6 Classical Optics Derived from Maxwell Equations

This section deals with what happens when a wave encounters a discontinuity between 2 semi-infinite media.

Assumptions made:

1. The media extend to infinity on either side of the interface, avoiding mulitple reflections.
2. The media are homogeneous, iotropic, stationary and lossless.
3. The boundary is infinitely thin so there is no diffraction etc.
4. The incident wave is plane and unifrom.

6.1 Boundary conditions

Suppose we have an interface between two diferent media (1 and 2) which contain electric and magnetic fields.

Using **Faradays Law**.

$$\oint_C \vec{E} \cdot \vec{dl} = -\frac{\partial \Phi}{\partial t} \tag{6.1.1}$$

$$\Rightarrow \quad E_{1t} = E_{2t} \tag{6.1.2}$$

ie the tangential component of the electric field across an interface is continuous.

Using Gauss' Law:

$$\oint_A \vec{D} \cdot \vec{dA} = Q \tag{6.1.3}$$

$$\Rightarrow \quad D_{n1} - D_{n2} = \sigma \tag{6.1.4}$$

ie the discontinuity in the component of the electric displacement normal to an interface is equal to the surface charge density at the interface. Similarly $\oint_A \vec{B} \cdot \vec{dA} = 0 \;\Rightarrow\; B_{n1} = B_{n2}$ ie the normal component of the magnetic flux density is continuous across the interface.

Finally, Ampere's circuital law $\Rightarrow$

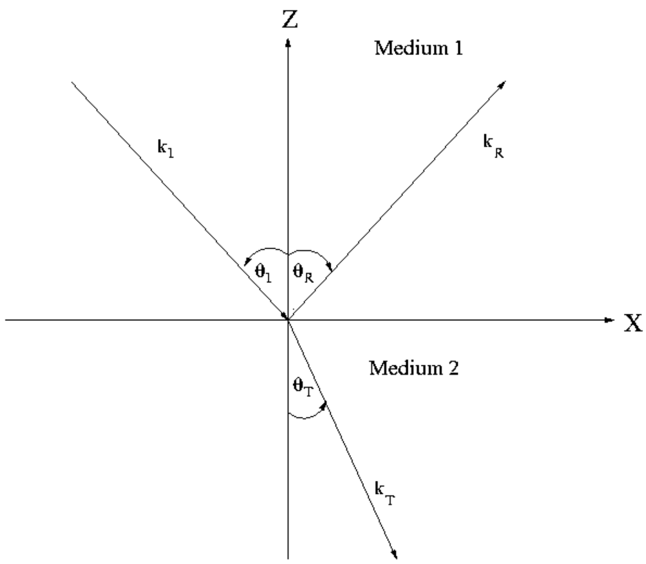
$$\oint_C \vec{H} \cdot \vec{dl} = I \tag{6.1.5}$$

$$\Rightarrow \quad H_{1t} = H_{2t} = J_\delta = a \tag{6.1.6}$$

where a is the surface current density.

Hence, there is a discontinuity at an interface in the component of the magnetic field parallel to the interface equal to the surface current density. Note, however, that a surface current can only exist on the surface of a perfect conductors (eg superconductors) where the conductivity $\sigma \to \infty$.

6.2 Reflection and refraction



n_1 = refractive index of medium 1, n_2 = refractive index of medium 2.

Assuming the incident wave is linearly polarized,

$$\vec{E}_I = \vec{E}_{I0} \exp(i(\omega_i t - \vec{k}_I \cdot \vec{r})) \tag{6.2.1}$$

$\vec{k}_i$ is *real* and points in the direction of propagation of the incident wave $\mid \vec{k}_I \mid = n_1 k_0$. The reflected and transmitted waves will be of the form:

$$\vec{E}_R = \vec{E}_{R0} \exp(i(\omega_R t - \vec{k}_R \cdot \vec{r})) \tag{6.2.2}$$

$$\vec{E}_T = \vec{E}_{R0} \exp(i(\omega_T t - \vec{k}_T \cdot \vec{r})) \tag{6.2.3}$$

Now recall:

$$\vec{\nabla}^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\epsilon} \vec{\nabla} \rho \tag{6.2.4}$$

with $\rho_f = 0$ and $\vec{J}_f = 0$ we can write:

$$\vec{\nabla}^2\vec{E}_R + \mu_1\epsilon_1\omega^2\vec{E}_R = \vec{\nabla}^2\vec{E}_R + k_1^2\vec{E}_R = 0 \tag{6.2.5}$$

where $k_1 = \omega\sqrt{\mu_1\epsilon_1}$. Now, since k_I and k_R are in the same medium, we have:

$$k_{Ix}^2 + k_{Iy}^2 + k_{Iz}^2 = k_{Rx}^2 + k_{Ry}^2 + k_{Rz}^2 = k_1^2 \tag{6.2.6}$$

and

$$k_{Tx}^2 + k_{Ty}^2 + k_{Tz}^2 = k_2^2 \tag{6.2.7}$$

Now the tangential component of $\vec{E}$ is continuous across the interface which implies that the tangential component of $\vec{E}_I + \vec{E}_R$ is equal to the tangential component of $\vec{E}_T$ at the interface. The same boundary condition applies to $\vec{H}$. The relationship must exist between $\vec{E}_I, \vec{E}_R$ and $\vec{E}_T$ at the interface for *all* times t and for *all* points $\vec{r}_{int}$ on the interface. Therefore:

$$\omega_I = \omega_R = \omega_T \tag{6.2.8}$$

and

$$\vec{k}_I \cdot \vec{r}_{int} = \vec{k}_R \cdot \vec{r}_{int} = \vec{k}_T \cdot \vec{r}_{int} \tag{6.2.9}$$

where r_{int} = direction vector of the interface between the two media.
These relationships exist for *all* values of x and y. Therefore, if our incident wave lies in the y = 0 plane, $k_{Iy} = 0$ and, hence, $k_{Ry} = k_{Ty} = 0$. This means that the incident, reflected and transmitted waves are *coplanar*. The x-components must also be equal, so:

$$k_{Rx} = k_{Tx} = kIx = k_1\sin(\theta_I) \tag{6.2.10}$$

By definition:

$$k_{Tx} = k_2\sin(\theta_T) \tag{6.2.11}$$

so

$$k_1\sin(\theta_I) = k_2\sin(\theta_T) \tag{6.2.12}$$

$$\Rightarrow \frac{\sin(\theta_I)}{\sin(\theta_T)} = \frac{k_2}{k_1} = \frac{n_2}{n_1} \tag{6.2.13}$$

Where θ_T is the angle of refraction. This equation is *Snell's Law*.
Since the x - and y-components of the incident and reflected waves are equal and the magnitudes of $\vec{k}_R$ and $\vec{k}_I$ are equal, $k_{Rz}^2 = k_{Iz}^2$ and, hence, $k_{Rz} = -k_{Iz}$. The minus sign arises since the reflected wave travels away from the interface.

Putting all this together, we can write the electric field in each region as:

$$\vec{E}_I = \vec{E}_{I0} \exp(i(\omega t - k_1(x \sin(\theta_I) - z \cos(\theta_I)) \tag{6.2.14}$$

$$\vec{E}_R = \vec{E}_{R0} \exp(i(\omega t - k_1(x \sin(\theta_I) + z \cos(\theta_I)) \tag{6.2.15}$$

$$\vec{E}_T = \vec{E}_{T0} \exp(i(\omega t - k_2(x \sin(\theta_T) - z \cos(\theta_T)) \tag{6.2.16}$$

$$\frac{\sin(\theta_I)}{\sin(\theta_T)} = \frac{k_2}{k_1} = \frac{n_2}{n_1} \tag{6.2.17}$$

6.3 Fresnel's Equations

We now find relations between $\vec{E}_{I0}, \vec{E}_{R0}$ and $\vec{E}_{T0}$. The equations of continuity require:

$$E_{Ix} + E_{Rx} = E_{Tx}, \qquad E_{Iy} + E_{Ry} = E_{Ty} \tag{6.3.1}$$

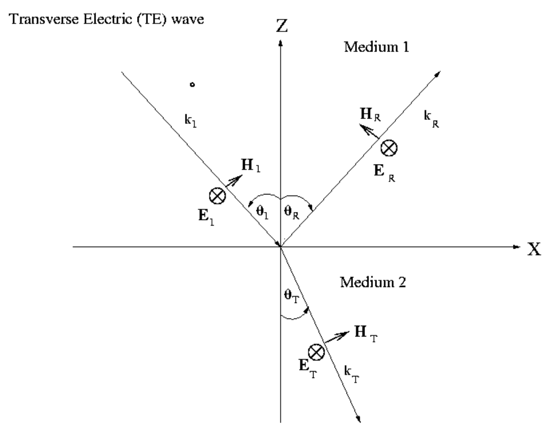
$$H_{Ix} + H_{Rx} = H_{Tx}, \qquad H_{Iy} + H_{Ry} = H_{Ty} \tag{6.3.2}$$

In addition because of the mutually perpendicular relationship between $\vec{E}, \vec{H}$ and $\vec{k}$,

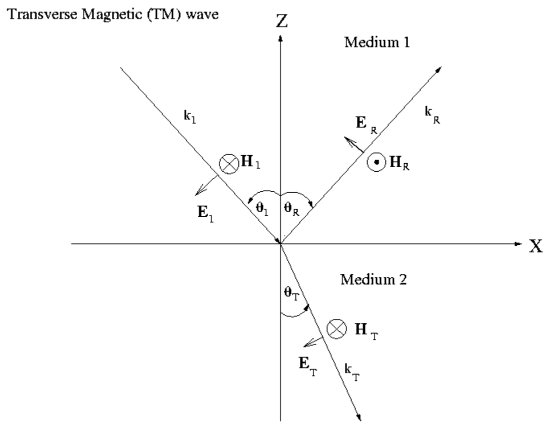
$$\vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \mu} \tag{6.3.3}$$

Thus once we have determined $\vec{E}$ we can find $\vec{H}$.
To find the amplitudes, we consider two orientention s (or polarizations) of $\vec{E}$ and $\vec{H}$. These are:

1. $\vec{E}$ vector normal to the plane of incidence, called *transverse electric* (TE) wave: 2. $\vec{E}$ vec-



tor parallel to the plane of incidence, called *transverse magnetic* (TM) wave: The $\vec{H}$ must be oriented in the direction shown so that $\vec{E} \times \vec{H}$ points in the direction of propagation.



Plane waves of arbitary polariation can be expressed as a sum of these orientations.

For **Transverse Electric (TE)** waves, continuity of the tangential component of E implies:

$$E_{I0} + E_{R0} = E_{T0} \tag{6.3.4}$$

Likewise, continuity of the tangential components of $\vec{H}$:

$$H_{I0}\cos(\theta_I) - H_{R0}\cos(\theta_I) = H_{T0}\cos(\theta_T) \tag{6.3.5}$$

then since impedence $Z = \dfrac{E}{H} = \dfrac{c\mu}{n}$, and so $Z = \dfrac{c\mu}{n}$, where n is the refractive index, we can therefore write the above equation as:

$$\frac{(E_{I0} - E_{R0})\cos(\theta_I)}{Z_1} = \frac{E_{T0}\cos(\theta_T)}{Z_2} \tag{6.3.6}$$

$$\Rightarrow \quad \frac{E_{I0} - E_{R0}}{E_{T0}} = \frac{Z_1\cos(\theta_T)}{Z_2\cos(\theta_I)} \tag{6.3.7}$$

Where we've made use of the relation $Z = \dfrac{E}{H} = \dfrac{c\mu}{n}$ so $Z_1 = \dfrac{c\mu}{n_1}$ and $Z_2 = \dfrac{c\mu}{n_2}$ and n = refractive index. Solving 6.3.4 and 6.3.7 gives the following expressions for the reflected and transmitted $\vec{E}_0$.

$$\left(\frac{E_{R0}}{E_{I0}}\right)_{TE} = \frac{Z_2\cos\theta_I - Z_1\cos\theta_T}{Z_2\cos\theta_I + Z_1\cos\theta_T} \tag{6.3.8}$$

$$\left(\frac{E_{T0}}{E_{I0}}\right)_{TE} = \frac{2Z_2\cos\theta_I}{Z_2\cos\theta_I + Z_1\cos\theta_T} \tag{6.3.9}$$

These are two of *Frenel's Equations.*

If $\vec{E}$ is parallel to the plane of incidence (**Transverse Magnetic (TM)**), we have:

$$H_{I0} - H_{R0} = H_{T0} \tag{6.3.10}$$

$$or \quad \frac{E_{I0} - E_{R0}}{Z_1} = \frac{E_{T0}}{Z_2} \tag{6.3.11}$$

and

$$(E_{I0} + E_{R0}) \cos \theta_I = E_{T0} \cos \theta_T \tag{6.3.12}$$

Then once again solving 6.3.11 and 6.3.12 gives:

$$\left(\frac{E_{R0}}{E_{I0}}\right)_{TM} = \frac{Z_2 \cos \theta_T - Z_1 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I} \tag{6.3.13}$$

$$\left(\frac{E_{T0}}{E_{I0}}\right)_{TM} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I} \tag{6.3.14}$$

which are the other two *Fresnel Equations.* At normal incidence the difference between these equations vanish since both the electric and magnetic fields are transverse to boundary. We then get:

$$\frac{E_{R0}}{\vec{E}_{I0}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{6.3.15}$$

$$\frac{E_{T0}}{\vec{E}_{I0}} = \frac{2Z_2}{Z_2 + Z_1} \tag{6.3.16}$$

If we consider non-magnetic nonconductors, then $\dfrac{Z_1}{Z_2} = \dfrac{n_2}{n_1}$ so for **TE** waves:

$$\left(\frac{E_{R0}}{E_{I0}}\right)_{TE} = \frac{\frac{n_1}{n_2} \cos \theta_I - \cos \theta_T}{\frac{n_1}{n_2} \cos \theta_I + \cos \theta_T} \tag{6.3.17}$$

$$\left(\frac{E_{T0}}{E_{I0}}\right)_{TE} = \frac{2\frac{n_1}{n_2} \cos \theta_I}{\frac{n_1}{n_2} \cos \theta_I + \cos \theta_T} \tag{6.3.18}$$

Now depending on the relative values of n_1 and n_2, the sign of the reflected wave can be positive or negative. The change of sign corresponds to a phase change of π between the incident and reflected waves. If $n_1 < n_2$ there will be a phase change of π , while for $n_1 > n_2$ there will be no phase change. The transmitted wave is always in phase.

For a **TM** wave:

$$\left(\frac{E_{R0}}{E_{I0}}\right)_{TM} = \frac{-\cos \theta_I + \frac{n_1}{n_2} \cos \theta_T}{\cos \theta_I + \frac{n_1}{n_2} \cos \theta_T} \tag{6.3.19}$$

$$\left(\frac{E_{T0}}{E_{I0}}\right)_{RM} = \frac{2\frac{n_1}{n_2} \cos \theta_I}{\cos \theta_I + \frac{n_1}{n_2} \cos \theta_T} \tag{6.3.20}$$

Again the transmitted wave is in phase. However E_{I0} and E_{R0} can be in or out of phase. They are in phase if:

$$\frac{n_1}{n_2}\cos\theta_T > cos\theta_I \tag{6.3.21}$$

$$or \quad sin(\theta_T - \theta_I)cos(\theta_T + \theta_I) > 0 \tag{6.3.22}$$

This requires either:

$$\theta_T > \theta_I \quad and \quad \theta_T + \theta_I < \frac{\pi}{2} \tag{6.3.23}$$

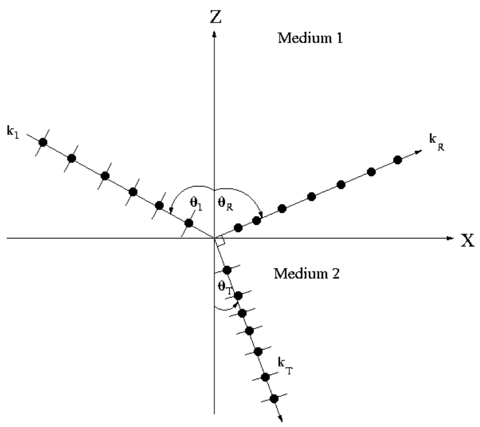
$$or \quad \theta_T < \theta_I \quad and \quad \theta_T + \theta_I > \frac{\pi}{2} \tag{6.3.24}$$

6.4 The Brewster Angle

If the expression $sin(\theta_T - \theta_I)cos(\theta_T + \theta_I) = 0$ or equivalently 6.3.19 is zero, then there is no reflected wave if $\vec{E}$ is parallel to the plane of incidence. This means that $\cos((\theta_T + \theta_I) = 0 \;\; \Rightarrow \theta_T + \theta_I = \pm\frac{\pi}{2}$. ie Geometrically this means the electric field of the transmitted wave is parallel to the direction of propagation of the reflected wave.
Note that the only other possible solution is $\theta_I - \theta_T = 0$ which is not a valid solution. The angle of incidence, θ_I, at which the condition is satisfied is called the *Brewster Angle* (θ_{IB}). At the Brewster angle

$$\frac{n_1}{n_2} = \frac{\sin\theta_T}{sin\theta_{IB}} = \frac{1}{\tan\theta_{IB}} \tag{6.4.1}$$

This effect can be used to produce polarized light. If unpolarized light is incident on the surface of a dielectric at the Brewster angle, then only the component of the wave with its electric field perpendicular to the plane of incidence will be reflected.



6.5 Coefficients of Reflection and Transmission

Setting $\mu_r = 1$ and looking at the time averaged Poynting vector gives:

$$\mathcal{P}_{Iav} = \frac{1}{2}\sqrt{\frac{\epsilon_1}{\mu_0}}E_{I0}^2\hat{n}_I \tag{6.5.1}$$

$$\mathcal{P}_{Rav} = \frac{1}{2}\sqrt{\frac{\epsilon_1}{\mu_0}}E_{R0}^2\hat{n}_R \tag{6.5.2}$$

$$\mathcal{P}_{Tav} = \frac{1}{2}\sqrt{\frac{\epsilon_2}{\mu_0}}E_{T0}^2\hat{n}_T \tag{6.5.3}$$

The unit vectors $\hat{n}_I$, $\hat{n}_R$, and $\hat{n}_T$ point in the direction of propagation of the incident, reflected and and transmitted waves respectively. For example:

$$\hat{n}_I = \frac{\vec{k}_1}{k_1} \tag{6.5.4}$$

Then the *coefficient of reflection*, **R**, is:

$$R = \left| \frac{\mathcal{P}_{Rav}\cdot\hat{n}}{\mathcal{P}_{Iav}\cdot\hat{n}} \right| = \frac{E_{R0}^2}{E_{I0}^2} \tag{6.5.5}$$

where $\hat{n}$ is a normal to the interface. Similarly, the *coefficient of transmission*, T, is:

$$T = \left| \frac{\mathcal{P}_{Tav}\cdot\hat{n}}{\mathcal{P}_{Iav}\cdot\hat{n}} \right| = \left(\frac{\epsilon_{r2}}{\epsilon_{r1}}\right)^{\frac{1}{2}} \frac{E_{T0}^2\cos\theta_T}{E_{I0}^2\cos\theta_I} = \frac{n_2 E_{T0}^2\cos\theta_T}{n_1 E_{I0}^2\cos\theta_I} \tag{6.5.6}$$

Therefore,

$$R_\perp = \left[\frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T}\right]^2 \tag{6.5.7}$$

$$T_\perp = \frac{4n_1\cos\theta_I\cos\theta_T}{\left[n_1\cos\theta_I + n_2\cos\theta_T\right]^2} \tag{6.5.8}$$

$$R_{II} = \left[\frac{-n_2\cos\theta_I + n_1\cos\theta_T}{n_2\cos\theta_I + n_1\cos\theta_T}\right]^2 \tag{6.5.9}$$

$$T_{II} = \frac{4n_1\cos\theta_I\cos\theta_T}{\left[n_2\cos\theta_I + n_1\cos\theta_T\right]^2} \tag{6.5.10}$$

It can be shown that

$$R + T = 1 \tag{6.5.11}$$

which is expected, and that at the Brewster Angle $R_{II} = 0$ and $T_{II} = 1$.

6.6 Non-uniform plane waves

The formula for electric and magnetic fields are:

$$\vec{E} = \vec{E}_0 \exp(i(\omega t - \vec{k} \cdot \vec{r})) \tag{6.6.1}$$

$$\vec{H} = \vec{H}_0 \exp(i(\omega t - \vec{k} \cdot \vec{r})) \tag{6.6.2}$$

We have investigated the case of interfaces where the vector $\vec{k}$ is real. If, however, we put

$$\vec{k} = \vec{\beta} - i\vec{\alpha} \tag{6.6.3}$$

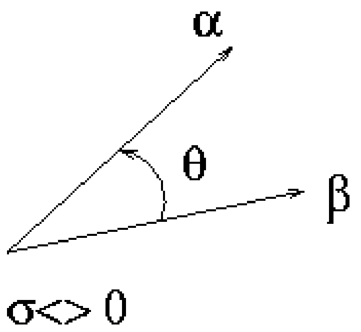
where $\vec{\alpha}$ and $\vec{\beta}$ are not in the same direction, then the wave decays exponentially in a direction which is not that in which the wave is travelling.
Now waves of constant *phase* are perpendicular to β and planes of constant *amplitude* are perpendicular to α. For example, since

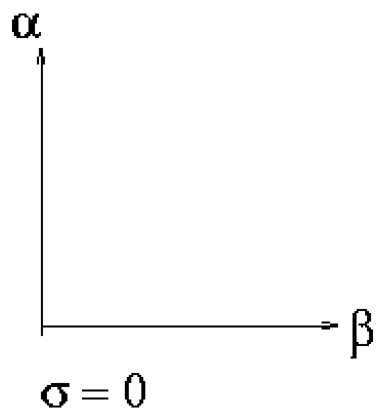
$$k^2 = \vec{k} \cdot \vec{k} = \omega^2 \epsilon\mu - i\omega\sigma\mu = (\beta^2 - \alpha^2 - 2i\vec{\alpha} \cdot \vec{\beta}) \tag{6.6.4}$$

then

$$\beta^2 - \alpha^2 = \omega^2 \epsilon\mu > 0 \Rightarrow \quad \beta > \alpha \tag{6.6.5}$$

Also $2\vec{\alpha} \cdot \vec{\beta} = \omega\sigma\mu \Rightarrow \theta \leq \frac{\pi}{2}$ and $\vec{\alpha} \perp \vec{\beta}$ if $\sigma = 0$.





Another useful, but "difficult to understand" concept is that of *complex angles*, ϕ, where

$$\sin\phi = \frac{\exp(i\phi) - \exp(-i\phi)}{2i} \tag{6.6.6}$$

and

$$\cos\phi = \frac{\exp(i\phi) + \exp(-i\phi)}{2} \tag{6.6.7}$$

Note that $\sin^2\phi+\cos^2\phi = 1$ still. Returning to Snell's Law, $n_1 \sin\theta_I = n_2 \sin\theta_T$ and noting that if $\frac{n_1}{n_2}\sin\theta_I > 1$, then $\sin\theta_T > 1$ and, thus θ_T must be *complex*, $\theta_T = a + ib$, so we always have:

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \tag{6.6.8}$$

using the real and imaginary parts we have:

$$\delta = \sqrt{\frac{2}{\sigma\mu\omega}} \quad so \tag{6.6.9}$$

$$\sin\theta_T = \frac{\exp(i(a+ib)) - exp(-i(a+ib))}{2i} = \frac{\exp(ia)\exp(-b) - \exp(-ia)\exp(b)}{2i} \tag{6.6.10}$$

also we have:

$$\cos\theta_T = \frac{\exp(i(a+ib)) + exp(-i(a+ib))}{2} = \frac{\exp(ia)\exp(-b) + \exp(-ia)\exp(b)}{2} \tag{6.6.11}$$

$\sin\theta_T$ is *real* (via 6.6.8) so a $= \frac{\pi}{2}$ (ie. $\theta_T = \frac{\pi}{2} + ib$) and so

$$\sin\theta_T = \frac{\exp(b) + \exp(-b)}{2} = \cosh b \tag{6.6.12}$$

$$\cos\theta_T = \frac{i(\exp(-b) - \exp(b))}{2} = -i\sinh b \tag{6.6.13}$$

If θ_T is complex then we have *total reflection*, the *critical angle of incidence*, θ_{Ic}, occurs when $\theta_T = 90\,\text{deg}$. ie.

$$\sin\theta_{IC} = \frac{n_2}{n_1} \tag{6.6.14}$$

The magnitude of the reflected wave is now unity (exercise) so we can write

$$\left(\frac{E_{R0}}{E_{I0}}\right)_\perp = \exp(i\phi_\perp) \tag{6.6.15}$$

where

$$\phi_\perp = 2\tan^{-1}\left[\frac{\sqrt{sin^2\theta_T - 1}}{\frac{n_1}{n_2}\cos\theta_I}\right] \tag{6.6.16}$$

This is the phase of the reflected wave with respect to the incident wave, the *phase shift on reflection*. The value of θ_T after total reflection can be readily obtained from the above equations:

$$a = \frac{\pi}{2}, \quad b = \cosh^{-1}(\frac{n_1}{n_2}\sin\theta_I) \tag{6.6.17}$$

The equations for the Incident, Reflected and Transmitted waves have the same forms as before:

$$\vec{E}_I = \vec{E}_{I0} \exp[i(\omega t - k_1(x \sin\theta_I - z\cos\theta_I)] \tag{6.6.18}$$

$$\vec{E}_R = \vec{E}_{R0} \exp[i(\omega t - k_1(x \sin\theta_I + z\cos\theta_I)] \tag{6.6.19}$$

$$\vec{E}_T = \vec{E}_{T0} \exp[i(\omega t - k_1(x \sin\theta_T - z\cos\theta_T)] \tag{6.6.20}$$

However now that θ_t is complex the transmitted wave does not represent energy transferred to medium 2.
We note that the wavenumber of the transmitted wave is:

$$\vec{k}_T = \vec{\beta}_T - i\vec{\alpha}_T = k_2(\sin\theta_T\hat{x} - \cos\theta_T\hat{z}) = k_2(\frac{n_1}{n_2}\sin\theta_I\hat{x} + i\sinh b\hat{z}) \tag{6.6.21}$$

Thus:

$$\vec{E}_T = \vec{E}_{T0}\exp(sinh(b)k_2 z)\exp(i(\omega t - k_2(\frac{n_1}{n_2}\sin\theta_I)x) \tag{6.6.22}$$

So no energy is transferred in the $-\hat{z}$ direction although there is some non-zero electric field for z $<$ 0. Instead the transmitted wave propagates in the $\hat{x}$ directions and decays exponentially into medium 2.
For TE waves ($\perp$) we can show:

$$\left(\frac{E_{T0}}{E_{I0}}\right)_\perp = \frac{2\cos\theta_I}{\sqrt{1-\frac{n_2}{n_1}}}\exp(i\frac{\phi_\perp}{2}) \tag{6.6.23}$$

We know:

$$\vec{H}_T = \frac{k}{\omega\mu_2}\hat{K}_T \times \vec{E}_T \tag{6.6.24}$$

so

$$\vec{E}_T \times \vec{H}_T^* = \frac{\vec{E}_T}{\omega\mu_2} \times (\hat{k}_T \times \vec{E}_T) = \frac{1}{\omega\mu_2}(\mid \vec{E}_T \mid^2 \vec{k}_T^* - (\vec{E}_T \cdot \vec{K}_T^*)\vec{E}_T^*) \tag{6.6.25}$$

For $\perp$ polarized waves, $\vec{E}_T$ is in the $\hat{y}$ direction so the second term is zero. Thus the Poynting vector points in the $\hat{x}$ direction. These waves are referred to as *evanescent* since they decay exponentially *into* the second medium and propagate along the interface.

6.7 Reflection and Transmission at the surface of a good conductor

Consider expressions for the reflected and trasmitted waves where medium 1 is a dielectric and medium 2 is now a conductor.

$$\vec{E}_R = \vec{E}_{R0}\exp(i(\omega t - k_1 x\sin\theta_I - k_1 z\cos\theta_i)) \tag{6.7.1}$$

$$
\begin{aligned}
\vec{E}_T &= \vec{E}_{T0}\exp(i(\omega t - k_2(x\sin\theta_T - z\cos\theta_T)) && (6.7.2)\\
&= \vec{E}_{T0}\exp(i(\omega t - k_1 x\sin\theta_I - k_2 z\cos\theta_T)) && (6.7.3)\\
&= \vec{E}_{T0}\exp(i(\omega t - k_1 x\sin\theta_I \pm k_2 z\sqrt{1-\frac{n_1}{n_2}^2\sin^2\theta_I})) && (6.7.4)
\end{aligned}
$$

From earlier work we have:

$$
\frac{k_1}{k_2} = \frac{\omega\sqrt{\epsilon_1\mu_1}\delta}{1-i} = \frac{\delta}{1-i}k_1 \tag{6.7.5}
$$

$$
\Rightarrow \quad k_2 = \frac{1-i}{\delta} \tag{6.7.6}
$$

Now since k_2 is a good conductor $\mid k_2 \mid \gg \mid k_1 \mid$ this implies

$$
\cos\theta_T = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_I} \approx 1 \tag{6.7.7}
$$

So $\theta_T \approx 0$ and hence the wave propagates"normally" into a good conductor. Our expression for $\vec{E}_T$ is now:

$$
\vec{E}_T \approx \vec{E}_{T0}\exp[i(\omega t - k_1 x\sin\theta_I \pm \frac{1+i}{\delta}z)] \tag{6.7.8}
$$

Since $k_1\sin\theta_I \ll \frac{1}{\delta}$ (exercise) we get:

$$
\vec{E}_T \approx \vec{E}_{T0}\exp[i(\omega t + \frac{z}{\delta}) + \frac{z}{\delta}] \tag{6.7.9}
$$

For $\vec{E}_{TM}$ (normal to plane of incidence), then since $\mid \frac{n_1}{n_2} \mid \ll 1$,

$$
\left(\frac{E_R}{E_I}\right)_{TM} = \frac{\frac{n_1}{n_2}\cos\theta_I - \cos\theta_T}{\frac{n_1}{n_2}\cos\theta_I + \cos\theta_T} \approx -1 \tag{6.7.10}
$$

This is true for any angle of incidence. Now we calculate the transmitted electric field, we also use $\cos\theta_T \approx 1$,

$$
\left(\frac{E_T}{E_I}\right)_{TM} = \frac{2\frac{n_1}{n_2}\cos\theta_I}{\frac{n_1}{n_2}\cos\theta_I + \cos\theta_T} \approx 2\frac{n_1}{n_2}\cos\theta_I \approx 0 \tag{6.7.11}
$$

This is also true for any angle of incidence.
Now for E_{TE} E parallel to the plane of incidence and using the same approximations:

$$
\left(\frac{E_R}{E_I}\right)_{TE} = \frac{\frac{n_1}{n_2}\cos\theta_T - \cos\theta_I}{\frac{n_1}{n_2}\cos\theta_T + \cos\theta_I} \approx \frac{\frac{n_1}{n_2} - \cos\theta_I}{\frac{n_1}{n_2} + \cos\theta_I} \approx -1 \tag{6.7.12}
$$

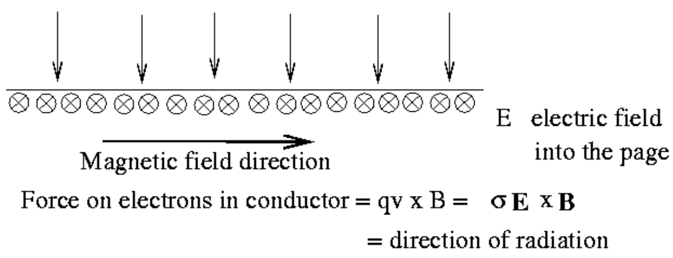
However this approximation is *not* valid at grazing incidence, where $\theta_I \approx 90\,\text{deg}$. Finally the transmitted component of the parallel component is given by

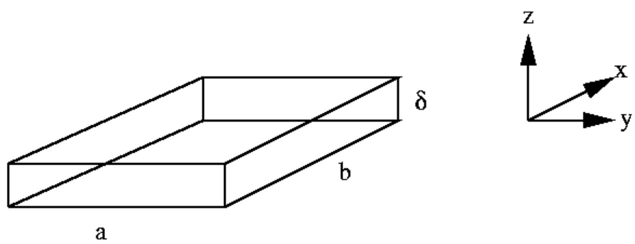
$$\left(\frac{E_T}{E_I}\right)_{TE} = \frac{2\frac{n_1}{n_2}\cos\theta_I}{\frac{n_1}{n_2}\cos\theta_T + \cos\theta_I} \approx 2\frac{n_1}{n_2} = \frac{2n_1\delta}{1-i} = n_1\delta(1+i) \tag{6.7.13}$$

Once again this approximation is not valid where $\theta_I \approx 90\,\text{deg}$.

6.8 Radiation Pressure

Consider a wave with it's $\vec{E}$ vector normal to the plane of incidence. In the conductor, the current density is $\sigma\vec{E}_T$. $\vec{H}$ is perpendicular to the moving electrons and we therefore have a $(Q\vec{v} \times \mu_0\vec{H})$ force.
This force is directed away from the interface and into the conductor. The resulting pressure is referred to as the *radiation pressure*. Now lets calculate it's effects.
Taking a piece of the material of side lengths a,b and depth δz, the total charge Q, is related to the charge density ρ, the current density $\vec{J}$ and electron velocity $\vec{v}$ via:





$$Q = \rho abdz = \frac{|\vec{J}|}{|\vec{v}|}abdz \tag{6.8.1}$$

$$\Rightarrow \quad Q\vec{v} = \vec{J}abdz \tag{6.8.2}$$

$$\Rightarrow \quad Q\vec{v} = \sigma\vec{E}_T abdz \tag{6.8.3}$$

So we get the force on this piece, $d\vec{F}$ as:

$$d\vec{F} = ab\sigma\vec{E}_T \times \mu_0\vec{H}dz \tag{6.8.4}$$

The pressure $d\vec{p}$ is just the force per unit area, so:

$$d\vec{p} = \sigma \vec{E}_T \times \mu_0 \vec{H} dz \tag{6.8.5}$$

Now we plug in our equations for $\vec{E}$:

$$\vec{E}_T = \vec{E}_{T0} \exp((i(\omega t + \frac{z}{\delta}) + \frac{z}{\delta}) \tag{6.8.6}$$

and we use:

$$H_T = \sqrt{\frac{\sigma}{\omega\mu_0}} \exp(i\frac{\pi}{4}) E_T = \frac{1-i}{\omega\mu_0\delta} E_T \tag{6.8.7}$$

The magnitude of the pressure is

$$\frac{dp}{dz} = \frac{\sigma\mu_0}{\omega\mu_0\delta} \mid E_{T0} \mid^2 \mid (1-i) \exp((i\omega t + \frac{z}{\delta}) + \frac{z}{\delta}) \mid^2 \tag{6.8.8}$$

Taking the time average gives:

$$\frac{dp_{av}}{dz} = \frac{\sigma}{2\omega\delta} \mid E_{T0\ av} \mid^2 \exp(2\frac{z}{\delta}) \tag{6.8.9}$$

To get the total pressure we integrate and get:

$$p_{av} = \frac{\sigma}{2\omega\delta} \mid E_{T0\ av} \mid^2 \int_{-\infty}^{0} \exp(2\frac{z}{\delta}) dz = \frac{\sigma}{4\omega} \mid E_{T0\ av} \mid^2 \tag{6.8.10}$$

Recall $\left(\frac{E_T}{E_I}\right)_\perp = 2\frac{n_1}{n_2} \cos\theta_I$ so in terms of the input field

$$p_{av} = \frac{\sigma}{4\omega} \mid \frac{n_1}{n_2} E_{Iav} \cos\theta_I \mid^2 \tag{6.8.11}$$

Finally expression in terms of input power flux:

$$p_{av} = \frac{2}{v_1} \cos^2\theta_I \mathcal{P}_{Iav} \tag{6.8.12}$$

where v_1 is the speed of the wave in medium 1, and $\mathcal{P}_{Iav}$ is the initial Poynting Vector. The derivation is left as an exercise. The analysis of waves parallel to the plane of incidence is more complex but leads to the same result.

6.9 Momentum density in an electromagentic wave

At normal incidence in a vacuum, $p_{av} = 2\frac{\mathcal{P}_{Iav}}{c}$. Since the conducting surface acts as a near perfect reflector the change in momentum of the wave is $2\frac{\mathcal{P}}{c}$ per unit time per unit area. Then:

$$Momentum\ Flux\ Density\ = \frac{\mathcal{P}_{Iav}}{c} \tag{6.9.1}$$

Now Momentum Flux Density = Momentum Volume Density$\times$ c. Therefore:

$$Momentum\ Volume\ Density = \frac{\mathcal{P}_{Iav}}{c^2} \tag{6.9.2}$$

Generalizing to a vector field:

$$M.V.D. = \vec{p} = \frac{\vec{E} \times \vec{H}}{c^2} \tag{6.9.3}$$