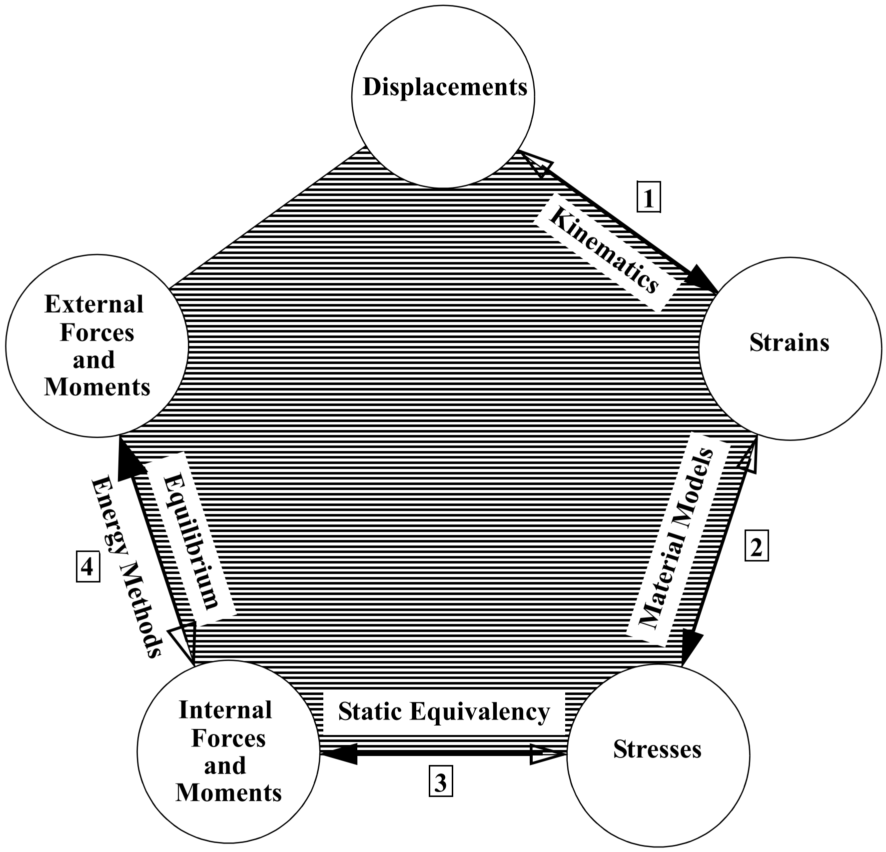
Energy Methods

- Minimum-energy principles are an alternative to statement of equilibrium equations.

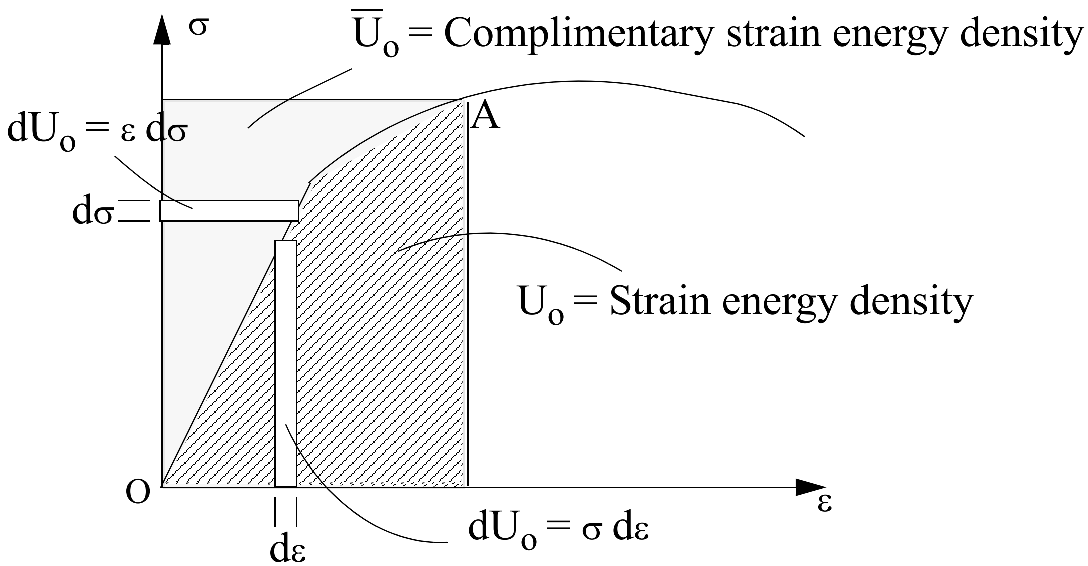


The learning objectives in this chapter are:

- Understand the perspective and concepts in energy methods.
- Learn the use of dummy unit load method and Castigliano's theorem for calculating displacements in statically determinate and indeterminate structures.

Strain Energy

- The energy stored in a body due to deformation is called the *strain energy.*
- The strain energy per unit volume is called the *strain energy density* and is the area underneath the stress-strain curve up to the point of deformation.



Strain Energy: $$U = \int_V U_o \; dV$$ [

Strain Energy Density: $$U_o = \int_0^{\varepsilon} \sigma d\varepsilon$$

Units: N-m / m^3, Joules / m^3, in-lbs / in^3, or ft-lb/$ft.^3$

Complimentary Strain Energy Density: $$\bar{U}_o = \int_0^{\sigma} \varepsilon d\sigma$$

Linear Strain Energy Density

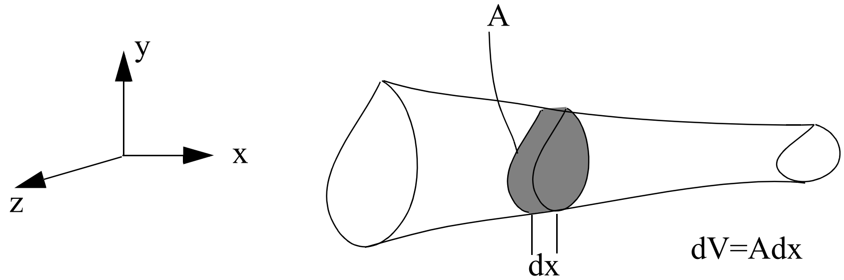
Uniaxial tension test: $$U_o = \int_0^{\varepsilon} \sigma d\varepsilon = \int_0^{\varepsilon} (E\varepsilon) d\varepsilon = \frac{E\varepsilon^2}{2} = \frac{1}{2}\sigma\varepsilon$$

$$U_o = \frac{1}{2}\tau\gamma$$

- Strain energy and strain energy density is a scaler quantity.

$$\boxed{U_o = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]}$$

1-D Structural Elements



Axial strain energy

- All stress components except σ_{xx} are zero.

$$\sigma_{xx} = E\varepsilon_{xx} \qquad \varepsilon_{xx} = \frac{du}{dx}(x)$$

$$U_A = \int_V \frac{1}{2}E\varepsilon_{xx}^2 dV = \int_L \left[\int_A \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dA\right]dx = \int_L \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 \int_A E dA\right]dx$$

$$U_A = \int_L U_a \; dx \qquad U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$$

- U_a is the strain energy per unit length.

$$\bar{U}_A = \int_L \bar{U}_a \; dx \qquad \bar{U}_a = \frac{1}{2}\frac{\boldsymbol{N}^2}{EA}$$

Torsional strain energy

- All stress components except $\tau_{x\theta}$ in polar coordinate are zero

$$\tau_{x\theta} = G\gamma_{x\theta} \qquad \gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$$

$$U_T = \int_V \frac{1}{2}G\gamma_{x\theta}^2 dV = \int_L \left[\int_A \frac{1}{2}G\left(\rho\frac{d\phi}{dx}\right)^2 dA\right]dx = \int_L \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 \int_A G\rho^2 dA\right]dx$$

$$U_T = \int_L U_t \; dx \qquad U_t = \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2$$

- U_t is the strain energy per unit length.

$$\bar{U}_T = \int_L \bar{U}_t \; dx \qquad \bar{U}_t = \frac{1}{2}\frac{\boldsymbol{T}^2}{GJ}$$

Strain energy in symmetric bending about z-axis

There are two non-zero stress components, σ_{xx} and τ_{xy}.

$$\sigma_{xx} = E\varepsilon_{xx} \qquad \varepsilon_{xx} = -y\frac{d^2 v}{dx^2}$$

$$U_B = \int_V \frac{1}{2}E\varepsilon_{xx}^2 dV = \int_L \left[\int_A \frac{1}{2}E\left(y\frac{d^2 v}{dx^2}\right)^2 dA\right]dx = \int_L \left[\frac{1}{2}\left(\frac{d^2 v}{dx^2}\right)^2 \int_A Ey^2 dA\right]dx$$

$$U_B = \int_L U_b \; dx \qquad U_b = \frac{1}{2}EI_{zz}\left(\frac{d^2 v}{dx^2}\right)^2$$

- where U_b is the bending strain energy per unit length.

$$\bar{U}_B = \int_L \bar{U}_b \; dx \qquad \bar{U}_b = \frac{1}{2}\frac{\boldsymbol{M_z}^2}{EI_{zz}}$$

The strain energy due to shear in bending is: $U_S = \int_V \frac{1}{2}\tau_{xy}\gamma_{xy} \; dV = \int_V \frac{1}{2}\frac{\tau_{xy}^2}{E} \; dV$

As $\tau_{max} \ll \sigma_{max}$ $\qquad U_s \ll U_B$

Table 7.1 Linear strain energy per unit length.

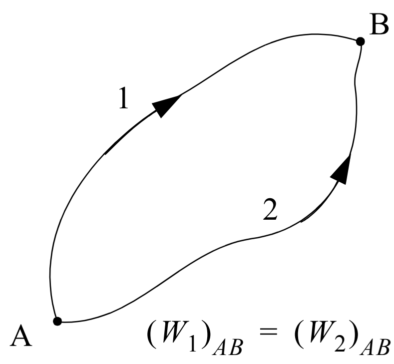
	Strain energy per unit length	Complementary strain energy per unit length
Axial	$U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$	$\bar{U}_a = \frac{1}{2}\frac{\boldsymbol{N}^2}{EA}$
Torsion of circular shafts	$U_t = \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2$	$\bar{U}_t = \frac{1}{2}\frac{\boldsymbol{T}^2}{GJ}$
Symmetric bending of beams	$U_b = \frac{1}{2}EI_{zz}\left(\frac{d^2 v}{dx^2}\right)^2$	$\bar{U}_b = \frac{1}{2}\frac{\boldsymbol{M_z}^2}{EI_{zz}}$

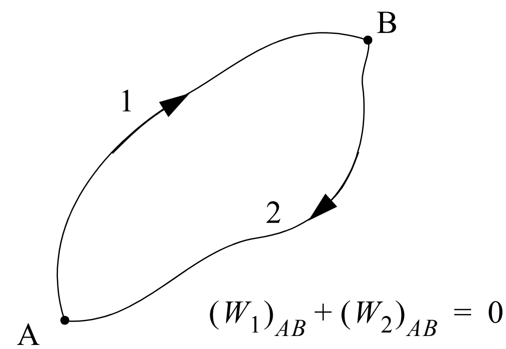
Work

- If a force moves through a distance, then work has been done by the force.

$$dW = F du$$

- Work done by a force is *conservative* if it is path independent.

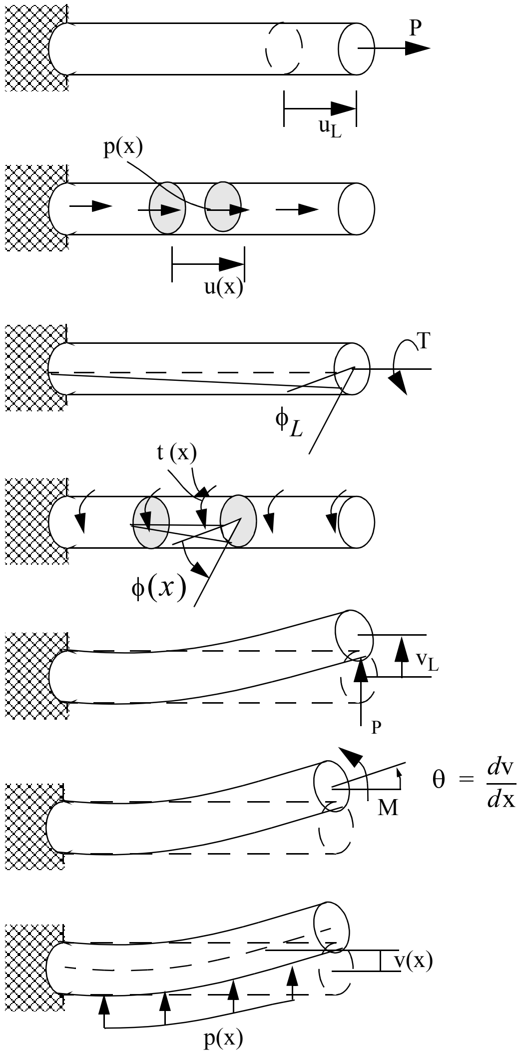




- In a conservative system, work done on a closed path is zero.
  work done to overcome friction varies with the length of path--- non-conservative.
  work done to create permanent deformation (plastic deforms)--- non-conservative.
- Elastic deformation is a conservative deformation while plastic deformation is non-conservative.
- Elastic deformation can however be linear or non-linear.
- Non-linear systems and non-conservative systems are two independent description of a system.

Work due to forces & moments acting on 1-D structural elements

Loading Mode



Work

$$\delta W = P\delta u_L$$

$$\delta W = \int_0^L p(x)\delta u(x)dx$$

$$\delta W = T\delta\phi_L$$

$$\delta W = \int_0^L t(x)\delta\phi(x)dx$$

$$\delta W = P\delta v_L$$

$$\delta W = M\delta\theta_L$$

$$\delta W = \int_0^L p(x)\delta v(x)dx$$

- Any variable that can be used for describing deformation is called the *generalized displacement*. The variable can be real or a mathematical.

$$v(x) = C_1 f_1(x) + C_2 f_2(x) + \cdots + C_i f_i(x) + \cdots + C_n f_n(x) = \sum_{i=1}^{n} C_i f_i(x)$$

- Any variable that can be used for describing the cause that produces deformation is called the *generalized force*. The variable can be real or a mathematical.

$$F_i = \int_0^L h_i(x)p(x)dx$$

$h_i(x)$ are called weighting function.

$$\boldsymbol{M}_z(x) = D_1 g_1(x) + D_2 g_2(x) + \cdots + D_i g_i(x) + \cdots + D_n g_n(x) = \sum_{i=1}^{n} D_i g_i(x)$$

Virtual Work

- Virtual work methods are applicable to linear and non-linear systems, to conservative as well as non-conservative systems.

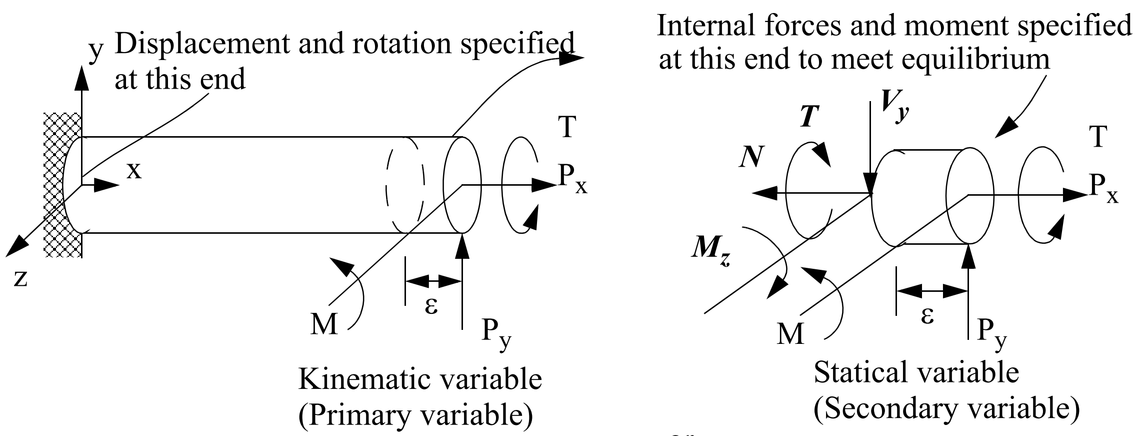
Principle of virtual work: *The total virtual work done on a body at equilibrium is zero.*

$$\delta W = 0$$

- Symbol δ will be used to designate a virtual quantity

$$\delta W_{ext} = \delta W_{int}$$

Types of boundary conditions



u ⟵ or ⟶ ***N***

ϕ ⟵ or ⟶ ***T***

v ⟵ or ⟶ V_y

$\theta = \frac{dv}{dx}$ ⟵ or ⟶ M_z

Geometric boundary conditions (Kinematic boundary conditions) (Essential boundary conditions): Condition specified on kinematic (primary) variable at the boundary.

Statical boundary conditions (Natural boundary conditions) Condition specified on statical (secondary) variable at the boundary.

Kinematically admissible functions

- Functions that are continuous and satisfies all the kinematic boundary conditions are called *kinematically admissible functions*.
- actual displacement solution is always a kinematically admissible function
- Kinematically admissible functions are not required to correspond to solutions that satisfy equilibrium equations.

Statically admissible functions

- Functions that satisfy satisfies all the static boundary conditions, satisfy equilibrium equations at all points, and are continuous at all points except where a concentrated force or moment is applied are called *statically admissible functions*.
- Actual internal forces and moments are always statically admissible.
- Statically admissible functions are not required to correspond to solutions that satisfy compatibility equations.

Degree of static redundancy = Number of unknown reactions
–Number of equilibrium equations

- In determining statically admissible internal forces and moments, the number of reactions that can be assigned arbitrary values is equal to the degree of static redundancy.

C 7.1 Determine a class of kinematically admissible displacement functions for the beam shown in Figure P7.1.

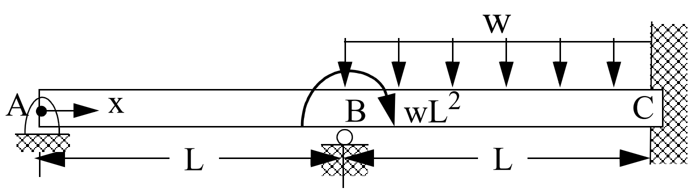
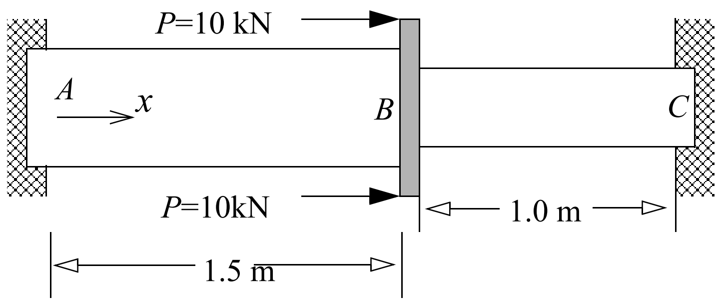


Figure P7.1

C 7.2 For the beam and loading shown in Figure P7.1 determine a statically admissible bending moment.

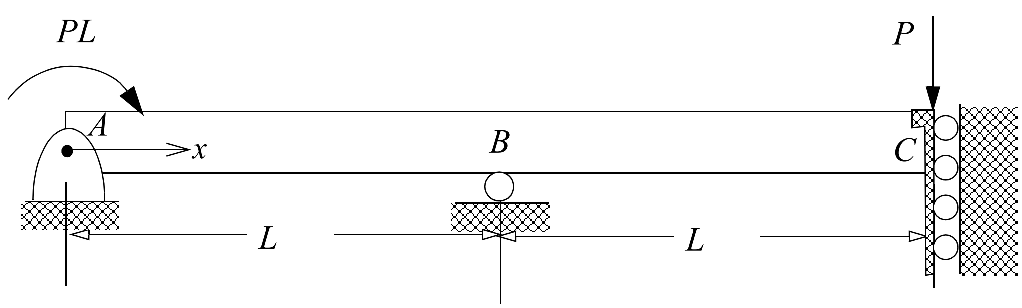
Class Problem 7.1

Determine a class of kinematically admissible axial displacement.(u)
Determine a statically admissible internal axial force.(***N***)



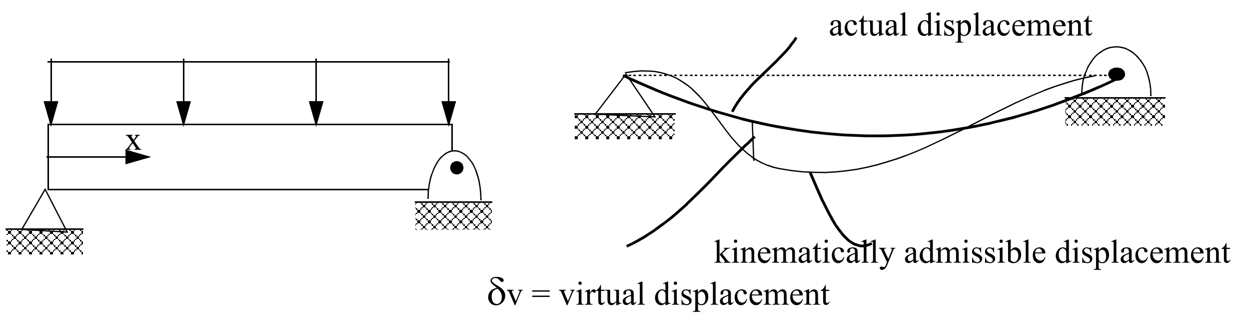
Class Problem 7.2

Determine a class of kinematically admissible bending displacement field (v)
Determine a statically admissible internal bending moment ($\boldsymbol{M}_z$).



Virtual displacement method

- The virtual displacement is an infinitesimal imaginary kinematically admissible displacement field imposed on a body.
- The word *infinitesimal* implies that neither the internal nor the external forces or moments change during the imposition of the virtual displacement.



- Of all the virtual displacements the one that satisfies the virtual work principle is the actual displacement field.
- In virtual displacement methods, zero virtual work implies all equilibrium equations are met.
- When the virtual work condition is approximately met we obtain an approximate solution to the displacement function such as in Rayleigh-Ritz method.

Virtual Force Method

- The virtual force is an infinitesimal imaginary statically admissible force field imposed on a body.
- The word *infinitesimal* implies that the body does not go through additional deformation due to the imposition of a virtual force or virtual moment.
- Of all the virtual force fields the one that satisfies the virtual work principle is the actual force field.
- In virtual force methods, zero virtual work implies all compatibility equations are met.

C 7.3 The roller at P shown in Figure P7.3 slides in the slot due to the force F = 20kN. Both bars have a cross-sectional area of $A = 100\ mm^2$ and a modulus of elasticity E = 200 GPa. Bar AP and BP have lengths of L_{AP}= 200 mm and L_{BP}= 250 mm respectively. Determine the axial stress in the member AP by virtual displacement method.

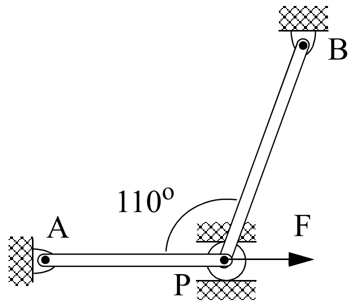


Figure P7.3

C 7.4 A force F = 20kN is applied to pin shown in Figure P7.4. Both bars have a cross-sectional area of $A = 100\ mm^2$ and a modulus of elasticity $E = 200$ GPa. Bar AP and BP have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm respectively. Using virtual force method determine the movement of pin in the direction of force F.

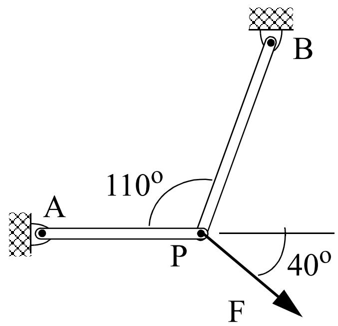


Figure P7.4

Dummy unit load method

- This is a virtual force method that is formalized.
- Can be used for axial, torsion or bending problems.

Application to axial members

Consider two axial rods.

Rod 1: Actual rod with actual internal axial force $N_1(x)$ and actual displacement $u_1(x)$.

Rod 2: A rod with *same supports* as rod 1 with a *unit force* placed at point x_p at which we want to calculate the displacement. $N_2(x)$ be the statically admissible bending moment and $u_2(x)$ be the kinematically admissible displacement for rod 2.

Note: No relationship between N_2 and u_2

The internal and external virtual work for rod 2:

$$\delta W_{int} = \int_0^L N_2(x)\, du_2 = \int_0^L N_2(x)\frac{du_2}{dx}\, dx$$

$$\delta W_{ext} = (\delta F = 1)u_2(x_P)$$

By theorem of virtual work: $$(\delta F = 1)u_2(x_P) = \int_0^L N_2(x)\frac{du_2}{dx}\, dx$$

$u_1(x)$ is a kinematically admissible displacement field, hence can be used for $u_2(x)$.

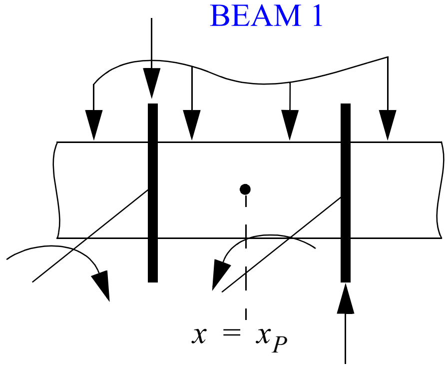
$$(\delta F = 1)u_1(x_P) = \int_0^L N_2(x)\frac{du_1}{dx}\, dx = \int_0^L \frac{N_2(x)N_1(x)}{EA}\, dx$$

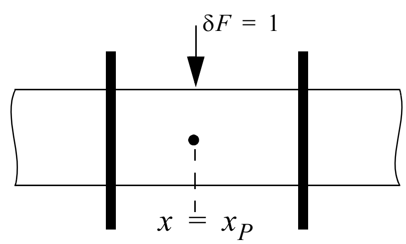
Application to beam bending

Deflection Calculations

BEAM 1: Actual beam with actual internal moment $\boldsymbol{M}_1(x)$ and actual displacement $v_1(x)$.

BEAM 2: A beam with *same supports* as beam 1 with a *unit force* placed at point x_p at which we want to calculate the displacement. $\boldsymbol{M}_2(x)$ be the statically admissible bending moment and $v_2(x)$ be the kinematically admissible displacement for beam 2.





Note: No relationship between $\boldsymbol{M}_2$ and v_2

The internal and external virtual work for beam 2:

$$\delta W_{int} = \int_0^L \boldsymbol{M_2}(x) d\theta_2 = \int_0^L \boldsymbol{M_2}(x)\frac{d\theta_2}{dx}\, dx = \int_0^L \boldsymbol{M_2}(x)\frac{d}{dx}\left(\frac{dv_2}{dx}\right)\, dx = \int_0^L \boldsymbol{M_2}(x)\frac{d^2 v_2}{dx^2}\, dx$$

$$\delta W_{ext} = (\delta F = 1) v_2(x_P)$$

By theorem of virtual work: $$(\delta F = 1) v_2(x_P) = \int_0^L \boldsymbol{M_2}(x)\frac{d^2 v_2}{dx^2}\, dx$$

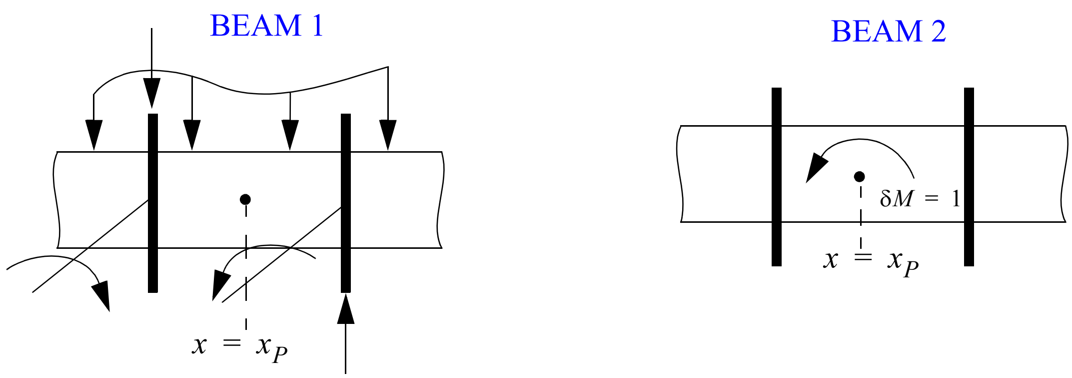
$v_1(x)$ is a kinematically admissible displacement field, hence can be used for $v_2(x)$.

$$(\delta F = 1) v_1(x_P) = \int_0^L \boldsymbol{M_2}(x)\frac{d^2 v_1}{dx^2}\, dx = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI} dx$$

Slope Calculations

BEAM 1: Actual beam with actual internal moment $\boldsymbol{M}_1(x)$ and actual displacement $v_1(x)$.

BEAM 2: A beam with *same supports* as beam 1 with a *unit moment* placed at point x_p at which we want to calculate the slope. $\boldsymbol{M}_2(x)$ be the statically admissible bending moment and $v_2(x)$ be the kinematically admissible displacement for beam 2.



Note: No relationship between $\boldsymbol{M}_2$ and v_2

The internal and external virtual work for beam 2:

$$\delta W_{int} = \int_0^L \boldsymbol{M_2}(x) d\theta_2 = \int_0^L \boldsymbol{M_2}(x)\frac{d\theta_2}{dx}\,dx = \int_0^L \boldsymbol{M_2}(x)\frac{d}{dx}\left(\frac{dv_2}{dx}\right)\,dx = \int_0^L \boldsymbol{M_2}(x)\frac{d^2v_2}{dx^2}\,dx$$

$$\delta W_{ext} = (\delta M = 1)\frac{dv_2}{dx}(x_P)$$

By theorem of virtual work: $$(\delta M = 1)\frac{d}{dx}(v_2)(x_P) = \int_0^L \boldsymbol{M_2}(x)\frac{d^2v_2}{dx^2}\,dx$$

$v_1(x)$ is a kinematically admissible displacement field, hence can be used for $v_2(x)$.

$$(\delta M = 1)\frac{dv_1}{dx}(x_P) = \int_0^L \boldsymbol{M_2}(x)\frac{d^2v_1}{dx^2}\,dx = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx$$

Table 1.2 Synopsis of Dummy Unit Load Method

Axial	$(\delta F = 1)u_1(x_P) = \int_0^L \frac{\boldsymbol{N}_1(x)\boldsymbol{N}_2(x)}{EA}dx$
Torsion	$(\delta T = 1)\phi_1(x_P) = \int_0^L \frac{\boldsymbol{T}_2(x)\boldsymbol{T}_1(x)}{GJ}dx$
Symmetric Bending	$(\delta F = 1)v_1(x_P) = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx$ $(\delta M = 1)\frac{dv_1}{dx}(x_P) = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx$

C 7.5 Using dummy unit load method, find the reaction force at A and deflection at B in terms of P, E,I, and L for the beam shown in Figure P7.5.

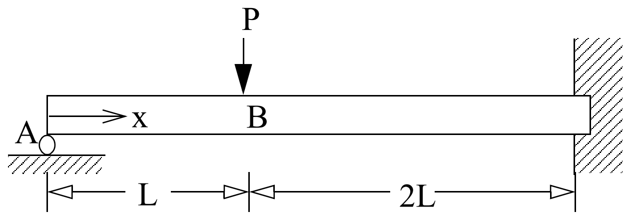
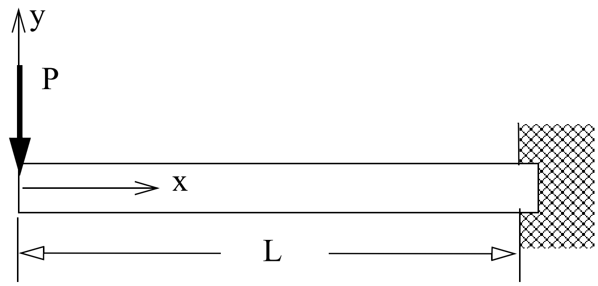


Figure P7.5

C 7.6 Using dummy unit load method determine the elastic curve for the beam and loading shown.



Castigliano's theorem

- Simple and more elegant way of finding reaction forces and/or moments for statically indeterminate structures.

Instead of a unit force we consider a force F applied at x_p in the dummy unit load method. For linear systems the corresponding statically admissible moment ($\tilde{\boldsymbol{M}}_2$) would be F multiplied by $\boldsymbol{M}_2$.

$$\tilde{\boldsymbol{M}}_2 = F\boldsymbol{M}_2 \qquad or \qquad \boldsymbol{M}_2 = \frac{\partial \tilde{M}_2}{\partial F}$$

$$v_1(x_P) = \int_0^L \frac{\boldsymbol{M}_2(x)\boldsymbol{M}_1(x)}{EI}dx = \int_0^L \frac{1}{EI}\left(\frac{\partial \tilde{\boldsymbol{M}}_2}{\partial F}\boldsymbol{M}_1(x)\right)dx$$

The actual moment is a statically admissible moment, and hence we can substitute $\tilde{\boldsymbol{M}}_2 = \boldsymbol{M}_1$ we obtain the following:

$$v_1(x_P) = \int_0^L \frac{1}{EI}\left(\frac{\partial \boldsymbol{M}_1}{\partial F}\boldsymbol{M}_1(x)\right)dx = \int_0^L \frac{1}{2EI}\left(\frac{\partial \boldsymbol{M}_1^2}{\partial F}\right)dx = \frac{\partial}{\partial F}\left[\int_0^L \frac{\boldsymbol{M}_1^2}{2EI}dx\right] = \frac{\partial \bar{U}_B}{\partial F}$$

- The derivative of the complimentary strain energy with respect to a force at x_p gives the deflection in the direction of the force at x_p.

$$\boxed{\frac{dv_1}{dx}(x_P) = \frac{\partial \bar{U}_B}{\partial M}}$$

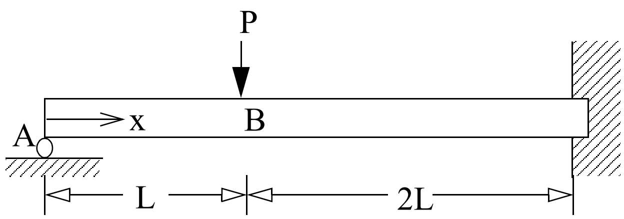
- The derivative of the complimentary strain energy with respect to a moment at x_p gives the slope in the direction of the moment at x_p.
- Performing the derivative with respect to force and moment before performing integration will generally result in less algebra.
- The integrals obtained after taking the derivative with respect to force and moment result in integrals that are identical to the dummy unit load method for finding reactions.

Table 1.3 Synopsis of Castigliano's Theorem

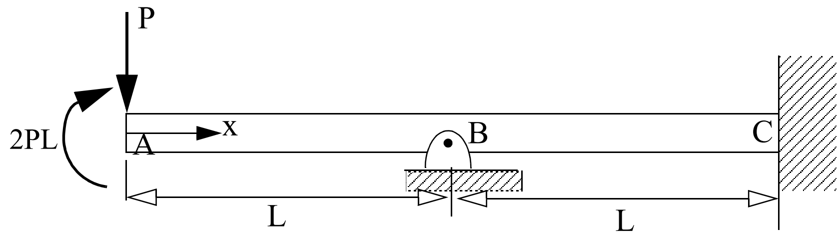
Axial	$u_1(x_P) = \dfrac{\partial \bar{U}_A}{\partial F}$	$\bar{U}_A = \displaystyle\int_0^L \frac{1}{2}\frac{\boldsymbol{N}^2}{EA}dx$
Torsion	$\phi_1(x_P) = \dfrac{\partial \bar{U}_T}{\partial T}$	$\bar{U}_T = \displaystyle\int_0^L \frac{1}{2}\frac{\boldsymbol{T}^2}{GJ}dx$
Symmetric Bending	$v_1(x_P) = \dfrac{\partial \bar{U}_B}{\partial F}$ $\dfrac{dv_1}{dx}(x_P) = \dfrac{\partial \bar{U}_B}{\partial M}$	$\bar{U}_B = \displaystyle\int_0^L \frac{1}{2}\frac{\boldsymbol{M}_z^{\,2}}{EI_{zz}}dx$

- The partial derivative of complementary strain energy of a structure with respect to a force is equal to the displacement at the point of application of the force, and partial derivative of complementary strain energy with respect to a moment is equal to the rotation at the point of application of the moment.

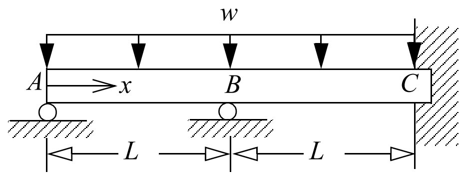
C 7.7 Using Castigliano's theorem, find the reaction force at A and deflection at B in terms of P, E,I, and L for the beam shown in Figure P7.5.



C 7.8 Determine the slope of the beam at B.

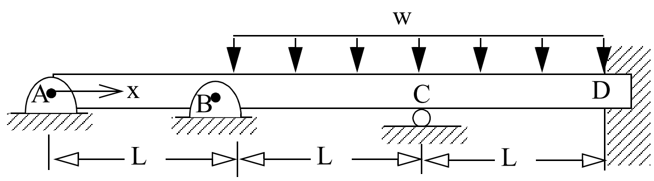


C 7.9 Determine the reaction forces at A and B.



Class Problem 7.3

Write the equations to determine the reaction forces at *A* and *B*. **Do not evaluate the integrals.**



Class Problem 7.4

Write the equations to determine the reaction forces at *A* and *B*. **Do not evaluate the integrals.**