LECTURE NOTES 7

ELECTROMAGNETIC WAVES IN CONDUCTORS

Inside a conductor, free charges can move/migrate around in response to *EM* fields contained therein, as we saw for the case of the longitudinal $\vec{E}$-field inside a current-carrying wire that had a static potential difference ΔV across its ends. Even in the static case of electric charge residing on the surface of a conductor, we saw that $\vec{E}_{inside}(\vec{r}) = 0$, but recall that this actually means (as we showed last semester) that the <u>*NET*</u> electric field inside the conductor is <u>zero</u>, *i.e.* $\underline{\underline{\vec{E}^{NET}_{inside}(\vec{r}) = 0}}$.

n.b. here, we assume {for simplicity's sake} that the conductor is linear/homogeneous/isotropic – *i.e.* <u>no</u> crystalline structure/<u>no</u> anisotropies/<u>no</u> inhomogenities/<u>no</u> non-uniformities/<u>no</u> voids/<u>no</u> defects…

From Ohm's Law, we know that the *<u>free</u>* current density $\vec{J}_{free}(\vec{r},t)$ is proportional to the (ambient) electric field inside the conductor: $\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$ where σ_C = conductivity of the metal conductor $\left(Siemens/m = Ohm^{-1}/m\right)$ and $\sigma_C = 1/\rho_C$ where ρ_C = resistivity of the metal conductor $(Ohm\text{-}m)$.

Thus inside such a conductor, we can assume that the linear/homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ. Maxwell's equations inside such a conductor {with $\vec{J}_{free}(\vec{r},t) \neq 0$ } are thus:

Using Ohm's Law: $\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu\vec{J}_{free}(\vec{r},t) + \mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t} = \mu\sigma_C\vec{E}(\vec{r},t) + \mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

Electric charge is (always) conserved, thus the continuity equation inside the conductor is:

$\boxed{\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) = -\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ <u>but</u>: $\boxed{\vec{J}_{free}(\vec{r},t) = \sigma_C\vec{E}(\vec{r},t)}$

$\therefore$ $\boxed{\sigma_C\left(\vec{\nabla}\cdot\vec{E}(\vec{r},t)\right) = -\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ <u>but</u>: $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon}$

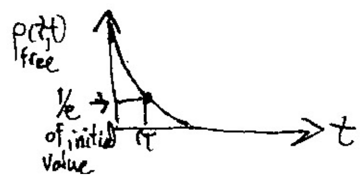
<u>thus</u>: $\boxed{\dfrac{\sigma_C\rho_{free}(\vec{r},t)}{\varepsilon} = -\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$ <u>or</u>: $\boxed{\dfrac{\partial\rho_{free}(\vec{r},t)}{\partial t} + \left(\dfrac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t) = 0}$ $\Leftarrow$ 1st order linear, homogeneous differential equation

The {physical} general solution of this differential equation for the free charge density is of the form:

$$\boxed{\rho_{free}(\vec{r},t) = \rho_{free}(\vec{r},t=0)e^{-\sigma_C t/\varepsilon} = \rho_{free}(\vec{r},t=0)e^{-t/\tau_{relax}}}$$ *i.e.* a <u>damped</u> exponential!!!

Thus, the continuity equation $\vec{\nabla}\bullet\vec{J}_{free}\left(\vec{r},t\right)=-\partial\rho_{free}\left(\vec{r},t\right)/\partial t$ inside a conductor tells us that any free charge density $\rho_{free}\left(\vec{r},t=0\right)$ initially present at time t = 0 is exponentially damped / dissipated in a characteristic time $\tau_{relax}\equiv\varepsilon/\sigma_C$ = charge relaxation time {*aka* time constant}, such that:

$$\boxed{\rho_{free}\left(\vec{r},t\right)=\rho_{free}\left(\vec{r},t=0\right)e^{-\sigma_C t/\varepsilon}=\rho_{free}\left(\vec{r},t=0\right)e^{-t/\tau_{relax}}}$$



Calculation of the Charge Relaxation Time for Pure Copper:

$$\boxed{\rho_{Cu}=1/\sigma_{Cu}=1.68\times10^{-8}\,\Omega\text{-m}}\;\Rightarrow\;\boxed{\sigma_{Cu}=1/\rho_{Cu}=5.95\times10^{7}\text{ Siemens/m}}$$

If we assume $\varepsilon_{Cu}\approx3\varepsilon_o=3\times8.85\times10^{-8}$ F/m for copper metal, then:

$$\boxed{\tau_{Cu}^{relax}=\varepsilon_{Cu}/\sigma_{Cu}=\rho_{Cu}\,\varepsilon_{Cu}=4.5\times10^{-19}\text{ sec}}\;\;!!!$$

However, recall that the characteristic/mean collision time of free electrons in pure copper is $\tau_{Cu}^{coll}\simeq\lambda_{Cu}^{coll}/v_{thermal}^{Cu}$ where $\lambda_{Cu}^{coll}\simeq3.9\times10^{-8}\,m$ = mean free path (between successive collisions) in pure copper, and $v_{thermal}^{Cu}\simeq\sqrt{3k_BT/m_e}\simeq12\times10^{5}\,m/sec$ and thus we obtain $\tau_{coll}^{Cu}\simeq3.2\times10^{-13}$ sec.

Hence we see that the calculated charge relaxation time in pure copper, $\tau_{Cu}^{relax}\simeq4.5\times10^{-19}$ sec is $\ll$ than the calculated collision time in pure copper, $\tau_{coll}^{Cu}\simeq3.2\times10^{-13}$ sec.

Furthermore, the experimentally measured charge relaxation time in pure copper is $\tau_{Cu}^{relax}\left(\text{exp't}\right)\simeq4.0\times10^{-14}$ sec, which is ≈5 orders of magnitude larger than the calculated charge relaxation time $\tau_{Cu}^{relax}\simeq4.5\times10^{-19}$ sec. The problem here is that {the macroscopic} Ohm's Law is simply out of its range of validity on such short time scales! Two additional facts here are that *both* ε and σ_C are frequency-dependent quantities { *i.e.* $\varepsilon=\varepsilon\left(\omega\right)$ and $\sigma_C=\sigma_C\left(\omega\right)$ }, which becomes increasingly important at the higher frequencies $\left(f=2\pi/\omega\sim1/\tau_{relax}\right)$ associated with short time-scale, transient-type phenomena!

So in reality, if we are willing to wait even a short time (*e.g.* $\Delta t\sim1\,\text{ps}=10^{-12}$ sec) then any initial free charge density $\rho_{free}\left(\vec{r},t=0\right)$ accumulated inside the conductor at t = 0 will have dissipated away/damped out, and from that time onwards, $\rho_{free}\left(\vec{r},t\right)=0$ *can* be safely assumed.

Thus, after many charge relaxation time constants, *e.g.* $20\tau_{relax} \le \Delta t \simeq 1\ ps = 10^{-12}\ sec$, then Maxwell's equations for a conductor become {with $\rho_{free}(\vec{r}, t \ge \Delta t) = 0$ from then onwards}:

Maxwell's equations for a ***charge-equilibrated*** conductor

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu\sigma_C\vec{E}(\vec{r},t) + \mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t} = \mu\left(\sigma_C\vec{E}(\vec{r},t) + \varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)}$

Now because these equations are different from the previous derivation(s) of monochromatic plane *EM* waves propagating in free space/vacuum and/or in linear/homogeneous/isotropic non-conducting materials {*n.b.* only equation 4) has changed}, we re-derive the wave equations for $\vec{E}$ & $\vec{B}$ from scratch. As before, we apply $\vec{\nabla}\times(\ \ )$ to equations 3) and 4):

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})}$$
$$= \boxed{\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma_C\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right)}$$
$$= \boxed{\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma_C\frac{\partial\vec{E}}{\partial t}}$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu\left(\sigma_C(\vec{\nabla}\times\vec{E})\right) + \varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})}$$
$$= \boxed{\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B} = -\mu\sigma_C\frac{\partial\vec{B}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}}$$
$$= \boxed{\nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma_C\frac{\partial\vec{B}}{\partial t}}$$

<u>Again</u>: $\boxed{\nabla^2\vec{E}(\vec{r},t) = \mu\varepsilon\dfrac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_C\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$ <u>and</u>: $\boxed{\nabla^2\vec{B}(\vec{r},t) = \mu\varepsilon\dfrac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_C\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

Note that these 3-D wave equations for $\vec{E}$ and $\vec{B}$ in a conductor have an additional term that has a single time derivative – which is analogous to a velocity-dependent <u>damping term</u>, *e.g.* for a mechanical harmonic oscillator.

The general solution(s) to the above wave equations are usually in the form of an oscillatory function * a damping term (*i.e.* a decaying exponential) – in the direction of the propagation of the *EM* wave, *e.g.* complex plane-wave type solutions for $\vec{E}$ and $\vec{B}$ associated with the above wave equation(s) are of the general form:

$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}}$ and: $\boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z-\omega t)} = \left(\dfrac{\tilde{k}}{\omega}\right)\hat{k}\times\tilde{\vec{E}}(z,t) = \dfrac{1}{\omega}\tilde{\vec{k}}\times\tilde{\vec{E}}(z,t)}$

with {frequency-dependent} <u>complex</u> wave number: $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$

where $k(\omega) = \Re e(\tilde{k}(\omega))$ and $\kappa(\omega) = \Im m(\tilde{k}(\omega))$ and corresponding <u>complex</u> wave vector $\tilde{\vec{k}}(\omega) = \tilde{k}(\omega)\hat{k} = \tilde{k}(\omega)\hat{z}$ (in the $+\hat{z}$ direction here), *i.e.* $\tilde{\vec{k}}(\omega) = (k(\omega) + i\kappa(\omega))\hat{z}$.

We plug $\tilde{\vec{E}}(z,t)=\tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}$ and $\tilde{\vec{B}}(z,t)=\tilde{\vec{B}}_o e^{i(\tilde{k}z-\omega t)}$ into their respective wave equations above, and obtain from each wave equation the same/identical *characteristic equation* – {*aka* a *dispersion relation*} between complex $\tilde{k}(\omega)$ and ω {please work this out yourselves!}:

$$\boxed{\tilde{k}^2(\omega)=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

Thus, since $\tilde{k}(\omega)=k(\omega)+i\kappa(\omega)$, then:

$$\boxed{\tilde{k}^2(\omega)=\left(k(\omega)+i\kappa(\omega)\right)^2=k^2(\omega)-\kappa^2(\omega)+2ik(\omega)\kappa(\omega)=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

If we {temporarily} suppress the ω-dependence of complex $\tilde{k}(\omega)$, this relation becomes:

$$\boxed{\tilde{k}^2=(k+i\kappa)^2=k^2-\kappa^2+2ik\kappa=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

We can solve this relation to determine $k(\omega)=\Re e\left(\tilde{k}(\omega)\right)$ and $\kappa(\omega)=\Im m\left(\tilde{k}(\omega)\right)$ as follows: First, separate this relation into *two* relations – *i.e.* separate out its real and imaginary parts:

$$\boxed{k^2-\kappa^2=\mu\varepsilon\omega^2} \quad \text{and:} \quad \boxed{2ik\kappa=i\mu\sigma_C\omega} \quad \text{or:} \quad \boxed{2k\kappa=\mu\sigma_C\omega}$$

We thus have *two* {separate/independent} equations $k^2-\kappa^2=\mu\varepsilon\omega^2$ and $2k\kappa=\mu\sigma_C\omega$, and we have *two* unknowns (k and k). Hence, we can solve these equations *simultaneously*!

From the latter relation, we see that: $\boxed{\kappa=\tfrac{1}{2}\mu\sigma_C\omega/k}$. Plug *this* result into the *other* relation:

$$\boxed{k^2-\kappa^2=k^2-\left(\tfrac{1}{2}\mu\sigma_C\omega/k\right)^2=k^2-\frac{1}{k^2}\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2=\mu\varepsilon\omega^2}$$

Then multiply by k^2 and rearrange the terms to obtain the following relation:

$$\boxed{k^4-\left(\mu\varepsilon\omega^2\right)k^2-\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2=0}$$

This may *look* like a scary equation to try to solve (*i.e.* a *quartic* equation - *eeekkk*!), but it's actually just a *quadratic* equation!

Define: $x\equiv k^2$, $a\equiv 1$, $b\equiv-\left(\mu\varepsilon\omega^2\right)$ and $c\equiv-\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2$, then this equation becomes "the usual" quadratic equation, of the form: $x^2+bx+c=0$, with solution(s)/root(s):

$$\boxed{x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}} \quad \text{or:} \quad \boxed{k^2=\frac{1}{2}\left[+\left(\mu\varepsilon\omega^2\right)\mp\sqrt{\left(\mu\varepsilon\omega^2\right)^2+4\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2}\right]}$$

This can be rewritten as:

$$\boxed{k^2=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1\mp\sqrt{1+\cancel{4}\frac{\left(\cancel{\mu^2}\sigma_C^2\cancel{\omega^2}\right)}{\cancel{4}\left(\cancel{\mu^2}\varepsilon^2\omega^{\cancel{4}2}\right)}}\right]=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1\mp\sqrt{1+\frac{\left(\sigma_C^2\right)}{\left(\varepsilon^2\omega^2\right)}}\right]=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1\mp\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]}$$

Now we can see that on physical grounds ($k^2>0$), we ***must*** select the + sign, hence:

$$\boxed{k^2=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1+\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]}\text{ and thus: }\boxed{k=\sqrt{k^2}=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2}=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

Having thus solved for k (or equivalently, k^2), then we can use *either* of our original *two* relations to solve for κ, *e.g.* $k^2-\kappa^2=\mu\varepsilon\omega^2$, then:

$$\boxed{\kappa^2=k^2-\mu\varepsilon\omega^2=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[1+\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]-\mu\varepsilon\omega^2=\frac{1}{2}\left(\mu\varepsilon\omega^2\right)\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]}$$

Thus, we obtain:

$$\boxed{k(\omega)=\Re e\left(\tilde{k}(\omega)\right)=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}\text{ and: }\boxed{\kappa(\omega)=\Im m\left(\tilde{k}(\omega)\right)=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}$$

Note that the imaginary part of $\tilde{k}$, $\kappa=\Im m\left(\tilde{k}\right)$ results in an *exponential* attenuation/damping of the monochromatic plane *EM* wave with increasing *z*:

$$\boxed{\tilde{\vec{E}}(z,t)=\tilde{\vec{E}}_o e^{-\kappa z}e^{i(kz-\omega t)}}\quad\text{and:}\quad\boxed{\tilde{\vec{B}}(z,t)=\tilde{\vec{B}}_o e^{-\kappa z}e^{i(kz-\omega t)}=\frac{1}{\omega}\tilde{\vec{k}}\times\tilde{\vec{E}}(z,t)=\frac{1}{\omega}\tilde{\vec{k}}\times\tilde{\vec{E}}_o e^{-\kappa z}e^{i(kz-\omega t)}}$$

n.b. these solutions satisfy the above wave equations for *any* choice of $\tilde{\vec{E}}_o$.

The characteristic distance over which $\vec{E}$ and $\vec{B}$ are attenuated/reduced to $1/e=e^{-1}=0.3679$ of their initial values (at *z* = 0) is known as the skin depth, $\boxed{\delta_{sc}(\omega)\equiv 1/\kappa(\omega)}$ (SI units: meters).

i.e. $$\boxed{\delta_{sc}(\omega)=\frac{1}{\kappa(\omega)}=\frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}}\Rightarrow\boxed{\begin{array}{l}\tilde{\vec{E}}(z=\delta_{sc},t)=\tilde{\vec{E}}_o e^{-1}e^{i(kz-\omega t)}\\ \tilde{\vec{B}}(z=\delta_{sc},t)=\tilde{\vec{B}}_o e^{-1}e^{i(kz-\omega t)}\end{array}}$$

The real part of $\tilde{k}$, *i.e.* $k(\omega) = \Re e\left(\tilde{k}(\omega)\right)$ determines the spatial wavelength $\lambda(\omega)$, the propagation speed $v(\omega)$ of the monochromatic *EM* plane wave in the conductor, and also the index of refraction:

$$\boxed{\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\Re e\left(\tilde{k}(\omega)\right)}}$$

$$\boxed{v(\omega) = \frac{\omega}{k(\omega)} = \frac{\omega}{\Re e\left(\tilde{k}(\omega)\right)}}$$

and: $$\boxed{n(\omega) = \frac{c}{v(\omega)} = \frac{ck(\omega)}{\omega} = \frac{c\Re e\left(\tilde{k}(\omega)\right)}{\omega}}$$

The above plane wave solutions satisfy the above wave equations(s) for <u>any</u> choice of $\tilde{\vec{E}}_o$. As we have also seen before, it can similarly be shown here that Maxwell's equations 1) and 2) $\left(\vec{\nabla}\bullet\vec{E} = 0 \text{ and } \vec{\nabla}\bullet\vec{B} = 0\right)$ rule out the presence of any {longitudinal} *z*-components for $\vec{E}$ and $\vec{B}$ (for *EM* waves propagating in the $+\hat{z}$ -direction) $\Rightarrow$ $\vec{E}$ and $\vec{B}$ are purely transverse waves (as before), even in a conductor!

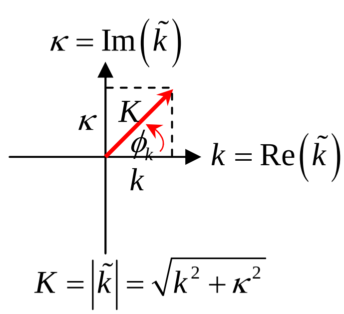
If we consider *e.g.* a linearly polarized monochromatic plane *EM* wave propagating in the $+\hat{z}$ -direction in a conducting medium, *e.g.* $\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}$, then:

$$\boxed{\tilde{\vec{B}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \left(\frac{k+i\kappa}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

$\Rightarrow$ $\tilde{\vec{E}}(z,t) \perp \tilde{\vec{B}}(z,t) \perp \hat{z}$ ($+\hat{z}$ = propagation direction)

The complex wavenumber $\tilde{k} = k + ik = Ke^{i\phi}$ where: $K \equiv \left|\tilde{k}\right| = \sqrt{k^2+\kappa^2}$ and $\phi_k \equiv \tan^{-1}\left(\kappa/k\right)$

In the complex $\tilde{k}$ -plane:



Then we see that: $\boxed{\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}}$ has: $\boxed{\tilde{E}_o = E_o e^{i\delta_E}}$ $\boxed{\tilde{k} = K e^{i\phi_k}}$

and that: $\boxed{\tilde{\vec{B}}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \frac{\tilde{k}}{\omega}\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$ has: $\boxed{\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega}\tilde{E}_o = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_E}}$

Thus, we see that: $\boxed{B_o e^{i\delta_B} = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_B} = \frac{K}{\omega} E_o e^{i(\delta_E+\phi_k)} = \frac{\sqrt{k^2+\kappa^2}}{\omega} E_o e^{i(\delta_E+\phi_k)}}$

i.e., inside a conductor, $\vec{E}$ and $\vec{B}$ are no longer in phase with each other!!!

Phases of $\vec{E}$ and $\vec{B}$: $\boxed{\delta_B = \delta_E + \phi_k}$

With phase *difference*: $\boxed{\Delta\varphi_{B-E} \equiv \delta_B - \delta_E = \phi_k}$ $\Leftarrow$ magnetic field ***lags*** behind electric field!!!

We also see that: $\boxed{\frac{B_o}{E_o} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \neq \frac{1}{c}}$

The real/physical $\vec{E}$ and $\vec{B}$ fields associated with linearly polarized monochromatic plane *EM* waves propagating in a conducting medium are *exponentially* damped:

$$\boxed{\vec{E}(z,t) = \Re e\left(\tilde{\vec{E}}(z,t)\right) = E_o e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{x}} \qquad \boxed{\delta_B = \delta_E + \phi_k}$$

$$\boxed{\vec{B}(z,t) = \Re e\left(\tilde{\vec{B}}(z,t)\right) = B_o e^{-\kappa z}\cos(kz-\omega t+\delta_B)\hat{y} = B_o e^{-\kappa z}\cos\left(kz-\omega t+\{\delta_E+\phi_k\}\right)\hat{y}}$$

$\boxed{\frac{B_o}{E_o} = \frac{K(\omega)}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{\frac{1}{2}}}$ where $\boxed{K(\omega) \equiv |\tilde{k}(\omega)| = \sqrt{k^2(\omega)+\kappa^2(\omega)} = \omega\left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2}}$

$\boxed{\delta_B = \delta_E + \phi_k}$, $\boxed{\phi_k(\omega) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right)}$ and $\boxed{\tilde{\vec{k}}(\omega) = \left(k(\omega)+i\kappa(\omega)\right)\hat{z}}$, $\boxed{\tilde{k}(\omega) = \left|\tilde{\vec{k}}(\omega)\right| = k(\omega)+i\kappa(\omega)}$

Definition of the ***skin depth*** $\delta_{sc}(\omega)$ in a conductor:

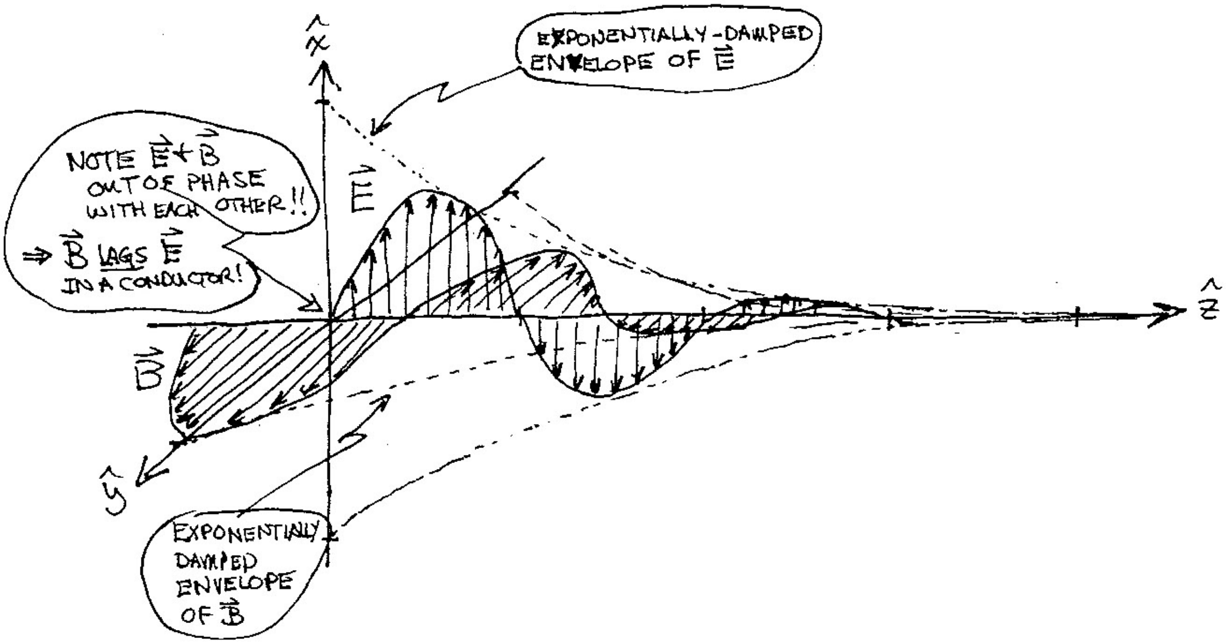
$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}} =$$

Distance over which the $\vec{E}$ and $\vec{B}$ fields fall to $1/e = e^{-1} = 0.3679$ of their initial values.

The *instantaneous* power *per unit volume* in the conductor {ultimately dissipated as heat!} is:

$$\boxed{p(z,t) = \vec{J}(z,t)\cdot\vec{E}(z,t) = \sigma_C\vec{E}(z,t)\cdot\vec{E}(z,t) = \sigma_C E^2(z,t) = E_o^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E) \quad \left(Watts/m^3\right)}$$

The *time*-*averaged* power *per unit volume* in the conductor is thus: $\boxed{\langle p(z,t)\rangle_t = \tfrac{1}{2}E_o^2 e^{-2\kappa z}}$



Special/Limiting Cases:

a) ***Good* conductors**: $\boxed{\sigma_C \gg \varepsilon\omega}$ Conductivity of good conductor $\boxed{\sigma_C \to \infty \ \left(i.e.\ \rho_C = 1/\sigma_C \to 0\right)}$.

Since $\tilde{k} = k + ik$ and $\sigma_C \gg \varepsilon\omega$, $i.e.$ $\left(\dfrac{\sigma_C}{\varepsilon\omega} \gg 1\right)$ then:

$$\boxed{k \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right]^{1/2} = \omega\sqrt{\frac{\cancel{\varepsilon}\,\mu\sigma_C}{2\cancel{\varepsilon}\,\omega}} = \sqrt{\frac{\omega\mu\sigma_C}{2}}}$$

and:

$$\boxed{\kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{\frac{\sigma_C}{\varepsilon\omega}}\right]^{1/2} = \omega\sqrt{\frac{\cancel{\varepsilon}\,\mu\sigma_C}{2\cancel{\varepsilon}\,\omega}} = \sqrt{\frac{\omega\mu\sigma_C}{2}}}$$

$\therefore$ In a good conductor $\boxed{\sigma_C \gg \varepsilon\omega}$:

$\boxed{k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}}}$ and skin depth: $\boxed{\delta_{sc}(\omega) \equiv \dfrac{1}{\kappa(\omega)} \simeq \sqrt{\dfrac{2}{\omega\mu\sigma_C}}}$.

FORMULAS FOR *EM* WAVE PROPAGATION IN A *GOOD* CONDUCTOR

$\boxed{k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}}}$ and: $\boxed{\delta_{sc}(\omega) = \text{skin depth} \equiv \dfrac{1}{\kappa(\omega)} \simeq \sqrt{\dfrac{2}{\omega\mu\sigma_C}}}$

Wavenumber, $\boxed{k(\omega) \equiv \dfrac{2\pi}{\lambda(\omega)}} \Rightarrow \boxed{\lambda(\omega) = \dfrac{2\pi}{k(\omega)} \simeq \dfrac{2\pi}{\kappa(\omega)} = 2\pi\delta_{sc}(\omega) = 2\pi\sqrt{\dfrac{2}{\omega\mu\sigma_C}}}$

---

n.b. in a *perfect* conductor: $\boxed{\sigma_C = \infty}$

$$\Rightarrow \boxed{k(\omega) \simeq \kappa(\omega) = \sqrt{\frac{\omega\mu\sigma_C}{2}} = \infty}$$

$$\Rightarrow \boxed{\lambda(\omega) = \frac{2\pi}{k(\omega)} = 0}$$

$$\boxed{\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} \simeq \sqrt{\frac{2}{\omega\mu\sigma_C}} = 0}$$

$$\boxed{\phi_k(\omega) \equiv (\delta_B - \delta_E) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) \simeq \tan^{-1}(1)}$$

But: $\boxed{\tan^{-1}(1) = 45^\circ = \dfrac{\pi}{4}}$

$$\Rightarrow \boxed{\phi = \delta_B - \delta_E = 45^\circ = \frac{\pi}{4}}$$

$\Rightarrow \vec{B}$ lags $\vec{E}$ by $\simeq 45^\circ$ in a *good* conductor.

n.b. In a *perfect* conductor: $\boxed{\sigma_C = \infty,\ \phi \equiv 45^\circ = \dfrac{\pi}{4}}$

In a typical *good* conductor (*e.g.* gold/silver/copper/…): $\boxed{(\sigma_C/\varepsilon\omega) \gg 1}$

For optical frequencies/visible light region: $\omega \simeq 10^{16}\ radians/sec$. A good conductor typically has $\sigma_C \simeq 10^7$ Siemens/m and $\varepsilon \simeq 3\varepsilon_o$, and at optical frequencies: $(\sigma_C/\varepsilon\omega) \simeq 37.7 \gg 1$ is satisfied.

If the conductor is *non*-*magnetic* (*e.g*. copper, aluminum, gold, silver, platinum… *etc*.)
$\Rightarrow \mu \simeq \mu_o = 4\pi\times10^{-7}\ Henrys/m$.

Then: $\boxed{k(\omega) \simeq \kappa(\omega) \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}} \approx \sqrt{\dfrac{\omega\mu_o\sigma_C}{2}} = \left[\dfrac{10^{16}\times4\pi\times10^{-7}\times10^{7}}{2}\right]^{1/2} \simeq 2.51\times10^{8}\ radians/m}$

And: $\boxed{\lambda(\omega) = 2\pi/k(\omega)}$ = wavelength in good conductor $\simeq \boxed{2.51\times10^{-8}\,m = 25.1\,nm}$

cf w/ vacuum wavelength: $\boxed{\lambda_o = \dfrac{2\pi}{k_o} = \dfrac{2\pi c}{\omega} = \dfrac{c}{f} \simeq \dfrac{2\pi\times3\times10^{8}}{10^{16}} = 1.885\times10^{-7}\,m = 188.5\,nm}$

$$\Rightarrow \boxed{\lambda(\omega) \simeq 25.1\,nm\left(\begin{smallmatrix}\text{good}\\ \text{conductor}\end{smallmatrix}\right) \ll \lambda_o = 188.5\,nm\left(\begin{smallmatrix}\text{vacuum}\\ \text{wavelength}\end{smallmatrix}\right)}$$

Vacuum/conductor λ-ratio: $\boxed{\left(\dfrac{\lambda_o}{\lambda(\omega)}\right) = \dfrac{188.5\,nm}{25.1\,nm} \simeq 7.52}$ at optical frequencies, $\omega \approx 10^{16}$ rad/sec.

Skin depth: $\boxed{\delta_{sc}(\omega) = \dfrac{1}{\kappa(\omega)} \simeq \dfrac{\lambda(\omega)}{2\pi} \simeq 4.0\times10^{-9}\,m = 4.0\,nm}$!!!

$\Rightarrow$ This explains why metals are opaque at optical frequencies, $\omega \approx 10^{16}\ radians/sec$
{and *also* explains why/how silvered sunglasses work!}

Compare these results for *EM* waves propagating in conductors at <u>*optical*</u> frequencies to those for *EM* waves propagating in conductors, but with <u>very low</u> frequencies – *e.g*. the AC line frequency, $f_{AC} = 60\,Hz \;\Rightarrow\; \omega_{AC} = 2\pi f_{AC} = 120\pi$ rad/sec, where the criterion for a good conductor, $(\sigma_C/\varepsilon\omega) \simeq 10^{15} \gg 1$ is certainly well-satisfied:

$$\text{At } f = 60\text{Hz:} \begin{cases} k_{AC} \simeq \kappa_{AC} \simeq \sqrt{\dfrac{\omega\mu\sigma_C}{2}} = \left[\dfrac{120\pi \times 4\pi \times 10^{-7} \times 10^{7}}{2}\right] = 48.7\ radians/m \\ \lambda_{AC} = \dfrac{2\pi}{k} = 0.129\,m = 12.9\,cm \\ \lambda_{o_{AC}} = 5\times 10^{6}\ m!! \\ \dfrac{\lambda_{o_{AC}}}{\lambda_{AC}} = \dfrac{5\times 10^{6}\,m}{0.129\,m} \simeq 3.87\times 10^{7}\,!! \\ \text{60 Hz AC skin depth: } \delta_{sc}^{AC} = \dfrac{\lambda_{AC}}{2\pi} \simeq 2.05\times 10^{-2}\ m = 2.05\,cm!! \end{cases}$$

$\Rightarrow$ Need <u>*at least*</u> 3-4$\times\delta_{sc} \approx$ several $\rightarrow 10\ cm$ to screen out unwanted 60 Hz AC signals !!

<u>*Instantaneous EM* Wave Energy Densities in a *Good* Conductor</u>:

$$\boxed{\left(\frac{\sigma_C}{\varepsilon\omega}\right) \gg 1}$$

$$\boxed{u_{EM} = u_E^{EM} + u_M^{EM} = \left(\frac{1}{2}\varepsilon E^2\right) + \left(\frac{1}{2\mu}B^2\right) = \left(\frac{1}{2}\varepsilon \vec{E}\bullet\vec{E}\right) + \left(\frac{1}{2\mu}\vec{B}\bullet\vec{B}\right)}$$

$$\boxed{\begin{array}{l}\phi_k \equiv (\delta_B - \delta_E) \simeq \frac{\pi}{4} = 45^\circ \\ \text{\{in a good conductor\}}\end{array}}$$

$$\boxed{\vec{E}(z,t) = E_o e^{-\kappa z}\cos(kz - \omega t + \delta_E)\,\hat{x}} \quad\text{and}\quad \boxed{\vec{B}(z,t) = B_o e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\,\hat{y}}$$

<u>Where</u>: $\boxed{B_o = \dfrac{K(\omega)}{\omega}E_o = \left[\varepsilon\mu\sqrt{1+\left(\dfrac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} E_o \simeq \sqrt{\dfrac{\cancel{\varepsilon}\,\mu\sigma_C}{\cancel{\varepsilon}\,\omega}} = \sqrt{\dfrac{\mu\sigma_C}{\omega}}E_o}$ for a good conductor,

<u>And</u>: $\boxed{k(\omega) \simeq \kappa(\omega) \approx \sqrt{\dfrac{\omega\mu\sigma_C}{2}}}$,

$\boxed{v(\omega) = \dfrac{\omega}{k(\omega)} \simeq \dfrac{\omega}{\sqrt{\dfrac{\omega\mu\sigma_C}{2}}} = \sqrt{\dfrac{2\omega}{\omega\sigma_C}} = \dfrac{c}{n(\omega)}}$ for a good conductor.

<u>Then</u>:

$$\boxed{u_E^{EM}(z,t) = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon\vec{E}\bullet\vec{E} = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E)} \quad\text{and:}$$

$$\boxed{u_M^{EM}(z,t) = \frac{1}{2\mu}B^2 = \frac{1}{2\mu}\vec{B}\bullet\vec{B} = \frac{1}{2\mu}B_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E + \phi_k) \simeq \frac{\sigma_C}{2\omega}E_o^2 e^{-2\kappa z}\cos^2(kz - \omega t + \delta_E + \phi_k)}$$

Time-averaging these quantities over one complete cycle: $\boxed{\langle u(z,t)\rangle \equiv \frac{1}{\tau}\int_0^\tau u(z,t)\,dt}$

$$\langle u_E^{EM}(z,t)\rangle = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\underbrace{\frac{1}{\tau}\int_0^\tau \cos^2(kz-\omega t+\delta_E)\,d\tau}_{=\frac{1}{2}} = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}$$

$$\langle u_M^{EM}(z,t)\rangle = \frac{\sigma_C}{2\omega}E_o^2 e^{-2\kappa z}\underbrace{\frac{1}{\tau}\int_0^\tau \cos^2(kz-\omega t+\delta_E+\phi_k)\,d\tau}_{=\frac{1}{2}} = \frac{1}{4}\left(\frac{\sigma_C}{\omega}\right)E_o^2 e^{-2\kappa z}$$

$$\therefore\ \boxed{\langle u_{Tot}^{EM}(z,t)\rangle = \langle u_E^{EM}(z,t)\rangle + \langle u_M^{EM}(z,t)\rangle = \frac{1}{4}\varepsilon\left(1+\frac{\sigma_C}{\varepsilon\omega}\right)E_o^2 e^{-2\kappa z}}$$ *n.b.* Exponentially attenuated in *z* !!!

But: $\boxed{\left(\frac{\sigma_C}{\varepsilon\omega}\right)\gg 1}$ for a good conductor, $\Rightarrow$ $\boxed{\langle u_{Tot}^{EM}(z,t)\rangle \simeq \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)\left[\frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\right]}$

i.e. the ratio: $\boxed{\frac{\langle u_M^{EM}(z,t)\rangle}{\langle u_E^{EM}(z,t)\rangle} = \left(\frac{\sigma_C}{\varepsilon\omega}\right)\gg 1}$ or $\boxed{\langle u_M^{EM}(z,t)\rangle \gg \langle u_E^{EM}(z,t)\rangle}$ for a good conductor.

$\Rightarrow$ Vast majority of *EM* wave energy is carried by the magnetic field in a good conductor !!!

Poynting's Vector: $\boxed{\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}}$ $\Rightarrow$ $\boxed{\langle \vec{S}(z,t)\rangle = \frac{1}{\mu}\langle \vec{E}\times\vec{B}\rangle = \frac{1}{2\mu}E_oB_oe^{-2\kappa z}\cos\phi_k\,\hat{z}}$ $\longleftarrow$ $\boxed{\phi_k = \frac{\pi}{4}}$

EM wave intensity (*aka* irradiance): $\boxed{I(z) = \langle|\vec{S}(z,t)|\rangle = \frac{1}{2\mu}E_oB_oe^{-2\kappa z}\cos\phi_k = \frac{1}{2\mu}E_o^2e^{-2\kappa z}\left(\frac{K}{\omega}\cos\phi_k\right)}$

But: $\boxed{\frac{K\cos\phi}{\omega} = \frac{k}{\omega} \simeq \frac{\sqrt{\frac{\omega\mu\sigma_C}{2}}}{\omega} = \sqrt{\frac{\mu\sigma_C}{2\omega}}}$ $\therefore$ $\boxed{I(z) = \langle|\vec{S}(z,t)|\rangle = \frac{1}{2\mu}\left(\frac{k}{\omega}\right)E_o^2e^{-2\kappa z} = \frac{1}{2}\sqrt{\frac{\sigma_C}{2\mu\omega}}E_o^2e^{-2\kappa z}}$

b.) **Special/Limiting Case of a *Fair* Conductor:** $\boxed{\sigma_C \approx \varepsilon\omega}$ $\Rightarrow$ Must use ***exact*** formulae!

c.) **Special/Limiting Case of a *Poor* Conductor:** (*i.e.* an *insulator*):

Here: $\boxed{\sigma_C \ll \varepsilon\omega}$, *i.e.* $\boxed{\left(\frac{\sigma_C}{\varepsilon\omega}\right)\ll 1}$. Conductivity of *poor* conductor: $\boxed{\sigma_C \to 0\ (i.e.\ \rho_C = 1/\sigma_C \to \infty)}$.

Complex wavenumber: $\boxed{\tilde{k} = k + i\kappa}$, with $\boxed{k = \mathrm{Re}(\tilde{k})}$ and $\boxed{\kappa = \mathrm{Im}(\tilde{k})}$.

$$\boxed{k(\omega) \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2+1\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[2+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2\right]^{1/2} \simeq \omega\sqrt{\varepsilon\mu}}$$

$\therefore$ $\boxed{k(\omega) \simeq \omega\sqrt{\varepsilon\mu}}$ for a *poor* conductor.

Likewise:

$$\boxed{\kappa(\omega) \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2} \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\cancel{1}+\frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2-\cancel{1}\right]^{1/2} = \omega\sqrt{\frac{\cancel{\varepsilon}\mu\sigma_C^2}{4\varepsilon^{\cancel{2}}\omega^2}} \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}$$

$\therefore$ $\boxed{\kappa(\omega) \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}$ for a ***poor*** conductor.

In a ***poor*** conductor $\boxed{\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$, the ratio: $\boxed{\left(\frac{\kappa(\omega)}{k(\omega)}\right) \simeq \frac{\frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}{\omega\sqrt{\varepsilon\mu}} = \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$ *i.e.* $\boxed{\kappa(\omega) \ll k(\omega)}$.

$\Rightarrow$ Complex wavenumber $\tilde{k} \equiv k + i\kappa$ is primarily ***real***, because $\kappa \ll k$ in a ***poor*** conductor.

Phase angle in a ***poor*** conductor: $\boxed{\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right) = \tan^{-1}\left(\frac{1}{2}\left(\frac{\omega_C}{\varepsilon\omega}\right)\right) \simeq \frac{1}{2}\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$

$\Rightarrow \delta_B = \delta_E + \phi_k \simeq \delta_E$, *i.e.* $\vec{B}$ and $\vec{E}$ are ***nearly*** in phase with each other in a ***poor*** conductor (*i.e.* losses very small in a poor conductor).

In a *typical* poor conductor, *e.g.* pure water:

Water has a ***huge*** static electric permittivity (due to permanent electric dipole moment of water molecule): $\varepsilon_{H_2O} \simeq 81\varepsilon_o$ (at zero Hz, *i.e.* $f = 0$) (at $P = 1\,ATM$ and $T = 20^\circ C$), however, at ***optical*** frequencies $(\omega \approx 10^{16}$ rad/sec$)$: $\varepsilon_{H_2O}(\omega) \approx 1.777\varepsilon_o$, where $\varepsilon_o = 8.85\times10^{-12}\ Farads/m$.

Since water is ***non-magnetic***: $\mu_{H_2O} \simeq \mu_o = 4\pi\times10^{-7}\ Henrys/m$

$\Rightarrow$ index of refraction: $n_{H_2O}(\omega) = \sqrt{\varepsilon_{H_2O}(\omega)\mu_{H_2O}/\varepsilon_o\mu_o} \simeq 1.333$ at ***optical*** frequencies.

The conductivity of pure water is: $\sigma_C^{H_2O} = 1/\rho_C^{H_2O} \simeq 1/2.5\times10^5\,\Omega\text{-}m = 4.0\times10^{-6}\ Siemens/m$ (at $P = 1\,ATM$ and $T = 20^\circ C$). Thus, the criteria for a ***poor*** conductor $(\sigma_C/\varepsilon\omega) \simeq 2.54\times10^{-11} \ll 1$ is certainly satisfied at ***optical*** frequencies.

The wavenumber in pure H_2O at ***optical*** frequencies is:

$$\boxed{k_{H_2O}(\omega) \simeq \omega\sqrt{\varepsilon\mu} \approx \omega\sqrt{\varepsilon\mu_o} = 10^{16}\sqrt{1.777\times8.85\times4\pi\times10^{-7}} \simeq 4.45\times10^7\ radians/m}$$

The wavelength in pure H_2O is: $\boxed{\lambda_{H_2O} = 2\pi/k_{H_2O} = 1.413\times10^{-7}\,m = 141.3\,nm}$ at ***optical*** frequencies.

cf w/ the ***vacuum*** wavelength: $\boxed{\lambda_o = c/f = 2\pi c/\omega = 1.885\times10^{-7}\,m = 188.5\,nm}$

Note that the *optical* wavelength ratio: $\boxed{\left(\dfrac{\lambda_o}{\lambda_{H_2O}}\right) = \dfrac{188.5\,nm}{141.3\,nm} = 1.333 \; = n_{H_2O}}$,

since $\lambda_{H_2O} = \lambda_o / n_{H_2O}$ in a *poor* conductor!!!

Skin depth: $\boxed{\delta_{sc}(\omega) \equiv \dfrac{1}{\kappa(\omega)} \simeq \dfrac{1}{\frac{1}{2}\sigma_C\sqrt{\mu/\varepsilon}}}$ for a *poor* conductor $\boxed{\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$.

For pure H_2O at *optical* frequencies:

$$\boxed{\kappa_{H_2O}(\omega) \simeq \frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}} \approx \frac{1}{2}\sigma_C\sqrt{\frac{\mu_o}{\varepsilon}} = \frac{1}{2}\left(\frac{1}{2.5\times10^5}\right)\sqrt{\frac{4\pi\times10^{-7}}{1.777\times8.85\times10^{-12}}} \simeq 5.65\times10^{-4}\,rad/m}$$

$$\boxed{\delta_{sc}^{H_2O}(\omega) \equiv \frac{1}{\kappa_{H_2O}} = 1.7688\times10^3\,m = 1.77\,km}$$

n.b. neglects/ignores Rayleigh scattering process – visible light photons elastically scattering off of H_2O molecules. $\lambda_{atten}^{vis} \simeq 10\,m$

Ratio: $\boxed{\left(\dfrac{\kappa_{H_2O}(\omega)}{k_{H_2O}(\omega)}\right) = \dfrac{\frac{1}{2}\sigma_C\sqrt{\frac{\mu}{\varepsilon}}}{\omega\sqrt{\varepsilon\mu}} = \dfrac{1}{2}\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) = \dfrac{1}{2}\left(\dfrac{1}{2.5\times10^5}\right)\dfrac{1}{1.777\times8.85\times10^{-12}\times10^{16}} = 1.27\times10^{-11} \ll 1}$

Phase difference: $\boxed{\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\dfrac{\kappa_{H_2O}}{k_{H_2O}}\right) \simeq 1.27\times10^{-11}\text{radians} \; (\ll 1)}$ *i.e.* $\boxed{\delta_B = \delta_E + \phi_k \simeq \delta_E}$

$\Rightarrow \vec{B}$ and $\vec{E}$ are *nearly* in phase with each other in pure H_2O at *optical* frequencies.

For pure H_2O at *low* frequencies – *e.g.* 60 Hz AC line frequency $(\omega_{AC} = 2\pi f_{AC} = 120\pi\,rad/sec)$:

The electric permittivity at $f = 60$ Hz is $\varepsilon_{H_2O}^{AC}(f \simeq 60Hz) \simeq 80\varepsilon_o = 80\times8.85\times10^{-12}\,Farads/m$ and $\mu_{H_2O}^{AC} \simeq \mu_o = 4\pi\times10^{-7}$ Henrys/m. Conductivity of pure H_2O: $\sigma_C^{H_2O} = 4.0\times10^{-6}\,Siemens/m$

Note that the criteria for a *poor* conductor: $\boxed{\left(\dfrac{\sigma_C}{\varepsilon_{H_2O}^{AC}\omega_{AC}}\right) \simeq \dfrac{4.0\times10^{-6}}{80\times8.85\times10^{-12}\cdot120\pi} \simeq 15 \ll 1}$

is ***not*** satisfied at the 60 Hz AC line frequency – *i.e.* at *low enough* frequencies, even *poor* conductors such as pure water are actually quite *good* conductors !!!

Thus, for the following, we must use the *good* conductor approximations:

$$\boxed{k_{AC}^{H_2O}(\omega) \simeq \kappa_{AC}^{H_2O}(\omega) \simeq \sqrt{\frac{\omega_{AC}\mu_{AC}^{H_2O}\sigma_C}{2}} \approx \sqrt{\frac{\omega_{AC}\mu_o\sigma_C}{2}} = \sqrt{\frac{120\pi\cdot4\pi\times10^{-7}\cdot4\times10^{-6}}{2}} = 3.08\times10^{-5}\,\text{rads/m}}$$

$\boxed{\lambda_{AC}^{H_2O}(\omega) \simeq \dfrac{2\pi}{k_{AC}^{H_2O}(\omega)} = 2.04\times10^5\,\text{m}}$ ***cf*** *w/* vacuum wavelength: $\boxed{\lambda_o = c/f_{AC} = \dfrac{2\pi c}{\omega_{AC}} = 5.00\times10^6\,\text{m}}$

Vacuum/good conductor wavelength ratio: $\boxed{\left(\dfrac{\lambda_o}{\lambda_{AC}^{H_2O}}\right) = \dfrac{5.00\times 10^6\,m}{2.04\times 10^5\,m} \simeq 24.495}$

<u>Skin depth for pure H_2O at 60 Hz AC line frequency</u>: $\delta_{H_2O}^{AC} \equiv 1/\kappa_{H_2O}^{AC} \simeq 3.25\times 10^4\,m = 32.5\,km$

This may seem like a large distance scale associated with the attenuation of the 60 Hz *EM* waves propagating in pure water, however compare the skin depth to the wavelength at this frequency: $\delta_{H_2O}^{AC} = 32.5\,km$ *vs.* $\lambda_{AC}^{H_2O} = 1.77\times 10^6\,m$, *i.e.* we see that $\delta_{H_2O}^{AC} \ll \lambda_{H_2O}^{AC}$, as we expect for the case of a <u>good</u> conductor !!!

The ratio $\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \simeq 1$ for pure H_2O at 60 Hz AC line frequency, which is what we expect for a <u>good</u> conductor {this ratio should be $\ll 1$ for a <u>poor</u> conductor}.

Thus, the phase difference is: $\boxed{\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\kappa_{H_2O}^{AC}/k_{H_2O}^{AC}\right) \simeq \tan^{-1}(1) = \dfrac{\pi}{4} = 45^\circ}$

which again is what we expect for a <u>good</u> conductor, *i.e.* $\vec{B}$ lags $\vec{E}$ by 45° !

***<u>Instantaneous EM</u>* <u>energy densities in a *poor* conductor</u>:** $\boxed{\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$

$$\boxed{u_{EM}(z,t) = u_E^{EM}(z,t) + u_M^{EM}(z,t) = \left(\frac{1}{2}\varepsilon E^2\right) + \left(\frac{1}{2\mu}B^2\right) = \left(\frac{1}{2}\varepsilon\vec{E}\bullet\vec{E}\right) + \left(\frac{1}{2\mu}\vec{B}\bullet\vec{B}\right)}$$

The physical $\vec{E}$ and $\vec{B}$ fields are:

$$\boxed{\vec{E}(z,t) = E_o e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{x}} \text{ and } \boxed{\vec{B}(z,t) = B_o e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi_k)\,\hat{y}}$$

<u>where</u>: $\boxed{B_o = \dfrac{K}{\omega}E_o = \left[\varepsilon\mu\sqrt{1+\left(\dfrac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} E_o \approx \sqrt{\varepsilon\mu}\;E_o}$ for a <u>poor</u> conductor, $\boxed{\left(\dfrac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$.

$\boxed{k \simeq \omega\sqrt{\varepsilon\mu} = \dfrac{\omega}{v}}$ <u>where</u>: $\boxed{v = \dfrac{1}{\sqrt{\varepsilon\mu}} = c/n}$ <u>and</u>: $\boxed{n = \sqrt{\dfrac{\varepsilon\mu}{\varepsilon_o\mu_o}}}$ for a <u>poor</u> conductor.

<u>and</u>: $\boxed{\kappa \simeq \dfrac{1}{2}\sigma_c\sqrt{\dfrac{\mu}{\varepsilon}} \ll k \simeq \omega\sqrt{\varepsilon\mu}}$, $\boxed{K \equiv |\tilde{k}| \simeq \omega\sqrt{\varepsilon\mu}}$ for a <u>poor</u> conductor.

<u>Then</u>: $\boxed{u_E^{EM}(z,t) = \dfrac{1}{2}\varepsilon E^2 = \dfrac{1}{2}c\vec{E}\bullet\vec{E} = \dfrac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E)}$ <u>and</u>:

$$\boxed{u_M^{EM}(z,t) = \frac{1}{2\mu}B^2 = \frac{1}{2\mu}\vec{B}\bullet\vec{B} = \frac{1}{2\mu}B_o^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_E+\phi_k)}$$

<u>*Time-averaging* these quantities</u>:

$$\boxed{\langle u_E^{EM}(z,t)\rangle = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}} \text{ and: } \boxed{\langle u_M^{EM}(z,t)\rangle = \frac{1}{4\mu}B_o^2 e^{-2\kappa z} \simeq \frac{1}{4\cancel{\mu}}(\varepsilon\cancel{\mu})E_o^2 e^{-2\kappa z} = \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}}$$

$$\therefore\ \boxed{\left\langle u_{Tot}^{EM}(z,t)\right\rangle = \left\langle u_{E}^{EM}(z,t)\right\rangle + \left\langle u_{M}^{E}(z,t)\right\rangle \simeq \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z} + \frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z} = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}}$$

Thus: $\boxed{\left\langle u_{Tot}^{EM}(z,t)\right\rangle = \frac{1}{2}\varepsilon E_o^2 e^{-2\kappa z}}$ for a poor conductor $\boxed{\left(\frac{\sigma_C}{\varepsilon\omega}\right) \ll 1}$.

The ratio of {time-averaged} electric/magnetic energy densities for a poor conductor:

$$\boxed{\frac{\left\langle u_{E}^{EM}(z,t)\right\rangle}{\left\langle u_{M}^{EM}(z,t)\right\rangle} \simeq \frac{\frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}}{\frac{1}{4}\varepsilon E_o^2 e^{-2\kappa z}} = 1} \quad \boxed{\phi_k \equiv \delta_B - \delta_E = \tan^{-1}\left(\frac{\kappa_{H_2O}}{k_{H_2O}}\right) \ll 1} \quad \boxed{\kappa_{H_2O} \simeq \frac{1}{2}\sigma_c\sqrt{\frac{\mu_o}{\varepsilon}} \ll k_{H_2O} \simeq \omega\sqrt{\varepsilon\mu_o}}$$

$\Rightarrow$ *EM* wave energy is shared $\approx$ equally by the $\vec{E}$ and $\vec{B}$ fields in a poor conductor!

***Instantaneous* Poynting's Vector** for *EM* waves propagating in a poor conductor:

$$\boxed{\vec{S}(z,t) = \frac{1}{\mu}\vec{E}(z,t)\times\vec{B}(z,t)} \Rightarrow \boxed{\left\langle \vec{S}(z,t)\right\rangle = \frac{1}{\mu}\left\langle \vec{E}(z,t)\times\vec{B}(z,t)\right\rangle \simeq \frac{\sqrt{\varepsilon\mu_o}}{2\mu_o}E_o^2 e^{-2\kappa z}\underbrace{\cos\phi_k}_{\approx 1}\hat{z}}$$

$$\therefore\ \boxed{\left\langle \vec{S}(z,t)\right\rangle \simeq \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_o}}E_o^2 e^{-2\kappa z}\hat{z}}$$ for a *poor* conductor.

Intensity of *EM* waves propagating in a *poor* conductor:

$$\boxed{I(z) = \left\langle \left|\vec{S}(z,t)\right|\right\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_o}}E_o^2 e^{-2\kappa z}}$$

Reflection of *EM* Waves at Normal Incidence from a Conducting Surface:

In the presence of free surface charges σ_{free} and/or free surface currents, $\vec{K}_{free}$ the boundary conditions obtained from (the integral forms of) Maxwell's equations for reflection and refraction at *e.g.* a dielectric-conductor interface become:

$\perp$ = normal to plane of interface
$\|\|$ = parallel to plane of interface

BC 1): (normal $\vec{D}$ at interface): $\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$

BC 2): (tangential $\vec{E}$ at interface): $\boxed{E_1^{\|} - E_2^{\|} = 0} \Rightarrow \boxed{E_1^{\|} = E_2^{\|}}$

BC 3): (normal $\vec{B}$ at interface): $\boxed{B_1^{\perp} - B_2^{\perp} = 0} \Rightarrow \boxed{B_1^{\perp} = B_2^{\perp}}$

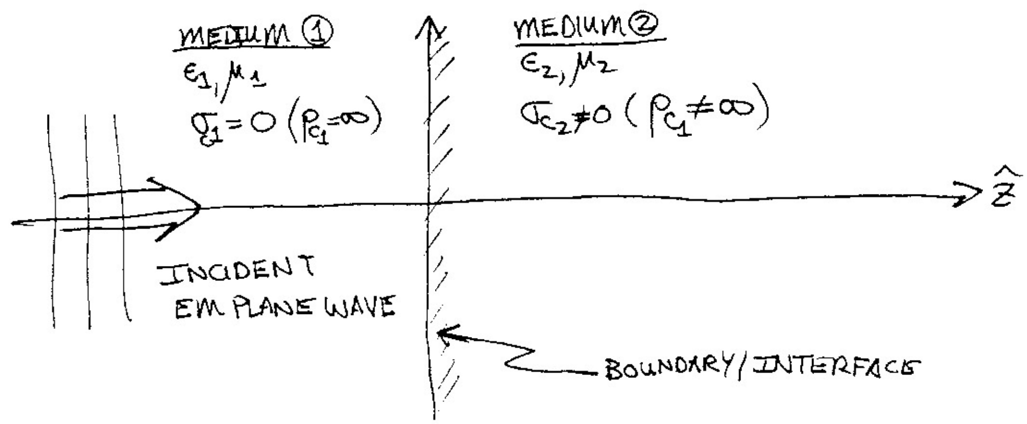
BC 4): (tangential $\vec{H}$ at interface): $\boxed{\dfrac{1}{\mu_1} B_1^{\|} - \dfrac{1}{\mu_2} B_2^{\|} = \vec{K}_{free} \times \hat{n}_{\underset{21}{\rightarrow}}}$

where $\hat{n}_{\underset{21}{\rightarrow}}$ is a unit vector $\perp$ to the interface, pointing <u>from</u> medium (2) <u>into</u> medium (1).

{*n.b.* do ***<u>not</u>*** confuse $\hat{n}_{\underset{21}{\rightarrow}}$ with the *EM* wave *<u>polarization</u>* *<u>vector</u>* $\hat{n}$!!!}

<u>Note</u>: For ***<u>Ohmic</u>*** conductors (*i.e.* "normal" conductors obeying <u>Ohm's Law</u> $\vec{J}_{free} = \sigma_C \vec{E}$) there can be ***<u>no</u>*** *<u>free</u>* ***<u>surface</u>*** currents, *i.e.* $\vec{K}_{free} = 0$ because $\vec{K}_{free} \neq 0$ would require an <u>infinite</u> $\vec{E}$-field at the boundary/interface!

Suppose $\exists$ a boundary/interface (located in the *x*-*y* plane at *z* = 0) between a non-conducting linear/homogeneous/isotropic medium (1) and a conductor (2). A monochromatic plane *EM* wave is incident on the interface, that is linearly polarized in $+\hat{x}$ direction, traveling in the $+\hat{z}$ direction, approaching the interface/boundary from the left, in medium (1) as shown in the figure below:



$$\boxed{\tilde{\vec{B}} = \frac{1}{v}\left(\hat{k}\times\tilde{\vec{E}}\right)}$$

Incident *EM* wave {medium (1)}: $\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{x}}$ and: $\boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{y}}$

Reflected *EM* wave {medium (1)}: $\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z=\omega t)}\hat{x}}$ and: $\boxed{\tilde{\vec{B}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z-\omega t)}\hat{y}}$

Transmitted *EM* wave {medium (2)}: $\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z-\omega t)}\hat{x}}$ and: $\boxed{\tilde{\vec{B}}_{trans}(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{o_{trans}} e^{i(\tilde{k}_2 z-\omega t)}\hat{y}}$

n.b. complex wavenumber in {conducting} medium (2): $\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$

In medium (1) *EM* fields are: $\boxed{\tilde{\vec{E}}_{Tot_1}(z,t) = \tilde{\vec{E}}_{inc}(z,t) + \tilde{\vec{E}}_{refl}(z,t)}$ and: $\boxed{\tilde{\vec{B}}_{Tot_1}(z,t) = \tilde{\vec{B}}_{inc}(z,t) + \tilde{\vec{B}}_{refl}(z,t)}$

In medium (2) *EM* fields are: $\boxed{\tilde{\vec{E}}_{Tot_2}(z,t) = \tilde{\vec{E}}_{trans}(z,t)}$ and: $\boxed{\tilde{\vec{B}}_{Tot_2}(z,t) = \tilde{\vec{B}}_{trans}(z,t)}$

Apply BC's at the $z = 0$ interface in the x-y plane:

BC 1): $\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$ but: $\boxed{E_1^{\perp} = \tilde{E}_{1_z} = 0}$ and: $\boxed{E_2^{\perp} = \tilde{E}_{2_z} = 0}$ $\therefore$ $\boxed{0 - 0 = \sigma_{free}}$ $\Rightarrow$ $\boxed{\sigma_{free} = 0}$

BC 2): $\boxed{E_1^{\parallel} = E_2^{\parallel}}$ $\therefore$ $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 3): $\boxed{B_1^{\perp} = B_2^{\perp}}$ but: $\boxed{B_1^{\perp} = B_{1_z} = 0}$ and: $\boxed{B_2^{\perp} = B_{2_z} = 0}$ $\Rightarrow$ $\boxed{0 = 0}$

BC 4): $\boxed{\frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = \vec{K}_{free}\times\hat{n}_{\vec{21}}}$ but: $\boxed{\vec{K}_{free} = 0}$ $\therefore$ $\boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) - \frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{o_{trans}} = 0}$

or: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta}\tilde{E}_{o_{trans}}}$ with: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$

Thus we obtain: $\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{o_{inc}}}$ and: $\boxed{\tilde{E}_{o_{trans}} = \frac{2}{\left(1+\tilde{\beta}\right)}\tilde{E}_{o_{inc}}}$

or: $\boxed{\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right) = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)}$ and: $\boxed{\left(\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}}\right) = \frac{2}{\left(1+\tilde{\beta}\right)}}$ with: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$

Note that these relations for reflection/transmission of *EM* waves at normal incidence on a non-conductor/conductor boundary/interface are identical to those obtained for reflection / transmission of *EM* waves at normal incidence on a boundary/interface between two non-conductors, except for the replacement of β with a now complex $\tilde{\beta}$ for the present situation.

For the case of a perfect conductor, the conductivity $\sigma_C = \infty$ {thus resistivity, $\rho_C = 1/\sigma_C = 0$}

$\Rightarrow$ *both* $\boxed{k_2 \simeq \kappa_2 \simeq \sqrt{\frac{\omega\mu_2\sigma_C}{2}} = \infty}$ and since: $\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$ then: $\boxed{\tilde{k}_2 = \infty + i\infty = \infty(1+i)}$

and since: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2} \Rightarrow \tilde{\beta} = \infty$

Thus, for a perfect conductor, we see that: $\boxed{\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}}$ and $\boxed{\tilde{E}_{trans} = 0}$ and thus for a perfect conductor the reflection and transmission coefficients are:

$$\boxed{R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = 1} \quad \text{and:} \quad \boxed{T = 1 - R = 0}$$

We also see that for a perfect conductor, for normal incidence, the reflected wave undergoes a 180^o phase shift with respect to the incident wave at the interface/boundary at $z = 0$ in the x-y plane. A perfect conductor screens out all *EM* waves from propagating in its interior.

For the case of a good conductor, the conductivity σ_C is finite-large, but not infinite. The reflection coefficient *R* for monochromatic plane *EM* waves at normal incidence on a good conductor is *not* unity, but close to it. {This is why good conductors make good mirrors!}

For a good conductor: $\boxed{R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*}$

Where: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$ and: $\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$. For a good conductor: $\boxed{k_2 \simeq \kappa_2 \simeq \sqrt{\frac{\omega\mu_2\sigma_C}{2}}}$

Thus: $\boxed{\tilde{\beta} = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2 = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\sqrt{\frac{\omega\mu_2\sigma_C}{2}}(1+i) = \mu_1 v_1\sqrt{\frac{\sigma_C}{2\mu_2\omega}}(1+i)}$

Define: $\boxed{\gamma \equiv \mu_1 v_1\sqrt{\frac{\sigma_C}{2\mu_2\omega}}}$ Then: $\boxed{\tilde{\beta} = \gamma(1+i)}$

Thus, the reflection coefficient *R* for monochromatic plane *EM* waves at normal incidence on a good conductor is:

$$\boxed{R = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \left[\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}\right]}$$

with: $\boxed{\gamma \equiv \mu_1 v_1\sqrt{\frac{\sigma_C}{2\mu_2\omega}}}$

Obviously, only a small fraction of the normally-incident monochromatic plane *EM* wave is transmitted into the good conductor, since $R < 1$ and $\boxed{T = 1 - R}$, *i.e.*:

$$\boxed{T = 1 - R = 1 - \left[\frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}\right] \quad (\ll 1)}$$

Note that the transmitted wave is exponentially attenuated in the *z*-direction; the $\vec{E}$ and $\vec{B}$ fields in the good conductor fall to $1/e$ of their initial $\{z = 0\}$ values (at/on the interface) after the monochromatic plane *EM* wave propagates a distance of one skin depth in *z* into the conductor:

$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \simeq \sqrt{\frac{2}{\omega\mu_2\sigma_C}}}$$

Note also that the energy associated with the transmitted monochromatic plane *EM* wave is ultimately dissipated in the conducting medium as heat.

In {bulk} metals, since the transmitted wave is {rapidly} absorbed/attenuated in the metal, we can only study/measure the reflection coefficient *R*. A full/detailed mathematical description of the physics of reflection from the surface of a metal conductor as a function of angle of incidence *i.e.* $R(\omega,\theta_{inc})$ requires the use of a complex dispersion relation $\tilde{k}(\omega) = \omega/\tilde{v}(\omega) = (\omega/c)\tilde{n}(\omega)$ with complex $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$ and complex propagation speed $\tilde{v}(\omega) = \mathrm{v}(\omega) + iv(\omega) = c/\tilde{n}(\omega)$ with accompanying complex index of refraction $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$, and thus is fairly complicated. So-called *ellipsometry* measurements of the *EM* radiation reflected from the surface of the metal as a function of angle of incidence yields information on the real and imaginary parts of the complex index of refraction of the metal $\tilde{n}(\omega) = n(\omega) + i\eta(\omega)$, and thus the real and imaginary parts of the complex dielectric constant and/or the complex electric susceptibility of the metal, since $\tilde{n}(\omega) = \sqrt{\tilde{\varepsilon}(\omega)/\varepsilon_o} = \sqrt{1 + \tilde{\chi}_e(\omega)}$ or $\tilde{n}^2(\omega) = \tilde{\varepsilon}(\omega)/\varepsilon_o = 1 + \tilde{\chi}_e(\omega)$.

If interested in learning more about this, *e.g.* please see/read ***Optics***, M.V. Klein, p. 588-592, Wiley, 1970 {P436 reference book on reserve in the Physics library}. Please see/read also the UIUC P402 Optics/Light Lab Ellipsometry Lab Handout C4 and especially the references at the end. Available at: **http://online.physics.uiuc.edu/courses/phys402/exp/C4/C4.pdf**

We will discuss the *dispersive* nature of dielectric, non-conducting materials in the next lecture…

Full Maxwell Equations in Matter:

The electromagnetic state of matter at a given observation point $\vec{r}$ at a given time *t* is described by four macroscopic quantities:

1.) The volume density of free charge: $\boxed{\rho_{free}(\vec{r},t)}$

2.) The volume density of electric dipoles: $\boxed{\vec{\mathrm{P}}(\vec{r},t)}$ $\Leftarrow$ *aka* electric polarization

3.) The volume density of magnetic dipoles: $\boxed{\vec{\mathrm{M}}(\vec{r},t)}$ $\Leftarrow$ *aka* magnetization

4.) The free electric current/unit area: $\boxed{\vec{J}_{free}(\vec{r},t)}$ $\Leftarrow$ *aka* {free} current density

All four of these quantities are macroscopically averaged - *i.e.* the microscopic fluctuations due to atomic/molecular makeup of matter have been smoothed out.

The four above quantities are related to the macroscopic $\vec{E}$ and $\vec{B}$ fields by the four Maxwell equations for matter (see Physics 435 Lect. Notes 24, p. 14):

1) Gauss' Law: $\boxed{\vec{\nabla}\bullet\vec{E} = \dfrac{\rho_{Tot}}{\varepsilon_o} = \dfrac{1}{\varepsilon_o}\left(\rho_{free} + \rho_{bound}\right)}$, where: $\boxed{\rho_{bound} = -\vec{\nabla}\bullet\vec{\mathrm{P}}}$

Auxiliary relation: $\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}}$ & constitutive relation: $\boxed{\vec{D} = \varepsilon\vec{E}}$

Electric polarization $\boxed{\vec{\mathrm{P}} = (\varepsilon - \varepsilon_o)\vec{E} = \varepsilon_0\chi_e\vec{E}}$, electric susceptibility $\boxed{\chi_e = \left(\dfrac{\varepsilon}{\varepsilon_o} - 1\right)}$

$$\boxed{\vec{\nabla}\bullet\vec{D} = \varepsilon_o\vec{\nabla}\bullet\vec{E} + \vec{\nabla}\bullet\vec{\mathrm{P}} = \rho_{free}}$$

2) No magnetic charges/monopoles: $\boxed{\vec{\nabla}\bullet\vec{B} = 0}$

Auxiliary relation: $\boxed{\vec{H} = \dfrac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}}$ $\Rightarrow$ $\boxed{\vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{\mathrm{M}}}$ & constitutive relation: $\boxed{\vec{B} = \mu\vec{H}}$

3) Faraday's Law: $\boxed{\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} = -\mu_0\dfrac{\partial\vec{H}}{\partial t} - \mu_0\dfrac{\partial\vec{\mathrm{M}}}{\partial t}}$

Magnetization: $\boxed{\vec{\mathrm{M}} - \left(\dfrac{\mu}{\mu_o - 1}\right)\vec{H} = \chi_m\vec{H}}$, magnetic susceptibility $\boxed{\chi_m = \left(\dfrac{\mu}{\mu_o} - 1\right)}$

4) Ampere's Law: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J}_{ToT} + \mu_o\vec{J}_D}$ with $\boxed{\vec{J}_D = \varepsilon_o\dfrac{\partial\vec{E}}{\partial t}}$

Total current density: $\boxed{\vec{J}_{ToT} = \vec{J}_{free} + \vec{J}^{mag}_{bound} + \vec{J}^{P}_{bound}}$ $\boxed{\vec{J}^{mag}_{bound} = \vec{\nabla}\times\vec{\mathrm{M}}}$ $\boxed{\vec{J}^{P}_{bound} = \dfrac{\partial\vec{\mathrm{P}}}{\partial t}}$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_o J_{free} + \mu_o\vec{\nabla}\times\vec{\mathrm{M}} + \mu_o\frac{\partial\vec{\mathrm{P}}}{\partial t} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}}$$

$$\boxed{\vec{\nabla}\times\vec{H} = \mu_o\vec{J}_{free} + \mu_o\frac{\partial\vec{D}}{\partial t}}$$

We also have Ohm's Law: $\boxed{\vec{J} = \sigma_C \vec{E}}$ and the 3 continuity eqn(s): $\boxed{\vec{\nabla}\cdot\vec{J}_\alpha = -\frac{\partial \rho_\alpha}{\partial t}}$ associated with α = free, bound and total electric charge conservation.

For many/most (but <u>not</u> all!!!) physics problems, *e.g.* in optics/condensed matter physics, materials of interest are frequently <u>non-magnetic</u> (or <u>*negligibly*</u> magnetic) and have no (free) charge densities present, *i.e.* $\rho_{free} = 0$. If $\mu \simeq \mu_o$, then $\vec{M} = 0$ and thus $\vec{H} = \vec{B}/\mu_o$ in such non-magnetic materials.

Then Maxwell's equations in matter, for $\rho_{free} = 0$ and $\vec{M} = 0$ reduce to:

1) <u>Gauss' Law</u>: $\boxed{\vec{\nabla}\cdot\vec{D} = 0}$ or: $\boxed{\vec{\nabla}\cdot\vec{E} = -\frac{1}{\varepsilon_o}\vec{\nabla}\cdot\vec{P} = \rho_{free}/\varepsilon_o}$

2) <u>No magnetic charges</u>: $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

3) <u>Faraday's Law</u>: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

4) <u>Ampere's Law</u>: $\boxed{\vec{\nabla}\times\vec{B} = \mu_o\varepsilon_o\frac{\partial \vec{E}}{\partial t} + \mu_o\frac{\partial \vec{P}}{\partial t} + \mu_o\vec{J}_{free}}$

We also have Ohm's Law $\boxed{\vec{J}_{free} = \sigma_c \vec{E}}$ and the Continuity eqn. $\boxed{\vec{\nabla}\cdot\vec{J}_{free} = 0}$ {here}.

<u>Then applying the curl operator to Faraday's Law</u>:

$$\boxed{\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) = -\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2} - \mu_o\frac{\partial^2\vec{P}}{\partial t^2} - \mu_o\frac{\partial \vec{J}_{free}}{\partial t} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} = \frac{1}{\varepsilon_o}\vec{\nabla}\rho_{bound} - \nabla^2\vec{E}}$$

We thus obtain the <u>*inhomogeneous*</u> wave equation:

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_o}\nabla\rho_{bound} + \mu_o\frac{\partial^2\vec{P}}{\partial t^2} + \mu_o\frac{\partial \vec{J}_{free}}{\partial t}}_{\text{source terms}}}$$

{and a similar/analogous one for $\vec{B}$ }

For nonconducting/poorly-conducting media, *i.e.* insulators/dielectrics, the first two terms on the RHS of the above equation are important – *e.g.* they explain many optical effects such as dispersion (wavelength/frequency-dependence of the index of refraction), absorption, double – refraction/bi-refringence, optical activity, ...

Note that the $\vec{\nabla}\rho_{bound} = -\vec{\nabla}\left(\vec{\nabla}\cdot\vec{P}\right)$ term is often zero, *e.g.* if the electric polarization $\vec{P}$ is uniform, or since: $\vec{\nabla}\cdot\vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$ and $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

e.g. for $\vec{P} \propto \vec{E}$ (i.e. $\vec{P}$ proportional to $\vec{E}$) where: $\vec{E}(z,t) = E_o\cos(kz - \omega t + \delta)\hat{x}$

For *good* conductors (*e.g.* metals), the *conduction* *term* $\boxed{\mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \mu_o \sigma_C \frac{\partial \vec{E}}{\partial t}}$ is the most important, because it explains the *opacity* of metals (*e.g.* in the visible light region) and also explains the *high* *reflectance* of metals.

All source terms on the RHS of the above inhomogeneous wave equation are of importance for semiconductors – however a proper/more complete physics description of *EM* wave propagation in semiconductors also requires the addition of quantum theory for rigorous treatment…