LECTURE NOTES 13

ELECTROMAGNETIC RADIATION

In P436 Lect. Notes 4-10.5 (Griffiths *ch*. 9-10}, we discussed the propagation of macroscopic *EM* waves, but we did not discuss *how* macroscopic *EM* waves are created. Using what we learned in P436 Lect. Notes 12, we can now discuss how macroscopic *EM* waves *are* created.

"Encrypted" into Maxwell's equations:

1) $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t) = -\dfrac{\rho_{tot}(\vec{r},t)}{\varepsilon_o}}$ 3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

2) $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t) = 0}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_o\vec{J}_{tot}(\vec{r},t) + \mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

is the physics associated with radiation of electromagnetic waves/electromagnetic energy, arising from the acceleration {and/or deceleration} of electric charges (and/or electric currents).

In the P436 Lecture Notes #12, we derived the (retarded) electromagnetic fields associated with a moving point charge q from the Liénard-Wiechert potentials:

$$\boxed{V_{\mathrm{r}}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{\kappa\mathfrak{r}}} \quad \text{where: } \boxed{\mathfrak{r} \equiv c\Delta t_r = c(t - t_{\mathrm{r}})}, \quad \boxed{\vec{\mathfrak{r}} \equiv \vec{r} - \vec{r}'(t_{\mathrm{r}})}$$

$$\boxed{\vec{A}_{\mathrm{r}}(\vec{r},t) = \frac{\mu_o q}{4\pi}\frac{\vec{v}(t_{\mathrm{r}})}{\kappa\mathfrak{r}}} \quad \text{and: } \boxed{\kappa \equiv 1 - \hat{\mathfrak{r}}\bullet\vec{v}(t_{\mathrm{r}})/c = 1 - \hat{\mathfrak{r}}\bullet\vec{\beta}(t_{\mathrm{r}})} = \text{"retardation" factor}$$

With: $\boxed{\vec{A}_{\mathrm{r}}(\vec{r},t) = \vec{\beta}(t_{\mathrm{r}})\left(V_{\mathrm{r}}(\vec{r},t)/c\right)}$ $\boxed{\vec{\beta}(t_{\mathrm{r}}) \equiv \vec{v}(t_{\mathrm{r}})/c}$ and: $\boxed{c^2 = 1/\varepsilon_o\mu_o}$

We also derived the corresponding {retarded} electric and magnetic fields associated with a moving point charge q:

$$\boxed{\vec{E}_{\mathrm{r}}(\vec{r},t) = -\vec{\nabla}V_{\mathrm{r}}(\vec{r},t) - \frac{\partial\vec{A}_{\mathrm{r}}(\vec{r},t)}{\partial t}}$$

$$\boxed{\vec{E}_{\mathrm{r}}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathfrak{r}}{\left(\vec{\mathfrak{r}}\cdot\vec{u}(t_r)\right)^3}\left[\overbrace{\left(c^2 - v^2(t_{\mathrm{r}})\right)\vec{u}(t_{\mathrm{r}})}^{\substack{\text{term for generalized}\\ \text{Coulomb field/velocity field}}} + \overbrace{\vec{\mathfrak{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)}^{\substack{\text{term for radiation/}\\ \text{acceleration field}}}\right]}$$

$$\boxed{\vec{B}_{\mathrm{r}}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\mathrm{r}}(\vec{r},t)} \quad \text{where: } \boxed{\vec{u}(t_{\mathrm{r}}) \equiv c\hat{\mathfrak{r}} - \vec{v}(t_{\mathrm{r}})}$$

$$\boxed{\vec{B}_{\mathrm{r}}(\vec{r},t) = \frac{1}{c}\frac{q}{4\pi\varepsilon_o}\frac{1}{\left(\vec{\mathfrak{r}}\bullet\vec{u}(t_{\mathrm{r}})\right)^3}\hat{\mathfrak{r}}\times\left[\overbrace{\left(c^2 - v^2(t_{\mathrm{r}})\right)\vec{u}(t_{\mathrm{r}})}^{\substack{\text{term for generalized}\\ \text{Coulomb field/velocity field}}} + \overbrace{\vec{\mathfrak{r}}\times\left(\vec{u}(t_{\mathrm{r}})\times\vec{a}(t_{\mathrm{r}})\right)}^{\substack{\text{term for radiation/}\\ \text{acceleration field}}}\right]} \quad \boxed{\vec{B}_{\mathrm{r}}(\vec{r},t) = \frac{1}{c}\hat{\mathfrak{r}}\times\vec{E}_{\mathrm{r}}(\vec{r},t)}$$

Microscopically:

The acceleration {and/or deceleration} of electric charges q and/or time-varying electric current densities $\left(e.g.\ \vec{J}=nq\vec{v};\ \partial\vec{J}/\partial t = nq\,\partial\vec{v}/\partial t \sim nq\vec{a}\right)$ "converts" (a portion of the) virtual photons (associated with the "static" Coulomb field, which individually have zero total energy/zero-frequency) to real photons (which individually have finite total energy/finite frequency f), which then freely propagate outward/away from the source of time-varying electric charge and/or electric current at the speed of light, c {in vacuum / free space}.

Since real photons individually carry energy/linear momentum/angular momentum, macroscopic *EM* waves carry energy/linear momentum angular momentum away from the source, in an irreversible manner – these *EM* waves propagate away from the source $\forall$ time. Energy/momentum must be input to the charged particle for this to happen – energy/momentum are {both} conserved in the radiation process.

{Note also that we can reverse the arrow of time $t \to -t$ in this process and thus learn about the absorption of energy/linear momentum/angular momentum by electric charges/currents from incoming/incident *EM* waves. . . .}

The total instantaneous power $P_r(\vec{r},t)$ associated with radiation of *EM* waves from a source (assumed to be localized) is obtained by integrating the retarded Poynting's vector $\vec{S}_r(\vec{r},t)$ over a large spherical shell of radius $\underline{r \gg a}$ = characteristic dimension of localized source – this is known as the "far-field" limit, when $r \to \infty$:

$$\boxed{P_r(\vec{r},t)=\oint_{S'}\vec{S}_r(\vec{r}',t)\cdot d\vec{a}'_\perp = \frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_r(\vec{r}',t)\times\vec{B}_r(\vec{r}',t)\right)\cdot d\vec{a}'_\perp}$$

The instantaneous power radiated is the limit of $P_r(\vec{r},t)$ as $r\to\infty$: $\boxed{P_r^{rad}(t)\equiv \lim_{r\to\infty} P_r(\vec{r},t)}$

The physical reason for this definition is simple. In the so-called "near-zone", when $r \lesssim a$, the (generalized) Coulomb field(s) (microscopically consisting of virtual photons) are dominant in this region – thus, time-varying but non-radiating $\vec{E}$ and $\vec{B}$ fields are present in proximity to the source. These near-zone *EM* fields fall off/decrease/diminish as $\sim 1/r^2$ from the source.

In reality, for finite r, there is always a mixture of radiating and non-radiating *EM* fields present that is associated with any source. Expressed in a graphical manner in terms of r/a:

r/a

$\underbrace{\ldots\ 10^{-4}\quad 10^{-3}\quad 10^{-2}\quad 10^{-1}\quad 10^{0}=1}_{\text{"near-zone" regime: } \left(\frac{r}{a}\lesssim 1\right)}\quad 10\quad \underbrace{100\quad 10^{3}\quad 10^{4}\quad 10^{5}\ldots\ \infty}_{\text{"far-zone" regime } \left(\frac{r}{a}\gg 1\right)}$

Generalized Coulomb field(s) dominant (virtual photons)

Radiation/acceleration field(s) dominant (freely propagating real photons)

The instantaneous *EM* power associated with the Generalized Coulomb field is:

$$\boxed{P_{\rm r}^{GCF}(\vec{r},t)=\oint_{S'}\vec{S}_{\rm r}^{GCF}(\vec{r}',t)\bullet d\vec{a}'_{\perp}=\frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_{\rm r}^{GCF}(\vec{r}',t)\times\vec{B}_{\rm r}^{GCF}(\vec{r}',t)\right)\bullet d\vec{a}'}$$

<u>But</u>: $\boxed{\vec{E}_{\rm r}^{GCF}(\vec{r},t)\sim 1/r^2}$ (even faster than this, if the <u>net</u> charge = 0, *e.g.* for higher order *EM* <u>moments</u> associated with <u>electric</u> dipoles, quadrupoles, octupoles, etc. …)

<u>And</u>: $\boxed{\vec{B}_{\rm r}^{GCF}(\vec{r},t)\sim 1/r^2}$ (even faster than this, if the <u>net</u> charge = 0, *e.g.* for higher order *EM* <u>moments</u> associated with <u>magnetic</u> dipoles, quadrupoles, octupoles, etc. …)

$\Rightarrow$ $\boxed{\vec{S}_{\rm r}^{GCF}(\vec{r},t)\sim 1/r^4}$ (ever faster, for high-order *EM* moments than a point charge distribution)

<u>But</u>: $\boxed{A_{\perp}^{\rm sphere}=4\pi r^2}$ = area of sphere of radius r.

$\therefore$ $\boxed{P_{\rm r}^{GCF}(\vec{r},t)\sim\frac{1}{r^4}\bullet r^2\sim\frac{1}{r^2}}$ $\Rightarrow$ *EM* Power associated with Generalized Coulomb fields is only appreciable <u>near</u> the source.

Note that $\boxed{\lim_{r\to\infty}P_{\rm r}^{GCF}(\vec{r},t)=0}$ *i.e.* no *EM* power is associated with G.C.F. at $r=\infty$

$\Rightarrow$ "static" sources do not radiate *EM* energy.

On the other hand, the instantaneous *EM* power associated with the radiation/acceleration fields is:

$$\boxed{P_{\rm r}^{rad}(\vec{r},t)=\oint_{S'}\vec{S}_{\rm r}^{rad}(\vec{r}',t)\bullet d\vec{a}'_{\perp}=\frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_{\rm r}^{rad}(\vec{r}',t)\times\vec{B}_{\rm r}^{rad}(\vec{r}',t)\right)\bullet d\vec{a}'_{\perp}}$$

<u>But</u>: $\boxed{\vec{E}_{\rm r}^{rad}\sim 1/r}$ and $\boxed{B_{\rm r}^{rad}\sim 1/r}$ $\Rightarrow$ $\boxed{\vec{S}_{\rm r}^{rad}(\vec{r},t)\sim 1/r^2}$, $\boxed{A_{\perp}^{sphere}\sim r^2}$

$\therefore$ $\boxed{P_{\rm r}^{rad}(\vec{r},t)\sim 1}$ (*i.e.* $P_{\rm r}^{rad}(\vec{r},t)$ is <u>independent</u> of the radius of the enclosing surface S')

Thus, we can <u>simply</u> pick $r\to\infty$ to <u>eliminate</u> the $P_{\rm r}^{GCF}(\vec{r},t)$ <u>contribution</u>!!!

{*n.b.* for <u>non-localized</u> sources of time-varying *EM* radiation – *e.g.* <u>infinite</u> planes, <u>infinitely</u> long wires, <u>infinite</u> solenoids, *etc*. this requires a different approach altogether… }

In general, <u>*arbitrary*</u> configurations of <u>localized</u>, time-dependent electric charge and/or electric current density distributions, $\boxed{\partial\rho(t_{\rm r})/\partial t_{\rm r}\equiv\dot{\rho}(t_{\rm r})}$ and $\boxed{\partial\vec{J}(t_{\rm r})/\partial t_{\rm r}\equiv\dot{\vec{J}}(t_{\rm r})}$ can/do produce *EM* radiation/freely-propagating *EM* waves.

As we learned in P435 (last semester), from the principal of linear superposition, we can always decompose an <u>arbitrary</u> electric charge and/or current distribution into a linear combination of *EM* moments of the electric charge/current distribution, *i.e.* electric monopole (electric charge), electric and magnetic dipole, electric and magnetic quadrupole, etc. … moments. This is true {separately} for both static and time-varying *EM* moments of the electric charge and/or current distribution(s).

For a point electric monopole field {E(0)}, *i.e.* $\boxed{\rho(\vec{r},t_r) = q\delta^3(t_r - (t - r/c))}$

$$\boxed{V_r^{(E0)}(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho(\vec{r}',t_r)}{r}d\tau' = \frac{q(t_r = (t - r/c))}{4\pi\varepsilon_o}}$$

Where $q(t)$ = total electric charge of the source at the time *t*. But electric charge is (always) conserved, and furthermore, (by definition) a <u>localized</u> source is one that does <u>*not*</u> have electric charge *q* flowing into or away from it. Therefore, the electric monopole moment contribution/portion associated with the (retarded) potential(s) and *EM* fields is of necessity <u>*static*</u> – *i.e.* the electric monopole moment *q* has <u>*no*</u> *EM* <u>radiation</u> associated with it. In other words, there can be no <u>*net*</u> <u>transversely</u> polarized *EM* radiation emitted from a <u>spherically-symmetric</u> charge distribution! {See *e.g.* J. D. Jackson *Classical Electrodynamics* 3rd ed. *p.* 410 for additional/further details.}

The lowest-order <u>*electric*</u> multipole moment capable of producing *EM* radiation is that associated with a time-varying <u>electric</u> <u>dipole</u> <u>moment</u>, $\boxed{\vec{p}(\vec{r}',t_r) = q\vec{d}(\vec{r}',t_r)}$.

$\Rightarrow$ Electric Dipole (E1) radiation originates from $\boxed{\dot{\rho}(\vec{r}',t_r)}$

The lowest-order <u>*magnetic*</u> multipole moment capable of producing *EM* radiation is that associated with a time-varying <u>magnetic</u> <u>dipole</u> <u>moment</u>, $\boxed{\vec{m}(\vec{r}',t_r) = I\vec{a}(\vec{r}',t_r)}$.

$\Rightarrow$ Magnetic Dipole (M1) radiation originates from $\boxed{\dot{\vec{J}}(\vec{r}',t_r)}$

Each time-varying, localized, higher-order *EM* moment contributes in alternating <u>succession</u> between $\underline{\dot{\rho}(\vec{r}',t_r)}$ and $\underline{\dot{\vec{J}}(\vec{r}',t_r)}$ (*i.e.* electric vs. magnetic):

Time-varying localized electric *EM* moments: $\dot{\rho}(\vec{r}',t_r)$	Time-varying localized magnetic *EM* moments: $\dot{\vec{J}}(\vec{r}',t_r)$
$\dot{E}(0)$ electric monopole $\dot{q}$ <u>NO</u>!	
	$\dot{M}(0)$ magnetic monopole $\dot{g}$ <u>NO</u>! (no magnetic charges anyways…)
$\dot{E}(1)$ electric dipole $\dot{\vec{p}} = q\dot{\vec{d}}$	
	$\dot{M}(1)$ magnetic dipole $\dot{\vec{m}} = I\dot{\vec{a}}$
$\dot{E}(2)$ electric quadrupole $\dot{\overleftrightarrow{Q}}_E = 2q\dot{\vec{d}}\vec{d}$	
	$\dot{M}(2)$ magnetic quadrupole $\dot{\overleftrightarrow{Q}}_m = 2I\dot{\vec{a}}\vec{a}$
$\dot{E}(3)$ electric octupole	
	$\dot{M}(3)$ magnetic octupole
$\dot{E}(4)$ electric sextupole	
… *etc*…	$\dot{M}(4)$ magnetic sextupole

$\therefore$ We will consider/discuss the case of *EM* radiation from an oscillating E(1) electric dipole and then discuss case of radiation from an arbitrary localized source consisting of an arbitrary linear combination of time-varying *EM* moments, $\sum_{n=1}^{\infty}\left(a_n\dot{E}(n)+b_n\dot{M}(n)\right)$, where $\dot{E}(n)$ and $\dot{M}(n)$ are n^{th}-order time-varying electric and magnetic multipole moments, respectively.

<u>E(1) Electric Dipole Radiation</u>:

Consider an oscillating (*i.e.* harmonic/sinusoidally time-varying) electric dipole: $\boxed{\vec{p}(t)=q\vec{d}(t)}$ where the charge separation distance varies in time as: $\boxed{\vec{d}(t)=\vec{d}(t)\hat{z}=d_o\cos(\omega t)\hat{z}}$, $\boxed{\omega=2\pi f}$

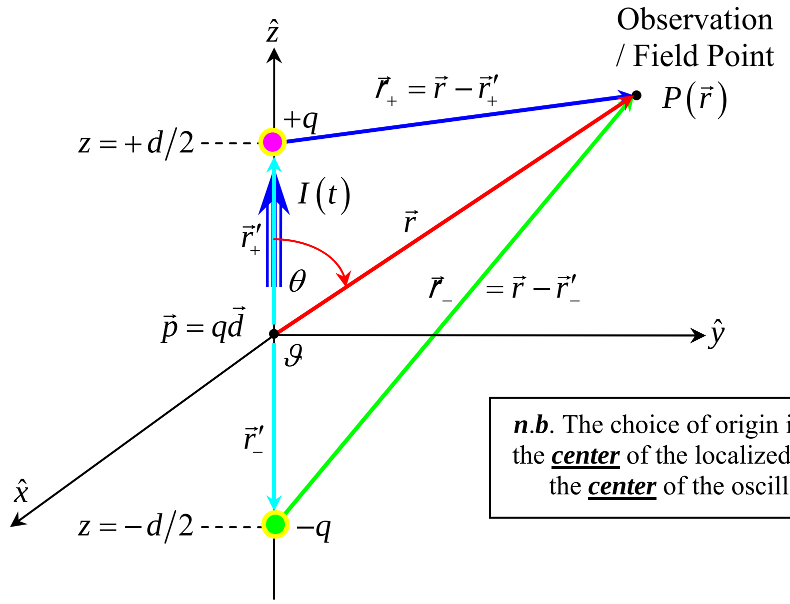
<u>Then</u>: $\boxed{\vec{p}(t)=qd_o\cos(\omega t)\hat{z}=p_o\cos(\omega t)\hat{z}}$, <u>with</u>: $\boxed{p_o=qd_o}$.

<u>Equivalently</u>, we *<u>can</u>* <u>alternatively</u> think of this as: $\boxed{\vec{p}(t)=q(t)\vec{d}}$, with $\boxed{\vec{d}=d\hat{z}}$ = constant, and with time-varying/oscillating electric charge: $\boxed{q(t)=q_o\cos(\omega t)}$.

<u>Then</u>: $\boxed{\vec{p}(t)=q_od\cos(\omega t)\hat{z}=p_o\cos(\omega t)\hat{z}}$, <u>with</u>: $\boxed{p_o=q_od}$. {*n.b.* same result!}

Either way one views this , the physical picture is of a harmonically time-varying/oscillating electric dipole moment $\boxed{\vec{p}(t)=p_o\cos(\omega t)\hat{z}=qd\cos\omega t\hat{z}}$ a picture of which, for a given moment/instant/snapshot in time is shown below, for $t=0$:

n.b. $\exists$ an electric <u>current</u> associated with the oscillating electric dipole: $\boxed{\vec{I}(t)=\frac{dq(t)}{dt}\hat{z}}$, $\boxed{\vec{I}(t=0)=0}$



n.b. The choice of origin is deliberately chosen at the ***<u>center</u>*** of the localized charge distribution – at the ***<u>center</u>*** of the oscillating electric dipole.

n.b. $\exists$ exist (as always) some <u>subtleties</u> associated with the calculation of retarded potentials associated with *<u>moving</u>* point charges – we will address these subsequently, but not right here / right now… we'll stick with the oscillating charge $\boxed{q(t)=q_o\cos(\omega t)}$ version for now…

Now $\vec{p}(t) = p\cos(\omega t)\hat{z}$ refers to the time-dependence associated with itself. An observer at field point $P(\vec{r})$ at $\vec{r}$ "sees" the effects of the time-varying $\vec{p}(t)$ manifest themselves at a finite time later, $\boxed{t = t_r + r/c}$ or: $\boxed{t_r = t - r/c}$ due to the retarded nature of this problem.

Thus, $\vec{p}(t)$ used in the formulae for the retarded scalar and vector potentials needs to be evaluated at the retarded time t_r, *i.e.* $\boxed{\vec{p}(t) \to \vec{p}(t_r)}$.

$$\boxed{V_r^{E(1)}(\vec{r},t) = \underbrace{\left(\frac{q_o}{4\pi\varepsilon_o}\right)\frac{\cos(\omega t_r^+)}{r_+}}_{\substack{\text{charge at}\\ +d/2\hat{z}}} - \underbrace{\left(\frac{q_o}{4\pi\varepsilon_o}\right)\frac{\cos(\omega t_r^-)}{r_-}}_{\substack{\text{charge at}\\ -d/2\hat{z}}} = \left(\frac{q_o}{4\pi\varepsilon_o}\right)\left[\frac{\cos(\omega t_r^+)}{r_+} - \frac{\cos(\omega t_r^-)}{r_-}\right]}$$

$$\boxed{\vec{A}_r^{E(1)}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right)\int \frac{I(t_r)}{r} d\vec{\ell}'} \quad \text{where: } \boxed{I(t_r) = -q_o\omega\sin(\omega t_r)} \quad \underline{\text{and}}\text{: } \boxed{d\vec{\ell}' = dz\hat{z}}$$

Explicitly putting in the retarded time: $\boxed{t_r = t - r/c}$:

$$\boxed{V_r^{E(1)}(r,t) = \left(\frac{q_o}{4\pi\varepsilon_o}\right)\left[\frac{\cos(\omega(t - r_+/c))}{r_+} - \frac{\cos(\omega(t - r_-/c))}{r_-}\right]}$$

$$\boxed{\vec{A}_r^{E(1)}(\vec{r},t) = -\left(\frac{\mu_o q_o \omega}{4\pi}\right)\int_{z=-d/2}^{z=+d/2} \frac{\sin[\omega(t - r/c)]}{r} dz\,\hat{z}}$$

Let us first focus our attention on calculating out $V_r^{E(1)}(r,t)$. From the law of cosines {see P435 Lecture Notes 8 *r.e.* the derivation of the *static* multipole moment expansion}:

$$\boxed{r_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}}$$

However, we want to investigate *EM* radiation in the "far zone" when $r \gg d$. For this situation:

$$\boxed{r_\pm = r\sqrt{1 \mp \left(\frac{d}{r}\right)\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2} \approx r\sqrt{1 \mp \left(\frac{d}{r}\right)\cos\theta}}$$

Or: $\boxed{r_\pm \simeq r\left(1 \mp \frac{1}{2}\left(\frac{d}{r}\right)\cos\theta\right)}$ for $r \gg d$.

Similarly/correspondingly:

$$\boxed{\frac{1}{r_\pm} \simeq \frac{1}{r\left(1 \mp \frac{1}{2}\left(\frac{d}{r}\right)\cos\theta\right)} \approx \frac{1}{r}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)} \quad \text{for } r \gg d.$$

Likewise, for $\cos\left(\omega\left(t - r_{\pm}/c\right)\right)$ we have, for the "far zone", when $\underline{\underline{r \gg d}}$:

$$\boxed{\begin{aligned}\cos\left(\omega\left(t - r_{\pm}/c\right)\right) &\simeq \cos\left[\omega\left(t - \frac{r}{c}\left(1 \mp \frac{d}{2r}\cos\theta\right)\right)\right] = \cos\left[\omega\left(t - \frac{r}{c}\right) \pm \left(\frac{\omega d}{2c}\right)\cos\theta\right] \\ &= \cos\left[\omega\left(t - \frac{r}{c}\right)\right]\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \pm \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right]\end{aligned}}$$

In order to proceed further we need to make an <u>additional</u> <u>simplifying</u> assumption, namely that the characteristic spatial dimension of the source (here, $a = d$) is $\ll$ wavelength λ of the emitted radiation, *i.e.* $d \ll \lambda$, where $\lambda = c/f$. Thus: $d \ll c/f$, where: $f = \omega/2\pi$, or: $d \ll 2\pi c/\omega$ or: $d \ll c/\omega$

n.b. This assumption is tantamount/physically equivalent to saying that we neglect any/all <u>time-retardation</u> effects associated with finite *EM* propagation delay times over the dimensions characteristic of/associated with the source – *i.e.* changes in charge/current are essentially <u>coherent/ instantaneous.</u>

Suppose we have a source (*e.g.* an atom) with $a = d = 1\,nm = \underline{\underline{10\,\mathring{A}}}$ emitting a $f = 1\,Hz$ sine-wave. Since *EM* radiation travels propagates at $1\,ft \simeq 30\,cm$ per nanosecond, a $1\,nm$ dimension source doesn't run into finite propagation decay time problems until:

$$c\Delta t \approx a = d \text{ (here) } \; i.e. \; \underline{\underline{c\Delta t \approx 1\,nm}} \Rightarrow \Delta t \simeq \frac{10^{-9}}{3\times 10^{8}} \simeq 0.3\times 10^{-17}\text{ sec} \Rightarrow \underline{\underline{f \simeq 3\times 10^{17}\,Hz}}$$

Thus, provided that we ***additionally*** <u>are</u> in the regime of $d \ll \lambda$, or $d \ll c/\omega$, *i.e.* $\boxed{\left(\frac{\omega d}{c}\right) \ll 1}$.

Then from the Taylor series expansion of $\cos(x) \approx 1$ and $\sin(x) \approx x$ for ***<u>very</u>*** small *x*, we see

<u>That</u>: $\boxed{\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \cos(0) \simeq 1}$ <u>and</u>: $\boxed{\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \left(\frac{\omega d}{2c}\right)\cos\theta}$

<u>Thus</u>: $\boxed{\cos\left(\omega\left(t - r_{\pm}/c\right)\right) \approx \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \pm \left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t - \frac{r}{c}\right)\right]}$

<u>Thus</u>:

$$\boxed{\begin{aligned}V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) \simeq \frac{q_o}{4\pi\varepsilon_o}\Bigg\{&\frac{1}{r}\left(1 + \left(\frac{d}{2r}\right)\cos\theta\right)\left[\cos\left[\omega\left(t - \frac{r}{c}\right)\right] - \left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right] \\ &- \frac{1}{r}\left(1 - \left(\frac{d}{2r}\right)\cos\theta\right)\left[\cos\left[\omega\left(t - \frac{r}{c}\right)\right] + \left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t - \frac{r}{c}\right)\right]\right]\Bigg\}\end{aligned}}$$

Expanding this out:

$$\boxed{\begin{aligned} V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) \simeq \frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\Bigg\{ & \cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \\ & -\left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \cancel{\frac{\omega d^2}{4rc}\cos^2\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \\ & -\cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \\ & -\left(\frac{\omega d}{2c}\right)\cos\omega\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \cancel{\frac{\omega d^2}{4rc}\cos^2\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}\Bigg\} \end{aligned}}$$

Thus:
$$\boxed{\begin{aligned} V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) &\simeq \frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left\{\left(\frac{d}{r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega d}{c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &= \frac{q_o d}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r}\right)\left\{\left(\frac{1}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \end{aligned}}$$

But: $\boxed{p_o = |\vec{p}| = q_o d}$

$$\therefore \quad \boxed{V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) \simeq \frac{p_o}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r}\right)\left\{-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+\left(\frac{1}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}}$$

In the "far-zone" $d \ll r$, with the *additional* restriction that we've also imposed on the source *EM* radiation: $d \ll \lambda$. We now *additionally* require/impose a *third* restriction that the "far-zone" also be such that $\lambda \ll r$, thus we have the hierarchical relation: $\boxed{d \ll \lambda \ll r}$ for "far-zone" *EM* radiation, namely that for $\boxed{\lambda \ll r} \rightarrow \boxed{\left(\frac{c}{\omega}\right) \ll r}$, then $\rightarrow \boxed{\left(\frac{\omega}{c}\right) \gg \left(\frac{1}{r}\right)}$ *i.e.* $\boxed{\frac{1}{\lambda} \gg \frac{1}{r} \text{ for } \lambda \ll r}$.

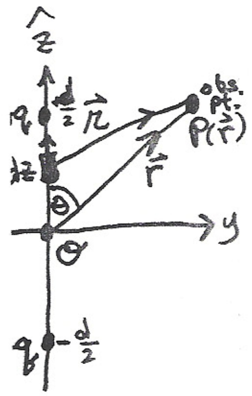
Thus for the far-zone, when $\boxed{d \ll \lambda \ll r}$ we can neglect the second term in the above expression for $V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t)$.

Then: $\boxed{V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) \simeq -\frac{p_o\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$ in the far-zone, for $\boxed{d \ll \lambda \ll r}$.

Note that in the static limit, when $\omega \rightarrow 0$ it *is* necessary to retain the second term in the above expression; we obtain in this limit: $\boxed{V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta) \simeq -\frac{p_o}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r^2}\right)}$ {*cf* w/ P435 Lect. Notes – same!}

Now let us focus our attention on calculating $\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t)$:

$$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = -\left(\frac{\mu_o q_o \omega}{4\pi}\right)\int_{z=-d/2}^{z=+d/2} \frac{\sin\left[\omega(t - \mathscr{r}/c)\right]}{\mathscr{r}}\, dz\, \hat{z}}$$



Because the integration itself introduces a factor of *d*, then to <u>first order</u> in $(d/r) \ll 1$: $\boxed{\mathscr{r} \simeq \sqrt{r^2 - 2rz\cos\theta + z^2}}$ <u>with</u>: $\boxed{|z| \le \frac{d}{2}}$

<u>Thus</u>: $$\boxed{\int_{z=-d/2}^{z=+d/2} \frac{\sin\left[\omega(t-\mathscr{r}/c)\right]}{\mathscr{r}}\, dz \simeq \frac{\sin\left[\omega(t-\mathscr{r}/c)\right]}{r}\, d}$$

<u>Then</u>: $$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t) \simeq -\frac{\mu_o (q_o d)\omega}{4\pi}\left(\frac{1}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{z}}$$ <u>but</u>: $\boxed{p_o = q_o d}$

<u>Thus</u>: $$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t) \simeq -\frac{\mu_o p_o \omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}}$$

Note that in the <u>static</u> <u>limit</u>, when $\omega \to 0$ then $\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t) \to 0$ as we expect.

Now that we have obtained the (retarded) scalar and vector potentials $V_{\mathrm{r}}^{\mathrm{E}(1)}(r,t)$ and $\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t)$ it is a "straight forward' exercise to compute the associated (retarded) EM fields, $\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t)$ and $\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)}(r,t)$:

$$\boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = -\vec{\nabla}V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) - \frac{\partial \vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t)}{\partial t}} \quad \text{and:} \quad \boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t)}$$

In <u>spherical</u> coordinates:

$$\begin{aligned}
\vec{\nabla}V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) &\simeq \left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right]\overbrace{\left\{\frac{-p_o\omega}{4\pi\varepsilon_o c}\frac{\cos\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}}^{\text{n.b. } V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) \text{ has no explicit } \hat{\varphi} \text{ dependence}} \\
&= -\frac{p_o\omega}{4\pi\varepsilon_o c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{rc}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{r} - \frac{1}{r^2}\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\} \\
&= +\frac{p_o\omega}{4\pi\varepsilon_o c r}\left\{\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r} + \left(\frac{1}{r}\right)\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} + \sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right]\right\}
\end{aligned}$$

But for "far-zone" *EM* radiation, $\underline{\underline{\boxed{d \ll \lambda \ll r}}}$ we have: $\boxed{\frac{\omega}{c} \gg \frac{1}{r}}$

$$\therefore \quad \boxed{\underbrace{\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\sim\vartheta(1)}\cos\theta \gg \left(\frac{1}{r}\right)\left\{\underbrace{\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\sim\vartheta(1)} + \underbrace{\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\sim\vartheta(1)}\right\}}$$

So we can neglect/drop the $\boxed{\left(\frac{1}{r}\right)\left\{\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}+\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right]\right\}}$ terms.

$$\therefore \quad \boxed{\vec{\nabla}V_{\rm r}^{\rm E(1)}(\vec{r},t)\simeq+\frac{p_o\omega^2\cos\theta}{4\pi\varepsilon_o c^2 r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}$$

And: $\boxed{\frac{\partial\vec{A}_{\rm r}^{\rm E(1)}(\vec{r},t)}{\partial t}\simeq-\frac{\mu_o p_o\omega}{4\pi}\left(\frac{1}{r}\right)\frac{\partial}{\partial t}\left(\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{z}=-\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}}$

But: $\boxed{\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}}$ in spherical coordinates.

$$\therefore \quad \boxed{\frac{\partial\vec{A}_{\rm r}^{\rm E(1)}(\vec{r},t)}{\partial t}\simeq-\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$$

Then for far-zone *EM* radiation, with $\boxed{d\ll\lambda\ll r}$: $\boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)=-\vec{\nabla}V_{\rm r}^{\rm E(1)}(\vec{r},t)-\frac{\partial\vec{A}_{\rm r}^{\rm E(1)}(\vec{r},t)}{\partial t}}$

$$\boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)\simeq-\frac{p_o\omega^2}{4\pi\varepsilon_o c^2 r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r}+\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$$

But: $\boxed{c^2=\frac{1}{\varepsilon_o\mu_o}}$ or: $\boxed{\frac{1}{c^2}=\varepsilon_o\mu_o}$

$$\therefore \quad \vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)\simeq\cancel{-\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r}}+\cancel{\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r}}-\frac{\mu_o p_o\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta\hat{\theta}$$

Or: $\boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)\simeq-\frac{\mu_o p_o\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$

Then: $\boxed{\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\rm r}^{\rm E(1)}(\vec{r},t)}$

with: $\boxed{\vec{A}_{\rm r}^{\rm E(1)}(r,t)\simeq-\frac{\mu_o p_o\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-{}^{r}\!/_{c}\right)\right]\hat{z}=-\frac{\mu_o p_o\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-{}^{r}\!/_{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$

Thus:

$$\boxed{\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\overset{=0}{\cancel{A_\varphi}}\right)-\overset{=0}{\cancel{\frac{\partial A_\theta}{\partial\varphi}}}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\overset{=0}{\cancel{\frac{\partial A_r}{\partial\varphi}}}-\frac{\partial}{\partial r}\left(r\overset{=0}{\cancel{A_\varphi}}\right)\right]\hat{\theta}+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}}$$

Thus:

$$\boxed{\begin{aligned}\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t) &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}\\ &\simeq \left(\frac{1}{r}\right)\left[-\frac{\mu_o p_o\omega}{4\pi}\right]\left\{\frac{\partial}{\partial r}\left(\frac{1}{\cancel{r}}\bullet\cancel{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right](-\sin\theta)\right)-\frac{1}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\frac{\partial\cos\theta}{\partial\theta}\right\}\hat{\varphi}\\ &\simeq -\frac{\mu_o p_o\omega}{4\pi r}\left\{+\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta+\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta\right\}\hat{\varphi}\\ &\simeq -\frac{\mu_o p_o\omega}{4\pi r}\left\{\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\underbrace{\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\therefore\text{ neglect}}\right\}\sin\theta\hat{\varphi}\end{aligned}}$$

Again, $\boxed{\left(\frac{\omega}{c}\right)\gg\left(\frac{1}{r}\right)}$ here, because $\boxed{\lambda\ll r}$, thus

$$\therefore\ \boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)\simeq-\frac{\mu_o p_o\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}\ \underline{\text{and}}\text{:}\ \boxed{\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)\simeq-\frac{\mu_o p_o\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

Now since $\boxed{\hat{r}\times\hat{\theta}=\hat{\varphi}}$, once again we see that: $\boxed{\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)=\frac{1}{c}\hat{r}\times\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)}$, *i.e.* $\vec{B}\perp\vec{E}$ and $\vec{B}\perp\hat{r}$

Note also that:

a.) $\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}$ and $\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}$ both vary as $\sim 1/r$.

b.) $\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)$ and $\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)$ are in phase with each other.

c.) $\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)$ and $\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)$ have the same angular dependence ($\sim\sin\theta$).

The *EM* radiation energy density, $u_{\mathrm{E(1)}}^{rad}(\vec{r},t)$ associated with the oscillating E(1) electric dipole for far-zone *EM* radiation $\{d\ll\lambda\ll r\}$ is:

$$\boxed{\begin{aligned}u_{\mathrm{E(1)}}^{rad}(\vec{r},t)&=u_{\mathrm{E(1)}}^{Erad}(\vec{r},t)+u_{\mathrm{E(1)}}^{Mrad}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)\bullet\vec{E}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)+\frac{1}{\mu_o}\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)\bullet\vec{B}_{\mathrm{r}}^{\mathrm{E(1)}}(\vec{r},t)\right)\\ &\simeq\frac{1}{2}\left(\frac{\varepsilon_o\mu_o^2p_o^2\omega^4}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\mu_o^{\cancel{2}}p_o^2\omega^4}{\cancel{\mu_o}16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\\ &\simeq\frac{1}{2}\left(\frac{\mu_o p_o^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\mu_o p_o^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\end{aligned}}\ Joules/m^3$$

n.b. $\boxed{u_{\mathrm{E(1)}}^{Erad}(\vec{r},t)=u_{\mathrm{E(1)}}^{Mrad}(\vec{r},t)}$ using: $\boxed{c^2=1/\varepsilon_o\mu_o}$ or: $\boxed{\varepsilon_o=1/\mu_o c^2}$.

$$\therefore\ \boxed{u_{\mathrm{E(1)}}^{rad}(\vec{r},t)\simeq\frac{\mu_o p_o^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]}\left(\frac{Joules}{m^3}\right)\ \underline{\text{for}}\text{:}\ \boxed{d\ll\lambda\ll r}\ \text{“far zone” limit}$$

The *EM* energy radiated by an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by Poynting's vector:

$$\boxed{\vec{S}^{rad}_{E(1)}(\vec{r},t) = \frac{1}{\mu_o}\left(\vec{E}^{E(1)}_{r}(\vec{r},t)\times\vec{B}^{E(1)}_{r}(\vec{r},t)\right)}$$

$$\boxed{\begin{aligned}\hat{r}\times\hat{\varphi} &= \hat{\varphi}\\ \hat{\theta}\times\hat{\varphi} &= \hat{r}\\ \hat{\varphi}\times\hat{r} &= \hat{\theta}\end{aligned}}$$

$$\boxed{\vec{S}^{rad}_{E(1)}(\vec{r},t) \simeq \left(\frac{1}{\cancel{\mu_o}}\right)\left(\frac{\cancel{\mu_o} p_o \omega^2}{4\pi r}\right)\left(\frac{\mu_o p_o \omega^2}{4\pi r c}\right)\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\left[\hat{\theta}\times\hat{\varphi}\right]}_{=+\hat{r}}}$$

<u>Or</u>: $$\boxed{\vec{S}^{rad}_{E(1)}(\vec{r},t) \simeq \frac{\mu_o p_o^2 \omega^4}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}} \left(\frac{Watts}{m^2}\right) \Leftarrow \text{ Radial outward flow of energy}$$

<u>for</u>: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

The *EM* radiation <u>linear</u> momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\wp}^{rad}_{E(1)}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}^{rad}_{E(1)}(\vec{r},t) = \frac{1}{c^2}\vec{S}^{rad}_{E(1)}(\vec{r},t)}$$

<u>Or</u>: $$\boxed{\vec{\wp}^{rad}_{E(1)}(\vec{r},t) \simeq \frac{\mu_o p_o^2 \omega^4}{16\pi^2 r^2 c^3}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\sin^2\theta\hat{r}} \left(\frac{kg}{m^2\text{-sec}}\right) \Leftarrow \text{ Radial outward } EM \text{ linear}$$

momentum flow <u>for</u>: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

The *EM* radiation <u>angular</u> momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\ell}^{rad}_{E(1)}(\vec{r},t) = \vec{r}\times\vec{\wp}^{rad}_{E(1)}(\vec{r},t)}$$

$$\boxed{\vec{\ell}^{rad}_{E(1)}(r,t) \simeq \frac{\mu_o p_o^2 \omega^4}{16\pi^2 r^2 c^3}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\sin^2\theta\left(\hat{r}\times\hat{r}\right) \equiv 0} \left(\frac{kg}{m\text{-sec}}\right) \Leftarrow \text{ \underline{No} angular momentum flow}$$

<u>for</u>: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

n.b. The <u>exact</u> $\vec{\ell}^{rad}_{E(1)}(\vec{r},t) \not\equiv 0$ *i.e.* ignore <u>restrictions</u> on far-zone limit, keep <u>all</u> higher-order terms . . . we have neglected $\vec{E}^{E(1)}_{r} \sim \hat{r}$ term which is non-negligible in the <u>near-zone</u> $(d \sim r)$ and also in the so-called <u>intermediate</u>, or <u>inductive zone</u> $(\lambda \sim r)$.

Time-Averaged Quantities for E(1) Radiation from an Oscillating Electric Dipole:

Recall the definition of time average: $\boxed{\langle A(t)\rangle \equiv \frac{1}{\tau}\int_{t=0}^{t=\tau} A(t)\,dt = \frac{1}{\tau}\int_{t=0}^{t=\tau} A_o \cos^2 \omega t\,dt = \frac{1}{2}A_o}$

The time-averaged *EM* radiation energy density associated with an oscillating electric dipole is:

$$\boxed{\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p_o^2 \omega^4}{32\pi^2 c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Joules}{m^3}\right) \quad \text{for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit}$$

The time-averaged |Poynting's vector|, which is also the intensity $I_{\mathrm{E}(1)}^{rad}$ of *EM* radiation associated with an oscillating electric dipole is:

$$\boxed{I_{\mathrm{E}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right|\right\rangle = \frac{1}{2}c\varepsilon_o\left\langle \left(E_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t)\right)^2\right\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right) \quad \text{for: } \boxed{d \ll \lambda \ll r \text{ "far-zone" limit}}$$

We also see that: $\boxed{I_{\mathrm{E}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right|\right\rangle = c\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle} \left(\frac{Watts}{m^2}\right)$.

The time-averaged *EM* radiated power associated with an oscillating electric dipole is:

$$\begin{aligned} \left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle &= \int_S \left\langle \vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \cdot d\vec{a}_\perp \simeq \frac{\mu_o p_o^2\omega^4}{32\pi^2 c\cancel{r^2}}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \cancel{r^2}\sin^2\theta \underbrace{\sin\theta d\theta}_{=d\cos\theta} d\varphi \\ &\simeq \frac{\mu_o p_o^2\omega^4}{{}_{16}\cancel{32}\pi^{\cancel{2}} c^2}\cancel{2\pi}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\, d\cos\theta = \frac{\mu_o p_o^2\omega^4}{16\pi c^2}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\, d\cos\theta \end{aligned}$$

Let $\boxed{u=\cos\theta}$, $\boxed{du=-d(\cos\theta)}$, $\boxed{\theta=0 \Rightarrow u=1}$, $\boxed{\theta=\pi \Rightarrow u=-1}$, $\boxed{\sin^2\theta = 1-\cos^2\theta = 1-u^2}$

$$\therefore \quad \boxed{\int_{-1}^{+1}(1-u^2)\,du = \left(u-\frac{1}{3}u^3\right)\Big|_{-1}^{+1} = +1-\frac{1}{3}+1-\frac{1}{3} = 2-\frac{2}{3} = \frac{4}{3}}$$

$\therefore$ The time-averaged radiated power is:

$$\boxed{\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{12\pi c}\right)} \text{ (Watts) for: } \boxed{d \ll \lambda \ll r} \text{ "far-zone" limit}$$

n.b. $\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle$ has no r-dependence!

Note that time-averaged radiated power varies as the fourth power of frequency!

The time-averaged *EM* radiation linear momentum density associated with an oscillating electric dipole is:

$$\boxed{\left\langle \vec{\wp}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c^2}\left\langle \vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \hat{r} \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}} \left(\frac{kg}{m^2\text{-sec}}\right) \quad \text{for: } \boxed{d \ll \lambda \ll r \text{ "far-zone" limit}}$$

The time-averaged *EM* radiation angular momentum density associated with an oscillating electric dipole is:

$$\boxed{\left\langle \vec{\ell}^{\,rad}_{E(1)}(\vec{r},t)\right\rangle = \vec{r}\times\left\langle \vec{\wp}^{\,rad}_{E(1)}(\vec{r},t)\right\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r}\right)(\hat{r}\times\hat{r}) \equiv 0} \left(\frac{kg}{m\text{-sec}}\right) \text{ for: } \boxed{\begin{array}{c} d \ll \lambda \ll r \\ \text{“far-zone” limit}\end{array}}$$

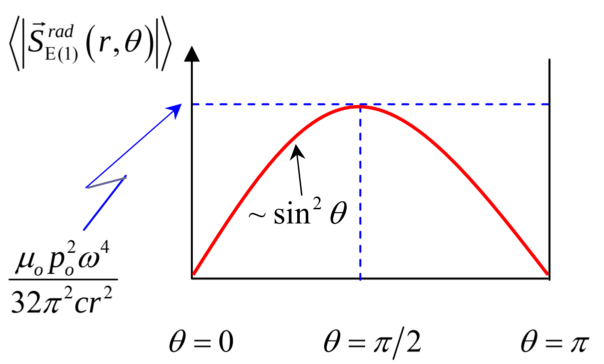
n.b. The exact $\left\langle \vec{\ell}^{\,rad}_{E(1)}(\vec{r})\right\rangle \not\equiv 0$ *i.e.* ignore restrictions on far-zone limit, keep all higher-order terms . . . we have neglected the $\vec{E}^{E(1)}_r \sim \hat{r}$ term which is non-negligible in the near-zone $(d \sim r)$ and also in the so-called intermediate, or inductive zone $(\lambda \sim r)$.

Note that because: $\boxed{I^{rad}_{E(1)}(\vec{r}) \equiv \left\langle \left|\vec{S}^{rad}_{E(1)}(\vec{r},t)\right|\right\rangle \simeq \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right)$

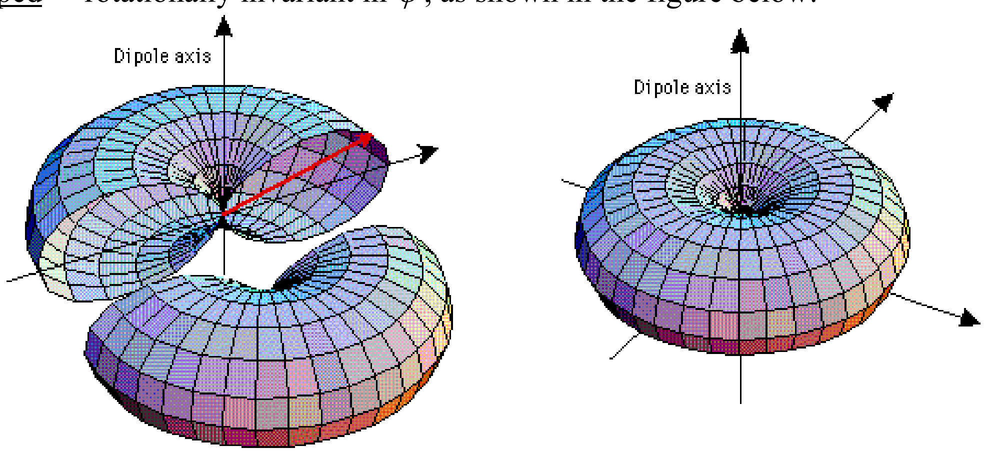
$\Rightarrow \boxed{\left\langle \vec{S}^{rad}_{E(1)}(r,\theta=0,\varphi)\right\rangle = \left\langle \vec{S}^{rad}_{E(1)}(r,\theta=\pi,\varphi)\right\rangle = 0}$ since: $\boxed{\sin^2 0 = \sin^2\pi = 0}$

i.e. no *EM* radiation occurs along the axis of the electric dipole ($\hat{z}$ axis)

EM radiation for E(1) electric dipole is peaked/maximum at $\theta = \pi/2$ (then $\sin^2\theta = 1$)
i.e. maximum *EM* radiation occurs $\perp$ to the axis of the electric dipole:



Thus, the intensity profile $I^{rad}_{E(1)}(\vec{r})$ in 3-D {for fixed r} for E(1) electric dipole radiation is donut-shaped - rotationally invariant in φ, as shown in the figure below:



Griffiths Example 11.1:

The time-averaged power for E(1) electric dipole radiation is $\boxed{\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \simeq \dfrac{\mu_o p_o^2 \omega^4}{12\pi c}}$.

Note that $\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \sim \omega^4$ (or $\sim f^4$, or $\sim \lambda^{-4}$)

For red light: $\boxed{\lambda_{\mathrm{red}} \simeq 780\,nm} \Rightarrow \boxed{f_{\mathrm{red}} = \dfrac{c}{\lambda_{\mathrm{red}}} = \dfrac{3\times10^8}{780\times10^{-9}} \simeq 3.85\times10^{14}\,Hz}$

For violet light: $\boxed{\lambda_{\mathrm{violet}} \simeq 350\,nm} \Rightarrow \boxed{f_{\mathrm{violet}} = \dfrac{c}{\lambda_{\mathrm{violet}}} = \dfrac{3\times10^8}{350\times10^{-9}} \simeq 8.57\times10^{14}\,Hz}$

<u>Hence</u>: $\boxed{\left(\dfrac{\left\langle P_{\mathrm{E}(1)}^{\mathrm{violet}}\right\rangle}{\left\langle P_{\mathrm{E}(1)}^{\mathrm{red}}\right\rangle}\right) = \left(\dfrac{f_{\mathrm{violet}}}{f_{\mathrm{red}}}\right)^4 = \left(\dfrac{8.57\times10^{14}}{3.85\times10^{14}}\right)^4 = (2.23)^4 \simeq 24.67}$.

$\Rightarrow \left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \sim \omega^4$ explains why the sky is blue! Sunlight {unpolarized light} incident on O_2 & N_2 molecules in the earth's atmosphere stimulates the O & N atoms – vibrates the {bound} atomic electrons at {angular} frequency ω, causing them to oscillate as electric dipoles! Solar *EM* radiation at a given angular frequency ω is thus absorbed and re-emitted in this *EM* radiation + atom <u>scattering</u> process.

The above formula for *EM* power radiated as E(1) electric dipole radiation by such atoms, by time-reversal invariance of the *EM* interaction, is also the *EM* power absorbed by atoms, thus we see that because of the ω^4-dependence of $\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle$, the higher frequency/shorter wavelength radiation (*i.e.* blue/violet light) is preferentially scattered <u>much</u> <u>more</u> <u>so</u> than the lower frequency/longer wavelength radiation (*i.e.* red light).

The Earth's sky appears blue {*e.g.* to an observer on the ground, or even *e.g.* a space shuttle astronaut in orbit around the earth} because the light from the sky is <u>scattered</u> (*i.e.* <u>re-radiated</u>) <u>light</u>, which is preferentially in the blue/violet portion of the visible light *EM* spectrum. The scattering of *EM* radiation off of atoms is known as <u>Rayleigh</u> <u>Scattering</u>.

Note that precisely same physics also simultaneously explains why the Sun appears <u>red</u> *e.g.* to an observer on the ground at <u>sunrise</u> and <u>sunset</u> – because at these times of the day, path that the sunlight takes through the atmosphere is the longest, relative to that associated *e.g.* with its position at {local} noon. If the higher-frequency blue/violet light is preferentially scattered <u>out</u> of the beam of sunlight, what is left <u>in</u> the beam of sunlight <u>after</u> traversing the entire thickness of the Earth's atmosphere is the lower-frequency, orange-red light.

Note that the Sun is a black-body radiator – its *EM* spectrum peaks in the infra-red region – thus it is NOT flat by any means {also is affected by frequency-dependent absorption in the atmosphere}:

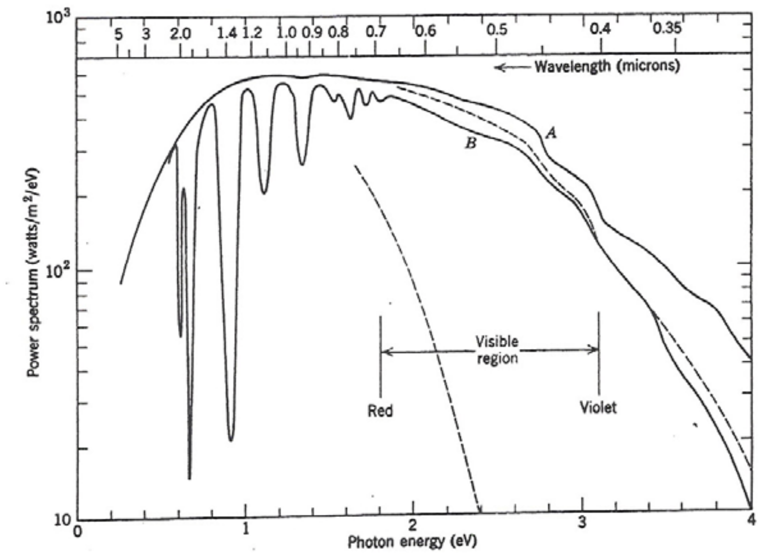
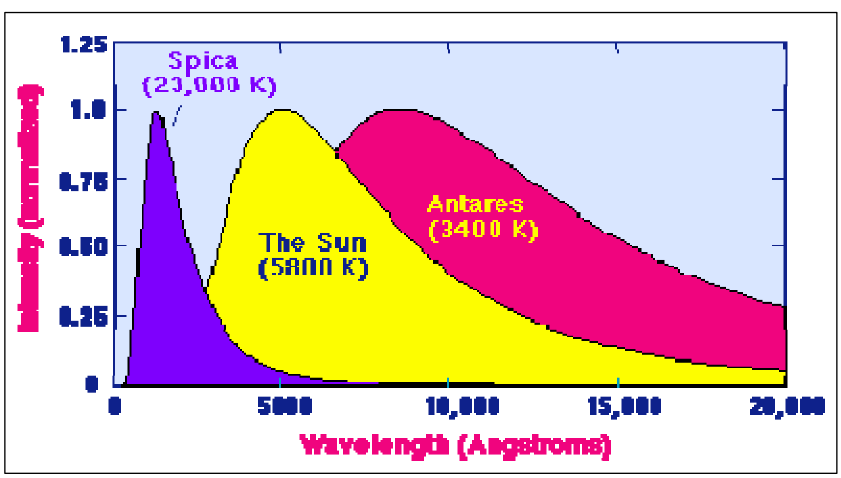


Figure 10.4 Power spectrum of solar radiation (in watts per square meter per electron volt) as a function of photon energy (in electron volts). Curve *A* is the incident spectrum above the atmosphere. Curve *B* is a typical sea-level spectrum with the sun at the zenith. The absorption bands below 2 eV are chiefly from water vapor and vary from site to site and day to day. The dashed curves give the expected sea-level spectrum at zenith and at sunrise-sunset if the only attenuation is from Rayleigh scattering by a dry, clean atmosphere.

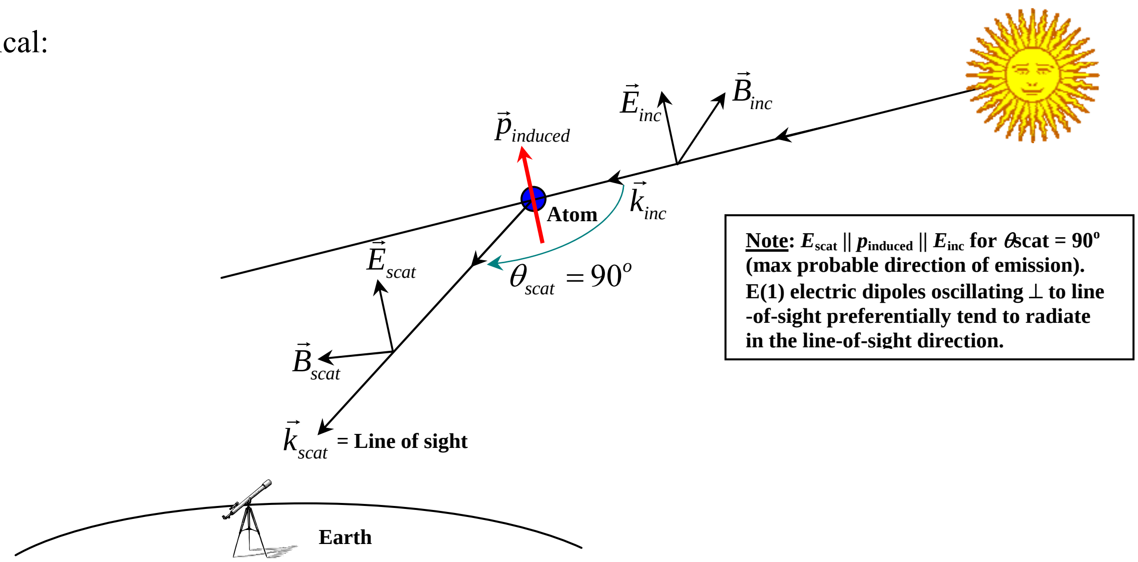
Note the log scale on the vertical axis! Thus, there is not much violet light in the Sun's *EM* spectrum, and hence there is a delicate "balancing" act of flux of *EM* radiation from the Sun {convoluted} with its black-body spectrum and the scattering of this radiation by atoms in the Earth's atmosphere – thus we see the sky as blue. Thus, if the black-body temperature of the sun was different, then the color of the Earth's sky in the visible portion of the *EM* spectrum would also be different – compare the black-body spectra of our Sun *e.g.* with that of Spica (260 *ly* away in the Virgo constellation) and Antares (a red giant 600 *ly* away in the Scorpio constellation):



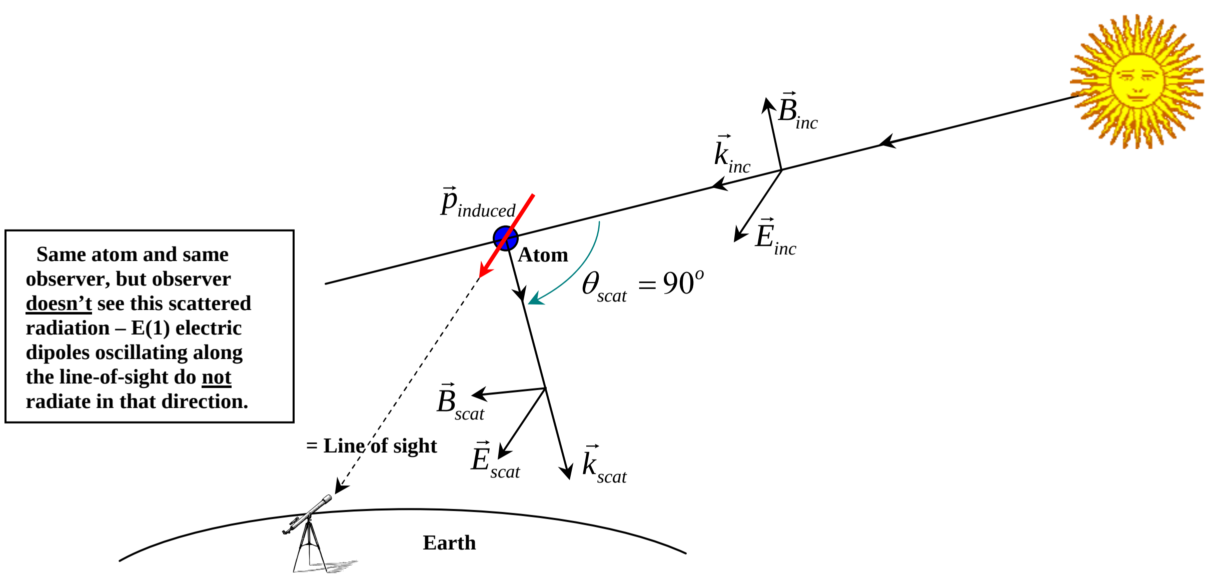
Light from the Sun is unpolarized (*i.e.* it consists of all polarizations, randomly oriented over time). However, because *EM* waves are transversely polarized (defined by the orientation of the $\vec{E}$-field vector) an incident *EM* plane wave from the Sun with polarization in a given direction ($\perp$ to $\vec{k}$-propagation direction) will (transitorily) induce electric dipole moments in gas atoms in earth's atmosphere, via $\vec{p}_{mol}(\omega) = \alpha_{mol}(\omega)\vec{E}$, where $\alpha_{mol}(\omega)$ is the ***molecular polarizability*** at {angular} frequency ω {see P435 Lect. Notes 12 and P436 Lect. Notes 7.5}.

The axis of induced electric dipole moments will be || to the plane of polarization of incident wave at that instant, hence the scattered radiation emitted by the atom will be preferentially at $\theta = 90^\circ = \pi/2$ (*i.e.* $\perp$) to the axis of the (induced) electric dipole of gas atoms in earth's atmosphere. There are two specific/limiting cases to consider – (*a*) when the incident $\vec{E}$-field vector is vertical and (*b*) when the incident $\vec{E}$-field vector is horizontal. Random polarization is then an arbitrary linear combination of these two limiting cases:

(*a*.) $\vec{E}_{inc}$ vertical:



(*b*.) $\vec{E}_{inc}$ horizontal:



Because the blue light an observer sees from a given portion of the sky is due to the preferential scattering of E(1) electric dipole-type Rayleigh scattering of sunlight/solar EM radiation off of gas atoms in the Earth's atmosphere, with $\vec{E}_{scat} \perp$ to the line-of-sight, this radiation has a net polarization – *i.e.* the light from the sky is polarized, especially so away from the sun, *i.e.* in the northern portions of the sky {in the northern hemisphere} !!! You can very easily observe/explicitly verify this using a pair of polaroid sunglasses – try it some time!!!

It is beneficial to wear polaroid sunglasses *e.g.* when out boating on a lake – in order to reduce "glare" from {polarized} sunlight reflected off of the surface of the water!!!

As mentioned above, at sunrise or sunset, the sun appears red when an observer is looking directly at the sun, because the blue/violet light is ~ 25× more preferentially scattered out of the beam of light incident from the sun {per unit thickness of atmosphere} than read light. Thus sunlight at the ground consists predominantly of what remains – red light.

Note that this is also true for moonrise and moonset – the moon will {likewise} have a reddish hue at these times, and note that this is also true *e.g.* for the case of an eclipse of the moon by the Earth.

One can also observe this same phenomenon *e.g.* using a glass pitcher of milk diluted with water – because milk molecules are efficient Rayleigh scatterers of visible light! Here's a simple experiment that you can carry out at home, *e.g.* using a flashlight: