Recall that the basic eqs. of magnetostatics are

$$\vec{\nabla}\cdot\vec{B}=0 \quad \text{and}$$

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j}.$$

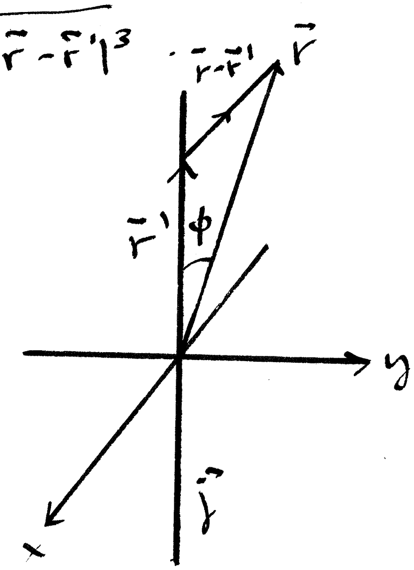
The vector potential is defined by $\vec{B}=\vec{\nabla}\times\vec{A}$.
gauge invariance allows us to impose Coulomb gauge,

$$\vec{\nabla}\cdot\hat{A}=0.$$

In Coulomb gauge, $\vec{\nabla}^2\vec{A}=-\frac{4\pi}{c}\vec{j}$. Therefore

$$\vec{A}=\frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}, \quad \text{which leads to}$$

$$\vec{B}(\vec{r})=\frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$




Examples

i) Infinitely long straight wire

$$\vec{j}(\vec{r}')=I\,\delta(x')\,\delta(y')\,\hat{e}_z$$

Therefore,

$$\vec{B}(\vec{r}) = \frac{1}{c}\int dz' \, I \, \hat{e}_z \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{I}{c}\int dz' \, \hat{e}_y \, \frac{\sin\phi}{x^2+y^2+(z-z')^2}$$

and because $\sin\phi = \sqrt{\dfrac{x^2+y^2}{x^2+y^2+(z-z')^2}}$

$$\vec{B}(\vec{r}) = \frac{I}{c}\,\frac{2\sqrt{x^2+y^2}}{x^2+y^2}\,\hat{e}_y = \frac{2I}{c\sqrt{x^2+y^2}} \equiv \frac{2I}{c\,r_\perp}$$

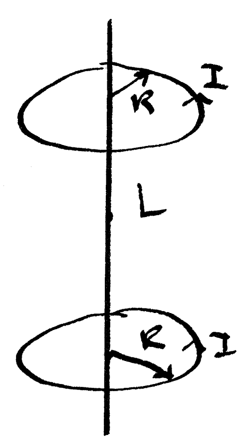
Of course, in this case it is simpler to use Ampère's circuital law,

$$2\pi r_\perp B \doteq \frac{4\pi}{c} I \quad \Rightarrow \quad B = \frac{2I}{c\,r_\perp}.$$

ii) Helmholtz coils

<u>Exercise 21</u>

a) Calculate the magnetic field along the $z$ axis in the following configuration:




b) Calculate $\dfrac{dB}{dz}$ at $z=0$.

## 7.11.4 Magnetic moment of a localized current.

As for the electric field, we can expand the field created by a current distribution in multipoles. Starting with

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

and Taylor expanding around $\vec{r}\,'=0$,

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|\vec{r}|} + \frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3} + \dots$$

we find

$$\vec{A}(\vec{r}) = \frac{1}{c|\vec{r}|}\int d^3r' \, \vec{j}(\vec{r}\,') + \frac{\vec{r}}{c|\vec{r}|^3}\cdot\int d^3r' \, \vec{j}(\vec{r}\,')\,\vec{r}\,' + \dots$$

<u>Exercise 22</u>

Show that $\int d^3r' \, \vec{j}(\vec{r}\,') = 0$,

$$\int d^3r' \, \vec{r}\,'\,\vec{j}(\vec{r}\,') = \frac{1}{2}\int d^3r' \left[\vec{r}\,'\,\vec{j}(\vec{r}\,') - \vec{j}(\vec{r}\,')\,\vec{r}\,'\right]$$

Therefore, we find

$$\vec{A}(\vec{r}) = \frac{1}{2c|\vec{r}|^3}\int d^3r' \; \vec{r}\cdot[\vec{r}'\vec{j}(\vec{r}') - \vec{j}(\vec{r}')\vec{r}'] + \cdots =$$

$$= \left[\frac{1}{2c}\int d^3r' \; \vec{r}' \times \vec{j}(\vec{r}')\right] \times \frac{\vec{r}}{|\vec{r}|^3} + \cdots$$

$\vec{M}(\vec{r}') = \frac{1}{2c}\vec{r}' \times \vec{j}(\vec{r}')$ is the magnetic moment density (magnetization), and

$\vec{\mu} = \frac{1}{2c}\int d^3r' \; \vec{r}' \times \vec{j}(\vec{r}')$ is the magnetic (dipole) moment.

We thus have $\vec{A}(\vec{r}) = \vec{\mu} \times \frac{\vec{r}}{|\vec{r}|^3} + \cdots$

Note that there is no contribution from the monopole: no analogue of "magnetic charge".

The magnetic field associated with a magnetic dipole is

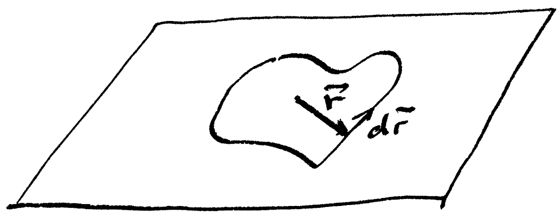
$$\vec{B} = \vec{\nabla} \times \left(\frac{\vec{\mu}\times\vec{r}}{|\vec{r}|^3}\right) = \vec{\mu}\,\vec{\nabla}\cdot\left(\frac{\vec{r}}{|\vec{r}|^3}\right) - (\vec{\mu}\cdot\vec{\nabla})\left(\frac{\vec{r}}{r^3}\right),$$

$$\vec{B}_{dm} = 4\pi\,\delta(\vec{r}) + \frac{3\hat{r}(\hat{r}\cdot\vec{\mu}) - \vec{\mu}}{|\vec{r}|^3}$$

## Exercise 23

Prove the last equation ■

Note that if a current is confined to a plane




$$\vec{\mu} = \frac{1}{2c}\int d^3r \; \vec{r} \times \vec{j}(\vec{r}) = \frac{I}{c}\,\frac{1}{2}\oint \vec{r} \times d\vec{r} = \frac{I}{c}\,\vec{S},$$

where $\vec{S} \perp$ to the plane and $|\vec{S}|$ is the surface spanned by the loop.

## 7.11.7. Force, torque and energy of a localized current

As we saw, a current placed in an external magnetic field experiences a force

$$\vec{F} = \frac{I}{c}\oint d\vec{r} \times B(\vec{r})$$

If the current is confined to the vicinity of $\vec{r} = 0$, we can expand

$$\vec{B} = \vec{B}(0) + (\vec{r}\cdot\vec{\nabla})\,\vec{B}\big|_0 + \cdots$$

Substituting, we find (after a few steps)

$$\vec{F} = \vec{\nabla}(\vec{B}\cdot\vec{\mu})\big|_0 + \cdots$$

Again, no contribution from a "magnetic charge"

The last eq. immediately gives the potential energy of a current in an external $\vec{B}$:

$$U = -\vec{B}\cdot\vec{\mu} + \cdots$$

Along the same lines, one can show that the torque acting on a current distribution,

$$\vec{\tau} = \frac{I}{c}\oint \vec{r}\times(d\vec{r}\times\vec{B}(\vec{r})) \Rightarrow$$

$$\vec{\tau} = \vec{\mu}\times\vec{B}\big|_0 + \cdots$$

# 8. Magnetization and Ferromagnetism

As before, if we are interested in the macroscopic magnetic field in matter (away from the vacuum) we need to take into account the magnetic properties of matter.

As before, we define a macroscopic magnetic field as a spatial average:

$$B(\vec{r}) = \langle \vec{b}(\vec{r}) \rangle = \int d^3r\, W(\vec{r}-\vec{r}\,')\, b(\vec{r}\,')$$

"microscopic" field.

We then average Maxwell's eqs (for the microscopic fields)

$$\vec{\nabla}\cdot\vec{b} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(homogeneous)}$$

$$\vec{\nabla}\times\vec{b} = \frac{4\pi}{c}\vec{j} \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle\vec{j}\rangle \qquad \text{(inhomogeneous)}.$$

The spatial average of the current is

$$\langle\vec{j}\rangle = \vec{J} + c\,\vec{\nabla}\times\vec{M} + \cdots$$

where

$$\vec{J} = \Big\langle \sum_{\substack{i \\ \text{free}}} q_i \vec{v}_i\, \delta(\vec{r}-\vec{r}_i) + \sum_{\substack{n \\ \text{molecules}}} q_n\, \vec{v}_n\, \delta(\vec{r}-\vec{r}_n) \Big\rangle$$

charge and CM velocity of molecule $n$

The magnetization is $\vec{M} = \langle \sum_n \quad \delta(\vec{r}-\vec{r}_n) \rangle$,

where $\vec{\mu}_n = \sum_{j(n)} \frac{q_j}{2c} (\vec{r}_{jn} \times \vec{v}_{jn})$ is the magnetic moment of the $n$-th molecule in the CM frame

Therefore, introducing $\vec{H} \equiv \vec{B} - 4\pi \vec{M}$ (this is mostly called the magnetic field though we reserve that name for $\vec{B}$) we arrive at

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J} \end{cases}$$

Macroscopic magnetostatics eqs.

In order to close the system a constitutive relation. For non-ferromagnetic materials we find

$$\vec{M} = \chi_m \vec{H}, \quad \text{where}$$

$\chi_m$ is the magnetic susceptibility.

Thus, $\vec{B} = \vec{H} + 4\pi \vec{M} = (1 + 4\pi \chi_m)\vec{H} = \mu \vec{H}$.

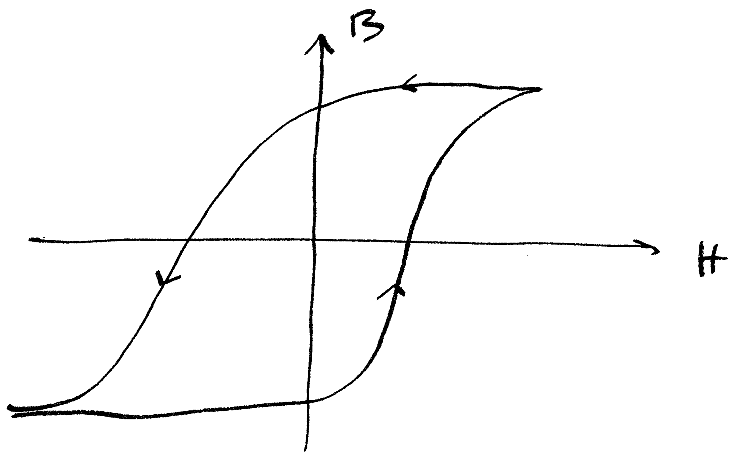
$\mu$ is known as the magnetic permeability

$$\begin{cases} \mu > 1 & \text{for paramagnetic materials} \\ \mu < 1 & \text{for diamagnetic materials} \end{cases}$$

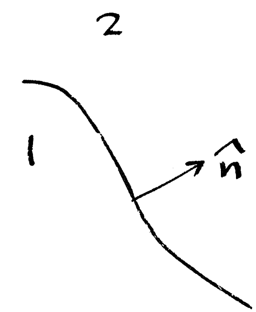
- Paramagnetic materials typically have permanent magnetic moments that align themselves with the magnetic field (recall $U = -\vec{\mu} \cdot \vec{B}$)

- Diamagnetic materials do not have permanent magnetic moments. The external magnetic field induces a magnetic moment anti-parallel to $\vec{B}$ (Lenz' rule).

For ferromagnetic moments $\vec{B}$ is not a single-valued function of $\vec{H}$, since it typically depends on the history of the material (hysteresis)



As in electrostatics, at the interface between two materials we need to impose appropriate junction conditions. Recall that




$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow (\vec{D}_2 - \vec{D}_1)\hat{n} = 4\pi\sigma$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0.$$

Then, by analogy

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow (\vec{B}_2 - \vec{B}_1)\hat{n} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} \Rightarrow \hat{n}\times(\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c}\vec{K},$$

where $\vec{K}$ is the (macroscopic) surface current density, which by definition is confined to the plane of the interface.