statics (18). As an exercise you may wish to check that indeed $\nabla.\widetilde{\mathbf{A}}(\mathbf{r}) = i\frac{\omega}{c^2}\widetilde{\Phi}(\mathbf{r})$.

Electric Dipole Radiation

To understand how these kinds of potentials can lead to radiation we shall examine the electric dipole term as an example. So consider

$$\widetilde{\mathbf{A}}(\mathbf{r}) = -i\frac{\mu_0\omega e^{ikr}}{4\pi}\frac{\widetilde{\mathbf{Q}}}{r}.$$

Using $\nabla r = \mathbf{r}/r := \mathbf{n}$, the unit vector in the radial direction,

$$\widetilde{\mathbf{B}}(\mathbf{r}) = \nabla\times\widetilde{\mathbf{A}}(\mathbf{r}) = -i\frac{\mu_0\omega e^{ikr}}{4\pi}\left(\frac{ik\mathbf{n}}{r} - \frac{\mathbf{n}}{r^2}\right)\times\widetilde{\mathbf{Q}} = \frac{\mu_0 k^2 c}{4\pi}\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)(\mathbf{n}\times\widetilde{\mathbf{Q}}).$$

The electric field can be evaluated either from $\widetilde{\mathbf{E}} = -\nabla\widetilde{\Phi} + i\omega\widetilde{\mathbf{A}}$ directly or by observing that Maxwell's equation

$$\nabla\times\mathbf{B} = \mu_0\epsilon_0\dot{\mathbf{E}} \qquad \Rightarrow \qquad \widetilde{\mathbf{E}} = \frac{ic^2}{\omega}\nabla\times\widetilde{\mathbf{B}} = \frac{ic}{k}\nabla\times\widetilde{\mathbf{B}}$$

and

$$\nabla\times\left(\frac{\mathbf{n}\times\widetilde{\mathbf{Q}}}{r}\right) = \nabla\times\left(\frac{\mathbf{r}\times\widetilde{\mathbf{Q}}}{r^2}\right) = -\frac{2\{\mathbf{n}\times(\mathbf{r}\times\widetilde{\mathbf{Q}})\}}{r^3} + \frac{1}{r^2}\nabla\times(\mathbf{r}\times\widetilde{\mathbf{Q}})$$

$$= -\frac{2\{\mathbf{n}\times(\mathbf{r}\times\widetilde{\mathbf{Q}})\}}{r^3} - \frac{(\nabla.\mathbf{r})}{r^2}\widetilde{\mathbf{Q}} + \frac{1}{r^2}(\widetilde{\mathbf{Q}}.\nabla)\mathbf{r} = -\frac{2\{\mathbf{n}\times(\mathbf{r}\times\widetilde{\mathbf{Q}})\}}{r^3} - \frac{2}{r^2}\widetilde{\mathbf{Q}} = -2\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}$$

so

$$
\begin{aligned}
&\widetilde{\mathbf{B}} = \frac{k^2}{4\pi\epsilon_0 c} e^{ikr}\left(1+\frac{i}{kr}\right)\frac{(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r} \\
\Rightarrow\quad &\widetilde{\mathbf{E}} = \frac{ic}{k}\nabla\times\widetilde{\mathbf{B}} \\
&= \frac{ic}{k}\frac{k^2}{4\pi\epsilon_0 c}\nabla\times\left(e^{ikr}\left(1+\frac{i}{kr}\right)\frac{(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r}\right) \\
&= \frac{ick}{4\pi\epsilon_0 c}e^{ikr}\left[ik\left(1+\frac{i}{kr}\right)\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r}+\nabla\times\left\{\left(1+\frac{i}{kr}\right)\frac{(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r}\right\}\right] \\
&= \frac{ick}{4\pi\epsilon_0 c}e^{ikr}\left[\left\{ik\left(1+\frac{i}{kr}\right)-\frac{i}{kr^2}\right\}\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r}-2\left(1+\frac{i}{kr}\right)\left(\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}\right)\right] \\
&= -\frac{k^2}{4\pi\epsilon_0}e^{ikr}\left[\left\{1+\frac{i}{kr}\left(1+\frac{i}{kr}\right)\right\}\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r}+\frac{2i}{k}\left(1+\frac{i}{kr}\right)\left(\frac{\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})}{r^2}\right)\right] \\
&= -\frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})+\frac{i}{kr}\left(1+\frac{i}{kr}\right)\left\{3\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})-\widetilde{\mathbf{Q}}\right\}\right],
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\nabla\times\left(\frac{\mathbf{n}\times\widetilde{\mathbf{Q}}}{r}\right) &= \nabla\times\left(\frac{\mathbf{r}\times\widetilde{\mathbf{Q}}}{r^2}\right) \\
&= -2\left(\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r^2}\right)+\frac{1}{r^2}\nabla\times(\mathbf{r}\times\widetilde{\mathbf{Q}}) \\
&= -2\left(\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r^2}\right)-\frac{1}{r^2}\widetilde{\mathbf{Q}}(\nabla.\mathbf{r})+\frac{1}{r^2}(\widetilde{\mathbf{Q}}.\nabla)\mathbf{r} \\
&= -2\left(\frac{\mathbf{n}\times(\mathbf{n}\times\widetilde{\mathbf{Q}})}{r^2}\right)-\frac{3}{r^2}\widetilde{\mathbf{Q}}+\frac{1}{r^2}\widetilde{\mathbf{Q}} \\
&= -2\left(\frac{\mathbf{n}.\widetilde{\mathbf{Q}}}{r^2}\right)\mathbf{n}
\end{aligned}
$$

Note that $\widetilde{\mathbf{E}}.\widetilde{\mathbf{B}} = 0$. These expressions are rather involved in general and it is instructive to examine two special limits:

i) **The near zone**, $kr << 1$ so r is small,

$$
\widetilde{\mathbf{B}} = \frac{ik}{4\pi\epsilon_0 c}\frac{\mathbf{n}\times\widetilde{\mathbf{Q}}}{r^2}, \qquad \widetilde{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{3\mathbf{n}(\mathbf{n}.\widetilde{\mathbf{Q}})-\widetilde{\mathbf{Q}}}{r^3},
$$

where the electric field dominates.

ii) **The far zone**, $kr >> 1$ so r is large,

$$\widetilde{\mathbf{B}} = \frac{k^2}{4\pi\epsilon_0 c}\frac{e^{ikr}}{r}(\mathbf{n}\times\widetilde{\mathbf{Q}}), \qquad \widetilde{\mathbf{E}} = -c(\mathbf{n}\times\widetilde{\mathbf{B}}),$$

where the electric and magnetic fields both fall off like $1/r$. Remember that this is a multipole expansion and these expression are only accurate when r is much greater than the largest dimension of the volume containing the charges and currents.

The physical electric and magnetic fields are then the real parts

$$\mathbf{E} = \Re(\widetilde{\mathbf{E}}e^{-iwt}) \qquad \mathbf{B} = \Re(\widetilde{\mathbf{B}}e^{-iwt}).$$

The far zone is particularly important for understanding radiation a long way away from the sources where the energy flux, averaged over a cycle of period $2\pi/\omega$, is given by (29)

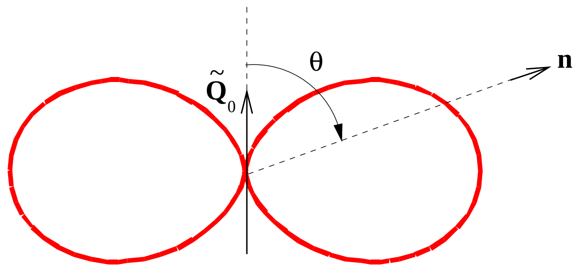
$$\begin{aligned}\bar{\mathbf{S}} &= \frac{1}{2\mu_0}\Re(\widetilde{\mathbf{E}}\times\widetilde{\mathbf{B}}^*) = \frac{1}{2\mu_0}\frac{1}{(4\pi\epsilon_0)^2}\frac{k^4}{cr^2}\{(\mathbf{n}\times\widetilde{\mathbf{Q}})\times\mathbf{n}\}\times(\mathbf{n}\times\widetilde{\mathbf{Q}}^*) \\ &= \frac{k^4 c}{2(4\pi)^2\epsilon_0}\frac{1}{r^2}\{(\widetilde{\mathbf{Q}}\times\mathbf{n}).(\widetilde{\mathbf{Q}}^*\times\mathbf{n})\}\mathbf{n}.\end{aligned}$$

The energy flux is therefore purely radial, in the $\mathbf{n}$ direction, and falls off like $1/r^2$.

Now suppose, for example, that the complex vector $\widetilde{\mathbf{Q}}$ has the same complex phase for each component, *i.e.* $\widetilde{\mathbf{Q}} = e^{i\alpha}\widetilde{\mathbf{Q}}_0$ where $\widetilde{\mathbf{Q}}_0$ is a real vector. In this particular case

$$\bar{\mathbf{S}} = \frac{k^4 c}{2(4\pi)^2\epsilon_0}\frac{\widetilde{Q}_0^2\sin^2\theta}{r^2}\mathbf{n} \tag{34}$$

where θ is the angle between $\widetilde{\mathbf{Q}}_0$ and $\mathbf{r}$ and $\widetilde{Q}_0^2 = \widetilde{\mathbf{Q}}_0.\widetilde{\mathbf{Q}}_0$. Most of the energy is radiated in the direction $\theta = \pi/2$, that is perpendicular to the direction of $\widetilde{\mathbf{Q}}_0$ and none is radiated parallel to $\widetilde{\mathbf{Q}}_0$.



The total time-averaged power radiated, $\bar{\mathcal{P}}$, is the integral of the energy flux through a sphere surrounding the dipole. Taking a sphere with large radius and using the radiation zone expressions for **E** and **B**

$$\bar{\mathcal{P}} = \frac{k^4 c\,\widetilde{Q}_0^2}{2(4\pi)^2\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\sin^2\theta}{r^2}\right)r^2\sin\theta d\theta d\phi = \frac{k^4 c\,\widetilde{Q}_0^2}{2(4\pi)^2\epsilon_0}\int_0^{2\pi}\int_{-1}^{1}(1-u^2)du = \frac{k^4 c\,\widetilde{Q}_0^2}{12\pi\epsilon_0},$$

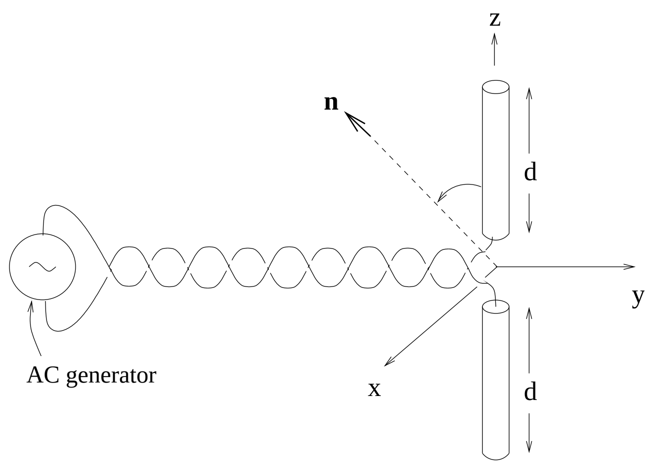
where $u = \cos\theta$. So the total power radiated through a sphere of large radius is

$$\boxed{\bar{\mathcal{P}} = \frac{\omega^4\widetilde{Q}_0^2}{12\pi\epsilon_0 c^3} = \frac{\ddot{Q}_0^2}{12\pi\epsilon_0 c^3},} \tag{35}$$

proportional to the square of the second derivative of $\widetilde{\mathbf{Q}}_0$ with respect to time.

Example: centre-fed linear antenna

A model for an antenna transmitting radio-waves is two collinear straight cylindrical rods of length d with constant circular cross-section made of some conducting material with an alternating current fed into a small gap between them (hence *centre-fed*).



We model the current as an oscillating function which decreases linearly (hence *linear*) from a maximum amplitude I_0 at the centre to zero at the end of the rods. Place the rod so as to be aligned along the z-axis with the central gap at the origin, then the physical current is the real part of

$$I(z,t) = \begin{cases} I_0\left(1 - \frac{|z|}{d}\right)e^{-i\omega t}, & |z| \le d \\ 0, & |z| > d. \end{cases}$$

Assuming the current density in the rods is independent of position, define J_0 by

$$I_0 = J_0\Delta A,$$

where ΔA is the cross-sectional area of the rods. Then we define a complex current density inside the antenna

$$\mathbf{J} = \frac{I_0}{\Delta A}\left(1 - \frac{|z|}{d}\right) e^{-i\omega t}\hat{\mathbf{z}}, \qquad -d \le z \le d$$

while $\mathbf{J} = 0$ outside the rods. Now

$$\nabla . \mathbf{J} = \pm \frac{I_0}{\Delta A d} e^{-i\omega t}$$

and conservation of charge

$$\nabla . \mathbf{J} = -\dot{\rho}$$

then implies a charge density, $\rho(\mathbf{r}, t) = \widetilde{\rho}(\mathbf{r}) e^{-i\omega t}$, with

$$\widetilde{\rho}(\mathbf{r}) = \pm \frac{iI_0}{\omega d \Delta A}$$

inside the antenna (plus for $0 < z \le d$ and minus for $-d \le z < 0$) while $\widetilde{\rho}$ vanishes outside the antenna. We can define a charge per unit length

$$\widetilde{\lambda} = \widetilde{\rho} \Delta A = \pm \frac{iI_0}{\omega d}$$

giving a dipole moment

$$\widetilde{Q}_z = \int_{-d}^{d} z \widetilde{\lambda}(z) dz = \frac{iI_0}{\omega d}\left(\int_0^d z dz - \int_{-d}^{0} z dz\right) = \frac{2iI_0}{\omega d}\int_0^d z dz = \frac{iI_0 d}{\omega},$$

while $\widetilde{Q}_x = \widetilde{Q}_y = 0$, so

$$\widetilde{\mathbf{Q}} . \widetilde{\mathbf{Q}}^* = \widetilde{Q}_0^2 = \left(\frac{I_0 d}{\omega}\right)^2 .$$

The time-averaged energy flux for $r >> d$ is now given by (34) to be

$$\bar{\mathbf{S}} = \frac{\omega^4}{2(4\pi)^2 \epsilon_0 c^3} \widetilde{Q}_0^2 \frac{\sin^2\theta}{r^2}\mathbf{n} = \frac{(\omega I_0 d)^2}{32\pi^2 \epsilon_0 c^3} \frac{\sin^2\theta}{r^2}\mathbf{n}.$$

The time-averaged power radiated through a large sphere with the antenna at the centre and $r >> d$ is now given by (35) to be

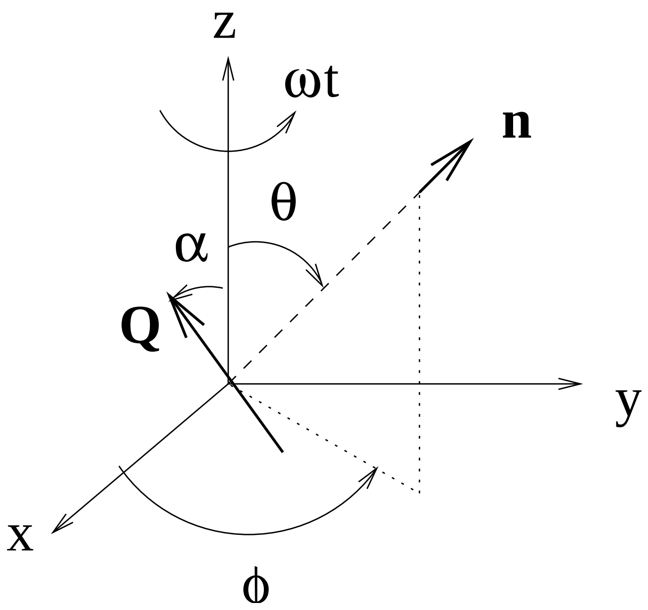
$$\bar{\mathcal{P}} = \frac{(\omega I_0 d)^2}{12\pi \epsilon_0 c^3}.$$

This is proportional to ω^2, so higher frequencies radiate more power for a given current I_0.

Example: rotating dipole

Our next example is a dipole of constant magnitude, rotating around an axis at a constant angle α to $\mathbf{Q}$. Choose the axis of rotation to be the z-axis with

$$\mathbf{Q} = Q_0 \sin\alpha\big(\cos(\omega t)\hat{\mathbf{x}} \pm \sin(\omega t)\hat{\mathbf{y}}\big) + Q_0 \cos\alpha\,\hat{\mathbf{z}} = Q_0 \sin\alpha \Re\big\{(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})e^{-i\omega t}\big\} + Q_0\cos\alpha\,\hat{\mathbf{z}}.$$



The last term on the right hand side is independent of time and will not radiate, so we can determine the radiation by focusing on

$$\widetilde{\mathbf{Q}} = Q_0 \sin\alpha(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}).$$

Expressing the unit radial vector $\mathbf{r}/r = \mathbf{n}$ in Cartesians,

$$\mathbf{n} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},$$

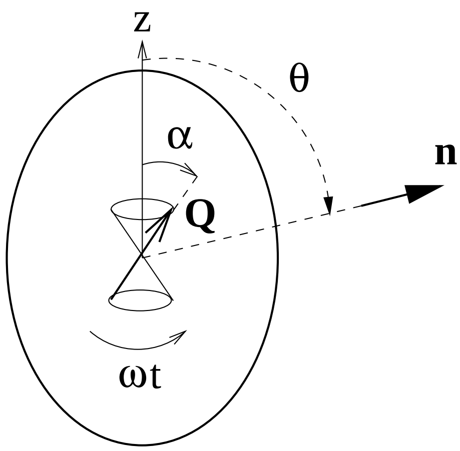
we can determine the Poynting vector from

$$\begin{aligned} &\widetilde{\mathbf{Q}} \times \mathbf{n} = Q_0 \sin\alpha\big\{(\sin\theta\sin\phi \pm i\sin\theta\cos\phi)\hat{\mathbf{z}} + (-\cos\theta)\hat{\mathbf{y}} \mp i\cos\theta\,\hat{\mathbf{x}}\big\} \\ \Rightarrow \quad &(\widetilde{\mathbf{Q}} \times \mathbf{n}).(\widetilde{\mathbf{Q}} \times \mathbf{n})^* = (\sin^2\theta + 2\cos^2\theta)Q_0^2\sin^2\alpha = (1+\cos^2\theta)Q_0^2\sin^2\alpha \end{aligned}$$

giving

$$\bar{\mathbf{S}} = \frac{\omega^4 Q_0^2 \sin^2\alpha(1+\cos^2\theta)}{32\pi^2\epsilon_0 c^3 r^2}\mathbf{n}$$

in the radiation zone $kr >> 1$. The radiation is most intense in the direction of the axis of rotation, the z-axis when $\theta = 0$ or π, but there is still some radiation (half the intensity of that along the z-axis) in the direction perpendicular to the axis of rotation, $\theta = \pi/2$.



Time-averaged energy flux from a rotating dipole.

The time-averaged power is then

$$\bar{\mathcal{P}} = \frac{\omega^4 Q_0^2 \sin^2\alpha}{16\pi\epsilon_0 c^3}\int_0^\pi (1+\cos^2\theta)\sin^2 d\theta = \frac{\omega^4 Q_0^2 \sin^2\alpha}{16\pi\epsilon_0 c^3}\int_{-1}^1 (1+u^2)du = \frac{\omega^4 Q_0^2 \sin^2\alpha}{6\pi\epsilon_0 c^3}.$$

Thus a rotating electric dipole radiates a time-averaged power proportional to the fourth power of the frequency.

A rotating magnetic dipole with

$$\mathbf{m} = m_0(\cos\omega t\hat{\mathbf{x}} - \sin\omega t\hat{\mathbf{y}})\sin\alpha + m_0\cos\alpha\hat{\mathbf{z}},$$

so $\mathbf{m}.\mathbf{m} = m_0^2$, leads to almost the same expression, except $\epsilon_0 \to 1/\mu_0$,

$$\bar{\mathcal{P}} = \frac{\omega^4 m_0^2 \mu_0 \sin^2\alpha}{6\pi c^3}.$$

A pulsar is a rotating neutron star with a magnetic dipole that is not aligned with the axis of rotation and this expression gives the time-averaged power radiated by a pulsar in electromagnetic (radio) waves. This loss of energy makes pulsars spin down with time.

6. Relativistic Formulation of Electromagnetism

From the special theory of relativity the Lorentz transformations between two inertial co-ordinates systems* (ct, x, y, z) and (ct', x', y', z'), written in matrix form, is

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(v) & -\gamma(v)\frac{v}{c} & 0 & 0 \\ -\gamma(v)\frac{v}{c} & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{36}$$

* We take the x, y and z-axis aligned with the x', y' and z'-axis respectively and the origins $(x, y, z) = (0, 0, 0)$ and $(x', y', z') = (0, 0, 0)$ co-incising at $t = t' = 0$.

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$. Equivalently, using an index notation $x^{\mu'} = (ct', x', y', z')$ and $x^\mu = (ct, x, y, z)$ with $\mu = 0, 1, 2, 3$,

$$x^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu}(v) x^\nu$$

where $L^{\mu'}{}_{\nu}(v)$ are the components of the 4×4 matrix in (36). Note that, as a matrix, $L(-v) = L^{-1}(v)$. Denote four dimensional vectors (4-vectors) by $\underset{\sim}{\mathbf{U}}$, with components U^μ in the x^μ co-ordinate system and $U^{\mu'}$ in the x'^μ co-ordinate system so

$$U^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu}(v) U^\nu.$$

Then an invariant "length squared" of $\underset{\sim}{\mathbf{U}}$, denoted by a dot product $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$, can be defined by first introducing a matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and defining

$$U_\mu := \sum_{\nu=0}^{3} \eta_{\mu\nu} U^\nu \qquad \Rightarrow \qquad (U_0, U_1, U_2, U_3) = (-U^0, U^1, U^2, U^3).$$

Also

$$U^\mu = \sum_{\nu=0}^{3} \left(\eta^{-1}\right)^{\mu\nu} U_\nu$$

where η^{-1} is the inverse matrix to η (in fact $\eta^{-1} = \eta$ since $\eta^2 = \mathbf{1}$). With this notation

$$\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}} := -\left(U^0\right)^2 + \left(U^1\right)^2 + \left(U^2\right)^2 + \left(U^3\right)^2 = -\left(U^0\right)^2 + \underline{\mathbf{U}}.\underline{\mathbf{U}} = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} U^\mu U^\nu = \sum_{\nu=0}^{3} U_\nu U^\nu,$$

where the 3 dimensional vector (3-vector) $\underline{\mathbf{U}}$ has components (U^1, U^2, U^3) in the x^μ co-ordinate system and (U'^1, U'^2, U'^3) in the $x^{\mu'}$ co-ordinate system.* Note that $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$ can be positive, negative or zero depending on whether $\left(U^0\right)^2 > \underline{\mathbf{U}}.\underline{\mathbf{U}}$ (time-like vector), $\left(U^0\right)^2 < \underline{\mathbf{U}}.\underline{\mathbf{U}}$ (space-like vector) or $\left(U^0\right)^2 = \underline{\mathbf{U}}.\underline{\mathbf{U}}$. (light-like or null vector).

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* Note that $\underline{\mathbf{U}}$ has no Lorentz invariant meaning, it is a different 3-vector in different reference frames. As an exercise, check that $\underset{\sim}{\mathbf{U}}.\underset{\sim}{\mathbf{U}}$ is the same in both reference frames but $\underline{\mathbf{U}}.\underline{\mathbf{U}}$ is not.

In this notation the differential form of charge conservation

$$\frac{\partial \rho}{\partial t} + \nabla.\mathbf{J} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \partial_i J^i = 0,$$

where $\partial_i = \partial/\partial x^i$, can be written succinctly by defining a 4-vector $\underset{\sim}{\mathbf{J}}$, with components

$$J^\mu = (c\rho, J^1, J^2, J^3)$$

in the x^u co-ordinates, so that

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \partial_i J^i = c\left(\frac{1}{c}\frac{\partial \rho}{\partial t}\right) + \sum_{i=1}^{3} \partial_i J^i = \sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = \sum_{\mu=0}^{3} \partial_\mu J^\mu = 0,$$

where $\partial_\mu = \partial/\partial x^\mu$. The 4-vector $\underset{\sim}{\mathbf{J}}$ is called the *4-current*.

Compare this with the wave equations for the potentials that follow from Maxwell's equations, with $\mu = \mu_0$, $\epsilon = \epsilon_0$ and $c^2 = 1/\epsilon_0\mu_0$ in the Lorentz gauge $\nabla.\mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$,

$$\begin{aligned} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{1}{\epsilon_0}\rho = -\mu_0 c^2 \rho \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J}. \end{aligned}$$

Combining $c\rho$ and $\mathbf{J}$ into a 4-vector then implies that it is also natural to combine Φ/c and $\mathbf{A}$ into a 4-potential

$$A^\mu = (\Phi/c, A^1, A^2, A^3)$$

which satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{\mathbf{A}} = \sum_{\mu,\nu=0}^{3} \left(\eta^{-1}\right)^{\mu\nu} \partial_\mu \partial_\nu \underset{\sim}{\mathbf{A}} = -\mu_0 \underset{\sim}{\mathbf{J}},$$

Denote by $\Box$ the second order differential operator

$$\Box = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right),$$

called the *wave operator*, or sometimes the *d'Alembertian*, then Maxwell's equations imply

$$\Box \underset{\sim}{\mathbf{A}} = -\mu_0 \underset{\sim}{\mathbf{J}}.$$

In this notation the Lorentz gauge condition is

$$\sum_{\mu=0}^{3} \partial_\mu A^\mu = 0.$$

What about the electric and magnetic fields themselves?

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \Rightarrow \qquad E^i = -c\frac{\partial A^0}{\partial x^i} - c\frac{\partial A_i}{\partial x^0} = c\big(\partial_i A_0 - \partial_0 A_i\big)$$

(note the sign change $A^0 = -A_0$) and

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \quad B^i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}\right) = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\big(\partial_j A_k - \partial_k A_j\big),$$

where ϵ^{ijk} is defined to be

$$\epsilon^{ijk} = \mathbf{e}_i.(\mathbf{e}_j \times \mathbf{e}_k)$$

with $\big\{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1\big\}$ a right-handed orthonormal basis.*

The 6 components of $\mathbf{E}$ and $\mathbf{B}$ can be combined into an anti-symmetric 4×4 matrix with components

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{37}$$

with $F_{\mu\nu} = -F_{\nu\mu}$. Then $E_i/c = F_{i0}$ and $F_{jk} = \sum_{k=1}^{3} \epsilon^{ijk} B_k$ and, as a matrix,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}.$$

The electric and magnetic fields are different to other 3-dimensional vectors that you have met in this regard. In relativity 3-momentum $\mathbf{P}$ is combined with energy E into the the 4-momentum $\underset{\sim}{\mathbf{P}} = (E/c, \mathbf{P})$ and current density $\mathbf{J}$ is combined with the charge density ρ into the 4-current $(\rho c, \mathbf{J})$. $\mathbf{E}$ and $\mathbf{B}$ do not become 4-vectors in relativity, they are the components of the anti-symmetric matrix $F_{\mu\nu}$ which is called the *electromagnetic field tensor*. Sometimes it is convenient to 'raise' the indices on $F_{\mu\nu}$ using η^{-1} thus, using a shorthand notation $\eta^{\mu\nu} = (\eta^{-1})^{\mu\nu}$,

$$F^{\mu\nu} = \sum_{\rho,\sigma=0}^{3} \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix},$$

* This is shorthand way of writing the components of a vector product: there are $3^3 = 27$ different possibilities for ϵ^{ijk} but 21 of these are zero (if any two of i, j or k are the same) so i, j and k must all be different leaving 6 possibilities, $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = +1$ and $\epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$.

or even just raise one index,

$$F^{\mu}{}_{\nu} = \sum_{\rho=0}^{3} \eta^{\mu\rho} F_{\rho\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

or

$$F_{\mu}{}^{\nu} = \sum_{\sigma=0}^{3} \eta^{\nu\sigma} F_{\mu\sigma} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0. \end{pmatrix}$$

Be careful of these signs, the notation of upper and lower indices is adopted here to account for the minus signs that arise in special relativity. A zero superscript always has the opposite sign to a zero subscript but there is no practical difference between an upper 1, 2, or 3 or a lower 1,2 or 3.

Maxwell's equations are now seen to be related to

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = \sum_{\mu=0}^{3} \partial_\mu (\partial^\mu A^\nu) - \sum_{\mu=0}^{3} \partial_\mu (\partial^\nu A^\mu) = \Box A^\nu - \partial^\nu \left(\sum_{\mu=0}^{3} (\partial_\mu A^\mu) \right) = \Box A^\nu = -\mu_0 J^\nu$$

(in the Lorentz gauge, note the sign change $\partial^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} \partial_\nu$ so $\partial^\mu = (-\partial_0, \partial_1, \partial_2, \partial_3)$). So the two Maxwell's equations involving sources

$$\nabla \times \mathbf{B} + \frac{1}{c^2} \dot{\mathbf{E}} = \mu_0 \mathbf{J}, \qquad \nabla . \mathbf{E} = \frac{\rho}{\epsilon_0}$$

are combined in a relativistic formulation into

$$\boxed{\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu}$$

(4 equations, one for each value of ν).

What about the other Maxwell's equations

$$\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla . \mathbf{B} = 0 \ ?$$

Consider the combination

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = \frac{1}{2} \left(\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} - \partial_\mu F_{\rho\nu} - \partial_\nu F_{\mu\rho} - \partial_\rho F_{\nu\mu} \right)$$

with μ, ν and ρ all different. There are $4 \times 3 \times 2 = 24$ possibilities, but only 4 of these are independent because, up to a sign, it does not matter what order the three indices are put in. With the choice $\mu = 1$, $\nu = 2$ and $\rho = 3$ this is

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \nabla . \mathbf{B},$$

with $\mu=0$, $\nu=1$ and $\rho=2$ it is

$$\frac{1}{c}\frac{\partial B_3}{\partial t}+\partial_1\left(\frac{E_2}{c}\right)+\partial_2\left(\frac{-E_1}{c}\right)=\frac{1}{c}(\nabla\times\mathbf{E})_3+\frac{1}{c}\frac{\partial B_3}{\partial t},$$

with $\mu=0$, $\nu=2$ and $\rho=3$ it is

$$\frac{1}{c}\frac{\partial B_1}{\partial t}+\partial_2\left(\frac{E_3}{c}\right)+\partial_3\left(\frac{-E_2}{c}\right)=\frac{1}{c}(\nabla\times\mathbf{E})_1+\frac{1}{c}\frac{\partial B_1}{\partial t},$$

with $\mu=0$, $\nu=3$ and $\rho=1$ it is

$$\frac{1}{c}\frac{\partial B_2}{\partial t}+\partial_3\left(\frac{E_1}{c}\right)+\partial_1\left(\frac{-E_3}{c}\right)=\frac{1}{c}(\nabla\times\mathbf{E})_2+\frac{1}{c}\frac{\partial B_2}{\partial t}.$$

Introducing the shorthand notation

$$\partial_{[\mu}F_{\nu\rho]}:=\frac{1}{3!}\big(\partial_\mu F_{\nu\rho}+\partial_\nu F_{\rho\mu}+\partial_\rho F_{\mu\nu}-\partial_\mu F_{\rho\nu}-\partial_\nu F_{\mu\rho}-\partial_\rho F_{\nu\mu}\big)$$

when μ, ν and ρ are all different* we have

$$\nabla\times\mathbf{E}-\frac{\partial\mathbf{B}}{\partial t}=0,\quad \nabla.\mathbf{B}=0\qquad\Leftrightarrow\qquad \partial_{[\mu}F_{\nu\rho]}=0.$$

In fact

$$\partial_{[\mu}F_{\nu\rho]}=0$$

is an automatic consequence of the fact that $F_{\mu\nu}$ can be derived from the potential A_μ, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, provided only that A_μ is at least twice differentiable.

In summary, Maxwell's equations can be written in a relativistic formulation as

$$\boxed{\begin{aligned}\sum_{\mu=0}^{3}\partial_\mu F^{\mu\nu}&=-\mu_0 J^\nu\\ \partial_{[\mu}F_{\nu\rho]}&=0,\end{aligned}}$$

with $J^\mu=(c\rho,\mathbf{J})$.

* The notation $[\mu\nu\rho]$ indicates that the three indices appear with all six possible permutations, with a plus sign for the three even permutations of the indices (*i.e* $\mu\nu\rho$, $\nu\rho\mu$ and $\rho\mu\nu$) and a minus sign for the three odd permutations (*i.e* $\mu\rho\nu$, $\nu\mu\rho$ and $\rho\nu\mu$). Such a linear combination is said to be *anti-symmetrised* under permutations.

Gauge invariance.

In relativistic notation the gauge transformation (30) can be written

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

where $\Lambda(x^\mu)$ is a differentiable function. Then the components of the electromagnetic field tensor are invariant

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \Lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

This is like a 4-dimensional version of the 3-dimensional analysis for $\mathbf{B}$,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{A}' = \mathbf{A} + \nabla \Lambda \qquad \Rightarrow \qquad \mathbf{B}' = \mathbf{B} \quad \text{since} \quad \nabla \times \nabla \Lambda = 0.$$

Indeed $F_{\mu\nu}$ is like a 4-dimensional 'curl' of A_μ.

Lorentz transformations

In this section we shall discuss how $\mathbf{E}$ and $\mathbf{B}$ transform under Lorentz transformations. To simplify notation let $\beta = v/c$ and $\gamma(\beta) = 1/\sqrt{1-\beta^2}$. Then

$$x^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) x^\nu$$

with

$$L^{\mu'}{}_\nu(\beta) = \begin{pmatrix} \gamma(\beta) & -\beta\gamma(\beta) & 0 & 0 \\ -\beta\gamma(\beta) & \gamma(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly J^μ are the components of a 4-vector so they transform as

$$J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) J^\nu$$

and A^μ are the components of a 4-vector so they transform as

$$A^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta) A^\nu$$

The 'divergence' of the 4-current

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = \underset{\sim}{\partial} . \underset{\sim}{J} = 0$$

is a scalar, not a vector, and so should be *invariant*,

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = \sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'} = 0,$$

charge is conserved in all reference frames. This dictates how ∂_μ should transform under Lorentz transformations, suppose

$$\partial_{\mu'} = \sum_{\rho=0}^{3} M^\rho{}_{\mu'}(\beta)\partial_\rho$$

for some $M^\rho{}_{\mu'}(\beta)$ then

$$\begin{aligned}\sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'} &= \sum_{\mu'=0}^{3} \left(\sum_{\rho=0}^{3} M^\rho{}_{\mu'}\partial_\rho\right)\left(\sum_{\nu=0}^{3} L^{\mu'}{}_\nu J^\nu\right) = \sum_{\nu,\rho=0}^{3}\left\{\sum_{\mu'=0}^{3}\big(M^\rho{}_{\mu'}L^{\mu'}{}_\nu\big)\partial_\rho J^\nu\right\}\\ &= \sum_{\nu,\rho=0}^{3} (ML)^\rho{}_\nu \partial_\rho J^\nu = \sum_{\nu=0}^{3} \partial_\nu J^\nu\end{aligned}$$

where ML is the product of the two matrices. This can only be true for any $\underset{\sim}{\mathbf{J}}$ if ML is the identity matrix, in components $(ML)^\rho{}_\nu = \delta^\rho{}_\nu$, so $M(\beta) = L^{-1}(\beta) = L(-\beta)$. Hence

$$J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_\nu(\beta)J^\nu, \qquad \partial_{\mu'} = \sum_{\nu=0}^{3} (L^{-1})^\nu{}_{\mu'}(\beta)\partial_\nu.$$

Indeed any vector with the index as a sub-script must transform with L^{-1}, *e.g* $J_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu}J^\nu$ transforms as

$$J_{\mu'} = \sum_{\nu=0}^{3} (L^{-1})^\nu{}_{\mu'} J_\nu$$

under Lorentz transformations. Vectors that transform with L are called *contra-variant* vectors (they have super-scripts) while vectors that transform with L^{-1} are called *co-variant* vectors (they have sub-scripts). The difference again amounts to some sign differences, since $L^{-1}(\beta) = L(-\beta)$.

We can now determine how $F_{\mu\nu}$, and hence $\mathbf{E}$ and $\mathbf{B}$, transform. Since

$$A_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^\nu{}_{\mu'} A_\nu \qquad \text{and} \qquad \partial_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^\nu{}_{\mu'} \partial_\nu,$$

we have

$$\begin{aligned}F_{\mu'\nu'} = \partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'} &= \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^\rho{}_{\mu'}\left(L^{-1}\right)^\sigma{}_{\mu'}\big(\partial_\rho A_\sigma - \partial_\sigma A_\rho\big)\\ &= \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^\rho{}_{\mu'}\left(L^{-1}\right)^\sigma{}_{\mu'} F_{\rho\sigma}..\end{aligned}$$

This can be re-written using the usual rules of matrix multiplication and the fact that L is a symmetric matrix $\left(L^{-1}\right)^T = L^{-1}$, in components $\left(L^{-1}\right)^{\rho}{}_{\mu'} = \left(L^{-1}\right)_{\mu'}{}^{\rho}$, so

$$F_{\mu'\nu'} = \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)^{\rho}{}_{\mu'} \left(L^{-1}\right)^{\sigma}{}_{\mu'} F_{\rho\sigma} = \sum_{\rho,\sigma=0}^{3} \left(L^{-1}\right)_{\mu'}{}^{\rho} F_{\rho\sigma} \left(L^{-1}\right)^{\sigma}{}_{\nu'}$$

or, in matrix notation,

$$F' = L^{-1} F L^{-1} \qquad \Leftrightarrow \qquad F = L F' L \tag{38}$$

where F is the co-variant matrix with components $F_{\mu\nu}$ and F' is the matrix with components $F_{\mu'\nu'}$.

As an illustration of (38) consider a point charge Q at rest at the origin of the $x^{\mu'}$ co-ordinate system. The electric and magnetic fields in the primed frame, with components $E_{i'}$ and $B_{i'}$, are

$$\mathbf{E}' = \frac{Q}{4\pi\epsilon_0} \frac{\mathbf{r}'}{(r')^2}, \qquad \mathbf{B} = 0 \tag{39}$$

so

$$F' = \begin{pmatrix} 0 & -E_{1'}/c & -E_{2'}/c & -E_{3'}/c \\ E_{1'}/c & 0 & 0 & 0 \\ E_{2'}/c & 0 & 0 & 0 \\ E_{3'}/c & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} F &= L F' L \\ &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_{1'}/c & -E_{2'}/c & -E_{3'}/c \\ E_{1'}/c & 0 & 0 & 0 \\ E_{2'}/c & 0 & 0 & 0 \\ E_{3'}/c & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{c} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta\gamma E_{1'} & -\gamma E_{1'} & -E_{2'} & -E_{3'} \\ \gamma E_{1'} & -\beta\gamma E_{1'} & 0 & 0 \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix} \\ &= \frac{1}{c} \begin{pmatrix} 0 & -(1-\beta^2)\gamma^2 E_{1'} & -\gamma E_{2'} & -\gamma E_{3'} \\ (1-\beta^2)\gamma^2 E_{1'} & 0 & \beta\gamma E_{2'} & \beta\gamma E_{3'} \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix} \\ &= \frac{1}{c} \begin{pmatrix} 0 & -E_{1'} & -\gamma E_{2'} & -\gamma E_{3'} \\ E_{1'} & 0 & \beta\gamma E_{2'} & \beta\gamma E_{3'} \\ \gamma E_{2'} & -\beta\gamma E_{2'} & 0 & 0 \\ \gamma E_{3'} & -\beta\gamma E_{3'} & 0 & 0 \end{pmatrix}. \end{aligned}$$

From this we can read off the components $\mathbf{E}$ and $\mathbf{B}$ in the unprimed frame and express them in terms of unprimed co-ordinates using the Lorentz transformation

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z$$

,

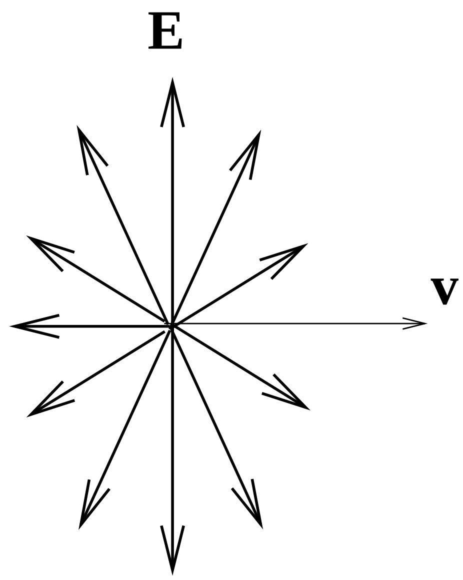
$$\begin{aligned}
E_1 = E_{1'} &= \frac{Q}{4\pi\epsilon_0}\frac{x'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma(x-vt)}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
E_2 = \gamma E_{2'} &= \frac{Q}{4\pi\epsilon_0}\frac{\gamma\, y'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma y}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
E_3 = \gamma E_{3'} &= \frac{Q}{4\pi\epsilon_0}\frac{\gamma\, z'}{(r')^3} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma z}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}
\end{aligned}$$

$$\begin{aligned}
B_1 &= 0 \\
B_2 &= -\beta\gamma E_{3'} = -\frac{Qv}{4\pi\epsilon_0 c}\frac{\gamma z}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}} \\
B_3 &= \beta\gamma E_{2'} = \frac{Qv}{4\pi\epsilon_0 c}\frac{\gamma y}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}.
\end{aligned}$$

Since Q is moving with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in the unprimed frame these can be more concisely written as

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\gamma(\mathbf{r}-\mathbf{v}t)}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}, \qquad \mathbf{B} = \frac{Q}{4\pi\epsilon_0 c}\frac{\gamma(\mathbf{v}\times\mathbf{r})}{\{\gamma^2(x-vt)^2+y^2+z^2\}^{3/2}}. \tag{40}$$

At $t = 0$ the electric field is reduced in the x-direction by a factor $1/\gamma^2$ relative to the usual spherically symmetric Coulomb field of a stationary charge and increased in the $y-z$ plane by a factor of γ,



and this picture moves to the right with constant speed v. There is a non-zero magnetic field in the unprimed frame, because Q is moving in that frame and therefore generating an electric current, which is everywhere perpendicular to $\mathbf{E}$ since $\mathbf{E}.\mathbf{B} = 0$.

Lorentz co-variance of Maxwell's equations.

Maxwell's equations are symmetric under Lorentz transformations, indeed this is how Lorentz transformations were first discovered, but nevertheless $\mathbf{E}$ and $\mathbf{B}$, and so $F_{\mu\nu}$, change — they are not invariant. Maxwell's equations are said to be *co-variant* under Lorentz transformations because their from is preserved even though the individual components change. To see what this means consider the relativistic form of Maxwell's equations in the unprimed frame

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \partial_{[\mu} F_{\nu\rho]} = 0.$$

In the primed frame

$$\partial_{\mu'} = \sum_{\nu=0}^{3} \left(L^{-1}\right)^{\nu}{}_{\mu'} \partial_\nu, \qquad J^{\mu'} = \sum_{\nu=0}^{3} L^{\mu'}{}_{\nu} J^\nu, \qquad \text{and} \qquad F^{\mu'\nu'} = \sum_{\rho,\sigma=0}^{3} L^{\mu'}{}_{\rho} L^{\nu'}{}_{\sigma} F^{\rho\sigma}$$

so

$$\sum_{\mu'=0}^{3} \partial_{\mu'} F^{\mu'\nu'} = \sum_{\mu,\sigma=0}^{3} L^{\nu'}{}_{\sigma} \partial_\mu F^{\mu\sigma} = -\mu_0 \sum_{\sigma=0}^{3} L^{\nu'}{}_{\sigma} J^\sigma = -\mu_0 J^{\nu'}$$

(a factor of L has canceled a factor L^{-1} in the first equation here) and

$$\begin{aligned}
\partial_{[\mu'} F_{\nu'\rho']} &= \sum_{\tau,\sigma,\rho=0}^{3} \left(L^{-1}\right)^{\tau}{}_{[\mu'} \left(L^{-1}\right)^{\sigma}{}_{\nu'} \left(L^{-1}\right)^{\lambda}{}_{\rho']} \partial_\tau F_{\sigma\rho} \\
&= \sum_{\tau,\sigma,\rho=0}^{3} \left(L^{-1}\right)^{\tau}{}_{\mu'} \left(L^{-1}\right)^{\sigma}{}_{\nu'} \left(L^{-1}\right)^{\lambda}{}_{\rho'} \partial_{[\tau} F_{\sigma\rho]} = 0.
\end{aligned}$$

Hence, in the primed frame, Maxwell's equations are

$$\sum_{\mu'=0}^{3} \partial_{\mu'} F^{\mu'\nu'} = -\mu_0 J^{\nu'}, \qquad \partial_{[\mu'} F_{\nu'\rho']} = 0,$$

exactly the same from as in the unprimed frame, even though the individual components are different. This is what is meant by co-variance and the statement above that Lorentz transformations are a symmetry of Maxwell's equations.

Since the components are different in different reference frames, it can sometimes be difficult to see symmetries when the individual components are written out, as in equation (40) for example. It is often useful to construct quantities that are genuinely invariant, *i.e.* they are the same in every reference frame. Such quantities can be evaluated in any inertial reference frame and we know that we would get the same answer in any other frame and sometimes calculations are easier in one particular frame so it is clearly easiest to use that frame. One way of constructing invariants is to 'contract' indices so that there are no free indices on our expressions. For example

$$\sum_{\mu=0}^{3} \partial_\mu J^\mu = 0 = \sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'}$$

is an invariant, it is the same in all reference frames (it happens to be zero).*

We can make an invariant out of **E** and **B** by considering the following quadratic expression in F,

$$\sum_{\mu,\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} = 2\sum_{i=1}^{3} F_{0i}F^{0i} + \sum_{i,j=1}^{3} F_{ij}F^{ij} = -\frac{2}{c^2}\mathbf{E}.\mathbf{E} + \sum_{i,j,k,l=1}^{3} \left(\epsilon_{ijk}B^k\right)\left(\epsilon^{ijl}B^l\right).$$

Now $\sum_{i,j=1}^{3} \epsilon_{ijk}\epsilon^{ijl} = 2\delta_k{}^l$, so the combination

$$\frac{1}{4}\sum_{\mu\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{B}.\mathbf{B} - \frac{\mathbf{E}.\mathbf{E}}{c^2}\right) = \frac{1}{2}\left(\mathbf{B}'.\mathbf{B}' - \frac{\mathbf{E}'.\mathbf{E}'}{c^2}\right)$$

is an invariant under Lorentz transformations, it is the same in all inertial references frames.† As an exercise you should check this for (39) and (40).

There is in fact a second quadratic invariant that can be constructed from $F_{\mu\nu}$. To show this we first need a 4-dimensional version of ϵ^{ijk}, which we denote by $\epsilon^{\mu\nu\rho\sigma}$. This is defined to be zero if any of the 4 indices μ, ν, ρ or σ are the same so, of the $4^4 = 256$ possibilities, 212 vanish and only $4! = 24$ are non-zero. The non-zero ones are all defined to be ± 1 and for these $\{\mu,\nu,\rho,\sigma\}$ must be some permutation of the four indices $\{0,1,2,3\}$. The permutation is called *even* if the sequence $\{\mu,\nu,\rho,\sigma\}$ can be obtained $\{0,1,2,3\}$ by an even number of interchanges of pairs and *odd* if $\{\mu,\nu,\rho,\sigma\}$ must be obtained $\{0,1,2,3\}$ by an odd number of interchanges of pairs. For example $\{0,1,2,3\}$, $\{1,0,3,2\}$, $\{0,2,3,1\}$ and $\{2,0,1,3\}$ are even permutations (there are 12 in all) while $\{1,0,2,3\}$, $\{0,1,3,2\}$, $\{2,0,3,1\}$ and $\{1,2,3,0\}$ are odd (again there are 12 of these). Equivalently one and only one index must be 0 for a non-zero value and

$$\epsilon^{0ijk} = -\epsilon^{i0jk} = \epsilon^{ij0k} = -\epsilon^{ijk0} = \epsilon^{ijk},$$

---

* It is crucial that one index is up and one is down here, because only then do we get a cancellation between L and L^{-1} in the primed expression $\sum_{\mu'=0}^{3} \partial_{\mu'} J^{\mu'}$. If both indices were sub-scripts, or both super-scripts, there would be no such cancellation, for example $\sum_{\mu'=0}^{3} \partial_{\mu'} J_{\mu'}$ is *not* Lorentz invariant.

† This is reminiscent of the energy density stored in the electro-magnetic field, $w = \frac{1}{2\mu_0}\left(\frac{\mathbf{E}.\mathbf{E}}{c^2} + \mathbf{B}.\mathbf{B}\right)$, but it is not the same, because of the sign difference. Energy is not Lorentz invariant.

with $i, j, k = 1, 2$ or 3, exhausts all possibilities. An important of consequence of this definition of $\epsilon^{\mu\nu\rho\sigma}$ is that it is Lorentz invariant. To see this consider the Lorentz transformed quantity

$$\epsilon^{0'1'2'3'} = \sum_{\mu,\nu,\rho,\sigma=0}^{3} L^{0'}{}_{\mu} L^{1'}{}_{\nu} L^{2'}{}_{\rho} L^{3'}{}_{\sigma} \epsilon^{\mu\nu\rho\sigma}.$$

The right hand side of this equation is nothing other than the definition of the determinant of the 4×4 matrix L, which evaluates to one

$$\epsilon^{0'1'2'3'} = \det L = 1,$$

hence

$$\epsilon^{0'1'2'3} = \epsilon^{0123}$$

and all the other components of $\epsilon^{\mu'\nu'\rho'\sigma'}$ follow from the usual properties of determinant (interchange two rows or two columns changes a sign, the determinant vanishes if any two rows or columns are identical). We conclude that

$$\epsilon^{\mu'\nu'\rho'\sigma'} = \sum_{\tau,\lambda,\eta,\zeta=0}^{3} L^{\mu'}{}_{\tau} L^{\nu'}{}_{\lambda} L^{\rho'}{}_{\eta} L^{\sigma'}{}_{\zeta} \epsilon^{\tau\lambda\eta\zeta}$$

has exactly the same components in every inertial reference frame, ± 1 or 0. Note that lowering the indices introduces minus sign, since one of them is necessarily the index 0, and $\epsilon_{0123} = -\epsilon^{0123} = -1$.

Now the combination $\sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ has no free indices and is a Lorentz invariant, again because the four factors of L^{-1} cancel against the four factors of L in $\sum_{\mu',\nu',\rho',\sigma'=0}^{3} F_{\mu'\nu'} F_{\rho'\sigma'} \epsilon^{\mu'\nu'\rho'\sigma'}$. Expanding this in terms of $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} \sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} &= 4 \sum_{i,j,k=1}^{3} F_{0i} F_{jk} \epsilon^{ijk} = -\frac{4}{c} \sum_{i,j,k=1}^{3} E^i \Big(\sum_{l=1}^{3} \epsilon_{jkl} B^l \Big) \epsilon^{ijk} \\ &= -\frac{4}{c} \sum_{i,l=1}^{3} E^i B^l \big(2\delta^i{}_l \big) = -\frac{8}{c} \sum_{i=1}^{3} E^i B^i = -\frac{8}{c} \mathbf{E}.\mathbf{B}. \end{aligned}$$

So

$$-\frac{1}{8} \sum_{\mu,\nu,\rho,\sigma=0}^{3} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = \frac{\mathbf{E}.\mathbf{B}}{c}$$

has the same value in all inertial reference frames.

It is convenient to define

$$\widetilde{F}^{\mu\nu} := \frac{1}{2} \sum_{\rho,\sigma=0}^{3} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$

called the *dual* of $F_{\mu\nu}$, which has components

$$\widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3/c & E_2/c \\ -B_2 & E_3/c & 0 & -E_1/c \\ -B_3 & -E_2/c & E_1/c & 0 \end{pmatrix},$$

so $E_i/c \to B_i$ and $B_i \to -E_i/c$, the operation of taking the dual essentially interchanges $\mathbf{E}$ and $\mathbf{B}$. In terms of the dual

$$-\frac{1}{4}\sum_{\mu\nu=0}^{3} F_{\mu\nu}\widetilde{F}^{\mu\nu} = \frac{\mathbf{E}.\mathbf{B}}{c}$$

and

$$\sum_{\nu=0}^{3} \partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\sum_{\mu,\rho,\sigma=0}^{3} \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = \frac{1}{2}\sum_{\mu,\rho,\sigma=0}^{3} \epsilon^{\mu\nu\rho\sigma}\partial_{[\mu}F_{\rho\sigma]} = 0.$$

Maxwell's equations are now succinctly written as

$$\boxed{\sum_{\nu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \sum_{\nu=0}^{3} \partial_\mu \widetilde{F}^{\mu\nu} = 0.}$$

When $J^\mu = 0$ Maxwell's equations are symmetric under the interchange

$$\widetilde{F}^{\mu\nu} \quad \leftrightarrow \quad F^{\mu\nu},$$

and in modern attempts to unify the fundamental forces of nature, such as string theory, this kind of duality symmetry plays a very important rôle. The symmetry is not there when $J^\mu \neq 0$ but it can be re-instated by postulating a dual current $\widetilde{J}^\mu$ such that

$$\sum_{\nu=0}^{3} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad \sum_{\nu=0}^{3} \partial_\mu \widetilde{F}^{\mu\nu} = -\mu_0 \widetilde{J}^\nu.$$

Since the duality operation interchanges electric and magnetic fields and J^μ is a current arising from electric charges $\widetilde{J}^\mu$ is a current arising from *magnetic* charges — re-instating full duality symmetry necessitates introducing magnetic monopoles. Such particles have never been observed, if they exist they must be both very rare, because we do not see any that may have been produced in high energy astrophysical processes, and very heavy, because we have not been able to produce any in the laboratory. If magnetic monopoles exist they may be as heavy as 10^{16} times the mass of a proton.