LECTURE NOTES 4

A Mini-Review of "Generic" Wave Phenomena:

Waves in 1-Dimension

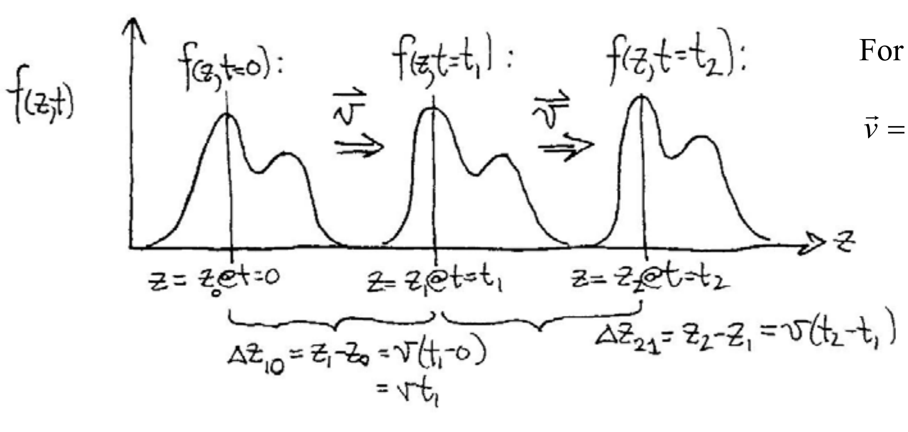
What is a 1-dimensional (1-D) wave?

- A (classical) 1-D *traveling* wave is a quasi-coherent collective phenomenon – that of a "disturbance" associated with a localized excess of energy (above ambient thermal / background energy) in a macroscopic, continuous medium, which propagates (*i.e.* translates in 1-D space) as time progresses.

  In a dissipationless (*i.e.* lossless), non-dispersive medium, the shape / profile /envelope (*i.e.* crests and troughs) of the wave propagates with constant velocity.

- In a dispersive medium, a traveling wave consisting of a linear combination of several / many different frequencies, the various frequency components of the wave will each propagate with different speed, thus the overall shape of the wave will change with time in a dispersive medium.
- In a dissipative (but non-dispersive) medium the wave amplitude(s) will decrease with time (or equivalently propagation distance), Often, real media are not only dissipative, but also dispersive, thus dissipation in a medium may also be frequency dependent.
- Classical media can be both dispersive and dissipative – one or both or neither.
- A (classical) 1-D *standing* wave = a linear superposition of two counter-propagating traveling waves (*e.g.* a standing wave on a stringed instrument.)
- Standing waves do not propagate in space, although they can / do evolve in time (due to dispersion, dissipation and other processes).

Let us consider a 1-dimensional transverse *traveling* wave, *e.g.* on a taught / tight string:



For $|\vec{v}|$ = constant

$$\vec{v} = v\hat{z} = \frac{\Delta z}{\Delta t}\hat{z} = \left(\frac{z_2 - z_1}{t_2 - t_1}\right)\hat{z}$$

For non-dissipative, non-dispersive media, the transverse shape {= transverse displacement from equilibrium shape of string} of a traveling wave $f(z,t)$ is invariant under space translations and time translations.

Mathematically, this means that:

$$\boxed{f(z,t_2) = f\left(z - v(t_2 - t_1), t_1\right) = \underbrace{f\left(z - v(t_2 - 0), 0\right)}_{= f(z - vt_2, 0)} = f(z - vt_2, 0)}$$

$$\boxed{f(z,t_1) = \underbrace{f\left(z - v(t_1 - 0), 0\right)}_{= f(z - vt_1, 0)} = f(z - vt_1, 0)}$$

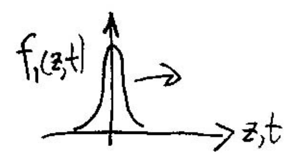
In general, $f(z,t)$ for a 1-D traveling wave at position z and time t = a <u>special</u> type of function $g(z - vt)$. The function $f(z,t)$ that mathematically describes the 1-D wave motion / wave propagation is <u>*not*</u> arbitrary / will-nilly – it is a very special / very specific <u>causal</u> relationship of the location(s) of the 1-D wave in <u>*both*</u> space and time: $f(z,t)$ is restricted to the <u>causal</u> subset of functions $g(z - vt)$, *i.e.* classical traveling waves <u>obey</u> causality.

This means that the <u>argument</u> of the causal g-functions, $(z - vt)$ = constant, independent of space (z) and time (t) , *i.e.* the argument $(z - vt)$ = constant $\forall$ (for all) allowed (z, t) .

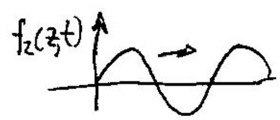
The following are some examples of mathematically acceptable / causal functions describing dissipationless, dipersionless classical traveling wave in 1-D (*n.b.* here, A and b = constants, *e.g.* independent of frequency):

For 1-D traveling wave propagation: $(z - vt)$ = constant

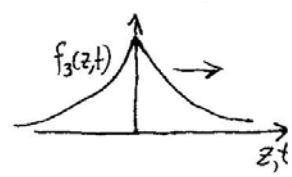
$f_1(z,t) = Ae^{-b(z-vt)^2}$ (Gaussian wave)



$f_2(z,t) = A \underbrace{\sin}_{\text{or cos}} \left(b(z - vt)\right)$ (Sine/Cosine Wave)



$f_3(z,t) = \dfrac{A}{b(z - vt)^2 + 1}$ ("Cusp" Wave)



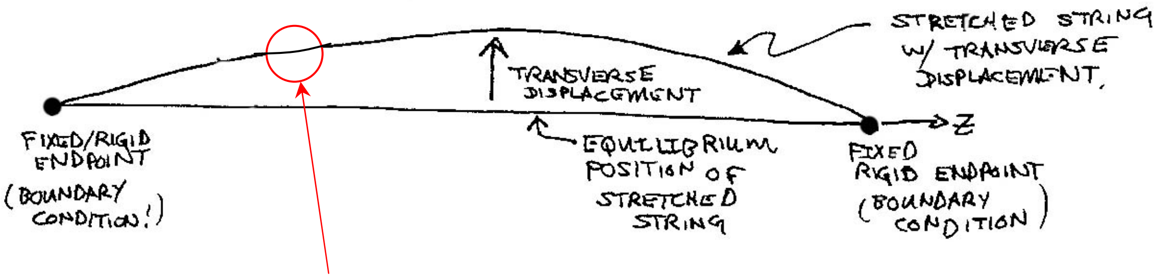
Some examples of mathematically <u>unacceptable</u> / <u>a-causal</u> 1-D "wave" functions:

$f_4(z,t) = Ae^{-b(z^2+vt)}$ (*n.b.* here again, A and b = constants)

$f_5(z,t) = A\sin(bz)\cos(bvt)^3$

Example: 1-D transverse mechanical traveling waves on a string obey Newton's 2nd Law: $\boxed{\vec{F} = m\vec{a}}$

If a stretched string is transversely displaced from its equilibrium position, as shown in the figure below, the transverse *displacement* of the string from its equilibrium position at a point z along the string at a given instant in time t is mathematically described by the function $f(z,t)$.



Let us investigate / analyze the forces acting on small/infinitesimal segment of the string:

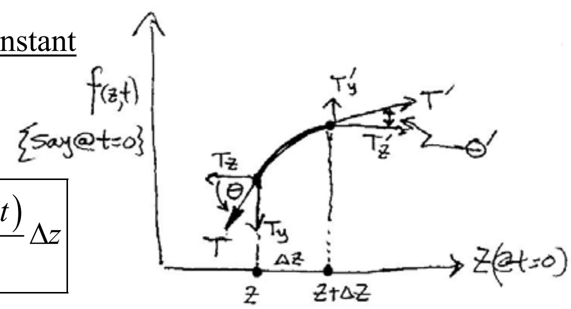
As can be seen from the figure below, at any given instant in time, t the *net* transverse force $\Delta F_y(z,t)$ acting on an infinitesimal string segment (of length Δz) between z and ($z+\Delta z$) on a string with tension T (Newtons) is: $\boxed{\Delta F_y(z,t) = T_y'(z+\Delta z,t) - T_y(z,t) = T\sin\theta' - T\sin\theta}$

For small transverse displacements: String tension T = constant

$\boxed{\sin\theta \approx \theta \approx \tan\theta}$

$\therefore \boxed{\Delta F_y(z,t) \cong T(\tan\theta' - \tan\theta)}$ $\quad$ *n.b.* $\tan\theta$ = slope

$$\boxed{\Delta F_y(z,t) = T\left(\left.\frac{\partial f(z,t)}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f(z,t)}{\partial z}\right|_{z}\right) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$$



Thus: $\boxed{\Delta F_y(z,t) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$ for small transverse displacements of the string from its equilibrium (*i.e.* zero excess energy) configuration.

If the mass per unit length of the string is: $\boxed{\mu = m_{string}/L}$ (*kg/m*) (where the total string mass = m_{string} and the total length of the string = L) then Newton's 2nd Law: $\boxed{\Delta F_y(z,t) = ma_y(z,t)}$ where $a_y(z,t)$ = transverse acceleration (in the $\hat{y}$-direction) at the point z at time t is

$$\boxed{a_y(z,t) = \frac{\partial^2 f(z,t)}{\partial t^2}}$$

The string segment of infinitesimal length Δz has mass $\boxed{m = \mu\Delta z}$ $\left(=\left[m_{string}/L\right]\Delta z\right)$

$$\therefore \boxed{\Delta F_y(z,t) = ma_y(z,t) \simeq \mu\Delta z\frac{\partial^2 f(z,t)}{\partial t^2}}$$

But: $\boxed{\Delta F_y(z,t) \simeq T\frac{\partial^2 f(z,t)}{\partial z^2}\Delta z}$ from the transverse force imbalance relation (above)

$\therefore$ for small displacements: $\boxed{T\dfrac{\partial^2 f(z,t)}{\partial z^2}\cancel{\Delta z} = \mu\cancel{\Delta z}\dfrac{\partial^2 f(z,t)}{\partial t^2}}$ or: $\boxed{\dfrac{\partial^2 f(z,t)}{\partial z^2} = \left(\dfrac{\mu}{T}\right)\dfrac{\partial^2 f(z,t)}{\partial t^2}}$

Note that, from dimensional analysis: $\boxed{\dfrac{T}{\mu} = \dfrac{\text{Force, Newtons}}{\text{mass}/\text{unit length}} = \dfrac{\cancel{\text{kg}}\text{-m}/\text{s}^2}{\cancel{\text{kg}}/\text{m}} = \dfrac{\text{m}^2}{\text{s}^2} = \left(\dfrac{\text{m}}{\text{s}}\right)^2}$

From conservation of energy associated with traveling waves propagating on a taught string, it can be shown that $\boxed{v = \sqrt{T/\mu}}$ = longitudinal speed of propagation of transverse waves on a string. For dispersionless media, note that v = constant ≠ fcn(frequency, f).

Thus we arrive at the 1-D wave equation for transverse traveling waves propagating *e.g.* on a taught/stretched string:

$$\boxed{\frac{\partial^2 f(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f(z,t)}{\partial t^2}} \quad \text{with:} \quad \boxed{v = \sqrt{T/\mu}}$$

We can re-arrange the wave equation into its more traditional form: $\boxed{\dfrac{\partial^2 f(z,t)}{\partial z^2} - \dfrac{1}{v^2}\dfrac{\partial^2 f(z,t)}{\partial t^2} = 0}$

Thus, we see that the wave equation is a 2nd order linear and homogeneous differential equation.

Solutions of wave equation are all functions $f(z,t)$ of the form where the longitudinal position z and time t are causally connected to each other by $(z - vt)$ = constant, *i.e.* all of the functions $g\,(z - vt)$ = constant.

The requirement / restriction that $f(z, t) = g\,(z - vt)$ explicitly means that: $u \equiv (z - vt) =$ argument of the g-functions, *i.e.* that:

$$\boxed{\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}} \quad \text{because} \quad \boxed{\frac{\partial u}{\partial z} = \frac{\partial(z - vt)}{\partial z} = 1}$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}} \quad \text{because} \quad \boxed{\frac{\partial u}{\partial t} = \frac{\partial(z - vt)}{\partial t} = -v}$$

And thus:

1.) $\boxed{\dfrac{\partial^2 f}{\partial z^2} = \dfrac{\partial}{\partial z}\left(\dfrac{dg}{du}\right) = \dfrac{d^2 g}{du^2}\dfrac{\partial u}{\partial z} = \dfrac{d^2 g}{du^2}}$ 2.) $\boxed{\dfrac{\partial^2 f}{\partial t^2} = -v\dfrac{\partial}{\partial t}\left(\dfrac{dg}{du}\right) - v\dfrac{d^2 g}{du^2}\dfrac{\partial u}{\partial t} = +v^2\dfrac{d^2 g}{du^2}}$

$\therefore$ $\boxed{\overset{1)}{\dfrac{d^2 g}{du^2}} = \overset{2)}{\dfrac{d^2 g}{du^2}}} \Rightarrow \boxed{\dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}}$ 1-D Wave Equation

Thus, $g\,(u)$ can be any differentiable function satisfying $\boxed{u = (z - vt)}$

Note that since the wave equation involves the square of the longitudinal propagation speed v, then another acceptable form of a solution is: $f(z, t) = g\,(z + vt)$

Thus there are two "generic" possible acceptable solutions:

a.) $f(z, t) = g\,(z - vt)$ where $(z - vt)$ = constant, thus if t increases $\rightarrow$ z also increases
b.) $f(z, t) = g\,(z + vt)$ where $(z + vt)$ = constant, thus if t increases $\rightarrow$ z decreases

Physically this means that:

a.) $f(z,t) = g(z - vt)$ represents a 1-D wave propagating in the $+\hat{z}$ direction

b.) $f(z,t) = g(z + vt)$ represents a 1-D wave propagating in the $-\hat{z}$ direction

The Linear Wave Equation & the Linear Superposition Principle

Provided that the initial (simplifying) assumption that the displacement from equilibrium is small, such that $\sin\theta \approx \theta \approx \tan\theta$ is valid, then the principle of linear superposition tells us that:

$$\boxed{f_{TOT}(z,t) = \sum_{i=1}^{n} f_i(z,t) = f_1(z,t) + f_2(z,t) + f_3(z,t) + \ldots}$$

is also a solution of the linear wave equation. Note that in general:

$$\boxed{f_{TOT}(z,t) = \underbrace{\sum_{i=1}^{n} g_i(z - vt)} + \underbrace{\sum_{j=1}^{m} h_j(z + vt)}}$$

Traveling waves propagating in the $+\hat{z}$ direction (*i.e.* to the right)

Traveling waves propagating in the $-\hat{z}$ direction (*i.e.* to the left)

Most generally, *i.e.* $f_{TOT}(z,t) =$ linear superposition of left & right-moving / propagating waves.

Standing Waves:

A standing wave (one which is *stationary* in space) is formed by superposing two identical traveling waves, except that one is a is left-going traveling wave and the other is right-going:

$$\boxed{f_{TOT}(z,t) = g(z - vt) + g(z + vt)} \text{ where } e.g.$$

$$\boxed{g(z - vt) = A\sin(k[z - vt]) = A\sin(kz - \omega t)} \text{ and: } \boxed{g(z + vt) = A\sin(k[z + vt]) = A\sin(kz + \omega t)}$$

Definition of nomencature used in wave propagation:

A = amplitude (= absolute value of maximum displacement from equilibrium) (m)

$v = f\lambda$ = longitudinal speed of propagation of wave (m/s)

f = frequency of vibration of wave (cycles per sec = *c.p.s.* = *Hz* (*Hertz*))

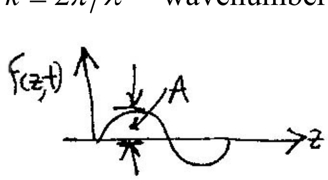
τ = period of wave $= 1/f$ (seconds, per cycle of oscillation)

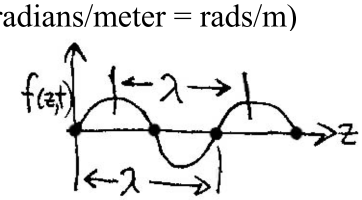
λ = wavelength (m) = spatial oscillation distance

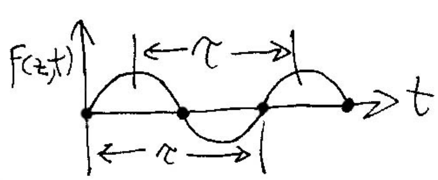
$\omega \equiv 2\pi f$ = "angular" frequency (radians/sec = rad/sec)

$k \equiv 2\pi/\lambda$ = wavenumber (radians/meter = rads/m)

$$\boxed{v = f\lambda = \frac{2\pi f}{2\pi/\lambda} = \omega/k}$$







Returning to the discussion of standing waves as a linear superposition of a left-going and a right-going traveling wave:

$$\boxed{f_{TOT}(z,t) = \overbrace{g(z-vt)}^{right} + \overbrace{g(z+vt)}^{left}}$$

<u>where</u>: $\boxed{g(z-vt) = A\sin(kz-\omega t)}$ and: $\boxed{g(z+vt) = A\sin(kz+\omega t)}$

<u>then</u>: $\boxed{f_{TOT}(z,t) = A\sin(kz-\omega t) + A\sin(kz+\omega t)}$

<u>now</u>: $\boxed{\sin(a\pm b) = \sin a\cos b \pm \cos a \sin b}$

$\therefore$ $\boxed{f_{TOT}(z,t) = A\left\{\sin kz\cos\omega t - \cancel{\cos kz\sin\omega t} + \sin kz\cos\omega t + \cancel{\cos kz\sin\omega t}\right\} \equiv 2A\sin kz\cos\omega t}$

<u>Thus</u>: $\boxed{f_{TOT}(z,t) = A'\sin kz\cos\omega t = 2A\sin kz\cos\omega t}$ *i.e.* define $\boxed{A' \equiv 2A}$

= 1-D standing wave (*i.e.* does <u>not</u> propagate/move in longitudinal $\pm\hat{z}$ -direction)

<u>Explicit check</u>: Does $f_{TOT}(z,t) = A'\sin kz\cos\omega t$ obey the wave equation?

i.e. does $\dfrac{\partial^2 f_{TOT}(z,t)}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f_{TOT}(z,t)}{\partial t^2}$??

$$\frac{\partial f(z,t)}{\partial z} = kA'\cos kz\cos\omega t \qquad \frac{\partial f(z,t)}{\partial t} = -\omega A'\sin kz\sin\omega t$$

$$\frac{\partial^2 f(z,t)}{\partial z^2} = -k^2 A'\sin kz\cos\omega t \qquad \frac{\partial^2 f(z,t)}{\partial t^2} = -\omega^2 A'\sin kz\cos\omega t$$

$$-k^2\cancel{A'}\cancel{\sin kz\cos\omega t} = \frac{1}{v^2}\left(-\omega^2\cancel{A'}\cancel{\sin kz\cos\omega t}\right)$$

$$-k^2 = -\omega^2/v^2 \quad\Rightarrow\quad v^2 = \omega^2/k^2 = \left(\omega/k\right)^2$$

$$\therefore\ v = \omega/k \qquad \text{Yes!}$$

<u>The Sinusoidal Traveling Wave</u>:

The most familiar wave: $f(z,t) = A\cos\left(k(z-vt)+\delta\right)$

Amplitude

wave number $k = 2\pi/\lambda$

speed of propagation

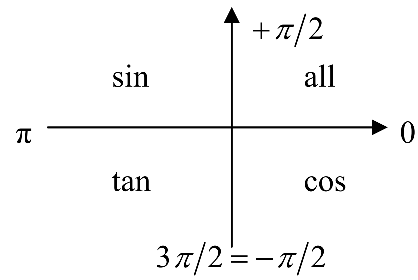
phase (radians) = constant (usually phase is defined between 0 and 2π)

<u>Note</u>: $\cos\left(k(z-vt)\pm\underbrace{\pi/2}_{=\delta}\right)$ but: $\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$

$$= \cos\left(k(z-vt)\right)\cos(\pm\pi/2) \mp \sin\left(k(z-vt)\right)\sin(\pi/2)$$

<u>But</u>: $\sin(+\pi/2) = +1$ and $\sin(-\pi/2) = -1$

$\therefore$ $\boxed{\cos(k(z-vt) \pm \pi/2) = \mp\sin(k(z-vt))}$



<u>Note also</u>: $f_{TOT}(z, t) = g(z - vt) + g(z + vt)$

Two waves in phase ($\delta = 0$) with each other

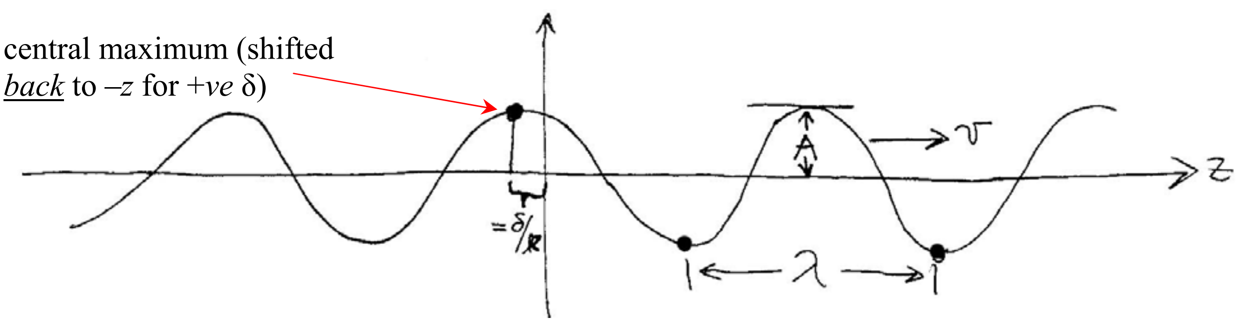
$f_{TOT}(z, t) = g(z - vt) - g(z + vt)$

180° (= π radians) out-of-phase ($\delta = 180^\circ = \pi$ radians)

Thus the function $f(z, t) = A\cos(k(z - vt) + \delta)$ at $t = 0$ appears as shown in the following figure:

$f(z, t = 0)$ = snapshot of wave at $t = 0$:

central maximum (shifted *<u>back</u>* to $-z$ for $+ve$ δ)



$$\delta/k = \frac{\delta}{\left(2\pi/\lambda\right)} = \left(\frac{\delta}{2\pi}\right)\lambda \qquad \left(\frac{\delta}{2\pi}\right) = \underline{\text{fractional}} \text{ phase}$$

Note the various alternate/equivalent mathematical forms describing the same traveling wave:

$$\begin{aligned} f(z,t) &= A\cos(k(z-vt)+\delta) \\ &= A\cos((kz-kvt)+\delta) \\ &= A\cos((kz-\omega t)+\delta) \\ &= A\cos\left(\left(\frac{2\pi z}{\lambda}+2\pi ft\right)+\delta\right) \\ &= A\cos\left(2\pi\left(\frac{z}{\lambda}-ft\right)+\delta\right) \\ &= A\cos\left[2\pi\left\{\left(\frac{z}{\lambda}-ft\right)+\frac{\delta}{2\pi}\right\}\right] \end{aligned}$$

$v = \lambda f = \omega/k$ (m/s)

$\omega = 2\pi f$ (rads/s)

$k = 2\pi/\lambda$ (1/m)

$\omega = kv$

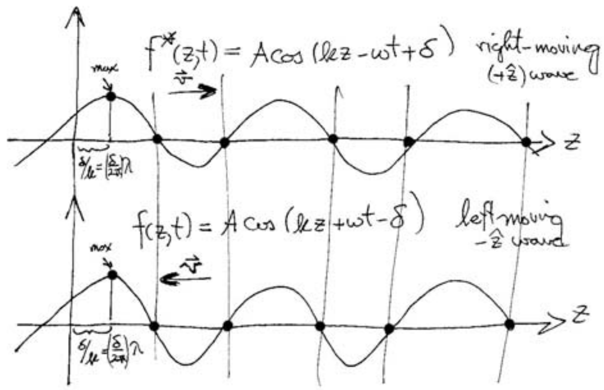
Note that because cos (x) is an even function of x, *i.e.* cos ($-x$) = cos (x)

Then: $f(z,t) = A\cos(kz + \omega t - \delta)$ = left-moving wave ($-\hat{z}$ direction)

$= A\cos(-kz - \omega t + \delta)$

But: $f^*(z,t) = A\cos(kz - \omega t + \delta)$ = right-moving wave ($+\hat{z}$ direction)

→ Switching the sign of k produces a wave with the same amplitude, phase, frequency and wavelength, but one which is traveling in the opposite direction.



Complex Notation:

Euler's Formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad \boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

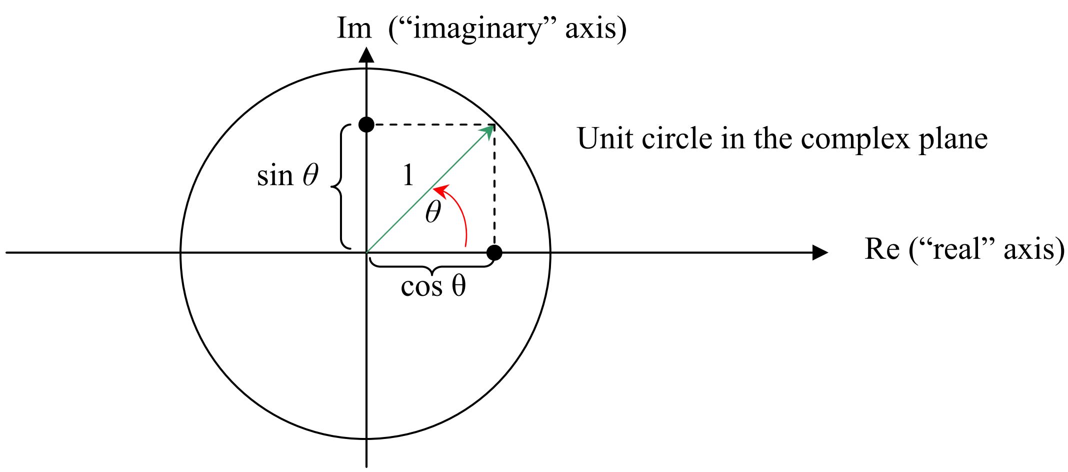
$$\boxed{\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)} \qquad \boxed{\sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$\boxed{i \equiv \sqrt{-1}} \qquad \boxed{i^* = -i = -\sqrt{-1}} \qquad \boxed{i^* i = i i^* = +1}$$

The magnitude of $e^{i\theta}$ is defined as $\left|e^{i\theta}\right|$:

$$\boxed{\left|e^{i\theta}\right| \equiv \sqrt{e^{i\theta}e^{-i\theta}} = 1 = \sqrt{(\cos\theta + i\sin\theta)^*(\cos\theta - i\sin\theta)} = \sqrt{\cos^2\theta + \sin^2\theta}}$$

Projections of a complex unit vector $e^{i\theta} = \cos\theta + i\sin\theta$ in the complex plane:



We will use the tilde symbol (~) over/above a physical variable to denote its <u>complex</u> nature:

Complex #: $\boxed{\tilde{z} = x + iy}$ $\boxed{\mathrm{Re}(\tilde{z}) = x}$ $\boxed{\mathrm{Im}(\tilde{z}) = y}$

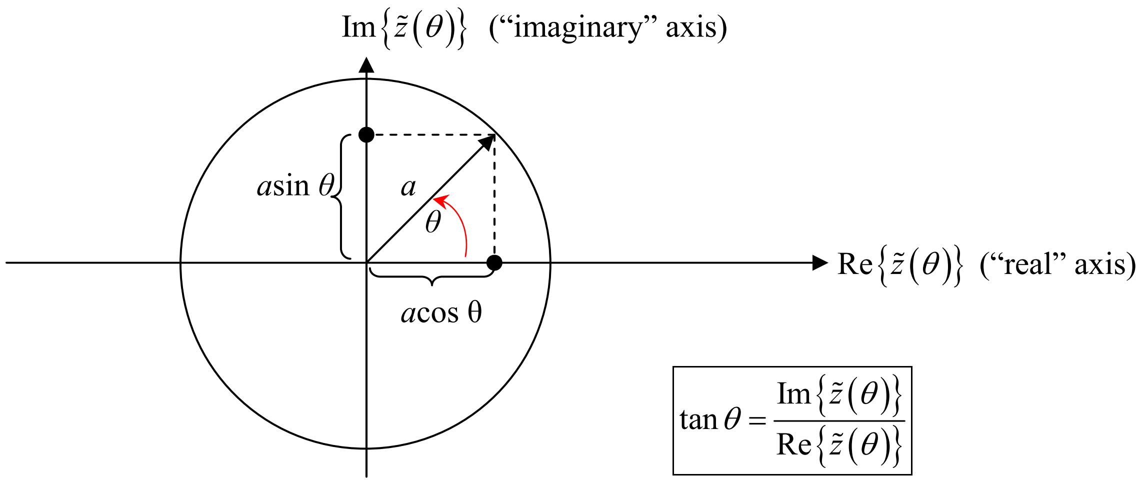
Complex conjugate ($i \rightarrow i^* = -i$): $\boxed{\tilde{z}^* = (x+iy)^* = x - iy}$ $\boxed{\mathrm{Re}(\tilde{z}^*) = x}$ $\boxed{\mathrm{Im}(\tilde{z}^*) = -y}$

<u>Suppose</u>: $\boxed{\tilde{z}(\theta) = ae^{i\theta} = \text{complex \#}}$ where $a = \text{real constant}$

$$\boxed{= a(\cos\theta + i\sin\theta) = a\cos\theta + ia\sin\theta}$$

The <u>magnitude</u> of $|\tilde{z}(\theta)|$: $\boxed{|\tilde{z}(\theta)| = a}$ $\boxed{|\tilde{z}(\theta)| = \sqrt{|\mathrm{Re}\{\tilde{z}(\theta)\}|^2 + |\mathrm{Im}\{\tilde{z}(\theta)\}|^2}}$

$\boxed{\mathrm{Re}\{\tilde{z}(\theta)\} = a\cos\theta}$ and $\boxed{\mathrm{Im}\{\tilde{z}(\theta)\} = a\sin\theta}$



$$\boxed{\tan\theta = \frac{\mathrm{Im}\{\tilde{z}(\theta)\}}{\mathrm{Re}\{\tilde{z}(\theta)\}}}$$

$$\boxed{|\tilde{z}(\theta)| = |ae^{i\theta}| = a|e^{i\theta}| = a = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a\sqrt{\cos^2\theta + \sin^2\theta} = a}$$

For a purely <u>real</u> wave function $f(z,t)$: $\boxed{f(z,t) = A\cos(kz - \omega t + \delta)}$ we can equivalently write this using complex notation as: $\boxed{f(z,t) = A\cos(kz - \omega t + \delta) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]}$

For a <u>complex</u> wave function $\tilde{f}(z,t)$: $\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} = \tilde{A}\cos(kz - \omega t) + i\tilde{A}\sin(kz - \omega t)}$

with <u>complex</u> amplitude $\tilde{A}$: $\boxed{\tilde{A} \equiv Ae^{i\delta}}$

A = <u>Real</u> number

<u>Then</u>: $\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} = Ae^{i\delta}e^{i(kz-\omega t)} = Ae^{i[(kz-\omega t)+\delta]} = A\{\cos(kz - \omega t + \delta) + i\sin(kz - \omega t + \delta)\}}$

Griffiths Example 9.1 - Linear Superposition of Two Sinusoidal Waves:

Suppose that we have a situation where two real sinusoidal traveling waves $f_1(z,t) = A_1\cos(kz - \omega t + \delta_1)$ and $f_2(z,t) = A_2\cos(kz - \omega t + \delta_2)$ are simultaneously present at the same point z that have the same frequency f (and thus same wavelength λ, angular frequency ω, and wavenumber k) but have different amplitudes A_1, A_2 and (absolute) phases δ_1, δ_2 {defined relative to a common chosen origin of time, $t = 0$}.

We *can* simply add the two waves together: $\boxed{f_3(z,t) = f_1(z,t) + f_2(z,t)}$ however, this approach will involve some rather tedious algebra and use of trigonometric identities to obtain $f_3(z,t)$.

A much easier method is to carry this out using complex notation:

$$\boxed{f_3(z,t) = f_1(z,t) + f_2(z,t) = \mathrm{Re}\{\tilde{f}_1(z,t)\} + \mathrm{Re}\{\tilde{f}_2(z,t)\} = \mathrm{Re}\{\tilde{f}_1(z,t) + \tilde{f}_2(z,t)\} = \mathrm{Re}\{\tilde{f}_3(z,t)\}}$$

with: $\boxed{\tilde{f}_3(z,t) = \tilde{f}_1(z,t) + \tilde{f}_2(z,t)}$ $\boxed{\tilde{f}_1(z,t) \equiv \tilde{A}_1 e^{i(kz-\omega t)}}$ $\boxed{\tilde{f}_2(z,t) \equiv \tilde{A}_2 e^{i(kz-\omega t)}}$ $\boxed{\tilde{f}_3(z,t) \equiv \tilde{A}_3 e^{i(kz-\omega t)}}$

$$\boxed{\tilde{f}_3(z,t) \equiv \tilde{A}_3 e^{i(kz-\omega t)} = \tilde{f}_1(z,t) + \tilde{f}_2(z,t) = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)}}$$

thus: $\boxed{\tilde{A}_3 e^{i(kz-\omega t)} = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)}} \Rightarrow \boxed{\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2}$ or: $\boxed{A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}}$

writing this last relation out in its explicit complex form:

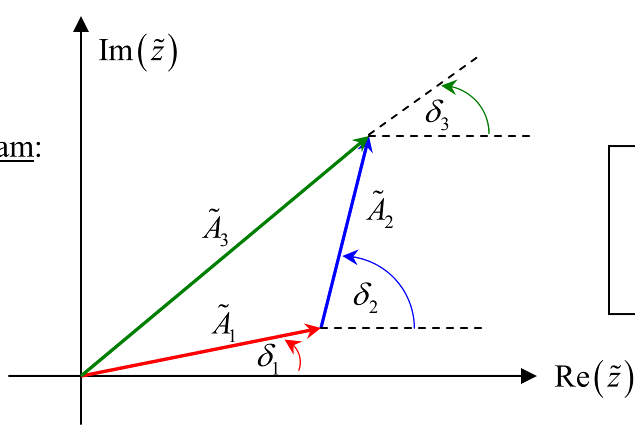
$$\boxed{A_3\cos\delta_3 + iA_3\sin\delta_3 = A_1\cos\delta_1 + iA_1\sin\delta_1 + A_2\cos\delta_2 + iA_2\sin\delta_2}$$

Thus we see that:

$$\boxed{\mathrm{Re}(\tilde{A}_3) = A_3\cos\delta_3 = A_1\cos\delta_1 + A_2\cos\delta_2} \qquad \boxed{\mathrm{Im}(\tilde{A}_3) = A_3\sin\delta_3 = A_1\sin\delta_1 + A_2\sin\delta_2}$$

We can either use the so-called *Phasor Diagram* in the complex plane to obtain $\mathrm{Re}(\tilde{A}_3)$ and $\mathrm{Im}(\tilde{A}_3)$, or wade through the tedious trigonometry and algebra.

Phasor Diagram:



n.b. phases are always defined relative to the Real axis

The use of the phasor diagram does not allow us to evade the use of algebra and trigonometry…

What we are essentially doing here is nothing more than adding two 2-dimensional vectors together, *i.e.* $\boxed{\vec{C}=\vec{A}+\vec{B}}$ where $\boxed{\vec{A}\equiv a_x\hat{x}+a_y\hat{y}=a\cos\theta_1\hat{x}+a\sin\theta_1\hat{y}}$ $\boxed{\vec{B}\equiv b_x\hat{x}+b_y\hat{y}=b\cos\theta_2\hat{x}+b\sin\theta_2\hat{y}}$ and $\boxed{\vec{C}\equiv c_x\hat{x}+c_y\hat{y}=c\cos\theta_3\hat{x}+c\sin\theta_3\hat{y}}$

Then: $\boxed{c_x=c\cos\theta_3=a_x+b_x=a\cos\theta_1+b\cos\theta_2}$ and $\boxed{c_y=c\sin\theta_3=a_y+b_y=a\sin\theta_1+b\sin\theta_2}$.

The magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are $\boxed{\left|\vec{A}\right|=a=\sqrt{a_x^2+a_y^2}}$, $\boxed{\left|\vec{B}\right|=b=\sqrt{b_x^2+b_y^2}}$ and $\boxed{\left|\vec{C}\right|=c=\sqrt{c_x^2+c_y^2}}$

The phase angles are: $\boxed{\delta_1=\tan^{-1}\left(a_y/a_x\right)}$, $\boxed{\delta_2=\tan^{-1}\left(b_y/b_x\right)}$ and $\boxed{\delta_3=\tan^{-1}\left(c_y/c_x\right)}$.

Thus, for the addition of two complex amplitudes, we see that:

$$\begin{aligned}\left|\tilde{A}_3\right|&=\sqrt{\tilde{A}_3\bullet\tilde{A}_3^*}=\sqrt{\left(\tilde{A}_1+\tilde{A}_2\right)\bullet\left(\tilde{A}_1+\tilde{A}_2\right)^*}=\sqrt{\left(A_1e^{i\delta_1}+A_2e^{i\delta_2}\right)\left(A_1e^{-i\delta_1}+A_2e^{-i\delta_2}\right)}\\&=\sqrt{A_1^2+A_2^2+A_1A_2\left(e^{i\delta_1}e^{-i\delta_2}+e^{-i\delta_1}e^{i\delta_2}\right)}=\sqrt{A_1^2+A_2^2+A_1A_2\left[2\cos\left(\delta_2-\delta_1\right)\right]}\\&=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\delta_{12}}\qquad\text{where }\ \Delta\delta_{21}\equiv\left(\delta_2-\delta_1\right)\end{aligned}$$

Then: $\boxed{\left|\tilde{A}_3\right|=A_3=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\delta_{12}}}$ with $\boxed{\Delta\delta_{21}\equiv\left(\delta_2-\delta_1\right)}$

The {absolute} phase angle δ_3 can be obtained from:

$$\boxed{\tan\delta_3=\frac{A_3\sin\delta_3}{A_3\cos\delta_3}=\frac{\sin\delta_3}{\cos\delta_3}=\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}}\ \textit{i.e.}\ \boxed{\delta_3=\tan^{-1}\left\{\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}\right\}}\ \text{radians}$$

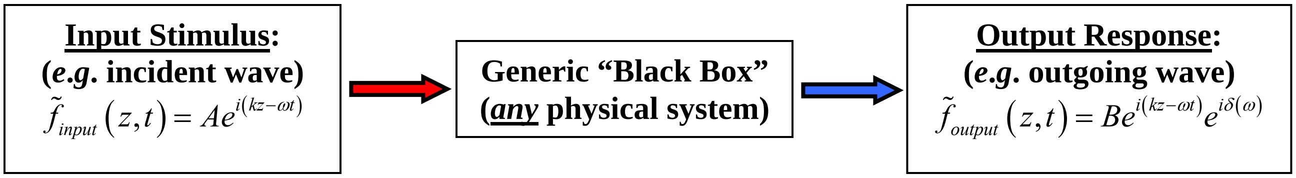
<u>Why Use Complex Notation?</u> $\boxed{\tilde{f}\left(z,t\right)=\tilde{A}e^{i(kz-\omega t)}=Ae^{i(kz-\omega t+\delta)}}$

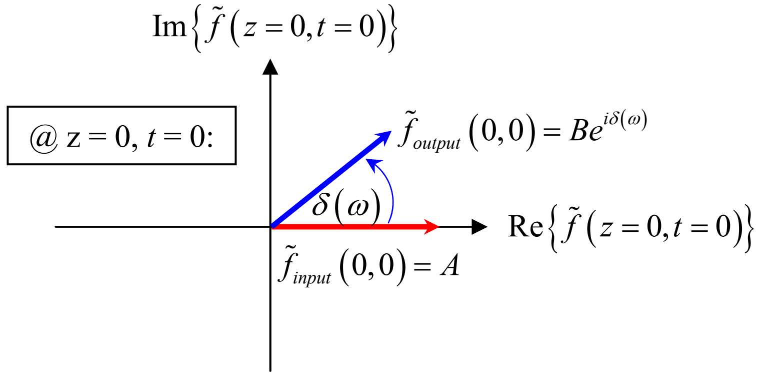
Whenever we have <u>two or more</u> waves, if ∃ (there exists) a <u>definite phase relation</u> (defined at some specific origin of time $t = t_o$) between them (*i.e.* the two or more waves are <u>coherent</u>) then the waves will <u>*interfere*</u> with each other at a given point in space, z.

Interference phenomena occurs at the <u>*amplitude*</u> level – *i.e.* wave amplitudes interfere for waves that are coherent / have a (well defined) definite phase-relation.

Using complex notation, phase information can be explicitly {and relatively easily} carried out mathematically properly describing the interference of two (or more) amplitudes of waves. ∃ does exist other ways to do this, but they are more tedious, algebraically…

Note also that the use of complex notation allows us to explicitly describe properly / mathematically the phase-shifts that can / do occur in the response of a system (a "black box") to an input stimulus / input signal:





Classical Systems – Interference of Wave Amplitudes

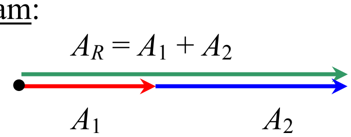
Consider two traveling waves interfering with each other in a non-dispersive medium with different frequencies and amplitudes. For a non-dispersive medium, this means that $\boxed{v = f_1\lambda_1 = f_2\lambda_2 = \omega_1/k_1 = \omega_2/k_2}$ with angular frequencies of $\boxed{\omega_1 = 2\pi f_1}$, $\boxed{\omega_2 = 2\pi\lambda_2}$ and wavenumbers $\boxed{k_1 = 2\pi/\lambda_1}$, $\boxed{k_2 = 2\pi/\lambda_2}$.

Then: $\boxed{\tilde{f}_{TOT}(z,t) = \tilde{f}_1(z,t) + \tilde{f}_2(z,t) = A_1e^{i(k_1z-\omega_1t+\delta_1)} + A_2e^{i(k_2z-\omega_2t+\delta_2)}}$

Easy cases of phase relations between the two waves:

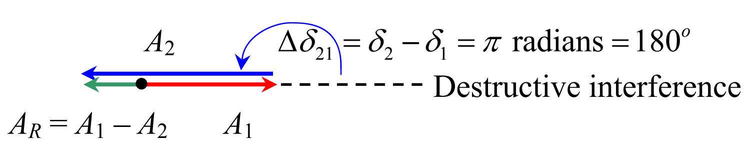
1.) $\delta_1 = \delta_2$ (in phase). Then $\Delta\delta_{21} = \delta_2 - \delta_1 = 0$ radians $= 0^o$

Phasor diagram:

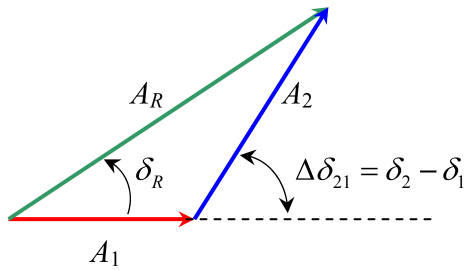


Constructive interference

2.) $\delta_2 = \delta_1 + \pi$ (180^o out of phase). Then: $\Delta\delta_{21} = \delta_2 - \delta_1 = \pi$ radians $= 180^o$



3.) <u>The General Case</u>:



From the diagram, we see that the magnitude of the resultant (*i.e.* net or total) amplitude A_R is:

$$\boxed{\begin{aligned} A_R = \left|\tilde{f}_{TOT}(z,t)\right| &= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\Delta\delta_{21})} \\ &= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_2 - \delta_1)} \end{aligned}}$$

(*n.b.* this is simply the Law of cosines!!!)

$$\boxed{A_R = \sqrt{\tilde{f}_{TOT}(z,t) * \tilde{f}^*_{TOT}(z,t)}}$$

Complex conjugate of $\tilde{f}_{TOT}(z,t)$

<u>Thus if</u>: $\boxed{\tilde{f}_1(z,t) = A_1 e^{i(kz-\omega t+\delta_1)}}$ and $\boxed{\tilde{f}_2(z,t) = A_2 e^{i(kz-\omega t+\delta_2)}}$

<u>Then</u>: $\boxed{\tilde{f}_{TOT}(z,t) = \tilde{f}_1(z,t) + \tilde{f}_2(z,t) = A_R e^{i(kz-\omega t+\delta_R)}}$

<u>Where</u>: $\boxed{A_R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_2 - \delta_1)} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}}$

n.b. Cosine is an even function of its argument, thus $\boxed{\Delta\delta_{21} = \Delta\delta}$

Then if the two waves have <u>equal</u> amplitudes, *i.e.* $\boxed{A_1 = A_2 = A}$:

1.) If $\delta_1 = \delta_2$ (in phase with each other), then $\Delta\delta = \delta_2 - \delta_1 = 0$ and hence:

$\cos(\delta_2 - \delta_1) = 1 \Rightarrow$ {total} <u>constructive</u> interference.

<u>Resultant Amplitude</u>: $\boxed{A_R = \sqrt{A^2 + A^2 + 2A^2\cos 0} = \sqrt{4A^2} = 2A}$

2.) If $\delta_2 = \delta_1 \pm \pi \Rightarrow \Delta\delta = \delta_2 - \delta_1 = \pm\pi = \pm 180^o$ out of phase with each other and hence:

$\cos(\delta_2 - \delta_1) = \cos(\pm\pi) = -1 \Rightarrow$ {total} <u>destructive</u> interference.

<u>Resultant Amplitude</u>: $\boxed{A_R = \sqrt{A^2 + A^2 + 2A^2\cos\pi} = \sqrt{A^2 + A^2 - 2A^2} = 0}$

Note that classical wave interference effects <u>can/do</u> occur at the amplitude level even if $f_1 \neq f_2$
e.g. Sound waves on strings, in air, …

Electronic signals
EM waves
etc.

$$\begin{cases} f_1 \approx f_2 \rightarrow \text{beats phenomena} & \text{(this is a form of interference)} \\ f_1 \gg f_2 \rightarrow \text{modulation phenomena} & \text{(also a form of interference)} \\ \text{(or vice versa)} & \end{cases}$$

Note also that amplitude interference effects occur in the world of quantum mechanics – *i.e.* matter waves – but is somewhat more complicated – *e.g.* by line width effects and/or uncertainty principle effects. Only identical particles with the exact same quantum numbers (external and internal) can interfere with each other…

Fourier's Theorem – Fourier Transforms

Any wave {whose derivatives exist / are well defined everywhere} can be expressed mathematically precisely as a linear combination of sine-type waves:

$$\boxed{\tilde{f}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)e^{i(kz-\omega t)}dk} \qquad \omega = vk$$

The integral over negative wavenumbers, $-k$ simply means that those waves are propagating in the $-\hat{z}$ direction. Thus technically speaking, we really should write λ = spatial wavelength = $\boxed{\lambda = 2\pi/|k|}$ and $\boxed{\omega = 2\pi f = |k|v}$.

The complex amplitudes $\tilde{A}(k)$ can be obtained from initial conditions @ $t = 0$:

$$f(z,t=0)= \underline{\qquad} \quad \text{and} \quad \dot{f}(z,t=0)\equiv\frac{\partial f(z,t=0)}{\partial t}= \underline{\qquad}$$

and use of the Fourier transform: $\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z,t)e^{-i(kz-\omega t)}dz}$

Obtaining: $\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(z,0)+\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}dz}$

See Griffiths Problem 9.32

Wave Intensity, *I*:

Wave intensities are proportional to $\left(\tilde{A}\tilde{A}^*\right)$. Thus, the total intensity $I \;\alpha \left(\tilde{f}_{TOT}(z,t)\bullet\tilde{f}^*_{TOT}(z,t)\right)$

In above previous example, the individual normalized intensities are: $\boxed{I_1 = A_1^2}$ and $\boxed{I_2 = A_2^2}$.

The corresponding resultant normalized intensity is: $\boxed{I_R = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos(\delta_2-\delta_1)}$

Then for the special case of $A_1 = A_2 = A$ or equivalently $I_1 = I_2 = I_o$, Then *e.g.* for ($z = 0$, $t = 0$):

1.) $\delta_2 = \delta_1$ then: $\Delta\delta = \delta_2 - \delta_1 = 0$, $\cos(\delta_2-\delta_1) = +1$: $I_R = 4\,I_o$ constructive interference.

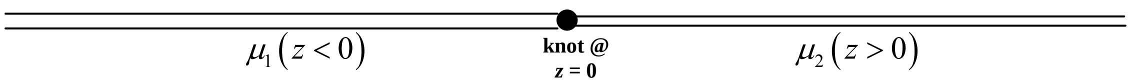
2.) $\delta_2 = \delta_1 \pm \pi$ then: $\Delta\delta = \delta_2 - \delta_1 = \pm\pi$, $\cos(\delta_2-\delta_1) = -1$: $I_R = 0$ destructive interference.

Let's consider our vibrating string problem again:

<u>Boundary Conditions (End Conditions), Wave Reflection and Wave Transmission</u>

Mechanical wave behavior / wave motion *e.g.* on a string as a function of time and space depends critically on the end conditions / boundary conditions – *i.e.* on how rigidly (or not) the string is attached at its ends.

Or, *e.g.* could have two dispersionless strings tied together in a knot (say @ $z = 0$), but one string has mass per unit length $\mu_1(z<0)$, the second string has mass per unit length $\mu_2(z>0)$. Both strings are stretched and rigidly attached at LHS and RHS ends (string is infinitely long) @ $z = \pm\infty$.



For this latter situation, the longitudinal speed of propagation of waves on a dispersionless string is $v = \sqrt{T/\mu}$ where the string tension T = <u>same</u> in <u>both</u> strings {otherwise $F \neq 0$ in <u>equilibrium</u> – this <u>can't</u> happen, because if $F \neq 0$, then Newton's 2nd Law $F = ma$ → <u>*something*</u> <u>accelerates</u> → therefore <u>*must*</u> have $F = 0$ in <u>equilibrium</u>}.

Thus, for the 1st string with $\mu_1(z<0)$: $\boxed{v_1 = \sqrt{T/\mu_1}\,(z<0)}$

and for the 2nd string with $\mu_2(z>0)$: $\boxed{v_2 = \sqrt{T/\mu_2}\,(z>0)}$

<u>But</u>: $\boxed{\dfrac{v_1}{v_2} = \dfrac{f_1\lambda_1}{f_2\lambda_2}}$

However, the frequencies of oscillation associated with a single vibrating string composed of 2 different strings types of tied together @ $z = 0$ are the <u>same</u> - *i.e.* $f_1 = f_2 = f$ and thus the angular frequencies are the same, *i.e.* $\omega_1 = 2\pi f_1 = \omega_2 = 2\pi f_2 = \omega$.

$$\therefore \boxed{\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1/2\pi}{\lambda_2/2\pi} = \frac{k_2}{k_1}}$$

Suppose that an <u>incident</u> traveling wave propagates in the $+\hat{z}$ direction, initially coming from the LHS portion ($z < 0$) of string, *i.e.* to left of the knot @ $z = 0$: $\boxed{\tilde{f}_{inc}(z,t) = \tilde{A}_{inc}e^{i(k_1 z-\omega t)}}$ ($z < 0$).

This wave is incident on the knot @ $z = 0$.

However, because of the mismatch / discontinuity in materials of the string to the left ($z < 0$) and to the right ($z > 0$) of $z = 0$, some portion of incident wave is reflected from knot and propagates in $-\hat{z}$ direction (*i.e.* traveling <u>backwards</u> along string 1): $\boxed{\tilde{f}_{refl}(z,t) = \tilde{A}_{refl}e^{i(-k_1 z-\omega t)}}$ ($z < 0$).

Some portion of the incident wave is transmitted past / through knot @ $z = 0$ and propagates in the $+\hat{z}$ direction (traveling along string 2): $\boxed{\tilde{f}_{trans}(z,t) = \tilde{A}_{trans}e^{i(k_2 z-\omega t)}}$ ($z > 0$).

Assuming that both strings are <u>ideal</u> (*i.e.* they are <u>dissipationless</u>), then energy and linear momentum are <u>both</u> conserved in this <u>scattering</u> process at the discontinuity / knot @ $z = 0$.

For simplicity's sake, if the incident wave is an infinitely long sinusoidal wave, the net disturbance (net / total displacement amplitude, using the superposition principle is:

$$\boxed{\tilde{f}_{TOT}^{LHS}(z,t) = \tilde{A}_{inc}e^{i(k_1 z-\omega t)} + \tilde{A}_{refl}e^{i(-k_1 z-\omega t)}} \qquad (z<0)$$

$$\boxed{\tilde{f}_{TOT}^{RHS}(z,t) = \tilde{A}_{trans}e^{i(k_2 z-\omega t)}} \qquad (z>0)$$

However @ $z = 0$: $f_{TOT}(z,t)$ $\left(\mathrm{Re}\left(\tilde{f}_{TOT}(z,t)\right)\right)$ must be continuous {recall that physically, $f_{TOT}(z,t)$ corresponds to the transverse displacement of the string at the space-time point (z,t)}. Mathematically, this translates to a Dirichlet-type boundary condition @ $z = 0$:
$\boxed{f_{TOT}^{LHS}(0,t) = f_{TOT}^{RHS}(0,t)}$ = transverse displacement = same value on both sides of "point" knot.

If the knot physically has zero (or negligible) mass, then the slopes of $f_{TOT}(z=0,t)$ are the same on both sides of the "point" knot. Mathematically, this translates to a Neumann-type boundary condition @ $z = 0$: $\boxed{\left.\frac{\partial f_{TOT}^{LHS}(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial f_{TOT}^{RHS}(z,t)}{\partial z}\right|_{z=0}}$

Note that the above boundary conditions (BC's) technically apply only to $\mathrm{Re}\left[\tilde{f}_{TOT}(z,t)\right]$ at $z = 0$, but note that $\mathrm{Im}\left[\tilde{f}_{TOT}(z,t)\right]$ differs from $\mathrm{Re}\left[\tilde{f}_{TOT}(z,t)\right]$ simply by replacing cosine ( ) with sine ( ). → Hence, the BC's apply to $\tilde{f}_{TOT}(z,t)$.

BC1 @ $z = 0$: The complex value of total amplitude @ $z = 0$: $\boxed{\tilde{f}_{TOT}^{LHS}(z=0,t) = \tilde{f}_{TOT}^{RHS}(z=0,t)}$

BC2 @ $z = 0$: The complex value of amplitude slopes @ $z = 0$: $\boxed{\left.\frac{\partial \tilde{f}_{TOT}^{LHS}(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial \tilde{f}_{TOT}^{RHS}(z,t)}{\partial z}\right|_{z=0}}$

Physically, continuity of the slope implies that there are no additional forces operative at the knot ($z = 0$).

From BC1 @ $z = 0$: $\boxed{\tilde{A}_{inc} + \tilde{A}_{refl} = \tilde{A}_{trans}}$ *n.b.* We thus have *two* equations

From BC2 @ $z = 0$: $\boxed{k_1\left(\tilde{A}_{inc} - \tilde{A}_{refl}\right) = k_2\tilde{A}_{trans}}$ and *three* unknowns $\left(\tilde{A}_{inc}, \tilde{A}_{refl}, \tilde{A}_{trans}\right)$

k_1 and k_2 are assumed to be known / given (*e.g.* $T = 100$ N, $f = 100$ Hz)

→ Can express $\tilde{A}_{refl}$ and $\tilde{A}_{trans}$ in terms of $\tilde{A}_{inc}$ – Solve BC1 and BC2 simultaneously to obtain:

$$\boxed{\tilde{A}_{refl} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_{inc} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)\tilde{A}_{inc}} \Leftarrow \text{using} \boxed{\left(\frac{k_2}{k_1}\right) = \left(\frac{v_1}{v_2}\right)}$$

$$\boxed{\tilde{A}_{trans} = \left(\frac{2k_1}{k_1 + k_2}\right)\tilde{A}_{inc} = \left(\frac{2v_2}{v_1 + v_2}\right)\tilde{A}_{inc}} \Leftarrow \text{using} \boxed{\left(\frac{k_2}{k_1}\right) = \left(\frac{v_1}{v_2}\right)}$$

The real amplitudes and the phases are thus related by:

$$\boxed{A_{refl}e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}}$$
$$\boxed{A_{trans}e^{i\delta_T} = \left(\frac{2v_2}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}}$$

If string 2 (on the RHS, $z > 0$) is lighter than string 1 (on the LHS, $z < 0$), *i.e.* $\mu_2 < \mu_1 \Rightarrow v_2 > v_1$ since $\boxed{v_2 = \sqrt{T/\mu_2}\,(z > 0)}$, $\boxed{v_1 = \sqrt{T/\mu_1}\,(z < 0)}$ then $\boxed{(v_2 - v_1) > 0} \rightarrow$ all three wave amplitudes have the same phase angle, *i.e.* $\boxed{\delta_I = \delta_R = \delta_T}$.

Thus for $\mu_2 < \mu_1$ (or $v_2 > v_1$), $\boxed{\delta_I = \delta_R = \delta_T}$ and hence:

$$\boxed{A_{refl}e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}} \Rightarrow \boxed{A_{refl} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc}}$$
$$\boxed{A_{trans}e^{i\delta_T} = \left(\frac{2v_2}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}} \Rightarrow \boxed{A_{trans} = \left(\frac{2v_2}{v_1 + v_2}\right) A_{inc}}$$

If string 2 (on the RHS, $z > 0$) is heavier than string 1 (on the LHS, $z < 0$), *i.e.* $\mu_2 > \mu_1 \Rightarrow v_2 < v_1$: then: $\boxed{(v_2 - v_1) < 0} \rightarrow$ the reflected wave is 180° (= π radians) out of phase with the incident wave *i.e.* $\cos(-k_1 z - \omega t + \delta_I - \pi) = -\cos(-k_1 z - \omega t + \delta_I)$. The polarity of reflected wave is flipped relative to incident wave, *i.e.* $\boxed{\delta_I = \delta_R \pm \pi = \delta_T}$! Note that: $\boxed{e^{\pm i\pi} = \cos\pi \pm i\sin\pi = -1}$

Thus for $\mu_2 > \mu_1$ (or $v_2 < v_1$), $\boxed{\delta_I = \delta_R \pm \pi = \delta_T}$ and hence:

$$\boxed{A_{refl}e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}} \Rightarrow \boxed{A_{refl} = -\left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc} = +\left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_{inc}}$$
$$\boxed{A_{trans}e^{i\delta_T} = \left(\frac{2v_2}{v_1 + v_2}\right) A_{inc}e^{i\delta_I}} \Rightarrow \boxed{A_{trans} = \left(\frac{2v_2}{v_1 + v_2}\right) A_{inc}}$$

If *e.g.* string 2 (on the RHS, $z > 0$) is infinitely massive, *i.e.* $\boxed{\mu_2 = \infty} \Rightarrow \boxed{v_2 = \sqrt{T/\mu_2} = 0}$ $(z > 0)$
Then we see that: $\boxed{A_{refl} = A_{inc}}$ and $\boxed{A_{trans} = 0}$.

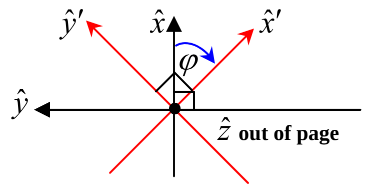
Wave Polarization

Depending (largely) on the type of wave and the nature of the medium that the waves are propagating in / on, the waves can have another degree of freedom known as polarization.

Waves propagating in $\pm\hat{z}$ direction with small transverse displacement amplitude on a string are known as transverse waves because the displacement of string (relative to its equilibrium position) is transverse to the direction of propagation – (*i.e.* $\vec{v}_{prop} = \pm\hat{z}$).

Thus the transverse displacement amplitude $\boxed{\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}}$ is oriented *e.g.* in $\pm\hat{x}$ and/or $\pm\hat{y}$ directions for a traveling transverse wave propagating in/along the $+\hat{z}$ direction.

→ Thus transverse traveling waves have two polarization states, either the $\pm\hat{x}$ or the $\pm\hat{y}$ direction, or equivalently 2 orthogonal {*i.e.* mutually-perpendicular} linear combinations of the $\pm\hat{x}$ and $\pm\hat{y}$ basis states for waves propagating in the $\pm\hat{z}$ direction:



Propagation of *e.g.* longitudinal sound waves in solid or non-solid media, *e.g.* normal gases, liquids and solids also has longitudinal polarization – because longitudinal sound waves have longitudinal displacements of atoms / molecules – *i.e.* along / against (*i.e.* parallel/anti-parallel to) the direction of propagation of the longitudinal wave, *e.g.* in the $\pm\hat{z}$ direction.

Here, the longitudinal displacement amplitude is (also) of the form: $\boxed{\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}}$

→ Longitudinal traveling waves have only one polarization state, *e.g.* the $\hat{z}$ direction.

Both longitudinal and transverse waves obey the same wave equation: $\boxed{\dfrac{\partial^2 \tilde{f}(z,t)}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 \tilde{f}(z,t)}{\partial t^2}}$

Longitudinal Waves: *e.g.* sound waves / acoustic waves – liquids, gases and solids and *e.g.* large amplitudes in strings (compression waves)

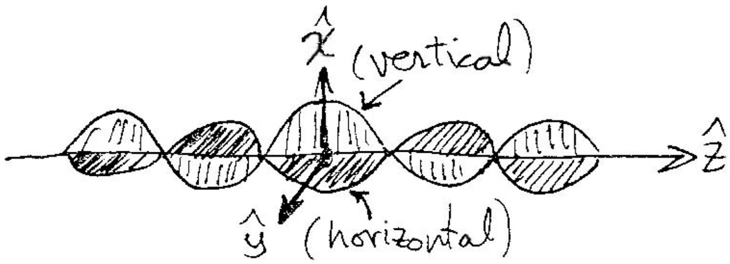
Transverse Waves: *e.g.* Small and large amplitudes in strings, long solid rods, solid bars, *etc.* (shear waves). *EM* waves are transverse waves.

$\exists$ Two orthogonal polarization states for transverse waves – thus, we can represent the transverse displacement amplitude as a vector quantity, indicating its polarization state:

Transverse displacement is *e.g.* in:

Vertical Plane – "vertical" polarization (up & down): $\boxed{\tilde{\vec{f}}_{vert}(z,t) = \tilde{A}e^{i(kz-wt)}\hat{x}}$

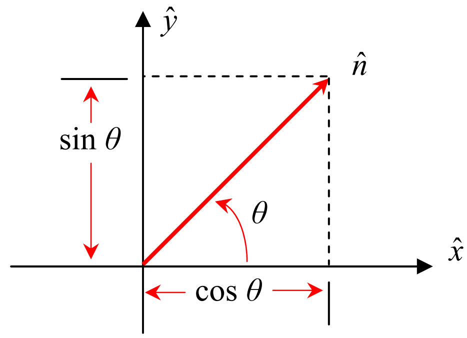
Horizontal Plane – "horizontal" polarization (sideways): $\boxed{\tilde{\vec{f}}_{horiz}(z,t) = \tilde{A}e^{i(kz-wt)}\hat{y}}$



(Direction of propagation)

The polarization unit vector $\hat{n}$ lying in the transverse plane defines the plane of polarization: (*i.e.* plane of transverse vibrations {here})

Note that: $\boxed{\hat{n}\bullet\hat{z} \equiv 0}$



Define the polarization angle θ wrt to $\hat{x}$ axis:

$$\boxed{\hat{n} = \cos\theta\hat{x} + \sin\theta\hat{y}}$$

Linearly Polarized Transverse Waves

A linearly polarized transverse wave that propagates in the $\hat{z}$ direction is such that both the $\hat{x}$ and $\hat{y}$ components of the vector transverse displacement amplitude of the wave are either precisely in-phase with each other (*i.e.* have 0° relative phase), or they can also be precisely ± 180° out-of-phase with each other.

Examples of Linearly Polarized Transverse Waves: $\boxed{\tilde{\vec{f}}(z,t) = \tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x} + \tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}}$

$$\boxed{\tilde{\vec{f}}(z,t) = \tilde{A}\cos\theta e^{i(kz-\omega t)}\hat{x} - \tilde{A}\sin\theta e^{i(kz-\omega t)}\hat{y}}$$

Circularly Polarized Transverse Waves

A circularly polarized transverse wave that propagates in the $\hat{z}$ direction is such that the $\hat{x}$ and $\hat{y}$ components of the transverse wave have *a*.) equal amplitudes, but *b*.) the $\hat{x}$ and $\hat{y}$ components of the vector transverse displacement amplitude of the wave differ in phase by $\pm 90^o = \pm\pi/2$ from each other (*e.g.* $\delta_{vert} = 0^o$, $\delta_{horiz} = \pm 90^o = \pm\pi/2$ radians)

Example # 1: $\boxed{\vec{f}_{vert}(z,t) = A\cos(kz-\omega t)\hat{x}}$

$$\boxed{\vec{f}_{horiz}(z,t) = A\cos\left[(kz-\omega t)+90^o\right]\hat{y} = -A\sin(kz-\omega t)\hat{y}}$$

But note that $\boxed{f_{vert}^2 + f_{horiz}^2 = A^2}$ thus $\boxed{\vec{f}(z,t) = \vec{f}_{vert}(z,t) + \vec{f}_{horiz}(z,t)}$ lies on a circle of radius A.

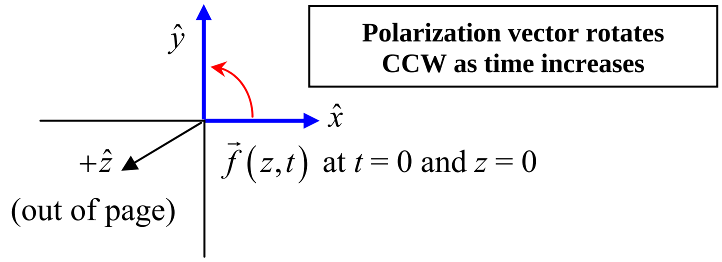
At time $t = 0$: $\boxed{\vec{f}(z,t=0) = A\cos(kz)\hat{x} - A\sin(kz)\hat{y}}$

At time $t = \frac{\pi}{2\omega}$: $\boxed{\vec{f}\left(z,t=\frac{\pi}{2\omega}\right) = A\cos(kz-90^o)\hat{x} - A\sin(kz-90^o)\hat{y} = A\sin(kz)\hat{x} + A\cos(kz)\hat{y}}$

→ At a fixed position in space, $\vec{f}(z,t)$ rotates counter-clockwise CCW (in the x-y plane) as time increases, for a wave propagating in the $+\hat{z}$ direction.

→ Known as LCP = Left Circular Polarization.

$\vec{f}(z,t)$ at $t = \pi/2\omega$ and $z = 0$:



Example # 2: $\boxed{f_{vert}(z,t) = A\cos(kz - wt)\hat{x}}$

$\boxed{f_{horiz}(z,t) = A\cos\left[(kz - wt) - 90^o\right]\hat{y} = +A\sin(kz - wt)\hat{y}}$

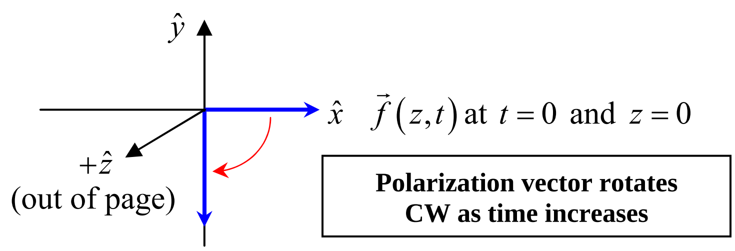
Again $\boxed{f_{vert}^2 + f_{horiz}^2 = A}$ and thus $\boxed{\vec{f}(z,t) = \vec{f}_{vert}(z,t) + \vec{f}_{horiz}(z,t)}$ lies on a circle of radius A.

At time $t = 0$: $\boxed{\vec{f}(z,t=0) = A\cos(kz)\hat{x} - A\sin(kz)\hat{y}}$

At time $t = \dfrac{\pi}{2\omega}$: $\boxed{f\left(z,t=\dfrac{\pi}{2\omega}\right) = A\cos\left(kz - 90^o\right)\hat{x} + A\sin\left(kz - 90^o\right)\hat{y} = A\sin(kz)\hat{x} - A\cos(kz)\hat{y}}$

→ At a fixed position in space, $f(z, t)$ rotates clockwise CW (in the x-y plane) as time increases, for a wave propagating in the $+\hat{z}$ direction

→ Known as RCP = Right Circular Polarization.



$\vec{f}(z,t)$ at $t = \dfrac{\pi}{2\omega}$ and $z = 0$

If interested, information on the propagation of acoustical waves, acoustic wave phenomena in general and ***e.g.*** solving the wave equation in 1-, 2- & 3-dimensions is available online (PDF format) from Professor Errede's lecture notes for the UIUC Physics 498 Physics of Music / Musical Instruments Course. The URL for the online lecture notes for the UIUC P498POM course is:

http://online.physics.uiuc.edu/courses/phys498pom/498pom_lectures.html