This course:

Advanced Electromagnetic Theory

· Classical electromagnetism relevant because:

- simplest example of a gauge theory

  recall that standard model gauge group is

  $SO(3) \otimes SU(2) \otimes U(1)$

  in electromagnetism, gauge group is $U(1)$

- most of what we know about the universe stems from electromagnetic phenomena

- practical applications

· . . . . .

Organization Details

Please check www.phy.syr.edu/~vcrmen/Teaching/PHY641S12

Novelty: · Pop quizzes count 10 %.

• Textbook: J. Franklin, <u>Classical Electrodynamics</u>.

And without further ado . . .

0. Maxwells equations

For sources in vacuum, Maxwell's eqs. are

$$\left.\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{1}{c}\,\partial_t \vec{B} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned}\right\} \text{homogeneous}$$

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \times \vec{B} \quad \frac{1}{c}\,\partial_t \vec{E} &= \frac{4\pi \vec{j}}{c} \end{aligned}\right\} \text{inhomogeneous.}$$

We work in the <u>gaussian system of units</u>

These four equations form the basis of ALL electromagnetic phenomena.

To start, we shall consider static fields created by <u>static charges</u>

1. Foundations of electrostatics

1.1. Coulomb's law.

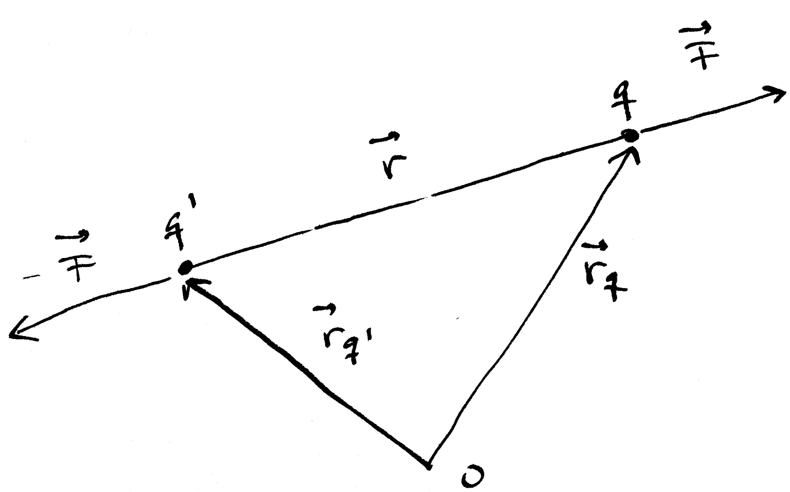
We shall consider static charges ⇒ no currents:

$$\rho \neq 0 \,, \quad \vec{j} = 0 .$$

According to <u>Coulomb's law</u>, the force experienced by two charges q and q' separated by $\vec{r}$ is

$$\vec{F} = \frac{q \cdot q'}{r^2} \hat{r} = \frac{q \cdot q'}{r^3} \vec{r}$$

Note: Equally charges repel each other:



$\vec{r} = \vec{r}_q - \vec{r}_{q'}$; $\hat{r} = \frac{\vec{r}}{r}$ unit vector in the direction of $\vec{r}$.

• In other systems of units, Coulomb's law reads $\vec{F} = \frac{k \cdot q \cdot q'}{r^3} \vec{r}$.

The force exerted by a set of charges q_n located at $\vec{r}_n$ on a charge q located at $\vec{r}$ obeys the superposition principle:

$$\vec{F} = \sum_n \frac{q q_n}{|\vec{r} - \vec{r}_n|^3} (\vec{r} - \vec{r}_n).$$

1.2. The electric field

It is extremely useful to think of the force experienced by a charge q as due to an electric field $\vec{E}$: $\vec{F} = q\,\vec{E}$.

Thus, for multiple charges,

$$\vec{E} = \sum_n \frac{q_n}{|\vec{r} - \vec{r}_n|^3} (\vec{r} - \vec{r}_n), \qquad (A1)$$

whereas for a distribution of charges

$$\vec{E} = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3} (\vec{r}-\vec{r}') \qquad (A2)$$

$\rho(\vec{r})$ is the <u>charge density</u> at $\vec{r}$.

For a set of point-like charges q_n at $\vec{r}_n$,

$$\rho(\vec{r}) = \sum_n q_n \, \delta^{(3)}(\vec{r}-\vec{r}_n) . \qquad (A3)$$

Recall that the "δ - function" obeys

$$\int d^3r \; f(\vec{r}) \; \delta(\vec{r}-\vec{r}_0) = f(\vec{r}).$$

Using this identity in (A2) with (A3) leads to (A1).

We shall later show that

$$\vec{\nabla}_r \cdot \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right) = 4\pi \, \delta^{(3)}(\vec{r}-\vec{r}').$$

Hence, taking the divergence of equation (A2) we find Coulomb's equation in differential form:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

Sometimes, people (Europeans) write $\text{div}\,\vec{E} = 4\pi\rho$

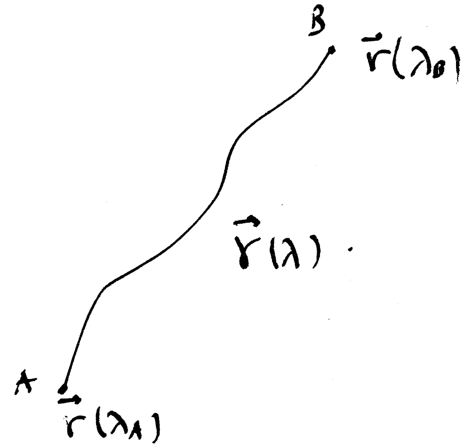
Recall that $\vec{\nabla}\cdot\vec{E} = \sum_{i=1}^{3} \frac{\partial E_i}{\partial x^i} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$.

1.3. Electric potential

Consider now the work exerted by an electric field on a charge q moving on a path $\vec{\gamma}(\lambda)$:

Since $dW = \vec{F}\cdot d\vec{r} = q\vec{E}\cdot d\vec{r}$,

$$W = q\int_{\gamma} \vec{E}\cdot d\vec{r} = q\int_{\lambda_A}^{\lambda_B} \vec{E}\cdot\frac{d\vec{r}}{d\lambda}\,d\lambda$$



Because

$$\frac{d}{d\lambda}\left(\frac{1}{|\vec{r}(\lambda)-\vec{r}'|}\right) =$$

$$\parallel$$

$$\frac{d}{d\lambda}\left(\frac{1}{\sqrt{(\vec{r}(\lambda)-\vec{r}')^2}}\right) = -\frac{1}{2}\frac{1}{\left[(\vec{r}(\lambda)-\vec{r}')^2\right]^{3/2}}\; 2(\vec{r}-\vec{r}')\cdot\frac{d\vec{r}}{d\lambda}$$

$$= -\frac{(\vec{r}(\lambda)-\vec{r}')}{|\vec{r}(\lambda)-\vec{r}'|^3}\cdot\frac{d\vec{r}}{d\lambda}$$

we find, using equation (A2),

$$W = -q \int_{\lambda_A}^{\lambda_B} d\lambda \, \frac{d}{d\lambda} \left(\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right) \equiv -q \left(\phi(\vec{r}_B) - \phi(\vec{r}_A) \right),$$

where we have defined the electric potential

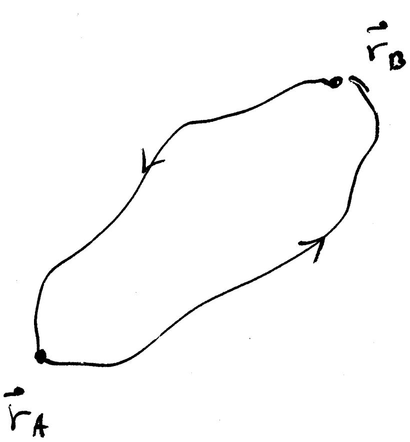
$$\phi(\vec{r}) = \int d^3 r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} . \qquad (A5)$$

In particular, for a closed path, $\vec{r}_A = \vec{r}_B$,

$$W = \oint \vec{E} \cdot d\vec{r} = 0 .$$

$\vec{E}$ (or $\vec{F}$) is a conservative force field

work exerted on a charge is path - independent



Conversely, given that $\oint \vec{E} \cdot d\vec{r} = 0$, we can define a potential $\phi(\vec{r})$ such that

$$\int_{r_A}^{r_B} \vec{E} \cdot d\vec{r} = -\left(\phi(\vec{r}_B) - \phi(\vec{r}_A)\right).$$

Say, choose an arbitrary point O, and assign to it the potential ϕ_0. Then, define

$$\phi(\vec{r}) \equiv \phi_0 - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{r} \qquad (A4)$$

well-defined, because integral path-independent.

given an arbitrary scalar function $\phi(\vec{r})$, we can define its gradient, $\vec{\nabla}\phi$ by demanding that for all paths $\vec{r}(\lambda)$.

$$\frac{d}{d\lambda}\phi(\vec{r}(\lambda)) \equiv \vec{\nabla}\phi \cdot \frac{d\vec{r}}{d\lambda}. \qquad \left(d\phi = \vec{\nabla}\phi \cdot d\vec{r}\right)$$

Clearly, from the chain rule

$$(\vec{\nabla}\phi)_i = \frac{\partial \phi}{\partial x^i}, \quad \text{or} \quad \vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right).$$

Using this relation in (A4), we immediately find

$$\vec{E} = -\vec{\nabla}\phi.$$

The same relation follows from equation (A5) using that

$$\vec{\nabla}_r\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

and comparing with (A2)

1.4. Gauss' law

The divergence theorem

Consider an arbitrary, smooth vector field $\vec{E}(\vec{r})$. Its divergence, $\vec{\nabla}\cdot\vec{E}$ is a scalar. According to the divergence theorem,

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int_{\partial V} \vec{E}\cdot d\vec{A}$$

$\int_V dV$ is a volume integral, and

$\int_{\partial V} d\vec{A}$ is an integral over the boundary of V,

a two-dim surface (in 3D). Recall that

$d\vec{A} = \vec{n} \cdot dA$, where $\vec{n}$ is the normal

to the area element.

Therefore,

$$\int_{\partial V} \vec{E} \cdot d\vec{A} = \int_V dV \, \vec{\nabla} \cdot \vec{E} = \int_V dV \, 4\pi\rho = 4\pi Q_{int},$$

where $Q_{int} = \int_V dV \, \rho$ is the total charge

inside the volume V.

$$\int_{\partial V} \vec{E} \cdot d\vec{A} = 4\pi Q_{int}$$ is Gauss' law.

The divergence theorem is a special case

of Stoke's theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

M: n-dimensional manifold; $d\omega$: exterior derivative

ω: $n-1$ differential form; of ω.