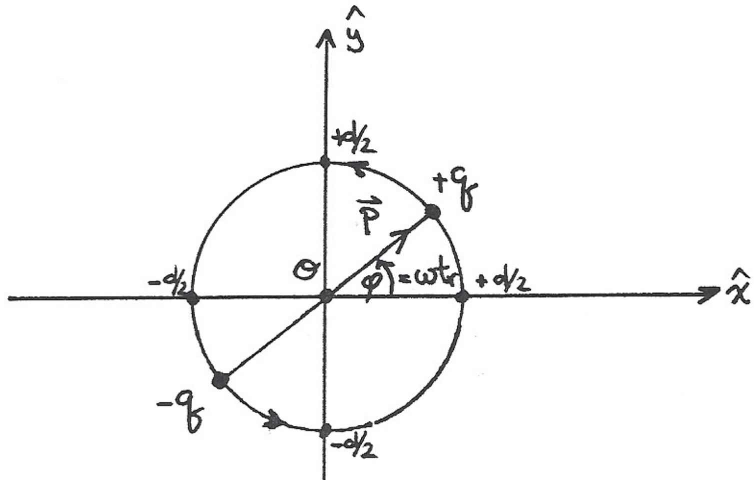
LECTURE NOTES 13.75

EM Radiation Fields Associated with a *Rotating* E(1) Electric Dipole

Griffiths Problem 11.4:

Consider an E(1) electric dipole that rotates CCW {as viewed from *above*} in the *x*-*y* plane with constant angular frequency $\omega = 2\pi f$, as shown in the figure below:



The {retarded} time-dependent position vector(s) describing the motion of the two point charges $+q(\vec{r}_+, t_r) \equiv q_+(\vec{r}_+, t_r)$ and $-q(\vec{r}_-, t_r) \equiv q_-(\vec{r}_-, t_r)$, with $\varphi = \omega t_r$ are given by:

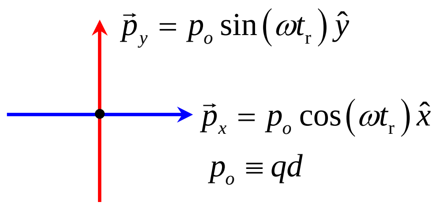
$$\boxed{\vec{r}_+(\vec{r}, t_r) = +\frac{d}{2}\cos\varphi\hat{x} + \frac{d}{2}\sin\varphi\hat{y} = +\frac{d}{2}\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right]}$$

$$\boxed{\vec{r}_-(\vec{r}, t_r) = -\frac{d}{2}\cos\varphi\hat{x} - \frac{d}{2}\sin\varphi\hat{y} = -\frac{d}{2}\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right]}$$

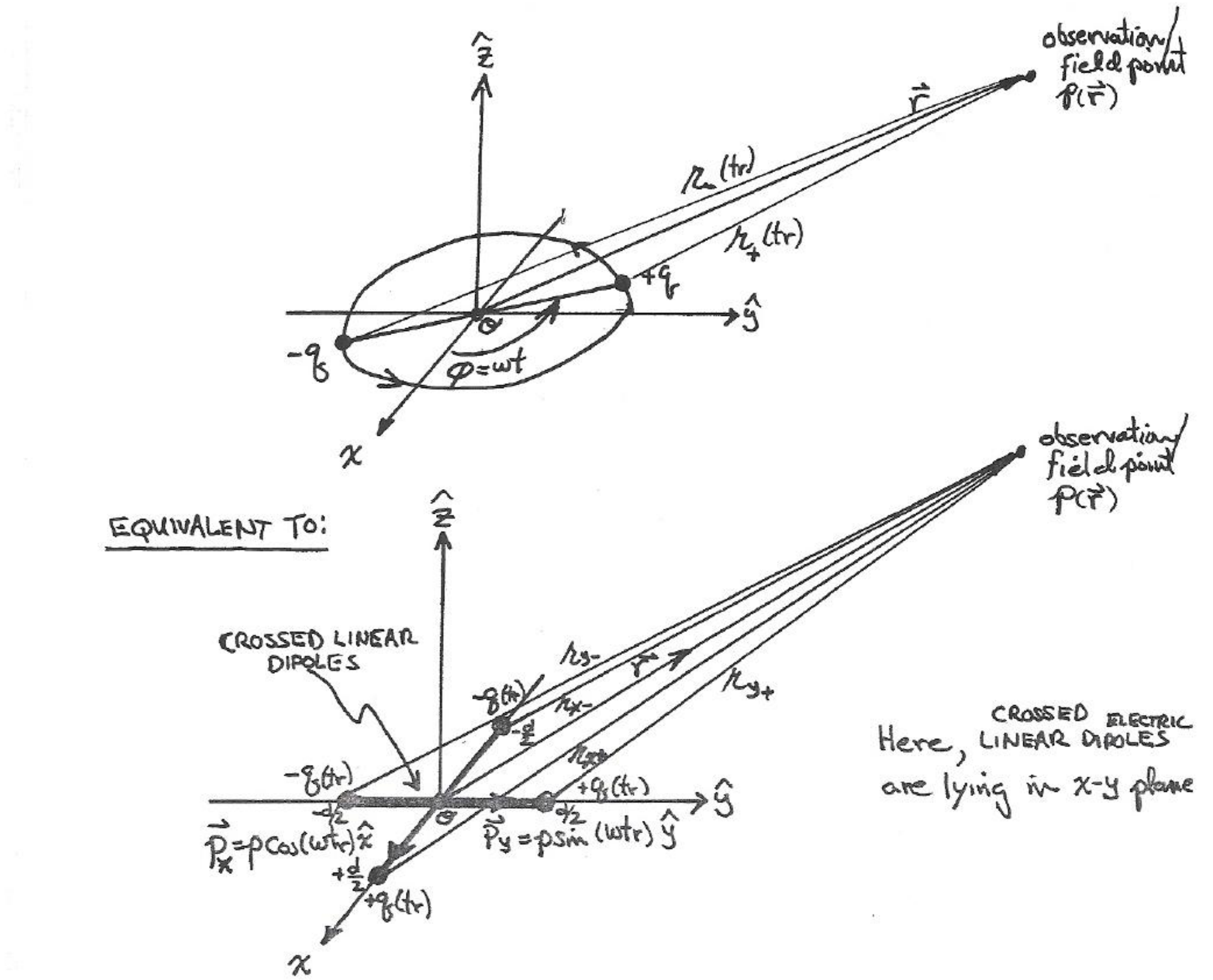
Then:

$$\begin{aligned}
\vec{p}(\vec{r}, t_r) &= q_+\vec{r}_+ + q_-\vec{r}_- = +q\vec{r}_+ - q\vec{r}_- \\
&= q\left(\frac{d}{2}\right)\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right] + q\left(\frac{d}{2}\right)\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right] \\
&= qd\cos(\omega t_r)\hat{x} + qd\sin(\omega t_r)\hat{y} = p_o\left[\cos(\omega t_r)\hat{x} + \sin(\omega t_r)\hat{y}\right] = \vec{p}_x(\vec{r}, t_r) + \vec{p}_y(\vec{r}, t_r)
\end{aligned}$$

$\Rightarrow \vec{p}(\vec{r}, t_r)$ = Superposition of two crossed (*i.e.* perpendicular/orthogonal) linear oscillating dipoles, one || to the $\hat{x}$ axis, one || to the $\hat{y}$ axis, the latter of which is $90^\circ = \pi/2$ radians out-of-phase with the former, as shown in the figure below:



Then in 3-D:



By the principle of linear superposition, we can add the separate contributions associated with $\vec{p}_x$ and $\vec{p}_y$ to obtain the {total(ly)} retarded scalar and vector potentials.

Recall for the linear oscillating electric dipole aligned along the $\hat{z}$ -axis $\boxed{\vec{p}_z(\vec{r},t_r) = p_o\cos(\omega t_r)\hat{z}}$ {in the "far-zone" limit, $d \ll \lambda \ll r$ } we obtained:

$$\boxed{V_{r_z}^{E(1)}(\vec{r},t) \simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] = -\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{r\cos\theta}{r^2}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

But: $z = r\cos\theta$

$$\therefore \text{ for } \boxed{\vec{p}_z(\vec{r},t_r) = p_o\cos(\omega t_r)\hat{z} \quad \begin{cases} V_{r_z}^{E(1)}(\vec{r},t) \simeq -\dfrac{p_o\omega}{4\pi\varepsilon_o c}\left(\dfrac{z}{r^2}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{r_z}^{E(1)}(\vec{r},t) \simeq -\dfrac{\mu_o p_o\omega}{4\pi}\left(\dfrac{1}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{z} \end{cases}}$$

Thus, since $x = r\sin\theta\cos\varphi$ and $y = r\sin\theta\sin\varphi$, then:

$$\boxed{\vec{p}_x(r,t_{\mathrm{r}}) = p_o\sin(\omega t_{\mathrm{r}})\,\hat{x}\quad \begin{cases} V_{\mathrm{r}_x}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\dfrac{p_o\omega}{4\pi\varepsilon_o c}\left(\dfrac{x}{r^2}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] = -\dfrac{p_o\omega}{4\pi\varepsilon_0 c}\left(\dfrac{\sin\theta\cos\varphi}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{\mathrm{r}_x}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\dfrac{\mu_o p_o\omega}{4\pi}\left(\dfrac{1}{r}\right)\sin\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{x} \end{cases}}$$

<u>And</u>:

$$\boxed{\vec{p}_y(r,t_{\mathrm{r}}) = p_o\sin(\omega t_{\mathrm{r}})\,\hat{y}\quad \begin{cases} V_{\mathrm{r}_y}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\dfrac{p_o\omega}{4\pi\varepsilon_o c}\left(\dfrac{y}{r^2}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right] = -\dfrac{p_o\omega}{4\pi\varepsilon_o c}\left(\dfrac{\sin\theta\sin\varphi}{r}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right] \\ \vec{A}_{\mathrm{r}_y}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\dfrac{\mu_o p_o\omega}{4\pi}\left(\dfrac{1}{r}\right)\cos\left[\omega\left(t-\dfrac{r}{c}\right)\right]\hat{y} \end{cases}}$$

Then the totally retarded potentials {in the "far-zone" limit, $d \ll \lambda \ll r$ } are:

$$\boxed{\begin{aligned} V_{\mathrm{r}_{tot}}^{\mathrm{E}(1)}(\vec{r},t) &= V_{\mathrm{r}_x}^{\mathrm{E}(1)}(\vec{r},t) + V_{\mathrm{r}_y}^{\mathrm{E}(1)}(\vec{r},t) \\ &\simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left\{\left(\frac{x}{r^2}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{y}{r^2}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &\simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left\{\left(\frac{\sin\theta\cos\varphi}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{\sin\theta\sin\varphi}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &\simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\sin\theta}{r}\right)\left\{\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \sin\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \end{aligned}}$$

<u>And</u>:

$$\boxed{\begin{aligned} \vec{A}_{\mathrm{r}_{tot}}^{\mathrm{E}(1)}(\vec{r},t) &= \vec{A}_{\mathrm{r}_x}^{\mathrm{E}(1)}(\vec{r},t) + \vec{A}_{\mathrm{r}_y}^{\mathrm{E}(1)}(\vec{r},t) \\ &\simeq -\frac{\mu_o p_o\omega}{4\pi}\left(\frac{1}{r}\right)\left\{\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{x} + \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{y}\right\} \end{aligned}}$$

<u>But</u>: $$\boxed{\begin{aligned} \hat{x} &= \sin\theta\cos\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} - \sin\varphi\hat{\varphi} \\ \hat{y} &= \sin\theta\sin\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} + \cos\varphi\hat{\varphi} \end{aligned}}$$

<u>Thus</u>: $\vec{A}_{\mathrm{r}_{tot}}^{\mathrm{E}(1)}(\vec{r},t) =$ big mess!!!

Instead of {mindlessly} bulldozing/grinding our way thru this, we can obtain

$$\boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t) = -\vec{\nabla}V^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t)}{\partial t}} \quad \text{and} \quad \boxed{\vec{B}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t)} \quad \text{by:}$$

a.) Using the already known form of $\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_z}(\vec{r},t)$ that we have previously obtained from the <u>single</u> oscillating dipole aligned along the $\hat{z}$ -axis, $\vec{p}_z = p_o\cos(\omega t_{\mathrm{r}})\hat{z}$ – *i.e.* we simply rotate the $\vec{E}^{E(1)}_{\mathrm{r}_z}(\vec{r},t)$ solution by 90° {and change the phase relation in the $\hat{y}$ -direction} to obtain $\vec{E}^{E(1)}_{\mathrm{r}_x}(\vec{r},t)$ and $\vec{E}^{E(1)}_{\mathrm{r}_y}(\vec{r},t)$ associated with $\vec{p}_x = p_o\cos(\omega t_{\mathrm{r}})\hat{x}$ and $\vec{p}_y = p_o\sin(\omega t_{\mathrm{r}})\hat{y}$, respectively, and then:

b.) obtain the corresponding/associated *B*-fields using the relation $\boxed{B^{\mathrm{E}(1)}_{\mathrm{r}}(\vec{r},t) = \frac{1}{c}\left(\hat{r}\times\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}}(\vec{r},t)\right)}$

Thus, recall for $\vec{p}_z = p_o\cos(\omega t_{\mathrm{r}})\hat{z}$ in the "far-zone" limit { $d \ll \lambda \ll r$ } that we obtained:

$$\boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_z}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

<u>however, note that</u>: $\boxed{\sin\theta\;\hat{\theta} = \cos\theta\hat{r} - \hat{z}}$

$$\therefore \quad \boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_z}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi r}\left[\cos\theta\hat{r} - \hat{z}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

, <u>but</u>: $\boxed{\cos\theta = \frac{z}{r}}$

$$\therefore \quad \boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_z}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi r}\left[\left(\frac{z}{r}\right)\hat{r} - \hat{z}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

$$\therefore \quad \boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_x}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi r}\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} \Leftarrow \text{for } \boxed{\vec{p}_x = p_o\cos(\omega t_{\mathrm{r}})\hat{x}}$$

<u>And</u>: $$\boxed{\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_y}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi r}\left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \Leftarrow \text{for } \boxed{\vec{p}_y = p_o\sin(\omega t_{\mathrm{r}})\hat{y}}$$

Thus, the totally retarded electric field {in the "far-zone" limit, $d \ll \lambda \ll r$ } is:

$$\boxed{\begin{aligned}\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t) &= \vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_x}(\vec{r},t) + \vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_y}(\vec{r},t)\\ &\simeq -\frac{\mu_o p_o \omega^2}{4\pi r}\left\{\left[\left(\frac{x}{r}\right)\hat{r} - \hat{x}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + \left[\left(\frac{y}{r}\right)\hat{r} - \hat{y}\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\end{aligned}}$$

<u>And</u>: $\boxed{B^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t) = \frac{1}{c}\left(\hat{r}\times\vec{E}^{\mathrm{E}(1)}_{\mathrm{r}_{tot}}(\vec{r},t)\right)}$

Thus, the totally retarded Poynting's vector for the rotating E(1) electric dipole is:

$$\boxed{\begin{aligned}\vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) &= \frac{1}{\mu_o}\left(\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\times\vec{B}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)\\ &= \frac{1}{\mu_o c}\left[\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\left(\hat{r}\times\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)\right]\end{aligned}}$$

<u>But</u>: $\boxed{\vec{A}\times\vec{B}\times\vec{C} = \vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)}$

$$\therefore \quad \boxed{\vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) = \frac{1}{\mu_o c}\left\{\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2\hat{r}-\left(\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\bullet\hat{r}\right)\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right\}}$$

<u>But</u>: $\boxed{\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}_{Tot}}(\vec{r},t)\bullet\hat{r}=0}$ <u>because</u>:
$$\boxed{\begin{aligned}\left[\left(\frac{x}{r}\right)\hat{r}-\hat{x}\right]\bullet\hat{r} &= \left(\frac{x}{r}\right)\hat{r}\bullet\hat{r}-\hat{x}\bullet\hat{r} = \frac{x}{r}-\frac{x}{r}=0\\ \left[\left(\frac{y}{r}\right)\hat{r}-\hat{y}\right]\bullet\hat{r} &= \left(\frac{y}{r}\right)\hat{r}\bullet\hat{r}-\hat{y}\bullet\hat{r} = \frac{y}{r}-\frac{y}{r}=0\end{aligned}}$$

$$\therefore \quad \boxed{\vec{S}^{rad}_{\mathrm{E(1)}_{tot}}(\vec{r},t) = \frac{1}{\mu_o c}\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2\hat{r}}$$

In the "far-zone" limit $\{ d \ll \lambda \ll r \}$:

$$\boxed{\begin{aligned}\left(E^{\mathrm{E(1)}}_{\mathrm{r}_{tot}}(\vec{r},t)\right)^2 \simeq \left(\frac{\mu_o p_o\omega^2}{4\pi r}\right)^2 &\left\{\left[\left(\frac{x}{r}\right)\hat{r}-\hat{x}\right]^2\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\left[\left(\frac{y}{r}\right)\hat{r}-\hat{y}\right]^2\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right.\\ &\left.+\underbrace{2\left[\left(\frac{x}{r}\right)\hat{r}-\hat{x}\right]\bullet\left[\left(\frac{y}{r}\right)\hat{r}-\hat{y}\right]\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\left(t-\frac{r}{c}\right)\right]}_{=\vec{p}_x\leftrightarrow\vec{p}_y\ \textit{interference}\ \text{term !!!}}\right\}\end{aligned}}$$

<u>Noting that</u>: $\boxed{\hat{x}\bullet\vec{r}=x}$ <u>and</u>: $\boxed{\hat{y}\bullet\vec{r}=y}$:

<u>Then</u>: $\boxed{\left[\left(\frac{x}{r}\right)\hat{r}-\hat{x}\right]^2 = \left[\left(\frac{x}{r^2}\right)\vec{r}-\hat{x}\right]\bullet\left[\left(\frac{x}{r^2}\right)\vec{r}-\hat{x}\right] = \left(\frac{x^2}{r^2}\right)-2\left(\frac{x}{r}\right)^2+1 = 1-\left(\frac{x}{r}\right)^2}$

<u>And</u>: $\boxed{\left[\left(\frac{y}{r}\right)\hat{r}-\hat{y}\right]^2 = \left[\left(\frac{y}{r^2}\right)\vec{r}-\hat{y}\right]\bullet\left[\left(\frac{y}{r^2}\right)\vec{r}-\hat{y}\right] = \left(\frac{y^2}{r^2}\right)-2\left(\frac{y}{r}\right)^2+1 = 1-\left(\frac{y}{r}\right)^2}$

<u>And</u>: $\boxed{\left[\left(\frac{x}{r}\right)\hat{r}-\hat{x}\right]\bullet\left[\left(\frac{y}{r}\right)\hat{r}-\hat{y}\right] = \frac{xy}{r^2}-\frac{xy}{r^2}-\frac{xy}{r^2} = -\frac{xy}{r^2}}$

Thus:

$$\boxed{\begin{aligned}\left(E_{r_{tot}}^{E(1)}(\vec r,t)\right)^2 \simeq \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \Bigg\{&\left(1-\left(\frac{x}{r}\right)^2\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(1-\left(\frac{y}{r}\right)^2\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right] \\ &-2\left(\frac{xy}{r^2}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\Bigg\}\end{aligned}}$$

Or:

$$\boxed{\begin{aligned}\left(E_{r_{tot}}^{E(1)}(\vec r,t)\right)^2 &\simeq \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \Bigg\{\overbrace{\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]}^{=1} \\ &\quad -\frac{1}{r^2}\left[x^2\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+2xy\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+y^2\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right]\Bigg\} \\ &= \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left\{1-\frac{1}{r^2}\left[x\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+y\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]^2\right\}\end{aligned}}$$

But: $x = r\sin\theta\cos\varphi$ and: $y = r\sin\theta\sin\varphi$

$$\therefore \boxed{\begin{aligned}\left(\vec E_{r_{tot}}^{E(1)}(\vec r,t)\right)^2 &\simeq \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left\{1-\sin^2\theta\underbrace{\left[\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\sin\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]^2}_{=\cos\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]}\right\} \\ &= \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left\{1-\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]\right\}\end{aligned}}$$

Thus, in the "far-zone" limit, where $d \ll \lambda \ll r$, the totally retarded Poynting's vector for the rotating E(1) electric dipole is:

$$\boxed{\vec S_{E(1)_{tot}}^{rad}(\vec r,t) \simeq \frac{1}{\mu_o c}\left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left\{1-\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)-\varphi\right]\right\}\hat r} \quad \left(\frac{Watts}{m^2}\right)$$

Then the <u>time-averaged</u> Poynting's vector in the "far-zone" limit { $d \ll \lambda \ll r$ } for the rotating E(1) electric dipole is:

$$\boxed{\left\langle \vec S_{E(1)_{tot}}^{rad}(\vec r)\right\rangle \simeq \frac{1}{\mu_o c}\left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left[1-\frac{1}{2}\sin^2\theta\right]\hat r} \quad \left(\frac{Watts}{m^2}\right)$$

The time-averaged total power radiated <u>per</u> <u>unit</u> <u>solid</u> <u>angle</u> in the "far-zone" limit { $d \ll \lambda \ll r$ } for the rotating dipole is:

$$\boxed{\frac{d\left\langle P_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle}{d\Omega} = r^2\left\langle \vec{S}_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle \bullet \hat{r} \simeq \frac{1}{\mu_o c}\left(\frac{\mu_o p_o \omega^2}{4\pi}\right)^2\left[1-\frac{1}{2}\sin^2\theta\right]} \quad \left(\frac{Watts}{steradian}\right)$$

Dipole radiation pattern. l=1, m=+/- 1

Note that the power angular distribution varies as: $\boxed{\left(1-\tfrac{1}{2}\sin^2\theta\right)}$

i.e. is associated with the $\ell=1,\ m=\pm 1$ spherical harmonic $\boxed{Y_{\ell=1}^{m=\pm 1}(\theta,\varphi)}$

$\Rightarrow$ *z*-component of *EM* angular momentum $\boxed{L_z \neq 0}$ here !

The <u>total</u> time-averaged power radiated into 4π steradians in the "far-zone" limit { $d \ll \lambda \ll r$ } is:

$$\boxed{\begin{aligned}
\left\langle P_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle &= \int \frac{d\left\langle P_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle}{d\Omega} d\Omega = \int_S \left\langle \vec{S}_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle \bullet d\vec{a}_\perp \\
&= \frac{\mu_o}{c}\left(\frac{p_o\omega^2}{4\pi}\right)^2 \int \frac{1}{\cancel{r^2}}\left(1-\frac{1}{2}\sin^2\theta\right)\cancel{r^2}\sin\theta d\theta d\varphi \\
&= \frac{\mu_o p_o^2\omega^4}{16\pi^2 c} 2\pi\left[\int_0^\pi \sin\theta d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta d\theta\right] \\
&= \frac{\mu_o p_o^2\omega^4}{8\pi c}\left(2-\frac{1}{2}\cdot\frac{4}{3}\right) = \frac{\mu_o p_o^2\omega^4}{8\pi c}\left(2-\frac{2}{3}\right) = \frac{\mu_o p_o^2\omega^4}{8\pi c}\left(\frac{4}{3}\right) = \frac{\mu_o p_o^2\omega^4}{6\pi c}
\end{aligned}}$$

Thus, we see that $\boxed{\left\langle P_{\mathrm{E}(1)_{tot}}^{rad}(\vec{r},t)\right\rangle = \frac{\mu_o p_o^2\omega^4}{6\pi c} = 2\times\left\langle P_{\mathrm{E}(1)_z}^{rad}(\vec{r},t)\right\rangle} = 2\times$ times the time-averaged radiated power (in the "far-zone" limit) for a <u>single</u> E(1) oscillating electric dipole.

Note that <u>in</u> <u>general</u>, using the principle of linear superposition: $\boxed{\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2}$ and $\boxed{\vec{B}_{tot} = \vec{B}_1 + \vec{B}_2}$ thus, the total Poynting's vector is:

$$\boxed{\begin{aligned}
\vec{S}_{tot} &= \frac{1}{\mu_o}\vec{E}_{tot}\times\vec{B}_{tot} = \frac{1}{\mu_o}\left[\left(\vec{E}_1+\vec{E}_2\right)\times\left(\vec{B}_1\times\vec{B}_2\right)\right] \\
&= \frac{1}{\mu_o}\left[\vec{E}_1\times\vec{B}_1 + \vec{E}_2\times\vec{B}_2 + \vec{E}_1\times\vec{B}_2 + \vec{E}_2\times\vec{B}_1\right]
\end{aligned}}$$

<u>Or</u>: $\boxed{\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2 + \frac{1}{\mu_o}\left(\vec{E}_1\times\vec{B}_2\right) + \frac{1}{\mu_o}\left(\vec{E}_2\times\vec{B}_1\right)}$

<u>In</u> <u>general</u> , the cross terms {interference terms} in Poynting's vector will ***<u>not</u>*** always cancel!!!

In the case (*<u>here</u>*) with the <u>rotating</u> physical electric dipole, they <u>do</u> vanish, because the fields of 1) and 2) are 90^o out of phase with each other, the cross-term(s) <u>vanish</u> in the time-averaging procedure. $\Rightarrow$ Total power: $\boxed{P_{ToT}^{E(1)} = P_1^{E(1)} + P_2^{E(2)}}$ {<u>*here*</u>}.

***EM* Radiation – Low-Order Angular Distributions:**

Table 9.1 Some Angular Distributions: $|\mathbf{X}_{lm}(\theta, \phi)|^2$

l	m = 0	m = ±1	m = ±2
1 Dipole	$\frac{3}{8\pi}\sin^2\theta$	$\frac{3}{16\pi}(1 + \cos^2\theta)$	
2 Quadrupole	$\frac{15}{8\pi}\sin^2\theta\cos^2\theta$	$\frac{5}{16\pi}(1 - 3\cos^2\theta + 4\cos^4\theta)$	$\frac{5}{16\pi}(1 - \cos^4\theta)$

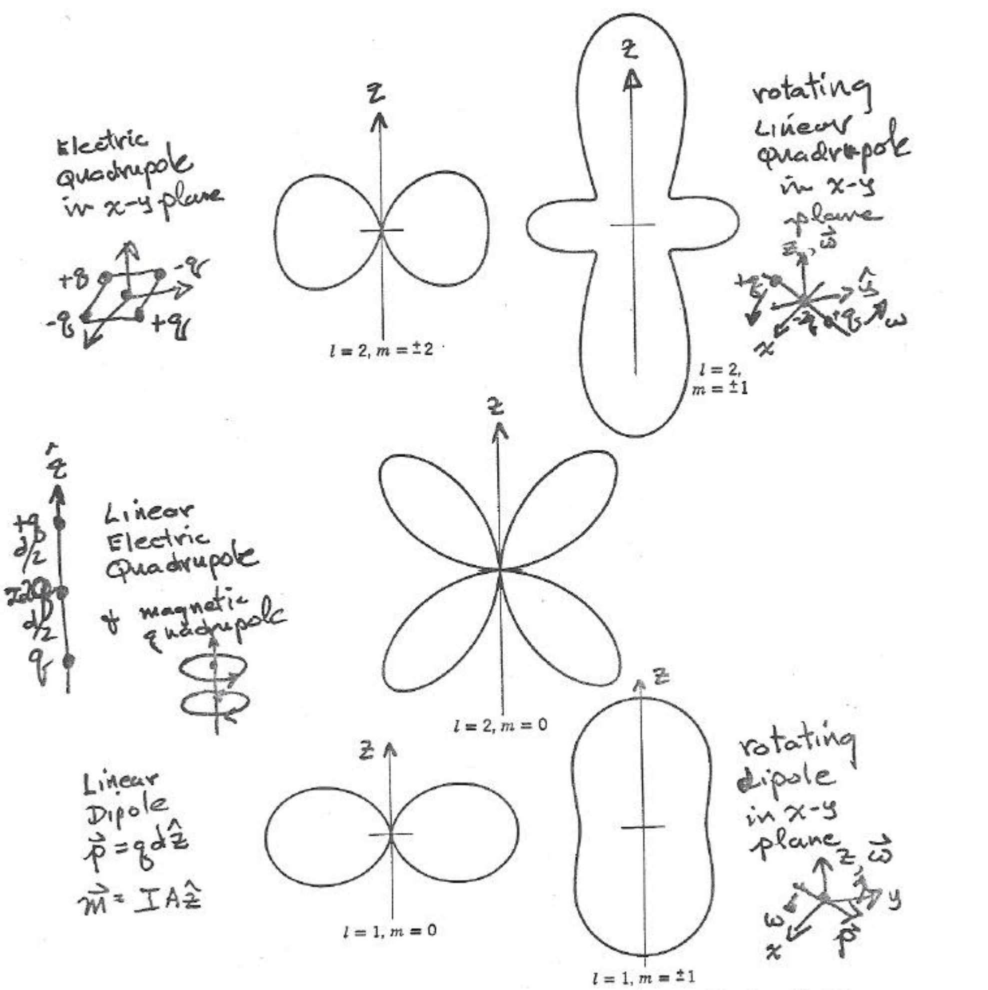
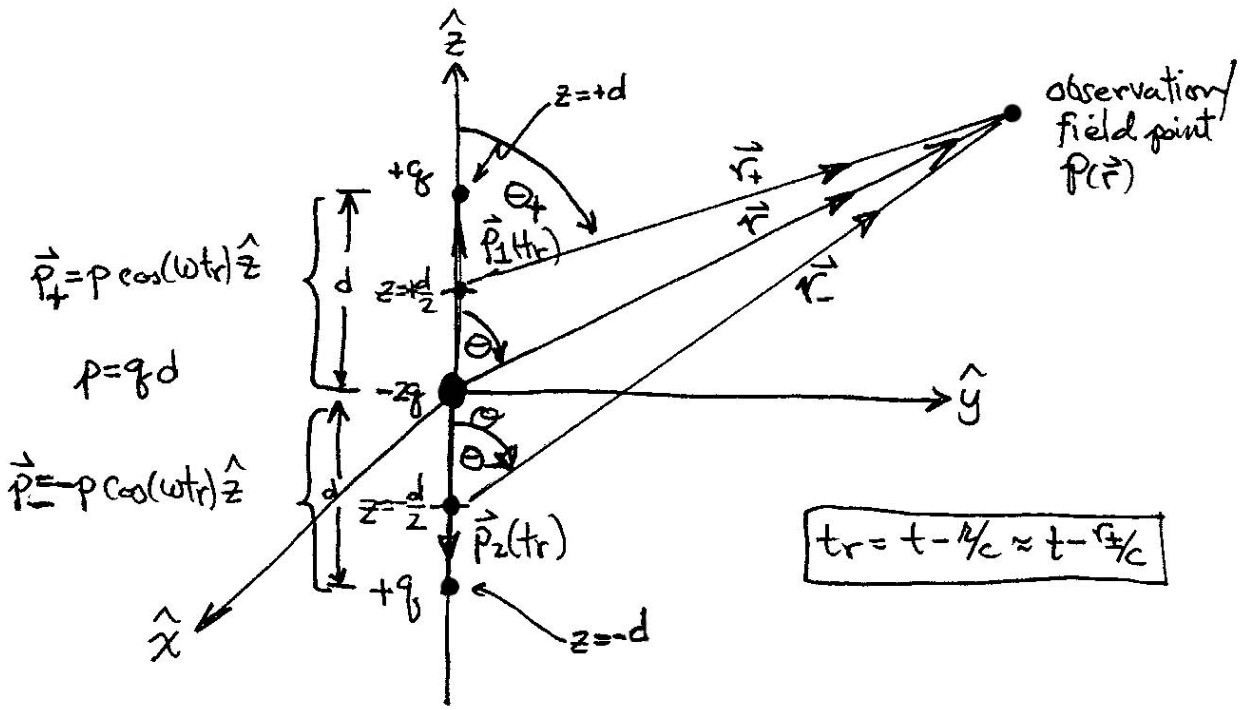


Figure 9.5 Dipole and quadrupole radiation patterns for pure (l, m) multipoles.

"Far-Zone" *EM* Radiation Fields Associated with a Oscillating Linear Electric Quadrupole, E(2)

Griffith's Problem 11.11:

Construct a linear electric quadrupole from two opposing linear electric dipoles.
Take two oppositely-oriented oscillating dipoles, one with $\vec{p}_{+}(t_r) = p\cos(\omega t_r)\hat{z}$ {where $p = qd$} with its center located at $z_1 = +d/2$ and another with $\vec{p}_{-}(t_r) = -p\cos(\omega t_r)\hat{z}$ with its center located at $z_2 = -d/2$ as shown in the figure below:



For *EM* radiation associated with this linear oscillating E(2) electric quadrupole in the "far-zone" limit { $d \ll \lambda \ll r$ }, keeping terms only to first order in d/r, *i.e.* $\boxed{\dfrac{\omega}{c} \gg \dfrac{1}{r}}$, using the principle of linear superposition, the total(ly) retarded scalar potential is: $\boxed{V_{r_{tot}}^{E(2)}(\vec{r},t) = V_{r_+}^{E(1)}(\vec{r},t) + V_{r_-}^{E(1)}(\vec{r},t)}$

where: $\boxed{V_{r_\pm}^{E(1)}(\vec{r},t) \simeq \mp \dfrac{p\omega}{4\pi\varepsilon_o c}\left(\dfrac{\cos\theta_\pm}{r_\pm}\right)\sin\left[\omega\left(t - \dfrac{r_\pm}{c}\right)\right]}$ (see P436 Lecture Notes 13.5, *p*. 10)

1) Now: $\boxed{r_\pm = \sqrt{r^2 + \left(d/2\right)^2 \mp 2r\left(d/2\right)\cos\theta} = r\sqrt{1 + \left(d/2r\right)^2 \mp \left(d/r\right)\cos\theta}}$

$\boxed{r_\pm \simeq r\sqrt{1 \mp \underbrace{\left(d/r\right)}_{\ll 1}\cos\theta}}$ but $\boxed{d/r \ll 1}$, we will keep only linear terms in $\left(d/r\right)$

$$\boxed{\frac{1}{r_\pm} \simeq \frac{1}{r\sqrt{1 \mp \underbrace{\left(d/r\right)}_{\ll 1}\cos\theta}} \approx \frac{1}{r}\left(1 \pm \left(d/2r\right)\cos\theta\right)}$$

2) And:

$$\boxed{\begin{aligned}\cos\theta_{\pm} &= \frac{r\cos\theta \mp \left(d/2\right)}{r_{\pm}} \simeq \cancel{r}\left(\cos\theta \mp \frac{d}{2r}\right)\frac{1}{\cancel{r}}\left(1 \pm \frac{d}{2r}\cos\theta\right)\\ &= \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r} - \underbrace{\left(\frac{\cancel{d}}{\cancel{2}r}\right)^2}_{\ll 1}\cos\theta \simeq \cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r}\\ &\simeq \cos\theta \mp \left(\frac{d}{2r}\right)\left(1-\cos^2\theta\right) = \cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta\end{aligned}}$$

3) Then:

$$\boxed{\begin{aligned}\sin\left[\omega\left(t - r_{\pm}/c\right)\right] &\simeq \sin\left\{\omega\left[t - \frac{r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right]\right\}\\ &\simeq \sin\left[\omega\left(t - r/c\right) \pm \left(\frac{\omega d}{2c}\right)\cos\theta\right] \quad \text{but:} \quad \sin\left(A \pm B\right) = \sin A\cos B \pm \cos A \sin B\\ &\simeq \sin\left[\omega\left(t - r/c\right)\right]\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \pm \cos\left[\omega\left(t - r/c\right)\right]\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right]\end{aligned}}$$

But: $\boxed{\left(\frac{\omega d}{c}\right) \ll 1}$ in the "far-zone" limit $\{ d \ll \lambda \ll r \}$ $\therefore$ $\boxed{\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx 1}$ since: $\boxed{\cos\left(\approx 0\right) \approx 1}$

And: $\boxed{\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \left(\frac{\omega d}{2c}\right)\cos\theta}$ since $\boxed{\sin\alpha \approx \alpha}$ for $\boxed{\alpha \ll 1}$.

$$\therefore \boxed{\sin\left[\omega\left(t - r_{\pm}/c\right)\right] \approx \sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]}$$

Then, keeping only terms linear in $\left(d/r\right)$:

$$\boxed{\begin{aligned}V^{\mathrm{E}(1)}_{r_{\pm}}\left(\vec{r},t\right) &\simeq \mp\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta_{\pm}}{r_{\pm}}\right)\sin\left[\omega\left(t - \frac{r_{\pm}}{c}\right)\right]\\ &\simeq \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)\left(\cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta\right)\\ &\qquad \times\left[\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left(\omega\left[t - r/c\right]\right)\right]\\ &\simeq \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\left(\cos\theta \mp \left(\frac{d}{2r}\right)\sin^2\theta \pm \left(\frac{d}{2r}\right)\cos^2\theta - \left(\frac{\cancel{d}}{\cancel{2}r}\right)^2\sin^2\theta\cos\theta\right)\right.\\ &\qquad \left.\times\left[\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right]\right\}\end{aligned}}$$

Thus:

$$\boxed{\begin{aligned} V_{r_\pm}^{E(1)}(\vec{r},t) &\simeq \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left(\cos\theta \pm\left(\frac{d}{2r}\right)\left(\cos^2\theta-\sin^2\theta\right)\right) \\ &\qquad\times\left[\sin\left[\omega\left(t-r/c\right)\right]\pm\left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t-r/c\right)\right]\right] \\ &\approx \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cos\theta\sin\left[\omega\left(t-r/c\right)\right]\pm\left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]\right. \\ &\qquad\pm\left(\frac{d}{2r}\right)\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right] \\ &\qquad\left.+\underbrace{\cancel{\left(\frac{d}{2r}\right)\left(\frac{\omega d}{2c}\right)}}_{\ll 1}\left(\cos^2\theta-\sin^2\theta\right)\cos\theta\cos\left[\omega\left(t-r/c\right)\right]\right\} \end{aligned}}$$

Finally:

$$\boxed{\begin{aligned} V_{r_\pm}^{E(1)}(\vec{r},t) &\approx \mp\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cos\theta\sin\left[\omega\left(t-r/c\right)\right]\pm\left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]\right. \\ &\qquad\left.\pm\left(\frac{d}{2r}\right)\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right]\right\} \end{aligned}}$$

Then: $\boxed{V_{r_{Tot}}^{E(2)}(\vec{r},t)=V_{r_+}^{E(1)}(\vec{r},t)+V_{r_-}^{E(1)}(\vec{r},t)}$

$$\boxed{\begin{aligned} V_{r_{tot}}^{E(2)}(\vec{r},t) &\simeq -\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cancel{\cos\theta\sin\left[\omega\left(t-r/c\right)\right]}+\left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]\right. \\ &\qquad\left.+\left(\frac{d}{2r}\right)\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right]\right\} \\ &\quad+\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\cancel{\cos\theta\sin\left[\omega\left(t-r/c\right)\right]}-\left(\frac{\omega d}{2c}\right)\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]\right. \\ &\qquad\left.-\left(\frac{d}{2r}\right)\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right]\right\} \\ &\simeq -\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\left(\frac{\omega d}{c}\right)\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]-\left(\frac{d}{r}\right)\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right]\right\} \\ &\simeq -\frac{p\omega^2 d}{4\pi\varepsilon_o c^2 r}\left\{\cos^2\theta\cos\left[\omega\left(t-r/c\right)\right]-\underbrace{\cancel{\left(\frac{c}{\omega r}\right)}}_{\ll 1}\left(\cos^2\theta-\sin^2\theta\right)\sin\left[\omega\left(t-r/c\right)\right]\right\} \end{aligned}}$$

In the "far-zone" limit { $d \ll \lambda \ll r$ } $\boxed{\left(\frac{c}{\omega}\right)\ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right)\ll 1}$, keep only linear terms!

$\therefore$ In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{V_{r_{tot}}^{E(2)}(\vec{r},t) \simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]}$$

<u>Now let's work on obtaining</u>: $\boxed{\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) = \vec{A}_{r_+}^{E(1)}(\vec{r},t) + \vec{A}_{r_-}^{E(1)}(\vec{r},t)}$

<u>Where</u>: $\boxed{\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) \simeq \mp\frac{\mu_o p\omega}{4\pi r_\pm c}\sin\left[\omega\left(t - r_\pm/c\right)\right]\hat{z}}$ (see P436 Lecture Notes 13.5, p.10)

Carrying out the same methodology as above, keeping only linear terms in $\left(\frac{d}{r}\right)$ and $\left(\frac{\omega d}{c}\right)$:

$$\begin{aligned}
\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) &\simeq \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)\left\{\sin\left[\omega(t - r/c\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right\} \\
&\simeq \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left\{\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right]\right. \\
&\qquad \left. \pm\left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right] + \underbrace{\cancel{\left(\frac{d}{2r}\right)\left(\frac{\omega d}{2c}\right)}}_{\ll 1}\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}$$

$$\boxed{\vec{A}_{r_\pm}^{E(1)}(\vec{r},t) \simeq \mp\frac{\mu_o p\omega\hat{z}}{4\pi r}\left\{\sin\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] \pm \left(\frac{d}{2r}\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right)\right\}}$$

Then in the "far-zone" limit $\{ d \ll \lambda \ll r \}$, with $\boxed{\left(\frac{c}{\omega}\right) \ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right) \ll 1}$:

$$\begin{aligned}
\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) &\simeq -\left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\cancel{\sin\left[\omega\left(t - r/c\right)\right]} + \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] + \left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\quad + \left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\cancel{\sin\left[\omega\left(t - r/c\right)\right]} - \left(\frac{\omega d}{2c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] - \left(\frac{d}{2r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\simeq -\left(\frac{\mu_o p\omega\hat{z}}{4\pi r}\right)\left\{\left(\frac{\omega d}{c}\right)\cos\theta\cos\left[\omega\left(t - r/c\right)\right] + \left(\frac{d}{r}\right)\cos\theta\sin\left[\omega\left(t - r/c\right)\right]\right\} \\
&\simeq -\left(\frac{\mu_o p\omega^2 d}{4\pi c r}\right)\cos\theta\left\{\cos\left[\omega\left(t - r/c\right)\right] + \underbrace{\cancel{\left(\frac{c}{\omega r}\right)}}_{\ll 1}\sin\left[\omega\left(t\, r/c\right)\right]\right\}\hat{z}
\end{aligned}$$

$\therefore$ In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\vec{A}_{r_{tot}}^{E(2)}(\vec{r},t) \simeq -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z}}$$

Then the totally retarded electric and magnetic fields associated with E(2) "far-zone" *EM* radiation from an oscillating linear electric quadrupole are:

$$\boxed{\vec{E}^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) = -\vec{\nabla}V^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t)}{\partial t}} \quad \text{and} \quad \boxed{\vec{B}^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t)}$$

Note that: $\boxed{V^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) \simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]}$ has no explicit φ-dependence

Note that: $\boxed{\vec{A}^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) \simeq -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z}}$ also has no explicit φ-dependence,

and also note that $A^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) = \{\ \}\hat{z}$, *i.e.* $A^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) \parallel \hat{z}$.

Now:

$$\begin{aligned}
\vec{\nabla}V^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) &= \frac{\partial V^{\mathrm{E(2)}}_{\mathrm{r}_{Tot}}(\vec{r},t)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V^{\mathrm{E(2)}}_{\mathrm{r}_{Tot}}(\vec{r},t)}{\partial \theta}\hat{\theta} \\
&\simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\cos^2\theta\left\{-\frac{1}{r^2}\cos\left[\omega\left(t - r/c\right)\right] + \frac{1}{r}\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t - r/c\right)\right]\right\}\hat{r} \\
&\quad -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\frac{-2\cos\theta\sin\theta}{r^2}\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta} \\
&\simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left\{\frac{1}{r}\left(\frac{\omega}{c}\right)\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} - \frac{1}{r^2}\cos^2\theta\cos\left[\omega\left(t - r/c\right)\right]\hat{r}\right. \\
&\qquad\left. - \frac{1}{r^2}\left(2\cos\theta\sin\theta\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\} \\
&\simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^2}\right)\left(\frac{1}{r}\right)\left(\frac{\omega}{c}\right)\left\{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r}\right. \\
&\qquad\left. - \underbrace{\left(\frac{c}{\omega r}\right)}_{\ll 1}\cos\theta\left[\cos\theta\hat{r} + 2\sin\theta\hat{\theta}\right]\cos\left[\omega\left(t - r/c\right)\right]\right\}
\end{aligned}$$

In the "far-zone" limit { $d \ll \lambda \ll r$ }, $\boxed{\left(\frac{c}{\omega}\right) \ll r}$ or: $\boxed{\left(\frac{c}{\omega r}\right) \ll 1}$.

$\therefore$ In the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\vec{\nabla}V^{\mathrm{E(2)}}_{\mathrm{r}_{tot}}(\vec{r},t) \simeq -\left(\frac{p\omega^2 d}{4\pi\varepsilon_o c^3}\right)\left(\frac{\cos^2\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{r}}$$

Next: $\boxed{\frac{\partial \vec{A}^{E(2)}_{r_{tot}}}{\partial t}(\vec{r},t) \simeq +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{z}}$ But: $\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$

$$\therefore \quad \boxed{\frac{\partial \vec{A}^{E(2)}_{r_{tot}}}{\partial t}(\vec{r},t) \simeq +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]}$$

Then: $\boxed{\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) = -\vec{\nabla}V^{E(2)}_{r_{tot}}(\vec{r},t) - \frac{\partial \vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)}{\partial t}}$

Thus:

$$\boxed{\begin{aligned}\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq &+\left(\frac{p\omega^3 d}{4\pi\varepsilon_o c^3 r}\right)\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} \\ &-\left(\frac{\mu_o p\omega^3 d}{4\pi c r}\right)\left\{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{r} - \cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\}\end{aligned}}$$

But: $\boxed{c^2 = \frac{1}{\varepsilon_o\mu_o}}$ or: $\boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$

$$\therefore \quad \boxed{\begin{aligned}\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq &\left(\frac{\mu_o p\omega^3 d}{4pcr}\right)\cancel{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]}\hat{r} \\ &-\left(\frac{\mu_o p\omega^3 d}{4\pi c r}\right)\left\{\cancel{\cos^2\theta\sin\left[\omega\left(t - r/c\right)\right]}\hat{r} - \cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}\right\}\end{aligned}}$$

n.b. $\vec{\nabla}V^{E(2)}_{r}$ term cancels with 1^{st} $\partial\vec{A}^{E(2)}_{r}/\partial t$ term!!!

Thus, in the "far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$, $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq +\left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}}$$

Now let's work on obtaining: $\boxed{\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) = \vec{\nabla}\times\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)}$

Note that $\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t)$ has only a $\hat{z}$ component and has no explicit φ-dependence.

Note also that: $\boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$.

In spherical coordinates:

$$\boxed{\begin{aligned}\vec{B}^{E(2)}_{r_{tot}} = \vec{\nabla}\times\vec{A}^{E(2)}_{r_{tot}} = &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\cancel{A_\varphi}\right) - \cancel{\frac{\partial A_\theta}{\partial\varphi}}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\varphi}} - \frac{\partial}{\partial r}\left(r\,\cancel{A_\varphi}\right)\right]\hat{\theta} \\ &+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}\end{aligned}}$$

$$\therefore \boxed{\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA^{E(2)}_{\theta}\right) - \frac{\partial A^{E(2)}_{r}}{\partial \theta}\right]\hat{\varphi}} \qquad \boxed{\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}}$$

<u>where</u>: $\boxed{\vec{A}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq -\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{z}}$ in the "far-zone" limit, { $d \ll \lambda \ll r$ }.

Thus, in the"far-zone" limit { $d \ll \lambda \ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$:

$$\boxed{\begin{aligned}\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) &\simeq -\left(\frac{\mu_o p\omega^2 d}{4\pi c r}\right)\left\{-\left(\frac{\omega}{c}\right)\cos\theta\sin\theta\sin\left[\omega\left(t - r/c\right)\right] + \frac{1}{r}(2\cos\theta\sin\theta)\cos\left[\omega\left(t - r/c\right)\right]\right\}\hat{\varphi} \\ &\simeq +\left(\frac{\mu_o p\omega^3 d}{4\pi c^2 r}\right)\cos\theta\sin\theta\left\{\sin\left[\omega\left(t - r/c\right)\right] - \underbrace{\left(\frac{c}{\omega r}\right)}_{\ll 1}2\cos\left[\omega\left(t - r/c\right)\right]\right\}\hat{\varphi}\end{aligned}}$$

$$\therefore \boxed{\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq \left(\frac{\mu_o p\omega^3 d}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\varphi}}$$

<u>Since</u>: $\boxed{\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t) \simeq \left(\frac{\mu_o p\omega^3 d}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]\hat{\theta}}$

<u>Note again that</u>: $\boxed{\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t)} \Leftarrow \boxed{\hat{r}\times\hat{\theta} = \hat{\varphi}}$

<u>Note also that</u>: $\boxed{\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t)\bullet\hat{r} = 0}$, $\boxed{\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t)\bullet\hat{r} = 0}$ and $\boxed{\vec{E}^{E(2)}_{r_{tot}}(\vec{r},t)\bullet\vec{B}^{E(2)}_{r_{tot}}(\vec{r},t) = 0}$

<u>Note also that</u>: $\vec{E}^{E(2)}_{r_{tot}}$ and $\vec{B}^{E(2)}_{r_{tot}}$ have the same angular dependence: <u>$\sim\cos\theta\sin\theta$</u>

$\Rightarrow$ E(2) electric quadrupole $\vec{E}^{E(2)}_{r_{tot}}$ and $\vec{B}^{E(2)}_{r_{tot}}$ fields both vanish for $\boxed{\theta = 0 = \frac{\pi}{2} = \pi}$!!!

<u>Note also that</u>: $\vec{E}^{E(2)}_{r_{tot}}$ and $\vec{B}^{E(2)}_{r_{tot}}$ both vary as <u>$\sim 1/r$</u>

<u>Note also that</u>: $\vec{E}^{E(2)}_{r_{tot}}$ and $\vec{B}^{E(2)}_{r_{tot}}$ are in-phase with each other $\sim\sin\left[\omega\left(t - r/c\right)\right]$

$\Rightarrow$ linearly polarized *EM* radiation from linear/axial electric quadrupole.
{*n.b.* <u>NOT</u> true for <u>all</u> types of electric quadrupoles.}

Now calculate: $u_{\mathrm{E}(2)}^{rad}(\vec{r},t)$, $\vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)$, $\vec{\wp}_{\mathrm{E}(2)}^{rad}(\vec{r},t)$, $\vec{\ell}_{\mathrm{E}(2)}^{rad}(\vec{r},t)$, $P_{\mathrm{E}(2)}^{rad}(\vec{r},t)$ *etc*. for "far-field" *EM* radiation associated with linear oscillating E(2) electric quadrupole:

EM Energy Density for E(2) Linear Oscillating Electric Quadupole:

$$\boxed{u_{\mathrm{E}(2)}^{rad}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\cdot\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right)+\frac{1}{\mu_o}\left(\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\cdot\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right)}\ \left(\frac{Joules}{m^3}\right)$$

$$\begin{aligned}u_{\mathrm{E}(2)}^{rad}(\vec{r},t)&\simeq\frac{1}{2}\left\{\varepsilon_o\left(\frac{\mu_o p\omega^2 d}{4\pi c}\right)^2\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right.\\&\qquad\left.+\frac{1}{\mu_o}\left(\frac{\mu_o p\omega^2 d}{4\pi c^2}\right)^2\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\\&\simeq\frac{1}{2}\left\{\underbrace{\frac{\mu_o p^2\omega^6 d^2}{16\pi^2c^4}}+\underbrace{\frac{\mu_o p^2\omega^6 d^2}{16\pi^2c^4}}\right\}\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-r/c\right)\right]\end{aligned}$$

but: $\boxed{\varepsilon_o=\frac{1}{\mu_o c^2}}$

n.b. EM radiation energy is {again} carried equally by $\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}$ and $\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}$

$$\therefore\quad\boxed{u_{\mathrm{E}(2)}^{rad}(\vec{r},t)\simeq\left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-r/c\right)\right]}\quad\left(\frac{Joules}{m^3}\right)$$

In the "far-zone" limit { $d\ll\lambda\ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

Poynting's Vector for E(2) Linear Oscillating Electric Quadrupole:

$$\boxed{\vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)=\frac{1}{\mu_o}\left(\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}\times\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}\right)\simeq\frac{1}{\mu_o}\left(\frac{\mu_o^2 p^2\omega^6 d^2}{16\pi^2c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-r/c\right)\right]\overbrace{\left(\hat{\theta}\times\hat{\varphi}\right)}^{=\hat{r}}}$$

$$\boxed{\vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\simeq\left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-r/c\right)\right]\hat{r}}\quad\left(\frac{Watts}{m^2}\right)$$

In the "far-zone" limit { $d\ll\lambda\ll r$ }, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

Note again that: $\boxed{\vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)=\vec{c}u_{\mathrm{E}(2)}^{rad}(\vec{r},t)}$ where $\boxed{\vec{c}\equiv c\hat{r}}$, $\boxed{\hat{r}\parallel\hat{k}}$

***EM* Linear Momentum Density for E(2) Linear Oscillating Electric Quadrupole:**

$$\boxed{\vec{\wp}^{\,rad}_{E(2)}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}^{rad}_{E(2)}(\vec{r},t) = \frac{1}{c^2}\vec{S}^{rad}_{E(2)}(\vec{r},t) \simeq \left(\frac{\mu_o p^2\omega^6 d^2}{16\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\sin^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}} \quad \left(\frac{kg}{m^2\text{-}sec}\right)$$

In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

***EM* Angular Momentum Density for E(2) Linear Oscillating Electric Quadrupole:**

$$\boxed{\vec{\ell}^{\,rad}_{E(2)}(\vec{r},t) = \vec{r}\times\vec{\wp}^{\,rad}_{E(2)}(\vec{r},t) = 0} \quad \left(\frac{kg}{m\text{-}sec}\right) \qquad n.b. \text{ exact } \vec{\ell}^{\,rad}_{E(2)}(\vec{r},t) \not\equiv 0.$$

In the "far-zone" limit $\{ d \ll \lambda \ll r \}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$.

Time-Averaged Quantities for E(2) Linear Oscillating Electric Quadrupole Radiation

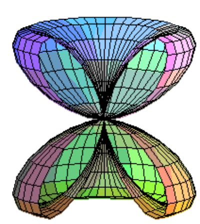
Recall: $\boxed{\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{\tau}\int_0^\tau \sin^2(\omega t)\,dt = \frac{1}{2}}$

Define electric quadrupole moment: $\boxed{Q^e_{zz} \equiv qdd = pd}$ $\left(Coulomb - m^2\right)$

$$\boxed{\left\langle u^{rad}_{E(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p^2 d^2\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right) = \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)} \quad \left(\frac{Joules}{m^3}\right)$$

$$\boxed{I^{rad}_{E(2)}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{E(2)}(\vec{r},t)\right|\right\rangle \simeq \left(\frac{\mu_o p^2 d^2\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r} = \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)} \quad \left(\frac{Watts}{m^2}\right)$$

$$\boxed{I^{rad}_{E(2)}(\vec{r}) \equiv \left\langle\left|\vec{S}^{rad}_{E(2)}(\vec{r},t)\right|\right\rangle = c\left\langle u^{rad}_{E(2)}(\vec{r},t)\right\rangle}$$



Time-averaged radiated power: $\boxed{\left\langle P^{rad}_{E(2)}(\vec{r},t)\right\rangle = \int_{S'}\left\langle\vec{S}^{rad}_{E(2)}(\vec{r},t)\right\rangle\bullet d\vec{a}_\perp}$

where: $\boxed{d\vec{a}_\perp = r^2 d\Omega\,\hat{r}}$ and: $\boxed{d\Omega = \sin\theta d\theta d\varphi}$

$$\left\langle P^{rad}_{E(2)}(\vec{r})\right\rangle \simeq \frac{\mu_o p^2 d^2\omega^6}{\cancel{32}\,\cancel{\pi^2}\,c^3\,\cancel{r^2}}\bullet\cancel{r^2}\bullet\cancel{2\pi}\int_{\theta=0}^{\theta=\pi}\left(\cos^2\theta\sin^2\theta\right)\sin\theta d\theta$$

$$\left\langle P^{rad}_{E(2)}(\vec{r},t)\right\rangle \simeq \frac{\mu_o p^2 d^2\omega^6}{16\pi c^3}\int_{\theta=0}^{\theta=\pi}\cos^2\theta\left(1-\cos^2\theta\right)\sin\theta d\theta = \frac{\mu_o p^2 d^2\omega^6}{16\pi c^3}\int_{\theta=0}^{\theta=\pi}\int_{\theta=0}^{\theta=\pi}\left[\cos^2\theta - \cos^4\theta\right]\sin\theta d\theta$$

Let: $\boxed{u = \cos\theta,\ \ du = -\sin\theta d\theta}$, then: $\boxed{\theta = 0:\ u = +1}$ and: $\boxed{\theta = \pi:\ u = -1}$

Then: $\boxed{\int_{\theta=0}^{\theta=\pi}\left[\cos^2\theta-\cos^4\theta\right]\sin\theta d\theta=\int_{u=-1}^{u=+1}\left(u^2-u^4\right)du=\left(\frac{1}{3}u^3-\frac{1}{5}u^5\right)\Big|_{-1}^{+1}=\frac{2}{3}-\frac{2}{5}=\frac{10-6}{15}=\frac{4}{15}}$

$$\therefore \quad \boxed{\left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle \simeq\left(\frac{\cancel{4}}{15}\right)\frac{\mu_o p^2 d^2\omega^6}{\underset{4}{\cancel{16}}\,\pi c^3}=\frac{\mu_o p^2 d^2\omega^6}{60\pi c^3}=\frac{\mu_o Q_{zz}^{e^2}\omega^6}{60\pi c^3}} \quad \text{(Watts)}$$

Note that time-averaged E(2) *EM* power radiated in the "far-zone" limit is proportional to the square of the electric quadrupole moment $\boxed{Q_{zz}^e=pd=qdd=qd^2}$ $\left(Coulomb-m^2\right)$

Note also that time averaged E(2) *EM* radiated power $\sim\omega^6$ (*cf*. with $\sim\omega^4$ for E(1) *EM* radiation.

The time-averaged *EM* angular power radiated by E(2) linear electric quadrupole $\left(\ell=2,\ m=0\ Y_\ell^m\right)$:

$$\boxed{\frac{d\left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle}{d\Omega}\equiv\left\langle \vec{S}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle\bullet r^2\hat{r}\simeq\left(\frac{\mu_o p^2 d^2\omega^6}{32\pi^2 c^4}\right)\cos^2\theta\sin^2\theta=\left(\frac{\mu_o Q_{zz}^{e^2}\omega^6}{32\pi^2 c^3}\right)\cos^2\theta\sin^2\theta} \quad \left(\frac{Watts}{steradian}\right)$$

Note that $\boxed{d\left\langle P_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle\big/d\Omega}$ has zeros when $\boxed{\theta=0=\pi/2=\pi}$!!!

The time-averaged E(2) *EM* linear momentum density in the "far-zone" limit is:

$$\boxed{\left\langle \vec{\wp}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle\simeq\left(\frac{\mu_o p^2\omega^6 d^2}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}=\left(\frac{\mu_o Q_{zz}^{e^2}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}} \quad \left(\frac{kg}{m^2\text{-}sec}\right)$$

The time-averaged E(2) *EM* angular momentum density in the "far-zone" limit is:

$$\boxed{\left\langle \vec{\ell}_{\mathrm{E}(2)}^{rad}(\vec{r},t)\right\rangle=0} \quad \left(\frac{kg}{m\text{-}sec}\right)$$

The characteristic impedance of an E(2) oscillating linear electric quadrupole antenna:

$$\boxed{Z_{antenna}^{\mathrm{E}(2)}(\vec{r})\equiv\frac{\left|\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}{\left|\vec{H}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}=\frac{\left|\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}{\frac{1}{\mu_o}\left|\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right|}=\mu_o c=\sqrt{\frac{\mu_o}{\varepsilon_o}}=Z_o=120\pi\,\Omega\simeq 377\,\Omega}$$

The radiation resistance of an E(2) oscillating linear electric quadrupole antenna:
Recall $I_o=q\omega$ for linear electric dipole (also true here)

$$\boxed{\left\langle P_{\mathrm{E}(2)}^{rad}\right\rangle\simeq\frac{\mu_o p^2 d^2\omega^6}{60\pi c^3}\equiv I_o^2 R_{\mathrm{E}(2)}^{rad}} \Rightarrow \boxed{R_{\mathrm{E}(2)}^{rad}=\frac{\left\langle P_{\mathrm{E}(2)}^{rad}\right\rangle}{I_o^2}\simeq\frac{\mu_o p^2 d^2\omega^6}{60\pi c^3 I_o^2}} \quad \underline{\text{but}}:\ \boxed{p=qd}$$

$$\therefore \quad \boxed{R_{\mathrm{E}(2)}^{rad}\simeq\frac{\mu_o \cancel{q^2} d^4\omega^{\cancel{6}\,4}}{60\pi c^3\,\cancel{q^2}\,\cancel{\omega^2}}\simeq\frac{\mu_o\omega^4 d^4}{60\pi c^3}=\frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4\mu_o c=\frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o}$$

Where: $\boxed{Z_o = \mu_o c = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega}$ = impedance of free space / vacuum

But: $\boxed{\left(\frac{\omega d}{c}\right) \ll 1}$ in the "far-zone" limit { $d \ll \lambda \ll r$ }, thus we see that:

$$\boxed{R_{\mathrm{E}(2)}^{rad} = \frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o \ll Z_o \simeq 377\,\Omega}$$

Comparison of *EM* Quantities for E(1) Oscillating Linear Electric Dipole vs. E(2) Oscillating Linear Electric Quadrupole in the "far-zone" limit $d \ll \lambda \ll r$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$

	$\boxed{\ell = 1,\ m = 0}$ **Oscillating E(1) Linear Electric Dipole**	$\boxed{\ell = 2,\ m = 0}$ **Oscillating E(2) Linear Electric Quadrupole**
Moments	$\vec{p}(t) = q(t)\vec{d},\ \vec{d} = dz,\ p = qd$	$\vec{\vec{Q}}^e_{zz} = q(t)\vec{d}\vec{d},\ Q^e_{zz} = qdd$
Retarded Scalar Potential	$V_r^{E(1)}(\vec{r},t) \simeq -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$	$V_r^{E(2)}(\vec{r},t) \simeq -\left(\frac{Q^e_{zz}\omega^2}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$
Retarded Vector Potential	$\vec{A}_r^{E(1)}(r,t) \simeq -\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$	$\vec{A}_r^{E(2)}(\vec{r},t) \simeq -\left(\frac{\mu_o Q^e_{zz}\omega^2}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$
	$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$	
Retarded Electric Field	$\vec{E}_r^{E(1)}(\vec{r},t) \simeq -\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$	$\vec{E}_r^{E(2)}(\vec{r},t) \simeq +\left(\frac{\mu_o Q^e_{zz}\omega^3}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Retarded Magnetic Field	$\vec{B}_r^{E(1)}(\vec{r},t) \simeq -\frac{\mu_o p\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$	$\vec{B}_r^{E(2)}(\vec{r},t) \simeq +\left(\frac{\mu_o Q^e_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Time-Avg'd *EM* Energy Density	$\left\langle u^{rad}_{E(1)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$\left\langle u^{rad}_{E(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting's Vect/Intensity	$I^{rad}_{E(1)}(\vec{r}) \equiv \left\langle\left\vert\vec{S}^{rad}_{E(1)}(\vec{r},t)\right\vert\right\rangle \simeq \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$I^{rad}_{E(2)}(\vec{r}) \equiv \left\langle\left\vert\vec{S}^{rad}_{E(2)}(\vec{r},t)\right\vert\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\left\langle P^{rad}_{E(1)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right)$	$\left\langle P^{rad}_{E(2)}(\vec{r},t)\right\rangle \simeq \frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}$
Time-Avg'd *EM* Linear Momentum Density	$\left\langle\vec{\wp}^{rad}_{E(1)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$	$\left\langle\vec{\wp}^{rad}_{E(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\left\langle\vec{\ell}^{rad}_{E(1)}(\vec{r},t)\right\rangle = 0$	$\left\langle\vec{\ell}^{rad}_{E(2)}(\vec{r},t)\right\rangle = 0$
Characteristic Antenna Impedance	$Z^{E(1)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$ =	$Z^{E(2)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$
Antenna Radiation Resistance	$R^{E(1)}_{rad} \simeq \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o$	$R^{E(2)}_{rad} = \frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o$

Note the ratios of *EM* power radiated:

$$\boxed{\left\langle P^{rad}_{\mathrm{E}(2)}\right\rangle \Big/ \left\langle P^{rad}_{\mathrm{E}(1)}\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}\right) \Big/ \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right) \simeq \frac{1}{5}\left(\frac{Q^e_{zz}}{p}\right)^2 \frac{\omega^2}{c^2} = \frac{1}{5}\left(\frac{qdd}{qd}\right)^2 \frac{\omega^2}{c^2} = \frac{1}{5}\left(\frac{\omega d}{c}\right)^2 \ll 1}$$

Recall/Compare to:

$$\boxed{\left\langle \mathrm{P}^{rad}_{\mathrm{M}(1)}\right\rangle \Big/ \left\langle \mathrm{P}^{rad}_{\mathrm{E}(1)}\right\rangle \simeq \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right) \Big/ \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right) = \left(\frac{\omega b}{c}\right)^2 \ll 1}$$ where $\boxed{p = qd}$ and $\boxed{m = \pi b^2 I_o}$,

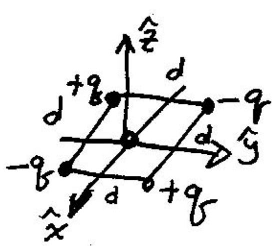
and $\boxed{I_o = q\omega}$ and $\boxed{d = \pi b}$ or: $\boxed{b = d/\pi}$.

Then: $$\boxed{\left\langle P^{rad}_{\mathrm{M}(1)}\right\rangle \Big/ \left\langle P^{rad}_{\mathrm{E}(2)}\right\rangle \simeq \left(\frac{\mu_o m^2\omega^4}{12\pi c^3}\right) \Big/ \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}\right) \simeq 5\left(\frac{b}{d}\right)^2 \approx 5\left(\frac{1}{\pi}\right)^2 \approx \frac{5}{\pi^2} \sim \frac{1}{2} \sim \vartheta(1)}$$!!!

General comments for the ℓ^{th}-order, $m = 0$ electric multipole in "far-zone" limit, $d \ll \lambda \ll r$:

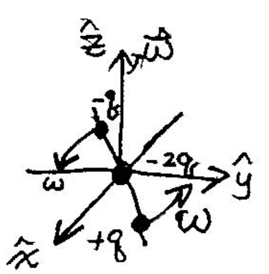
Each successive/higher power of ℓ brings in a multiplicative factor of $(\omega d/c) \ll 1$ to the retarded *EM* potentials and retarded *EM* fields, and thus brings in a multiplicative factor of $(\omega d/c)^2 \ll 1$ to the retarded *EM* energy densities, Poynting's vector, *EM* power radiated/ *EM* intensity, *EM* linear momentum density, etc.

- By similar methodology of above {plus suitable space-rotations}, we can obtain all of the above results for the oscillating E(2) quadrupole *e.g.* lying in the *x*-*y* plane as shown in the figure below:

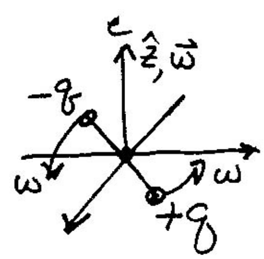


$\ell = 2,\ m = \pm 1$

- Similarly, we can also *e.g.* take the linear E(2) electric quadrupole (along $\hat{z}$ axis) $\Rightarrow$ place it in the *x*-*y* plane and have it rotate at angular frequency ω:



$\Rightarrow$ Get E(2) $\ell = 2, m = \pm 1$ results, analogous to linear E(1) electric dipole rotating in *x*-*y* plane $(\ell = 1, m = \pm 1)$:

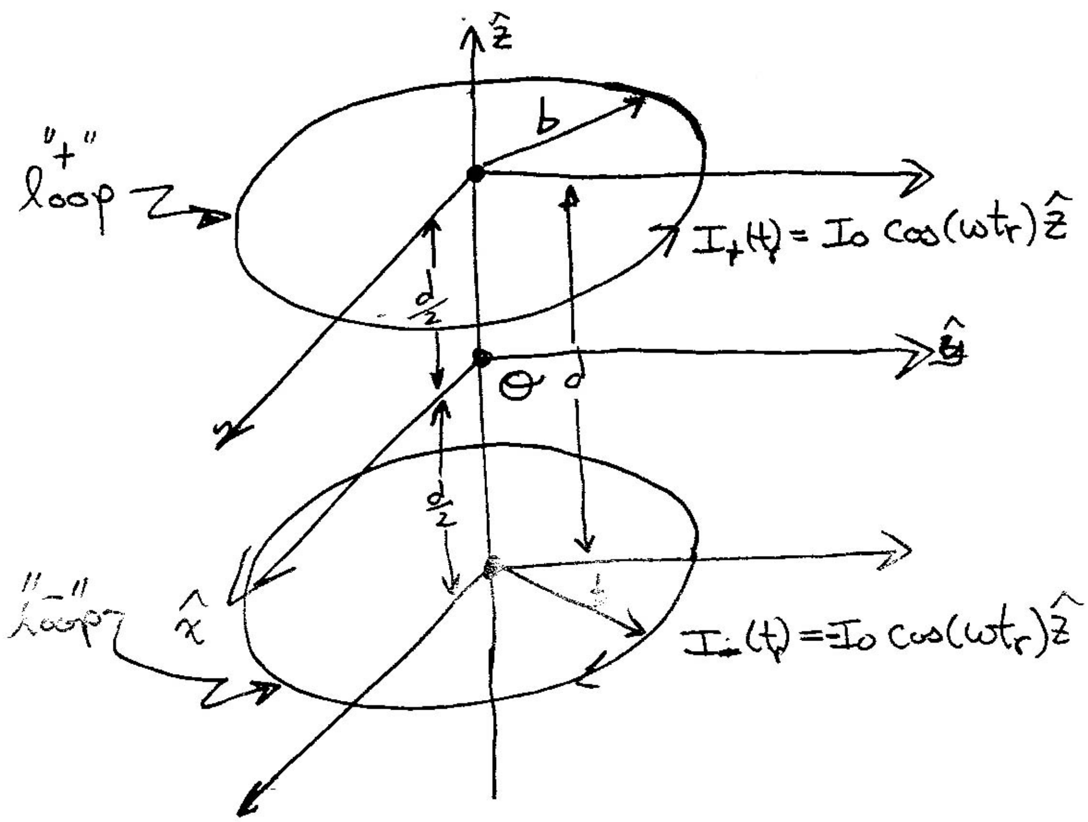


$\Rightarrow$ See angular distribution radiation patterns on page 8 of these P436 Lecture Notes.

"Far-Zone" *EM* Radiation Fields Associated with a Oscillating Linear Magnetic Quadrupole, M(2)

Instead of blindly/mindlessly grinding out the "far-zone" *EM* radiation field results for the oscillating linear magnetic quadrupole, via use of the duality transform, we can use the results from the oscillating E(2) linear electric quadrupole to obtain results for oscillating M(2) linear magnetic quadrupole, *i.e.* we will use the duality transform on the E(2) electric charge/current density distributions/*EM* moments and the "far-zone" E(2) electric and magnetic fields:

$$\boxed{\begin{gathered} Q_{zz}^{e} \Rightarrow Q_{zz}^{m}/c \\ \left\{\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t), c\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t)\right\} \Rightarrow \left\{\vec{E}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t), c\vec{B}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)\right\} \end{gathered}}$$



From P435 Lecture Notes #18 page 7-9:

$$\boxed{\begin{pmatrix}\vec{E}' \\ c\vec{B}'\end{pmatrix} = R(\varphi)\begin{pmatrix}\vec{E} \\ \vec{B}\end{pmatrix} = \begin{pmatrix}\cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\vec{E} \\ \vec{B}\end{pmatrix}} \quad \underline{\text{where}}: \boxed{\varphi = 90^\circ = \pi/2}.$$

Thus: $\boxed{\begin{pmatrix}\vec{E}' \\ c\vec{B}'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\vec{E} \\ c\vec{B}\end{pmatrix}} \Rightarrow \boxed{\begin{pmatrix}\vec{E}' = c\vec{B} \\ c\vec{B}' = -\vec{E}\end{pmatrix}}$ or: $\boxed{\begin{pmatrix}\vec{E}' \\ c\vec{B}'\end{pmatrix} = \begin{pmatrix}c\vec{B} \\ -\vec{E}\end{pmatrix}}$

n.b. φ is *not* a physical/space angle here!

<u>Recall that</u>:

$\boxed{q = g_m / c}$ or: $\boxed{g_m = qc}$ for electric vs. magnetic charges / monopole moments {E(0) & M(0)}.

$\boxed{p = m/c}$ or: $\boxed{m = pc}$ for electric vs. magnetic dipole moments {E(1) & M(1)}.

$\boxed{Q_{zz}^{e} = Q_{zz}^{m}/c}$ or: $\boxed{Q_{zz}^{m} = Q_{zz}^{e} c}$ for electric vs. magnetic quadrupole moments {E(2) & M(2)}.

Note also that, as we saw for the case of the M(1) magnetic dipole, where the scalar potential was $V_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) = 0$, likewise, for the case of the M(2) magnetic quadrupole, the scalar potential is also zero, *i.e.* $V_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t) = 0$.

We can then {easily} obtain $\vec{A}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)$ from $\vec{E}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)$, since:

$$\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t) = -\vec{\nabla} \underbrace{\cancel{V_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)}}_{=0} - \frac{\partial \vec{A}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)}{\partial t} = -\frac{\partial \vec{A}_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t)}{\partial t}}$$

Comparison of *EM* Quantities for E(2) Oscillating Linear Electric Quadrupole vs. M(2) Oscillating Linear Magnetic Quadrupole in the "far-zone" limit $d \ll \lambda \ll r$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$

	$\ell = 2,\ m = 0$ Oscillating E(2) Linear Electric Quadrupole	$\ell = 2,\ m = 0$ Oscillating M(2) Linear Magnetic Quadrupole
Moments	$\vec{Q}^{e}_{zz} = q(t)\vec{d}\vec{d},\ Q^{e}_{zz} = qdd$ $Q^{e}_{zz} = Q^{m}_{zz}/c$	$\vec{Q}^{m}_{zz} = I_o \vec{A}_{\ell oop},\ \vec{A}_{\ell oop} = \pi b^2 \hat{z},\ Q^{m}_{zz} = I_o \pi^2 b$
Retarded Scalar Potential	$V_{\mathrm{r}}^{\mathrm{E}(2)}(\vec{r},t) \simeq -\left(\frac{Q^{e}_{zz}\omega^2}{4\pi\varepsilon_o c^2}\right)\left(\frac{\cos^2\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$	$V_{\mathrm{r}}^{\mathrm{M}(2)}(\vec{r},t) = 0$
Retarded Vector Potential	$\vec{A}_{\mathrm{r}}^{\mathrm{E}(2)} \simeq -\left(\frac{\mu_o Q^{e}_{zz}\omega^2}{4\pi c}\right)\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$ $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$	$\vec{A}_{\mathrm{r}}^{\mathrm{M}(2)} \simeq -\left(\frac{\mu_o Q^{m}_{zz}\omega^2}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Electric Field	$\vec{E}_{\mathrm{r}}^{\mathrm{E}(2)} \simeq +\left(\frac{\mu_o Q^{e}_{zz}\omega^3}{4\pi c}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$	$\vec{E}_{\mathrm{r}}^{\mathrm{M}(2)} \simeq +\left(\frac{\mu_o Q^{m}_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Magnetic Field	$\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \simeq +\left(\frac{\mu_o Q^{e}_{zz}\omega^3}{4\pi c^2}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$	$\vec{B}_{\mathrm{r}}^{\mathrm{E}(2)} \simeq -\left(\frac{\mu_o Q^{m}_{zz}\omega^3}{4\pi c^3}\right)\left(\frac{\cos\theta\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Time-Avg'd *EM* Energy Density	$\left\langle u^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^4}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$	$\left\langle u^{rad}_{\mathrm{M}(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^6}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting's Vect/Intensity	$I^{rad}_{\mathrm{E}(2)}(\vec{r}) \equiv \left\langle \left\lvert \vec{S}^{rad}_{\mathrm{E}(2)} \right\rvert \right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^3}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$	$I^{rad}_{\mathrm{M}(2)}(\vec{r}) \equiv \left\langle \left\lvert \vec{S}^{rad}_{\mathrm{M}(2)} \right\rvert \right\rangle \simeq \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\left\langle P^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle \simeq \frac{\mu_o Q^{e^2}_{zz}\omega^6}{60\pi c^3}$	$\left\langle P^{rad}_{\mathrm{M}(2)}(\vec{r},t)\right\rangle \simeq \frac{\mu_o Q^{m^2}_{zz}\omega^6}{60\pi c^5}$
Time-Avg'd *EM* Linear Momentum Density	$\left\langle \vec{\wp}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{e^2}_{zz}\omega^6}{32\pi^2 c^5}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$	$\left\langle \vec{\wp}^{rad}_{\mathrm{M}(2)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o Q^{m^2}_{zz}\omega^6}{32\pi^2 c^7}\right)\left(\frac{\cos^2\theta\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\left\langle \vec{\ell}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle = 0$	$\left\langle \vec{\ell}^{rad}_{\mathrm{E}(2)}(\vec{r},t)\right\rangle = 0$
Characteristic Antenna Impedance	$Z^{\mathrm{E}(2)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$ =	$Z^{\mathrm{M}(2)}_{rad} = Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,\Omega \simeq 377\,\Omega$
Antenna Radiation Resistance	$R^{\mathrm{E}(2)}_{rad} = \frac{1}{60\pi}\left(\frac{\omega d}{c}\right)^4 Z_o$	$R^{\mathrm{M}(2)}_{rad} = \frac{1}{60\pi}\left(\frac{\omega\pi b}{c}\right)^6 Z_o$

In the "far-zone" limit $\{\pi b = d \ll \lambda \ll r\}$, to leading order in $\left(\frac{d}{r}\right)$, $\left(\frac{\omega d}{c}\right)$ and $\left(\frac{c}{\omega r}\right)$ we see (again) that:

$$\boxed{\vec{B}_{\rm r}^{\rm E(2)}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}_{\rm r}^{\rm E(2)}(\vec{r},t)} \Leftarrow \boxed{\hat{r}\times\hat{\theta}=\hat{\varphi}} \quad \underline{\text{and}}: \boxed{\vec{B}_{\rm r}^{\rm M(2)}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}_{\rm r}^{\rm M(2)}(\vec{r},t)} \Leftarrow \boxed{\hat{r}\times\hat{\varphi}=-\hat{\theta}}$$

Ratio of *EM* radiation resistances: $\boxed{R_{\rm M(2)}^{rad} \big/ R_{\rm E(2)}^{rad} = \left(\frac{\omega\pi b}{c}\right)^2 = \left(\frac{\omega d}{c}\right)^2 \ll 1}$

Ratio of time-averaged *EM* power radiated: $\boxed{\left\langle P_{\rm M(2)}^{rad}\right\rangle \big/ \left\langle P_{\rm E(2)}^{rad}\right\rangle = \left(\frac{\omega\pi b}{c}\right)^2 = \left(\frac{\omega d}{c}\right)^2 \ll 1}$