Recall that a Lorentz transformation can be cast as

$$\begin{cases} t' = \cosh\theta \cdot t - \sinh\theta \cdot x \\ x' = -\sinh\theta\, t + \cosh\theta\, x \\ y' = y \\ z' = z \end{cases},$$

where $\underset{\substack{\uparrow \\ \text{Rapidity}}}{\cosh\theta} \equiv \gamma \equiv \dfrac{1}{\sqrt{1-v^2}}$.

This is reminiscent of a rotation. The analogy goes much further:

- On 3-dimensional (flat) space, there is a special class of coordinate systems in which the distance between two points satisfies

$$dl^2 = dx^2 + dy^2 + dz^2 \qquad \underline{\text{Cartesian coordinates}}$$

A "straight" line satisfies $\dfrac{d^2\vec{x}}{dl^2} = 0$.

The set of all rotations $R \in SO(3)$ is a subgroup of those coordinate transformations that preserve this line element: rotations & translations.

similarly

- In (flat) 4-dimensional spacetime, there is a special class of coordinate systems in which the spacetime interval between two events is

$$(S) \quad d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$ Inertial frame

A force-free objects follows a trajectory with

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \qquad (\mu = 0, 1, 2, 3)$$

Lorentz transformations form a subgroup of all coordinate transformations that preserve (S).

The latter define the

$$\text{Poincare group:} \; \left\{ \begin{array}{l} \text{Spacetime translations} \\ \\ \underbrace{\text{Rotations}, \text{ Boosts (T)}}_{\text{Lorentz group}} \end{array} \right.$$

(10 parameters)

Exercise 39

Show that spacetime translations $x^\mu \to x^\mu + c^\mu$, rotations: $\vec{x} \to R\vec{x}$ and boosts (T) preserve (5).

∎

Exercise 40 (Relativistic addition of velocities)

A particle moves at speed $\vec{v}'$ in inertial frame O', which moves itself at constant speed $\vec{v}$ in inertial frame O.

i) What is the speed of the particle in O?

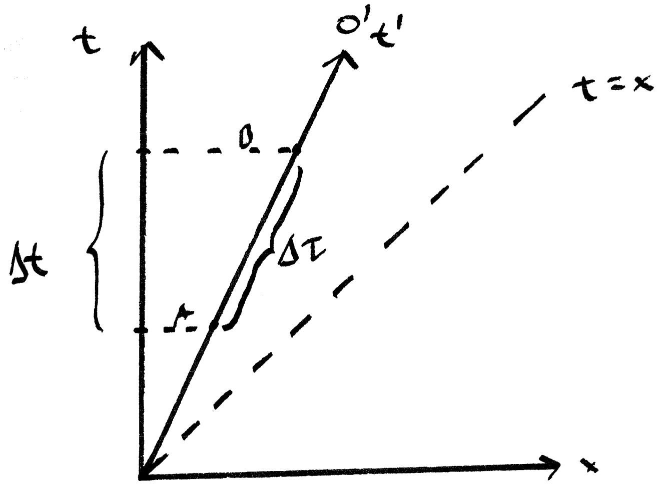
ii) Suppose now that $\vec{v}' \parallel \vec{v} \parallel \hat{e}_x$. The particle defines an inertial frame O''. How is the rapidity θ'' of the LT between O and O'' related to those belonging to the LTs between O and O', and O' and O''?

The spacetime interval

Consider now the spacetime interval between two events, $\Delta \tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$.

. If $\Delta \tau^2 > 0$ we say that the interval is <u>timelike</u>

An observer can move between the two events at less than c. There is hence an inertial frame in which the two events happen at the same location. $\Delta \tau$ is then the <u>proper time</u>



between the two events, i.e. in such a frame

$$\Delta t' = \Delta \tau$$

Example : Time dilation

in O' : $\Delta \tau'^2 = \Delta t'^2 - \Delta x'^2 = \Delta t'^2$

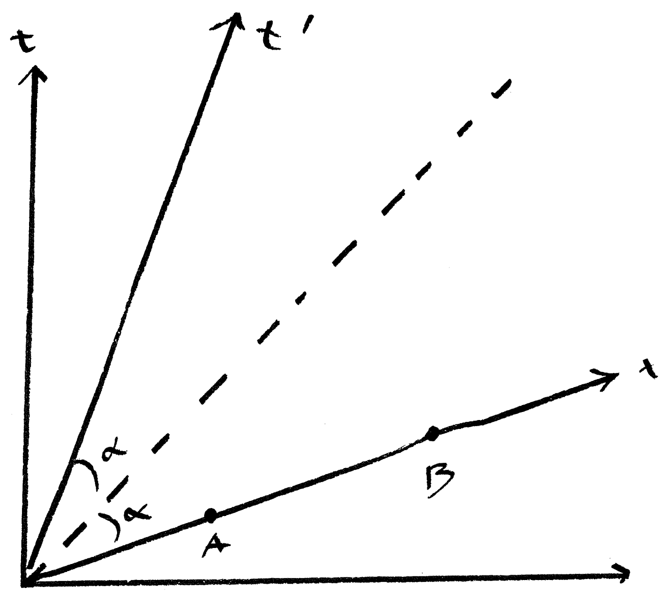
$\qquad\qquad \shortparallel$

in O : $\Delta \tau^2 = \Delta t^2 - \Delta x^2 = \Delta t^2 - v^2 \Delta t^2 = \Delta t^2 (1 - v^2)$

It follows that $\Delta \tau = \dfrac{\Delta t}{\sqrt{1 - v^2}}$ time dilation

• If $\Delta\tau^2 < 0$ we say that the interval is <u>spacelike</u>

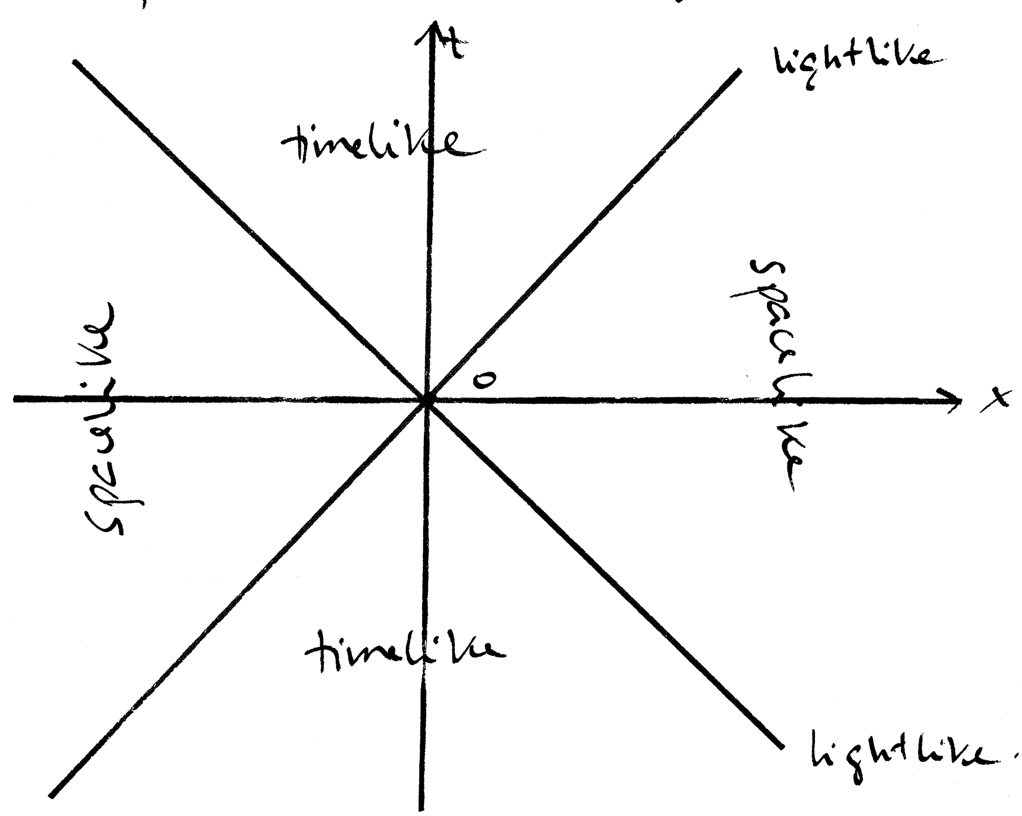
An observer present at both events would have to travel faster than light (impossible). There is



a coordinate system in which the two events are simultaneous. In such coordinate system

$\Delta s = \sqrt{-\Delta\tau^2}$ is the <u>proper distance</u> between the two events.

• If $\Delta\tau^2 = 0$, the interval is <u>light-like</u>



6. Four Vectors

Writing down a rotation can be very time consuming. It is much easier to write $\vec{x}' = R\vec{x}$, $R \in SO(3)$.

Some physicists prefer to be explicit, so they write

$$x'^i = R^i{}_j \, x^j \quad , \text{ where } \quad i = 1,2,3 \quad j = 1,2,3 \quad \text{and}$$

Einstein's Summation Convention is assumed:

Sum over repeated indices in opposite locations

Rotations are special because they preserve the line element:

$$dl'^2 = \delta_{ij} \, dx'^i \, dx'^j = \delta_{ij} \, R^i{}_k \, dx^k \, R^j{}_l \, dx^l =$$

$$= R_{jk} \, R^j{}_l \, dx^k dx^l = \underbrace{(R^T)_{kj} \, R^j{}_l}_{\delta_{kl} \text{ since } R^T R = \mathbb{1}} \, dx^k dx^l = dl^2$$

Similarly, we collect the four spacetime coords $t, \vec{x}$ in a "four-vector" x^μ, with components $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, which under a Lorentz transf. Λ transforms as

$$\underline{x'^\mu = \Lambda^\mu{}_\nu \, x^\nu}$$

By definition, Lorentz transformations preserve the spacetime interval

$$d\tau^2 \equiv \eta_{\mu\nu} \, dx^\mu dx^\nu, \quad \text{where}$$

$\eta = \text{diag}(1, -1, -1, -1)$. Therefore

$$d\tau'^2 = \eta_{\mu\nu} \, dx'^\mu dx'^\nu = \eta_{\mu\nu} \, \Lambda^\mu{}_\rho \, dx^\rho \, \Lambda^\nu{}_\sigma \, dx^\sigma \overset{!}{=} \eta_{\rho\sigma} \, dx^\rho dx^\sigma$$

$$\Rightarrow \quad \underline{\eta_{\mu\nu} \, \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$$

- Objects that under change of inertial frame transform like the coordinates x^μ are called (contravariant) <u>four-vectors</u>. For instance, the four velocity $u^\mu \equiv \frac{dx^\mu}{d\tau}$ transforms like a four-vector: $u'^\mu = \Lambda^\mu{}_\nu \, u^\nu$, because $d\tau$ is a Lorentz scalar (invariant under Lorentz transf.)

Note that $d\tau^2 \equiv dt^2 - v^2 dt^2 = dt^2(1 - v^2) \Rightarrow d\tau = dt\sqrt{1-v^2}$

· given a (contravariant) four-vector v^μ, we can define a <u>covariant four-vector</u> v_μ by contraction with the metric:

$$\underline{v_\mu = \eta_{\mu\nu} v^\nu}$$

Under Lorentz transformations

$$v'_\mu = \eta_{\mu\nu} \Lambda^\nu{}_\rho v^\rho = \eta_{\mu\nu} \Lambda^\nu{}_\rho \eta^{\rho\sigma} \eta_{\sigma\delta} v^\delta ,$$

where $\eta^{\rho\sigma}$ is the <u>inverse</u> of $\eta_{\sigma\delta}$: $\eta^{\rho\sigma}\eta_{\sigma\delta} = \delta^\rho{}_\delta$.

Therefore, $\underline{v_\mu' = \Lambda_\mu{}^\nu v_\nu}$, where

$$\Lambda_\mu{}^\nu = \eta_{\mu\rho} \Lambda^\rho{}_\sigma \eta^{\sigma\nu} .$$

Note that $\eta^{\mu\nu}$ has the same components as $\eta_{\mu\nu}$.

It follows from the last equations that

$$\Lambda_\mu{}^\nu \Lambda^\mu{}_\gamma = \eta_{\mu\rho} \Lambda^\rho{}_\sigma \eta^{\sigma\nu} \Lambda^\mu{}_\gamma = \eta_{\gamma\sigma} \eta^{\sigma\nu} = \delta_\gamma{}^\nu .$$

Hence, $\Lambda_\mu{}^\nu$ is the transposed inverse of $\Lambda^\mu{}_\nu$:

$$\underline{x^\nu = \Lambda_\mu{}^\nu x'^\mu}$$

In particular, $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$ transforms like

a covariant vector:

$$\frac{\partial}{\partial x'^\mu} \equiv \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu ,$$

chain rule

and <u>the contraction of a covariant and a contravariant vector is Lorentz-invariant (a Lorentz scalar)</u>

$$u'_\mu v'^\mu = \Lambda_\mu{}^\nu u_\nu \Lambda^\mu{}_\rho v^\rho = \delta_\rho{}^\nu u_\nu u^\rho = u_\nu u^\nu .$$

(Alternatively,

$$u'_\mu v'^\mu = \eta_{\mu\nu} u'^\mu v'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho u^\rho \Lambda^\nu{}_\sigma v^\sigma = \eta_{\rho\sigma} u^\rho v^\sigma$$
$$= u_\sigma v^\sigma .)$$

• Note that given a covariant four-vector v_μ, we can define a contravariant four-vector by "raising an index" with the inverse metric:

$$v^\mu \equiv \eta^{\mu\nu} v_\nu$$

Exercise 41

Show that if $v_\nu = \eta_{\nu\sigma} v^\sigma$, $v'^\mu = \eta^{\mu\nu} v_\sigma = v^\mu$.

Finally, we can define <u>tensors</u> by multiplying the components of covariant and contravariant vectors, which transform according to the location of their indices

Examples:

$\Pi_{\mu\nu} \equiv p_\mu p_\nu$ transforms as $\Pi'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \Pi_{\rho\sigma}$

$\Pi^\mu{}_\nu \equiv p^\mu p_\nu$ " " $\Pi'^\mu{}_\nu = \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma \Pi^\rho{}_\sigma$

$\Pi^{\mu\nu} \equiv p^\mu p^\nu$ " " $\Pi'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \Pi_{\rho\sigma}$

Note that we can think of

$\Pi_{\mu\nu}$: map from contr. vectors to cov. vectors

$\Pi^\nu{}_\nu$: " " contr. vectors " contr. "

$\Pi^{\mu\nu}$: " " cov. vectors " contr. "