7 The multipole expansion

7.1 Multipole expansion of the scalar wave equation

Consider the emission and scattering of electromagnetic radiation. This type of problem involves solving the vector wave equation. The solutions of this equation in free space are conveniently written as an expansion in orthogonal spherical waves. This expansion is known as the *multipole expansion*. Let us examine this expansion in more detail.

Before considering the vector wave equation, let us consider the somewhat simpler scalar wave equation. A scalar field $\psi(\boldsymbol{r}, t)$ satisfying the homogeneous wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{7.1}$$

can be Fourier analyzed in time

$$\psi(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} \psi(\boldsymbol{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\,\omega t}\, d\omega \tag{7.2}$$

with each Fourier harmonic satisfying the Helmholtz wave equation

$$(\nabla^2 + k^2)\,\psi(\boldsymbol{r}, \omega) = 0, \tag{7.3}$$

where $k^2 = \omega^2/c^2$. We can write the Helmholtz equation in terms of spherical polar coordinates $(r,\, \theta,\, \varphi)$:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\,\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + k^2\right]\psi = 0. \tag{7.4}$$

As is well known, it is possible to solve this equation via the separation of variables:

$$\psi(\boldsymbol{r}, \omega) = \sum_{l,m} f_{lm}(r)\, Y_{lm}(\theta, \varphi). \tag{7.5}$$

Here, we restrict our attention to physical solutions which are well behaved in the angular variables θ and φ. The spherical harmonics $Y_{lm}(\theta, \varphi)$ satisfy the following

equations:

$$-\frac{\partial^2 Y_{lm}}{\partial\varphi^2} = m^2\, Y_{lm}, \tag{7.6a}$$

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{lm} = l(l+1)\,Y_{lm}, \tag{7.6b}$$

where l is a non-negative integer, and m is an integer which satisfies the inequality $|m|\leq l$. The radial functions $f_{lm}(r)$ satisfy

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]f_l(r)=0, \tag{7.7}$$

where there is no dependence on m. With the substitution

$$f_l(r)=\frac{u_l(r)}{r^{1/2}}, \tag{7.8}$$

Eq. (7.7) is transformed into

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(l+1/2)^2}{r^2}\right]u_l(r)=0. \tag{7.9}$$

It can be seen, by comparison with Eq. (5.39), that this is a type of Bessel's equation of half-integer order $l+1/2$. Thus, we can write the solution for $f_{lm}(r)$ as

$$f_{lm}(r)=\frac{A_{lm}}{r^{1/2}}\,J_{l+1/2}(kr)+\frac{B_{lm}}{r^{1/2}}\,Y_{l+1/2}(kr), \tag{7.10}$$

where A_{lm} and B_{lm} are arbitrary constants. The half-integer order Bessel functions $J_{l+1/2}(z)$ and $Y_{l+1/2}(z)$ have analogous properties to the integer order Bessel functions $J_m(z)$ and $Y_m(z)$. In particular, the $J_{l+1/2}(z)$ are well behaved in the limit $|z|\rightarrow 0$, whereas the $Y_{l+1/2}(z)$ are badly behaved. The asymptotic expansions (5.43) remain valid when $m\rightarrow l+1/2$.

It is convenient to define the *spherical Bessel functions* $j_l(r)$ and $y_l(r)$, where

$$j_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{l+1/2}(z), \tag{7.11a}$$

$$y_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} Y_{l+1/2}(z). \tag{7.11b}$$

It is also convenient to define the spherical Hankel functions

$$h_l^{(1,2)}(z) = j_l(z) \pm \mathrm{i}\, y_l(z). \tag{7.12}$$

For real z, $h_l^{(2)}(z)$ is the complex conjugate of $h_l^{(1)}(z)$. It turns out that the spherical Bessel functions can be expressed in the closed form

$$j_l(z) \;=\; (-z)^l \left(\frac{1}{z}\frac{d}{dz}\right)^l \left(\frac{\sin z}{z}\right), \tag{7.13a}$$

$$y_l(z) \;=\; -(-z)^l \left(\frac{1}{z}\frac{d}{dz}\right)^l \left(\frac{\cos z}{z}\right). \tag{7.13b}$$

In the limit of small argument

$$j_l(z) \;\to\; \frac{z^l}{(2l+1)!!}\left[1 + O(z^2)\right], \tag{7.14a}$$

$$y_l(z) \;\to\; -\frac{(2l-1)!!}{z^{l+1}}\left[1 + O(z^2)\right], \tag{7.14b}$$

where $(2l+1)!! = (2l+1)(2l-1)(2l-3)\cdots 5\cdot 3\cdot 1$. In the limit of large argument

$$j_l(z) \;\to\; \frac{\sin(z - l\pi/2)}{z}, \tag{7.15a}$$

$$y_l(z) \;\to\; -\frac{\cos(z - l\pi/2)}{z}, \tag{7.15b}$$

and

$$h_l^{(1)} \to (-\mathrm{i})^{l+1}\,\frac{\mathrm{e}^{\mathrm{i}\,z}}{z}. \tag{7.16}$$

The inhomogeneous Helmholtz equation is conveniently solved using the Green's function $G_\omega(\boldsymbol{r}, \boldsymbol{r}')$, which satisfies (see Eq. (2.109))

$$(\nabla^2 + k^2)\, G_\omega(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{7.17}$$

The solution of this equation, subject to the *Sommerfeld radiation condition*, which ensures that sources radiate waves instead of absorbing them, is written

(see Section 2.13)

$$G_\omega(\boldsymbol{r},\boldsymbol{r}') = \frac{\mathrm{e}^{\,\mathrm{i}\,k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{7.18}$$

The spherical harmonics satisfy the completeness relation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta'). \tag{7.19}$$

Now the three dimensional delta function can be written

$$\delta(\boldsymbol{r}-\boldsymbol{r}') = \frac{\delta(r-r')}{r^2}\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta'). \tag{7.20}$$

It follows that

$$\delta(\boldsymbol{r}-\boldsymbol{r}') = \frac{\delta(r-r')}{r^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi). \tag{7.21}$$

Let us expand the Green's function in the form

$$G_\omega(\boldsymbol{r},\boldsymbol{r}') = \sum_{l,m} g_l(r,r')\,Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi). \tag{7.22}$$

Substitution of this expression into Eq. (7.17) yields

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]g_l = -\frac{\delta(r-r')}{r^2}. \tag{7.23}$$

The appropriate boundary conditions are that g_l is finite at the origin and corresponds to an *outgoing* wave at infinity (*i.e.*, $g \propto \mathrm{e}^{\,\mathrm{i}\,kr}$ in the limit $r\rightarrow\infty$). The solution of the above equation which satisfies these boundary conditions is

$$g_l(r,r') = A\,j_l(kr_<)\,h_l^{(1)}(kr_>), \tag{7.24}$$

where $r_<$ and $r_>$ are the greater and the lesser of r and r', respectively. The correct discontinuity in slope at $r=r'$ is assured if $A=\mathrm{i}\,k$, since

$$\frac{dh_l^{(1)}(z)}{dz}\,j_l(z) - h_l^{(1)}(z)\,\frac{dj_l(z)}{dz} = \frac{\mathrm{i}}{z^2}. \tag{7.25}$$

Thus, the expansion of the Green's function is

$$\frac{\mathrm{e}^{\,\mathrm{i}\,k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} = \mathrm{i}\,k\sum_{l=0}^{\infty} j_l(kr_<)\,h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi). \tag{7.26}$$

This is a particularly useful result, as we shall discover, since it easily allows us to express the general solution of the inhomogeneous wave equation as a multipole expansion.

It is well known in quantum mechanics that Eq. (7.6b) can be written in the form

$$L^2\,Y_{lm} = l(l+1)\,Y_{lm}. \tag{7.27}$$

The differential operator L^2 is given by

$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2, \tag{7.28}$$

where

$$\boldsymbol{L} = -\mathrm{i}\,\boldsymbol{r}\wedge\nabla \tag{7.29}$$

is $1/\hbar$ times the orbital angular momentum operator of wave mechanics.

The components of $\boldsymbol{L}$ can be conveniently written in the combinations

$$L_+ = L_x + \mathrm{i}\,L_y = \mathrm{e}^{\,\mathrm{i}\,\varphi}\left(\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\,\frac{\partial}{\partial\varphi}\right), \tag{7.30a}$$

$$L_- = L_x - \mathrm{i}\,L_y = \mathrm{e}^{-\mathrm{i}\,\varphi}\left(-\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\,\frac{\partial}{\partial\varphi}\right), \tag{7.30b}$$

$$L_z = -\mathrm{i}\,\frac{\partial}{\partial\varphi}. \tag{7.30c}$$

We note that $\boldsymbol{L}$ operates only on angular variables and is independent of r. From the definition (7.29) it is evident that

$$\boldsymbol{r}\cdot\boldsymbol{L} = 0 \tag{7.31}$$

holds as an operator equation. It is easily demonstrated from Eqs. (7.30) that

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{7.32}$$

The following results are well known in quantum mechanics:

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)}\, Y_{l,m+1}, \tag{7.33a}$$

$$L_- Y_{lm} = \sqrt{(l+m)(l-m+1)}\, Y_{l,m-1}, \tag{7.33b}$$

$$L_z Y_{lm} = m\, Y_{lm}. \tag{7.33c}$$

In addition,

$$L^2 \boldsymbol{L} = \boldsymbol{L}\, L^2, \tag{7.34a}$$

$$\boldsymbol{L} \wedge \boldsymbol{L} = \mathrm{i}\, \boldsymbol{L}, \tag{7.34b}$$

$$L_j \nabla^2 = \nabla^2 L_j, \tag{7.34c}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\, r^2 \frac{\partial}{\partial r} - \frac{L^2}{r^2}. \tag{7.35}$$

7.2 Multipole expansion of the vector wave equation

Maxwell's equations in free space reduce to

$$\nabla\cdot\boldsymbol{E} = 0, \tag{7.36a}$$

$$\nabla\cdot c\boldsymbol{B} = 0, \tag{7.36b}$$

$$\nabla\wedge\boldsymbol{E} = \mathrm{i}\, k\, c\boldsymbol{B}, \tag{7.36c}$$

$$\nabla\wedge c\boldsymbol{B} = -\mathrm{i}\, k\, \boldsymbol{E}, \tag{7.36d}$$

assuming an $\mathrm{e}^{-\mathrm{i}\,\omega t}$ time dependence of all field quantities. Here, $k = \omega/c$. Eliminating $\boldsymbol{E}$ between Eqs. (7.36c) and (7.36d), we obtain the following equations for $\boldsymbol{B}$:

$$(\nabla^2 + k^2)\boldsymbol{B} = 0, \tag{7.37a}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{7.37b}$$

with $\boldsymbol{E}$ given by

$$\boldsymbol{E} = \frac{\mathrm{i}}{k}\nabla\wedge c\boldsymbol{B}. \tag{7.38}$$

Alternatively, $\boldsymbol{B}$ can be eliminated to give

$$(\nabla^2 + k^2)\boldsymbol{E} = 0, \tag{7.39a}$$

$$\nabla\cdot\boldsymbol{E} = 0, \tag{7.39b}$$

with $\boldsymbol{B}$ given by

$$c\boldsymbol{B} = -\frac{\mathrm{i}}{k}\nabla\wedge\boldsymbol{E}. \tag{7.40}$$

It is clear that each Cartesian component of $\boldsymbol{B}$ and $\boldsymbol{E}$ satisfies the Helmholtz wave equation (7.3). Hence, these components can be written in a general expansion of the form

$$\psi(\boldsymbol{r}) = \sum_{l,m}\left[A_{lm}^{(1)}\,h_l^{(1)}(kr) + A_{lm}^{(2)}\,h_l^{(2)}(kr)\right]Y_{lm}(\theta,\varphi), \tag{7.41}$$

where ψ stands for any Cartesian component of $\boldsymbol{E}$ or $c\boldsymbol{B}$. Note, however, that the three Cartesian components of $\boldsymbol{E}$ or $\boldsymbol{B}$ are not entirely independent, since they must also satisfy the constraints $\nabla\cdot\boldsymbol{E} = 0$ and $\nabla\cdot\boldsymbol{B} = 0$. Let us examine how these constraints can be satisfied with the minimum labour.

Consider the scalar $\boldsymbol{r}\cdot\boldsymbol{A}$, where $\boldsymbol{A}$ is a well behaved vector field. It is easily verified that

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A}) = \boldsymbol{r}\cdot(\nabla^2\boldsymbol{A}) + 2\nabla\cdot\boldsymbol{A}. \tag{7.42}$$

It follows from Eqs. (7.37) and (7.39) that the scalars $\boldsymbol{r}\cdot\boldsymbol{E}$ and $\boldsymbol{r}\cdot\boldsymbol{B}$ both satisfy the Helmholtz wave equation:

$$(\nabla^2 + k^2)(\boldsymbol{r}\cdot\boldsymbol{E}) = 0, \tag{7.43a}$$

$$(\nabla^2 + k^2)(\boldsymbol{r}\cdot\boldsymbol{B}) = 0. \tag{7.43b}$$

Thus, the general solutions for $\boldsymbol{r}\cdot\boldsymbol{E}$ and $\boldsymbol{r}\cdot c\boldsymbol{B}$ can be written in the form (7.41).

Let us define a *magnetic multipole* field of order (l,m) by the conditions

$$\boldsymbol{r}\cdot c\boldsymbol{B}_{lm}^{(M)} = \frac{l(l+1)}{k}\,g_l(kr)\,Y_{lm}(\theta,\varphi), \tag{7.44a}$$

$$\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(M)} = 0, \tag{7.44b}$$

where

$$g_l(kr) = A_l^{(1)}\, h_l^{(1)}(kr) + A_l^{(2)}\, h_l^{(2)}(kr). \tag{7.45}$$

The presence of the factor $l(l+1)/k$ is for later convenience. Equation (7.40) yields

$$k\,\boldsymbol{r}\cdot c\boldsymbol{B} = -\mathrm{i}\,\boldsymbol{r}\cdot(\nabla\wedge\boldsymbol{E}) = -\mathrm{i}\,(\boldsymbol{r}\wedge\nabla)\cdot\boldsymbol{E} = \boldsymbol{L}\cdot\boldsymbol{E}, \tag{7.46}$$

where $\boldsymbol{L}$ is given by Eq. (7.29). With $\boldsymbol{r}\cdot\boldsymbol{B}$ given by Eq. (7.44a), the electric field associated with a magnetic multipole must satisfy

$$\boldsymbol{L}\cdot\boldsymbol{E}_{lm}^{(M)}(r,\theta,\varphi) = l(l+1)\, g_l(kr)\, Y_{lm}(\theta,\varphi) \tag{7.47}$$

and $\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(M)} = 0$. Note that the operator $\boldsymbol{L}$ acts only on the angular variables (θ,φ). This means that the radial dependence of $\boldsymbol{E}_{lm}^{(M)}$ must be given by $g_l(kr)$. Note also, from Eqs. (7.33), that the operator $\boldsymbol{L}$ acting on Y_{lm} transforms the m value but does not change the l value. It is easily seen from Eqs. (7.27) and (7.31) that the solution to Eqs. (7.44b) and (7.47) can be written in the form

$$\boldsymbol{E}_{lm}^{(M)} = g_l(kr)\, \boldsymbol{L}\, Y_{lm}(\theta,\varphi). \tag{7.48}$$

Thus, the angular dependence of $\boldsymbol{E}_{lm}^{(M)}$ consists of some linear combination of $Y_{l,m-1}$, Y_{lm}, and $Y_{l,m+1}$. Equation (7.48), together with

$$c\boldsymbol{B}_{lm}^{(M)} = -\frac{\mathrm{i}}{k}\,\nabla\wedge\boldsymbol{E}_{lm}^{(M)}, \tag{7.49}$$

specifies the electromagnetic fields of a *magnetic* multipole of order (l,m). Note from Eq. (7.31) that the electric field given by Eq. (7.48) is transverse to the radius vector. Thus, magnetic multipole fields are sometimes termed *transverse electric* (TE) multipole fields.

The fields of an *electric* or *transverse magnetic* (TM) multipole of order (l,m) are specified by the conditions

$$\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(E)} = -\frac{l(l+1)}{k}\, f_l(kr)\, Y_{lm}(\theta,\varphi), \tag{7.50a}$$

$$\boldsymbol{r}\cdot\boldsymbol{B}_{lm}^{(E)} = 0. \tag{7.50b}$$

It follows that the fields of an electric multipole are given by

$$c\boldsymbol{B}_{lm}^{(E)} = f_l(kr)\,\boldsymbol{L}\,Y_{lm}(\theta,\varphi), \tag{7.51a}$$

$$\boldsymbol{E}_{lm}^{(E)} = \frac{\mathrm{i}}{k}\nabla\wedge c\boldsymbol{B}_{lm}^{(E)}. \tag{7.51b}$$

The radial function $f_l(kr)$ is given by an expression like (7.45).

The two sets of multipole fields (7.48), (7.49), and (7.51), form a complete set of vector solutions to Maxwell's equations in free space. Since the vector spherical harmonic $\boldsymbol{L}\,Y_{lm}$ plays an important role in multipole fields, it is convenient to introduce the normalized form

$$\boldsymbol{X}_{lm}(\theta,\varphi) = \frac{1}{\sqrt{l(l+1)}}\,\boldsymbol{L}\,Y_{lm}(\theta,\varphi). \tag{7.52}$$

It can be demonstrated that the vector spherical harmonics possess the orthogonality properties

$$\int \boldsymbol{X}_{l'm'}^{*}\cdot\boldsymbol{X}_{lm}\,d\Omega = \delta_{ll'}\,\delta_{mm'}, \tag{7.53a}$$

$$\int \boldsymbol{X}_{l'm'}^{*}\cdot(\boldsymbol{r}\wedge\boldsymbol{X}_{lm})\,d\Omega = 0, \tag{7.53b}$$

for all l, l', m, and m'.

By combining the two types of fields we can write the general solution to Maxwell's equations in free space in the form

$$c\boldsymbol{B} = \sum_{l,m}\left[a_E(l,m)\,f_l(kr)\,\boldsymbol{X}_{lm} - \frac{\mathrm{i}}{k}\,a_M(l,m)\,\nabla\wedge g_l(kr)\boldsymbol{X}_{lm}\right], \tag{7.54a}$$

$$\boldsymbol{E} = \sum_{l,m}\left[\frac{\mathrm{i}}{k}\,a_E(l,m)\,\nabla\wedge f_l(kr)\boldsymbol{X}_{lm} + a_M(l,m)\,g_l(kr)\,\boldsymbol{X}_{lm}\right], \tag{7.54b}$$

where the coefficients $a_E(l,m)$ and $a_M(l,m)$ specify the amounts of electric (l,m) and magnetic (l,m) multipole fields. The radial functions $f_l(kr)$ and $g_l(kr)$ are of the form (7.45). The coefficients $a_E(l,m)$ and $a_M(l,m)$, as well as the relative proportions in (7.45), are determined by the sources and the boundary conditions.

Equations (7.54) yield

$$\begin{aligned} \boldsymbol{r}\cdot c\boldsymbol{B} &= \frac{1}{k}\sum_{l,m} a_M(l,m)\, g_l(kr)\, \boldsymbol{L}\,\boldsymbol{X}_{lm} \\ &= \frac{1}{k}\sum_{l,m} a_M(l,m)\, g_l(kr)\sqrt{l(l+1)}\, Y_{lm}, \end{aligned} \tag{7.55}$$

and

$$\begin{aligned} \boldsymbol{r}\cdot \boldsymbol{E} &= -\frac{1}{k}\sum_{l,m} a_E(l,m)\, f_l(kr)\, \boldsymbol{L}\,\boldsymbol{X}_{lm} \\ &= -\frac{1}{k}\sum_{l,m} a_E(l,m)\, f_l(kr)\sqrt{l(l+1)}\, Y_{lm}, \end{aligned} \tag{7.56}$$

where use has been made of Eqs. (7.27), (7.29), and (7.31). It follows from the well known orthogonality property of the spherical harmonics that

$$a_M(l,m)\, g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*\, \boldsymbol{r}\cdot c\boldsymbol{B}\, d\Omega, \tag{7.57a}$$

$$a_E(l,m)\, f_l(kr) = -\frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*\, \boldsymbol{r}\cdot \boldsymbol{E}\, d\Omega. \tag{7.57b}$$

Thus, knowledge of $\boldsymbol{r}\cdot\boldsymbol{B}$ and $\boldsymbol{r}\cdot\boldsymbol{E}$ at two different radii in a source free region permits a complete specification of the fields, including the relative proportions of $h_l^{(1)}$ and $h_l^{(2)}$ in f_l and g_l.

7.3 Properties of multipole fields

Let us examine some of the properties of the multipole fields (7.48), (7.49), and (7.51). Consider, first of all, the so-called *near zone*, for which $kr \ll 1$. In this region $f_l(kr)$ is proportional to $y_l(kr)$, given by the asymptotic expansion (7.14b), unless its coefficient vanishes identically. Excluding this possibility, the limiting behaviour of the magnetic field for an electric (l, m) multipole is

$$c\boldsymbol{B}_{lm}^{(E)} \to -\frac{k}{l}\,\boldsymbol{L}\,\frac{Y_{lm}}{r^{l+1}}, \tag{7.58}$$

where the proportionality coefficient is chosen for later convenience. To find the electric field we must take the curl of the right-hand side. The following operator identity is useful

$$\mathrm{i}\,\nabla\wedge\boldsymbol{L} = \boldsymbol{r}\,\nabla^2 - \nabla\left(1 + r\frac{\partial}{\partial r}\right). \tag{7.59}$$

The electric field (7.51b) is

$$\boldsymbol{E}_{lm}^{(E)} \to \frac{-\mathrm{i}}{l}\,\nabla\wedge\boldsymbol{L}\left(\frac{Y_{lm}}{r^{l+1}}\right). \tag{7.60}$$

Since Y_{lm}/r^{l+1} is a solution of Laplace's equation, the first term in (7.59) vanishes. Consequently, the electric field at close distances for an electric (l, m) multipole is

$$\boldsymbol{E}_{lm}^{(E)} \to -\nabla\left(\frac{Y_{lm}}{r^{l+1}}\right). \tag{7.61}$$

This, of course, is an electrostatic multipole field. Such a field is obtained in a more straightforward manner by observing that $\boldsymbol{E} \to -\nabla\phi$, where $\nabla^2\phi = 0$, in the near zone. Solving Laplace's equation by separation of variables in spherical polar coordinates, and demanding that ϕ be well behaved as $|\boldsymbol{r}| \to \infty$, yields

$$\phi(r,\theta,\varphi) = \sum_{l,m}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}. \tag{7.62}$$

Note that the magnetic field (7.58) (normalized with respect to c^{-1}) is smaller than the electric field (7.61) by a factor of order kr. Thus, in the near zone

the magnetic field associated with an electric multipole is always much smaller than the corresponding electric field. For magnetic multipole fields it is evident from Eqs. (7.48), (7.49), and (7.51) that the roles of $\boldsymbol{E}$ and $\boldsymbol{B}$ are interchanged according to the transformation

$$\boldsymbol{E}^{(E)} \rightarrow -c\boldsymbol{B}^{(M)}, \tag{7.63a}$$

$$c\boldsymbol{B}^{(E)} \rightarrow \boldsymbol{E}^{(M)}. \tag{7.63b}$$

In the so-called *far zone* or *radiation zone*, for which $kr \gg 1$, the multipole fields depend on the boundary conditions imposed at infinity. For definiteness, let us consider the case of outgoing waves at infinity, which is appropriate to radiation by a localized source. For this case, the radial function $f_l(kr)$ is proportional to the spherical Hankel function $h_l^{(1)}(kr)$. From the asymptotic form (7.16), it is clear that in the radiation zone the magnetic field of an electric (l, m) multipole goes as

$$c\boldsymbol{B}_{lm}^{(E)} \rightarrow (-\mathrm{i})^{l+1} \frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{kr} \boldsymbol{L}\, Y_{lm}. \tag{7.64}$$

Using Eq. (7.51b), the electric field can be written

$$\boldsymbol{E}_{lm}^{(E)} = \frac{(-\mathrm{i})^l}{k^2} \left[\nabla\left(\frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{r}\right) \wedge \boldsymbol{L}\, Y_{lm} + \frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{r} \nabla \wedge \boldsymbol{L}\, Y_{lm}\right]. \tag{7.65}$$

Neglecting terms which fall off faster than $1/r$, the above expression reduces to

$$\boldsymbol{E}_{lm}^{(E)} = -(-\mathrm{i})^{l+1} \frac{\mathrm{e}^{\,\mathrm{i}\,kr}}{kr} \left[\boldsymbol{n} \wedge \boldsymbol{L}\, Y_{lm} - \frac{1}{k}(\boldsymbol{r}\nabla^2 - \nabla) Y_{lm}\right], \tag{7.66}$$

where use has been made of the identity (7.59), and $\boldsymbol{n} = \boldsymbol{r}/r$ is a unit vector pointing in the radial direction. The second term in square brackets is smaller than the first term by a factor of order $1/kr$, and can therefore be neglected in the limit $kr \gg 1$. Thus, we find that the electric field in the radiation zone is given by

$$\boldsymbol{E}_{lm}^{(E)} = c\boldsymbol{B}_{lm}^{(E)} \wedge \boldsymbol{n}, \tag{7.67}$$

where $\boldsymbol{B}_{lm}^{(E)}$ is given by Eq. (7.64). These fields are typical radiation fields; *i.e.*, they are transverse to the radius vector, mutually orthogonal, and fall off like $1/r$. For magnetic multipoles we merely make the transformation (7.63).

Consider a linear superposition of electric (l,m) multipoles with different m values, but all possessing a common l value. It follows from Eqs. (7.54) that

$$c\boldsymbol{B}_l = \sum_l a_E(l,m)\,\boldsymbol{X}_{lm}\,h_l^{(1)}(kr)\,\mathrm{e}^{-\mathrm{i}\,\omega t}, \tag{7.68a}$$

$$\boldsymbol{E}_l = \frac{\mathrm{i}}{k}\nabla\wedge c\boldsymbol{B}_l. \tag{7.68b}$$

For harmonically varying fields the time averaged energy density is given by

$$u = \frac{\epsilon_0}{4}\,(\boldsymbol{E}\cdot\boldsymbol{E}^* + c\boldsymbol{B}\cdot c\boldsymbol{B}^*). \tag{7.69}$$

In the radiation zone the two terms are equal. It follows that the energy density contained in a spherical shell between radii r and $r + dr$ is

$$dU = \frac{\epsilon_0\,dr}{2k^2}\sum_{m,m'} a_E^*(l,m')\,a_E(l,m)\int \boldsymbol{X}_{lm'}^*\cdot\boldsymbol{X}_{lm}\,d\Omega, \tag{7.70}$$

where the asymptotic form (7.16) of the spherical Hankel function has been used. Making use of the orthogonality relation (7.53a), we obtain

$$\frac{dU}{dr} = \frac{\epsilon_0}{2k^2}\sum_m |a_E(l,m)|^2, \tag{7.71}$$

which is clearly independent of the radius. For a general superposition of electric and magnetic multipoles the sum over m becomes a sum over l and m, and $|a_E|^2$ becomes $|a_E|^2 + |a_M|^2$. Thus, the total energy in a spherical shell in the radiation zone is an *incoherent sum* over all multipoles.

The time averaged angular momentum density of harmonically varying electromagnetic fields is given by

$$\boldsymbol{m} = \frac{\epsilon_0}{2}\,\mathrm{Re}\,[\boldsymbol{r}\wedge(\boldsymbol{E}\wedge\boldsymbol{B}^*)]. \tag{7.72}$$

For a superposition of electric multipoles the triple product can be expanded and the electric field (7.68b) substituted, to give

$$\boldsymbol{m} = \frac{\epsilon_0 c}{2k}\,\mathrm{Re}\,[\boldsymbol{B}^*(\boldsymbol{L}\cdot\boldsymbol{B})]. \tag{7.73}$$

Thus, the angular momentum in a spherical shell lying between radii r and $r+dr$ in the radiation zone is

$$d\boldsymbol{M} = \frac{\epsilon_0 c\,dr}{2k^3}\,\mathrm{Re}\sum_{m,m'} a_E^*(l,m')\,a_E(l,m)\int(\boldsymbol{L}\cdot\boldsymbol{X}_{lm'})^*\,\boldsymbol{X}_{lm}\,d\Omega. \tag{7.74}$$

It follows from Eqs. (7.27) and (7.52) that

$$\frac{d\boldsymbol{M}}{dr} = \frac{\epsilon_0 c}{2k^3}\,\mathrm{Re}\sum_{m,m'} a_E^*(l,m')\,a_E(l,m)\int Y_{lm'}^*\,\boldsymbol{L}\,Y_{lm}\,d\Omega. \tag{7.75}$$

According to Eqs. (7.33), the Cartesian components of $d\boldsymbol{M}/dr$ can be written:

$$\begin{aligned}
\frac{dM_x}{dr} &= \frac{\epsilon_0 c}{4k^3}\,\mathrm{Re}\sum_m\Big[\sqrt{(l-m)(l+m+1)}\,a_E^*(l,m+1) \\
&\quad +\sqrt{(l+m)(l-m+1)}\,a_E^*(l,m-1)\Big]\,a_E(l,m), && (7.76\text{a})\\
\frac{dM_y}{dr} &= \frac{\epsilon_0 c}{4k^3}\,\mathrm{Im}\sum_m\Big[\sqrt{(l-m)(l+m+1)}\,a_E^*(l,m+1) \\
&\quad -\sqrt{(l+m)(l-m+1)}\,a_E^*(l,m-1)\Big]\,a_E(l,m), && (7.76\text{b})\\
\frac{dM_z}{dr} &= \frac{\epsilon_0 c}{2k^3}\sum_m m\,|a_E(l,m)|^2. && (7.76\text{c})
\end{aligned}$$

Thus, for a general lth order electric multipole that consists of a superposition of different m values, only the z component of the angular momentum takes a relatively simple form.

7.4 Sources of multipole radiation

Let us now examine the connection between multipole fields and their sources. Suppose that there exist localized distributions of electric change $\rho(\boldsymbol{r}, t)$, true current $\boldsymbol{j}(\boldsymbol{r}, t)$, and magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$. We assume that the time dependence can be analyzed into its Fourier components, and we therefore only consider harmonically varying sources, $\rho(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\,\omega t}$, $\boldsymbol{j}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\,\omega t}$, and $\boldsymbol{M}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\,\omega t}$, where it is understood that we take the real parts of complex quantities.

Maxwell's equations can be written

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{7.77a}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{7.77b}$$

$$\nabla\wedge\boldsymbol{E} - \mathrm{i}\,k\,c\boldsymbol{B} = 0, \tag{7.77c}$$

$$\nabla\wedge c\boldsymbol{B} + \mathrm{i}\,k\,\boldsymbol{E} = \mu_0 c\,(\boldsymbol{j} + \nabla\wedge\boldsymbol{M}), \tag{7.77d}$$

with the continuity equation

$$\mathrm{i}\,\omega\rho = \nabla\cdot\boldsymbol{j}. \tag{7.78}$$

It is convenient to deal with divergenceless fields. Thus, we use as the field variables, $\boldsymbol{B}$ and

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{\mathrm{i}}{\epsilon_0\omega}\,\boldsymbol{j}. \tag{7.79}$$

In the region outside the sources $\boldsymbol{E}'$ reduces to $\boldsymbol{E}$. When expressed in terms of these fields, Maxwell's equations become

$$\nabla\cdot\boldsymbol{E}' = 0, \tag{7.80a}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{7.80b}$$

$$\nabla\wedge\boldsymbol{E}' - \mathrm{i}\,k\,c\boldsymbol{B} = \frac{\mathrm{i}}{\epsilon_0\omega}\nabla\wedge\boldsymbol{j}, \tag{7.80c}$$

$$\nabla\wedge c\boldsymbol{B} + \mathrm{i}\,k\,\boldsymbol{E}' = \mu_0 c\,\nabla\wedge\boldsymbol{M}. \tag{7.80d}$$

The curl equations can be combined to give two inhomogeneous Helmholtz wave equations:

$$(\nabla^2 + k^2)c\boldsymbol{B} = -\mu_0 c\,\nabla\wedge(\boldsymbol{j} + \nabla\wedge\boldsymbol{M}), \tag{7.81}$$

and

$$(\nabla^2 + k^2)\boldsymbol{E}' = -\mathrm{i}\, k\, \mu_0 c\, \nabla\wedge\left(\boldsymbol{M} + \frac{\nabla\wedge\boldsymbol{j}}{k^2}\right). \tag{7.82}$$

These equations, together with $\nabla\cdot\boldsymbol{B} = 0$, and $\nabla\cdot\boldsymbol{E}' = 0$, and the curl equations giving $\boldsymbol{E}'$ in terms of $\boldsymbol{B}$ and *vice versa*, are the analogues to Eqs. (7.37)–(7.40) when sources are present.

Since the multipole coefficients in Eqs. (7.54) are determined according to Eqs. (7.57) from the scalars $\boldsymbol{r}\cdot\boldsymbol{B}$ and $\boldsymbol{r}\cdot\boldsymbol{E}'$, it is sufficient to consider wave equations for these quantities, rather than the vector fields $\boldsymbol{B}$ and $\boldsymbol{E}'$. From Eqs. (7.42), (7.81), (7.82), and the identity

$$\boldsymbol{r}\cdot(\nabla\wedge\boldsymbol{A}) = (\boldsymbol{r}\wedge\nabla)\cdot\boldsymbol{A} = \mathrm{i}\,\boldsymbol{L}\cdot\boldsymbol{A} \tag{7.83}$$

for any vector field $\boldsymbol{A}$, we obtain the inhomogeneous wave equations

$$(\nabla^2 + k^2)\,\boldsymbol{r}\cdot c\boldsymbol{B} \;=\; -\mathrm{i}\,\mu_0 c\,\boldsymbol{L}\cdot(\boldsymbol{j} + \nabla\wedge\boldsymbol{M}), \tag{7.84a}$$

$$(\nabla^2 + k^2)\,\boldsymbol{r}\cdot\boldsymbol{E}' \;=\; k\,\mu_0 c\,\boldsymbol{L}\cdot\left(\boldsymbol{M} + \frac{\nabla\wedge\boldsymbol{j}}{k^2}\right). \tag{7.84b}$$

Now the Green's function for the inhomogeneous Helmholtz equation (defined by Eq. (7.17)), subject to the boundary condition of outgoing waves at infinity, is given by Eq. (7.18). It follows that Eqs. (7.84) can be inverted to give

$$\boldsymbol{r}\cdot c\boldsymbol{B}(\boldsymbol{r}) \;=\; \frac{\mathrm{i}\,\mu_0 c}{4\pi}\int \frac{\mathrm{e}^{\,\mathrm{i}\,k\,|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\boldsymbol{L}'\cdot[\boldsymbol{j}(\boldsymbol{r}') + \nabla'\wedge\boldsymbol{M}(\boldsymbol{r}')]\; d^3\boldsymbol{r}', \tag{7.85a}$$

$$\boldsymbol{r}\cdot\boldsymbol{E}'(\boldsymbol{r}) \;=\; -\frac{k\,\mu_0 c}{4\pi}\int \frac{\mathrm{e}^{\,\mathrm{i}\,k\,|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\boldsymbol{L}'\cdot\left[\boldsymbol{M}(\boldsymbol{r}') + \frac{\nabla'\wedge\boldsymbol{j}(\boldsymbol{r}')}{k^2}\right]\; d^3\boldsymbol{r}'. \tag{7.85b}$$

In order to evaluate the multipole coefficients by means of Eqs. (7.57), we first observe that the requirement of outgoing waves at infinity makes $A_l^{(2)} = 0$ in Eq. (7.45). Thus, we choose $f_l(kr) = g_l(kr) = h_l^{(1)}(kr)$ in Eqs. (7.54) as the radial eigenfunctions of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the source free region. Next, let us consider

the expansion (7.26) of the Green's function for the Helmholtz equation in terms of spherical harmonics. We assume that the point $\boldsymbol{r}$ lies outside some spherical shell which completely encloses the sources. It follows that $r_< = r'$ and $r_> = r$ in all of the integrations. Making use of the orthogonality property of the spherical harmonics, it follows from Eq. (7.26) that

$$\int Y_{lm}^*(\theta, \varphi)\, \frac{\mathrm{e}^{\,\mathrm{i}\,k\,|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\, d\Omega = \mathrm{i}\,k\,h_l^{(1)}(kr)\,j_l(kr')\,Y_{lm}^*(\theta', \varphi'). \tag{7.86}$$

Finally, Eqs. (7.57), (7.85), and (7.86) yield

$$a_E(l,m) = \frac{\mu_0 c\,\mathrm{i}\,k^3}{\sqrt{l(l+1)}} \int j_l(kr)\,Y_{lm}^*\,\boldsymbol{L}\cdot\left(\boldsymbol{M} + \frac{\nabla\wedge\boldsymbol{j}}{k^2}\right) d^3\boldsymbol{r}, \tag{7.87a}$$

$$a_M(l,m) = -\frac{\mu_0 c\,k^2}{\sqrt{l(l+1)}} \int j_l(kr)\,Y_{lm}^*\,\boldsymbol{L}\cdot(\boldsymbol{j} + \nabla\wedge\boldsymbol{M})\,d^3\boldsymbol{r}. \tag{7.87b}$$

The expressions (7.87) give the strengths of the various multipole fields outside the source in terms of integrals over the source densities $\boldsymbol{j}$ and $\boldsymbol{M}$. They can be transformed into more useful forms by means of the following arguments. The results

$$\boldsymbol{L}\cdot\boldsymbol{A} = \mathrm{i}\,\nabla\cdot(\boldsymbol{r}\wedge\boldsymbol{A}), \tag{7.88a}$$

$$\boldsymbol{L}\cdot(\nabla\wedge\boldsymbol{A}) = \mathrm{i}\,\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A}) - \mathrm{i}\,\frac{1}{r}\frac{\partial(r^2\,\nabla\cdot\boldsymbol{A})}{\partial r} \tag{7.88b}$$

follow from the definition (7.29) of $\boldsymbol{L}$, and simple vector identities. Substituting into Eq. (7.87a), we obtain

$$a_E(l,m) = -\frac{\mu_0 c\;k^3}{\sqrt{l(l+1)}} \int j_l(kr)\,Y_{lm}^*\left[\nabla\cdot(\boldsymbol{r}\wedge\boldsymbol{M}) + \frac{\nabla^2(\boldsymbol{r}\cdot\boldsymbol{j})}{k^2} - \mathrm{i}\,\frac{c}{k\,r}\frac{\partial(r^2\rho)}{\partial r}\right] d^3\boldsymbol{r}, \tag{7.89}$$

where use has been made of Eq. (7.78). Use of Green's theorem on the second term replaces ∇^2 by $-k^2$ (since we can neglect the surface terms, and $j_l(kr)\,Y_{lm}^*$

is a solution of the Helmholtz equation). A radial integration by part on the third term (again neglecting surface terms) casts the radial derivative over onto the spherical Bessel function. The result for the *electric multipole coefficient* is

$$a_E(l,m) = \frac{\mu_0 c\, k^2}{\mathrm{i}\sqrt{l(l+1)}} \int Y_{lm}^* \left[c\rho\, \frac{d[r\, j_l(kr)]}{dr} + \mathrm{i}\, k\, (\boldsymbol{r}\cdot\boldsymbol{j})\, j_l(kr) \right. \\ \left. - \mathrm{i}\, k\, \nabla\cdot(\boldsymbol{r}\wedge\boldsymbol{M})\, j_l(kr) \right] d^3\boldsymbol{r}. \tag{7.90}$$

The analogous set of manipulations using Eq. (7.87b) leads to an expression for the *magnetic multipole coefficient*:

$$a_M(l,m) = \frac{\mu_0 c\, k^2}{\mathrm{i}\sqrt{l(l+1)}} \int Y_{lm}^* \left[\nabla\cdot(\boldsymbol{r}\wedge\boldsymbol{j})\, j_l(kr) + \nabla\cdot\boldsymbol{M}\, \frac{d[r\, j_l(kr)]}{dr} \right. \\ \left. - k^2\, (\boldsymbol{r}\cdot\boldsymbol{M})\, j_l(kr) \right] d^3\boldsymbol{r}. \tag{7.91}$$

Both the above results are exact, and are valid for arbitrary wavelength and source size.

In the limit in which the source dimensions are very small compared to a wavelength (*i.e.*, $kr \ll 1$) the expressions for the multipole coefficients can be considerably simplified. Using the asymptotic form (7.14a), and keeping only lowest powers in kr for terms involving ρ, $\boldsymbol{j}$, and $\boldsymbol{M}$, we obtain the approximate electric multipole coefficient

$$a_E(l,m) \simeq \frac{\mu_0 c\, k^{l+2}}{\mathrm{i}\,(2l+1)!!} \left(\frac{l+1}{l}\right)^{1/2} (Q_{lm} + Q_{lm}'), \tag{7.92}$$

where the multipole moments are

$$Q_{lm} = \int r^l\, Y_{lm}^*\, c\rho\, d^3\boldsymbol{r}, \tag{7.93a}$$

$$Q_{lm}' = \frac{-\mathrm{i}\, k}{l+1} \int r^l\, Y_{lm}^*\, \nabla\cdot(\boldsymbol{r}\wedge\boldsymbol{M})\, d^3\boldsymbol{r}. \tag{7.93b}$$

The moment Q_{lm} has the same form as a conventional electrostatic multipole moment. The moment Q_{lm}' is an induced electric multipole moment due to the

magnetization. It is generally a factor kr smaller than the normal moment Q_{lm}. For the magnetic multipole coefficient $a_M(l,m)$ the corresponding long wavelength approximation is

$$a_M(l,m) \simeq \frac{\mu_0 c\, \mathrm{i}\, k^{l+2}}{(2l+1)!!} \left(\frac{l+1}{l}\right)^{1/2} (\mathcal{M}_{lm} + \mathcal{M}'_{lm}), \tag{7.94}$$

where the magnetic multipole moments are

$$\mathcal{M}_{lm} = -\frac{1}{l+1} \int r^l\, Y_{lm}^*\, \nabla\cdot(\boldsymbol{r} \wedge \boldsymbol{j})\, d^3\boldsymbol{r}, \tag{7.95a}$$

$$\mathcal{M}'_{lm} = -\int r^l\, Y_{lm}^*\, \nabla\cdot\boldsymbol{M}\, d^3\boldsymbol{r} \tag{7.95b}$$

Note that for a system with intrinsic magnetization the magnetic moments $\mathcal{M}_{lm}$ and $\mathcal{M}'_{lm}$ are generally of the same order of magnitude.

Thus, in the long wavelength limit the electric multipole fields are determined by the charge density ρ, whereas the magnetic multipole fields are determined by the magnetic moment densities $\boldsymbol{r} \wedge \boldsymbol{j}/2$ and $\boldsymbol{M}$.

7.5 Radiation from a linear centre-fed antenna

As an illustration of the use of a multipole expansion for a source whose dimensions are comparable to a wavelength, consider the radiation from a linear centre-fed antenna. We assume that the antenna lies along the z-axis, and extends from $z = -d/2$ to $z = d/2$. The current flowing along the antenna vanishes at the end points, and is an even function of z. Thus, we can write

$$I(z,t) = I(|z|)\, \mathrm{e}^{-\mathrm{i}\,\omega\, t}, \tag{7.96}$$

where $I(d/2) = 0$. Since the current flows radially, $\boldsymbol{r} \wedge \boldsymbol{j} = 0$. Furthermore, there is no intrinsic magnetization. Thus, according to Eq. (7.91), all of the magnetic multipole coefficients $a_M(l,m)$ vanish. In order to calculate the electric multipole coefficients $a_E(l,m)$, we need expressions for the charge and current densities. In

spherical polar coordinates the current density $\boldsymbol{j}$ can be written in the form

$$\boldsymbol{j}(\boldsymbol{r}) = \hat{\boldsymbol{r}}\,\frac{I(r)}{2\pi r^2}\,[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]\,, \tag{7.97}$$

for $r < d/2$, where the delta functions cause the current to flow only upwards and downwards along the z-axis. From the continuity equation (7.78), the charge density is given by

$$\rho(\boldsymbol{r}) = \frac{1}{\mathrm{i}\,\omega}\,\frac{dI(r)}{dr}\left[\frac{\delta(\cos\theta - 1) - \delta(\cos\theta + 1)}{2\pi r^2}\right], \tag{7.98}$$

for $r < d/2$.

These expressions for $\boldsymbol{j}$ and ρ can be substituted into Eq. (7.90) to give

$$\begin{aligned} a_E(l, m) \;\; &= \;\; \frac{\mu_0 c\, k^2}{2\pi\sqrt{l(l+1)}}\int_0^{d/2} dr\left\{kr\, j_l(kr)\, I(r) - \frac{1}{k}\,\frac{dI(r)}{dr}\,\frac{d[r\, j_l(kr)]}{dr}\right\} \\ &\qquad \int d\Omega\, Y_{lm}^{*}\,[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]\,. \end{aligned} \tag{7.99}$$

The angular integral has the value

$$\int d\Omega\, Y_{lm}^{*}\,[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)] = 2\pi\,\delta_{m,0}\,\left[Y_{l0}(0) - Y_{l0}(\pi)\right], \tag{7.100}$$

showing that only $m = 0$ multipoles occur. This is hardly surprising given the cylindrical symmetry of the antenna. The $m = 0$ spherical harmonics are even (odd) about $\theta = \pi/2$ for l even (odd). Hence, the only nonvanishing multipoles have l odd. So, the angular integral takes the value

$$\int d\Omega\, Y_{lm}^{*}\,[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)] = \sqrt{4\pi(2l + 1)}, \tag{7.101}$$

for l odd and $m = 0$. After some slight rearrangement, Eq. (7.99) can be written

$$\begin{aligned} a_E(l, 0) \;\; &= \;\; \frac{\mu_0 c\, k}{2\pi}\left[\frac{4\pi(2l+1)}{l(l+1)}\right]^{1/2}\int_0^{d/2}\left\{-\frac{d}{dr}\left[r j_l(kr)\,\frac{dI}{dr}\right]\right. \\ &\qquad \left. + r j_l(kr)\left(\frac{d^2 I}{dr^2} + k^2\, I\right)\right\}\, dr. \end{aligned} \tag{7.102}$$

kd	$a_E(1,0)$	$a_E(3,0)/a_E(1,0)$	$a_E(5,0)/a_E(1,0)$
π	$4\sqrt{6\pi}\,(\mu_0 c\,I/4\pi d)$	4.95×10^{-2}	1.02×10^{-3}
2π	$4\pi\sqrt{6\pi}\,(\mu_0 c\,I/4\pi d)$	0.325	3.09×10^{-2}

Table 3: The first few electric multipole coefficients for a half-wave and a full-wave antenna

In order to evaluate the integral (7.102) we need to specify the current $I(z)$ along the antenna. In the absence of radiation, the sinusoidal time variation at frequency ω implies a sinusoidal space variation with wavenumber $k = \omega/c$. However, the emission of radiation generally modifies the current distribution. The correct current $I(z)$ can only be found be solving a complicated boundary value problem. For the sake of simplicity, we assume that $I(z)$ is a known function; specifically,

$$I(z) = I\,\sin(kd/2 - k\,|z|), \tag{7.103}$$

for $z < d/2$, where I is the peak current. With a sinusoidal current the second term in curly brackets in Eq. (7.102) vanishes. The first term is a perfect differential. Consequently, Eqs. (7.102) and (7.103) yield

$$a_E(l,0) = \frac{\mu_0 c\,I}{\pi d}\left[\frac{4\pi(2l+1)}{l(l+1)}\right]^{1/2}\left(\frac{kd}{2}\right)^2 j_l(kd/2), \tag{7.104}$$

for l odd.

Let us consider the special cases of a half-wave antenna ($kd = \pi$; *i.e.*, the length of the antenna is half a wavelength) and a full-wave antenna ($kd = 2\pi$). For these two values of kd the $l = 1$ coefficient is tabulated in Table 3, along with the relative values for $l = 3$, 5. It is clear from the table that the coefficients decrease rapidly in magnitude as l increases, and that higher l coefficients are more important the larger the source dimensions. However, even for a full-wave antenna it is generally adequate to retain only the $l = 1$ and $l = 3$ coefficients in order to calculate the angular distribution of the radiation. It is certainly adequate to keep only these two harmonics in order to calculate the total power radiated (which depends on the sum of the squares of the coefficients).

In the radiation zone the multipole fields (7.54) reduce to

$$\begin{aligned} c\boldsymbol{B} &\simeq \frac{\mathrm{e}^{\,\mathrm{i}\,(kr-\omega t)}}{kr} \sum_{l,m} (-\mathrm{i})^{l+1} \left[a_E(l,m)\, \boldsymbol{X}_{lm} \right. \\ &\qquad \left. + a_M(l,m)\, \boldsymbol{n} \wedge \boldsymbol{X}_{lm} \right], && (7.105\text{a}) \\ \boldsymbol{E} &\simeq c\boldsymbol{B} \wedge \boldsymbol{n}, && (7.105\text{b}) \end{aligned}$$

where use has been made of the asymptotic form (7.16). The time-averaged power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{\mathrm{Re}\,(\boldsymbol{n}\cdot\boldsymbol{E} \wedge \boldsymbol{B}^*)\, r^2}{2\mu_0}, \tag{7.106}$$

or

$$\frac{dP}{d\Omega} = \frac{1}{2\,\mu_0 c\, k^2} \left| \sum_{l,m} (-\mathrm{i})^{l+1} \left[a_E(l,m)\, \boldsymbol{X}_{lm} + a_M(l,m)\, \boldsymbol{n} \wedge \boldsymbol{X}_{lm} \right] \right|^2. \tag{7.107}$$

Retaining only the $l=1$ and $l=3$ electric multipole coefficients, the angular distribution of the radiation from the antenna is given by

$$\frac{dP}{d\Omega} = \frac{|a_E(l,0)|^2}{4\,\mu_0 c\, k^2} \left| \boldsymbol{L}Y_{1,0} - \frac{a_E(3,0)}{\sqrt{6}\, a_E(1,0)}\, \boldsymbol{L}Y_{3,0} \right|^2, \tag{7.108}$$

where use has been made of Eq. (7.52). The various factors in the absolute square are

$$\begin{aligned} |\boldsymbol{L}Y_{1,0}|^2 &= \frac{3}{4\pi} \sin^2\theta, && (7.109\text{a}) \\ |\boldsymbol{L}Y_{3,0}|^2 &= \frac{63}{16\pi} \sin^2\theta\, (5\cos^2\theta - 1)^2, && (7.109\text{b}) \\ (\boldsymbol{L}Y_{1,0})^* \cdot (\boldsymbol{L}Y_{3,0}) &= \frac{3\sqrt{21}}{8\pi} \sin^2\theta\, (5\cos^2\theta - 1). && (7.109\text{c}) \end{aligned}$$

With these angular factors, Eq. (7.108) becomes

$$\frac{dP}{d\Omega} = \lambda\, \frac{3\,\mu_0 c\, I^2}{\pi^3} \frac{3\,\sin^2\theta}{8\pi} \left| 1 - \sqrt{\frac{7}{8}} \frac{a_E(3,0)}{a_E(1,0)}\, (5\cos^2\theta - 1) \right|^2, \tag{7.110}$$

where λ equals 1 for a half-wave antenna and $\pi^2/4$ for a full-wave antenna. The coefficient in front of $(5\cos^2\theta - 1)$ is 0.0463 and 0.304 for the half-wave and full-wave antenna, respectively. It turns out that the radiation pattern from the two-term multipole expansion given above is almost indistinguishable from the exact result for the case of a half-wave antenna. For the case of a full-wave antenna the two-term expansion yields a radiation pattern which differs from the exact result by less than 5%.

The total power radiated by the antenna is

$$P = \frac{1}{2\,\mu_0 c\, k^2} \sum_{l\,\mathrm{odd}} |a_E(l,0)|^2, \tag{7.111}$$

where use has been made of Eq. (7.71). It is evident from Table 3 that a two-term multipole expansion gives an accurate expression for the radiated power for both a half-wave and a full-wave antenna. In fact, a one-term multipole expansion gives a fairly accurate result for the case of a half-wave antenna.

It is clear from the above analysis that the multipole expansion converges rapidly when the source dimensions are of order the wavelength of the radiation. It is also clear that if the source dimensions are much less than the wavelength then the multipole expansion is likely to be completely dominated by the term corresponding to the lowest value of l.

7.6 Spherical wave expansion of a vector plane wave

In discussing the scattering or absorption of electromagnetic radiation by localized systems, it is useful to be able to express a plane electromagnetic wave as a superposition of spherical waves.

Consider, first of all, the expansion of a scalar plane wave as a set of scalar spherical waves. This expansion is conveniently obtained from the expansion (7.26) for the Green's function of the scalar Helmholtz equation. Let us take the limit $\boldsymbol{r}' \to \infty$ of this equation. We can make the substitution $|\boldsymbol{r} - \boldsymbol{r}'| \simeq r' - \boldsymbol{n}\cdot\boldsymbol{r}$ on the left-hand-side, where $\boldsymbol{n}$ is a unit vector pointing in the direction of $\boldsymbol{r}'$. On the right-hand side, $r_< = r$ and $r_> = r'$. Furthermore, we can use the asymptotic

form (7.16) for $h_l^{(1)}(kr)$. Thus, we obtain

$$\frac{\mathrm{e}^{\mathrm{i}\,kr'}}{4\pi\,r'}\,\mathrm{e}^{-\mathrm{i}\,k\boldsymbol{n}\cdot\boldsymbol{r}} = \mathrm{i}\,k\,\frac{\mathrm{e}^{\mathrm{i}\,kr'}}{kr'}\sum_{l,m}(-\mathrm{i})^{l+1} j_l(kr)\,Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi). \tag{7.112}$$

Canceling the factor $\mathrm{e}^{\mathrm{i}\,kr'}/r'$ on either side, and taking the complex conjugate, we get the following expansion for a scalar plane wave,

$$\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi\sum_{l=0}^{\infty}\mathrm{i}^l\,j_l(kr)\sum_{m=-l}^{+l}Y^*_{lm}(\theta,\varphi)\,Y_{lm}(\theta',\varphi'), \tag{7.113}$$

where $\boldsymbol{k}$ is the wave vector with the spherical coordinates k, θ', φ'. The well known *addition theorem* for the spherical harmonics states that

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-1}^{+l}Y^*_{lm}(\theta,\varphi)\,Y_{lm}(\theta',\varphi'), \tag{7.114}$$

where γ is the angle subtended between the vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$. Consequently,

$$\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\varphi-\varphi'). \tag{7.115}$$

It follows from Eqs. (7.113) and (7.114) that

$$\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}\mathrm{i}^l\,(2l+1)\,j_l(kr)\,P_l(\cos\gamma), \tag{7.116}$$

or

$$\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}\mathrm{i}^l\sqrt{4\pi\,(2l+1)}\,j_l(kr)\,Y_{l,0}(\gamma), \tag{7.117}$$

since

$$Y_{l,0}(\theta) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta). \tag{7.118}$$

Let us now make an equivalent expansion for a circularly polarized plane wave incident along the z-axis:

$$\boldsymbol{E}(\boldsymbol{r}) = (\hat{\boldsymbol{x}}\pm\mathrm{i}\,\hat{\boldsymbol{y}})\,\mathrm{e}^{\mathrm{i}\,kz}, \tag{7.119a}$$

$$c\boldsymbol{B}(\boldsymbol{r}) = \hat{\boldsymbol{z}}\wedge\boldsymbol{E} = \mp\mathrm{i}\,\boldsymbol{E}. \tag{7.119b}$$

Since the plane wave is finite everywhere (including the origin), its multipole expansion (7.54) can only involve the well behaved radial eigenfunctions $j_l(kr)$. Thus,

$$\boldsymbol{E} = \sum_{l,m}\left[a_\pm(l,m)\, j_l(kr)\, \boldsymbol{X}_{lm} + \frac{\mathrm{i}}{k}\, b_\pm(l,m)\, \nabla\wedge j_l(kr)\boldsymbol{X}_{lm}\right], \tag{7.120a}$$

$$c\boldsymbol{B} = \sum_{l,m}\left[\frac{-\mathrm{i}}{k}\, a_\pm(l,m)\, \nabla\wedge j_l(kr)\boldsymbol{X}_{lm} + b_\pm(l,m)\, j_l(kr)\, \boldsymbol{X}_{lm}\right]. \tag{7.120b}$$

To determine the coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$ we make use of a slight generalization of the standard orthogonality properties (7.53) of the vector spherical harmonics:

$$\int [f_l(r)\boldsymbol{X}_{l'm'}]^* \cdot [g_l(r)\boldsymbol{X}_{lm}]\, d\Omega = f_l^*\, g_l\, \delta_{ll'}\, \delta_{mm'}, \tag{7.121a}$$

$$\int [f_l(r)\boldsymbol{X}_{l'm'}]^* \cdot [\nabla\wedge g_l(r)\boldsymbol{X}_{lm}]\, d\Omega = 0. \tag{7.121b}$$

The first of these follows directly from Eq. (7.53a). The second follows from Eqs. (7.31), (7.53b), (7.59), and the identity

$$\nabla = \frac{\boldsymbol{r}}{r}\frac{\partial}{\partial r} - \frac{\mathrm{i}}{r^2}\,\boldsymbol{r}\wedge\boldsymbol{L}. \tag{7.122}$$

The coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$ are obtained by taking the scalar product of Eqs. (7.120) with $\boldsymbol{X}_{lm}^*$ and integrating over all solid angle, making use of the orthogonality relations (7.121). This yields

$$a_\pm(l,m)\, j_l(kr) = \int \boldsymbol{X}_{lm}^* \cdot \boldsymbol{E}\, d\Omega, \tag{7.123a}$$

$$b_\pm(l,m)\, j_l(kr) = \int \boldsymbol{X}_{lm}^* \cdot c\boldsymbol{B}\, d\Omega. \tag{7.123b}$$

Substitution of Eqs. (7.52) and (7.120a) into Eq. (7.123a) gives

$$a_\pm(l,m)\,j_l(kr) = \int \frac{(L_\mp\,Y_{lm})^*}{\sqrt{l(l+1)}}\,\mathrm{e}^{\,\mathrm{i}\,kz}\,d\Omega, \tag{7.124}$$

where the operators $L_\pm$ are defined in Eqs. (7.30). Making use of Eqs. (7.33), the above expression reduces to

$$a_\pm(l,m)\,j_l(kr) = \frac{\sqrt{(l\pm m)(l\mp m+1)}}{\sqrt{l(l+1)}}\int Y^*_{l,m\mp 1}\,\mathrm{e}^{\,\mathrm{i}\,kz}\,d\Omega. \tag{7.125}$$

If the expansion (7.117) is substituted for $\mathrm{e}^{\,\mathrm{i}\,kz}$, and use is made of the orthogonality properties of the spherical harmonics, then we obtain the result

$$a_\pm(l,m) = \mathrm{i}^l\,\sqrt{4\pi\,(2l+1)}\,\delta_{m,\pm 1}. \tag{7.126}$$

It is clear from Eqs. (7.119b) and (7.123b) that

$$b_\pm(l,m) = \mp\,\mathrm{i}\,a_\pm(l,m). \tag{7.127}$$

Thus, the general expansion of a circularly polarized plane wave takes the form

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\mathrm{i}^l\sqrt{4\pi(2l+1)}\left[j_l(kr)\boldsymbol{X}_{l,\pm 1} \pm \frac{1}{k}\,\nabla\wedge j_l(kr)\boldsymbol{X}_{l,\pm 1}\right], \tag{7.128a}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\mathrm{i}^l\sqrt{4\pi(2l+1)}\left[\frac{-\mathrm{i}}{k}\,\nabla\wedge j_l(kr)\boldsymbol{X}_{l,\pm 1} \mp \mathrm{i}\,j_l(kr)\boldsymbol{X}_{l,\pm 1}\right]. \tag{7.128b}$$

The expansion for a linearly polarized plane wave is easily obtained by taking the appropriate linear combination of the above two expansions.

7.7 Mie scattering

Consider a plane electromagnetic wave incident on a spherical obstacle. In general, the wave is scattered, to some extent, by the obstacle. Thus, far away from

the sphere the electromagnetic fields can be expressed as the sum of a plane wave and a set of outgoing spherical waves. There may be absorption by the obstacle, as well as scattering. In this case, the energy flow away from the obstacle is less than the total energy flow towards it: the difference represents the absorbed energy.

The fields outside the sphere can be written as the sum of incident and scattered waves:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{inc}} + \boldsymbol{E}_{sc}, \tag{7.129a}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_{\mathrm{inc}} + \boldsymbol{B}_{sc}, \tag{7.129b}$$

where $\boldsymbol{E}_{\mathrm{inc}}$ and $\boldsymbol{B}_{\mathrm{inc}}$ are given by (7.128). Since the scattered fields are outgoing waves at infinity, their expansions must be of the form

$$\begin{aligned}\boldsymbol{E}_{\mathrm{sc}} &= \frac{1}{2}\sum_{l=1}^{\infty} \mathrm{i}^l \sqrt{4\pi\,(2l+1)} \qquad (7.130\mathrm{a}) \\ &\qquad \left[\alpha_\pm(l)\, h_l^{(1)}(kr)\, \boldsymbol{X}_{l,\pm 1} \pm \frac{\beta_\pm(l)}{k}\, \nabla \wedge h_l^{(1)}(kr)\, \boldsymbol{X}_{l,\pm 1}\right],\end{aligned}$$

$$\begin{aligned}c\boldsymbol{B}_{\mathrm{sc}} &= \frac{1}{2}\sum_{l=1}^{\infty} \mathrm{i}^l \sqrt{4\pi\,(2l+1)} \qquad (7.130\mathrm{b}) \\ &\qquad \left[\frac{-\mathrm{i}\,\alpha_\pm(l)}{k}\, \nabla \wedge h_l^{(1)}(kr)\boldsymbol{X}_{l,\pm 1} \mp \mathrm{i}\,\beta_\pm(l)\, h_l^{(1)}(kr)\, \boldsymbol{X}_{l,\pm 1}\right].\end{aligned}$$

The coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ are determined by the boundary conditions on the surface of the sphere. In general, it is necessary to sum over all m harmonics in the above expressions. However, for the restricted class of spherically symmetric scatterers only $m = \pm 1$ harmonics need be retained (since only these harmonics occur in the spherical wave expansion of the incident plane wave (see Eqs. (7.128)), and a spherically symmetric scatterer does not couple different m harmonics).

The angular distribution of the scattered power can be written in terms of the coefficients $\alpha(l)$ and $\beta(l)$ using the scattered electromagnetic fields evaluated on

the surface of a sphere of radius a surrounding the scatterer. In fact, it is easily demonstrated that

$$\begin{aligned}\frac{dP_{\rm sc}}{d\Omega} &= \frac{a^2}{2\mu_0}\,{\rm Re}\,[\boldsymbol{n}\cdot\boldsymbol{E}_{\rm sc}\wedge\boldsymbol{B}_{\rm sc}^{\ast}]_{r=a}\\ &= -\frac{a^2}{2\mu_0}\,{\rm Re}\,[\boldsymbol{E}_{\rm sc}\cdot(\boldsymbol{n}\wedge\boldsymbol{B}_{\rm sc}^{\ast})]_{r=a},\end{aligned}\tag{7.131}$$

where $\boldsymbol{n}$ is a radially directed outward normal. The differential scattering cross section is defined as the ratio of $dP_{\rm sc}/d\Omega$ to the incident flux $1/\mu_0 c$. Hence,

$$\frac{d\sigma_{\rm sc}}{d\Omega} = -\frac{a^2}{2}\,{\rm Re}\,[\boldsymbol{E}_{\rm sc}\cdot(\boldsymbol{n}\wedge c\boldsymbol{B}_{\rm sc}^{\ast})]_{r=a}.\tag{7.132}$$

We need to evaluate this expression using the electromagnetic fields specified in Eqs. (7.128), (7.129), and (7.130). The following identity, which can be established with the aid of Eqs. (7.29), (7.52), and (7.59), is helpful in this regard:

$$\nabla\wedge f(r)\boldsymbol{X}_{lm} = \boldsymbol{n}\,\frac{{\rm i}\sqrt{l(l+1)}}{r}\,f(r)\,Y_{lm} + \frac{1}{r}\frac{d[r f(r)]}{dr}\,\boldsymbol{n}\wedge\boldsymbol{X}_{lm}.\tag{7.133}$$

For instance, using this result we can write $\boldsymbol{n}\wedge c\boldsymbol{B}_{sc}$ in the form

$$\begin{aligned}\boldsymbol{n}\wedge c\boldsymbol{B}_{\rm sc} &= \frac{1}{2}\sum_{l=1}^{\infty}{\rm i}^l\sqrt{4\pi\,(2l+1)}\\ &\qquad\left[\frac{{\rm i}\,\alpha_\pm(l)}{k}\,\frac{1}{r}\,\frac{d[r h_l^{(1)}(kr)]}{dr}\boldsymbol{X}_{l,\pm1}\mp{\rm i}\,\beta_\pm(l)\,h_l^{(1)}(kr)\,\boldsymbol{n}\wedge\boldsymbol{X}_{l,\pm1}\right].\end{aligned}\tag{7.134}$$

It can be demonstrated, after considerable algebra, that

$$\frac{d\sigma_{\rm sc}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l\sqrt{2l+1}\,[\alpha_\pm(l)\,\boldsymbol{X}_{l,\pm1}\pm{\rm i}\,\beta_\pm(l)\,\boldsymbol{n}\wedge\boldsymbol{X}_{l,\pm1}]\right|^2.\tag{7.135}$$

In obtaining this formula, use has been made of the standard result

$$\frac{df_l(z)}{dz}\,f_{l'}^{\ast}(z) - \frac{df_{l'}^{\ast}(z)}{dz}\,f_l(z) = \frac{2\,{\rm i}}{z^2},\tag{7.136}$$

where $f_l(z) = \mathrm{i}^l\, h_l^{(1)}(z)$. The total scattering cross section is obtained by integrating Eq. (7.135) over all solid angle, making use of the following orthogonality relations for the vector spherical harmonics (see Eqs. (7.53)):

$$\int \boldsymbol{X}^\ast_{l'm'}\cdot\boldsymbol{X}_{lm}\,d\Omega = \delta_{ll'}\,\delta_{mm'}, \tag{7.137a}$$

$$\int \boldsymbol{X}^\ast_{l'm'}\cdot(\boldsymbol{n}\wedge\boldsymbol{X}_{lm})\,d\Omega = 0, \tag{7.137b}$$

$$\int (\boldsymbol{n}\wedge\boldsymbol{X}^\ast_{l'm'})\cdot(\boldsymbol{n}\wedge\boldsymbol{X}_{lm})\,d\Omega = \delta_{ll'}\,\delta_{mm'}. \tag{7.137c}$$

Thus,

$$\sigma_{\mathrm{sc}} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right]. \tag{7.138}$$

According to Eqs. (7.135) and (7.138), the total scattering cross section is independent of the polarization of the incident radiation (*i.e.*, it is the same for both the $\pm$ signs). However, the differential scattering cross section in any particular direction is, in general, different for different circular polarizations of the incident radiation. This implies that if the incident radiation is linearly polarized then the scattered radiation is elliptically polarized. Furthermore, if the incident radiation is unpolarized then the scattered radiation exhibits partial polarization, with the degree of polarization depending on the angle of observation.

The total power absorbed by the sphere is given by

$$\begin{aligned} P_{\mathrm{abs}} &= -\frac{a^2}{2\mu_0}\,\mathrm{Re}\int[\boldsymbol{n}\cdot\boldsymbol{E}\wedge\boldsymbol{B}^\ast]_{r=a}\,d\Omega \\ &= \frac{a^2}{2\mu_0}\,\mathrm{Re}\int[\boldsymbol{E}\cdot(\boldsymbol{n}\wedge\boldsymbol{B}^\ast)]_{r=a}\,d\Omega. \end{aligned} \tag{7.139}$$

A similar calculation to that outlined above yields the following expression for the absorption cross section,

$$\sigma_{\mathrm{abs}} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[2 - |\alpha_\pm(l)+1|^2 - |\beta_\pm(l)+1|^2\right]. \tag{7.140}$$

The total or extinction cross section is the sum of $\sigma_{\rm sc}$ and $\sigma_{\rm abs}$:

$$\sigma_{\rm t} = -\frac{\pi}{k^2}\sum_l (2l+1)\,{\rm Re}\,[\alpha_\pm(l)+\beta_\pm(l)]. \tag{7.141}$$

Not surprisingly, the above expressions for the cross sections closely resemble those obtained in quantum mechanics from partial wave expansions.

Let us now consider the boundary conditions at the surface of the sphere (whose radius is a, say). For the sake of simplicity, let us suppose that the sphere is a perfect conductor. In this case, the appropriate boundary condition is that the tangential electric field is zero at $r = a$. According to Eqs. (7.128), (7.129), and (7.133), the tangential electric field is given by

$$\begin{aligned}\boldsymbol{E}_{\rm tan} &= \sum_l {\rm i}^l\sqrt{4\pi(2l+1)}\left\{\left[j_l + \frac{\alpha_\pm(l)}{2}\,h_l^{(1)}\right]\boldsymbol{X}_{l,\pm 1}\right.\\ &\qquad\left.\pm\,\frac{1}{x}\frac{d}{dx}\left[x\left(j_l+\frac{\beta_\pm(l)}{2}\,h_l^{(1)}\right)\right]\boldsymbol{n}\wedge\boldsymbol{X}_{l,\pm 1}\right\},\end{aligned} \tag{7.142}$$

where $x = ka$, and all of the spherical Bessel functions have the argument x. Thus, the boundary condition yields

$$\alpha_\pm(l)+1 = -\frac{h_l^{(2)}(ka)}{h_l^{(1)}(ka)}, \tag{7.143a}$$

$$\beta_\pm(l)+1 = -\left[\frac{(x\,h_l^{(2)}(x)\,)'}{(x\,h_l^{(1)}(x)\,)'}\right]_{x=ka}, \tag{7.143b}$$

where $'$ denotes d/dx. Note that $\alpha_\pm(l)+1$ and $\beta_\pm(l)+1$ are both numbers of modulus unity. This implies, from Eq. (7.140), that there is no absorption for the case of a perfectly conducting sphere (in general, there is some absorption if the sphere has a finite conductivity). We can write $\alpha_\pm(l)$ and $\beta_\pm(l)$ in the form

$$\alpha_\pm(l) = {\rm e}^{2\,{\rm i}\,\delta_l} - 1, \tag{7.144a}$$

$$\beta_\pm(l) = {\rm e}^{2\,{\rm i}\,\delta_l'} - 1, \tag{7.144b}$$

where the phase angles δ_l and δ_l' are called *scattering phase shifts*. It follows from Eqs. (7.143) that

$$\tan\delta_l = \frac{j_l(ka)}{y_l(ka)}, \tag{7.145a}$$

$$\tan\delta_l' = \left[\frac{(x\,j_l(x))'}{(x\,y_l(x))'}\right]_{x=ka}. \tag{7.145b}$$

Let us specialize to the limit $ka \ll 1$, in which the wavelength of the radiation is much greater than the radius of the sphere. The asymptotic expansions (7.14) yield

$$\alpha_\pm(l) \simeq -\frac{2\,\mathrm{i}\,(ka)^{2l+1}}{(2l+1)[(2l-1)!!]^2},$$

$$\beta_\pm(l) \simeq -\frac{(l+1)}{l}\,\alpha_\pm(l), \tag{7.146a}$$

for $l \geq 1$. It is clear that the scattering coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ become small very rapidly as l increases. In the very long wavelength limit only the $l = 1$ coefficients need be retained. It is easily seen that

$$\alpha_\pm(1) = -\frac{\beta_\pm(1)}{2} \simeq -\frac{2\,\mathrm{i}}{3}\,(ka)^3. \tag{7.147}$$

In this limit, the differential scattering cross section (7.135) reduces to

$$\frac{d\sigma_{\rm sc}}{d\Omega} \simeq \frac{2\pi}{3}\,a^2(ka)^4\,|\boldsymbol{X}_{1,\pm1} \mp 2\,\mathrm{i}\,\boldsymbol{n}\wedge\boldsymbol{X}_{1,\pm1}|^2\,. \tag{7.148}$$

It can be demonstrated that

$$|\boldsymbol{n}\wedge\boldsymbol{X}_{1,\pm1}|^2 = |\boldsymbol{X}_{1,\pm1}|^2 = \frac{3}{16\pi}\,(1+\cos^2\theta), \tag{7.149}$$

and

$$[\pm\mathrm{i}\,(\boldsymbol{n}\wedge\boldsymbol{X}^*_{1,\pm1})\cdot\boldsymbol{X}_{1,\pm1}] = -\frac{3\pi}{8}\,\cos\theta. \tag{7.150}$$

Thus, in long wavelength limit the differential scattering cross section limits to

$$\frac{d\sigma_{\rm sc}}{d\Omega} \simeq a^2 (ka)^4 \left[\frac{5}{8}\,(1+\cos^2\theta) - \cos\theta\right]. \tag{7.151}$$

The scattering is predominately backwards, and is independent of the state of polarization of the incident radiation. The total scattering cross section is given by

$$\sigma_{\rm sc} = \frac{10\pi}{3}\, a^2\,(ka)^4. \tag{7.152}$$

This well known result was first obtained by Mie and Debye. Note that the cross section scales as the inverse fourth power of the wavelength of the incident radiation. This scaling is generic to all scatterers whose dimensions are much smaller than the wavelength. In fact, it was first derived by Rayleigh using dimensional analysis.