11.4 Causal Relation between $\vec{D}$ and $\vec{E}$

With $\epsilon = \epsilon(\omega)$, the relation between $\vec{D}$ and $\vec{E}$ changes. By definition, in Fourier space we have

$$\vec{D}(\omega, \vec{x}) = \epsilon(\omega)\, \vec{E}(\omega, \vec{x})$$
$$= [1 + \epsilon(\omega) - 1]\, \vec{E}(\omega, \vec{x}).$$

Dropping the spatial coordinates, we find, in real time space:

$$\vec{D}(t, \vec{x}) = \vec{E}(t, \vec{x}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, [\epsilon(\omega) - 1]\, \vec{E}(\omega, \vec{x})\, e^{-i\omega t}$$

And since

$$\vec{E}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tilde{t}\, \vec{E}(\tilde{t})\, e^{i\omega \tilde{t}}$$ we get

$$\vec{D}(t, \vec{x}) = \vec{E}(t, \vec{x}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, d\tilde{t}\, [\epsilon(\omega) - 1]\, \vec{E}(\tilde{t}, \vec{x})\, e^{i\omega(\tilde{t} - t)}$$
$$= \vec{E}(t, \vec{x}) + \int_{-\infty}^{\infty} d\tilde{t}\, g(t - \tilde{t})\, \vec{E}(\tilde{t}),$$ where

$$g(t - \tilde{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, [\epsilon(\omega) - 1]\, e^{-i\omega(t - \tilde{t})}$$ is

the convolution kernel.

Causality demands that $\vec{D}(t,\vec{x})$ be determined in terms of $\vec{E}(\tilde{t},\vec{x})$ for $\tilde{t} < t$, but, certainly, $\vec{D}(t,\vec{x})$ should not depend on $\vec{E}(\tilde{t},\vec{x})$ for $\tilde{t} > t$.

Therefore:

$$\underline{\text{causality: } \quad g(\Delta t) \overset{!}{=} 0 \quad \text{for} \quad \Delta t < 0}$$

Exercise 30

Recall our derivation of $\chi_e = \frac{e^2}{m} \frac{N}{\omega_0^2 - \omega^2 - i\omega\gamma}$ in a simple model of an e^- bound to a nucleus by a harmonic potential. Show that in this case $g(\Delta t) = 0$ for $\Delta t < 0$

Hint: Cauchy's theorem ∎

Using causality, we can write

$$\epsilon(\omega) = 1 + \int\limits_{\underset{!}{\uparrow}0}^{\infty} g(\Delta t)\, e^{i\omega \Delta t}\, d\Delta t \qquad (*)$$

Because g is real, this implies

$$\epsilon^*(\omega^*) = \epsilon(-\omega).$$

Furthermore, equation (*) implies that $\epsilon(\omega)$ is an analytic function of ω in the upper half plane (provided $g(\Delta t)$ remains finite for all Δt), since the integral converges. Similarly, $\epsilon(\omega)$ is analytic on the real axis if $g(\Delta t) \to 0$ as $\Delta t \to \infty$.

This is true for dielectrics. Say, in the simplified model

$$\epsilon(\omega) - 1 = 4\pi \frac{e^2}{m} \frac{N}{\omega_0^2 - \omega^2 - i\omega\gamma}$$, analytic for $\mathrm{Im}(\omega) \geq 0$.

Conductors however have free electrons, unbound, so for conductors $\omega_0 = 0$. Then,

$$\epsilon(\omega) - 1 = \frac{4\pi N e^2}{m} \frac{i}{\omega(\gamma - i\omega)},$$

which has a pole at $\omega = 0$

Integration by parts in equation (*) gives

$$\epsilon(\omega) - 1 = \int_0^\infty g(\Delta t)\, \frac{1}{i\omega} \frac{d}{d\Delta t}\left(e^{i\omega \Delta t}\right) d\Delta t =$$

$$= \frac{1}{i\omega}\left(\cancel{g(\Delta t=\infty)}^{\,0}\, e^{i\omega\infty} - g(\Delta t = 0) \right) - \frac{1}{i\omega} \int_0^\infty g'(\Delta t)\, e^{i\omega\Delta t} =$$

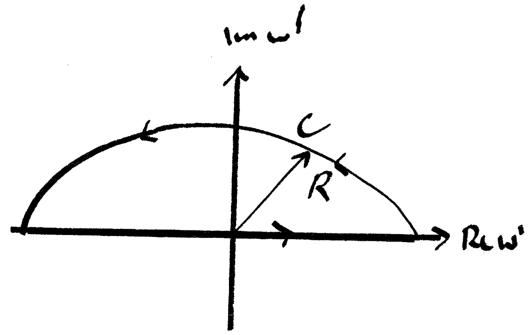
$$= \frac{i}{\omega}\, g(\Delta t = 0) - \frac{1}{\omega^2}\, g'(\Delta t = 0) + \cdots$$

Since $g(\Delta t) = 0$ for $\Delta t < 0$, by continuity, $g(\Delta t) = 0$.

Hence, $$\epsilon(\omega) - 1 = -\frac{1}{\omega^2}\, g'(\Delta t = 0) + \cdots$$

Now, using Cauchy's theorem,

$$\epsilon(z) = 1 + \frac{1}{2\pi i} \oint_C \frac{\epsilon(\omega') - 1}{\omega' - z}\, d\omega'$$



Since $(\epsilon(\omega) - 1) \propto \frac{1}{\omega^2}$ contribution of arc vanishes as $R \to \infty$.

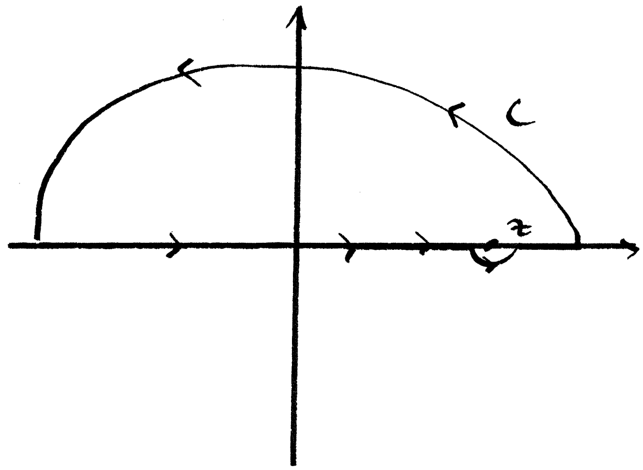
Then

$$\epsilon(z) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega') - 1}{\omega' - z}.$$

The integrand has a pole along the real line. Since $\epsilon(\omega')$ analytic in upper half plane, regulate by writing

$$z \to z + i\epsilon \quad , \text{with } \epsilon > 0$$

This amounts to a deformation of the integration contour:



We have now $\epsilon(\omega) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega') - 1}{\omega' - \omega - i\epsilon}$

<u>Exercise 31</u>

Show that

$$\frac{1}{\omega' - \omega \mp i\epsilon} = P\left(\frac{1}{\omega' - \omega}\right) \pm i\pi\,\delta(\omega' - \omega) \qquad \text{(in the limit } \epsilon \to 0)$$

Using this relation, we arrive at

$$\epsilon(\omega) = 1 + \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{\epsilon(\omega') - 1}{\omega' - \omega} d\omega'$$

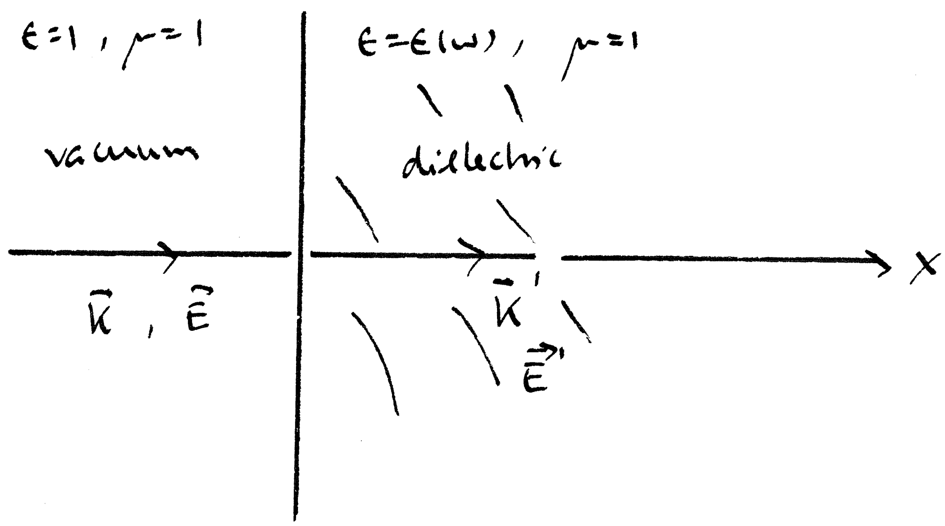
Taking real and imaginary parts,

$$\left.\begin{aligned} \mathrm{Re}\,\epsilon(\omega) &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\epsilon(\omega')}{\omega' - \omega} d\omega' \\ \mathrm{Im}\,\epsilon(\omega) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\epsilon(\omega') - 1}{\omega' - \omega} d\omega' \end{aligned}\right\} \quad \text{Kramers-Kronig dispersion relations}$$

These relations connect absorption ($\mathrm{Im}\,\epsilon$) with dispersion ($\mathrm{Re}\,\epsilon$).

11.6.3. No superluminal propagation

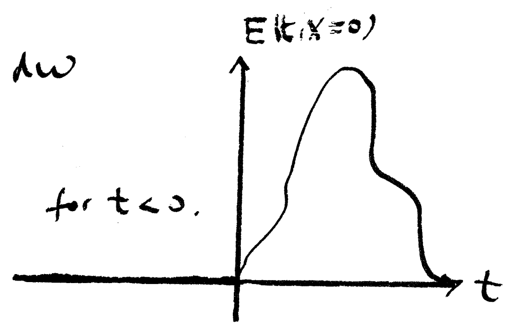
As an application of these ideas, let us consider propagation in a dielectric medium



From exercise 29, for normal incidence

$$\vec{E}'(t, \vec{x}) = \int_{-\infty}^{\infty} \left(\frac{2}{1+n(\omega)} \right) \cdot A(\omega)\, e^{ik(\omega)x - i\omega t}\, d\omega$$

where $A(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \vec{E}(t, x=0)\, e^{i\omega t}$,



is the amplitude of the incident wave in Fourier space, and $k = |\vec{k}| = \dfrac{n(\omega)}{c}\,\omega = \dfrac{\sqrt{\epsilon}}{c}\,\omega$.

From the previous asymptotic expansion, $\epsilon(\omega) - 1 = -\dfrac{g'(\Delta t)}{\omega^2} + \ldots$

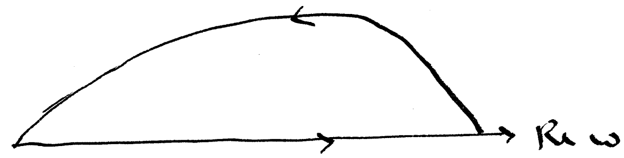
- $\epsilon(\omega) \to 1$ and $n(\omega) \to$ as $\omega \to \infty$.

Therefore, at large ω, integrand in $\vec{E}'(t,\vec{x})$ becomes proportional to

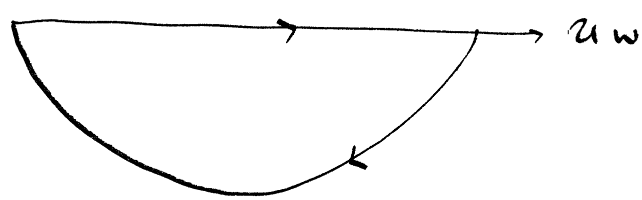
$$e^{i\frac{\omega}{c}(x-ct)}$$

Evaluate integral by contour integration:

$x - ct > 0$:



$x - ct < 0$:



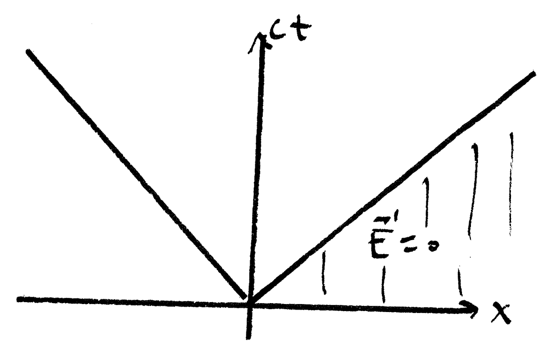
$\vec{E}(t, x=0) = 0$ for $t < 0$

$\Downarrow$

since $A(\omega)$ and $n(\omega)$ analytic in upper half-plane,

it follows that

$$\vec{E}'(t,\vec{x}) = 0 \quad \text{for} \quad x - ct > 0$$



causality $\Rightarrow$ No signal propagates faster than light

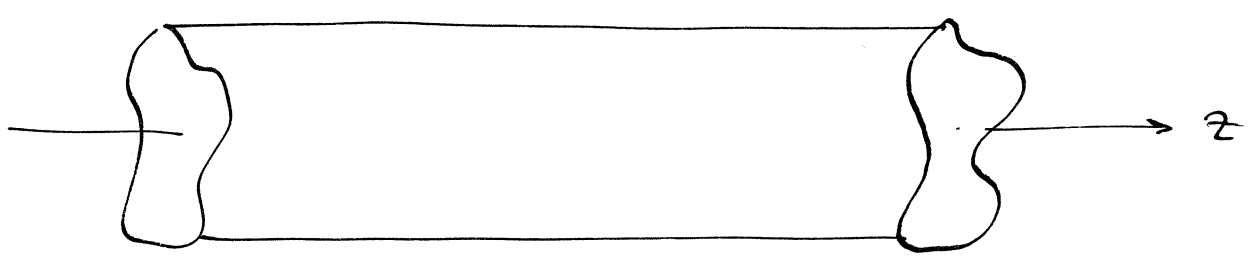
(in a dielectric medium).

12. Wave Guides (and Cavities)

For practical purposes (transmission of electromagnetic signals), it is useful to study wave propagation in wave guides

12.1. Cylindrical Wave Guides

A cylindrical wave guide is a (hollow) conducting cylinder of any cross-sectional shape



A wave travelling along such a wave guide can be written, by symmetry as

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r}_T,t)\, e^{i(kz-\omega t)},$$ where

$\vec{r}_\perp \cdot \hat{z} = 0$

By definition, the wave satisfies the wave equation

$$\vec{\nabla}^2 \vec{E} - \frac{\epsilon\mu}{c^2} \partial_t^2 \vec{E} = 0,$$ which gives

$$\left(\vec{\nabla}_T^2 - k^2 + \frac{\epsilon\mu\omega^2}{c^2} \right) \bar{E}(\vec{r}_T)\, \cancel{e^{i(kz-\omega t)}} = 0$$

cancel out.

Here, $\vec{\nabla}_T^2$ is the Laplacian along the transverse direction. Say, in

- cartesian coordinates: $\vec{\nabla}_T^2 = \partial_y^2 + \partial_z^2$.
- cylindrical " : $\vec{\nabla}_T^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$

Therefore, $\bar{E}(\bar{r}_T)$ satisfies the eigenvalue equation

$$\vec{\nabla}_T^2 \bar{E}(\bar{r}_T) = -\gamma^2 \bar{E}(\bar{r}_T),$$ with

$$\gamma^2 = \frac{\epsilon\mu\omega^2}{c^2} - k^2,$$ and boundary conditions determined by Maxwell's equations.

Same considerations apply for $\vec{B}$:

$$\vec{\nabla}_T^2 \vec{B}(r_T) = -\gamma^2 \vec{B}(r_T),$$ with

appropriate bc.