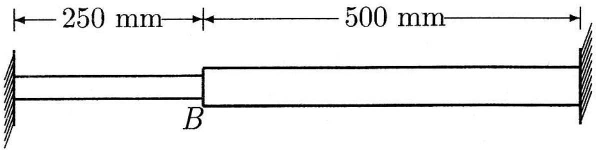
1. The compound bar shown undergoes a temperature change of −20°C. Both parts are assumed to remain elastic.



$A = 50\ \text{mm}^2$	$A = 100\ \text{mm}^2$	$\Delta T = -20^\circ C$
$E = 190$ GPa	$E = 70$ GPa	
$\alpha = 12.5 \times 10^{-6}/^\circ C$	$\alpha = 19.0 \times 10^{-6}/^\circ C$	

Determine the stress in each part of the bar, and the displacement (magnitude and direction) of the joint B. Indicate whether you are using the force method or the displacement method.

Force method: unknown is $P = \sigma_1 A_1 = \sigma_2 A_2$

Compatibility: $\varepsilon_1 L_1 + \varepsilon_2 L_2 = 0$

$$\left(\alpha_1 \Delta T + \frac{P}{E_1 A_1}\right)L_1 + \left(\alpha_2 \Delta T + \frac{P}{E_2 A_2}\right)L_2 = 0$$

$$\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}\right)P = (\alpha_1 L_1 + \alpha_2 L_2)(-\Delta T)$$

$$\left(\frac{250}{190 \times 50} + \frac{500}{70 \times 100}\right)\left(\frac{mm}{kN}\right)P = (12.5 \times 250 + 19 \times 500)(10^{-6}\ mm)(20)$$

$$P = 2.583\ kN$$

$$\sigma_1 = \frac{P}{A_1} = 51.7\ MPa, \quad \sigma_2 = \frac{P}{A_2} = 25.8\ MPa$$

$$\Delta_B = \varepsilon_1 L_1 = \left(\frac{P}{E_1 A_1} + \alpha_1 \Delta T\right)L_1 = \left(\frac{2.583}{190 \times 50} - 20 \times 12.5 \times 10^{-6}\right) \times 250\ mm$$

$$= 5.47 \times 10^{-3}\ mm$$

Displacement method: the unknown is Δ_B

Equilibrium: $\sigma_1 A_1 = \sigma_2 A_2$; $\quad \varepsilon_1 = \frac{\Delta_B}{L_1}, \quad \varepsilon_2 = -\frac{\Delta_B}{L_2}$

$$E_1 A_1\left(\frac{\Delta_B}{L_1} - \alpha_1 \Delta T\right) = E_2 A_2\left(-\frac{\Delta_B}{L_2} - \alpha_2 \Delta T\right)$$

$$\left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}\right)\Delta_B = (E_2 A_2 \alpha_2 - E_1 A_1 \alpha_1)(-\Delta T)$$

$$\left(\frac{190 \times 50}{250} + \frac{70 \times 100}{500}\right)(GPa \cdot mm)\Delta_B = (70 \times 19 \times 100 - 190 \times 12.5 \times 50)(10^{-6}\ GPa\ mm)(20)$$

$$\Delta_B = 5.47 \times 10^{-3}\ mm$$

$$\sigma_1 = E_1\left(\frac{\Delta_B}{L_1} - \alpha_1 \Delta T\right) = 51.7\ MPa$$

$$\sigma_2 = E_2\left(-\frac{\Delta_B}{L_2} - \alpha_2 \Delta T\right) = 25.8\ MPa$$

2. A cylindrical pressure vessel of mean diameter 1000 mm and thickness 5 mm is pressurized to 600 kPa. It is found that the longitudinal and circumferential strains are 2.0×10^{-4} and 7.0×10^{-4}, respectively. Assuming linear isotropic elasticity, calculate the Young's modulus E (in GPa) and the Poisson's ratio ν.

$$\varepsilon_\ell = 2.0\times10^{-4}, \quad \varepsilon_c = 7.0\times10^{-4}$$

$$\sigma_\ell = \frac{pr}{2t} = \frac{600\,\text{kPa}\times 500\,\text{mm}}{2\times 5\,\text{mm}} = 30\ \text{MPa}$$

$$\sigma_c = \frac{pr}{t} = 60\ \text{MPa}$$

$$\varepsilon_\ell = \frac{1}{E}(30-\nu 60)(\text{MPa}) = 2.0\times10^{-4}$$

$$\varepsilon_c = \frac{1}{E}(60-\nu 30)(\text{MPa}) = 7.0\times10^{-4}$$

$$\frac{\varepsilon_c}{\varepsilon_\ell} = 3.5 = \frac{60-\nu 30}{30-\nu 60} = \frac{2-\nu}{1-2\nu}$$

$$3.5(1-2\nu) = 2-\nu$$

$$(3.5-2) = (7-1)\nu \Rightarrow \nu = \frac{1}{4}$$

$$E = \frac{(30-15)(\text{MPa})}{2.0\times10^{-4}} = \underline{75.0\ \text{GPa}}$$

3. A circular bar of radius c is subject to a combined bending moment M and torque T.

(a) Assuming elastic behavior, find the principal stresses at the most highly stressed points in the bar.

(b) If the yield stress in tension or compression is σ_{yp}, find the condition for **initial yielding** in terms of M and T according to (i) the Tresca criterion, (ii) the Mises criterion. (Note: $M_{\text{yp}} = \pi c^3 \sigma_{\text{yp}}/4$.)

(a) $\sigma_{max} = \pm \frac{Mc}{I} = \pm \frac{4M}{\pi c^3}$; $\tau_{max} = \frac{Tc}{J} = \frac{2T}{\pi c^3}$

$$\sigma_{1,2} = \frac{\sigma_{max}}{2} \pm \sqrt{\left(\frac{\sigma_{max}}{2}\right)^2 + \tau_{max}}$$

$$= \frac{2}{\pi c^3}\left(\pm M \pm \sqrt{M^2+T^2}\right)$$

Tension side: $\sigma_{1,2} = \frac{2}{\pi c^3}\left(M \pm \sqrt{M^2+T^2}\right)$

Compression side: $\sigma_{1,2} = \frac{2}{\pi c^3}\left(-M \pm \sqrt{M^2+T^2}\right)$

(b) (i) Tresca: $\max\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right) = \sigma_{yp}$

$$\frac{4}{\pi c^3}\sqrt{M^2+T^2} = \sigma_{yp}$$

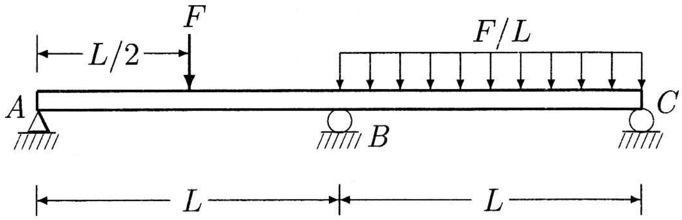
$$\rightarrow \underline{M^2 + T^2 = M_{yp}^2}$$

(ii) Mises: $\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{yp}^2$

$$\left(\frac{2}{\pi c^3}\right)^2 \left[M^2 - \cancel{2M\sqrt{M^2+T^2}} + M^2 + T^2 - \cancel{M^2} + \cancel{M^2} + T^2 + M^2 + \cancel{2M\sqrt{M^2+T^2}} + M^2 + T^2\right]$$

$$\underline{4M^2 + 3T^2 = 4M_{yp}^2}$$

4. For the beam shown, with EI = constant:

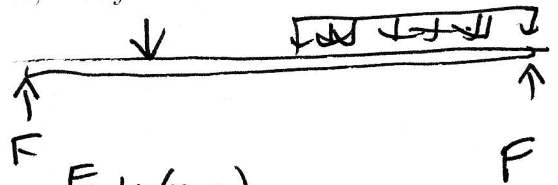


(a) Using singularity functions and superposition, find the reaction force at B.

(b) Using statics, find the bending moment at B.

(c) Sketch the deflected shape of the beam, indicating where you expect to find maxima or minima and inflection points, if any.

(a) Remove support:



$$q = -F\delta(x-\tfrac{L}{2}) - \tfrac{F}{L} H(x-L)$$

$$V = F - FH(x-\tfrac{L}{2}) - \tfrac{F}{L}\langle x-L\rangle$$

$$M = Fx - F\langle x-\tfrac{L}{2}\rangle - \tfrac{F}{2L}\langle x-L\rangle^2$$

$$EIv' = A + \tfrac{F}{2}x^2 - \tfrac{F}{2}\langle x-\tfrac{L}{2}\rangle^2 - \tfrac{F}{6L}\langle x-L\rangle^3$$

$$EIv = Ax + \tfrac{F}{6}x^3 - \tfrac{F}{6}\langle x-\tfrac{L}{2}\rangle^3 - \tfrac{F}{24L}\langle x-L\rangle^4$$

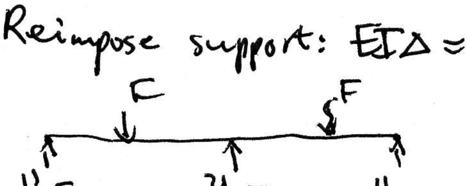
$$EIv(2L) = 0 = 2AL + \tfrac{F}{6}(2L)^3 - \tfrac{F}{6}(\tfrac{3L}{2})^3 - \tfrac{FL^3}{24}$$

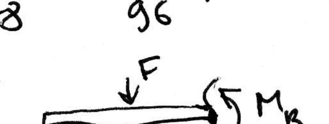
$$\rightarrow A = -\tfrac{FL^2}{96}(64-27-2) = -\tfrac{35}{96}FL^2$$

$$EIv(L) - \tfrac{35}{96}FL^3 + \tfrac{FL^3}{6} - \tfrac{F}{6}(\tfrac{L}{2})^3 = -\tfrac{FL^3}{96}(35-16+2) = -\tfrac{21}{96}FL^3$$

Reimpose support: $EI\Delta = \frac{R_B(2L)^3}{48} = \frac{21}{96}FL^3 \rightarrow R_B = \frac{21}{16}F$

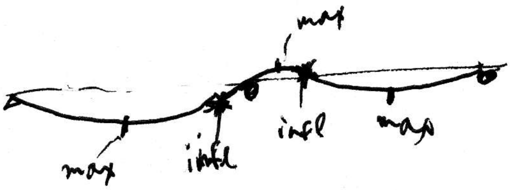
(b)



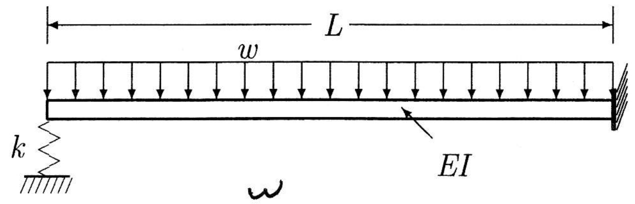


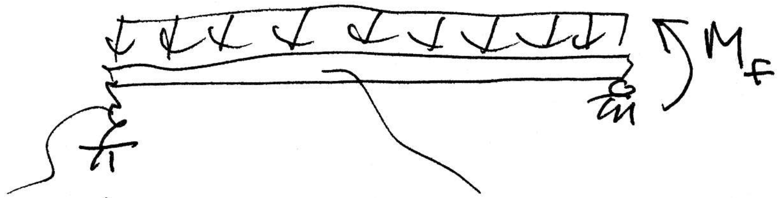
$$M_B = \tfrac{11}{32}FL - F\tfrac{L}{2} = -\tfrac{5}{32}FL$$

(c)



5. Using an energy method (Castigliano's theorem or virtual work), find the moment at the fixed end.





$$F = \frac{wL}{2} + \frac{M_f}{L} \qquad M = \frac{w}{2}x(L-x) + \frac{M_f x}{L}$$

$$\bar{U} = \frac{F^2}{2k} + \int_0^L \frac{M^2}{2EI}\,dx$$

$$\frac{\partial \bar{U}}{\partial M_f} = \frac{F}{k}\frac{\partial F}{\partial M_f} + \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_f}\,dx$$

$$= \frac{1}{k}\left(\frac{wL}{2} + \frac{M_f}{L}\right)\frac{1}{L} + \frac{1}{EI}\int_0^L \left[\frac{w}{2}x(L-x) + \frac{M_f x}{L}\right]\frac{x}{L}\,dx$$

$$= 0$$

$$M_f\left(\frac{1}{kL} + \frac{1}{EIL^2}\int_0^L x^2\,dx\right)$$

$$= -\frac{w}{2}\left[\frac{[\text{illegible}]}{k} + \frac{1}{EIL}\int_0^L x^2(L-x)\,dx\right]$$

$$M_f\left(\frac{1}{kL^2} + \frac{L}{3EI}\right) = -\frac{w}{2}\left(\frac{1}{k} + \frac{L^3}{12EI}\right)$$

$$M_f = -\frac{wL^2}{2} \cdot \frac{1 + \frac{kL^3}{12EI}}{1 + \frac{kL^3}{3EI}}$$

6. A free-standing (cantilever) rectangular post of cross-sectional dimensions a and b $(b > a)$ and height L is made of lumber with compressive crushing strength σ_B and Young's modulus E. Find L such that the post is equally likely to fail by crushing and by buckling.

$$P_{cr} = \frac{\pi^2 EI}{4L^2}, \qquad I = \frac{a^3 b}{12} \quad (\text{weak})$$

$$\sigma_{cr} = \frac{\pi^2 E \cdot a^3 b}{48 L^2 \cdot ab} = \sigma_B$$

$$L^2 = \frac{\pi^2 E a^2}{48 \sigma_B}$$

$$L = \frac{\pi a}{4} \sqrt{\frac{E}{3\sigma_B}}$$