LECTURE NOTES 20

RELATIVISTIC ELECTRODYNAMICS of MOVING MACROSCOPIC MEDIA

We have discussed the relativistic aspects of electrodynamics associated with moving electric charges and electric currents: $\boxed{J^{\mu} = \left(c\rho, \vec{J} \right)}$.

For <u>macroscopic</u> media, restricting ourselves here (for simplicity/clarity's sake) to linear, homogeneous, isotropic materials, such as "class A" dielectrics and magnetic materials that are described in classical electrodynamics by the auxiliary 3-D macroscopic fields:

$$\boxed{\vec{D} \equiv \varepsilon \vec{E}} \qquad \boxed{\vec{H} \equiv \frac{1}{\mu} \vec{B}}$$

n.b. $\varepsilon, \mu, K_e, K_m, \chi_e, \chi_m$ are **<u>not</u>** Lorentz invariant quantities !!!

c, ε_o, μ_o **<u>are</u>** Lorentz invariant !!!

ε = <u>scalar</u> quantity
    = macroscopic electric permittivity
      of the material.
	μ = <u>scalar</u> quantity
	    = macroscopic magnetic permeability
	      of the material.

$\boxed{K_e \equiv \varepsilon/\varepsilon_o}$ = <u>scalar</u> quantity = relative electric permittivity of the material (*a.k.a.* dielectric "constant").	$\boxed{K_m \equiv \mu/\mu_o}$ = <u>scalar</u> quantity = relative magnetic permeability of the material.
$\boxed{\chi_e \equiv K_e - 1}$ = <u>scalar</u> quantity = electric susceptibility of the material. $\boxed{\chi_e = \varepsilon/\varepsilon_o - 1}$ <u>or</u>: $\boxed{\varepsilon = \varepsilon_o \left(1 + \chi_e \right)}$	$\boxed{\chi_m \equiv K_m - 1}$ = <u>scalar</u> quantity = magnetic susceptibility of the material. $\boxed{\chi_m = \mu/\mu_o - 1}$ <u>or</u>: $\boxed{\mu = \mu_o \left(1 + \chi_m \right)}$

<u>Note that</u>: $\boxed{\varepsilon, \mu, K_e, K_m, \chi_e, \chi_m}$ are all defined in the **<u>*rest/proper frame*</u>** of the linear material.

<u>Note also that</u>: $\boxed{c = 1/\sqrt{\varepsilon_o \mu_o}}$, ε_o μ_o and hence: $\boxed{Z_o = \sqrt{\mu_o/\varepsilon_o}}$ are <u>all</u> Lorentz invariant quantities:

$\boxed{c = 1/\sqrt{\varepsilon_o \mu_o}}$ = speed of light/EM waves in free space/vacuum = $\underline{3.0 \times 10^{-8} \ m/s}$.

$\varepsilon_o \equiv$ macroscopic electric permittivity of free space/vacuum = $\underline{8.85 \times 10^{-12} \ F/m}$.

$\mu_o \equiv$ macroscopic magnetic permeability of free space/vacuum = $\underline{4\pi \times 10^{-7} \ H/m}$.

$\boxed{Z_o = \sqrt{\mu_o/\varepsilon_o}}$ = macroscopic impedance of free space/vacuum = $\underline{120\pi \ \Omega \simeq 377 \ \Omega}$.

Thus, for <u>linear</u> macroscopic materials, the following 3-D vector quantities are defined in the **<u>*rest/proper frame*</u>** of the linear material:

$$\boxed{\vec{D}\left(\vec{r},t\right) = \varepsilon_o \left(1 + \chi_e \right) \vec{E}\left(\vec{r},t\right)} \ \underline{\text{and}}: \ \boxed{\vec{B}\left(\vec{r},t\right) = \mu_o \left(1 + \chi_m \right) \vec{H}\left(\vec{r},t\right)}$$

$\boxed{\vec{\mathrm{P}}\left(\vec{r},t\right) \equiv \varepsilon_o \chi_e \vec{E}\left(\vec{r},t\right) = n\left\langle \vec{p}\left(\vec{r},t\right) \right\rangle}$ = electric polarization = electric dipole moment per unit volume.

$\boxed{\vec{\mathrm{M}}\left(\vec{r},t\right) \equiv \chi_m \vec{H}\left(\vec{r},t\right) = n\left\langle \vec{m}\left(\vec{r},t\right) \right\rangle}$ = magnetization = magnetic dipole moment per unit volume.

Thus, we obtain the {usual} constitutive/auxilliary relations for linear media defined in the **_rest/proper frame_** of the linear material:

$$\boxed{\vec{D}(\vec{r},t) = \varepsilon_o \vec{E}(\vec{r},t) + \vec{\mathrm{P}}(\vec{r},t)}\;\; \underline{\text{and:}}\;\; \boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu_o}\vec{B}(\vec{r},t) - \vec{\mathrm{M}}(\vec{r},t)}$$

In order to go over to a relativistic formulation of these relations, we <u>must</u> be careful/precise in their physical meaning. In particular, the electric permittivity, ε and magnetic permeability, μ are defined in the rest frame of the material – i.e. ε is the _proper_ electric permittivity and μ is the _proper_ magnetic permeability

Thus, the macroscopic fields $\vec{D},\vec{\mathrm{P}},\vec{E}$ and $\vec{H},\vec{\mathrm{M}},\vec{B}$ are all defined in the <u>rest/proper</u> frame of the linear material – i.e. they are _proper_ macroscopic electromagnetic fields.

Because we already know/understand the Lorentz transformation properties of $\vec{E}$ and $\vec{B}$, we can write the auxiliary/constitutive relations as:

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_o}\left(\vec{D}(\vec{r},t) - \vec{\mathrm{P}}(\vec{r},t)\right)}\;\; \underline{\text{and:}}\;\; \boxed{\vec{B}(\vec{r},t) = \mu_o\left(\vec{H}(\vec{r},t) + \vec{\mathrm{M}}(\vec{r},t)\right)}$$

The relativistic _EM_ field tensor $F^{\mu\nu}$ is:

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

and its relativistic _dual_ $G^{\mu\nu}$ is:

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

We construct a rank-2 anti-symmetric tensor $D^{\mu\nu}$ (analogous to $F^{\mu\nu}$) for the $\vec{D}$ and $\vec{H}$ fields. It describes the (relativistic six-component) "electromagnetic displacement" field, $D^{\mu\nu}$.

{n.b. again, comparing forms in different textbooks, variations for $D^{\mu\nu}$ will be found that depend on the choice/definition of the metric $g^{\mu\nu}$ used and conventions r.e. overall constants, etc. !!! }.

Taken together, the auxiliary/constitutive relations $\left\{\begin{array}{l}\vec{E} = \dfrac{1}{\varepsilon_o}\left(\vec{D} - \vec{\mathrm{P}}\right) \\ \vec{B} = \mu_o\left(\vec{H} + \vec{\mathrm{M}}\right)\end{array}\right\}$ suggest replacing $\left\{\begin{array}{l}\vec{E} \to \dfrac{1}{\varepsilon_o}\vec{D} \\ \vec{B} \to \mu_o H\end{array}\right\}$

However, it is a little "cleaner" if we also divide through by μ_o, i.e. replace $\left\{\begin{array}{l}\vec{E} \to \dfrac{1}{\mu_o\varepsilon_o}\vec{D} = c^2\vec{D} \\ \vec{B} \to \vec{H}\end{array}\right\}$.

Then $D^{\mu\nu}$ becomes:

$$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix}$$

SI units check:

$$D = \frac{Coul}{m^2}, \quad H = \frac{Amps}{m}$$

$$cD = \frac{Coul}{m^2}\frac{m}{\sec} = \frac{Amp}{m}$$

We also construct the relativistic _dual_ of $D^{\mu\nu}$, which is $H^{\mu\nu}$ (the analog of $G^{\mu\nu}$), defined as:

$$H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}D_{\lambda\sigma}$$

The relativistic dual tensor $H^{\mu\nu}$ may also be obtained directly from $D^{\mu\nu}$ by carrying out an appropriate <u>duality transformation</u> on the $\vec{D}$ and $\vec{H}$-fields, analogous to that for the $\vec{E}$ and $\vec{B}$-fields:

$$\begin{cases} E' = E\cos\varphi_D + cB\sin\varphi_D \\ cB' = cB\cos\varphi_D - E\sin\varphi_D \end{cases} \quad \varphi_D = -90° = -\frac{\pi}{2} \quad \underline{or:} \quad \begin{pmatrix} E' \\ cB' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D + \sin\varphi_D \\ -\sin\varphi_D \cos\varphi_D \end{pmatrix}\begin{pmatrix} E \\ cB \end{pmatrix}$$

$$\begin{cases} cD' = cD\cos\varphi_D + H\sin\varphi_D \\ H' = H\cos\varphi_D - cD\sin\varphi_D \end{cases} \quad \varphi = -90° = -\frac{\pi}{2} \quad \underline{or:} \quad \begin{pmatrix} cD' \\ H' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D + \sin\varphi_D \\ -\sin\varphi_D \cos\varphi_D \end{pmatrix}\begin{pmatrix} cD \\ H \end{pmatrix}$$

Then for $\varphi_D = -90°$, we replace: $\boxed{cD \to +H}$ and: $\boxed{H \to -cD}$ in $\boxed{D^{\mu\nu} \to H^{\mu\nu}}$:

$$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix} \Rightarrow H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}D_{\lambda\sigma}$$

Likewise, we may analogously define/construct a rank-2 anti-symmetric tensor $P^{\mu\nu}$ for the $\vec{P}$ and $\vec{M}$ fields describing the (relativistic six-component) electromagnetic "polarization" field $P^{\mu\nu}$. Since $\vec{P}$ has the same SI units as $\vec{D}\left(\dfrac{\text{Coulombs}}{\text{meter}}\right)$ and $\vec{M}$ has the same SI units as $\vec{H}\left(\dfrac{\text{Amps}}{\text{meter}}\right)$ we see that:

$$P^{\mu\nu} \equiv \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{pmatrix}$$

Note the <u>minus sign reversal</u> here for M_x, M_y, M_z relative to P_x, P_y, P_z due to

$$\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}} \text{ vs. } \boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}} !!!$$

Again, we can construct a relativistic _dual_ of $P^{\mu\nu}$ which we call $M^{\mu\nu}$ {the analog of $G^{\mu\nu}$, for $F^{\mu\nu}$ and/or $H^{\mu\nu}$, for $D^{\mu\nu}$}, defined as:

$$M^{\mu\nu} \equiv \begin{pmatrix} 0 & -M_x & -M_y & -M_z \\ M_x & 0 & -cP_z & cP_y \\ M_y & cP_z & 0 & -cP_x \\ M_z & -cP_y & cP_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}P_{\lambda\sigma}$$

Again, note the <u>minus sign reversal</u> here for M_x, M_y, M_z relative to P_x, P_y, P_z due to

$$\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}} \text{ vs. } \boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}} \text{ !!!}$$

Again, the relativistic dual tensor $M^{\mu\nu}$ can be obtained directly from $P^{\mu\nu}$ by carrying out the same duality transformation as above, but on the $\vec{P}$ and $\vec{M}$ fields. Since $\vec{P}$ has the same SI units as $\vec{D}$, and $\vec{M}$ has the same SI units as $\vec{H}$ we see that:

$$\left.\begin{matrix} -cP' = -cP\cos\varphi_D + M\sin\varphi_D \\ M' = +M\cos\varphi_D + cP\sin\varphi_D \end{matrix}\right\} \varphi = -90° = -\frac{\pi}{2} \quad \underline{\text{or:}} \quad \begin{pmatrix} -cP' \\ M' \end{pmatrix} = \begin{pmatrix} \cos\varphi_D + \sin\varphi_D \\ -\sin\varphi_D \cos\varphi_D \end{pmatrix}\begin{pmatrix} -cP \\ M \end{pmatrix}$$

Then for $\varphi_D = -90°$, we replace: $\boxed{cP \to -M}$ and: $\boxed{M \to +cP}$ in $\boxed{P^{\mu\nu} \to M^{\mu\nu}}$:

$$P^{\mu\nu} \equiv \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{pmatrix} \Rightarrow M^{\mu\nu} \equiv \begin{pmatrix} 0 & -M_x & -M_y & -M_z \\ M_x & 0 & -cP_z & cP_y \\ M_y & cP_z & 0 & -cP_x \\ M_z & -cP_y & cP_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}P_{\lambda\sigma}$$

From the classical electrodynamic auxiliary/constitutive relations: $\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}}$ <u>and</u>: $\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}}$

or: $\boxed{\vec{E} = \frac{1}{\varepsilon_o}(\vec{D} - \vec{P})}$ <u>and</u>: $\boxed{\vec{B} = \mu_o(\vec{H} + \vec{M})}$ we see that their (combined) relativistic equivalents, in terms of the *EM* field tensors and their <u>duals</u> are:

$$\boxed{D^{\mu\nu} = \frac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu}} \quad \underline{\text{and:}} \quad \boxed{H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}} \text{ n.b. } \textbf{\textit{\underline{must}}} \text{ be + here !!!}$$

<u>or</u>: $\boxed{F^{\mu\nu} = \mu_o(D^{\mu\nu} - P^{\mu\nu})}$ <u>and</u>: $\boxed{G^{\mu\nu} = \mu_o(H^{\mu\nu} - M^{\mu\nu})}$

n.b. If we <u>define</u> $\boxed{D^{\mu\nu} \equiv \frac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu}}$ then we <u>**_must_**</u> have the + sign in $\boxed{H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}}$

because the <u>dual</u> *EM* field tensor $\boxed{G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}}$

connects to: $\boxed{G^{\mu\nu} = \mu_o(H^{\mu\nu} - M^{\mu\nu}) = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma} = \frac{1}{2}\mu_o\varepsilon^{\mu\nu\lambda\sigma}(D_{\lambda\sigma} - P_{\lambda\sigma})}$.

Thus the macroscopic relativistic *EM* field tensors and their duals are:

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$$

$$D^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix}$$

$$H^{\mu\nu} \equiv \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}D_{\lambda\sigma}$$

$$P^{\mu\nu} \equiv \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{pmatrix}$$

$$M^{\mu\nu} \equiv \begin{pmatrix} 0 & -M_x & -M_y & -M_z \\ M_x & 0 & -cP_z & cP_y \\ M_y & cP_z & 0 & -cP_x \\ M_z & -cP_y & cP_x & 0 \end{pmatrix} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}P_{\lambda\sigma}$$

Let us explicitly check the correctness/validity of these relations – assuming things are internally correct within *each* of the *EM* field tensors $T^{\mu\nu}$, then we only need to check two non-zero elements:

For the T^{01} components: For the T^{12} components:

$$D^{01} = \frac{1}{\mu_o}F^{01} + P^{01}$$

$$cD_x = \frac{1}{\mu_0}\frac{E_x}{c} + cP_x$$

$$D_x = \frac{1}{\mu_o c^2}E_x + P_x$$

$$D_x = \varepsilon_o E_x + P_x$$

or:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

$$H^{01} = \frac{1}{\mu_o}G^{01} + M^{01}$$

$$H_x = \frac{1}{\mu_o}B_x - M_x$$

or:

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$$

$$D^{12} = \frac{1}{\mu_o}F^{12} + P^{12}$$

$$H_z = \frac{1}{\mu_o}B_z - M_z$$

or:

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$$

$$H^{12} = \frac{1}{\mu_o}G^{12} + M^{12}$$

$$-cD_z = -\frac{1}{\mu_o c}E_z - cP_z$$

$$D_z = \frac{1}{\mu_o c^2}E_z + P_z$$

$$D_z = \varepsilon_o E_z + P_z$$

or:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

Thus, we obtain the relativistic auxiliary/constitutive relations and their *duals*:

$$D^{\mu\nu} = \frac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu} \quad \text{and:} \quad H^{\mu\nu} = \frac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}$$

or: $$F^{\mu\nu} = \mu_o\left(D^{\mu\nu} - P^{\mu\nu}\right) \quad \text{and:} \quad G^{\mu\nu} = \mu_o\left(H^{\mu\nu} - M^{\mu\nu}\right)$$

Lorentz Transformations of the Relativistic Macroscopic
Electrodynamical Fields $\{F^{\mu\nu}, D^{\mu\nu}, P^{\mu\nu}\}$ and Their Duals $\{G^{\mu\nu}, H^{\mu\nu}, M^{\mu\nu}\}$

We already know/have shown that in the the relativistic *EM* fields in frame IRF(*S'*) are related to those in IRF(*S*) via: $\boxed{F'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}F^{\lambda\sigma}}$ and $\boxed{G'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}G^{\lambda\sigma}}$, where $\Lambda^{\mu\nu}$ = the Lorentz transformation tensor.

For <u>linear</u> macroscopic media, we have <u>linear</u> relations between $\{\vec{D}, \vec{E}, \vec{P}\}$ and also between

$\{\vec{H}, \vec{B}, \vec{M}\}$, which hold relativistically between $\boxed{D^{\mu\nu} = \dfrac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu}}$ and also $\boxed{H^{\mu\nu} = \dfrac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}}$.

$\therefore$ We explicitly see using the Lorentz transformations of individual field quantities defined in IRF(*S*) to the frame IRF(*S'*): $\boxed{D'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}D^{\lambda\sigma}}$, $\boxed{F'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}F^{\lambda\sigma}}$ and: $\boxed{P'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}P^{\lambda\sigma}}$

<u>That</u>: $\boxed{D^{\mu\nu} = \dfrac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu}}$ $\Rightarrow$ $\boxed{\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}\left[D^{\lambda\sigma} = \dfrac{1}{\mu_o}F^{\mu\nu} + P^{\mu\nu}\right]}$

$\Rightarrow$ $\boxed{\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}D^{\lambda\sigma} = \dfrac{1}{\mu_o}\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}F^{\lambda\sigma} + \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}P^{\lambda\sigma}}$ $\Rightarrow$ $\boxed{D'^{\mu\nu} = \dfrac{1}{\mu_o}F'^{\mu\nu} + P'^{\mu\nu}}$ holds in IRF(*S'*).

Likewise, for the <u>*dual*</u> tensors we explicitly see using the Lorentz transformations of individual field quantities defined in IRF(*S*) to the frame IRF(*S'*): $\boxed{H'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}H^{\lambda\sigma}}$, $\boxed{G'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}G^{\lambda\sigma}}$ and: $\boxed{M'^{\mu\nu} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}M^{\lambda\sigma}}$

<u>That</u>: $\boxed{H^{\mu\nu} = \dfrac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}}$ $\Rightarrow$ $\boxed{\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}\left[H^{\mu\nu} = \dfrac{1}{\mu_o}G^{\mu\nu} + M^{\mu\nu}\right]}$

$\Rightarrow$ $\boxed{\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}H^{\lambda\sigma} = \dfrac{1}{\mu_o}\Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}G^{\lambda\sigma} + \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}M^{\lambda\sigma}}$ $\Rightarrow$ $\boxed{H'^{\mu\nu} = \dfrac{1}{\mu_o}G'^{\mu\nu} + M'^{\mu\nu}}$ holds in IRF(*S'*).

Equivalently, we can <u>*alternatively*</u> express these as a <u>simple</u> Lorentz transformation of the macroscopic $\{\vec{E}, \vec{D}, \vec{P}\}$ and $\{\vec{B}, \vec{H}, \vec{M}\}$ fields, e.g. for a Lorentz transformation from the <u>rest/proper</u> frame IRF(*S*) where the macroscopic $\{\vec{E}, \vec{D}, \vec{P}\}$ and $\{\vec{B}, \vec{H}, \vec{M}\}$ fields are defined, to another IRF(*S'*) which is moving with velocity $\vec{v} \parallel$ to one of the $\hat{x}, \hat{y},$ or $\hat{z}$ axes (e.g. $\vec{v} = +v\hat{x}$).

We have already shown {in P436 Lecture Notes 19 p. 1-3} that the components of $\vec{E}$ and $\vec{B}$ that are $\parallel$ and $\perp$ to the boost direction $(\vec{v})$ transform as:

IRF(S'):   IRF(S):   IRF(S'):   IRF(S):

$$\boxed{E'^{\parallel} = E^{\parallel}}$$ $$\boxed{B'^{\parallel} = B^{\parallel}}$$

$$\boxed{\begin{aligned} E'^{\perp} &= \gamma\left(E^{\perp} + \left(\vec{v}\times\vec{B}\right)^{\perp}\right) \\ &= \gamma\left(E^{\perp} + \left(\vec{\beta}\times c\vec{B}\right)^{\perp}\right) \end{aligned}}$$ $$\boxed{\begin{aligned} B'^{\perp} &= \gamma\left(B^{\perp} - \frac{1}{c^2}\left(\vec{v}\times\vec{E}\right)^{\perp}\right) \\ &= \gamma\left(B^{\perp} - \left(\vec{\beta}\times\frac{\vec{E}}{c}\right)^{\perp}\right) \end{aligned}}$$ $\underline{\text{where:}}$ $\boxed{\gamma = \dfrac{1}{\sqrt{1-\beta^2}}}$ $\underline{\text{and:}}$ $\boxed{\begin{aligned}\beta &= \frac{v}{c} \\ \vec{\beta} &= \frac{\vec{v}}{c}\end{aligned}}$

$\underline{\text{Thus:}}$

IRF(S'):   IRF(S):   IRF(S'):   IRF(S):

$$\boxed{D'^{\parallel} = D^{\parallel}}$$ $$\boxed{H'^{\parallel} = H^{\parallel}}$$

$$\boxed{\begin{aligned} D'^{\perp} &= \gamma\left(D^{\perp} + \frac{1}{c^2}\left(\vec{v}\times\vec{H}\right)^{\perp}\right) \\ &= \gamma\left(D^{\perp} + \frac{1}{2}\left(\vec{\beta}\times\vec{H}\right)^{\perp}\right) \end{aligned}}$$ $$\boxed{\begin{aligned} H'^{\perp} &= \gamma\left(H^{\perp} - \left(\vec{v}\times\vec{D}\right)^{\perp}\right) \\ &= \gamma\left(H^{\perp} - c\left(\vec{\beta}\times\vec{D}\right)^{\perp}\right) \end{aligned}}$$

$\underline{\text{And similarly:}}$

IRF(S'):   IRF(S):   IRF(S'):   IRF(S):

$$\boxed{P'^{\parallel} = P^{\parallel}}$$ $$\boxed{M'^{\parallel} = M^{\parallel}}$$

$$\boxed{\begin{aligned} P'^{\perp} &= \gamma\left(P^{\perp} - \frac{1}{c^2}\left(\vec{v}\times\vec{M}\right)^{\perp}\right) \\ &= \gamma\left(P^{\perp} - \frac{1}{c}\left(\vec{\beta}\times\vec{M}\right)^{\perp}\right) \end{aligned}}$$ $$\boxed{\begin{aligned} M'^{\perp} &= \gamma\left(M^{\perp} + \left(\vec{v}\times\vec{P}\right)^{\perp}\right) \\ &= \gamma\left(M^{\perp} + c\left(\vec{\beta}\times\vec{P}\right)^{\perp}\right) \end{aligned}}$$ $\boxed{\begin{array}{c}\text{n.b. note the sign reversals here} \\ \text{relative to the above relations !!!}\end{array}}$

$\underline{\text{And we also have:}}$

$\boxed{\begin{array}{l}\text{n.b. We see from these two relations that in the } \underline{\text{rest/proper}} \text{ frame} \\ \text{IRF(S_0) of a polarized dielectric with electric polarization } \vec{P}_0 \\ \text{(or a magnetized material with magnetization } \vec{M}_0 \text{) if the linear} \\ \text{material is moving e.g. with velocity } \vec{v} = +v\hat{x} \text{ in the lab frame} \\ \text{IRF(S), an observer in the lab frame will "see" a } \underline{\text{combination}} \text{ of} \\ \text{electric polarization } \vec{P} \text{ and magnetization } \vec{M} \text{ in } \underline{\text{both}} \text{ cases!}\end{array}}$

In IRF(S'):   In IRF(S):

$$\boxed{\vec{D}' = \varepsilon_o\vec{E}' + \vec{P}'}$$ $$\boxed{\vec{D} = \varepsilon_o\vec{E} + \vec{P}}$$

$$\boxed{\vec{H}' = \frac{1}{\mu_o}\vec{B}' - \vec{M}'}$$ $$\boxed{\vec{H} = \frac{1}{\mu_o}\vec{B} - M}$$

n.b. $\boxed{\varepsilon_o, \mu_o, c = \dfrac{1}{\sqrt{\varepsilon_o\mu_o}}}$ are all Lorentz invariant quantities.

Relativistic Invariants Associated with Macroscopic Relativistic *EM* Fields

We have already shown/discussed that for the $\vec{E}$ and $\vec{B}$-field tensors $F^{\mu\nu}$ and $G^{\mu\nu}$, there are two and *only* two relativistic Lorentz invariant quantities:

1) $$F^{\mu\nu}F_{\mu\nu} = F_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}G_{\mu\nu} = -G_{\mu\nu}G^{\mu\nu} = 2\left(B^2 - \left(\frac{E}{c}\right)^2\right)$$

2) $$F^{\mu\nu}G_{\mu\nu} = F_{\mu\nu}G^{\mu\nu} = G^{\mu\nu}F_{\mu\nu} = G_{\mu\nu}F^{\mu\nu} = -\frac{4}{c}\left(\vec{E}\bullet\vec{B}\right)$$

Since the (inner) product of a <u>contravariant</u> tensor of rank-n with <u>any other covariant</u> tensor of the same rank-n is a Lorentz invariant quantity (i.e. same/valid in *any* inertial reference frame, we see that for $\{F^{\mu\nu}, D^{\mu\nu}, \mathrm{P}^{\mu\nu}\}$ and their duals $\{G^{\mu\nu}, H^{\mu\nu}, \mathrm{M}^{\mu\nu}\}$ we can literally have a "field day" / go *wild {!!!}* and form <u>many</u> additional Lorentz invariant quantities, such as:

3) $$D^{\mu\nu}D_{\mu\nu} = D_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}H_{\mu\nu} = -H_{\mu\nu}H^{\mu\nu} = 2\left(H^2 - (cD)^2\right)$$

4) $$D^{\mu\nu}H_{\mu\nu} = D_{\mu\nu}H^{\mu\nu} = H^{\mu\nu}D_{\mu\nu} = H_{\mu\nu}D^{\mu\nu} = -4c\left(\vec{D}\bullet\vec{H}\right)$$

5) $$\mathrm{P}^{\mu\nu}\mathrm{P}_{\mu\nu} = \mathrm{P}_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{M}^{\mu\nu}\mathrm{M}_{\mu\nu} = -\mathrm{M}_{\mu\nu}\mathrm{M}^{\mu\nu} = 2\left(\mathrm{M}^2 - (c\mathrm{P})^2\right)$$

6) $$\mathrm{P}^{\mu\nu}\mathrm{M}_{\mu\nu} = \mathrm{P}_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}\mathrm{P}_{\mu\nu} = \mathrm{M}_{\mu\nu}\mathrm{P}^{\mu\nu} = +4c\left(\vec{\mathrm{P}}\bullet\vec{\mathrm{M}}\right)$$ ⟵ See these lecture notes, p.10 for details.

We can also form the "crossed" Lorentz invariants:

7) $$F^{\mu\nu}D_{\mu\nu} = F_{\mu\nu}D^{\mu\nu} = D^{\mu\nu}F_{\mu\nu} = D_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}H_{\mu\nu} = -G_{\mu\nu}H^{\mu\nu} = -H^{\mu\nu}G_{\mu\nu} = -H_{\mu\nu}G^{\mu\nu}$$

8) $$F^{\mu\nu}\mathrm{P}_{\mu\nu} = F_{\mu\nu}\mathrm{P}^{\mu\nu} = \mathrm{P}^{\mu\nu}F_{\mu\nu} = \mathrm{P}_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}\mathrm{M}_{\mu\nu} = -G_{\mu\nu}\mathrm{M}^{\mu\nu} = -\mathrm{M}^{\mu\nu}G_{\mu\nu} = -\mathrm{M}_{\mu\nu}G^{\mu\nu}$$

9) $$D^{\mu\nu}\mathrm{P}_{\mu\nu} = D_{\mu\nu}\mathrm{P}^{\mu\nu} = \mathrm{P}^{\mu\nu}D_{\mu\nu} = \mathrm{P}_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}\mathrm{M}_{\mu\nu} = -H_{\mu\nu}\mathrm{M}^{\mu\nu} = -\mathrm{M}^{\mu\nu}H_{\mu\nu} = -\mathrm{M}_{\mu\nu}H^{\mu\nu}$$

<u>As well as</u>:

10) $$F^{\mu\nu}\mathrm{M}_{\mu\nu} = F_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}F_{\mu\nu} = \mathrm{M}_{\mu\nu}F^{\mu\nu} = -G^{\mu\nu}\mathrm{P}_{\mu\nu} = -G_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{P}^{\mu\nu}G_{\mu\nu} = -\mathrm{P}_{\mu\nu}G^{\mu\nu}$$

11) $$D^{\mu\nu}G_{\mu\nu} = D_{\mu\nu}G^{\mu\nu} = G^{\mu\nu}D_{\mu\nu} = G_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}F_{\mu\nu} = -H_{\mu\nu}F^{\mu\nu} = -F^{\mu\nu}H_{\mu\nu} = -F_{\mu\nu}H^{\mu\nu}$$

12) $$D^{\mu\nu}\mathrm{M}_{\mu\nu} = D_{\mu\nu}\mathrm{M}^{\mu\nu} = \mathrm{M}^{\mu\nu}D_{\mu\nu} = \mathrm{M}_{\mu\nu}D^{\mu\nu} = -H^{\mu\nu}\mathrm{P}_{\mu\nu} = -H_{\mu\nu}\mathrm{P}^{\mu\nu} = -\mathrm{P}^{\mu\nu}H_{\mu\nu} = -\mathrm{P}_{\mu\nu}H^{\mu\nu}$$

Thus, we can form a total of 12 unique <u>*bi-linear*</u> Lorentz invariant quantities using the relativistic macroscopic *EM* field tensors $\{F^{\mu\nu}, D^{\mu\nu}, \mathrm{P}^{\mu\nu}\}$ and their <u>*duals*</u> $\{G^{\mu\nu}, H^{\mu\nu}, \mathrm{M}^{\mu\nu}\}$.

We explicitly work out the first <u>four</u> of these new Lorentz invariant quantities:

3)
$$
\begin{aligned}
D^{\mu\nu}D_{\mu\nu} = &+D^{00}D^{00} - D^{01}D^{01} - D^{02}D^{02} - D^{03}D^{03} \\
&- D^{10}D^{10} + D^{11}D^{11} + D^{12}D^{12} + D^{13}D^{13} \\
&- D^{20}D^{20} + D^{21}D^{21} + D^{22}D^{22} + D^{23}D^{23} \\
&- D^{30}D^{30} + D^{31}D^{31} + D^{32}D^{32} + D^{33}D^{33}
\end{aligned}
$$

$$
\begin{array}{cccc}
= & 0 & -c^2 D_x^2 & -c^2 D_y^2 & -c^2 D_z^2 \\
& -c^2 D_x^2 & +0 & +H_z^2 & +H_y^2 \\
& -c^2 D_y^2 & +H_z^2 & 0 & +H_x^2 \\
& -c^2 D_z^2 & +H_y^2 & +H_x^2 & +0
\end{array}
$$

$$
= 2\left(H^2 - c^2 D^2\right)
$$

4)
$$
\begin{aligned}
D^{\mu\nu}H_{\mu\nu} = &+D^{00}H^{00} - D^{01}H^{01} - D^{02}H^{02} - D^{03}H^{03} \\
&- D^{10}H^{10} + D^{11}H^{11} + D^{12}H^{12} + D^{13}H^{13} \\
&- D^{20}H^{20} + D^{21}H^{21} + D^{22}H^{22} + D^{23}H^{23} \\
&- D^{30}H^{30} + D^{31}H^{31} + D^{32}H^{32} + D^{33}H^{33}
\end{aligned}
$$

$$
\begin{array}{cccc}
= & 0 & -cD_x H_x & -c^2 D_y H_x & -cD_z H_z \\
& -c^2 D_x^2 & +0 & +H_z^2 & +H_y^2 \\
& -c^2 D_y^2 & +H_z^2 & 0 & +H_x^2 \\
& -c^2 D_z^2 & +H_y^2 & +H_x^2 & +0
\end{array}
$$

$$
= -4c\left(\vec{D}\cdot\vec{H}\right)
$$

5)
$$
\begin{aligned}
P^{\mu\nu}P_{\mu\nu} = &+P^{00}P^{00} - P^{01}P^{01} - P^{02}P^{02} - P^{03}P^{03} \\
&- P^{10}P^{10} + P^{11}P^{11} + P^{12}P^{12} + P^{13}P^{13} \\
&- P^{20}P^{20} + P^{21}P^{21} + P^{22}P^{22} + P^{23}P^{23} \\
&- P^{30}P^{30} + P^{31}P^{31} + P^{32}P^{32} + P^{33}P^{33}
\end{aligned}
$$

$$
\begin{array}{cccc}
= & 0 & -c^2 P_x^2 & -c^2 P_y^2 & -c^2 P_z^2 \\
& -c^2 P_x^2 & +0 & +M_z^2 & +M_y^2 \\
& -c^2 P_y^2 & +M_z^2 & +0 & +M_x^2 \\
& -c^2 P_z^2 & +M_y^2 & +M_x^2 & +0
\end{array}
$$

$$
= 2\left(M^2 - c^2 P^2\right)
$$

6)
$$
\begin{aligned}
P^{\mu\nu}M_{\mu\nu} &= +P^{00}M^{00} - P^{01}M^{01} - P^{02}M^{02} - P^{03}M^{03} \\
&\quad - P^{10}M^{10} + P^{11}M^{11} + P^{12}M^{12} + P^{13}M^{13} \\
&\quad - P^{20}M^{20} + P^{21}M^{21} + P^{22}M^{22} + P^{23}M^{23} \\
&\quad - P^{30}M^{30} + P^{31}M^{31} + P^{32}M^{32} + P^{33}M^{33} \\
&= \quad 0 \quad\;\; + P_xM_x \;\; + c^2P_yM_y + cP_zM_z \\
&\quad + cP_xM_x \;\; + 0 \quad\;\; + cP_zM_z \;\; + cP_yM_y \\
&\quad + cP_yM_y + cP_zM_z \;\; + 0 \quad\;\; + cP_xM_x \\
&\quad + cP_zM_z + cP_yM_y \;\; + cP_xM_x \quad + 0 \\
&= +4c\left(\vec{P}\cdot\vec{M}\right)
\end{aligned}
$$

We can also form "higher-order" tri-linear and quadri-linear invariants from combinations of $\{F^{\mu\nu}, D^{\mu\nu}, P^{\mu\nu}\}$ and $\{G^{\mu\nu}, H^{\mu\nu}, M^{\mu\nu}\}$, e.g.:

$\boxed{F^{\mu\nu}D_{\mu\lambda}P_{\nu}^{\ \lambda}}$ (and all allowed permutations): $\boxed{F^{\mu\nu}D_{\mu\lambda}P_{\nu\sigma}G^{\lambda\sigma}}$ (and all allowed permutations):

Shorthand: Shorthand:

   FFF GGG FFG - - - - - - → *etc.*   *FFFF*   *etc.*- - - →

   FFD GGH FFH   *FFFD*

   *FFP GGM FF*M   *FFF*P

   FDD GHH   *FFFG*

   *FDD GH*M   *FFFH*

   FPP GMM   *FFF*M

   DDD HHH

   PPP MMM *etc.*   *etc.*

e.g. *F D* P *G* H and all allowed permutations
e.g. *F D* P *G* H M and all allowed permutations

Note: Obviously, not all of these combinations of fields will be truly unique!

e.g.

$$
\boxed{F^{\mu\nu}F_{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma} = \left(F^{\mu\nu}F_{\mu\nu}\right)\left(F^{\lambda\sigma}F_{\lambda\sigma}\right) = \left[2\left(B^2 - \left(\frac{E}{c}\right)^2\right)\right]^2}
$$

and:

$$
\boxed{\begin{aligned}
F^{\mu\nu}F_{\mu\nu}P^{\lambda\sigma}P_{\lambda\sigma} &= \left(F^{\mu\nu}F_{\mu\nu}\right)\left(P^{\lambda\sigma}P_{\lambda\sigma}\right) = 2\left(B^2 - \left(\frac{E}{c}\right)^2\right)\cdot 2\left(M^2 - c^2P^2\right) \\
&= 4\left(B^2 - \left(\frac{E}{c}\right)^2\right)\left(M^2 - c^2P^2\right)
\end{aligned}}
$$

Maxwell's Equations in Tensor Notation for Macroscopic Linear Media

The classical electrodynamics version of Maxwell's equations for macroscopic linear media are:

1.) Gauss' Law: $\boxed{\vec{\nabla} \bullet \vec{E} = \dfrac{1}{\varepsilon_o} \rho_{ToT} = \dfrac{1}{\varepsilon_o}\left(\rho_{free} + \rho_{bound}\right)}$ where: $\boxed{\rho_{ToT} = \rho_{free} + \rho_{bound}}$

The auxiliary/constitutive relation $\boxed{\vec{D} = \varepsilon_o \vec{E} + \vec{\mathrm{P}}}$ gives: $\boxed{\vec{\nabla} \bullet \vec{D} = \varepsilon_o \vec{\nabla} \bullet \vec{E} + \vec{\nabla} \bullet \vec{P}}$

$\boxed{\vec{\nabla} \bullet \vec{E} = \dfrac{1}{\varepsilon_o} \rho_{ToT}} \Rightarrow \boxed{\vec{\nabla} \bullet \vec{D} = \rho_{free}}$ and $\boxed{\vec{\nabla} \bullet \vec{P} = -\rho_{bound}}$

$\boxed{\vec{D} = \varepsilon \vec{E}}, \boxed{\vec{\mathrm{P}} = \varepsilon_o \chi_e \vec{E}} \Rightarrow \boxed{\varepsilon = \varepsilon_o\left(1 + \chi_e\right)}$

2.) No magnetic monopoles: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$

3.) Faraday's Law: $\boxed{\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}}$

4.) Ampere's Law: $\boxed{\vec{\nabla} \times \vec{B} - \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} = \mu_o \vec{J}_{ToT}}$ where: $\boxed{\vec{J}_{ToT} = \vec{J}_{free} + \vec{J}^M_{bound} + \vec{J}^P_{bound}}$

The auxiliary/constitutive relation $\boxed{\vec{H} = \dfrac{1}{\mu_o}\vec{B} - \vec{\mathrm{M}}}$ gives: $\boxed{\vec{\nabla} \times \vec{H} = \dfrac{1}{\mu_o}\vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{\mathrm{M}}}$

Maxwell's displacement current term $\boxed{\vec{J}_D = \varepsilon_o \dfrac{\partial \vec{E}}{\partial t}}$, polarization current: $\boxed{\vec{J}^P_{bound} = \dfrac{\partial \vec{\mathrm{P}}}{\partial t}}$

$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \dfrac{\partial \vec{D}}{\partial t}}$ and $\boxed{\vec{\nabla} \times \vec{\mathrm{M}} = J^M_{bound}}$ where: $\boxed{\dfrac{\partial \vec{D}}{\partial t} = \varepsilon_o \dfrac{\partial \vec{E}}{\partial t} + \dfrac{\partial \vec{P}}{\partial t} = \vec{J}_D + \vec{J}^P_{bound}}$

$\boxed{\vec{H} = \dfrac{1}{\mu}\vec{B}}, \boxed{\vec{\mathrm{M}} = \chi_m \vec{H}} \Rightarrow \boxed{\mu = \mu_o\left(1 + \chi_m\right)}$

Relativistically, we have seen that Maxwell's four equations for free space / the vacuum {i.e. no matter present} are contained within the two tensor relations:

$\boxed{\dfrac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o J^\mu_{ToT}}$ and: $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$

Given that: $\boxed{D^{\mu\nu} = \dfrac{1}{\mu_o}F^{\mu\nu} + \mathrm{P}^{\mu\nu}}$ and: $\boxed{H^{\mu\nu} = \dfrac{1}{\mu_o}G^{\mu\nu} + \mathrm{M}^{\mu\nu}}$

We see that the <u>inhomogenous</u> relativistic Maxwell relations are:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_o J^{\mu}_{ToT}$$

$$\frac{\partial D^{\mu\nu}}{\partial x^{\nu}} = J^{\mu}_{free} \quad = \left(c\rho_{free}, \vec{J}_{free}\right) \quad \begin{cases} \vec{\nabla}\bullet\vec{D} = \rho_{free} \\ \vec{\nabla}\bullet\vec{H} - \dfrac{\partial\vec{D}}{\partial t} = \vec{J}_{free} \end{cases}$$

$$\frac{\partial P^{\mu\nu}}{\partial x^{\nu}} = J^{\mu}_{bound} = \left(-c\rho_{bound}, \vec{J}_{bound}\right) \quad \begin{cases} \vec{\nabla}\bullet\vec{P} = -\rho_{bound} \\ \vec{J}_{bound} = J^{P}_{bound} + \vec{J}^{M}_{bound} = \dfrac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M} \end{cases}$$

Whereas the <u>homogenous</u> relativistic Maxwell relations are:

$$\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0 \begin{cases} \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \end{cases} \Rightarrow \frac{\partial H^{\mu\nu}}{\partial x^{\nu}} = -\frac{\partial M^{\mu\nu}}{\partial x^{\nu}} \begin{cases} \vec{\nabla}\bullet\vec{H} = -\vec{\nabla}\bullet\vec{M} \\ \vec{\nabla}\times\vec{D} = -\dfrac{1}{c^2}\dfrac{\partial\vec{H}}{\partial t} \\ \vec{\nabla}\times\vec{P} = +\dfrac{1}{c^2}\dfrac{\partial\vec{M}}{\partial t} \end{cases}$$

The relativistic constitutive relations for linear, homogeneous, uniform, isotropic media {i.e. the relativistic analogs of $\boxed{\vec{D} = \varepsilon\vec{E}}$ and $\boxed{\vec{B} = \mu\vec{H}}$} were proposed by H. Minkowski as:

$$\boxed{D^{\mu\nu}\eta_{\nu} = c^2\varepsilon F^{\mu\nu}\eta_{\nu}} \text{ and: } \boxed{H^{\mu\nu}\eta_{\nu} = \frac{1}{\mu}G^{\mu\nu}\eta_{\nu}}$$

<u>n.b.</u>
The relativistic *EM* field tensors are <u>contracted</u> with the <u>proper</u> 4-velocity, η^{μ}.

The relativistic *EM* field tensors are defined in the rest/proper frame IRF(*S*) of the linear material.

ε is the <u>proper</u> electric permittivity and μ is the <u>proper</u> magnetic permeability.
Both quantities are defined in the <u>rest/proper</u> frame IRF(*S*) of the linear material.

η^{μ} is the (contravariant) <u>proper</u> 4-velocity of the material.

We can easily show that these formulae give the correct constitutive relations when the linear material is at rest, i.e. when $\boxed{\eta_v = (-c, 0, 0, 0)}$. Here the sum over v collapses to a single term:

$$\boxed{\begin{array}{c}\text{In the} \\ \text{rest/proper} \\ \text{frame of the} \\ \text{linear material:}\end{array}} \left\{\begin{array}{l}\boxed{D^{\mu v}\eta_v = c^2\varepsilon F^{\mu v}\eta_v} \Rightarrow \boxed{D^{\mu 0}\eta_0 = c^2\varepsilon F^{\mu 0}\eta_0} \Rightarrow \boxed{-c\vec{D} = -c^2\varepsilon\left(\vec{E}/c\right)} \Rightarrow \boxed{\vec{D} = \varepsilon\vec{E}} \\[3mm] \boxed{H^{\mu v}\eta_v = \dfrac{1}{\mu}G^{\mu v}\eta_v} \Rightarrow \boxed{H^{\mu 0}\eta_0 = \dfrac{1}{\mu}G^{\mu 0}\eta_0} \Rightarrow \boxed{-\vec{H} = -\dfrac{1}{\mu}\vec{B}} \Rightarrow \boxed{\vec{H} = \dfrac{1}{\mu}\vec{B}}\end{array}\right.$$

If the linear medium is moving e.g. in the lab frame IRF(*S*) with <u>ordinary velocity</u> $\vec{u}$ then: $\boxed{\eta_v = (-\gamma c, \gamma\vec{u}) = \gamma(-c, \vec{u})}$. Then for $\vec{u} \neq 0$: $\boxed{D^{\mu v}\eta_v = c^2\varepsilon F^{\mu v}\eta_v}$ and: $\boxed{H^{\mu v}\eta_v = \dfrac{1}{\mu}G^{\mu v}\eta_v}$

<u>For $\mu = 0$:</u>

$$\boxed{D^{0v}\eta_v = D^{00}\eta_0 + D^{01}\eta_1 + D^{02}\eta_2 + D^{03}\eta_3 = 0 + cD_x(\gamma u_x) + cD_y(\gamma u_y) + cD_z(\gamma u_z) = \gamma c(\vec{D}\bullet\vec{u})}$$

$$\boxed{F^{0v}\eta_v = F^{00}\eta_0 + F^{01}\eta_1 + F^{02}\eta_2 + F^{03}\eta_3 = 0 + \dfrac{E_x}{c}(\gamma u_x) + \dfrac{E_y}{c}(\gamma u_y) + \dfrac{E_z}{c}(\gamma u_z) = \gamma\dfrac{1}{c}(\vec{E}\bullet\vec{u})}$$

$\therefore \boxed{D^{0v}\eta_v = c^2\varepsilon F^{0v}\eta_v} \Rightarrow \boxed{\gamma c(\vec{D}\bullet\vec{u}) = c^2\varepsilon\left(\dfrac{\gamma}{c}\right)(\vec{E}\bullet\vec{u})}$ <u>or:</u> $\boxed{\vec{D}\bullet\vec{u} = \varepsilon(\vec{E}\bullet\vec{u})}$

<u>And:</u>

$$\boxed{H^{0v}\eta_v = H^{00}\eta_0 + H^{01}\eta_1 + H^{02}\eta_2 + H^{03}\eta_3 = H_x(\gamma u_x) + H_y(\gamma u_y) + H_z(\gamma u_z) = \gamma(\vec{H}\bullet\vec{u})}$$

$$\boxed{G^{0v}\eta_v = G^{00}\eta_0 + G^{01}\eta_1 + G^{02}\eta_2 + G3\eta_3 = B_x(\gamma u_x) + B_y(\gamma u_y) + B_z(\gamma u_z) = \gamma(\vec{B}\bullet\vec{u})}$$

$\therefore \boxed{H^{0v}\eta_v = \dfrac{1}{\mu}G^{0v}\eta_v} \Rightarrow \boxed{\gamma(\vec{H}\bullet\vec{u}) = \dfrac{1}{\mu}\gamma(\vec{B}\bullet\vec{u})}$ <u>or:</u> $\boxed{\vec{H}\bullet\vec{u} = \dfrac{1}{\mu}(\vec{B}\bullet\vec{u})}$

<u>Similarly for $\mu = 1$:</u>

$$\boxed{D^{1v}\eta_v = D^{10}\eta_0 + D^{11}\eta_1 + D^{12}\eta_2 + D^{13}\eta_3 = +\gamma c^2 D_x + \gamma H_z u_y - \gamma H_y u_z = \gamma\left(c^2 D_x + \left(\vec{u}\times\vec{H}\right)_x\right)}$$

$$\boxed{F^{1v}\eta_v = F^{10}\eta_0 + F^{11}\eta_1 + F^{12}\eta_2 + F^{13}\eta_3 = \gamma E_x + \gamma B_z u_y - \gamma B_y u_z = \gamma\left(E_x + \left(\vec{u}\times\vec{B}\right)_x\right)}$$

$\therefore \boxed{D^{1v}\eta_v = c^2\varepsilon F^{12}\eta_v} \Rightarrow \boxed{\gamma\left(c^2 D_x\left(\vec{u}\times\vec{H}\right)_x\right) = \gamma\left(E_x + \left(\vec{u}\times\vec{B}\right)_x\right)c} \Rightarrow \boxed{\vec{D} + \dfrac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + (\vec{u}\times\vec{B})\right]}$

<u>And:</u>

$$\boxed{H^{1v}\eta = H^{10}\eta_0 + H^{11}\eta_1 + H^{12}\eta_2 + H^{13}\eta_3 = +\gamma cH_x - \gamma cD_z u_y + \gamma cD_y u_z = \gamma c\left(H_x - \left(\vec{u}\times\vec{D}\right)_z\right)}$$

$$\boxed{G^{1v}\eta = G^{10}\eta_0 + G^{11}\eta_1 + G^{12}\eta_2 + G^{13}\eta_3 = +\gamma cBx - \gamma\dfrac{E_z}{c}u_y + \gamma\dfrac{E_y}{c}u_z = \dfrac{\gamma}{c}\left[c^2 B_x - \left(\vec{u}\times\vec{E}\right)_x\right]}$$

$\therefore \boxed{H^{01}\eta_v = \dfrac{1}{\mu}G^{1v}} \Rightarrow \boxed{\gamma c\left(H_x - \left(\vec{u}\times\vec{D}\right)_z\right) = \dfrac{\gamma}{c}\left[c^2 B_x - \left(\vec{u}\times\vec{E}\right)_x\right]\dfrac{1}{\mu}} \Rightarrow \boxed{\vec{H} - \left(\vec{u}\times\vec{D}\right) = \dfrac{1}{\mu}\left[\vec{B} - \dfrac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$

<u>Thus for:</u> $\boxed{D^{\mu\nu}\eta_\nu = c^2\varepsilon F^{\mu\nu}\eta_\nu}$ and: $\boxed{H^{\mu\nu}\eta_\nu = \dfrac{1}{\mu}G^{\mu\nu}\eta_\nu}$

<u>We obtain:</u>

$$\boxed{\vec{D} + \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right]}$$ <u>and:</u> $\boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) = \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$ **

<u>Insert:</u> $\boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) + \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$

<u>Into:</u> $\boxed{\vec{D} + \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right) = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right]}$ <u>Solve for $\vec{D}$:</u>

$$\boxed{\vec{D} + \frac{1}{c^2}\vec{u}\times\left(\vec{u}\times\vec{D}\right) + \frac{1}{c^2\mu}\left(\vec{u}\times\vec{B}\right) - \frac{1}{c^4\mu}\vec{u}\times\left(\vec{u}\times\vec{E}\right) = \varepsilon\vec{E} + \varepsilon\left(\vec{u}\times\vec{B}\right)}$$

Use the $\vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$ rule on the first triple cross-product:

$$\boxed{\vec{D} + \frac{1}{c^2}\left[\vec{u}\left(\vec{u}\bullet\vec{D}\right) = -u^2\vec{D}\right] = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{\mu c^2}\left(\vec{u}\times\vec{B}\right) + \frac{1}{\mu c^4}\left[\vec{u}\times\left(\vec{u}\times\vec{E}\right)\right]}$$

<u>But:</u> $\boxed{\left(\vec{u}\cdot\vec{D}\right) = \varepsilon\left(\vec{u}\cdot\vec{E}\right)}$

<u>Thus:</u>

$$\boxed{\begin{aligned}\vec{D}\left(1 - \frac{u^2}{c^2}\right) &= -\frac{\varepsilon}{c^2}\vec{u}\left(\vec{u}\bullet\vec{E}\right) + \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{\mu c^2}\left(\vec{u}\times\vec{B}\right) + \frac{1}{\mu c^4}\left[\left(\vec{E}\bullet\vec{u}\right)\vec{u} - u^2\vec{E}\right] \\ &= \varepsilon\left\{\left[1 - \frac{u^2}{\varepsilon\mu c^4}\right]\vec{E} - \frac{1}{c^2}\left[1 - \frac{1}{\varepsilon\mu c^2}\right]\left(\vec{E}\bullet\vec{u}\right)\vec{u} + \left(\vec{u}\times\vec{B}\right)\left[1 - \frac{1}{\varepsilon\mu c^2}\right]\right\}\end{aligned}}$$

<u>Define:</u> $\boxed{\gamma_u \equiv \dfrac{1}{\sqrt{1 - u^2/c^2}}}$ and: $\boxed{v \equiv \dfrac{1}{\sqrt{\varepsilon\mu}}}$ = speed of propagation of *EM* waves in the linear medium.

<u>Then:</u> $\boxed{\vec{D} = \gamma_u^2\varepsilon\left\{\left(1 - \frac{u^2v^2}{c^4}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\left[\left(\vec{u}\times\vec{B}\right) - \frac{1}{c^2}\left(\vec{E}\bullet\vec{u}\right)\vec{u}\right]\right\}}$

From equation ** (RHS up above):

$$\boxed{\vec{H} = \left(\vec{u}\times\vec{D}\right) = \frac{1}{\mu}\left[\vec{B} = \frac{1}{c^2}\left(\vec{u}\times\vec{E}\right)\right]}$$

<u>Insert:</u> $\boxed{\vec{D} = \varepsilon\left[\vec{E} + \left(\vec{u}\times\vec{B}\right)\right] - \frac{1}{c^2}\left(\vec{u}\times\vec{H}\right)}$ <u>Solve for $\vec{H}$:</u>

$\therefore$ $\boxed{\vec{H} - \vec{u} \times \left\{ -\frac{1}{c^2}\left(\vec{u} \times \vec{H}\right) + \varepsilon\left[\vec{E} + \left(\vec{u} \times \vec{B}\right)\right]\right\} = \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u} \times \vec{E}\right)\right]}$

$\boxed{\vec{H} + \frac{1}{c^2}\left[\left(\vec{u}\cdot\vec{H}\right)\vec{u} - u^2\vec{H}\right] = \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u} \times \vec{E}\right)\right] + \varepsilon\left(\vec{u} \times \vec{E}\right) + \varepsilon\left[\vec{u} \times \left(\vec{u} \times \vec{B}\right)\right]}$

<u>But</u>: $\boxed{\left(\vec{H}\cdot\vec{u}\right) = \left(\vec{u}\cdot\vec{H}\right) = \frac{1}{\mu}\left(\vec{u}\cdot\vec{B}\right)}$

$\therefore$ $\boxed{\begin{aligned}\vec{H}\left(1 - \frac{u^2}{c^2}\right) &= \frac{1}{\mu c^2}\left(\vec{B}\cdot\vec{u}\right)\vec{u} + \frac{1}{\mu}\left[\vec{B} - \frac{1}{c^2}\left(\vec{u} \times \vec{E}\right)\right] + \varepsilon\left(\vec{u} \times \vec{E}\right) + \varepsilon\left[\left(\vec{B}\cdot\vec{u}\right)\vec{u} - u^2\vec{B}\right] \\ &= \frac{1}{\mu}\left\{\left[1 - \mu\varepsilon u^2\right]\vec{B} + \left(\varepsilon\mu - \frac{1}{c^2}\right)\left[\left(\vec{u} \times \vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\}\end{aligned}}$

<u>But</u>: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1 - u^2/c^2}}}$ <u>and</u>: $\boxed{v \equiv \frac{1}{\sqrt{\varepsilon\mu}}}$

$\therefore$ $\boxed{\vec{H} = \frac{\gamma_u^2}{\mu}\left\{\left(1 - \frac{u^2}{v^2}\right)\vec{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left[\left(\vec{u} \times \vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\}}$

Thus, the macroscopic $\vec{D}$ and $\vec{H}$-fields in a linear medium moving with ordinary velocity $\vec{u}$ in IRF(S) in terms of the $\vec{E}$ and $\vec{B}$ fields present <u>in</u> IRF(S) are:

$\boxed{\vec{D} = \gamma_u^2\varepsilon\left\{\left(1 - \frac{u^2 v^2}{c^4}\right)\vec{E} + \left(1 - \frac{v^2}{c^2}\right)\left[\left(\vec{u} \times \vec{B}\right) - \frac{1}{c^2}\left(\vec{E}\cdot\vec{u}\right)\vec{u}\right]\right\}}$ <u>with</u>: $\boxed{\gamma_u \equiv \frac{1}{\sqrt{1 - u^2/c^2}}}$

$\boxed{\vec{H} = \frac{\gamma_u^2}{\mu}\left\{\left(1 - \frac{u^2}{v^2}\right)\vec{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left[\left(\vec{u} \times \vec{E}\right) + \left(\vec{B}\cdot\vec{u}\right)\vec{u}\right]\right\}}$ <u>and</u>: $\boxed{v \equiv \frac{1}{\sqrt{\varepsilon\mu}}}$

Note that ε and μ are the <u>proper</u> electric permittivity and the <u>proper</u> magnetic permeability of the linear medium – i.e. they are defined in the <u>rest/proper</u> frame of the linear medium.

When $\boxed{u = |\vec{u}| \ll c}$, then $\boxed{\gamma_u \simeq 1}$ {i.e. the non-relativistic limit}.
Thus, keeping only up to terms <u>linear</u> in $\vec{u}$:

$\boxed{\vec{D} \simeq \varepsilon\vec{E} + \left(1 - (v/c)^2\right)\left(\vec{u} \times \vec{B}\right) \approx \varepsilon\vec{E}}$ and: $\boxed{\vec{H} \simeq \frac{1}{\mu}\vec{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\left(\vec{u} \times \vec{E}\right) \approx \frac{1}{\mu}\vec{B}}$

Usually very small   Usually very small