LECTURE NOTES 15

Electrodynamics and Relativity

In the macroscopic "everyday" world $(v \ll c)$ we are accustomed to living in, we know that the classical laws of mechanical physics obey Galileo's notion (or principle) of classical relativity, as long as we are always in an inertial reference frame (*i.e.* a non-accelerating reference frame).

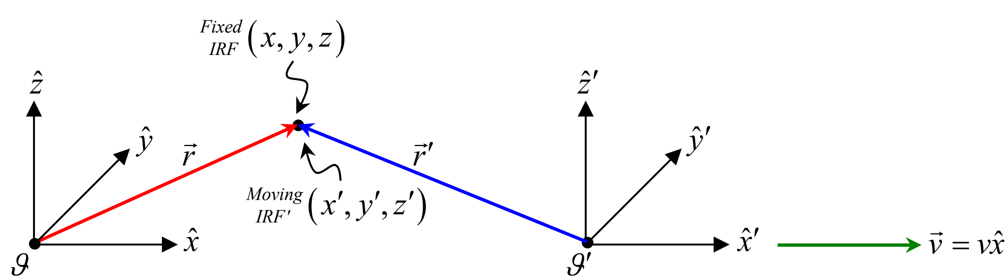
Newton's First Law of Motion holds in an inertial reference frame (IRF):

> "An object at rest remains at rest, and an object moving with (constant) speed v remains moving at (constant) speed v and in the same direction (*i.e.* $v =$constant in an IRF), unless acted upon by a net/non-zero/ unbalanced force"

$$\Rightarrow \quad \vec{F}_{net} = 0 \text{ in an IRF } (\vec{a} = 0), \qquad \underbrace{\vec{F}_{net} \neq 0 = m\vec{a}}_{\text{Newton's 2nd Law.}} \text{ in a non-IRF } (\vec{a} \neq 0)$$

In a Galilean transformation between two inertial references frames, *e.g.* one fixed $(v = 0)$ and one moving $(v' \neq 0)$ along the $\hat{x}$-axis. The two reference frames coincide at time $t = 0$:

Fixed IRF: Moving IRF′:



Fixed IRF: Moving IRF′:

$\boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$ $\boxed{\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}' = (x - vt)\hat{x} + y\hat{y} + z\hat{z}}$ $\boxed{\hat{x}' \parallel \hat{x}}$, $\boxed{\hat{y}' \parallel \hat{y}}$, $\boxed{\hat{z}' \parallel \hat{z}}$

$\boxed{x' = x - vt}$, $\boxed{y' = y}$, $\boxed{z' = z}$ and: $\boxed{t' \equiv t}$

In a Galilean transformation, separation distances (spatial intervals): $\boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}}$ and time differences (temporal intervals): $\boxed{\Delta t = t_2 - t_1}$ are the same / identical in *all* inertial reference frames, *i.e.* $\boxed{\Delta d' = \Delta d}$, and $\boxed{\Delta t' = \Delta t}$.

Thus: $\boxed{\Delta d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}} = \boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}}$

And: $\boxed{\Delta t' = t_2' - t_1'} = \boxed{\Delta t = t_2 - t_1}$

No matter how fast an object is moving, in a Galilean transformation, spatial separation distances are unchanged/constant and the rate of passage of time is also unchanged/constant.

Note that in Galilean physics $\exists$ there also exists no notion of an absolute IRF.

Does the principle of relativity also apply to electrodynamics? *i.e.* are the physical laws of *E & M* also valid/same in all inertial reference frames?

One might initially be tempted to say *no*, because *e.g.* a stationary/fixed charge in one IRF_1 has only a static electric field $\vec{E}$=constant associated with it, whereas an observer in another IRF_2 moving at constant velocity, $\vec{v}$ with respect to the first/fixed IRF_1 would see a magnetic field $\vec{B}$ associated with the (moving) charge. $\Rightarrow$ *EM* theory pre-supposes $\exists$ a unique IRF (stationary) from which/with respect to which all velocities should be measured.

However, this notion is wrong/incorrect!

Another example: Suppose we have a static magnetic field $\boxed{\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})}$ in (fixed) IRF_1 *e.g.* due to permanent bar magnet, $\boxed{\left|\vec{B}(\vec{r})\right| \sim 1/r^3}$ where r = observer distance from the bar magnet. An observer in a moving IRF_2 passing by the permanent magnet will observe a (time varying) electric field due to $\boxed{\vec{E}(\vec{r}') = -\partial\vec{A}(\vec{r}')/\partial t}$ in her/his IRF_2.

Another example: In a fixed IRF_1, move a circular conducting loop (of radius *a*) *e.g.* at a constant velocity from a $\vec{B} = 0$ region into a $\vec{B} \neq 0$ region. Get an induced *EMF* $\boxed{\varepsilon_1^{IRF_1}(t) = -d\Phi_m(t)/dt}$. However, in IRF_2 of the loop (*e.g.*, imagine observer is now sitting in the center of loop as the loop is moved from the $\vec{B} = 0$ region into the $\vec{B} \neq 0$ region), the observer in IRF_2 will "see" a time-varying magnetic field, which (by Faraday's Law) creates an electric field which induces {precisely} the same *EMF* (voltage/potential difference around the loop, $\boxed{\varepsilon_2^{IRF_2}(t') \equiv \varepsilon_1^{IRF_1}(t)}$.

Physicists in Maxwell's time (mid/late 1800's $\rightarrow$ early 1900's) grappled with the principle of relativity and electrodynamics – the consensus thinking at that time was that the $\vec{E}$ and $\vec{B}$-fields were "strains" in an invisible, all-pervasive/all-permeating medium known as the æther (a "jelly-like" substance, which also simultaneously had to be ~ infinitely rigid {because the of speed of light, $c = 1/\sqrt{\varepsilon_o \mu_o} = 3\times10^8\ m/s$ was already known to be very high at that time}. Transverse electromagnetic waves could not propagate without being immersed in such a medium, or so they thought….

The "absolute" IRF, then, was the one in which the æther medium was at *rest*, *i.e.* the rest frame of the æther.

Michelson and Morley's famous æther drift experiment carried out in the late 1880's – to accurately measure the earth's speed *w.r.t.* æther – this was a null result!! They found that the speed of light *c* was the same in all directions. This situation was not resolved for ~ 20 years, despite many theoretical and experimental efforts. All kinds of (crazy) things were proposed theoretically and investigated experimentally…. (*n.b.* ~ 100 years from now, perhaps some of *today's* current "theories" may *also* be viewed to be just as crazy …. $\Leftarrow$ think about this !!!)

It is certainly a credit to the genius and intellect of Albert Einstein, taking in all of what was then currently known theoretically and experimentally, to successfully develop his initial theory of special relativity (IRF's only) and then later, to general relativity (including non-IRF's).

Einstein's two *postulates* of special relativity:

1) Principle of Relativity: Laws of physics apply/are the same in all IRF's

2) Speed of light $\boxed{c = 1/\sqrt{\varepsilon_o \mu_o} = 3\times 10^8\ m/s}$ (in vacuum) ≡ same in *all* IRF's for *all* observers, *regardless* of the motion (*i.e.* the speed) of the source.

Einstein's 1st postulate elevates Galileo's Principle of Relativity (for classical mechanics) to encompass *all* physics. ⇒ ∃ NO æther medium / ∃ NO no absolute IRF for which *EM* waves "need" to propagate in.

Einstein's 2nd postulate has {even more} "brain-numbing" consequences for "mere mortals":

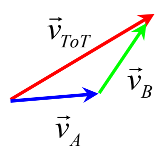
a) Spatial intervals $\boxed{\Delta d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 z_1)^2}}$ vs. $\boxed{\Delta d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}}$ are NOT the same in different IRF's, *i.e.* $\boxed{\Delta d \neq \Delta d'}$!!!

b) Temporal intervals $\boxed{\Delta t = t_2 - t_1}$ vs. $\boxed{\Delta t' = t_2' - t_1'}$ are NOT the same in different IRF's, *i.e.* $\boxed{\Delta t \neq \Delta t'}$!!!

In special relativity, space *and* time are treated on an *equal* footing with each other, and so what *is* conserved/preserved is the so-called *space-time* interval, as defined below:

$$\boxed{I \equiv (\Delta d)^2 - (c\Delta t)^2} = \boxed{I' \equiv (\Delta d')^2 - (c\Delta t')^2}$$

In Galilean Relativity (Euclidean Space): The Velocity Addition Rule is Simple Vector Addition:



$$\boxed{\vec{v}_{tot} = \vec{v}_A + \vec{v}_B}$$

e.g. A man walking down corridor of train at 5 mph relative to the train, but the train is moving at 50 mph relative to the ground:

$\boxed{v_{\text{ground}}^{\text{man}} = v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}$ parallel or anti-parallel velocity vector addition:

$$v_{\text{ground}}^{\text{man}} = 50 \pm 5 \text{ mph} = \begin{cases} 45 \text{ mph (man walking to the back of the train)} \\ 55 \text{ mph (man walking to the front of the train)} \end{cases}$$

In Galilean Relativity, a beam of light emitted from a flashlight on a moving train will travel faster (or slower) than a beam of light shone from a flashlight on the ground:

Flashlight on train:	$v_{\text{ground}}^{\text{light}} = v_{\text{ground}}^{\text{train}} + c$	$+\hat{x}$ direction	*vs.*	Flashlight on ground:	$v_{\text{ground}}^{\text{light}} = +c$	$+\hat{x}$ direction
	$v_{\text{ground}}^{\text{light}} = v_{\text{ground}}^{\text{train}} - c$	$-\hat{x}$ direction			$v_{\text{ground}}^{\text{light}} = -c$	$-\hat{x}$ direction

Einstein's 2nd Postulate of Special Relativity says this ***doesn't*** happen!

$\Rightarrow$ Correct Einsteinian / Special Relativity Velocity Addition Formula (1-Dimension) is:

$$\boxed{v_{\text{ground}}^{\text{man}} = \frac{v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \left(\frac{v_{\text{ground}}^{\text{train}} \cdot v_{\text{train}}^{\text{man}}}{c^2}\right)\right]}}$$ for parallel and/or anti-parallel velocity addition.

n.b. If $\boxed{v' \ll c}$, then in *this* limit, we obtain the Galilean Velocity Addition Rule – everyday world !

If the train is moving at the speed of light (with respect to the ground) then $\boxed{v_{\text{ground}}^{\text{train}} = c,}$ and:

$$\boxed{v_{\text{ground}}^{\text{man}} = \frac{c \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \frac{c \cdot v_{\text{train}}^{\text{man}}}{c^2}\right]} = \frac{\left(c \pm v_{\text{train}}^{\text{man}}\right)}{\left(1 \pm \frac{v_{\text{train}}^{\text{man}}}{c}\right)} = \frac{c\left(c \pm v_{\text{train}}^{\text{man}}\right)}{c\left(1 \pm \frac{v_{\text{train}}^{\text{man}}}{c}\right)} = \frac{c\,\cancel{\left(c \pm v_{\text{train}}^{\text{man}}\right)}}{\cancel{\left(c \pm v_{\text{train}}^{\text{man}}\right)}} = c}\ !!!$$

While Einstein was initially motivated by relativity issues concerning electrodynamics, relativity actually addresses fundamental nature of space-time aspects of the universe in which we live – thereby encompassing all physics, all fundamental forces of nature!

As a consequence of this, the "speed of light", c (in vacuum) is not just the maximum speed of *EM* waves /*EM* signals {only}, c is the maximum speed of any/all waves/signals and/or particles, irrespective of their nature – because this maximum speed of propagation has to do with the nature of space-time.

Thus, the relativity of space-time tells us that: $\boxed{c = c_{EM} = c_{\text{grav}} = c_{\text{weak}} = c_{\text{strong}}}$
i.e. the speed of "light" c is independent of/does not depend on the type of force!

However, in E&M, we do have the relation $\boxed{c = 1/\sqrt{\varepsilon_o \mu_o} = c_{EM}}$ where ε_o and μ_o are macroscopic *EM* properties of the vacuum / "empty", matter-free space:

$\boxed{\mathscr{C} = C/\ell}$: $\Rightarrow$ $\boxed{\varepsilon_o = 8.85\times10^{-12}\ \text{Farads}/m}$ = electric permittivity of free space/vacuum

$\boxed{\mathscr{L} = L/\ell}$: $\Rightarrow$ $\boxed{\mu_o = 4\pi\times10^{-7}\ \text{Henrys}/m}$ = magnetic permeability of free space/vacuum

And: $\boxed{\mathbb{Z}_o = \sqrt{\mu_o/\varepsilon_o} = 120\pi\ \Omega \simeq 377\ \Omega}$ = impedance of free space/vacuum

The macroscopic *EM* parameters of free space, ε_o and μ_o *are* {intimately} related/connected to the microscopic QED properties of the vacuum – *i.e.* the electrically-charged, virtual, particle-antiparticle pairs: e^+e^-, $\mu^+\mu^-$, $\tau^+\tau^-$, $d\bar{d}$, $u\bar{u}$, $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, $t\bar{t}$ and W^+W^-.

However, the virtual particle-antiparticle pairs of the vacuum *also* carry weak charges & weak moments (*all* spin-½ leptons & quarks, the spin-1 $W^{\pm}$, Z^0 bosons), strong charges & strong moments (*all* quark-antiquark pairs $\left(q\bar{q},\ q = d,u,s,c,b,t\right)$, and these particles also have mass.

Thus: Q-Gravity: QWD: QCD: QED:

$$\boxed{c_{\text{grav}} = 1\Big/\sqrt{\varepsilon_{\text{g}}\mu_{\text{g}}}\,,\ = c_{\text{weak}} = 1\Big/\sqrt{\varepsilon_{\text{w}}\mu_{\text{w}}}\,, = c_{\text{strong}} = 1\Big/\sqrt{\varepsilon_{\text{s}}\mu_{\text{s}}}\,, = c_{EM} = 1\Big/\sqrt{\varepsilon_o\mu_o} = c}$$

and: $$\boxed{\mathbb{Z}_{\text{grav}} = \sqrt{\frac{\mu_{\text{g}}}{\varepsilon_{\text{g}}}} = ??,\ \mathbb{Z}_{\text{weak}} = \sqrt{\frac{\mu_{\text{w}}}{\varepsilon_{\text{w}}}} = ??,\ \mathbb{Z}_{\text{strong}} = \sqrt{\frac{\mu_{\text{s}}}{\varepsilon_{\text{s}}}} = ??,\ \mathbb{Z}_o = \sqrt{\frac{\varepsilon_o}{\mu_o}} = 120\pi\,\Omega}$$

$\Rightarrow$ Implies *deep* connections between the four fundamental forces of nature (and inter-relations between them – unification?) – ***and*** – space-time!!!

The Geometry of Relativity

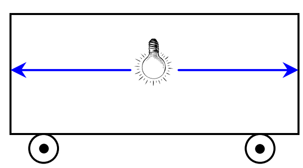
As mentioned earlier, Einstein's postulates (*esp*. #2) have several striking, "non-everyday" consequences:

a) Space intervals Δd are *not* the same in all IRF's $\Rightarrow$ Lorentz contraction in (boosted) IRF's
b) Time intervals Δt are *not* the same in all IRF's $\Rightarrow$ time dilation, and the relativity of simultaneity

In order to elucidate these phenomena, we consider a series of "gedanken" (thought) experiments:

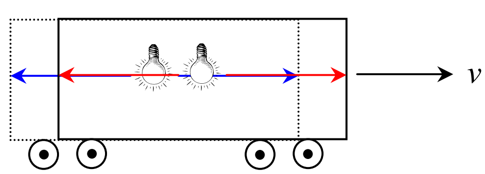
The Relativity of Simultaneity

1st Gedanken Experiment: Consider *e.g*. a freight train moving at constant speed, v along a smooth, straight railroad track. In the center of one boxcar of the freight train is a light bulb, as shown in the figure below:

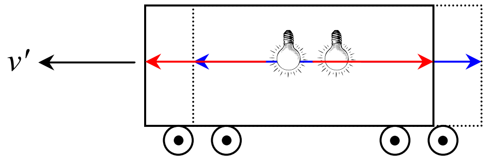


When the light bulb is switched on, light from the bulb spreads out in all directions / 4π steradians at the speed of light, c. Since the light bulb is equidistant from the two ends of the freight car, an observer on the train (*i.e*. in the IRF′ of the train) will find that the light reaches the front end of the freight car simultaneously with light reaching the back end of the freight car (consistent with our "everyday" world/Galilean experience) – these two "events" are simultaneous in this IRF.

However, an observer on the ground, watching the train go by with the light bulb turned on will *not* see these two "events" as being simultaneous. From his/her perspective, he/she sees the train moving forward, and therefore in this observer's IRF, since the beam of light heading towards the back of the freight car has a shorter distance to travel than the beam of light heading towards the front of the freight car, the observer on the ground will see the light hit the back of the freight car *before* the light hits the front of the freight car, as shown in the figure below:



Another, 3rd observer on an express train (*e.g.* moving much faster than the freight train and moving in the same direction) "sees" the freight train from his/her IRF$''$ as "going backwards", hence would "see" the light from the light bulb hit the front of the freight car before the light hits the back of the freight car, as shown in the figure below:



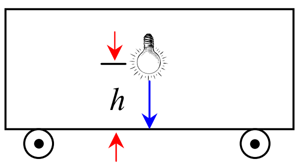
$\Rightarrow$ Two "events" that are simultaneous in one IRF are not in general simultaneous in other IRF's

n.b. If c was *e.g.* 100 m/s (and not 3×10^8 m/s) we all would have noticed/realized this, long ago...

Time Dilation:

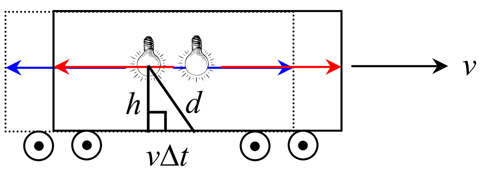
2nd Gedanken Experiment: How long a time interval does it take for light to go from the light bulb *e.g.* to the floor of the freight car?

To an observer on the train (*i.e.* in the rest frame/IRF$'$ of the train):



$\boxed{h = c\Delta t'}$ where h = height of light bulb above floor of train, thus: $\boxed{\Delta t' = h/c}$

To a stationary observer on the ground/in the ground/ "laboratory" IRF, he/she sees:



$\boxed{v\Delta t}$ = horizontal distance train moves in time interval Δt

Light (from the ground observer's perspective) has to travel a distance d to reach the floor:

$\boxed{d = \sqrt{h^2 + (v\Delta t)^2}}$, which takes a time interval $\boxed{\Delta t = \frac{d}{c} = \frac{\sqrt{h^2+(v\Delta t)^2}}{c} > \Delta t' = \frac{h}{c}}$ for the light to hit the floor of freight car, from the ground observer's perspective.

$\Rightarrow$ ***Moving*** clocks run ***slow*** $\Rightarrow$ time dilation !!!

Solve for the time interval Δt: $\boxed{(c\Delta t)^2 = d^2 = h^2 + (v\Delta t)^2}$ $\Rightarrow$ $\boxed{(c^2 - v^2)\Delta t^2 = h^2}$

Thus: $\boxed{\Delta t = \frac{h}{\sqrt{c^2 - v^2}} = \frac{h/c}{\sqrt{1-(v/c)^2}} = \frac{h/c}{\sqrt{1-\beta^2}} = \gamma\, h/c}$ where: $\boxed{\beta \equiv v/c}$ and: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$

But: $\boxed{\Delta t' = h/c}$ $\therefore$ $\boxed{\Delta t = \dfrac{\Delta t}{\sqrt{1-\beta^2}} = \gamma \Delta t'}$ or: $\boxed{\Delta t' = \dfrac{1}{\gamma}\Delta t}$ where: $\boxed{1 \le \gamma \le \infty}$

$\boxed{\beta = 0}$ $\boxed{\beta = 1}$

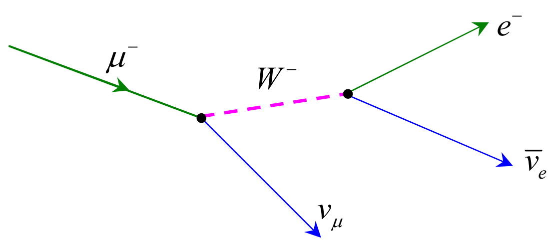
Thus: $\boxed{\Delta t = \gamma \Delta t' \ge \Delta t'}$ or: $\boxed{\Delta t' = \dfrac{1}{\gamma}\Delta t \le \Delta t}$

Time Dilation:

Griffiths Example 12.1:

A muon $\left(\mu^{\pm}\right)$ has a mean lifetime (it is unstable) of $\boxed{\tau'_\mu \simeq 2.2 \mu s}$ (in its own rest frame, IRF′)

Muons $\left.\begin{array}{l} \mu^+ \to e^+ \nu_e \bar{\nu}_\mu \\ \mu^- \to e^- \bar{\nu}_e \nu_\mu \end{array}\right\}$ decay via the charged weak interaction (*i.e.* mediated via the $W^{\pm}$ boson):



> The muon is a heavier "cousin" of the electron – it has a rest mass of
> $m_\mu \simeq 105.66\ MeV/c^2$
> $\left(m_e = 0.511\ MeV/c^2\right)$
> $\left(1\ MeV/c^2 = 10^6\ eV/c^2\right)$

If a muon is traveling at 3/5 of the speed of light in the laboratory (*i.e.* the ground) reference frame, what is the mean lifetime of muon observed in the lab reference frame {IRF}?

From above: $\boxed{\Delta t = \gamma \Delta t'}$ $\Rightarrow$ $\boxed{\tau_\mu = \gamma_\mu \tau'_\mu}$, where: $\boxed{\tau'_\mu \simeq 2.2 \mu s}$. What is γ_μ?

$\boxed{\gamma_\mu = \dfrac{1}{\sqrt{1-\beta_\mu^2}}}$, and: $\boxed{\beta_\mu = \dfrac{v}{c} = \dfrac{3}{5}}$

$$\therefore \quad \boxed{\gamma_\mu = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{1}{\sqrt{1-(9/25)}} = \frac{1}{\sqrt{16/25}} = \frac{1}{\sqrt{(4/5)^2}} = \frac{5}{4} \quad (\ge 1)}$$

$\therefore$ $\boxed{\tau_\mu = \gamma_\mu \tau'_\mu = \dfrac{5}{4}\tau'_\mu}$ *i.e.* $\boxed{\tau_\mu = \dfrac{5}{4} \times 2.2 \mu s = 2.75 \mu s}$ in the lab frame {IRF}.

At Fermilab, beams of muons *e.g.* with momentum of $p_\mu = 211.32\ GeV/c$ $\left(1\,GeV = 10^9\,eV\right)$ can easily be produced. What is the mean lifetime of these muons, as observed in the lab frame?

Again: $\boxed{\tau_\mu = \gamma_\mu \tau'_\mu}$, and the relativistic momentum of the muon in the lab frame {IRF} is:

$$\boxed{p_\mu c = \gamma_\mu \beta_\mu m_\mu c^2 = 211.32\,GeV}$$

$$\therefore \quad \boxed{\gamma_\mu \beta_\mu = \frac{p_\mu c}{m_\mu c^2} = \frac{211.32\,GeV}{105.66\,MeV} = 2.0\times 1000 = 2000}$$

Thus: $\boxed{\gamma_\mu \beta_\mu = \dfrac{\beta_\mu}{\sqrt{1-\beta_\mu^2}} = 2000}$ or: $\boxed{\dfrac{\beta_\mu^2}{1-\beta_\mu^2} = (2000)^2}$

Solve for β_μ: $\boxed{\beta_\mu^2 = (2000)^2\left(1-\beta_\mu^2\right) = (2000)^2 - (2000)^2\beta_\mu^2} \Rightarrow \boxed{\left\{(2000)^2+1\right\}\beta_\mu^2 = (2000)^2}$

Thus: $\boxed{\beta_\mu^2 = \dfrac{(2000)^2}{(2000)^2+1}} \Rightarrow \boxed{\beta_\mu = \sqrt{\dfrac{(2000)^2}{(2000)^2+1}}}$ or: $\boxed{\beta_\mu = 0.\underbrace{999999}_{\text{6 nines}} = \dfrac{v_\mu}{c}} \Rightarrow \boxed{v_\mu = 0.999999875c}$

$$\therefore \quad \boxed{\gamma_\mu = \frac{1}{\sqrt{1-\beta_\mu^2}} \simeq \gamma_\mu\beta_\mu} = 2000 \text{ !!!}$$

$$\therefore \quad \boxed{\tau_\mu = \gamma_\mu \tau'_\mu = 2000\times 2.2\mu s = 4.4\,ms = 0.0044\sec} \text{ in the lab IRF !!!}$$

$\Rightarrow$ A $p_\mu = 211.32\;GeV/c$ muon lives on average 2000× longer in the lab frame {IRF} than in its own rest frame {IRF'}.

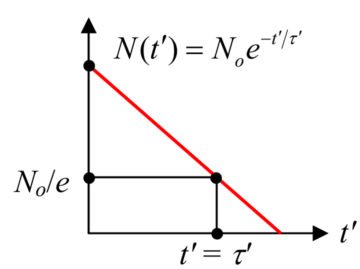
The "proper" decay length of an unstable particle (such as the muon) is $\boxed{\ell' \equiv c\tau'}$, where τ' = mean lifetime {*a.k.a.* "proper" lifetime}of the particle in its own rest frame {IRF'}.

n.b. In the particle's own rest frame {IRF'}, the particle is at *rest*, so it has no decay length!

Thus, the proper decay length $\boxed{\ell' \equiv c\tau'}$ has no meaning for a particle at rest/in its own rest frame.

The decay time distribution in the unstable particle's own rest frame is: $\boxed{N(t') = N_o e^{-t'/\tau'}}$ where N_o = # of particles at $t = 0$.

$$\boxed{\text{Semi-log Plot of } N(t') \text{ vs. } t'}$$



For a muon with mean/proper lifetime $\boxed{\tau'_\mu \simeq 2.2\mu s}$ the "proper" decay length of the muon is:
$\boxed{\ell'_\mu \equiv c\tau'_\mu = 3\times 10^8\,m/s \times 2.2\mu s = 6.6\times 10^2\,m = 660 \text{ meters}}$.

Since $\boxed{\tau = \gamma\tau'}$, the decay time distribution of an unstable particle moving in the lab frame is:
$\boxed{N(t) = N_o e^{-t/\tau} = N_o e^{-t/\gamma\tau'}}$.

At the end of these lecture notes (see *p*.23), we show that the corresponding decay length for an unstable particle moving in the lab frame is: $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau'}$. Note that when $\boxed{v \to 0}$, $\boxed{\beta = v/c \to 0}$, $\boxed{\gamma = 1/\sqrt{1-\beta^2} \to 1}$ and thus $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau' \to 0}$ {as it should}.

Thus at Fermilab, for a beam of muons with $p_\mu = 211.32\ \ GeV/c$, they will travel a mean distance of: $\boxed{\ell_\mu = \gamma_\mu\beta_\mu\ell'_\mu = \gamma_\mu\beta_\mu c\tau'_\mu = 2000\ell'_\mu = 2000\times 660\, m = 13.2\, km}$ before the beam of muons decay to $1/e = 0.368$ of their initial number, as seen in the lab frame.

$\Rightarrow$ Expains why FNAL has lots of shielding to "range out" (*i.e.* absorb) the muons after passing through the HEP experiments that are using them for studies.

n.b. Since $\boxed{\ell = \gamma\beta\ell' = \gamma\beta c\tau'}$, then $\boxed{\ell = \ell' = c\tau'}$ occurs when $\boxed{\gamma\beta = 1}$ and thus $\boxed{\ell = \ell' = c\tau'}$ is the distance a beam of unstable particles travel in the lab frame before their # falls to $1/e = 0.368$ their initial #, traveling with $\boxed{\gamma\beta = 1}$.

Note further that since $\boxed{pc = \gamma\beta mc^2}$, we also see that: $\boxed{\gamma\beta = 1}$ corresponds to: $\boxed{\gamma\beta = pc/mc^2 = 1}$.

Griffiths Example 12.2 The Twin Paradox: Time Travel Into The Future *Is* Possible!!!

A pair of identical twins celebrate their 21st birthday by saying goodbye to each other. One is a medical doctor, the other, an astronaut. The astronaut blasts off in a rocket ship at a {constant} speed of $\boxed{v = \beta c = \tfrac{12}{13}c}$ and heads out towards α -centauri. After 5 years on her watch, she turns around and heads back to earth at the same {constant} speed to rejoin her twin sister doctor, who stayed at home on earth.

How old is each twin at their reunion?

The traveling astronaut twin has aged 5 + 5 = 10 years in her own IRF' in making this round trip, and thus she arrives back home on her 31st birthday.

However, viewed from her twin sister's earth-bound "IRF", the astronaut's clock has been running slower by a factor of:

$$\boxed{\gamma = \frac{\Delta t}{\Delta t'} = 1\Big/\sqrt{1-\beta^2} = 1\Big/\sqrt{1-\left(\tfrac{12}{13}\right)^2} = 1\Big/\sqrt{\left(\tfrac{169-144}{169}\right)^2} = \frac{13}{5} = 2.6}$$

The elapsed time for the astronaut twin to make the round trip, as viewed from the earth's "IRF" is therefore:

$$\boxed{\Delta t = \gamma\Delta t' = \frac{13}{5}\times 10 = \frac{130}{5} = 26 \text{ years}}$$

Thus the astronaut's earth-bound twin sister (the doctor, who stayed on earth) is now herself 21 + 26 = 47 years old, *i.e.* doctor twin is celebrating her 47th birthday.

$\Rightarrow$ Thus the earth-bound doctor twin is 47 – 31 = 16 years older than her astronaut twin!!!

The paradox associated with this situation arises when it is viewed from the astronaut twin's IRF – the astronaut twin sees the earth fly off away from her at $\boxed{v = \beta c = \tfrac{12}{13}c}$, then turn around after 5 years, and come back toward her at the same speed.

From the astronaut twin's perspective, she is at rest and her doctor twin is the one who is in motion – so shouldn't it be that the doctor twin is younger? (*i.e.* < 31 years old)?

This paradox *is* resolved by consideration of which of the twins *actually* experienced accelerations during the experiment. If both twins had stayed on earth, neglecting the earth's rotation and gravity, then they would have experienced/undergone no accelerations.
However, the astronaut twin *does* experience accelerations/decelerations while on her rocket ship trip to α -centauri and back – she has to go from going out: $0 \to \tfrac{12}{13}c$, $\tfrac{12}{13}c \to 0$, then coming back: $0 \to \tfrac{12}{13}c$, $\tfrac{12}{13}c \to 0$.

Thus, the traveling twin is not in an IRF at all times during her trip, while her earth-bound sister is in an "IRF" (neglecting earth's rotation and gravity) at all times.

The astronaut twin *cannot* claim to be a stationary observer/in an IRF at all times, because of the accelerations/ decelerations she experiences on her journey towards α -centauri and back to earth.

See/work Griffiths problem 12.16 on how to analyze this problem correctly from astronaut twin's perspective.

An Actual Twin Paradox Experiment:

A twin paradox experiment was carried out in the early 1970's using very high-precision cesium beam atomic clocks. Four commercial aircraft were flown around the world twice, two going east, and two going west. The atomic clocks were compared before and after each journey with identical {stationary} atomic clocks at the U.S. Naval Observatory.

Making allowances for the earth's rotation (*n.b.* we're actually living in a *non*-IRF!!!) and the gravitational red shift (due to the earth's gravitational field – a general relativistic effect!), the average observed *vs*. calculated time differences of the aircraft-based clock *vs*. the ground-based clock, in nanoseconds was:

Eastward Trip: $\boxed{\Delta t_E^{obs} = -59 \pm 10 \text{ ns (observed)}}$ vs. $\boxed{\Delta t_E^{pred} = -40 \pm 23 \text{ ns (predicted)}}$

Westward Trip: $\boxed{\Delta t_W^{obs} = +273 \pm 7 \text{ ns (observed)}}$ vs. $\boxed{\Delta t_W^{pred} = +275 \pm 21 \text{ ns (predicted)}}$

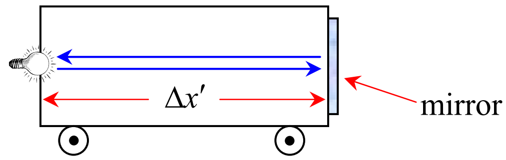
The kinematic effect of special relativity is $\approx$ comparable to that associated with the general relativistic effect.

$\Rightarrow$ Moral of the story: Always fly eastward – you will live {hundreds of nanoseconds} longer !!!

See/read J.C. Hafele and R.E. Keating, Science 177, *p*.166-168 (**1972**) for further details.

Lorentz Contraction:

3rd Gedanken Experiment: In the IRF′ of the freight train's boxcar, set up the boxcar such that the light bulb is at one end of the boxcar, a mirror at the other end, such that light signal can be sent down and back, as shown in the figure below:



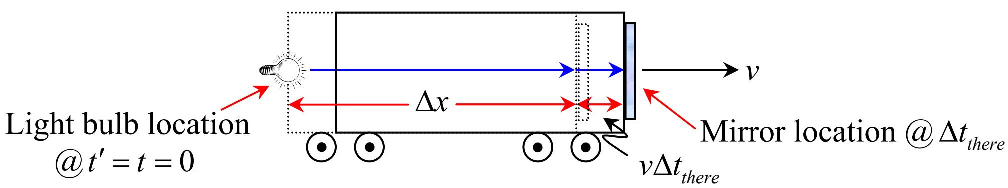
In the IRF′ of the freight train's boxcar, how long does it take the light signal to complete a round trip?

For an observer at rest on the train/in the train's IRF′, the round-trip time is $\boxed{\Delta t' = \Delta t'_{there} + \Delta t'_{back}}$:

$$\boxed{\Delta t' = \Delta t'_{there} + \Delta t'_{back} = \frac{\Delta x'}{c} + \frac{\Delta x'}{c} = 2\frac{\Delta x'}{c}}$$

For a stationary observer on the ground/in the ground IRF, the round-trip time is $\boxed{\Delta t = \Delta t_{there} + \Delta t_{back}}$:

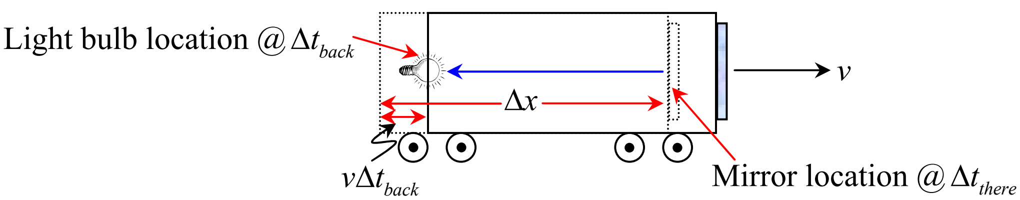
a.) The time interval required for the light signal to travel from the light bulb to the mirror = Δt_{there} :



$$\boxed{\Delta t_{there} = \frac{\Delta x}{c} + \frac{v\Delta t_{there}}{c} = \frac{\Delta x + v\Delta t_{there}}{c}} \quad n.b. \quad \boxed{\Delta x \neq \Delta x'} \text{ and } \boxed{\Delta t \neq \Delta t'} \text{ !!!}$$

Solve for Δt_{there} : $\boxed{\Delta t_{there} - \frac{v}{c}\Delta t_{there} = \left(1 - \frac{v}{c}\right)\Delta t_{there} = (1-\beta)\Delta t_{there} = \frac{\Delta x}{c}}$ $\therefore$ $\boxed{\Delta t_{there} = \frac{\Delta x}{c}\frac{1}{(1-\beta)}}$

b.) The time required for the light signal to travel from the mirror back to the light bulb = Δt_{back} :



$$\boxed{\Delta t_{back} = \frac{\Delta x}{c} - \frac{v\Delta t_{back}}{c} = \frac{\Delta x - v\Delta t_{back}}{c}}$$

Solve for Δt_{back} : $\boxed{\Delta t_{back} + \frac{v}{c}\Delta t_{back} = \left(1 + \frac{v}{c}\right)\Delta t_{back} = (1+\beta)\Delta t_{back} = \frac{\Delta x}{c}}$ $\therefore$ $\boxed{\Delta t_{back} = \frac{\Delta x}{c}\frac{1}{(1+\beta)}}$

$\therefore$ For a stationary observer on the ground/in the ground IRF, the round-trip time is:

$$\boxed{\begin{aligned}\Delta t &= \Delta t_{there} + \Delta t_{back} = \frac{\Delta x}{c}\frac{1}{(1-\beta)} + \frac{\Delta x}{c}\frac{1}{(1+\beta)} = \left[\frac{1}{(1-\beta)} + \frac{1}{(1+\beta)}\right]\frac{\Delta x}{c} \\ &= \left[\frac{(1+\beta)+(1-\beta)}{(1-\beta)(1+\beta)}\right]\frac{\Delta x}{c} = \left[\frac{1+\cancel{\beta}+1-\cancel{\beta}}{(1-\beta^2)}\right]\frac{\Delta x}{c} \\ &= 2\frac{1}{(1-\beta^2)}\frac{\Delta x}{c} = 2\gamma^2\frac{\Delta x}{c}\end{aligned}}$$

with: $\boxed{\begin{aligned}\beta &\equiv \frac{v}{c} \\ \gamma &\equiv \frac{1}{\sqrt{1-\beta^2}}\end{aligned}}$

Round trip time as seen by stationary observer on ground/in the ground IRF: $\boxed{\Delta t = 2\gamma^2\left(\frac{\Delta x}{c}\right)}$

Round trip time as seen by stationary observer on the train/in the train IRF′: $\boxed{\Delta t' = 2\left(\frac{\Delta x'}{c}\right)}$

<u>But the Time Dilation Formula is</u>: $\boxed{\Delta t = \gamma \Delta t'}$ <u>or</u>: $\boxed{\Delta t' = \frac{1}{\gamma}\Delta t}$

on ground on train on train on ground

<u>Since</u>: $\boxed{\Delta t = 2\gamma^2\left(\frac{\Delta x}{c}\right)}$ = Round trip time as seen by a stationary observer on the <u>ground</u>.

<u>And</u>: $\boxed{\Delta t' = 2\left(\frac{\Delta x'}{c}\right)}$ = Round trip time as seen by a stationary observer on train, and $\boxed{\Delta x \neq \Delta x'}$

<u>And</u>: $\boxed{\Delta t = \gamma \Delta t'}$ = Time dilation formula.

$\Rightarrow$ $\boxed{\Delta t = \gamma \Delta t' = 2\gamma^2\left(\frac{\Delta x}{c}\right)}$ <u>or</u>: $\boxed{\Delta t' = 2\gamma\left(\frac{\Delta x}{c}\right)}$ <u>but</u>: $\boxed{\Delta t' = 2\left(\frac{\Delta x'}{c}\right)}$ $\therefore$ $\boxed{2\left(\frac{\Delta x'}{c}\right) = 2\gamma\left(\frac{\Delta x}{c}\right)}$

$\Rightarrow$ $\boxed{\Delta x' = \gamma \Delta x}$ <u>or</u>: $\boxed{\Delta x = \frac{1}{\gamma}\Delta x'}$

on train on ground on ground on train

$\boxed{\Delta t' = \frac{1}{\gamma}\Delta t}$ <u>or</u>: $\boxed{\Delta t = \gamma \Delta t'}$

Therefore, for a stationary observer on the ground/in the ground IRF:

<u>Lorentz Contraction</u>: A <u>moving</u> meter stick is <u>shortened</u>: $\boxed{\Delta x = \frac{1}{\gamma}\Delta x'}$ ***n.b.*** <u>only</u> along the direction of motion !!!

<u>Time Dilation</u>: A <u>moving</u> clock runs <u>slow</u>: $\boxed{\Delta t = \gamma \Delta t'}$

<u>Where</u>: $\Delta x,\ \Delta t =$ length and time interval in the rest frame of meter stick and clock. = ground IRF {here}

However, for a stationary observer in the rest frame {IRF′} of the <u>railroad car</u> – his/her meter sticks are <u>contracted</u> by the same Lorentz factor γ, so all of his/her <u>spatial</u> measurements that he/she makes in the rest frame of the box car will come out the <u>same</u> as if the box car were at rest (with respect to the ground) !!!

From the stationary observer's perspective/IRF′ on the <u>train</u> it's the objects on the <u>ground</u> that are <u>shortened</u>!

How is it possible that both observers (A on the train, B on the ground) could be correct??

<u>They both are !!!</u> <u>Huh???</u>

We simply need to examine the details of the process whereby a length is <u>actually</u> <u>measured</u>.

In order to measure the length of a board at <u>rest</u> (*w.r.t.* the measurer/observer), one simply lays a ruler down along it and measures it – *i.e.* record the readings of the ruler at each end of the board and subtract them: $\boxed{\Delta x = x_2 - x_1}$.

However, if the board is <u>moving</u>, then one <u>must</u> read the ends of the board at the <u>same instant of time</u> (in the measurer/observer's reference frame).

⇒ Due to the simultaneity of relativity, two observers (in two <u>*different*</u> IRF's) will <u>disagree</u> on what constitutes "<u>the same instant of time</u>" !!!

When stationary person on the <u>ground</u> measures the length of a <u>moving</u> box car, he/she reads position of the two ends of the boxcar at the same instant of time in his/her IRF.

When a stationary person on the <u>train</u> watches the person on the ground making this measurement, the train observer see the <u>ground person</u> reading the <u>front-end</u> of the box car first, then the <u>rear-end</u> of the box car second – *i.e.* observer on train sees these ground-based measurements take place at different times/non-simultaneously !!!

Both observers measure lengths in their own IRF's correctly and each discover the other's meter stick to be shortened !!!

⇒ There is no inconsistency in this from the perspective of special relativity – it is {simply} a feature of special relativity !!!

Griffiths Example 12.3: The Barn & Ladder Paradox – Another Gedanken Experiment:

n.b. ∃ no *direct* experimental macroscopic confirmation of Lorentz contraction !!!
(unlike time dilation – which *has* been experimentally verified).

The technology doesn't yet exist to accelerate macroscopic objects to relativistic speeds, $v \approx c$.

A farmer has a ladder that is too long to store in his barn.

The farmer figures that if he gets *e.g.* his daughter (who run much faster than the farmer) to move the ladder into barn at a high enough speed, the ladder would Lorentz-contract, and thus the fast-moving ladder would then fit into barn!

The farmer would then slam the barn door shut at instant the whole ladder fits into the barn.

However, the farmer's daughter pointed out to him that from her moving reference frame {IRF′}, the barn would Lorentz contract, not the ladder!! ⇒ Thus, from her perspective (*i.e.* her IRF) the problem of fitting the ladder into barn would be aggravated by running relativistically fast into the barn with the ladder !!!

Who's right??? Will the ladder fit or won't it???

Again, they're both correct!!!

The statement: "the ladder is in the barn" means that all parts of the ladder (including both ends) are inside the barn at the same instant of time. ⇐ This a condition that depends on the observer – *i.e.* his/her IRF.

There are two relevant "events" in ladder – barn problem:

a.) The back end of the ladder makes it in the door of barn.

b.) The front end of the ladder hits far wall of barn.

The farmer, at rest in the barn's IRF, sees *a*.) occur before *b*.) !!!

The daughter, at rest in the ladder's IRF′, sees *b*.) occur before *a*.) !!!

Contradiction??? No !!!

Just due to a difference in the simultaneity of "event" times due to different IRF's!!!

The Nature of Lorentz Contraction:

A moving object is shortened *only* along (*i.e.* parallel to) its direction of motion/velocity vector.

Spatial dimensions transverse/perpendicular to the direction of motion/velocity vector are not contracted/not affected.

A Gedanken Experiment for Lorentz-Noncontraction in the Transverse Direction(s):

Is the height of a railroad boxcar {running along straight, horizontal railroad track} the same in all IRF's???

Build a tall wall along the side of the railroad tracks. Bring up the railroad car and stop it at the wall and mark its height h on the wall, *e.g.* using blue paint. Now back off the train, get a running relativistic start and use *e.g.* red paint to mark the height of the train boxcar on the wall as the train flies past the wall.

If the transverse directions are Lorentz-contracted, a stationary observer on the ground would predict the red line to be *lower* than the blue one, whereas an observer on the train would predict the opposite – *i.e.* the red line would be higher than the blue line. Which line is lower?

The principle of relativity says both observers/both IRF's are equally justified, but (here) both cannot be correct.

The answer: The red and blue lines exactly coincide $\Rightarrow$ there is no Lorentz contraction (or expansion) in the transverse direction.

Lorentz contraction only occurs in the longitudinal direction/direction of motion!

Lorentz Transformations:

Any physical process consists of one or more "events".

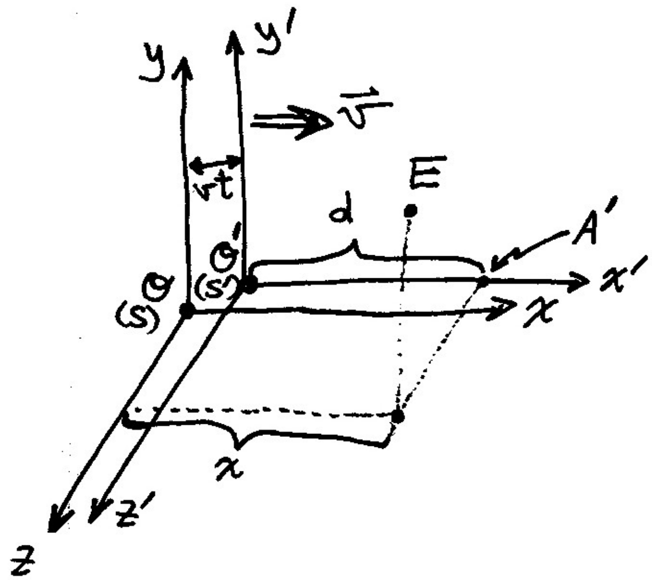
An "event" is something that occurs at a specific location (x, y, z) in space at a precise instant in time (t), *e.g.* the explosion of a firecracker.

A Lorentz Transformation is the mathematical prescription for (properly) transforming the physics associated with an "event" as seen/observed in one IRF to another IRF.

e.g. Suppose we have {4-dimensional} space-time coordinates (x, y, z, t) in IRF(S) for an event, "E", and we want to know the space-time coordinates (x', y', z', t') for the same event "E" in another IRF(S').

Suppose IRF(S') is moving at a velocity $\boxed{\vec{v} = v\hat{x}}$ relative to IRF(S). The axes of the two coordinate systems coincide at time $t = 0$, *i.e.* the instant when the two origins ϑ and ϑ' coincide, as shown in the figure below:

The Situation as Seen by a Stationary Observer in IRF(S) at Time t, for Event "E":



At time t, origin $\vartheta'(\text{in } S')$ will be a distance vt from origin $\vartheta(\text{in } S)$.
$\therefore \boxed{x = d + vt}$ where d = distance from A' to ϑ' at time t, when event "E" occurs, as measured/observed in IRF(S).

If we carry out a Galilean transformation from IRF(S) to IRF(S'):

$\boxed{d = x' = x - vt}$ ← no spatial contraction along direction of motion.
$\boxed{y' = y}$
$\boxed{z' = z}$
$\boxed{t' = t}$ ← no time dilation.

Where does the Galilean transformation go wrong? $\boxed{d = x'}$ {and $\boxed{t' = t}$}.

d = distance from A' to ϑ' as measured in IRF(S)
x' = distance from A' to ϑ' as measured in IRF(S')

Factually: $\boxed{d \neq x'}$ (because of (special) relativity) !!!

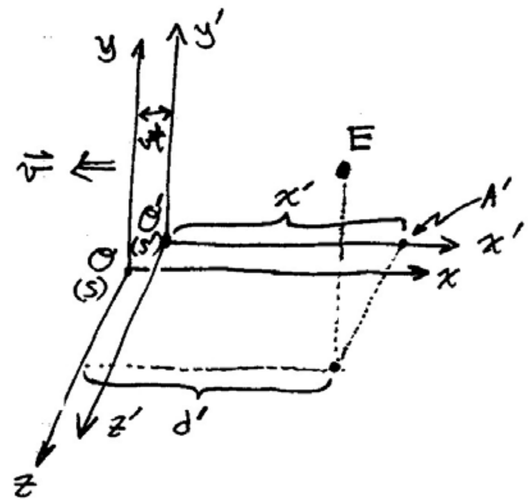
Because A' and ϑ' are at ***rest*** in IRF(S'), then from the perspective of an observer the lab frame IRF(S), x' is the "moving meter stick", which appears Lorentz-contracted in IRF(S).

Thus: $\boxed{d = \frac{1}{\gamma} x'}$ where: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$ and: $\boxed{\beta = \frac{v}{c}}$

Then: $\boxed{d = \frac{1}{\gamma} x' = x - vt}$ or: $\boxed{x' = \gamma(x - vt)}$

However, the same argument can be made from the perspective of an observer in IRF(S'), as shown in the figure below:

The Situation as Seen by a Stationary Observer in IRF(S') at Time t', for Event "E":



If observers in IRF(S) and IRF(S') both start their clocks $t = t' = 0$ when the origins ϑ and ϑ' coincide, then at time t' in IRF(S'), origin ϑ will be a distance vt from ϑ' and thus: $\boxed{x' = d' - vt'}$ where d' = the distance from A' and ϑ' as measured in IRF(S') when even "E" occurs at time t'.

The times t (in IRF(S)) and t' (in IRF(S') represent the same physical instant at event "E", viewed from/in these two IRF's, respectively.

x = distance from A' and ϑ as measured in IRF(S).
d' = distance from A' and ϑ as measured in IRF(S').

Because A' and ϑ are at rest in IRF(S), here x is the "moving meter stick" which appears Lorentz-contracted in IRF(S').

Thus: $\boxed{d' = \frac{1}{\gamma} x}$ where: $\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}}$ and: $\boxed{\beta = \frac{v}{c}}$

Then: $\boxed{x' = d' - vt' = \frac{1}{\gamma} x - vt'}$ or: $\boxed{x = \gamma\left(x' + vt'\right)}$

Time Dilation:

We have: (1) $\boxed{x' = \gamma\left(x - vt\right) = \gamma x - \gamma vt}$

(2) $\boxed{x = \gamma\left(x' + vt\right) = \gamma x' + \gamma vt}$

Insert (1) into (2) and solve for t' in terms of t and x:

$$\boxed{x = \gamma x' + \gamma vt' = \gamma\left(\gamma x - \gamma vt\right) + \gamma vt' = \gamma^2 x - \gamma^2 vt + \gamma vt'}$$

Or: $\boxed{\gamma vt' = \gamma^2 vt - \gamma^2 x + x}$ $\Leftarrow$ divide both sides by γv

$$\boxed{t' = \gamma t - \frac{(\gamma^2 - 1)}{\gamma v} x} \quad \Leftarrow \text{ multiply both sides by } c$$

$$\boxed{ct' = \gamma ct - \frac{(\gamma^2 - 1)}{\gamma}\left(\frac{c}{v}\right) x = \gamma ct - \frac{(\gamma^2 - 1)}{\gamma\beta} x} \quad \underline{\text{where}}\text{: } \boxed{\beta = \frac{v}{c}} \quad \underline{\text{and}}\text{: } \boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}, \quad \boxed{\gamma^2 = \frac{1}{1-\beta^2}}$$

$$\boxed{ct' = \gamma ct - \frac{\left(\frac{1}{1-\beta^2} - 1\right)}{\gamma\beta} x = \gamma ct - \frac{\left(\frac{1}{1-\beta^2} - \frac{1-\beta^2}{1-\beta^2}\right)}{\gamma\beta} x}$$

$$\boxed{ct' = \gamma ct - \frac{\left(\frac{\cancel{1} - \cancel{1} + \beta^2}{1-\beta^2}\right)}{\gamma\beta} x = \gamma ct - \frac{\left(\frac{\beta^2}{1-\beta^2}\right)}{\gamma\beta} x = \gamma ct - \frac{\gamma^{\cancel{2}}\beta^{\cancel{2}}}{\cancel{\gamma}\,\cancel{\beta}} x = \gamma ct - \gamma\beta x = \gamma(ct - \beta x)}$$

$$\therefore \quad \boxed{ct' = \gamma(ct - \beta x)}$$

Likewise, if we insert (2) into (1) and {instead} solve for t in terms of t' and x', we obtain:

$$\boxed{ct = \gamma(ct' + \beta x')}$$

Thus, we now have all the ingredients needed for specifying our Lorentz transformation to/from IRF(S) $\rightleftharpoons$ IRF(S').

<u>Lorentz Transformation from IRF(S) $\to$ IRF(S'):</u>

$$\boxed{\begin{array}{ll} x' = \gamma(x - \beta ct) & \vec{v} = +v\hat{x} \text{ in IRF}(S) \\ y' = y & \beta = \dfrac{v}{c}, \quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ z' = z & \\ ct' = \gamma(ct - \beta x) & \end{array}}$$

<u>Lorentz Transformation from IRF(S') $\to$ IRF(S):</u>

$$\boxed{\begin{array}{ll} x = \gamma(x' + \beta ct') & \vec{v} = -v\hat{x} \text{ in IRF}(S') \\ y = y' & \beta = \dfrac{v}{c}, \quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ z = z' & \\ ct = \gamma(ct' + \beta x') & \end{array}}$$

Griffiths Example 12.4: Simultaneity, Synchronization and Time Dilation

In lab frame IRF(*S*), suppose event "*A*" occurs at $\boxed{x_A = 0,\ t_A = 0}$ and event "*B*" occurs at $\boxed{x_B = b,\ t_B = 0}$. The two events "*A*" and "*B*" ***are*** simultaneous in IRF(*S*), because both occur at $\boxed{t_A = t_B = t = 0}$.

However, events "*A*" and "*B*" are ***not*** simultaneous in IRF(*S′*) {rest frame moving with relative velocity $\beta\hat{x}$ to lab frame} because the Lorentz transformation from IRF(*S*) to IRF(*S′*) gives:

$$\boxed{\begin{aligned} x'_A &= \gamma\left(x_A - \beta c t_A\right) = 0 \\ y'_A &= y_A \\ z'_A &= z_A \\ ct'_A &= \gamma\left(ct_A - \beta x_A\right) = 0 \end{aligned}} \quad \underline{\text{and}}: \quad \boxed{\begin{aligned} x'_B &= \gamma\left(x_B - \beta c t_B\right) = \gamma b \\ y'_B &= y_B \\ z'_B &= z_B \\ ct'_B &= \gamma\left(ct_B - \beta x_B\right) = -\gamma\beta b \end{aligned}}$$

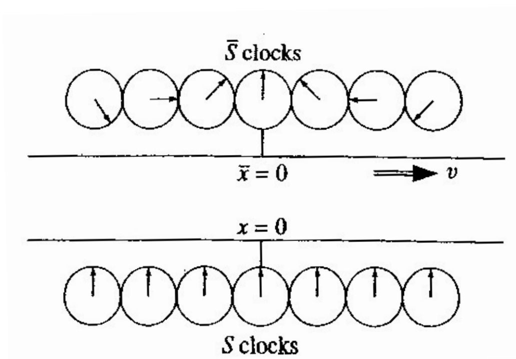
Thus, in IRF(*S*) {lab frame}: $\boxed{x_A = 0,\ t_A = 0}$ and: $\boxed{x_B = b,\quad t_B = 0}$

Whereas in IRF(*S′*) {rest frame} : $\boxed{x'_A = 0,\ t'_A = 0}$ but: $\boxed{x'_B = \gamma b,\ t'_B = -\gamma\beta b/c}$

Thus, we see that in IRF(*S′*), event "*B*" occurred *before* event "*A*" !!!

n.b. Event "*A*" occurs at $t_A = 0$ and $t'_A = 0$ (simultaneously) in IRF(*S*) and IRF(*S′*), respectively, because the origins $\vartheta(S)$ and $\vartheta'(S')$ coincide (in space) at $t = t_A = t_B = 0$.

⇒ Clocks that are synchronized in one IRF(*S*) are not synchronized in another IRF(*S′*), as can be seen in the following figure:



Suppose in IRF(*S*) at the time *t* = 0, an observer in IRF(*S*) decides to examine all clocks in IRF(S'). He/she discovers that clocks in IRF(*S′*) all read different times, due to each of their seven different *x*-locations in IRF(*S*), and will vary/differ according to:

$$\boxed{t' = \gamma\left(t - \beta\frac{x}{c}\right) = -\gamma\beta\left(\frac{x}{c}\right)}$$ {since *t* = 0 in IRF(*S*) for all seven *x*-points}

⇒ For *x* < 0, clocks to the left of the origin $(x' < 0)$ in IRF(*S′*) are increasingly ahead $(t' > 0)$.

⇒ For *x* > 0, clocks to the right of the origin $(x' > 0)$ in IRF(*S′*) are increasingly behind $(t' < 0)$.

$\Rightarrow$ Non-synchronization of clocks in an IRF(S') follows directly from the Lorentz transformation from (the synchronized) IRF(S) $\rightarrow$ IRF(S').

n.b. From the viewpoint of an observer in IRF(S'), if his/her clocks are synchronized in IRF(S') (*e.g.* all $t' = 0$) then it will be the clocks in IRF(S) that are non-synchronized:

$$\boxed{t = \gamma\left(t' + \beta\frac{x'}{c}\right) = +\gamma\beta\left(\frac{x'}{c}\right)} \quad \{\text{since } t' = 0 \text{ in IRF}(S')\}$$

$\Rightarrow$ For $x' < 0$, clocks to the left of the origin $(x < 0)$ in IRF(S) are increasingly behind $(t < 0)$.

$\Rightarrow$ For $x' > 0$, clocks to the right of the origin $(x > 0)$ in IRF(S) are increasingly ahead $(t > 0)$.

If observer in lab frame IRF(S) focuses his/her attention on a single clock in IRF(S'), *e.g.* the clock located at $\boxed{x' = a}$ and watches it over a time interval Δt, how much time elapses on the moving clock? Here x' is fixed in IRF(S') and: $\boxed{\Delta t \equiv t_2 - t_1}$, $\boxed{\Delta t' \equiv t_2' - t_1'}$.

Then: $\boxed{t = \gamma\left(t' + \beta\frac{x'}{c}\right)}$ $\qquad$ $\boxed{x' = a}$ is fixed in IRF(S')

Thus: $\boxed{t_2 = \gamma\left(t_2' + \beta\frac{a}{c}\right)}$

And: $\boxed{t_1 = \gamma\left(t_1' + \beta\frac{a}{c}\right)}$

$$\therefore \quad \boxed{\begin{aligned}\Delta t = t_2 - t_1 &= \gamma\left(t_2' + \beta\left(\frac{a}{c}\right)\right) - \gamma\left(t_1' + \beta\left(\frac{a}{c}\right)\right) \\ &= \gamma t_2' + \cancel{\gamma\beta\left(\frac{a}{c}\right)} - \gamma t_1' - \cancel{\gamma\beta\left(\frac{a}{c}\right)} = \gamma t_2' - \gamma t_1' \\ &= \gamma\left(t_2' - t_1'\right) = \gamma\Delta t'\end{aligned}}$$

$$\Rightarrow \quad \boxed{\Delta t = \gamma\Delta t'}, \ \text{or:} \ \boxed{\Delta t' = \frac{1}{\gamma}\Delta t}$$

Time Dilation Formulae

Griffiths Example 12.5: Lorentz Contraction

Consider a stick moving with velocity $\vec{v} = +v\hat{x}$.

Its rest length (as measured in the moving frame IRF(S')) is: $\boxed{\Delta x' \equiv x_2' - x_1'}$

An observer in lab frame IRF(S) wants to measure the length of this stick in his/her reference frame, *e.g.* at the same instant $t = 0$ in his/her IRF(S). Thus in IRF(S): $\boxed{\Delta x \equiv x_2 - x_1}$ at $t = 0$.

Then in IRF(S'): $\boxed{x_2' = \gamma\left(x_2 - vt\right) = \gamma x_2}$

$\boxed{x_1' = \gamma\left(x_1 - vt\right) = \gamma x_1}$

$$\therefore \quad \boxed{\Delta x' = x_2' - x_1' = \gamma x_2 - \gamma x_1 = \gamma\left(x_2 - x_1\right) = \gamma \Delta x}$$

$$\Rightarrow \quad \boxed{\Delta x' = \gamma \Delta x} \quad \text{or:} \quad \boxed{\Delta x = \frac{1}{\gamma}\Delta x'}$$

Lorentz Contraction Formulae

Example: Show invariant interval(s) *are* invariant/independent of inertial reference frame:

For the most general case in space-time:

In lab frame IRF(S): Event 1 is at $\left(x_1, t_1\right)$, Event 2 is at $\left(x_2, t_2\right)$, corresponding to:

In moving frame IRF(S'): Event 1 is at $\left(x_1', t_1'\right)$, Event 2 is at $\left(x_2', t_2'\right)$.

Using the 1-D Lorentz transformations:

$\boxed{x_1' = \gamma x_1 - \gamma\beta c t_1}$ and: $\boxed{x_2' = \gamma x_2 - \gamma\beta c t_2}$ $\Rightarrow$ $\boxed{\Delta x' = \left(x_2' - x_1'\right) = \gamma\left(x_2 - x_1\right) - \gamma\beta c\left(t_2 - t_1\right) = \gamma\Delta x - \gamma\beta c\Delta t}$

$\boxed{ct_1' = \gamma c t_1 - \gamma\beta x_1}$ and: $\boxed{ct_2' = \gamma c t_2 - \gamma\beta x_2}$ $\Rightarrow$ $\boxed{c\Delta t' = c\left(t_2' - t_1'\right) = \gamma c\left(t_2 - t_1\right) - \gamma\beta\left(x_2 - x_1\right) = \gamma c\Delta t - \gamma\beta\Delta x}$

Does the invariant interval $I' = I$???

$$\boxed{\begin{aligned}
I' &\equiv \left(\Delta x'\right)^2 - \left(c\Delta t'\right)^2 = \gamma^2\left(\Delta x - \beta c\Delta t\right)^2 - \gamma^2\left(c\Delta t - \beta\Delta x\right)^2 \\
&= \gamma^2\left(\Delta x\right)^2 - \cancel{2\gamma^2\beta\left(\Delta x c\Delta t\right)} + \gamma^2\beta^2\left(c\Delta t\right)^2 - \gamma^2\left(c\Delta t\right)^2 + \cancel{2\gamma^2\beta\left(\Delta x c\Delta t\right)} - \gamma^2\beta^2\left(\Delta x\right)^2 \\
&= \underbrace{\gamma^2\left(1-\beta^2\right)}_{=1}\left(\Delta x\right)^2 - \underbrace{\gamma^2\left(1-\beta^2\right)}_{=1}\left(c\Delta t\right)^2 \quad \text{but: } \gamma^2 = 1\big/\left(1-\beta^2\right) \\
&= \left(\Delta x\right)^2 - \left(c\Delta t\right)^2 \equiv I \quad \text{Yes!!!}
\end{aligned}}$$

Griffiths Example 12.6: Einstein's 1-D Velocity Addition Rule

n.b. IRF(S') is moving with velocity $\vec{v} = v\hat{x}$ relative to IRF(S)

Suppose a particle moves a distance *dx* in a time *dt* in IRF(S).

The speed of the particle as observed in ***lab*** frame IRF(S) is then: $\boxed{u = \dfrac{dx}{dt}}$ (in IRF(S))

However, in IRF(S') it has moved a distance: $\boxed{dx' = \gamma\left(dx - \beta c dt\right)}$ in a time: $\boxed{dt' = \gamma\left(dt - \beta\dfrac{dx}{c}\right)}$

Thus, the speed of the particle as observed in the ***moving*** frame IRF(S') is therefore:

$$\boxed{u' = \frac{dx'}{dt'} = \frac{\cancel{\gamma}\left(dx - \beta c dt\right)}{\cancel{\gamma}\left(dt - \beta\left(\frac{dx}{c}\right)\right)} = \frac{dx - \beta c dt}{dt - \beta\left(\frac{dx}{c}\right)} = \frac{\frac{dx}{dt} - \beta c}{1 - \left(\frac{\beta}{c}\right)\frac{dx}{dt}} = \frac{u - \beta c}{1 - u\frac{\beta}{c}} = \frac{u - v}{1 - \frac{uv}{c^2}}} \quad \text{(in IRF}(S'))$$

Einstein's 1-D Velocity Addition Rule for IRF(S') moving with velocity $\vec{v} = v\hat{x}$ relative to IRF(S):

$$\boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \quad \text{or:} \quad \boxed{\beta'_{u'} = \frac{\beta_u - \beta}{1 - \beta_u\beta}} \quad \text{where:} \quad \boxed{\beta = \frac{v}{c}}, \quad \boxed{\beta_u = \frac{u}{c}}, \quad \boxed{\beta'_{u'} = \frac{u'}{c}}$$

If the situation is ***reversed*** for the two IRF's, then the speed of the particle as observed in IRF(S) in terms of its speed as observed in IRF(S') is given by:

Einstein's 1-D Velocity Addition Rule for IRF(S) moving with velocity $\vec{v}' = -v\hat{x}$ relative to IRF(S'):

$$\boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}} \quad \text{or:} \quad \boxed{\beta_u = \frac{\beta'_{u'} + \beta}{1 - \beta'_{u'}\beta}}$$

n.b. Compare *e.g.* this last 1-D velocity addition result to that originally given at top of *p*. 4 above – they are the same/identical for $\vec{v} = \pm v\hat{x}$:

$$\boxed{v_{\text{ground}}^{\text{man}} = \frac{v_{\text{ground}}^{\text{train}} \pm v_{\text{train}}^{\text{man}}}{\left[1 \pm \left(\frac{v_{\text{ground}}^{\text{train}} \cdot v_{\text{train}}^{\text{man}}}{c^2}\right)\right]}}$$

Derivation of Lab Frame Decay Length and Decay Time Relations $\boxed{\ell = \gamma\beta c\tau'}$ and $\boxed{\tau = \gamma\tau'}$ for Unstable Relativistic Particles

In the ***rest*** frame IRF(S') of an unstable particle (*e.g.* a muon), this particle is:

Created at the IRF(S') space-time coordinate: $\boxed{(x_1', y_1', z_1', t_1') = (0,0,0,0)}$

Decays at the IRF(S') space-time coordinate: $\boxed{(x_2', y_2', z_2', t_2') = (0,0,0,\tau')}$

Where τ' = the decay time of the particle in IRF(S')

And: $\boxed{\Delta x' \equiv x_2' - x_1' = 0}$ and: $\boxed{\Delta t' \equiv t_2' - t_1' = \tau'}$. {*n.b.* the unstable particle is ***not*** moving in IRF(S').}

In the ***lab*** frame IRF(S), if the unstable particle has velocity ***only*** in the $\hat{x}$-direction, *i.e.* $\boxed{\vec{v} = v\hat{x}}$, then if the two reference frames IRF(S) and IRF(S') coincide at $\boxed{t_1 = t_1' = 0}$, then in the ***lab*** frame IRF(S), this particle is:

Created at the IRF(S) space-time coordinate: $\boxed{(x_1, y_1, z_1, t_1) = (0,0,0,0)}$

Decays at the IRF(S) space-time coordinate: $\boxed{(x_2, y_2, z_2, t_2) = (\ell,0,0,\tau)}$

Where τ = the decay time and ℓ = decay length of the particle in IRF(S)

And: $\boxed{\Delta x \equiv x_2 - x_1 = \ell}$ and: $\boxed{\Delta t \equiv t_2 - t_1 = \tau}$.

Next, we need to use the Lorentz Transformation from IRF(S') $\rightarrow$ IRF(S):

$$\boxed{\begin{array}{ll} x = \gamma(x' + \beta ct') & \vec{v} = -v\hat{x} \text{ in IRF}(S') \\ y = y' & \beta = \dfrac{v}{c}, \quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ z = z' & \\ ct = \gamma(ct' + \beta x') & \end{array}}$$

Then:

$\boxed{x_2 = \gamma(x_2' + \beta ct_2') = \gamma(0 + \beta c\tau') = \gamma\beta c\tau'}$ and: $\boxed{ct_2 = \gamma(ct_2' + \beta x_2') = \gamma(c\tau' + \beta 0) = \gamma c\tau'}$ or: $\boxed{t_2 = \gamma\tau'}$

$\boxed{x_1 = \gamma(x_1' + \beta ct_1') = \gamma(0 + \beta c0) = 0}$ and: $\boxed{ct_1 = \gamma(ct_1' + \beta x_1') = \gamma(c0 + \beta 0) = 0}$ or: $\boxed{t_1 = t_1' = 0}$

Thus in the lab frame IRF(S): $\boxed{\ell = \Delta x \equiv x_2 - x_1 = \gamma\beta c\tau'}$ and: $\boxed{\tau = \Delta t \equiv t_2 - t_1 = \gamma\tau'}$

$\therefore$ The decay length and decay time of an unstable particle in the ***lab*** frame IRF(S) are respectively: $\boxed{\ell = \gamma\beta c\tau'}$ and: $\boxed{\tau = \gamma\tau'}$, where τ' = the decay time of the particle in its own ***rest*** frame IRF(S').