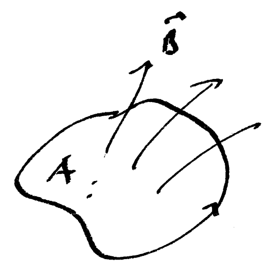
9. Maxwell's equations

We proceed to study now time varying electric and magnetic fields.

9.1. Faraday's law

Faraday observed that a change in the magnetic flux

$$\Phi \equiv \int_A \vec{B} \cdot d\vec{A}$$



through a circuit causes an electromotive force

$$\mathcal{E} = \oint_{\partial A} \vec{E} \cdot d\vec{r} \, .$$

Quantitatively, $\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt}$

The minus sign captures Lenz's law:

The induced current opposes the change of flux through the circuit.

Therefore, Faraday's law reads

$$\oint_{\partial A} \vec{E} \cdot d\vec{r} = -\frac{1}{c} \frac{d}{dt} \int_A \vec{B} \cdot d\vec{A} ,$$

which can be turned into a differential relation using Stokes' theorem:

$$\int_A \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\frac{1}{c} \frac{d}{dt} \int_A \vec{B} \cdot d\vec{A} \qquad \forall A .$$

Therefore
$$\boxed{\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0}$$

This equation displays the "intimate" relation between electric and magnetic fields.

9.3. Displacement current

So far, the equations for the microscopic fields $\vec{E}$ and $\vec{B}$ read

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi \rho \\ \vec{\nabla} \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} \end{cases}$$

But these equations are <u>not</u> compatible with current conservation:

From Ampère's law:

$$\vec{\nabla}\cdot\vec{j} = \frac{c}{4\pi}\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$$, instead of the expected

$$\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}.$$

Maxwell realized this inconsistency and introduced a modification in Ampère's law to solve it: the displacement current $\frac{1}{4\pi}\partial_t\vec{E} = \vec{j}_{displ}$.

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}.$$

In this form, $0 \overset{\text{Maxwell}}{=} \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} \overset{\text{Coulomb}}{=}$

$$= \frac{4\pi}{c}\left[\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t}\right] \checkmark$$

These are the final (microscopic) equations of electromagnetism

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{e} + \frac{1}{c}\,\partial_t\vec{b} = 0$$
$$\vec{\nabla}\cdot\vec{b} = 0$$
$$\vec{\nabla}\times\vec{b} - \frac{1}{c}\,\partial_t\vec{e} = \frac{4\pi}{c}\,\vec{j}$$

microscopic Maxwell's equations.

In order to obtain the macroscopic equations we need to take spatial averages, as before. The result is

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\,\partial_t\vec{B} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\,\partial_t\vec{D} = \frac{4\pi}{c}\,\vec{J}$$

macroscopic Maxwell's equations.

These equations automatically imply charge conservation,

$$\frac{\partial\rho_t}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 .$$

In principle, the Lorentz force has to be postulated separately:

$$\vec{F} = q\left(\vec{e} + \frac{\vec{v}}{c}\times\vec{b}\right)$$

Vector and Scaler Potentials

Consider the microscopic Maxwell equations.

Since $\vec{\nabla}\cdot\vec{b}=0$, we can write $\underline{\vec{b}=\vec{\nabla}\times\vec{A}}$.

Then, $\vec{\nabla}\times\vec{e}+\frac{1}{c}\partial_t\vec{b}=0$ becomes $\vec{\nabla}\times(\vec{e}+\frac{1}{c}\partial_t\vec{A})=0$.

Hence, we can write $\underline{\vec{e}+\frac{1}{c}\partial_t\vec{A}=-\vec{\nabla}\phi}$

Substituting these relations in the inhomogeneous

eqs. then gives

$$\left\{\begin{array}{l}\nabla^2\phi+\frac{1}{c}\partial_t\vec{\nabla}\cdot\vec{A}=-4\pi\rho \qquad \text{and}\\[2em] \nabla^2\vec{A}-\frac{1}{c}\frac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}\right)=-\frac{4\pi}{c}\vec{j}\end{array}\right.$$

Again, $\vec{b}$ and $\vec{e}$ are invariant under the

gauge transformations

$$\vec{A}\to\vec{A}'=\vec{A}+\vec{\nabla}\chi \qquad \text{and}$$

$$\phi\to\phi'=\phi-\frac{1}{c}\partial_t\chi\,, \qquad \text{since}$$

$$\vec{e}=-\vec{\nabla}\phi-\frac{1}{c}\partial_t\vec{A}\to\vec{e}'=-\vec{\nabla}\phi'-\frac{1}{c}\partial_t\vec{A}'=-\vec{\nabla}\phi-\frac{1}{c}\partial_t\chi+\frac{1}{c}\partial_t\vec{\nabla}\chi-$$

$$-\frac{1}{c}\partial_t\vec{\nabla}\chi=\vec{e}.$$

We can use gauge invariance to impose convenient conditions on ϕ and $\vec{A}$.

9.8 Magnetic monopoles

It is natural to ask how Maxwell's equations would look like if magnetic monopoles existed. From the conventional Maxwell's eqs. one would guess

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 4\pi \rho_e \\ \vec{\nabla} \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = -\frac{4\pi}{c} \vec{j}_m \\ \vec{\nabla} \cdot \vec{B} = 4\pi \rho_m \\ \vec{\nabla} \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{j}_e \end{array} \right. \qquad (\mathrm{MM})$$

Exercise 25

Show that these eqs. imply the conservation of electric and magnetic charge ∎

Exercise 26

i) Show that the equations (MM) are invariant under the <u>duality transformation</u>

$$\vec{E}' = \vec{E}\cos\theta + \vec{B}\sin\theta \qquad \rho_e' = \rho_e\cos\theta + \rho_m\sin\theta$$

$$\vec{B}' = -\vec{E}\sin\theta + \vec{B}\cos\theta \qquad \rho_m' = -\rho_e\sin\theta + \rho_m\cos\theta$$

and similarly for $\vec{j}_e$, $\vec{j}_m$.

ii) Show that the "Lorentz force"

$$\vec{F} = q_e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) + q_m\left(\vec{B} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

is also invariant under such transformations

This implies that the electric and magnetic charge of a particle is a matter of convention.

If all particles have the same ratio $\frac{\rho_m}{\rho_e}$, then we can "rotate" by a duality transf. so that $\rho_m = 0$ for all of them.

Otherwise we can only assume that one of them (e.g. the electron) has $\rho_m = 0$.

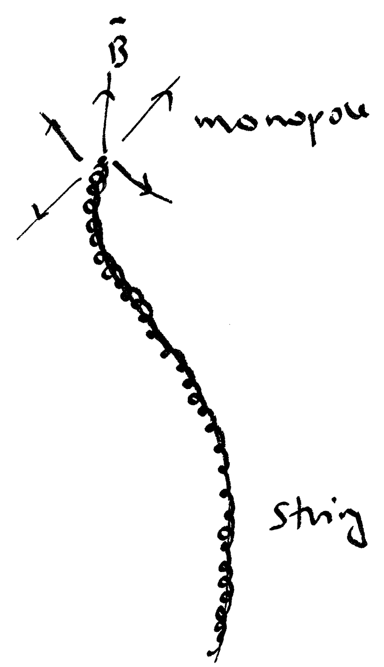
In 1931 Dirac showed that the existence of a single magnetic monopole (of charge $\rho_m = g$; $\rho_e = 0$) would imply the quantization of electric charge e:

$$\frac{eg}{\hbar c} = \frac{n}{2}, \quad n \in \mathbb{Z}$$

(Dirac quantization condition)

Dirac modeled a monopole as a "string" of aligned dipoles extending to infinity, or a tightly wound solenoid along a string:

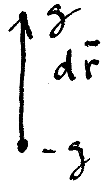
or



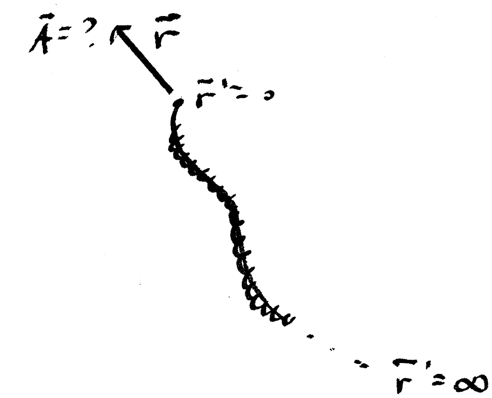
From Lecture 12, the vector potential (in Coulomb gauge) created by a dipole $d\vec{\mu}$ located at $\vec{r}\,'$ is

$$\vec{A}(\vec{r}) = d\vec{\mu}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} \qquad \left[\text{we had } \vec{A}(\vec{r}) = \vec{\mu} \times \frac{\vec{r}}{|\vec{r}|^2} \text{ for } \vec{r}\,'=0\right]$$

$$= -d\vec{\mu} \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

With $d\vec{\mu} = g\, d\vec{r}\,'$, 

, the potential becomes



$$\vec{A}(\vec{r}) = -g \int_{\text{string}} d\vec{r}\,' \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

To calculate $\vec{B}$ we use Jackson's identities:

$$\vec{B}(\vec{r}) \equiv \vec{\nabla} \times \vec{A} = -g \int_{\vec{r}\,'=\infty}^{\vec{r}\,'=0} \Big[d\vec{r}\,' \underbrace{\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)}_{-4\pi\,\delta(\vec{r}-\vec{r}\,')} + \underbrace{(d\vec{r}\,'\cdot\vec{\nabla}')}_{\text{derivative along string}}\, \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\Big]$$

Thus, away from the string.

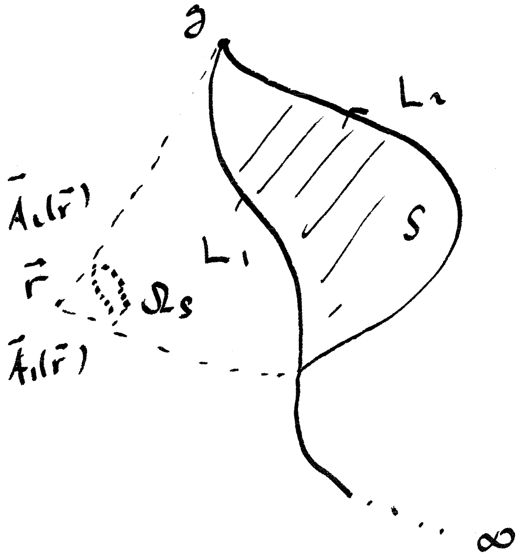
$$\vec{B}(\vec{r}) = -g\,\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = g\,\frac{\vec{r}}{|\vec{r}|^2},$$ the field of a magnetic monopole of charge g!

In order for Dirac's monopole to be physically sensible, the location of the string should be irrelevant. One can indeed show that the location of the string can be changed by a gauge transformation:

Consider two different strings. Then, the vector potential of the two strings at $\vec{r}$ satisfies

$$\vec{A}_2(\vec{r}) = \vec{A}_1(\vec{r}) + g\,\vec{\nabla}\Omega_S(\vec{r}),$$

where Ω_S is the subtended by S.



If we now consider the Hamiltonian of an electron in an electromagnetic field:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

it is easy to check that if $i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$,

then, the solution of $i\hbar\frac{\partial|\psi'\rangle}{\partial t} = H'|\psi'\rangle$,

H' being the transformed Hamiltonian

$$H' = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}'\right)^2 + e\,\phi' \quad , \text{ with}$$

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \quad \text{and} \quad \phi' = \phi - \frac{1}{c}\partial_t \chi \quad \text{is}$$

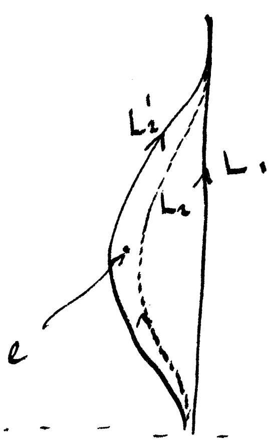
$$\langle \vec{r} \,|\, \psi' \rangle = \exp\left[ie\chi/\hbar c\right] \langle \vec{r} \,|\, \psi \rangle .$$

Therefore, the wave functions of an electron in the field of two different strings are related by

$$\langle \vec{r} \,|\, \psi_2 \rangle = \exp\left[ieg\,\Omega_s(\vec{r})/\hbar c\right] \langle \vec{r} \,|\, \psi_1 \rangle .$$

But Ω_s changes by 4π when the surface moves from slightly above s to slightly below.

We avoid the ambiguity in the wave function if



$$\frac{ieg \cdot 4\pi}{\hbar c} = i n \cdot 2\pi \, , \quad n \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{eg}{\hbar c} = \frac{n}{2} \, , \quad n \in \mathbb{Z}$$

In "spontaneously broken" gauge theories there exist regular monopole solutions "'t Hooft - Polyakov" monopole homotopy groups, ...