CHAPTER EIGHT
STRESS TRANSFORMATION

Learning objectives

1. Learn the equations and procedures of relating stresses in different coordinate systems (on different planes) at a point.
2. Visualize planes passing through a point on which stresses are given or are being found, in particular the planes of maximum normal and shear stress.

Figure 8.1 shows failure surfaces of aluminum and cast iron members under axial and torsional loads. Why do different materials under similar loading produce different failure surfaces? If we had a combined loading of axial and torsion, then what would be the failure surface, and which stress component would cause the failure? The answer to this question is critical for the successful design of structural members that are subjected to combined axial, torsional, and bending loads. In Chapter 10 we will study combined loading and failure theories that relate maximum normal and shear stresses to material strength. In this chapter we develop procedures and equations that transform stress components from one coordinate system to another at a *given point*.

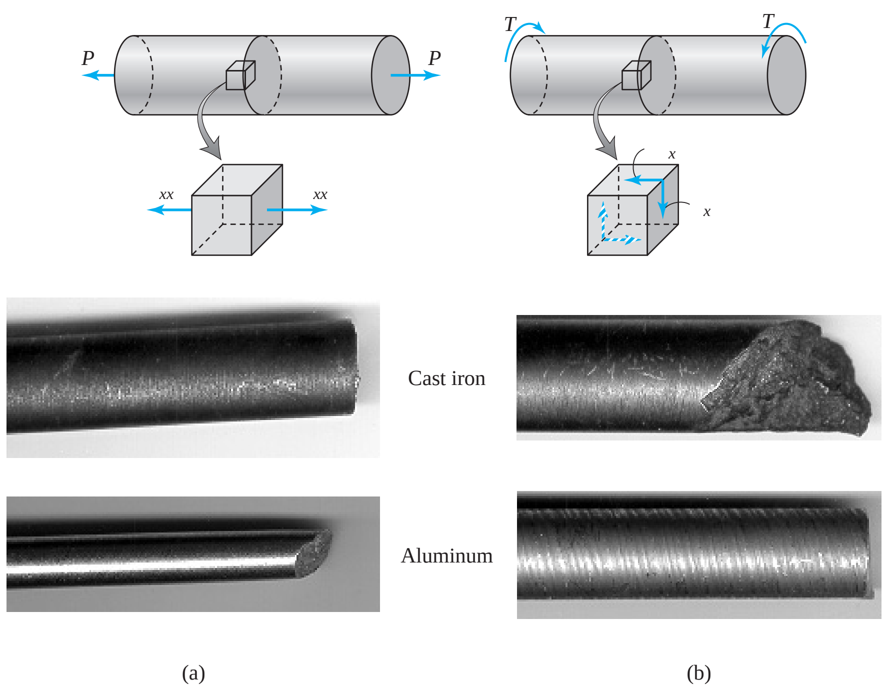


Figure 8.1 Failure surfaces. (*a*) Axial load. (*b*) Torsional load. (Specimens courtesy Professor J. B. Ligon.)

Stress transformation can also be viewed as relating stresses on different planes that pass through a point. The outward normals of the planes form the axes of a coordinate system. Thus relating stresses on different planes is equivalent to relating stresses in different coordinate systems. We will use both viewpoints in this chapter of stress transformation.

8.1 PRELUDE TO THEORY: THE WEDGE METHOD

In this chapter we will study three methods of stress transformation. The *wedge method*, described in this section, is used to derive stress transformation equations in the next section. The stress transformation equations are then manipulated to generate a graphical procedure called *Mohr's circle*, which is described in Section 8.3.

Two coordinate systems will be used in this chapter. First, the entire problem is described in a fixed reference coordinate system called the **global coordinate system**. We usually relate internal forces and moments to external forces and moments in the global coordinate system. The internal quantities are then used to obtain stresses, such as axial stress, torsional shear stress, and bending normal and shear stresses. And second, a **local coordinate system** that can be fixed at any point on the body. The orientation of the local coordinate system is defined with respect to the global coordinate. In all two-dimensional problems in this book, the local coordinate system will be the *n*, *t*, *z* coordinate system.

- The *n* direction is the direction of the *outward normal* to the plane on which we are finding the stresses.
- The *z* direction is identical to the *z* direction of the global coordinate system.
- The tangent *t* direction can be found from the right-hand rule, as shown in Figure 8.2. Just as *x* cross *y* yields *z*, in a similar manner *n* cross *t* yields *z*. With the thumb of the right hand pointed in the known *z* direction, the curl of the fingers is from the known *n* direction toward the *t* direction.

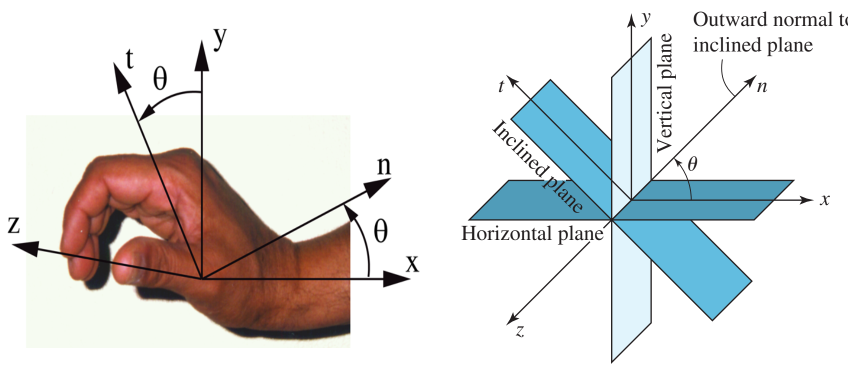


Figure 8.2 Local and global coordinate systems.

Alternatively, the positive *t* direction can be found by curling the fingers of the right hand from the *z* direction toward the *n* direction. The positive *t* direction is then given by the direction of the thumb. With the *n* direction as positive in the outward normal direction, positive shear stress τ_{nt} is in the positive tangent direction and negative τ_{nt} will be in the negative tangent direction.

In this section we restrict ourselves to plane stress problems (see Sections 1.3.2 and 3.6). We will consider only those inclined planes that can be obtained by rotation about the *z* axis, as shown in Figure 8.2*b*.

8.1.1 Wedge Method Procedure

The wedge method has five steps shown below, and elaborated by applications of Examples 8.1 and 8.2.

Step 1: A stress cube with the plane on which stresses are to be found, or are given, is constructed.

Step 2: A wedge is constructed from the following three planes:

1. A vertical plane that has an outward normal in the *x* direction.
2. A horizontal plane that has an outward normal in the *y* direction.
3. The specified inclined plane on which we either seek or are given the stresses.

Establish a local *n*, *t*, *z* coordinate system using the outward normal of the inclined plane as the *n* direction. All the known and unknown stresses are shown on the wedge. The diagram so constructed will be called a *stress wedge.*

Step 3: Multiply the stress components by the area of the planes on which the stress components are acting, to obtain the forces acting on that plane. The wedge with the forces drawn will be referred to as the *force wedge*.

Step 4: Balance forces in *any* two directions to determine the unknown stresses. We can write equilibrium equations on the force wedge because the wedge represents a point on a body that is in equilibrium.

Step 5: Check the answer intuitively by considering each stress component individually. By inspection, we decide whether the stress component will produce tensile or compressive normal stress on the incline and whether it will produce positive or negative shear stress on the incline.

EXAMPLE 8.1

A steel beam in a bridge was repaired by welding along a line that is 35° to the axis of the beam. The normal stress near the bottom of the beam is estimated using beam theory and is shown on the stress cube. Determine the normal and shear stress on the plane containing the weld line.

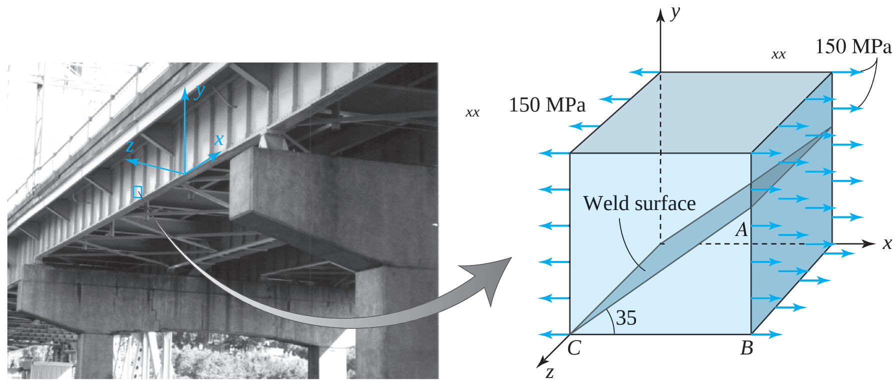


Figure 8.3 Stress cube at a point on a bridge.

PLAN

Step 1 of the procedure outlined in Section 8.1.1 is complete, as shown in Figure 8.3. We follow the remaining steps to solve the problem.

SOLUTION

Step 2: We construct a wedge from the horizontal, vertical, and inclined plane, as shown in Figure 8.4*a*. The outward normal to the inclined plane is drawn, and knowing the positive *z* direction, we establish the positive *t* direction using the right-hand rule for the *n*, *t*, *z* coordinates. On the inclined plane we can show the normal stress σ_{nn} and the shear stress τ_{nt}. From triangle *ABC* we note that $\Delta y = \Delta t \sin 35°$.

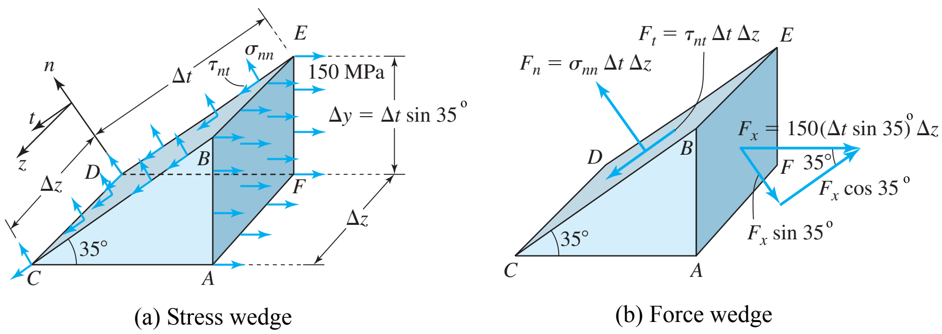


Figure 8.4 Wedges in Example 8.1.

Step 3: We multiply the stresses σ_{nn} and τ_{nt} by the area of the incline *BCDE* to obtain the forces in the *n* and *t* directions, respectively. Similarly, we multiply the stress of 150 MPa by the area of the plane *ABEF* to obtain the force in the *x* direction. These forces are shown on the force wedge in Figure 8.4*b*.

Step 4: As the unknowns are in the *n* and *t* directions, we balance the forces in the *n* and *t* directions. The components of force F_x in the *n* and *t* directions are shown on the force wedge in Figure 8.4*b*. Balancing forces in the *n* direction, we obtain

$$\sigma_{nn}\Delta t\,\Delta z - (150\,\Delta t \sin 35°\,\Delta z)\sin 35° = 0 \quad \text{or} \quad (\sigma_{nn} - 150\sin 35°\sin 35°)\Delta t\,\Delta z = (\sigma_{nn} - 49.35)\Delta t\,\Delta z = 0 \tag{E1}$$

ANS. $\sigma_{nn} = 49.35 \text{ MPa (T)}$

In a similar manner, balancing the forces in the *t* direction, we obtain

$$\tau_{nt}\Delta t\,\Delta z - (150\,\Delta t \sin 35°\,\Delta z)\cos 35° = 0 \quad \text{or} \quad (\tau_{nt} - 150\sin 35°\cos 35°)\Delta t\,\Delta z = (\tau_{nt} - 70.48)\Delta t\,\Delta z = 0 \tag{E2}$$

ANS. $\tau_{nt} = 70.48 \text{ MPa (T)}$

Step 5: We check the answer using intuitive arguments. The surface *ABC* in Figure 8.3 tends to move away from the rest of the cube. Hence the material resistance opposing it results in a tensile stress, as seen. A more visual way is to imagine the inclined plane in Figure

8.3 as a glued surface. Because of σ_{xx}, the two surfaces on either side of the glue are pulled apart; hence the glue is put into tension. Similar σ_{xx} will cause the wedge ABC to slide upward relative to the rest of the cube; hence the material resistance (like friction) will be downward, resulting in a positive shear stress, as seen.

COMMENTS

1. In Equations (E1) and (E2) the dimensions Δt and Δz were common factors and did not affect the final answer. In other words, the dimensions of the stress cube are immaterial. This is not surprising, as the stress cube is a visualization aid symbolically representing a point. Only the relative orientation of the plane is important.
2. The stress cube in Figure 8.3 and the stress and force wedges in Figure 8.4 can be represented in two dimensions, as shown in Figure 8.5. These are easier to draw and work with. But once more it must be emphasized that stress is a distributed force and not a vector, as depicted in Figure 8.5*b*. Force equilibrium can be done only on the force wedge

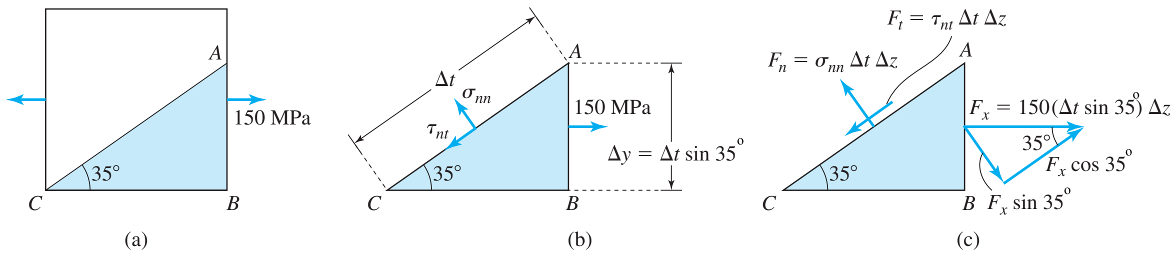


Figure 8.5 Wedge method in Example 8.1. (*a*) stress cube; (*b*) stress wedge; (*c*) force wedge.

3. In constructing the stress wedge we took the lower wedge. An alternative approach is to take the upper wedge, as shown in Figure 8.6. This is possible as the dimensions of the stress cube are immaterial. Only the orientation of the planes is important.

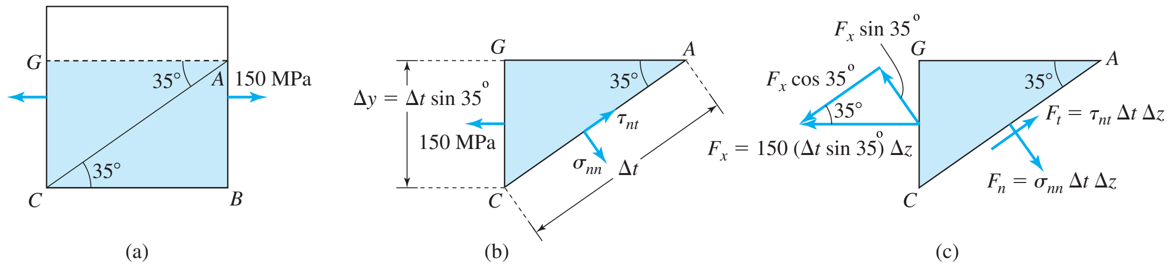


Figure 8.6 Alternative approach in Example 8.1. (*a*) stress cube; (*b*) stress wedge; (*c*) force wedge.

4. Some writing could be saved by taking $\Delta t = 1$ and $\Delta z = 1$, as these terms always drop out. But the geometric visualization may become more difficult in the process.

EXAMPLE 8.2

Fibers are oriented at 30° to the x axis in a lamina of a composite[1] plate, as shown Figure 8.7. Stresses at a point in the lamina were found by the finite-element method[2] as

$$\sigma_{xx} = 30\ \text{MPa (T)} \qquad \sigma_{yy} = 60\ \text{MPa (C)} \qquad \tau_{xy} = 50\ \text{MPa}$$

In order to assess the strength of the interface between the fiber and the resin, determine the normal and shear stresses on the plane containing the fiber.

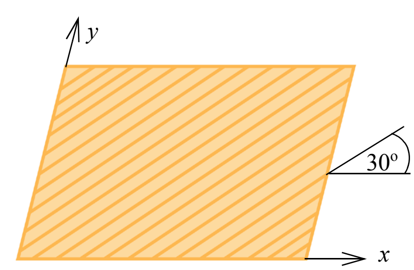


Figure 8.7 Stresses in lamina in Example 8.2.

PLAN

In Step 1 of the procedure outlined Section 8.1.1, we can draw the stress cube with an plane inclined at 30° and then follow the remaining steps of the procedure.

[1]See Section 3.12.3 for a brief description of composite materials.
[2]See Section 4.8 for a brief description of the finite-element method.

SOLUTION

Step 1: We draw a stress cube in two dimensions for the given state of stress, and the plane inclined at 30° counterclockwise to the x axis, as shown in Figure 8.8*a*

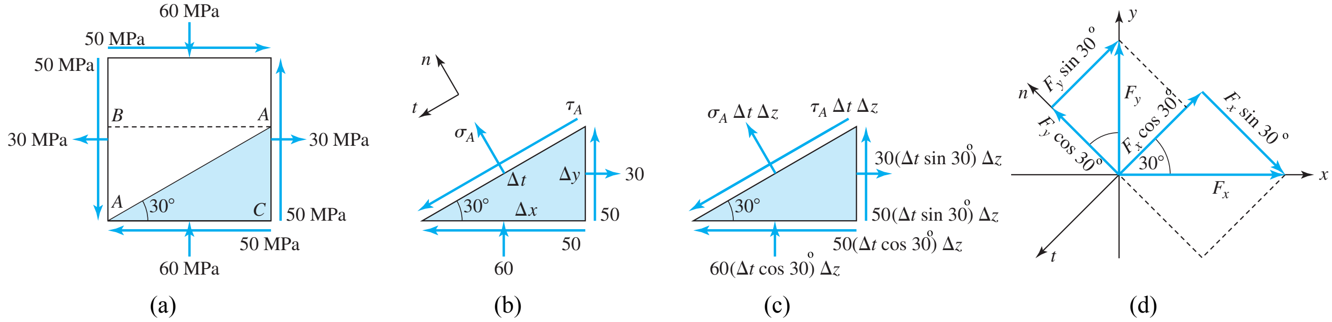


Figure 8.8 (*a*) Stress cube. (*b*) Stress wedge. (*c*) Force wedge (d) Resolution of force components.

Step 2: We can choose wedge *ACA* or wedge *ABA* as a stress wedge. Figure 8.8*b* shows the stress wedge *ACA* with a local n, t, z coordinate system.

Step 3: We assume the length of the inclined plane to be Δt. From geometry we see that $\Delta x = \Delta t \cos 30^o$ and $\Delta y = \Delta t \sin 30^o$. If we assume that the dimension of the cube out of the paper is Δz, we get the following areas: inclined plane $\Delta t\, \Delta z$, vertical plane $\Delta y\, \Delta z$, and horizontal plane $\Delta x\, \Delta z$. The stresses are converted into forces by multiplying by the area of the plane, and a force wedge is drawn as shown in Figure 8.8*c*.

Step 4: We can balance forces in any two directions. We choose to balance forces in the n and t directions as the unknowns are in the n and t directions. Figure 8.8*d* shows the resolution of the forces in the x, y coordinates to n, t coordinates.

The forces in the x and y directions in Figure 8.8*c* that need resolution are:

$$F_x = [(30\ \text{MPa})(\Delta t \sin 30°)\Delta z - (50\ \text{MPa})(\Delta t \cos 30°)\Delta z] = -(28.301\ \text{MPa})(\Delta t\, \Delta z) \tag{E1}$$

$$F_y = [(60\ \text{MPa})(\Delta t \cos 30°)\Delta z + (50\ \text{MPa})(\Delta t \sin 30°)\,\Delta z] = (76.961\ \text{MPa})(\Delta t\, \Delta z) \tag{E2}$$

From Figure 8.8c and Figure 8.8*d*, the equilibrium of forces in the n direction yields,

$$\sigma_A(\Delta t\, \Delta z) - F_x \sin 30° + F_y \cos 30° = 0 \quad \text{or}$$

$$\sigma_A(\Delta t\, \Delta z) - [-(28.301\ \text{MPa})(\Delta t\, \Delta z)] \sin 30° + [(76.961\ \text{MPa})(\Delta t\, \Delta z)] \cos 30° = 0 \quad \text{or}$$

$$\sigma_A(\Delta t\, \Delta z) + (14.15\ \text{MPa})(\Delta t\, \Delta z) + (66.65\ \text{MPa})(\Delta t\, \Delta z) = [\sigma_A + 80.8\,\text{MPa}](\Delta t\, \Delta z) = 0 \tag{E3}$$

ANS. $\sigma_A = 80.8\ \text{MPa (C)}$

The shear stress can be similarly calculated by equilibrium in the t direction,

$$\tau_A(\Delta t\ \Delta z) - F_x \cos 30° - F_y \sin 30° = 0 \quad \text{or}$$

$$\tau_A(\Delta t\ \Delta z) - [-(28.301\ \text{MPa})(\Delta t\, \Delta z)] \cos 30° - [(76.961\ \text{MPa})(\Delta t\, \Delta z)] \sin 30° = 0 \quad \text{or}$$

$$\tau_A(\Delta t\ \Delta z) + (24.509\ \text{MPa})(\Delta t\ \Delta z) - (38.481\ \text{MPa})(\Delta t\ \Delta z) = [\tau_A - (13.971\,\text{MPa})](\Delta t\ \Delta z) = 0 \tag{E4}$$

ANS. $\tau_A = 14.0\ \text{MPa}$

Step 5: We can check the answers intuitively. Consider each stress component individually and visualize the inclined plane as a glue line. The rectangles shown in Figure 8.9 are for purposes of explanation. One can go through the arguments mentally without drawing these rectangles.

Figure 8.9 shows that the right surface (wedge) and the left surface will move:

- Apart due to σ_{xx}—putting the glue in *tension*
- Into each other due to σ_{yy}—putting the glue in *compression*
- Into each other due to τ_{xy}—putting the glue in *compression*
- Into each other due to τ_{yx}—putting the glue in *compression*.

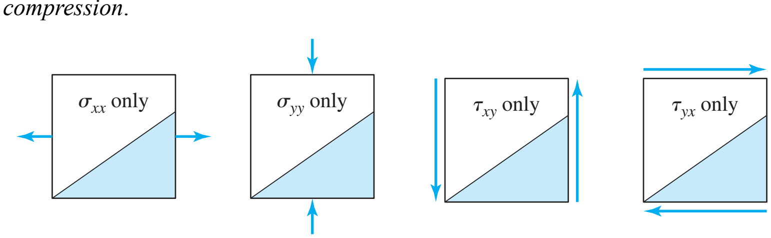


Figure 8.9 Intuitive check.

Thus the normal stress in the glue (on the inclined plane) is expected to be in compression, which is consistent with our answer.

Figure 8.9 shows that the right surface (shaded wedge), with respect to the left surface, will slide:

- Upward due to σ_{xx}; therefore the shaded wedge will have a *positive* (downward) shear stress
- Upward due to σ_{yy}; therefore the right wedge will have a *positive* (downward) shear stress
- Upward due to τ_{xy}; therefore the right wedge will have a *positive* (downward) shear stress

- Downward due to τ_{yx}; therefore the right wedge will have a *negative* (upward) shear stress.

Thus the shear stress on the incline is expected to be positive, which is consistent with our answer.

COMMENTS

1. In the intuitive check, three of the components gave one answer, whereas the fourth gave an opposite answer. What happens if two intuitive deductions are positive, two intuitive deductions are negative, and the stress components are nearly equal in magnitude? The question emphasizes that intuitive reasoning is a quick and important check on results, but one must be cautious with the conclusions.
2. We could have balanced forces in the x and y directions, in which case we have to find the x and y components of the normal and tangential forces on the force wedge. After removing the common factors $\Delta t\,\Delta z$ we would obtain

$$\sigma_A \sin 30° + \tau_A \cos 30° = -(50 \text{ MPa})\cos 30° + (30 \text{ MPa})\sin 30° = -28.30 \text{ MPa}$$

$$\sigma_A \cos 30° - \tau_A \sin 30° = -(50 \text{ MPa})\sin 30° - (60 \text{ MPa})\cos 30° = -76.96 \text{ MPa}$$

Solving these two equations, we obtain the values of σ_A and τ_A as before. By balancing forces in the n, t directions we generated one equation per unknown but did extra computation in finding components of forces in the n and t directions. By balancing forces in the x and y directions, we did less work finding the components of forces, but we did extra work in solving simultaneous equations. This shows that the important point is to balance forces in any two directions, and the direction chosen for balancing the forces is a matter of preference.

3. Figure 8.8 is useful in reducing the algebra when forces are balanced in the n and t directions. But you may prefer to resolve components of individual forces, as shown in Figure 8.10, and then write the equilibrium equations. The method is a little more tedious, but has the advantage that the intuitive check can be conducted as one writes the equilibrium equations as follows.

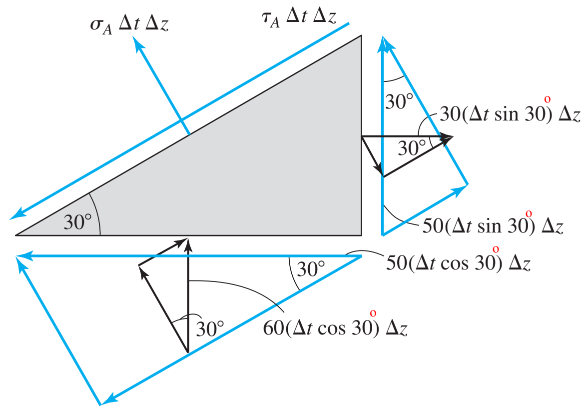


Figure 8.10 Alternative force resolution.

The normal stress σ_A on the incline will be:

- Tensile due to σ_{xx}; Compressive due to σ_{yy}; Compressive due to τ_{xy};Compressive due to τ_{yx}.

As σ_{xx} is the smallest stress component, it is not surprising that the total result is a compressive normal stress on the inclined plane.

The shear stress on the incline will be:

- Positive due to σ_{xx};Positive due to σ_{yy}; Positive due to τ_{xy};Negative due to τ_{yx}.

We expect the net result to be positive shear stress on the incline.

PROBLEM SET 8.1

Stresses by inspection

8.1 In Figure P8.1, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

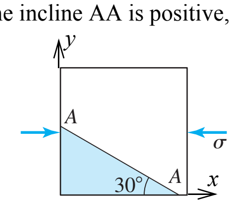


Figure P8.1

8.2 In Figure P8.2 determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

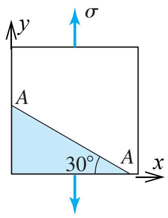


Figure P8.2

8.3 In Figure P8.3, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

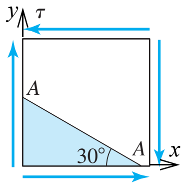


Figure P8.3

8.4 In Figure P8.4, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

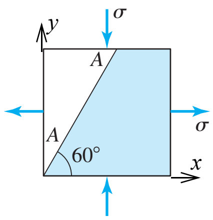


Figure P8.4

8.5 In Figure P8.5, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

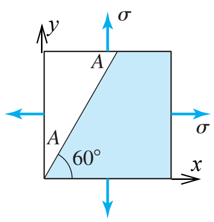


Figure P8.5

8.6 In Figure P8.6, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

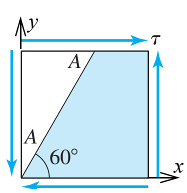


Figure P8.6

8.7 In Figure P8.7, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

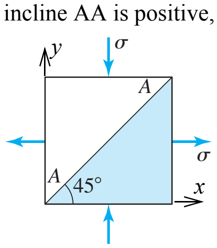


Figure P8.7

8.8 In Figure P8.8, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

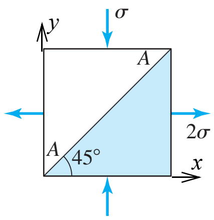


Figure P8.8

8.9 In Figure P8.9, determine by inspection: (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

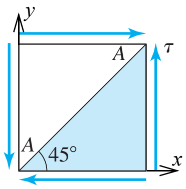


Figure P8.9

8.10 Determine the normal and shear stresses on plane AA in Problem 8.1 for $\sigma = 10$ ksi.

8.11 Determine the normal and shear stresses on plane AA in Problem 8.4 for $\sigma = 10$ ksi.

8.12 Determine the normal and shear stresses on plane AA in Problem 8.6 for $\tau = 10$ ksi.

8.13 Determine the normal and shear stresses on plane AA in Problem 8.7 for $\sigma = 60$ MPa.

8.14 Determine the normal and shear stresses on plane AA in Problem 8.9 for $\tau = 60$ MPa.

8.15 A shaft is adhesively bonded along the seam as shown in Figure P8.15. By inspection determine whether the adhesive will be in tension or in compression.

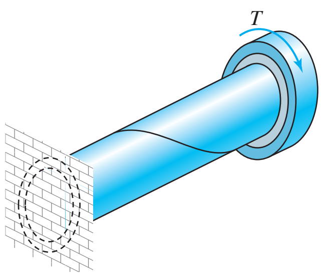


Figure P8.15

8.16 A shaft is adhesively bonded along the seam as shown in Figure P8.16. By inspection determine whether the adhesive will be in tension or in compression .

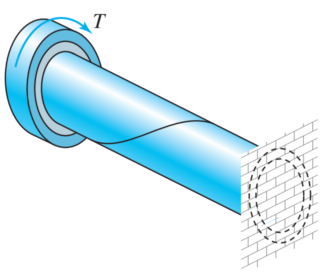


Figure P8.16

8.17 A shaft is adhesively bonded along the seam as shown in Figure P8.17. By inspection determine whether the adhesive will be in tension or in compression.

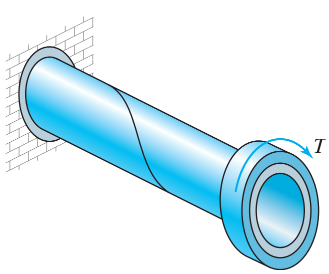


Figure P8.17

8.18 A shaft is adhesively bonded along the seam as shown in Figure P8.18. By inspection determine whether the adhesive will be in tension or in compression.

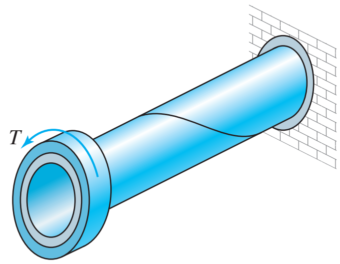


Figure P8.18

8.19 Determine the normal and shear stresses on plane AA shown in Figure P8.19.

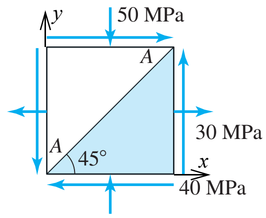


Figure P8.19

8.20 Determine the normal and shear stresses on plane AA shown in Figure P8.20.

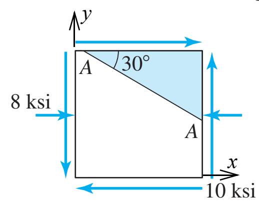


Figure P8.20

8.21 Determine the normal and shear stresses on plane AA shown in Figure P8.21.

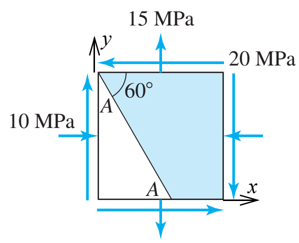


Figure P8.21

8.22 The stresses at a point in plane stress are σ_{xx} = 45 MPa (T), σ_{yy} = 15 MPa (T), and τ_{xy} = –20 MPa. Determine the normal and shear stresses on a plane passing through the point at 28° counterclockwise to the x axis.

8.23 The stresses at a point in plane stress are σ_{xx} = 45 MPa (T), σ_{yy} = 15 MPa (C), and τ_{xy} = –20 MPa. Determine the normal and shear stresses on a plane passing through the point at 38° clockwise to the x axis.

8.24 The stresses at a point in plane stress are σ_{xx} = 10 ksi (C), σ_{yy} = 20 ksi (C), and τ_{xy} = 30 ksi. Determine the normal and shear stresses on a plane passing through the point that is 42° counterclockwise to the x axis.

8.25 A cast-iron shaft of 25-mm diameter fractured along a surface that is 45° to the axis of the shaft. The shear stress τ due to torsion is as shown in Figure P8.25. If the ultimate normal stress for the brittle cast-iron material is 330 MPa (T), determine the torque that caused the fracture.

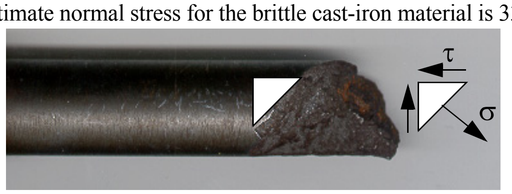


Figure P8.25

Design problems

8.26 In a wooden structure a member was adhesively bonded along a plane 40° to the horizontal plane, as shown in Figure P8.26. The stresses at a point on the bonded plane due to a load P on the structure, were estimated as shown, where P is in lb. If the adhesive strength in tension is 500 psi and its strength in shear is 200 psi, determine the maximum permissible load the structure can support without breaking the adhesive joint.

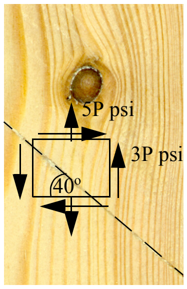


Figure P8.26

Stretch yourself

In three dimensions, the area of the inclined plane A can be related to the areas of the surfaces of the stress cube using the direction cosines of the outward normals, as shown in Figure P8.26.

$$n_x = \cos\theta_x \qquad A_x = n_x A \qquad n_y = \cos\theta_y \qquad A_y = n_y A \qquad n_z = \cos\theta_z \qquad A_z = n_z A$$

These relationships can be used to convert the stress wedge into a force wedge. Using this information, solve Problems 8.27 and 8.28. (Hint: A component of a vector in a given direction can be found by taking the dot [scalar] product of the vector with a unit vector in the given direction.)

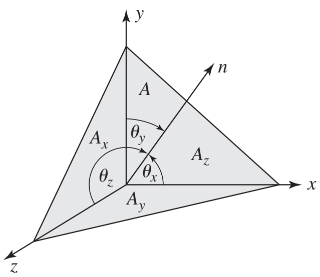


Figure P8.26

8.27 The stresses at a point are σ_{xx} = 8 ksi (T), σ_{yy} = 12 ksi (T), and σ_{zz} = 8 ksi (C). Determine the normal stress on a plane that has outward normals at 60°, –60°, and 45° to the x, y, and z directions, respectively.

8.28 The stresses at a point are τ_{xy} = 125 MPa and τ_{xz} = –150 MPa. Determine the normal stress on a plane that has outward normals at 72.54°, 120°, and 35.67° to the x, y, and z directions, respectively.

8.2 STRESS TRANSFORMATION BY METHOD OF EQUATIONS

We follow the wedge method procedure described in Section 8.1.1 with variables in place of numbers to develop equations that relate the stresses in the Cartesian coordinate system to the stresses on an arbitrary inclined plane. We once more consider only those planes that can be obtained by rotating about the z axis, as shown in Figure 8.2a. The *outward normal* to the inclined plane makes an angle θ with the x axis. The angle θ is considered *positive counterclockwise from the x axis*, as shown in Figure 8.11a.

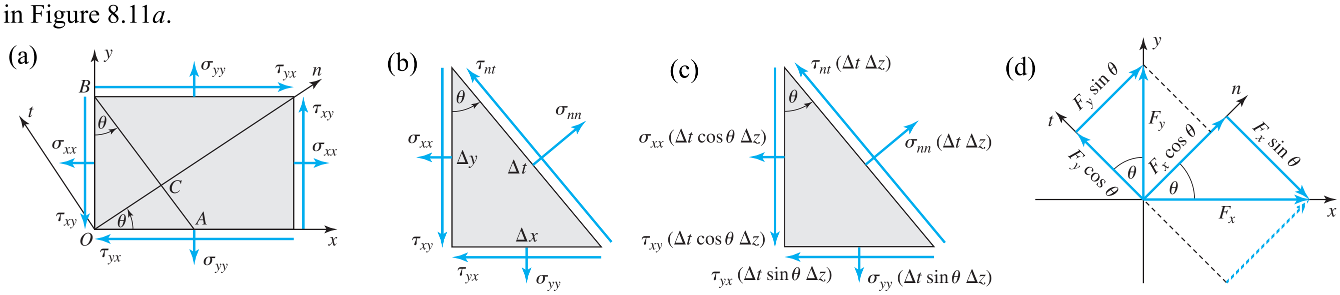


Figure 8.11 (a) Stress cube. (b) Stress wedge. (c) Force wedge. (d) Resolution of force components.

Step 1: We draw the stress cube with all positive stress components as shown in Figure 8.11a. From triangle OAC in Figure 8.11a we deduce that the angle OAC is 90° − θ. From triangle OAB we conclude that the angle OBA is θ.

Step 2: The stress wedge *OAB* is drawn as shown in Figure 8.11*b*. Positive normal σ_{nn} and shear stress τ_{nt} are drawn on the inclined plane.
Step 3: We obtain the force wedge by multiplying the stresses by the areas of the planes on which they act as shown in Figure 8.11*c*.
Step 4: To write the equilibrium equations we use Figure 8.11*d* for resolving forces from the *x* and *y* direction to *n* and *t* direction. The forces in the *x* and *y* direction in Figure 8.11*c* that need resolving are:

$$F_x = -\sigma_{xx}\cos\theta\,\Delta t\,\Delta z - \tau_{yx}\sin\theta\,\Delta t\,\Delta z = -(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)$$
$$F_y = -\sigma_{yy}\sin\theta\,\Delta t\,\Delta z - \tau_{xy}\cos\theta\,\Delta t\,\Delta z = -(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)$$

By equilibrium of forces in the *n* direction on the force wedge in Figure 8.11*c*, we obtain

$$\sigma_{nn}(\Delta t\,\Delta z) + F_x\cos\theta + F_y\sin\theta = 0 \quad \text{or}$$
$$\sigma_{nn}(\Delta t\,\Delta z) + [-(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)]\cos\theta + [-(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)]\sin\theta = 0$$

Because $\Delta t\,\Delta z$ is a common factor, these equations simplify to

$$\boxed{\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta} \tag{8.1}$$

Similarly by equilibrium of forces in the *t* direction on the force wedge in Figure 8.11*c*, we obtain the shear stress,

$$\tau_{nt}\,\Delta t\,\Delta z - F_x\sin\theta + F_y\cos\theta = 0 \quad \text{or}$$
$$\tau_{nt}\,\Delta t\,\Delta z - [-(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)]\sin\theta + +[-(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)]\cos\theta = 0\text{, or}$$

$$\boxed{\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)} \tag{8.2}$$

We can find σ_{tt} by substituting $90 + \theta$ in place of θ into Equation (8.1),

$$\boxed{\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta} \tag{8.3}$$

Equations (8.1) through (8.3) transform stresses from the *x* and *y* coordinate system into *n* and *t* coordinate system that is obtained by rotating by an angle θ in the counterclockwise direction.

8.2.1 Maximum Normal Stress

In Section 3.1we observed that a brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material. Cracks in the material propagate due to tensile stress. Adhesively bonded material debonds due to tensile normal stress, which is called *peel stress*. Similarly, failure may occur due to a maximum compressive normal stress because of the phenomenon called *buckling,* which is discussed in Chapter 11. In this section we develop equations for maximum tensile and compressive normal stresses.

In Equation (8.1) the stresses σ_{xx}, σ_{yy}, and τ_{xy} are assumed known. Thus Equation (8.1) expresses σ_{nn} as a function of θ. From calculus we know that the maximum or minimum of a function exists where the first derivative is zero. Before performing the differentiation we rewrite Equations (8.1) and (8.2) in terms of the double angles of 2θ[3] as

$$\sigma_{nn} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{8.4}$$

$$\tau_{nt} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{8.5}$$

Let $\theta = \theta_p$ be the angle of the outward normal of the plane on which the maximum or minimum normal stress exists. Differentiating Equation (8.4), we obtain

$$\left.\frac{d\sigma_{nn}}{d\theta}\right|_{\theta=\theta_p} = -2\frac{(\sigma_{xx} - \sigma_{yy})\sin 2\theta}{2} + 2\tau_{xy}\cos 2\theta\bigg|_{\theta=\theta_p} = 0 \quad \text{or}$$

$$\boxed{\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}} \tag{8.6}$$

[3] $\cos^2\theta = (1 + \cos 2\theta)/2$, $\sin^2\theta = (1 - \cos 2\theta)/2$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and $\cos\theta\sin\theta = (\sin 2\theta)/2$.

We note that tan $(180 + 2\theta_p) = \tan 2\theta_p$. Thus there are two angles—180° apart—that satisfy Equation (8.6), as shown in Figure 8.12: $\theta_1 = \theta_p$ and $\theta_2 = 90 + \theta_p$. Then from Figure 8.12 we obtain the following where the plus sign is taken with subscript 1 and the minus sign with subscript 2:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sin 2\theta_{1,2} = \pm\tau_{xy}/R \qquad \cos 2\theta_{1,2} = \pm\frac{\sigma_{xx} - \sigma_{yy}}{2}/R$$

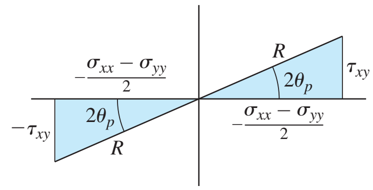


Figure 8.12 Two angles of principal planes.

Let the normal and shear stresses on planes with outward normals in the θ_1 and θ_2 directions be represented by σ_1, τ_1, and σ_2, τ_2, respectively. Substituting the sines and cosines of $2\theta_1$ and $2\theta_2$ into Equations (8.4) and (8.5), we obtain

$$\boxed{\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}} \tag{8.7}$$

$$\tau_{1,2} = 0 \tag{8.8}$$

where $\sigma_{1,2}$ represents the two stresses σ_1 and σ_2 with the plus sign to be taken with σ_1 and the minus sign with σ_2. Equation (8.8) shows that the planes on which σ_1 and σ_2 act are planes with zero shear stress. Planes with zero shear stresses are called **principal planes**. The normal direction to the principal planes is referred to as the **principal direction** or *principal axis*, and the angles the principal directions makes with the global coordinate system are called **principal angles**.

The normal stress on a principal plane is called the **principal stress** and the greatest principal stress is called **principal stress 1**. In defining greatest principal stress both the magnitude and the sign are considered. A stress of −2 MPa is greater than −10 MPa. Alternatively, if normal stresses are shown on an axis with negative values to the left of the origin and positive values to the right, then the rightmost normal stress is principal stress 1 denoted by σ_1.

The stresses in Equation (8.7) represent the maximum or minimum normal stress at a point. This implies that principal stresses are the maximum and minimum normal stresses at a point. Furthermore, the plane of principal stress 1 (θ_1) is 90° away from the plane of principal stress 2 (θ_2). In other words, principal planes are *orthogonal*. Adding Equation (8.1), Equation (8.3), and the principal stresses in Equation (8.7), we obtain

$$\boxed{\sigma_{nn} + \sigma_{tt} = \sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2} \tag{8.9}$$

Equation (8.9) shows that the sum of the normal stresses in an orthogonal coordinate system at a point does not depend on the orientation of the coordinate system in other words is *invariant*.

In summary:

- The sum of the normal stresses is invariant with the coordinate transformation.
- Principal stresses are maximum or minimum normal stresses
- Principal planes and principal directions are orthogonal.

8.2.2 Procedure for determining principal angle and stresses

Equation (8.6) will give us either θ_1 or θ_2. Thus it is not clear whether the principal angle found from Equation (8.6) is associated with σ_1 or σ_2. The problem can be resolved by the following procedure:

Step 1: Find θ_p from Equation (8.6).
Step 2: Substitute θ_p in Equation (8.1) to find a principal stress.
Step 3: Find the other principal stress from Equation (8.9).
Step 4: Decide which of the two principal stresses is principal stress 1.
Step 5: If the stress obtained from substituting θ_p into Equation (8.1) yields principal stress 1, then we report θ_p as principal angle 1 θ_1, otherwise we subtract (or add) 90° from θ_p and report the result as principal angle 1.

Step 6: Use Equation (8.7) as a check on the results.

From the definition of plane stress,[4] the plane with the outward normal in the z direction has zero shear stress. Therefore this plane is a principal plane and the normal stress σ_{zz} is a principal stress of zero value. In plane strain, the shear strain and hence shear stresses with subscript z are also zero. Hence in plane strain too, σ_{zz} is the third principal stress, but it is not zero. Using Figure 3.27, we can summarize as

$$\boxed{\sigma_3 = \sigma_{zz} = \begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_{xx} + \sigma_{yy}) = \nu(\sigma_1 + \sigma_2) & \text{plane strain} \end{cases}} \tag{8.10}$$

The value of the third principal stress affects the maximum shear stress at a point, as will be seen in the next two sections.

8.2.3 In-Plane Maximum Shear Stress

Ductile materials yield when the maximum shear stress exceeds the yield stress. In bonded members, such as lap joints, the loads are transferred from one member to another through shear and are designed on the basis of the shear strength of the adhesive. In this section we develop equations for maximum shear stress.

In determining the maximum shear stress from Equation (8.2) we are considering only planes that can be obtained from rotation about the z axis, as shown in Figure 8.2. Thus we are not considering all possible planes that may pass through the point. The maximum shear stress on a plane that can be obtained by rotating about the z axis is called **in-plane maximum shear stress.** Let $\theta = \theta_s$ be the plane at which the in-plane maximum shear stress exists. By differentiating Equation (8.5) we get

$$\left.\frac{d\tau_{nt}}{d\theta}\right|_{\theta=\theta_s} = -2\frac{(\sigma_{xx} - \sigma_{yy})\cos 2\theta}{2} - 2\tau_{xy}\sin 2\theta\Big|_{\theta=\theta_s} = 0 \qquad \text{or} \qquad \tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}} \tag{8.11}$$

Once more, two angles can satisfy Equation (8.11). Letting $\bar{\theta}_1 = \theta_s$ and $\bar{\theta}_2 = 90 + \theta_s$, then from Figure 8.13 we obtain

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sin 2\bar{\theta}_{1,2} = \mp\frac{\sigma_{xx} - \sigma_{yy}}{2}\Big/R \qquad \cos 2\bar{\theta}_{1,2} = \pm\tau_{xy}/R$$

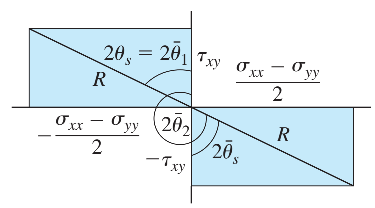


Figure 8.13 Two angles of maximum shear stress planes.

Let τ_{12} and τ_{21} be the shear stresses on the two planes defined by the angles $\bar{\theta}_1$ and $\bar{\theta}_2$. We can find the sines and cosines of $2\bar{\theta}_1$ and $2\bar{\theta}_2$, as shown in Figure 8.13, and substitute these quantities into Equations (8.4) and (8.5) to obtain

$$\sigma_{\text{av}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \qquad |\tau_p| = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \tag{8.12}$$

where τ_p is the in-plane maximum shear stress obtained from the magnitude of the equation $\tau_{12} = -\tau_{21} = R$.

From Equations (8.6) and (8.11) we can obtain

$$\tan 2\theta_s = \frac{-1}{\tan 2\theta_p} = \tan(90° + 2\theta_p)$$

Therefore $\theta_s = 45 + \theta_p$. In other words, maximum in-plane shear stress exists on two planes, each of which is 45° away from the principal planes.

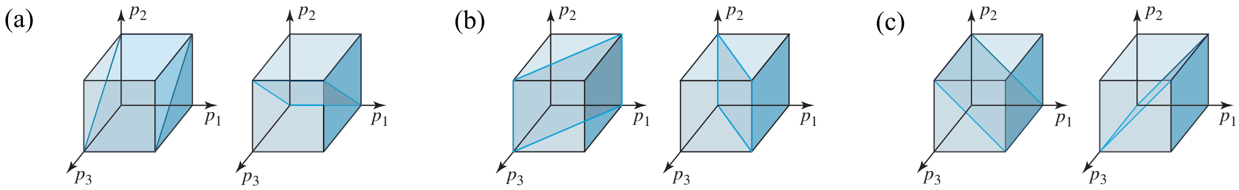
[4]See Section 1.3.2 for a definition of plane stress and Section 2.5.1 for a definition of plane strain. See Section 3.6 for the difference between plane stress and plane strain.

8.2.4 Maximum Shear Stress

The **maximum shear stress** at a point is the *absolute* maximum shear stress that acts on any plane passing through the point.

In the previous section we saw that as we rotate the coordinate system about the z axis (the third principal axis), the shear stress varies from a zero value, at a principal plane. It reaches a maximum value, given by Equation (8.10), on a plane that is 45° to a principal plane. Will this observation also be true if we rotate about principal axis 1 or 2? The answer is yes, because there is no distinction between the three principal planes passing through a point. On a cube six possible diagonal surfaces are at 45° to the cube surfaces. We consider each of the three rotations and show all the possibilities of maximum shear stress on the stress cube in Figures 8.14(a) through 8.14(c).

Figure 8.14 shows the six possible planes that are 45° to principal planes on which the maximum shear stress may exist if rotation is restricted about one of the three principal axis.



$$\tau_{23} = -\tau_{32} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{31} = -\tau_{13} = \frac{\sigma_3 - \sigma_1}{2} \qquad \tau_{21} = -\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}$$

Figure 8.14 Planes of maximum shear obtained by (a) rotating about principal axis 1. (b) by rotating about principal axis 2. (c) by rotating about principal axis 3.

The maximum shear stress at a point is the largest in magnitude of the three values obtained from Figures 8.14. It is written conveniently as

$$\boxed{\tau_{max} = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right)} \tag{8.13}$$

Equation (8.13) shows that the maximum shear stress value depends on principal stress 3. Equation (8.10) shows that the value of principal stress 3 depends on whether a plane stress or plane strain exists. In other words, the maximum shear stress value may be different in plane stress and in plane strain.

EXAMPLE 8.3

Solve Example 8.2 using Equations (8.1) and (8.2). Also determine the principal stresses, principal angle 1, and the maximum shear stress at the point.

PLAN

(a) We can determine the angle of the outward normal to the inclined plane containing the fiber. We can then use Equations (8.1) and (8.2) to find the normal and shear stresses on the plane. (b) Principal stress 3 is zero because the point is in plane stress. We follow the procedure in Section 8.2.2 to determine the principal angle 1 and principal stresses 1 and 2. (c) We can find the maximum shear stress from Equation (8.13).

SOLUTION

(a) The plane AA containing the fiber is at an angle of 30° from the x axis. Hence the direction of the outward normal is $\theta = 120°$, as shown in Figure 8.15. Substituting in Equations (8.1) and (8.2), we obtain

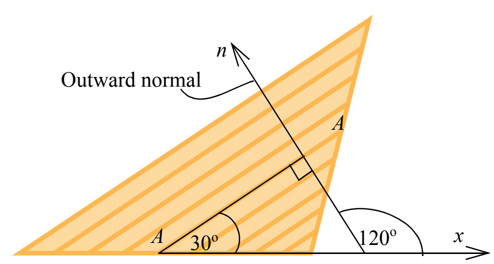


Figure 8.15 Outward normal to plane in Example 8.3.

$$\sigma_A = (30\text{ MPa})\cos^2 120° + (-60\text{ MPa})\sin^2 120° + 2(50\text{ MPa})\sin 120°\cos 120° = -80.80\text{ MPa} \quad \text{(E1)}$$

$$\tau_A = -(30\text{ MPa})\sin 120°\cos 120° + (-60\text{ MPa})\sin 120°\cos 120° + (50\text{MPa})(\cos^2 120° - \sin^2 120°) = 13.97\text{ MPa}$$

ANS. $\sigma_A = 80.8\text{ MPa }(C)$ $\tau_A = 14.0\text{ MPa}$

(b) We follow procedure in Section 8.2.2.
Step 1: Find the principal angle from Equation (8.6),

$$\theta_p = \frac{1}{2}\arctan\left(\frac{50\text{ MPa}}{[30\text{ MPa} - (-60\text{ MPa})]/2}\right) = \frac{1}{2}\arctan\left(\frac{50}{45}\right) = 24.01° \quad \text{(E2)}$$

Step 2: Substitute the principal angle into Equation (8.1) to obtain one of the principal stresses,

$$\sigma_p = (30\text{ MPa})\cos^2 24.01° + (-60\text{ MPa})\sin^2 24.01° + 2(50\text{ MPa})\sin 24.01°\cos 24.01° = 52.26\text{ MPa} \quad \text{(E3)}$$

Step 3: Note that $\sigma_{xx} + \sigma_{yy}$ = 30 MPa– 60 MPa= –30 MPa. From Equations (8.9) and (E3) the other principal stress is –82.26 MPa.
Step 4: The principal stress in Equation (E3) is greater than the stress in *step 3*, therefore it is principal stress 1.
Step5: The angle in Equation (E2) is principal angle 1.
Step 6: We can check our calculations of σ_1 and σ_2 using as shown,

$$\sigma_{1,2} = \frac{30\text{MPa} + (-60\text{ MPa})}{2} \pm \sqrt{\left[\frac{30\text{ MPa} - (-60\text{ MPa})}{2}\right]^2 + (50\text{ MPa})^2} = -15\text{ MPa} \pm 67.26\text{ MPa} \text{ or}$$

$$\sigma_1 = -15\text{ MPa} + 67.26\text{ MPa} = 52.26\text{ MPa} \qquad \sigma_2 = -15\text{ MPa} - 67.26\text{ MPa} = -82.26\text{ MPa}\text{-----Checks}$$

We report our answers as

ANS. $\sigma_1 = 52.3\text{ MPa (T)}$ $\sigma_2 = 82.3\text{ MPa (C)}$ $\sigma_3 = 0$ $\theta_1 = 24.0°\text{ ccw}$

(c) The maximum shear stress at the point is half the maximum difference between the principal stresses, as per Equation (8.13), which in this problem is between σ_1 and σ_2,

$$\tau_{max} = \frac{52.26\text{ MPa} - (-82.26\text{ MPa})}{2} = 67.26\text{ MPa}$$

ANS. $\tau_{max} = 67.3\text{ MPa}$

COMMENTS

1. If the principal stress in step 3 was greater than the stress in Equation (E3), then it would be principal stress 1 and we could either add or subtract 90° from θ_p in Equation (E2) to report the principal angle one θ_1.
2. In finding normal stress σ_A and shear stress τ_A on the inclined plane, we substituted $\theta = 120°$ as the angle of the outward normal. It can be checked that if we substituted $\theta = 300°$, $\theta = -60°$, or $\theta = -240°$ into Equations (8.1) and (8.2), we would obtain the same values of σ_A and τ_A. This is illustrated in Figure 8.16. A plane passing through a point has two sides. The stresses on either side are the same, and hence the outward normal direction to either side can be used for computing normal and shear stresses on the plane. The direction of the outward normal can be measured by going counterclockwise (positive direction) or by going clockwise (negative direction) from the x axis. Equations (8.1) and (8.2) reflect this observation, as substitution of $(\theta + 180°)$, $(\theta - 180°)$, $(\theta + 360°)$, or $(\theta - 360°)$ in place of θ in Equations (8.1) and (8.2) results in the same expressions for the two equations. In other words, the values of the *stresses on a plane through a point are unique* and depend on the orientation of the plane only and not on how its orientation is described or measured.

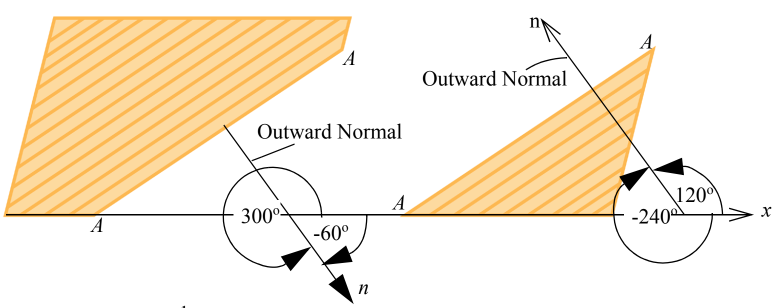


Figure 8.16 Different values of θ in Example 8.3.

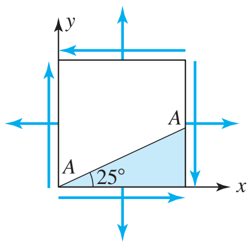
3. If the point were in plane strain on a material with a Poisson ratio of $\frac{1}{3}$, then the third principal stress would be $\sigma_3 = \nu(\sigma_1 + \sigma_2) = -10$ MPa. Thus in this problem $\sigma_1 > \sigma_3 > \sigma_2$, and hence for this problem the maximum shear stress would be unaffected. But if the third principal stress value were not in between principal stresses 1 and 2, then by Equation (8.12) the maximum stress value would be affected.

QUICK TEST 8.1 **Time: 15 minutes/Total: 20 points**

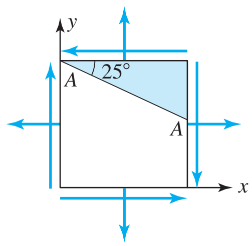
Each question is worth 2 points. Use the solutions given in Appendix E to grade yourself.

In Questions 1 through 3, what is the value of θ you would substitute in the stress transformation equations to find the normal and shear stresses on plane *AA*?

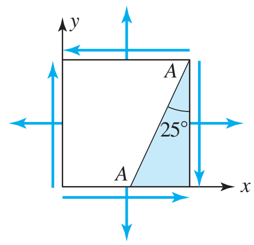
1.



2.



3.



4. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (C). What is the value of principal stress 1?

5. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (C) and 20 ksi (C). What is the value of principal stress 1?

6. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (T). What is the value of the maximum shear stress at that point?

7. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (C). What is the value of the maximum shear stress at that point?

8. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (C) and 20 ksi (C). What is the value of the maximum shear stress at that point?

In Questions 9 and 10, the angle θ_p from $\tan 2\theta_p = 2\tau_{xy}/(\sigma_{xx} - \sigma_{yy})$ was found to be $-35°$. On substituting this value into Equation (8.1), the normal stress was found to be 100 MPa (T). If the other principal stress is as given, then what is the value of principal angle 1?

9. 125 MPa (T).

10. 125 MPa (C).

8.3 STRESS TRANSFORMATION BY MOHR'S CIRCLE

In this section we develop a graphical technique for determining stresses on different planes passing through a point. Squaring Equations (8.4) and (8.5) and adding the result to eliminate θ, we obtain

$$\left(\sigma_{nn} - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 \qquad (8.14)$$

Consider the equation of a circle: $(x - a)^2 + y^2 = R^2$. We thus see that Equation (8.14) represents a circle with center coordinates $(a, 0)$ and radius R, where $a = (\sigma_{xx} + \sigma_{yy})/2$ and $R = \sqrt{[(\sigma_{xx} - \sigma_{yy})/2]^2 + \tau_{xy}^2}$. The circle is called **Mohr's circle** for stress and the coordinates of each point on the circle are the stresses (σ_{nn}, τ_{nt}). These are the normal and shear stresses on an arbitrarily oriented plane that is passing through the point at which the stresses σ_{xx}, σ_{yy}, and τ_{xy} are specified. Thus:

- Each point on Mohr's circle represents a unique plane that passes through the point at which the stresses are specified.
- The coordinates of the point on Mohr's circle are the normal and shear stresses on the plane represented by the point.

8.3.1 Construction of Mohr's Circle

We can construct Mohr's circle in five steps:

Step 1: Show the stresses σ_{xx}, σ_{yy}, and τ_{xy} on a stress cube and label the vertical plane V and the horizontal plane H, as shown in Figure 8.17*a*.

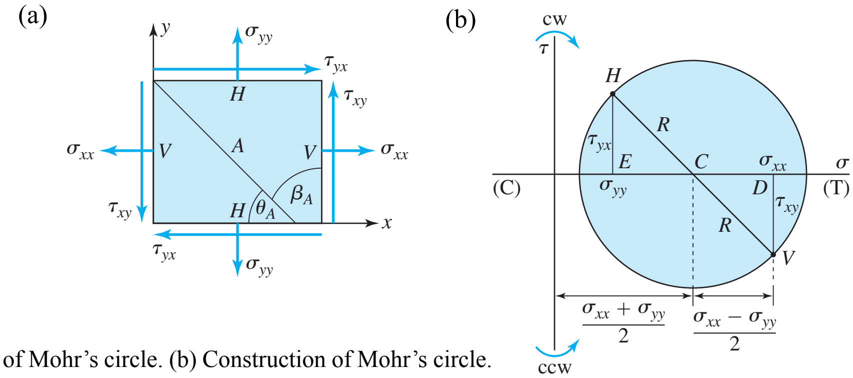


Figure 8.17 (a) Stress cube for construction of Mohr's circle. (b) Construction of Mohr's circle.

Step 2: Write the coordinates of points V and H as $V(\sigma_{xx},\ \tau_{xy})$ and $H(\sigma_{yy},\ \tau_{yx})$. The rotation arrow next to the shear stresses corresponds to the rotation of the cube caused by the set of shear stresses on planes V and H.

Step 3: Draw the horizontal axis with the tensile normal stress to the right and the compressive normal stress to the left, as shown in Figure 8.17*b*. Draw the vertical axis with the clockwise direction of shear stress up and the counterclockwise direction of rotation down.

Step 4: Locate points V and H and join the points by drawing a line. Label the point at which line VH intersects the horizontal axis as C.

Step 5: With C as the center and CV or CH as the radius, draw Mohr's circle.

To justify our construction, note the two triangles VCD and HCE in Figure 8.17*b* are identical, because

- angle VCD = angle HCE
- right angle CDV = right angle CEH
- side HE = side DV from the symmetry of shear stresses.

Thus side CE = side CD. In other words, C is the midpoint of DE, and the coordinates of the center point C are the mean values of the coordinates of points D and E, $[(\sigma_{xx} + \sigma_{yy})/2,\ 0]$, which represents the center of the circle, as in Equation (8.14). Since the length of side CD is the difference between the coordinates of D and C, we obtain the radius of the circle from the Pythagorean theorem as $\sqrt{[(\sigma_{xx} - \sigma_{yy})/2]^2 + \tau_{xy}^2}$, which is consistent with Equation (8.14).

An important point to remember is the differentiation made in Step 2 between τ_{xy} and τ_{yx}. Equations (8.4) and (8.5) tell us that the stresses on different planes are related by twice the angle between the planes. The vertical plane V and the horizontal plane H are 90° apart on the stress cube. Thus these planes must be 180° apart on Mohr's circle, as each point on Mohr's circle represents a unique plane. This implies that if the vertical plane V is located above the σ axis, then the horizontal plane H should be is located below the σ axis to maintain the 180° difference on Mohr's circle. If we use the conventional method of using the upper half-plane for positive values of shear stress and the lower half-plane for negative values of shear stress, then V and H will both be either in the upper half or in the lower half because the shear stresses $\tau_{xy} = \tau_{yx}$. By associating the clockwise and counterclockwise rotation, we can satisfy the requirement that the horizontal plane and the vertical plane on Mohr's circle be 180° apart. In summary:

- Angles between planes on a stress cube are doubled when plotted on Mohr's circle.
- The sign of shear stress cannot be determined directly from Mohr's circle, as the only information from Mohr's circle is that the shear stress causes the plane to rotate clockwise or counterclockwise.

8.3.2 Principal Stresses from Mohr's Circle

Figure 8.18 shows that the shear stresses are zero at points P_1 and P_2, and hence these points represent planes that are the principal planes. The normal stresses at these points are principal stresses, and these principal stresses can be found by inspection as the average normal stress plus or minus the radius. The stress at point P_1 is principal stress 1. In other words, principal stress 1 is the rightmost normal stress on Mohr's circle.

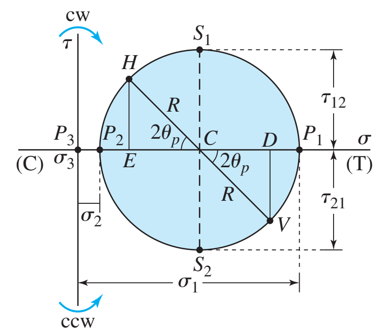


Figure 8.18 Principal stresses and in-plane maximum shear.

The angle between lines CV and CP_1 is $2\theta_p$, because all angles on Mohr's circle are double the actual angle between planes. The value of angle $2\theta_p$ can be found from the known dimensions of triangle VCD or triangle HCE, and we confirm the relationship given in Equation (8.6).

Is θ_p same as θ_l? In Figure 8.18 it would seem so. Figure 8.19 shows two examples in which the principal angle θ_p is different from principal angle θ_1. To clarify which angle we need consider that θ_1 is the angle of principal plane 1 (P_1) from the x direction. The outward normal to the vertical plane (V) is the x direction. On the Mohr's circle CV represents the x direction and CP_1 represents the principle direction 1. The angle between CV and CP_1 on the Mohr's circle is $2\theta_l$ in all cases, with counterclockwise rotation from CV as positive. As a final check, the value of θ_1 calculated from the Mohr's circle can be substituted into Equation (8.1), and the result should be principal stress one σ_1.

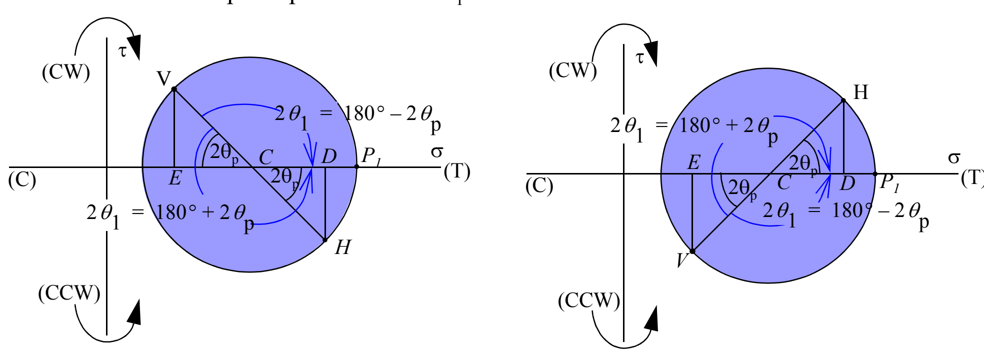


Figure 8.19 Examples of θ_1 different from θ_p.

Inspection of Figure 8.18 confirms that the maximum and the minimum normal stresses will be principal stresses. The observation that the principal planes are orthogonal is also obvious from Figure 8.18, as points P_1 and P_2, which represent principal planes, are at 180° on Mohr's circle. The coordinates of the center of the circle are the mean value of the normal stresses of any two points that are on a diameter of the circle. This confirms that the sum of normal stresses on orthogonal planes is invariant with respect to coordinate transformation.

8.3.3 Maximum In-Plane Shear Stress

The maximum in-plane shear stress will exist on the plane represented by points S_1 and S_2 in Figure 8.18, and its value is the radius of Mohr's circle, which is consistent with Equation (8.12). Points S_1 and S_2 are at 90° from points P_1 and P_2 on Mohr's circle, which is consistent with the earlier observation that the maximum in-plane shear stress exists on two planes which are at 45° to the principal planes.

8.3.4 Maximum Shear Stress

The circle in Figure 8.18 is the in-plane Mohr's circle, as the coordinate axes *n* and *t* are always in the *xy* plane. The in-plane circle represent all the planes that are obtained by rotating about principal axis 3. Let point P_3 represent principal plane 3. For plane stress problems, point P_3 coincides with the origin, as shown in Figure 8.20.

We can draw two more circles, one between P_1 and P_3 and the second between P_2 and P_3, as shown in Figure 8.20. These two circles represent rotation about principal axis 2 and principal axis 1, respectively, and are termed out-of-plane circles. The three circles together represent the complete state of stress at a point. *The maximum shear stress at a point is the radius of the biggest circle* [see Equation (8.13)]. This observation is also valid for plane strain. The difference is that the value of σ_3 will have to be found by using Equation (8.10), plotted on the horizontal axis, and labeled P_3.

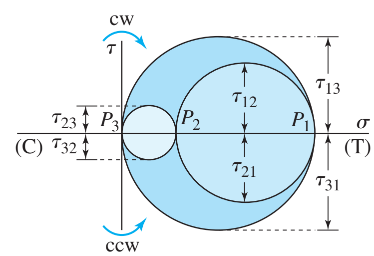


Figure 8.20 Maximum shear stress in plane stress.

8.3.5 Principal Stress Element

The principal stress element is a visualization aid used in the prediction of failure surfaces. Potential failure surfaces are the planes on which maximum normal or maximum shear stress acts—in other words, the principal planes and the plane of maximum shear. A **principal stress element** shows stresses on a wedge constructed from the principal planes and the plane of maximum shear stress.

We can describe our construction in terms of stress cubes, although they are not required once the method is understood:

Step 1: Draw a square and label the vertical side *V* and Horizontal side *H* as shown in Figure 8.21.

Step 2: Rotate the coordinate axis by an angle θ_1 and the cube along with it. The vertical plane (*V*) rotates to principal plane 1 (P_1). The horizontal plane (*H*) rotates to principal plane 2 (P_2) as shown in Figure 8.21.

Step 3: Draw the diagonals, that is planes 45° to the principal planes P_1 and P_2, representing the plane of maximum shear stress. Label the plane S_1 and S_2. Plane S_1 is 45° counterclockwise from plane P_1 in Figure 8.21*a*. On the Mohr's circle in Figure 8.18, if we rotate counterclockwise direction by 90° starting from point P_1, we reach point S_1. The stress wedge obtained is shown in Figure 8.21*b*. Similarly, starting from point P_1 on the Mohr's circle and rotating in the clockwise direction by 90°, we reach S_2 in Figure 8.18. The corresponding stress wedge is shown in Figure 8.21*c*.

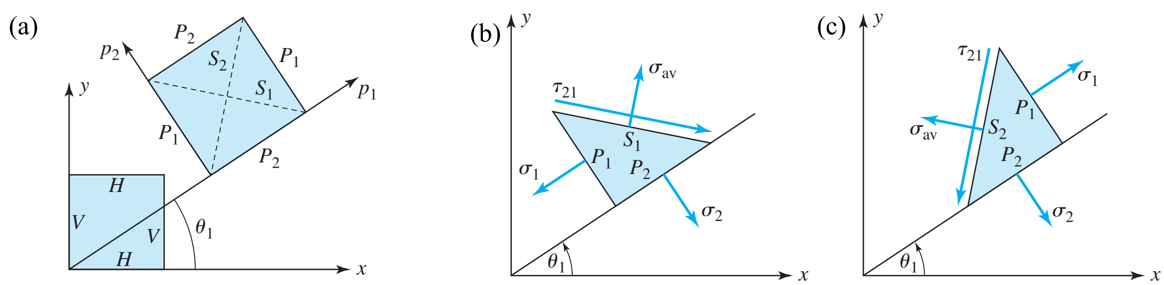


Figure 8.21 (a) Principal planes and planes of maximum in-plane shear stress. (b, c) Principal elements.

Step 4: Show principal stress 1 on plane P_1 and principal stress 2 on plane P_2. On the inclined plane show the maximum in-plane shear stress in the clockwise (or counterclockwise) direction if the inclined plane corresponds to the point in the upper (lower) half of Mohr's circle. Also show the average normal stress value on the inclined plane.

Figure 8.22 shows Mohr's circle and the principal element associated with the axial loading of a circular bar. Cast iron, a brittle material, fails from maximum tensile stress—that is, due to principal stress 1—and the failure surface is the principal plane 1. Aluminum, a ductile material, fails from maximum shear stress, and the failure surface is the plane of maximum shear S_2. Local imperfections dictated that the failure surface was S_2 rather than S_1, which from our theory is equally likely. An explanation of the failure surfaces due to torsion shown in Figure 8.1 is left to Problem 8.35.

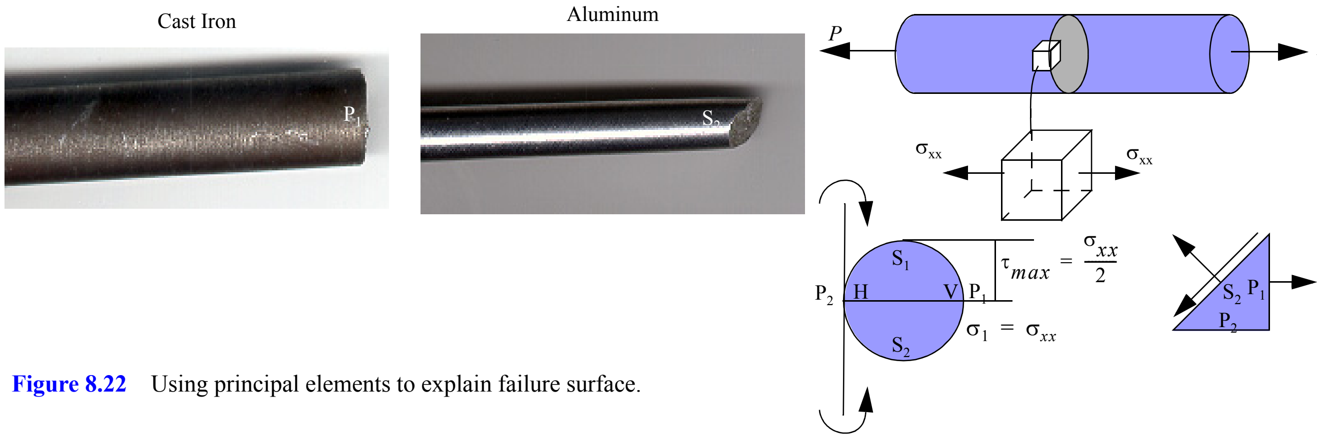


Figure 8.22 Using principal elements to explain failure surface.

8.3.6 Stresses on an Inclined Plane

The stresses on an inclined plane are found by first locating the plane on Mohr's circle and then determining the coordinates of the point representing the plane. This is achieved as follows:

Step 1: Draw the inclined plane on the stress cube and label it A, as shown in Figure 8.17.

Step 2: Locate the inclined plane on Mohr's circle as will be described later and label it A, as shown in Figure 8.23.

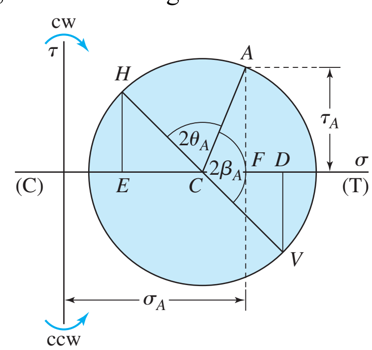


Figure 8.23 Stresses on inclined plane.

Step 3: Calculate the coordinates of point A.

Step 4: Determine the sign of shear stress.

There are two alternatives in Step 2.

1. On the stress cube in Figure 8.17, the inclined plane A is at an angle θ_A from the horizontal plane in the clockwise direction. Starting from line CH on Mohr's circle in Figure 8.23, we rotate by an angle $2\theta_A$ in the clockwise direction and then draw the line CA. Point A represents plane A.
2. On the stress cube, the inclined plane A is at an angle β_A from the vertical plane in the counterclockwise direction. Starting from line CV we rotate by an angle $2\beta_A$ in the counterclockwise direction and draw the line CA. Point A represents plane A.

We note from the stress cube that $\theta_A + \beta_A = 90°$, and from Mohr's circle we see that $2(\theta_A + \beta_A) = 180°$. This once more confirms that each point on Mohr's circle represents a unique plane, and it is immaterial how we locate that point on the circle.

Step 3 is the reverse of Step 2 in the construction of Mohr's circle and is a simple problem in geometry. Angle FCA can be found from the known angles. Radius CA of the circle is known, and lengths FA and CF can be found from triangle FCA. The coordinates of point A are $(\sigma_A,\ \tau_A \circlearrowright)$. The direction of rotation is recorded as clockwise because point A is in the upper plane in Figure 8.23. If point A had been in the lower plane, we would have recorded a counterclockwise rotation with the shear stress.

To determine the sign of shear stress, we start by drawing the shear stress such that the inclined plane A rotates in the same direction as was recorded with the coordinates in Step 3. A local coordinate system is established, and if the shear stress is in the positive tangent direction, then it is positive. The two possibilities are shown in Figure 8.24. In both cases the shear stress is negative.

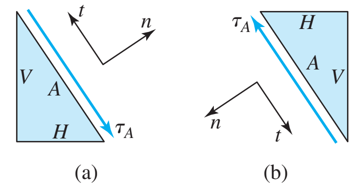


Figure 8.24 Sign of shear stress on an incline.

EXAMPLE 8.4

For each of the two states of stress below, plot the normal stress and the shear stress on a plane versus θ—the angle of the outward normal of the plane—draw Mohr's circle for each state of stress, on each diagram identify the planes at $\theta_A = 30^\circ$, $\theta_B = 75^\circ$, $\theta_D = 105^\circ$, and $\theta_E = 150^\circ$.
Case I: The uniaxial stress state is $\sigma_{xx} = \sigma_0$, and all other stress components are zero.
Case II: The state of pure shear is $\tau_{xy} = \tau_0$, and all other stress components are zero.

PLAN

We can substitute the given states of stress into Equations (8.1) and (8.2) to obtain σ_{nn} and τ_{nt} as a function of θ and plot them. For each state of stress we can draw the stress cube, write the coordinates of planes V and H, and draw Mohr's circle. Starting from point V on Mohr's circle we can rotate by twice the angle in the counterclockwise direction to get the various points on the circle.

SOLUTION

Case I: Substituting the stress components for the uniaxial stress states into Equations (8.1) and (8.2), we obtain

$$\sigma_{nn} = \sigma_0 \cos^2\theta \qquad \text{or} \qquad \frac{\sigma_{nn}}{\sigma_0} = \cos^2\theta \tag{E1}$$

$$\tau_{nt} = -\sigma_0 \sin\theta\cos\theta \qquad \text{or} \qquad \frac{\tau_{nt}}{\sigma_0} = -\sin\theta\cos\theta \tag{E2}$$

Equations (E1) and (E2) can be plotted as shown in Figure 8.25*a*. We can also draw the stress cube showing uniaxial tension and record the coordinates of points V and H (Figure 8.25*b*). With no shear stress, the two points V and H are on the horizontal axis forming the diameter of Mohr's circle with the center at C and radius $R = \sigma_0/2$. Starting from point V on Mohr's circle we rotate counterclockwise by twice the angle θ to get the inclined planes, as shown in Figure 8.25*c*.

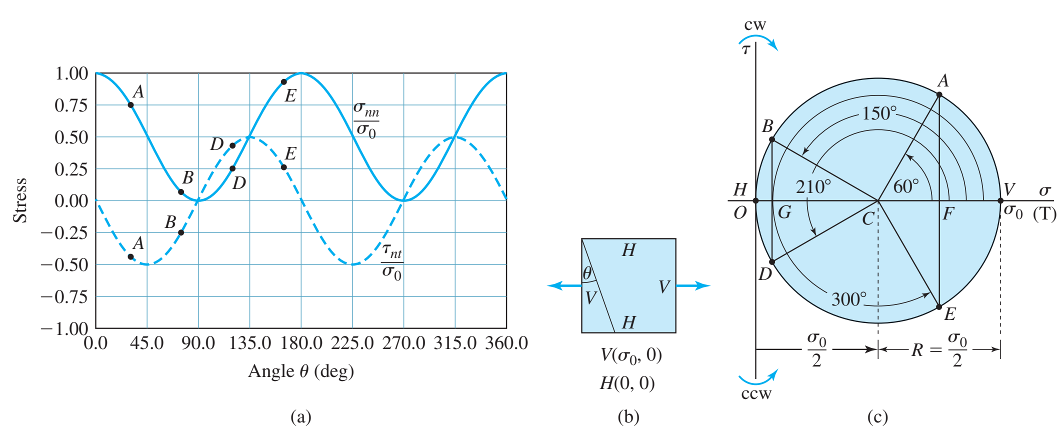


Figure 8.25 Stresses on inclined plane in uniaxial state of stress.

Case II: Substituting the stress components for state of pure shear in Equations (8.1) and (8.2), we obtain

$$\sigma_{nn} = 2\tau_0 \sin\theta\ \cos\theta \qquad \text{or} \qquad \frac{\sigma_{nn}}{\tau_0} = 2\sin\theta\ \cos\theta \tag{E3}$$

$$\tau_{nt} = \tau_0(\cos^2\theta - \sin^2\theta) \qquad \text{or} \qquad \frac{\tau_{nt}}{\tau_0} = \cos^2\theta - \sin^2\theta \tag{E4}$$

Equations (E3) and (E4) can be plotted as shown in Figure 8.26. We can also draw the stress cube showing pure shear and record the coordinates of points V and H. With normal stress, the two points V and H are on the vertical axis forming the diameter of Mohr's circle with center at C and radius $R = \tau_0$. Starting from point V on Mohr's circle we rotate counterclockwise by twice the angle θ to get the inclined planes, as shown in Figure 8.26.

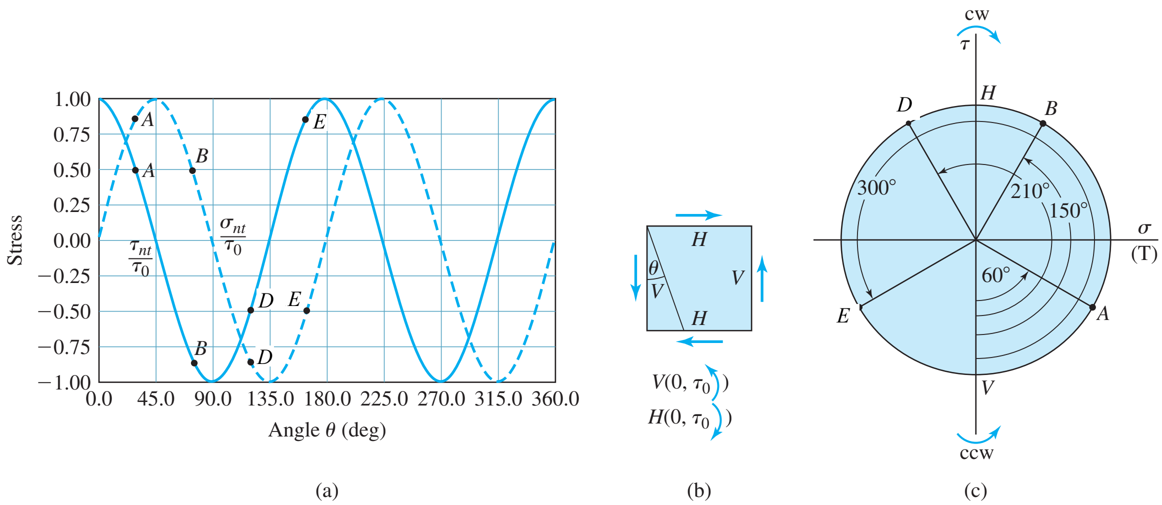


Figure 8.26 Stresses on inclined plane in state of pure shear stress.

COMMENTS

1. The example shows the relationship of planes on a graph and on a Mohr's circle, emphasizing that angles are double when plotted on a Mohr's circle.
2. Recall that in axial members and on top and bottom surface of beam, the only non-zero stress component is σ_{xx}. For both these cases, Figure 8.25 shows that the plane of maximum shear is 45° to the axis of the member and the value of maximum shear stress is half the value the normal stress at that point.
3. Recall that in torsion of circular shafts only non-zero stress component is $\tau_{x\theta}$. For a shaft in torsion, Figure 8.26 shows that the principal planes are 45° to the axis of the shaft and the magnitude of the principal stresses is same as the magnitude of torsional shear stress at that point.
4. Observations in comments 3 and 4 should be remembered as these can be used in strength based design of brittle and ductile materials.

EXAMPLE 8.5

(a) Solve Example 8.2 using Mohr's circle. Determine (b) the principal stresses, and principal angle 1; (c) the maximum shear stress at the point. (d) Show the results on a principal element.

PLAN

We can follow the steps outlined for the construction of Mohr's circle in Section 8.3.1 and calculate the various quantities from geometry.

SOLUTION

Step 1: We draw the stress cube and label the vertical and horizontal planes V and H, as shown in Figure 8.27*a*.

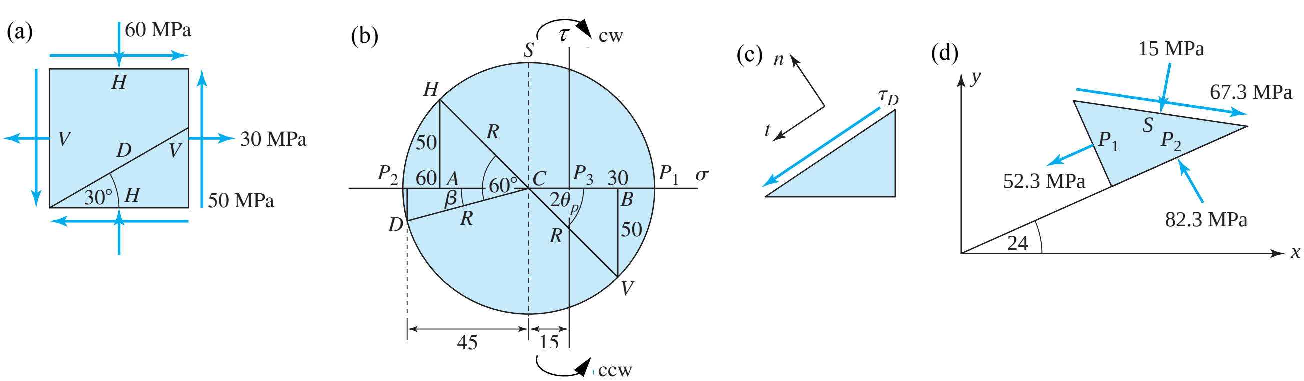


Figure 8.27 (a) Stress cube. (b) Mohr's circle. (c) Sign of shear stress. (d) Principal element

Step 2: From Figure 8.27*a* we note the coordinates of points V and H as

$$V(30, 50) \qquad H(-60, 50)$$

Step 3: We draw the axes for Mohr's circle, as shown in Figure 8.27*b*.

Step 4: We locate points V and H and join the two points. The coordinates of the center C are the mean value of the coordinates of points A and B, that is, $[30 + (-60)]/2 = -15$. The distance AC is $[30 - (-15)] = 45$, from which the radius as: $R = \sqrt{45^2 + 50^2} = 67.27$. The angle θ_p can then be calculated from triangle VCB as

$$\tan 2\theta_p = \frac{50}{45} = 1.1111 \qquad \text{or} \qquad 2\theta_p = 48.01° \tag{E1}$$

(a) Plane D is 30° counterclockwise from the horizontal plane in the stress cube in Figure 8.27*a*. We rotate by twice the angle (i.e., by 60°) from line CH on Mohr's circle in Figure 8.27*b* and draw the line CD. The coordinates (σ_D, τ_D) of point D are calculated from geometry,

$$\beta = 60° - 2\theta_p = 11.99° \qquad \sigma_D = -(15 + R\cos\beta) = -80.8 \qquad \tau_D = R\sin\beta = 13.97 \tag{E2}$$

We next draw the plane and determine the sign of τ_D. The shear stress must cause the plane to rotate counterclockwise, because point D on Mohr's circle was in the lower half of the plane. We establish a local coordinate system and determine that the shear stress has a positive sign (Figure 8.27*c*). The results are

ANS. $\sigma_D = 80.8\text{ MPa (C)} \qquad \tau_D = 14.0\text{ MPa}$

(b) The principal stresses are the coordinates of points P_1 and P_2,

$$\sigma_1 = -15 + R = 52.27 \qquad \sigma_2 = -15 - R = -82.27 \tag{E3}$$

$$\theta_p = \frac{48.02}{2} = 24.01° \tag{E4}$$

The principal stresses and principal angle 1 for the problem are

ANS. $\sigma_1 = 52.3\text{ MPa (T)} \qquad \sigma_2 = 82.3\text{ MPa (C)} \qquad \sigma_3 = 0 \qquad \theta_1 = 24.0°\text{ ccw}$

(c) The circle between P_1 and P_3 and the one between P_2 and P_3 are both inscribed within the in-plane circle shown in Figure 8.27*b*. Thus in this problem the in-plane maximum shear stress and the maximum shear stress at the given point are the same,

ANS. $\tau_{max} = 67.3\text{ MPa}$

(d) Figure 8.27*d* shows the principal element which is drawn following the procedure in Section 8.3.5.

COMMENTS

1. The Mohr's circle method looks longer than the method of equations because of the explanation needed for the geometric constructions. However, computationally the difference between the two methods is small. The advantage of the method of equations is that the equations can be programmed and solved by computer. The advantage of using Mohr's circle is that it helps in the intuitive understanding of stress transformation.
2. The maximum shear stress shown in Figure 8.27 is negative, as can be deduced by establishing a local n, t coordinate system.

EXAMPLE 8.6

A 30-in.-long thin cylindrical tube is to transmit a torque of 25π in.·kips. The tube is to be fabricated by butt welding a $\frac{1}{16}$-in.-thick steel plate ($G = 12{,}000$ ksi) along a spiral seam, as shown in Figure 8.28. Buckling considerations limit the allowable stress in steel to 10 ksi in compression. The allowable shear stress in the weld is 12 ksi, and the allowable tensile stress in the weld is 20 ksi. Stiffness considerations limit the relative rotation of the two ends to 3°. Determine the minimum outer radius of the tube to the nearest $\frac{1}{16}$ in.

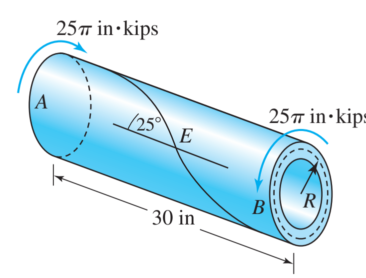


Figure 8.28 Geometry of shaft and loading in Example 8.6.

PLAN

We are required to find R to satisfy four limitations. By inspection we see that the weld material would be put into compression, and hence we can ignore the constraint on the maximum tensile stress in the weld. We can use the thin-tube approximation for computing J in terms of R, as given in Problem 5.28. For this simple loading we can determine the internal torque by inspection as $\boldsymbol{T} = +25\pi$ in.·kips. We can find $\phi_B - \phi_A$ in terms of R using Equation (5.12) and find one limit on R. Since the tube is thin, we can further assume that the torsional shear stress will not vary significantly from the inside to the outside. Hence we can evaluate it at the centerline radius. We can find the torsional shear stress in terms of R using Equation (5.10). We can then find the maximum compressive stress in steel and the shear stress in the seam using either Mohr's circle or the method of equations and find two other limits on R. We choose R that satisfies all limits and round it upward to the nearest $\frac{1}{16}$ in.

SOLUTION

For thin tubes, from Problem 5.28, we have

$$J = 2\pi R^3 t \tag{E1}$$

Substituting Equation (E1), $\boldsymbol{T} = 25\pi$ in.·kips, $G = 12{,}000$ ksi, $t = (1/16)$ in., and $L = 30$ in. into Equation (5.12), we obtain the rotation of the section at B with respect to the section at A, which should be less than 3° = 0.0524 rad. We thus find one limit on R,

$$\phi_B - \phi_A = \frac{(25\pi \text{ in.}\cdot\text{kips})(30 \text{ in.})}{(12{,}000 \text{ kips/in.}^2)[2\pi R^3(1/16) \text{ in.}]} = \frac{0.5 \text{ in.}^3}{R^3} \le 0.0524 \qquad \text{or} \qquad R \ge 2.12 \text{ in.} \tag{E2}$$

Substituting Equation (E1), $\boldsymbol{T} = 25\pi$ in.·kips, and $\rho = R$ into Equation (5.10),

$$\tau_{x\theta} = \frac{(25\pi \text{ in.}\cdot\text{kips})R}{[2\pi R^3(1/16) \text{ in.}]} = \frac{200 \text{ kips}}{R^2} \tag{E3}$$

The direction of shear stress at point E can be determined using subscripts or intuitively, as shown in Figure 8.29. A two-dimensional representation of the stress cube is shown in Figure 8.29*c*. The directions of shear stress on the other surfaces are determined using the fact that pairs shear stresses either point toward the corner or away from the corner, as shown in Figure 8.29*c*

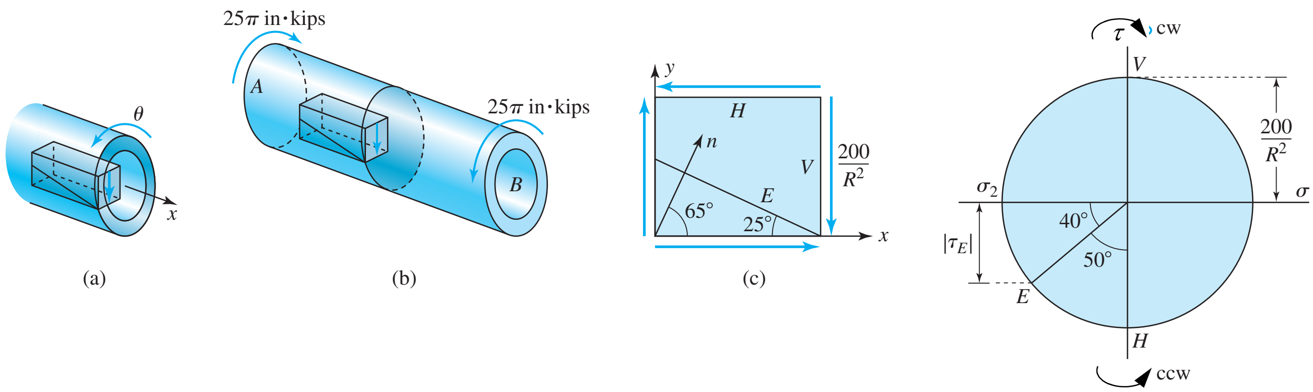


Figure 8.29 Direction of shear stress (a) by subscript.(b) by inspection. (c) Stress cube. (d) Stresses on the inclined plane using Mohr's circle.

We can determine the maximum compressive stress and the shear stress on the inclined plane using either Mohr's circle for stress or the method of equations.

Mohr's circle method: We record the coordinates of point V as $V(0, 200/R^2)$ and the coordinates of point H as $H(0, 200/R^2)$ and draw Mohr's circle as shown in Figure 8.29*d*. We determine principal stress 2,

$$\sigma_2 = \frac{(200 \text{ kips})}{R^2} \text{ (C)} \tag{E4}$$

We then locate point E on Mohr's circle and determine the shear stress,

$$|\tau_E| = \frac{(200 \text{ kips})}{R^2}\sin 40° = \frac{(128.56 \text{ kips})}{R^2} \tag{E5}$$

Method of equations: We note from Figure 8.29*c* that $\tau_{xy} = -(200 \text{ kips})/R^2$ ksi, the normal stresses are zero, and the angle of the normal to the inclined plane is 65°. Substituting this information into Equations (8.7) and (8.2), we obtain principal stress 2,

$$\sigma_2 = 0 - \sqrt{0 + \left(\frac{200 \text{ kips}}{R^2}\right)^2} = \frac{(200 \text{ kips})}{R^2} \text{ (C)} \tag{E6}$$

and the shear stress on the inclined plane,

$$\tau_E = \left(-\frac{200 \text{ kips}}{R^2}\right)(\cos^2 65° - \sin^2 65°) \qquad \text{or} \qquad |\tau_E| = \frac{(128.56 \text{ kips})}{R^2} \tag{E7}$$

We now consider the limitation on the compressive stress in steel and the shear stress in the weld and find two other limitations on R,

$$\sigma_2 = \frac{(200 \text{ kips})}{R^2} \le (10 \text{ kips/in.}^2) \qquad \text{or} \qquad R \ge 4.472 \text{ in.} \tag{E8}$$

$$|\tau_E| = \frac{(128.56 \text{ kips})}{R^2} \le (12 \text{ kips/in.}^2) \qquad \text{or} \qquad R \ge 3.273 \text{ in.} \tag{E9}$$

Comparing Equations (E2), (E8), and (E9), we see that the minimum value of R that will satisfy all three conditions is 4.472 in., given by Equation (E8). Rounding upward to the closest $\frac{1}{16}$ in., we obtain the value of the centerline circle radius.

ANS. $R = 4\frac{1}{2}$ in.

COMMENTS

1. Consider the error due to the thin-tube approximation for J. The outer radius for the tube is $R_o = R + t/2 = 4\frac{17}{32}$ in., and the inner radius of the tube is $R_i = R - t/2 = 4\frac{15}{32}$ in. . Thus the value of exact $J = \pi(R_o^4 - R_i^4)/2$ would be 35.786 in.^{4}. The value of approximate $J = 2\pi R^3 t$ for thin tubes is 35.785 in.^{4}, a percentage difference from exact J of 0.003%, which is negligible for any engineering calculation.
2. Consider the approximation of uniform shear stress in the tube. If we substitute $\rho = 4\frac{17}{32}$ in. into Equation (5.10), we obtain a value of 9.945 ksi at the outer surface. If we substitute $\rho = 4\frac{1}{2}$ in. into Equation (5.10), we obtain a value of 9.876 ksi at the centerline, for a difference of 0.69%, which is also negligible.
3. If we do not use the thin-tube approximation, then we have to find roots of nonlinear equations by numerical methods (see Problem 8.70). The thin-tube approximation can be used if $t < R/10$.
4. In this problem the direction (sign of τ_{xy}) of shear stress is important only to ensure that the weld is subjected to compressive stress and not tensile stress. The magnitude of the shear stress in the weld is unaffected by the direction (sign of τ_{xy}) of shear stress. (This will not be true in combined loading problems.)

EXAMPLE 8.7

A T-section beam is constructed by gluing two pieces of wood together, as shown in Figure 8.30. The maximum normal stress in the glue joint is to be limited to 2 MPa in tension and the maximum shear stress is to be limited to 1.7 MPa. Determine the maximum value for load w.

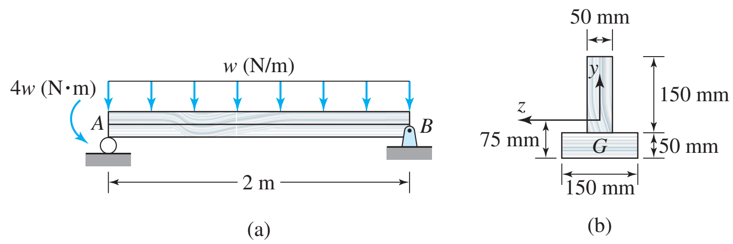


Figure 8.30 Beam and loading in Example 8.7.

PLAN

We are given that the principal stress 1 in the glue cannot exceed 2 MPa, and the maximum shear stress in the glue cannot exceed 1.7 MPa. We can draw the shear force and bending moment diagrams in terms of w. We then find the bending normal stress in glue and the bending shear stress in glue, considering in the sections where the moment M_z is maximum and the shear force V_y is maximum. We can draw stress cubes at the various sections and find principal stress 1 and the maximum shear stress in terms of w. Using the limiting values we can find the value of w.

SOLUTION

We can find the reaction forces at A and B by considering the free-body diagram of the entire beam. We then draw the shear force and bending moment diagrams, as shown in Figure 8.31a. The maximum shear force and bending moment are given by

$$(V_y)_{max} = -3w \text{ N} \qquad M_{max} = -4w \text{ N}\cdot\text{m} \tag{E1}$$

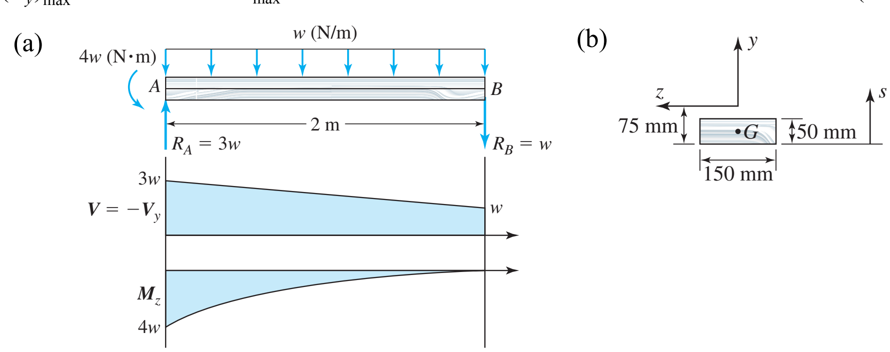


Figure 8.31 (a) Shear force and bending moment diagrams. (b) Calculation of Q_z.

Since the maximum bending moment and the maximum shear force exist in the section at A, the maximum principal flexural normal stress in glue and the maximum shear stress in glue will also exist in the section at A. The area moment of inertia of the cross section can be calculated as $I_{zz} = 53.125(10^6)\ \text{mm}^4$. Substituting $y_G = -25$ mm and Equation (E1) into Equation (6.12), we obtain the flexural normal stress at G as

$$\sigma_G = -\frac{-(4w\ \text{N}\cdot\text{m})[-25(10^{-3})\ \text{m}]}{53.125(10^{-6}\ \text{m}^4)} = -1882w\ \text{N/m}^2 \tag{E2}$$

We can draw the area A as the area between the bottom surface and point G, as shown in Figure 8.31b, and find Q_z,

$$Q_G = (150\ \text{mm})(50\ \text{mm})(-50\ \text{mm}) = -375(10^3)\ \text{mm}^3 \tag{E3}$$

Substituting Equations (E1) and (E3) into Equation (6.27), we obtain the shear stress,

$$\tau_{xs} = -\frac{(-3w\ \text{N})[-375(10^{-6})\ \text{m}^3]}{[53.125(10^{-6})\ \text{m}^4][50(10^{-3})\ \text{m}]} = -423.5w\ \text{N/m}^2 \tag{E4}$$

We can draw the stress cube and show on it the stresses in Equations (E2) and (E4). Note that with s positive upward, the shear stress τ_{xs} on the surface with the outward normal in the positive x direction will be downward to reflect the negative sign in Equation (4). Alternatively, the sign of the shear stress τ_{xy} is negative as the shear force V_y is negative. We can draw Mohr's circle as shown in Figure 8.32. The radius R is given by

$$R = \sqrt{(941w\ \text{N/m}^2)^2 + (423.5w\ \text{N/m}^2)^2} = 1032w\ \text{N/m}^2 \tag{E5}$$

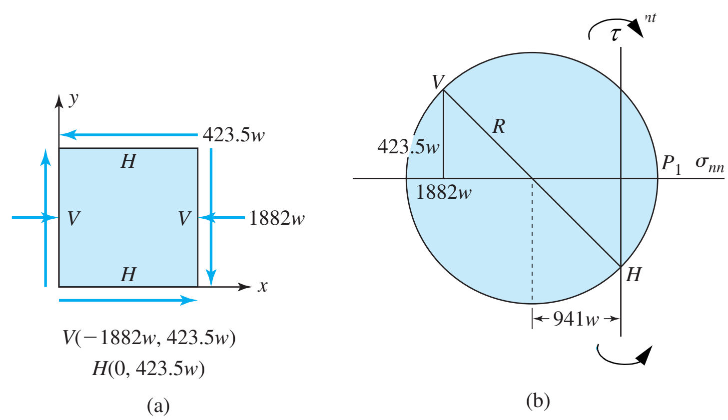


Figure 8.32 Mohr's circle in Example 8.7.

Principal stress 1 σ_1 can be found, and the limiting value on σ_1 yields one limit on w,

$$\sigma_1 = (-941w\ \text{N/m}^2) + (1032w\ \text{N/m}^2) = 91w\ \text{N/m}^2 \le 2(10^6)\ \text{N/m}^2 \quad \text{or} \quad w \le 22{,}000\ \text{N/m} \tag{E6}$$

The maximum shear stress τ_{max} is the radius of the circle, and the limiting value on τ_{max} yields the other limit on w,

$$\tau_{max} = 1032w\ \text{N/m}^2 = (1.7)10^6\ \text{N/m}^2 \quad \text{or} \quad w \le 1647\ \text{N/m} \tag{E7}$$

Comparing Equations (E6) and (E7), we conclude that the maximum permissible value of w is

ANS. $w_{max} = 1647\ \text{N/m}$

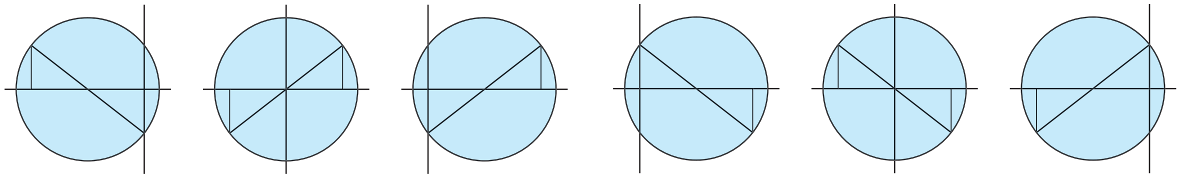
COMMENTS

1. The maximum normal stress in glue is $\sigma_2 = -941w\ \text{N/m}^2 - 1032w\ \text{N/m}^2 = -1973w\ \text{N/m}^2$, which in magnitude is nearly 20 times greater than σ_1. However it is not even considered, because it is compressive and so does not affect the failure of glue.
2. The maximum bending normal stress in wood is at the top of the beam at section A and its value is $\sigma_{xx} = -9411.8w\ \text{N/m}^2$. The maximum bending shear stress is at the neutral axis and its value is $\tau_{xy} = -441.2w\ \text{N/m}^2$. At the top of the beam the only nonzero stress is σ_{xx}. Thus, from Figure 8.25, the maximum shear stress is $\tau_{max} = \sigma_{xx}/2 = -4705.9w\ \text{N/m}^2$, which is an order of magnitude greater than the maximum bending shear stress. If we had to consider shear strength failure of wood, then we would use the maximum value of $4705.9w\ \text{N/m}^2$ in our calculation.
3. Comment 2 emphasizes the difference between maximum stresses in a material and maximum bending stresses. Comment 1 emphasizes that it is not the magnitude of the maximum stress but the type of stress that causes failure in a material and is important in design.

QUICK TEST 8.2 **Time: 20 minutes/Total: 20 points**

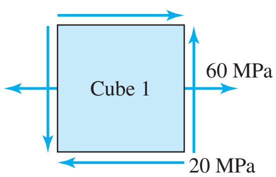
Each question is worth 2 points. Use the solutions given in Appendix E to grade yourself.

In Questions 1 through 3, associate the stress cubes with the appropriate Mohr's circle given:

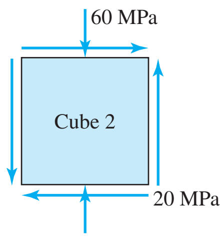


Circle *A* Circle *B* Circle *C* Circle *D* Circle *E* Circle *F*

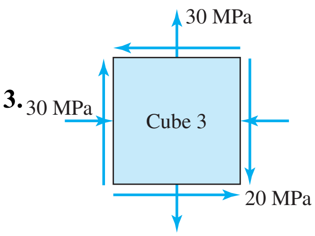
1.



2.

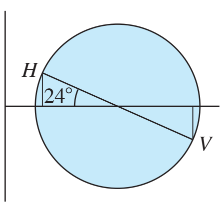


3.

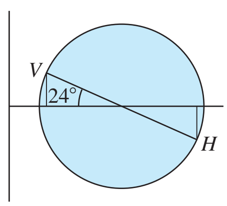


In Questions 4 through 6, Mohr's circles correspond to a plane state of stress. Determine the two possible values of principal angle 1 θ_1 in each question.

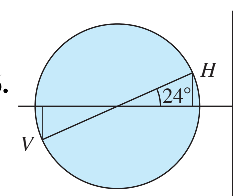
4.



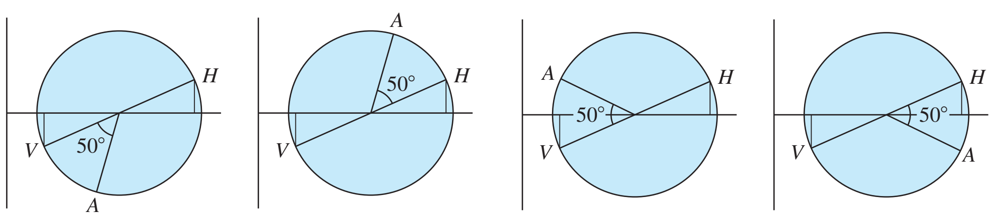
5.



6.

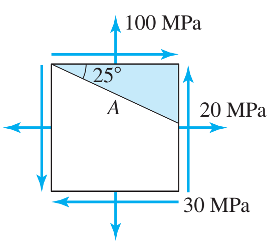


In Questions 7 through 9, Mohr's circle corresponds to the state of stress shown. Associate plane *A* on the stress cube with the corresponding Mohr's circles showing plane *A*, which are given:

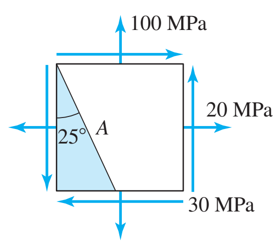


Circle *A* Circle *B* Circle *C* Circle *D*

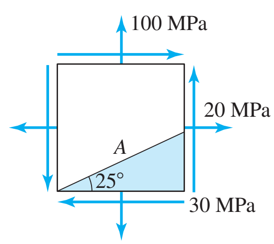
7.



8.



9.

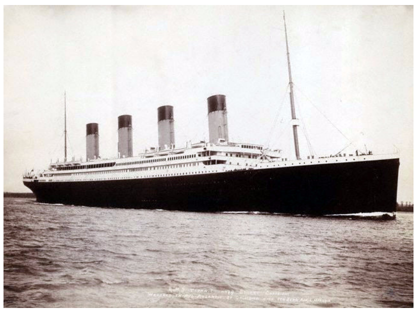


10. Plane *E* passes through a point that has the state of stress given in Question 7. The normal and shear stresses on plane *E* were found to be $\sigma_E = 90\text{ MPa (T)}$ and $\tau_E = -40\text{ MPa}$. What are the normal stress and the shear stress on the plane that is 90° counterclockwise from plane *E*?

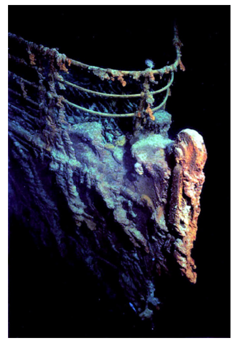
MoM in Action: Sinking of Titanic

On April 14, 1912, on the fourth day of her maiden voyage, the *RMS Titanic* (Figure 8.33*a*) struck an iceberg in Atlantic ocean. The ship dubbed "the unsinkable" sank in less than three hours (Figure 8.33*b*), with a loss of 1500 people, but it was not the last startling catastrophic failure. On January 15, 1919, a large molasses tank burst in Boston, killing 21 people and injuring another 150. On March 12, 1928, near Los Angeles, the St. Francis Dam failed suddenly, and the resulting flood killed more than 600 people. And on April 28, 1988, on Aloha flight 243 between Hilo and Honolulu in Hawaii, the fuselage of the aircraft blew open at 24,000 feet, killing a flight attendant and injuring another eight. The initiating cause of each tragedy was different. The final mechanism of catastrophic failure, however, was the same in each case—*brittle fracture*.

(a)



(b)



(c)

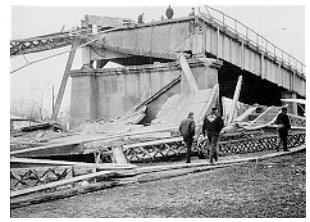


Figure 8.33 (a) RMS Titanic (b) Titanic bow at bottom of ocean. (c) Sliver Bridge.

To highlight some fracture issues, consider a crack in a windshield. It can stay that way for years, grow slowly, or grow suddenly into a crisscross pattern. A crack must reach critical length that is material and stress dependent before it starts growing. The slow growth is due to *ductile fracture,* while the rapid growth is due to *brittle fracture*.

In ductile materials, the high stresses at the crack tip causes plastic deformation, thus blunting the tip of the crack. Subsequent growth depends on there being sufficient energy in the deformed solid to create new crack surfaces, resulting in ductile fracture. On December 15, 1967, for example, Silver Bridge (Figure 8.33*b*), which spanned the Ohio River, collapsed owing to ductile fracture of a pin, killing 46 and injuring 9 other people. National Bridge Inspection Standards (NBIS) were established soon after and now require periodic inspection of all bridges.

In brittle materials, once a crack reaches critical length, fracture proceeds at extremely high speeds (in the neighborhood of 7000 ft/sec). The breaking in two of the tanker *S.S. Schenectady* (see Figure 1.1) is a vivid example. Brittle fracture can also occur in ductile materials. Polymer in a windshield, a ductile material, may become brittle due to sunlight and aging. The four days of sailing in icy cold water of Atlantic made the metal of *Titanic* more brittle, increasing its propensity to brittle fracture. Failure due to fatigue (see Section 3.10) always produces brittle fracture surfaces, even in ductile materials.

Tensile principal stress one (σ_1) is the dominant cause of crack growth. Though brittle fracture can cause catastrophe, it can also be used productively. By scoring glass with a diamond cutter, or a plasterboard with a utility knife, we introduce a sharp tipped crack in the material. By bending the glass and plasterboard we create tensile stress at the crack tip and thus produce a clean surface break. *Brittle coating method* is an experimental technique in which a brittle material is spray coated onto a machine component. As the component is loaded, cracks perpendicular to principal direction one (θ_1) start appearing in the coating.

A good design ensures that the critical crack length in a structure is never smaller then what can be detected by instruments. Regular inspections can then provide the warning needed to fix the crack. Drilling a hole to blunt the crack tip, putting obstacles like rivets in path of crack growth, or inserting glue as in windshield cracks are some of the ways of arresting crack growth from becoming catastrophic.

Brittle fracture is in itself neither good nor bad. It is, however, nature's constant reminder of a fundamental engineering lesson: *to command nature we need to obey it first*.

PROBLEM SET 8.2

8.29 Show that Equations (8.7) and (8.8) are correct by substituting the values of sines and cosines following Figure 8.12 into Equations (8.4) and (8.5).

8.30 Show that Equation (8.12) is correct by substituting the appropriate sines and cosines following Figure 8.13 into Equations (8.4) and (8.5).

8.31 Derive Equation (8.7) by starting from Equation (8.3).

8.32 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.32.

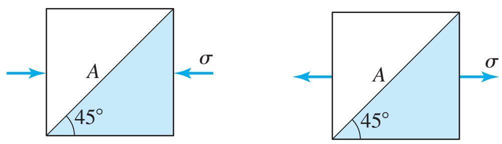


Figure P8.32 (a) (b)

8.33 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.33.

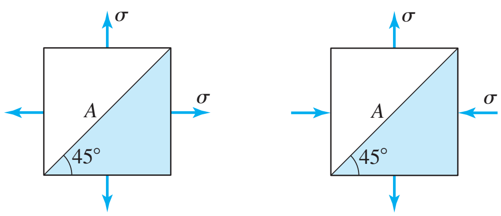


Figure P8.33 (a) (b)

8.34 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.34.

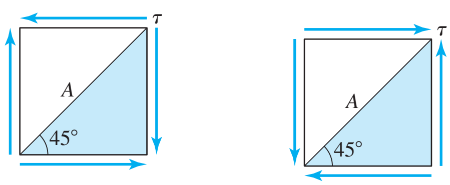


Figure P8.34 (a) (b)

8.35 Explain the failure surfaces due to torsion that are shown in Figure 8.1.

8.36 Solve Problem 8.19 by the method of equations.

8.37 Solve Problem 8.19 by Mohr's circle.

8.38 Solve Problem 8.20 by the method of equations.

8.39 Solve Problem 8.20 by Mohr's circle.

8.40 Solve Problem 8.21 by the method of equations.

8.41 Solve Problem 8.21 by Mohr's circle.

8.42 In a thin body (plane stress) the stress element is as shown in Figure P8.42. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

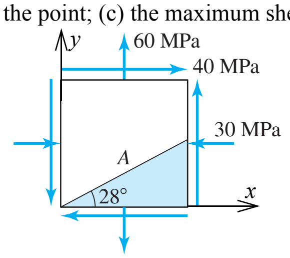


Figure P8.42

8.43 In a thin body (plane stress) the stress element is as shown in Figure P8.43. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

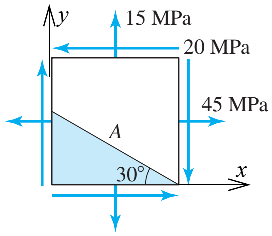


Figure P8.43

8.44 In a thin body (plane stress) the stress element is as shown in Figure P8.44. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

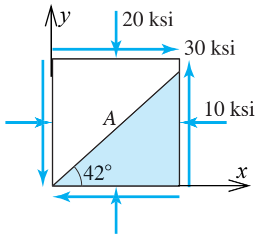


Figure P8.44

8.45 In a thick body (plane strain) the stress element is as shown in Figure P8.45. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

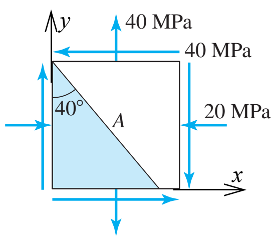


Figure P8.45

8.46 In a thick body (plane strain) the stress element is as shown in Figure P8.46. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

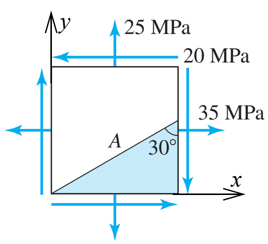


Figure P8.46

8.47 In a thick body (plane strain) the stress element is as shown in Figure P8.47. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

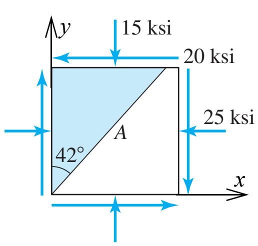


Figure P8.47

8.48 A thin plate (E=30,000 ksi, $\nu = 0.25$) is subjected to a uniform stress $\sigma = 10$ ksi as shown in Figure P8.48. Assuming plane stress, determine the maximum shear stress in the plate.

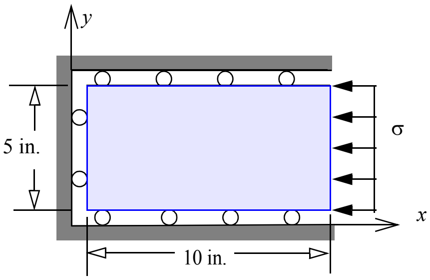


Figure P8.48

8.49 The strains at a point in plane stress and the material properties are as given below. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.79.)

$$\varepsilon_{xx} = 500\ \mu \qquad \varepsilon_{yy} = 400\ \mu \qquad \gamma_{xy} = -300\ \mu \qquad E = 200\ \text{GPa} \qquad \nu = 0.32$$

8.50 The strains at a point in plane stress and the material properties are as given below.Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.80.)

$$\varepsilon_{xx} = -3000\ \mu \qquad \varepsilon_{yy} = 1500\ \mu \qquad \gamma_{xy} = 2000\ \mu \qquad E = 70\ \text{GPa} \qquad G = 28\ \text{GPa}$$

8.51 The strains at a point in plane stress and the material properties are as given below. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.81.)

$$\varepsilon_{xx} = -800\ \mu \qquad \varepsilon_{yy} = -1000\ \mu \qquad \gamma_{xy} = -500\ \mu \qquad E = 30{,}000\ \text{ksi} \qquad \nu = 0.3$$

8.52 A rectangle inscribed on an aluminum (10,000 ksi, $\nu = 0.25$) plate is observed to deform into the colored shape shown in Figure P8.52. Determine the principal stresses, principal angle 1, and the maximum shear stress.

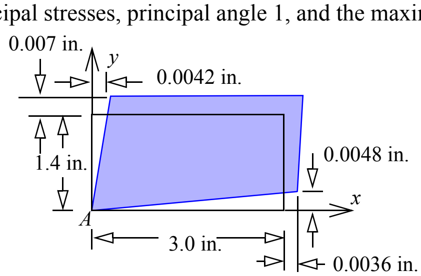


Figure P8.52

In Problems 8.53 through 8.55, the difference in the principal stresses $\sigma_1 - \sigma_2$ and the principal direction 1 θ_1 from the x axis were measured by photoelasticity (see Section 8.4.1*) at several points and are given in each problem. The sum of the principal stresses $\sigma_1 + \sigma_2$ was found from elasticity[5] and is also given. Assuming plane stress, determine the stresses σ_{xx}, σ_{yy}, and τ_{xy} at the point.*

Problem	$\sigma_1 - \sigma_2$	θ_1	$\sigma_1 + \sigma_2$
8.53	10 ksi	-15°	6 ksi
8.54	3 ksi	+25°	-17 ksi.
8.55	5 ksi	+35°	5 ksi.

8.56 A broken 2-in.× 6-in. wooden bar was glued together as shown in Figure P8.56. Determine the normal and shear stresses in the glue when $F = 12$ kips.

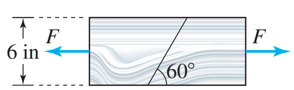


Figure P8.56

8.57 A 10-lb picture is hung using a wire of diameter of 1/8 in. as shown in Figure P8.57. What is the maximum shear stress in the wire?

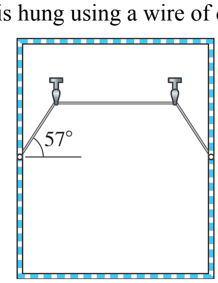
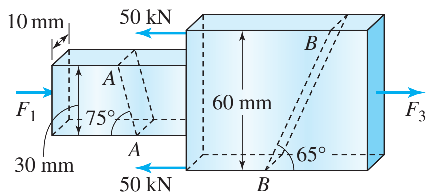


Figure P8.57

[5]Equations of elasticity show that $(\partial^2/\partial x^2)(\sigma_1 + \sigma_2) + (\partial^2/\partial y^2)(\sigma_1 + \sigma_2) = 0$. This differential equation can be solved numerically or analytically with the appropriate boundary conditions.

8.58 Two rectangular bars of thickness 10 mm are loaded as shown in Figure P8.58. For a force $F_1 = 25$ kN, determine the normal and shear stress on planes *AA* and *BB*.



FigureP8.58

8.59 An aluminum rod (*E*= 70 GPa) and a steel rod (*E*= 210 GPa) are securely fastened to a rigid plate that does not rotate during the application of the load *P* as shown in Figure P8.59. The diameter of aluminum and steel rods are 20 mm and 10 mm, respectively. If the applied force *P*=25 kN, determine the maximum shear stress in aluminum and steel.

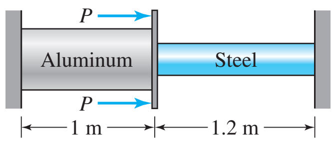


Figure P8.59

8.60 Determine the normal and shear stresses in the seam of the shaft passing through point *A* at an angle $\theta = 60^\circ$ to the axis of a solid shaft of 2-in. diameter, as shown in Figure P8.60.

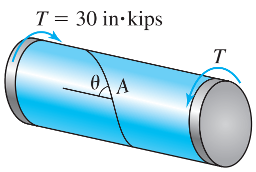


Figure P8.60

8.61 Two circular steel shafts (G =12,000 ksi) of diameter 2 in. are securely connected to an aluminum shaft (G =4,000 ksi) of diameter 1.5 in. as shown in Figure P8.61. Determine (a) the maximum normal stress in the shaft; (b) the normal and shear stress on a weld running through point E.

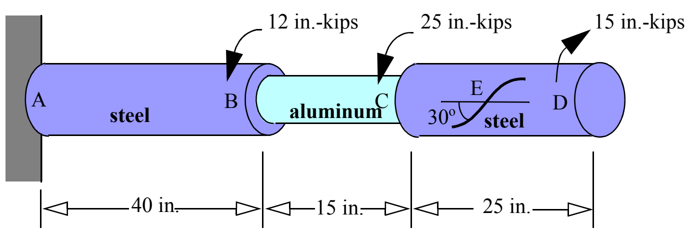


Figure P8.61

8.62 Two pieces of solid shafts of diameter 75 mm are securely connected to a rigid wheel as shown in Figure P8.62. The shaft material has a modulus of rigidity $G = 80$ GPa. If the applied torque $T = 8$ kN-m, determine the maximum normal stress in the shaft.

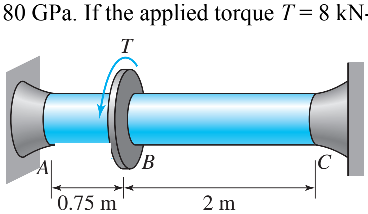


Figure P8.62

8.63 If the applied force in Figure P8.63 is P = 1.8 kN, determine the maximum normal and shear stress at points *A*, *B*, and *C* which are on the surface of the beam.

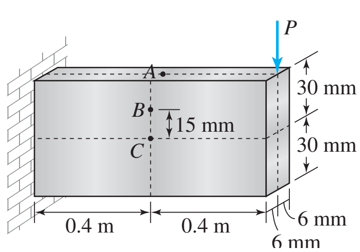


Figure P8.63

8.64 If the applied force in Figure P8.64 is P = 1.8 kN, determine the maximum normal and shear stress at points at points A, B, and C which are on the surface of the beam.

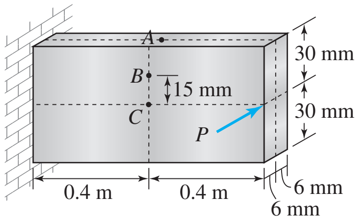


Figure P8.64

8.65 Two pieces of lumber are glued together to form the beam shown in Figure P8.65. The intensity of the distributed load is w = 25 lb/in. Determine (a) the maximum shear stress in the beam; (b) the maximum normal stress in the glue.

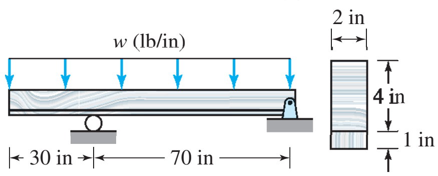


Figure P8.65

8.66 Two pieces of wood are glued together to form a beam, as shown in Figure P8.66. The applied moment M_{ext}= 9 in.-kips. Determine (a) the maximum shear stress in the beam; (b) the maximum normal stress in the glue.

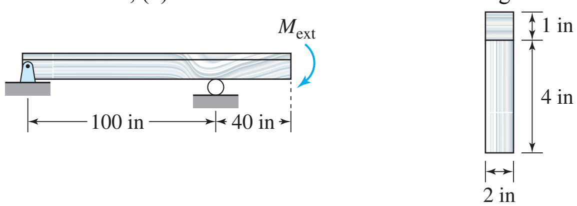


Figure P8.66

Design problems

8.67 A broken 2 in. x 6 in. wooden bar is glued together as shown in Figure P8.56. The allowable normal and shear stress in the glue are 600 psi (T) and 400 psi, respectively. Determine the maximum force F to the nearest pound that can be transmitted by the bar.

8.68 A rigid bar ABC is supported by two aluminum cables (E = 10,000 ksi) as shown in Figure P8.68. The allowable shear stress in aluminum is 20 ksi. If the applied force P = 10 kips, determine the minimum diameter of cables CE and BD to the nearest 1/16 in. Both cables have the same diameter.

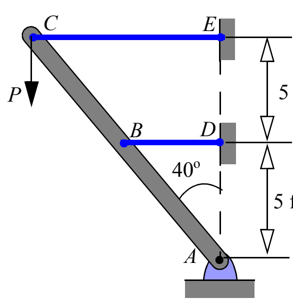


Figure P8.68

8.69 A thin tube of $\frac{1}{8}$-in. thickness has a mean diameter of 6 in. What is the maximum torque the tube can transmit if the allowable normal stress in compression is 10 ksi?

8.70 Solve Example 8.6 again, but without the thin-tube approximation.

8.71 An aluminum rod (E_{al} = 70 GPa) and a steel rod (E_s = 210 GPa) are securely fastened to a rigid plate that does not rotate during the application of load P, as shown in Figure P8.59. The diameters of the aluminum and steel rods are 20 mm and 10 mm, respectively. The allowable shear stresses in aluminum and steel are 120 MPa and 150 MPa. Determine the maximum force P that can be applied to the rigid plate.

8.72 A shaft is welded along the seam that makes an angle of θ = 60° to the axis of a 2 in. diameter shaft as shown in Figure P8.72. The allowable normal and shear stresses in the weld are 15 ksi (T) and 10 ksi, respectively. Determine the maximum torque T_{ext} that can be applied.

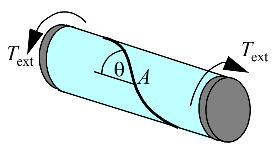


Figure P8.72

8.73 Two pieces of solid shaft of 75-mm diameter are securely connected to a rigid wheel, as shown in Figure P8.62. The shaft material has a modulus of shear rigidity $G = 80$ GPa and an allowable normal stress in tension or compression of 90 MPa. Determine the maximum torque T that can act on the wheel.

8.74 A cantilever beam is constructed by gluing three pieces of timber, as shown in Figure P8.74. The allowable shear stress in the glue is 300 psi and the allowable tensile stress is 200 psi. The allowable tensile or compressive stress in wood is 2000 psi. Determine the maximum intensity of the distributed load w.

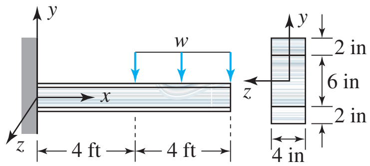


Figure P8.74

8.75 Two pieces of lumber are glued together to form the beam shown in Figure P8.66. The allowable shear stress in the in the wood is 600 psi and the allowable tensile stress in the glue is 650 psi (T). Determine the maximum moment M_{ext} that can be applied.

8.76 Determine the thickness of a steel plate required for a thin cylindrical boiler with a centerline diameter of 2.5 m, if the maximum tensile stress is not to exceed 100 MPa and the maximum shear stress is not to exceed 60 MPa, when the pressure in the boiler is 1800 kPa.

8.77 A thin cylindrical tank is fabricated by butt welding a $\frac{1}{2}$-in.-thick plate, as shown in Figure P8.77. The centerline diameter of the tank is 4 ft. The maximum tensile stress of the plate cannot exceed 30 ksi. The normal and shear stresses in the weld are limited to 25 ksi and 18 ksi, respectively. What is the maximum pressure the tank can hold?

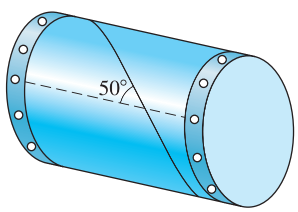


Figure P8.77

Stretch yourself

8.78 By multiplying the matrices, show that the following matrix equation is the same as Equations (8.1), (8.2), and (8.3):

$$[\sigma]_{nt} = [T]^T[\sigma][T] \qquad \text{where} \qquad [\sigma]_{nt} = \begin{bmatrix} \sigma_{nn} & \tau_{nt} \\ \tau_{tn} & \sigma_{tt} \end{bmatrix} \qquad [T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

$[T]^T$ represents the transpose of the matrix $[T]$. The matrix $[T]$ is the transformation matrix that relates the x and y coordinates to the n and t coordinates.

8.79 Show that the eigenvalues of the matrix $[\sigma]$ are the principal stresses given by Equation (8.7).

8.80 Using the wedge shown in Figure P8.26, show that the normal stress on an inclined plane is related to the stresses in Cartesian coordinates by the equation

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2\tau_{xy}n_xn_y + 2\tau_{yz}n_yn_z + 2\tau_{zx}n_zn_x \tag{8.15}$$

8.81 Figure P8.81 show eight (octal) planes that make equal angles with the principal planes. These planes are called *octahedral planes*. Though the signs of the direction cosines change with each plane, the magnitude of the direction cosines is the same for all eight planes; that is, $|n_x| = 1/\sqrt{3}$, $|n_y| = 1/\sqrt{3}$, and $|n_z| = 1/\sqrt{3}$. The normal stress and the shear stress on the octahedral planes σ_{oct} and τ_{oct} are given by equations below. Using Equation (8.15), obtain Equation (8.16).

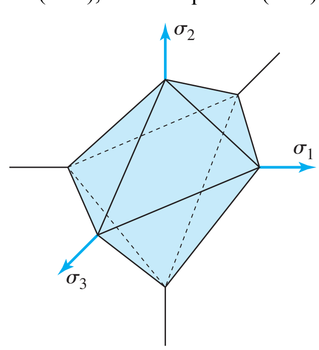


Figure P8.81

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{8.16}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{8.17}$$

Computer problems

8.82 On a machine component made of steel ($E = 30{,}000$ ksi, $G = 11{,}600$ ksi) the following strains were found: $\varepsilon_{xx} = [100(2x + y) + 50]\ \mu$, $\varepsilon_{yy} = -100(2x + y)\ \mu$, and $\gamma_{xy} = 200(x - 2y)\ \mu$. Assuming plane stress, determine the principal stresses, principal angle 1, and the maximum shear stress every 30° on a circle of radius 1 around the origin. Use a spreadsheet or write a computer program for the calculations.

8.83 The stresses in polar coordinates around a hole in a very large plate subject to a uniform stress σ (Figure P8.83) are given by equations below. On a ship deck with a manhole having a diameter of 2 ft, it was estimated that $\sigma = 10$ ksi. Calculate the principal stresses every 15° at a radius of 18 in. Use a spreadsheet or write a computer program for the calculations.

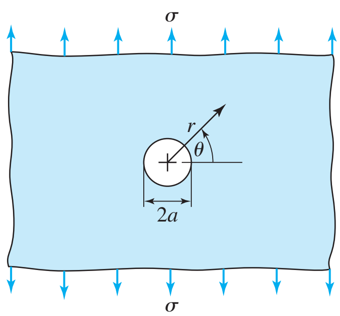


Figure P8.83

$$\sigma_{rr} = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{8.18a}$$

$$\sigma_{\theta\theta} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{8.18b}$$

$$\tau_{r\theta} = \frac{\sigma}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta \tag{8.18c}$$

QUICK TEST 8.3

Time: 15 minutes/Total: 20 points

Answer true or false. If false, give the correct explanation. Each question is worth 2 points. Use the solution given in Appendix E to grade yourself.

1. In plane stress there are two principal stresses and in plane strain there are three principal stresses.
2. Principal planes are always orthogonal.
3. For a given state of stress at a point, the principal stresses depend on the material.
4. Depending on the coordinate system used for finding stresses at a point, the values of the stress components differ. Hence the principal stress at that point will depend on the coordinate system in which the stresses were found.
5. Planes of maximum shear stress are always at 90° to principal planes.
6. The sum of the normal stresses in an orthogonal coordinate system is independent of the orientation of the coordinate system.
7. If principal stress 1 is tensile and principal stress 2 is compressive, then the in-plane maximum shear stress and the maximum shear stress are the same for plane *stress* problems.
8. If principal stress 1 is tensile and principal stress 2 is compressive, then the in-plane maximum shear stress and the maximum shear stress are the same for plane *strain* problems.
9. Two planes passing through a point can be represented by the same point on Mohr's circle.
10. Two points on Mohr's circle can represent the same plane.

*8.4 CONCEPT CONNECTOR

Photoelasticity is an experimental method for deducing stress information from observing the effects on light as it passes through a transparent material that is stressed. The analysis is complex, and you will study it in advanced courses, but the principles behind this remarkable visual representation of stress are simpler. To explain photoelasticity, we must first understand the transmission of light.

8.4.1 Photoelasticity

The color of light depends on the frequency of the wave, and it may be affected by transmission through different-colored materials. White light is a mixture of different frequencies. However, even waves of the same frequency can travel differently in different materials. That is because light waves may lie in different planes at a point. (In quantum mechanics, we learn to think of light also as particles, called *photons*, which can then vibrate in different planes). Light of a single frequency is *monochromatic*, and light with one plane of vibrations is *polarized*.

A *polarizer* is a material that permits light rays to pass only if their plane of vibration lies parallel to one axis, called the *polarizing* axis. If two polarizers are used, then the second polarizer is called an *analyzer*. If the polarizer and the analyzer have polarizing axes arranged perpendicular to each other, then no light will pass through, and a *dark field* will be produced. If the polarizer and the analyzer have polarizing axes parallel to each other, then all the light that passes through the first polarizer will pass through the analyzer, too, and a *light field* will be produced.

The velocity of light depends on the material through which it is passing. Usually the velocity of light is the same in all directions, and the material is said to be *isotropic*. However, when some transparent materials are stressed, light travels through the material at different speeds along different planes of vibration. These materials have *two* polarizing axes, at right angles to each other. And, unlike in a polarizer, neither axis simply selects the light. Rather, the velocities of light along the two polarizing axes are different. This behavior of light is called *birefringence*, and materials in which light velocities are different in the two polarizing axes are called *birefringent*.

When light passes through birefringent material, a ray along one axis takes longer to pass through it than a ray through the other axis. In other words, the ray along the slows axis reaches the same amplitude as the faster ray after a time Δt. The time difference Δt is called *retardation* time.

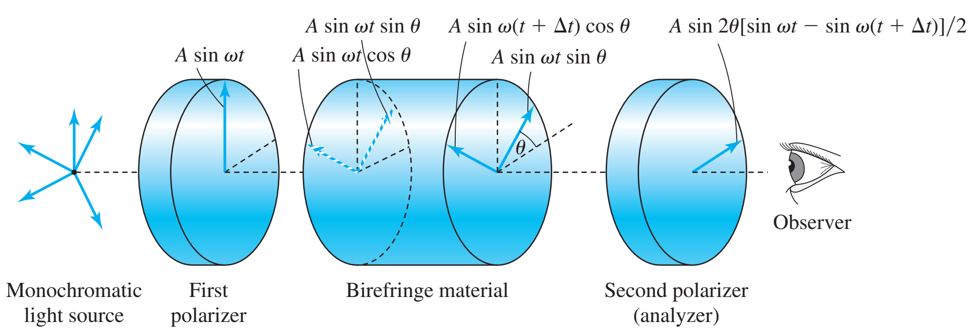


Figure 8.34 Transmission of light in photoelasticity.

Figure 8.34 shows light originating from a monochromatic source. It then encounters not just a polarizer and an analyzer, but also a birefringent material placed between them. As light passes first through the polarizer, it emerges with its plane of vibration parallel to the vertical axis. Now suppose that the two axes of the birefringent material are set at an angle θ with respect to the vertical. Light passing through the polarizer will have components along each of these axes, and the component of light along the slow axis is retarded by time Δt. In other words, two rays emerge from the birefringent material. The observer sees only the horizontal components of these two rays, because the second polarizer will pass only light that is parallel to the horizontal axis. In sum, the light that the observer sees depends on two variables—the angle θ of the birefringent material's fast axis with respect to the analyzer axis and the retardation time Δt.

What makes *photoelasticity* possible is that birefringence is directly related to stress, as observed by James Clerk Maxwell, the Scottish mathematician and physicist, in 1857:

1. The principal axes of stresses in a birefringent material are the fast and slow axes, with the fast axis corresponding to principal stress 1.

2. The retardation time is proportional to the difference of principal stresses.

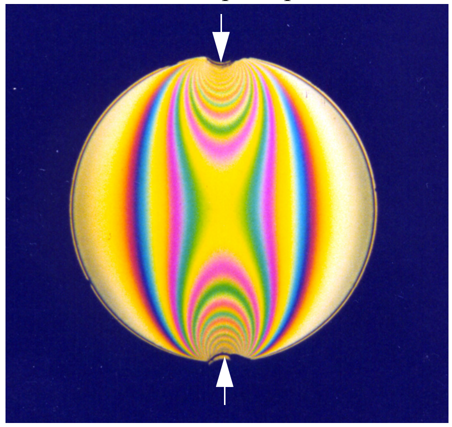


Figure 8.35 Photoelastic fringes showing principal stress difference. (Courtesy Professor I. Miskioglu.)

Suppose we start with a light field, so that the polarizer and analyzer axes are parallel. Note that $\sin(n\pi)$ is zero. We conclude that the observer will see dark spots where $\theta = n\pi/2$. Lines that connect these dark spots, called *isoclinic lines*, thus give us the direction of principal stress 1 at different points. If we start instead with a dark field, then the isoclinic lines join points of maximum transmission of light, where $\sin 2\theta = 1$.

If we start with a light field, the observer will also not see any light if the term in brackets in Figure 8.34 equals zero. At these points $\Delta t = 2n\pi/\omega$, and lines connecting these points are called *fringes*. Because Δt is related to the difference in principal stresses, the fringes thus yield the values of $\sigma_1 - \sigma_2$. Figure 8.35 shows these fringes in a disc subjected to diametrically opposite compressive forces.

By choosing different orientations for the polarizing axes of the polarizer, the analyzer, and the birefringe material, we can obtain different isoclinic lines and different fringes. Through a succession of such combinations of axes, we can then obtain a visual representation of stress in a material. Photographs are taken for each isoclinic line and fringe, and a composite photograph that shows all isoclinic lines and fringes is made.

Actually, to describe plane stress completely, we need three pieces of information. Photoelasticity yields only two—the orientation of principal axis 1 and the difference of principal stresses. However, on a free surface we know that one of the principal stresses is zero. Thus photoelasticity will give a complete state of stress for a point on a free surface. In the interior, if we know the sum of the principal stresses, then we can obtain the complete state of stress at that point. To obtain the sum of the principal stresses may require a mix of analytical, numerical, and other experimental methods.

8.5 CHAPTER CONNECTOR

In this chapter we studied the relationship of stresses in different coordinate systems and methods to determine the maximum tensile normal stress, maximum compressive normal stress, and maximum shear stress. In Chapter 10 we shall study various failure theories, including maximum-normal-stress and maximum-shear-stress theories. We also apply these theories to the design and failure analysis of simple structures and machines.

POINTS AND FORMULAS TO REMEMBER

- Stress transformation equations relate stresses *at a point* in different coordinate systems:
- $$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \quad (8.1)$$

- $$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad (8.2)$$

- where σ_{xx}, σ_{yy}, and τ_{xy} are the stresses in x, y, z coordinate system, σ_{nn}, σ_{tt}, and τ_{nt} are the stresses in n, t, z coordinate system and θ is measured from the x axis in the counterclockwise direction to the n direction.
- The values of stresses on a plane through a point are unique and depend on the orientation of the plane only and not on how its orientation is described or measured.
- Planes on which the shear stresses are zero are called *principal planes*.
- Principal planes are orthogonal.
- The normal direction to the principal planes is referred to as the principal direction or *principal axis*.
- The angles the principal axis makes with the global coordinate system are called *principal angles*.
- Normal stress on a principal plane is called *principal stress*.
- The greatest principal stress is called *principal stress 1*.
- Principal stresses are the maximum and minimum normal stresses at a point.
- The maximum shear stress on a plane that can be obtained by rotating about the z axis is called *in-plane maximum shear stress*.
- The maximum shear stress at a point is the *absolute* maximum shear stress that is on any plane passing through the point.
- Maximum in-plane shear stress exists on two planes which are at 45° to the principal planes.

- $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (8.6) \qquad \sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad (8.7)$$

- $$|\tau_p| = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \quad (8.12)$$

- where θ_p is the angle to either principal plane 1 or 2, σ_1 and σ_2 are the principal stresses, τ_p is the in-plane maximum shear stress.

- $$\sigma_{nn} + \sigma_{tt} = \sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2 \quad (8.9)$$

- $$\sigma_3 = \sigma_{zz} = \begin{cases} 0, & \text{plane stress} \\ \nu(\sigma_{xx} + \sigma_{yy}), & \text{plane strain} \end{cases} \quad (8.10)$$

- $$\tau_{max} = \left|\max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right)\right| \quad (8.13)$$

- Each point on Mohr's circle represents a unique plane that passes through the point at which the stresses are specified.
- The coordinates of the point on Mohr's circle are the normal and shear stresses on the plane represented by the point.
- Angles between planes on a stress cube are doubled when plotted on Mohr's circle.
- The sign of shear stress cannot be determined directly from Mohr's circle, which tells only that the shear stress causes the plane to rotate clockwise or counterclockwise.
- The maximum shear stress at a point is the radius of the biggest circle.
- A principal stress element shows stresses on a wedge constructed from principal planes and the plane of maximum shear stress.