Last time we saw that

$$\vec{A}(t,\vec{r}) = \frac{e^{i\frac{\omega}{c}|\vec{r}|}}{c|\vec{r}|}\,\dot{\vec{p}}\,, \quad \text{where}$$

$$\vec{p} = \int d^3r'\,\vec{r}\,'\rho(\vec{r}\,')\,, \qquad \underline{\text{dipole radiation}}.$$

The electric and magnetic fields are transverse:

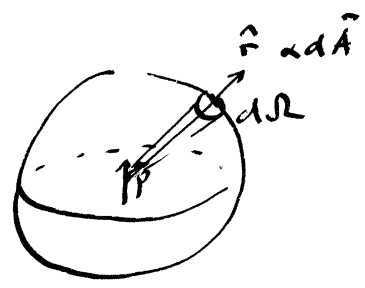
$$\vec{B} = \frac{\omega^2}{c^2}\,\hat{r}\times\vec{p}\,\frac{e^{i\frac{\omega}{c}|\vec{r}|}}{|\vec{r}|} \qquad \text{and} \qquad \vec{E} = -\hat{r}\times\vec{B}$$

(we are assuming $\vec{p} \propto e^{-i\omega t}$).

To calculate the power radiated by the oscillating dipole, we consider the averaged Poynting vector $\langle\vec{S}\rangle = \frac{c}{8\pi}\,\vec{E}^*\times\vec{B}$, which describes the flux of EM energy per unit area.

Therefore, the energy crossing an area element $d\vec{A}$ per unit time is

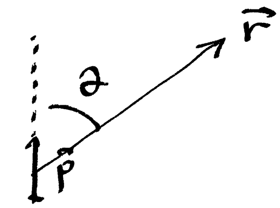
$$dP = \langle\vec{S}\rangle\cdot d\vec{A} = \langle\vec{S}\rangle\cdot\hat{r}\,r^2\,d\Omega$$



Using that

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \underset{\vec{b}}{(-\hat{r}} \times \underset{\vec{c}}{\vec{B}^*)} \times \underset{\vec{a}}{\vec{B}} = \frac{c}{8\pi} |\vec{B}|^2 \hat{r} = \frac{c}{8\pi} \frac{\omega^4}{c^4} \frac{|\hat{r} \times \vec{p}|^2}{|\vec{r}|^2} \hat{r}$$

we get



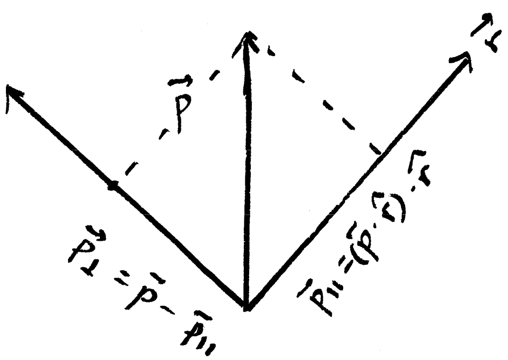
$$dP = \frac{c}{8\pi} \frac{\omega^4}{c^4} |\vec{p}|^2 \sin^2\theta \, d\Omega ,$$

which does not depend on $|\vec{r}|$. The total radiated power is

$$\mathcal{P} = \int dP = \frac{c k^4}{3} |\vec{p}|^2 \quad , \text{ since } \quad |\vec{k}| = \frac{\omega}{c} .$$

↑ Problem for a classical atom!

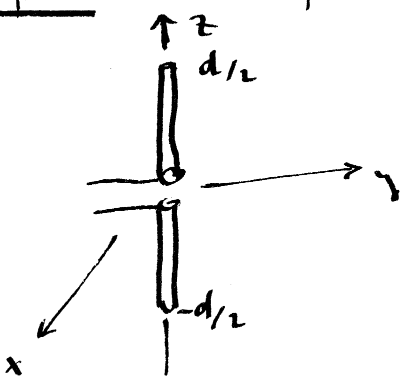
Note that because $\vec{E} = \vec{B} \times \hat{r} = k^2 \frac{e^{ik|\vec{r}|}}{|\vec{r}|} (\overset{\vec{b}}{\hat{r}} \times \overset{\vec{c}}{\vec{p}}) \times \overset{\vec{a}}{\hat{r}}$,



$$\vec{E} = k^2 \frac{e^{ik|\vec{r}|}}{|\vec{r}|} [\vec{p} - \hat{r}(\vec{p} \cdot \hat{r})] \propto \vec{p}_T$$

radiation is polarized in the $\vec{r} - \vec{p}$ plane.

<u>Example</u>: Centerfed linear antenna.



A simple model for the current in the antenna is

$$I(z) = I_0 \left(1 - \frac{2|z|}{d}\right) e^{-i\omega t}$$

Exercise 35

Calculate the power radiated by such an antenna

13.9 Magnetic Dipole and Electric Quadrupole Radiation

Let's go back to the original approximation

$$\vec{A}(t,\vec{r}) = \frac{e^{i(k|\vec{r}|-\omega t)}}{c|\vec{r}|} \int d^3r' \exp(-ik\,\hat{r}\cdot\vec{r}')\,\vec{j}(\vec{r}').$$

Consider now the $n=1$ term in the expansion of the exponential:

$$\int d^3r'\,\vec{j}(\vec{r}')\,(-ik\,\hat{r}\cdot\vec{r}') = -ik\hat{r}\cdot\int d^3r'\,\vec{r}'\,\vec{j}(\vec{r}')$$

As in Exercise 22 (Lecture 16), we write the latter as symmetric and antisymmetric components:

$$\mathcal{J}_s(k\hat{r}) = -\frac{ik\hat{r}}{2}\cdot\int d^3r'\,(\vec{r}'\,\vec{j}(\vec{r}') + \vec{j}(\vec{r}')\,\vec{r}')$$

$$\mathcal{J}_a(k\hat{r}) = -\frac{ik\hat{r}}{2}\cdot\int d^3r'\,(\vec{r}'\,\vec{j}(\vec{r}') - \vec{j}(\vec{r}')\,\vec{r}')$$

As in Lecture 16, we find that the antisymm component gives

$$\vec{J}_a = \frac{ik}{2}\, \hat{r} \times \int d^3r' \; \vec{r}' \times \vec{j}(\vec{r}') \equiv ikc\, \hat{r} \times \vec{\mu},$$

where $\vec{\mu} = \frac{1}{2c} \int d^3r' \; \vec{r}' \times \vec{j}\,(\vec{r}')$ is the magnetic dipole moment.

Therefore, we find

$$\vec{A}(t, \vec{r}) = \frac{e^{i(k|\vec{r}|-\omega t)}}{c|\vec{r}|} \, ikc\, \hat{r} \times \vec{\mu}, \text{ which gives}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -k^2 \frac{e^{i(k|\vec{r}|-\omega t)}}{|\vec{r}|} \hat{r} \times (\hat{r} \times \vec{\mu}) = \frac{k^2 e^{i(k|\vec{r}|-\omega t)}}{|\vec{r}|} [\vec{\mu} - \hat{r}(\hat{r}\cdot\vec{\mu})]$$

and

$$\vec{E} = -\hat{r} \times \vec{B} = -\frac{k^2 e^{i(k|\vec{r}|-\omega t)}}{|\vec{r}|} \hat{r} \times \vec{\mu}.$$

Same as electric dipole radiation, with $\vec{p} \to \vec{\mu}$, $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$.

Therefore, in this case $\vec{E}$ is perpendicular to the $\vec{\mu} - \vec{r}$ plane.

Let us turn our attention now to the symmetric part, $\vec{J}_s$:

Exercise 36

i) Show that $\vec{J}_s = -i \frac{k\hat{r}}{2} \cdot \int d^3r' \, \vec{r}' \vec{r}' \, \rho(\vec{r}') \cdot (-i\omega)$

ii) Show that the corresponding fields are

$$\vec{B} = -i \frac{k^3}{6} \frac{e^{ik|\vec{r}|}}{|\vec{r}|} \hat{r} \times (Q\hat{r}) \quad \text{and}$$

$$\vec{E} = -i \frac{k^3}{6} \frac{e^{ik|\vec{r}|}}{|\vec{r}|} [Q\hat{r} - (\hat{r} \cdot Q\hat{r})\hat{r}],$$

where $Q_{\alpha\beta} = \int d^3r' \, (3r'_\alpha r'_\beta - |\vec{r}'|^2 \delta_{\alpha\beta}) \rho(\vec{r}')$

is the quadrupole tensor of the charge density.

iii) Show that the radiated power is

$$\frac{dP}{d\Omega} = \frac{ck^6}{288\pi} [(Q\hat{r})^2 - (\hat{r} \cdot Q\hat{r})^2]$$

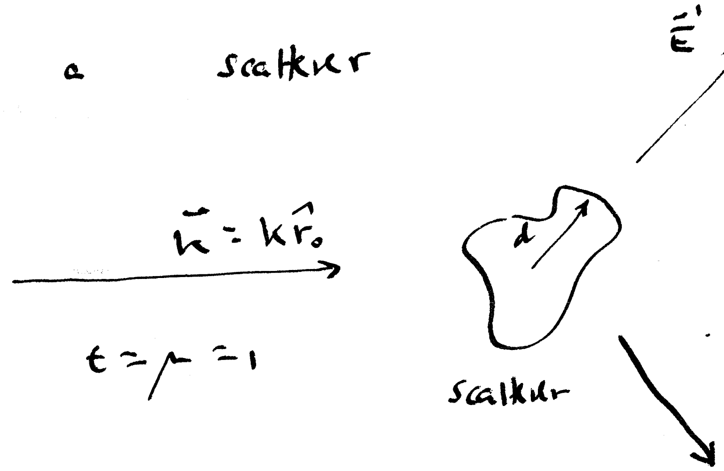
For obvious reasons this is known as electric quadrupole radiation.

iv) Compare the power radiated in quadrupole radiation to that of (electric) dipole radiation.

13.11 Scattering of Electromagnetic Radiation

Our radiation formulas can be also used to understand scattering of radiation:

Suppose a plane wave $\vec{E} = \vec{E}_0 \, e^{i(kz - \omega t)}$ propagating in a medium with $\epsilon = \mu = 1$ hits a scatterer



The electric field of the incident wave will typically induce an electric and magnetic dipole in the scatterer, which will therefore radiate.

From our previous results, if $\vec{p}$ and $\vec{\mu}$ are the induced dipoles, the fields far away from the scatterer (at $\vec{r} \approx 0$) are

$$\vec{E}' = k^2 \frac{e^{ik|\vec{r}|}}{|\vec{r}|} \left[(\hat{r} \times \vec{p}) \times \hat{r} - \hat{r} \times \vec{\mu} \right], \quad \vec{B}' = \hat{r} \times \vec{E}'$$

We define the differential scattering cross section $\frac{d\sigma}{d\Omega}$ as the power radiated in the direction $\hat{r}$ with polarization $\hat{\epsilon}$ per unit solid angle, and per unit incident flux in the direction $\hat{r}_0$ with polarization $\hat{\epsilon}_0$:

$$\frac{d\sigma}{d\Omega}(\hat{r}, \hat{\epsilon}; \hat{r}_0, \hat{\epsilon}_0) \equiv \frac{r^2 \frac{c}{8\pi} |\hat{\epsilon}^* \vec{E}'|^2}{\frac{c}{8\pi} |\hat{\epsilon}_0^* \cdot \vec{E}|^2}$$

Therefore, since $\hat{\epsilon}^* \cdot \hat{r} = 0$,

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{|\hat{\epsilon}_0^* \cdot \vec{E}_0|^2} \, |\hat{\epsilon}^* \cdot \vec{p} + (\hat{r} \times \hat{\epsilon}^*) \cdot \vec{\mu}|^2$$

Note that this expression is only valid in the limit

$$d \ll \lambda \ll r.$$

The k^4 dependence is known as Rayleigh's law, and is almost universal for scattering at long wavelengths. It is the basis of Rayleigh's famous explanation of the blue sky and the red sunset.

Scattering by a (small) dielectric sphere

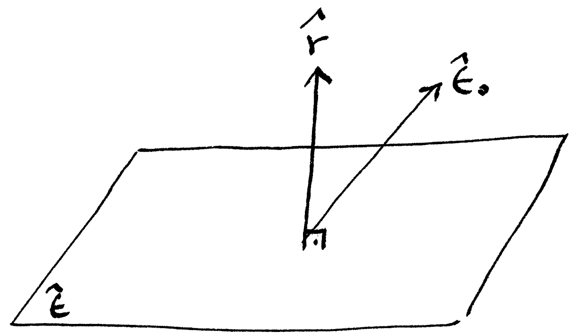
As a simple example, suppose that the scatterer consists of a small dielectric sphere of dielectric constant $\epsilon(\omega)$ and $\mu = 1$. Then, from Lecture 10, the dipole moment of the sphere is

$$\vec{p} = \frac{\epsilon - 1}{\epsilon + 2} R^3 \vec{E}.$$

With $\vec{E} = \hat{\epsilon}_0 E_0 e^{i(kz - \omega t)}$ we get

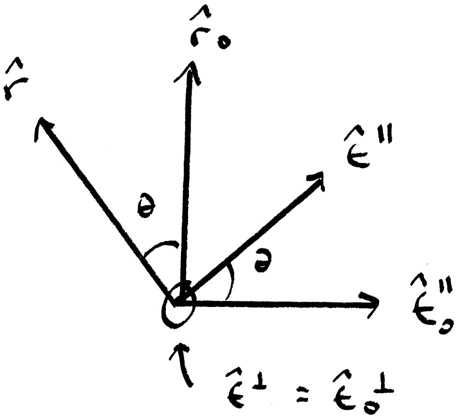
$$\frac{d\sigma}{d\Omega} = k^4 R^6 \left|\frac{\epsilon - 1}{\epsilon + 2}\right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2.$$

Therefore, light is linearly polarized in the plane defined by $\hat{r}$ and $\hat{\epsilon}_0$:



In practice this implies that when unpolarized light becomes (partially) polarized after scattering:

Scattering plane $\hat{r} - \hat{r}_0$



Light polarized along $\hat{\epsilon}_0^{\parallel}$: $\frac{d\sigma_\parallel}{d\Omega} \propto \cos^2\theta$; $\frac{d\sigma_\perp}{d\Omega} \propto 0$

Light polarized along $\hat{\epsilon}_0^{\perp}$: $\frac{d\sigma_\parallel}{d\Omega} \propto 0$; $\frac{d\sigma_\perp}{d\Omega} \propto 1$

Average over initial polarizations:

with weights $p_\parallel = p_\perp = \frac{1}{2}$

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{1}{2} k^4 R^6 \left|\frac{\epsilon-1}{\epsilon+2}\right|^2 \cos^2\theta$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{2} k^4 R^6 \left|\frac{\epsilon-1}{\epsilon+2}\right|^2$$

At $\theta = \frac{\pi}{2}$ light is 100 % polarized $\perp$ scattering plane.

Finally, let us consider scattering by a point particle (an electron). With

$$m\frac{d^2\vec{x}}{dt^2} = -e\vec{E}_0 e^{-i\omega t}, \quad \vec{x} = \frac{e\vec{E}_0}{m\omega^2} e^{-i\omega t}, \quad \vec{p} = \frac{e^2\vec{E}_0}{m\omega^2} e^{-i\omega t}$$

we get

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{1+\cos^2\theta}{2}.$$

This is the cross section for <u>Thomson scattering</u>, which agrees in the non-relativistic limit $\omega \ll m$ with the corresponding calculation in quantum field theory