14.10 Covariant Electromagnetism

Let us finally address whether Maxwell's eqs. satisfy the principle of special relativity, and, in particular, whether they are invariant under Lorentz transformations.

Let us start with the conservation of charge:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \qquad (c)$$

(Recall that this follows from Maxwell's eqs.)

If we introduce the four-current

$j^\mu = (\rho, \vec{j})$, then (c) reads

$\underline{\partial_\mu j^\mu = 0}$, which is manifestly invariant iff j^μ transforms like a vector.

The question therefore is whether j^μ transforms like a contravariant four-vector:

$$j'^\mu \stackrel{?}{=} \Lambda^\mu{}_\nu \, j^\nu .$$

Exercise 47

i) Show that $j^\mu = \sum_i (q_i, q_i \vec{v}_i) \, \delta^{(3)}(\vec{x} - \vec{x}_i(t))$

can be cast as

$$j^\mu = \sum_i \int d\tau_i \quad q_i \, u_i^\mu \, \delta^{(4)}(x^\nu - x_i^\nu(\tau)) .$$

ii) Show that $\delta^{(4)}(x^\nu - x_0^\nu)$ is a Lorentz scalar.

As a result, j^μ does transform like a vector.

Let us turn our attention to the Maxwell eqs. themselves.

Recall (Lecture 20), that in ~~Lorentz~~ Lorenz gauge ($\partial_t \phi + \vec{\nabla} \cdot \vec{A} = 0$), Maxwell's eqs. simplify to

$$\begin{cases} \vec{\nabla}^2 \phi - \partial_t^2 \phi = -4\pi \rho \\ \vec{\nabla}^2 \vec{A} - \partial_t^2 \vec{A} = -4\pi \vec{j} \end{cases} \quad \text{inhomogeneous.}$$

This suggest the introduction of a four-potential

$$A^\mu \equiv (\phi, \vec{A})$$

and a spacetime Laplacian

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \equiv \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 .$$

In terms of these quantities, A^μ satisfies the manifestly invariant equation

$$\underline{\Box A^\mu = 4\pi j^\mu}$$

Note that the Lorentz condition can be also cast in invariant form:

$$\underline{\partial_\mu A^\mu = 0} \qquad \text{(Lorentz gauge)}$$

To obtain the electric and magnetic fields, we need to calculate first derivatives of A^μ:

$$\begin{cases} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

Recall that these fields are invariant under the gauge transformations

$$\phi \to \phi' - \partial_t \chi \, , \quad \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi$$

It is therefore more natural to consider the covariant vector potential A_μ, which under gauge transformations transforms as

$$A_\mu \to A_\mu' = A_\mu + \partial_\mu \chi$$

The only gauge-invariant expression (tensor) we can construct out of first derivatives of A_μ is the field-strength tensor (antisymmetric)

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \, ,$$

which is clearly invariant under $A_\mu \to A_\mu + \partial_\mu \chi$.

Note that
$$F_{0i} = \dot{A}_i - \partial_i A_0 = -\dot{A}^i - \partial_i A^0 = E^i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \overbrace{\epsilon_{ijk}\,\epsilon^{kmn}}^{\delta_i^m \delta_j^n - \delta_j^m \delta_i^n}\underbrace{\partial_m A_n}_{-\epsilon^{km}{}_n \partial_m A^n = -(\vec{\nabla}\times\vec{A})^k} = -\epsilon_{ijk} B^k$$

In other words, $F_{\mu\nu}$ has components:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}$$

In terms of $F_{\mu\nu}$, the inhomogeneous eqs. simply read

$$\underline{\partial_\mu F^{\mu\nu} = 4\pi j^\nu}, \quad \text{since}$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu \underbrace{(\partial_\mu A^\mu)}_{0 \text{ in Lorentz gauge.}}$$

Because $F_{\mu\nu}$ contains $\vec{B}$ and $\vec{E}$, we can also write the homogeneous Maxwell's eqs. in terms of $F_{\mu\nu}$. For that purpose introduce the <u>dual</u> tensor

$$F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$

where ϵ is totally antisymmetric and $\epsilon^{0123} = 1$.

Note that $F^{*\mu\nu}$ is also antisymmetric.

The components of $F^{*\mu\nu}$ are:

$$F^{*0i} = \frac{1}{2} \epsilon^{0ijk} F_{jk} = -\frac{\epsilon^{0ijk}}{2} \epsilon_{jk} B^m = -\delta^i{}_m B^m = -B^i$$

and

$$F^{*ij} = \frac{1}{2}(\epsilon^{ij0k} F_{0k} + \epsilon^{ijk0} F_{k0}) = \epsilon^{ijk0}(-E_k) = -\epsilon^{ij}{}_k E^k.$$

In other words, in F^* the roles of $\vec{B}$ and $\vec{E}$ are reversed (wrt F).

The homogeneous eqs. then read

$$\underline{\partial_\mu F^{*\mu\nu} = 0},$$

which is identically satisfied, because

$$\partial_\mu F^{*\mu\nu} = \frac{1}{2} \partial_\mu [\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}] = \frac{1}{2}\partial_\mu [\epsilon^{\mu\nu\rho\sigma} 2 \partial_\rho A_\sigma] =$$

$$= \epsilon^{\mu\nu\rho\sigma} \partial_\mu \partial_\rho A_\sigma = 0.$$

Finally, we just need to address the Lorentz force:

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \text{ or } F^i = q(E^i + \epsilon^i{}_{jk} v^j B^k)$$

Recalling that $u^\mu = \gamma(1, \vec{v})$, we note that

$$F^0{}_\mu u^\mu = F^0{}_i u^i = \gamma \vec{E} \cdot \vec{v}.$$

$$F^i{}_\mu u^\mu = F^i{}_0 u^0 + F^i{}_j u^j = \gamma E^i + \epsilon^i{}_{jk} B^k \gamma v^j =$$

$$= \gamma \left(E^i + (\vec{v} \times \vec{B})^i \right)$$

$q \cdot \vec{E} \cdot \vec{v}$ is the change in energy per unit time.

$q(\vec{E} + \vec{v} \times \vec{B})$ is the change of momentum per unit time.

Hence:

$$\gamma \frac{dE}{dt} = \gamma \cdot q \vec{E} \cdot \vec{v} = q F^0{}_\mu u^\mu$$

$$\gamma \frac{dp^i}{dt} = \gamma q \left(E^i + (\vec{v} \times \vec{B})^i \right) = q F^i{}_\mu u^\mu.$$

Since $\gamma = \dfrac{dt}{d\tau}$ and $E = p^0$, we find

$$\underline{\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu u^\nu} \quad , \quad \text{manifestly Lorentz-inv.}$$

Therefore, <u>all</u> the content of EM can be summarized in the Maxwell's eqs.:

$$\begin{cases} \partial_\mu F^{\mu\nu} = 4\pi j^\mu & \text{inhomogeneous} \\ \partial_\mu F^{*\mu\nu} = 0 & \text{homogeneous} \end{cases}$$

$$\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu u^\nu \qquad \text{Lorentz force.}$$

Note that charge conservation easily follows from the inhomogeneous eqs.: $\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{0 \text{ since } F^{\mu\nu} \text{ antisymmetric}} = 4\pi \partial_\nu j^\nu$.

For many purposes (e.g. quantization) it is very important that these eqs. can be derived by extremizing a manifestly invariant action principle:

$$S_{em} = \int d^4x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right],$$

where $S_{em} = S_{em}[A_\mu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This action is a Lorentz scalar:

$$S_{EM}[A_\mu] = S_{EM}[\Lambda_\mu{}^\nu A_\nu].$$

Exercise 48

i) Show that S_{EM} is a Lorentz scalar.

ii) Derive Maxwell's eqs. (in covariant form)

by extremizing S_{EM} wrt variations of $A_\mu(x)$.

iii) Derive the Lorentz force eq. by

extremizing the action

$$S_p = -\int d\lambda \left[m\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} + q A_\mu \frac{dx^\mu}{d\lambda} \right]$$

hint: recall Exercise 45