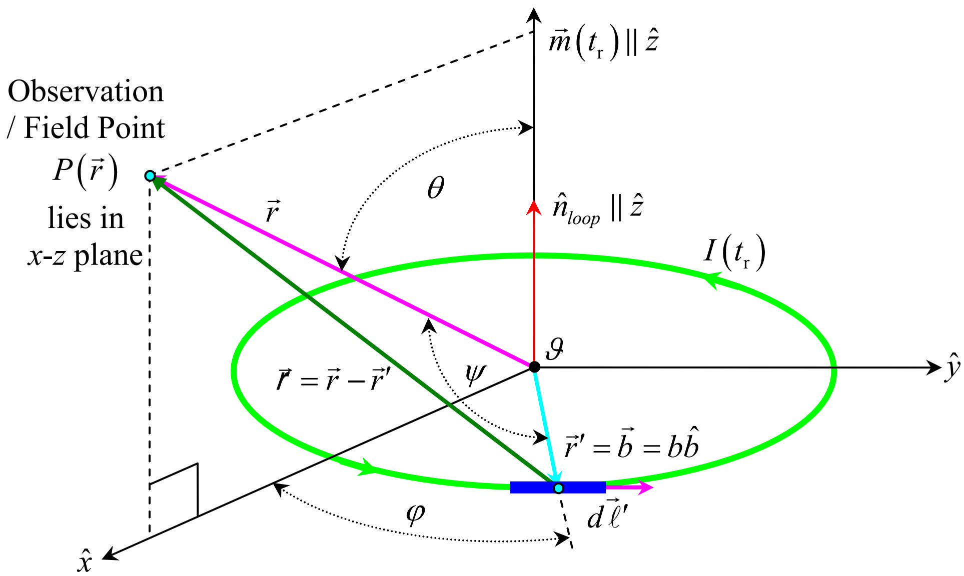
LECTURE NOTES 13.5

M(1) Magnetic Dipole Radiation:

A time-varying current $I(t_r) = I_o \cos(\omega t_r)$ flows in a circular loop of radius b {chosen for convenience's sake in to lie in the *x-y* plane} as shown in the figure below, and has associated with it an oscillating magnetic dipole moment:

$\boxed{\vec{m}(t_r) = I(t_r)\vec{A}_{loop} = \pi b^2 I_o \cos(\omega t_r)\hat{z}}$ where: $\boxed{\vec{A}_{loop} = A_{loop}\hat{n}_{loop} = \pi b^2 \hat{z}}$ with: $\boxed{\hat{n}_{loop} = \hat{z}}$.

Or: $\boxed{\vec{m}(t_r) = m_o \cos(\omega t_r)\hat{z}}$ where: $\boxed{m_o \equiv \pi b^2 I_o}$



Note that there is no volume electric charge density $\rho(\vec{r}, t_r)$ associated with the current flowing in the loop, thus the retarded scalar potential $V_r^{M(1)}(\vec{r}, t) = 0$ and thus $\vec{\nabla} V_r^{M(1)}(\vec{r}, t) = 0$ {here}.

The retarded vector potential is:

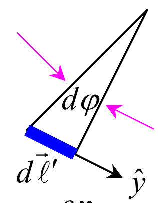
$\boxed{\vec{A}_r^{M(1)}(\vec{r}, t) = \left(\frac{\mu_o}{4\pi}\right)\int \frac{I_o \cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} d\vec{\ell}'}$ with: $\boxed{t_r = t - \mathscr{r}/c}$ and: $\boxed{\vec{\mathscr{r}} = \vec{r} - \vec{r}'(t_r)}$

n.b. We know that $\vec{A}_r^{M(1)}(\vec{r}, t)$ follows the direction of (conventional) current flow

$\Rightarrow \vec{A}_r^{M(1)}(\vec{r}, t) \parallel \hat{\varphi}$ -direction.

For convenience's sake, the observation / field point $P(\vec{r})$ is chosen directly above the $\hat{x}$-axis in the *x-z* plane (see above figure). From (azimuthal / circular symmetry) associated with this problem, one can see that *e.g.* the $\hat{x}$-component contributions to $\vec{A}_r^{M(1)}(\vec{r},t)$ from symmetrically-placed current segments $I(t)d\vec{\ell}'$ on either side of the $\hat{x}$-axis will cancel each other, $\vec{A}_r^{M(1)}(\vec{r},t)$ at this observation point in the *x-z* plane will point in the $\hat{y}$-direction, but for an arbitrary field point, we see that $\vec{A}_r^{M(1)}(\vec{r},t)$ points in the $\hat{\varphi}$-direction since:

$\boxed{\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}}$ (in spherical-polar and/or cylindrical coordinates)
(*n.b.* the angle $\varphi = 0$ for points in the *x-z* plane!)



"$s = r\theta$"
$d\ell' = bd\varphi$

Thus, for the choice of the observation / field point $P(\vec{r})$ in *x-z* plane (see figure above): $\boxed{d\vec{\ell}' = bd\varphi\,\hat{y}\cdot\cos\varphi = b\cos\varphi d\varphi\,\hat{y}}$ (*n.b.* the $\cos\varphi$ term simply picks off the $\hat{y}$-component of $d\vec{\ell}'$), but note that $d\vec{\ell}'$ is actually $\parallel$ to $\hat{\varphi}$.

Then: $$\boxed{\vec{A}_r^{M(1)}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right)I_o b\int_{\varphi=0}^{\varphi=2\pi}\frac{\cos\left[\omega\left(t - \mathscr{r}/c\right)\right]}{\mathscr{r}}\cos\varphi d\varphi\,\hat{\varphi}}$$

From the Law of Cosines: $\boxed{\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi}}$ where: $\boxed{\psi = \cos^{-1}\left(\hat{r}\cdot\hat{b}\right)}$ {See above figure}

$\vec{r}$ lies in the *x-z* plane: $\boxed{\vec{r} = r\sin\theta\hat{x} + r\cos\theta\hat{z}}$ and $\vec{b}$ lies in *x-y* plane: $\boxed{\vec{b} = b\cos\varphi\hat{x} + b\sin\varphi\hat{y}}$

$$\therefore \boxed{\vec{r}\cdot\vec{b} = rb\cos\psi = \left(r\sin\theta\hat{x} + r\cos\theta\hat{z}\right)\cdot\left(b\cos\varphi\hat{x} + b\sin\varphi\hat{y}\right) = rb\sin\theta\cos\varphi}$$
$$\therefore \boxed{\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\varphi}}$$

Here again, we are interested in far-zone *EM* radiation solutions, *i.e.* the observer is far away from source, such that the characteristic dimension of the source is such that $\boxed{b \ll r}$.

Then keeping only the 1st non-trivial term in the Taylor series expansion of $\mathscr{r}$ with $b \ll r$:

$$\boxed{\mathscr{r} = r\sqrt{1 + \left(\frac{b}{r}\right)^2 - 2\left(\frac{b}{r}\right)\sin\theta\cos\varphi} \simeq r\sqrt{1 - 2\left(\frac{b}{r}\right)\sin\theta\cos\varphi}} \text{ with: } \boxed{\left(\frac{b}{r}\right) \ll 1} \text{ and: } \boxed{\sqrt{1-\epsilon} \simeq 1 - \frac{\epsilon}{2}}$$

$$\therefore \boxed{\mathscr{r} \simeq r\left(1 - \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \text{ for: } \boxed{\left(\frac{b}{r}\right) \ll 1}$$

And: $$\boxed{\frac{1}{\mathscr{r}} \simeq \frac{1}{r\left(1 - \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \approx \frac{1}{r}\left(1 + \left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)} \text{ for: } \boxed{\left(\frac{b}{r}\right) \ll 1}$$

Now:

$$\boxed{\begin{aligned}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] &\simeq \cos\left[\omega\left(t-\frac{r}{c}\right)+\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right]\\ &= \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)-\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)\end{aligned}}$$

Here (again) we assume for "far-zone" radiation that $\boxed{b \ll \lambda}$ and $\boxed{\lambda \ll r}$, *i.e.* $\boxed{b \ll \lambda \ll r}$

and since $\boxed{b \ll \lambda}$ and $\boxed{\lambda = c/f = 2\pi c/\omega}$ $\Rightarrow$ $\boxed{b \ll \frac{c}{\omega}}$ or: $\boxed{\frac{\omega b}{c} \ll 1}$.

Then:

$$\boxed{\begin{aligned}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] &\simeq \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\overbrace{\cos 0}^{=1}-\sin\left[\omega\left(t-\frac{r}{c}\right)\right]*\left(\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\right)\\ &\simeq \cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\end{aligned}}$$

Thus:

$$\boxed{\begin{aligned}\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t) &= \left(\frac{\mu_o}{4\pi}\right)I_o b\int_{\varphi=0}^{\varphi=2\pi}\frac{\cos\left[\omega\left(t-r/c\right)\right]}{r}\cos\varphi d\varphi\,\hat{\varphi}\\ &\simeq \left(\frac{\mu_o}{4\pi}\right)\frac{I_o b}{r}\int_{\varphi=0}^{\varphi=2\pi}\left(1+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\right)\\ &\qquad *\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos\varphi d\varphi\,\hat{\varphi}\\ &= \left(\frac{\mu_o}{4\pi}\right)\frac{I_o b}{r}\int_{\varphi=0}^{\varphi=2\pi}\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right.\\ &\qquad \left.+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\underbrace{\cancel{\left(\frac{\omega b^2}{rc}\right)}\sin^2\theta\cos^2\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}\right\}\cos\varphi d\varphi\,\hat{\varphi}\end{aligned}}$$

However, note that: $\boxed{\left(\frac{b}{r}\right)\ll 1}$ and: $\boxed{\left(\frac{\omega b}{c}\right)\ll 1}$, thus: $\boxed{\left(\frac{\omega b^2}{rc}\right)=\left(\frac{b}{r}\right)\left(\frac{\omega b}{c}\right)\lll 1}$,

i.e. $\boxed{\left(\frac{\omega b^2}{rc}\right)=\left(\frac{b}{r}\right)\left(\frac{\omega b}{c}\right)}$ is a 2nd-order term, so we drop/neglect it !!!

$\therefore$

$$\boxed{\begin{aligned}\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{I_o b}{r}\int_{\varphi=0}^{\varphi=2\pi}&\left\{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega b}{c}\right)\sin\theta\cos\varphi\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right.\\ &\left.+\left(\frac{b}{r}\right)\sin\theta\cos\varphi\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\cos\varphi\, d\varphi\,\hat{\varphi}\end{aligned}}$$

Note that the first term in the above integral: $\boxed{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos\varphi\, d\varphi = 0}$

The second term is: $\boxed{-\left(\frac{\omega b}{c}\right)\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi}$

The third term is: $\boxed{+\left(\frac{b}{r}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi}$

But: $\boxed{\int_{\varphi=0}^{\varphi=2\pi}\cos^2\varphi\, d\varphi = \pi}$

Thus: $\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{I_o b^2}{r}(\pi\sin\theta)\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi}}$

But: $\boxed{m_o \equiv \pi b^2 I_o = A_{loop} I_o}$

$\therefore$ $\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{m_o}{r}\sin\theta\left\{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi}}$ **

Note that in the static limit ($\omega \to 0$), only the 1st term in the { … } brackets survives:

$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r}) \simeq \frac{\mu_o m_o}{4\pi r^2}\sin\theta\hat{\varphi}}$ = vector potential for a (static) magnetic dipole.

Note also that in the "far-zone" for M(1) *EM* radiation, with $\boxed{b \ll \lambda \ll r}$ or: $\boxed{\frac{c}{\omega} \ll r}$

If $\boxed{\frac{c}{\omega} \ll r}$, then $\boxed{\frac{\omega}{c} \gg \left(\frac{1}{r}\right)}$ $\therefore$ Drop the 1st term in { … } in the above expression for $\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t)$:

$\therefore$ $\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq \left(\frac{\mu_o}{4\pi}\right)\frac{m_o}{r}\sin\theta\left\{\overbrace{\cancel{\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}}^{\simeq 0}-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\hat{\varphi}}$

Thus: $\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq -\left(\frac{\mu_o}{4\pi}\right)\frac{\omega m_o}{cr}\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$

Or: $\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq -\frac{\mu_o m_o \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$

Then: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) = \overbrace{\cancel{-\vec{\nabla}V_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t)}}^{=0} - \frac{\partial\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t)}{\partial t} = -\frac{\partial\vec{A}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t)}{\partial t}}$

Thus: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq +\frac{\mu_o m_o \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$

And:

$$\boxed{\begin{aligned}&\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)\\&=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)-\underbrace{\cancel{\frac{\partial A_\theta}{\partial\varphi}}}_{=0}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\underbrace{\cancel{\frac{\partial A_r}{\partial\theta}}}_{=0}-\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta}+\frac{1}{r}\left[\underbrace{\cancel{\frac{\partial}{\partial r}(rA_\theta)}}_{=0}-\underbrace{\cancel{\frac{\partial A_r}{\partial\theta}}}_{=0}\right]\hat{\varphi}\end{aligned}}$$

Or:

$$\boxed{\begin{aligned}\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\simeq{}&\frac{1}{r\sin\theta}\left(-\frac{\mu_o m_o\omega}{4\pi cr}\right)\frac{\partial}{\partial\theta}(\sin^2\theta)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}\\&-\frac{1}{r}\left(\frac{-\mu_o m_o\omega}{4\pi c}\right)\sin\theta\frac{\partial}{\partial r}\left(\left(\frac{\cancel{r}}{\cancel{r}}\right)\sin\left[\omega\left(t-{}^{r}\!/_{c}\right)\right]\right)\hat{\theta}\end{aligned}}$$

Or:

$$\boxed{\begin{aligned}\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\simeq{}&-\frac{\mu_o m_o\omega}{4\pi cr^2}\frac{2\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}\\&+\frac{\mu_o m_o\omega}{4\pi cr}\left(\frac{-\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\end{aligned}}$$

Or:

$$\boxed{\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\simeq-\left(\frac{\mu_o m_o\omega}{4\pi cr}\right)\left\{2\left(\frac{1}{r}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}+\left(\frac{\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}}$$

Again, $\boxed{\frac{\omega}{c}\gg\left(\frac{1}{r}\right)}$ $\therefore$ Drop the 1st term in { … } in the above expression for $\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)$

$$\boxed{\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\simeq-\left(\frac{\mu_o m_o\omega}{4\pi cr}\right)\left\{\cancel{2\left(\frac{1}{r}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}+\left(\frac{\omega}{c}\right)\sin\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}}$$

Thus: $$\boxed{\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)\simeq-\frac{\mu_o m_o\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

Note that in the static limit ($\omega\to0$), first going back to using the expression for $\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)$ as given in ** on the previous page, and then using $\boxed{\vec{B}_{\rm r}^{\rm M(1)}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\rm r}^{\rm M(1)}(\vec{r},t)}$, we do indeed obtain the familiar result: $\boxed{\vec{B}_{\rm r}^{\rm M(1)}(\vec{r})\simeq-\left(\frac{\mu_o m_o}{4\pi r^3}\right)\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)}$.

Therefore, for M(1) magnetic dipole radiation, in the "far-zone", with $\boxed{b \ll \lambda \ll r}$, we have:

$$\boxed{V_r^{\mathrm{M}(1)}(\vec{r},t) = 0}$$

$$\boxed{\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t) \simeq -\frac{\mu_o m_o \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}} \quad \text{where: } \boxed{m_o = \pi b^2 I_o}$$

$$\boxed{\vec{E}_r^{\mathrm{M}(1)}(\vec{r},t) \simeq -\frac{\partial \vec{A}_r^{\mathrm{M}(1)}(\vec{r},t)}{\partial t} \simeq +\frac{\mu_o m_o \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$$

$$\boxed{\vec{B}_r^{\mathrm{M}(1)}(\vec{r},t) = \vec{\nabla}\times\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t) = -\frac{\mu_o m_o \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

Here again, note that: $\boxed{\vec{B}_r^{\mathrm{M}(1)}(\vec{r},t) = \frac{1}{c}\hat{r}\times\vec{E}_r^{\mathrm{M}(1)}(\vec{r},t)}$ with: $(\hat{r}\times\hat{\varphi} = -\hat{\theta})$

{since: $\hat{r}\times\hat{\theta} = \hat{\varphi}$ $\quad \hat{\varphi}\times\hat{r} = \hat{\theta}$ $\quad \hat{\theta}\times\hat{\varphi} = \hat{r}$ }

Note that $\vec{E}_r^{\mathrm{M}(1)}(\vec{r},t)$ and $\vec{B}_r^{\mathrm{M}(1)}(\vec{r},t)$:

a) <u>both</u> have same $\underline{\underline{\sim 1/r}}$ dependence (as does $\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t)$).

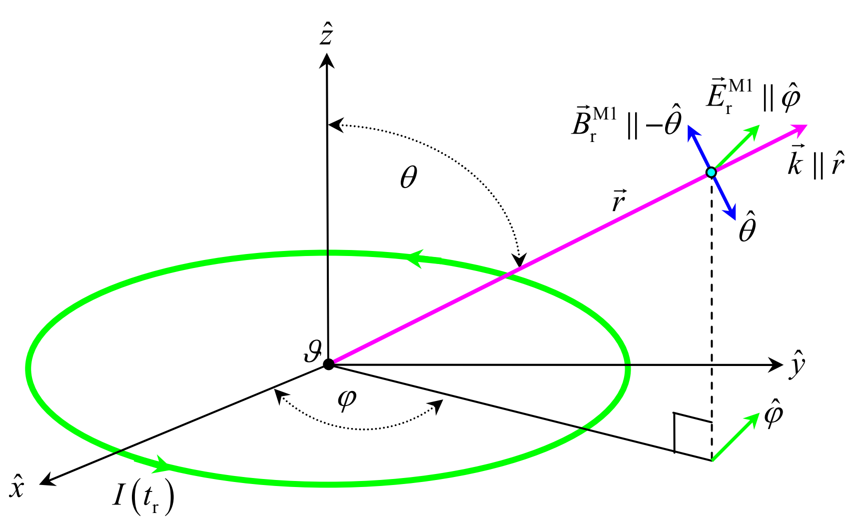
b) <u>both</u> have same ~ sin θ dependence (as does $\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t)$).

c) <u>both</u> are in <u>phase</u> with each other – <u>both</u> have same $\cos\left[\omega\left(t - r/c\right)\right]$ factors.

d) <u>both</u> are 90° <u>out of phase</u> with $\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t)$.

e) $\vec{B}_r^{\mathrm{M}(1)}(\vec{r},t)$ is $\perp$ to $\vec{E}_r^{\mathrm{M}(1)}(\vec{r},t)$ as it must be.

<u>Note also that</u>: $\vec{A}_r^{\mathrm{M}(1)}(\vec{r},t)$, $\vec{E}_r^{\mathrm{M}(1)}(\vec{r},t)$ and $\vec{B}_r^{\mathrm{M}(1)}(\vec{r},t)$ vanish (*i.e.* = 0) when $\theta = 0$ and $\theta = \pi$ *i.e.* at the poles, along the $\hat{z}$ -axis (as we also saw in the case of E(1) electric dipole radiation).



The *EM* radiation energy density $u_{\mathrm{M}(1)}^{rad}(\vec{r},t)$ associated with the oscillating M1 magnetic dipole for far-zone *EM* radiation $\{b \ll \lambda \ll r\}$ is:

$$\boxed{\begin{aligned} u_{\mathrm{M}(1)}^{rad}(\vec{r},t) &= u_{\mathrm{M}(1)}^{E}(\vec{r},t) + u_{\mathrm{M}(1)}^{M}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o \vec{E}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t)\cdot\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) + \frac{1}{\mu_0}\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t)\cdot\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t)\right) \\ &\simeq \frac{1}{2}\left\{\varepsilon_o \frac{\mu_o^2 m_o^2\omega^4}{16\pi^2 c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{1}{\cancel{\mu_o}}\frac{\mu_o^{\cancel{2}} m_o^2\omega^4}{16\pi^2 c^4}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \end{aligned}}$$

But: $\boxed{c^2 = \dfrac{1}{\varepsilon_o\mu_o}}$ or: $\boxed{\varepsilon_o = \dfrac{1}{\mu_o c^2}}$, so again we see that $u_{\mathrm{M}(1)}^{E}(\vec{r},t) = u_{\mathrm{M}(1)}^{M}(\vec{r},t)$ in the "far-zone" limit $b \ll \lambda \ll r$, and thus:

$$\boxed{u_{\mathrm{M}(1)}^{rad}(\vec{r},t) \simeq \left(\frac{\mu_o m_o^2\omega^4}{16\pi^2 c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]}\left(\frac{Joules}{m^3}\right) \text{ for } \boxed{b \ll \lambda \ll r} \text{ with } \boxed{m_o \equiv \pi b^2 I_o}$$

The *EM* energy radiated by oscillating magnetic dipole in the far-zone limit $\{b \ll \lambda \ll r\}$ is given by Poynting's vector:

$$\boxed{\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t)\times\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) \simeq -\frac{1}{\cancel{\mu_o}}\frac{\mu_o^{\cancel{2}} m_o^2\omega^4}{16\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\overbrace{\left[\hat{\varphi}\times\hat{\theta}\right]}^{=-\hat{r}}} \quad \boxed{\begin{aligned}\hat{r}\times\hat{\theta}&=\hat{\varphi}\\ \hat{\theta}\times\hat{\varphi}&=\hat{r}\\ \hat{\varphi}\times\hat{r}&=\hat{\theta}\end{aligned}}$$

$$\boxed{\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t) \simeq +\frac{\mu_o m_o^2\omega^4}{16\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}\left(\frac{Watts}{m^2}\right) \Leftarrow \text{Radial outward flow of } EM \text{ energy}$$

for: $\boxed{b \ll \lambda \ll r}$ "far zone" limit

The *EM* radiation linear momentum density associated with an oscillating magnetic dipole, in the far zone limit $\{b \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\wp}_{\mathrm{M}(1)}^{rad}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t) = \frac{1}{c^2}\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t)}$$

Or: $$\boxed{\vec{\wp}_{\mathrm{M}(1)}^{rad}(\vec{r},t) \simeq +\frac{\mu_o m_o^2\omega^4}{16\pi^2 c^5}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}\left(\frac{kg}{m^2\text{-sec}}\right) \Leftarrow \text{Radial outward } EM \text{ linear}$$

momentum flow for: $\boxed{b \ll \lambda \ll r}$ "far zone" limit

The EM radiation angular momentum density associated with an oscillating magnetic dipole, in the far zone $\{b \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\ell}_{\mathrm{M}(1)}^{rad}(\vec{r},t) = \vec{r}\times\vec{\wp}_{M(1)}^{rad}(\vec{r},t) = \frac{\mu_o m_o^2\omega^4}{16\pi^2 c^5}\left(\frac{\sin^2\theta}{r}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\left[\hat{r}\times\hat{r}\right]}_{=0} = 0}\left(\frac{kg}{m\text{-sec}}\right)$$

$\Rightarrow$ No angular momentum flow for: $\boxed{b \ll \lambda \ll r}$ "far zone" limit *n.b.* Again, the exact $\vec{\ell}_{\mathrm{E}(1)}^{rad}(\vec{r},t) \not\equiv 0$ *i.e.* ignore restrictions on far-zone limit, keep all higher-order terms . . . we have neglected $\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)} \sim \hat{r}$ term which is non-negligible in the near-zone $(d \sim r)$ and also in the so-called intermediate, or inductive zone $(\lambda \sim r)$.

Time-Averaged Quantities for M(1) Radiation from an Oscillating Magnetic Dipole:

The time-averaged *EM* radiation energy density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle u_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Joules}{m^3}\right) \text{ for: } \boxed{b \ll \lambda \ll r} \text{ “far-zone” limit}$$

The time-averaged |Poynting's vector|, which is also the intensity $I_{\mathrm{M}(1)}^{rad}$ of *EM* radiation associated with an oscillating magnetic dipole is:

$$\boxed{I_{\mathrm{M}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r})\right|\right\rangle = \frac{1}{2}c\varepsilon_o \left\langle \left(E_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t)\right)^2\right\rangle \simeq \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{“far-zone” limit}\end{array}}$$

We also see that: $\boxed{I_{\mathrm{M}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right|\right\rangle = c\left\langle u_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle} \left(\frac{Watts}{m^2}\right)$.

The time-averaged *EM* radiated power associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle = \int_S \left\langle \vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle \cdot d\vec{a}_\perp \simeq \frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3} \underbrace{\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \sin^2\theta \sin\theta \, d\theta \, d\varphi}_{=\frac{4}{3}2\pi=\frac{8\pi}{3}}}$$

$\therefore$ The time-averaged radiated power is:

$$\boxed{\left\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o m_o^2 \omega^4}{12\pi c^3}\right)} (Watts) \text{ for: } \boxed{b \ll \lambda \ll r} \text{ “far-zone” limit}$$

n.b. $\left\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle$ has no *r*-dependence!

The time-averaged *EM* radiation linear momentum density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle \vec{\wp}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c^2}\left\langle \vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle \hat{r} \simeq \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}} \left(\frac{kg}{m^2\text{-sec}}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{“far-zone” limit}\end{array}}$$

The time-averaged *EM* radiation angular momentum density associated with an oscillating magnetic dipole is:

$$\boxed{\left\langle \vec{\ell}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle = \vec{r} \times \left\langle \vec{\wp}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^5}\right)\left(\frac{\sin^2\theta}{r}\right)(\hat{r}\times\hat{r}) \equiv 0} \left(\frac{kg}{m\text{-sec}}\right) \text{ for: } \boxed{\begin{array}{c} b \ll \lambda \ll r \\ \text{“far-zone” limit}\end{array}}$$

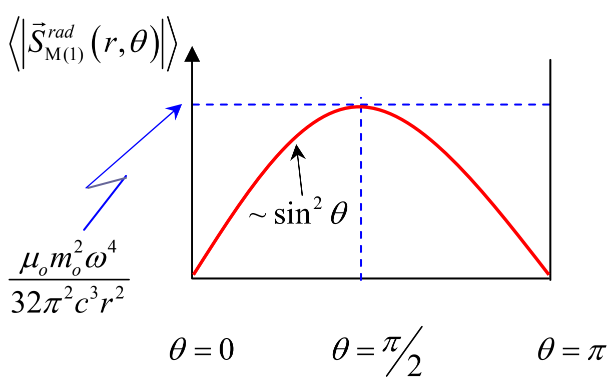
n.b. Again, the exact $\left\langle \vec{\ell}_{\mathrm{M}(1)}^{\,rad}(\vec{r}) \right\rangle \not\equiv 0$ *i.e.* ignore restrictions on far-zone limit, keep all higher-order terms . . . we have neglected the $\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)} \sim \hat{r}$ term which is non-negligible in the near-zone $(d \sim r)$ and also in the so-called intermediate, or inductive zone $(\lambda \sim r)$.

Again, note that because: $\boxed{I_{\mathrm{M}(1)}^{rad}(\vec{r}) \equiv \left\langle \left| \vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r},t) \right| \right\rangle \simeq \left(\frac{\mu_o m_o^2 \omega^4}{32\pi^2 c^3} \right) \left(\frac{\sin^2\theta}{r^2} \right)} \left(\frac{Watts}{m^2} \right)$

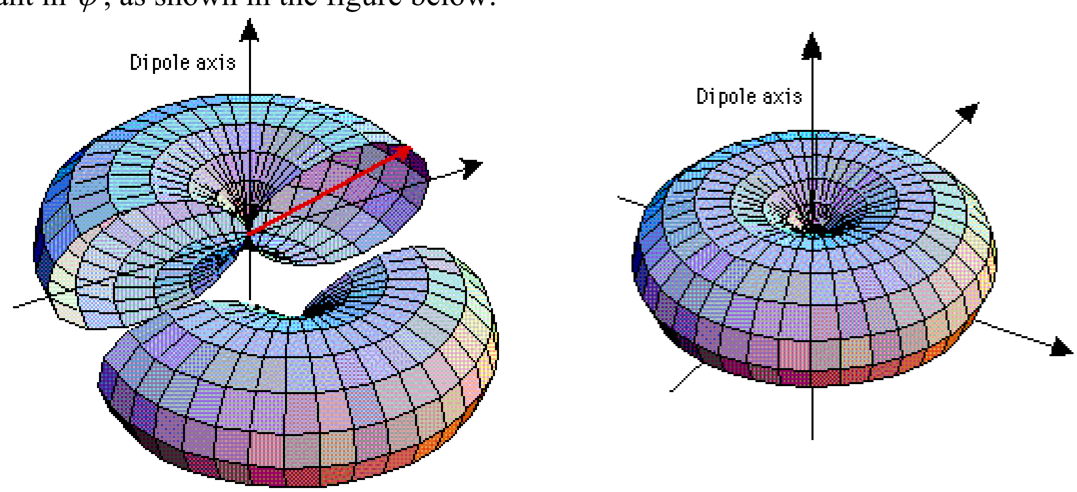
$\Rightarrow \boxed{\left\langle \vec{S}_{\mathrm{M}(1)}^{rad}(r,\theta=0,\varphi) \right\rangle = \left\langle \vec{S}_{\mathrm{M}(1)}^{rad}(r,\theta=\pi,\varphi) \right\rangle = 0}$ since: $\boxed{\sin^2 0 = \sin^2 \pi = 0}$

i.e. no *EM* radiation occurs along the axis of the magnetic dipole ($\hat{z}$ axis)

EM radiation for M(1) electric dipole is {also} peaked/maximum at $\theta = \pi/2$ (then $\sin^2\theta = 1$) *i.e.* maximum *EM* radiation occurs $\perp$ to the axis of the magnetic dipole:



Thus, the intensity profile $I_{\mathrm{M}(1)}^{rad}(\vec{r})$ in 3-D {for fixed r} for M(1) magnetic dipole *EM* radiation is {again} donut-shaped {as in the case of E(1) electric dipole *EM* radiation} - it is rotationally invariant in φ, as shown in the figure below:



It is useful / illuminating to compare sources, retarded potentials, retarded fields, energy densities, Poynting's Vector, power radiated, linear and angular-momentum densities associated with E(1) electric dipole radiation *vs*. M(1) magnetic dipole radiation <u>in the far-zone limit</u>, with $\boxed{d = \pi b \ll \lambda \ll r}$:

	<u>E(1) Oscillating Electric Dipole</u>	**<u>M(1) Oscillating Magnetic Dipole</u>**
<u>Source Charge</u>:	$q(\vec{r},t_{\mathrm{r}}) = q_o\delta(z \pm d/2)\cos(\omega t_{\mathrm{r}})$	$q(\vec{r},t_{\mathrm{r}}) = 0$
<u>Source Currents</u>:	$I(\vec{r},t_{\mathrm{r}}) = -q_o\omega\sin(\omega t_{\mathrm{r}})$ {on $\hat{z}$ -axis, $\lvert z\rvert < d/2$ }	$I(\vec{r},t_{\mathrm{r}}) = I_o\cos(\omega t_{\mathrm{r}})$ {in *x*-*y* plane, radius *b*}
<u>*EM* Moments</u>:	$\vec{p}(\vec{r},t_{\mathrm{r}}) = q(\vec{r},t_{\mathrm{r}})\vec{d},\ \vec{d} = d\hat{z}$ $p_o = q_o d = \lvert\vec{p}\rvert = q_o\lvert\vec{d}\rvert$	$\vec{m}(\vec{r},t_{\mathrm{r}}) = I(\vec{r},t_r)\vec{A}_{loop} = \pi b^2 I(\vec{r},t_{\mathrm{r}})$ $m_o = \pi b^2 I_o = \lvert\vec{m}\rvert$
Retarded Scalar Potential	$V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$	$V_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) = 0$
Retarded Vector Potential	$\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t) \simeq -\frac{\mu_o p_o\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}$	$\vec{A}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) \simeq -\frac{\mu_o m_o\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Electric Field	$\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o p_o\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$	$\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) \simeq +\frac{\mu_o m_o\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$
Retarded Magnetic Field	$\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \simeq -\frac{\mu_o p_o\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}$	$\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)}(\vec{r},t) \simeq -\frac{\mu_o m_o\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$
Time-Avg'd *EM* Energy Density	$\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$\langle u_{\mathrm{M}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^4}\right)\left(\frac{\sin^2\theta}{r^2}\right)$
Time-Avg'd Poynting Vector/Intensity	$I_{\mathrm{E}(1)}^{rad}(\vec{r}) \equiv \langle\lvert\vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\rvert\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)$	$I_{\mathrm{M}(1)}^{rad}(\vec{r}) \equiv \langle\lvert\vec{S}_{\mathrm{M}(1)}^{rad}(\vec{r})\rvert\rangle \simeq \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)$
Time-Avg'd Radiated *EM* Power	$\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{12\pi c}\right)$	$\langle P_{\mathrm{M}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o m_o^2\omega^4}{12\pi c^3}\right)$
Time-Avg'd *EM* Linear Momentum Density	$\langle\vec{\wp}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$	$\langle\vec{\wp}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\rangle \simeq \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$
Time-Avg'd *EM* Angular Momentum Density	$\langle\vec{\ell}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\rangle = 0$	$\langle\vec{\ell}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\rangle = 0$

Note that for "equal" strength *EM* moments, where $\boxed{p_o = q_o d}$ and $\boxed{m_o = \pi b^2 I_o}$, and $\boxed{I_o = q_o \omega}$ and we let $\boxed{d = \pi b}$:

<u>Then</u>: $$\boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle} = \left(\frac{\mu_o m_o^2 \omega^4}{12\pi c^3}\right) \bigg/ \left(\frac{\mu_o p_o^2 \omega^4}{12\pi c}\right) = \frac{m_o^2}{p_o^2 c^2} = \left(\frac{m_o}{p_o c}\right)^2 = \left(\frac{\pi b^2 I_o}{q_o d c}\right)^2}$$

$$\therefore \quad \boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle} = \left(\frac{\omega b}{c}\right)^2}$$

But in the "far-zone" limit $\boxed{d = \pi b \ll \lambda \ll r}$ we have: $\boxed{\left(\frac{\omega b}{c}\right) \ll 1}$!!!

$$\therefore \quad \boxed{\frac{\left\langle \mathrm{P}_{\mathrm{M}(1)}^{rad}(\vec{r},t)\right\rangle}{\left\langle \mathrm{P}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle} = \left(\frac{\omega b}{c}\right)^2 \ll 1}$$

Thus, for "equal" strength *EM* moments (as defined above), the oscillating E(1) electric dipole radiates vastly more power in the form of EM waves than does an oscillating M(1) magnetic dipole.

$\Rightarrow$ This is why *e.g.* all commercial radio & television stations use electric dipole antennae to broadcast their signals!

Note also that the structure of the $\vec{E}$ and $\vec{B}$ fields for E(1) electric dipole vs. M(1) magnetic dipole radiation, in the "far-zone" limit $\boxed{d = \pi b \ll \lambda \ll r}$ are very similar, except that the $\vec{E}$ and $\vec{B}$ field vectors for the M(1) case are ***<u>rotated</u>*** by 90$^{\mathrm{o}}$ (*i.e.* $\hat{\theta}$ and $\hat{\varphi}$ are interchanged), compared to the E(1) case:

<u>E(1)</u>: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)} \sim \hat{\theta}}$ $\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)} \sim \hat{\varphi}}$

<u>B(1)</u>: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M}(1)} \sim -\hat{\varphi}}$ $\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{M}(1)} \sim -\hat{\theta}}$

The Characteristic Impedance of an Antenna:

The characteristic impedance of an antenna is exactly as we defined the characteristic impedance of a waveguide; noting here that we are dealing with manifestly transverse waves for *EM* wave radiation from *e.g*. either an E(1) electric dipole or M(1) magnetic dipole antenna:

$$\boxed{Z_{antenna}(\vec{r}) \equiv \frac{\left|\vec{E}_{\perp}^{rad}(\vec{r})\right|}{\left|\vec{H}_{\perp}^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}_{\perp}^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}_{\perp}^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}^{rad}(\vec{r})\right|}}$$

Let's check the SI units of this definition {referring to the trusty 1-page handout that I gave out at beginning of semester}:

$$\boxed{\left(\frac{E}{B/\mu_o}\right) = \frac{\left(Volts/m\right)}{\left(Teslas\Big/\frac{Henrys}{m}\right)} = \frac{\left(Volts/m\right)}{\left(N/A\text{-}m\right)\Big/\left(N/A^2\right)} = \frac{Volts}{Amps} = Ohms}$$

For both E(1) electric dipole and M(1) magnetic dipole radiation, we see that the characteristic impedances of these antennae in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$) with $\boxed{c = 1\big/\sqrt{\varepsilon_o \mu_o}}$ are:

$$\boxed{Z_{antenna}^{\mathrm{E}(1)}(\vec{r}) = Z_{antenna}^{\mathrm{M}(1)}(\vec{r}) = \mu_o c = \sqrt{\frac{\mu_o}{\varepsilon_o}} \equiv Z_o = 120\pi\ \Omega = 377\ \Omega}$$

Where:

$\mu_o = 4\pi \times 10^{-7}$ Henrys/m = magnetic permeability of free space / vacuum

$\varepsilon_o = 8.85 \times 10^{-12}$ Farads/m = electric permittivity of free space / vacuum

And: $\boxed{Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}\,\mathrm{Henrys/m}}{8.85 \times 10^{-12}\,\mathrm{Farads/m}}} = 120\pi\ \Omega = 377\ \Omega}$ is the characteristic impedance of free space/the vacuum.

Thus we see that E(1) electric dipole and M(1) magnetic dipole antennae (in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$)) are *perfectly* impedance-matched for propagation of E(1) and/or M(1) *EM* waves into free space / vacuum!

Note also that in the "far-zone" ($d = \pi b \ll \lambda \ll r$) that $Z_{antenna}^{\mathrm{E}(1)}(\vec{r})$, $Z_{antenna}^{\mathrm{M}(1)}(\vec{r})$ have no $\vec{r}$ -dependence.

The Radiation Resistance of an Antenna:

The radiation resistance of an antenna R_{rad} is defined relative to the antenna power P_{rad} and the amplitude of current flowing in the antenna I_o:

$$\boxed{P_{rad}^{antenna} \equiv I_o^2 R_{rad}^{antenna}} \quad \text{or:} \quad \boxed{R_{rad}^{antenna} \equiv \frac{P_{rad}^{antenna}}{I_o^2}} \quad \text{(Ohms)}$$

$p = qd$ For E(1) electric dipole antenna: $I_o = q_o\omega$ = amplitude of current flowing in dipole
$m = \pi b^2 I_o$ For M(1) magnetic dipole antenna: I_o = amplitude of current flowing in loop

In the "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E(1)}} \simeq \frac{\mu_o p_o^2 \omega^4}{12\pi c^3 I_o^2} = \frac{\mu_o \cancel{q_o^2} d^2 \omega^{\cancel{4}}}{12\pi c \cancel{q_o^2} \cancel{\omega^2}} = \frac{\mu_o \omega^2 d^2}{12\pi c}(\Omega)}$$

$$\boxed{R_{rad}^{\mathrm{M(1)}} \simeq \frac{\mu_o m_o^2 \omega^4}{12\pi c^3 I_o^2} = \frac{\mu_o \pi^{\cancel{2}} b^4 \cancel{I_o^2} \omega^4}{12 \cancel{\pi} c^3 \cancel{I_o^2}} = \frac{\mu_o \pi \omega^4 b^4}{12 c^3}(\Omega)}$$

n.b. $R_{rad}^{\mathrm{E(1),M(1)}}$ are both frequency-dependent

In the "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E(1)}} \simeq \frac{\mu_o \omega^2 d^2}{12\pi c} = \frac{\omega^2 d^2}{12\pi c^2}(\mu_o c) = \frac{\omega^2 d^2}{12\pi c^2}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{\omega^2 d^2}{12\pi c^2} Z_{rad}^{\mathrm{E(1)}}}$$

$$\boxed{R_{rad}^{\mathrm{M(1)}} \simeq \frac{\mu_o \pi \omega^4 b^4}{12 c^3} = \frac{\pi \omega^4 b^4}{12 c^4}(\mu_o c) = \frac{\pi \omega^4 b^4}{12 c^4}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{\pi \omega^4 b^4}{12 c^4} Z_{rad}^{\mathrm{M(1)}}}$$

But: $\boxed{Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = Z_{rad}^{\mathrm{E(1)}} = Z_{rad}^{\mathrm{M(1)}}}$

$\therefore$ In the "far-zone" $d = \pi b \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E(1)}} \simeq \frac{\omega^2 d^2}{12\pi c^2} Z_o = \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o} \quad \text{and:} \quad \boxed{R_{rad}^{\mathrm{M(1)}} \simeq \frac{\pi \omega^4 b^4}{12 c^4} Z_o = \frac{1}{12\pi^3}\left(\frac{\omega \pi b}{c}\right)^4 Z_o}$$

However, in the "far-zone" ($d = \pi b \ll \lambda \ll r$) we have: $\boxed{\left(\frac{\omega d}{c}\right) = \left(\frac{\omega \pi b}{c}\right) \ll 1}$

Thus, we see that the radiation resistances $R_{rad}^{\mathrm{E(1),M(1)}}$ associated with E(1) electric dipole and M(1) magnetic dipole antennae in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$) are much less than the characteristic impedances $\boxed{Z_{rad}^{\mathrm{E(1),M(1)}} = Z_o = 120\pi\ \Omega \simeq 377\ \Omega}$ of these antennae:

$$\boxed{R_{rad}^{\mathrm{E(1)}} \simeq \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o \ll Z_o = 377\,\Omega} \quad \text{and:} \quad \boxed{R_{rad}^{\mathrm{M(1)}} \simeq \frac{1}{12\pi^3}\left(\frac{\omega \pi b}{c}\right)^4 Z_o \ll Z_o = 377\,\Omega}$$

Taking the ratio of these *EM* radiation resistances {in "far-zone" limit, *i.e.* $d = \pi b \ll \lambda \ll r$ } we also see that for $d = \pi b$:

$$\boxed{\left(\frac{R_{rad}^{\mathrm{M(1)}}}{R_{rad}^{\mathrm{E(1)}}}\right) = \frac{1}{\pi^2}\left(\frac{\omega d}{c}\right)^2 \ll 1} \quad \text{or:} \quad \boxed{R_{rad}^{\mathrm{M(1)}} = \frac{1}{\pi^2}\left(\frac{\omega d}{c}\right)^2 R_{rad}^{\mathrm{E(1)}} \ll R_{rad}^{\mathrm{E(1)}}}$$

i.e. the M(1) magnetic dipole *EM* radiation resistance is much less than the E(1) electric dipole *EM* radiation resistance, for "equal" strength moments, as defined by $\boxed{I_o = q_o\omega}$ and $\boxed{d = \pi b}$.

Polarization of E(1) Electric Dipole and M(1) Magnetic Dipole *EM* Radiation:

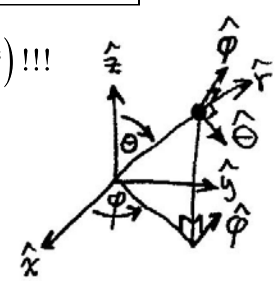
The *EM* radiation from E(1) electric dipole and/or M(1) magnetic dipole is linearly polarized in the "far-zone" limit $d = \pi b \ll \lambda \ll r$:

$$\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \simeq -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\theta}} \quad \boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \simeq +\frac{\mu_o m_o \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\varphi}}$$

But: $\boxed{k = \omega/c}$ and note that cos (x) = cos ($-x$) = even fcn (x).

$\therefore$ $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) \sim -\cos(kr - \omega t)\hat{\theta}}$ and: $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M(1)}}(\vec{r},t) \sim \cos(kr - \omega t)\hat{\varphi}}$

Note that: $\boxed{\cos(kr - \omega t)}$ is associated with ***spherical*** outgoing waves, $\left(\vec{k} = k\hat{r}\right)$!!!

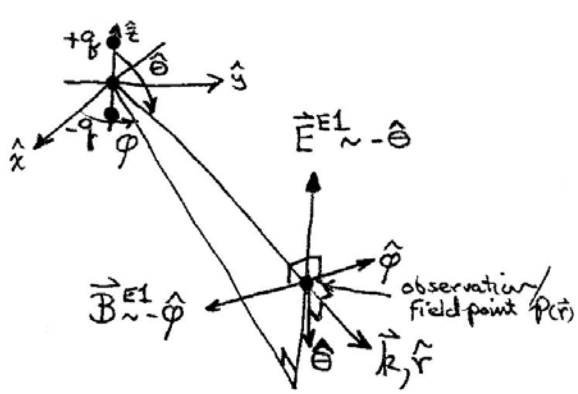


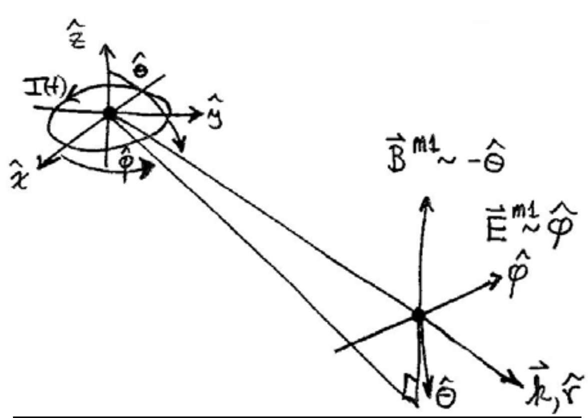
However for $r \to \infty$, spherical outgoing waves $\to$ plane outgoing waves.

If: $\hat{r}$ = propagation direction, *e.g.* $\hat{r} = \hat{z}$, then: cos ($kr - \omega t$) $\to$ cos ($kz - \omega t$).

Thus, in the "far-zone" limit, ($d = \pi b \ll \lambda \ll r$) for $r \to \infty$ (*i.e.* $r \gg \lambda$):

Polarization of E(1) Electric Dipole Radiation: $\perp$ to Polarization of M(1) Magnetic Dipole Radiation:





$\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}} \parallel -\hat{\theta} \ \{\parallel \vec{p} = q_o d\hat{z} \text{ when } \theta = 90^o\}}$ $\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{M(1)}} \parallel \hat{\varphi} \ \{\& \vec{B}_{\mathrm{r}}^{\mathrm{M(1)}} \parallel \vec{m} = m_o\hat{z} \text{ when } \theta = 90^o\}}$

$\boxed{\vec{S}_{\mathrm{r}}^{\mathrm{E(1)}} \sim -\hat{\theta} \times -\hat{\varphi} = +\hat{r}}$ $\boxed{\vec{S}_{\mathrm{r}}^{\mathrm{M(1)}} \sim \hat{\varphi} \times -\hat{\theta} = +\hat{r}}$

Recall from P435 Lecture Notes 8 on the {generalized} multipole expansion of $V(\vec{r},t)$ {and $\vec{A}(\vec{r},t)$} the order ℓ of the multipole was related to the spherical harmonic $Y_{\ell,m}(\theta,\varphi)$.

E(1) linear electric dipole and M(1) linear magnetic dipole radiation corresponds to the $\ell = 1, m = 0$ terms in the multipole expansion.

Rotating electric and magnetic dipoles (see *e.g*. Griffiths Problem 11.4, *p*. 450) correspond to $\ell = 1, m = \pm 1$ terms in the multipole expansion.

Electric and magnetic quadrupoles {of various kinds} correspond to $\ell = 2, m = \pm 2, \pm 1, 0$ terms in the multipole expansion, and so on…

The polarization of the *EM* radiation associated with each such multipole therefore depends on the ℓ & m values, and thus on the associated spherical harmonic $Y_{\ell,m}(\theta,\varphi)$, and thus can be linearly polarized (LP), or circularly polarized (RCP and/or LCP) !!!

The Time-Averaged Power Radiated by an *EM* Source:

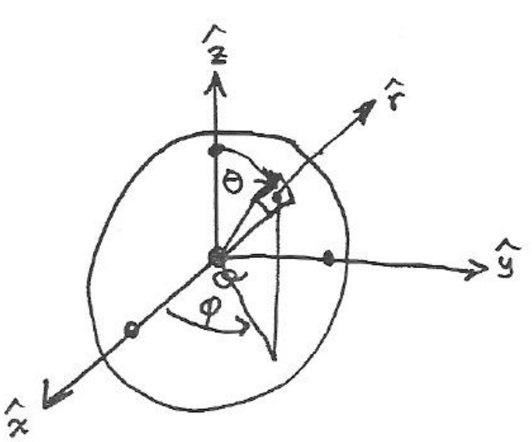
The time-averaged *EM* radiated power associated with an oscillating electric and/or magnetic multipole of order ℓ {in the "far-zone" limit $d = \pi b \ll \lambda \ll r$ }is:

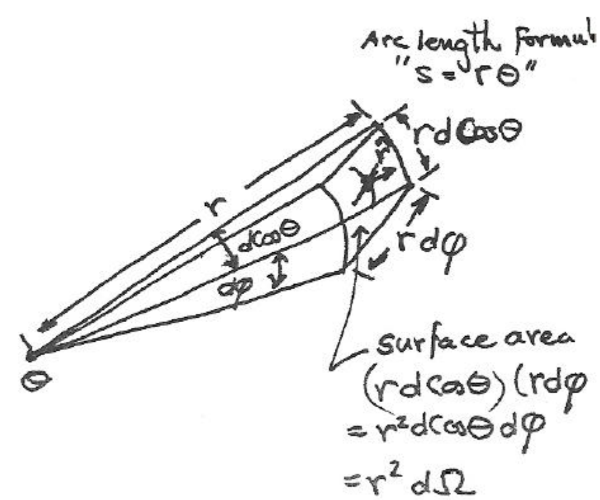
$$\boxed{\left\langle P^{rad}_{\ell-pole}(\vec{r},t)\right\rangle = \int_S \left\langle \vec{S}^{rad}_{\ell-pole}(\vec{r},t)\right\rangle \cdot d\vec{a}_\perp}$$

Where: $\boxed{d\vec{a}_\perp = r^2 \sin\theta d\theta d\varphi \hat{r} = r^2 d\cos\theta d\varphi \hat{r} = r^2 d\Omega \hat{r}}$

And where: $\boxed{d\Omega(\theta,\varphi) = d\cos\theta d\varphi = \sin\theta d\theta d\varphi}$ = solid angle (units = *steradians*)

And: $\boxed{\int d\Omega(\theta,\varphi) = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} d\cos\theta d\varphi = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \sin\theta d\theta d\varphi = 4\pi}$ (*steradians*)





<u>Since</u>: $$\boxed{\left\langle P_{\ell-pole}^{rad}\left(\vec{r},t\right)\right\rangle = \int_{\Omega}\left(\frac{d\left\langle P_{\ell-pole}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right)d\Omega = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\left(\frac{d\left\langle P_{\ell-pole}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right)d\cos\theta d\varphi} \quad (Watts)$$

We see that the <u>angular power</u> associated with an ℓ^{th}-order multipole is:

$$\boxed{\frac{d\left\langle P_{\ell-pole}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega} = I_{\ell-pole}^{rad}\left(\vec{r}\right)r^2 = \left\langle\left|\vec{S}_{\ell-pole}^{rad}\left(\vec{r},t\right)\right|\right\rangle r^2 = \left\langle\vec{S}_{\ell-pole}^{rad}\left(\vec{r},t\right)\right\rangle\bullet r^2\hat{r}} \quad \left(\frac{Watts}{steradian}\right)$$

Then, for the "far-zone" limit $\{d = \pi b \ll \lambda \ll r\}$:

$$\boxed{\frac{d\left\langle P_{E(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega} = I_{E(1)}^{rad}\left(\vec{r}\right)r^2 = \left\langle\left|\vec{S}_{E(1)}^{rad}\left(\vec{r},t\right)\right|\right\rangle r^2 \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{\cancel{r^2}}\right)\cancel{r^2} = \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\sin^2\theta} \quad \left(\frac{Watts}{steradian}\right)$$

$$\boxed{\frac{d\left\langle P_{M(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega} = I_{M(1)}^{rad}\left(\vec{r}\right)r^2 = \left\langle\left|\vec{S}_{M(1)}^{rad}\left(\vec{r},t\right)\right|\right\rangle r^2 \simeq \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{\cancel{r^2}}\right)\cancel{r^2} = \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\sin^2\theta} \quad \left(\frac{Watts}{steradian}\right)$$

<u>Again, the ratio of</u>:

$$\boxed{\left(\frac{d\left\langle P_{M(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right)\Bigg/\left(\frac{d\left\langle P_{E(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right) = \left(\frac{m_o}{p_o c}\right)^2 = \left(\frac{\pi b^2 I_o}{q_o d c}\right)^2 = \left(\frac{\cancel{\pi} b^{\cancel{2}}\cancel{q_o}\omega}{\cancel{q_o}\cancel{\pi}\cancel{b}c}\right) = \left(\frac{\omega b}{c}\right) \ll 1} \text{ for } \boxed{\begin{array}{l} I_o = q_o\omega \\ d = \pi b \end{array}}$$

in the "far-zone" limit $\{d = \pi b \ll \lambda \ll r\}$.

Thus, we see that for the same θ and φ, the {time-averaged} angular power radiated by an M(1) magnetic dipole is $\ll$ than the angular power radiated by an E(1) electric dipole in the "far-zone" limit $\{d = \pi b \ll \lambda \ll r\}$, for "equal" strength moments, as defined by $\boxed{I_o = q_o\omega}$ and $\boxed{d = \pi b}$:

$$\boxed{\left(\frac{d\left\langle P_{M(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right) = \left(\frac{\omega b}{c}\right)\left(\frac{d\left\langle P_{E(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right) \ll \left(\frac{d\left\langle P_{E(1)}^{rad}\left(\vec{r},t\right)\right\rangle}{d\Omega}\right)} \text{ for } \boxed{I_o = q_o\omega} \text{ and } \boxed{d = \pi b}$$

in the "far-zone" limit $\{d = \pi b \ll \lambda \ll r\}$.

