CHAPTER 3
DIPOLE AND QUADRUPOLE MOMENTS

3.1 *Introduction*

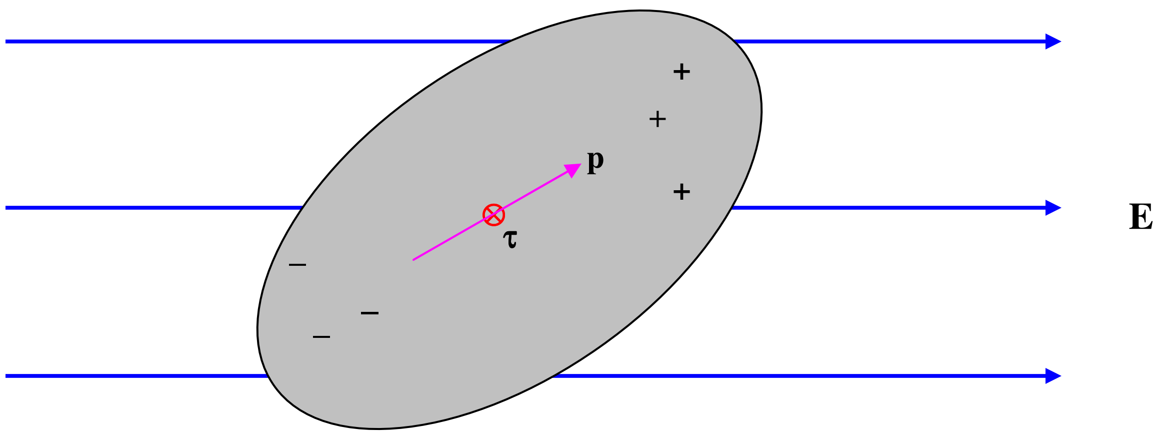


FIGURE III.1

Consider a body which is on the whole electrically neutral, but in which there is a separation of charge such that there is more positive charge at one end and more negative charge at the other. Such a body is an *electric dipole.*

Provided that the body as a whole is electrically neutral, it will experience no *force* if it is placed in a uniform external electric field, but it will (unless very fortuitously oriented) experience a *torque.* The magnitude of the torque depends on its orientation with respect to the field, and there will be two (opposite) directions in which the torque is a *maximum*.

The maximum torque that the dipole experiences when placed in an external electric field is its *dipole moment*. This is a vector quantity, and the torque is a maximum when the dipole moment is at right angles to the electric field. At a general angle, the torque $\boldsymbol{\tau}$, the dipole moment $\mathbf{p}$ and the electric field $\mathbf{E}$ are related by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}. \qquad 3.1.1$$

The SI units of dipole moment can be expressed as N m (V/m)$^{-1}$. However, work out the dimensions of p and you will find that its dimensions are Q L. Therefore it is simpler to express the dipole monent in SI units as coulomb metre, or C m.

Other units that may be encountered for expressing dipole moment are cgs esu, debye, and atomic unit. I have also heard the dipole moment of thunderclouds expressed in kilometre coulombs. A cgs esu is a centimetre-gram-second electrostatic unit. I shall describe the cgs esu system in a later chapter; suffice it here to say that a cgs esu of dipole moment is about 3.336×10^{-12} C m, and a debye (D) is 10^{-18} cgs esu. An atomic unit of electric dipole moment is a_0e, where a_0 is the radius of the first Bohr orbit for hydrogen and e is the magnitude of the electronic charge. An atomic unit of dipole moment is about 8.478×10^{-29} C m.

I remark in passing that I have heard, distressingly often, some such remark as “The molecule has a dipole”. Since this sentence is not English, I do not know what it is intended to mean. It would be English to say that a molecule is a dipole or that it has a dipole moment.

3.2 *Mathematical Definition of Dipole Moment*

In the introductory section 3.1 we gave a *physical* definition of dipole moment. I am now about to give a *mathematical* definition.

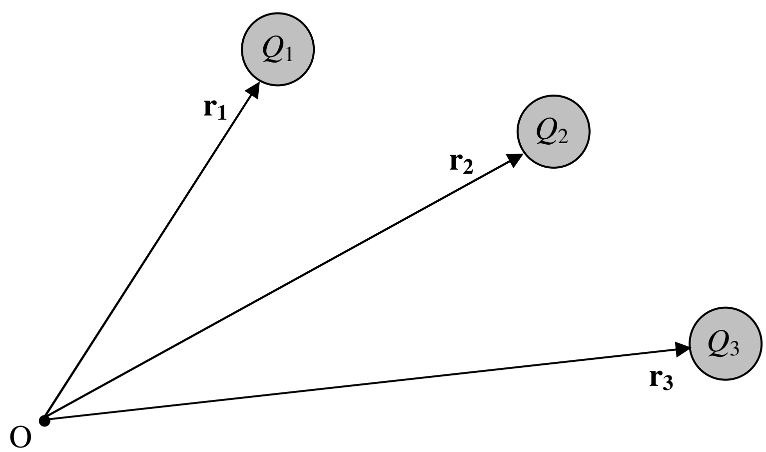


FIGURE III.2

Consider a set of charges Q_1, Q_2, Q_3 ... whose position vectors with respect to a point O are $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$ ... with respect to some point O. The vector sum

$$\mathbf{p} = \sum Q_i \mathbf{r_i}$$

is the dipole moment of the system of charges with respect to the point O. You can see immediately that the SI unit has to be C m.

Exercise. Convince yourself that if the system as a whole is electrically neutral, so that there is as much positive charge as negative charge, the dipole moment so defined is independent of the position of the point O. One can then talk of "the dipole moment of the system" without adding the rider "with respect to the point O".

Exercise. Convince yourself that if any electrically neutral system is place in an external electric field $\mathbf{E}$, it will experience a torque given by $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$, and so the two definitions of dipole moment – the physical and the mathematical – are equivalent.

Exercise. While thinking about these two, also convince yourself (from mathematics or from physics) that the moment of a simple dipole consisting of two charges, $+Q$ and $-Q$ separated by a distance l is Ql. We have already noted that C m is an acceptable SI unit for dipole moment.

3.3 *Oscillation of a Dipole in an Electric Field*

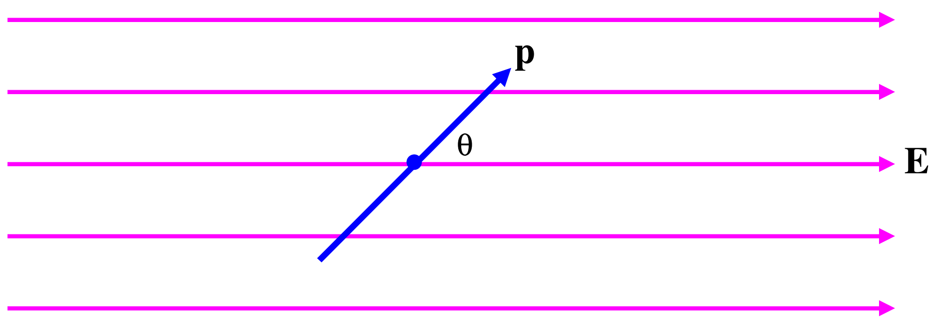


FIGURE III.3

Consider a dipole oscillating in an electric field (figure III.3). When it is at an angle θ to the field, the magnitude of the restoring torque on it is $pE \sin \theta$, and therefore its equation of motion is $I\ddot{\theta} = -pE\sin\theta$, where I is its rotational inertia. For small angles, this is approximately $I\ddot{\theta} = -pE\theta$, and so the period of small oscillations is

$$P = 2\pi\sqrt{\frac{I}{pE}}\,. \qquad 3.3.1$$

Would you expect the period to be long if the rotational inertia were large? Would you expect the vibrations to be rapid if p and E were large? Is the above expression dimensionally correct?

3.4 *Potential Energy of a Dipole in an Electric Field*

Refer again to figure III.3. There is a torque on the dipole of magnitude $pE \sin\theta$. In order to increase θ by $\delta\theta$ you would have to do an amount of work $pE \sin\theta\ \delta\theta$. The amount of work you would have to do to increase the angle between **p** and **E** from 0 to θ would be the integral of this from 0 to θ, which is $pE(1 - \cos\theta)$, and this is the potential energy of the dipole, provided one takes the potential energy to be zero when **p** and **E** are parallel. In many applications, writers find it convenient to take the potential energy (P.E.) to be zero when **p** and **E** perpendicular. In that case, the potential energy is

$$\text{P.E.} = -pE\cos\theta = -\mathbf{p}\bullet\mathbf{E}. \qquad 3.4.1$$

This is negative when θ is acute and positive when θ is obtuse. You should verify that the product of p and E does have the dimensions of energy.

3.5 *Force on a Dipole in an Inhomogeneous Electric Field*

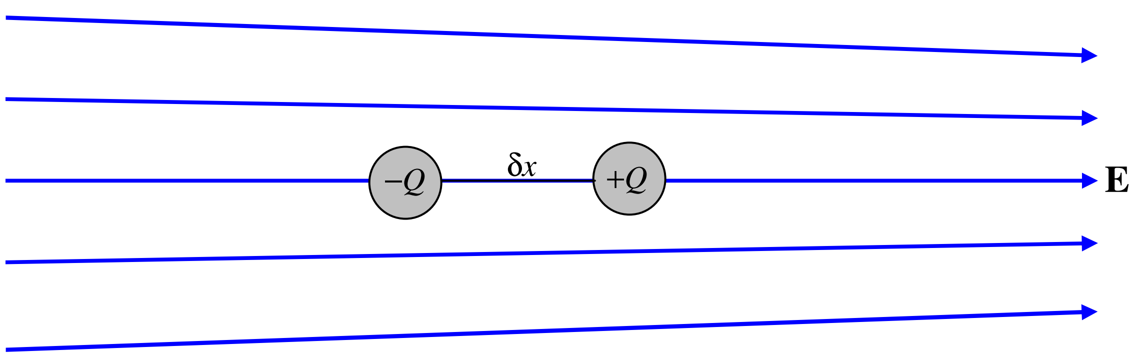


FIGURE III.4

Consider a simple dipole consisting of two charges $+Q$ and $-Q$ separated by a distance δx, so that its dipole moment is $p = Q\ \delta x$. Imagine that it is situated in an inhomogeneous electrical field as shown in figure III.4. We have already noted that a dipole in a *homogeneous* field experiences no net force, but we can see that it *does* experience a net force in an *inhomogeneous* field. Let the field at $-Q$ be E and the field at $+Q$ be $E + \delta E$. The force on $-Q$ is QE to the left, and the force on $+Q$ is $Q(E + \delta E)$ to the right. Thus there is a net force to the right of $Q\ \delta E$, or:

$$\text{Force} = p\frac{dE}{dx}. \qquad 3.5.1$$

Equation 3.5.1 describes the situation where the dipole, the electric field and the gradient are all parallel to the *x*-axis. In a more general situation, all three of these are in different directions. Recall that electric field is minus potential gradient. Potential is a scalar function, whereas electric field is a vector function with three component, of which the *x*-component, for example is $E_x = -\frac{\partial V}{\partial x}$. Field *gradient* is a symmetric *tensor* having nine components (of which, however, only six are distinct), such as $\frac{\partial^2 V}{\partial x^2}$, $\frac{\partial^2 V}{\partial y \partial z}$, etc. Thus in general equation 3.5.1 would have to be written as

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -\begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{xy} & V_{yy} & V_{yz} \\ V_{xz} & V_{yz} & V_{zz} \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}, \qquad 3.5.2$$

in which the double subscripts in the potential gradient tensor denote the second partial derivatives.

3.6 *Induced Dipoles and Polarizability*

We noted in section 1.3 that a charged rod will attract an *uncharged* pith ball, and at that time we left this as a little unsolved mystery. What happens is that the rod *induces a dipole moment* in the uncharged pith ball, and the pith ball, which now has a dipole moment, is attracted in the *inhomogeneous field* surrounding the charged rod.

How may a dipole moment be induced in an uncharged body? Well, if the uncharged body is metallic (as in the gold leaf electroscope), it is quite easy. In a metal, there are numerous free electrons, not attached to any particular atoms, and they are free to wander about inside the metal. If a metal is placed in an electric field, the free electrons are attracted to one end of the metal, leaving an excess of positive charge at the other end. Thus a dipole moment is induced.

What about a nonmetal, which doesn't have free electrons unattached to atoms? It may be that the individual molecules in the material have permanent dipole moments. In that case, the imposition of an external electric field will exert a torque on the molecules, and will cause all their dipole moments to line up in the same direction, and thus the bulk material will acquire a dipole moment. The water molecule, for example, has a permanent dipole moment, and these dipoles will align in an external field. This is why pure water has such a large dielectric constant.

But what if the molecules do not have a permanent dipole moment, or what if they do, but they cannot easily rotate (as may well be the case in a solid material)? The bulk material can still become polarized, because a dipole moment is induced in the individual molecules, the electrons inside the molecule tending to be pushed towards one end of the molecule. Or a molecule such as CH_4, which is symmetrical in the absence of an external electric field, may become distorted from its symmetrical shape when placed in an electric field, and thereby acquire a dipole moment.

Thus, one way or another, the imposition of an electric field may induce a dipole moment in most materials, whether they are conductors of electricity or not, or whether or not their molecules have permanent dipole moments.

If two molecules approach each other in a gas, the electrons in one molecule repel the electrons in the other, so that each molecule induces a dipole moment in the other. The two molecules then attract each other, because each dipolar molecule finds itself in the inhomogeneous electric field of the other. This is the origin of the van der Waals forces.

Some bodies (I am thinking about individual molecules in particular, but this is not necessary) are more easily polarized that others by the imposition of an external field. The ratio of the induced dipole moment to the applied field is called the *polarizability* α of the molecule (or whatever body we have in mind). Thus

$$\mathbf{p} = \alpha \mathbf{E}. \qquad 3.6.1$$

The SI unit for α is C m (V m^{-1})$^{-1}$ and the dimensions are $\mathrm{M^{-1}T^2Q^2}$.

This brief account, and the general appearance of equation 3.6.1, suggests that **p** and **E** are in the same direction – but this is so only if the electrical properties of the molecule are isotropic. Perhaps most molecules – and, especially, long organic molecules – have *anisotropic polarizability.* Thus a molecule may be easy to polarize with a field in the *x*-direction, and much less easy in the *y*- or *z*-directions. Thus, in equation 3.6.1, the polarizability is really a symmetric *tensor*, **p** and **E** are not in general parallel, and the equation, written out in full, is

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{xy} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{xz} & \alpha_{yz} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \qquad 3.6.2$$

(Unlike in equation 3.5.2, the double subscripts are not intended to indicate second partial derivatives; rather they are just the components of the polarizability tensor.) As in several analogous situations in various branches of physics (see, for example, section 2.17 of Classical Mechanics and the inertia tensor) there are three mutually orthogonal directions (the eigenvectors of the polarizability tensor) for which **p** and **E** will be parallel.

3.7 *The Simple Dipole*

As you may expect from the title of this section, this will be the most difficult and complicated section of this chapter so far. Our aim will be to calculate the field and potential surrounding a simple dipole.

A simple dipole is a system consisting of two charges, $+Q$ and $-Q$, separated by a distance $2L$. The dipole moment of this system is just $p = 2QL$. We'll suppose that the dipole lies along the x-axis, with the negative charge at $x = -L$ and the positive charge at $x = +L$. See figure III.5.

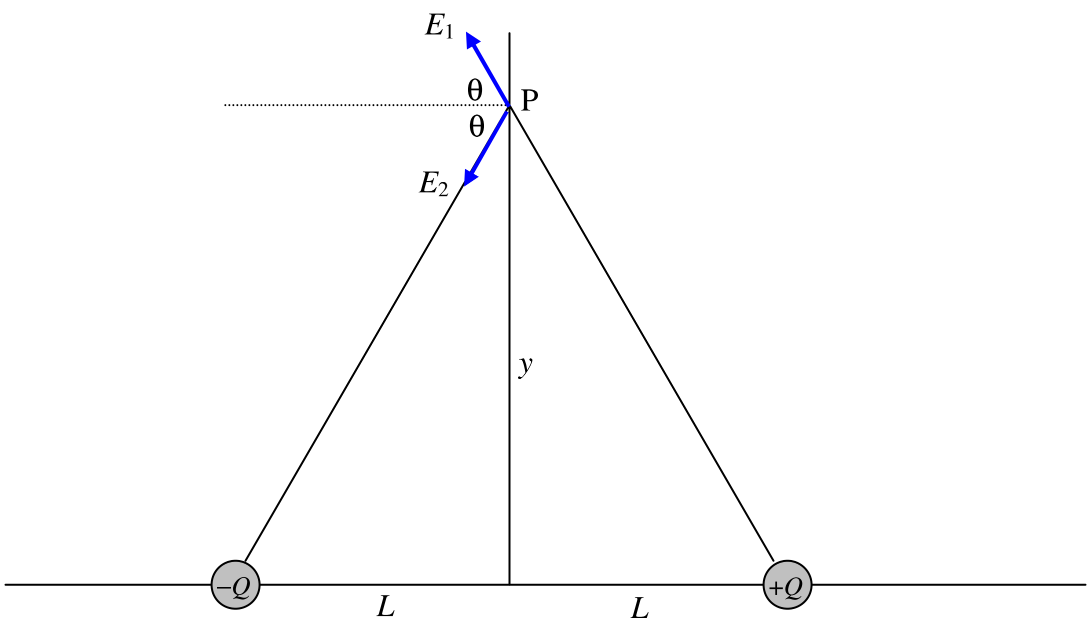


FIGURE III.5

Let us first calculate the electric field at a point P at a distance y along the y-axis. It will be agreed, I think, that it is directed towards the left and is equal to $E_1 \cos\theta + E_2 \cos\theta$,

where $E_1 = E_2 = \dfrac{Q}{4\pi\epsilon_0(L^2 + y^2)}$ and $\cos\theta = \dfrac{L}{(L^2 + y^2)^{1/2}}$.

Therefore $$E = \frac{2QL}{4\pi\epsilon_0(L^2 + y^2)^{3/2}} = \frac{p}{4\pi\epsilon_0(L^2 + y^2)^{3/2}} . \qquad 3.7.1$$

For large y this becomes

$$E = \frac{p}{4\pi\epsilon_0 y^3}. \qquad 3.7.2$$

That is, the field falls off as the cube of the distance.

To find the field on the x-axis, refer to figure III.6.

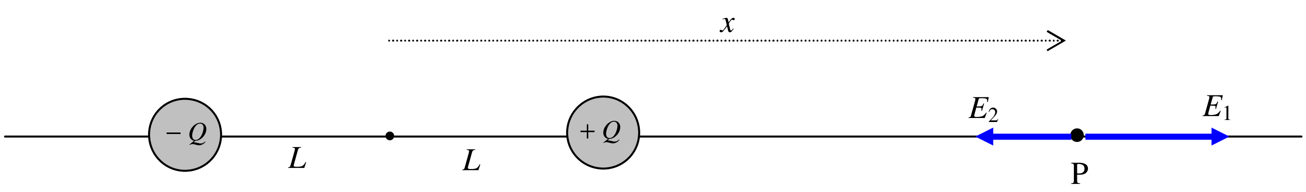


FIGURE III.6

It will be agreed, I think, that the field is directed towards the right and is equal to

$$E = E_1 - E_2 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{(x-L)^2} - \frac{1}{(x+L)^2}\right). \qquad 3.7.3$$

This can be written $\frac{Q}{4\pi\epsilon_0 x^2}\left(\frac{1}{(1-L/x)^2} - \frac{1}{(1+L/x)^2}\right)$, and on expansion of this by the binomial theorem, neglecting terms of order $(L/x)^2$ and smaller, we see that at large x the field is

$$E = \frac{2p}{4\pi\epsilon_0 x^3}. \qquad 3.7.4$$

Now for the field at a point P that is neither on the axis (x-axis) nor the equator (y-axis) of the dipole. See figure III.7.

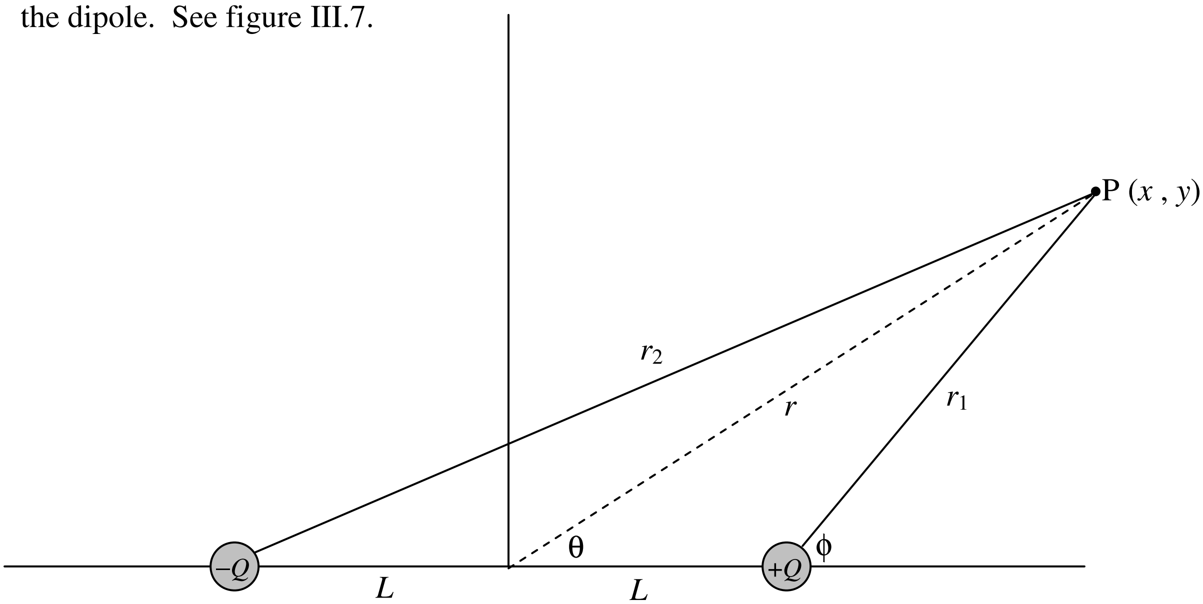


FIGURE III.7

It will probably be agreed that it would not be particularly difficult to write down expressions for the contributions to the field at P from each of the two charges in turn. The difficult part then begins; the two contributions to the field are in different and awkward directions, and adding them vectorially is going to be a bit of a headache.

It is much easier to calculate the *potential* at P, since the two contributions to the potential can be added as scalars. Then we can find the x- and y-components of the field by calculating $\partial V/\partial x$ and $\partial V/\partial y$.

Thus $$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\{(x-L)^2 + y^2\}^{1/2}} - \frac{1}{\{(x+L)^2 + y^2\}^{1/2}}\right). \qquad 3.7.5$$

To start with I am going to investigate the potential and the field at a large distance from the dipole – though I shall return later to the near vicinity of it.

At large distances from a small dipole (see figure III.8), we can write $r^2 = x^2 + y^2$,

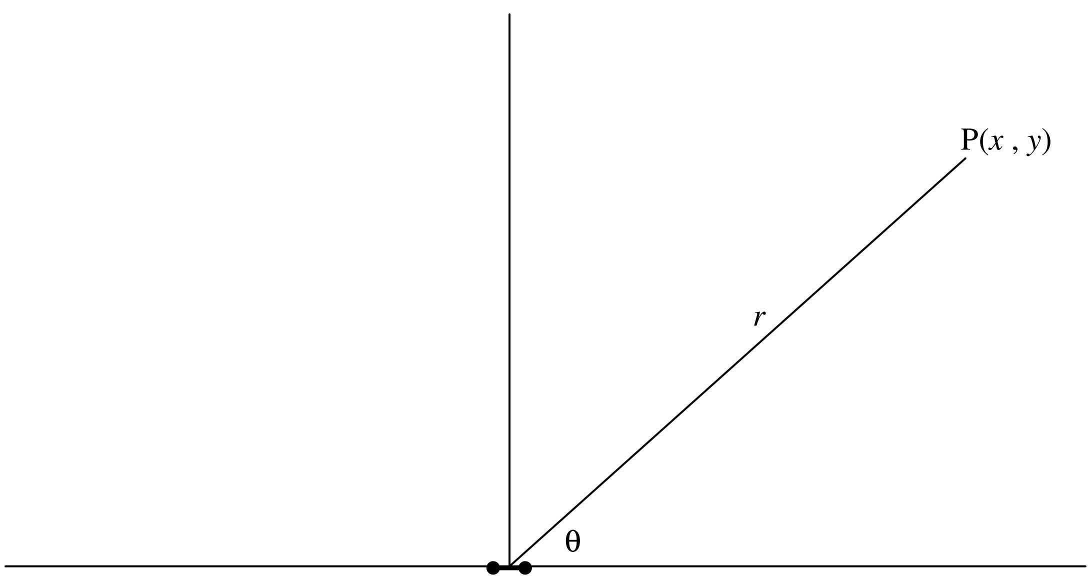


FIGURE III.8

and, with $L^2 << r^2$, the expression 3.7.5 for the potential at P becomes

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{(r^2 - 2Lx)^{1/2}} - \frac{1}{(r^2 + 2Lx)^{1/2}}\right) = \frac{Q}{4\pi\epsilon_0 r}\left((1 - 2Lx/r^2)^{-1/2} - (1 + 2Lx/r^2)^{-1/2}\right).$$

When this is expanded by the binomial theorem we find, to order L/r , that the potential can be written in any of the follwing equivalent ways:

$$V = \frac{2QLx}{4\pi\epsilon_0 r^3} = \frac{px}{4\pi\epsilon_0 r^3} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}. \qquad 3.7.6$$

Thus the equipotentials are of the form

$$r^2 = c\cos\theta, \qquad 3.7.7$$

where $$c = \frac{p}{4\pi\epsilon_0 V}. \qquad 3.7.8$$

Now, bearing in mind that $r^2 = x^2 + y^2$, we can differentiate $V = \dfrac{px}{4\pi\epsilon_0 r^3}$ with respect to x and y to find the x- and y-components of the field.

Thus we find that

$$E_x = \frac{p}{4\pi\epsilon_0}\left(\frac{3x^2 - r^2}{r^5}\right) \quad \text{and} \quad E_y = \frac{pxy}{4\pi\epsilon_0 r^5}. \qquad 3.7.9\text{a,b}$$

We can also use polar coordinates find the radial and transverse components from $E_r = -\dfrac{\partial V}{\partial r}$ and $E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial\theta}$ together with $V = \dfrac{p\cos\theta}{4\pi\epsilon_0 r^2}$ to obtain

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \quad \text{and} \quad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}. \qquad 3.7.10\text{a,b}$$

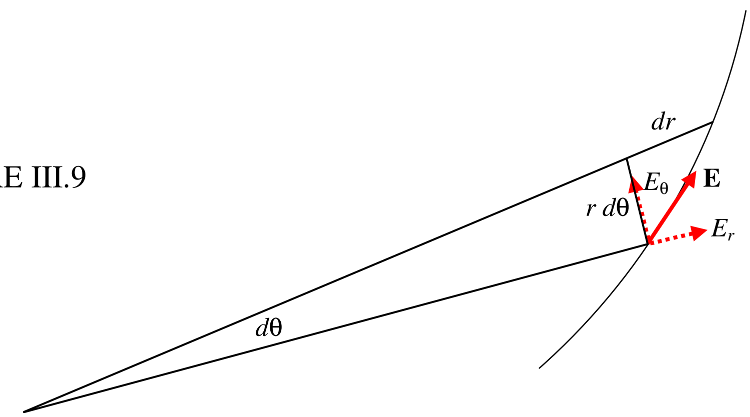
For those who enjoy vector calculus, we can also say $\mathbf{E} = -\dfrac{1}{4\pi\epsilon_0}\nabla\left(\dfrac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$, from which, after a little algebra and quite a lot of vector calculus, we find

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right). \qquad 3.7.11$$

This equation contains all the information that we are likely to want, but I expect most readers will prefer the more explicit rectangular and polar forms of equations 3.7.9 and 3.7.10.

Equation 3.7.7 gives the equation to the equipotentials. The equation to the lines of force can be found as follows. Referring to figure III.9, we see that the differential equation to the lines of force is

FIGURE III.9



$$r\frac{d\theta}{dr} = \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta} = \tfrac{1}{2}\tan\theta, \qquad 3.7.12$$

which, upon integration, becomes

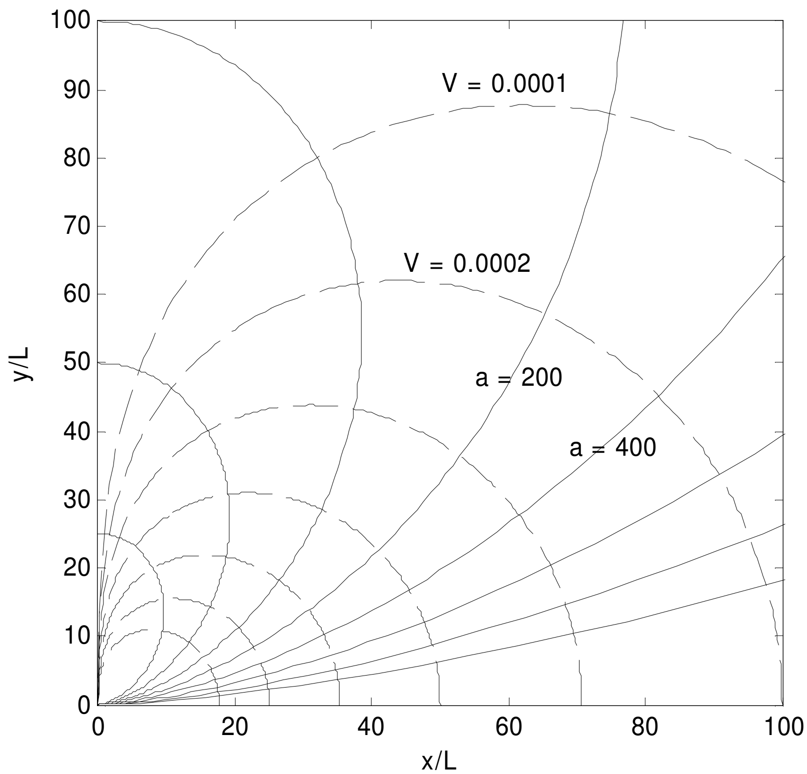
$$r = a\sin^2\theta. \qquad 3.7.13$$

Note that the equations $r^2 = c\cos\theta$ (for the equipotentials) and $r = a\sin^2\theta$ (for the lines of force) are orthogonal trajectories, and either can be derived from the other. Thus, given that the differential equation to the lines of force is $r\frac{d\theta}{dr} = \tfrac{1}{2}\tan\theta$ with solution $r = a\sin^2\theta$, the differential equation to the orthogonal trajectories (i.e. the equipotentials) is $-\frac{1}{r}\frac{dr}{d\theta} = \tfrac{1}{2}\tan\theta$, with solution $r^2 = c\cos\theta$.

In figure III.10, there is supposed to be a tiny dipole situated at the origin. The unit of length is L, half the length of the dipole. I have drawn eight electric field lines (continuous), corresponding to a = 25, 50, 100, 200, 400, 800, 1600, 3200. If r is expressed in units of L, and if V is expressed in units of $\frac{Q}{4\pi\epsilon_0 L}$, the equations 3.7.7 and

3.7.8 for the equipotentials can be written $r = \sqrt{\dfrac{2\cos\theta}{V}}$, and I have drawn seven equipotentials (dashed) for V = 0.0001, 0.0002, 0.0004, 0.0008, 0.0016, 0.0032, 0.0064. It will be noticed from equation 3.7.9a, and is also evident from figure III.10, that E_x is zero for $\theta = 54^\circ\ 44'$.

FIGURE III.10



Equipotentials near to the dipole

These, then, are the field lines and equipotentials at a large distance from the dipole. We arrived at these equations and graphs by expanding equation 3.7.5 binomially, and neglecting terms of higher order than L/r. We now look close to the dipole, where we cannot make such an approximation. Refer to figure III.7.

We can write 3.7.5 as

$$V(x, y) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \qquad 3.7.14$$

where $r_1^2 = (x-L)^2 + y^2$ and $r_2^2 = (x+L)^2 + y^2$. If, as before, we express distances in terms of L and V in units of $\frac{Q}{4\pi\epsilon_0 L}$, the expression for the potential becomes

$$V(x,y) = \frac{1}{r_1} - \frac{1}{r_2}, \qquad 3.7.15$$

where $r_1^2 = (x+1)^2 + y^2$ and $r_2^2 = (x-1)^2 + y^2$.

One way to plot the equipotentials would be to calculate V for a whole grid of (x, y) values and then use a contour plotting routine to draw the equipotentials. My computing skills are not up to this, so I'm going to see if we can find some way of plotting the equipotentials directly.

I present two methods. In the first method I use equation 3.7.15 and endeavour to manipulate it so that I can calculate y as a function of x and V. The second method was shown to me by J. Visvanathan of Chennai, India. We'll do both, and then compare them.

First Method.

To anticipate, we are going to need the following:

$$r_1^2 r_2^2 = (x^2 + y^2 + 1)^2 - 4x^2 = B^2 - A, \qquad 3.7.16$$

$$r_1^2 + r_2^2 = 2(x^2 + y^2 + 1) = 2B, \qquad 3.7.17$$

and

$$r_1^4 + r_2^4 = 2[(x^2 + y^2 + 1)^2 + 4x^2] = 2(B^2 + A), \qquad 3.7.18$$

where

$$A = 4x^2 \qquad 3.7.19$$

and

$$B = x^2 + y^2 + 1. \qquad 3.7.20$$

Now equation 3.7.15 is $r_1 r_2 V = r_2 - r_1$. In order to extract y it is necessary to square this twice, so that r_1 and r_2 appear only as r_1^2 and r_2^2. After some algebra, we obtain

$$r_1^2 r_2^2 [2 - V^4 r_1^2 r_2^2 + 2V^2(r_1^2 + r_2^2)] = r_1^4 + r_2^4. \qquad 3.7.21$$

Upon substitution of equations 3.7.16,17,18, for which we are well prepared, we find for the equation to the equipotentials an equation which, after some algebra, can be written as a quartic equation in B:

$$a_0 + a_1B + a_2B^2 + a_3B^3 + a_4B^4 = 0, \qquad 3.7.22$$

where $$a_0 = A(4 + V^4A), \qquad 3.7.23$$

$$a_1 = 4V^2A, \qquad 3.7.24$$

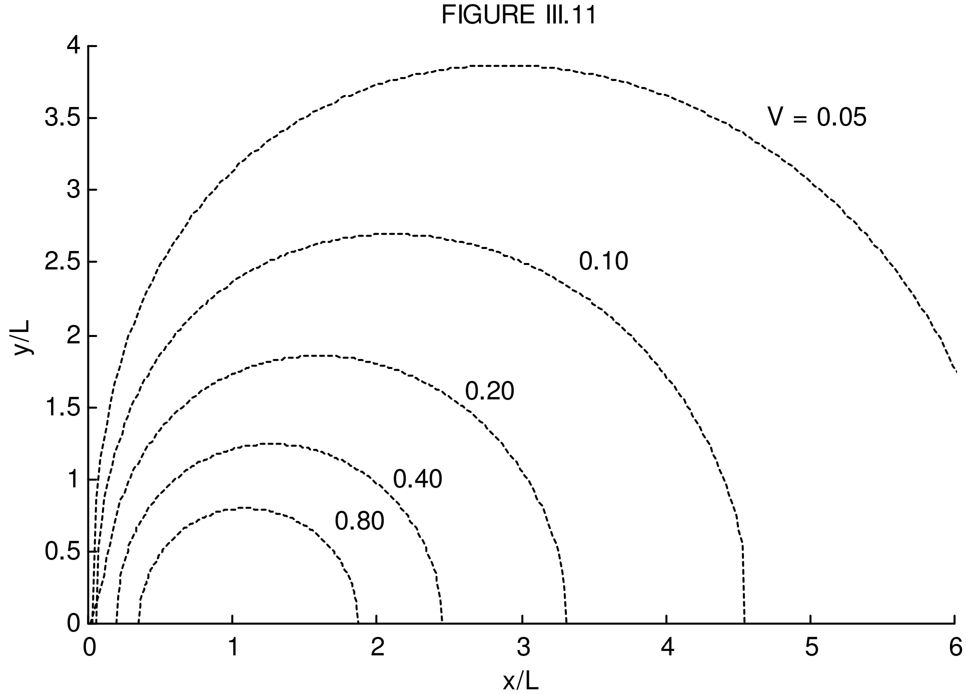
$$a_2 = -2V^2A, \qquad 3.7.25$$

$$a_3 = -4V^2, \qquad 3.7.26$$

and $$a_4 = V^4. \qquad 3.7.27$$

The algorithm will be as follows: For a given V and x, calculate the quartic coefficients from equations 3.7.23-27. Solve the quartic equation 3.7.22 for B. Calculate y from equation 3.7.20. My attempt to do this is shown in figure III.11. The dipole is supposed to have a negative charge at (−1 , 0) and a positive charge at (+1 , 0). The equipotentials are drawn for $V = 0.05$, 0.10, 0.20, 0.40, 0.80.

FIGURE III.11



Second method (J. Visvanathan).

In this method, we work in polar coordinates, but instead of using the coordinates (r, θ), in which the origin, or pole, of the polar coordinate system is at the centre of the dipole (see figure III.7), we use the coordinates (r_1, ϕ) with origin at the positive charge.

From the triangle, we see that

$$r_2^2 = r_1^2 + 4L^2 + 4Lr_1 \cos\phi. \qquad 3.7.28$$

For future reference we note that

$$\frac{\partial r_2}{\partial r_1} = \frac{r_1 + 2L\cos\phi}{r_2}. \qquad 3.7.29$$

Provided that distances are expressed in units of L, these equations become

$$r_2^2 = r_1^2 + 4r_1 \cos\phi + 4, \qquad 3.7.30$$

$$\frac{\partial r_2}{\partial r_1} = \frac{r_1 + 2\cos\phi}{r_2}. \qquad 3.7.31$$

If, in addition, electrical potential is expressed in units of $\frac{Q}{4\pi\epsilon_0 L}$, the potential at P is given, as before (equation 3.17.15), by

$$V(r_1, \phi) = \frac{1}{r_1} - \frac{1}{r_2}. \qquad 3.7.32$$

Recall that r_2 is given by equation 3.7.30, so that equation 3.7.32 is really an equation in just V, r_1 and ϕ.

In order to plot an equipotential, we fix some value of V; then we vary ϕ from 0 to π, and, for each value of ϕ we have to try to calculate r_1.This can be done by the Newton-Raphson process, in which we make a guess at r_1 and use the Newton-Raphson process to obtain a better guess, and continue until successive guesses converge. It is best if we can make a fairly good first guess, but the Newton-Raphson process will often converge very rapidly even for a poor first guess.

Thus we have to solve the following equation for r_1 for given values of V and ϕ,

$$f(r_1) = \frac{1}{r_1} - \frac{1}{r_2} - V = 0, \qquad 3.7.33$$

bearing in mind that r_2 is given by equation 3.7.31.

By differention with respect to r_1, we have

$$f'(r_1) = -\frac{1}{r_1^2} + \frac{1}{r_2^2}\frac{\partial r_2}{\partial r_1} = -\frac{1}{r_1^2} + \frac{r_1 + 2\cos\phi}{r_2^3}, \qquad 3.7.34$$

and we are all set to begin a Newton-Raphson iteration: $r_1 = r_1 - f/f'$. Having obained r_1, we can then obtain the (x, y) coordinates from $x = 1 + r_1\cos\phi$ and $y = r_1\sin\phi$.

I tried this method and I got exactly the same result as by the first method and as shown in figure III.11.

So which method do we prefer? Well, anyone who has worked through in detail the derivations of equations 3.7.16 -3.7.27, and has then tried to program them for a computer, will agree that the first method is very laborious and cumbersome. By comparison Visvanathan's method is much easier both to derive and to program. On the other hand, one small point in favour of the first method is that it involves no trigonometric functions, and so the numerical computation is potentially faster than the second method in which a trigonometric function is calculated at each iteration of the Newton-Raphson process. In truth, though, a modern computer will perform the calculation by either method apparently instantaneously, so that small advantage is hardly relevant.

3.8 *Quadrupole Moment*

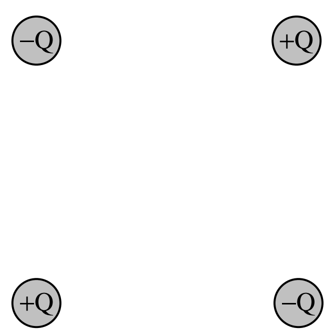


FIGURE III.12

Consider the system of charges shown in figure III.12. It has no net charge and no net dipole moment. Unlike a dipole, it will experience neither a net force nor a net torque in *any uniform* field. It may or may not experience a net force in an external nonuniform field. For example, if we think of the quadrupole as two dipoles, each dipole will experience a force proportional to the local field gradient in which it finds itself. If the field gradients at the location of each dipole are equal, the forces on each dipole will be equal but opposite, and there will net force on the quadrupole. If, however, the field gradients at the positions of the two dipoles are unequal, the forces on the two dipole will be unequal, and there will be a net force on the quadruople. Thus there will be a net force if there is a non-zero gradient of the field gradient. Stated another way, there will be no net force on the quadrupole if the mixed second partial derivatives of the field componets (the third derivatives of the potential!) are zero.   Further, if the quadrupole is in a nonuniform field, increasing, say, to the right, the upper pair will experience a force to the right and the lower pair will experience a force to the left; thus the system will experience a net torque in an inhomogeneous field, though there will be not net force unless the field gradients on the two pairs are unequal.

The system possess what is known as a *quadrupole moment*.   While a single charge is a scalar quantity, and a dipole moment is a vector quantity, the quadrupole moment is a second order symmetric tensor.

The dipole moment of a system of charges is a vector with three components given by $p_x = \sum Q_i x_i$, $p_y = \sum Q_i y_i$, $p_z = \sum Q_i z_i$. The quadrupole moment $\mathbf{q}$ has nine components (of which six are distinct) defined by $q_{xx} = \sum Q_i x_i^2$, $q_{xy} = \sum Q_i x_i y_i$, etc., and its matrix representation is

$$\mathbf{q} = \begin{pmatrix} q_{xx} & q_{xy} & q_{xz} \\ q_{xy} & q_{yy} & q_{yz} \\ q_{xz} & q_{yz} & q_{zz} \end{pmatrix}. \qquad 3.8.1$$

For a continuous charge distribution with charge density ρ coulombs per square metre, the components will be given by $q_{xx} = \int \rho\, x^2 d\tau$, etc., where $d\tau$ is a volume element, given in rectangular coordinates by $dxdydz$ and in spherical coordinates by $r^2 \sin\theta\, dr d\theta d\phi$. The SI unit of quadrupole moment is C m^2, and the dimensions are L^{2}Q,

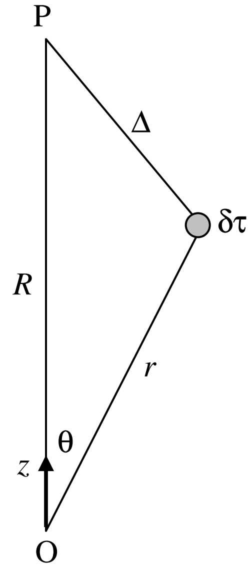
By suitable rotation of axes, in the usual way (see for example section 2.17 of Classical Mechanics), the matrix can be diagonalized, and the diagonal elements are then the eigenvalues of the quadrupole moment, and the trace of the matrix is unaltered by the rotation.

3.9 *Potential at a Large Distance from a Charged Body*

We wish to find the potential at a point P at a large distance R from a charged body, in terms of its total charge and its dipole, quadrupole, and possibly higher-order moments. There will be no loss of generality if we choose a set of axes such that P is on the z-axis.

We refer to figure III.13, and we consider a volume element $\delta\tau$ at a distance r from some origin. The point P is at a distance r from the origin and a distance Δ from $\delta\tau$. The potential at P from the charge in the element $\delta\tau$ is given by

FIGURE III.13



$$4\pi\epsilon_0 \delta V \;=\; \frac{\rho\,\delta\tau}{\Delta} \;=\; \frac{\rho}{R}\left(1 + \frac{r^2}{R^2} - \frac{2r}{R}\cos\theta\right)^{-1/2}\delta\tau\,, \qquad 3.9.1$$

and so the potential from the charge on the whole body is given by

$$4\pi\epsilon_0 V \;=\; \frac{1}{R}\int\rho\left(1 + \frac{r^2}{R^2} - \frac{2r}{R}\cos\theta\right)^{-1/2}\delta\tau. \qquad 3.9.2$$

On expanding the parentheses by the binomial theorem, we find, after a little trouble, that this becomes

$$4\pi\epsilon_0 V \;=\; \frac{1}{R}\int\rho\,d\tau + \frac{1}{R^2}\int\rho\,r\,P_1(\cos\theta)\,d\tau + \frac{1}{2!R^3}\int\rho\,r^2 P_2\,(\cos\theta)\,d\tau$$
$$+ \frac{1}{3!R^4}\int\rho\,r^3 P_3\,(\cos\theta)\,d\tau + \ldots\,, \qquad 3.9.3$$

where the polynomials P are the Legendre polynomials given by

$$P_1(\cos\theta) \;=\; \cos\theta, \qquad 3.9.4$$

$$P_2(\cos\theta) \;=\; \tfrac{1}{2}(3\cos^2\theta - 1), \qquad 3.9.5$$

and $$P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta). \qquad 3.9.6$$

We see from the forms of these integrals and the definitions of the components of the dipole and quadrupole moments that this can now be written:

$$4\pi\epsilon_0 V = \frac{Q}{R} + \frac{p}{R^2} + \frac{1}{2R^3}(3q_{zz} - \operatorname{Tr}\mathbf{q}) + \ldots, \qquad 3.9.7$$

Here Tr $\mathbf{q}$ is the trace of the quadrupole moment matrix, or the (invariant) sum of its diagonal elements. Equation 3.9.7 can also be written

$$4\pi\epsilon_0 V = \frac{Q}{R} + \frac{p}{R^2} + \frac{1}{2R^3}[2q_{zz} - (q_{xx} + q_{yy})] + \ldots . \qquad 3.9.8$$

The quantity $2q_{zz} - (q_{xx} + q_{yy})$ of the diagonalized matrix is often referred to as "the" quadrupole moment. It is zero if all three diagonal components are zero or if $q_{zz} = \tfrac{1}{2}(q_{xx} + q_{yy})$. If the body has cylindrical symmetry about the z-axis, this becomes $2(q_{zz} - q_{xx})$.

Exercise.

Show that the potential at (r, θ) in the vicinity of the linear quadrupole of figure III.14 is

$$V = \frac{QL^2(3\cos^2\theta - 1)}{4\pi\epsilon_0 r^3}.$$

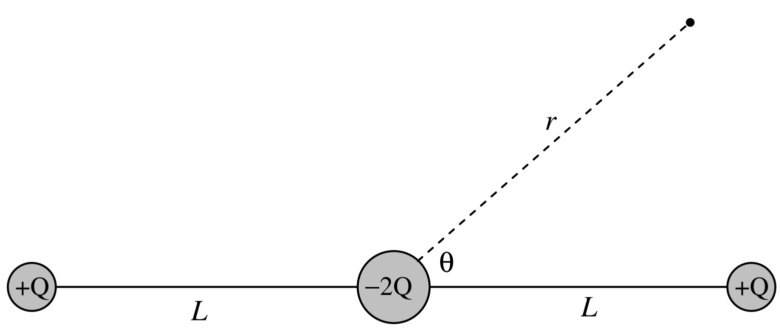


FIGURE III.14