LECTURE NOTES 6

ELECTROMAGNETIC WAVES IN MATTER

Electromagnetic Wave Propagation in Linear Media

We now consider *EM* wave propagation inside matter, but only in regions where there are NO ***free*** charges and/or ***free*** currents (*e.g.* the medium is an insulator/non-conductor).

For this situation, Maxwell's equations become:

1) $\boxed{\vec{\nabla}\cdot\vec{D}(\vec{r},t)=0}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{H}(\vec{r},t)=\frac{\partial\vec{D}(\vec{r},t)}{\partial t}}$

The medium in which *EM* waves propagate is assumed to be linear, homogeneous and isotropic, thus the following relations are valid in this medium:

$$\boxed{\vec{D}(\vec{r},t)=\varepsilon\vec{E}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{H}(\vec{r},t)=\frac{1}{\mu}\vec{B}(\vec{r},t)}$$

Where:

ε = electric permittivity of the medium.

$\varepsilon=\varepsilon_o(1+\chi_e)$, χ_e = electric susceptibility of the medium.

μ = magnetic permeability of the medium.

$\mu=\mu_o(1+\chi_m)$, χ_m = magnetic susceptibility of the medium.

ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.

μ_o = magnetic permeability of free space = $4\pi\times10^{-7}$ Henrys/m.

Thus, Maxwell's equations for the $\vec{E}$ and $\vec{B}$ fields inside this linear, homogeneous and isotropic non-conducting medium become:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\varepsilon\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

Note that the above four relations are (almost) identical to those for *EM* waves in free space $\{cf$ with eqns. 1) - 4) on page 1 of P436 Lect. Notes 5$\}$. We simply replace the macroscopic *EM* parameters associated with the vacuum $\{\varepsilon_o,\mu_o\}$ with those associated with the linear, homogeneous and isotropic medium $\{\varepsilon,\mu\}$.

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- In free space/vacuum, the speed of propagation of *EM* waves is:

$\boxed{v_{prop} = \dfrac{1}{\sqrt{\varepsilon_o\mu_o}} = c} = 3\times10^8\ \text{m/s}$, the $\vec{E}$ and $\vec{B}$ fields in vacuum obey the wave equation:

$$\boxed{\nabla^2\vec{E}(\vec{r},t) = \varepsilon_o\mu_o\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \quad \boxed{\nabla^2\vec{B}(\vec{r},t) = \varepsilon_o\mu_o\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = \frac{1}{c^2}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

- In a linear /homogeneous/isotropic medium, the speed of propagation of *EM* waves is:

$\boxed{v'_{prop} = \dfrac{1}{\sqrt{\varepsilon\mu}}}$ and the $\vec{E}$ and $\vec{B}$ fields in the medium obey the following wave equation:

$$\boxed{\nabla^2\vec{E}(\vec{r},t) = \varepsilon\mu\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \quad \boxed{\nabla^2\vec{B}(\vec{r},t) = \varepsilon\mu\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

<u>For linear / homogeneous / isotropic media</u>:

$$\boxed{\begin{array}{ll} \varepsilon = K_e\varepsilon_o = (1+\chi_e)\varepsilon_o & K_e = \dfrac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity} \\ \mu = K_m\mu_0 = (1+\chi_m)\mu_o & K_m = \dfrac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability} \end{array}}$$

$$\therefore \boxed{v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e\varepsilon_o K_m\mu_o}} = \frac{1}{\sqrt{K_eK_m}}\frac{1}{\sqrt{\varepsilon_o\mu_o}} = \frac{1}{\sqrt{K_eK_m}}c} \quad i.e. \quad \boxed{v'_{prop} = \frac{1}{\sqrt{K_eK_m}}c}$$

<u>Now ***if***</u>:

$$\boxed{K_e = \left(\frac{\varepsilon}{\varepsilon_o}\right) = (1+\chi_e) \ge 1} \ \underline{\text{and}}\ \boxed{K_m = \left(\frac{\mu}{\mu_o}\right) = (1+\chi_m) \ge 1} \quad \underline{\text{or if}}\text{:} \ \boxed{K_eK_m \ge 1}$$

{true for a wide variety of common/everyday materials – gases, liquids & solids}

<u>Then</u>: $\boxed{\sqrt{K_eK_m} \ge 1}$ <u>thus</u>: $\boxed{\dfrac{1}{\sqrt{K_eK_m}} \le 1} \ \Rightarrow \ \boxed{v'_{prop} = \dfrac{1}{\sqrt{K_eK_m}}c \le c}$

Note also that since $\boxed{K_e = \dfrac{\varepsilon}{\varepsilon_o}}$ and $\boxed{K_m = \dfrac{\mu}{\mu_o}}$ are <u>dimensionless</u> quantities, then so is $\boxed{\dfrac{1}{\sqrt{K_eK_m}}}$.

We can now define the <u>index of refraction</u> {*n.b.* a dimensionless quantity} of the linear / homogeneous / isotropic medium as:

$$\boxed{n \equiv \sqrt{K_eK_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o\mu_o}}}$$

Thus, for linear / homogeneous / isotropic media: $\boxed{v'_{prop} = c/n\ (\le c)}$ because $\boxed{n \ge 1}$ {<u>*usually*</u>}.

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n.b. We will find out {soon!} that ε and μ are in fact ***not*** constants, instead they are {very often} frequency-dependent quantities, *i.e.* $\boxed{\varepsilon = \varepsilon(\omega)}$ and $\boxed{\mu = \mu(\omega)}$, $\boxed{\omega = 2\pi f}$.

Thus: $$\boxed{K_e = K_e(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_o} = 1 + \chi_e(\omega)} \text{ and } \boxed{K_m = K_m(\omega) = \frac{\mu(\omega)}{\mu_o} = 1 + \chi_m(\omega)}$$

Hence: $$\boxed{n = n(\omega) = \sqrt{K_e(\omega) K_m(\omega)} = \sqrt{\frac{\varepsilon(\omega)\mu(\omega)}{\varepsilon_o \mu_o}} = \sqrt{(1+\chi_e(\omega))(1+\chi_m(\omega))}}$$

For now, we will ignore/neglect any/all frequency-dependent effects, for simplicity, *i.e.*

$$\boxed{v'_{prop} = \frac{c}{n}} = \text{constant} \qquad \boxed{n = \sqrt{k_e k_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o\mu_o}} = \sqrt{(1+\chi_e)(1+\chi_m)}} = \text{constant}$$

Now for many (but not all) linear/homogeneous/isotropic materials: $\boxed{\mu = \mu_o(1+\chi_m) \simeq \mu_o}$

(*e.g.* true for many paramagnetic and diamagnetic-type materials) $\Rightarrow$ $\boxed{|\chi_m| \sim \vartheta(10^{-8}) \sim 0}$

Thus: $$\boxed{K_m = \frac{\mu}{\mu_o} = (1+\chi_m) \simeq 1} \Rightarrow \boxed{n \simeq \sqrt{K_e}} \text{ and } \boxed{v'_{prop} = \frac{c}{n} \simeq \frac{c}{\sqrt{K_e}}}.$$

- The instantaneous *EM* energy density associated with a linear/homogeneous/isotropic material:

$$\boxed{u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \frac{1}{\mu}B^2(\vec{r},t)\right) = \frac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t) + \vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right)} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with $\boxed{\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)}$ and $\boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$.

- The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material:

$$\boxed{\vec{S}(\vec{r},t) = \frac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right)} \left(\frac{\text{Watts}}{\text{m}^2}\right) \text{ with } \boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$$

- For monochromatic (*i.e.* sinusoidal, single frequency) plane *EM* waves propagating in a linear/homogeneous/isotropic medium, $\vec{E}$ and $\vec{B}$ satisfy/obey the wave equation:

$$\boxed{\nabla^2\vec{E}(\vec{r},t) = \varepsilon\mu\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}} \qquad \boxed{\nabla^2\vec{B}(\vec{r},t) = \varepsilon\mu\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}}\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}}$$

E and *B*-field solutions for a linearly polarized plane *EM* wave with polarization vector $\hat{n} \perp \hat{k}$ propagating in this linear/homogeneous/isotropic medium are of the form:

$$\boxed{\vec{E}(\vec{r},t) = E_o\cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)\hat{n}} \text{ and } \boxed{\vec{B}(\vec{r},t) = B_o\cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)\left(\hat{k}\times\hat{n}\right)}$$

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With: $\boxed{\vec{B}(\vec{r},t)=\frac{1}{v'_{prop}}\hat{k}\times\vec{E}(\vec{r},t)}$, thus: $\boxed{\left|\vec{B}(\vec{r},t)\right|=\frac{1}{v'_{prop}}\left|\vec{E}(\vec{r},t)\right|}$, *i.e.* $\boxed{B_o=\frac{1}{v'_{prop}}E_o}$

And: $\boxed{v_{prop}=f\lambda=\omega/k}$ with angular frequency $\boxed{\omega=2\pi f}$ and wavenumber $\boxed{k=2\pi/\lambda}$.

- The *intensity* of an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{I(\vec{r})\equiv\left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle=v'_{prop}\left\langle u_{EM}(\vec{r},t)\right\rangle=\frac{1}{2}v'_{prop}\varepsilon E_o^2(\vec{r})=\frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_o^2(\vec{r})=\left(\frac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Where $\boxed{E_{o_{rms}}\equiv\frac{1}{\sqrt{2}}E_o}$. The RMS *intensity* of the *EM* wave is:

$$\boxed{I_{rms}(\vec{r})\equiv\left\langle\left|\vec{S}_{rms}(\vec{r},t)\right|\right\rangle=v'_{prop}\left\langle u_{EM_{rms}}(\vec{r},t)\right\rangle=\frac{1}{2}v'_{prop}\varepsilon E_{o_{rms}}^2(\vec{r})=\frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})}\quad\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

i.e. $\boxed{I_{rms}(\vec{r})=\tfrac{1}{2}I(\vec{r})}$, $\boxed{\left\langle\left|\vec{S}_{rms}(\vec{r},t)\right|\right\rangle=\tfrac{1}{2}\left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle}$, $\boxed{\left\langle u_{EM_{rms}}(\vec{r},t)\right\rangle=\tfrac{1}{2}\left\langle u_{EM}(\vec{r},t)\right\rangle}$, *etc.*

- The instantaneous linear momentum density associated with an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t)=\varepsilon\mu\vec{S}(\vec{r},t)=\frac{1}{v'^2_{prop}}\vec{S}(\vec{r},t)=\varepsilon\cancel{\mu}\frac{1}{\cancel{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)=\varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\quad\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

- The instantaneous angular momentum density associated with an *EM* wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t)=\vec{r}\times\vec{\wp}_{EM}(\vec{r},t)=\varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\quad\left(\frac{\text{kg}}{\text{m-sec}}\right)$$

- And of course, an *EM* wave propagating in this medium has:

Total instantaneous *EM* energy: $\boxed{U_{EM}(t)=\int_v u_{EM}(\vec{r},t)\,d\tau}$ (Joules)

Total instantaneous linear momentum: $\boxed{\vec{p}_{EM}(t)=\int_v\vec{\wp}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power: $\boxed{P_{EM}(t)=\frac{\partial U_{EM}(t)}{\partial t}=-\oint_S\vec{S}(\vec{r},t)\cdot d\vec{a}}$ (Watts)

n.b. through a closed surface

Total instantaneous angular momentum: $\boxed{\vec{\mathcal{L}}_{EM}(t)=\int_v\vec{\ell}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$

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QUESTION:

What happens when an *EM* wave passes from one linear/homogeneous/isotropic medium into another (*e.g*. vacuum → gas; air → water; water → oil; glass → plastic; etc…?

As we saw in the case of mechanical transverse traveling waves propagating on the taught string which had two different mass-per-unit-lengths $(\mu_1 \text{ and } \mu_2)$, we anticipate that *EM* wave reflection and wave transmission phenomena will also occur at the interface/boundary between two different linear/homogeneous/isotropic media.

However, in the *EM* wave situation, what actually happens at the boundary/interface between two linear/homogeneous/isotropic media depends on the electro-dynamical versions of the boundary conditions on the $\vec{E}$ and $\vec{B}$-fields at that interface {as we derived last semester in P435 from the integral form of Maxwell's equations}:

BC 1) The NORMAL component of $\vec{D}$ is continuous across the interface (true only when there are no free surface charges present @ the interface):

$$\boxed{D_1^{\perp}(\vec{r},t)\Big|_{intf} = D_2^{\perp}(\vec{r},t)\Big|_{intf}} \Rightarrow \boxed{\varepsilon_1 E_1^{\perp}(\vec{r},t)\Big|_{intf} = \varepsilon_2 E_2^{\perp}(\vec{r},t)\Big|_{intf}} \text{ since } \boxed{\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)}$$

BC 2) The TANGENTIAL component of $\vec{E}$ is {always} continuous across the interface:

$$\boxed{E_1^{\parallel}(\vec{r},t)\Big|_{intf} = E_2^{\parallel}(\vec{r},t)\Big|_{intf}}$$

BC 3) The NORMAL component of $\vec{B}$ is {always} continuous across the interface:

$$\boxed{B_1^{\perp}(\vec{r},t)\Big|_{intf} = B_2^{\perp}(\vec{r},t)\Big|_{intf}}$$

BC 4) The TANGENTIAL component of $\vec{H}$ is continuous across the interface (true only when there are no free surface currents flowing @ the interface):

$$\boxed{H_1^{\parallel}(\vec{r},t)\Big|_{intf} = H_2^{\parallel}(\vec{r},t)\Big|_{intf}} \Rightarrow \boxed{\frac{1}{\mu_1}B_1^{\parallel}(\vec{r},t)\Big|_{intf} = \frac{1}{\mu_2}B_2^{\parallel}(\vec{r},t)\Big|_{intf}} \text{ since } \boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$$

Note {again} that the above boundary condition relations were all obtained from the integral form(s) of Maxwell's equations.

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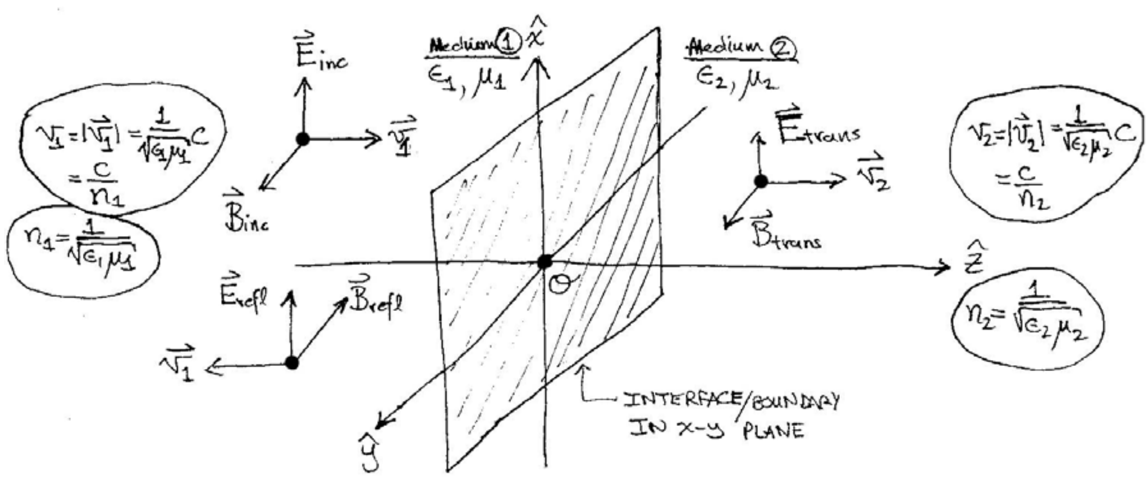
Reflection & Transmission of Linear Polarized Plane *EM* Waves at Normal Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

As shown in the figure below, a boundary between two linear / homogeneous / isotropic media lies in *x-y* plane, with a monochromatic plane *EM* wave of frequency ω propagating in the $+\hat{z}$ -direction, which is linearly polarized in $+\hat{x}$ -direction. Thus this *EM* wave approaches the boundary from the left and is at normal incidence to boundary:



We write down the complex amplitudes for the $\vec{E}$ and $\vec{B}$ -fields:

Incident *EM* plane wave (in medium 1):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}}$$

Reflected *EM* plane wave (in medium 1):

Propagates in the $-\hat{z}$ -direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\boxed{\hat{n}_{refl} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{refl} = \left|\vec{k}_{refl}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}}$$

Transmitted *EM* plane wave (in medium 2):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\boxed{\hat{n}_{trans} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2}$$

$$\boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}}$$

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Note that {*here*, in *this* situation} the $\vec{E}$-field / polarization vectors are all oriented in the *same* direction, *i.e.* $\boxed{\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}}$ or equivalently: $\boxed{\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)}$.

At the interface / boundary between the two linear / homogeneous / isotropic media, *i.e.* at $z = 0$ {in the x-y plane} the boundary conditions 1) – 4) *must* be satisfied for the total $\vec{E}$ and $\vec{B}$-fields immediately present on either side of the interface between the two media:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$
(*n.b.* $\perp$ refers to the x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$
(*n.b.* $\parallel$ refers to the x-y boundary, *i.e.* in the x-y plane)

BC 3) Normal $\vec{B}$ continuous: $\boxed{B^{\perp}_{1_{Tot}} = B^{\perp}_{2_{Tot}}}$ ($\perp$ to x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 4) Tangential $\vec{H}$ continuous: $\boxed{\frac{1}{\mu_1} B^{\parallel}_{1_{Tot}} = \frac{1}{\mu_2} B^{\parallel}_{2_{Tot}}}$ ($\parallel$ to x-y boundary, *i.e.* in x-y plane)

For plane *EM* waves at normal incidence on the boundary at $z = 0$ lying in the x-y plane, note that no components of $\vec{E}$ or $\vec{B}$ (incident, reflected or transmitted waves) are allowed to be along the $\pm\hat{z}$ propagation direction(s) because of the $\vec{E}$ and $\vec{B}$-field transversality requirement(s) on the propagation of *EM* waves {arising from constraints imposed by Maxwell's equations}.

Thus, because of this, we see that BC 1) and BC 3) impose no restrictions {here} on such *EM* waves since: { $E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0$; $E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0$ } and { $B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0$; $B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0$ }

$\Rightarrow$ The only restrictions on plane *EM* waves propagating with normal incidence on the boundary at $z = 0$ {lying in the x-y plane} are imposed by BC 2) and BC 4).

$\therefore$ At $z = 0$ in medium 1) (*i.e.* $z \le 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}(z=0,t) = \tilde{\vec{E}}_{inc}(z=0,t) + \tilde{\vec{E}}_{refl}(z=0,t)} \text{ and}$$
$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}(z=0,t) = \frac{1}{\mu_1}\tilde{\vec{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}_{refl}(z=0,t)}$$

While at $z = 0$ in medium 2) (*i.e.* $z \ge 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{2_{Tot}}(z=0,t) = \tilde{\vec{E}}_{trans}(z=0,t)} \text{ and}$$
$$\boxed{\frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}_{trans}(z=0,t)}$$

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Then BC 2) (Tangential $\vec{E}$ is continuous @ $z = 0$) requires that:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \tilde{\vec{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \quad \text{or:} \quad \boxed{\tilde{\vec{E}}_{inc}(z=0,t) + \tilde{\vec{E}}_{refl}(z=0,t) = \tilde{\vec{E}}_{trans}(z=0,t)}.$$

Then BC 4) (Tangential $\vec{H}$ is continuous @ $z = 0$) requires that:

$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \quad \text{or:} \quad \boxed{\frac{1}{\mu_1}\tilde{\vec{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}_{refl}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}_{trans}(z=0,t)}$$

Inserting the explicit expressions for the complex $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ fields

$$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}}\, e^{i(k_1 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}}\, e^{i(k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\hat{x}} \qquad \boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}}\, e^{i(k_2 z - \omega t)}\hat{y}}$$

into the above boundary condition relations, these equations become:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}}\cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}}\cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}}\cancel{e^{-i\omega t}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}}\cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}}\cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}\cancel{e^{-i\omega t}}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations, we have at $z = 0$ {*n.b.* <u>*everywhere*</u> in the *x*-*y* plane, which must be independent of/valid for <u>*any*</u> time *t*}:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Assuming that {μ_1 and μ_2} and {v_1 and v_2} are known / given for the two media, we have <u>two</u> equations {from BC 2) and BC 4)} and <u>*three*</u> unknowns { $\tilde{E}_{o_{inc}}$, $\tilde{E}_{o_{refl}}$, $\tilde{E}_{o_{trans}}$ }

→ Solve above equations <u>simultaneously</u> for { $\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}$ } in terms of / scaled to $\tilde{E}_{o_{inc}}$.

First (for convenience) let us define: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$

Then BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$) relation becomes: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}}$

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BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}}$ with $\boxed{\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}}$

Add BC 2) and BC 4) relations: $\boxed{2\tilde{E}_{o_{inc}} = (1+\beta)\, \tilde{E}_{o_{trans}}}$ $\Rightarrow$ $\boxed{\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}}$ (2+4)

Subtract (BC 2) – BC 4)) relations: $\boxed{2\tilde{E}_{o_{refl}} = (1-\beta)\tilde{E}_{o_{trans}}}$ $\Rightarrow$ $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{2}\right)\tilde{E}_{o_{trans}}}$ (2–4)

Insert the result of eqn. (2+4) into eqn. (2–4): $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{\cancel{2}}\right)\left(\dfrac{\cancel{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\dfrac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}}$

$\therefore$ $\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}}$ and $\boxed{\tilde{E}_{o_{trans}} = \left(\dfrac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}}$

<u>Now</u>: $\boxed{\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}}$ and: $\boxed{v_1 = \dfrac{c}{n_1}}$, $\boxed{v_2 = \dfrac{c}{n_2}}$ where: $\boxed{n_1 = \sqrt{\dfrac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}}$ and $\boxed{n_2 = \sqrt{\dfrac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}}$

$$\therefore\ \boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}}$$

Now if the two media are <u>both</u> paramagnetic and/or diamagnetic, such that $\boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$

i.e. $\boxed{\mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o}$ and: $\boxed{\mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o}$

{very common for many (but not all) non-conducting linear/homogeneous/isotropic media}

<u>Then</u>: $\boxed{\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\dfrac{v_1}{v_2}\right) = \left(\dfrac{n_2}{n_1}\right)}$ for $\boxed{\mu_1 \approx \mu_2 \approx \mu_o}$ or $\boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$

<u>Then</u>:

$$\left.\begin{aligned} &\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{1-(v_1/v_2)}{1+(v_1/v_2)}\right)\tilde{E}_{o_{inc}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}} \\ &\boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{2}{1+(v_1/v_2)}\right)\tilde{E}_{o_{nc}} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}} \end{aligned}\right\}$$

n.b. these relations are <u>identical</u> to the those we obtained for traveling transverse waves on a taught string with $\mu_1 = m_1/L_1$ and $\mu_2 = m_2/L_2$ with a {massless} knot at $z = 0$ {see p. 16, P436 Lect. Notes 4}!!!

We can <u>alternatively</u> express these relations in terms of the <u>indices of refraction</u> n_1 & n_2:

$\boxed{\tilde{E}_{o_{refl}} = \left(\dfrac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{o_{inc}}}$ and $\boxed{\tilde{E}_{o_{trans}} = \left(\dfrac{2n_1}{n_1 + n_2}\right)\tilde{E}_{o_{inc}}}$

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Now since: $\boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$

$\boxed{\tilde{E}_{o_{refl}} = E_{o_{refl}} e^{i\delta}}$ δ = phase angle (in radians) defined at the zero of time, $t = 0$

$\boxed{\tilde{E}_{o_{trans}} = E_{o_{trans}} e^{i\delta}}$

Then for the purely real amplitudes ($E_{o_{inc}}$, $E_{o_{refl}}$, $E_{o_{trans}}$) these relations become:

Monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \quad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}}$$

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

For a monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$ note the following points:

- If $v_2 > v_1$ (*i.e.* $n_2 < n_1$) {*e.g.* medium 1) = glass $\Rightarrow$ medium 2) = air}:

$$\boxed{E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \Rightarrow$$

> $E_{o_{refl}}$ <u>is precisely in-phase with</u> $E_{o_{inc}}$ <u>because</u> $(v_2 - v_1) > 0$.

- If $v_2 < v_1$ (*i.e.* $n_2 > n_1$) {*e.g.* medium 1) = air $\Rightarrow$ medium 2) = glass}:

$$\boxed{E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \Rightarrow$$

> $E_{o_{refl}}$ <u>is 180° out-of-phase with</u> $E_{o_{inc}}$ <u>because</u> $(v_2 - v_1) < 0$.

i.e.

$$\boxed{E_{o_{refl}} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{o_{inc}} = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{o_{inc}}} \Rightarrow$$

> The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (i.e. $n_2 > n_1$) !!!

- $E_{o_{trans}}$ is <u>always</u> in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $(n_1 \,\&\, n_2)$ because:

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}}$$

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What fraction of the incident *EM* wave energy is reflected?

What fraction of the incident *EM* wave energy is transmitted?

In a given linear/homogeneous/isotropic medium with $\boxed{v = \sqrt{\frac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = c/n}$:

The time-averaged energy density in the *EM* wave is: $\boxed{\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r})}\left(\frac{\text{Joules}}{\text{m}^3}\right)$

The time-averaged Poynting's vector is: $\boxed{\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{\mu}\langle \vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\rangle}\left(\frac{\text{Watts}}{\text{m}^2}\right)$

The intensity (*aka* irradiance) of the *EM* wave is:

$$\boxed{I(\vec{r}) \equiv \langle|\vec{S}(\vec{r},t)|\rangle = v\langle u_{EM}(\vec{r},t)\rangle = v\left(\frac{1}{2}\varepsilon E_o^2(\vec{r})\right) = \frac{1}{2}\varepsilon v E_o^2(\vec{r}) = \varepsilon v E_{o_{rms}}^2(\vec{r})}\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Note that the three Poynting's vectors associated with this problem are such that $\vec{S}_{inc} \parallel (+\hat{z})$, $\vec{S}_{refl} \parallel (-\hat{z})$ and $\vec{S}_{trans} \parallel (+\hat{z})$.

For a monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$:

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)E_{o_{inc}}} \quad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right)E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right)E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right)E_{o_{inc}}}$$

Take the ratios $\left(E_{o_{refl}}/E_{o_{inc}}\right)$ and $\left(E_{o_{trans}}/E_{o_{inc}}\right)$, then square them:

$$\boxed{\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \quad \text{and} \quad \boxed{\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Define the reflection coefficient as:

$$\boxed{R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\langle|\vec{S}_{refl}(\vec{r},t)|\rangle}{\langle|\vec{S}_{inc}(\vec{r},t)|\rangle} = \frac{v_1\langle u_{EM}^{refl}(\vec{r},t)\rangle}{v_1\langle u_{EM}^{inc}(\vec{r},t)\rangle} = \frac{\langle u_{EM}^{refl}(\vec{r},t)\rangle}{\langle u_{EM}^{inc}(\vec{r},t)\rangle} = \frac{\frac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2(\vec{r})}{\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})} = \frac{E_{o_{refl}}^2(\vec{r})}{E_{o_{inc}}^2(\vec{r})}}$$

Define the transmission coefficient as:

$$\boxed{T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\langle|\vec{S}_{trans}(\vec{r},t)|\rangle}{\langle|\vec{S}_{inc}(\vec{r},t)|\rangle} = \frac{v_2\langle u_{EM}^{trans}(\vec{r},t)\rangle}{v_1\langle u_{EM}^{inc}(\vec{r},t)\rangle} = \frac{\left(\frac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})\right)}{\left(\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})\right)} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})}{\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})}}$$

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For a linearly-polarized monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$:

Reflection coefficient:
$$\boxed{R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o\,inc}(\vec{r})}\right)^2}$$

Transmission coefficient:
$$\boxed{T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2}$$

But:
$$\boxed{\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \quad \&$$
$$\boxed{\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Thus:

Reflection coefficient:
$$\boxed{R(\vec{r}) \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

Transmission coefficient:
$$\boxed{T(\vec{r}) \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Now: $\boxed{\dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \dfrac{\dfrac{\varepsilon_2 \mu_2 v_2}{\mu_2}}{\dfrac{\varepsilon_1 \mu_1 v_1}{\mu_1}}}$ but: $\boxed{\begin{array}{l} v_2^2 = \dfrac{1}{\varepsilon_2 \mu_2} \;\Rightarrow\; \varepsilon_2 \mu_2 = \dfrac{1}{v_2^2} \\ v_1^2 = \dfrac{1}{\varepsilon_1 \mu_1} \;\Rightarrow\; \varepsilon_1 \mu_1 = \dfrac{1}{v_1^2} \end{array}}$

$$\therefore \quad \boxed{\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\left(\frac{1}{v_2^2}\cdot v_2\right)\Big/\mu_2}{\left(\frac{1}{v_1^2}\cdot v_1\right)\Big/\mu_2} = \frac{1\big/\mu_2 v_2}{1\big/\mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta} \;\text{!!!} \quad i.e. \quad \boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}}$$

for $\boxed{\mu_1 \approx \mu_2 \approx \mu_o}$

$$\therefore \quad \boxed{T(\vec{r}) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2 + v_1)^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

Thus:
$$\boxed{R(\vec{r}) + T(\vec{r}) = \frac{(1-\beta)^2}{(1+\beta)^2} + \frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2 + 4\beta}{(1+\beta)^2} = \frac{1 - 2\beta + \beta^2 + 4\beta}{(1+\beta)^2} = \frac{1 + 2\beta + \beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1}$$

$\therefore$ $\boxed{R(\vec{r}) + T(\vec{r}) = 1}$ $\Rightarrow$ *EM* energy is conserved at the interface/boundary between two L/H/I media in this process !!!

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For a linearly-polarized monochromatic plane *EM* wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$:

$\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

Reflection coefficient: $$\boxed{R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{(1-\beta)^2}{(1+\beta)^2} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

Transmission coefficient: $$\boxed{T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \beta\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2 + v_1)^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

$\boxed{R(\vec{r}) + T(\vec{r}) = 1}$ and $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_2 v_1}}$ $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

EXAMPLE:

A monochromatic plane *EM* wave is incident on an air-glass interface at normal incidence:

Indices of refraction for air and glass (*n.b.* both are non-magnetic materials) $\begin{pmatrix} n_1 = n_{air} & \simeq 1.0 \\ n_2 = n_{glass} & \simeq 1.5 \end{pmatrix}$

Reflection coefficient: $$\boxed{R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1.0 - 1.5}{1.0 + 1.5}\right)^2 = \left(\frac{-0.5}{2.5}\right)^2 = \left(-\frac{1}{5}\right)^2 = \frac{1}{25} = 0.04 = 4\%}$$

Transmission coefficient: $$\boxed{T = \frac{4n_1 n_2}{(n_1 + n_2)^2} = \frac{4 \cdot 1.0 \cdot 1.5}{(1.0 + 1.5)^2} = \frac{6.0}{(2.5)^2} = \frac{6.0}{6.25} = 0.96 = 96\%}$$

$$\boxed{R + T = 0.04 + 0.96 = 1.00}$$

QUESTION: Is *EM* linear momentum conserved in this process?

The time-averaged linear momentum densities associated with the 3 *EM* waves are:

$$\boxed{\left\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t) \right\rangle = +\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{inc}}^2(\vec{r})\right)\hat{z} = +\frac{1}{v_1}\left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle \hat{z}}$$

$$\boxed{\left\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t) \right\rangle = -\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{refl}}^2(\vec{r})\right)\hat{z} = -\frac{1}{v_1}\left\langle u_{EM}^{refl}(\vec{r},t) \right\rangle \hat{z}}$$

$$\boxed{\left\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t) \right\rangle = +\frac{1}{v_2}\left(\frac{1}{2}\varepsilon_2 E_{o_{trans}}^2(\vec{r})\right)\hat{z} = +\frac{1}{v_2}\left\langle u_{EM}^{trans}(\vec{r},t) \right\rangle \hat{z}}$$

In order that *EM* linear momentum be conserved at the interface, we must have the time-averaged initial *EM* linear momentum at the interface = the time-averaged final *EM* linear momentum at the interface, *i.e.* $\left\langle \vec{p}_{EM}^{\,inital}(\vec{r},t) \right\rangle\big|_{z=0} = \left\langle \vec{p}_{EM}^{\,final}(\vec{r},t) \right\rangle\big|_{z=0}$.

{*n.b.* we (again) use time-averages here, in order to make direct comparisons with experimental measurements of these quantities}.

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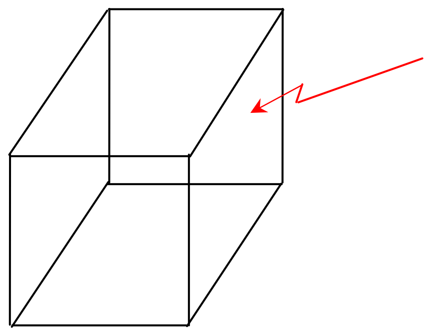
Now: $\boxed{\langle \vec{p}(\vec{r},t)\rangle = \int_v \langle \vec{\wp}(\vec{r},t)\rangle d\tau = \langle \vec{\wp}(\vec{r},t)\rangle * \text{Volume}, \Delta V}$ where the volume associated with the *EM* wave over the time interval Δt is $\boxed{\Delta V = \ell A_\perp = v\Delta t A_\perp}$



Thus:
$$\boxed{\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\rangle = \langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle \Delta V_{inc} = \langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle v_1 \Delta t A_\perp}$$
$$\boxed{\langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\rangle = \langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle \Delta V_{refl} = \langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle v_1 \Delta t A_\perp}$$
$$\boxed{\langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\rangle = \langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle \Delta V_{trans} = \langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle v_2 \Delta t A_\perp}$$

Then:
$$\boxed{\langle \vec{p}_{EM}^{\,inital}(\vec{r},t)\rangle\big|_{z=0} = \langle \vec{p}_{EM}^{\,final}(\vec{r},t)\rangle\big|_{z=0}}$$

$\Rightarrow$
$$\boxed{\langle \vec{p}_{EM}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} = \langle \vec{p}_{EM}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} + \langle \vec{p}_{EM}^{\,trans}(\vec{r},t)\rangle\big|_{z=0}}$$

Thus:
$$\boxed{\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle \Delta V_{inc}\big|_{z=0} = \langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle \Delta V_{refl}\big|_{z=0} + \langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle \Delta V_{trans}\big|_{z=0}}$$

or:
$$\boxed{\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle v_1 \Delta t A_\perp\big|_{z=0} = \langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle v_1 \Delta t A_\perp\big|_{z=0} + \langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle v_2 \Delta t A_\perp\big|_{z=0}}$$

i.e:
$$\boxed{v_1 \langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle\big|_{z=0} = v_1 \langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle\big|_{z=0} + v_2 \langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle\big|_{z=0}}$$

But:
$$\boxed{\langle \vec{\wp}_{EM}^{\,inc}(\vec{r},t)\rangle = +\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{inc}}^2(\vec{r})\right)\hat{z}}$$
$$\boxed{\langle \vec{\wp}_{EM}^{\,refl}(\vec{r},t)\rangle = -\frac{1}{v_1}\left(\frac{1}{2}\varepsilon_1 E_{o_{refl}}^2(\vec{r})\right)\hat{z}}$$
$$\boxed{\langle \vec{\wp}_{EM}^{\,trans}(\vec{r},t)\rangle = +\frac{1}{v_2}\left(\frac{1}{2}\varepsilon_2 E_{o_{trans}}^2(\vec{r})\right)\hat{z}}$$

Thus:
$$\boxed{\frac{\cancel{v_1}}{\cancel{v_1}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_1 E_{o_{inc}}^2(\vec{r})\right)\Bigg|_{z=0} = -\frac{\cancel{v_1}}{\cancel{v_1}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_1 E_{o_{refl}}^2(\vec{r})\right)\Bigg|_{z=0} + \frac{\cancel{v_2}}{\cancel{v_2}}\left(\frac{\cancel{1}}{\cancel{2}}\varepsilon_2 E_{o_{trans}}^2(\vec{r})\right)\Bigg|_{z=0}}$$

or:
$$\boxed{\varepsilon_1\left(E_{o_{inc}}^2(\vec{r}) + E_{o_{refl}}^2(\vec{r})\right)\Big|_{z=0} = \varepsilon_2 E_{o_{trans}}^2(\vec{r})\Big|_{z=0}}$$

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Divide this relation on both sides by $E_{o_{inc}}^2(\vec{r})$

$$\boxed{1+\underbrace{\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2}_{=R(\vec{r})}=\frac{\varepsilon_2}{\varepsilon_1}\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2=\left(\frac{v_1}{v_2}\right)\underbrace{\left(\frac{\varepsilon_1 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2}_{\equiv T(\vec{r})}}$$

Thus: $\boxed{1+R(\vec{r})=\left(\frac{v_1}{v_2}\right)T(\vec{r})}$ But: $\boxed{R(\vec{r})+T(\vec{r})=1}$ or: $\boxed{R(\vec{r})=1-T(\vec{r})}$

$\therefore$ $\boxed{1+\left(1-T(\vec{r})\right)=\left(\frac{v_1}{v_2}\right)T(\vec{r})}$ or: $\boxed{2-T(\vec{r})=\left(\frac{v_1}{v_2}\right)T(\vec{r})}$ or: $\boxed{2=\left[1+\left(\frac{v_1}{v_2}\right)\right]T(\vec{r})}$

Thus: $\boxed{T(\vec{r})=\frac{2}{\left[1+(v_1/v_2)\right]}=\frac{2v_2}{(v_1+v_2)}}$

But: $\boxed{T(\vec{r})=\frac{4\beta}{(1+\beta)^2}\simeq\frac{4v_2v_1}{(v_2+v_1)^2}\neq\frac{2v_2}{(v_1+v_2)}}$ {from above} !!! where $\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_2 v_1}}$

$\Rightarrow$ Linear momentum carried by *EM* wave is ***NOT*** conserved in/at interface between two linear / homogeneous / isotropic media !!! Why???? How???

The physical reason for this is because {again} we're not "counting *all* of the beans" here…

The *EM* waves that are present in each of the linear / homogeneous / isotropic media (*i.e.* the *EM* waves that exist in medium 1 and medium 2) polarize the atoms/molecules in that medium and create an *additional* co-traveling momentum in that medium – which results from the {mechanical} momentum of the electrons associated with the atomic/molecular induced electric dipole moments that arise in response to the induced polarization associated with the incident/reflected/transmitted traveling *EM* waves! Please see/read P436 Lect. Notes 7.5….

Thus, overall linear momentum *is* conserved when the *EM* wave and its co-traveling electron / atom / molecule induced electric dipole mechanical momentum associated with the medium is included

In medium 1: $\boxed{\left\langle\vec{p}_{Tot}^{\,inc}(\vec{r},t)\right\rangle=\left\langle\vec{p}_{EM}^{\,inc}(\vec{r},t)\right\rangle+\left\langle\vec{p}_{e^-dipole}^{\,inc}(\vec{r},t)\right\rangle}$

In medium 1: $\boxed{\left\langle\vec{p}_{Tot}^{\,refl}(\vec{r},t)\right\rangle=\left\langle\vec{p}_{EM}^{\,refl}(\vec{r},t)\right\rangle+\left\langle\vec{p}_{e^-dipole}^{\,refl}(\vec{r},t)\right\rangle}$

In medium 2: $\boxed{\left\langle\vec{p}_{Tot}^{\,trans}(\vec{r},t)\right\rangle=\left\langle\vec{p}_{EM}^{\,trans}(\vec{r},t)\right\rangle+\left\langle\vec{p}_{e^-dipole}^{\,trans}(\vec{r},t)\right\rangle}$

It is curious that the time-averaged energy in *EM* waves (alone) *is* conserved, whereas time-averaged *EM* field linear momentum is *not* conserved at the interface of two L/H/I media. Microscopically, note that a photon's energy $E_\gamma = hf$ is unchanged in such a medium, whereas a photon's momentum $p_\gamma = h/\lambda_\gamma$ *is* changed. Since macroscopic *EM* field linear momentum is *not* conserved at the interface of two L/H/I media, neither will *EM* field angular momentum / *EM* field angular momentum density be conserved {only}, since $\boxed{\vec{\ell}_{EM}(\vec{r},t)=\vec{r}\times\vec{\wp}_{EM}(\vec{r},t)}$.

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For further details on this subject, see/read:
1.) J.D. Jackson, *Classical Electrodynamics*, *p*. 262, 3rd Ed. Wiley, NY
2.) R.E. Peierls, *Proc. Roy. Soc. London* **347**, *p*. 475 (1976).
3.) R.E. Peierls, *Proc. Roy. Soc. London* **355**, *p*. 141 (1971).
4.) R. Loudon, L. Allen and D.F. Nelson, *Phys. Rev. E* **55**, *p*. 1071 (1997).

Arbitrary/Generalized Polarization States of a Plane *EM* Wave; Elliptical, Circular and Linear Polarization

As we saw in the previous discussion, a monochromatic, linearly-polarized plane *EM* wave *e.g.* propagating in the $+\hat{z}$ direction in medium 1, which is also at normal incidence to a boundary between two linear / homogenous / isotropic media {located as before at $z = 0$ in the *x*-*y* plane} has the following mathematical forms {for linear polarization in the $+\hat{x}$ direction} for the complex $\vec{E}$ and $\vec{B}$ fields:

Incidentmonochromatic, linearly-polarized *EM* plane wave (in medium 1):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with linear polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{inc}^{LP}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}} \quad \text{with: } \boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1} \quad \text{and: } \boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$$

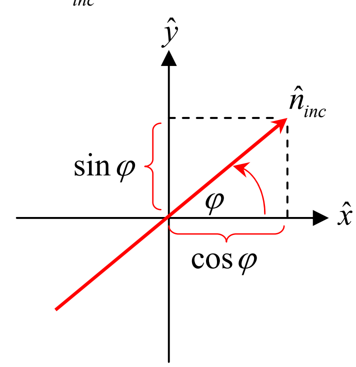
$$\boxed{\tilde{\vec{B}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}} \quad \text{since: } \boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z}\times\hat{x} = +\hat{y}}$$

In general, this monochromatic, linearly-polarized *EM* plane wave incident on the boundary between two linear / homogenous / isotropic media can be polarized in *any* direction in the *x*-*y* plane. More generally then, we can write the polarization vector $\hat{n}_{inc}$ as:

$$\boxed{\hat{n}_{inc} = \cos\varphi\hat{x} + \sin\varphi\hat{y}} \quad \text{where} \quad \boxed{0 \le \varphi < 2\pi}$$

$\varphi = 0^o$: $\Rightarrow$ LP in $+\hat{x}$ -direction

$\varphi = 90^o$: $\Rightarrow$ LP in $+\hat{y}$ -direction



Thus, more generally, we can write the complex $\vec{E}$ and $\vec{B}$ fields for the incident monochromatic, but arbitrarily linearly-polarized *EM* plane wave (in medium 1) as:

Incident monochromatic, arbitrarily linearly-polarized *EM* plane wave (in medium 1):

$+\hat{z}$ propagation direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), arbitrary linear polarization $\boxed{\hat{n}_{inc} = \cos\varphi\hat{x} + \sin\varphi\hat{y}}$

$$\boxed{\tilde{\vec{E}}_{inc}^{LP}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{n}_{inc} = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\left[\cos\varphi\hat{x} + \sin\varphi\hat{y}\right]}$$

with: $\boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$ and: $\boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$

$$\boxed{\tilde{\vec{B}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\left(\hat{k}_{inc} \times \hat{n}_{inc}\right)}$$

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But: $\boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times [\cos\varphi \hat{x} + \sin\varphi \hat{y}] = \cos\varphi(\hat{z} \times \hat{x}) + \sin\varphi(\hat{z} \times \hat{y}) = +\cos\varphi \hat{y} - \sin\varphi \hat{x}}$

Very Useful Table:

$\hat{x} \times \hat{y} = +\hat{z}$	$\hat{y} \times \hat{x} = -\hat{z}$
$\hat{y} \times \hat{z} = +\hat{x}$	$\hat{z} \times \hat{y} = -\hat{x}$
$\hat{z} \times \hat{x} = +\hat{y}$	$\hat{x} \times \hat{z} = -\hat{y}$

Thus, the complex $\vec{B}$-field can be equivalently written as:

$$\boxed{\tilde{\vec{B}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}^{LP}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}(\hat{k}_{inc} \times \hat{n}_{inc}) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}[\cos\varphi \hat{y} - \sin\varphi \hat{x}]}$$

As always, the *physical* $\vec{E}$ and $\vec{B}$ fields associated with this *EM* wave are of the form:

$$\begin{aligned}
\vec{E}_{inc}^{LP}(z,t) &= \mathrm{Re}\left\{\tilde{\vec{E}}_{inc}^{LP}(z,t)\right\} = \mathrm{Re}\left\{\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}[\cos\varphi \hat{x} + \sin\varphi \hat{y}]\right\}, \quad \text{but: } \tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta} \\
&= \mathrm{Re}\left\{E_{o_{inc}} e^{i\delta} e^{i(k_1 z - \omega t)}[\cos\varphi \hat{x} + \sin\varphi \hat{y}]\right\} = \mathrm{Re}\left\{E_{o_{inc}} e^{i(k_1 z - \omega t + \delta)}[\cos\varphi \hat{x} + \sin\varphi \hat{y}]\right\} \\
&= E_{o_{inc}} \mathrm{Re}\left\{e^{i(k_1 z - \omega t + \delta)}\right\}[\cos\varphi \hat{x} + \sin\varphi \hat{y}] \\
&= E_{o_{inc}} \mathrm{Re}\left\{\cos(k_1 z - \omega t + \delta) + i\sin(k_1 z - \omega t + \delta)\right\}[\cos\varphi \hat{x} + \sin\varphi \hat{y}] \\
&= E_{o_{inc}} \cos(k_1 z - \omega t + \delta)[\cos\varphi \hat{x} + \sin\varphi \hat{y}]
\end{aligned}$$

$$\begin{aligned}
\vec{B}_{inc}^{LP}(z,t) &= \mathrm{Re}\left\{\tilde{\vec{B}}_{inc}^{LP}(z,t)\right\} = \frac{1}{v_1}\hat{k}_{inc} \times \vec{E}_{inc}^{LP}(z,t) = \frac{1}{v_1}E_{o_{inc}} \cos(k_1 z - \omega t + \delta)(\hat{k}_{inc} \times \hat{n}_{inc}) \\
&= \frac{1}{v_1}E_{o_{inc}} \cos(k_1 z - \omega t + \delta)[\cos\varphi \hat{y} - \sin\varphi \hat{x}]
\end{aligned}$$

Now, for a circularly-polarized monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence, the *physical* $\vec{E}$ and $\vec{B}$ fields can be written mathematically as follows:

$$\boxed{\vec{E}_{inc}^{CP}(z,t) = E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{x} \pm \sin(k_1 z - \omega t + \delta)\hat{y}\right]} \quad \text{with} \quad \boxed{\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}}$$

$$\begin{aligned}
\vec{B}_{inc}^{CP}(z,t) &= \tfrac{1}{v_1}\hat{k}_{inc} \times \vec{E}_{inc}^{CP}(z,t) = \tfrac{1}{v_1}E_{o_{inc}}\left\{\hat{z} \times \left[\cos(k_1 z - \omega t + \delta)\hat{x} \pm \sin(k_1 z - \omega t + \delta)\hat{y}\right]\right\} \\
&= \tfrac{1}{v_1}E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)(\hat{z} \times \hat{x}) \pm \sin(k_1 z - \omega t + \delta)(\hat{z} \times \hat{y})\right] \\
&= \tfrac{1}{v_1}E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{y} \mp \sin(k_1 z - \omega t + \delta)\hat{x}\right]
\end{aligned}$$

Note that the $\pm$ signs between the 90° out-of-phase $\hat{x}$ and $\hat{y}$ components for $\vec{E}$ (and the corresponding $\mp$ signs for $\vec{B}$) denote the *handedness* of the circularly polarized *EM* wave – *i.e.* whether it is right- or left-circularly polarized!

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A <u>right- (left-) circularly-polarized</u> monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence, the <u>*physical*</u> $\vec{E}$ and $\vec{B}$ fields can be written mathematically as follows:

RCP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{RCP}(z,t) = E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{x} + \sin(k_1 z - \omega t + \delta)\hat{y}\right]} \\ &\boxed{\vec{B}_{inc}^{RCP}(z,t) = \frac{1}{v_1} E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{y} - \sin(k_1 z - \omega t + \delta)\hat{x}\right]} \end{aligned} \right.$$

LCP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{LCP}(z,t) = E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{x} - \sin(k_1 z - \omega t + \delta)\hat{y}\right]} \\ &\boxed{\vec{B}_{inc}^{RCP}(z,t) = \frac{1}{v_1} E_{o_{inc}}\left[\cos(k_1 z - \omega t + \delta)\hat{y} + \sin(k_1 z - \omega t + \delta)\hat{x}\right]} \end{aligned} \right.$$

Note that at $(z,t) = (0,0)$ these *EM* fields at that point/at that time are:

RCP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{RCP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} + \sin\delta\hat{y}\right]} \\ &\boxed{\vec{B}_{inc}^{RCP}(0,0) = \frac{1}{v_1} E_{o_{inc}}\left[\cos\delta\hat{y} - \sin\delta\hat{x}\right]} \end{aligned} \right.$$

LCP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{LCP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} - \sin\delta\hat{y}\right]} \\ &\boxed{\vec{B}_{inc}^{LCP}(0,0) = \frac{1}{v_1} E_{o_{inc}}\left[\cos\delta\hat{y} + \sin\delta\hat{x}\right]} \end{aligned} \right.$$

Or more generally for circularly-polarized *EM* waves (right- or left-handed):

CP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{CP}(0,0) = E_{o_{inc}}\left[\cos\delta\hat{x} \pm \sin\delta\hat{y}\right]} \quad (+ = \text{RCP},\ - = \text{LCP}) \\ &\boxed{\vec{B}_{inc}^{CP}(0,0) = \frac{1}{v_1} E_{o_{inc}}\left[\cos\delta\hat{y} \mp \sin\delta\hat{x}\right]} \quad (- = \text{RCP},\ + = \text{LCP}) \end{aligned} \right.$$

If we compare these formulae to their equivalents for <u>arbitrarily linearly-polarized</u> *EM* waves, with $\boxed{\hat{n}_{LP} = \hat{n}_{inc} \equiv \cos\varphi\hat{x} + \sin\varphi\hat{y}}$:

LP *EM* Wave

$$\left\{ \begin{aligned} &\boxed{\vec{E}_{inc}^{LP}(0,0) = E_{o_{inc}}\cos\delta\left[\cos\varphi\hat{x} + \sin\varphi\hat{y}\right] = E_{o_{inc}}\cos\delta\,\hat{n}_{LP} = E_{o_{inc}}\cos\delta\,\hat{n}_{inc}} \\ &\boxed{\vec{B}_{inc}^{LP}(0,0) = \frac{1}{v_1} E_{o_{inc}}\cos\delta\left[\cos\varphi\hat{y} - \sin\varphi\hat{x}\right] = \frac{1}{v_1} E_{o_{inc}}\cos\delta\left(\hat{k} \times \hat{n}_{inc}\right)} \end{aligned} \right.$$

Then we see that we can {analogously} define right- and left-circular transverse polarization <u>unit vectors</u> (*i.e.* lying in the *x*-*y* plane, $\perp$ to the direction of propagation {here, in the $+\hat{z}$ direction}):

<u>RCP *EM* Wave</u>: $\boxed{\hat{n}_{RCP} = \hat{n}_{+} \equiv \cos\delta\hat{x} + \sin\delta\hat{y}}$

<u>LCP *EM* Wave</u>: $\boxed{\hat{n}_{LCP} = \hat{n}_{-} \equiv \cos\delta\hat{x} - \sin\delta\hat{y}}$

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Thus, we can write the *physical* $\vec{E}$ and $\vec{B}$ fields at $(z,t)=(0,0)$ associated with a right- (left-) circularly-polarized monochromatic plane *EM* wave, propagating in the $+\hat{z}$ direction in medium 1 incident on the boundary between two linear / homogenous / isotropic media at normal incidence as follows, for $\boxed{\hat{n}_{RCP}=\hat{n}_{+}\equiv\cos\delta\hat{x}+\sin\delta\hat{y}}$ and $\boxed{\hat{n}_{LCP}=\hat{n}_{-}\equiv\cos\delta\hat{x}-\sin\delta\hat{y}}$:

RCP *EM* Wave

$$\boxed{\vec{E}_{inc}^{RCP}(0,0)=E_{o_{inc}}\left[\cos\delta\hat{x}+\sin\delta\hat{y}\right]=E_{o_{inc}}\hat{n}_{RCP}=E_{o_{inc}}\hat{n}_{+}}$$

$$\boxed{\vec{B}_{inc}^{RCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y}-\sin\delta\hat{x}\right]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{RCP}\right)=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{+}\right)}$$

LCP *EM* Wave

$$\boxed{\vec{E}_{inc}^{LCP}(0,0)=E_{o_{inc}}\left[\cos\delta\hat{x}-\sin\delta\hat{y}\right]=E_{o_{inc}}\hat{n}_{LCP}=E_{o_{inc}}\hat{n}_{-}}$$

$$\boxed{\vec{B}_{inc}^{LCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y}+\sin\delta\hat{x}\right]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{LCP}\right)=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{-}\right)}$$

Or more generally for circularly-polarized *EM* waves (right- or left-handed):

CP *EM* Wave

$$\boxed{\vec{E}_{inc}^{RCP}(0,0)=E_{o_{inc}}\left[\cos\delta\hat{x}\pm\sin\delta\hat{y}\right]=E_{o_{inc}}\hat{n}_{\pm}}$$

$$\boxed{\vec{B}_{inc}^{RCP}(0,0)=\frac{1}{v_1}E_{o_{inc}}\left[\cos\delta\hat{y}\mp\sin\delta\hat{x}\right]=\frac{1}{v_1}E_{o_{inc}}\left(\hat{k}_{inc}\times\hat{n}_{\pm}\right)}$$

Defining right and left complex circular-polarization unit vectors, respectively as:

$$\boxed{\hat{\epsilon}_{RCP}=\hat{\epsilon}_{-}\equiv\tfrac{1}{\sqrt{2}}\left[\hat{x}-i\hat{y}\right]} \quad \text{and} \quad \boxed{\hat{\epsilon}_{LCP}=\hat{\epsilon}_{+}\equiv\tfrac{1}{\sqrt{2}}\left[\hat{x}+i\hat{y}\right]}:$$

The corresponding complex CP (RCP or LCP) *EM* waves are of the following forms

RCP *EM* Wave

$$\boxed{\tilde{\vec{E}}_{inc}^{RCP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\left[\hat{x}-i\hat{y}\right]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{RCP}=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{-}}$$

$$\boxed{\tilde{\vec{B}}_{inc}^{RCP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{RCP}(z,t)}$$

LCP *EM* Wave

$$\boxed{\tilde{\vec{E}}_{inc}^{LCP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\left[\hat{x}+i\hat{y}\right]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{LCP}=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{+}}$$

$$\boxed{\tilde{\vec{B}}_{inc}^{LCP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{LCP}(z,t)}$$

CP *EM* Wave

$$\boxed{\tilde{\vec{E}}_{inc}^{CP}(z,t)=E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\left[\hat{x}\mp i\hat{y}\right]=\sqrt{2}E_{o_{inc}}e^{i(k_1z-\omega t+\delta)}\hat{\epsilon}_{\mp}}$$

$$\boxed{\tilde{\vec{B}}_{inc}^{CP}(z,t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}^{CP}(z,t)}$$

(*n.b.* $-$ = RCP, $+$ = LCP here !!!)

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At a fixed point in space (*e.g.* $z = 0$), an observer looking into the oncoming/incident LCP *EM* wave sees the electric field vector $\vec{E}_{inc}^{LCP}(z=0,t)$ spinning/rotating counter-clockwise (CCW) in a circle at angular frequency ω for a LCP *EM* wave as time progresses. A LCP *EM* wave is said to have positive helicity, because a LCP *EM* wave propagating in the $+\hat{z}$ direction has positive angular momentum density, *i.e.* $\boxed{\ell_{EM}^{LCP}(z,t) = +\ell_{EM}\hat{z}}$.

Similarly, at a fixed point in space (*e.g.* $z = 0$), an observer looking into the oncoming/incident RCP *EM* wave sees the electric field vector $\vec{E}_{inc}^{RCP}(z=0,t)$ spinning/rotating clockwise (CW) in a circle at angular frequency ω for RCP light as time progresses. A RCP *EM* wave is said to have negative helicity, because a RCP *EM* wave propagating in the $+\hat{z}$ direction has negative angular momentum density, *i.e.* $\boxed{\ell_{EM}^{RCP}(z,t) = -\ell_{EM}\hat{z}}$.

Note that both linearly-polarized and circularly-polarized *EM* waves are limiting/special cases of the more general class of elliptically-polarized *EM* waves.

For a generally-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$, if the $\hat{x}$ and $\hat{y}$ components of the complex electric field have the same phase, *i.e.* $\boxed{\tilde{E}_{ox} = E_{ox}e^{i\delta}}$ and $\boxed{\tilde{E}_{oy} = E_{oy}e^{i\delta}}$, then this is a linearly-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction: $\boxed{\tilde{\vec{E}}^{LP}(z,t) = \left[E_{ox}\hat{x} + E_{oy}\hat{y}\right]e^{i(k_1 z - \omega t + \delta)}}$. If the $\hat{x}$ and $\hat{y}$ components of the complex electric field have the same amplitude and the same phase, *i.e.* $\boxed{\tilde{E}_{ox} = E_o e^{i\delta}}$ and $\boxed{\tilde{E}_{oy} = E_o e^{i\delta}}$, then this is monochromatic plane *EM* wave a linearly-polarized at +45° (wrt the $\hat{x}$-axis) propagating in the $+\hat{z}$ direction: $\boxed{\tilde{\vec{E}}^{LP}(z,t) = E_o\left[\hat{x} + \hat{y}\right]e^{i(k_1 z - \omega t + \delta)}}$. Other special cases of linear polarization, such as LP in the $\hat{x}$-only, or the $\hat{y}$-only direction, or the –45° (wrt the $\hat{x}$-axis) can also be easily worked out.

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$ of the generally-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction have different phases, *i.e.* $\boxed{\tilde{E}_{ox} = E_{ox}e^{i\delta_x}}$ and $\boxed{\tilde{E}_{oy} = E_{oy}e^{i\delta_y}}$, then this *EM* wave is elliptically-polarized.

If the $\hat{x}$ and $\hat{y}$ components of the complex electric field $\boxed{\tilde{\vec{E}}(z,t) = \left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right]e^{i(k_1 z - \omega t)}}$ of the generally-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction have the same amplitudes {*i.e.* $\boxed{E_{ox} = E_{oy} = E_o}$} *and* their phases differ by $\boxed{\delta_x - \delta_y = \pm 90^o = \pm\pi/2}$ radians, *i.e.* $\boxed{\tilde{E}_{ox} = E_o e^{i\delta_x}}$ and $\boxed{\tilde{E}_{oy} = E_o e^{i\delta_y} = E_o e^{i(\delta_x \mp \pi/2)} = E_o e^{i\delta_x}e^{\mp i\pi/2} = \mp iE_o e^{i\delta_x} = \mp i\tilde{E}_{ox}}$ {since $\boxed{e^{\mp i\pi/2} = \cos(\pi/2) \mp i\sin(\pi/2) = \mp i}$}, hence $\boxed{\left[\tilde{E}_{ox}\hat{x} + \tilde{E}_{oy}\hat{y}\right] = E_o\left[\hat{x} \mp i\hat{y}\right] = \sqrt{2}E_o\,\hat{\epsilon}_{\mp}}$ and thus we see that this *EM* wave is circularly-polarized.

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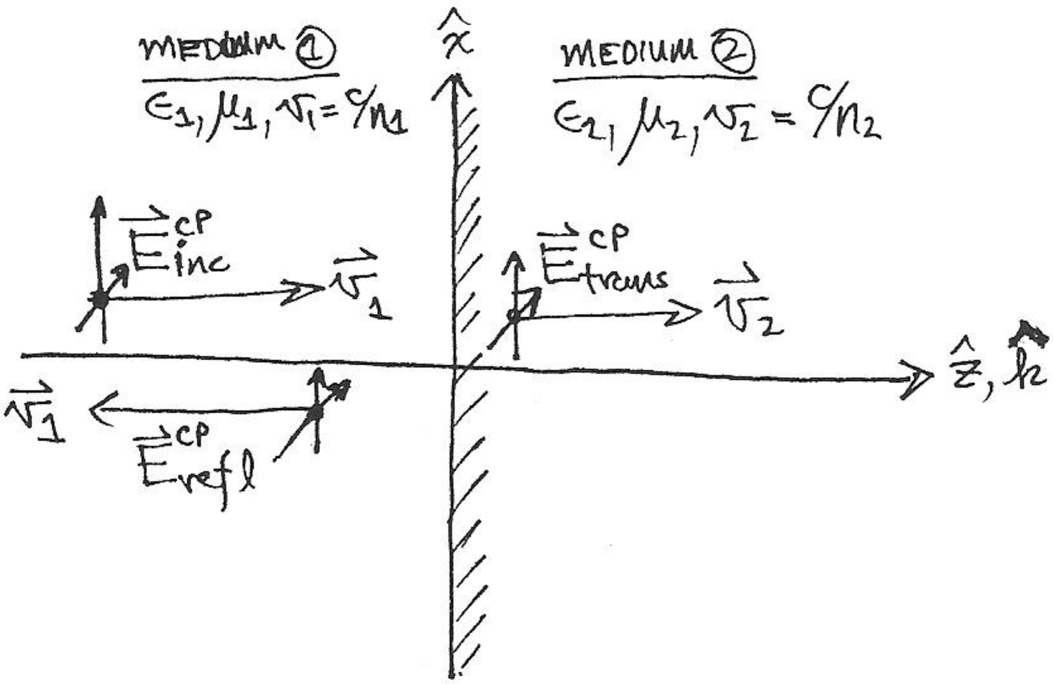
Reflection & Transmission of Circularly Polarized Plane *EM* Waves at Normal Incidence at a Boundary Between Two Linear / Homogeneous / Isotropic Media

A circularly-polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction is normally incident on a boundary {in the *x*-*y* plane} between two linear, homogeneous and isotropic media as shown in the figure below:



The complex amplitudes for the CP $\vec{E}$ and $\vec{B}$ fields are summarized below:

<u>Incident CP monochromatic plane *EM* wave:</u>

$$\boxed{\tilde{\vec{E}}^{CP}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\left[\hat{x} \mp i\hat{y}\right] = E_{o_{inc}} e^{i(k_1 z - \omega t + \delta)}\left[\hat{x} \mp i\hat{y}\right]} \quad n.b. \quad \boxed{\hat{k}_{inc} = \hat{k}_1 = +\hat{z}}$$

$$\boxed{\tilde{\vec{B}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\left[\hat{y} \pm i\hat{x}\right] = \tfrac{1}{v_1} E_{o_{inc}} e^{i(k_1 z - \omega t + \delta)}\left[\hat{y} \pm i\hat{x}\right]}$$

<u>Reflected CP monochromatic plane *EM* wave:</u>

$$\boxed{\tilde{\vec{E}}^{CP}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(k_1 z - \omega t)}\left[\hat{x} \mp i\hat{y}\right] = E_{o_{refl}} e^{i(k_1 z - \omega t + \delta)}\left[\hat{x} \mp i\hat{y}\right]} \quad n.b. \quad \boxed{\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}}$$

$$\boxed{\tilde{\vec{B}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(k_1 z - \omega t)}\left[-\hat{y} \mp i\hat{x}\right] = -\tfrac{1}{v_1} E_{o_{refl}} e^{i(k_1 z - \omega t + \delta)}\left[\hat{y} \pm i\hat{x}\right]}$$

<u>Transmitted CP monochromatic plane *EM* wave:</u>

$$\boxed{\tilde{\vec{E}}^{CP}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\left[\hat{x} \mp i\hat{y}\right] = E_{o_{trans}} e^{i(k_2 z - \omega t + \delta)}\left[\hat{x} \mp i\hat{y}\right]} \quad n.b. \quad \boxed{\hat{k}_{trans} = \hat{k}_2 = +\hat{z}}$$

$$\boxed{\tilde{\vec{B}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\left[\hat{y} \pm i\hat{x}\right] = \tfrac{1}{v_2} E_{o_{trans}} e^{i(k_2 z - \omega t + \delta)}\left[\hat{y} \pm i\hat{x}\right]}$$

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The boundary conditions on the CP $\vec{E}$ and $\vec{B}$ fields @ $z = 0$ in the x-y plane are summarized below:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$
(*n.b.* $\perp$ refers to the x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$
(*n.b.* $\parallel$ refers to the x-y boundary, *i.e.* in the x-y plane)

BC 3) Normal $\vec{B}$ continuous: $\boxed{B^{\perp}_{1_{Tot}} = B^{\perp}_{2_{Tot}}}$ ($\perp$ to x-y boundary, *i.e.* in the $+\hat{z}$ direction)

BC 4) Tangential $\vec{H}$ continuous: $\boxed{\dfrac{1}{\mu_1} B^{\parallel}_{1_{Tot}} = \dfrac{1}{\mu_2} B^{\parallel}_{2_{Tot}}}$ ($\parallel$ to x-y boundary, *i.e.* in x-y plane)

Thus, at $z = 0$:

Again, because the transversality requirements (from Maxwell's equations) of the $\vec{E}$ and $\vec{B}$ fields, we see that BC 1) and BC 3) impose no restrictions {here} on such CP *EM* waves since:
{ $E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0$; $E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0$ } and { $B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0$; $B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0$ }

$\Rightarrow$ Again, the only restrictions on plane *EM* waves propagating with normal incidence on the boundary at $z = 0$ {lying in the x-y plane} are imposed by BC 2) and BC 4).

$\therefore$ At $z = 0$ in medium 1) (*i.e.* $z \le 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}(z=0,t) = \tilde{\vec{E}}^{CP}_{inc}(z=0,t) + \tilde{\vec{E}}^{CP}_{refl}(z=0,t)} \text{ and}$$
$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}(z=0,t) = \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{refl}(z=0,t)}$$

While at $z = 0$ in medium 2) (*i.e.* $z \ge 0$) we must have:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{2_{Tot}}(z=0,t) = \tilde{\vec{E}}^{CP}_{trans}(z=0,t)} \text{ and}$$
$$\boxed{\frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}^{CP}_{trans}(z=0,t)}$$

Then BC 2) (Tangential $\vec{E}$ is continuous @ $z = 0$) requires that:

$$\boxed{\tilde{\vec{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \tilde{\vec{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \text{ or: } \boxed{\tilde{\vec{E}}^{CP}_{inc}(z=0,t) + \tilde{\vec{E}}^{CP}_{refl}(z=0,t) = \tilde{\vec{E}}^{CP}_{trans}(z=0,t)}.$$

Then BC 4) (Tangential $\vec{H}$ is continuous @ $z = 0$) requires that:

$$\boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\tilde{\vec{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \text{ or: } \boxed{\frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{inc}(z=0,t) + \frac{1}{\mu_1}\tilde{\vec{B}}^{CP}_{refl}(z=0,t) = \frac{1}{\mu_2}\tilde{\vec{B}}^{CP}_{trans}(z=0,t)}$$

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Inserting the explicit expressions for the complex $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ fields

$$\boxed{\tilde{\vec{E}}^{CP}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}[\hat{x} \mp i\hat{y}]} \quad \boxed{\tilde{\vec{B}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}^{CP}_{inc}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}[\hat{y} \pm i\hat{x}]}$$

$$\boxed{\tilde{\vec{E}}^{CP}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(k_1 z-\omega t)}[\hat{x} \mp i\hat{y}]} \quad \boxed{\tilde{\vec{B}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}^{CP}_{refl}(z,t) = \tfrac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(k_1 z-\omega t)}[-\hat{y} \mp i\hat{x}]}$$

$$\boxed{\tilde{\vec{E}}^{CP}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}[\hat{x} \mp i\hat{y}]} \quad \boxed{\tilde{\vec{B}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}^{CP}_{trans}(z,t) = \tfrac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}[\hat{y} \pm i\hat{x}]}$$

into the above boundary condition relations, these equations become:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations, we have at $z = 0$ { *n.b.* <u>everywhere</u> in *x*-*y* plane, independent of/valid for any time t}:

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Note that these last two relations for <u>circularly-polarized</u> plane *EM* waves are <u>identical</u> to those we obtained for the <u>linearly-polarized</u> monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction is normally incident on a boundary {in the *x*-*y* plane} between two linear, homogeneous and isotropic media.

$\Rightarrow$ The BC constraints on the $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ are decoupled from their polarization states!

Thus, we obtain precisely the same reflection and transmission coefficients for the <u>circularly-polarized</u> plane *EM* wave as we did for the <u>linearly-polarized</u> monochromatic plane *EM* wave propagating in the $+\hat{z}$ direction is normally incident on a boundary {in the *x*-*y* plane} between two linear, homogeneous and isotropic media:

$\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

<u>Reflection coefficient</u>: $\boxed{R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{(1-\beta)^2}{(1+\beta)^2} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$

<u>Transmission coefficient</u>: $\boxed{T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \beta\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2 + v_1)^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$

$\boxed{R(\vec{r}) + T(\vec{r}) = 1}$ and $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}}$ $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

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