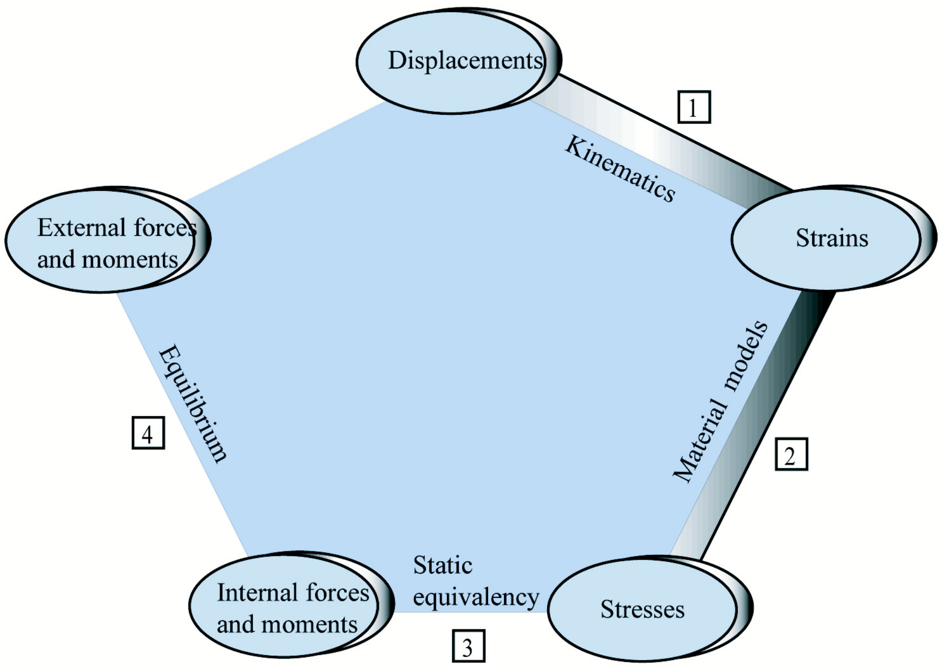
Basic Structural Analysis



The learning objectives in this chapter are:

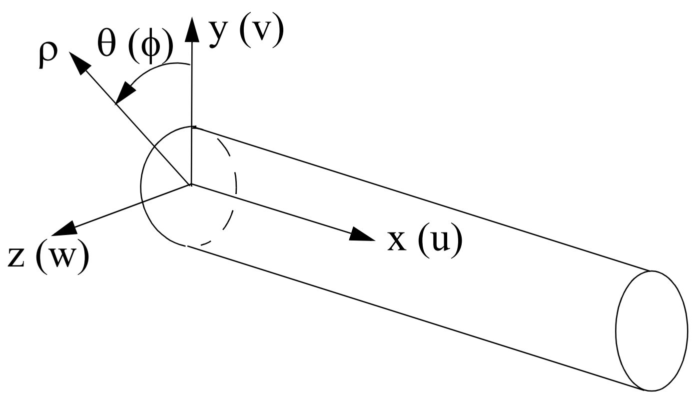
- Understand the limitations of basic theory and how complexities may be added to the basic theories of axial members, torsion of circular shafts, and symmetric bending of beams.
- Understand the concept and use of discontinuity functions in analysis of structural members subjected to discontinuous loads.

Preliminaries

Limitations

- The length of the member is significantly greater (approximately 10 times) then the greatest dimension in the cross-section. Approximation across the cross-section are now possible as the region of approximation is small.
- We are away from regions of stress concentration, where displacements and stresses can be three-dimensional.
- The variation of external loads or changes in the cross-sectional area is gradual except in regions of stress concentration.
- The external loads are such that the axial, torsion and bending problems can be studied individually.

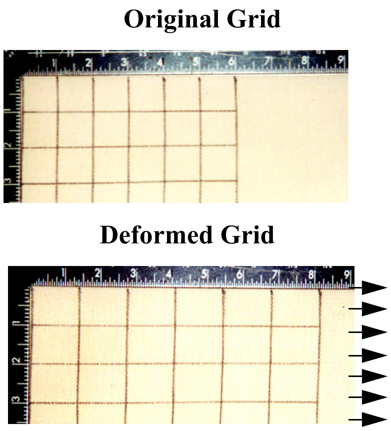
Convention



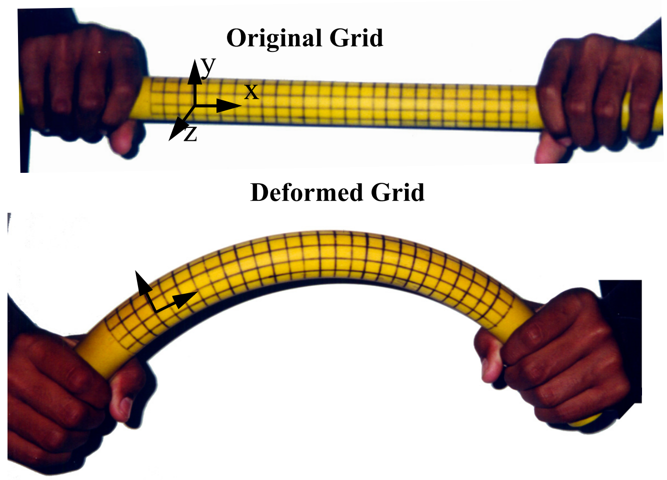
- The displacements u, v, and w will be considered positive in the positive x, y, and z -direction, respectively.
- The rotation ϕ of the cross-section will be considered positive counter-clockwise with respect to the x-axis.
- The external distributed torque per unit length t(x) is positive counter-clockwise with respect to the x-axis.
- The external distributed force per unit length $p_x(x)$ and $p_y(x)$ are considered positive in the positive x and y direction, respectively.

Deformations

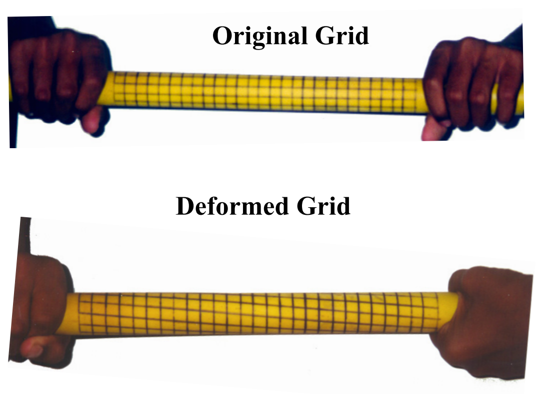
(a) **Axial**



(b) **Bending**



(c) **Torsion**



	Axial	Bending	Torsion
Assumption 1	Deformations are not function of time.		
Assumptions	2-A: Plane sections remain plane and parallel.	2a-B: Squashing deformation is significantly smaller than deformation due to bending. 2b-B: Plane sections before deformation remain plane after deformation. 2c-B: Plane perpendicular to the beam axis remain *nearly* perpendicular after deformation	2a-T: Plane sections perpendicular to the axis remain plane during deformation. 2b-T: All radials lines rotate by equal angle during deformation on a cross-section. 2c-T: Radials lines remain straight during deformation.
	$u = u_o(x)$ **(3.1-A)**	$v = v(x)$ **(3.1a-B)** $u = -y\frac{dv}{dx}$ **(3.1b-B)**	$\phi = \phi(x)$ **(3.1-T)**

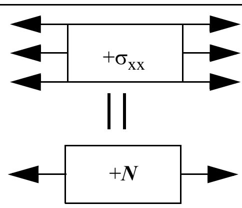
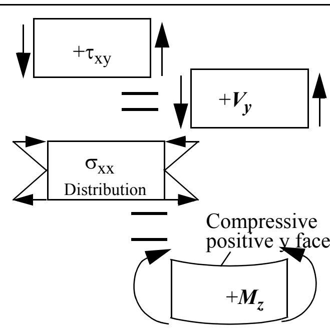
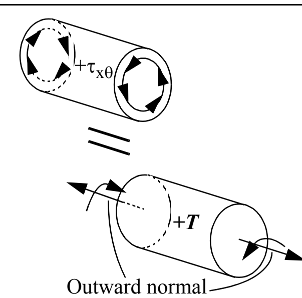
Strains

	Axial	Bending	Torsion
Assumption 3	The strains are small.		
	$\varepsilon_{xx} = \frac{du_o}{dx}(x)$ **(3.2-A)**	$\varepsilon_{xx} = -y\frac{d^2 v}{dx^2}(x)$ **(3.2-B)**	$\gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$ **(3.2-T)**

Stresses

	Axial	Bending	Torsion
Assumption 4	Material is isotropic.		
Assumption 5	There are no inelastic strains.		
Assumption 6	Material is elastic.		
Assumption 7	Stress and strains are linearly related.		
Using Hooke's law	$\sigma_{xx} = E\frac{du_o}{dx}(x)$ **(3.3-A)**	$\sigma_{xx} = -Ey\frac{d^2 v}{dx^2}(x)$ **(3.3-B)**	$\tau_{x\theta} = G\rho\frac{d\phi}{dx}(x)$ **(3.3-T)**

Internal Forces and Moments

	Axial	Bending	Torsion
Static equivalency	$N = \int_A \sigma_{xx} dA$ **(3.4a-A)** $M_z = -\int_A y\sigma_{xx} dA = 0$ **(3.4b-A)** $M_y = -\int_A z\sigma_{xx} dA = 0$ **(3.4c-A)**	$N = \int_A \sigma_{xx} dA = 0$ **(3.4a-B)** $M_z = -\int_A y\sigma_{xx} dA$ **(3.4b-B)** $V_y = \int_A \tau_{xy} dA$ **(3.4c-B)**	$T = \int_A \rho\tau_{x\theta} dA$ **(3.4-T)**
Sign convention			

Formulas

Substituting stresses into equations of internal forces and moments and Noting $\frac{du_o}{dx}$, $\frac{d^2 v}{dx^2}$, and $\frac{d\phi}{dx}$ are functions of x only while the integration is with respect to y and z.

	Axial	Bending	Torsion
Origin Location	$\int_A yEdA = 0$ **(3.5-A)**	$\int_A yEdA = 0$ **(3.5-B)**	
	$N = \frac{du_o}{dx}\int_A EdA$ **(3.6-A)**	$M_z = \frac{d^2 v}{dx^2}\int_A Ey^2 dA$ **(3.6-B)**	$T = \frac{d\phi}{dx}\int_A G\rho^2 dA$ **(3.6-T)**
Assumption 8 Material is homogenous across the cross-section.			
Origin is at the centroid of the cross-section	$\int_A ydA = 0$ **(3.7-A)**	$\int_A ydA = 0$ **(3.7-B)**	
	$\frac{du_o}{dx} = \frac{N}{EA}$ **(3.8-A)**	$\frac{d^2 v}{dx^2} = \frac{M_z}{EI_{zz}}$ **(3.8-B)**	$\frac{d\phi}{dx} = \frac{T}{GJ}$ **(3.8-T)**
	A = Area of cross-section EA = Axial Rigidity	I_{zz} = Second area moment of inertia EI_{zz} = Bending rigidity	J= Polar moment of the area. GJ = Torsional rigidity

Stress formulas

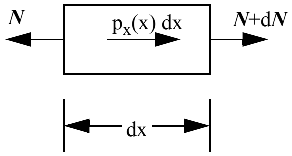
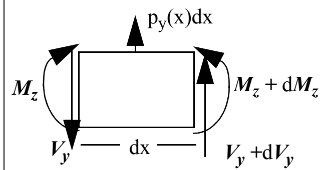
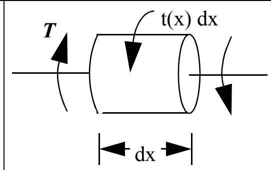
Substituting Equations (3.8-A), (3.8-B), and (3.8-T) into Equations (3.3-A), (3.3-B), and (3.3-T)

	Axial	Bending	Torsion
	$\sigma_{xx} = \frac{N}{A}$ **(3.9-A)**	$\sigma_{xx} = -\left(\frac{M_z y}{I_{zz}}\right)$ **(3.9-B)** See section... for shear stresses in bending.	$\tau_{x\theta} = \frac{T\rho}{J}$ **(3.9-T)**

Deformation formulas

	Axial	Bending	Torsion
Assumption 9 Material is homogenous between x_1 and x_2. Assumption 10 The structural member is not tapered between x_1 and x_2. Assumption 11 The external loads do not change with x between x_1 and x_2. Integrating Equations (3.8-A) and (3.8-T)			
	$u_2 - u_1 = \frac{N(x_2 - x_1)}{EA}$ **(3.10-A)**	See Section 3.2.4 for beam deflection.	$\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}$ **(3.10-T)**

Equilibrium Equations

	Axial	Bending	Torsion
			
	$\frac{dN}{dx} = -p_x(x)$ **(3.11-A)**	$\frac{dV_y}{dx} = -p_y(x)$ **(3.11a-B)** $\frac{d\boldsymbol{M}_z}{dx} = -\boldsymbol{V}_y$ **(3.11b-B)**	$\frac{d\boldsymbol{T}}{dx} = -t(x)$ **(3.11-T)**

Differential Equations

Substituting Equations (3.8-A), (3.8-B), and (3.8-T) into Equations (3.11-A), (3.11a-B), (3.11b-B), and (3.11-T)

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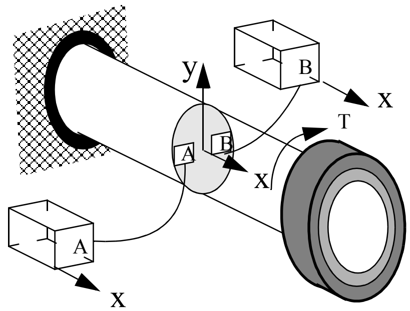
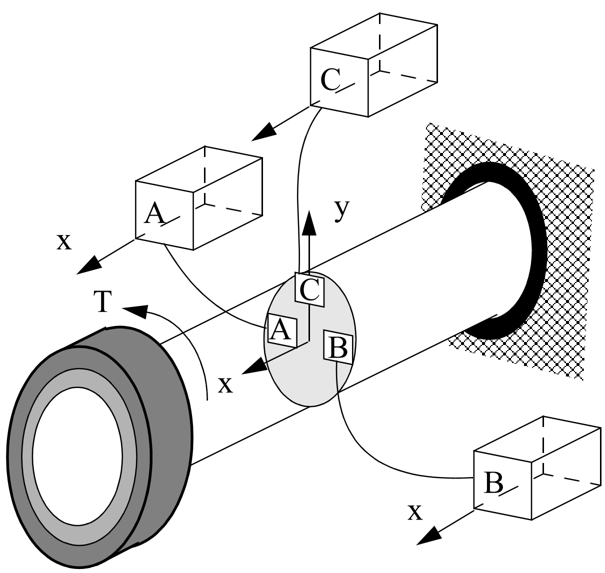
	$\frac{d}{dx}\left(EA\frac{du_o}{dx}\right) = -p_x(x)$ **(3.12-A)**	$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = p_y(x)$ **(3.12-B)**	$\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) = -t(x)$ **(3.12-T)**

C3.1 Draw the shear stress due to torsion on the stress cubes at points A and B shown. Is the shear stress positive or negative τ_{xy}?



Class Problem 3.1

Determine the direction of shear stress at points A, B and C (a) by inspection, and (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy} or τ_{xz}



C3.2 Determine the contraction of a column shown in Figure C3.2 due to its own weight. The specific weight is $\gamma = 0.28\ lb/in^3$, the modulus of elasticity is $E = 3,600\ ksi$, the length is $L = 120\ in$, and the radius varies as $R = \sqrt{240-x}$, where, R and x are in inches.

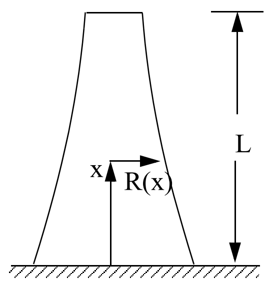
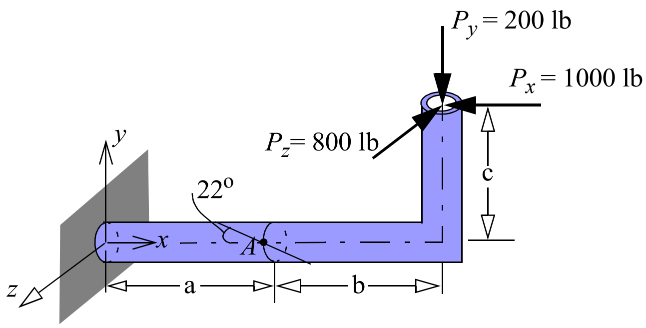
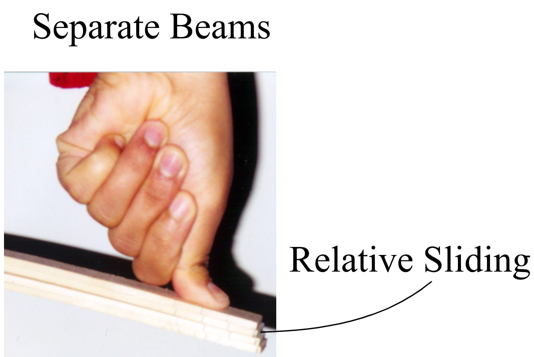


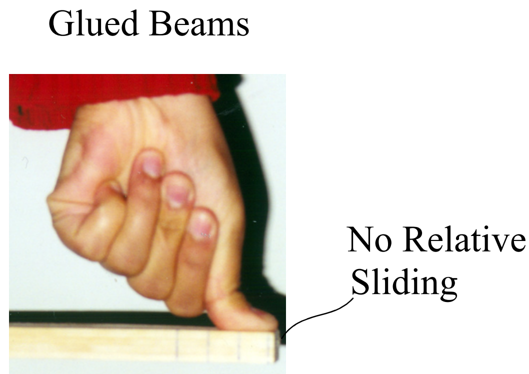
Figure C3.2

C3.3 A through crack of 0.07 inch was observed on a plane 22^{o} to the axis of a bent pipe at point A as shown. The pipe has an outside diameter of 2 in., a wall thickness of $\frac{1}{4}$ in. and the critical stress intensity factor for the material of $22\ ksi\sqrt{in}$ If a = 16 in., b = 16 in.,and c = 10 in., determine the factor of safety.



Shear Stress in Thin Symmetric Beams

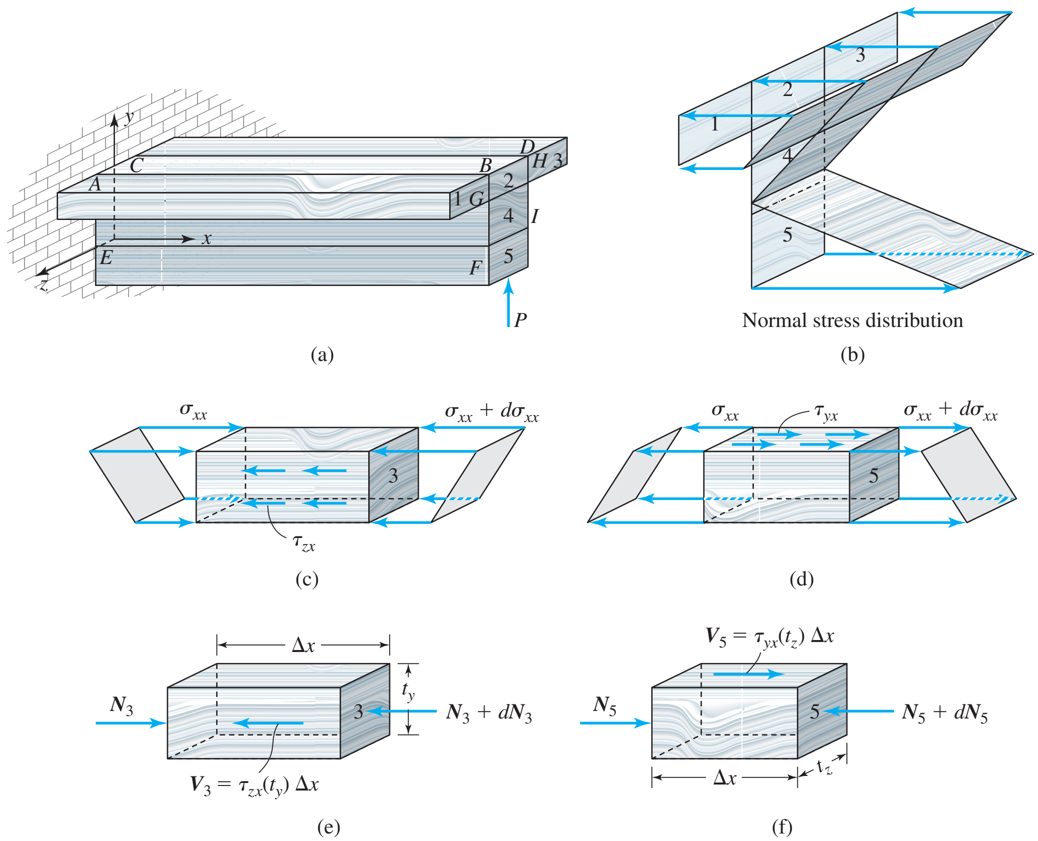


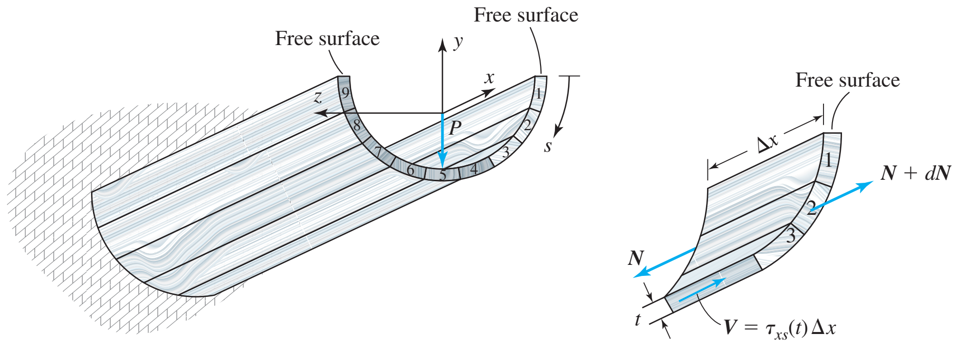


- Assumption of plane section perpendicular to the axis remain perpendicular during bending requires the following limitation.

  Maximum bending shear stress must be an order of magnitude less than maximum bending normal stress.

Shear stress direction





(a) (b)

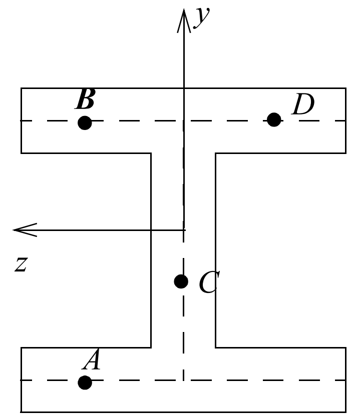
Shear Flow: $q = \tau_{xs} t$

- The units of shear flow ‘q’ are force per unit length.

The shear flow along the center-line of the cross-section is drawn in such a direction as to satisfy the following rules:

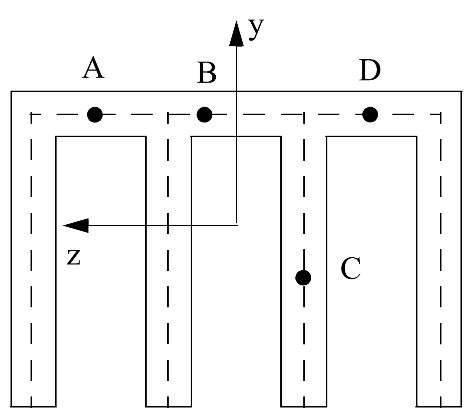
- the resultant force in the y-direction is in the same direction as V_y.
- the resultant force in the z-direction is zero.
- it is symmetric about the y-axis. This requires shear flow will change direction as one crosses the y-axis on the center-line.

C3.4 Assuming a positive shear force V_y, (a) sketch the direction of the shear flow along the center-line on the thin cross-sections shown. (b) At points A, B, C, and D, determine if the stress component is τ_{xy} or τ_{xz} and if it is positive or negative.

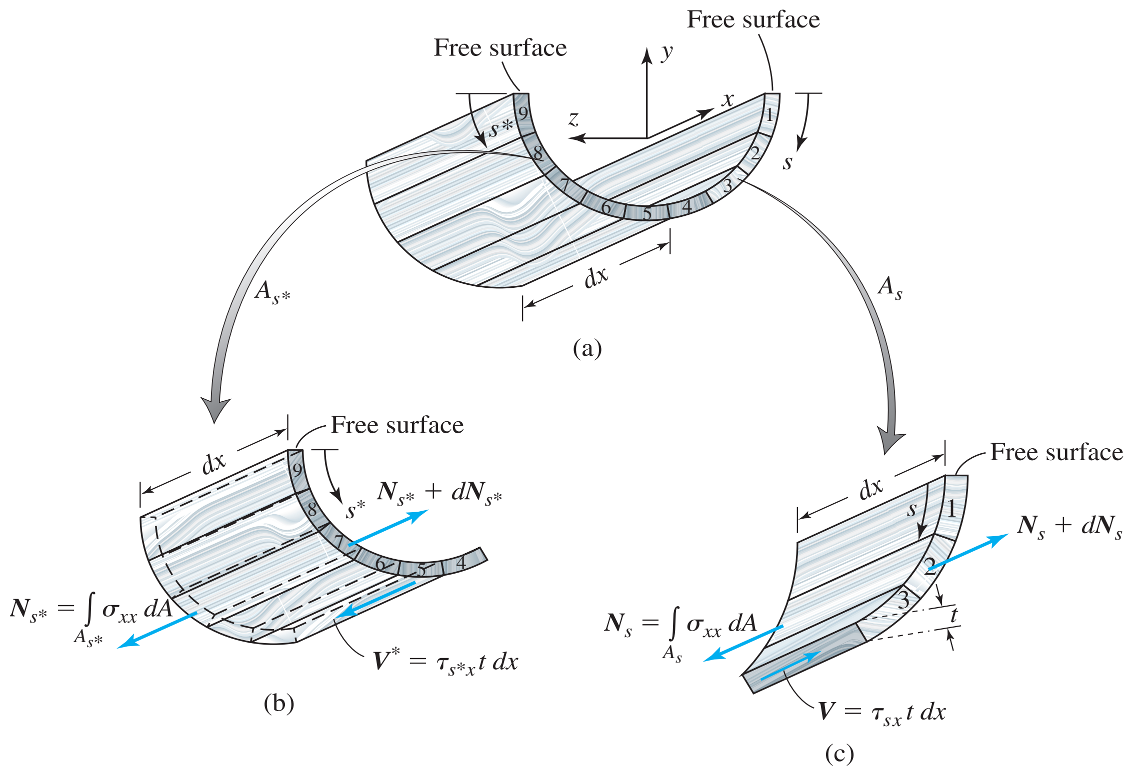


Class Problem 3.2

Assuming a positive shear force V_y, (a) sketch the direction of the shear flow along the center-line on the thin cross-sections shown.(b) At points A, B, C, and D, determine if the stress component is τ_{xy} or τ_{xz} and if it is positive or negative.



Bending Shear Stress Formula



$$(N_s + dN_s) - N_s + \tau_{sx} t dx = 0 \qquad \tau_{sx} t = -\frac{dN_s}{dx}$$

$$\tau_{sx} t = -\frac{d}{dx}\int_{A_s} \sigma_{xx} dA = -\frac{d}{dx}\int_{A_s}\left(-\frac{M_z y}{I_{zz}}\right) dA = \frac{d}{dx}\left[\frac{M_z}{I_{zz}}\int_{A_s} y dA\right]$$

- A_s is the area between the free surface and the point where shear stress is being evaluated.

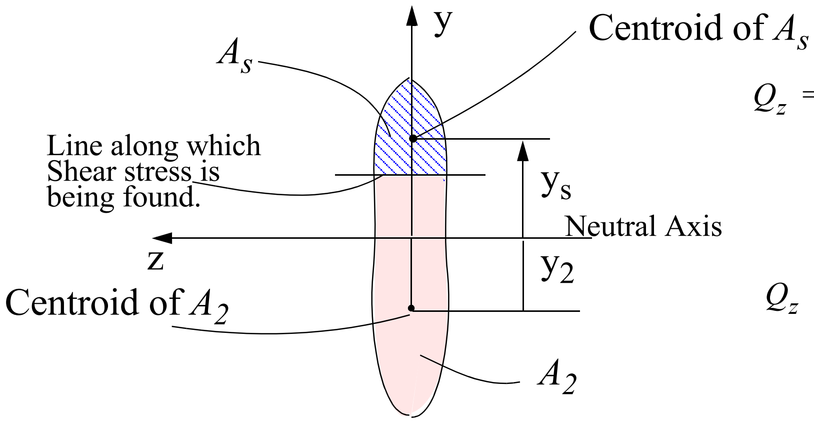
Define: $Q_z = \int_{A_s} y dA \qquad \tau_{sx} t = \frac{d}{dx}\left[\frac{M_z Q_z}{I_{zz}}\right]$

Assumption 1 The beam is not tapered.

$$q = t\tau_{sx} = \left(\frac{Q_z}{I_{zz}}\right)\frac{dM_z}{dx} = -\left(\frac{Q_z V_y}{I_{zz}}\right) \qquad \boxed{\tau_{sx} = \tau_{xs} = -\left(\frac{V_y Q_z}{I_{zz} t}\right)}$$

Calculation of $Q_z = \int_{A_s} y\,dA$

- A_s is the area between the free surface and the point where shear stress is being evaluated.
- Q_z is zero at the top surface as the enclosed area A_s is zero.
- Q_z is zero at the bottom surface (A_s=A) by definition of centroid.



$$Q_z = A_s y_s$$

$$Q_z = A_2 y_2$$

- Q_z is maximum at the neutral axis.

Bending shear stress at a section is maximum at the neutral axis.

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Table 3.2. Stresses and Strains

Axial		Symmetric Bending			Torsion	
		About z-axis	About y-axis	Strains		
Stresses	Strains	Stresses	Stresses		Stresses	Strains
$\sigma_{xx} = \frac{N}{A}$	$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$	$\sigma_{xx} = -\left(\frac{M_z y}{I_{zz}}\right)$	$\sigma_{xx} = -\left(\frac{M_y z}{I_{yy}}\right)$	$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$	$\sigma_{xx} = 0$	$\varepsilon_{xx} = 0$
$\sigma_{yy} = 0$	$\varepsilon_{yy} = -\left(\frac{\nu\sigma_{xx}}{E}\right)$	$\sigma_{yy} = 0$	$\sigma_{yy} = 0$	$\varepsilon_{yy} = -\left(\frac{\nu\sigma_{xx}}{E}\right)$	$\sigma_{yy} = 0$	$\varepsilon_{yy} = 0$
$\sigma_{zz} = 0$	$\varepsilon_{zz} = -\left(\frac{\nu\sigma_{xx}}{E}\right)$	$\sigma_{zz} = 0$	$\sigma_{zz} = 0$	$\varepsilon_{zz} = -\left(\frac{\nu\sigma_{xx}}{E}\right)$	$\sigma_{zz} = 0$	$\varepsilon_{zz} =$
$\tau_{xy} = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$	$\gamma_{xy} = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$	$\tau_{xs} = -\left(\frac{V_y Q_z}{I_{zz} t}\right)$ $\tau_{yz} = 0$	$\tau_{xs} = -\left(\frac{V_z Q_y}{I_{yy} t}\right)$ $\tau_{yz} = 0$	$\gamma_{xs} = \frac{\tau_{xs}}{G}$ $\gamma_{yz} = 0$	$\tau_{x\theta} = \frac{T\rho}{J}$ $\tau_{yz} = 0$	$\gamma_{x\theta} = \frac{\tau_{x\theta}}{G}$ $\gamma_{yz} = 0$

C3.5 A positive shear force V_y= 10 kN acts on the thin cross-section shown in Figure C3.5 (not drawn to scale). The cross-section has a uniform thickness of 10 mm. Determine the equation of shear flow along the center lines and sketch it.

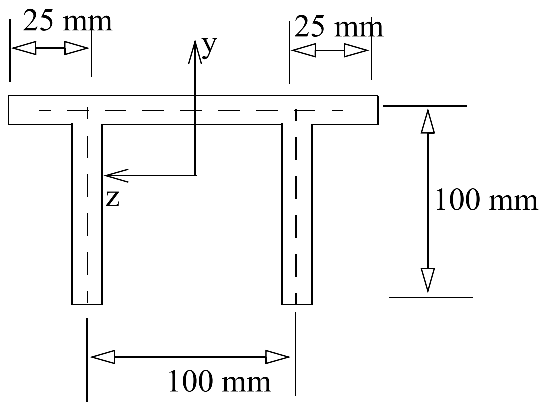


Figure C3.5

C3.6 A bending moment of M_z = 30 kN-m and a shear force of V_y= 10 kN acts on a thin cross-section shown in Figure C3.6 (not drawn to scale).The cross-section has a uniform thickness of 10 mm and the material has a modulus of elasticity of 200 GPa and a Poisson's ratio of 0.25. Determine the principal strains at point A which is just below the flange.

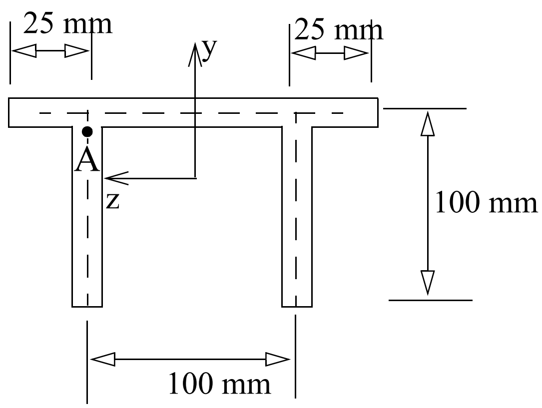
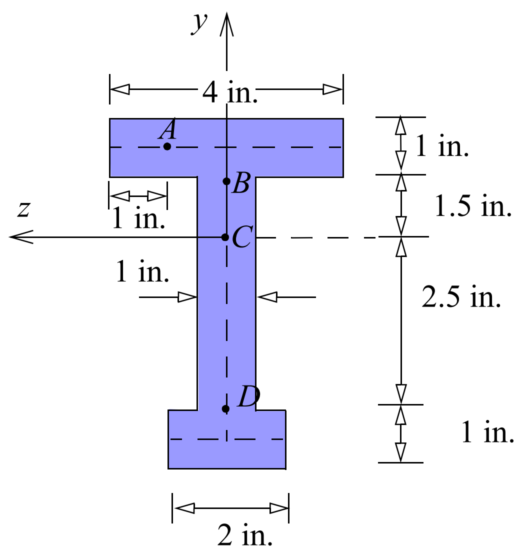


Figure C3.6

Class Problem 3.3

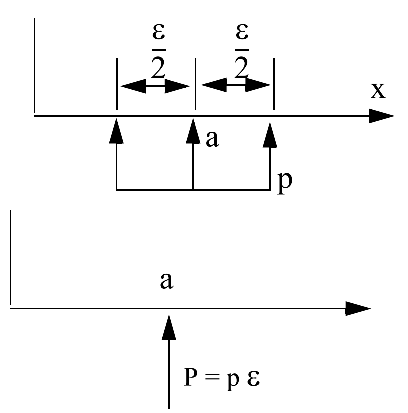
Identify the area A_s that will be used in calculation of shear stress at points *A*,*B*, *D* and the maximum shear stress. Also show direction of s.

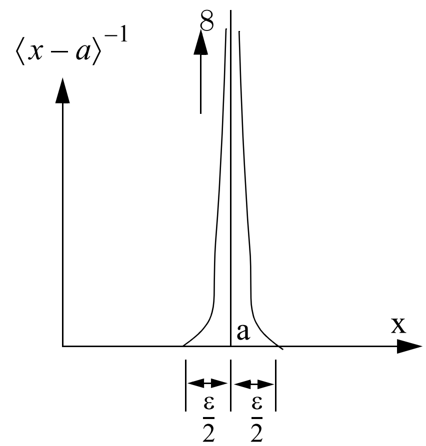


C3.7 In Timoshenko beams the assumption of planes remaining perpendicular to the axis of the beam is dropped to account for shear by permitting the cross section to rotate by an angle ψ from the vertical. Obtain the differential equations for vibration of Timoshenko beam by starting with the following displacement field

$$u = -y\psi(x, t) \qquad \mathrm{v} = \mathrm{v}(x, t)$$

Discontinuity Functions





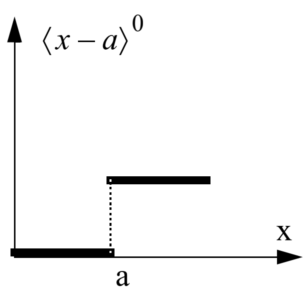
$$P = \lim_{p\to\infty}\ \lim_{\varepsilon\to 0}(p\varepsilon)$$ or

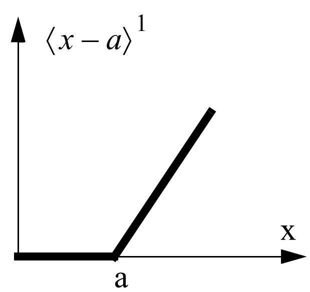
$$\langle x-a\rangle^{-1} = \begin{Bmatrix} 0 & x \neq a \\ \infty & x \to a \end{Bmatrix} \qquad \int_{(a-\varepsilon)}^{(a+\varepsilon)} \langle x-a\rangle^{-1} dx = 1$$

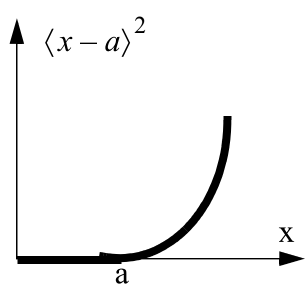
Delta Function: $\langle x-a\rangle^{-1}$

$$\int_{-\infty}^{x} \langle x-a\rangle^{-1} dx = \int_{-\infty}^{(a-\varepsilon)} \langle x-a\rangle^{-1} dx + \int_{(a-\varepsilon)}^{(a+\varepsilon)} \langle x-a\rangle^{-1} dx + \int_{(a+\varepsilon)}^{x} \langle x-a\rangle^{-1} dx = 1$$

$$\langle x-a\rangle^{0} = \int_{-\infty}^{x} \langle x-a\rangle^{-1} dx = \begin{Bmatrix} 0 & x < a \\ 1 & x > a \end{Bmatrix}$$







$$\langle x-a\rangle^{n} = \begin{cases} 0 & x \le a \\ (x-a)^{n} & x > a \end{cases}$$

$$\int_{-\infty}^{x} \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{(n+1)} \qquad n \geq 0$$

Doublet Function: $\langle x-a\rangle^{-2} = \begin{cases} 0 & x \neq a \\ \infty & x \to a \end{cases}$ $\qquad \int_{-\infty}^{x} \langle x-a\rangle^{-2} dx = \langle x-a\rangle^{-1}$

$$\frac{d\langle x-a\rangle^{-1}}{dx} = \langle x-a\rangle^{-2} \qquad \frac{d\langle x-a\rangle^{0}}{dx} = \langle x-a\rangle^{-1}$$

$$\frac{d\langle x-a\rangle^{n}}{dx} = n\langle x-a\rangle^{n-1} \qquad n \geq 1$$

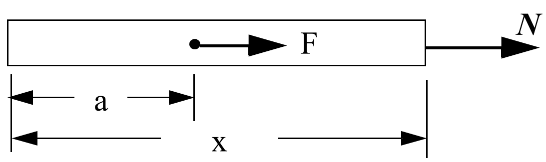
- The function delta function $\langle x-a\rangle^{-1}$ and the doublet function $\langle x-a\rangle^{-2}$ become infinite at x = a. Alternatively stated these functions are singular at x = a. and are referred to as *singularity functions*.
- The entire class of functions $\langle x-a\rangle^{n}$ for positive and negative ‘n’ are called the *discontinuity functions*.

Axial Displacement

$$\frac{du}{dx} = \frac{N}{EA} \quad \frac{dN}{dx} = -p_x(x)$$

Differential Equation: $\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -p_x(x)$

Boundary Conditions u or N

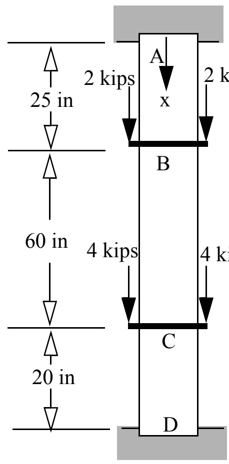


Template equations

$$N = -F\langle x-a\rangle^{0}$$

$$p_x = F\langle x-a\rangle^{-1}$$

Example 3.7



$$p_x = (10 + 4000\langle x-25\rangle^{-1} + 8000\langle x-85\rangle^{-1}) \ lb/in \quad \text{(E1)}$$

- Differential equation

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -[10 + 4000\langle x-25\rangle^{-1} + 8000\langle x-85\rangle^{-1}] \quad \text{(E2)}$$

- Boundary Conditions

$$u(0) = 0 \quad \text{(E3)}$$

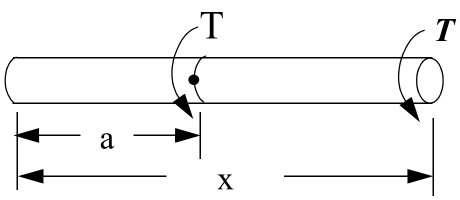
$$u(105) = 0 \quad \text{(E4)}$$

Torsional Rotation

$$\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ} \quad \frac{d\boldsymbol{T}}{dx} = -t(x)$$

Differential Equation: $\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) = -t(x)$

Boundary Conditions ϕ or $\boldsymbol{T}$



Template equations

$$T = -T\langle x-a\rangle^{0}$$

$$t = T\langle x-a\rangle^{-1}$$

C3.8 The external torque on a drill bit varies as a quadratic function to a maximum intensity of q in-lb/in as shown Figure C3.8. If the drill bit diameter is d, its visible length L, and modulus of rigidity G, determine (a) the maximum shear stress on the drill bit. (b) the relative rotation of the end of the drill bit with respect to the chuck.

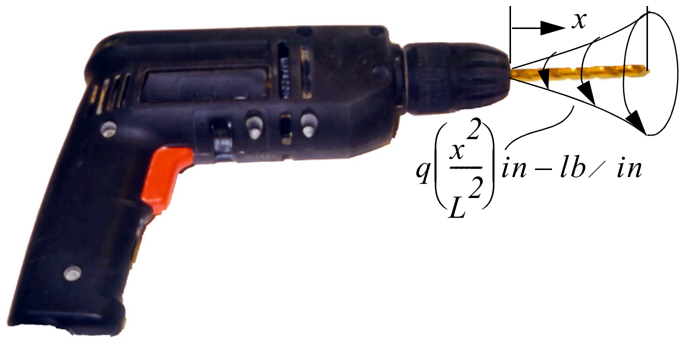
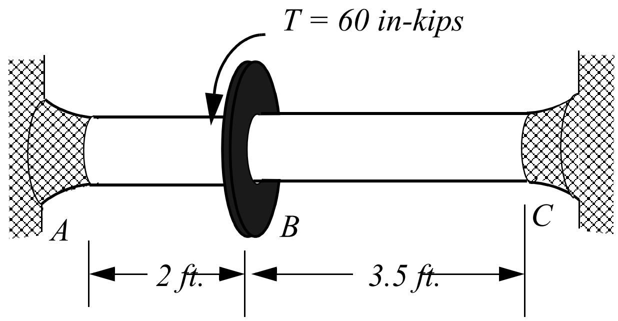


Figure C3.8

C3.9 An aluminum alloy (G = 28 GPa) hollow shaft has a critical stress intensity factor of $22\ ksi\sqrt{in}$. The shaft has a thickness of 1/4 in. and an outer diameter of 2 in. and is loaded as shown in Figure C3.9. What is the critical crack length at which the shaft be taken out of service?

Figure C3.9



Beam Deflection

2nd order differential equation: $$\frac{d^2 v}{dx^2} = \frac{\boldsymbol{M}_z}{EI_{zz}}$$

Equilibrium equations: $$\frac{d\boldsymbol{V}_y}{dx} = -p_y(x) \qquad \frac{d\boldsymbol{M}_z}{dx} = -\boldsymbol{V}_y$$

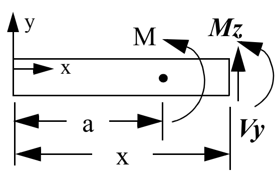
4th order differential Equation: $$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) = p_y(x)$$

Boundary Conditions

- Group 1 v or $\boldsymbol{V}_y$

and

- Group 2 $\frac{dv}{dx}$ or $\boldsymbol{M}_z$

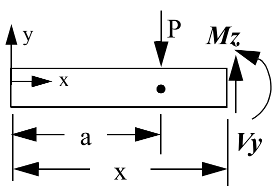


$$\boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -M & x > a \end{cases}$$

Template equations

$$\boldsymbol{M}_z = -M\langle x-a\rangle^0$$

$$p_y = -M\langle x-a\rangle^{-2}$$

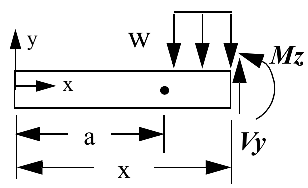


$$\boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -P(x-a) & x > a \end{cases}$$

Template equations

$$\boldsymbol{M}_z = -P\langle x-a\rangle^1$$

$$p_y = -P\langle x-a\rangle^{-1}$$



$$\boldsymbol{M}_z = \begin{cases} 0 & x < a \\ -\dfrac{w(x-a)^2}{2} & x > a \end{cases}$$

Template equations

$$\boldsymbol{M}_z = -w\frac{\langle x-a\rangle^2}{2}$$

$$p_y = -w\langle x-a\rangle^0$$

C3.10 The displacement of the beam in the y-direction, in section AB of the beam shown in Figure C3.10 is given by $v_1 = 5(x^3 - 20x^2)\ (10^{-6})\ in$ and in section BC is given $v_2 = 5(x^3 - 800x + 8000)\ (10^{-6})\ in$. If the bending rigidity (EI) is 135 (10^6) lbs-in_2, determine the moment M_B and the reaction force at B.

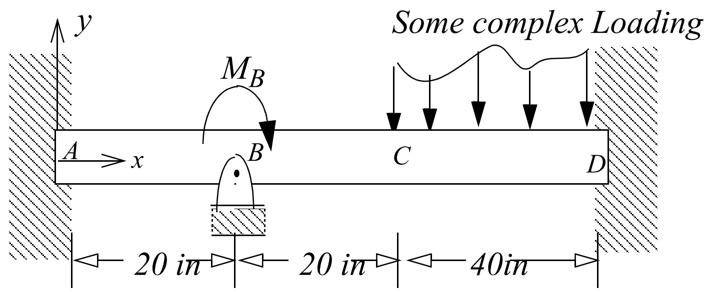


Figure C3.10

C3.11 In terms of w, L, E, and I, determine the deflection and slope at $x = L$ of the beam shown in Figure C3.11.

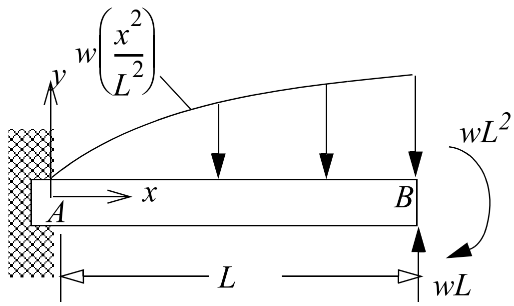


Figure C3.11

C3.12 (a) Determine the deflection of the beam at point *C* in terms of *E*, *I*, *w*, and *L* for the beam shown in Figure C3.12.(b) Determine the maximum bending moment and shear force.

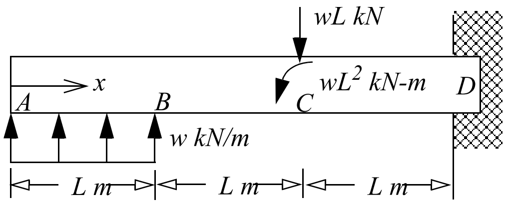


Figure C3.12