4.2.4 Multipole Expansion

We can expand the potential created by a confined charge distribution as

$$\phi(\vec{r}) = \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}},$$

where

$$q_{lm} = \int d^3r' \; Y^*_{lm}(\theta',\varphi') \, (r')^l \, \rho(\vec{r}\,').$$

The $l=1$ multipole describes the <u>dipole</u>

$$\vec{p} = \int d^3r' \; \vec{r}\,' \cdot \rho(\vec{r}\,').$$

For $l=2$, with

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\varphi}; \quad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \, e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

we find

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3r' \quad (x'-iy')^2 \rho(\vec{r}')$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int d^3r' \quad z'(x'-iy')\rho(\vec{r}')$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3r' \quad (3z'^2 - r'^2)\rho(\vec{r}')$$

These are just the "spherical harmonic" components of the <u>quadrupole tensor</u>

$$Q_{ij} = \int d^3r' \quad (3x_i' x_j' - r'^2 \delta_{ij})\rho(\vec{r}') \qquad [i,j=1,2,3]$$

which is <u>symmetric</u> and <u>traceless</u> and thus transforms in the $l=2$ representation of $SO(3)$:

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{11} - 2i\,Q_{12} - Q_{22}); \quad q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} (Q_{13} - i\,Q_{23})$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}.$$

Exercise 14

i) Show that

$$\phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2} Q_{ij} \frac{r^i r^j}{r^5} + \dots$$

by expanding $G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}$ in $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

in a Taylor series around the origin $\vec{r}\,' = 0$.

ii) Show that $\phi(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{r^3}$ is the field created by two charges q and $-q$ separated by $\vec{L}$ in the limit $\vec{L} \to 0$ with $\vec{p} = q\vec{L} = \text{const}$.

iii) Calculate the electric field created by such a dipole

■

2.3.2. Forces and Torques

We can use multipole expansions to calculate potentials, forces and torques of charge distributions in external fields:

- Consider for instance the potential energy:

$$U = \int d^3r \;\; \rho(\vec{r})\, \phi(\vec{r}).$$

If charge is localized around $\vec{r}=0$, expand

$$\phi(\vec{r}) = \phi(0) + \vec{r}\cdot\vec{\nabla}\phi\big|_0 + \frac{1}{2}\, r^i r^j \frac{\partial^2 \phi}{\partial r^i \partial r^j}\Big|_0 + \cdots$$

Therefore, $\quad U = Q\,\phi(0) - \vec{p}\cdot\vec{E}\big|_0 + \cdots$

- The force acting on a charge distribution is

$$\vec{F} = \int d^3r \;\; \rho(\vec{r})\, E(\vec{r}).$$

As before, we find

$$\vec{F} = Q\,\vec{E}\big|_0 + (\vec{p}\cdot\vec{\nabla})\vec{E}\big|_0 + \cdots$$

- The torque acting on a charge distribution is

$$\vec{\tau} = \int d^3r \;\; \vec{r}\times(\rho\,\vec{E}(\vec{r}))$$

Expanding $\quad \vec{E} = \vec{E}(0) + \cdots$

$$\vec{\tau} = \vec{p}\times\vec{E}\big|_0 + \cdots$$

Addendum: Complex Variables & Conformal Maps

In two-dimensional problems, or problems with translation symmetry along one spatial direction, there is a very powerful technique to solve Laplace's equation, which relies on complex variables and analytic functions.

In two-dimensions, we can use $z = x + iy$ as a (complex) coordinate on the plane.

Any <u>analytic function</u> $f(z) = u(x,y) + i v(x,y)$ satisfies the Cauchy-Riemann equations:

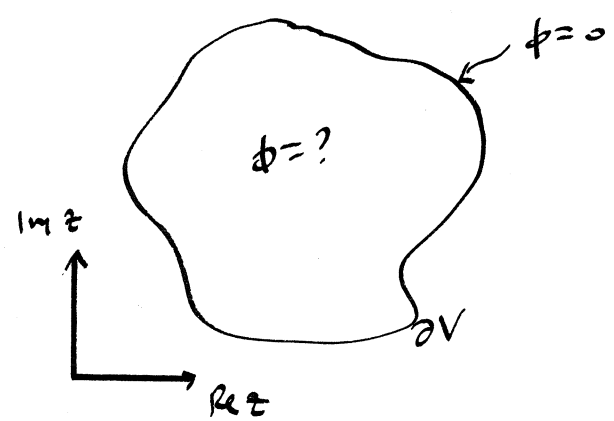
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

These then imply that both $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation:

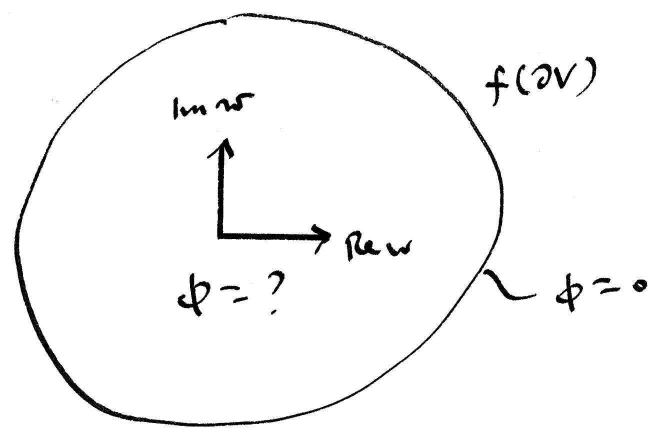
$$\begin{cases} \nabla^2 u \doteq \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ \nabla^2 v \doteq \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \end{cases}$$

Therefore, any analytic function "contains" two solutions of Laplace's eq.

Suppose now we need to solve a "complicated" Dirichlet problem in 2 dim:



Imagine now that we find an analytic function $w=f(z)$ that maps the complicated geometry into a "simple" geometry that we can solve:



Let us suppose we have solved the dirichlet problem in the w plane. Then, the solution can be written as the real part of an analytic function $F(w)$, with $\operatorname{Re} F(w) = 0$ if $w \in f(\partial V)$ (the boundary).

Consider now the "pull-back" of $F(w)$ to the z-plane: $\tilde{F}(z) = F(f(z))$.

Since F and f are analytic, so is $\tilde{F}$. Hence, its real part satisfies Laplace's equation. Moreover,

for $z \in \partial V$, $\operatorname{Re} \tilde{F}(z) = \operatorname{Re} F(f(z)) = \operatorname{Re} F(w) = 0$
since $w \in f(\partial V)$.

Thus, <u>$\operatorname{Re} \tilde{F}(z)$ is the solution to our "complicated" problem.</u> (we can also use this technique to generate "complicated" solutions from "simple" ones)

Note: The analytic function $f(z)$ that maps the geometry in the z-plane to the geometry in the w-plane defines a <u>conformal map</u>: a map that preserves angles and (local) proportions.

As we shall see later on, electromagnetism is a <u>conformally invariant</u> theory (classically), in part because the photon is <u>massless</u> [this is why we need to solve $\nabla^2 \phi = 0$, as opposed to $(\nabla^2 - m^2)\phi = 0$].

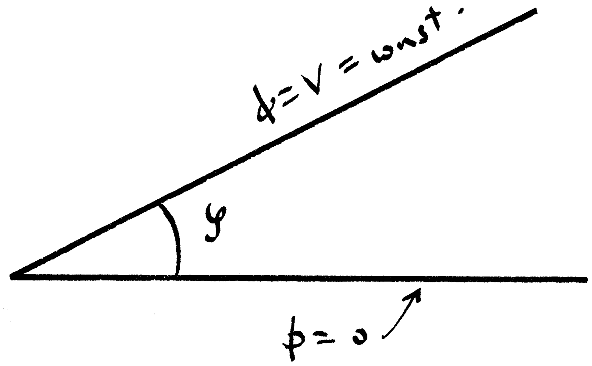
The set of coordinate transformations that preserves local angles and proportions is known as the <u>conformal group</u>. In $d \geq 3$ dimensions this group has a finite number of generators:

- translations, rotations
- dilatations
- special conformal transformations.

In $d=2$ this group has an infinite number of generators: it consists of all analytic functions, as we have seen.

Exercise 15

Find the potential in the following Dirichlet problem:



Hint: Consider the effects of the map

$$w = \log(z + V)$$

∎