Last time: Index gymnastics

- Under Lorentz transformations, a contravariant four-vector transforms according to

$$u'^\mu = \Lambda^\mu{}_\nu \, u^\nu .$$

- The spacetime metric is invariant:

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} .$$

- The covariant four-vector $u_\mu \equiv \eta_{\mu\nu} u^\nu$ transforms according to

$$u'_\mu = \Lambda_\mu{}^\nu \, u_\nu , \quad \text{where}$$

$$\Lambda_\mu{}^\nu \equiv \eta_{\mu\rho} \Lambda^\rho{}_\sigma \, \eta^{\sigma\nu} \quad \text{and}$$

- $\eta^{\mu\nu}$ is the inverse of $\eta_{\rho\sigma}$:

$$\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu{}_\rho .$$

- $\Lambda_\mu{}^\nu$ is the (transposed) inverse of $\Lambda^\rho{}_\sigma$:

$$\Lambda_\mu{}^\nu \Lambda^\mu{}_\gamma = \eta_{\mu\rho} \Lambda^\rho{}_\sigma \eta^{\sigma\nu} \Lambda^\mu{}_\gamma = \eta_{\gamma\sigma} \eta^{\sigma\nu} = \delta_\gamma{}^\nu$$

Therefore, if $x'^\nu = \Lambda^\nu{}_\mu x^\mu$,

$$x^\mu = \Lambda_\nu{}^\mu x'^\nu .$$

<u>Example</u>: $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$ transforms like a covariant four-vector:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu \quad \checkmark$$

Hence

- <u>The contraction of a covariant and a contravariant vector is Lorentz-invariant</u> (a Lorentz scalar):

$$u'_\mu v'^\mu = \Lambda_\mu{}^\nu u_\nu \Lambda^\mu{}_\rho v^\rho = \delta_\rho{}^\nu u_\nu v^\rho = u_\nu v^\nu \quad \checkmark$$

Alternatively:

$$u'_\mu v'^\mu = \eta_{\mu\nu} u'^\nu v'^\mu = \eta_{\mu\nu} \Lambda^\nu{}_\rho u^\rho \Lambda^\mu{}_\sigma v^\sigma = \eta_{\sigma\rho} u^\rho v^\sigma$$

$$= u_\sigma v^\sigma \quad \checkmark$$

- The object $v^\mu \equiv \eta^{\mu\nu} v_\nu$ transforms like a contravariant four-vector

<u>Exercise 41</u>

Show that if $v_\nu = \eta_{\nu\sigma} v^\sigma$ and $\tilde{v}^\mu = \eta^{\mu\nu} v_\nu$,

$$\tilde{v}^\mu = v^\mu .$$

■

- Finally, we can define <u>tensors</u> (multilinear maps) by multiplying the components of covariant and contravariant vectors, which transform according to the location of their indices.

<u>Examples:</u>

$\Pi_{\mu\nu} \equiv p_\mu p_\nu$ transforms as $\Pi'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \; \Pi_{\rho\sigma}$

$\Pi^\mu{}_\nu \equiv p^\mu p_\nu$ " " $\Pi'^\mu{}_\nu = \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma \; \Pi^\rho{}_\sigma$

$\Pi^{\mu\nu} \equiv p^\mu p^\nu$ " " $\Pi'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \; \Pi^{\rho\sigma}$

Note that we can think of

$\Pi_{\mu\nu}$: map from contr. vectors to cov. vectors

$\Pi^\mu{}_\nu$: " " contr. vectors to contr. vectors

$\Pi^{\mu\nu}$: " " cov. vectors to cov. vectors

Exercise 42

Show that the contraction $T_{\mu\nu}{}^{\mu\nu}$ of a rank-four tensor is a Lorentz scalar.

14.6. Relativistic Kinematics

Recall our definition of four-velocity

$$u^\mu = \frac{dx^\mu}{d\tau},$$

which has components

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau} \right) = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{dt} \frac{dt}{d\tau} \right) = \gamma(1, \vec{v}),$$

since $d\tau^2 = dt^2(1 - v^2)$.

Exercise 43

Show that $u_\mu u^\mu = 1$.

In an inertial frame, a force-free object moves with constant speed, that is,

$$\frac{d^2\vec{x}}{dt^2} = 0.$$

In particular, in the rest frame of the object, $t = \tau$ and

$$\frac{d^2 t}{dt^2} = 0 \quad ; \quad \frac{d^2 \vec{x}}{dt^2} = 0 \quad \Rightarrow \quad \frac{d^2 t}{d\tau^2} = 0 \, , \quad \frac{d^2 \vec{x}}{d\tau^2} = 0$$

since $t \equiv \tau$ in that frame.

Therefore, in an arbitrary inertial frame a force-free object moves along

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \, , \quad \text{manifestly Lorentz invariant!}$$

Exercise 44

Show that $$\frac{d^2 x^\mu}{d\tau^2} = \underbrace{u^\nu \partial_\nu u^\mu}_{\text{derivative of } u^\mu \text{ along } u^\nu} = \frac{du^\mu}{d\tau} .$$

■

Consider now the related quantity

$$p^\mu \equiv m u^\mu ,$$

which transforms like a contravariant four-vector, and has dimensions of momentum (or energy, since $c=1$).

By definition,

- $p^2 = m^2$, where $p^2 = p_\mu p^\mu$ (a Lorentz scalar)
- $\dfrac{dp^\mu}{d\tau} = 0$ (conserved)
- $p^\mu = (m\gamma, m\gamma\vec{v})$.

What is the meaning of p^μ? Let us look at the time component:

$$p^0 = m\gamma = \frac{m}{\sqrt{1-v^2}} \simeq m\left(1 + \frac{1}{2}v^2 + \dots\right)$$

$\frac{m}{2}v^2$ is the kinetic energy, so we identify p^0 with the energy of the particle (in the corresponding inertial frame).

For a particle at rest $p^0 = m$ $(= mc^2)$.

This is the rest energy: $E_0 = mc^2$

Similarly,

$p^i = m\gamma v^i = mv^i(1 + \dots)$, so we identify p^i with the spatial momentum of the particle.

Since $\underset{\overset{\parallel}{m^2}}{p^2} = p^{0\,2} - \vec{p}^{\,2} = E^2 - \vec{p}^{\,2}$, we can write

$$E^2 = m^2 + \vec{p}^{\,2}, \quad \text{or} \quad E = \sqrt{m^2 + \vec{p}^{\,2}},$$

which in the non-relativistic limit reads

$$E = m\sqrt{1 + \vec{p}^{\,2}/m^2} \simeq m\left(1 + \frac{1}{2}\frac{\vec{p}^{\,2}}{m^2}\right) = \underset{\substack{\uparrow \\ \text{Rest energy}}}{m} + \underset{\substack{\uparrow \\ \text{kinetic energy}}}{\frac{\vec{p}^{\,2}}{2m}} + \cdots$$

Note that Lorentz transformations mix "energy" and "momentum".

Exercise 45

Show that the equation of motion $\frac{d^2x^\mu}{d\tau^2} = 0$ can be derived by extremizing the action

$$S = \int d\tau \equiv \int \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu} \equiv \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

(Proper time along the trajectory).

Hint/Caution: RHS integral is invariant under reparametrizations.

Exercise 46

Our atmosphere is often struck by particles known as "cosmic rays." Some of these particles reach energies (in our rest frame) of around $E \simeq 10$ eV. We do not know what these high-energy particles are, nor where they are coming from. Suppose that they are neutrons. A neutron has a lifetime of about 15 min (in its own rest frame). In that case, how far could the source of these neutrons be at most? Answer in pc (parsecs). ∎

To conclude, let us calculate the energy of a particle with momentum p^μ measured by an observer with four-velocity u^μ. In a rest frame O' in which the observer is at rest,

$E' = p'^0$, and because $u'^\mu = \delta^\mu{}_0$, we have

$$E' = p'_\mu u'^\mu .$$

But the last expression is Lorentz invariant, so

$$E = p_\mu u^\mu .$$

<u>Example</u>: Rest energy

With $u^\mu = \frac{p^\mu}{m}$ (velocity of an observer moving with the particle),

$$E = p_\mu u^\mu = \frac{p_\mu p^\mu}{m} = \frac{m^2}{m} = m \quad \checkmark .$$

14.10 Covariant Electromagnetism

Let us finally address whether Maxwell's eqs. satisfy the principle of special relativity, and, in particular, whether they are invariant under Lorentz transformations.

Let us start with the conservation of charge,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 . \qquad (C),$$

If we introduce the four-current

$j^\mu = (\rho, \vec{j})$, then, (C) reads

$\partial_\mu j^\mu = 0$, which is manifestly covariant.
(takes the same form in all inertial frames.)

The question is whether j^μ transforms like a contravariant four-vector.

$$j'^\mu \stackrel{?}{=} \Lambda^\mu{}_\nu \, j^\nu .$$

Exercise 47

- Show that $j^\mu = \sum_i (q_i, q_i \vec{v}_i)\, \delta^{(3)}(\vec{x} - \vec{x}_i(t))$

can be cast as

$$j^\mu = \sum_i \int d\tau_i \; q_i u_i^\mu \; \delta^{(4)}(x^\nu - x_i^\nu(\tau))$$

- Show that $\delta^{(4)}(x^\nu - x_i^\nu)$ is a Lorentz scalar.