2.2. Electrostatic energy

The energy of a given charge distribution is

$$u = \frac{1}{2}\int d^3r \int d^3r' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad \text{(last time)}$$

Since $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$, this becomes

$$u = \frac{1}{2}\int d^3r \, \rho(\vec{r})\, \phi(\vec{r}).$$

Use now Poisson's eq: $\nabla^2\phi = -4\pi\rho$,

$$u = -\frac{1}{8\pi}\int d^3r \, \vec{\nabla}^2\phi \, \phi(\vec{r}) = \frac{1}{8\pi}\int d^3r \, \vec{\nabla}\phi\cdot\vec{\nabla}\phi \text{, or}$$

Integrate by parts (or use divergence theorem)

$$u = \frac{1}{8\pi}\int d^3r \, \vec{E}^2.$$

The last equation suggests that an electric field carries energy:

The energy density of an electric field $\vec{E}$ is

$$u = \frac{1}{8\pi} \vec{E}^2$$

We shall derive this expression again later on.

Exercise 6

Find the potential energy of a uniformly charged sphere of charge Q and radius R. What happens as $R \to 0$? ∎

3. Methods of Solution in Electrostatics

The two main equations of electrostatics are

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \times \vec{E} = 0 \end{cases}$$

These follow from Maxwell's eqs. in the electrostatic limit, or are a consequence of Coulomb's law.

3.1.1. Uniqueness theorem

Do the electrostatic eqs. uniquely determine the electric field $\vec{E}$?

Consider the eq. $\vec{\nabla} \times \vec{E} = 0$. According to the <u>Poincaré lemma</u>, we can write $\vec{E} = -\vec{\nabla}\phi$ <u>locally</u>. If we are dealing with a contractible space (simply connected, with a connected boundary) we can also write $\vec{E} = -\vec{\nabla}\phi$ <u>globally</u>.

With $\vec{E} = -\vec{\nabla}\phi$, Maxwell's eqs. reduce to

$$\Delta \phi = -4\pi\rho.$$

Imagine we have two different solutions ϕ_1 and ϕ_2 of this eq. Then $\psi = \phi_1 - \phi_2$ satisfies

$$\Delta \psi = 0.$$

Applying the divergence theorem to the vector field $\psi \vec{\nabla} \psi$ we find

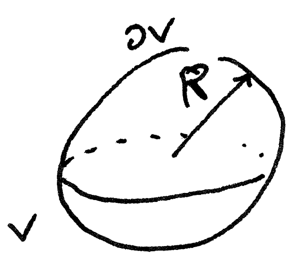
$$\int_V \vec{\nabla}\psi \cdot \vec{\nabla}\psi + \psi \nabla^2 \psi = \oint_{\partial V} (\psi \vec{\nabla}\psi) \cdot d\vec{A}$$

using $\nabla^2 \psi = 0$ and $\delta \vec{E} = -\vec{\nabla}\psi \equiv \vec{E}_1 - \vec{E}_2$

$$\int_V \delta \vec{E}^2 \, dV = -\oint_{\partial V} \psi \, \delta \vec{E} \cdot d\vec{A}$$

where we have defined $\vec{E}_1 = -\vec{\nabla}\phi_1$, $\vec{E}_2 = -\vec{\nabla}\phi_2$.

The solution is unique, $\delta\vec{E} = \vec{E}_1 - \vec{E}_2 = 0$ if the rhs integral vanishes. In the *absence of boundaries* we can take the volume V to infinity, and $\oint_{\partial V} \psi \, \delta\vec{E} \cdot d\vec{A}$ vanishes if we demand that the electric fields decay as $\frac{1}{R^2}$



In the presence of boundaries the solution is unique if we specify appropriate boundary conditions:

i) Dirichlet boundary conditions:

Specify ϕ on the boundary ∂V. $(\Rightarrow \psi = 0)$

ii) Neumann boundary conditions.

Specify $\vec{E}_\perp$ (normal of $\vec{E}$ to ∂V) on the boundary $(\Rightarrow \delta \vec{E} \cdot d\vec{A} = 0)$.

Note that on the boundary of a conductor

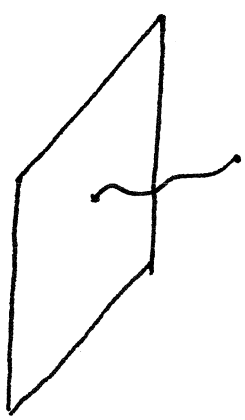
$$\phi = \text{const} \,, \quad \int_{\partial V} \vec{E}_\perp \cdot d\vec{A} = \oint_{\partial V} \vec{E} \, d\vec{A} = Q_{int} .$$

Therefore, the solution is unique if we specify the potential or the charge of each conductor.

Offshot: If we find a solution of Poisson's equation with the desired boundary conditions, that's <u>the</u> solution

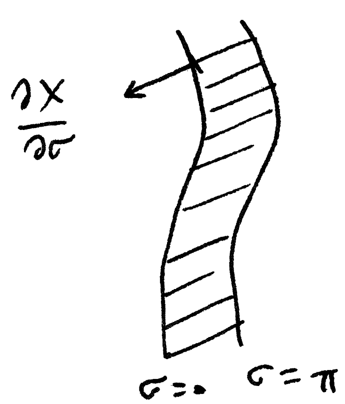
<u>Comment</u>

Dirichlet and Neumann boundary conditions appear in other physical contexts. E.g. boundary conditions on the endpoints of a string



$$X^i(\tau,\sigma) = X^i_0 \quad \text{at} \quad \sigma = 0 \qquad \text{(Dirichlet)}$$

$$\left.\frac{\partial X^i}{\partial \sigma}\right|_{\sigma=0} = 0 \qquad \text{(Neumann)}$$



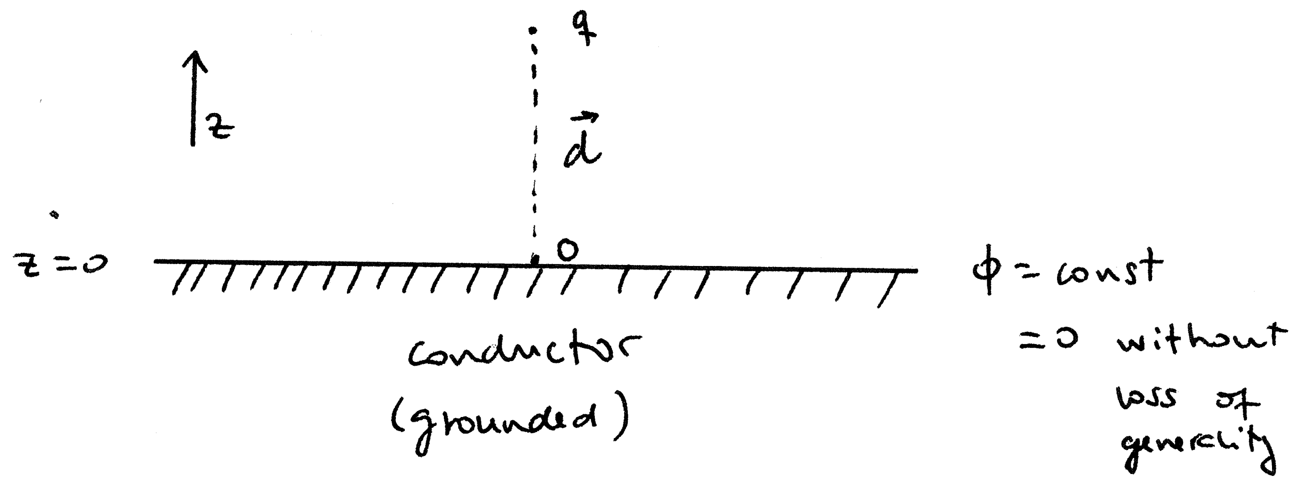
3.2. Method of Images

The uniqueness theorem justifies an important method to solve electrostatic problems in the presence of boundary surfaces.

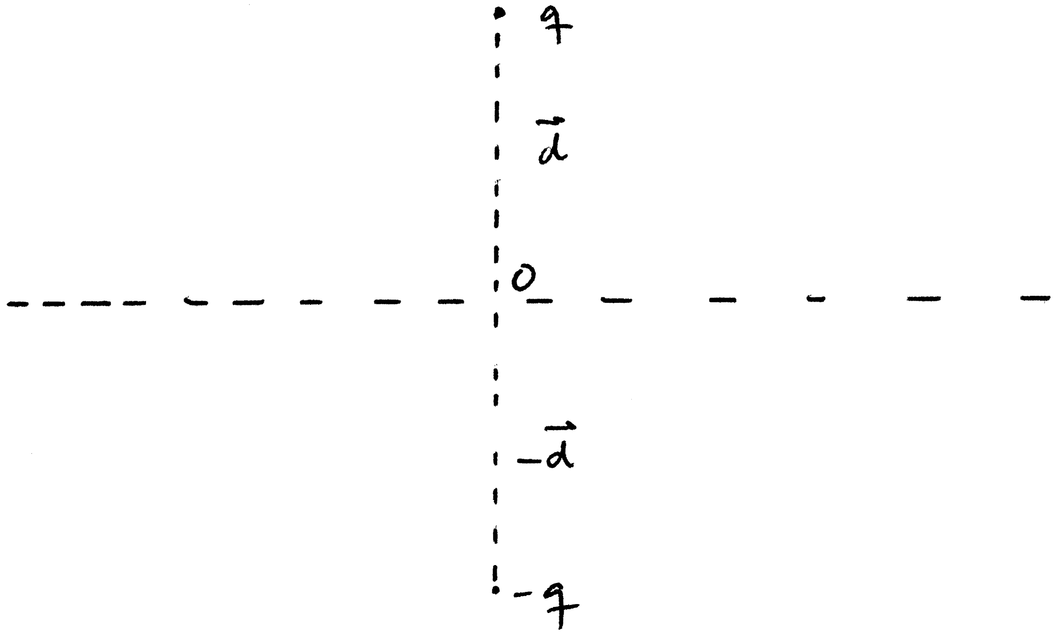
The idea is to introduce fictitious "image charges" to reproduce the desired boundary conditions on the boundary

Example

Calculate the potential $\phi(\vec{r})$ at $z \geq 0$



Solution: Use boundary charge to reproduce boundary condition $\phi(z=0) = 0$:



By reflection symmetry, $\phi = 0$ at $z = 0$. In particular, the solution is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{d}\,|} - \frac{q}{|\vec{r} - \vec{d}\,|}$$

We can use this method then to calculate the (yet unknown) charge distribution on the surface of the conductor:

$$\sigma = \frac{E_\perp}{4\pi}$$

Exercise 7

i) Calculate the potential created by a charge q at a distance d from the center of a grounded conducting sphere of radius a

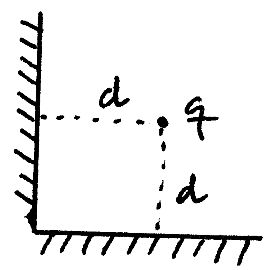
ii) Calculate the induced charge density on the surface of the sphere and the induced total charge.

iii) Find the energy of this system.

Repeat the same calculation (when appropriate) with an insulated conducting sphere.

Exercise 8

What configuration of image charges leads to the field created by two infinite grounded conducting planes meeting perpendicularly when a charge q is placed at a distance d from each?



3.3 Laplace's equation

In the following we are going to study electrostatic boundary problems in the absence of sources, $\rho = 0$. We must thus solve Laplace's equation

$$\nabla^2 \phi = 0$$

with appropriate boundary conditions.

It is often extremely useful to choose a coordinate system that matches the symmetries of the problem. Then, often the equation can be solved by separation of variables

3.3.1. Example : Cartesian coordinates

Calculate the potential inside a hollow box with the prescribed boundary conditions: