LECTURE NOTES 18.5

The Lorentz Transformation of $\vec{E}$ and $\vec{B}$ Fields:

We have seen that one observer's $\vec{E}$-field is another's $\vec{B}$-field (or a mixture of the two), as viewed from different inertial reference frames (IRF's).

What are the mathematical rules / physical laws of {special} relativity that govern the transformations of $\vec{E} \rightleftarrows \vec{B}$ in going from one IRF(S) to another IRF(S') ???

In the immediately preceding lecture notes, the reader may have noticed some tacit / implicit assumptions were made, which we now make explicit:

1) Electric charge q (like c, speed of light) *is* a Lorentz invariant *scalar* quantity.
   No matter how fast / slow an electrically-charged particle is moving, the *strength* of its electric charge is always the same, viewed from any/all IRF's: $e = 1.602 \times 10^{-19}$ Coulombs

   {n.b. electric charge is also a *conserved* quantity, valid in any / all IRF's.}

   Since the speed of light c is a Lorentz invariant quantity, then since $c = 1/\sqrt{\varepsilon_o \mu_o}$ then so is $c^2 = 1/\varepsilon_o \mu_o$ and thus ε_o and μ_o *must* be *separately* Lorentz invariant quantities, i.e.

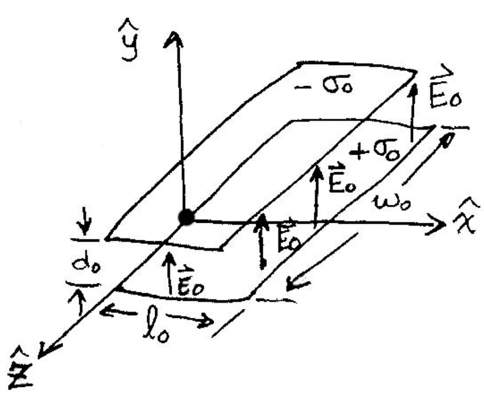
$$\left.\begin{aligned} \varepsilon_o &= 8.85 \times 10^{-12} \text{ Farads/meter} \\ \mu_o &= 4\pi \times 10^{-7} \text{ Henrys/meter} \end{aligned}\right\} \quad \boxed{\text{same in any / all IRF's}}$$

2) The Lorentz transformation rules for $\vec{E}$ and $\vec{B}$ are the same, no matter *how* the $\vec{E}$ and $\vec{B}$ fields are produced - e.g. from sources: q (charges) and/or I (currents); or from fields: e.g. $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$, etc.

The Relativistic Parallel-Plate Capacitor:

The simplest possible electric field: Consider a large ||-plate capacitor at rest in IRF(S_0). It carries surface charges $\pm\sigma_0$ on the top/bottom plates and has plate dimensions ℓ_0 and w_0 {in IRF(S_0)!} separated by a small distance $d \ll \ell_0, w_0$.

In IRF(S_0):



Electric field as seen in IRF(S_0):

$$\boxed{\vec{E}_0 = \frac{\sigma_0}{\varepsilon_o}\hat{y}}$$

No $\vec{B}$-field present in IRF(S_0):

$\boxed{\vec{B}_0 = 0}$ no currents present!

n.b. $\vec{E}_0$ is non-zero only in the gap region between ||-plates

Now consider examining this same capacitor setup from a <u>different</u> IRF(S), which is moving to the <u>right</u> at speed v_0 (as viewed from the rest frame IRF(S_0) of the $\|$-plate capacitor) i.e. $\vec{v} = +v_0\hat{x}$ is the velocity of IRF(S) relative to IRF(S_0).

Viewed from the moving frame IRF(S), the $\|$-plate capacitor is moving to the <u>left</u> (i.e. along the $-\hat{x}$-axis) with speed $-v_0$. The plates <u>along</u> the direction of motion have also Lorentz-contracted by a factor of $\boxed{\gamma_0 \equiv 1\big/\sqrt{1-(v_0/c)^2}}$, i.e. the length of plates in IRF(S) is now $\ell = \ell_0/\gamma_0 = \sqrt{1-(v_0/c)^2}\,\ell_0$

{n.b. the plate separation d and plate width w are <u>unchanged</u> in IRF(S) since both d and w are $\perp$ to direction of motion!!}

<u>Since</u>: $\boxed{\sigma \equiv \dfrac{Q_{TOT}}{Area} = \dfrac{Q_{TOT}}{\ell * w}}$ <u>but</u>: Q_{TOT} = Lorentz invariant quantity

<u>And</u>: $\boxed{\sigma_0 = \dfrac{Q_{TOT}}{A_0} = \dfrac{Q_{TOT}}{\ell_0 * w_0}}$ <u>but</u>: $w = w_0$ and $d = d_0$ since both $\perp$ to direction of motion.

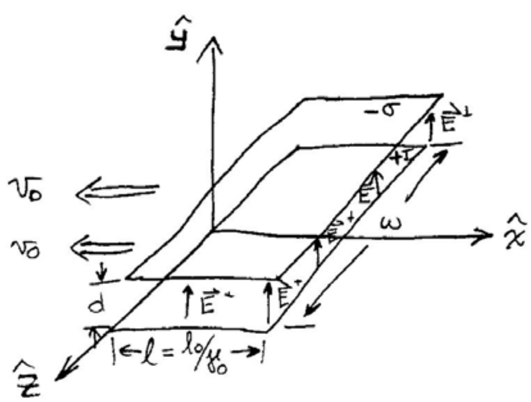
<u>Thus</u>: $\boxed{Q_{TOT} = \sigma\ell\cancel{w} = \sigma_0\ell_0\cancel{w_0}} \Rightarrow \boxed{\sigma\ell = \sigma_0\ell_0} \Rightarrow \boxed{\sigma = \sigma_0(\ell_0/\ell)}$

<u>But</u>: $\boxed{\ell = \ell_0/\gamma} \Rightarrow \boxed{\sigma = \sigma_0\left(\dfrac{\cancel{\ell_0}}{\cancel{\ell_0}/\gamma_0}\right) = \gamma_0\sigma_0}$

<u>Thus</u>: $\boxed{\sigma = \gamma_0\sigma_0}$ <u>but since</u>: $\boxed{\gamma_0 > 1} \Rightarrow \boxed{\sigma > \sigma_0}$

> The surface charge density on the plates of capacitor in IRF(S) is higher than in IRF(S_0) by factor of $\gamma_0 = 1\big/\sqrt{1-(v_0/c)^2}$

To an observer in IRF(S), the plates of the $\|$-plate capacitor are moving in the $-v_0\hat{x}$ direction.



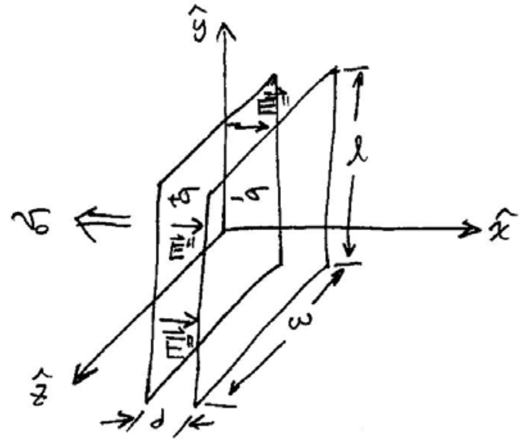
Thus the electric field $\vec{E}$ in IRF(S) is:

$\boxed{\vec{E}^{\perp} = \dfrac{\sigma}{\varepsilon_o}\hat{y} = \gamma_0\dfrac{\sigma_0}{\varepsilon_o}\hat{y} = \gamma_0\vec{E}_0^{\perp}}$ <u>where</u>: $\boxed{\sigma = \gamma_0\sigma_0}$

$\boxed{\vec{E}^{\perp} = \gamma_0\vec{E}_0^{\perp}}$, $\boxed{\vec{E}_0^{\perp} = \dfrac{\sigma_0}{\varepsilon_o}\hat{y}}$

The superscript $\perp$ is to <u>explicitly</u> remind us that $\boxed{\vec{E}^{\perp} = \gamma_0\vec{E}_0^{\perp}}$ is for $\vec{E}$-fields $\perp$ to the direction of motion. Here, $\vec{v} = +v_0\hat{x}$ between IRF's.

Now consider what happens when we rotate the {isolated} ||-plate capacitor by 90° in IRF(S_0), then $\boxed{\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{x}}$ in IRF(S_0), but in the moving frameIRF(S):



The electric field in IRF(S) is:

$$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_o}\hat{x} \equiv \vec{E}^{\parallel}}$$

<u>But</u>: $\boxed{\sigma = \frac{Q_{TOT}}{Area} = \frac{Q_{TOT}}{\ell w} = \frac{Q_{TOT}}{\ell_0 w_0} = \sigma_0}$

$\therefore$ $\boxed{\vec{E}^{\parallel} = \frac{\sigma}{\varepsilon_o}\hat{x} = \frac{\sigma_0}{\varepsilon_o}\hat{x} = \vec{E}_0^{\parallel}} \leftarrow \vec{E}$-field in IRF($S_0$)

$\therefore$ $\boxed{\vec{E}^{\parallel} = \vec{E}_0^{\parallel}}$

d is now Lorentz-contracted in IRF(S): $\boxed{d = d_0/\gamma_0}$ but has <u>no</u> effect on $\vec{E}$ {in IRF(S)}, because $\vec{E}$ does not depend on d! Why??? Because <u>here</u>, the ||-plates were first charged up (e.g. from a battery) and then <u>disconnected</u> from the battery!

$\Rightarrow$ Potential difference $\boxed{\Delta V(\text{IRF}(S)) \neq \Delta V_0(\text{IRF}(S_0))}$!!!

<u>Since</u>: $\boxed{\left(E^{\parallel} = \frac{\Delta V}{d}\right) = \left(E_0^{\parallel} = \frac{\Delta V_0}{d_0}\right)} \Rightarrow \boxed{\frac{\Delta V^{\parallel}}{d} = \frac{\Delta V_0^{\parallel}}{d_0}}$ <u>but</u>: $\boxed{d = d_0/\gamma_0} \Rightarrow \boxed{\frac{\Delta V^{\parallel}}{d_0/\gamma_0} = \frac{\Delta V_0^{\parallel}}{d_0}}$

$\therefore$ $\boxed{\Delta V(\text{IRF}(S)) = \gamma_0 \Delta V_0(\text{IRF}(S_0))}$

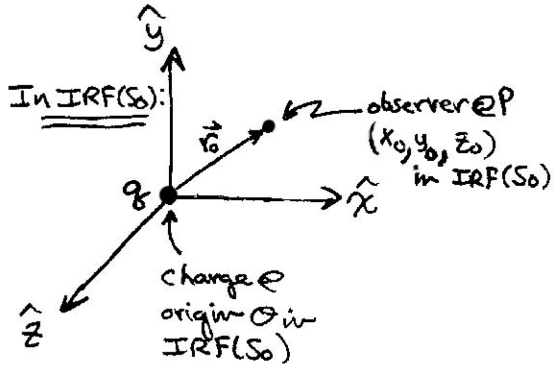
The ||-plate capacitor is <u>deliberately</u> not connected to an external battery (which would keep ΔV = constant, but then we would have $\sigma = \sigma_0$ in the $\perp$ case and $\sigma \neq \sigma_0$ in the || case. Currents would then flow (transitorially) in <u>both</u> situations.

Note that we also want to hang on to/utilize the Lorentz-invariant nature of Q_{TOT}, which is another reason why the battery is disconnected…

<u>Griffiths Example 12.13</u>: <u>The Electric Field of a Point Charge in Uniform Motion</u>

A point charge q is at rest in IRF(S_0). An observer is in IRF(S), which moves to the <u>right</u> (i.e. in the $+\hat{x}$ direction) at speed v_0 relative to IRF(S_0). What is the $\vec{E}$-field of the electric charge q, as viewed from the moving frame IRF(S)?

In the rest frame IRF(S_0) of the point charge q, the electric field of the point charge q is:



$$\boxed{\vec{E}_0 = \frac{1}{4\pi\varepsilon_o}\frac{q}{r_0^2}\hat{r}_0}$$ where: $\boxed{r_0^2 = x_0^2 + y_0^2 + z_0^2}$

$$\boxed{E_{0_x} = \frac{q}{4\pi\varepsilon_o}\frac{x_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}}$$

$$\boxed{E_{0_y} = \frac{q}{4\pi\varepsilon_o}\frac{y_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}}$$

$$\boxed{E_{0_z} = \frac{q}{4\pi\varepsilon_o}\frac{z_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}}$$

But: $\boxed{E^{\perp} = \gamma_0 E_0^{\perp}}$ and: $\boxed{E^{\parallel} = E_0^{\parallel}}$ $\boxed{\gamma_0 = 1\Big/\sqrt{1-(u_0/c)^2}}$

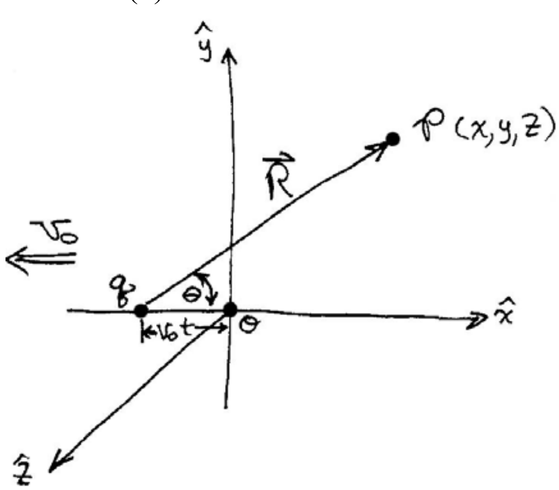
Then in IRF(S), which is moving to right (i.e. in the $+\hat{x}$ direction) at speed v_0 relative to IRF(S):

$$\left.\begin{array}{l}
\boxed{E_x = E_{0_x} = \dfrac{q}{4\pi\varepsilon_o}\dfrac{x_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}} \\
\boxed{E_y = \gamma_0 E_{0_y} = \dfrac{q}{4\pi\varepsilon_o}\dfrac{\gamma_0 y_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}} \\
\boxed{E_z = \gamma_0 E_{0_z} = \dfrac{q}{4\pi\varepsilon_o}\dfrac{\gamma_0 z_0}{\left(x_0^2+y_0^2+z_0^2\right)^{3/2}}}
\end{array}\right\}$$

n.b. These relations are currently expressed in terms of the IRF(S_0) coordinate (x_0, y_0, z_0) of the field point P.

However we want/need the IRF(S) $\vec{E}$ expressed in terms of the IRF(S) coordinate (x, y, z) of the field point P. $\Rightarrow$ Use the inverse Lorentz transformation on coordinates:

$\therefore$ In IRF(S) at time t:



Observation / field point in IRF(S) at time t:

Inverse Lorentz Transformation:

$\boxed{x_0 = \gamma_0\left(x + v_0 t\right) = \gamma_0 R_x}$ $\boxed{\vec{R} = R_x\hat{x} + R_y\hat{y} + R_z\hat{z}}$

$\boxed{y_0 = y \equiv R_y}$ $\boxed{\gamma_0 = 1\Big/\sqrt{1-(v_0/c)^2}}$

$\boxed{z_0 = z \equiv R_z}$

Then: $\boxed{x_0^2+y_0^2+z_0^2=r_0^2} = \boxed{\gamma_0^2 R_x^2+R_y^2+R_z^2} = \boxed{\gamma_0^2 R^2\cos^2\theta+R^2\sin^2\theta}$

From above figure: $\boxed{R_x = R\cos\theta} \Rightarrow \boxed{R_x^2 = R^2\cos^2\theta}$.

Then since: $\boxed{R^2 = R_x^2+\left(R_y^2+R_z^2\right)} \Rightarrow \boxed{\left(R_y^2+R_z^2\right)=R^2\sin^2\theta}$ Since: $\boxed{R^2 = R^2\left(\cos^2\theta+\sin^2\theta\right)}$.

$\therefore$ In IRF(S):

$$\boxed{E_x = \frac{q}{4\pi\varepsilon_o}\frac{\gamma_0 R_x}{\left(\gamma_0^2 R^2\cos^2\theta+R^2\sin^2\theta\right)^{3/2}}}$$

Picked up Lorentz factor γ_0 from Lorentz transformation of coordinate

$$\boxed{E_y = \frac{q}{4\pi\varepsilon_o}\frac{\gamma_0 R_y}{\left(\gamma_0^2 R^2\cos^2\theta+R^2\sin^2\theta\right)^{3/2}}}$$

$$\boxed{E_z = \frac{q}{4\pi\varepsilon_o}\frac{\gamma_0 R_z}{\left(\gamma_0^2 R^2\cos^2\theta+R^2\sin^2\theta\right)^{3/2}}}$$

Picked up Lorentz factor γ_0 from Lorentz transformation of field

$$\therefore \quad \boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma_0 \vec{R}}{\left(\gamma_0^2 R^2\cos^2\theta+R^2\sin^2\theta\right)^{3/2}}} \quad \text{with: } \boxed{\vec{R} = R_x\hat{x}+R_y\hat{y}+R_z\hat{z}}$$

Or: $$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma_0}{\left(\gamma_0^2\cos^2\theta+\sin^2\theta\right)^{3/2}}\left(\frac{\vec{R}}{R^3}\right)} \quad \text{where: } \boxed{\frac{\vec{R}}{R^3}=\frac{\hat{R}}{R^2}} \quad \text{since: } \boxed{\vec{R}=R\hat{R}} \quad \text{or: } \boxed{\hat{R}=\frac{\vec{R}}{R}}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\cancel{\gamma_0}}{\gamma_0^{\cancel{3}/2}\left(\cos^2\theta+\frac{1}{\gamma_0^2}\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}} \quad \text{with: } \boxed{\frac{1}{\gamma_0^2}=\left(1-\left(\frac{v_0}{c}\right)^2\right)}$$

Thus: $$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\left(1-\left(\frac{v_0}{c}\right)^2\right)}{\left(1-\sin^2\theta+\left(1-\left(\frac{v_0}{c}\right)^2\right)\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}}$$

Or: $$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\left(1-\left(\frac{v_0}{c}\right)^2\right)}{\left(1-\cancel{\sin^2\theta}+\cancel{\sin^2\theta}-\left(\frac{v_0}{c}\right)^2\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}}$$

Thus: $$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\left(1-\left(\frac{v_0}{c}\right)^2\right)}{\left(1-\left(\frac{v_0}{c}\right)^2\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}}$$

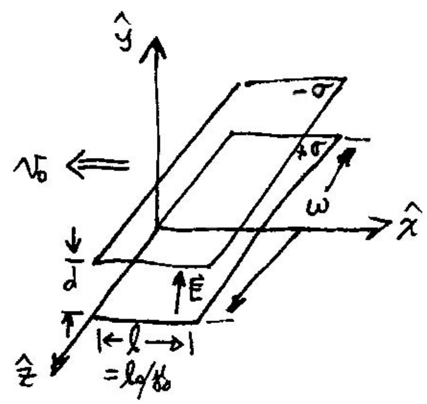
≡ Heaviside expression for the retarded electric field $\vec{E}_{ret}(\vec{r},t)$ expressed in terms of the present time t !!!

The unit vector $\hat{R}$ points along the line from the present position of the charged particle at time t !!! (See Griffiths Ch. 10, Equation 10.68, page 439)

$\Rightarrow$ $\boxed{\vec{E} \parallel \vec{R}}$ because the γ_0 factor is present in the numerator of all of the $\hat{x}, \hat{y}, \hat{z}$ components !!!

<u>But wait!!!</u> This isn't the *entire* story for the *EM* fields in IRF(*S*) !!!

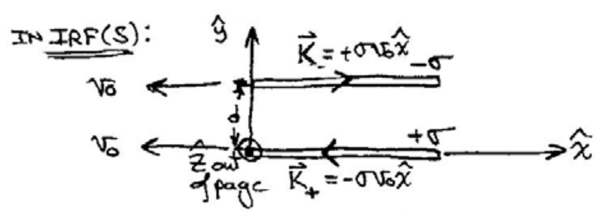
In the first example of the "horizontal" $\parallel$-plate capacitor, which was moving with relative velocity $\vec{v} = -v_0\hat{x}$ as seen by an observer in IRF(S), shown in the figure below:

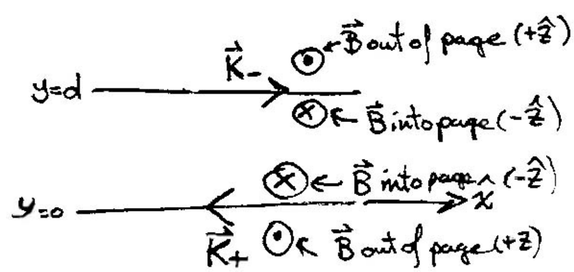


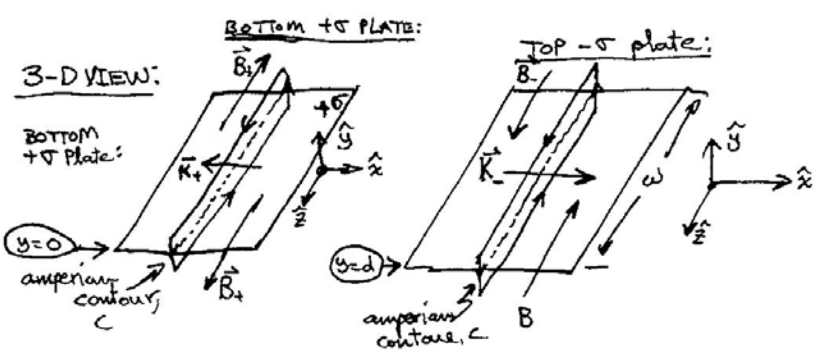
The moving surface-charged plates of the $\parallel$-plate capacitor as viewed by an observer in IRF(*S*) constitute <u>surface currents</u>: $\boxed{\vec{K}_{\pm} = \mp\sigma v_0\hat{x}}$ <u>with</u>: $\boxed{\sigma = \gamma_0\sigma_0}$

These two surface currents create a magnetic field in IRF(*S*) <u>between</u> the plates of the $\parallel$-plate capacitor !!!

<u>Side / edge-on view</u>: <u>*B*-fields produced (use right-hand rule)</u>:







<u>Ampere's Circuital Law</u>:

$$\oint_C \vec{B}\bullet d\vec{\ell} = \mu_o I_{encl}$$
$$= 2Bw = \pm\mu_o K w$$

$$\boxed{\vec{B}_+(y>0) = -\frac{\mu_o K_+}{2}\hat{z} = -\frac{\mu_o \sigma v_0}{2}\hat{z} \quad (y>0)}$$
$$\boxed{\vec{B}_+(y<0) = +\frac{\mu_o K_+}{2}\hat{z} = +\frac{\mu_o \sigma v_0}{2}\hat{z} \quad (y<0)}$$

$$\boxed{\vec{B}_-(y>d) = +\frac{\mu_o K_-}{2}\hat{z} = +\frac{\mu_o \sigma v_0}{2}\hat{z}}$$
$$\boxed{\vec{B}_-(y<d) = -\frac{\mu_o K_-}{2}\hat{z} = -\frac{\mu_o \sigma v_0}{2}\hat{z}}$$

Add $\vec{B}_+$ and $\vec{B}_-$ to get the total $\vec{B}$-field:

In IRF(S):
$$\begin{cases} \boxed{\vec{B}_{TOT}(y>d) = -\frac{\mu_o \sigma v_0}{2}\hat{z} + \frac{\mu_o \sigma v_0}{2}\hat{z} = 0} \\ \boxed{\vec{B}_{TOT}(0 \le y \le d) = -\frac{\mu_o \sigma v_0}{2}\hat{z} - \frac{\mu_o \sigma v_0}{2}\hat{z} = -\mu_o \sigma v_0 \hat{z} = -\mu_o K \hat{z}} \\ \boxed{\vec{B}_{TOT}(y<0) = +\frac{\mu_o \sigma v_0}{2}\hat{z} - \frac{\mu_o \sigma v_0}{2}\hat{z} = 0} \end{cases}$$

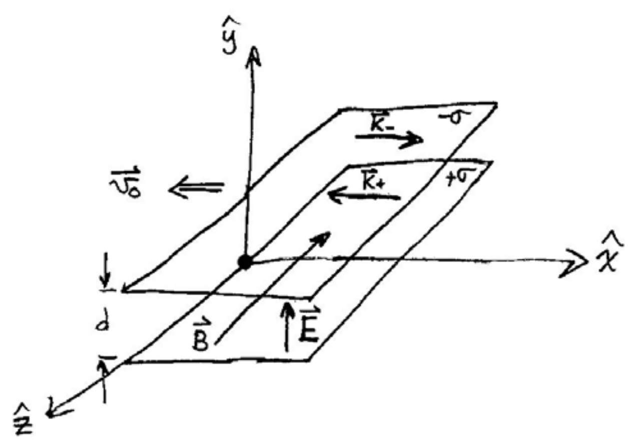
$\boxed{K = \sigma v_0}$ and: $\boxed{\sigma = \gamma_0 \sigma_0}$

Thus, in IRF(S) {which moves with velocity $\vec{v} = +v_0\hat{x}$ relative to IRF(S_0)} we have for the horizontal $||$-plate capacitor:

$\vec{E}$ only exists in region $0 \le y \le d$: $\boxed{\vec{E} = \frac{\sigma}{\varepsilon_o}\hat{y} = \gamma_0 \frac{\sigma_0}{\varepsilon_o}\hat{y}}$ <u>where</u>: $\boxed{\sigma = \gamma_0 \sigma_0}$ <u>and</u>: $\boxed{\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}}$

$\vec{B}$ only exists in region $0 \le y \le d$: $\boxed{\vec{B} = -\mu_o \sigma v_0 \hat{z} = -\gamma_0 \mu_o \sigma_0 v_0 \hat{z}}$ $\boxed{\beta_0 = v_0/c}$

<u>In IRF(S)</u>:



The fact that $\vec{B}$ exists / is non-zero *<u>only</u>* where $\vec{E}$ exists / is non-zero is *<u>not</u>* an accident / not a "mere" coincidence!

The space-time properties associated with rest frame IRF(S_0) are rotated (Lorentz-transformed) in going to IRF(S).

$\vec{E}_0$ in IRF(S_0) <u>only</u> exists between plates in IRF(S_0)
Gets space-time rotated (Lorentz-transformed) in going to IRF(S)
$\rightarrow$ $\vec{E}$ and $\vec{B}$ in IRF(S) only exist between the capacitor plates in IRF(S).

$\vec{E}$ and $\vec{B}$ between plates in IRF(S) comes from / is associated with $\vec{E}_0$ between plates in IRF(S_0)

Point is: *EM* field energy density u_{EM} (x,y,z,t) *must* be non-zero in a given IRF in order to have *EM* fields present at space-time point (x,y,z,t)!

EM field energy densities u_0 and $u \neq 0$ *only* in region between plates of || -plate capacitor.

In IRF(S_0): $\boxed{u_0(\vec{r}_0,t_0) = \frac{1}{2}\varepsilon_o E_0^2(\vec{r}_0,t_0)}$ $\leftarrow \vec{E}_0$ only

In IRF(S): $\boxed{u(\vec{r},t) = \frac{1}{2}\varepsilon_o E^2(\vec{r},t) + \frac{1}{2\mu_o}B^2(\vec{r},t)}$

Space-time $(\vec{r},t)$ point functions

n.b. If $u_0 = 0$ in one IRF(S_0) $\rightarrow$ $u = 0$ in another IRF(S).

Momentary Aside: Hidden Momentum Associated with the Relativistic Parallel Plate Capacitor.

Between plates of || -plate capacitor in IRF(S):

$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y} = \gamma_0\frac{\sigma_0}{\varepsilon_0}\hat{y}}$ where: $\boxed{\sigma = \gamma_0\sigma_0}$ and: $\boxed{\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}}$

$\boxed{\vec{B} = -\mu_o\sigma v_0\hat{z} = -\gamma_0\mu_o\sigma_0 v_0\hat{z}}$ $\boxed{\beta_0 = v_0/c}$

Between plates of || -plate capacitor (only!): $\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = -\gamma_0^2\frac{\sigma_0^2 v_0}{\varepsilon_0}\overbrace{(\hat{y}\times\hat{z})}^{\hat{x}} = -\frac{\gamma_0^2\sigma_0^2 v_0}{\varepsilon_0}\hat{x}}$

$\boxed{\begin{array}{c}\hat{x}\times\hat{y}=\hat{z}\\ \hat{y}\times\hat{z}=\hat{x}\\ \hat{z}\times\hat{x}=\hat{y}\end{array}}$

Poynting's vector points in the direction of motion of the || -plate capacitor, as seen by an observer in IRF(S), and only exists/is non-zero in the region between plates of the || -plate capacitor.

EM field linear momentum density in IRF(S): $\boxed{\vec{\wp}_{EM} = \varepsilon_o\mu_o\vec{S} = -\gamma_0^2\mu_o\sigma_0^2 v_0\hat{x}}$.
Points in the direction of motion of the || -plate capacitor, as seen by an observer in IRF(S), only exists/is non-zero in the region between plates of the || -plate capacitor.

EM field linear momentum in IRF(S): $\boxed{\vec{p}_{EM} = \vec{\wp}_{EM} * \text{Volume(IRF}(S)) = \vec{\wp}_{EM}V_S}$

Where the volume (in IRF(S)): $\boxed{V_S \equiv \ell w d = \frac{\ell_0}{\gamma_0}w_0 d_0}$, where: $\boxed{\ell = \ell_0/\gamma_0}$, $\boxed{w = w_0}$ and: $\boxed{d = d_0}$ {here}.

Define the volume (in IRF(S_0)): $\boxed{V_0 \equiv \ell_0 w_0 d_0}$. Thus: $\boxed{V_S = V_0/\gamma_0}$.

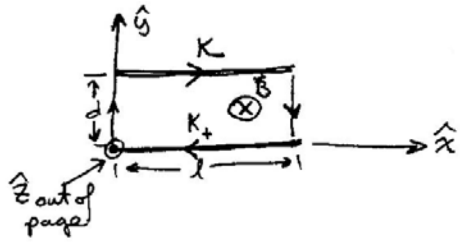
$$\therefore \boxed{\vec{p}_{EM} = -\gamma_0^2\mu_o\sigma_0^2 v_0 V_S\hat{x} = -\gamma_0^2\mu_o\sigma_0^2 v_0 * \frac{\ell_0 w_0 d_0}{\gamma_0}\hat{x} = -\gamma_0\mu_0\sigma_0^2 v_0\underbrace{(\ell_0 w_0 d_0)}_{\equiv V_0}\hat{x} = -\gamma_0\mu_0\sigma_0^2 v_0 V_0\hat{x}}$$

The "hidden momentum" is: $\boxed{\vec{p}_{hidden} = \frac{1}{c^2}(\vec{m}\times\vec{E})}$.

Note that: $\boxed{\vec{m} \parallel \vec{B}}$ and: $\boxed{\vec{m} = -IA_\perp\hat{z}}$ with: $\boxed{I = \int\vec{K}\bullet d\vec{\ell}_\perp = Kw}$ where: $\boxed{K = \sigma v_0}$.

The cross-sectional area is: $\boxed{A_\perp = \ell\bullet d = \left(\frac{\ell_0}{\gamma_0}\right)d_0}$ since $\boxed{\ell = \ell_0/\gamma_0}$ and $\boxed{d = d_0}$, $\boxed{w = w_0}$.

Side view (in IRF(S)):



$$\boxed{\begin{aligned}\vec{m} &= -IA_{\perp}\hat{z} = -Kw\ell d = -(\sigma v_0)\, w_0 \frac{\ell_0}{\gamma_0} d_0 \hat{z} \\ &= -(\cancel{\gamma_0}\sigma_0 v_0)\, w_0 \frac{\ell_0}{\cancel{\gamma_0}} d_0\hat{z} = -(\sigma_0 v_0)(\ell_0 w_0 d_0)\,\hat{z}\end{aligned}} \quad \boxed{\gamma_0\sigma_0}$$

$$\boxed{\vec{E} = \gamma_0 \frac{\sigma_0}{\varepsilon_o}\hat{y}}$$

$\therefore$ In IRF(S):

$$\boxed{\vec{p}_{hidden} = -\frac{1}{c^2}\sigma_0 v_0 (\ell_0 w_0 d_0) * \gamma_0 \frac{\sigma_0}{\varepsilon_o}\overbrace{(\hat{z}\times\hat{y})}^{=-\hat{x}} = +\frac{\gamma_0\sigma_0^2 v_0}{c^2\varepsilon_o}\underbrace{(\ell_0 w_0 d_0)}_{\equiv V_0}\hat{x} = +\gamma_0\mu_o\sigma_0^2 v_0 V_0 \hat{x}} \text{ using } \boxed{\frac{1}{c^2} = \varepsilon_o\mu_o}$$

Thus In IRF(S): $\boxed{\vec{p}_{hidden} = \frac{1}{c^2}(\vec{m}\times\vec{B}) = +\gamma_0\mu_o\sigma_0^2 v_0 V_0\hat{x}}$ But: $\boxed{\vec{p}_{EM} = \varepsilon_o(\vec{E}\times\vec{B}) * V_S = -\gamma_0\mu_o\sigma_0^2 v_0 V_0 \hat{x}}$

Thus, we (again) see that: $\boxed{\vec{p}_{hidden} = -\vec{p}_{EM}}$ for the relativistic $||$-plate capacitor !!!

Note that in IRF(S_0): $\boxed{\vec{B}_0 = 0} \Rightarrow \boxed{\vec{S}_0 = \frac{1}{\mu_o}\vec{\mathrm{E}}_0 \times \vec{B}_0 = 0} \Rightarrow \boxed{\vec{p}_{EM} = 0}$.

But note that: $\boxed{\vec{p}_{hidden} = \frac{1}{c^2}(\vec{m}_0 \times \vec{B}_0) = 0}$, thus $\boxed{\vec{p}_{hidden} = -\vec{p}_{EM}}$ is *also* valid in IRF(S_0).

$\Rightarrow$ The numerical value of hidden momentum is reference frame dependent, i.e. it is not a Lorentz-invariant quantity, just as relativistic momentum $\vec{p}$ {in general} is reference frame dependent/is not a Lorentz-invariant quantity.

Now let us return to task of determining the Lorentz transformation rules for $\vec{E}$ and $\vec{B}$:

For the case of the relativistic horizontal $||$-plate capacitor, let us consider a third IRF(S') that travels to the right (i.e. in the $+\hat{x}$-direction) with velocity $\vec{v} = +v\hat{x}$ relative to IRF(S).

In IRF(S'), the *EM* fields are: $\boxed{\vec{E}' = \frac{\sigma'}{\varepsilon_o}\hat{y}}$ and: $\boxed{\vec{B}' = -\mu_o\sigma' v'\hat{z}}$

From use of Einstein's velocity addition rule: $\vec{v}'$ is the velocity of IRF(S') relative to IRF(S_0):

$$\boxed{v' = \frac{v+v_0}{1+\frac{vv_0}{c^2}}} \text{ with: } \boxed{\gamma' = \frac{1}{\sqrt{1-(v'/c)^2}}} \text{ and: } \boxed{\sigma' = \gamma'\sigma_0}$$

We need to express $\vec{E}'$ and $\vec{B}'$ {defined in IRF(S')} in terms of $\vec{E}$ and $\vec{B}$ {defined in IRF(S)}.

$$\boxed{\vec{E}' = \frac{\sigma'}{\varepsilon_o}\hat{y} = \frac{\gamma'\sigma_0}{\varepsilon_o}\hat{y}} \quad \underline{\text{where}}: \boxed{\gamma' = \frac{1}{\sqrt{1-(v'/c)^2}}} \quad \underline{\text{and}}: \boxed{\sigma_0 = \frac{\sigma}{\gamma_0}} \quad \underline{\text{and}}: \boxed{\gamma_0 = \frac{1}{\sqrt{1-(v_0/c)^2}}}$$

$$\therefore \quad \boxed{\vec{E}' = \frac{\gamma'\sigma}{\gamma_0\varepsilon_o}\hat{y}} \quad \underline{\text{but}}: \boxed{\vec{E} = \frac{\sigma}{\varepsilon_o}\hat{y}} \ \{\text{in IRF(S)}\} \quad \therefore \quad \boxed{\vec{E}' = \left(\frac{\gamma'}{\gamma_0}\right)\vec{E} = \left(\frac{\gamma'}{\gamma_0}\right)\frac{\sigma}{\varepsilon_o}\hat{y}}$$

$$\boxed{\vec{B}' = -\mu_o\sigma'v'\hat{z} = -\gamma'\mu_o\sigma_0 v'\hat{z}} \quad \underline{\text{but}}: \boxed{\sigma_0 = \frac{\sigma}{\gamma_0}}$$

$$\therefore \quad \boxed{\vec{B}' = -\left(\frac{\gamma'}{\gamma_0}\right)\mu_o\sigma v'\hat{z}} \quad \underline{\text{but}}: \boxed{\vec{B} = -\gamma_0\mu_o\sigma_0 v_0\hat{z} = -\mu_o\sigma v_0\hat{z}} \ \{\text{in IRF}(S)\}$$

Now (after some algebra): $\boxed{\left(\frac{\gamma'}{\gamma_0}\right) = \frac{\sqrt{1-(v_0/c)^2}}{\sqrt{1-(v'/c)^2}} = \frac{1+vv_0/c^2}{\sqrt{1-(v/c)^2}} = \gamma\left(1+\frac{vv_0}{c^2}\right)}$ $\underline{\text{where}}$: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}}$

$\underline{\text{But}}$: $\boxed{v' = \frac{(v+v_0)}{(1+vv_0/c^2)}}$ $\underline{\text{and}}$: $\boxed{\left(\frac{\gamma'}{\gamma_0}\right) = \gamma\left(1+vv_0/c^2\right)}$ $\underline{\text{with}}$: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}}$

$\therefore$ $\underline{\text{In IRF}(S')}$:

$$\boxed{\vec{E}' = \left(\frac{\gamma'}{\gamma_0}\right)\vec{E} = \left(\frac{\gamma'}{\gamma_0}\right)\frac{\sigma}{\varepsilon_o}\hat{y} = \gamma\left(1+vv_0/c^2\right)\frac{\sigma}{\varepsilon_o}\hat{y}}$$

$$\boxed{\vec{B}' = -\left(\frac{\gamma'}{\gamma_0}\right)\mu_o\sigma v'\hat{z} = -\gamma\left(\cancel{1+vv_0/c^2}\right)\mu_o\sigma\left(\frac{v+v_0}{\cancel{1+vv_0/c^2}}\right)\hat{z} = -\gamma\mu_o\sigma\left(v+v_0\right)\hat{z}}$$

Compare these to the $\vec{E}$ and $\vec{B}$ fields in IRF(S):

$\underline{\text{In IRF}(S)}$:

$$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y} = \gamma_0\frac{\sigma_0}{\varepsilon_0}\hat{y}} \quad \underline{\text{where}}: \boxed{\sigma = \gamma_0\sigma_0} \quad \underline{\text{and}}: \boxed{\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}}$$

$$\boxed{\vec{B} = -\mu_o\sigma v_0\hat{z} = -\gamma_0\mu_o\sigma_0 v_0\hat{z}} \qquad \boxed{\beta_0 = v_0/c}$$

Using: $\boxed{\mu_o = 1/c^2\varepsilon_o}$ we can rewrite $\vec{E}'$ in IRF(S') as:

$$\boxed{\vec{E}' = \gamma\frac{\sigma}{\varepsilon_o}\hat{y} + \gamma\mu_o vv_0\sigma\hat{y} = \gamma\underbrace{\left(\frac{\sigma}{\varepsilon_o}\hat{y}\right)}_{\substack{=E_y \\ \text{in IRF}(S)}} - \gamma v\underbrace{\left(-\mu_o\sigma v_0\right)}_{\substack{=B_z \\ \text{in IRF}(S)}}\hat{y}}$$

$\underline{\text{where}}$: $\boxed{\vec{v} = v\hat{x}}$ $\underline{\text{and}}$: $\boxed{\vec{B} = B_z\hat{z}}$ $\Rightarrow \boxed{\vec{v}\times\vec{B} = vB_z\underbrace{(\hat{x}\times\hat{z})}_{=-\hat{y}} = -vB_z\hat{y}}$

Very Useful Table(s):

$\hat{x}\times\hat{y} = \hat{z}$	$\hat{y}\times\hat{x} = -\hat{z}$
$\hat{y}\times\hat{z} = \hat{x}$	$\hat{z}\times\hat{y} = -\hat{x}$
$\hat{z}\times\hat{x} = \hat{y}$	$\hat{x}\times\hat{z} = -\hat{y}$

$\therefore$ $\boxed{\vec{E}' = E'_y\hat{y} = \left(\gamma E_y - \gamma v B_z\right)\hat{y} = \gamma\left(E_y - vB_z\right)\hat{y}}$ Or simply: $\boxed{E'_y = \gamma\left(E_y - vB_z\right)}$

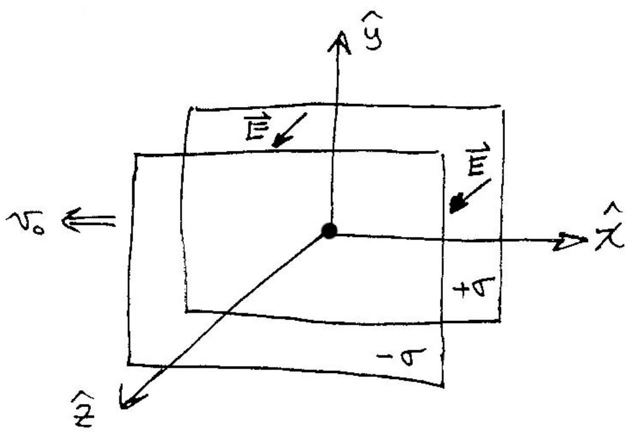
Likewise, again using: $\boxed{\mu_o = 1/c^2\varepsilon_o}$ we can rewrite $\vec{B}'$ in IRF(S') as:

$$\boxed{\vec{B}' = -\gamma\mu_o\sigma\left(v + v_0\right)\hat{z} = -\gamma\mu_o\sigma v_0\hat{z} - \gamma\mu_o\sigma v\hat{z} = \gamma\underbrace{\left(-\mu_o\sigma v_0\hat{z}\right)}_{\substack{=\vec{B}=B_z\hat{z} \\ \text{in IRF}(S)}} - \gamma\frac{v}{c^2}\underbrace{\left(\frac{\sigma}{\varepsilon_o}\right)}_{\substack{=E_y \\ \text{in IRF}(S)}}\hat{z}}$$

$\therefore$ $\boxed{\vec{B}' = B'_z\hat{z} = \gamma B_z\hat{z} - \gamma\frac{v}{c^2}E_y\hat{z} = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\hat{z}}$ Or simply: $\boxed{B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)}$

Thus, we now know how the E_y and B_z fields transform.

Next, in order to obtain the Lorentz transformation rules for E_z and B_y, we align the capacitor plates parallel to x-y plane instead of x-z plane as shown in the figure below:



In IRF(S) the {now} rotated fields are:

$$\boxed{\vec{E}^r = \frac{\sigma}{\varepsilon_o}\hat{z} = E^r_z\hat{z}}$$

$$\boxed{\vec{B}^r = +\mu_o\sigma v_0\hat{y} = B^r_y\hat{y}}$$

Use the right-hand rule to get correct sign !!!

The corresponding $\vec{E}'$ and $\vec{B}'$ fields in IRF(S') are (repeating the above methodology):

$\boxed{E'_z = \gamma\left(E_z + vB_y\right)}$ and: $\boxed{B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)}$

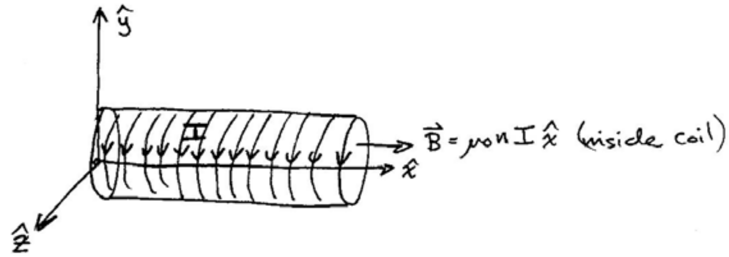
As we have already seen (by orienting the plates of capacitor parallel to y-z plane):

$\boxed{E'_x = E_x}$ $\leftarrow$ n.b. there was no accompanying $\vec{B}$ -field in this case!

$\Rightarrow$ Thus we are not able to deduce the Lorentz transformation for B_x ($||$ to direction of motion) from the $||$-plate capacitor problem.

So alternatively, let us consider the long solenoid problem, with solenoid {and current flow} oriented as shown in the figure below:

In IRF(S):



In IRF(S):

$\boxed{\vec{E} = 0}$

$\boxed{\vec{B} = B_x \hat{x} = \mu_o n I \hat{x}}$

We want to view this from IRF(S'), which is moving with velocity $\boxed{\vec{v} = +v\hat{x}}$ relative to IRF(S).

In IRF(S): $\boxed{n = N/L}$ = # turns/unit length, N = total # of turns, L = length of solenoid in IRF(S).

Viewed by an observer in IRF(S'), the solenoid length contracts: $\boxed{L' = L/\gamma}$ in IRF(S')

$\therefore$ In IRF(S'): $\boxed{n' = \dfrac{N}{L'} = \gamma \dfrac{N}{L} = \gamma n}$ = # turns/unit length in IRF(S'), where $\boxed{\gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-(v/c)^2}}}$

However, time also dilates in IRF(S') relative to IRF(S) – affects currents:

$\boxed{I = \dfrac{dQ}{dt}}$ in IRF(S)$\Rightarrow$ $\boxed{I' = \dfrac{dQ}{dt'} = \left(\dfrac{dt}{dt'}\right)\dfrac{dQ}{dt} = \left(\dfrac{dt}{dt'}\right) I}$ in IRF(S'). But: $\boxed{\left(\dfrac{dt}{dt'}\right) = \dfrac{1}{\gamma}}$ $\therefore$ $\boxed{I' = \dfrac{1}{\gamma} I}$

$$\therefore \quad \boxed{\vec{B}' = B'_x \hat{x} = \mu_o n' I' \hat{x} = \mu_o \left(\cancel{\gamma} n\right)\left(\frac{1}{\cancel{\gamma}} I\right)\hat{x} = \mu_o n I \hat{x} = B_x \hat{x} = \vec{B}} \quad \therefore \boxed{B'_x = B_x}$$

Longitudinal / parallel-to-boost direction B-field does not change !!!

Thus, we now have a complete set of Lorentz transformation rules for $\vec{E}$ and $\vec{B}$, for a Lorentz transformation from IRF(S) to IRF(S'), where IRF(S') is moving with velocity $\boxed{\vec{v} = +v\hat{x}}$ relative to IRF(S):

$$\boxed{E'_x = E_x} \qquad \boxed{B'_x = B_x} \qquad \boxed{\gamma = 1\big/\sqrt{1-\beta^2}}$$

$$\boxed{E'_y = \gamma\left(E_y - vB_z\right)} \qquad \boxed{B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)} \qquad \boxed{\beta = \frac{v}{c}}$$

$$\boxed{E'_z = \gamma\left(E_z - vB_y\right)} \qquad \boxed{B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)}$$

Just stare at/ponder these relations for a while – take your (proper) {space-}time…

Do you possibly see a wee bit of Maxwell's equations afoot here ??? ;)

<u>Two limiting cases warrant special attention</u>:

1.) If $\boxed{\vec{B}=0}$ in lab IRF(S), then in IRF(S') we have: $\boxed{\vec{B}'=\gamma\frac{v}{c^2}\left(E_z\hat{y}-E_y\hat{z}\right)=\frac{v}{c^2}\left(E'_z\hat{y}-E'_y\hat{z}\right)}$

<u>But</u>: $\boxed{\vec{v}=+v\hat{x}}$ $\therefore$ $\boxed{\vec{B}'=-\frac{1}{c^2}\left(\vec{v}\times\vec{E}'\right)}$ in IRF(S') !!!

2.) If $\boxed{\vec{E}=0}$ in lab IRF(S), then in IRF(S') we have: $\boxed{\vec{E}'=-\gamma v\left(B_z\hat{y}-B_y\hat{z}\right)=-v\left(B'_z\hat{y}-B'_y\hat{z}\right)}$

<u>But</u>: $\boxed{\vec{v}=+v\hat{x}}$ $\therefore$ $\boxed{\vec{E}'=\vec{v}\times\vec{B}'}$ in IRF(S') $\leftarrow$ i.e. magnetic part of Lorentz force law !!!

<u>Griffiths Example 12.14</u>: <u>Magnetic Field of a Point Charge in Uniform Motion</u>

A point electric charge q moves with constant velocity $\boxed{\vec{v}=+v\hat{z}}$ in the lab frame IRF(S).

Find the magnetic field $\vec{B}$ in IRF(S) associated with this moving point electric charge.

Note that in the <u>rest</u> <u>frame</u> of the electric charge {IRF(S_0)} that $\boxed{\vec{B}_0=0}$ everywhere.

$\therefore$ In IRF(S) the electric charge q is moving with velocity $\boxed{\vec{v}=+v\hat{z}}$ and: $\boxed{\vec{B}=-\frac{1}{c^2}\left(\vec{v}\times\vec{E}\right)}$.

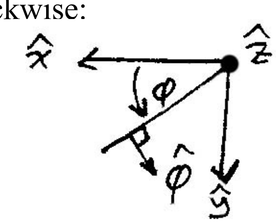
But the electric field of moving point charge q in IRF(S) is: $\boxed{\vec{E}=\frac{q}{4\pi\varepsilon_o}\frac{\left(1-\left(v/c\right)^2\right)}{\left(1-\left(v/c\right)^2\sin^2\theta\right)^{3/2}}\frac{\hat{R}}{R^2}}$ $\boxed{\vec{E}\parallel\hat{R}}$

(See pages 3-6 above/Griffiths Example 12.13)

$$\therefore \quad \boxed{\vec{B}=-\frac{1}{c^2}\left(\vec{v}\times\vec{E}\right)=\frac{\mu_o}{4\pi}\frac{qv\left(1-\left(v/c\right)^2\right)\sin\theta}{\left(1-\left(v/c\right)^2\sin^2\theta\right)^{3/2}}\left(\frac{1}{R^2}\right)\hat{\varphi}}$$

<u>where</u>: $\boxed{\theta=\cos^{-1}\left(\hat{v}\bullet\hat{R}\right)}$, $\boxed{\vec{v}\times\hat{R}=v\sin\theta\,\hat{\varphi}}$.

If the point electric charge q is heading directly towards you (i.e. along the $+\hat{z}$ direction) then $\hat{\varphi}$ aims counter-clockwise:



<u>NOTE</u>: In the non-relativistic limit ($v \ll c$): $\boxed{\vec{B}=\left(\frac{\mu_o}{4\pi}\right)q\left(\frac{\vec{v}\times\hat{R}}{R^2}\right)}$ $\Leftarrow$ The Biot-Savart Law for a moving point charge !!!

<u>NOTE ALSO</u>: H.C. Øersted discovered the link between electricity and magnetism in 1820. It wasn't until 1905, with Einstein's special relativity paper that a handful of humans on this planet finally understood the profound nature of this relationship – a timespan of 85 years – approximately a human lifetime passed! (see / read handout on Øersted)

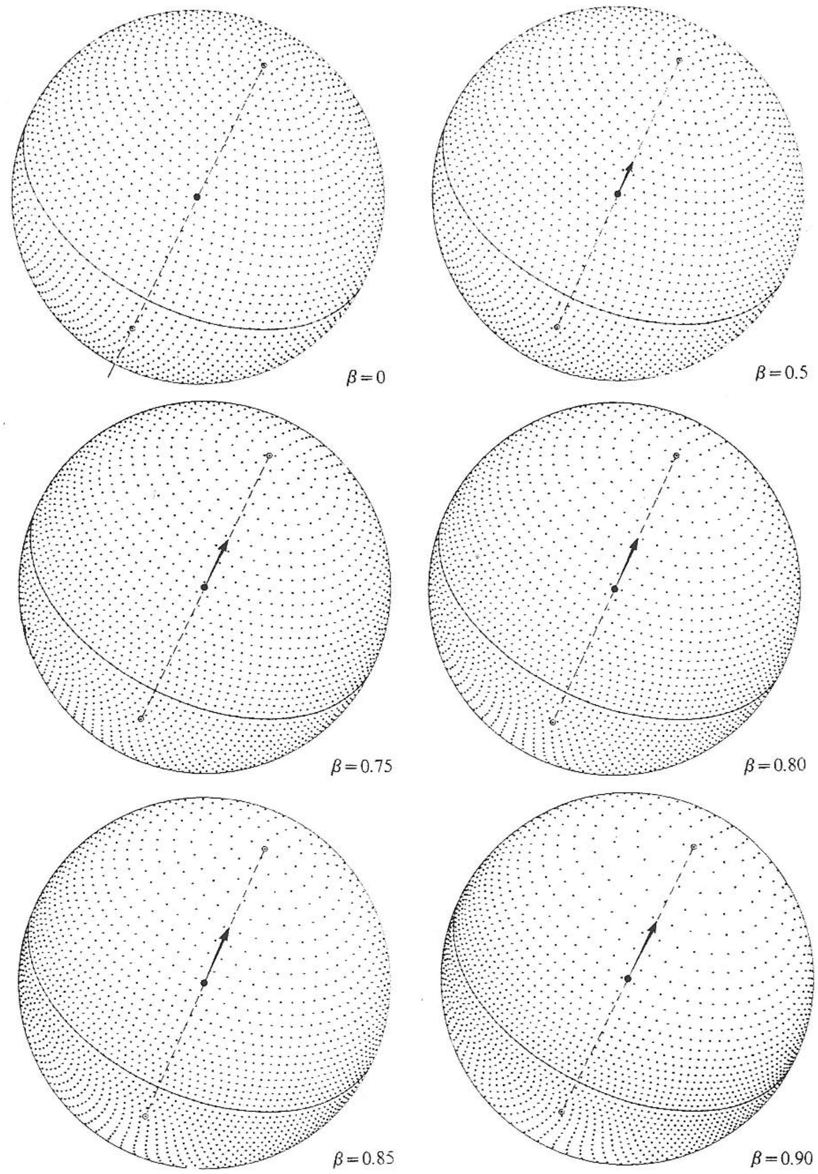
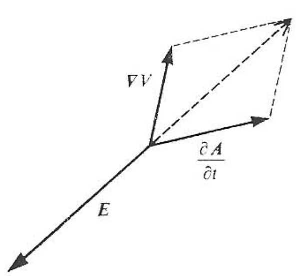


Figure 6-9. Lines of force for a charge Q moving along the diameter of an imaginary sphere. The dots show where lines emerge from the sphere at the instant when the charge is at its center. The density of the dots is a measure of the electric field intensity. The total number of dots is the same in all six figures, so as to satisfy Gauss's law (Section 6.8). Note how the field shifts to the region of $\theta = 90°$ as the velocity increases. For $\mathcal{V} = c$ the field is all concentrated at $\theta = 90°$.



$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}.$$

Figure 6-14. The electric field intensity is the vector sum of $-\nabla V$ and $-\partial A/\partial t$.

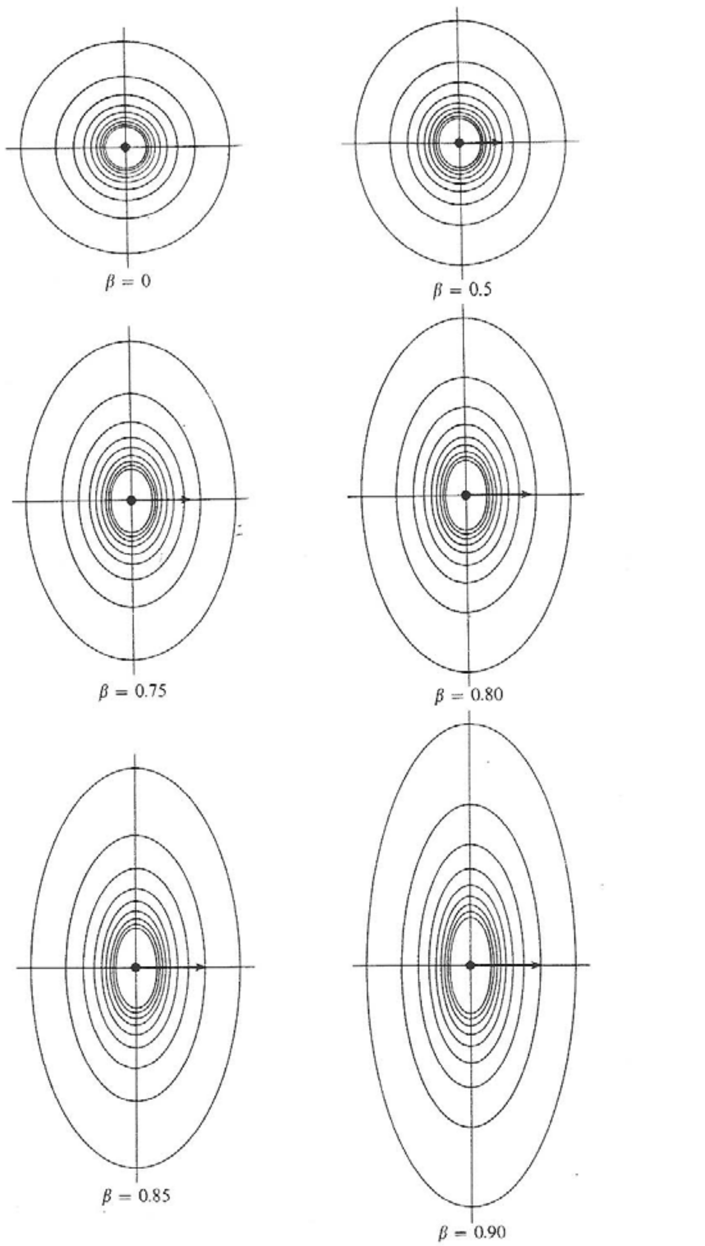


Figure 6-15. Equipotential surfaces $V =$ Constant for a point charge Q moving either to the right or to the left. The equipotentials near Q are not shown because they are too close together.

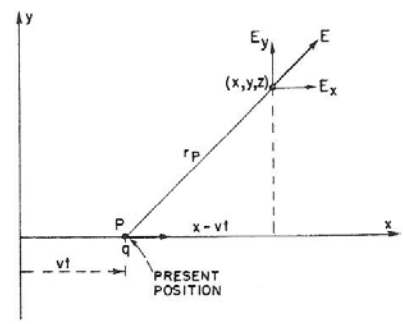


Fig. 26–3. For a charge moving with constant speed, the electric field points radially from the "present" position of the charge.

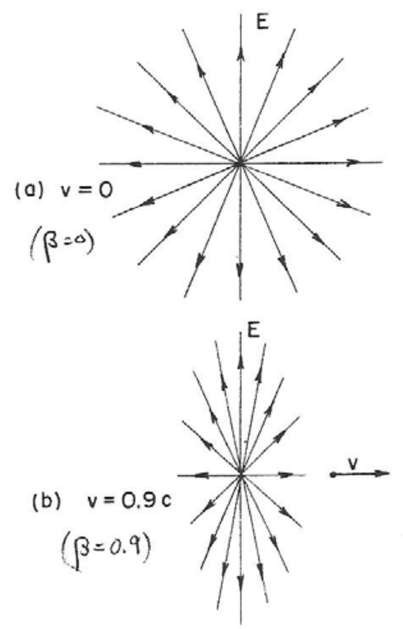


Fig. 26–4. The electric field of a charge moving with the constant speed $v = 0.9c$, part (b), compared with the field of a charge at rest, part (a).

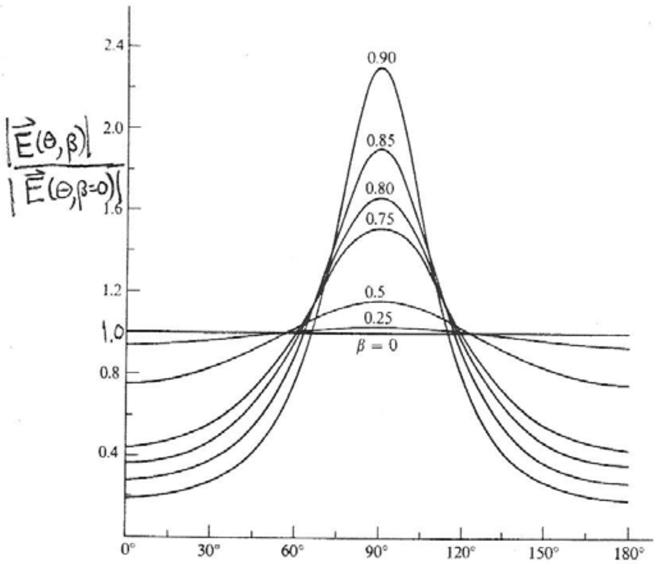


Figure 6-10. The electric field intensity E of a moving point charge as a function of the polar angle θ of Figure 6-6, for seven values of $\beta = \mathcal{V}/c$. The observer is stationary and sees the charge moving at the uniform velocity $\mathcal{V}$. For $\beta = 0$ the field is isotropic. It is hardly disturbed at $\beta = 0.25$. As the velocity increases the field increases near $\theta = 90°$ and decreases both ahead of the charge (near $\theta = 0$) and behind it (near $\theta = 180°$). At extremely high velocities most of the electric field is concentrated near $\theta = 90°$. These curves explain *qualitatively* the validity of Gauss's law for moving charges: as the velocity increases, the flux of $\boldsymbol{E}$ shifts from the regions where $\theta \approx 0$ and $\approx 180°$ to $\theta \approx 90°$ and the total flux of $\boldsymbol{E}$ remains constant. (Then why are the areas under the curves not equal?) Note that the electric field is always symmetrical about 90°. This means that there is no way of telling, from the shape of the field, whether the charge is moving to the right or to the left. The vertical scale gives E divided by $Q/4\pi\epsilon_0 r^2$.

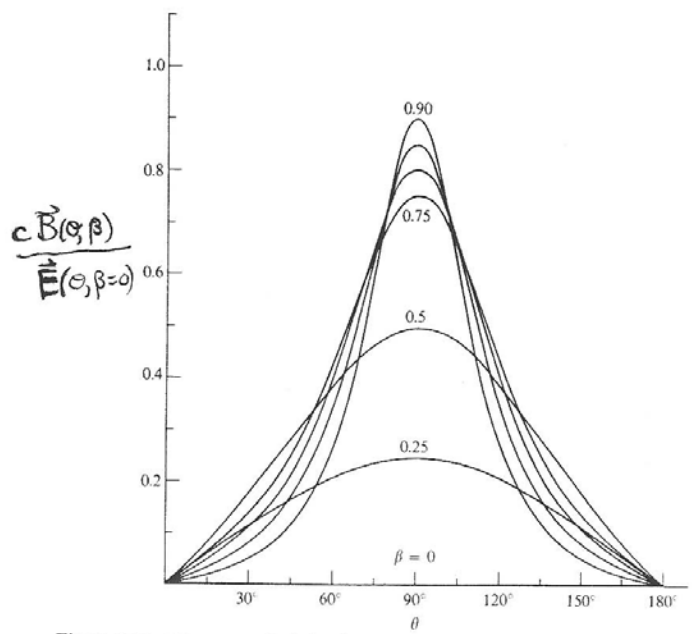


Figure 6-11. The magnetic induction B of a moving point charge as a function of the polar angle θ for seven values of β. For $\beta = 0$ there is no magnetic field. As β increases, B first increases at all angles. Then B continues to increase near $\theta = 90°$, while *de*creasing both ahead of the charge and behind it. At extremely high velocities, most of the magnetic field is concentrated near the plane $\theta = 90°$. Note that the magnetic field is symmetrical with respect to $\theta = 90°$. The vertical scale gives B divided by $\mu Q\gamma c/4\pi r^2$, and thus the maximum ordinate on any curve is β.

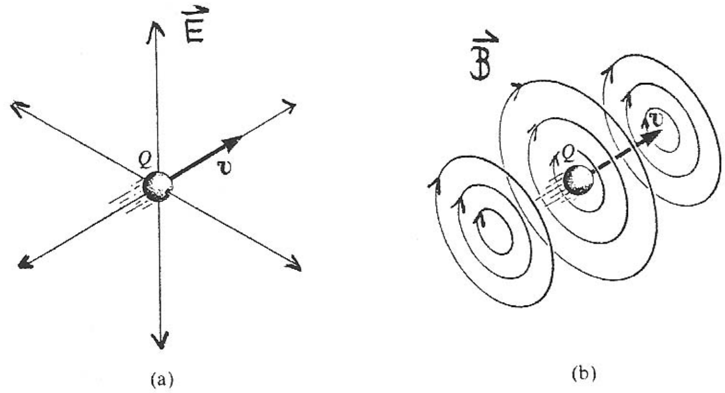


Figure 6-8. (a) Typical line of $\boldsymbol{E}$, and (b) typical lines of $\boldsymbol{B}$ for a charge Q moving at a constant velocity $\boldsymbol{v}$, as seen by a stationary observer. The electric field is radial. The lines of $\boldsymbol{B}$ are circles centered on the trajectory and perpendicular to it.

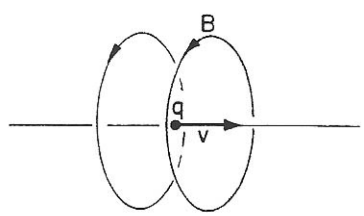


Fig. 26–5. The magnetic field near a moving charge is $\mathbf{v} \times \mathbf{E}$. (Compare with Fig. 26–4.)

The Electromagnetic Field Tensor $F^{\mu\nu}$

The relations for the Lorentz transformation of $\vec{E}$ and $\vec{B}$ in the lab frame IRF(S) to $\vec{E}'$ and $\vec{B}'$ in IRF(S'), where IRF(S') is moving with velocity $\boxed{\vec{v} = +v\hat{x}}$ relative to IRF(S) are:

$E^{\parallel}$ components: $\boxed{E'_x = E_x}$

$E^{\perp}$ components: $\boxed{E'_y = \gamma\left(E_y - \beta c B_z\right)}$, $\boxed{E'_z = \gamma\left(E_z + \beta c B_y\right)}$

$B^{\parallel}$ components: $\boxed{B'_x = B_x}$

$B^{\perp}$ components: $\boxed{B'_y = \gamma\left(B_y + \beta\, E_z/c\right)}$, $\boxed{B'_z = \gamma\left(B_z - \beta\, E_y/c\right)}$

$\boxed{\gamma = 1\big/\sqrt{1-\beta^2}}$ $\boxed{\beta = v/c}$

It is readily apparent that the $\vec{E}$ and $\vec{B}$ field certainly do not transform like the spatial / 3-D vector parts of e.g. two separate contravariant 4-vectors (E^{μ} and B^{μ}), because the (orthogonal) components of $\vec{E}$ and $\vec{B}$ $\perp$ to the direction of the Lorentz transformation are mixed together {as seen in the case for $\vec{B} = 0$ in IRF(S) resulting in $\boxed{\vec{B}' = -\frac{1}{c^2}\left(\vec{v} \times \vec{E}'\right)}$ in IRF(S') and the case for $\vec{E} = 0$ in IRF(S) resulting in $\boxed{\vec{E}' = \vec{v} \times \vec{B}'}$ in IRF(S')}.

Note also that the form of the Lorentz transformation for || vs. $\perp$ components is "switched" {here} for the fields !!!

Recall that true relativistic 4-vectors transform by the rule $\boxed{a'^{\mu} = \Lambda^{\mu}_{\gamma} a^{\nu}}$, e.g. for a Lorentz boost from IRF(S) $\rightarrow$ IRF(S') along $\boxed{\vec{v} = +v\hat{x}}$.

Specifically, recall that $\boxed{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}}$ is explicitly:

$$\boxed{ct' = \gamma\left(ct - \beta x\right)}$$

Parallel components: $\boxed{x' = \gamma\left(x - \beta ct\right)}$ $\boxed{E'_x = E_x}$ $\boxed{B'_x = B_x}$

Perpendicular Components: $\boxed{y' = y}$ $\boxed{E'_y = \gamma\left(E_y - \beta c B_z\right)}$ $\boxed{B'_y = \gamma\left(B_y + \beta E_z / c\right)}$

$\boxed{z' = z}$ $\boxed{E'_z = \gamma\left(E_z + \beta c B_y\right)}$ $\boxed{B'_z = \gamma\left(B_z - \beta E_y / c\right)}$

For a Lorentz transformation $\boxed{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}}$ from IRF(S) $\rightarrow$ IRF(S') along $\boxed{\vec{v} = +v\hat{x}}$, the Λ^{μ}_{ν} tensor has the form:

Row index

$$\boxed{\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Column index

Thus, there is *no* way that the $\vec{E}$ and $\vec{B}$ fields can be construed as being the spatial / 3-D vector components of contravariant 4-vectors E^{μ} and B^{μ}.

(What would be their temporal components/ scalar counterparts: E^0 and B^0 = ???)

It turns out that the 3-D spatial vectors $\vec{E}$ and $\vec{B}$ are the components of a 4 × 4 rank-two EM field tensor, $F^{\mu\nu}$!!!

A 4 × 4 rank two tensor $t^{\lambda\sigma}$ Lorentz transforms via *two* Λ-factors (one for *each* index):

$$\boxed{t'^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} t^{\lambda\sigma}}$$

Where $t^{\lambda\sigma}$ is:

Row index Column index

$$t^{\lambda\sigma} = \begin{array}{c|c} \text{Column \#}\quad 0 \quad 1 \quad 2 \quad 3 & \text{Row \#} \\ \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

The 16 components of the 4 × 4 second rank tensor $t^{\lambda\sigma}$ need not all be different (e.g. Λ^{μ}_{ν})
– the 16 components of $t^{\lambda\sigma}$ may have symmetry / anti-symmetry properties:

Symmetric 4 × 4 rank two tensor: $\boxed{t^{\lambda\sigma} = +t^{\sigma\lambda}}$
Anti-symmetric 4 × 4 rank two tensor: $\boxed{t^{\lambda\sigma} = -t^{\sigma\lambda}}$

- For the case of a <u>symmetric</u> 4 × 4 rank two tensor $\boxed{t^{\lambda\sigma} = +t^{\sigma\lambda}}$, of the 16 total components, 10 are <u>unique</u>, but 6 are <u>repeats</u>:

$$t^{01} = t^{10}, \quad t^{02} = t^{20}, \quad t^{03} = t^{30}$$
$$t^{12} = t^{21}, \quad t^{13} = t^{31}$$
$$t^{32} = t^{23}$$

$$\boxed{t^{\lambda\sigma}_{sym.} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{01} & t^{11} & t^{12} & t^{13} \\ t^{02} & t^{12} & t^{22} & t^{23} \\ t^{03} & t^{13} & t^{23} & t^{33} \end{pmatrix}}$$

- For the case of an <u>anti-symmetric</u> 4 × 4 rank two tensor $\boxed{t^{\lambda\sigma} = -t^{\sigma\lambda}}$, of the 16 total components, only 6 are <u>unique</u>, 6 are <u>repeats</u> (but with a minus sign!) and the 4 diagonal elements <u>*must*</u> all be ≡ 0 (i.e. $t^{00} = t^{11} = t^{22} = t^{33} \equiv 0$):

$$t^{01} = -t^{10}, \quad t^{02} = -t^{20}, \quad t^{03} = -t^{30}$$
$$t^{12} = -t^{21}, \quad t^{13} = -t^{31}$$
$$t^{32} = -t^{23}$$

$$\boxed{t^{\lambda\sigma}_{\substack{anti\\sym.}} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}}$$

So it would seem that the spatial / 3-D $\vec{E}$ and $\vec{B}$ vectors can be represented by an <u>anti-symmetric</u> 4 × 4 rank two contravariant tensor !!!

Let's investigate how the Lorentz transformation rule $\boxed{t'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}}$ for a 4 × 4 rank two anti-symmetric tensor $t^{\lambda\sigma}_{\substack{anti\\sym.}}$ works (6 unique non-zero components). Starting with $\boxed{t'^{01} = \Lambda^{0}_{\lambda}\Lambda^{1}_{\sigma}t^{\lambda\sigma}}$:

$$\Lambda^{\mu}_{\lambda}\left(\text{or }\Lambda^{\nu}_{\sigma}\right) = \begin{array}{c} \underline{\text{Column \# }\quad 0 \quad 1 \quad 2 \quad 3 \quad \text{Row \#}} \\ \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$\boxed{t^{\lambda\sigma}_{\substack{anti\\sym.}} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}}$$

We see that $\Lambda^0_{\lambda} = 0$, unless $\lambda = 0$ or 1: $\underset{\gamma}{\Lambda^0_0}$ or $\underset{-\gamma\beta}{\Lambda^0_1}$

Row # (upper index) · Column # (lower index)

We also see that $\Lambda^1_{\sigma} = 0$, unless $\sigma = 0$ or 1: $\underset{-\gamma\beta}{\Lambda^1_0}$ or $\underset{\gamma}{\Lambda^1_1}$

Row # (upper index) · Column # (lower index)

$\therefore$ There are only 4 non-zero terms in the sum: $\boxed{t'^{01} = \Lambda^0_{\lambda}\Lambda^1_{\sigma}t^{\lambda\sigma} = \Lambda^0_0\Lambda^1_0 t^{00} + \Lambda^0_0\Lambda^1_1 t^{01} + \Lambda^0_1\Lambda^1_0 t^{10} + \Lambda^0_1\Lambda^1_1 t^{11}}$

<u>But</u>: $t^{00} = 0$ and $t^{11} = 0$, whereas $t^{01} = -t^{10}$.

$$\therefore \boxed{t'^{01} = \left(\Lambda^0_0\Lambda^1_1 - \Lambda^0_1\Lambda^1_0\right)t^{01} = \left(\gamma\cdot\gamma - (-\gamma\beta)(-\gamma\beta)\right)t^{01} = \left(\gamma^2 - \gamma^2\beta^2\right)t^{01} = \gamma^2\left(1-\beta^2\right)t^{01}}$$

<u>But</u>: $\boxed{\gamma^2 = 1/1-\beta^2}$ $\therefore \boxed{t'^{01} = t^{01}}$

One (e.g. <u>*you*</u>!!!) can work through the remaining 5 of the 6 distinct cases for $\boxed{t'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}}$ (or for completeness' sake, why not work out all 16 cases explicitly!!!) . . .

The complete set of <u>*six*</u> rules for the Lorentz transformation of a 4 × 4 rank two contravariant anti-symmetric tensor $\boxed{t'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}}$ are:

$$\boxed{t'^{01} = t^{01}} \quad \leftrightarrow \quad \boxed{E'_x = E_x}$$
$$\boxed{t'^{02} = \gamma\left(t^{02} - \beta t^{12}\right)} \quad \leftrightarrow \quad \boxed{E'_y = \gamma\left(E_y - \beta cB_z\right)}$$
$$\boxed{t'^{03} = \gamma\left(t^{03} - \beta t^{31}\right)} \quad \leftrightarrow \quad \boxed{E'_z = \gamma\left(E_z + \beta cB_y\right)}$$
$$\boxed{t'^{23} = t^{23}} \quad \leftrightarrow \quad \boxed{B'_x = B_x}$$
$$\boxed{t'^{31} = \gamma\left(t^{31} + \beta t^{03}\right)} \quad \leftrightarrow \quad \boxed{B'_y = \gamma\left(B_y + \beta E_z/c\right)}$$
$$\boxed{t'^{12} = \gamma\left(t^{12} - \beta t^{02}\right)} \quad \leftrightarrow \quad \boxed{B'_z = \gamma\left(B_z - \beta E_y/c\right)}$$

As can be seen, the above Lorentz transformation rules for this tensor are indeed precisely those we derived on physical grounds for the *EM* fields!

Thus, we can now explicitly construct the <u>electromagnetic</u> <u>field</u> <u>tensor</u> $F^{\mu\nu}$ – a 4 × 4 rank two contravariant anti-symmetric tensor:

$$\boxed{F^{\mu\nu} \equiv \begin{pmatrix} F^{00}\equiv 0 & F^{01} & F^{02} & F^{03} \\ F^{10} = -F^{01} & F^{11}\equiv 0 & F^{12} & F^{13} = -F^{31} \\ F^{20} = -F^{02} & F^{21} = -F^{12} & F^{22}\equiv 0 & F^{23} \\ F^{30} = -F^{03} & F^{31} & F^{32} = -F^{23} & F^{33}\equiv 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{pmatrix}}$$

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{pmatrix}}$$

E-field representation of $F^{\mu\nu}$ (SI units: Volts/m). Note the space-time structure of anti-symmetric $F^{\mu\nu}$ – all non-zero elements have space-time attributes – there are no time-time or space-space components!!!

Note that there are simple alternate / equivalent forms of the field tensor $F^{\mu\nu}$.
If we divide our $F^{\mu\nu}$ by c, we get Griffith's definition of $F^{\mu\nu}$:

i.e. $$\boxed{F^{\mu\nu}_{Griffiths} = \frac{1}{c}F^{\mu\nu}_{Ours} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}}$$

B-field representation of $F^{\mu\nu}$ (SI units: Tesla)

Note that there exists an equivalent, but different way of *embedding* $\vec{E}$ and $\vec{B}$ in an anti-symmetric 4 × 4 rank two tensor:

Instead of comparing:

a) $\boxed{E_x' = E_x}$, $\boxed{E_y' = \gamma\left(E_y - \beta cB_z\right)}$, $\boxed{E_z' = \gamma\left(E_z + \beta cB_y\right)}$

with: a') $\boxed{t'^{01} = t^{01}}$, $\boxed{t'^{02} = \gamma\left(t^{02} - \beta t^{12}\right)}$, $\boxed{t'^{03} = \gamma\left(t^{03} + \beta t^{31}\right)}$

and also comparing:

b) $\boxed{cB_x' = cB_x}$, $\boxed{cB_y' = \gamma\left(cB_y + \beta E_z\right)}$, $\boxed{cB_z' = \gamma\left(cB_z - \beta E_y\right)}$

with: b') $\boxed{t'^{23} = t^{23}}$, $\boxed{t'^{31} = \gamma\left(t^{31} + \beta t^{03}\right)}$, $\boxed{t'^{12} = \gamma\left(t^{12} - \beta t^{02}\right)}$

We could instead compare {a) with b')} and {b) with a')}, and thereby obtain the so-called anti-symmetric rank two *dual tensor* $G^{\mu\nu}$:

$$\boxed{G^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}}$$

E-field representation of $G^{\mu\nu}$ (SI units: Volts/m)

Note that the $G^{\mu\nu}$ elements can be obtained *directly* from $F^{\mu\nu}$ by carrying out a duality transformation (!!!): $\boxed{\vec{E} \to c\vec{B},\ \ c\vec{B} \to -\vec{E}}$

Note that the duality transformation leaves the Lorentz transformation of $\vec{E}$ and $\vec{B}$ unchanged !!! (i.e. ϕ duality = 90°).

Again, note that our representation of the dual tensor $G^{\mu\nu}$ is simply related to Griffiths by dividing ours by c, i.e.:

n.b. For the sake of compatibility with Griffiths book, we will use his definitions of the field strength tensors. Please be aware that these definitions *do* vary in different books!

$$\boxed{G^{\mu\nu}_{Griffiths} = \frac{1}{c}G^{\mu\nu}_{ours} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}}$$

B-field representation of $G^{\mu\nu}$ (SI units: Tesla)

With Einstein's 1905 publication of his paper on special relativity, physicists of that era soon realized that the $\vec{E}$ and $\vec{B}$-fields were indeed intimately related to each other, because of the unique structure of space-time that is associated with the universe in which we live.

Prior to Ørsted's 1820 discovery that there was indeed a relation between electric vs. magnetic phenomena, physicists thought that electricity and magnetism were separate / distinct entities. Maxwell's theory of electromagnetism "unified" these two phenomena as one, but it was not until Einstein's 1905 paper, that physicists truly understood why they were so related!

It is indeed amazing that the electromagnetic field is a 4 × 4 rank two anti-symmetric contravariant tensor, $F^{\mu\nu}$ (or equivalently, $G^{\mu\nu}$) !!! (*six* components of which are unique.)

- We have learned that the 4-vector product of any two (bona-fide) relativistiv 4-vectors, e.g. $\boxed{a^\mu b_\mu = a_\mu b^\mu}$ = Lorentz invariant quantity (i.e. $\boxed{a^\mu b_\mu = a_\mu b^\mu}$ = same value in all / any IRF)

- Likewise, the tensor product $\boxed{A^{\mu\nu}B_{\mu\nu} = A_{\mu\nu}B^{\mu\nu}}$ of any two relativistic rank-2 tensors is also a Lorentz invariant quantity.

- Thus, the tensor products of *EM* field tensors $F^{\mu\nu}$ and $G^{\mu\nu}$, namely $\boxed{F^{\mu\nu}F_{\mu\nu}}$, $\boxed{G^{\mu\nu}G_{\mu\nu}}$, and $\boxed{F^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}G^{\mu\nu}}$ are *all* Lorentz invariant quantities.

- Specifically, it can be shown that (see Griffiths Problem 12.50, page 537 – P436 HW#14):

$$\boxed{F^{\mu\nu}F_{\mu\nu} = -G^{\mu\nu}G_{\mu\nu} = 2\left(B^2 - \frac{1}{c^2}E^2\right) \propto \left(u_M - u_E\right)} \leftarrow \text{Lorentz invariant quantity !!!}$$

And: $$\boxed{F^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}G^{\mu\nu} = -\frac{4}{c}\left(\vec{E}\cdot\vec{B}\right)} \leftarrow \text{Lorentz invariant quantity !!!}$$

- Obviously, $\boxed{F^{\mu\nu}F_{\mu\nu} = F_{\mu\nu}F^{\mu\nu}}$, $\boxed{G^{\mu\nu}G_{\mu\nu} = G_{\mu\nu}G^{\mu\nu}}$ and $\boxed{F^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}G^{\mu\nu}}$ are the *only* Lorentz invariants that can be formed / constructed using $F^{\mu\nu}$ and $G^{\mu\nu}$!!!

- Note that this is by no means as obvious / clear, trying to form Lorentz invariant quantities starting *directly* from the $\vec{E}$ and $\vec{B}$ fields themselves !!!

⇒ n.b. Since *EM* energy density $\boxed{u_{EM} = \tfrac{1}{2}\varepsilon_o E^2 + \tfrac{1}{2\mu_o}B^2}$, Poynting's vector $\boxed{\vec{S} = \tfrac{1}{\mu_o}\vec{E}\times\vec{B}}$, *EM* field linear momentum density $\boxed{\vec{\wp}_{EM} = \varepsilon_o\mu_o\vec{S}}$ and *EM* field angular momentum density $\boxed{\vec{\ell}_{EM} = \vec{r}\times\vec{\wp}_{EM}}$ *cannot* be formed from *any* linear combinations of $\boxed{F^{\mu\nu}F_{\mu\nu}}$, $\boxed{G^{\mu\nu}G_{\mu\nu}}$ or $\boxed{F^{\mu\nu}G_{\mu\nu}}$, then u_{EM}, $\vec{S}$, $\vec{\wp}_{EM}$ and $\vec{\ell}_{EM}$ *cannot* be Lorentz invariant quantities!

