Lecture notes for 640-343 ELECTRODYNAMICS.

1 Summary of Electrostatics

1.1 Coulomb's Law

Force between two point charges

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2 \hat{r}_{12}}{\mid \vec{r}_1 - \vec{r}_2 \mid^2} \tag{1.1.1}$$

1.2 Electric Field

$$\vec{E}(r) = \frac{\vec{F}(\vec{r})}{Q_1} \tag{1.2.1}$$

For a charge distribution:

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\int_{volume} \frac{\rho(\vec{r'})\hat{R}}{R^2} dv \tag{1.2.2}$$

$$\vec{R} = \vec{r} - \vec{r'} \tag{1.2.3}$$

1.3 Electric Potential

The scalear or Electric Potenetial V is defined by:

$$\vec{E}(\vec{r}) = -\nabla(V) \tag{1.3.1}$$

Where V = Electric Potential. For a static electric field.

$$\nabla \times \vec{E} = 0 \qquad \Rightarrow \qquad \oint_{line} \vec{E} \cdot \vec{dl} = 0 \tag{1.3.2}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Rightarrow \qquad \oint_{surface} \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon_0} \tag{1.3.3}$$

In a static field, conductors are equipotentials.
Electric fields are perpendicular to equipotentials.

Electric potential due to a dipole:

$$V(\vec{r}) = \frac{\vec{P}\cdot\hat{r}}{4\pi\epsilon_0 r^2}, \qquad \vec{P} = Q\vec{d} \tag{1.3.4}$$

1.4 Dielectrics

An external electric field induces a Polarization density field $\vec{P}$ in a class of materials called Dielectrics.
This in turn induces a bound surface charge density:

$$\rho_{ps} = \vec{P}\cdot\hat{r} \tag{1.4.1}$$

on the surfaces of the dielectric. It also produces a bound volume charge density:

$$\rho_p = -\nabla\cdot\vec{P} \tag{1.4.2}$$

The electric flux density or electric displacement $\vec{D}$ is defined:

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \tag{1.4.3}$$

$$\Rightarrow \nabla\cdot\vec{D} = \rho_{free} \tag{1.4.4}$$

Independent of dielectric materials.
For simple dielectrics:

$$\vec{D} = \epsilon_0\epsilon r\vec{E} = \epsilon\vec{E} \tag{1.4.5}$$

ϵ_r = relative permittivity; ϵ = absolute permittivity.
The Dielectric Strength is the maximum electric field that does not cause an insulator to break down.

1.5 Boundaries between Dielectrics

At the boundary between dielectrics, tangetial componets of the electric field are equal.

$$E_{1t} = E_{2t} \tag{1.5.1}$$

$$\frac{D_{1t}}{\epsilon} = \frac{D_{2t}}{\epsilon} \tag{1.5.2}$$

The normal components of the $\vec{D}$ field are given by:

$$D_{1n} - D_{2n} = \rho_s \tag{1.5.3}$$

where ρ_s = free surface charge density.
The capacitance of a system of conductors is defined to be:

$$Q = CV \tag{1.5.4}$$

1.6 Laplace's Equation

Poissson's Equation: $\nabla^2 V = -\frac{\varrho}{\epsilon}$
Laplace's Equation: $\nabla^2 V = 0$ where there are no free charges.
Uniqueness Theorem:
Any solution of Laplace's equation that satisfies the boundary conditions is the unique solution for the specified situation..

1.7 Method of Images

Involves placing ficticious image charges outside the system of interest in such a way to minic the boundary conditions of the problem. The uniqueness theorem tells us that if the boundary conditions can be satisfied by a combination of image charges then the solution found is the only correction.

1.8 General Solution of Laplaces Equation

Use the method of seperation of variables.

$$\begin{aligned} V(x,y,z) &= X(x)Y(y)Z(z) \\ V(r,\theta,\phi) &= R(r)\Theta(\theta)\Phi(\phi) \end{aligned}$$

In cartesian coords the solutions are sums of exponentials:

$$X(x) = \sum_{i=1}^{\infty} A_i e^{k_i x} + B_i e^{-k_i x} \tag{1.8.1}$$

In spherical coords the solutions are the sums of Legendre Polynomials.

$$\begin{aligned} R(R) &= \sum_{n=1}^{\infty} R_n(r) = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-(n+1)}) \\ \Theta(\theta) &= \sum_{n=1}^{\infty} \Theta_n(\theta) = \sum_{n=1}^{\infty} C_n P_n(cos\theta) \end{aligned}$$

1.9 Numerical Methods

Use finite difference equations. Calculate V(x,y) on a discrete grid of points (x_i,y_j).
get itteration equation:

$$V_{ij}^{N+1} = \frac{1}{2(1+d)}(V_{i+1,j}^{N} + V_{i-1,j}^{N+1} + d(V_{i,j+1}^{N} + V_{i,j-1}^{N+1})) \tag{1.9.1}$$

$$d = (\frac{\Delta x}{\Delta y})^2 \tag{1.9.2}$$

2 Magnetic Fields

2.1 Definitions

Magnetic force arises between 2 charges that move relative to one another.

$$\vec{F}_m = \frac{\mu_0}{4\pi}\frac{qq_1}{r^2}\vec{v} \times (\vec{v}_1 \times \frac{\vec{r}}{\mid \vec{r} \mid}), \qquad By\ Definition: \quad \frac{\mu_0}{4\pi} = 10^{-7} \tag{2.1.1}$$

$$Then \quad \vec{F}_m = q\vec{v} \times \vec{B} \tag{2.1.2}$$

Where $\vec{B}$ = Magnetic Flux density caused by q_1 moving with velocity $\vec{v}$.

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_1}{r^2}\vec{v}_1 \times \frac{\vec{r}}{\mid \vec{r} \mid} \tag{2.1.3}$$

The Lorentz force on a particle is

$$\vec{F} = \vec{F}_e + \vec{F}_m = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2.1.4}$$

2.2 Current density

The magnetic flux density $\vec{B}$ of a steady state current I in a curent conductor is given by the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{line\ of\ current} \frac{\vec{dl} \times (\vec{r} - \vec{r}')}{\mid \vec{r} - \vec{r}' \mid^3} \tag{2.2.1}$$

The <u>current density</u> $\vec{J}$ is the current flow per unit area. The direction of $\vec{J}$ is the direction normal to the element of area $\Delta\vec{A}$.

$$\lim_{\Delta A \to 0} \vec{J}(\vec{r}) = \frac{\Delta I(\vec{r})}{\Delta A}\Delta\hat{A} \tag{2.2.2}$$

($\vec{J}$ direction parallel to local current direction).

The total current flowing through an arbitary surface S is:

$$I = \int_S \vec{J} \cdot \vec{ds} \tag{2.2.3}$$

2.3 Ampere's Law

The condition for conservation of charge is:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial\rho}{\partial t} \tag{2.3.1}$$

For steady currents:

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{2.3.2}$$

General Biot-Savart Law is:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r'}) \times (\vec{r} - \vec{r'})}{\mid \vec{r} - \vec{r'} \mid^3} dV' \tag{2.3.3}$$

$$Found: \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{2.3.4}$$

$$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \mu_0 I \tag{2.3.5}$$

These are **Ampere's Circuital Law**.
The mgnetic forace on a current in a awire is:

$$\vec{F} = I \int_C \vec{dl} \times \vec{B} \tag{2.3.6}$$

Where C= contour of the wire.

2.4 Vector Potential

Since there are no magnetic monopoles:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.4.1}$$

$\Rightarrow$ Exists a field, the magnetic vector potential $\vec{A}$, such that:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.4.2}$$

$$and \qquad \vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{2.4.3}$$

The vector Potential due to a magnetic dipole is:

$$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{r}}{4\pi r^2} \tag{2.4.4}$$

Where $\vec{m}$ = area $\times$ current is the dipole loop. Direction is the direction of area. $\vec{m}$ = magnetic dipole moment.

$$\Rightarrow \vec{B} = \frac{\mu_0 \mid m \mid}{4\pi R^3} (\hat{r} 2 \cos(\theta) + \hat{\theta} \sin(\theta)) \tag{2.4.5}$$

For steady currents $\vec{\nabla} \cdot \vec{A} = 0$ for steady currents.

2.5 Magnetic properties of materials

Define:

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{m}_k}{\Delta v} \tag{2.5.1}$$

as the local magnetisation density. This $\vec{M}$ induces bound volume and surface current densities:

$$\vec{J}_m = \vec{\nabla} \times \vec{M} \tag{2.5.2}$$

is the bound volume current density,

$$\vec{J}_{ms} = \vec{M} \times \hat{s} \tag{2.5.3}$$

is the bound surface current density. $\hat{s}$ is the normal to the surface.
Define $\vec{H}$ = Magnetic field intensity as:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \tag{2.5.4}$$

Then $\vec{\nabla} \times \vec{H} = \vec{J}_{free}$ where $\vec{J}_{free}$ is the free current density.
In linear magnetic materials, $\vec{M} = \chi_m \vec{H}$ where χ_m = magnetic susceptibility.
Then

$$\vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H} \tag{2.5.5}$$

where $\mu_r = 1 + \chi_m = \dfrac{\mu}{\mu_0}$ = relative permeability of a medium. μ = Absolute permeability of this medium.
At the interface of 2 magnetic media $B_{1n} = B_{2n}$ (normal component of $\vec{B}$ is continuous) and $H_{1t} = H_{2t}$ (tangential component of $\vec{H}$ is continuous).

A material is Diamagnetic if $\mu_r \leq 1$, Paramagnetic if $\mu \geq$ and Ferromagnetic if $\mu_r \gg 1$.

Ferromagnetic materials are highly <u>non</u> linear and $\vec{B}$ depends on the history of the material as well as $\vec{H}$. This is called hysterises.

3 Time varying fields - revison

3.1 Electromagnetic Induction

$$Magnetic\ Flux: \quad \Phi = \int_S \vec{B} \cdot \vec{ds} = \oint_C \vec{A} \cdot \vec{dl} \tag{3.1.1}$$

The Inductance of a circuit is defined as $L = \dfrac{N\Phi}{I}$ where N = number of turns.

$$Faradays\ Law: \quad V = -\frac{d\Phi}{dt} \tag{3.1.2}$$

where V = emf induced in every loop of a circuit enclosing the magnetic flux Φ.
This is generalized to:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.1.3}$$

This is generalized **Faradays Law**.

$$Magnetic\ Energy\ density \quad W_m = \frac{1}{2}\vec{H} \cdot \vec{B} = \frac{B^2}{2\mu} = \frac{1}{2}\mu H^2 \quad Joules/m^3 \tag{3.1.4}$$

$$Electric\ Energy\ density \quad W_E = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\epsilon E^2 = \frac{1}{2\epsilon}B^2 \quad Joules/m^3 \tag{3.1.5}$$

Conservation of charge in a changing magnetic field leads to the generalization of Ampere's circuital law. This becomes:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{3.1.6}$$

3.2 Maxwell's equations

We now have all of Maxwell's equations.

$$\begin{array}{cccc}
Differential & Integral & & \\
\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \oint \vec{E} \cdot d\vec{l} = -\dfrac{\partial \Phi}{\partial t} & Faradays\ \ Law & (3.2.1) \\
\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \dfrac{\partial \vec{D}}{\partial t} \cdot d\vec{s} & Ampere's\ \ Circuital\ \ Law & (3.2.2) \\
\vec{\nabla} \cdot \vec{D} = \rho & \oint_S \vec{D} \cdot d\vec{s} = Q & Gauss'\ \ Law & (3.2.3) \\
\vec{\nabla} \cdot \vec{B} = 0 & \oint \vec{B} \cdot d\vec{s} = 0 & No\ \ Magnetic\ \ Monopoles & (3.2.4)
\end{array}$$

3.3 Electromagnetic Waves

Combine these equations in a source free region to get the wave equations for $\vec{E}$ and $\vec{B}$.

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{3.3.1}$$

$$\nabla^2 \vec{B} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \tag{3.3.2}$$

Get an E/M wave that propagates at velocity $v = \dfrac{1}{\sqrt{\mu\epsilon}} = c$ in free space.

The plane wave solutions are:

$$\vec{E} = \vec{E}_0 \cos(\omega t - \vec{k} \cdot \vec{r}), \quad \vec{B} = \vec{B}_0 \cos(\omega t - \vec{k} \cdot \vec{r}) \tag{3.3.3}$$

Direction of $\vec{k}$ is the direction of propagation of the wave. $\mid k \mid = \dfrac{2\pi}{\lambda}$ where λ = wavelength of the wave, $\omega = \dfrac{2\pi \times frequency}{2\pi}$. Phase velocity $\mid v \mid = \dfrac{\omega}{\mid k \mid}$.

In an E/M wave $\vec{E} \perp \vec{B}$. The poynting Vector:

$$\mathcal{P} = \vec{E} \times \vec{H} \tag{3.3.4}$$

is the power density of the wave. The direction of $\mathcal{P}$ is the direction of wave propagation.