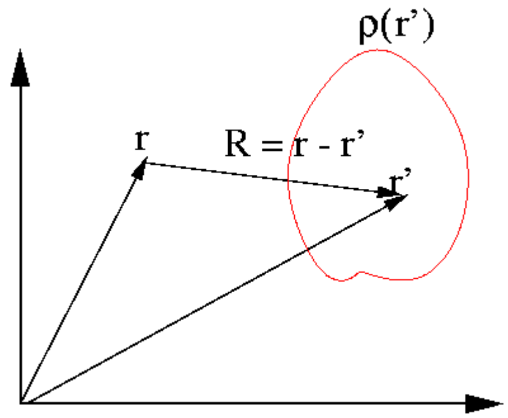
4 Potentials

4.1 Scalar Potential

Our general definition of the electric field is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\vec{r'})\hat{R} \ \ dV'}{\mid R \mid^2} \tag{4.1.1}$$

$$= \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\vec{r'})\vec{R} \ \ dV'}{\mid R \mid^3} \tag{4.1.2}$$



Now $\vec{R} = \vec{r} - \vec{r'}$ and we have $-\vec{\nabla}(\frac{1}{\mid \vec{r} - \vec{r'} \mid}) = \frac{\vec{r} - \vec{r'}}{\mid \vec{r} - \vec{r'} \mid^3}$ (exercise to prove this!) so:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V -\vec{\nabla}(\frac{1}{\mid \vec{r} - \vec{r'} \mid})\rho(\vec{r'}) \ \ dV' \tag{4.1.3}$$

Now since the gradient operator is a partial derivative with respect to $\vec{r}$ NOT $\vec{r'}$.

$$\Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla}(\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'}) \ \ dV'}{\mid \vec{r} - \vec{r'} \mid}) \tag{4.1.4}$$

Define $V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r'}) \ \ dV'}{\mid \vec{r} - \vec{r'} \mid}$ as the **Electric or Scalar Potential.**

$$\text{So} \qquad \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{R})$$

It is often easier to determine the scalar function V rather than the vector field $\vec{E}$. We can find $\vec{E}$ by taking the gradient of V.

Now given that:

$$\begin{aligned}\vec{E} &= -\vec{\nabla}V && (4.1.5)\\ \Rightarrow \vec{\nabla}\times\vec{E} &= -\vec{\nabla}\times(\vec{\nabla}V) && (4.1.6)\\ \Rightarrow \vec{\nabla}\times\vec{\mathbf{E}} &= \mathbf{0} && (4.1.7)\end{aligned}$$

It is important to note that this is for a **STATIC** electric field. Applying Stokes theorem:

$$\int_{Surface}\vec{\nabla}\times\vec{E}\cdot\vec{ds} = \oint_{Circuit}\vec{E}\cdot\vec{dl} = 0 \tag{4.1.8}$$

So **STATIC** electric field's are **CONSERVATIVE**. The work done or Energy gained in moving a charged particle around a closed circuit is $\oint_{circuit}\vec{F}\cdot\vec{dl} = \oint_{circuit} q\vec{E}\cdot\vec{dl} = 0$. This needs to be modified for time varying fields.

4.2 The Vector Potential

Since there are no magnetic monopoles, there are no sources of magnetic fields. This means that **FOR ALL** magnetic fields:

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4.2.1}$$

This is true for both static and time varying fields. Then since:

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0 \tag{4.2.2}$$

for all vector fields $\vec{F}$:

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \tag{4.2.3}$$

Then $\vec{A}$ is the **Vector Potential**. This equation is always valid even for time varying fields. How can $\vec{A}$ be calculated and visualized?

Start from the Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r'})\times(\vec{r}-\vec{r'})dV'}{\mid\vec{r}-\vec{r'}\mid^3} \tag{4.2.4}$$

write: $\dfrac{\vec{r}-\vec{r'}}{\mid\vec{r}-\vec{r'}\mid^3} = -\vec{\nabla}(\dfrac{1}{\mid\vec{r}-\vec{r'}\mid})$

$$\Rightarrow \vec{B}(\vec{r}) = -\frac{\mu}{4\pi}\int_{V'}\vec{J}(\vec{r})\times\vec{\nabla}(\frac{1}{\mid\vec{r}-\vec{r'}\mid})dV' \tag{4.2.5}$$

Recall the Vector Identity: For a vector field $\vec{F}$ and scalar field ϕ,

$$\vec{\nabla} \times (\Phi \vec{F}) = \Phi \vec{\nabla} \times \vec{F} - \vec{F} \times \vec{\nabla}(\Phi) \tag{4.2.6}$$

$$So \quad -\vec{J}(\vec{r'}) \times \vec{\nabla}(\frac{1}{\mid \vec{r} - \vec{r'} \mid}) = -\frac{1}{\mid \vec{r} - \vec{r'} \mid} \vec{\nabla} \times \vec{J}(\vec{r'}) + \vec{\nabla} \times (\frac{1}{\mid \vec{r} - \vec{r'} \mid} \vec{J}(\vec{r'})) \tag{4.2.7}$$

Since $\vec{J}(\vec{r'})$ does not depend on $\vec{r}$, $\vec{\nabla} \times \vec{J}(\vec{r'}) = 0$.

$$\Rightarrow -\vec{J}(\vec{r'}) \times \vec{\nabla}(\frac{1}{\mid \vec{r} - \vec{r'} \mid}) = \vec{\nabla} \times \frac{\vec{J}(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} \tag{4.2.8}$$

$$\Rightarrow \vec{B}(\vec{r}) = -\frac{\mu}{4\pi} \int_{V'} \vec{J}(\vec{r'}) \times \vec{\nabla}(\frac{1}{\mid \vec{r} - \vec{r'} \mid}) dV' \tag{4.2.9}$$

$$\Rightarrow \vec{B}(\vec{r}) = -\frac{\mu}{4\pi} \int_{V'} \vec{\nabla} \times \frac{\vec{J}(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.2.10}$$

Since $\vec{\nabla}$ is with respect to $\vec{r}$ and the integral is with respect to $\vec{r'}$ we can take $\vec{\nabla}$ outside the integral and get:

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.2.11}$$

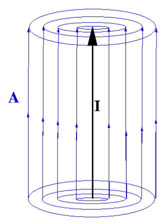
So we can identify:

$$\vec{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.2.12}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{4.2.13}$$

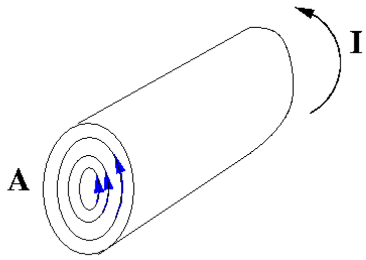
Note that from this equation we can visualize the vector potential. It is generally parallel to the current density and decreases as $\frac{1}{R}$ from the source.

For a long, straight wire, $\vec{A}$ is parallel to the wire and decreases as $\frac{1}{R}$ from the wire.



Vector Potential due to a long straight wire

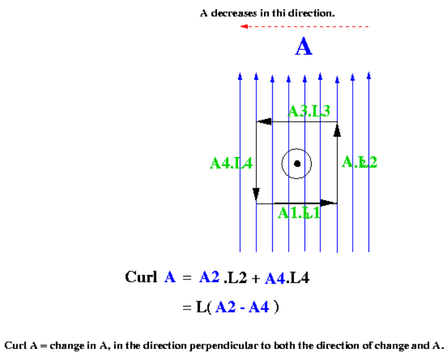
For a solenoid, $\vec{A}$, runs around in loops parallel to the current flow.



Vector Potential in a solenoid

We can use Stokes theorem to visualize how the curl differential operator works.

$$\int_{surface} \vec{B} \cdot \vec{ds} = \int_{surface} (\vec{\nabla} \times \vec{A}) \cdot \vec{ds} = \oint_{circuit} \vec{A} \cdot \vec{dl} \tag{4.2.14}$$



The curl is the rate of change in the Vector field as measured by a line integral around a loop in the field. The direction of the curl is perpendicular to the direction of vector field change and to the vector itself.

4.3 The Vector Potential and $\vec{E}$

Now we note that Faradays law gives:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.3.1}$$

then since $\vec{B} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$,

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \tag{4.3.2}$$

$$\Rightarrow \vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \tag{4.3.3}$$

Since $\vec{\nabla} \times \vec{\nabla} \cdot \vec{F} = 0 \ \ \forall \ \vec{F}$, and $\vec{E} = -\vec{\nabla} V$, for static fields,

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \tag{4.3.4}$$

So the generalized potential equations for time varying fields are:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{4.3.5}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{4.3.6}$$

4.4 Gauge Invariance

We note that since $\vec{B} = \vec{\nabla} \times \vec{A}$, and $\vec{\nabla} \times \vec{\nabla} \cdot \vec{A} = 0$ we can modify $\vec{A}$ by:

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{4.4.1}$$

where Λ is **ANY** scalar function and the magnetic field, $\vec{B}$, remains unchanged. However to keep $\vec{E}$ unchanged requires a simultaneous modification of the scalar potential:

$$V' = V - \frac{\partial \Lambda}{\partial t} \tag{4.4.2}$$

So that:

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\Lambda = \vec{B} \tag{4.4.3}$$

$$-\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla} V + \vec{\nabla}(\frac{\partial \Lambda}{\partial t}) - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \vec{\nabla}\Lambda}{\partial t} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \vec{B} \tag{4.4.4}$$

So the Electric and Magnetic fields are unchanged under the combined transformations:

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \ \ , \ \ V' = V - \frac{\partial \Lambda}{\partial t} \tag{4.4.5}$$

Where Λ is **ANY** scalar function.

These transformations are called Gauge Transformations and the invariance of $\vec{E}$ and $\vec{B}$ after a Gauge Transformation is called Gauge Invariance. The fact that $\vec{E}$ and $\vec{B}$ are unchanged under Gauge Transformations means that there is an arbitrariness in their definition. The full implications of are profound. In the sense that invariance under spatial translations gives conservation of momentum, invariance under time translations gives conservation of energy, invariance under Gauge Transformations implies the electromagnetic force. Maxwell's equations can be shown to be the consequence of Gauge Invariance and Special Relativity.

4.5 Maxwell's Equations and Potentials

The Maxwell Equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon} \tag{4.5.1}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu} + \epsilon \frac{\partial \vec{E}}{\partial t} = \vec{J} \tag{4.5.2}$$

can be written in terms of the potentials:

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = \frac{\rho}{\epsilon} \tag{4.5.3}$$

and

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \mu\epsilon \frac{\partial V}{\partial t}) = -\mu \vec{J} \tag{4.5.4}$$

These equations are coupled, meaning that each equation contains terms of both V and $\vec{A}$. We now exploit Gauge Invariance to choose $\vec{A'}$ and V' such that:

$$\vec{\nabla} \cdot \vec{A'} + \mu\epsilon \frac{\partial V'}{\partial t} = 0 \tag{4.5.5}$$

This uncouples the two equations to give:

$$\nabla^2 V' - \mu\epsilon \frac{\partial^2 V'}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{4.5.6}$$

$$\nabla^2 \vec{A'} - \mu\epsilon \frac{\partial^2 \vec{A'}}{\partial t^2} = -\mu \vec{J} \tag{4.5.7}$$

The are the *non homogeneous wave equations.* The condition required to decouple the equations is known as the **Lorentz Condition**.

4.6 The Lorentz Gauge

The Gauge Transformation required to decouple the equations in the scalar and magnetic potentials was as follows:

$$\vec{A'} = \vec{A} + \vec{\nabla}\Lambda \;\; and \;\; V' = V - \frac{\partial \Lambda}{\partial t} \tag{4.6.1}$$

with the requirement that the scalar field Λ satisfies:

$$\nabla^2\Lambda - \mu\epsilon\frac{\partial^2\Lambda}{\partial t^2} = -(\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t}) \tag{4.6.2}$$

We can verify that this satisfies the Lorentz Condition.

First make the gauge transformation $\vec{A} \rightarrow \vec{A}'$:

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\Lambda) \tag{4.6.3}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda \tag{4.6.4}$$

Now make the required corresponding transformation $V \rightarrow V'$:

$$V' = V - \frac{\partial\Lambda}{\partial t} \quad \Rightarrow \quad \mu\epsilon\frac{\partial V'}{\partial t} = \mu\epsilon\frac{\partial}{\partial t}(V - \frac{\partial\Lambda}{\partial t}) \tag{4.6.5}$$

$$\Rightarrow \mu\epsilon\frac{\partial V'}{\partial t} = \mu\epsilon\frac{\partial V}{\partial t} - \mu\epsilon\frac{\partial^2\Lambda}{\partial t^2} \tag{4.6.6}$$

Put them together and apply the condition on Λ.

$$\begin{aligned} So\ \ \vec{\nabla}\cdot\vec{A}' + \mu\epsilon\frac{\partial V'}{\partial t} &= \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \mu\epsilon\frac{\partial V}{\partial t} - \mu\epsilon\frac{\partial^2\Lambda}{\partial t^2} && (4.6.7)\\ &= \vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t} + (\nabla^2\Lambda - \mu\epsilon\frac{\partial^2\Lambda}{\partial t^2}) && (4.6.8)\\ &= 0 && (4.6.9) \end{aligned}$$

So if $\vec{A}, V$ satisfy the Lorentz condition, then any transformations $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, $V' = V + \frac{\partial\Lambda}{\partial t}$ where:

$$\nabla^2\Lambda - \mu\epsilon\frac{\partial^2\Lambda}{\partial t^2} = 0 \tag{4.6.10}$$

preserves the condition. All potentials in this restricted class are said to belong to the so-called "Lorentz Gauge". We have substantial freedom to modify the potentials. This can be exploited to provide a simpler description of the potentials. There are a number of other specific "Gauges" that exploit some symmetry to provide a simpler description of the Potentials.

4.7 The Coulomb Gauge

The "Coulomb Gauge" requires that:

$$\vec{\nabla}\cdot\vec{A} = 0 \tag{4.7.1}$$

Then we take the divergence of $\vec{E}$:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}) \tag{4.7.2}$$

$$\Rightarrow \quad \frac{\rho}{\epsilon} = -\nabla^2 V - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} \tag{4.7.3}$$

$$\Rightarrow \quad \nabla^2 V = -\frac{\rho}{\epsilon} \tag{4.7.4}$$

Which is the equation for a static scalar field. Note that through the appropriate choice of Gauge we have recovered a convenient form for V.

Now any equation of the form: $\nabla^2 a = -f$ has a solution,

$$a(\vec{r}) = \frac{1}{4\pi} \int_{V'} \frac{f(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.7.5}$$

So the solution of $\nabla^2 V = -\frac{\rho}{\epsilon}$ is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.7.6}$$

Which is the **instantaneous** Coulomb Potential due to the charge density $\rho(\vec{r})$. Hence, the name *Coulomb Gauge*.
The vector potential satisfies:

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \mu\epsilon \vec{\nabla} \frac{\partial V}{\partial t} \tag{4.7.7}$$

For a **static** field this becomes:

$$\nabla^2 \vec{A} = -\mu \vec{J} \tag{4.7.8}$$

Which has the familiar solution,

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r'})}{\mid \vec{r} - \vec{r'} \mid} dV' \tag{4.7.9}$$

Of course this equation is only valid for **static** fields. The Coulomb Gauge makes calculation of $\vec{A}$ for a time varying field *really* horrible.

4.8 Retarded Potentials and the Lorentz Gauge

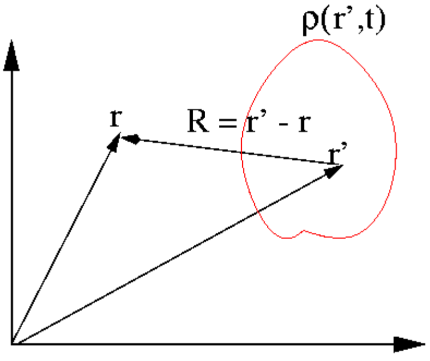
The above equations are clearly only valid for static fields. However the modifications for time varying fields are not so great. Simply one has to take account of the finite time for a change in charge or current densities to propagate to our point of measurement. This occurs at the speed of light.

So the equations for the potentials in a time varying field are:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho(\vec{r}',t-\frac{R}{c})}{R}dV' \tag{4.8.1}$$

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}',t-\frac{R}{c})}{R}dV' \tag{4.8.2}$$

where $R = \mid \vec{r}' - \vec{r} \mid$, $c =$ speed of light.



These equations are called *Retarded Potentials* because they explicitly show the effect of the finite propagation time of changes in the charge and current density distributions. The calulation of the potential requires that the densities be evaluated at earlier times depending on the distance to the point of measurement.

These equations for the potentials also satisfy the the Lorentz Condition that:

$$\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t} = 0 \tag{4.8.3}$$

So that these potentials are members of the Lorentz Gauge. Explicitly:

$$\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t} = \vec{\nabla}\cdot\frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}',t-\frac{R}{c})}{R}dV' + \mu\epsilon\frac{\partial}{\partial t}\frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho(\vec{r}',t-\frac{R}{c})}{R}dV' \tag{4.8.4}$$

Since $\vec{\nabla}$, operates with respect to $\vec{r}$, not $\vec{r}'$:

$$= \frac{\mu}{4\pi}\int_{V'}\frac{\vec{\nabla}\cdot\vec{J}(\vec{r}',t-\frac{R}{c})}{R}dV' + \frac{\mu}{4\pi}\int_{V'}\frac{\frac{\partial}{\partial t}\rho(\vec{r}',t-\frac{R}{c})}{R}dV' \tag{4.8.5}$$

Then from conservation of charge, $\vec{\nabla}\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t}$ substituting we get

$$= \frac{\mu}{4\pi}\int_{V'}\frac{-\frac{\partial}{\partial t}\rho(\vec{r}',t-\frac{R}{c})}{R}dV' + \frac{\mu}{4\pi}\int_{V'}\frac{\frac{\partial}{\partial t}\rho(\vec{r}',t-\frac{R}{c})}{R}dV' \tag{4.8.6}$$

$$= 0 \tag{4.8.7}$$

as required.