LECTURE NOTES 17

Proper Time and *Proper* Velocity

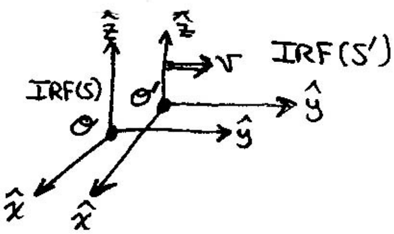
- As you progress along your world line (moving with "ordinary" velocity $\vec{u}$ in IRF(S)) in the *ct* vs. *x* Minkowski/space-time diagram, your watch runs *slow* (i.e. in *your* own rest frame IRF(S')) in comparison to clocks on the wall in the lab frame IRF(S).

- The clocks on the wall in the lab frame IRF(S) tick off a time interval dt, whereas in your reference frame (your rest frame) IRF(S') the time interval is: $\boxed{dt' = dt/\gamma_u = \sqrt{1-\beta_u^2}\,dt}$

- n.b. this is the exact same time dilation formula that we obtained earlier, with:

$$\boxed{\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1}{\sqrt{1-\beta_u^2}}} \text{ and: } \boxed{\beta_u \equiv (u/c)}$$

- Here, we use $u = |\vec{u}| =$ relative speed of an object as observed in an inertial reference frame (here, u = speed of you, as observed in the lab IRF(S)).



- We will henceforth use $v = |\vec{v}| =$ relative speed between two inertial systems – e.g. IRF(S')) relative to IRF(S):

- Because the time interval, dt' occurs in your own REST FRAME (IRF(S')) we give it a special name: $\boxed{d\tau' = dt'}$ = PROPER time interval (in your rest frame) $\boxed{\tau' = t'}$ = proper TIME (in your own rest frame).

- The name "proper" time is due to a mis-translation of the French word "*propre*", meaning "own".

- Proper time τ' is different than "ordinary" time, *t*.

  Proper time τ' is a Lorentz-invariant quantity, whereas "ordinary" time *t* depends on the choice of IRF.

  The Lorentz-invariant interval: $\boxed{dI \equiv dx'_\mu dx'^\mu = dx'^\mu dx'_\mu = ds'^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2}$

  Proper time interval: $\boxed{d\tau' \equiv \sqrt{-dI/c^2} = \sqrt{-ds'^2/c^2} = \sqrt{dt'^2 - dx'^2/c^2 - dy'^2/c^2 - dz'^2/c^2} = \sqrt{dt'^2} = dt'}$

  Proper time: $\boxed{\tau' \equiv \tau'_2 - \tau'_1 \equiv \int_{\tau'_1}^{\tau'_2} d\tau' = \int_{t'_1}^{t'_2} dt' = t'_2 - t'_1 = \Delta t'}$

- Because $d\tau'$ and τ' are Lorentz-invariant quantities: $\boxed{d\tau' = d\tau}$ and: $\boxed{\tau' = \tau}$ {i.e. drop primes}.

- In terms of 4-D space-time, proper time is analogous to arc length *S* in 3-D Euclidean space.

- Special designation is given to being in the REST FRAME of an object.

- The rest frame of an object = the proper frame.

Consider a situation where you are on an airplane flight from NYC to LA. The pilot comes on the loudspeaker and announces in mid-flight that the jet stream is flowing backwards today, and that the plane's present velocity is $u = 0.8c$, due west, $(\beta_u = 0.8!!)$

What the pilot means by "velocity" is the spatial displacement $d\vec{\ell}$ per unit time interval, dt.

The pilot is referring to the plane's velocity relative to the ground (we assume here that the earth is non-rotating/non-moving – let's keep it somewhat simple, eh, so we can use IRF's, eh?)

Thus, $d\vec{\ell}$ and dt are meant to be understood as quantities as quantities as measured by an observer on the ground (e.g. an airplane flight controller, using RADAR) in the ground-based IRF(S).

Thus: $\boxed{\vec{u} = \dfrac{d\vec{\ell}}{dt}}$ = "ordinary" velocity in IRF(S) $\quad$ $d\vec{\ell}$ and dt are measured in the ground-based IRF(S)

You, on the other hand are in *your* own rest frame (IRF(S')) in the airplane, sitting in your seat.

You know that the distance from NYC to LA is: $L = 2763$ miles (referring to your trusty Rand-McNally Road Atlas (back pages) that you brought along with you for your trip).

So you, from *your* perspective, might be more interested in the quantity known as your proper velocity $\vec{\eta}$, defined as:

Proper 3-Velocity: $\boxed{\vec{\eta} \equiv \dfrac{d\vec{\ell}}{d\tau}}$ = ***hybrid*** measurement = Spatial displacement, as measured on the ground (in IRF(S)) per unit time interval as measured in your (or an object's) rest frame (in IRF(S')).

Since: $\boxed{d\tau = dt' = \dfrac{1}{\gamma_u} dt = \sqrt{1-\beta_u^2}\, dt = \sqrt{1-(u/c)^2}\, dt}$ and: $\boxed{\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}}$, $\boxed{\beta_u \equiv (u/c)}$

Then: $\boxed{\vec{\eta} \equiv \dfrac{d\vec{\ell}}{d\tau} = \dfrac{d\vec{\ell}}{\frac{1}{\gamma_u} dt} = \gamma_u \dfrac{d\vec{\ell}}{dt}}$, but: $\boxed{\vec{u} \equiv \dfrac{d\vec{\ell}}{dt}}$ $\therefore$ $\boxed{\vec{\eta} = \gamma_u \vec{u} = \dfrac{1}{\sqrt{1-\beta_u^2}} \vec{u} = \dfrac{1}{\sqrt{1-(u/c)^2}} \vec{u}}$

Of course, for non-relativistic speeds $\boxed{u \ll c}$ then $\boxed{\vec{\eta} \simeq \vec{u}}$ to a high degree.

From a theoretical perspective, an appealing aspect of proper 3-velocity, $\vec{\eta}$ is that it Lorentz-transforms simply from one IRF to another IRF.

$\vec{\eta}$ = 3-D spatial component(s) of a relativistic 4-vector, η^μ

The {contravariant} proper 4-velocity is: $\boxed{\eta^\mu \equiv \dfrac{dx^\mu}{d\tau}}$ whose zeroth/temporal component is:

$$\boxed{\eta^0 \equiv \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\frac{dt}{\frac{1}{\gamma_u} dt} = \gamma_u c \frac{\cancel{dt}}{\cancel{dt}} = \gamma_u c = \frac{c}{\sqrt{1-\beta_u^2}} = \frac{c}{\sqrt{1-(u/c)^2}}}$$ with: $\boxed{\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}} \quad \beta_u \equiv (u/c)}$

The proper 4-velocity is:

$$\boxed{\eta^{\mu} \equiv \frac{dx^{\mu}}{d\tau} = \left(\eta^0, \vec{\eta}\right) = \left(\gamma_u c, \vec{\eta}\right)} \quad \text{or:} \quad \boxed{\eta^{\mu} = \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{pmatrix}}$$

The numerator of the proper 4-velocity, dx^u is the displacement 4-vector (as measured in the ground-based IRF(S). The denominator of the proper 4-velocity, $d\tau$ = proper time interval (as measured in your (an object's) rest frame IRF(S').

The Lorentz Transformation of a Proper 4-Velocity, η^{μ} :

Suppose we want to Lorentz transformation your proper 4-velocity from the lab IRF(S) to *another* (different) IRF(S'') along a common $\hat{x}$ -axis, in which IRF(S'') is moving with relative velocity $\boxed{\vec{v} = v\hat{x}}$ with respect to IRF(S):

Most generally, in tensor notation: $\boxed{\eta''^{\mu} = \Lambda^{\mu}_{\nu}\eta^{\nu}}$ with Λ^{μ}_{ν} = Lorentz boost tensor. Thus:

$$\boxed{\eta''^{\mu} = \Lambda^{\mu}_{\nu}\eta^{\nu}} \Rightarrow \boxed{\begin{pmatrix} \eta''^0 \\ \eta''^1 \\ \eta''^2 \\ \eta''^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix}} \Rightarrow \boxed{\begin{pmatrix} \eta''^0 \\ \eta''^1 \\ \eta''^2 \\ \eta''^3 \end{pmatrix} = \begin{pmatrix} \gamma\left(\eta^0 - \beta\eta^1\right) \\ \gamma\left(\eta^1 - \beta\eta^0\right) \\ \eta^2 \\ \eta^3 \end{pmatrix}} \ \text{with:} \ \boxed{\begin{matrix} \gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} \\ \beta \equiv \dfrac{v}{c} \end{matrix}}$$

Where: $\boxed{\eta''^{\mu} \equiv \dfrac{dx''^{\mu}}{d\tau}}$ and: $\boxed{\eta^{\mu} \equiv \dfrac{dx^{\mu}}{d\tau}}$

Compare this result to the same Lorentz transformation of "ordinary" 3-velocities, along a common $\hat{x}$ -axis. We can simply use the Einstein velocity addition rule:

$$\boxed{\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}} \qquad \boxed{u''_x = \frac{dx''}{dt''} = \frac{u_x - v}{1-\left(u_x v/c^2\right)}}$$

$$\boxed{\vec{u}'' = u''_x\hat{x} + u''_y\hat{y} + u''_z\hat{z}} \qquad \boxed{u''_y = \frac{dy''}{dt''} = \frac{u_y}{\gamma\left(1-\left(u_x v/c^2\right)\right)}} \ \text{with:} \ \boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}} \ \text{and:} \ \boxed{\beta \equiv \frac{v}{c}}$$

$$\boxed{u''_z = \frac{dz''}{dt''} \quad \frac{u_z}{\gamma\left(1-\left(u_x v/c^2\right)\right)}}$$

{See Griffiths Example 12.6 (p. 497-98) and Griffiths Problem 12/14 (p.498)}

Now we can see why Lorentz transformation of "ordinary" velocities is more cumbersome than Lorentz transformation of proper 4-velocities:

- For "ordinary" 3-velocities $\boxed{\vec{u} = \frac{d\vec{\ell}}{dt}}$, we must Lorentz transform both $\left\{\begin{matrix}\text{numerator, } d\vec{\ell} \\ \text{denominator, } dt\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} d\vec{\ell}'' \\ dt'' \end{matrix}\right\}$
- For proper 4-velocities $\boxed{\eta^\mu \equiv \frac{dx^\mu}{d\tau}}$ we only need to transform the numerator, $d\vec{\ell} \Rightarrow d\vec{\ell}''$.

Relativistic Energy and Momentum: 4-Momentum:

In classical mechanics, the 3-D vector momentum, $\vec{p} =$ mass $\times$ velocity $\vec{v}$, i.e. $\boxed{\vec{p} = m\vec{v}}$.
How do we extend this to relativistic mechanics?

Should we use "ordinary" velocity, $\boxed{\vec{u} \equiv \frac{d\vec{\ell}}{dt}}$ for $\vec{v}$,

or should we use proper velocity, $\boxed{\vec{\eta} \equiv \frac{d\vec{\ell}}{d\tau}}$ for $\vec{v}$??

In classical mechanics, $\vec{\eta}$ and $\vec{u}$ are identical.
In relativistic mechanics, $\vec{\eta}$ and $\vec{u}$ are *not* identical.

We must use the proper velocity $\vec{\eta}$ in relativistic mechanics, because otherwise, the law of conservation of momentum would be inconsistent with the principle of relativity {= the laws of physics are the same in all IRF's} if we *were* to define relativistic 3-momentum as: $p = m\vec{u}$. No!!

Thus, we define the relativistically-correct 3-momentum as:

$$\boxed{\vec{p} \equiv m\vec{\eta} = \gamma_u m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\beta_u^2}} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}} \quad \text{with: } \boxed{\gamma_\mu \equiv \frac{1}{\sqrt{1-\beta_u^2}}} \quad \text{and: } \boxed{\beta_u \equiv \left(\frac{u}{c}\right)}$$

Relativistic 3-momentum $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u}}$ is the spatial part of a relativistic 4-momentum vector: $\boxed{p^\mu \equiv m\eta^\mu}$, i.e. $\boxed{p^\mu = \left(p^0, \vec{p}\right)}$.

The temporal/zeroth/scalar component of the relativistic 4-momentum vector is:

$$\boxed{p^0 \equiv m\eta^0 = \gamma_u mc = \frac{mc}{\sqrt{1-\beta_u^2}} = \frac{mc}{\sqrt{1-(u/c)^2}}} \quad \text{with: } \boxed{\gamma_\mu \equiv \frac{1}{\sqrt{1-\beta_u^2}}} \quad \text{and: } \boxed{\beta_u \equiv (u/c)}$$

Einstein called $\boxed{m_{rel} \equiv \gamma_u m = \frac{m}{\sqrt{1-\beta_u^2}} = \frac{m}{\sqrt{1-(u/c)^2}}}$ = relativistic mass.

Thus: $\boxed{p^0 = m\eta^0 = m_{rel}c}$, $\boxed{\eta^0 = \gamma_u c = \frac{c}{\sqrt{1-\beta_u^2}} = \frac{c}{\sqrt{1-(u/c)^2}}}$ $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u} = m_{rel}\vec{u}}$

n.b. Modern usage has abandoned the use of relativistic mass in favor of the relativistic energy, *E*.

Relativistic Energy: $\boxed{E \equiv \gamma_u mc^2 = \dfrac{mc^2}{\sqrt{1-\beta_u^2}} = \dfrac{mc^2}{\sqrt{1-(u/c)^2}}}$ with: $\boxed{\gamma_\mu \equiv \dfrac{1}{\sqrt{1-\beta_u^2}}}$ and: $\boxed{\beta_u \equiv (u/c)}$

Thus, we see that relativistic energy $\boxed{E \equiv \gamma_u mc^2 = m_{rel}c^2}$ and thus: $\boxed{p^0 \equiv E/c}$

Therefore, the components of the relativistic 4-momentum are: $\boxed{p^\mu = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \equiv \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}}$

Note that relativistic energy of an massive object is non-zero even when an object is stationary (i.e. in its own rest frame), i.e. when $\boxed{\gamma_u = 1\big/\sqrt{1-\beta_u^2} = 1}$ and $\boxed{\beta_u = 0}$.

Then for $\boxed{\gamma_u = 1}$, $\boxed{\beta_u = 0}$: $\boxed{E_{rest} = mc^2}$ = rest energy = rest mass * c^2. $\Leftarrow$Einstein's famous formula!

The remainder of relativistic energy (if $\beta_u \neq 0$) is attributable to the motion of the particle – i.e. it is relativistic *kinetic* energy, E_{kin}.

Total Relativistic Energy: $\boxed{E \equiv E_{Tot} = E_{kin} + E_{rest} = \gamma_u mc^2}$ but: $\boxed{E_{rest} = mc^2}$

$\therefore$ $\boxed{E_{kin} = E_{ToT} - E_{rest} = \gamma_u mc^2 - mc^2 = (\gamma_u - 1)mc^2}$

Relativistic Kinetic Energy: $\boxed{E_{kin} = (\gamma_u - 1)mc^2 = \left(\dfrac{1}{\sqrt{1-\beta_u^2}} - 1\right)mc^2 = \left(\dfrac{1}{\sqrt{1-(u/c)^2}} - 1\right)mc^2}$

In the non-relativistic regime, $\boxed{u \ll c}$: $\boxed{E_{kin} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \ldots \approx \frac{1}{2}mu^2}$ (classical formula).

But for $\boxed{u \ll c}$, then: $\boxed{p = mu}$, and thus: $\boxed{E_{kin} \simeq \dfrac{p^2}{2m}}$ for $\boxed{u \ll c}$ (classical formula).

n.b. The total relativistic energy, E_{ToT} (= *E*) and total relativistic momentum, $p_{ToT} = |\vec{p}_{ToT}|$ are conserved in a closed system/every closed system.

If the system is not closed, (e.g. $\exists$ external forces present) then $E = E_{ToT}$ and $p_{ToT} = |\vec{p}|$ will not necessarily be conserved. $\Rightarrow$ Simply expand/enlarge the definition of "system" until it IS closed, then the (new) E_{ToT} and $p_{ToT} = |\vec{p}_{ToT}|$ will be conserved.

Relativistic mass, $\boxed{m_{rel} = \gamma_u m = \dfrac{m}{\sqrt{1-\beta_u^2}} = \dfrac{m}{\sqrt{1-(u/c)^2}}}$ is also conserved in a/every closed system,

because $\boxed{m_{r\ell} = E_{ToT}/c^2}$ or: $\boxed{E_{ToT} = \gamma_u mc^2 = m_{r\ell}c^2}$ $\Rightarrow$ conservation of relativistic mass $\equiv$ conservation of total relativistic energy, i.e. this is simply redundant information.

Same in all inertial reference frames

Note the distinction between a Lorentz-invariant quantity and a conserved quantity.

Same before vs. after
a process/an "event"

Rest mass is a Lorentz-invariant quantity, but it is not {necessarily} a conserved quantity.

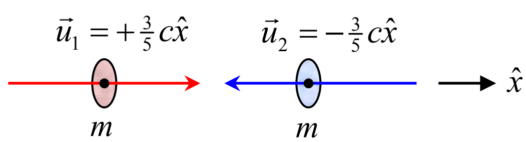
Example: The {unstable} charged pi-meson decays (via weak charged-current interaction, with mean/proper lifetime $\tau_{\pi^+} = 26.0\,ns$) to a muon and muon neutrino: $\pi^+ \to \mu^+ \nu_\mu$. The charged pion mass m_{π^+} is not conserved in the decay, however the total relativistic energy of the charged pion $\boxed{E_{\pi^+} = \sqrt{p_{\pi^+}^2 c^2 + m_{\pi^+}^2 c^4}}$ *is* a conserved, but not Lorentz-invariant quantity.

The scalar product of *any* relativistic 4-vector a^μ with itself is a Lorentz-invariant quantity (i.e. = same numerical value in *any* IRF): e.g. $\boxed{p_\mu p^\mu = p^\mu p_\mu = -m^2c^2}$.

$$\boxed{p_\mu p^\mu = p^\mu p_\mu = -\left(p^0\right)^2 + \left(\vec{p}\bullet\vec{p}\right) = -\left(E/c\right)^2 + p^2 = -\cancel{p}^2 - m^2c^2 + \cancel{p}^2 = -m^2c^2}$$

Griffiths Example 12.7: Relativistic Kinematics

Two relativistic lumps of clay {each of rest mass m} collide head-on with each other. Each lump of clay is traveling at relativistic speed $u = \frac{3}{5}c$ as shown in the figure below:



The two relativistic lumps of clay stick together (i.e. this is an *inelastic* collision).

What is the total mass M of the composite lump of clay *after* the collision?

Conservation of momentum before vs. after:

Since the two lumps of clay have identical rest masses and equal, but opposite velocities:

$\boxed{\vec{p}_{ToT}^{\,before} = \vec{p}_1 + \vec{p}_2}$ but: $\boxed{\vec{p}_1 = -\vec{p}_2 = \gamma_u m\vec{u}_1}$ where: $\boxed{\gamma_u = \dfrac{1}{\sqrt{1-\beta_u^2}}}$ $\therefore$ $\boxed{\vec{p}_{ToT}^{\,before} = 0}$

Conservation of energy before vs. after:

Before: Each lump of clay has total energy: $\boxed{E = \gamma_u mc^2 = \frac{mc^2}{\sqrt{1-\beta_u^2}} = \frac{mc^2}{\sqrt{1-(u/c)^2}}}$

$$\therefore \boxed{E = \frac{mc^2}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{mc^2}{\sqrt{1-\frac{9}{25}}} = \frac{mc^2}{\sqrt{\frac{16}{25}}} = \frac{5}{4}mc^2}$$

Thus: $\boxed{E_{ToT}^{before} = E_{ToT_1} + E_{ToT_2} = 2\gamma_u mc^2 = \frac{10}{4}mc^2 = \frac{5}{2}mc^2}$

However, E_{ToT} is {always} conserved in a closed system. $\Rightarrow$ $\boxed{E_{ToT}^{after} = E_{ToT}^{before} = \frac{5}{2}mc^2}$

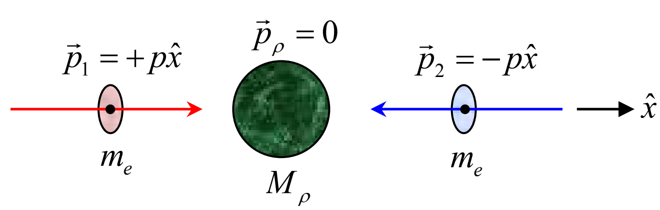
And $\vec{p}_{ToT}$ is also {always} separately conserved in a closed system. $\Rightarrow$ $\boxed{\vec{p}_{ToT}^{after} = \vec{p}_{ToT}^{before} = 0}$

$\Rightarrow \boxed{\vec{u}^{after} = 0}$ since: $\boxed{\vec{p}_{ToT}^{after} = \gamma_{u_{after}} M\vec{u}^{after} = 0}$. n.b. $\Rightarrow$ $\boxed{\gamma_{u_{after}} = \frac{1}{\sqrt{1-\beta_{u_{after}}^2}} = \frac{1}{\sqrt{1-(u_{after}/c)^2}} = 1}$

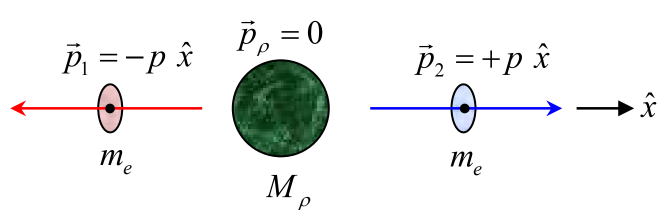
Then: $\boxed{E_{ToT}^{after} = \gamma_{u_{after}} Mc^2 = Mc^2 = \frac{5}{2}mc^2 \left(= E_{ToT}^{before}\right)}$ $\therefore$ $\boxed{M = \frac{5}{2}m \neq 2m}$!!! Does this sound crazy??

This *is* what happens in the "everyday" world of particle physics! It's perfectly OK !!!

e.g. The production of a neutral rho meson in electron-positron collisions: $\boxed{e^+ + e^- \to \rho^0}$.
The rest mass of the neutral rho meson is: $\boxed{M_\rho = 770\, MeV/c^2}$ Electron rest mass: $\boxed{m_e = 0.511\, MeV/c^2}$



Run the collision process *backwards* in time, e.g. the decay of a neutral rho meson: $\boxed{\rho^0 \to e^+ + e^-}$



The production of a neutral rho meson $\boxed{e^+ + e^- \to \rho^0}$ manifestly involves the *EM* interaction. Similarly, the time-reversed situation: the decay of a neutral rho meson $\boxed{\rho^0 \to e^+ + e^-}$ manifestly *also* involves the *EM* interaction.

The *EM* interaction is *invariant* under time-reversal, i.e. $t \to -t$, thus {in the rest frame of the neutral rho meson} the transition rate $\boxed{\Gamma\left(e^+ + e^- \to \rho^0\right)}$ (#/sec) *vs*. the decay rate $\boxed{\Gamma\left(\rho^0 \to e^+ + e^-\right)}$ (#/sec) are identical {for the same/identical electron / positron momenta in neutral rho meson production vs. decay}. Experimentally: $\boxed{\Gamma\left(\rho^0 \to e^+ + e^-\right) = 7.02 \text{ KeV} = 1.70\times 10^{18}\text{ sec}^{-1}}$.

For our above macroscopic inelastic collision problem, microscopically what would the new matter of the macroscopic mass M be made up of, since $\boxed{\Delta M = M - 2m = \tfrac{5}{2}m - 2m = \tfrac{1}{2}m}$???

In a classical analysis of the inelastic collision of two relativistic macroscopic lumps of clay {each of mass m} the composite / stuck-together single lump of clay of mass $\boxed{M = \tfrac{5}{2}m > 2m}$ would be very *hot* – it would have a *great deal* of thermal energy in fact !!!

$$\boxed{Mc^2 = \frac{5}{2}mc^2 = \underbrace{2mc^2}_{\substack{\text{classical mass}\\ \text{of composite}\\ \text{lump}}} + E_{thermal}} \Rightarrow \underline{\underline{E_{thermal} = 0.5mc^2}}!!! \quad E = mc^2 = \text{Einstein's energy-mass formula}$$

Connection Between Conserved Quantities and Lorentz-Invariant Quantities:

Before: $\boxed{p^\mu_{before} = \left(E_{before}/c, \vec{p}_{before}\right)}$ After: $\boxed{p^\mu_{after} = \left(E_{after}/c, \vec{p}_{after}\right)}$. However, total relativistic energy E and total relativistic momentum $\vec{p}$ are *separately* conserved quantities, thus:

$\boxed{E_{after} = E_{before} = Mc^2}$ and $\boxed{\vec{p}_{after} = \vec{p}_{before} = 0}$. But: $\boxed{p_\mu p^\mu = p^\mu p_\mu = -M^2c^2}$ is Lorentz invariant:

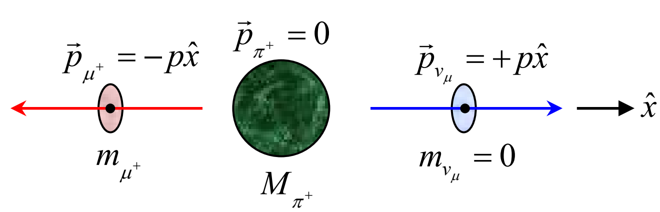
$$\boxed{p_\mu p^\mu = p^\mu p_\mu = -\left(p^0\right)^2 + \left(\vec{p}\bullet\vec{p}\right) = -\left(E/c\right)^2 + p^2 = -M^2c^2 + 0 = -M^2c^2}$$ holds before & after!

Griffiths Example 12.8: Relativistic kinematics associated with $\pi^+ \to \mu^+ \nu_\mu$ decay.

Pion rest mass: $\boxed{m_{\pi^+} = 139.57\, MeV/c^2}$ Pion mean lifetime: $\boxed{\tau_{\pi^+} = 26.033\text{ nsec} = 26.033\times 10^{-9}\text{ sec}}$

Muon rest mass: $\boxed{m_{\mu^+} = 105.66\, MeV/c^2}$ Muon neutrino rest mass: $\boxed{m_{\nu_\mu} = 0}$ (assumed).

In the rest frame of the π^+ meson:



Energy Conservation: Momentum Conservation:

Before: $\boxed{E^{before}_{ToT} = m_{\pi^+}c^2}$ $\boxed{\vec{p}^{\,before}_{ToT} = 0}$

After: $\boxed{E^{after}_{ToT} = E_{\mu^+} + E_{\nu_\mu} = m_{\pi^+}c^2}$ $\boxed{\vec{p}^{\,after}_{ToT} = \vec{p}_{\mu^+} + \vec{p}_{\nu_\mu} = 0} \Rightarrow \boxed{\vec{p}_{\mu^+} = -\vec{p}_{\nu_\mu} = -p\hat{x}}$

But: $\boxed{E_{\nu_\mu} = p_{\nu_\mu} c = |\vec{p}_{\nu_\mu}| c}$ since: $\boxed{m_{\nu_\mu} = 0}$. $\Rightarrow \boxed{p_{\mu^+} = |\vec{p}_{\mu^+}|} = \boxed{p_{\nu_\mu} = |\vec{p}_{\nu_\mu}|}$

And: $\boxed{E_{\mu^+}^2 = p_{\mu^+}^2 c^2 + m_{\mu^+}^2 c^4}$ or: $\boxed{p_{\mu^+}^2 c^2 = E_{u^+}^2 - m_{\mu^+}^2 c^4} \Rightarrow \boxed{p_{\mu^+} c = \sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4}}$

$\therefore \boxed{p_{\mu^+} = |\vec{p}_{\mu^+}|} = \boxed{p_{\nu_\mu} = |\vec{p}_{\nu_\mu}|} = \boxed{\sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4} \Big/ c}$

Then: $\boxed{E_{ToT}^{after} = E_{\mu^+} + E_{\nu_\mu} = E_{\mu^+} + p_{\nu_\mu} c}$ but: $\boxed{p_{\nu_\mu} = p_{\mu^+} = \dfrac{\sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4}}{c}}$

$\therefore \boxed{E_{ToT}^{after} = E_{\mu^+} + E_{\nu_\mu} = E_{\mu^+} + p_{\nu_\mu} c = E_{\mu^+} + \sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4} = E_{ToT}^{before} = m_{\pi^+} c^2}$

$\therefore \boxed{E_{\mu^+} + \sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4} = m_{\pi^+} c^2}$ Solve for E_{μ^+} :

$\boxed{E_{\mu^+}^2 - m_{\mu^+}^2 c^4 = \left(m_{\pi^+} c^2 - E_{\mu^+}\right)^2 = m_{\pi^+}^2 c^4 - 2\left(m_{\pi^+} c^2 E_{\mu^+}\right) + E_{\mu^+}^2}$ or: $\boxed{2 m_{\pi^+} c^2 E_{\mu^+} = m_{\pi^+}^2 c^4 - m_{\mu^+}^2 c^4}$

Thus: $\boxed{E_{\mu^+} = \dfrac{m_{\pi^+}^2 c^4 - m_{\mu^+}^2 c^4}{2 m_{\pi^+} c^2} = \dfrac{\left(m_{\pi^+}^2 - m_{\mu^+}^2\right) c^2}{2 m_{\pi^+}}}$ and: $\boxed{p_{\nu_\mu} = p_{\mu^+} = \dfrac{\sqrt{E_{\mu^+}^2 - m_{\mu^+}^2 c^4}}{c}}$ with: $\boxed{\vec{p}_{\mu^+} = -\vec{p}_{\nu_\mu}}$

as viewed from the rest frame of the π^+ meson.

- In *classical* collisions, total 3-momentum $\vec{p}_{ToT}$ and total mass, m_{ToT} are *always* conserved: $\boxed{\vec{p}_{ToT}^{before} = \vec{p}_{ToT}^{after}}$, $\boxed{m_{ToT}^{before} = m_{ToT}^{after}}$. Total kinetic energy E_{kin}^{ToT} is not conserved $\Rightarrow$ *inelastic* collision.

- An *inelastic* (i.e. a “sticky”) collision generates heat at the expense of kinetic energy.

- An *inelastic* collision of an electron (e^-) with an atom {initially in its ground state} may leave the atom in an excited state, or even ionized, kicking out a once-bound atomic electron!
  $\Rightarrow$ Internal {quantum} degrees of freedom can be excited in inelastic e^- - atom collisions.

- An “explosive” collision generates *kinetic* energy at the expense of *chemical* (i.e. *EM*) energy, or nuclear (i.e. strong-force) energy, or weak-force energy. . . .

- If *kinetic* energy *is* conserved (classically), $\Rightarrow$ *elastic* (i.e. billiard-ball) collision.

- In *relativistic* collisions, total 3-momentum and total energy are *always* conserved (in a closed system) but total mass and total kinetic energy are not, in general, conserved.

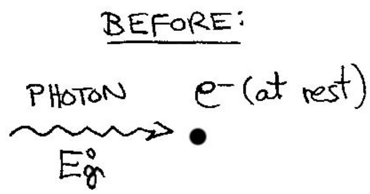
  * Once again, in relativistic collisions, a process is called elastic if the total kinetic energy *is* conserved $\Rightarrow$ total mass is also conserved in relativistic elastic collisions.

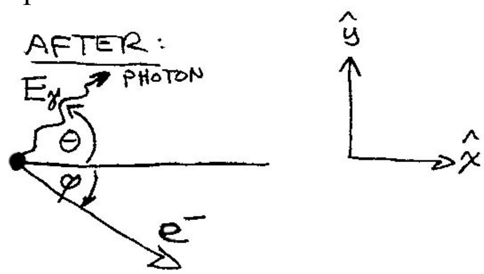
  * A relativistic collision is called inelastic if the total kinetic energy is not conserved.
  $\Rightarrow$ Total mass is not conserved in a relativistic inelastic collision.

Griffiths Example 12.9:

Compton scattering = Relativistic Elastic Scattering of Photons with Electrons.

An incident photon of energy $\boxed{E_\gamma^0 = p_\gamma^0 c}$ elastically scatters (i.e. "bounces") off of an electron, which is initially at rest (in the lab frame). Find the final energy E_γ of the outgoing scattered photon as a function of the scattering angle, θ of the photon:



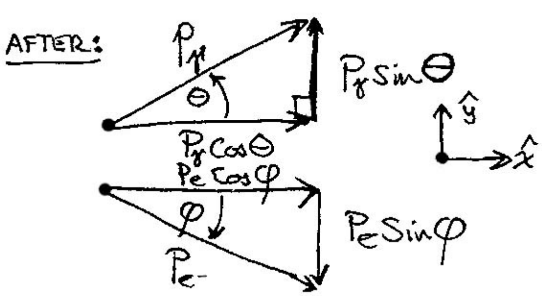


Consider conservation of total relativistic momentum in the transverse ($\perp$) (i.e. $\hat{y}$-axis) direction:

$$\boxed{p_{\perp_{TOT}}^{before} = 0 = p_{\perp_{TOT}}^{after}}$$

$$\boxed{p_{\perp_{TOT}}^{before} = p_{\perp_\gamma}^{before} + p_{\perp_{e^-}}^{before} = 0 + 0 = 0}$$

$$\boxed{p_{\perp_{TOT}}^{after} = p_{\perp_\gamma}^{after} + p_{\perp_{e^-}}^{after} = 0} \Rightarrow \boxed{\overbrace{p_{\perp_\gamma}^{after}}^{+\hat{y}\text{ direction}} = -\overbrace{p_{\perp_{e^-}}^{after}}^{-\hat{y}\text{ direction}}}$$



Since: $\boxed{\overbrace{p_{\perp_\gamma}^{after}}^{+\hat{y}\text{ direction}} = -\overbrace{p_{\perp_{e^-}}^{after}}^{-\hat{y}\text{ direction}}}$

Or: $\boxed{\left|p_{\perp_\gamma}^{after}\right| = \left|p_{\perp_{e^-}}^{after}\right|}$

Or: $\boxed{p_\gamma \sin\theta = p_{e^-} \sin\varphi}$

But: $\boxed{p_\gamma = E_\gamma / c}$

$\therefore$ $\boxed{\dfrac{E_\gamma}{c}\sin\theta = p_{e^-}\sin\varphi}$

Solve for $\sin\varphi$: $\boxed{\sin\varphi = \left(\dfrac{E_\gamma}{p_{e^-} c}\right)\sin\theta}$

Conservation of total relativistic momentum in the longitudinal (i.e. $\hat{x}$) direction gives:

$\boxed{p_{\parallel_{TOT}}^{before} = \dfrac{E_\gamma^0}{c}}$ (n.b. $p_{e^-}^{before} = 0$, since e$^-$ initially at rest, hence $p_{\parallel e^-}^{before} = 0$)

$$\boxed{p_{\parallel_{TOT}}^{after} = p_{\parallel_\gamma}^{after} + p_{\parallel_{e^-}}^{after} = p_\gamma \cos\theta + p_{e^-}\cos\varphi}$$

$\therefore$ Since: $\boxed{p_{\parallel_{TOT}}^{before} = p_{\parallel_{TOT}}^{after}}$ then: $\boxed{E_\gamma^0 / c = p_\gamma \cos\theta + p_{e^-}\cos\varphi}$

But: $\boxed{\sin\varphi = \left(\frac{E_\gamma}{p_{e^-}c}\right)\cos\theta}$ thus: $\boxed{\cos\varphi = \sqrt{1-\sin^2\varphi} = \sqrt{1-\left(\frac{E_\gamma}{P_{e^-}c}\right)^2\sin^2\theta}}$

$$\therefore \quad \boxed{\frac{E_\gamma^0}{c} = p_\gamma\cos\theta + p_{e^-}\sqrt{1-\left(\frac{E_\gamma}{p_{e^-}c}\right)^2\sin^2\theta}}$$

Or: $\boxed{p_{e^-}^2c^2 = \left(E_\gamma^0 - E_\gamma\cos\theta\right)^2 + E_\gamma^2\sin^2\theta = E_\gamma^{0^2} - 2E_\gamma^0E_\gamma\cos\theta + E_\gamma^2}$

Conservation of total energy: $\boxed{E_{TOT}^{before} = E_{TOT}^{after}}$ $\Rightarrow$ $\boxed{\overbrace{E_\gamma^0 + m_ec^2}^{E_{TOT}^{before}} = \overbrace{E_\gamma + E_{e^-}}^{E_{TOT}^{after}} = E_\gamma + \sqrt{p_e^2c^2 + m_e^2c^4}}$

$$\therefore \quad \boxed{E_\gamma^0 + m_ec^2 = E_\gamma + \sqrt{E_\gamma^{0^2} - 2E_\gamma^0E_\gamma\cos\theta + E_\gamma^2 + m_e^2c^4}}$$

Solve for E_γ (after some algebra): $\boxed{E_\gamma = \frac{1}{\left[(1-\cos\theta)/m_ec^2 + 1/E_\gamma^0\right]}}$

E_γ = energy of recoil photon in terms of initial photon energy E_γ^0, scattering angle of photon θ and rest energy of electron m_ec^2.

Can alternatively express this relation in terms of photon wavelengths:

Before: $\boxed{E_\gamma^0 = hf_\gamma^0 = hc/\lambda_\gamma^0}$

After: $\boxed{E_\gamma = hf_\gamma = hc/\lambda_\gamma}$

Get: $\boxed{\lambda_\gamma = \lambda_0 + \left(\frac{hc}{m_ec^2}\right)(1-\cos\theta)}$

Useful constants:

$\boxed{hc = 1239.841\text{ eV-nm} \simeq 1240\text{ eV-nm}}$

$\boxed{m_ec^2 \simeq 0.511\text{ MeV} = 0.511\times10^6\text{ eV}}$

Define the so-called Compton wavelength of the electron: $\boxed{\lambda_e \equiv \left(\frac{hc}{m_ec^2}\right) = 2.426\times10^{-12}\text{ m}}$

Then: $\boxed{\lambda_\gamma = \lambda_0 + \lambda_e(1-\cos\theta)}$

The Compton Differential Scattering Cross Section:

As we learned in P436 Lecture Notes 14.5 (p. 9-22) non-relativistic photon-free electron scattering $\left(E_\gamma^0 \ll m_ec^2\right)$ is adequately described by the classical *EM* physics-derived differential Thomson scattering cross section:

$$\boxed{\frac{d\sigma_{T_{e^-}}^{unpol}(\theta,\varphi)}{d\Omega} \simeq \frac{1}{2}r_e^2\left(1+\cos^2\theta\right)} \quad \text{where: } \boxed{r_e \equiv \frac{e^2}{4\pi\varepsilon_o m_ec^2} \simeq 2.82\times10^{-15}\,m}$$

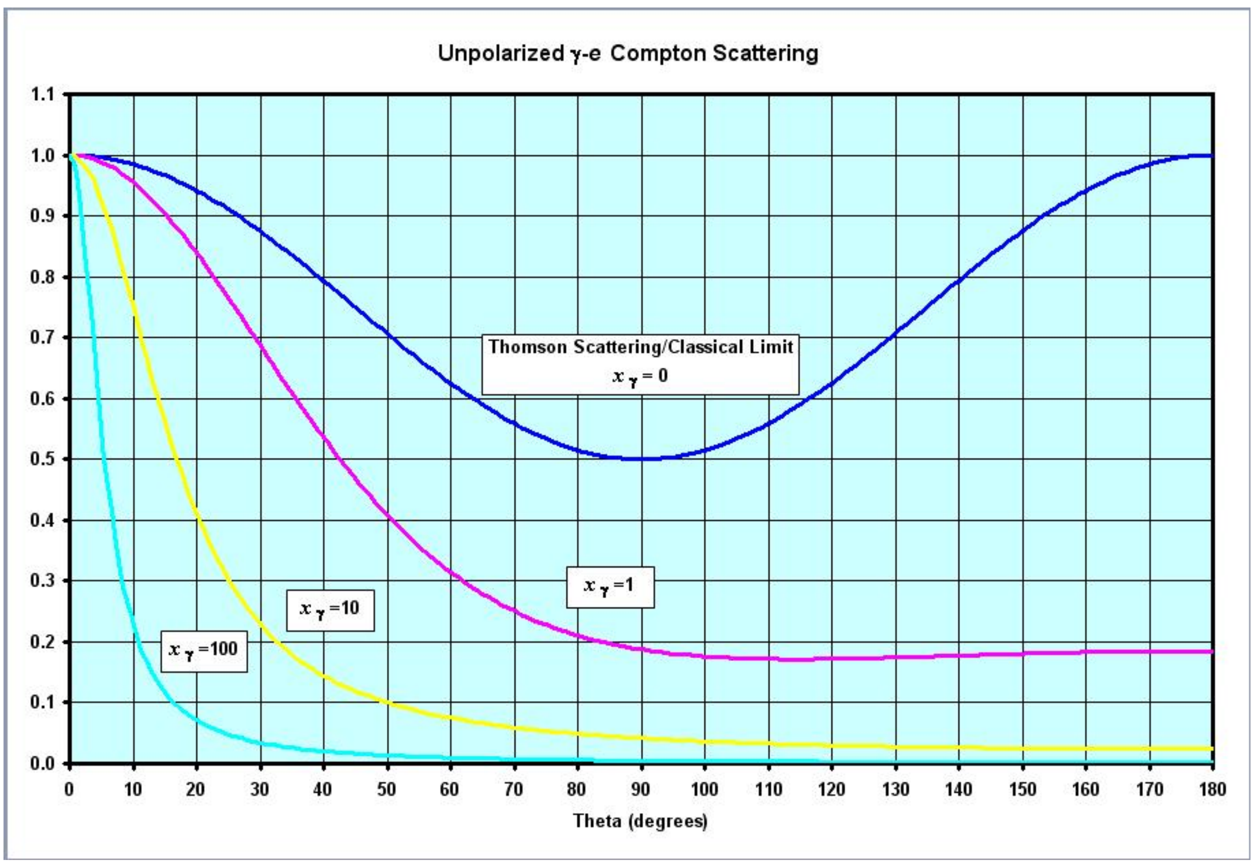
Classical electron radius

However, when $E_\gamma^0 \geq m_e c^2$ from the above discussion of the relativistic kinematics of photon-electron scattering, it is obvious that the classical theory is not valid in this regime. The fully-relativistic quantum mechanical theory – that of quantum electrodynamics (*QED*) – is required to get it right. Without going into the gory details, the results of the *QED* calculation associated with the two Feynman graphs {the so-called *s*- and *u*-channel graphs} on p. 5 of P436 Lect. Notes 14.5 for the Compton differential scattering cross section – known as the ***Klein-Nishina formula*** is, for photon-electron scattering:

$$\boxed{\frac{d\sigma_{C_{e^-}}^{unpol}(\theta,\varphi)}{d\Omega} = \frac{1}{2} r_e^2 \left(1+\cos^2\theta\right) \frac{1}{\left[1+x_\gamma(1-\cos\theta)\right]^2} \left[1 + \frac{x_\gamma^2 (1-\cos\theta)^2}{\left(1+\cos^2\theta\right)\left[1+x_\gamma(1-\cos\theta)\right]}\right]}$$

where: $\boxed{x_\gamma \equiv E_\gamma^0 / m_e c^2 = hf_\gamma^0 / m_e c^2}$. In the non-relativistic limit $x_\gamma \to 0$, the Compton scattering cross section agrees with the classical Thomson scattering cross section, as shown in the figure below of the normalized differential scattering cross section $d\sigma_{C_{e^-}}^{unpol}(\theta,\varphi) / r_e^2 d\Omega$ vs. θ.

Note that as $x_\gamma \to \infty$ the Compton differential scattering cross section becomes increasingly sharply peaked in the forward direction, $\theta \to 0$.



Relativistic Dynamics

Newton's 1st Law of Motion ("An object at rest remains at rest, an object moving with speed *v* remains moving at speed *v*, unless acted upon by a net/non-zero/unbalanced force" – the Law of Inertia) is built into/incorporated in the Principle of Relativity.

Newton's 2nd law of motion {classical mechanics}: $\boxed{\vec{F}(\vec{r},t)=\frac{d\vec{p}(\vec{r},t)}{dt}\ \left(=m\vec{a}(\vec{r},t)\right)}$ retains its validity in relativistic mechanics, provided relativistic momentum is used.

Griffiths Example 12.10: 1-D Relativistic Motion Under a *Constant* Force.

A particle of (rest) mass m is subject to a *constant* force: $\boxed{\vec{F}(\vec{r},t)=\vec{F}=F\hat{x}=\text{constant vector}}$.

If the particle starts from rest at the origin at time $t = 0$, find its position $x(t)$ as a function of t.

Since the relativistic motion is 1-D, then: $\boxed{F=\frac{dp(t)}{dt}}$ = constant, or: $\boxed{\frac{dp(t)}{dt}=F}$ = constant.

$\Rightarrow$ $\boxed{p(t)=Ft\ +\ \text{constant of integration}}$. The particle starts from rest at $t = 0$. $\therefore$ $\boxed{p(t=0)=0}$

$\Rightarrow$ constant of integration = 0. $\therefore$ $\boxed{p(t)=Ft}$ {here}

Relativistically:

$$\boxed{p(t)=\gamma_u(t)\,mu(t)=\frac{mu(t)}{\sqrt{1-\left(\frac{u(t)}{c}\right)^2}}=Ft}\quad \text{where: } \boxed{\gamma_u(t)=\frac{1}{\sqrt{1-\left(\frac{u(t)}{c}\right)^2}}}$$

Solve for $\vec{u}(\vec{r},t)$: $\boxed{m^2u^2=F^2t^2\left(1-\left(\frac{u^2}{c^2}\right)\right)=F^2t^2-\frac{F^2t^2}{c^2}u^2}$ $\Rightarrow$ $\boxed{\left(m^2+\frac{F^2t^2}{c^2}\right)u^2=F^2t^2}$

Or: $\boxed{u^2=\frac{F^2t^2}{m^2+\frac{F^2t^2}{c^2}}=\frac{(F/m)^2t^2}{1+(F/mc)^2t^2}}$ $\Rightarrow$ $\boxed{u(t)=\frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}}$ = Relativistic particle velocity for constant applied force $\vec{F}$

n.b. when: $\boxed{Ft/m\ll c}$ then: $\boxed{u(t)\simeq Ft/m}$ $\Leftarrow$ Classical dynamics answer.

Note also that as $t\to\infty$: $\boxed{u(t\to\infty)\to c}$!!! (Relativistic denominator ensures this!)

Since: $\boxed{u(t)=\frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}=\frac{dx(t)}{dt}}$

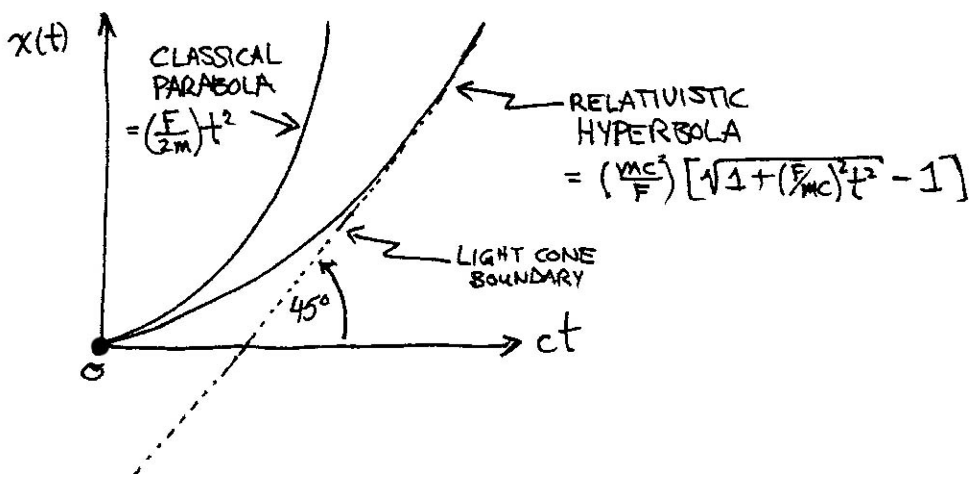
Then: $\boxed{x(t)=\int_0^t u(t')\,dt'=(F/m)\int_0^t\frac{t'}{\sqrt{1+(F/mc)^2\,t'^2}}dt'}$

The motion is hyperbolic: $\boxed{x(t) = \left(\frac{F}{m}\right)\left(\frac{mc}{F}\right)^2 \sqrt{1+(F/mc)^2 t'^2}\Big|_0^t = \left(\frac{mc^2}{F}\right)\left[\sqrt{1+(F/mc)^2 t^2} - 1\right]}$

n.b. Had we done this in classical dynamics, the result would have been parabolic motion:

$$\boxed{x(t) = \left(\frac{F}{2m}\right)t^2}$$

Thus in relativistic dynamics – e.g. a charged particle placed in a uniform electric field $\vec{E}$, the resulting motion under a constant force $\boxed{\vec{F} = q\vec{E}}$ is hyperbolic motion (not parabolic motion, as in classical dynamics) – see/compare two cases, as shown in figure below:



Relativistic Work:

Relativistic work is defined the same as classical work: $\boxed{W \equiv \int \vec{F} \bullet d\vec{\ell}}$

The Work-Energy Theorem (the net work done *on* a particle = increase in particle's kinetic energy) also holds relativistically:

$$\boxed{W = \int \vec{F} \bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt} \bullet d\vec{\ell} = \int \frac{d\vec{p}}{dt} \bullet \frac{d\vec{\ell}}{dt} dt = \int \frac{d\vec{p}}{dt} \bullet \vec{u}\, dt = \Delta E_{kin}} \quad \text{since: } \boxed{\vec{u} = \frac{d\vec{\ell}}{dt}}$$

But: $\boxed{\left(\frac{d\vec{p}}{dt} \bullet \vec{u}\right) = \frac{d}{dt}\frac{m\vec{u}}{\sqrt{1-(u/c)^2}} \bullet \vec{u}}$ since: $\boxed{\vec{p} = \gamma_u m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\beta_u^2}} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}}$

Thus:

$$\boxed{\begin{aligned}\left(\frac{d\vec{p}}{dt}\cdot\vec{u}\right) &= \frac{m}{\sqrt{1-(u/c)^2}}\left(\frac{d\vec{u}}{dt}\right)\cdot\vec{u} + \frac{\left(m\vec{u}/c^2\right)u}{\left[1-(u/c)^2\right]^{3/2}}\left(\frac{du}{dt}\right)\cdot\vec{u} = \frac{m\vec{u}}{\sqrt{1-(u/c)^2}}\cdot\frac{d\vec{u}}{dt} + \frac{m\left(u/c\right)^2\vec{u}}{\left[1-(u/c)^2\right]^{3/2}}\cdot\frac{d\vec{u}}{dt} \\ &= \left\{\frac{1}{\sqrt{1-(u/c)^2}} + \frac{(u/c)^2}{\left[1-(u/c)^2\right]^{3/2}}\right\}m\vec{u}\cdot\frac{d\vec{u}}{dt} = \left\{\frac{1-\cancel{(u/c)^2}+\cancel{(u/c)^2}}{\left[1-(u/c)^2\right]^{3/2}}\right\}m\vec{u}\cdot\frac{d\vec{u}}{dt} = \frac{m\vec{u}}{\left[1-(u/c)^2\right]^{3/2}}\cdot\frac{d\vec{u}}{dt} \\ &= \frac{mu}{\left[1-(u/c)^2\right]^{3/2}}\frac{du}{dt} = \frac{d}{dt}\left\{\frac{mc^2}{\sqrt{1-(u/c)^2}}\right\}\end{aligned}}$$

But: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}}$ $\therefore$ $\boxed{\left(\frac{d\vec{p}}{dt}\cdot\vec{u}\right) = \frac{d}{dt}\left\{\gamma_u mc^2\right\} = \frac{dE_{tot}}{dt}}$

Thus:

$$\boxed{\begin{aligned}W &= \int\vec{F}\cdot d\vec{\ell} = \int\frac{d\vec{p}}{dt}\cdot d\vec{\ell} = \int\frac{d\vec{p}}{dt}\cdot\frac{d\vec{\ell}}{dt}dt = \int\frac{d\vec{p}}{dt}\cdot\vec{u}dt = \Delta E_{kin} \\ &= \int\frac{dE_{tot}}{dt}\cdot dt = E_{tot}^{final} - E_{tot}^{initial} = \Delta E_{tot}\end{aligned}}$$

But: $\boxed{E_{tot} = E_{kin} + E_{rest} = E_{kin} + mc^2}$ n.b. $\boxed{E_{tot} = \gamma_u mc^2 = \underbrace{(\gamma_u - 1)mc^2}_{E_{kin}} + mc^2}$, $\boxed{E_{kin} = (\gamma_u - 1)mc^2}$

$\therefore$ $\boxed{\underbrace{E_{TOT}^{final} - E_{TOT}^{initial}}_{=\Delta E_{TOT}} = \left(E_{kin}^{final} + \cancel{mc^2}\right) - \left(E_{kin}^{initial} + \cancel{mc^2}\right) = \underbrace{E_{kin}^{final} - E_{kin}^{initial}}_{=\Delta E_{kin}}}$

i.e. $\boxed{W = \Delta E_{TOT} = E_{TOT}^{final} - E_{TOT}^{initial} = \Delta E_{kin} = E_{kin}^{final} - E_{kin}^{initial}}$ ← (final-initial) difference in total energy = (final-initial) difference in kinetic energy = work done **on** particle.

As we have already encountered elsewhere in *E&M*, Newton's 3rd Law of Motion ("For every action (force) there is an equal and opposite reaction") does NOT (in general) extend to the relativistic domain, because e.g. if two objects are separated in 3-D space, the 3rd Law is incompatible with the relativity of simultaneity.

Suppose the 3-D force of *A* acting on *B* at some instant *t* is: $\boxed{\vec{F}_{\overline{AB}}(\vec{r}_B, t) = +\vec{F}(\vec{r}_B, t)}$

and the 3-D force of *B* acting on *A* at the same instant *t* is: $\boxed{\vec{F}_{\overline{BA}}(\vec{r}_A, t) = -\vec{F}(\vec{r}_A, t)}$

} As observed e.g. in lab IRF(*S*)

Then Newton's 3rd Law *does* apply in this reference frame.

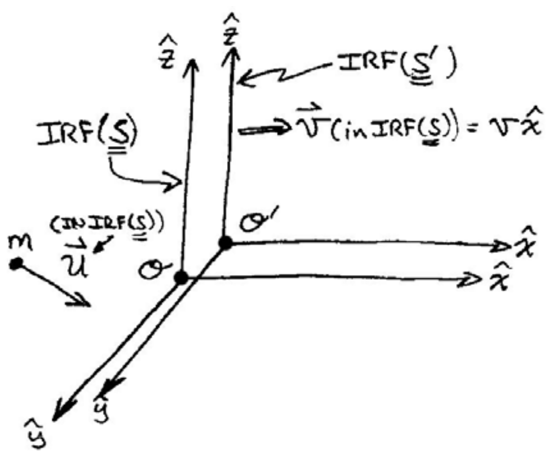
However, a moving observer {moving relative to the above IRF(*S*)} will report that these equal-but-opposite 3-D forces occurred at different times as seen from his/her IRF(*S′*), thus in his/her IRF(*S′*), Newton's 3rd Law is violated (the two 3-D forces $\vec{F}'_{\overline{AB}}(\vec{r}'_B, t')$ and $\vec{F}'_{\overline{BA}}(\vec{r}'_A, t')$ at the same time t' in IRF(*S′*) are quite unlikely to be equal *and* opposite, e.g. if they are changing in time in IRF(*S*)).

Only in the case of <u>contact</u> <u>interactions</u> (i.e. 2 point particles at same point in space-time = (x_A, t_A)) where the two 3-D forces $\vec{F}_{\overline{AB}}(\vec{r}_B, t)$ and $\vec{F}_{\overline{BA}}(\vec{r}_A, t)$ are applied at the <u>same</u> point (x_A) at the <u>same</u> time, and in the {trivial} case where forces are <u>constant</u>, does Newton's 3rd Law hold!

$$\boxed{\vec{F}(\vec{r},t) = \frac{d\vec{p}(\vec{r},t)}{dt}}$$

The observant student may have noticed that because $\vec{F}(\vec{r},t)$ is the derivative of the (relativistic) momentum $\vec{p}(\vec{r},t)$ with respect to the <u>ordinary</u> (and not the <u>proper</u>) time t, it "suffers" from the same "ugly" behavior that "ordinary" velocity does, in Lorentz-transforming it from one IRF to another: both <u>numerator</u> and <u>denominator</u> of $\dfrac{d\vec{p}(\vec{r},t)}{dt}$ must be transformed.

Thus, if we carry out a Lorentz transformation from IRF(S) to IRF(S'), along the $\hat{x}$-axis where $\boxed{\vec{v} = v\hat{x}}$ is velocity vector of IRF(S') as observed in IRF(S), and $\vec{u}$ is the velocity vector of a particle of mass m as observed in IRF(S):



<u>Then</u>: $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}}$ <u>where</u>: $\boxed{\beta \equiv \dfrac{v}{c}}$ with: $\boxed{\vec{v} = v\hat{x}}$

The γ and β factors are needed for the Lorentz transformation of kinematic quantities from IRF(S) → IRF(S').

First, let us work out the $\hat{y}'$ and $\hat{z}'$ (i.e. the <u>transverse</u>) components of the 3-D force $\vec{F}'(\vec{r}', t')$ as seen in IRF(S') {they are simpler / easier to obtain. . . }:

<u>Noting that</u>: $\boxed{\vec{F} = \dfrac{d\vec{p}}{dt}}$, $\boxed{\vec{F}' = \dfrac{d\vec{p}'}{dt'}}$ <u>and that</u>: $\boxed{dt' = \gamma dt - \dfrac{\gamma\beta}{c}dx}$ <u>and</u>: $\boxed{u_x = \dfrac{dx}{dt}}$

<u>In IRF(S')</u>: $$\boxed{F'_y = \frac{dp'_y}{dt'} = \frac{dp_y}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_y}{dt}}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma\left(1-(\beta u_x/c)\right)}}$$

<u>Similarly</u>: $$\boxed{F'_z = \frac{dp'_z}{dt'} = \frac{dp_z}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_z}{dt}}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_z}{\gamma\left(1-(\beta u_x/c)\right)}}$$

Now calculate the $\hat{x}'$ -component of the force $\vec{F}'(\vec{r}',t')$ in IRF(S'):

In IRF(S'): $\boxed{F'_x = \frac{dp'_x}{dt'} = \frac{\cancel{\gamma} dp_x - \cancel{\gamma}\beta dp^0}{\cancel{\gamma} dt - \frac{\cancel{\gamma}\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - (\beta/c)\frac{dE_{TOT}}{dt}}{1-(\beta u_x/c)}}$ where: $\boxed{p^0 = \frac{E_{TOT}}{c}}$

But we have calculated $\frac{dE_{TOT}}{dt}$ above / earlier: $\boxed{\frac{dE_{TOT}}{dt} = \frac{d\vec{p}}{dt}\bullet\vec{u} = \vec{F}\bullet\vec{u} = \vec{u}\bullet\vec{F}}$ since: $\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$

$$\therefore \quad \left. \begin{array}{l} \boxed{F'_x = \dfrac{F_x - \beta(\vec{u}\bullet\vec{F})/c}{(1-(\beta u_x/c))}} \\ \boxed{F'_y = \dfrac{F_y}{\gamma(1-(\beta u_x/c))}} \\ \boxed{F'_z = \dfrac{F_z}{\gamma(1-(\beta u_x/c))}} \end{array} \right\}$$

Relativistic "ordinary" x, y, z force components observed in IRF(S') acting on particle of mass m, for a Lorentz transformation from lab IRF(S) to IRF(S').

IRF(S') moving with velocity $\vec{v} = v\hat{x}$ relative to IRF(S) (as seen in IRF(S)), $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta \equiv v/c$.

Particle of mass m is moving with "ordinary" velocity $\vec{u}$ as seen in IRF(S).

We see that only when the particle of mass m is instantaneously at rest in IRF(S) (i.e. $\vec{u}(t) = 0$) will we then have a "simple" Lorentz transformation of the "ordinary" force $\vec{F} \to \vec{F}'$:

$$\boxed{\vec{u} = 0}: \left\{ \begin{array}{l} \boxed{F'_x = F_x} \quad \Rightarrow \boxed{F'_{\parallel} = F_{\parallel}} \\ \left. \begin{array}{l} \boxed{F'_y = F_y/\gamma} \\ \boxed{F'_z = F_z/\gamma} \end{array} \right\} \Rightarrow \boxed{F'_{\perp} = F_{\perp}/\gamma} \end{array} \right.$$

$\leftarrow$ n.b. || force components are same/identical !!!

Where the subscripts || ($\perp$) refer to the parallel (perpendicular) components of the force with respect to the motion of IRF(S') relative to IRF(S), respectively.

Note that for $\vec{u} = 0$, the component of $\vec{F}$ || to the Lorentz boost direction is unchanged.
For $\vec{u} = 0$, the component of $\vec{F}$ $\perp$ to the Lorentz boost direction is reduced by the factor $1/\gamma$.

Proper Force – The Minkowski Force:

In analogy to the definition of the proper time interval $d\tau$ and the proper velocity $\boxed{\vec{\eta} = d\vec{\ell}/d\tau}$ versus the "ordinary" time interval dt and the "ordinary" velocity $\boxed{\vec{u} = d\vec{\ell}/dt}$, we define a proper force $\vec{K}$ (also known as the Minkowski force), which is the derivative of the relativistic momentum $\vec{p}$ with respect to proper time $d\tau$:

$$\boxed{\vec{K} \equiv \frac{d\vec{p}}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{d\vec{p}}{dt}} \quad \text{but:} \quad \boxed{\frac{dt}{d\tau} = \frac{dt}{dt'} = \gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}}$$

$$\therefore \quad \boxed{\vec{K} \equiv \frac{d\vec{p}}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{d\vec{p}}{dt} = \gamma_u \vec{F}} \quad \text{where:} \quad \boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

$$\text{Thus:} \quad \boxed{\vec{K} = \gamma_u \vec{F} = \frac{1}{\sqrt{1-\beta_u^2}}\vec{F} = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\vec{F}} \quad \text{where:} \quad \boxed{\gamma_u \equiv \frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}} \quad \text{and:} \quad \boxed{\beta_u \equiv \frac{u}{c}}$$

We can "4-vectorize" the Minkowski Force, because it's plainly / clearly a 4-vector:

ct: $\boxed{K^0 \equiv \dfrac{dp^0}{d\tau} = \dfrac{1}{c}\dfrac{dE_{TOT}}{d\tau}}$ since: $\boxed{p^0 = \dfrac{E_{TOT}}{c}}$ $\Rightarrow K^0 =$ Proper rate at which energy of particle increases (or decreases) = (Proper power delivered to the particle)/c !

x: $\boxed{K^1 \equiv \dfrac{dp^1}{d\tau} = \gamma_u F^1 = \dfrac{1}{\sqrt{1-\beta_u^2}}F^1 = \dfrac{1}{\sqrt{1-(u/c)^2}}F^1}$

y: $\boxed{K^2 \equiv \dfrac{dp^2}{d\tau} = \gamma_u F^2 = \dfrac{1}{\sqrt{1-\beta_u^2}}F^2 = \dfrac{1}{\sqrt{1-(u/c)^2}}F^2}$

z: $\boxed{K^3 \equiv \dfrac{dp^3}{d\tau} = \gamma_u F^3 = \dfrac{1}{\sqrt{1-\beta_u^2}}F^3 = \dfrac{1}{\sqrt{1-(u/c)^2}}F^3}$

with: $\boxed{\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}} = \dfrac{1}{\sqrt{1-(u/c)^2}}}$

Thus: $\boxed{K^\mu \equiv \dfrac{dp^\mu}{d\tau}}$ ← Minkowski 4-vector force = proper 4-vector force.

Relativistic dynamics can be formulated in terms of either "ordinary" quantities or "proper" (particle rest frame) quantities. The latter is much neater / elegant, but it is (by its nature) restricted to the particle's rest frame IRF(S') {n.b. We can always easily Lorentz boost this "proper" result to any other inertial reference frame. . . }

There is a very simple reason for this! Since we humans live in the lab frame IRF(S) – we want to know everything about particle's trajectory, the forces acting on it, etc. in the lab because this is the only IRF that we can (easily) make physical measurements in – often, it is not possible to make physical measurements in a particle's rest frame / proper frame, especially if the particles are in relativistic motion (e.g. at Fermilab…).

In the long run, we will (usually) be interested in the particle's trajectory as a function of "ordinary" time, so in fact the "ordinary" 4-force $\boxed{F^{\mu} \equiv dp^{\mu}/dt}$ is often more useful, even if it is more painful / cumbersome to calculate / compute…

We want to obtain the relativistic generalization of the classical Lorentz force law $\boxed{\vec{F}_C = q\vec{E} + q\vec{u} \times \vec{B}}$ {$\vec{u}$ = particle's "ordinary" velocity in IRF(S)}. Does the classical formula $\vec{F}_C$ correspond to the "ordinary" relativistic force $\vec{F}$, or to the proper / Minkowski force $\vec{K}$?

Thus, for the relativistic Lorentz force, should we write: $\boxed{\vec{F} = q\vec{E} + q\vec{u} \times \vec{B} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)}$???

Or rather, should the relativistic Lorentz force relation be: $\boxed{\vec{K} = q\vec{E} + q\vec{u} \times \vec{B} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)}$???

Since proper time and "ordinary" time are identical in *classical* physics / Euclidean / Galilean 3-space, classical physics can't tell us the answer.

It turns out that the Lorentz force law is an "ordinary" relativistic force law: $\boxed{\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)}$

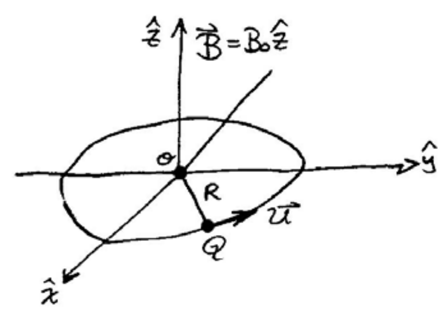
We'll see why shortly…

We'll also construct the proper / Minkowski electromagnetic force law, as well . . .

But first, some examples:

Griffiths Example 12.11: Relativistic Charged Particle Moving in a Uniform Magnetic Field

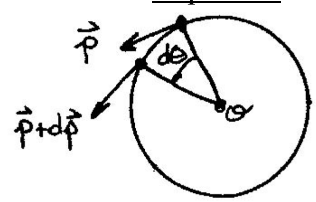
We've discussed this before, from a classical dynamics point of view:



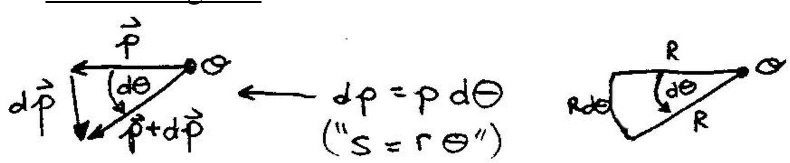
The typical trajectory of a charged particle (charge Q, mass m) moving in a uniform magnetic field is cyclotron motion. If the velocity of particle ($\vec{u}$) lies in the x-y plane and $\vec{B} = B_o\hat{z}$, then $\boxed{\vec{F} = Q\vec{u} \times \vec{B} = QuB_o\left(-\hat{r}\right) = -QuB_o\hat{r}}$ as shown on the right:

The magnetic force points radially inward – it provides the centripetal acceleration needed to sustain the circular motion. However, in special relativity the centripetal force is *not* mu^2/R (as it is in classical mechanics). Rather, it is: $\boxed{F = \frac{dp}{dt} = p\frac{d\theta}{dt} = p\frac{R}{R}\frac{d\theta}{dt} = p\frac{1}{R}\left(\frac{Rd\theta}{dt}\right) = p\frac{u}{R}}$.

Top View:



Vector Diagram:



$\boxed{\vec{F} = p\frac{u}{R}\left(-\hat{r}\right)}$ (n.b. Classically: $\boxed{\vec{p} = m\vec{u}}$ thus, classically: $\boxed{\vec{F} = m\frac{u^2}{R}\left(-\hat{r}\right)}$)

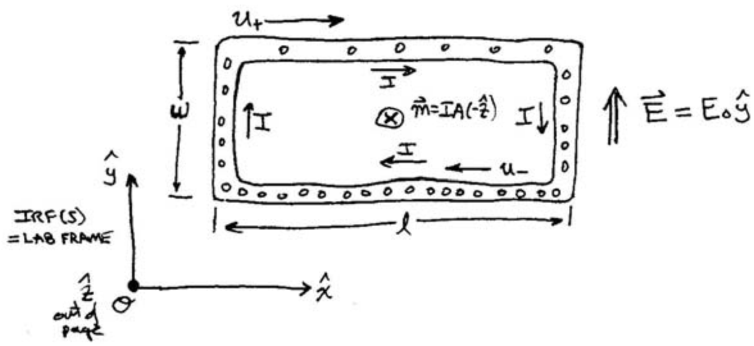
Thus, relativistically: $\boxed{QuB_o(-\hat{r}) = p\dfrac{u}{R}(-\hat{r})}$ or: $\boxed{QuB_o = p\dfrac{u}{R}}$ or: $\boxed{p = QB_oR}$

The relativistic cyclotron formula is identical to classical / non-relativistic formula!

However here, p is understood to be the relativistic 3-momentum, $\boxed{\vec{p} = m\vec{\eta} = \gamma_u m\vec{u}}$.

Griffiths Example 12.12: Hidden Momentum

Consider a magnetic dipole moment $\vec{m}$ modeled as a rectangular loop of wire (dimensions $\ell \times w$) carrying a steady current I. Imagine the current as a uniform stream of non-interacting positive charges flowing freely through the wire at constant speed u. (i.e. a fictitious kind of superconductor.) A uniform electric field $\vec{E}$ is applied as shown in the figure below:



The application of the external uniform electric field $\vec{E} = E_o\hat{y}$ changes the physics – the electric charges are accelerated in the left segment of the loop and decelerated in the right segment of the loop. [n.b. admittedly this is not a very realistic model, but other more realistic models *do* lead to the same result – see V. Hnizdo, Am. J. Phys. **65**, 92 (1997)].
Find the total momentum of all of the charges in the loop.

The momenta associated with the electric charges in the left and right segments of the loop cancel each other (i.e. $\vec{p}$ (in left segment) = $-\vec{p}$ (in right segment), so we only need to consider the momenta associated with the electric charges flowing in the top and bottom segments of the loop.

Suppose there are N_+ charges flowing in the top segment of the loop, moving in $+\hat{x}$ direction with speed $u_+ > u\left(\vec{E}=0\right)$ {because they underwent acceleration traveling on the LHS segment} and N_- charges flowing in the bottom segment of the loop, moving in the $-\hat{x}$ direction with speed $u_- < u\left(\vec{E}=0\right)$ {because they underwent deceleration traveling on the RHS segment}.

Note that the current $I = \lambda u$ *must* be the same in all four segments of the loop, otherwise charges would be piling up somewhere.

In particular: $I = I_+$ (top segment of loop) = I_- (bottom segment of loop), i.e. $\boxed{I = I_+ = I_-}$.

Since: $\boxed{\lambda \equiv \dfrac{Q_{TOT}}{\ell} = \dfrac{NQ}{\ell}}$ then: $\boxed{I_+ = \lambda_+ u_+ = N_+\left(\dfrac{Q}{\ell}\right)u_+} = \boxed{I_- = \lambda_- u_- = N_-\left(\dfrac{Q}{\ell}\right)u_-}$

$$\therefore \quad \boxed{N_+\left(\frac{Q}{\ell}\right)u_+ = N_-\left(\frac{Q}{\ell}\right)u_- = I} \;\Rightarrow\; \boxed{N_+u_+ = N_-u_- = \frac{I\ell}{Q}}$$

Classically, the linear momentum of each electric charge is $\vec{p}_{classical} = m_Q\vec{u}$ where m_Q = mass of the charged particle.

The total classical linear momentum of the charged particles flowing to the right in the top segment of the loop is: $\boxed{\vec{p}_{+_{classical}}^{\text{ top segment}} = \sum_{i=1}^{N_+} m_Q\vec{u}_+ = N_+m_Qu_+\left(+\hat{x}\right)}$

The total classical linear momentum of the charged particles flowing to the left in the bottom segment of the loop is: $\boxed{\vec{p}_{-_{classical}}^{\text{ bottom segment}} = \sum_{i=1}^{N_-} m_Q\vec{u}_- = N_-m_Qu_-\left(-\hat{x}\right)}$

The NET (or total) classical linear momentum of the charged particles flowing in the loop is:

$$\boxed{\begin{aligned}\vec{p}_{classical}^{\,TOT} &= \cancel{\vec{p}_{classical}^{\text{ left segment}}} + \vec{p}_{+_{classical}}^{\text{ top segment}} + \cancel{\vec{p}_{classical}^{\text{ right segment}}} + \vec{p}_{-_{classical}}^{\text{ bottom segment}} = \vec{p}_{+_{classical}}^{\text{ top segment}} + \vec{p}_{-_{classical}}^{\text{ bottom segment}} \\ &= N_+m_Qu_+\hat{x} - N_-m_Qu_-\hat{x} = \left(N_+u_+ - N_-u_-\right)m_Q\hat{x} = \left(I\ell/Q - I\ell/Q\right)m_Q\hat{x} = 0 \;\;!!!\end{aligned}}$$

Thus, $\boxed{\vec{p}_{classical}^{\,TOT} = 0}$ as we expected, since we know the loop is not moving.

However, now let us consider the relativistic momentum:

$\boxed{\vec{p}_{rel} = \gamma_u m_Q\vec{u}}$ (even if $|\vec{u}| = u \ll c$) where: $\boxed{\gamma_u \equiv \dfrac{1}{\sqrt{1-\beta_u^2}} = \dfrac{1}{\sqrt{1-\left(u/c\right)^2}}}$

The total relativistic linear momentum of the charged particles flowing to the right in the top segment of the loop is: $\boxed{\vec{p}_{+_{rel}}^{\text{ top segment}} = \gamma_{u^+}N_+m_Qu_+\left(+\hat{x}\right)}$ where: $\boxed{\gamma_{u^+} \equiv \dfrac{1}{\sqrt{1-\beta_+^2}} = \dfrac{1}{\sqrt{1-\left(u_+/c\right)^2}}}$.

The total relativistic linear momentum of the charged particles flowing to the left in the bottom segment of the loop is: $\boxed{\vec{p}_{-_{rel}}^{\text{ bottom segment}} = \gamma_{u^-}N_-m_Qu_-\left(-\hat{x}\right)}$ where: $\boxed{\gamma_{u^-} \equiv \dfrac{1}{\sqrt{1-\beta_-^2}} = \dfrac{1}{\sqrt{1-\left(u_-/c\right)^2}}}$.

The net / total relativistic momentum is:

$$\boxed{\vec{p}_{rel}^{\,TOT} = \vec{p}_{+_{rel}}^{\text{ top segment}} + \vec{p}_{-_{rel}}^{\text{ bottom segment}} = \left(\gamma_{u^+}N_+m_Qu_+ - \gamma_{u^-}N_-m_Qu_-\right)\hat{x} = \left(\gamma_{u^+}N_+u_+ - \gamma_{u^-}N_-u_-\right)m_Q\hat{x}}$$

But $\boxed{I = I_+ = I_-}$ gave us: $\boxed{N_+u_+ = N_-u_- = \dfrac{I\ell}{Q}}$ $\therefore$ $\boxed{\vec{p}_{rel}^{\,TOT} = \left(\gamma_{u^+} - \gamma_{u^-}\right)m_Q\left(\dfrac{I\ell}{Q}\right)\hat{x} \neq 0}$ because $\boxed{\gamma_{u^+} \neq \gamma_{u^-}}$!!!

Charged particles flowing in the top segment of the loop are moving faster than those flowing in the bottom segment of the loop.

The gain in energy $\left(\gamma_u mc^2\right)$ of the charged particles going up the left segment of the loop = the work done *on* the charges by the electric force ($W = QE_o w$) (w = height of the rectangle).

Thus, for a charged particle going up the left segment of the loop, the energy gain is:

$$\boxed{\Delta E = \gamma_{u^+} m_Q c^2 - \gamma_{u^-} m_Q c^2 = \left(\gamma_{u^+} - \gamma_{u^+}\right) m_Q c^2 = W = QE_o w} \Rightarrow \boxed{\left(\gamma_{u^+} - \gamma_{u^-}\right) = \frac{QE_o w}{m_Q c^2}}$$

Where E_o = the magnitude of the {uniform/constant} electric field.

$$\therefore \quad \boxed{\vec{p}_{rel}^{\,TOT} = \left(\gamma_{u^+} - \gamma_{u^-}\right) m_Q \left(\frac{I\ell}{Q}\right)\hat{x} = \left(\frac{\cancel{Q} E_o w}{\cancel{m}_Q c^2}\right)\cancel{m}_Q \left(\frac{I\ell}{\cancel{Q}}\right)\hat{x} = \left(\frac{E_o I \ell w}{c^2}\right)\hat{x}}$$

But: $\boxed{\ell w = A}$ = area of the loop. $\therefore$ $\boxed{\vec{p}_{rel}^{\,TOT} = \frac{E_o IA}{c^2}\hat{x}}$ but: $\boxed{m = |\vec{m}| = IA}$ $\Rightarrow$ $\boxed{\vec{p}_{rel}^{\,TOT} = \frac{mE_o}{c^2}\hat{x}}$

But: $\boxed{\vec{m} = m\left(-\hat{z}\right)}$ (see picture above) and: $\boxed{\vec{E} = E_o\hat{y}}$ i.e. $\boxed{\vec{m} \perp \vec{\mathrm{E}}}$ here.

Thus, vectorially we {actually} have: $\boxed{\vec{p}_{rel}^{\,TOT} = \frac{1}{c^2}\left(\vec{m}\times\vec{E}\right)}$ where: $\boxed{\left(\vec{m}\times E\right) = mE_o \overbrace{\left(-\hat{z}\times\hat{y}\right)}^{=+\hat{x}}}$

Thus a magnetic dipole moment $\vec{m}$ in the presence of an electric field $\vec{E}$ carries relativistic linear momentum $\vec{p}$, even though it is not moving !!!

n.b. it also (therefore) carries relativistic angular momentum $\boxed{\vec{L}_{rel} = \vec{r}\times\vec{p}_{rel}}$.

How big is this effect? Explicit numerical example - use "everyday" values:

E_o = 1000 V/m
I = 1 Amp
A = (10 cm)^2 = 0.01 m^2
m = IA = 0.01 A-m^2

$$\boxed{\left|\vec{p}_{rel}^{\,TOT}\right| = \frac{mE_o}{c^2} = \frac{10^{-2}\times10^{+3}}{\left(3\times10^8\right)^2} = 10^{-16}\,\text{kg m/s}}$$ Tiny !!! The $1/c^2$ factor kills this effect !!!

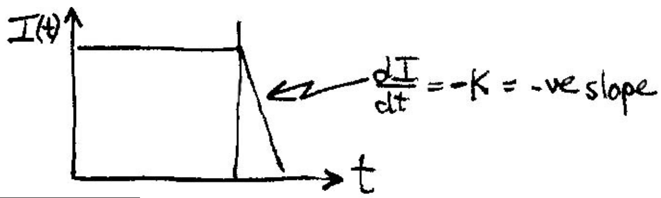
This so-called macroscopic hidden linear momentum is strictly relativistic, purely mechanical But note that it precisely cancels the electromagnetic linear momentum stored in the $\vec{E}$ and $\vec{B}$ fields!!! (Microscopically, the momentum imbalance arises from the imbalance of virtual photon emission on top segment of the loop vs. the bottom segment of the loop.)

Likewise, the corresponding hidden angular momentum precisely cancels the electromagnetic angular momentum stored in the $\vec{E}$ and $\vec{B}$ fields.

→ Now go back and take another look at Griffiths Example 8.3, pages 356-57. (the coax cable carrying uniform charge / unit length λ and steady current I flowing down / back cable.)

Let's pursue this problem a little further…

Suppose there is a <u>change</u> in the current, e.g. suppose the current drops / decreases to zero. For simplicity's sake, assume $\frac{dI}{dt} = -K$ (i.e. the current decreases linearly with time)



<u>Classically:</u> $\boxed{I(t) = I_+(t) = I_-(t)}$ (as before)

$$\boxed{I_+(t) = N_+(t)\left(\frac{Q}{\ell}\right)u_+} \qquad \boxed{I_-(t) = N_-(t)\left(\frac{Q}{\ell}\right)u_-}$$

$$\boxed{N_+(t)\left(\frac{Q}{\ell}\right)u_+ = N_-(t)\left(\frac{Q}{\ell}\right)u_-}$$

We assume that u, u_+ and u_- are unaffected by the change in the current with time.

<u>Then:</u> $\boxed{\frac{dI}{dt} = \frac{dI_+(t)}{dt} = \frac{dI_-(t)}{dt}} \Rightarrow \boxed{\left(\frac{Q}{\ell}\right)u_+\frac{dN_+(t)}{dt} = \left(\frac{Q}{\ell}\right)u_-\frac{dN_-(t)}{dt} = -K}$

$\therefore$ $\boxed{\frac{dN_+(t)}{dt}u_+ = \frac{dN_-(t)}{dt}u_- = -\frac{K\ell}{Q}}$ = constant (no time dependence on RHS of equation)

<u>Then:</u>

$$\boxed{\frac{d\vec{p}_{+_{classical}}(t)}{dt} = \frac{dN_+(t)}{dt}m_Q u_+(+\hat{x}) = -\frac{K\ell m_Q}{Q}(+\hat{x}) = -\frac{K\ell m_Q}{Q}\hat{x}}$$

$$\boxed{\frac{d\vec{p}_{-_{classical}}(t)}{dt} = \frac{dN_-(t)}{dt}m_Q u_-(-\hat{x}) = -\frac{K\ell m_Q}{Q}(-\hat{x}) = +\frac{K\ell m_Q}{Q}\hat{x}}$$

<u>Constant</u>

$\therefore$ The net / total classical time-rate of change of <u>linear</u> momentum is:

$$\boxed{\vec{F}^{tot}_{classical}(t) = \frac{d\vec{p}^{tot}_{classical}(t)}{dt} = \frac{d\vec{p}_{+_{classical}}(t)}{dt} + \frac{d\vec{p}_{-_{classical}}(t)}{dt} = -\frac{K\ell m_Q}{Q}\hat{x} + \frac{K\ell m_Q}{Q}\hat{x} = 0}$$

<u>Thus</u>: $\boxed{\vec{F}^{tot}_{classical}(t) = \frac{d\vec{p}^{tot}_{classical}(t)}{dt} = 0}$ as we expected, since the loop is not moving.

Now, let's investigate this situation <u>relativistically</u>:

<u>Since</u>: $\boxed{\vec{p}_{rel} = \gamma_u m_Q \vec{u}} \Rightarrow \boxed{\vec{p}_{+_{rel}} = \gamma_{u^+} m_Q \vec{u}_+}$ <u>and</u>: $\boxed{\vec{p}_{-_{rel}} = \gamma_{u^-} m_Q \vec{u}_-}$

For individual charges with mass m_Q

Then: $\boxed{\vec{p}_{+_{rel}}^{\text{top segment}}(t) = \gamma_{u^+} N_+(t) m_Q u_+ (+\hat{x})}$ where: $\boxed{\gamma_{u^+} \equiv \dfrac{1}{\sqrt{1-\beta_+^2}} = \dfrac{1}{\sqrt{1-(u_+/c)^2}}}$.

And: $\boxed{\vec{p}_{-_{rel}}^{\text{bottom segment}}(t) = \gamma_{u^-} N_-(t) m_Q u_- (-\hat{x})}$ where: $\boxed{\gamma_{u^-} \equiv \dfrac{1}{\sqrt{1-\beta_-^2}} = \dfrac{1}{\sqrt{1-(u_-/c)^2}}}$.

And: $$\boxed{\frac{d\vec{p}_{+_{rel}}^{\text{top segment}}(t)}{dt} = \gamma_{u^+} m_Q u_+ \frac{dN_+(t)}{dt}(\hat{x}) = \text{constant} = -\gamma_{u^+}\left(\frac{K\ell m_Q}{Q}\right)\hat{x}}$$

And: $$\boxed{\frac{d\vec{p}_{-_{rel}}^{\text{bottom segment}}(t)}{dt} = \gamma_{u^-} m_Q u_- \frac{dN_-(t)}{dt}(-\hat{x}) = \text{constant} = +\gamma_{u^-}\left(\frac{K\ell m_Q}{Q}\right)\hat{x}}$$

The net / total time rate of change of relativistic momentum is:

$$\boxed{\begin{aligned}\frac{d\vec{p}_{rel}^{\,TOT}(t)}{dt} &= \frac{d\vec{p}_{+_{rel}}^{\text{top segment}}(t)}{dt} + \frac{d\vec{p}_{-_{rel}}^{\text{bottom segment}}(t)}{dt} = -\gamma_{u^+}\left(\frac{K\ell m_Q}{Q}\right)\hat{x} + \gamma_{u^-}\left(\frac{K\ell m_Q}{Q}\right)\hat{x} \\ &= \left(\gamma_{u^+} - \gamma_{u^-}\right)\left(\frac{K\ell m_Q}{Q}\right)\hat{x} \neq 0 \qquad \left(\gamma_{u^+} \neq \gamma_{u^-}\right)\end{aligned}}$$

From above (p. 20): $\boxed{\left(\gamma_{u^+} - \gamma_{u^-}\right) = \dfrac{QE_o w}{m_Q c^2}}$ where: E_o = electric field amplitude

$$\boxed{\frac{d\vec{p}_{rel}^{\,TOT}(t)}{dt} = -\left(\frac{\cancel{Q}E_o w}{\cancel{m}_Q c^2}\right)\left(\frac{K\ell \cancel{m}_Q}{\cancel{Q}}\right)\hat{x} = -\frac{E_o K\ell w}{c^2}\hat{x} = -\frac{E_o K A}{c^2}\hat{x}}$$ $\boxed{A = \ell \times w = \text{cross-sectional area of the loop}}$

Now: $\boxed{\dfrac{dI}{dt} = -K}$ and: $\boxed{m = IA}$ $\rightarrow$ $\boxed{\dfrac{dm}{dt} = \dfrac{dI}{dt}A}$ (Since A = constant).

$\therefore$ $\boxed{\dfrac{dm}{dt} = -KA = \dfrac{dI}{dt}A}$ = time rate of change of the magnetic dipole moment of the loop.

$\Rightarrow$ $\boxed{\dfrac{d\vec{p}_{rel}^{\,TOT}(t)}{dt} = \dfrac{1}{c^2}\dfrac{dm(t)}{dt}E_o\hat{x}}$ but: $\underline{\left(\vec{m} = m(-\hat{z})\right) \perp \left(\vec{E} = E_o\hat{y}\right)}$

$\therefore$ $\boxed{\vec{F}_{rel}(t) = \dfrac{d\vec{p}_{rel}^{\,TOT}(t)}{dt} = \dfrac{1}{c^2}\left(\dfrac{d\vec{m}(t)}{dt} \times \vec{E}\right) \neq 0}$ (assuming external $\vec{E}$-field is constant in time)

Thus, $\exists$ a net "hidden" force acting on the magnetic dipole, when $dI/dt \neq 0$.

One might think that this net "hidden" force would be exactly cancelled / compensated for by a countering force due to the electromagnetic fields, as we saw in the static case ($dI/dt = 0$), with a steady current *I*. But it isn't't!! Why??

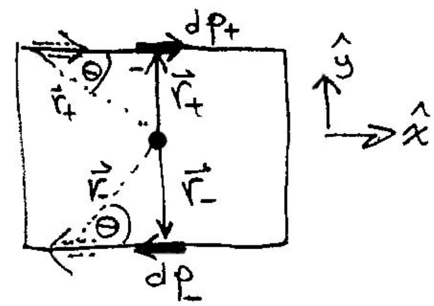
As we saw for M(1) magnetic dipole radiation, a time-varying current in a loop produces *EM* radiation. Essentially there is a radiation reaction / back-force that acts on the "antenna" – a radiation pressure – much like the recoil / impulse from firing a bullet out of a gun – the short explosive "pulse" launches the bullet, but the gun is also kicked backwards, too.

The same thing happens here when $dI/dt \neq 0$ - the far zone *EM* radiation fields are produced (i.e. real photons) while $dI/dt \neq 0$ and carry away linear momentum, and since $dI/dt \neq 0$, $\exists$ a net force imbalance on the radiating object! (n.b. – e.g. by linear momentum conservation, a laser pen has a recoil force acting on it from emitting the laser radiation – a radiation back reaction)

Likewise, the net "hidden" time rate of change of relativistic angular momentum is:

$$\boxed{\frac{d\vec{L}_{rel}^{TOT}(t)}{dt} = \vec{r} \times \frac{d\vec{p}_{rel}^{TOT}(t)}{dt} = \frac{1}{c^2}\frac{dm(t)}{dt} E_o\left(\vec{r} \times \hat{x}\right)}$$

Which will also not be exactly cancelled either, for the same reason – the *EM* radiation field can / will carry away angular momentum…



In reality, in order to calculate $\frac{d\vec{L}_{rel}^{TOT}(t)}{dt}$, we need to go back and integrate infinitesimal contributions along the (short) segments of upper and lower / top and bottom segments of the loop because $|\vec{r} \times \vec{p}| = rp\sin\theta$, $\theta = \measuredangle$ between $\vec{r}$ and $\vec{p}$.

Same for $\left|\vec{r} \times \frac{d\vec{p}}{dt}\right| = r\frac{dp}{dt}\sin\theta$.

Will get result that has geometrical factor of order ≤ 1.
→ Conclusions won't be changed by this, just actual #.

As we know, the time rate of change of angular momentum: $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$ = torque.

Thus, the time rate of change of the net / total "hidden" relativistic angular momentum $\frac{d\vec{L}_{rel}^{TOT}(t)}{dt}$ = net "hidden" relativistic torque, $\vec{\tau}_{rel}^{TOT}(t)$.

Thus: $$\boxed{\vec{\tau}_{rel}^{TOT}(t) = \frac{d\vec{L}_{rel}^{TOT}(t)}{dt} = \vec{r} \times \frac{d\vec{p}_{rel}^{TOT}(t)}{dt} = \vec{r} \times \vec{F}_{rel}^{TOT}(t) = \frac{1}{c^2}\vec{r} \times \left(\frac{d\vec{m}(t)}{dt} \times \vec{E}\right) \neq 0}$$

Which is *not* completely / exactly cancelled when $dI(t)/dt \neq 0$!!!

Linear momentum, angular momentum, energy, etc. are all conserved for this whole system, it's just that the *EM* radiation emitted from the antenna is free-streaming, carrying away all these quantities with it!

In the static situation I = constant, the "hidden" relativistic linear momentum and angular momentum is exactly cancelled by the linear momentum and angular momentum (respectively) carried by the (macroscopic) static electromagnetic fields $\vec{E}$ and $\vec{B}$. Microscopically, the field linear and angular momentum is carried by the static, virtual photons associated with the macroscopic $\vec{E}$ and $\vec{B}$ fields, cancelling the (macroscopic) "hidden" linear and angular relativistic momentum of the magnetic dipole in a uniform $\vec{E}$-field.

In the non-static situation $dI(t)/dt \neq 0$, virtual photons undergo space-time rotation, becoming real photons, which carry away real linear and angular momentum. "Hidden" relativistic linear and angular momentum is no longer exactly cancelled by the (now) real field linear and angular momentum associated with the *EM* radiation fields. It is only partially cancelled by remaining / extant virtual / near-zone / inductive zone *EM* fields.

Griffiths Problem 12.36: Relativistic "Ordinary" Force

In classical mechanics Newton's 2nd Law is: $\boxed{\vec{F} = m\vec{a}}$.

The relativistic "ordinary" force relation; $\boxed{\vec{F}_{rel} = \frac{d\vec{p}}{dt}}$ cannot be so simply expressed.

$$\boxed{\vec{F}_{rel} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma_u m\vec{u}) = \frac{d}{dt}\left(\frac{1}{\sqrt{1-(u/c)^2}}m\vec{u}\right)} \quad \text{where: } \boxed{\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}}}$$

$$\boxed{\vec{F}_{rel} = m\left\{\frac{\frac{d\vec{u}}{dt}}{\sqrt{1-(u/c)^2}} + \vec{u}\left(-\frac{1}{2}\right)\frac{\frac{1}{c^2}2\vec{u}\bullet\frac{d\vec{u}}{dt}}{\left(1-(u/c)^2\right)^{3/2}}\right\}} \quad \text{where: } \boxed{\vec{a} \equiv \frac{d\vec{u}}{dt}} = \text{"ordinary" acceleration.}$$

$$\therefore \quad \boxed{\vec{F}_{rel} = \frac{m}{\sqrt{1-(u/c)^2}}\left\{\vec{a} + \frac{\vec{u}(\vec{u}\bullet\vec{a})}{c^2\left(1-(u/c)^2\right)}\right\} = \frac{m}{\sqrt{1-(u/c)^2}}\left\{\vec{a} + \frac{\vec{u}(\vec{u}\bullet\vec{a})}{(c^2-u^2)}\right\}} \quad \text{Q.E.D.}$$

Griffiths Problem 12.38: *Proper* Acceleration

We define the proper four-vector acceleration in the obvious way, as:

$$\boxed{\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}} \quad \text{where: } \boxed{\eta^\mu \equiv \frac{dx^\mu}{d\tau}} = \text{proper four-velocity}$$

a) Find α^0 and $\vec{\alpha}$ in terms of $\vec{u}$ and $\vec{a}$ (= "ordinary" velocity, "ordinary" acceleration):

$$\boxed{\alpha^0 = \frac{d\eta^0}{d\tau} = \frac{d\eta^0}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-(u/c)^2}}\frac{d}{dt}\left(\frac{c}{\sqrt{1-(u/c)^2}}\right)} \quad \text{since: } \boxed{d\tau = \frac{1}{\gamma_u}dt} \Rightarrow \boxed{\frac{dt}{d\tau} = \gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}}$$

$$\boxed{\alpha^0 = \frac{c}{\sqrt{1-(u/c)^2}} \frac{\left(-\frac{1}{\cancel{2}}\right)\left(-\frac{1}{c^2}\right)\cancel{2}\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^{3/2}} = \frac{1}{c}\frac{\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^2}}$$ <u>where</u>: $\boxed{\vec{a} \equiv \frac{d\vec{u}}{dt}}$

<u>Similarly</u>:

$$\boxed{\vec{\alpha} = \frac{d\vec{\eta}}{d\tau} = \frac{d\vec{\eta}}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-(u/c)^2}}\frac{d}{dt}\left(\frac{\vec{u}}{\sqrt{1-(u/c)^2}}\right)}$$ <u>since</u>: $\boxed{\vec{\eta} = \gamma_u \vec{u}}$ <u>and</u>: $\boxed{\gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}}$

$$\boxed{\vec{\alpha} = \frac{1}{\sqrt{1-(u/c)^2}}\left\{\frac{\vec{a}}{\sqrt{1-(u/c)^2}} + \vec{u}\frac{\left(-\frac{1}{\cancel{2}}\right)\left(-\frac{1}{c^2}\right)\cancel{2}\vec{u}\bullet\vec{a}}{\left(1-(u/c)^2\right)^{3/2}}\right\}}$$

$$\boxed{\vec{\alpha} = \frac{1}{\left(1-(u/c)^2\right)}\left\{\vec{a} + \frac{\vec{u}\left(\vec{u}\bullet\vec{a}\right)}{\left(c^2-u^2\right)}\right\} = \gamma_u\left(\frac{\vec{F}_{rel}}{m}\right)}$$ ← see Problem 12.36 above.

b) Express $\alpha_\mu \alpha^\mu$ in terms of $\vec{u}$ and $\vec{a}$:

$$\begin{aligned}\alpha_\mu\alpha^\mu &= -\left(\alpha^0\right)^2 + \vec{\alpha}\bullet\vec{\alpha} = -\frac{1}{c^2}\frac{\left(\vec{u}\bullet\vec{a}\right)^2}{\left(1-(u/c)^2\right)^4} + \frac{1}{\left(1-(u/c)^2\right)^4}\left[\vec{a}\left(1-(u/c)^2\right) + \frac{1}{c^2}\vec{u}\left(\vec{u}\bullet\vec{a}\right)\right]^2 \\ &= \frac{1}{\left(1-(u/c)^2\right)^4}\left\{-\frac{1}{c^2}\left(\vec{u}\bullet\vec{a}\right)^2 + a^2\left(1-(u/c)^2\right)^2 + \frac{2}{c^2}\left(1-(u/c)^2\right)\left(\vec{u}\bullet\vec{a}\right)^2 + \frac{1}{c^4}u^2\left(\vec{u}\bullet\vec{a}\right)^2\right\} \\ &= \frac{1}{\left(1-(u/c)^2\right)^4}\left\{a^2\left(1-(u/c)^2\right)^2 + \frac{\left(\vec{u}\bullet\vec{a}\right)^2}{c^2}\left(-1+2-2\left(\frac{u}{c}\right)^2 + \left(\frac{u}{c}\right)^2\right)\right\}\end{aligned}$$

<u>Or</u>: $$\boxed{\alpha_\mu\alpha^\mu = \frac{1}{\left(1-(u/c)^2\right)^4}\left[a^2 + \frac{\left(\vec{u}\bullet\vec{a}\right)^2}{\left(c^2-u^2\right)}\right]}$$ ← n.b. Lorentz invariant quantity – same in all IRFs.

c) Show $\boxed{\eta^\mu \alpha_\mu = 0}$.

Recall that the "dot-product" of (<u>*any*</u>) two relativistic four-vectors is a <u>Lorentz-invariant quantity</u>.

Thus, if we deliberately/consciously choose to <u>evaluate</u> $\boxed{\eta_\mu\eta^\mu = \eta^\mu\eta_\mu = -\left(\eta^0\right)^2 + \vec{\eta}\bullet\vec{\eta}}$ in the <u>rest</u> <u>frame</u> of an object, we see that $\boxed{\vec{\eta}\bullet\vec{\eta} = 0}$ in the <u>rest</u> <u>frame</u> of the object, and therefore:

$$\boxed{\eta_\mu\eta^\mu = \eta^\mu\eta_\mu = -\left(\eta^0\right)^2 + \underbrace{\vec{\eta}\bullet\vec{\eta}}_{=0} = -\left(\eta^0\right)^2 = -c^2} = \underline{\text{constant}}.$$

Note that $\boxed{\eta^\mu \alpha_\mu = \eta^\mu \frac{d\eta_\mu}{d\tau}}$ is <u>also</u> the "dot-product" of two relativistic four vectors $\{\eta^\mu$ and $\alpha_\mu\}$.

<u>Note also that</u>: $\boxed{\frac{d}{d\tau}\left(\eta^\mu \eta_\mu\right) = \frac{d\eta^\mu}{d\tau}\eta_\mu + \eta^\mu \frac{d\eta_\mu}{d\tau} = \alpha^\mu \eta_\mu + \eta^\mu \alpha_\mu = 2\alpha^\mu \eta_\mu}$

<u>But</u>: $\boxed{\eta^\mu \eta_\mu = -c^2}$ = <u>constant</u> (from above). <u>Thus</u>: $\boxed{\frac{d}{d\tau}\left(\eta^\mu \eta_\mu\right) = \frac{d}{d\tau}\left(-c^2\right) = 0} \rightarrow \therefore \boxed{\alpha^\mu \eta_\mu = 0}$.

d) Write the Minkowski / <u>proper</u> force version of Newton's 2nd law, $\boxed{K^\mu = \frac{dp^\mu}{d\tau}}$ in terms of the <u>proper</u> acceleration α^μ.

$$\boxed{K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(m\eta^\mu\right) = m\frac{d\eta^\mu}{d\tau} = m\alpha^\mu}$$

e) Evaluate the Lorentz-invariant 4-product $\boxed{K^\mu \eta_\mu}$:

$\boxed{K^\mu \eta_\mu = m\alpha^\mu \eta_\mu}$ <u>but</u>: $\boxed{\alpha^\mu \eta_\mu = 0}$ from part c) above.

$$\therefore \boxed{K^\mu \eta_\mu = 0}$$

