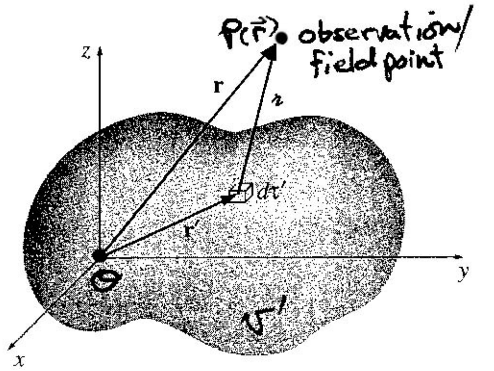
LECTURE NOTES 14

EM RADIATION FROM AN ARBITRARY SOURCE:

We now apply the formalism/methodology that we have developed in the previous lectures on low-order multipole *EM* radiation {E(1), M(1), E(2), M(2)} to an arbitrary configuration of electric charges and currents, only restricting these to be localized charge and current distributions, contained within a finite volume v' near the origin:

$$\boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_r)}$$
$$\boxed{\mathscr{r} = \left|\vec{r} - \vec{r}'(t_r)\right|}$$
$$\boxed{\mathscr{r} = \sqrt{r^2 + r'^2(t_r) - 2\vec{r}\cdot\vec{r}'(t_r)}}$$
$$\boxed{\Delta t = t - t_r = \mathscr{r}/c}$$
$$\boxed{t_r = t - \mathscr{r}/c}$$



For arbitrary, localized {total} electric charge and current density distributions $\rho_{Tot}(\vec{r}', t_r)$ and $\vec{J}_{Tot}(\vec{r}', t_r)$, the retarded scalar and vector potentials, respectively are:

$$\boxed{V_r(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{Tot}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{Tot}(\vec{r}',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau'} \quad \text{with} \quad \boxed{t_r = t - \mathscr{r}/c}$$

$$\boxed{\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{Tot}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{Tot}(\vec{r}',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau'} \quad \text{and} \quad \boxed{\mathscr{r} = \sqrt{r^2 + r'^2(t_r) - 2\vec{r}\cdot\vec{r}'(t_r)}}$$

Again, for *EM* radiation, we assume that the observation / field point $\vec{r}$ is far away from the localized source charge / current distribution, such that: $r'_{\max} \ll r$ or: $r'_{\max}/r \ll 1$.

Then keeping only up to terms linear in $\left(\frac{r'}{r}\right)$: $\boxed{\mathscr{r} = r\sqrt{1 + \left(\frac{r'(t_r)}{r}\right)^2 - \frac{2\vec{r}\cdot\vec{r}'(t_r)}{r^2}} \simeq r\sqrt{1 - \frac{2\vec{r}\cdot\vec{r}'(t_r)}{r^2}}}$

But: $\boxed{\sqrt{1-\epsilon} \simeq 1 - \frac{1}{2}\epsilon}$ for $\boxed{\epsilon \ll 1}$ $\Rightarrow$ $\therefore$ $\boxed{\mathscr{r} \simeq r\left(1 - \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right) = r\left(1 - \frac{\hat{r}\cdot\vec{r}'(t_r)}{r}\right)}$

And: $\boxed{\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\frac{1}{\left(1 - \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right)} \approx \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'(t_r)}{r^2}\right) = \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}'(t_r)}{r}\right)}$

Now: $\boxed{\rho_{Tot}(\vec{r}',t_r)=\rho_{Tot}\left(\vec{r}',t-\frac{r}{c}\right)\simeq\rho_{Tot}\left(\vec{r}',t-\frac{r}{c}\left(1-\frac{\hat{r}\bullet\vec{r}'(t_r)}{r^2}\right)\right)\approx\rho_{Tot}\left(\vec{r}',t-\frac{r}{c}+\frac{\hat{r}\bullet\vec{r}'(t_r)}{c}\right)}$

Expand $\rho_{Tot}(\vec{r}',t_r)$ as a Taylor series in the present time t about the retarded time, at the origin $\{\vec{r}'=0\}$:

Defining: $\boxed{t_o\equiv t-\frac{r}{c}}$

Then: $\boxed{\rho_{Tot}(\vec{r}',t_r)\simeq\rho_{Tot}(\vec{r}',t_o)+\dot{\rho}_{Tot}(\vec{r}',t_o)\left(\frac{\hat{r}\bullet\vec{r}'}{c}\right)+\frac{1}{2!}\ddot{\rho}_{Tot}(\vec{r}',t_o)\left(\frac{\hat{r}\bullet\vec{r}'}{c}\right)^2+\frac{1}{3!}\dddot{\rho}_{Tot}(\vec{r}',t_o)\left(\frac{\hat{r}\bullet\vec{r}'}{c}\right)^3+\ldots}$

Where: $\boxed{\dot{\rho}_{Tot}(\vec{r}',t_o)\equiv\frac{d\rho_{Tot}(\vec{r}',t_o)}{dt_r}}$ *etc.*

We can drop / neglect ***all*** higher-order terms ***beyond*** the $\dot{\rho}$ term, ***provided that***:

$\boxed{r'_{\max}\ll\frac{c}{|\ddot{\rho}/\dot{\rho}|},\ \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}},\ \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}},\ \ldots}$ is satisfied…

For a *harmonically* oscillating system (*i.e.* one with angular frequency ω), each of these ratios, *e.g.* $\frac{c}{|\ddot{\rho}/\dot{\rho}|}$, *etc.* is $=\frac{c}{\omega}$ and thus we have: $\boxed{r'_{\max}\ll\frac{c}{\omega}}$ if $\boxed{r_{\max}=d\ll\frac{c}{\omega}}$, then $\boxed{\left(\frac{\omega d}{c}\right)\ll 1}$,

or equivalently {here}: $\boxed{\left(\frac{\omega r'_{\max}}{c}\right)\ll 1}$.

The two approximations $\boxed{\frac{r'_{\max}}{r}\ll 1}$ and $\boxed{\frac{\omega r'_{\max}}{c}\ll 1}$, or more generally: $\boxed{\frac{r'_{\max}|\ddot{\rho}/\dot{\rho}|}{c}\ll 1}$ *etc*… amount to keeping only the first-order {the lowest-order, non-negligible} terms in r'.

The retarded scalar potential $V_r(\vec{r},t)$ then becomes:

$$V_r(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho_{Tot}(\vec{r}',t_r)}{r}\,d\tau'=\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{1}{r}\rho_{Tot}(\vec{r}',t_r)\,d\tau'$$

$$\simeq\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{1}{r}\left(1+\frac{\hat{r}\bullet\vec{r}'(t_o)}{r}\right)\left(\rho_{Tot}(\vec{r}',t_o)+\dot{\rho}_{Tot}(\vec{r}',t_o)\left(\frac{\hat{r}\bullet\vec{r}'(t_o)}{c}\right)+\ldots\right)d\tau'$$

$$\approx\frac{1}{4\pi\varepsilon_o r}\left[\int_{v'}\rho_{Tot}(\vec{r}',t_o)\,d\tau'+\left(\frac{\hat{r}}{r}\right)\bullet\int_{v'}\vec{r}'(t_o)\rho_{Tot}(\vec{r}',t_0)\,d\tau'+\left(\frac{\hat{r}}{c}\right)\bullet\int_{v'}\vec{r}'(t_o)\dot{\rho}_{Tot}(\vec{r}',t_o)\,d\tau'+\ldots\right]$$

$$\boxed{V_r(\vec{r},t)\simeq\frac{1}{4\pi\varepsilon_o r}\left[\overbrace{\int_{v'}\rho_{Tot}(\vec{r}',t_o)\,d\tau'}^{=Q_{Tot}(t_o)}+\left(\frac{\hat{r}}{r}\right)\bullet\overbrace{\int_{v'}\vec{r}'(t_o)\rho_{Tot}(\vec{r}',t_o)\,d\tau'}^{=\vec{p}_{Tot}(t_o)}+\left(\frac{\hat{r}}{c}\right)\bullet\frac{d}{dt}\overbrace{\int_{v'}\vec{r}'(t_o)\rho_{Tot}(\vec{r}',t_o)\,d\tau'}^{=\vec{p}_{Tot}(t_o)}+\ldots\right]}$$

Or: $$\boxed{V_r(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{Q_{Tot}(t_o)}{r} + \frac{\hat{r}\bullet\vec{p}_{Tot}(t_o)}{r^2} + \frac{\hat{r}\bullet\dot{\vec{p}}_{Tot}(t_o)}{cr} + \ldots\right]}$$

In the static limit: monopole term; dipole term; vanishes in the static limit

The retarded vector potential, to first order in r' ($r \simeq r$) {with $t_o \equiv t - r/c$} then becomes:

$$\boxed{\vec{A}_r(\vec{r},t) = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{Tot}(\vec{r}',t_r)}{r}\,d\tau' = \frac{\mu_o}{4\pi}\int_{v'}\frac{\vec{J}_{Tot}(\vec{r}',t-r/c)}{r}\,d\tau' \simeq \frac{\mu_o}{4\pi r}\int_{v'}\vec{J}_{Tot}(\vec{r}',t_o)\,d\tau'}$$

Griffiths Problem 5.7 (*p*. 214) showed that for *localized* electric charge / current distributions contained in the source volume v', that:

$$\boxed{\int_{v'}\vec{J}_{Tot}(\vec{r}',t_r)\,d\tau' = \frac{d\vec{p}_{Tot}(\vec{r},t)}{dt} = \dot{\vec{p}}_{Tot}(\vec{r},t)}$$

Thus: $$\boxed{\vec{A}_r(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\dot{\vec{p}}_{Tot}(t_o)}{r}}$$

Note that $\vec{p}_{Tot}(t_o)$ is *already* first order in r' $\Rightarrow$ any *additional* refinements are therefore *second order* in r'; thus, the higher-order terms can be neglected/ignored (*here*).

Next, we calculate the retarded $\vec{E}$ and $\vec{B}$ fields. Since we want the *EM* radiation fields (the "far-zone" limit), we drop / neglect $1/r^2$, $1/r^3$, $1/r^4$, *etc.* terms, and keep only the $1/r$ radiation-field terms.

Note that the radiation terms come entirely from those terms in the Taylor series expansions for $\rho_{Tot}(\vec{r}',t_o)$ and $\vec{J}_{Tot}(\vec{r}',t_o)$ in which we differentiate the argument t_o of $\rho_{Tot}(\vec{r}',t_o)$, $\vec{J}_{Tot}(\vec{r}',t_o)$.

Since: $\boxed{t_o \equiv t - r/c}$ then: $\boxed{\vec{\nabla}t_o = -\frac{1}{c}\vec{\nabla}r}$ but: $\boxed{\vec{\nabla}r = \hat{r}}$ $\therefore$ $\boxed{\vec{\nabla}t_o = -\frac{1}{c}\hat{r}}$

Thus: $$\boxed{\vec{\nabla}V_r(\vec{r},t) \simeq \vec{\nabla}\left[\frac{1}{4\pi\varepsilon_o}\frac{\hat{r}\bullet\dot{\vec{p}}_{Tot}(t_o)}{cr}\right] \approx \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\bullet\ddot{\vec{p}}_{Tot}(t_o)}{cr}\right]\vec{\nabla}t_o = -\frac{1}{4\pi\varepsilon_o c^2}\frac{\left[\hat{r}\bullet\ddot{\vec{p}}_{Tot}(t_o)\right]}{r}\hat{r}}$$

And: $$\boxed{\frac{\partial\vec{A}_r(\vec{r},t)}{\partial t} \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{\ddot{\vec{p}}_{Tot}(t_o)}{r}}$$

Then the retarded electric field for *EM* radiation in the "far-zone" limit is:

$$\boxed{\vec{E}_{\rm r}(\vec{r},t) = -\vec{\nabla}V_{\rm r}(\vec{r},t) - \frac{\partial \vec{A}_{\rm r}(\vec{r},t)}{\partial t} \simeq +\frac{1}{4\pi\varepsilon_o c^2}\frac{\left[\hat{r}\bullet\ddot{\vec{p}}_{Tot}(t_o)\right]}{r}\hat{r} - \frac{\mu_o}{4\pi}\frac{\ddot{\vec{p}}_{Tot}(t_o)}{r}} \quad \text{but: } \boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$$

$$\therefore \boxed{\vec{E}_{\rm r}(\vec{r},t) \simeq \frac{\mu_o}{4\pi r}\left[\left(\hat{r}\bullet\ddot{\vec{p}}_{Tot}(t_o)\right)\hat{r} - \ddot{\vec{p}}_{Tot}(t_o)\right] = \frac{\mu_o}{4\pi r}\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}_{Tot}(t_o)\right)\right]} \quad \{\text{using the BAC-CAB rule}\}$$

where the second time-derivative of the total electric dipole moment $\ddot{\vec{p}}_{Tot}(t_o)$ is evaluated at the retarded time $t_o \equiv t - r/c$ and computed from the origin, ϑ $\{\vec{r}\,' = 0\}$: $\ddot{\vec{p}}_{Tot}(t_o) = \ddot{\vec{p}}_{Tot}\left(0, t - r/c\right)$.

The retarded magnetic field for *EM* radiation in the "far-zone" limit is:

$$\boxed{\begin{aligned}\vec{B}_{\rm r}(\vec{r},t) &= \vec{\nabla}\times\vec{A}_{\rm r}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\vec{\nabla}\times\frac{\dot{\vec{p}}_{Tot}(t_o)}{r} \approx \left(\frac{\mu_o}{4\pi r}\right)\left[\vec{\nabla}\times\dot{\vec{p}}(t_o)\right] \\ &= \left(\frac{\mu_o}{4\pi}\right)\underbrace{\left[\vec{\nabla}t_o\times\ddot{\vec{p}}_{Tot}(t_o)\right]} = -\frac{\mu_o}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}_{Tot}(t_o)\right]\end{aligned}}$$

Where in first step we have used the relation $\vec{\nabla}\times\vec{v}(t_{\rm r}) = -\vec{a}(t_{\rm r})\times\vec{\nabla}t_{\rm r}$ {see "term (3)" P436 Lect. Notes 12 *p*. 11 and/or Griffiths Equation 10.55, *p*. 436} and in the last step on the RHS we have {again} used the relation $\vec{\nabla}t_o = -\frac{1}{c}\hat{r}$.

$$\therefore \boxed{\vec{B}_{\rm r}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}_{Tot}(t_o)\right]}$$

where the second time-derivative of the total electric dipole moment $\ddot{\vec{p}}_{Tot}(t_o)$ is evaluated at the retarded time $t_o \equiv t - r/c$ and computed from the origin, ϑ $\{\vec{r}\,' = 0\}$: $\ddot{\vec{p}}_{Tot}(t_o) = \ddot{\vec{p}}_{Tot}\left(0, t - r/c\right)$.

If we use spherical-polar coordinates, with the $\hat{z}$-axis $\parallel$ $\ddot{\vec{p}}_{Tot}(t_o)$, then noting that:

$$\boxed{\hat{r}\times\ddot{\vec{p}}_{Tot}(t_o) = \ddot{p}_{Tot}(t_o)\left[\hat{r}\times\hat{z}\right]} \quad \text{but: } \boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}} \qquad \boxed{\hat{r}\times\hat{r} = 0}$$

$$= \boxed{\ddot{p}_{Tot}(t_o)\left[\hat{r}\times\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\right]} \qquad \boxed{\hat{r}\times\hat{\theta} = \hat{\varphi}}$$

$$= \boxed{-\ddot{p}_{Tot}(t_o)\sin\theta\hat{\varphi}} \qquad \boxed{\hat{\theta}\times\hat{\varphi} = \hat{r}}$$

$$\boxed{\hat{\varphi}\times\hat{r} = \hat{\theta}}$$

Thus: $\boxed{\vec{E}_{\mathrm{r}}(r,\theta,t) \simeq \frac{\mu_o \ddot{p}(t_o)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}} \leftarrow \boxed{\hat{r}\times(-\hat{\varphi}) = \hat{\theta}}$

And: $\boxed{\vec{B}_{\mathrm{r}}(r,\theta,t) \approx \frac{\mu_o \ddot{p}_{Tot}(t_o)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\varphi}}$

and we also see that {again} $\boxed{\vec{B}_{\mathrm{r}}(r,\theta,t) = \frac{1}{c}\hat{r}\times\vec{E}_{\mathrm{r}}(r,\theta,t)}$, $\boxed{\vec{B}_{\mathrm{r}} \perp \vec{E}_{\mathrm{r}} \perp \hat{r}\ \left\{\| \hat{k}\right\}}$

The instantaneous retarded *EM* radiation energy density $u(r,\theta,t)$ in the "far-zone" limit is:

$$\begin{aligned}
u_{\mathrm{r}}^{rad}(r,\theta,t) &= \frac{1}{2}\left(\varepsilon_o E_{\mathrm{r}}^2(r,\theta,t) + \frac{1}{\mu_0}B_{\mathrm{r}}^2(r,\theta,t)\right) \\
&\simeq \frac{1}{2}\left[\frac{\varepsilon_o\mu_o^2\ddot{p}^2(t_o)}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right) + \frac{1}{\mu_o}\frac{\mu_o^2\ddot{p}^2(t_o)}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\right] \quad \text{but: } \varepsilon_o = \frac{1}{\mu_o c^2} \\
&= \frac{1}{2}\left[\frac{\mu_o\ddot{p}^2(t_0)}{16\pi^2c^2} + \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c^2}\right]\left(\frac{\sin^2\theta}{r^2}\right) = \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)
\end{aligned}$$

Thus: $\boxed{u_{\mathrm{r}}^{rad}(r,\theta,t) \simeq \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)}$ (Joules)

The instantaneous retarded Poynting's vector in the "far-zone" limit is:

$$\boxed{\vec{S}_{\mathrm{r}}^{rad}(r,\theta,t) = \frac{1}{\mu_o}\vec{E}_{\mathrm{r}}(r,\theta,t)\times B_{\mathrm{r}}(r,\theta,t)} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\boxed{\vec{S}_{\mathrm{r}}^{rad}(r,\theta,t) \simeq \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c}\left(\frac{\sin^2\theta}{r^2}\right)\underbrace{(\hat{\theta}\times\hat{\varphi})}_{=\hat{r}} = \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} = \vec{c}u_{\mathrm{r}}^{rad}(r,\theta,t)} \text{ with } \boxed{\begin{matrix}\vec{c}\equiv c\hat{r} \\ \hat{r}\,\|\,\hat{k}\end{matrix}}$$

The instantaneous retarded *EM* power radiated per unit solid angle in the "far-zone" limit is:

$$\boxed{\frac{dP_{\mathrm{r}}^{rad}(r,\theta,t)}{d\Omega} = \vec{S}^{rad}(r,\theta,t)\bullet r^2\hat{r} \simeq \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c}\sin^2\theta} \quad \left(\frac{\text{Watts}}{\text{steradian}}\right)$$

The total instantaneous retarded *EM* power radiated into 4π steradians, with vector area element $d\vec{a}_{\perp} = r^2\sin\theta d\theta d\varphi\hat{r} = r^2 d\Omega\hat{r}$ in the "far-zone" limit is:

$$\boxed{P_{\mathrm{r}}^{rad}(t) = \int_S \vec{S}_{\mathrm{r}}^{rad}(r,\theta,t)\bullet d\vec{a}_{\perp} \simeq \frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\sin\theta d\theta d\varphi = \frac{\mu_o\ddot{p}^2(t_o)}{6\pi^2c}} \text{ (Watts)}$$

The instantaneous retarded *EM* radiation <u>linear</u> momentum density in the "far-zone" limit is:

$$\boxed{\vec{\wp}_{\mathrm{r}}^{rad}(r,\theta,t)=\frac{1}{c^2}\vec{S}_{\mathrm{r}}^{rad}(r,\theta,t)\simeq\frac{\mu_o\ddot{p}^2(t_o)}{16\pi^2c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}}\quad\left(\frac{\mathrm{kg}}{\mathrm{m}^2\,\mathrm{sec}}\right)$$

The instantaneous retarded EM radiation <u>angular</u> momentum density in the "far-zone" limit is:

$$\boxed{\vec{\ell}_{\mathrm{r}}^{rad}(r,\theta,t)=\vec{r}\times\vec{\wp}_{\mathrm{r}}^{rad}(r,\theta,t)=0}$$

The characteristic impedance of the antenna associated with this lowest-order *EM* radiation is:

$$\boxed{Z_{rad}=\frac{|\vec{E}_{\mathrm{r}}|}{|\vec{H}_{\mathrm{r}}|}=\frac{|\vec{E}_{\mathrm{r}}|}{\frac{1}{\mu_o}|\vec{B}_{\mathrm{r}}|}=\mu_o c=\sqrt{\frac{\mu_o}{\varepsilon_o}}=Z_o=120\pi\ \Omega\simeq 377\ \Omega}$$

The radiation resistance of the antenna associated with this lowest-order *EM* radiation is:

$$\boxed{R_{rad}=\frac{\mathrm{P}_{rad}}{I_o^2}=\frac{\mu_o\ddot{p}^2(t_o)}{6\pi c I_o^2}}$$

Note that in the above, we deliberately/conciously neglected the electric monopole {E(0)} term in the retarded scalar potential for "far-zone" limit, $r'_{\max}\ll r$:

$$\boxed{V_r^{\mathrm{E}(0)}(\vec{r},t)\simeq\frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int_{v'}\rho(\vec{r}',t_o)\,d\tau'=\frac{1}{4\pi\varepsilon_o}\frac{Q_{tot}(t_0)}{r}}$$

As mentioned previously (P436 Lect. Notes 13, *p*. 4), that because of electric charge conservation, a spherically-symmetric electric monopole moment cannot radiate transversely-polarized *EM* waves – spherical symmetry of the monopole moment restricts oscillations only to the radial direction – thus one could get radiation of one polarization from a certain $d\Omega$ solid angle element, but then radiation from other $d\Omega$'s on the sphere also contribute, such that the ***net*** *EM* radiation from the entire sphere = 0 – total destructive interference. (Gauss' Law - $\int_{S'}\vec{E}\bullet d\vec{a}=Q_{tot}^{encl}/\varepsilon_o$ independent of the size of the spherically symmetric charge distribution enclosed by the surface S'.

Note also that for *EM* radiation, $\vec{B}$ must be $\perp$ to $\vec{E}$, and with both $\vec{E}$ and $\vec{B}$ $\perp$ to $\hat{k}$, the propagation direction. How do you do this for a <u>spherically</u>- <u>symmetric</u> source, where $\hat{k}=\hat{r}$?

Note that if electric charge were <u>***not***</u> conserved, then we <u>***would***</u> get a retarded electric monopole field proportional to $1/r$: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E}(0)}(\vec{r},t)\simeq\frac{1}{4\pi\varepsilon_o c}\frac{\dot{Q}(t_o)}{r}\hat{r}}$ ← *n.b.* this says nothing about the physical size of the spherically-symmetric charge distribution.

Contrast the behavior of transverse waves associated with *EM* radiation from a spherically-symmetric source (an oscillating electric monopole moment) ($\equiv$ no *EM* radiation) to that of longitudinal sound waves / acoustic waves radiated from a spherically symmetric oscillating acoustic monopole sound source – *e.g.* a radially inward / outward oscillating sphere (a breathing bubble) – the latter of which very definitely can propagate / create sound precisely because sound waves are longitudinal, not transverse waves!!

Now think about the electron – for *EM* radiation fields, electric dipole / quadrupole / *etc.* higher *EM* moments *break* the rotational invariance / rotational symmetry associated with the spherical monopole electric charge distribution of the source – thus transverse *EM* waves (*EM* radiation) *can* couple to such electric monopole {E(0)} sources – and also ones that lack rotational invariance!!!

In the above Taylor series expansions for $\rho(\vec{r}',t_r)$ and $\vec{J}(\vec{r}',t_r)$, we only kept terms to first-order in r' in these expansions and demonstrated that the first-order "far-zone" limit radiation terms were associated with the electric dipole moment {E(1)} .

For E(1) electric dipole *EM* radiation to first-order in r' for $r'_{\max} \ll r$ the retarded instantaneous scalar and vector potentials, electric and magnetic fields are:

$$\boxed{V_r^{E(1)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\bullet\dot{\vec{p}}(t_o)}{cr}\right]}$$

$$\boxed{\vec{A}_r^{E(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\dot{\vec{p}}(t_o)}{r}\right]}$$

n.b. proportional to $\dot{\vec{p}}(t_0)$ (first time derivative of $\vec{p}(t_o)$ - "velocity")

$$\boxed{\vec{E}_r^{E(1)}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}(t_o)\right)}{r}\right]}$$

$$\boxed{\vec{B}_r^{E(1)}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\ddot{\vec{p}}(t_o)}{r}\right]}$$

n.b. proportional to $\ddot{\vec{p}}(t_0)$ (second time derivative of $\vec{p}(t_o)$ - "acceleration")

Suppose the (localized) charge / current distributions are such that there is no (time-varying) E(1) electric dipole moment, $\vec{p}(\vec{r}',t_r)=0$ and/or: $\dot{\vec{p}}(\vec{r}',t_r)=0$, $\ddot{\vec{p}}(\vec{r}',t_r)=0$.

Then the Taylor series expansion of $\rho(\vec{r}',t_r)$ and $\vec{J}(\vec{r}',t_r)$ to first order in r' would give nothing for potentials and fields associated with "far-zone" *EM* radiation. However, higher-order terms in these expansions might give rise to non-vanishing potentials and fields.

The second order terms in r' correspond to M(1) magnetic dipole and E(2) electric quadrupole *EM* radiation terms – in order to see/verify this, the second-order contribution needs to be / can be separated out into M(1) and E(2) terms.

Indeed, if we compare *e.g.* the ratio of *EM* power radiated for M(1) magnetic dipole *vs*. E(2) electric quadrupole radiation (in the "far-zone" limit):

$$\boxed{\frac{\langle P^{rad}_{M(1)}\rangle}{\langle P^{rad}_{E(2)}\rangle} \simeq \frac{\left(\dfrac{\mu_o m_o^2 \omega^4}{12\pi c^3}\right)}{\left(\dfrac{\mu_o Q_{zz}^{e^2}\omega^6}{60\pi c^3}\right)}} \quad \text{where:} \quad \boxed{\begin{aligned} &m_o = \pi b^2 I_o = \pi b^2 q\omega \\ &I_o = q\omega \\ &Q_{zz}^e = qdd = \pi^2 b^2 q \\ &d = \pi b \end{aligned}}$$

Thus: $$\boxed{\frac{\langle P^{rad}_{M(1)}\rangle}{\langle P^{rad}_{E(2)}\rangle} \simeq \frac{\left(\dfrac{\cancel{\mu_o}\pi^2 b^4 \cancel{\omega^6} \cancel{q^2}}{\cancel{12}\cancel{\pi}\cancel{c^3}}\right)}{\left(\dfrac{\cancel{\mu_o}\cancel{q^2} d^4 \cancel{\omega^6}}{\cancel{60}_5\cancel{\pi}\cancel{c^3}}\right)} = \frac{5\pi^2 b^4}{d^4} = \frac{5}{\pi^2} \simeq \frac{1}{2} \approx \mathcal{G}(1)}$$

Similarly, the third order terms in r' in the Taylor series expansion of $\rho(\vec{r}',t_r)$ and $\vec{J}(\vec{r}',t_r)$ correspond to M(2) magnetic quadrupole and E(3) electric octupole radiation terms – *i.e.* the third-order contribution needs to be / can be separated out into M(2) and E(3) terms!

Similarly, the fourth order terms in r' in the Taylor series expansion of $\rho(\vec{r}',t_r)$ and $\vec{J}(\vec{r}',t_r)$ correspond to M(3) magnetic octupole and E(4) electric sextupole radiation terms – *i.e.* the fourth-order contribution can be separated out into M(3) and E(4) terms!

And so on, for each successive higher-order term r' in the Taylor series expansion of $\rho(\vec{r}',t_r)$ and/or $\vec{J}(\vec{r}',t_r)$!!!

Griffiths Example 11.2:

a.) An oscillating (*i.e.* harmonically varying) electric dipole has time-dependent dipole moment:

$$\boxed{p(t_r) = p_o\cos(\omega t_r)} \quad \text{where:} \quad \boxed{\vec{p}(t_r) = p(t_r)\hat{z} = p_o\cos(\omega t_r)\hat{z}}$$

$$\boxed{\dot{p}(t_r) = \frac{dp(t_r)}{dt_r} = -\omega p_o\sin(\omega t_r)}$$

$$\boxed{\ddot{p}(t_r) = \frac{d\dot{p}(t_r)}{dt_r} = \frac{d^2p(t_r)}{dt_r^2} = -\omega^2 p_o\cos(\omega t_r)}$$

Then: $\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$ with: $\boxed{t_o \equiv t - r/c}$

$$\boxed{V_r^{E(1)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_o}\left[\frac{\hat{r}\cdot\dot{\vec{p}}(t_o)}{cr}\right] = \frac{-\omega p_o}{4\pi\varepsilon_o}\left[\frac{\hat{r}\cdot\hat{z}}{cr}\right]\sin(\omega t_o) = \frac{-\omega p_o}{4\pi\varepsilon_o cr}\cos\theta\sin(\omega t_o)}$$

And:

$$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\dot{\vec{p}}(t_o)}{r}\right] = -\frac{\mu_o \omega p_o}{4\pi r}\sin(\omega t_o)\,\hat{z}}$$

$$\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$$

$$\boxed{\vec{\mathrm{E}}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times(\hat{r}\times\ddot{\vec{p}}(t_o))}{r}\right] = \frac{\mu_o}{4\pi}\left[\frac{\hat{r}\times(\hat{r}\times\hat{z})}{r}\right](-\omega^2 p_o)\cos(\omega t_o)}$$

$$\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\ddot{\vec{\mathrm{P}}}(t_o)}{r}\right] = -\frac{\mu_o}{4\pi c}\left[\frac{\hat{r}\times\hat{z}}{r}\right](-\omega^2 p_o)\cos(\omega t_o)}$$

But: $\boxed{(\hat{r}\times\hat{z}) = \hat{r}\times(\cancel{\cos\theta\hat{r}} - \sin\theta\hat{\theta}) = -\sin\theta(\hat{r}\times\hat{\theta}) = -\sin\theta\hat{\varphi}}$

And: $\boxed{\hat{r}\times(\hat{r}\times\hat{z}) = (\hat{r}\times\hat{\varphi})*(-\sin\theta) = -\hat{\theta}(-\sin\theta) = +\sin\theta\hat{\theta}}$

Thus:

$$\boxed{V_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq \frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

with: $\boxed{t_o \equiv t - r/c}$

$$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}}$$

where: $\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$

$$\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

$$\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$$

Compare these results for the E(1) electric dipole *EM* radiation "far-zone" limit case with those we obtained P436 Lecture Notes 13 {see pages 8-11}, and/or P436 Lecture Notes 13.5 {the E(1)/M(1) summary / comparison page 11} – they (of course) are identical!

b.) A single, point electric charge q can have (by definition) an electric dipole moment $\vec{p}(t_{\mathrm{r}}) = q\vec{d}(t_{\mathrm{r}})$ where $\vec{d}(t_{\mathrm{r}})$ is the position vector of the point electric charge q at the retarded time t_{r} with respect to the (local) origin ϑ. (*n.b.* subject to all the caveats *r.e.* choice of origin for an EDM having a net charge – see P435 Lecture Notes. . .)

However: $\boxed{\dot{\vec{p}}(t_{\mathrm{r}}) = \frac{d\vec{p}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\frac{d\{\vec{d}(t_{\mathrm{r}})\}}{dt_{\mathrm{r}}} = q\vec{v}(t_{\mathrm{r}})}$

And: $\boxed{\ddot{\vec{p}}(t_r) = \frac{d\dot{\vec{p}}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\frac{d\vec{v}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} = q\vec{a}(t_{\mathrm{r}})}$

n.b. these two quantities do ***not*** depend on the choice of origin !!!

$\vec{v}(t_{\mathrm{r}})$ = velocity vector of point electric charge q at the (retarded) time t_{r}

$\vec{a}(t_{\mathrm{r}})$ = acceleration vector of point electric charge q at the (retarded) time t_{r}

Everything goes through as before – get the same retarded scalar and vector potentials, retarded $\vec{E}$ and $\vec{B}$ fields, u, $\vec{S}$, P, *etc*.

In particular, the radiated *EM* E(1) power associated with a moving point charge q is:

$$\boxed{P_q \simeq \frac{\mu_o \ddot{p}^2(t_o)}{6\pi c}} \text{ (Watts)} \quad \underline{\text{But}}: \quad \boxed{\ddot{p}(t_o) = qa(t_o)}$$

$$\therefore \boxed{P_q \simeq \frac{\mu_o q^2 a^2(t_o)}{6\pi c}}$$ ← Famous Larmour Formula (*EM* power radiated from a point charge q)

Note that the E(1) *EM* power radiated by a point charge q is proportional to the *square* of the acceleration a and also to the *square* of the electric charge q.

This is the origin of statement: "Whenever one accelerates an electric charge q, it radiates away *EM* energy in the form of (real) photons". It is the E(1) electric dipole term which dominates this radiation process.

n.b. This is also true for *decelerating* charged particles – the *time-reversed* situation!!!
$P_q \sim a^2$ ← doesn't care about sign of $\vec{a}$ {The *EM* interaction is time-reversal invariant}!!!

Radiation from accelerated / decelerated $+q$ vs. $-q$ charges is the same if $|+q| = |-q|$.
(P_q doesn't care about the sign of q!)

But: $P_q \sim q^2$ → so if double q → then P_q increases by factor of 4×!

⇒ For the *same* acceleration/deceleration, high-Z nuclei radiate *EM* energy {in the form of photons} much more than *e.g.* a proton (= hydrogen nucleus) – process is known as bremmstrahlung {= "braking radiation", auf Deutsch}.

e.g. Uranium ($Z_u = 92$) gives $92^2 = 8464\times$ more *EM* radiation than a proton for the *same* acceleration, a.

EM Power Radiated by a Moving Point Electric Charge:

The (retarded) electric field of an electric charge q in arbitrary motion is:

$$\boxed{\vec{E}_{\rm r}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]}$$

where: $\boxed{\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}(t_{\rm r})}$

$\boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_{\rm r}) = \vec{r} - \mathrm{w}(t_{\rm r})}$

$\boxed{\mathscr{r} = |\vec{r} - \vec{r}'(t_{\rm r})| = c\Delta t = c(t - t_{\rm r})}$

or: $\boxed{t_{\rm r} = t - \mathscr{r}/c}$.

The associated (retarded) magnetic field is:

$$\boxed{\vec{B}_{\rm r}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}_{\rm r}(\vec{r},t)}$$

As mentioned before, the *first* term in $\vec{E}_{\rm r}(\vec{r},t)$, $\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u}\right]$ is known as the generalized Coulomb field, or velocity field.

The *second* term in $\vec{E}_r(\vec{r},t)$, $\dfrac{q}{4\pi\varepsilon_o}\dfrac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$ is known as the <u>acceleration</u> field (*a.k.a.* the <u>radiation field</u>).

The retarded Poynting's vector is: $\boxed{\vec{S}_r(\vec{r},t)=\dfrac{1}{\mu_o}\left(\vec{E}_r(\vec{r},t)\times\vec{B}_r(\vec{r},t)\right)}$ where: $\boxed{\vec{B}_r(\vec{r},t)=\dfrac{1}{c}\hat{\mathscr{r}}\times\vec{E}_r(\vec{r},t)}$

Use the $\boxed{\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})}$ rule:

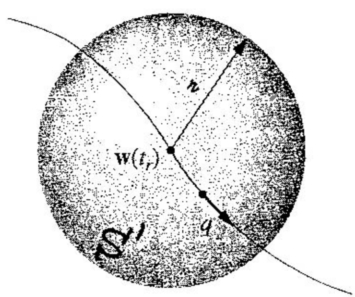
$$\boxed{\vec{S}_r(\vec{r},t)=\frac{1}{\mu_o c}\left[\vec{E}_r(\vec{r},t)\times\left(\hat{\mathscr{r}}\times\vec{E}_r(\vec{r},t)\right)\right]=\frac{1}{\mu_o c}\left[E_r^2(\vec{r},t)\,\hat{\mathscr{r}}-\left(\hat{\mathscr{r}}\cdot\vec{E}_r\right)\vec{E}_r(\vec{r},t)\right]}$$

However, note that not all of this *EM* energy flux constitutes *EM* <u>radiation</u> (<u>real</u> photons) – some of it is still in the form of <u>virtual</u> photons, $\boxed{\vec{S}_r(\vec{r},t)=\vec{S}_r^{virt}(\vec{r},t)+\vec{S}_r^{rad}(\vec{r},t)}$

The metaphor Griffiths uses, that of flies "attached" to a moving garbage truck, is a reasonable picture to imagine here….

- *n.b.* – In order to "detect" the total *EM* power radiated by a moving point charge q, we draw a huge sphere of radius $\mathscr{r}$ centered on the position of the charged particle $\mathrm{w}(t_r)$ at the retarded time $\boxed{t_r=t-\mathscr{r}/c}$ and wait the appropriate time interval $\boxed{\Delta t=t-t_r=\mathscr{r}/c}$ for the *EM* radiation radiated at the retarded time t_r to arrive at the surface of the sphere.

Note that the retarded time t_r is the correct retarded time for <u>all</u> points on the surface of the sphere S'.



- Again, since the area of the sphere, $A_{sphere}=\pi\mathscr{r}^2$ ($\sim\mathscr{r}^2$) then any term in $\vec{S}_r(\vec{r},t)$ that varies as $1/\mathscr{r}^2$ will yield a finite answer for radiated *EM* power, $P_{rad}=\oint_{S'}\vec{S}_r(\vec{r},t)\cdot d\vec{a}_\perp$.
- However, note that terms in $\vec{S}_r(\vec{r},t)$ that vary as $1/\mathscr{r}^3$, $1/\mathscr{r}^4$, $1/\mathscr{r}^5$ … *etc.* will contribute nothing to P_{rad} in the limit $\mathscr{r}\to\infty$.
- For this reason, only the acceleration fields represent true *EM* radiation (<u>real</u> photons) – hence their other name, that of <u>radiation</u> <u>fields</u>:

$$\boxed{\vec{E}_{rad}(\vec{r},t)=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]}$$

The *EM* velocity fields do indeed carry *EM* energy – as the charged particle moves through space-time, this *EM* energy is dragged along with it – but it is ***not*** in the form of *EM* ***radiation***.

Note that $\vec{E}_{rad}(\vec{r},t)$ is $\perp\ \hat{\mathscr{r}}$ (due to the $\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$ term)

$\Rightarrow$ The second term in $\vec{S}_{rad}(\vec{r},t)$ vanishes:

$$\boxed{\vec{S}_{rad}(\vec{r},t)=\frac{1}{\mu_o c}\left[E_{rad}^2(\vec{r},t)\,\hat{\mathscr{r}}-\left(\cancel{\hat{\mathscr{r}}\bullet\vec{E}_{rad}(\vec{r},t)}\right)\vec{E}_{rad}(\vec{r},t)\right]=\frac{1}{\mu_o c}E_{rad}^2(\vec{r},t)\,\hat{\mathscr{r}}}$$

Now if the point charge q *happened* to be at rest ($\vec{v}(t_r)=0$) at the retarded time t_r,

then: $\boxed{\vec{u}(t_r)=c\hat{\mathscr{r}}-\overset{=0}{\cancel{\vec{v}(t_r)}}=c\hat{\mathscr{r}}}$ {*here*}. Then in *this* case:

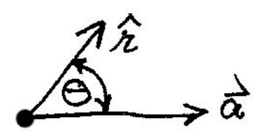
$$\begin{aligned}\vec{E}_{rad}(\vec{r},t)&=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\bullet\vec{u})^3}\left[\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]=\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\bullet c\hat{\mathscr{r}})^3}\left[\vec{\mathscr{r}}\times(c\hat{\mathscr{r}}\times\vec{a})\right]\\&=\frac{q}{4\pi\varepsilon_o c^2}\frac{1}{\mathscr{r}^2}\left[\vec{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})\right]=\frac{q}{4\pi\varepsilon_o c^2}\frac{1}{\mathscr{r}}\left[\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})\right]\\&=\frac{\mu_o q}{4\pi\mathscr{r}}\left[\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})\right]\qquad\text{since }\frac{1}{c^2}=\varepsilon_o\mu_o\\&=\frac{\mu_o q}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\bullet\vec{a})\,\hat{\mathscr{r}}-\underbrace{(\hat{\mathscr{r}}\bullet\hat{\mathscr{r}})}_{=1}\vec{a}\right]=\frac{\mu_o q}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\bullet\vec{a})\,\hat{\mathscr{r}}-\vec{a}\right]\end{aligned}$$

Then {*here*} in this case {$\vec{v}(t_r)=0$}:

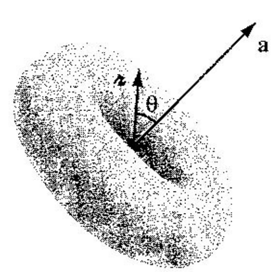
$$\boxed{\vec{S}_{rad}(\vec{r},t)=\frac{1}{\mu_o c}E_{rad}^2(\vec{r},t)\,\hat{\mathscr{r}}=\frac{\mu_o q^2}{4\pi c\,\mathscr{r}^2}\left[a^2-(\hat{\mathscr{r}}\bullet\vec{a})^2\right]\hat{\mathscr{r}}}$$

But: $\hat{\mathscr{r}}\bullet\vec{a}=a\cos\theta$ where θ = opening angle between $\hat{\mathscr{r}}$ and acceleration $\vec{a}$.

$$\therefore\ \boxed{\vec{S}_{rad}(\vec{r},t)=\frac{\mu_o q^2 a^2}{4\pi c\,\mathscr{r}^2}\left[1-\cos^2\theta\right]\hat{\mathscr{r}}=\frac{\mu_o q^2 a^2}{4\pi c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}}}$$



Here again, we see that no power is radiated in the forward-backward directions ($\theta=0$ and $\theta=\pi$) – radiated power is maximum when $\theta=\pi/2=90^o$, *i.e.* when $\hat{\mathscr{r}}\perp\vec{a}$ - get a donut-shaped intensity pattern about the instantaneous acceleration vector $\vec{a}(t_r)$:



The power radiated by this point charge (which is instantaneously at rest at time t_r) is:

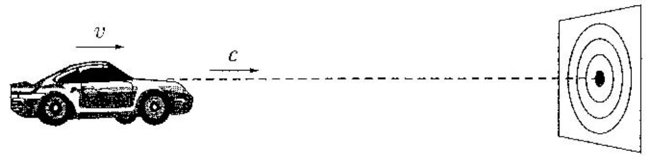
$$\boxed{\begin{aligned} P_{rad}(t) &= \oint_{S'} \vec{S}_{rad}(\vec{r},t)\cdot d\vec{a}_\perp = \frac{\mu_o q^2 a^2(t_r)}{16\pi^2 c}\int \frac{\sin^2\theta}{\cancel{r^2}}\cancel{r^2}\sin\theta\, d\theta\, d\varphi \\ &= \frac{\mu_o q^2 a^2(t_r)}{\cancel{16}_{8}\,\cancel{\pi^2}\,c}\cdot\cancel{2\pi}\underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{=4/3} = \frac{\mu_o q^2 a^2(t_r)}{\cancel{8}_{2}\,\pi}\cdot\frac{\cancel{4}}{3} = \frac{\mu_o q^2 a^2(t_r)}{6\pi c} \end{aligned}}$$

$$\boxed{P_{rad}(t) = \frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \Leftarrow \text{ Larmour Power Formula \{again\} !!!}$$

This formula was derived assuming $\vec{v}(t_r) = 0$, but in fact, we get the ***same*** formula as long as $v(t_r) \ll c$ (non-relativistic motion).

- An exact treatment of $\vec{v}(t_r) \neq 0$ is (much) more difficult / tedious.
- Note that in special relativity {inertial (non-accelerated) reference frames}, the choice $\vec{v}(t_r) = 0$ merely represents a judicious choice of an (inertial) reference frame, with ***no*** loss of generality.
- If we can determine how $P_{rad}(t)$ transforms from one reference frame to another, then we can deduce the more general $\vec{v}(t_r) \neq 0$ result (Liénard) from the (Larmour) $\vec{v}(t_r) = 0$ result. (See *e.g.* Griffiths problem 12.69, *p*. 545).
- For the case $\vec{v}(t_r) \neq 0$, $\vec{E}_{rad}(\vec{r},t)$ is more complicated (than the $\vec{v}(t_r) = 0$ case).
- For the case $\vec{v}(t_r) \neq 0$, $\vec{S}_{rad}(\vec{r},t)$ = the rate of energy passing through the (imaginary) large-radius surface S' of the sphere, $\vec{S}_{rad}(\vec{r},t)$ is NOT the same as the rate of energy when it left the charged particle, at the retarded time t_r.

Consider the example of a person firing a stream of bullets (photons) out the window of a moving car, parallel to the direction of motion of the car:



The rate at which the bullets strike a target, R_{tgt} (#/sec) is ***not*** the same as the rate of bullets leaving the gun, R_{gun} (#/sec) because of the relative motion of the car with respect to the target. This is again related to the Doppler effect. It is purely a geometrical factor (*i.e.* it is ***not*** due to special relativity). For bullets moving ***parallel*** to the car's velocity vector:

$$\boxed{R_{gun} = \left(1-\beta(t_r)\right)R_{tgt}} \quad \text{or:} \quad \boxed{R_{tgt} = \frac{1}{1-\beta(t_r)}R_{gun}} \quad \text{where:} \quad \boxed{\beta(t_r) \equiv v(t_r)/c}$$

Whereas for bullets moving ***anti-parallel*** to the car's velocity vector:

$$\boxed{R_{gun} = \left(1+\beta(t_r)\right) R_{tgt}} \quad \underline{\text{or}}: \quad \boxed{R_{tgt} = \frac{1}{1+\beta(t_r)} R_{gun}} \quad \underline{\text{where}}: \quad \boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$$

And for ***arbitrary*** directions, with $\hat{\mathscr{r}} \equiv$ unit vector from car to target:

$$\boxed{R_{gun} = \left(1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)\right) R_{tgt}} \quad \underline{\text{or}}: \quad \boxed{R_{tgt} = \frac{1}{\left(1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)\right)} R_{gun}} \quad \underline{\text{where}}: \quad \boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$$

So if $\dfrac{dW}{dt}$ = rate of energy passing through sphere of radius $\mathscr{r}$ then the rate at which energy leaves the charge q is: $\boxed{\dfrac{dW}{dt_r} = \dfrac{dW}{dt}\bullet\dfrac{dt}{dt_r} = \dfrac{dW}{dt}\Big/\dfrac{\partial t_r}{\partial t} = \left(\dfrac{\vec{\mathscr{r}}\bullet\vec{u}}{c}\right)\dfrac{dW}{dt}}$ $\underline{\text{since}}$: $\boxed{\dfrac{\partial t_r}{\partial t} = \dfrac{c}{\hat{\mathscr{r}}\bullet\vec{u}} = \dfrac{\mathscr{r}c}{\vec{\mathscr{r}}\bullet\vec{u}}}$ with $\boxed{\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}(t_r)}$.

(see P436 Lect. Notes 12, *p*. 14-15, and/or Griffiths problem 10.17, *p*. 441)

$\underline{\text{But}}$: $\boxed{\dfrac{\hat{\mathscr{r}}\bullet\vec{u}}{c} = \dfrac{\hat{\mathscr{r}}\bullet\left(c\hat{\mathscr{r}} - \vec{v}(t_r)\right)}{c} = 1-\hat{\mathscr{r}}\bullet\vec{v}(t_r)/c = 1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r) \equiv \kappa}$ = ***retardation factor*** $\boxed{\vec{\beta}(t_r) \equiv \dfrac{\vec{v}(t_r)}{c}}$

$\underline{\text{Then}}$: $\boxed{\dfrac{dW}{dt_r} = \left(\dfrac{\hat{\mathscr{r}}\bullet\vec{u}}{c}\right)\dfrac{dW}{dt} = \left(1-\hat{\mathscr{r}}\bullet\vec{\beta}(t_r)\right)\dfrac{dW}{dt} = \kappa\dfrac{dW}{dt}}$

Thus, the power radiated into a patch of area $da = \mathscr{r}^2 \sin\theta d\theta d\varphi = \mathscr{r}^2 d\Omega$ on the sphere S', where $d\Omega = \sin\theta d\theta d\varphi$ = solid angle into which the *EM* power is radiated into area element da on the surface of the sphere S', with $\boxed{\vec{E}_{rad}(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_o}\dfrac{\mathscr{r}}{\left(\vec{\mathscr{r}}\bullet\vec{u}\right)^3}\left[\vec{\mathscr{r}}\times\left(\vec{u}\times\vec{a}\right)\right]}$ is given by:

$$\begin{aligned}
\frac{dP_{rad}(t_r)}{d\Omega} &= \left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_r)}{c}\right) S_{rad}(\vec{r},t) * \mathscr{r}^2 = \left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_r)}{c}\right)\frac{1}{\mu_o c}E_{rad}^2 \mathscr{r}^2 \\
&= \frac{q^2}{16\pi^2\varepsilon_o^2}\frac{1}{\mu_o c}\left(\frac{\hat{\mathscr{r}}\bullet\vec{u}(t_r)}{c}\right)\frac{\mathscr{r}^2}{\left(\vec{\mathscr{r}}\bullet\vec{u}(t_r)\right)^6}\left|\vec{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right|^2 \\
&= \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{\mathscr{r}}\bullet\vec{u}(t_r)\right)^5}
\end{aligned}$$

$\underline{\text{Thus}}$: $$\boxed{\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{\mathscr{r}}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{\mathscr{r}}\bullet\vec{u}(t_r)\right)^5}}$$

Integrating $\int_{S'} \frac{dP_{rad}(t)}{d\Omega} d\Omega$ over the sphere S' (*i.e.* over θ and φ angles) is a pain….
However, the result of this integration {again!} yields the famous Liénard Formula:

$$\boxed{P_{rad}(t) = \frac{\mu_o q^2}{6\pi c}\gamma^6\left(a^2(t_r) - \left|\frac{\vec{v}(t_r)\times\vec{a}(t_r)}{c}\right|^2\right) = \frac{\mu_o q^2}{6\pi c}\gamma^6\left(a^2(t_r) - \left|\vec{\beta}(t_r)\times\vec{a}(t_r)\right|^2\right)}$$

Where: $\boxed{\vec{\beta}(t_r) \equiv \vec{v}(t_r)/c}$ and $\boxed{\gamma(t_r) \equiv \frac{1}{\sqrt{1-\beta^2(t_r)}}}$ = Lorentz Factor.

$\boxed{0 \le \beta \le 1}$ $\boxed{1 \le \gamma \le \infty}$

Note that when $v \to c$, the γ^6 factor goes "berserk" – as the charged particle travels closer and closer to the speed of light c, the more one tries to accelerate it (to make it travel even closer to the speed of light, c), it radiates away more and more of the (absorbed) energy as $v \to c$!!!

$\Rightarrow$ *very* high energy electron accelerators are problematic in this regard, because the electron is so light, mass-wise, *e.g.* relative to the proton: $m_e = 0.511\,MeV/c^2$ whereas $m_p = 938.28\,MeV/c^2$.

Griffiths Example 11.3:

Suppose $\vec{v}(t_r)$ and $\vec{a}(t_r)$ are instantaneously collinear (*i.e.* parallel to each other). Find the angular distribution of radiated power $\frac{dP_{rad}(t)}{d\Omega}$ when $\boxed{\vec{v}(t_r)\cdot\vec{a}(t_r) = v(t_r)\,a(t_r)}$ (*i.e.* when $\vec{v}(t_r)\parallel\vec{a}(t_r)$)

Then in this case: $\boxed{\vec{u}(t_r) = \left(c\hat{r} - \vec{v}(t_r)\right)}$ because $\vec{v}(t_r)\parallel\vec{a}(t_r)$

Thus: $\boxed{\vec{u}(t_r)\times\vec{a}(t_r) = \left(c\hat{r} - \vec{v}(t_r)\right)\times\vec{a}(t_r) = c\hat{r}\times\vec{a}(t_r) - \overbrace{\cancel{\vec{v}(t_r)\times\vec{a}(t_r)}}^{=0} = c\hat{r}\times\vec{a}(t_r)}$

Then: $\boxed{\frac{dP_{rad}(t)}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{r}\times\left(\vec{u}(t_r)\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{r}\cdot\vec{u}(t_r)\right)^5} = \frac{q^2c^2}{16\pi^2\varepsilon_o}\frac{\left|\hat{r}\times\left(\hat{r}\times\vec{a}(t_r)\right)\right|^2}{\left(\hat{r}\cdot\vec{u}(t_r)\right)^5}}$ $\boxed{\vec{\beta}(t_r) \equiv \frac{\vec{v}(t_r)}{c}}$

Work on denominator term: $\boxed{\hat{r}\cdot\vec{u}(t_r) = \hat{r}\cdot\left(c\hat{r} - \vec{v}(t_r)\right) = c - \hat{r}\cdot\vec{v}(t_r) = c\left(1 - \hat{r}\cdot\vec{\beta}(t_r)\right) = \kappa c}$ $\boxed{\kappa \equiv 1 - \hat{r}\cdot\vec{\beta}(t_r)}$

Work on numerator term: $\boxed{\hat{r}\times\left(\hat{r}\times\vec{a}(t_r)\right) = \left(\hat{r}\cdot\vec{a}(t_r)\right)\hat{r} - \overbrace{\left(\hat{r}\cdot\hat{r}\right)}^{=1}\vec{a}(t_r) = \left(\hat{r}\cdot\vec{a}(t_r)\right)\hat{r} - \vec{a}(t_r)}$

Thus: $\boxed{\left|\hat{r}\times\left(\hat{r}\times\vec{a}(t_r)\right)\right|^2 = a^2(t_r) - \left(\hat{r}\cdot\vec{a}(t_r)\right)^2}$

Then: $$\boxed{\frac{dP_{rad}(t)}{d\Omega}=\frac{q^2c^2}{16\pi^2\varepsilon_o}\frac{\left[a^2(t_r)-\left(\hat{r}\bullet\vec{a}(t_r)\right)^2\right]}{c^5\left(1-\hat{r}\bullet\vec{\beta}(t_r)\right)^5}=\frac{q^2}{16\pi^2\varepsilon_o c^3}\frac{\left[a^2(t_r)-\left(\hat{r}\bullet\vec{a}(t_r)\right)^2\right]}{\left(1-\hat{r}\bullet\vec{\beta}(t_r)\right)^5}}$$

If we let the $\hat{z}$-axis point along $\vec{v}(t_r)$ – along $\vec{\beta}(t_r)=\vec{v}(t_r)/c$ {and hence also along $\vec{a}(t_r)$}:

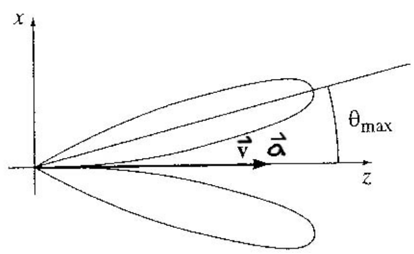
Then: $\boxed{\hat{r}\bullet\vec{a}=a\cos\theta}$ and: $\boxed{\hat{r}\bullet\vec{v}=v\cos\theta}$ or: $\boxed{\hat{r}\bullet\vec{\beta}=\beta\cos\theta}$ where θ = opening angle between $\hat{r}$ and acceleration $\vec{a}$, as shown on page 12 above.

Thus: $$\boxed{\frac{dP_{rad}(t)}{d\Omega}=\frac{q^2a^2(t_r)}{16\pi^2\varepsilon_o c^3}\frac{\left(1-\cos^2\theta\right)}{\left(1-\beta(t_r)\cos\theta\right)^5}}\quad\text{but:}\quad\boxed{\frac{1}{c^2}=\varepsilon_o\mu_o}$$

$$\therefore\quad\boxed{\frac{dP_{rad}(t)}{d\Omega}=\frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}}\quad\text{with:}\quad\boxed{\beta(t_r)=\frac{v(t_r)}{c}}$$

When $\beta\to 0$: $$\boxed{\frac{dP_{rad}(t)}{d\Omega}=\frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\sin^2\theta=\vec{S}_{rad}^{\vec{v}=0}(\vec{r},t)\bullet r^2\hat{r}}$$

When $\beta\to 1$: The donut of *EM* radiation intensity is *folded* ***forward*** by the factor $1/\left(1-\beta\cos\theta\right)^5$:



Note that there is still ***no*** radiation *precisely* in the forward direction, rather it is in a *cone* which becomes increasingly narrow as $\beta\to 1$, of half-angle:

$$\boxed{\theta_{\max}\approx\sqrt{(1-\beta)/2}}\quad\text{\{see Griffiths problem 11.15, p. 465\}}$$

The total *EM* power radiated into 4π steradians by the point charge for $\vec{v}\parallel\vec{a}$ is:

$$\boxed{\begin{aligned}P_{rad}(t)&=\int\frac{dP_{rad}(t)}{d\Omega}d\Omega=\frac{\mu_o q^2a^2(t_r)}{16\pi^2 c}\int\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}\sin\theta d\theta d\varphi\\&=\frac{\mu_o q^2a^2(t_r)}{8\pi c}\int_{\theta=0}^{\theta=\pi}\frac{\sin^2\theta}{\left(1-\beta(t_r)\cos\theta\right)^5}\sin\theta d\theta\end{aligned}}$$

Let: $u=\cos\theta$ $\quad\theta=0\to u=+1$
$du=-\sin\theta d\theta$ $\quad\theta=\pi\to u=-1$

Then: $\boxed{P_{rad}(t) = \frac{\mu_o q^2 a^2(t_r)}{8\pi c}\int_{-1}^{1}\frac{(1-u^2)}{(1-\beta u)^5}\,du}$ Integrate by parts: $\boxed{\int v\,du = uv - \int u\,dv}$

$$\boxed{P_{rad}(t) = \frac{\mu_o q^2 a^2(t_r)}{8\pi c}\left[\frac{4}{3}\left(1-\beta^2(t_r)\right)^{-3}\right] = \frac{\mu_o q^2 a^2(t_r)}{6\pi c}\left[\frac{1}{\left(1-\beta^2(t_r)\right)^3}\right]}$$

But: $\boxed{\gamma(t_r) \equiv \frac{1}{\sqrt{1-\beta^2(t_r)}}}$ with: $\boxed{\vec{\beta}(t_r) \equiv \vec{v}(t_r)/c}$

$$\therefore \quad \boxed{P_{rad}(t) = \frac{\mu_o q^2 a^2(t_r)}{6\pi c}\gamma^6(t_r)}$$

This is the same/identical result as obtained *directly* from the Liénard formula when $\vec{v}(t_r) \parallel \vec{a}(t_r)$. It is also known as the classical formula for bremmstrahlung.

Again, note that because $P_{rad}(t) \sim a^2(t_r)$, the *EM* power radiated doesn't depend on the ***sign*** of $\vec{a}(t_r)$ – *i.e.* whether the charged particle is ***accelerating*** or ***decelerating***.

Now it can also be shown that the Lorentz factor $\boxed{\gamma = E/mc^2}$, where $\boxed{E = \sqrt{(pc)^2 + (mc^2)^2}}$ = total relativistic energy associated with a charged particle moving with $\vec{\beta}(t_r) \equiv \vec{v}(t_r)/c$.
Thus, when $v \to c$, for a given {high} total energy *E*, then $\boxed{\gamma \sim 1/mc^2}$ and thus: $\boxed{P_{rad}(t) \sim 1/m^6}$.

Comparing *EM* bremmstrahlung radiation from an accelerated electron { $m_e = 0.511\,MeV/c^2$ } *vs*. that of *e.g*. an accelerated muon { $m_\mu = 105.66\,MeV/c^2$ }, for the same total energy *E*, an electron will radiate $(m_\mu/m_e)^6 \simeq (206.8)^6 = 7.8\times 10^{13}$ times more *EM* energy than a muon.
This explains why muons have such high penetrating power in traversing matter – they lose relatively little energy via bremmstrahlung, whereas high-energy electrons radiate *EM* energy "like crazy" in matter.

The Radiation Reaction on a Radiating Charged Particle

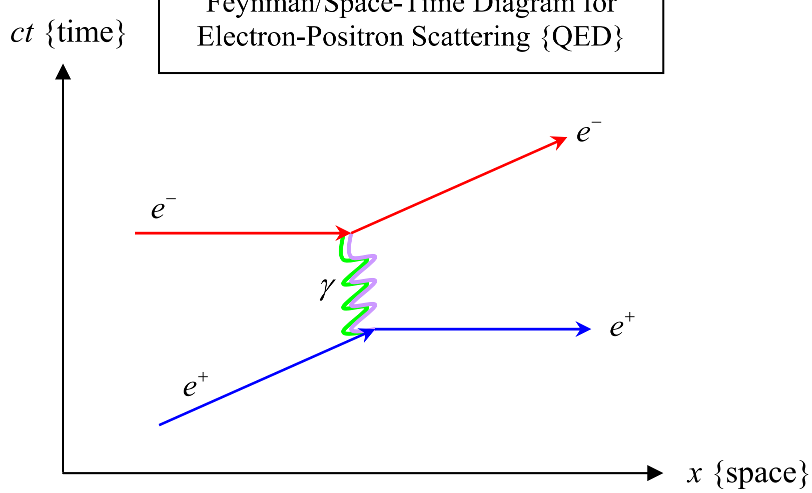
According to the laws of classical electrodynamics, an accelerating electric charge radiates electromagnetic energy in the form of real photons (= quanta of the *EM* radiation field).

By conservation of energy, the *EM* radiation carries off / carries away energy – which must come at the expense of the charged particle's *kinetic* energy {since its rest mass cannot change}.

In other words, one puts in energy to accelerate the charged particle, but the charged particle winds up being accelerated *less* than *e.g.* an electrically neutral particle {of the same rest mass of the charged particle}, for the same amount of input energy!

The devil is in the *microscopic* details of precisely how this is accomplished in both cases. At the microscopic level, an electrically charged particle of mass *m* is accelerated/increases its {*kinetic*} energy $\boxed{T = (\gamma - 1)mc^2}$ by absorbing *EM* energy (either in the form of virtual or real photons) from a source of *EM* field(s). In order to accelerate/increase the {*kinetic*} energy $\boxed{T = (\gamma - 1)mc^2}$ of an electrically neutral particle, it too must interact, at the microscopic level, via one of the four fundamental forces of nature, with a source {of fields} associated with that fundamental force.

Feynman/Space-Time Diagram for Electron-Positron Scattering {QED}



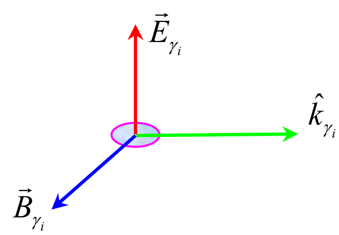
In the electromagnetic case, if an electrically charged particle is decelerated and radiates *EM* energy away in the form of {real} photons, by energy conservation, the change in the kinetic energy of the charged particle ***must*** equal the sum of the energies associated with each of the *n* individual {real} photons radiated by the charged particle, :

$$\boxed{\Delta KE_q = \sum_{i=1}^{n} E_{\gamma_i} = \sum_{i=1}^{n} hf_{\gamma_i}}$$

This implies that the radiation must {somehow!} exert a force, $\vec{F}_{rad}$ ***back*** on the electrically charged particle – *i.e.* a recoil force, analogous to that associated with firing a bullet from a gun. Thus, linear momentum p must ***also*** conserved in this process. In the emission of *EM* radiation {real photons}, linear momentum $\boxed{p_{\gamma_i} = h/\lambda_{\gamma_i} = hf_{\gamma_i}/c}$ is also carried away by each of the {real} photons. This comes at the expense of the charged particle's momentum $\vec{p}_q$ and (non-relativistically, for $v_q \ll c$): $\boxed{KE_q = p_q^2/2m}$

$$\boxed{\Delta\vec{p}_q^{\,recoil} c = \sum_{i=1}^{n} \vec{p}_{\gamma_i} c = \sum_{i=1}^{n} \frac{h}{\lambda_{\gamma_i}} c\hat{k}_{\gamma_i} = \sum_{i=1}^{n} hf_{\gamma_i}\hat{k}_{\gamma_i}}$$

$\hat{k}_{\gamma_i}$ = wave vector for the i^{th} photon



Thus, if a similarly accelerated/decelerated neutral particle doesn't radiate force quanta {of some kind} because it is accelerated/decelerated, then because the electrically-charged particle does radiate *EM* quanta {real photons} in the acceleration/deceleration process, then we can see that the final-state $KE_q < KE_o$ for similarly accelerated / decelerated neutral particle of the same mass *m* and initial/original kinetic energy as that of the electrically-charged particle.

The Radiation Reaction Force on a Charged Particle

For a non-relativistic particle ($v_q \ll c$) the Larmour Formula for the total instantaneous *EM* radiated power is:

$$\boxed{P_{rad}(t) \simeq \frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \quad \text{(Watts)}$$

Conservation of energy would then imply that this radiated *EM* power = the instantaneous rate at which the charged particle loses energy, due to the effect of the *EM* radiation back-reaction / recoil force $\vec{F}_{rad}(t_r)$:

$$\boxed{P_q(t_r) = \frac{dW(t_r)}{dt} = \vec{F}_{rad}(t_r)\bullet\vec{v}(t_r) = -\frac{\mu_o q^2 a^2(t_r)}{6\pi c}} \quad \text{(Watts)}$$

This relation / equation is actually ***wrong***. Why???

The reason is, that we calculated the radiated *EM* power by integrating Poynting's vector $\vec{S}_{rad}(\vec{r},t)$ for the *EM* radiation associated with the accelerating point charged particle over an "infinite" sphere of radius r; in this calculation the *EM* velocity fields played *no* role, since they fall off too rapidly as a function of r to make any contribution to $P_{rad}(t)$. However, the *EM* velocity fields *do* carry energy – because the total retarded electric field associated with the electrically charged particle is the sum of *two* terms – the *EM* velocity field *and* the *EM* acceleration field terms:

$$\boxed{\vec{E}_r^{tot}(\vec{r},t) = \vec{E}_r^{v}(\vec{r},t) + \vec{E}_r^{a}(\vec{r},t)}$$

The total retarded *EM* energy density associated with the total retarded electric field is:

$$\boxed{\begin{aligned} u_{\mathrm{E}}^{tot}(\vec{r},t) &= \tfrac{1}{2}\varepsilon_o E_{\mathrm{r}}^{tot^2}(\vec{r},t) = \tfrac{1}{2}\varepsilon_o\left(\vec{E}_{\mathrm{r}}^{v}(\vec{r},t)+\vec{E}_{\mathrm{r}}^{a}(\vec{r},t)\right)^2 \\ &= \tfrac{1}{2}\varepsilon_o\left[E_{\mathrm{r}}^{v^2}(\vec{r},t)+2\vec{E}_{\mathrm{r}}^{v}(\vec{r},t)\bullet\vec{E}_{\mathrm{r}}^{a}(\vec{r},t)+E_{\mathrm{r}}^{a^2}(\vec{r},t)\right]\end{aligned}}$$

Energy stored in velocity field only (virtual photons)

Generalized Coulomb fields only

Cross term!!! Energy stored in mixture of velocity and acceleration field (both virtual & real photons!!)

"Conversion" field virtual → real photons {and vice versa!}

Energy stored in acceleration field only (real photons)

Radiation fields only

Note that:

The Generalized Coulomb fields vary as $\sim 1/r^4$

The "Conversion" fields vary as $\sim 1/r^3$

The Radiation fields vary as $\sim 1/r^2$

Neither the Generalized Coulomb field nor the "Conversion" field contribute to *EM* radiation in the "far-zone" limit $r' \ll r$

Clearly, the first two terms in the *EM* energy density formula associated with the electric field have energy associated with them. However, this energy stays with the charged particle – it is not radiated away.

As the charged particle accelerates / decelerates, energy is exchanged between the charged particle and the velocity and acceleration fields. For the latter term (the last/ 3rd term in $u_{\mathrm{E}}^{tot}(\vec{r},t)$ above), this energy is irretrievably carried away (by real photons) out to $r = \infty$.

Thus, $\boxed{P_q(t_{\mathrm{r}}) = \dfrac{dW(t_{\mathrm{r}})}{dt} = \vec{F}_{rad}(t_{\mathrm{r}})\bullet\vec{v}(t_{\mathrm{r}}) = -\dfrac{\mu_o q^2 a^2(t_{\mathrm{r}})}{6\pi c}}$ only accounts for the last / 3rd term ($E_{\mathrm{r}}^{a^2}(\vec{r},t)$) in $u_{\mathrm{E}}^{tot}(\vec{r},t)$ above.

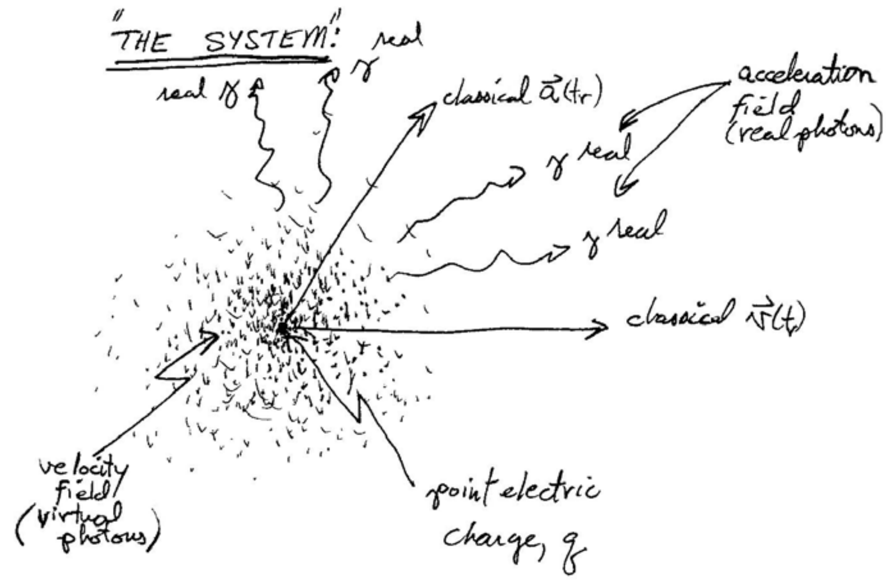
If we want to know the ***total*** recoil force exerted by the *EM* velocity ***and*** the *EM* acceleration fields on the point charge, then we need to know the ***total*** instantaneous power lost, not ***just*** the ***radiation-only*** contribution.

Thus, in this sense, the term "radiation" (back)-reaction is a misnomer because it should more appropriately be called an *EM* field (back)-reaction. Note further that this *EM* field (back)-reaction is also intimately connected with the issue of the so-called "hidden" *EM* momentum.

Shortly, we'll see that $\vec{F}_{rad}(t_{\mathrm{r}})$ is determined by the time derivative of the acceleration $\vec{a}(t_{\mathrm{r}})$, and can be non-zero even when the acceleration $\vec{a}(t_{\mathrm{r}})$ is instantaneously zero!
(Thus the charged particle is ***not*** radiating at that {retarded} instant in time!)

By energy conservation, the energy ***lost*** by the electrically charged particle in a given (retarded) time interval $\Delta t_r \equiv t_{r_2} - t_{r_1}$ $\left(t_{r_2} > t_{r_1}\right)$ ***must*** equal the energy carried away by the *EM* radiation, plus whatever extra energy has been pumped into the *EM* velocity/generalized Coulomb field.

If we consider time intervals $\Delta t_r = t_{r_2} - t_{r_1}$ such that "the system" (consisting of the point-charged particle *q* and the *EM* velocity field – see drawing on following page) returns to its initial state, then (assuming that the energy in the *EM* velocity fields is the same at time t_{r_2} as at time t_{r_1}, then the only net energy loss *is* in the form of *EM* radiation (due to the emission of *n* real photons).



Thus, while instantaneously $\boxed{P_q(t_r) = \frac{dW(t_r)}{dt} = \vec{F}_{rad}(t_r) \bullet \vec{v}(t_r) = -\frac{\mu_o q^2 a^2(t_r)}{6\pi c}}$ ***is*** incorrect, by suitably ***averaging*** this relation over a finite time interval, it ***is*** valid, with the restriction that state of "the system" is ***identical*** at the {retarded} times t_{r_1} and t_{r_2}:

$$\boxed{\frac{1}{\Delta t_r}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \vec{F}_{rad}(t'_r) \bullet \vec{v}(t'_r)\, dt'_r = -\frac{1}{\Delta t_r}\frac{\mu_o q^2}{6\pi c}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} a^2(t'_r)\, dt'_r}$$

For the case of periodic / harmonic motion, this means that the above integrals must be carried out over at least one (or more) complete / full cycles, $\Delta t_r \equiv t_{r_2} - t_{r_1} = n\,\tau$, $n = 1, 2, 3, \ldots$

For non-periodic motion, the condition that "the system" be identical at times t_{r_1} and t_{r_2} is more difficult to achieve – it is ***not*** enough that the instantaneous velocities and accelerations be equal at t_{r_1} and t_{r_2}, since the (retarded) fields farther out (at the {present} time $t = t_r + r/c$) depend on $\vec{v}(t_r)$ and $\vec{a}(t_r)$ at the earlier {retarded} time t_r !!!

For non-periodic motion, the condition that "the system" be identical at times t_{r_1} and t_{r_2} technically requires that not only $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$, but *all* higher derivatives of $\vec{v}(t_r)$ must ***also*** likewise be equal at times t_{r_1} and t_{r_2} !!!

However, in practice, for non-periodic motion, since the *EM* velocity fields fall off rapidly with r, it is *sufficient* that $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$, for a *brief* time interval, $\Delta t_r = t_{r_2} - t_{r_1}$.

The RHS of the above equation can be integrated by parts:

$$\boxed{\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} a^2(t'_r)\,dt'_r = \int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\frac{d\vec{v}(t'_r)}{dt_r}\right)\left(\frac{d\vec{v}(t'_r)}{dt_r}\right)dt'_r = \left(\vec{v}(t'_r)\bullet\frac{d\vec{v}(t'_r)}{dt_r}\right)\Bigg|_{t_{r_1}}^{t_{r_2}} - \int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \underbrace{\frac{d^2\vec{v}(t'_r)}{dt_r^2}}_{\equiv\dot{\vec{a}}(t'_r)}\bullet\vec{v}(t'_r)\,dt'_r}$$

Because of the restriction on $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$ at the time endpoints t_{r_1} and t_{r_2},

The term: $$\boxed{\left(\vec{v}(t'_r)\bullet\frac{d\vec{v}(t'_r)}{dt_r}\right)\Bigg|_{t_{r_1}}^{t_{r_2}} = \vec{v}(t'_r)\bullet\vec{a}(t'_r)\Big|_{t_{r_1}}^{t_{r_2}} = 0}$$

Thus: $$\boxed{\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\vec{F}_{rad}(t'_r)\bullet\vec{v}(t'_r)\right)dt'_r = +\frac{\mu_o q^2}{6\pi c}\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\dot{\vec{a}}(t'_r)\bullet\vec{v}(t'_r)\right)dt'_r}$$

Or: $$\boxed{\int_{t'_r=t_{r_1}}^{t'_r=t_{r_2}} \left(\vec{F}_{rad}(t'_r) - \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)\right)\bullet\vec{v}(t'_r)\,dt'_r = 0}$$

Mathematically, there are lots of ways this integral equation can be satisfied, but it will certainly be satisfied if:

$$\boxed{\vec{F}_{rad}(t'_r) = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)} \Leftarrow \text{Abraham-Lorentz formula}$$

This relation is known as the Abraham-Lorentz formula for the *EM* "radiation reaction" force. $\boxed{\vec{F}_{rad}(t'_r) = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t'_r)}$ is the *simplest* possible form the *EM* radiation reaction force can take.

Physically, note that this formula tells us only about the time-averaged force {albeit} over a very brief time interval $\Delta t_r = t_{r_2} - t_{r_1}$, of the force component parallel to $\vec{v}(t_r)$ - because of the original term $\left(\vec{F}_{rad}(t'_r)\bullet\vec{v}(t'_r)\right)$. As such, it tells us ***nothing*** about $\vec{F}_{rad_\perp}(t_r)$ $\perp$ to $\vec{v}(t_r)$.

n.b. These averages are also restricted to time intervals such that $\Delta t_r = t_{r_2} - t_{r_1}$ is chosen to ensure that $\vec{v}(t_{r_1}) = \vec{v}(t_{r_2})$ and $\vec{a}(t_{r_1}) = \vec{a}(t_{r_2})$.

The Abraham-Lorentz radiation reaction force $\vec{F}_{rad}\left(t_r'\right)=\dfrac{\mu_o q^2}{6\pi c}\dot{\vec{a}}\left(t_r'\right)$ also has disturbing, seemingly unphysical implications that are still not fully understood today, despite the passage of nearly a century!

Suppose a charged particle is subject to <u>*NO*</u> external forces. Then Newton's 2[nd] law says that:

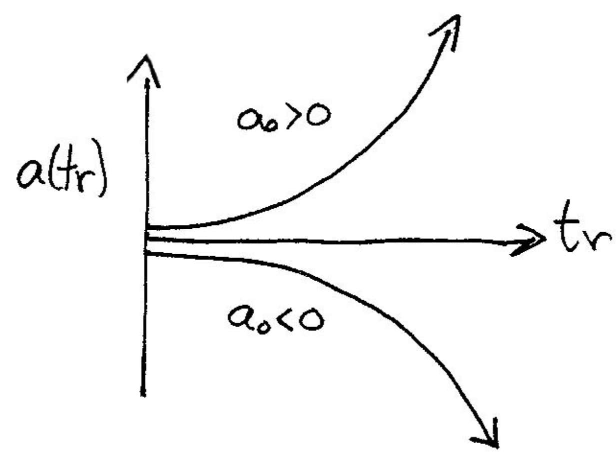
$$\boxed{\vec{F}_{rad}\left(t_r'\right)=\frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}\left(t_r'\right)=m\vec{a}\left(t_r\right)}$$ where m = (real) rest mass of the charged particle.

<u>Then</u>: $$\boxed{\vec{F}_{rad}\left(t_r'\right)=\frac{\mu_o q^2}{6\pi mc}m\dot{\vec{a}}\left(t_r'\right)=m\tau\dot{\vec{a}}\left(t_r'\right)=m\vec{a}\left(t_r\right)}$$ <u>or</u>: $$\boxed{\frac{\mu_o q^2}{6\pi mc}\dot{\vec{a}}\left(t_r\right)=\tau\dot{\vec{a}}\left(t_r\right)=\vec{a}\left(t_r\right)}$$

The solution to this linear, first-order homogeneous differential equation is: $\boxed{a\left(t_r\right)=a_o e^{+t_r/\tau}}$

where a_o = acceleration at the {retarded} zero of time, $t_r = 0$, and $\boxed{\tau\equiv\left(\dfrac{\mu_o q^2}{6\pi mc}\right)}$, which for the electron is a time constant of : $\boxed{\tau_e\simeq 6\times10^{-24}\,\text{sec}}$.

If $a_o \neq 0$, the acceleration exponentially increases ($+ve$, if $a_o > 0$, $-ve$, if $a_o < 0$) as time progresses! This is a runaway solution, which is *CRAZY* !!! This can only be avoided if $a_o \equiv 0$.



However, if the runaway solutions are <u>excluded</u> on physical grounds, then the charged particle develops an <u>acausal</u> behavior – *e.g.* if an external force is applied, the charged particle responds <u>before</u> the force acts!! This acausal "pre-acceleration" "jumps the gun" by only a short time $\tau_e \simeq 6\times10^{-24}$ sec, and since we know that quantum mechanics and uncertainty principle are operative on short distance/short timescales, perhaps this classical behavior shouldn't be <u>too</u> unsettling to us. Nevertheless, to many it is…. (see Griffiths Problem 11.19, *p*. 469 for more aspects/ramifications of the Abraham-Lorentz formula…)

Such difficulties also persist in the <u>fully-relativistic</u> version of the Abraham-Lorentz equation.

<u>Griffiths Example 11.4 – *EM* Radiation Damping</u>:

Calculate the *EM* radiation damping of an electrically charged particle attached to a spring of natural angular frequency ω_o with driving frequency = ω

The 1-dimensional equation of motion is:

$$\boxed{\begin{aligned} m\ddot{x}\left(t_r\right) &= F_{spring}\left(t_r\right)+F_{rad}\left(t_r\right)+F_{driving}\left(t_r\right)\\ &= -m\omega_o^2 x\left(t_r\right)+m\tau\dddot{x}\left(t_r\right)+F_{driving}\left(t_r\right)\end{aligned}}$$

With the system oscillating at the driving frequency ω:

Instantaneous position: $\boxed{x(t_r) = x_o \cos(\omega t_r + \delta)}$

Instantaneous velocity: $\boxed{\dot{x}(t_r) = -\omega x_o \sin(\omega t_r + \delta)}$

Instantaneous acceleration: $\boxed{\ddot{x}(t_r) = -\omega^2 x_o \cos(\omega t_r + \delta)}$

Instantaneous jerk: $\boxed{\dddot{x}(t_r) = +\omega^3 x_o \sin(\omega t_r + \delta) = -\omega^2 \underbrace{\left(-\omega x_o \sin(\omega t_r + \delta)\right)}_{=\dot{x}(t_r)}}$

Thus: $\boxed{\dddot{x}(t_r) = -\omega^2 \dot{x}(t_r)}$

Thus: $\boxed{m\ddot{x}(t_r) + \tau m\omega^2 \dot{x}(t_r) + m\omega_o^2 x(t_r) = F_{driving}(t_r)}$

Define the damping constant: $\boxed{\gamma \equiv \omega^2 \tau}$ (SI units: $1/\text{sec}$)

Then: $\boxed{m\ddot{x}(t_r) + m\gamma\dot{x}(t_r) + m\omega_o^2 x(t_r) = F_{driving}(t_r)}$ $\leftarrow$ 2nd-order linear inhomogeneous diff. eqn.

n.b. In this situation, the *EM* radiation damping is proportional to $\ddot{v}(t_r)$. Compare this to *e.g.* "normal" mechanical damping, which is proportional to $v(t_r)$ (*e.g.* friction / dissipation).

The Physical Basis of the Radiation Reaction

We derived the Abraham-Lorentz *EM* radiation reaction force $\vec{F}_{rad}(t_r') = \frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t_r')$ from consideration of conservation of energy in the *EM* radiation process, from what was observable in the far-field region, $\mathscr{r} \to \infty$.

Classically, if one tries to determine this radiation reaction force at the radiating point charge, we run into mathematical difficulties due to the mathematical point-behavior of the electric charge (*e.g.* at its origin) where the (static) electric field and corresponding scalar potential become singular, this problem correspondingly has infinite energy density at the point charge.

This singular nature is also the present for the retarded *EM* fields associated with a moving point charge:

$$\boxed{\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\bullet\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right]}$$

$$\boxed{\vec{B}_r(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}} \times \vec{E}_r(\vec{r},t)} \quad \text{with:} \quad \boxed{\vec{u} = c\hat{\mathscr{r}} - \vec{v}}$$

Today, we know that quantum mechanics is operative, *e.g.* from the Heisenberg uncertainty principle on {*e.g.* 1-dimensional} distance scales of:

$\boxed{\Delta x \Delta p_x \le \hbar}$ where: $\boxed{\hbar = h/2\pi}$ = Planck's Constant / 2π, $\boxed{h = 6.626 \times 10^{-34}\text{ J-sec}}$

Then: $\boxed{\Delta x \le \hbar/\Delta p_x}$ but: $\boxed{\Delta p_x c < m_e c^2}$ {for electrons}

Note that: $hc = 1240$ eV-nm and: $1\text{ nm} = 10^{-9}$ m

$$\therefore \quad \boxed{\Delta x \le \frac{\hbar c}{m_e c^2} = \frac{1240\text{ eV-nm}/2\pi}{0.511\text{ meV}} \simeq 0.386\times 10^{-12}\,\text{m} = 386\text{ fm}} \qquad (1\text{ fm} = 10^{-15}\text{ m})$$

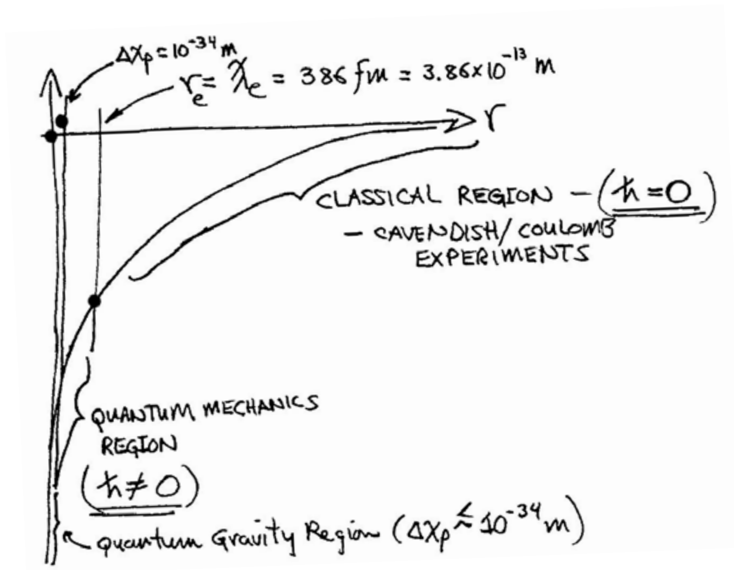
The quantity $\boxed{\bar{\lambda}_e \equiv \frac{\lambda_e}{2\pi} \equiv \frac{\hbar c}{m_e c^2} = \ 386\text{ fm}}$ = reduced Compton wavelength of the electron

And: $\boxed{\lambda_e \equiv \frac{hc}{m_e c^2} = 2427\text{ fm}}$ = Compton wavelength of electron

Thus: $\boxed{\Delta x \le \bar{\lambda}_e = \frac{\hbar c}{m_e c^2} = 386\text{ fm} = 386\times 10^{-15}\,\text{m}}$

for short distance scales of order $\boxed{\Delta x \le \bar{\lambda}_e = \hbar c/m_e c^2 = 386\text{ fm}}$ {and less} the behavior of an electron will be manifestly quantum mechanical in nature. Thus, we should not be surprised that when extrapolating classical *EM* theory into this short-distance regime, we obtain erroneous answers – we have no reason to expect classical theory to {continue to} hold in this quantum domain !!!

$$V_e(r) = -\frac{q}{4\pi\varepsilon_o}\frac{1}{r}$$



Similarly, we have no business extrapolating quantum mechanics to distance scales less than:

$$\boxed{r_e^{BH} = \frac{2G_N m_e}{c^2} = \frac{2\times 6.673\times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-1}\times 9.109\times 10^{-31}\,\text{kg}}{\left(3\times 10^8\,\text{m/s}\right)^2} \simeq 1.35\times 10^{-57}\,\text{m}} = \boxed{\text{Schwartzschild radius of electron (event horizon)}}$$

Where G_N = Newton's gravitational constant

The electron is a black hole at this distance scale – the Schwartzschild radius/event horizon of an electron is where space & time interchange roles!

However, long before this regime is reached, at distance scales corresponding to the Planck energy / Planck mass $\boxed{m_p c^2 = \sqrt{\hbar c^3/G_N} \simeq 2.2\times 10^{-8}\,kg = 1.2\times 10^{19}\,\text{GeV} = 1.2\times 10^{28}\,\text{eV}}$

{1 GeV = 10^9 eV}, is the regime of quantum gravity, where space-time itself becomes "foam-like" (not continuous) – quantized {somehow}. The distance scale where quantum gravity is operative is the Planck length $\boxed{L_P = \sqrt{\hbar G_N / c^3} \simeq 1.6\times 10^{-35}\,\text{m}}$. Note that the Planck length corresponds to a time-scale {known as the Planck time} of $\boxed{t_P = L_P/c = \sqrt{\hbar G_N/c^5} \simeq 5.4\times 10^{-44}\,\text{sec}}$.

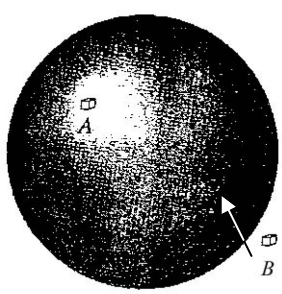
Nevertheless, back in the early 1900's, ignorance of quantum gravity and quantum mechanics did not stop Abraham, Lorentz, Poincaré {and many others} from applying classical *EM* theory - electrodynamics to calculate the self-force / radiation back-reaction on a point electric charge.

These efforts by-and-large modeled the point electron as {some kind of} spatially-extended electric charge distribution (of finite, but very small size), calculations could then be carried out and then (at the end of the calculation) the limit of the size of the charge distribution → 0.

In general (as we have already encountered this before in electrodynamics), the {retarded} classical/macroscopic *EM* force of one part (*A*) acting on another part (*B*) is *not* equal and opposite to the force of *B* acting on *A*, Newton's 3rd Law is seemingly violated:

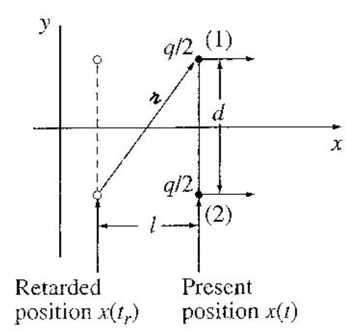
$$\boxed{\vec{F}_{\text{r}}^{\overrightarrow{AB}}\left(\vec{r}_B,t\right) \neq -\vec{F}_{\text{r}}^{\overrightarrow{BA}}\left(\vec{r}_A,t\right)}$$

Adding up the imbalances of such force pairs, we obtain the net force (imbalance) of a charge on itself – the "self-force" acting on the charge.



H.A. Lorentz originally calculated the self-force using a spherical charge distribution – tedious – see J.D. Jackson's Classical Electrodynamics, 3rd ed., sec. 16.3 and beyond if interested in these details….

A "less realistic" model of a charge is to use a rigid dumbbell in which the total charge q is divided into 2 halves separated by a fixed distance d (simplest possible charge arrangement to elucidate the self-force mechanism):



Assume that the dumbbell moves in $\hat{x}$-direction and (for simplicity) assume that the dumbbell is instantaneously at rest at the retarded time t_r. Then the retarded electric field at (1) due to (2) is:

$$\boxed{\vec{E}_{\text{r}}^{\overrightarrow{21}}\left(\vec{r}_1,t\right) = \frac{\frac{1}{2}q}{4\pi\varepsilon_o}\frac{\boldsymbol{\mathscr{r}}}{\left(\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{u}\right)^3}\left[\left(c^2+\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{a}\right)\vec{u}-\left(\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{u}\right)\vec{a}\right] = \frac{q}{8\pi\varepsilon_o}\frac{\boldsymbol{\mathscr{r}}}{\left(\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{u}\right)^3}\left[\left(c^2+\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{a}\right)\vec{u}-\left(\vec{\boldsymbol{\mathscr{r}}}\bullet\vec{u}\right)\vec{a}\right]}$$

Here: $\boxed{\vec{u} = c\hat{\mathscr{r}}}$ {because $\vec{v}(t_r) = 0$ }

Note that: $\boxed{\vec{\mathscr{r}} = \ell\hat{x} + d\hat{y} = \mathscr{r}\hat{\mathscr{r}}}$ and thus: $\boxed{\mathscr{r} = \sqrt{\ell^2 + d^2}}$.

Note also that: $\boxed{\ell = fcn(\vec{a})}$, and $\boxed{\vec{a}(t_r) = a(t_r)\hat{x}}$.

$\therefore$ $\boxed{\vec{\mathscr{r}}\cdot\vec{u}(t_r) = (\ell\hat{x} + d\hat{y})\cdot c\hat{\mathscr{r}} = \mathscr{r}\hat{\mathscr{r}}\cdot c\hat{\mathscr{r}} = c\mathscr{r}}$ and: $\boxed{\vec{\mathscr{r}}\cdot\vec{a}(t_r) = (\ell\hat{x} + d\hat{y})\cdot a(t_r)\hat{x} = a(t_r)\ell}$

We are in fact only interested in the $\hat{x}$-component of $\vec{E}_r^{21}(\vec{r},t)$, since the $\hat{y}$-components of $\vec{E}_r^{21}(\vec{r},t)$ and $\vec{E}_r^{12}(\vec{r},t)$ will cancel when we add forces on the two ends of the dumbbell.

Note further that since the two charges on the dumbbell are both moving in the same direction / parallel to each other, the magnetic forces associated one charge acting on the other will also cancel, thus Newton's 3rd Law is manifestly obeyed {here}, in this particular situation / configuration.

If $\boxed{\vec{u} = c\hat{\mathscr{r}}}$, then: $\boxed{\vec{u}_x = \vec{u}\cdot\hat{x} = \frac{c\vec{\mathscr{r}}}{\mathscr{r}}\cdot\hat{x}}$ and since: $\boxed{\vec{\mathscr{r}} = \ell\hat{x} + d\hat{y}}$ then: $\boxed{u_x = \frac{c}{\mathscr{r}}(l\hat{x} + d\hat{y})\cdot\hat{x} = \frac{c\ell}{\mathscr{r}}}$

Thus: $\boxed{E_{r_x}^{21}(\vec{r}_1,t) = \frac{q}{8\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\cdot\vec{u}(t_r)\right)^3}\left[\left(c^2 + \vec{\mathscr{r}}\cdot\vec{a}(t_r)\right)u_x - \left(\vec{\mathscr{r}}\cdot\vec{u}(t_r)\right)a_x(t_r)\right]}$

And: $\boxed{\left(\vec{\mathscr{r}}\cdot\vec{u}(t_r)\right) = c\mathscr{r}}$ and: $\boxed{\left(\vec{\mathscr{r}}\cdot\vec{a}(t_r)\right) = (\ell\hat{x} + d\hat{y})\cdot a(t_r)\hat{x} = a(t_r)\ell}$ since: $\boxed{\vec{a}(t_r) = a(t_r)\hat{x} = a_x(t_r)\hat{x}}$

Then:

$$
\begin{aligned}
E_{r_x}^{21}(\vec{r}_1,t) &= \frac{q}{8\pi\varepsilon_o}\frac{\mathscr{r}}{c^3\mathscr{r}^3}\left[\left(c^2 + a(t_r)\ell\right)\frac{c\ell}{\mathscr{r}} - c\mathscr{r}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathscr{r}^2}\left[\left(c^2 + a(t_r)\ell\right)\frac{\ell}{\mathscr{r}} - \mathscr{r}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathscr{r}^2}\left[\frac{\ell c^2}{\mathscr{r}} + \frac{a(t_r)\ell^2}{\mathscr{r}} - \mathscr{r}a(t_r)\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathscr{r}^2}\left[\frac{\ell c^2}{\mathscr{r}} + \frac{a(t_r)\ell^2 - \mathscr{r}^2 a(t_r)}{\mathscr{r}}\right] \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathscr{r}^3}\left[\ell c^2 - \underbrace{\left(\mathscr{r}^2 - \ell^2\right)}_{=d^2}a(t_r)\right] \quad \text{but: } \mathscr{r}^2 = \ell^2 + d^2 \quad \text{or: } \mathscr{r}^2 - \ell^2 = d^2 \\
&= \frac{q}{8\pi\varepsilon_o}\frac{1}{c^2\mathscr{r}^3}\left[\ell c^2 - d^2 a(t_r)\right]
\end{aligned}
$$

Thus: $\boxed{E_{r_x}^{21}(\vec{r}_1,t) = \frac{q}{8\pi\varepsilon_o}\frac{\left(\ell c^2 - d^2 a(t_r)\right)}{c^2\left(\ell^2 + d^2\right)^{3/2}}}$ since: $\boxed{\mathscr{r} = \sqrt{\ell^2 + d^2}}$

Then by symmetry: $\boxed{E_{\mathrm{r}_x}^{\overline{21}}(\vec{r}_1,t)=E_{\mathrm{r}_x}^{\overline{12}}(\vec{r}_2,t)}$

$\therefore$ The net {retarded} force on the rigid dumbbell is:

$$\boxed{\vec{F}_{\mathrm{r}}^{self}(\vec{r},t)=\vec{F}_{\mathrm{r}}^{\overline{21}}(\vec{r}_1,t)+\vec{F}_{\mathrm{r}}^{\overline{12}}(\vec{r}_2,t)=\tfrac{1}{2}q\vec{E}_{\mathrm{r}}^{\overline{21}}(\vec{r}_1,t)+\tfrac{1}{2}q\vec{E}_{\mathrm{r}}^{\overline{12}}(\vec{r}_2,t)=\frac{q^2}{8\pi\varepsilon_o c^2}\frac{\left(\ell c^2-d^2a(t_{\mathrm{r}})\right)}{\left(\ell^2+d^2\right)^{3/2}}\hat{x}}\Leftarrow\underline{\text{Exact}}$$

We now expand $\vec{F}_{\mathrm{r}}^{self}(\vec{r},t)$ in powers of *d*. Then when the size *d* of the electrically-charged dumbbell is taken to its limit of $d\to 0$, all positive powers will disappear.

Taylor's Theorem:

$$\boxed{x(t)=x(t_{\mathrm{r}})+\dot{x}(t_{\mathrm{r}})(t-t_{\mathrm{r}})+\frac{1}{2!}\ddot{x}(t_{\mathrm{r}})(t-t_r)^2+\frac{1}{3!}\dddot{x}(t_{\mathrm{r}})(t-t_{\mathrm{r}})^3+\dots}$$

Recall that: $\boxed{\dot{x}(t_{\mathrm{r}})=v(t_{\mathrm{r}})=0}$ and that: $\boxed{\ell=fcn\left(\vec{a}(t_{\mathrm{r}})\right)}$

Then: $\boxed{\ell=\left[x(t_{\mathrm{r}})-x(t_{\mathrm{r}})\right]=\frac{1}{2}a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2+\frac{1}{6}\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^3+\dots}$ where: $\boxed{\Delta t_{\mathrm{r}}\equiv(t-t_{\mathrm{r}})}$

But: $\boxed{c\Delta t_{\mathrm{r}}=\boldsymbol{r}=\sqrt{\ell^2+d^2}}\Rightarrow\boxed{c^2\Delta t_{\mathrm{r}}^2=\boldsymbol{r}^2=\ell^2+d^2}$

Or:

$$\boxed{\begin{aligned}d&=\sqrt{\boldsymbol{r}^2-\ell^2}=\sqrt{c^2\Delta t_{\mathrm{r}}^2-\ell^2}=\sqrt{c^2\Delta t_{\mathrm{r}}^2-\left(\frac{1}{2}a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2+\frac{1}{6}\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^3+\dots\right)^2}\\&=\sqrt{c^2\Delta t_{\mathrm{r}}^2-c^2\Delta t_{\mathrm{r}}^2\left(\frac{a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}}{2c}+\frac{\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2}{6c}+\dots\right)^2}=c\Delta t_{\mathrm{r}}\sqrt{1-\left(\frac{a(t_{\mathrm{r}})\Delta t_{\mathrm{r}}}{2c}+\frac{\dot{a}(t_{\mathrm{r}})\Delta t_{\mathrm{r}}^2}{6c}+\dots\right)^2}\\&=c\Delta t_{\mathrm{r}}-\frac{a^2(t_{\mathrm{r}})}{8c}\Delta t_{\mathrm{r}}^3+\{\ \}\Delta t_{\mathrm{r}}^4+\dots\end{aligned}}$$

We want Δt_{r} in terms of *d*. From above, it can be seen that we can solve for *d* in terms of Δt_{r}. But we can solve for Δt_{r} in terms of *d* using the reversion of series technique, which is a formal way / formal method that can be used to obtain an approximate value of Δt_{r} by ignoring all higher powers of Δt_{r}, to first order, we have:

$\boxed{d\simeq c\Delta t_{\mathrm{r}}}\Rightarrow\boxed{\Delta t_{\mathrm{r}}\simeq\frac{d}{c}}$ $\leftarrow$ use this as an approximation for obtaining a cubic correction term:

$$\boxed{d\simeq c\Delta t_{\mathrm{r}}-\frac{a^2(t_{\mathrm{r}})}{8c}\left(\frac{d}{c}\right)^3}\Rightarrow\boxed{\Delta t_{\mathrm{r}}\simeq\frac{d}{c}+\frac{a^2(t_{\mathrm{r}})d^3}{8c^5}}$$

Keep going. . . $\boxed{\Delta t_{\mathrm{r}}\simeq\frac{1}{c}d+\frac{a^2(t_{\mathrm{r}})}{8c^5}d^3+\{\ \}d^4+\dots}$

Thus: $$\boxed{\ell = \left[x(t) - x(t_r)\right] = \frac{1}{2}a(t_r)\Delta t_r^2 + \frac{1}{6}\dot{a}(t_r)\Delta t_r^3 + \ldots \simeq \frac{a(t_r)}{2c}d^2 + \frac{\dot{a}(t_r)}{6c^3}d^3 + \{\ \}d^4 + \ldots}$$

Then: $$\boxed{\vec{F}_r^{self}(\vec{r},t) = \frac{q^2}{8\pi\varepsilon_o c^2}\frac{\left(\ell c^2 - d^2 a(t_r)\right)}{\left(\ell^2 + d^2\right)^{3/2}}\hat{x} \simeq \frac{q^2}{4\pi\varepsilon_o}\left[-\frac{a(t_r)}{4c^2 d} + \frac{\dot{a}(t_r)}{12c^3} + \{\ \}d + \ldots\right]\hat{x}}$$

{Note that $a(t_r)$ and $\dot{a}(t_r)$ are evaluated at the retarded time t_r.}

Using the Taylor series expansion of $a(t_r)$, we can rewrite this result in terms of the present time:

$$\boxed{a(t_r) = a(t) + \dot{a}(t)(t_r - t) + \ldots = a(t) - \dot{a}(t)\Delta t_r + \ldots = a(t) - \dot{a}(t)\frac{d}{c} + \ldots}$$

Then: $$\boxed{\vec{F}_r^{self}(\vec{r},t) \simeq \frac{q^2}{4\pi\varepsilon_o}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + \{\ \}d + \ldots\right]\hat{x}}$$

The first term inside the brackets on the RHS is proportional to acceleration of the charge q. If we put it on LHS, then by Newton's 2nd Law $\vec{F} = m\vec{a}$, we see that it *adds* to the mass m of the dumbbell – there is *inertia* associated with accelerating an electrically-charged particle.

The total inertial mass of the dumbbell is therefore:

$$\boxed{m_{tot} = m_{dumbbell} + \frac{1}{4\pi\varepsilon_o}\left(\frac{q^2}{4dc^2}\right) = m_{dumbbell} + \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{dc^2}\right)}$$

Or: $$\boxed{m_{tot}c^2 = m_{dumbbell}c^2 + \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)}$$ $\Leftarrow$ rest mass energy, $\boxed{E = mc^2}$

Note that the {repulsive} electrostatic potential energy associated with this dumbbell is:

$$\boxed{U_E(r = d) = \left(\tfrac{1}{2}q\right)V(r = d) = \frac{\left(\frac{1}{2}q\right)^2}{4\pi\varepsilon_o d} = \frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)} \text{ (Joules)}$$

The fact that this works out "perfectly" is simply due to the fact that the initial choice of the dumbbell's orientation was deliberately/consciously chosen to be transverse to the direction of motion. For a longitudinally-oriented dumbbell, the *EM* mass correction is half this amount. For a spherical charge distribution, the *EM* mass correction is a factor of ¾ !!

The second term inside the brackets on the RHS of the $\vec{F}_r^{self}(\vec{r},t)$ relation is the *EM* radiation reaction term:

$$\boxed{\vec{F}_{rad}^{\,int}(\vec{r},t) = \frac{q^2\dot{a}(t)}{12\pi\varepsilon_o c^3}\hat{x} = \frac{\mu_o q^2\dot{a}(t)}{12\pi c}\hat{x}}$$

Note that $\vec{F}_{rad}^{\,\text{int}}(\vec{r},t)$ differs from Abraham-Lorentz result by a factor of 2×:

$$\boxed{\vec{F}_{rad}^{\,A\text{-}L}(\vec{r},t)=\frac{\mu_o q^2}{6\pi c}\dot{\vec{a}}(t)}$$

The reason for the factor of 2 difference is that physically, $\vec{F}_{rad}^{\,\text{int}}(\vec{r},t)$ is force of one end of the dumbbell acting on the on other – an *EM* interaction between the two ends.

There is also a force of <u>each</u> <u>end</u> of the dumbbell acting on <u>itself</u> – an *EM* <u>self-interaction</u> $\vec{F}_{rad}^{\,self}(\vec{r},t)$ for <u>each</u> end. When the *EM* self-interactions for each end are included (see Griffiths Problem 11.20, *p*. 473), then the <u>total</u> *EM* radiation reaction becomes:

$$\boxed{\vec{F}_{rad}^{\,tot}(\vec{r},t)=\vec{F}_{rad}^{\,\text{int}}(\vec{r},t)+2\vec{F}_{rad}^{\,self}(\vec{r},t)=\frac{\mu_o q^2\dot{a}(t)}{6\pi c}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right]\hat{x}=\frac{\mu_o q^2\dot{a}(t)}{6\pi c}\hat{x}}$$

which agrees perfectly with Abraham-Lorentz radiation-reaction force formula.

Thus, physically we see that the *EM* radiation reaction is due to the force of the charge acting on itself – an {apparent} <u>self-force</u>!

Note also that $\vec{F}_{rad}(\vec{r},t)$ does <u>NOT</u> depend on d ($\vec{F}_{rad}(\vec{r},t)$ is valid/well-behaved in limit of the size of the dumbbell, $d \to 0$).

<u>However, note that</u>: $\boxed{m_{tot}c^2=m_{dumbbell}c^2+\frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)\to\infty}$ when $d \to 0$!!!

The inertial mass of the classical electron becomes <u>infinite</u> when when $d \to 0$, because:

$$\boxed{U_E(r=d)=\left(\tfrac{1}{2}q\right)V(r=d)=\frac{\left(\frac{1}{2}q\right)^2}{4\pi\varepsilon_o d}=\frac{1}{4\pi\varepsilon_o}\left(\frac{\left(\frac{1}{2}q\right)^2}{d}\right)\to\infty} \text{ when } d\to 0 \text{ !!!}$$

{But we already knew this, as we learned long ago, in P435/last semester…}

Note that this unpleasant/awkward problem also persists in the fully-relativistic, quantum electrodynamical theory {QED}. Infinities/singularities there are dealt with/side-stepped by a process known as <u>mass renormalization</u>, so as to avoid such infinities – look only at mass differences / energy differences…