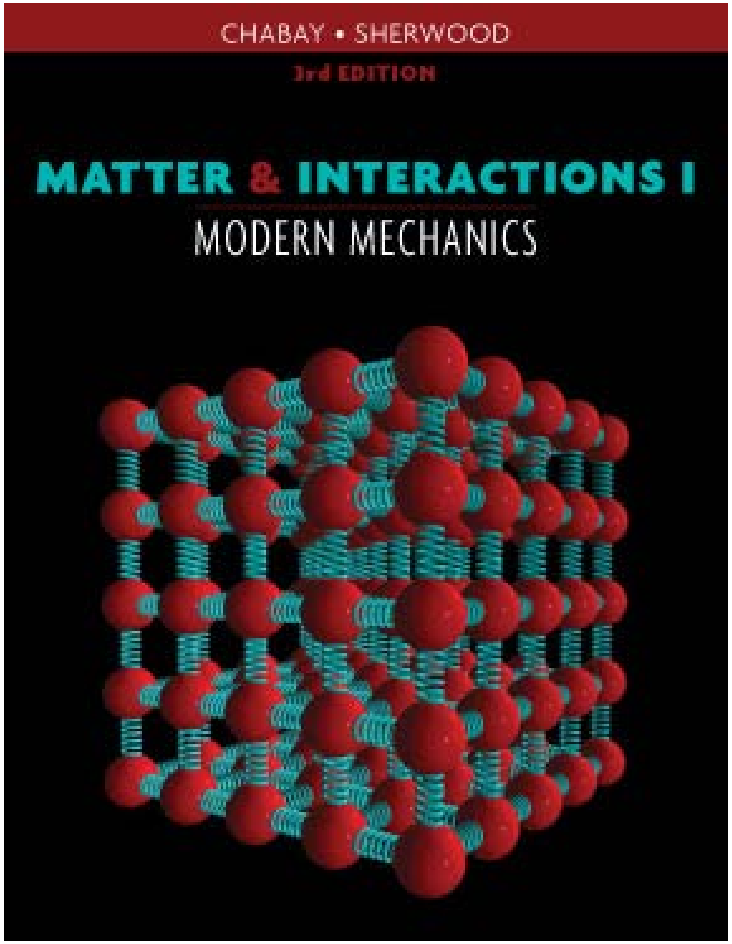

PHY1004W 2012
Modern Mechanics
Part 4

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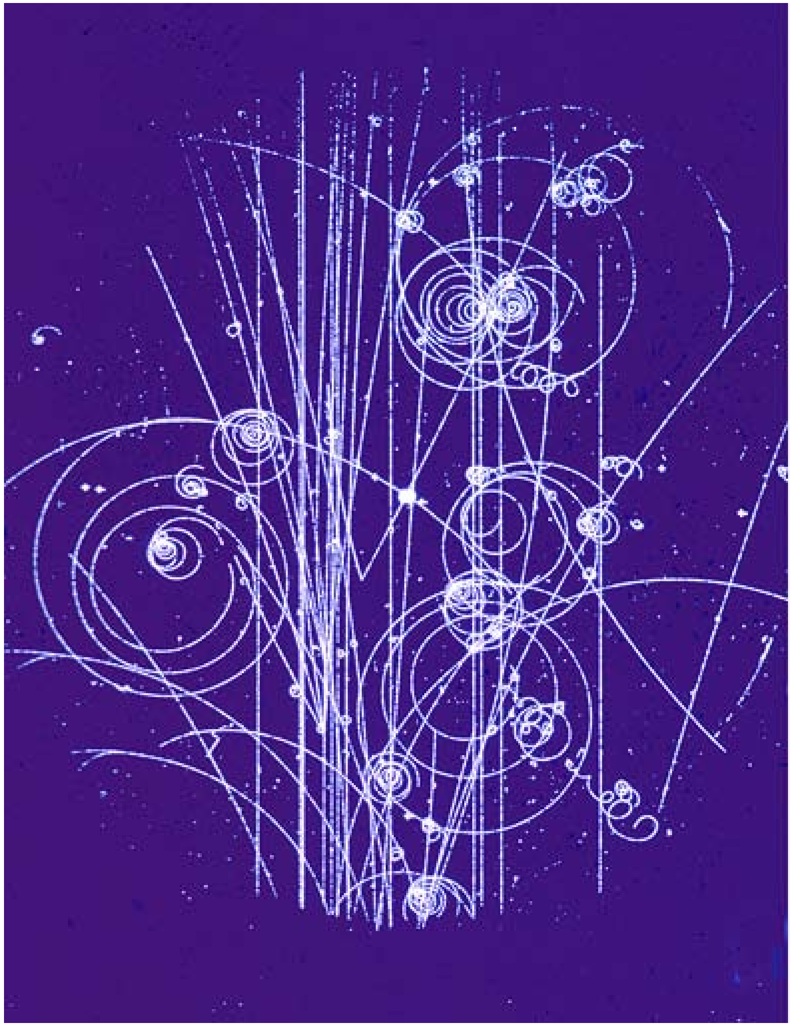
These slides have benefited from significant guidance from the notes of Roger Fearick (UCT Physics) and the resources provided by the textbook authors.

These slides are available on … UCT OpenContent

M&I

Chapter 9

Multiparticle Systems



The motion of the centre of mass

Two basic principles for forces acting on particles:

Momentum principle: $$\frac{d\vec{\mathbf{p}}_{sys}}{dt} = \vec{\mathbf{F}}_{net}$$

... leads to: **conservation of momentum**.

Energy principle: $\Delta E_{\text{system}} = W_{\text{surr}} +$ other energy transfers

... leads to: **conservation of energy**.

We want to extend these to multiparticle systems.

Centre of mass

We define the centre of mass for a system of particles as:

$$\vec{\mathbf{r}}_{CM} = \sum \frac{m_i \vec{\mathbf{r}}_i}{M_{total}} \qquad \text{total mass} = \sum m_i$$

$$x_{CM} = \sum \frac{m_i x_i}{M_{total}} \; ; \quad y_{CM} = \sum \frac{m_i y_i}{M_{total}} \; ; \quad z_{CM} = \sum \frac{m_i z_i}{M_{total}}$$

then $\vec{\mathbf{r}}_{CM} = x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} + z_{CM}\hat{\mathbf{k}}$

For an extended body (a continuous mass distribution) :

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M_{total}} \int \vec{\mathbf{r}} dm = \frac{1}{M_{total}} \int x dm \; \hat{\mathbf{i}} + \frac{1}{M_{total}} \int y dm \; \hat{\mathbf{j}} + \frac{1}{M_{total}} \int z dm \; \hat{\mathbf{k}}$$

Velocity of the centre of mass

The importance of centre of mass lies in the fact that the motion of the centre of mass for a system of particles (or extended body) can often be described simply since it is related to net force on the system.

Consider n particles of total mass M which remains constant.

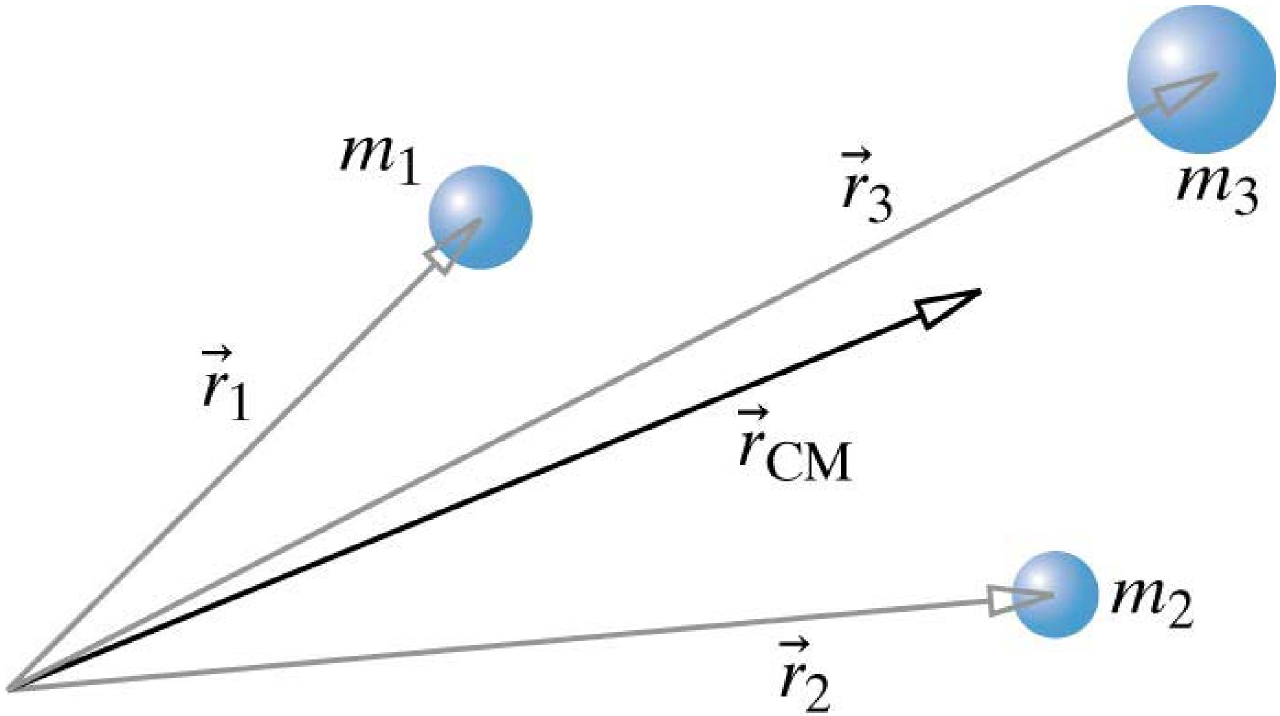
Then: $$M_{total}\vec{\mathbf{r}}_{CM} = \sum m_i \vec{\mathbf{r}}_i$$

$$\therefore M_{total}\frac{d\vec{\mathbf{r}}_{CM}}{dt} = \sum m_i \frac{d\vec{\mathbf{r}}_{\mathrm{i}}}{dt}$$

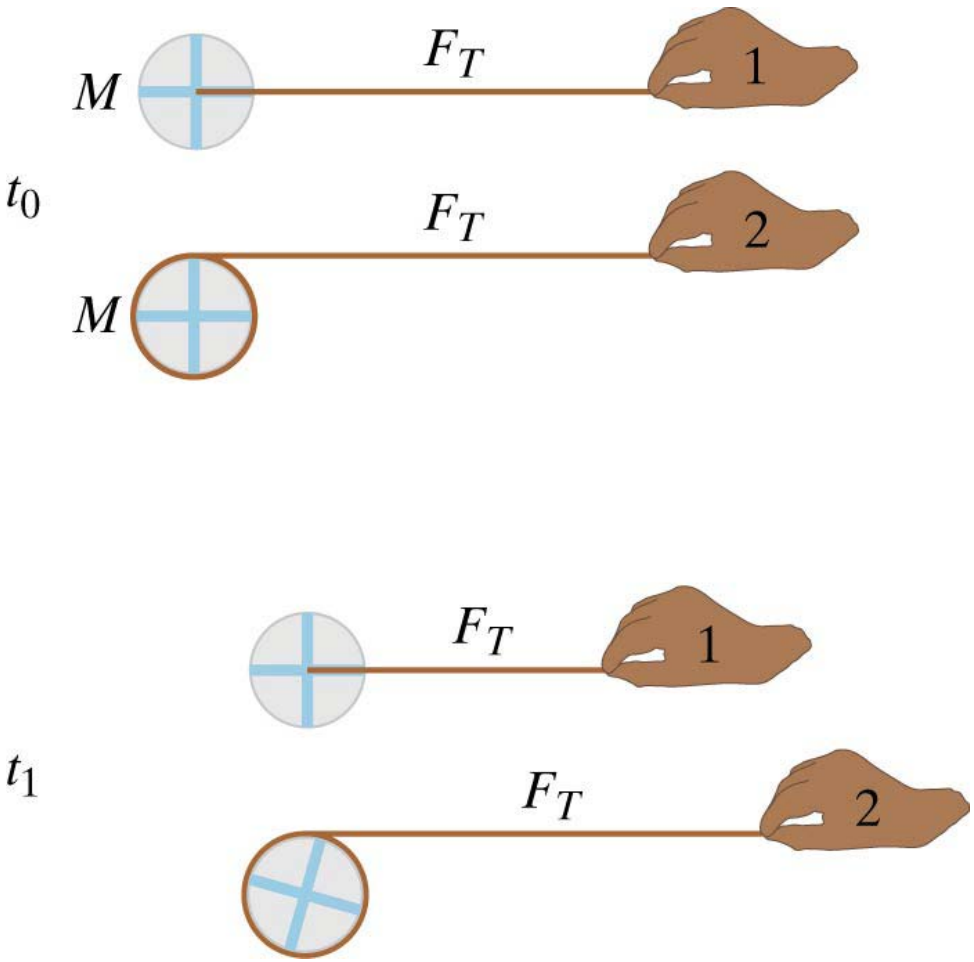
$$\therefore \vec{\mathbf{p}}_{sys} = M_{total}\vec{\mathbf{v}}_{CM} = \sum m_i \vec{\mathbf{v}}_i$$

Velocity of centre of mass

Velocity of i^{th} particle of mass m


m_1
$\vec{r}_3$
m_3
$\vec{r}_1$
$\vec{r}_{CM}$
m_2
$\vec{r}_2$

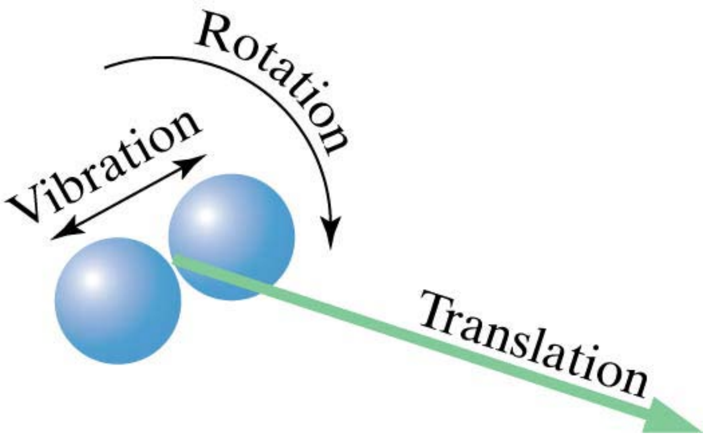
Application: Pull on two hockey pucks



One puck is pulled by a string attached to its centre.
The other puck is pulled by a string wrapped around its edge, which unrolls as the puck is pulled.

See twopucks.py

Separation of multiparticle system energy



For a complex system

$$K_{tot} = K_{trans} + \underbrace{K_{rot} + K_{vib}}_{K_{relative}}$$

(to centre of mass)

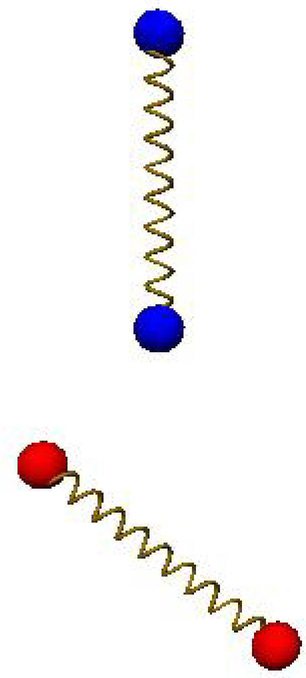
$$K_{trans} = \tfrac{1}{2} M_{total} v_{CM}^2 = \frac{p_{sys}^2}{2M_{total}}$$

The total energy of a translating, rotating, vibrating oxygen molecule can be written as

$$E_{tot} = K_{trans} + K_{rot} + K_{vib} + \tfrac{1}{2} k_s s^2 + 2mc^2$$

1	2	3	4	5

See krel.py



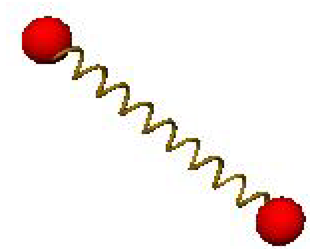
Which object has the greater total momentum (magnitude)?

(1) Top object (blue)
(2) Bottom object (red)
(3) Their total momentum is the same

1	2	3	4	5

See krel.py

Which object has the greater total kinetic energy?

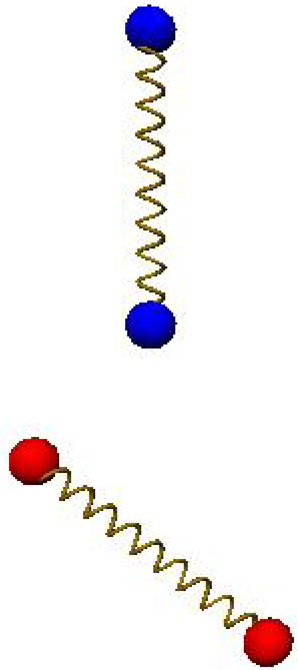


(1) Top object (blue)
(2) Bottom object (red)
(3) Their total kinetic energy is the same

1	2	3	4	5

See krel.py

Which object has the greater translational kinetic energy (K_{TRANS})?



(1) Top object (blue)
(2) Bottom object (red)
(3) Their total translational kinetic energy is the same

See RotateVibrateTranslate.py

CASE = 0: left ball fixed -- think about vcm & vrel
CASE = 1: v_CM=0, vibration, zero external force
CASE = 2: initially in motion, rotation + translation, zero external force
CASE = 3: initially in motion, rot + vib + trans, zero external force
CASE = 4: weaker spring, 0 initial stretch, const force on b2
CASE = 5: initially slightly stretched, at angle, const force
CASE = 6: vibration perp to translation; zero net force

Gravitational energy of a multiparticle system

How does the centre of mass influence energy?

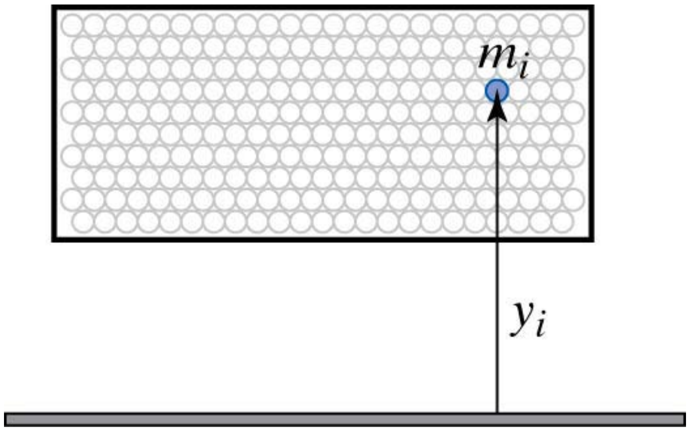
e.g. the change in potential energy near the surface of the Earth is ...

$$\Delta U_g = gm_1y_1 + gm_2y_2 + gm_3y_3 + \ldots$$

$$= g(m_1y_1 + m_2y_2 + m_3y_3 + \ldots)$$

$$= gMy_{CM}$$

near the Earth's surface



(the centre of mass is sometimes called the **centre of gravity** in this context).

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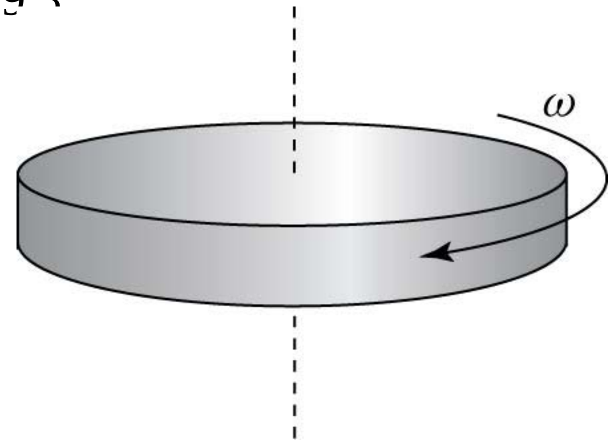
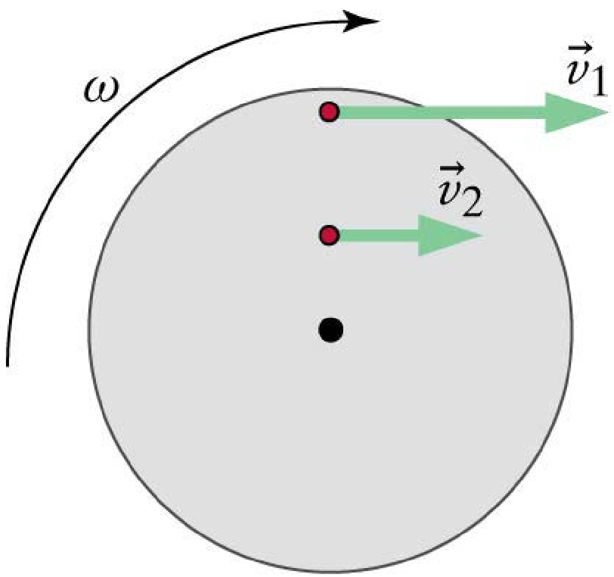
Rotational kinetic energy



Consider a rigid disk rotating at constant period *T* around a fixed axis of rotation.

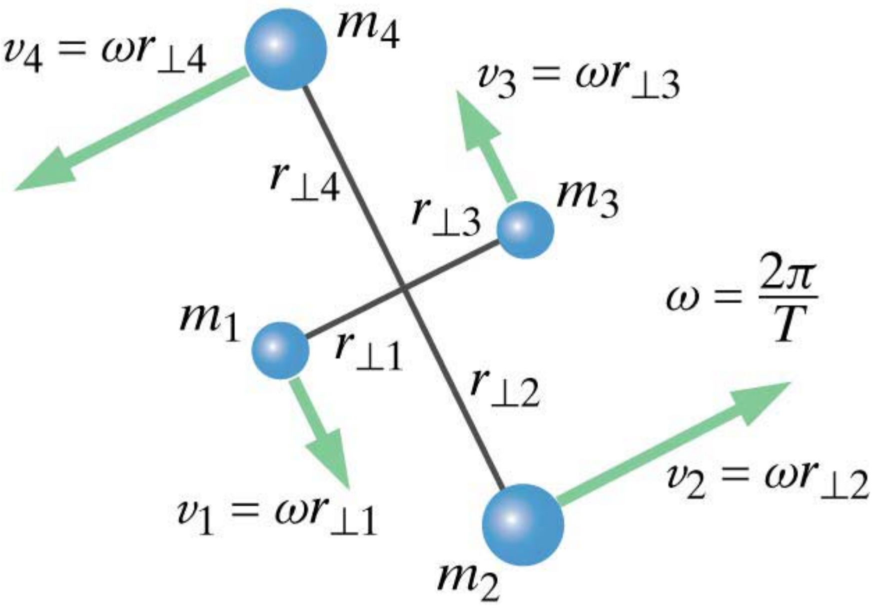
Then the angular speed $\omega = \dfrac{2\pi}{T}$ (radians per second)



The translational speed of an atom on the disk can be expressed as

$$v_i = \frac{2\pi r_i}{T} = \left(\frac{2\pi}{T}\right) r_i = \omega r_i$$

Now consider four point masses held rigid as shown, rotating at constant ω ...



$$K_{rot} = \frac{1}{2}\left[m_1(\omega r_{\perp 1})^2 + m_2(\omega r_{\perp 2})^2 + m_3(\omega r_{\perp 3})^2 + m_4(\omega r_{\perp 4})^2\right]$$

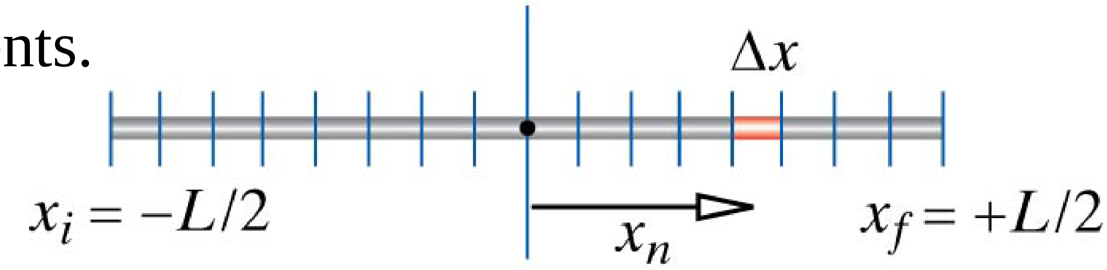
$$= \frac{1}{2}\left[m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + m_3 r_{\perp 3}^2 + m_4 r_{\perp 4}^2\right]\omega^2 \quad = \frac{1}{2} I \omega^2$$

where $I = m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + m_3 r_{\perp 3}^2 + m_4 r_{\perp 4}^2$ is the "**moment of inertia**"

In general for a collection of *i* point particles $I = \sum_i m_i r_{\perp i}^2$

Moment of inertia of a thin rod

Slice up the rod of length L and total mass M into N segments.



Since $\Delta x = \dfrac{L}{N}$

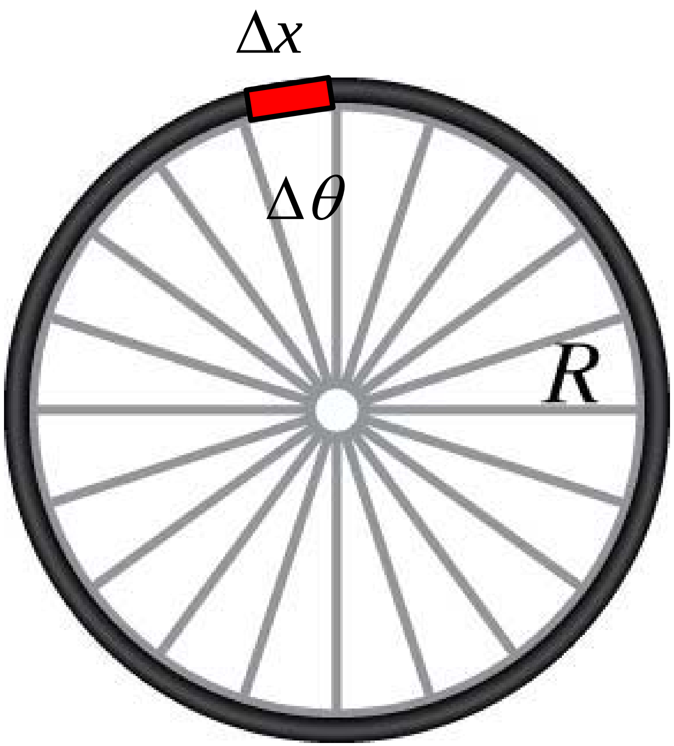
then $\Delta M = \dfrac{M}{N} = \dfrac{M}{L/\Delta x} = M\dfrac{\Delta x}{L}$

Then $\Delta I = (\Delta M)\, x_n^2 = \left(M\dfrac{\Delta x}{L}\right) x_n^2 = \dfrac{M}{L} x_n^2 \Delta x$

Total moment of inertia $I = \sum_{n=1}^{N} \Delta I = \dfrac{M}{L}\sum_{n=1}^{N} x_n^2 \Delta x$

Let $\Delta x \to 0$ then $I = \dfrac{M}{L}\int_{-L/2}^{L/2} x^2 dx$ ... $= \dfrac{1}{12} ML^2$

Moment of inertia of a ring (bicycle wheel)



Slice up the ring of radius R and total mass M into N segments.

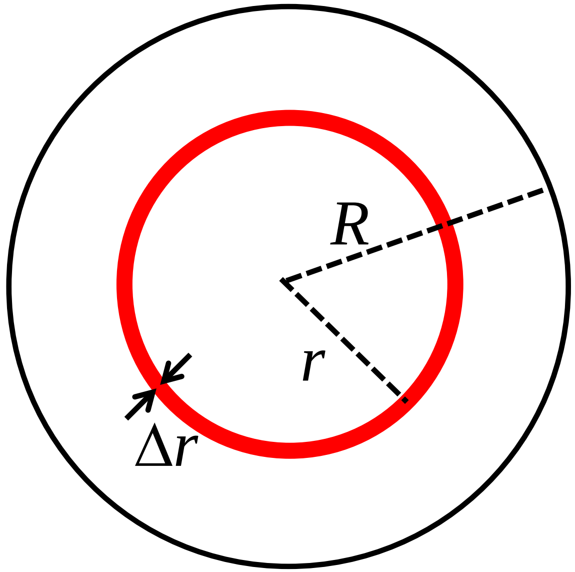
Since $\Delta x = \dfrac{2\pi R}{N}$ and $\dfrac{\Delta x}{R} = \Delta\theta$

then $$\Delta M = \frac{M}{N} = \frac{M}{2\pi R/\Delta x} = M\frac{\Delta\theta}{2\pi}$$

Then $$\Delta I = (\Delta M)R^2 = \left(M\frac{\Delta\theta}{2\pi}\right)R^2$$

Let $\Delta\theta \to 0$ then $$I = \frac{MR^2}{2\pi}\int_0^{2\pi} d\theta \;=\; \frac{MR^2}{2\pi}2\pi = MR^2$$

Moment of inertia of a thin disk



Slice up the disk of radius R and total mass M and area A into N concentric rings, each of thickness Δr .

Then $\Delta M = \dfrac{M}{A}\Delta A = \dfrac{M}{\pi R^2} 2\pi r \Delta r$

Then $\Delta I = (\Delta M) r^2 = \left(\dfrac{M}{\pi R^2} 2\pi r \Delta r \right) r^2$

Let $\Delta r \to 0$ then $I = \dfrac{2M}{R^2}\displaystyle\int_0^R r^3 dr = \dfrac{2M}{R^2}\dfrac{R^4}{4} = \dfrac{1}{2} MR^2$

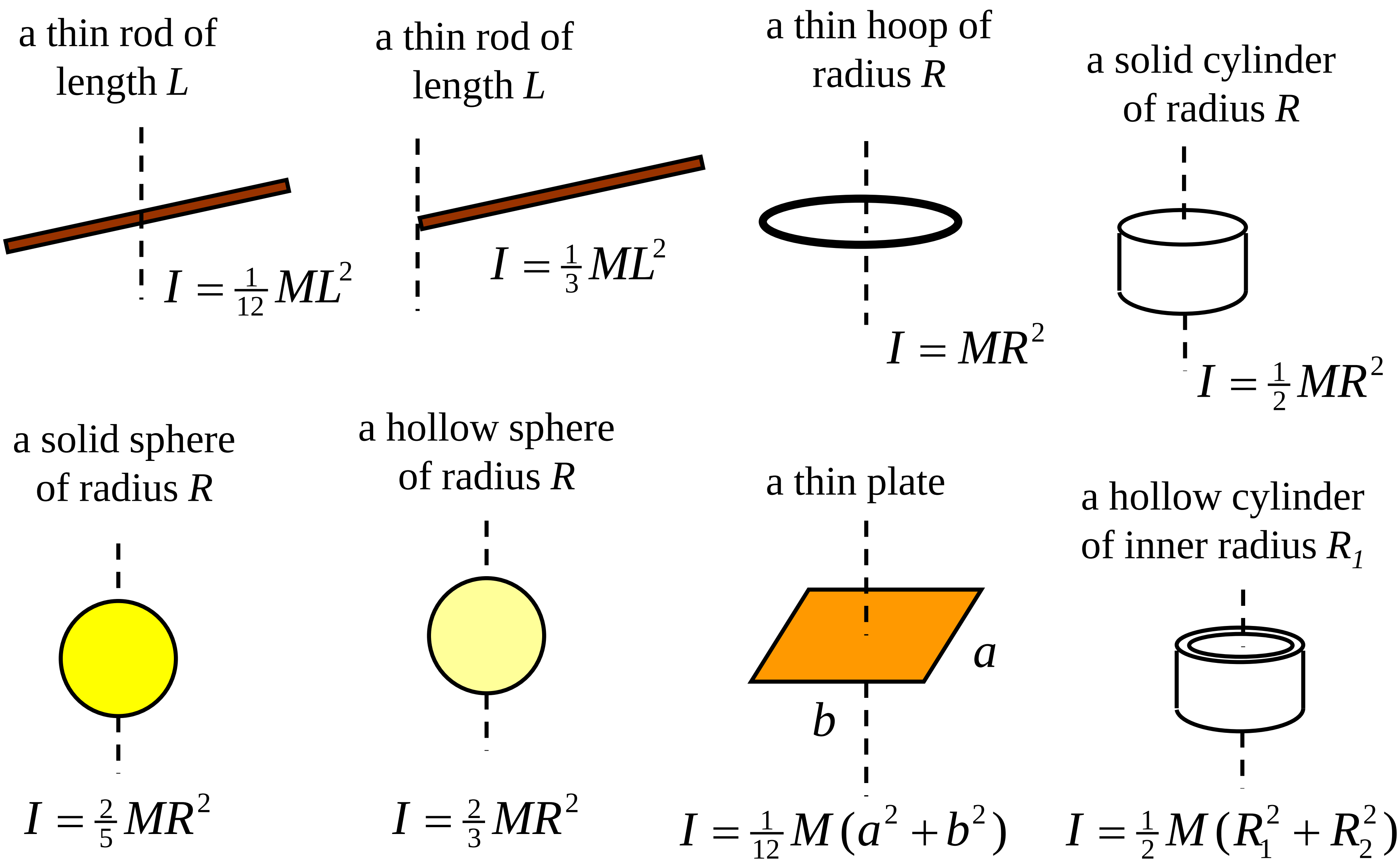
So we can see that the moment of inertia I is a measure of the rotational inertia of a body, and plays the same role for rotational motion that the mass does for translational motion. We can see that I depends both on the mass of the body and how that mass is distributed.

Moments of inertia can be calculated for any shape of body for rotation about any axis from the formula

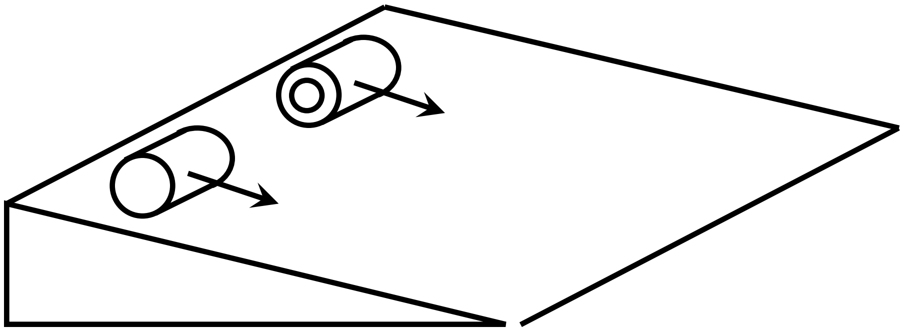
$$I = \int r^2 dm$$

Some moments of inertia for rigid bodies

- - - axis of rotation



Demonstration



A solid cylinder and a hollow cylinder are raced down an incline. If the outer radii are the same and the masses are the same, then which reaches the bottom first, and why?

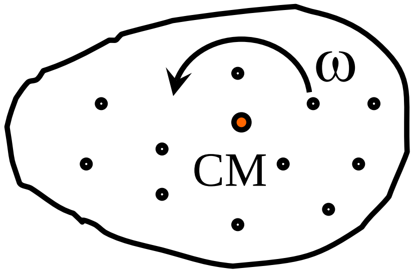
Rotational Kinetic Energy

$$K_{rot} = \tfrac{1}{2}m_1 v_1^{\,2} + \tfrac{1}{2}m_2 v_2^{\,2} + \tfrac{1}{2}m_3 v_3^{\,2} + \ldots$$

$$= \tfrac{1}{2}m_1 r_1^{\,2}\omega^2 + \tfrac{1}{2}m_2 r_2^{\,2}\omega^2 + \tfrac{1}{2}m_3 r_3^{\,2}\omega^2 + \ldots$$

$$= \tfrac{1}{2}\left(\sum mr^2\right)\ \omega^2$$

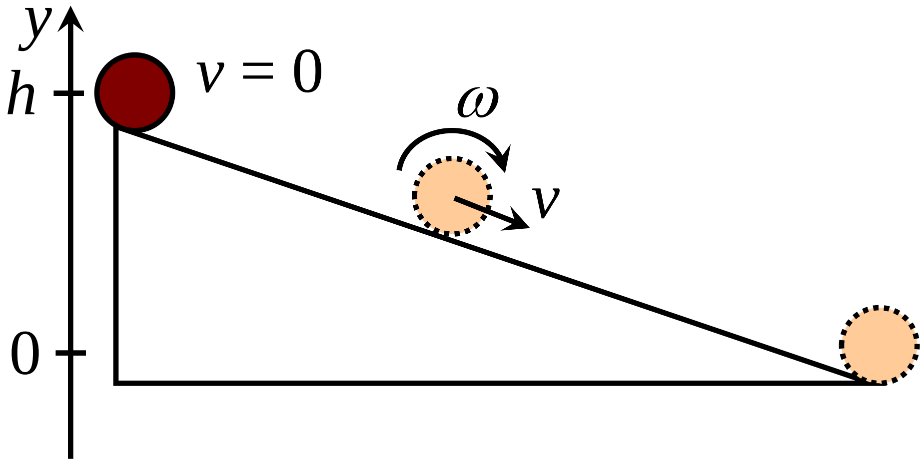
$$\therefore K_{rot} = \tfrac{1}{2}I\omega^2$$



For a <u>rigid body</u> rotating about fixed axis

A body that rotates while its CM undergoes translational motion will have K_{Total}:

$$K_{Total} = K_{trans} + K_{rot} \;=\; \tfrac{1}{2}Mv^2_{CM} \;+\; \tfrac{1}{2}I_{CM}\,\omega^2$$



Consider a solid sphere (mass m and radius r_0) rolling without slipping down an incline of height h.

Total energy at height y is $K_{trans} + K_{rot} + U_g \quad = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy$

Total energy at the top ($v = 0$, $\omega = 0$) $\quad = \; m\,g\,h$

Total energy at the bottom $\; = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

I for a solid sphere for axis of rotation through centre = $\frac{2}{5}mr_0{}^2$

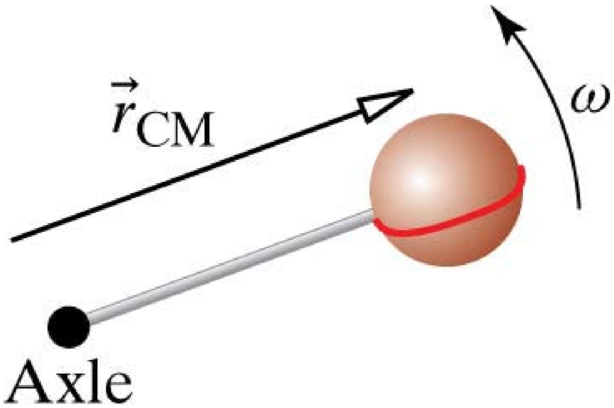
and since $v = v_{\text{tangential}} = r_0\omega$

Conservation of energy: $\quad mgh \; = \; \frac{1}{2}mv^2 \; + \; \frac{1}{2}(\frac{2}{5}mr_0{}^2)\dfrac{v^2}{r_0{}^2}$

... giving ... $\quad v = \sqrt{\frac{10}{7}gh}$

If the ball slipped (no friction): $v = \sqrt{2gh} \quad > \quad v_{roll} = \sqrt{\frac{10}{7}gh}$

Rigid rotation about a point which is not the centre of mass



A object with known I_{CM} is connected to a low mass rod and rotates about an axle.

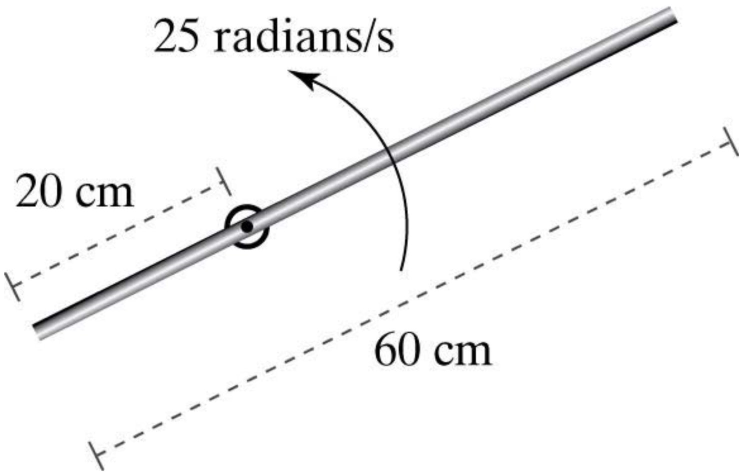
$$K_{trans} = \tfrac{1}{2} M v_{CM}^2 = \tfrac{1}{2} M \left(\omega r_{CM}\right)^2 = \tfrac{1}{2}\left(M r_{CM}^2\right)\omega^2$$

$$K_{Total} = K_{trans} + K_{rot} = \tfrac{1}{2}\left(M r_{CM}^2\right)\omega^2 + \tfrac{1}{2} I_{CM}\omega^2$$

$$= \tfrac{1}{2}\left(M r_{CM}^2 + I_{CM}\right)\omega^2$$

$I = M r_{CM}^2 + I_{CM}$ "**parallel axis theorem**"

Example



A thin rod of mass 140 g and 60 cm long rotates at an angular speed of 25 radians per second about an axle that is 20 cm from one end.

$$K_{Total} = K_{trans} + K_{rot} = \tfrac{1}{2}\left(Mr_{CM}^2 + I_{CM}\right)\omega^2$$

$$= \tfrac{1}{2}M\left(r_{CM}^2 + \tfrac{1}{12}L^2\right)\omega^2$$

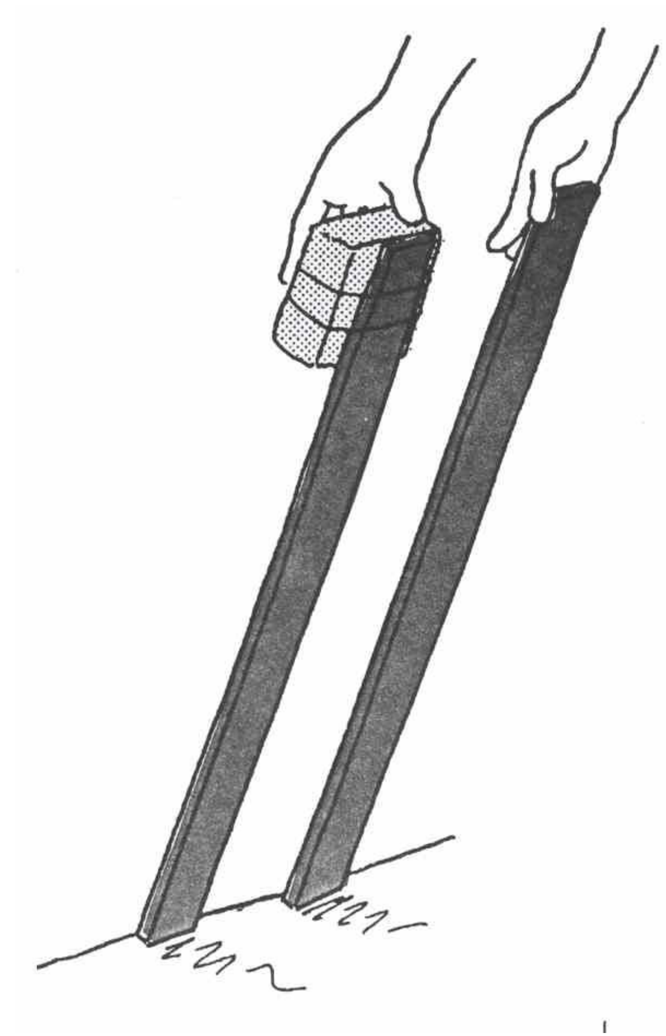
$r_{CM} = 0.1\ \text{m}$

Get $K_{Total} = 1.75\ \text{J}$

1	2	3	4	5

A pair of upright metre sticks, with lower ends against a wall, are allowed to fall to the floor.
One is bare, and the other has a heavy weight attached to its upper end.
The stick to hit the floor first is the ...

(A) bare stick
(B) weighted stick
(C) …both the same

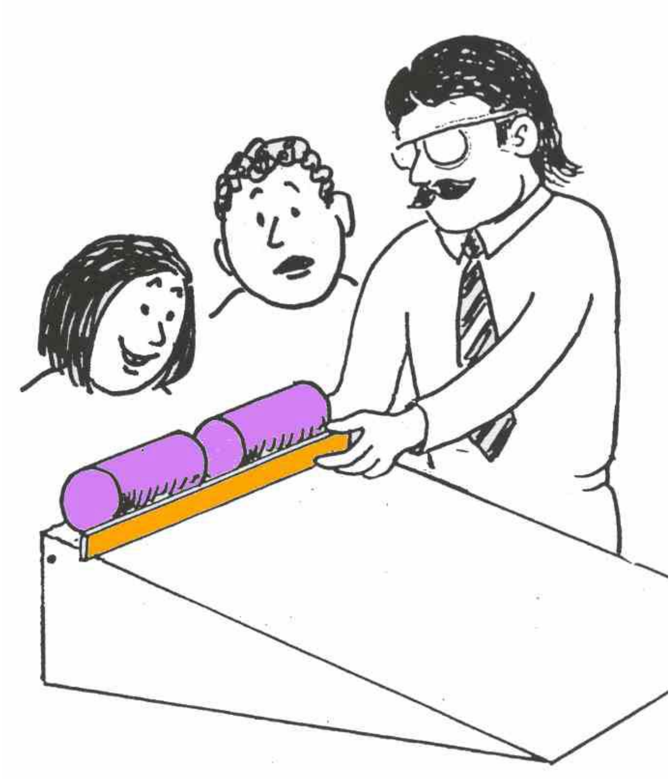


1	2	3	4	5

Roll a pair of identical cans of carbonated cooldrink down an incline. You won't be surprised to find they roll at the same rate. Now shake one of them so bubbles form inside, then repeat the experiment.

Now...

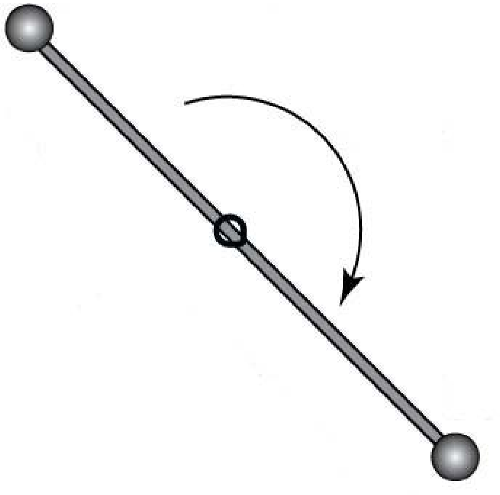
(A) the shaken can wins the race
(B) the shaken can loses the race
(C) both cans still roll together.



1	2	3	4	5

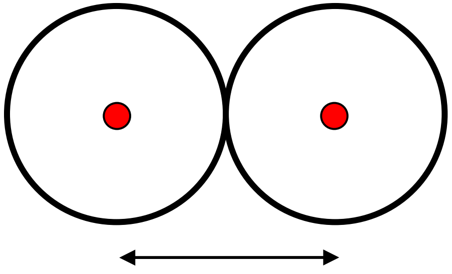
Two balls of mass 0.7 kg are connected by a low mass rigid rod of length 0.4 m. The object rotates around a pivot at its center, with angular speed 13 radians/s. What is the rotational kinetic energy of this object?

(1) 484 J
(2) 4.73 J
(3) 2.37 J
(4) 0.056 J
(5) 0 J



1	2	3	4	5

A diatomic molecule such as molecular nitrogen (N_2) consists of two atoms each of mass *M*, whose nuclei are a distance *d* apart. What is the moment of inertia of the molecule about its center of mass?



(1) Md^2

(2) $2Md^2$

(3) $\frac{1}{2}Md^2$

(4) $\frac{1}{4}Md^2$

(5) $4Md^2$

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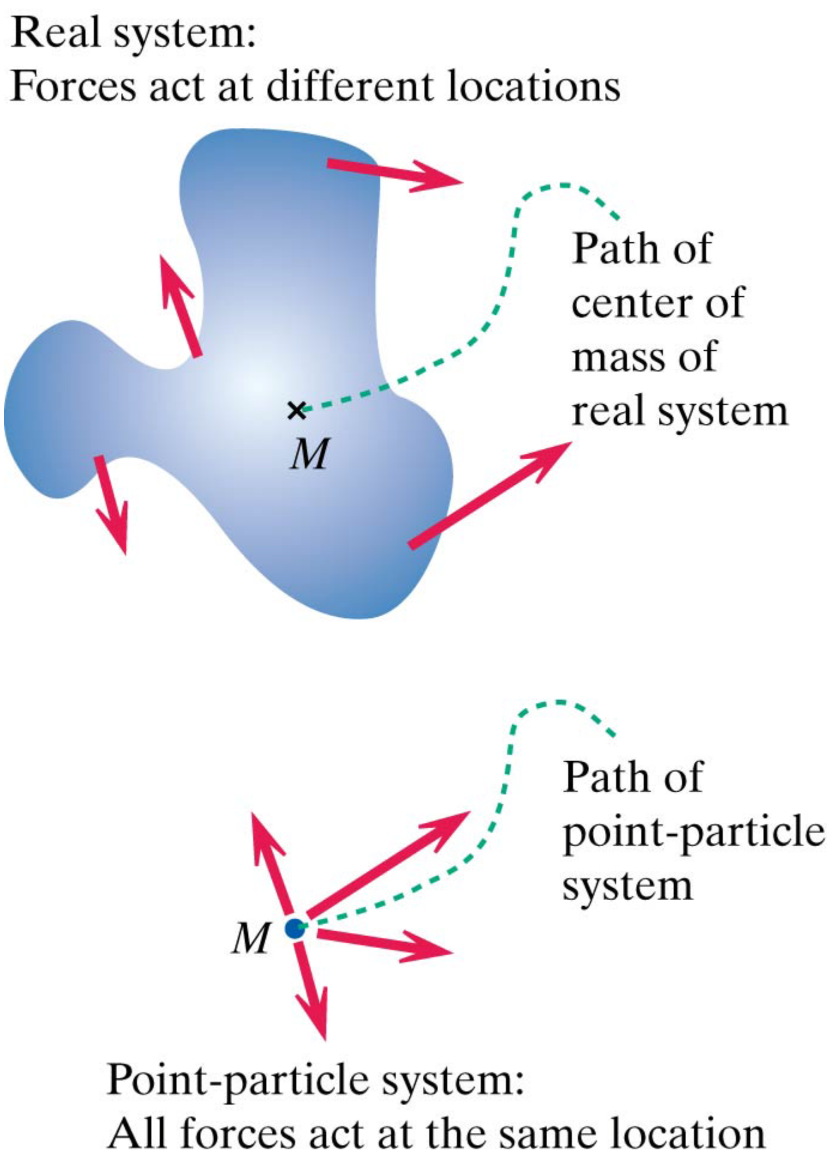
The point particle system



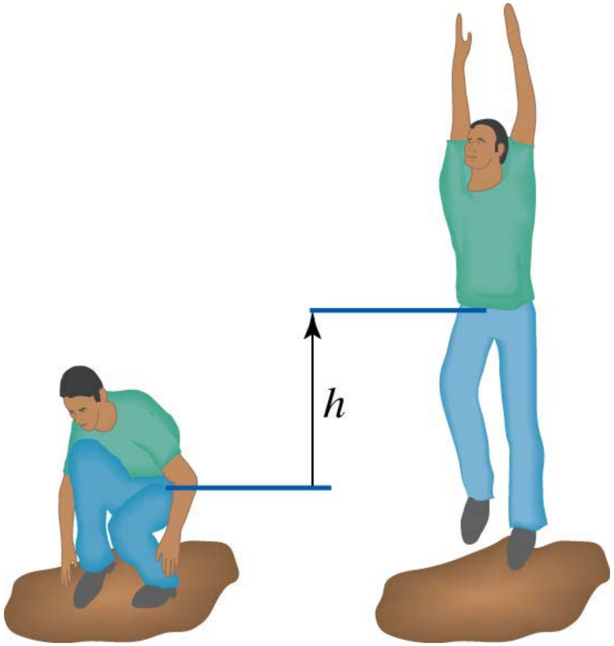
A point particle system has the same mass as a real multiparticle system, but all its mass is concentrated into a point particle located at the centre of mass of the real system.

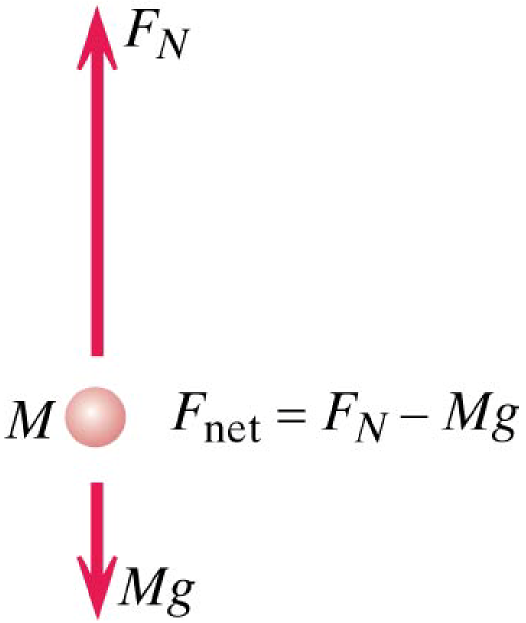
For a point particle system:

$$\frac{d\vec{\mathbf{p}}_{sys}}{dt} = \vec{\mathbf{F}}_{net}$$

$$\Delta K_{trans} = \int_i^f \vec{\mathbf{F}}_{net} \bullet d\vec{\mathbf{r}}_{CM}$$

The point particle system : jumping up





$$\Delta K_{trans} = \frac{1}{2} M v_{CM}^2 - 0 = \int_i^f \vec{\mathbf{F}}_{net} \bullet d\vec{\mathbf{r}}_{CM} = \left(F_N - mg\right) h$$

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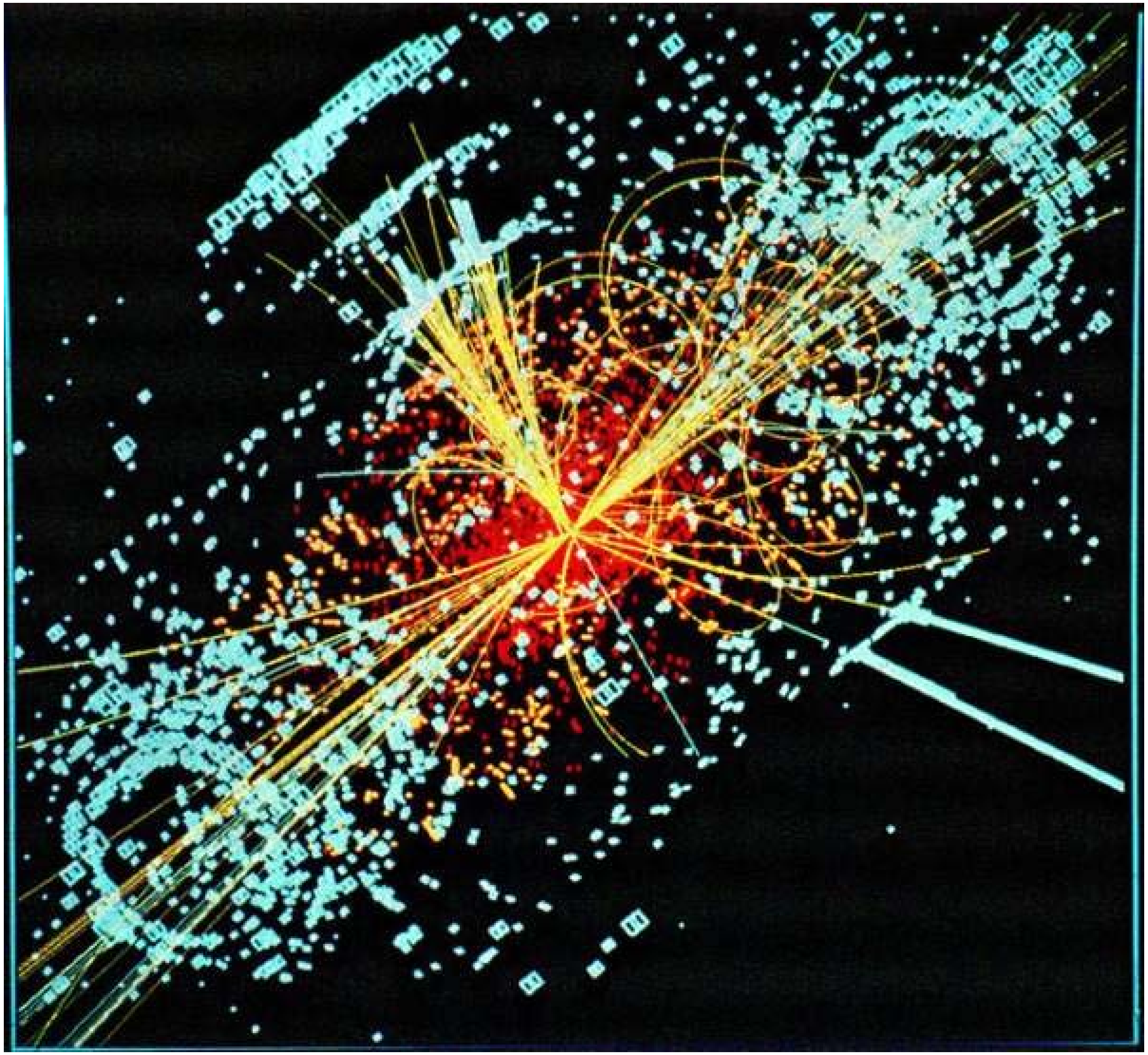
Chapter 10

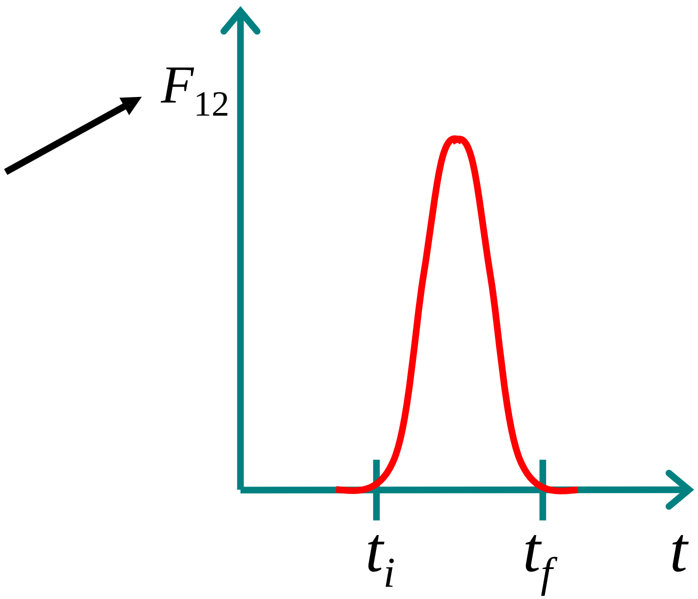
Collisions



Internal interactions in collisions

A collision is when two bodies interact over a short time interval. The forces that the bodies exert on each other are usually so strong during the collision that all forces acting on a body may be ignored.

During a collision between two bodies (1 and 2), the contact force exerted by one body on the other jumps from zero to a very large value and then abruptly drops to zero again.



The time interval $\Delta t = t_f - t_i$ is usually very small.

Note that $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ for the collision

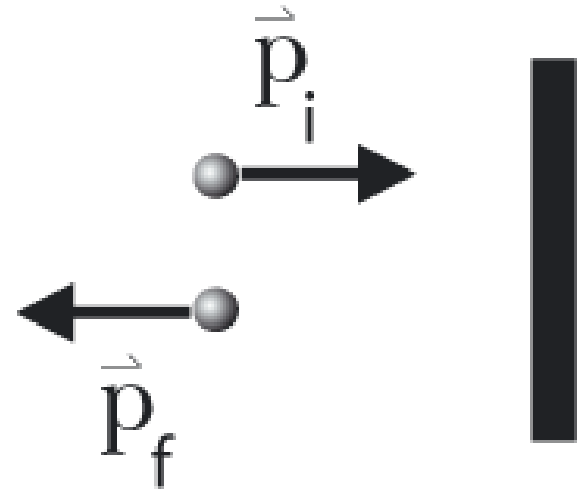
1	2	3	4	5

Whenever an interaction occurs in a system, forces occur in equal and opposite pairs.
Which of the following do not always occur in equal and opposite pairs?

(A) Impulses
(B) Accelerations
(C) Momentum changes

1	2	3	4	5

A ball bounces off a wall.
mass 0.1 kg
$v_i = 6$ m/s
$v_f = 6$ m/s

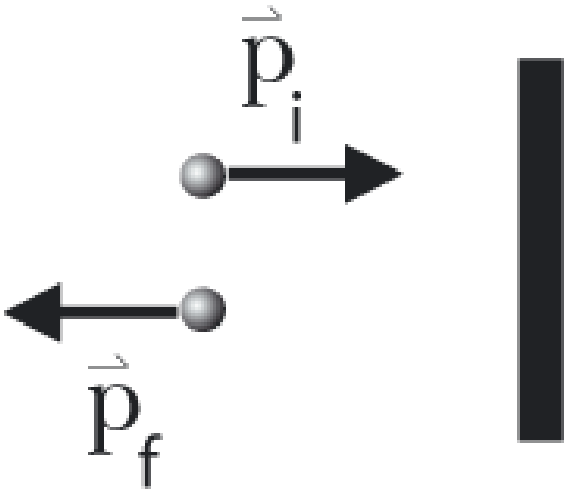


What is the change in p_x of the ball?

(1) 0 kg m/s
(2) –1.2 kg m/s
(3) +1.2 kg m/s
(4) +0.6 kg m/s
(5) –0.6 kg m/s

1	2	3	4	5

A ball bounces off a wall.
mass 0.1 kg
$v_i = 6$ m/s
$v_f = 6$ m/s



What is the change in p_x of the Earth?

(1) 0 kg m/s
(2) –1.2 kg m/s
(3) +1.2 kg m/s
(4) +0.6 kg m/s
(5) –0.6 kg m/s

Elastic and inelastic collisions

Elastic collision: no change in the internal energy of the interacting objects ... i.e. no thermal energy rise, no lasting deformations, no new vibrations, etc.

$$\Delta E_{\text{int}} = 0 \quad \text{therefore} \quad K_f = K_i$$

inelastic collision: $\Delta E_{\text{int}} \neq 0 \quad \text{and} \quad K_f \neq K_i$

maximally inelastic collision: Maximum energy dissipation and the objects stick together

In all cases above, momentum still conserved: $\Delta \vec{\mathbf{p}}_{sys} = 0$

since $\vec{\mathbf{F}}_{net} = 0$

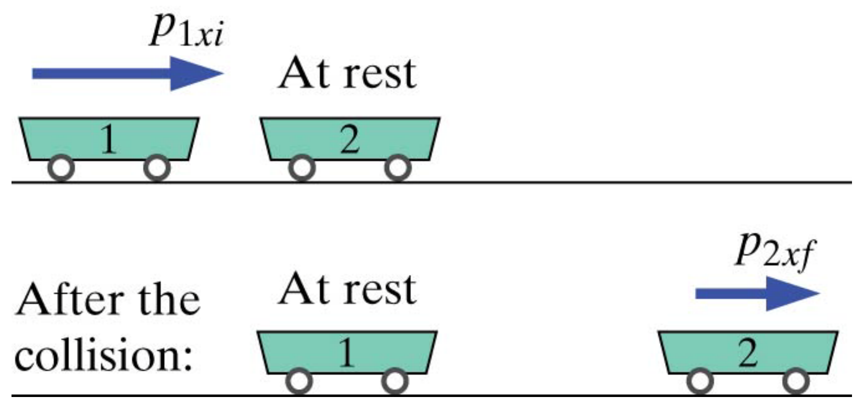
A head-on collision of equal masses in 1D

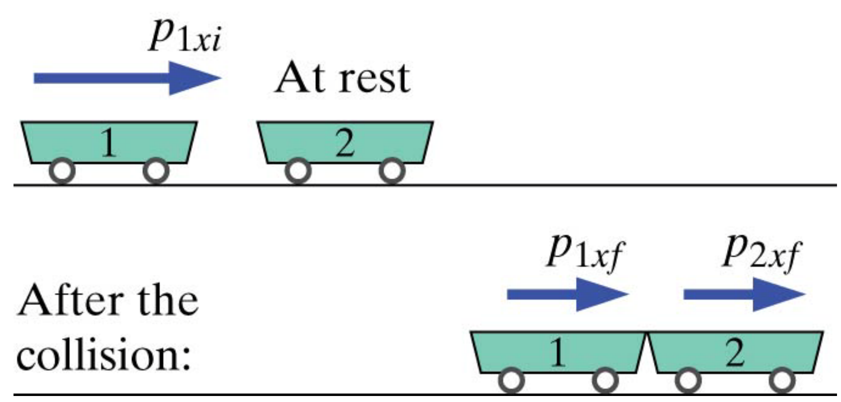
Two extreme cases:

elastic collision:

$$p_{2xf} = p_{1xi}$$

$$K_{2f} = K_{1i}$$





Maximally inelastic collision:

$$p_{2xf} + p_{1xf} = p_{1xi}$$

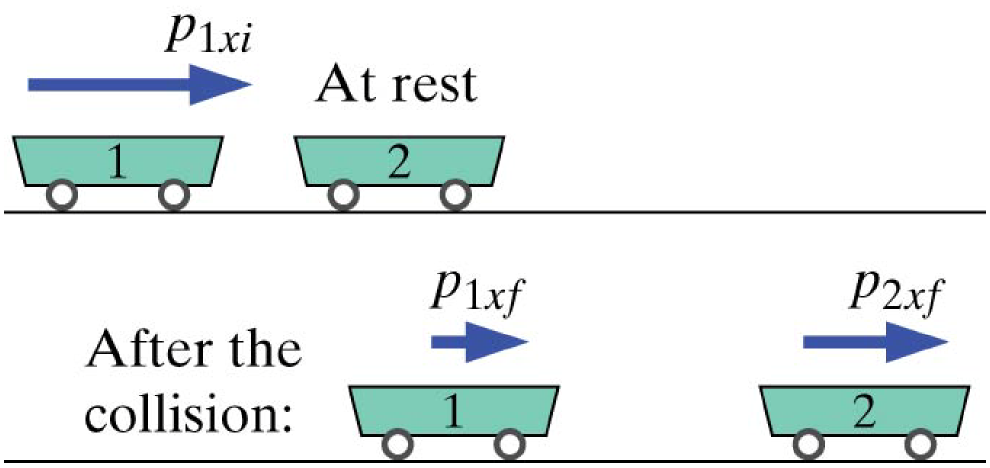
$$K_{1f} + K_{2f} = \tfrac{1}{2} K_{1i}$$

An elastic collision of unequal masses in 1D

$$\vec{\mathbf{p}}_{1i} = \vec{\mathbf{p}}_{2f} + \vec{\mathbf{p}}_{1f}$$

$$K_{1i} = K_{1f} + K_{2f}$$

$$\therefore \frac{p_{1i}^2}{2m_1} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2}$$



... solve to find ...

$$p_{1f} = \frac{m_1 \pm m_2}{m_1 + m_2} p_{1i}$$

$$p_{2f} = \frac{2m_1}{m_1 + m_2} p_{1i}$$

M&I
3E 10.4

A head-on collision of unequal masses

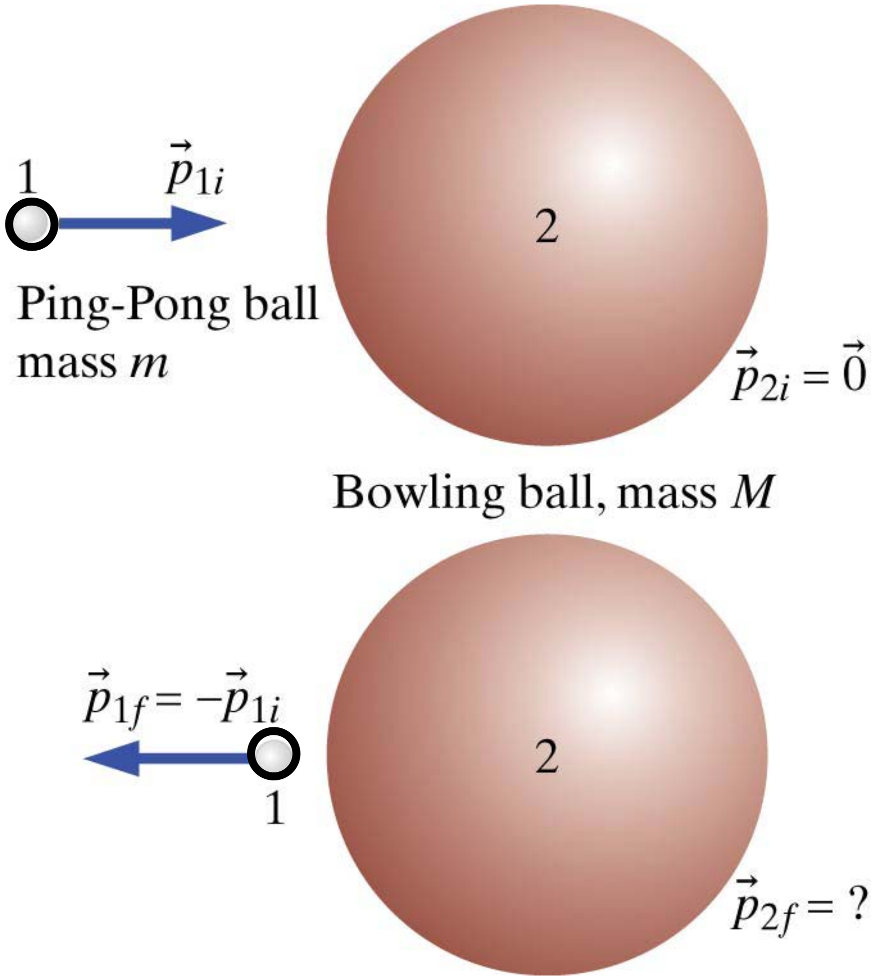
$$\vec{\mathbf{p}}_{1i} + \vec{\mathbf{p}}_{2i} = \vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f}$$

... find ...

$$\vec{\mathbf{p}}_{2f} = 2\vec{\mathbf{p}}_{1i}$$

and

$$v_{2f} = 2\left(\frac{m_1}{m_2}\right)v_{1i}$$



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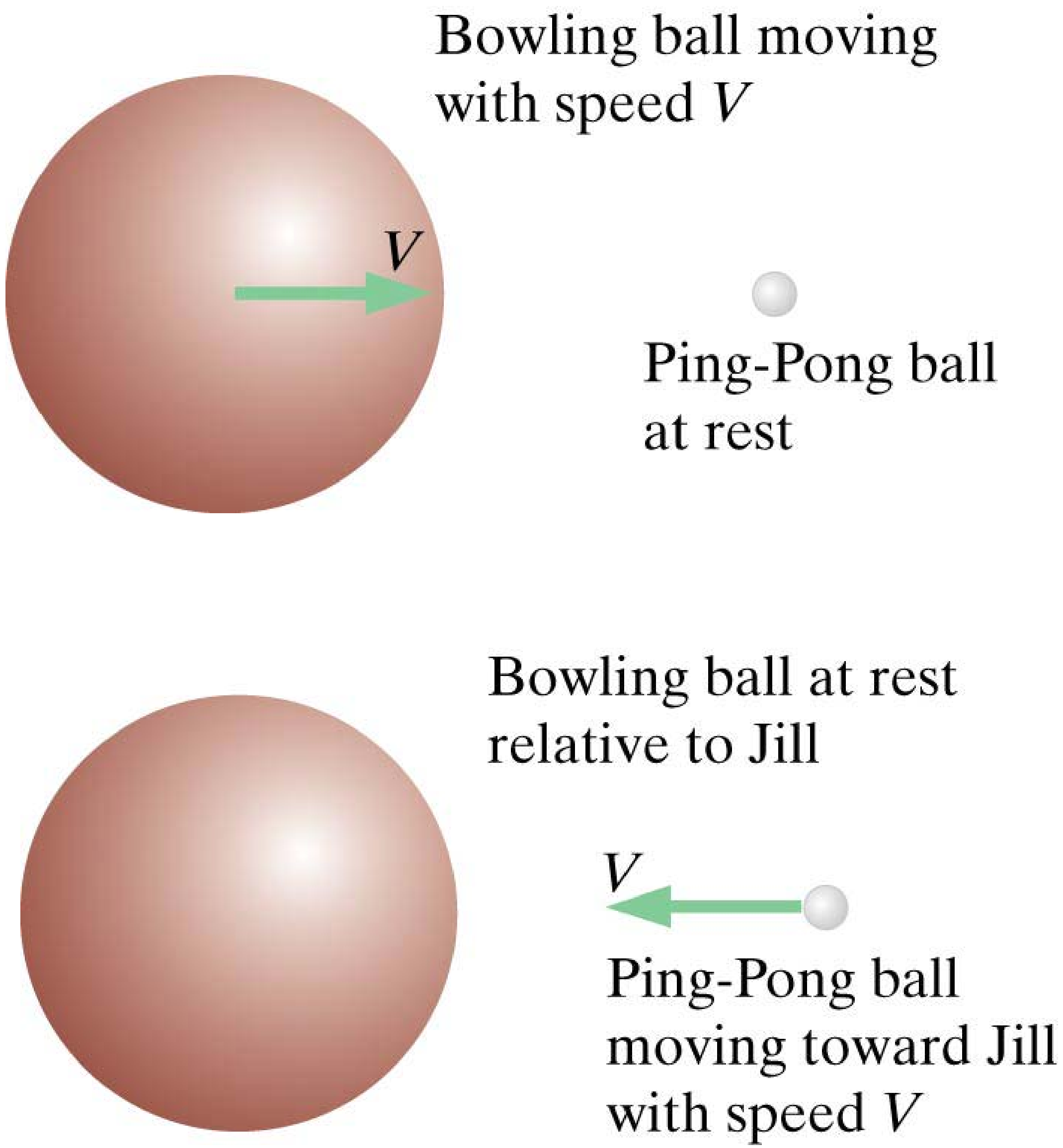
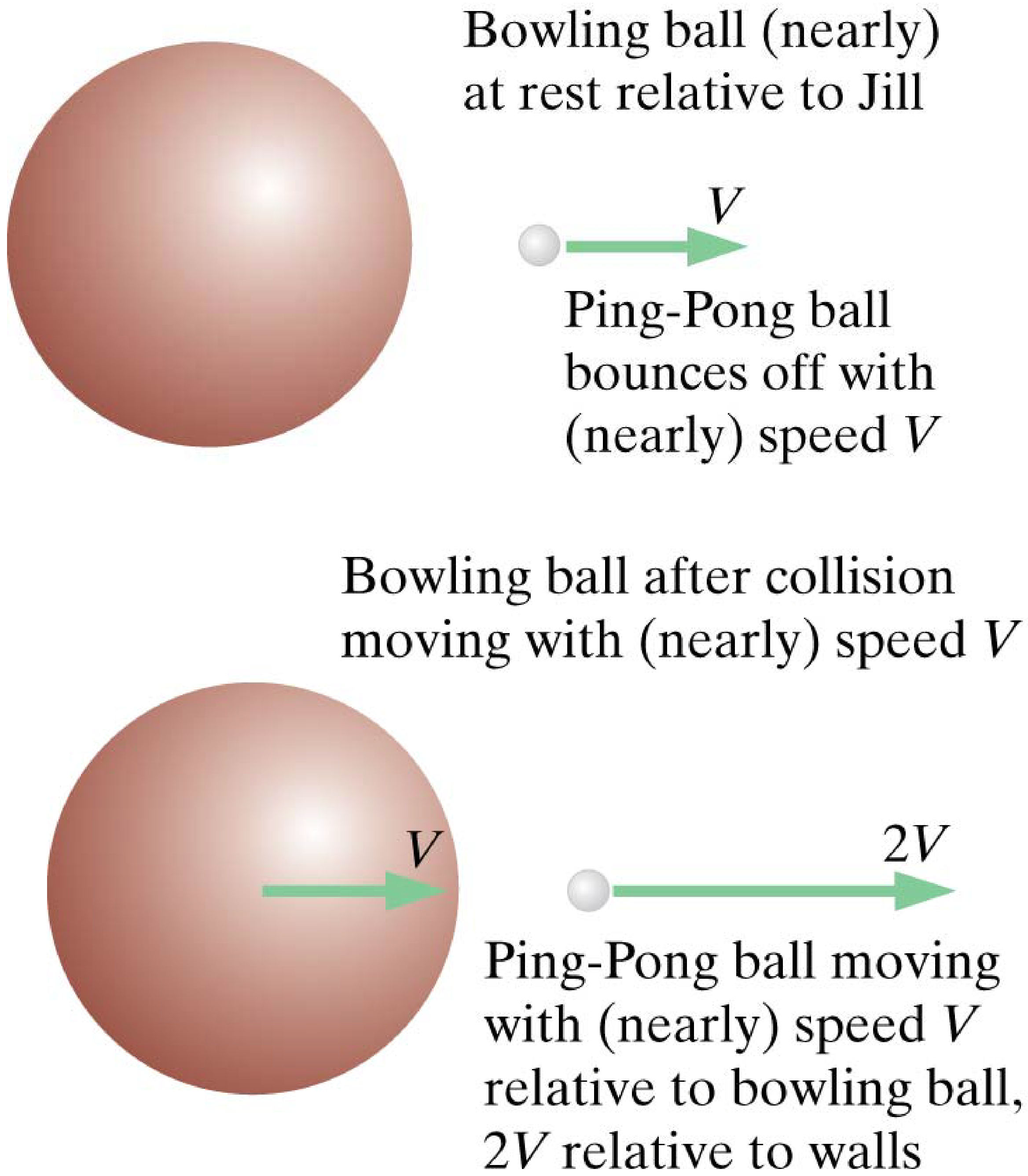
3E 10.5

Frame of reference

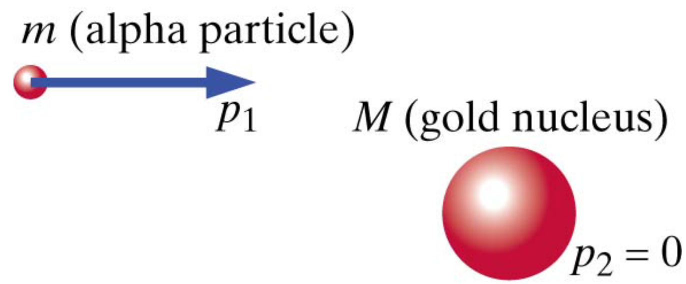




See:
Reference_frames.py

M&I
3E 10.6

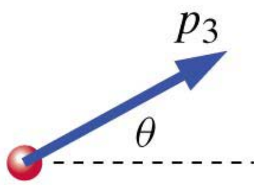
Scattering: collisions in 2D and 3D



$$\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4$$

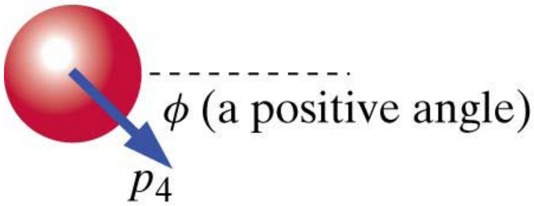
$$p_1 = p_3 \cos\theta + p_4 \cos\phi$$

$$0 = p_3 \sin\theta - p_4 \sin\phi$$



$$K_1 = K_3 + K_4$$

$$\frac{p_1^2}{2m_1} = \frac{p_3^2}{2m_1} + \frac{p_4^2}{2m_2}$$



Elastic scattering: identical particles, one at rest

$$\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4$$

hence

$$\vec{\mathbf{p}}_1 \bullet \vec{\mathbf{p}}_1 = \left(\vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4\right) \bullet \left(\vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4\right)$$

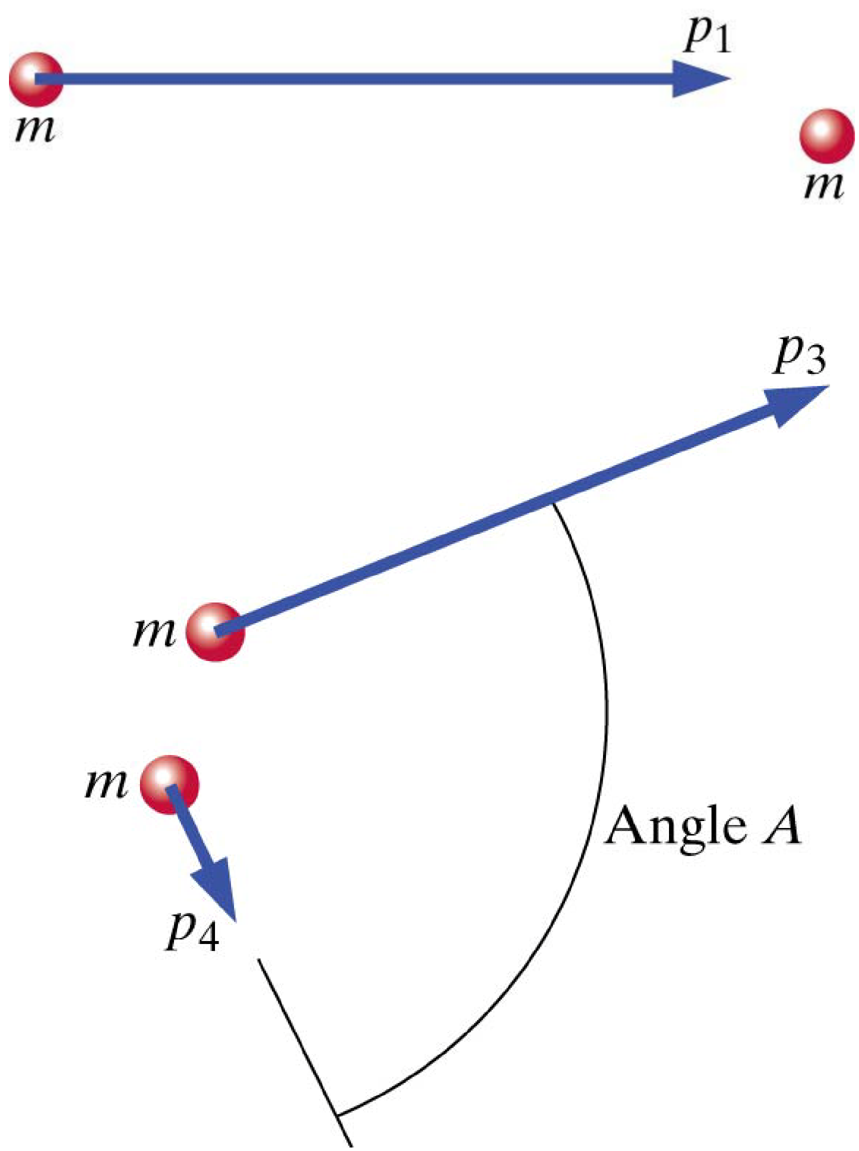
$$p_1^2 = p_3^2 + p_4^2 + 2\,p_3\,p_4 \cos A$$

... divide by $2m$ ...

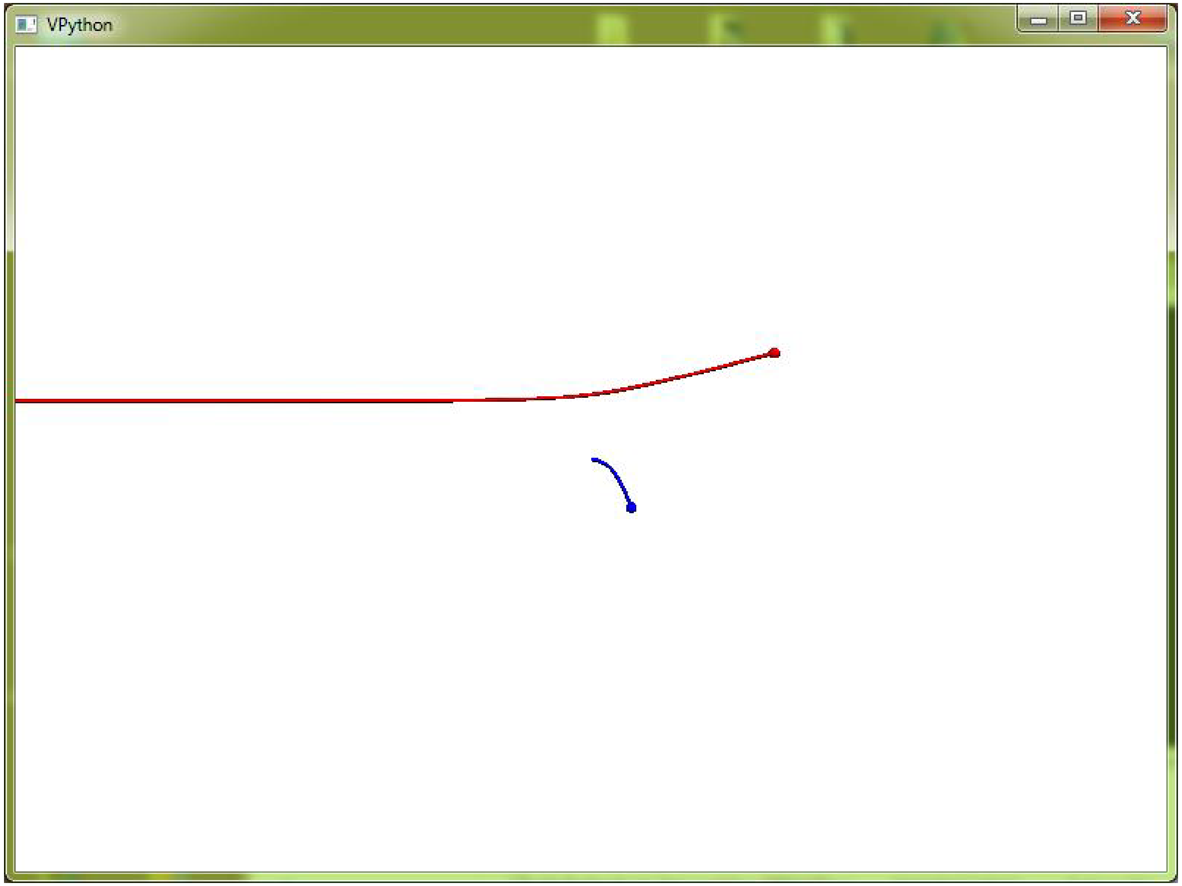
$$\frac{p_1^2}{2m} = \frac{p_3^2}{2m} + \frac{p_4^2}{2m} + \frac{2\,p_3\,p_4 \cos A}{2m}$$

$$\therefore K_1 = K_3 + K_4 + \frac{2\,p_3\,p_4 \cos A}{2m}$$

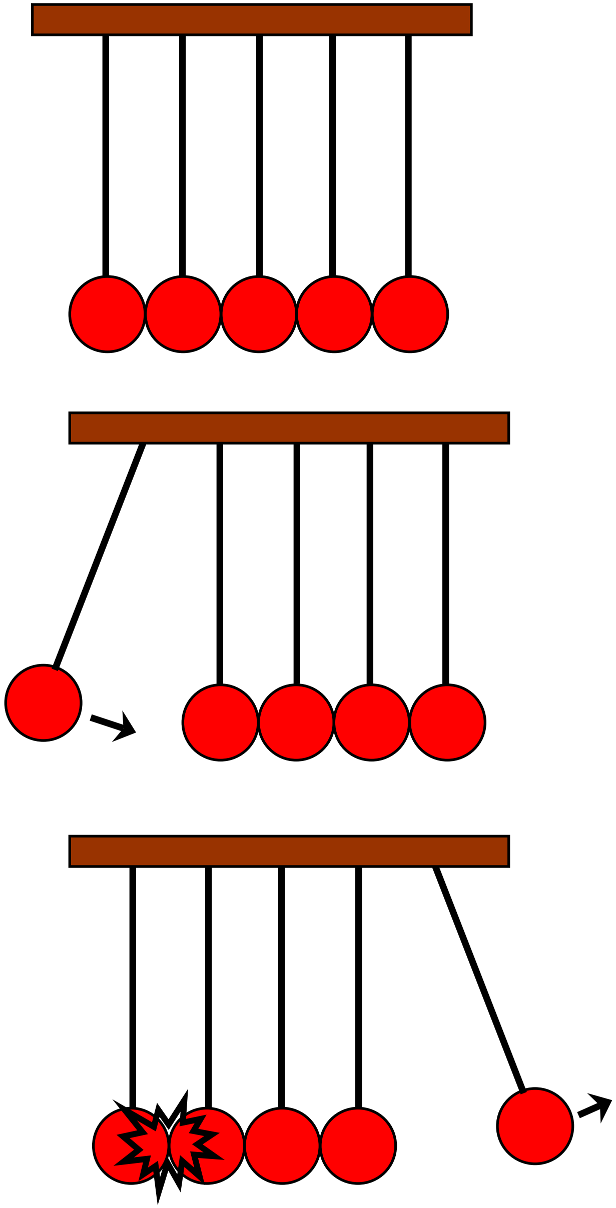
$$\therefore \cos A = 0 \qquad \text{or} \qquad A = 90°$$



See alpha_on_alpha.py
alpha_on_electron.py
alpha_on_gold.py



Demonstration: “Newton’s cradle”



Five spheres of equal mass hang at the end of strings.

One ball is pulled back and released to strike the other stationary balls...

... and one ball flies off on the other side.

Kinetic energy is conserved so the collision is elastic.
But why don’t two balls fly out with half the speed?

If two balls fly off with half the speed of the incoming ball, then that would conserve momentum, since

$$mv = \tfrac{1}{2}mv + \tfrac{1}{2}mv$$

But it wouldn't conserve kinetic energy.

The incoming ball has kinetic energy $\tfrac{1}{2}mv^2$

and two outgoing balls with half the speed have kinetic energy:

$$\tfrac{1}{2}m(\tfrac{1}{2}v)^2 + \tfrac{1}{2}m(\tfrac{1}{2}v)^2 = \tfrac{1}{4}mv^2 \neq \tfrac{1}{2}mv^2$$

1	2	3	4	5

Two lead bricks moving in the $+x$ and $-x$ directions, each with kinetic energy K, smash into each other and come to a stop. What happened to the energy?

(1) The kinetic energy of the system remained constant.

(2) The kinetic energy changed into thermal energy.

(3) The total energy of the system decreased by an amount $2K$.

(4) Since the blocks were moving in opposite directions, the initial kinetic energy of the system was zero, so there was no change in energy.

1 | 2 | 3 | 4 | 5

A squishy clay ball collides in midair with a baseball, and sticks to the baseball.
The stuck-together objects keep moving.

Initial kinetic energies: $K_i = K_{1clay} + K_{1baseball}$
Final kinetic energy of stuck clay+ball: $K_f = K_{(clay+ball)}$

Which must be true for this collision?

(1) $K_f = K_i$
(2) $K_f > K_i$
(3) $K_f < K_i$

1	2	3	4	5

Which of the following is a property of all “elastic” collisions?

(1) The colliding objects interact through springs.

(2) The kinetic energy of one of the objects doesn’t change.

(3) The total kinetic energy is constant at all times
- before, during, and after the collision.

(4) The total kinetic energy after the collision is equal to the total kinetic energy before the collision.

(5) The elastic spring energy after the collision is greater than the elastic spring energy before the collision.

1	2	3	4	5

Which of the following is true for both “elastic” and “inelastic” collisions?

(1) The internal energy of the system after the collision is different from what it was before the collision.

(2) The total momentum of the system doesn’t change.

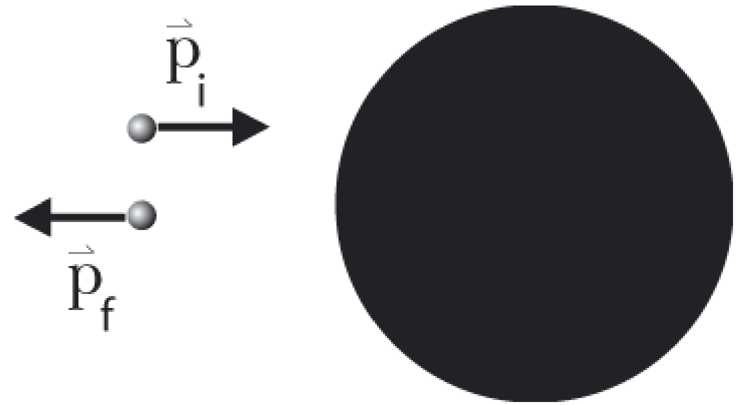
(3) The total kinetic energy of the system doesn’t change.

1	2	3	4	5

A ping-pong ball bounces elastically off a bowling ball which is initially at rest.

After the collision the ping-pong ball's kinetic energy is K_p.

What is the kinetic energy of the bowling ball?



(1) K_p
(2) $-K_p$
(3) much greater than K_p
(4) negligibly small (nearly zero)

1	2	3	4	5

A ball of mass m_1 hits a stationary target of mass m_2 head-on. The total initial and final kinetic energies are the same. Which of the following statements is *false*?

(1) If $m_1 << m_2$, the momentum of the ball hardly changes.

(2) If $m_1 < m_2$, the ball bounces straight back.

(3) If $m_1 < m_2$, the ball bounces straight back with less kinetic energy than it had originally.

(4) If $m_1 >> m_2$, the ball keeps going without change of direction.

1	2	3	4	5

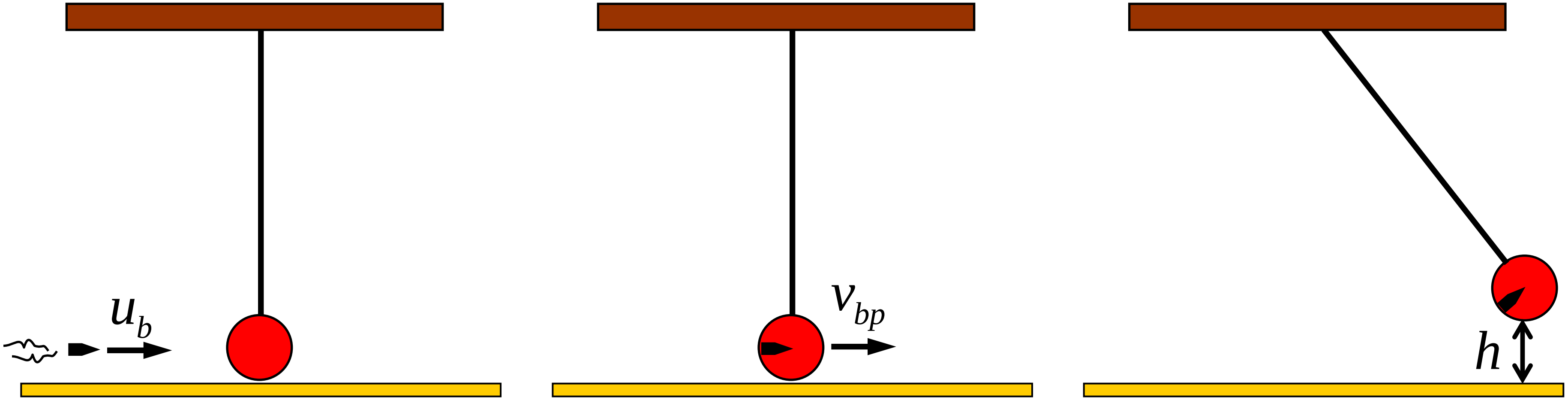
A bullet of mass *m* traveling horizontally at a very high speed *v* embeds itself in a block of mass *M* that is sitting at rest on a nearly frictionless surface.

Which must be true for the system of bullet + block in this collision?

(1) $K_f = K_i$
(2) $K_f > K_i$
(3) $K_f < K_i$

Demonstration: The ballistic pendulum

1. A bullet is fired into a pendulum, which is initially at rest.
2. The bullet lodges in the pendulum, which moves to the right.
3. The bullet and pendulum swing to a height *h*.



$$m_b u_b + m_p u_p = (m_b + m_p) v_{bp}$$

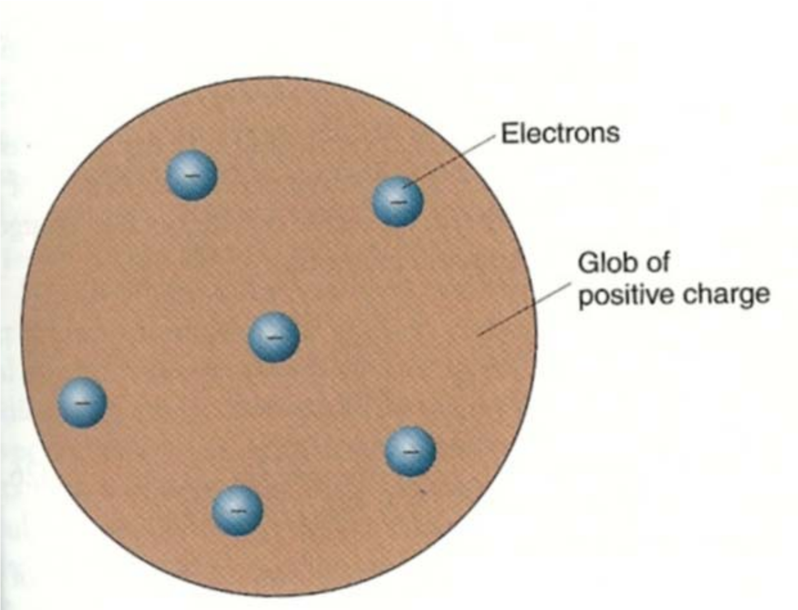
$$m_b u_b + 0 = (m_b + m_p) v_{bp}$$

$$\tfrac{1}{2}(m_b + m_p) v_{bp}^2 = (m_b + m_p) g h$$

Discovering the nucleus inside atoms

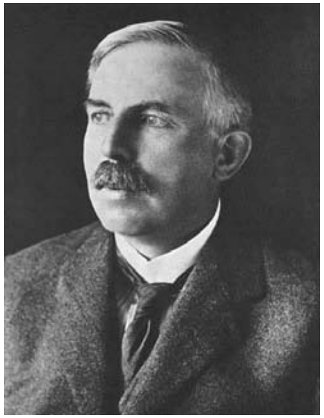
JJ Thomson (Nobel Prize in Physics, 1906)
"Plum pudding" model.

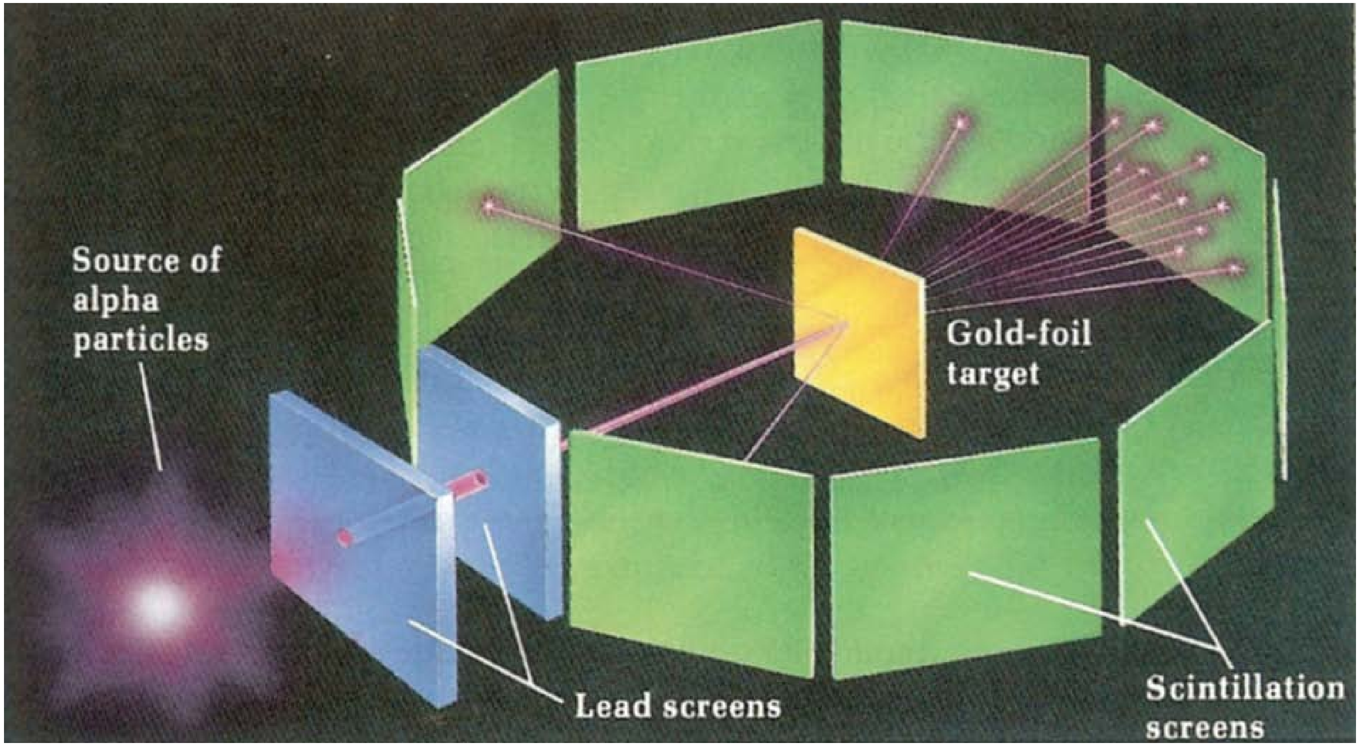
Poor agreement with experiment.



The Rutherford model

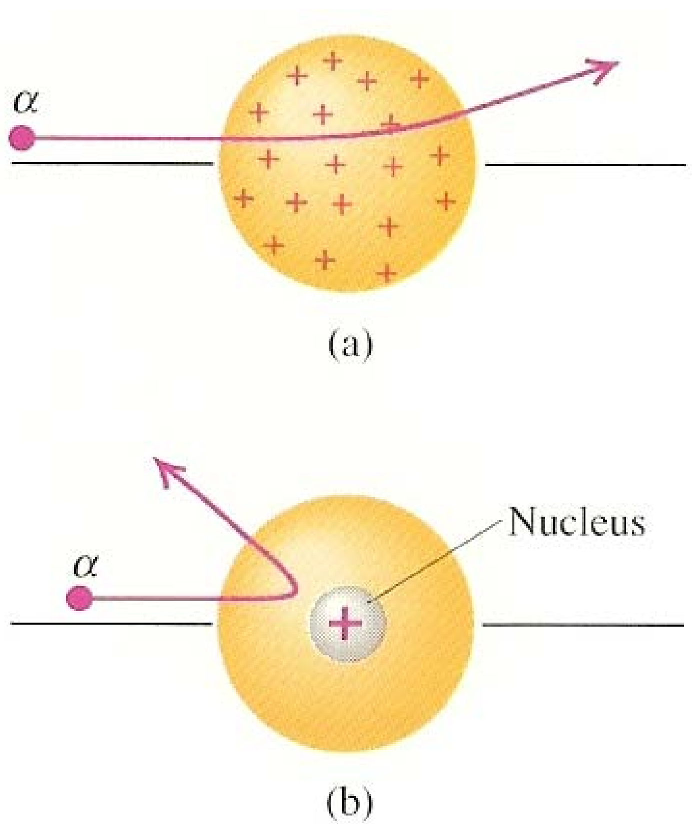
Ernest Rutherford
(Nobel Prize in Chemistry, 1908)



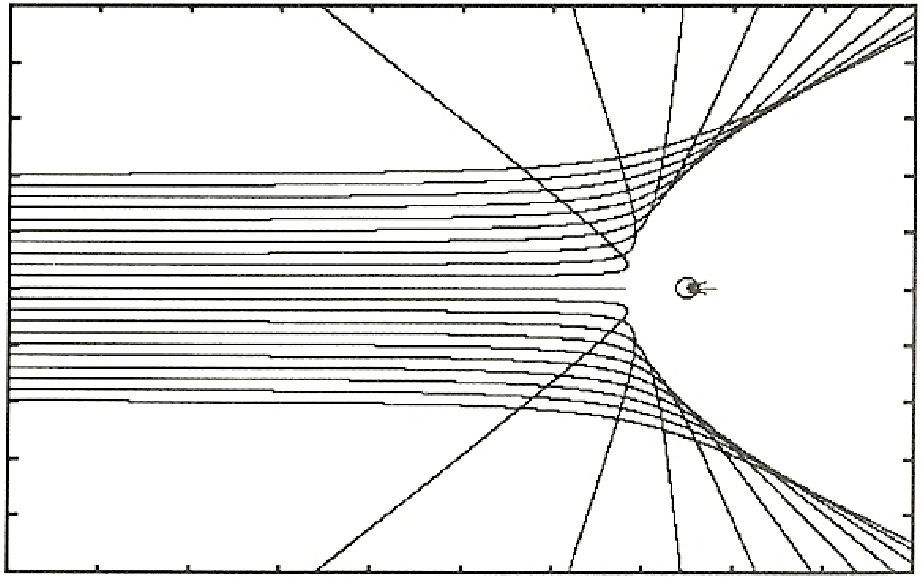


Scattering of a ^{4}He nucleus off …

(a) Thomson's atom



(b) Rutherford's atom



Simulation of Rutherford scattering off a gold nucleus

See Rutherford.py

Rutherford: “It was the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15 inch shell at a piece of tissue paper and it came back and hit you.”

Rutherford model (1911): The atom has a small hard central core (nucleus) where all the positive charge is concentrated. The negative charge inhabitants the nearly empty space around the nucleus.

Thus most of the alpha particles migrate through the gold foil with some or no (Coulomb) interaction, but some will experience a “head-on” collision with a nucleus and return in a backwards direction.

But there were still unanswered questions …

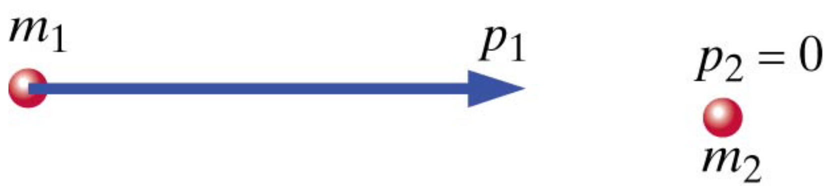
… why does the nucleus (all positive charge) not fly apart due to Coulomb repulsion?

… why do the negative charges not radiate energy, spiral inwards and collapse into the nucleus due to Coulomb attraction?

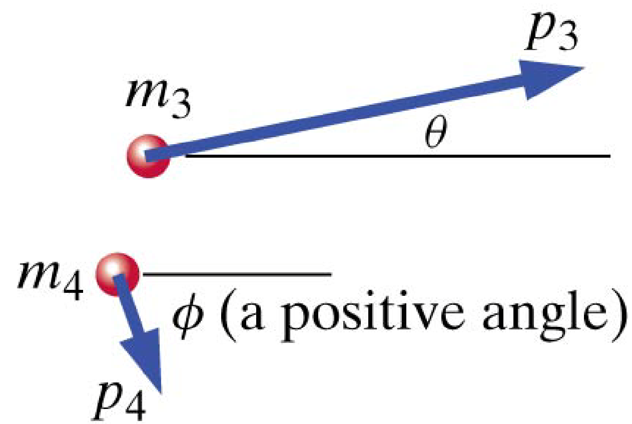
… the model did also not explain existing experimental observations.

M&I
3E 10.9

Relativistic momentum and energy (collisions in a particle accelerator)



$$\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_4$$



$$p_1 = p_3 \cos\theta + p_4 \cos\phi$$

$$0 = p_3 \sin\theta - p_4 \sin\phi$$

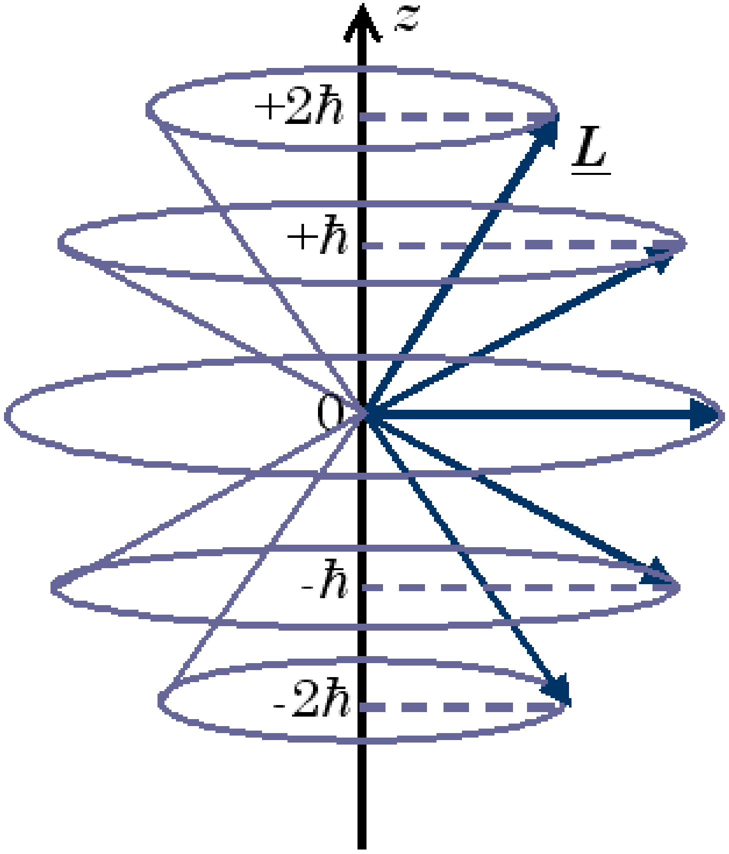
$$E_1 + m_2 c^2 = E_3 + E_4$$

$$\sqrt{(p_1 c)^2 + (m_1 c^2)^2} + m_2 c^2 = \sqrt{(p_3 c)^2 + (m_3 c^2)^2} + \sqrt{(p_4 c)^2 + (m_4 c^2)^2}$$

M&I

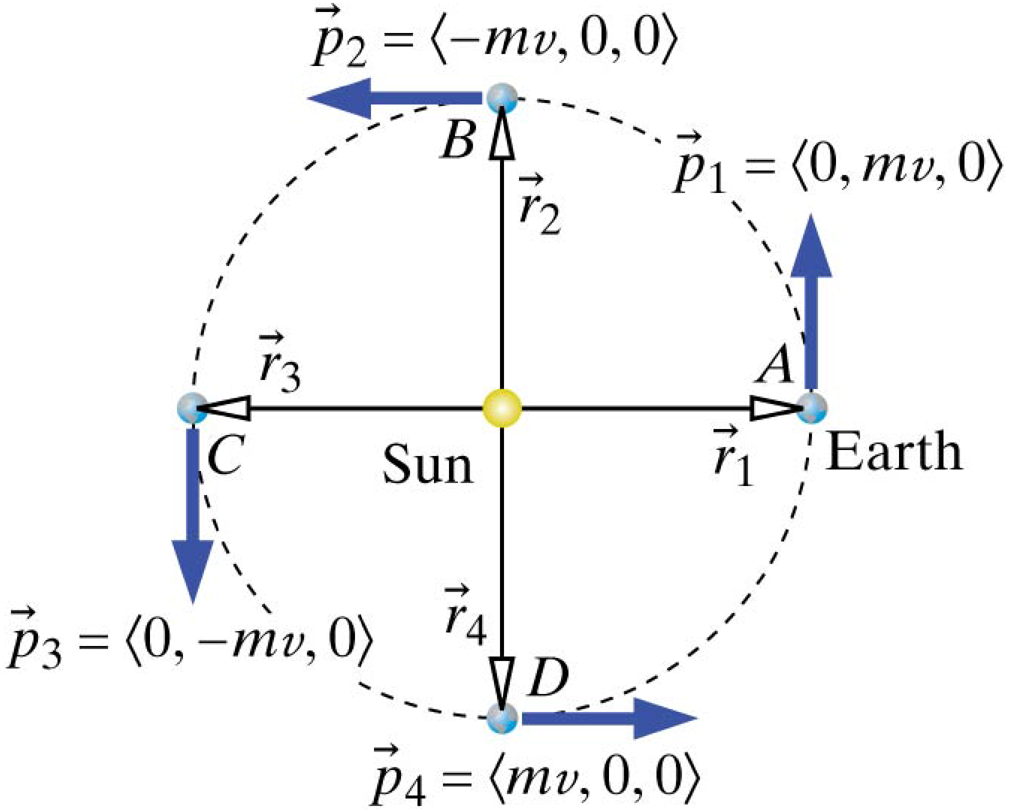
Chapter 11

Angular Momentum



Translational angular momentum

Consider the momentum $\vec{\mathbf{p}}$ of the Earth at four positions as it moves around the Sun.

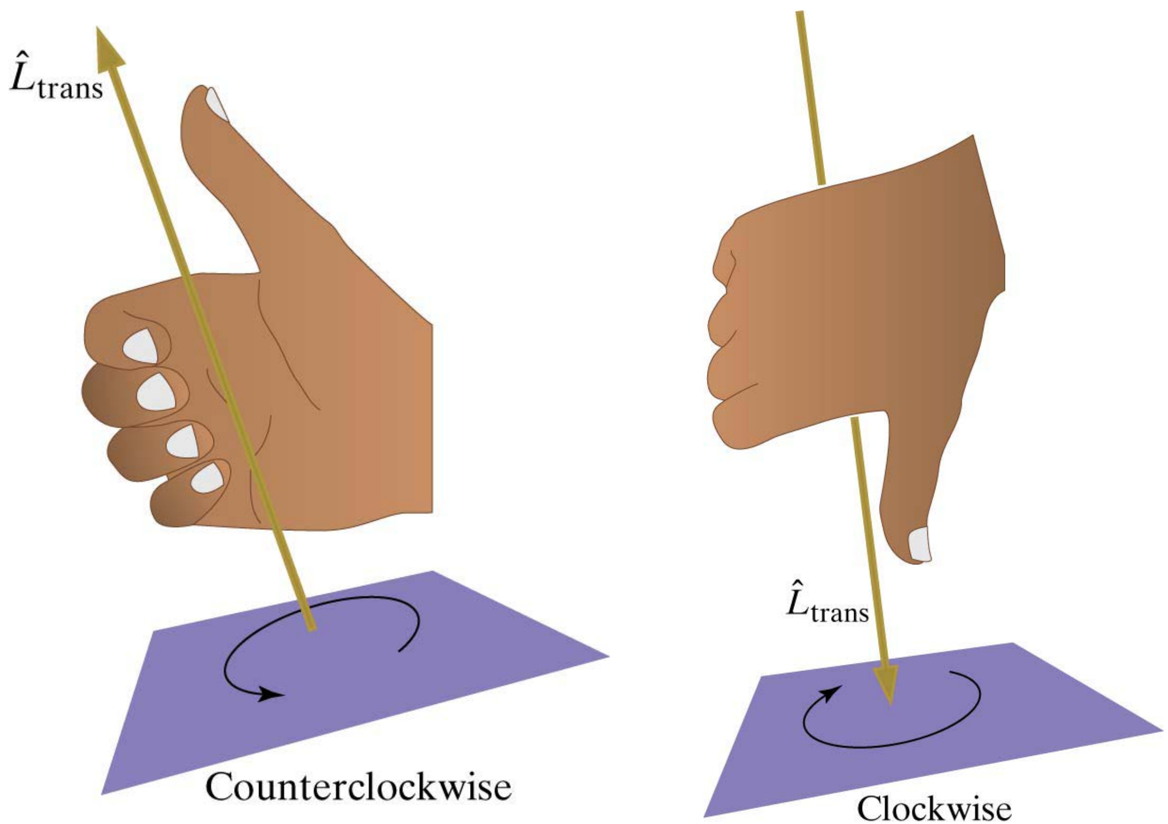


The magnitude of the translational angular momentum is defined as $\left|\vec{\mathbf{L}}_{trans,Sun}\right| = \left|\vec{\mathbf{r}}\right|\left|\vec{\mathbf{p}}\right|\sin\theta$

Translational angular momentum: direction

$$\left|\vec{\mathbf{L}}_{trans,Sun}\right| = |\vec{\mathbf{r}}||\vec{\mathbf{p}}|\sin\theta$$

What about the direction of $\vec{\mathbf{L}}_{trans,Sun}$?



The direction of $\hat{\mathbf{L}}_{trans}$ depends on the direction of the rotation and is perpendicular to the plane defined by $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

The vector cross product

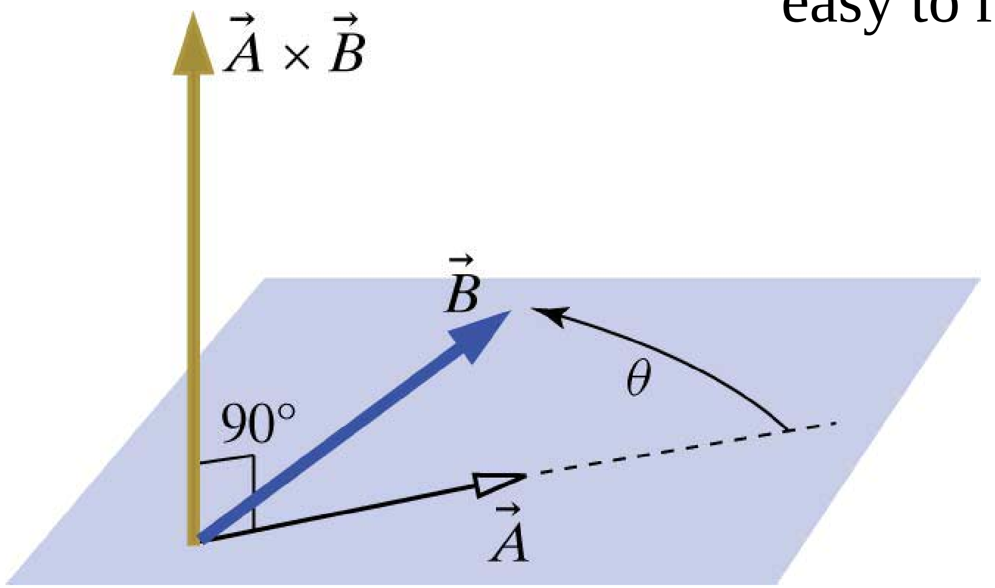
$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

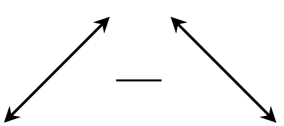
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

easy to remember:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



always

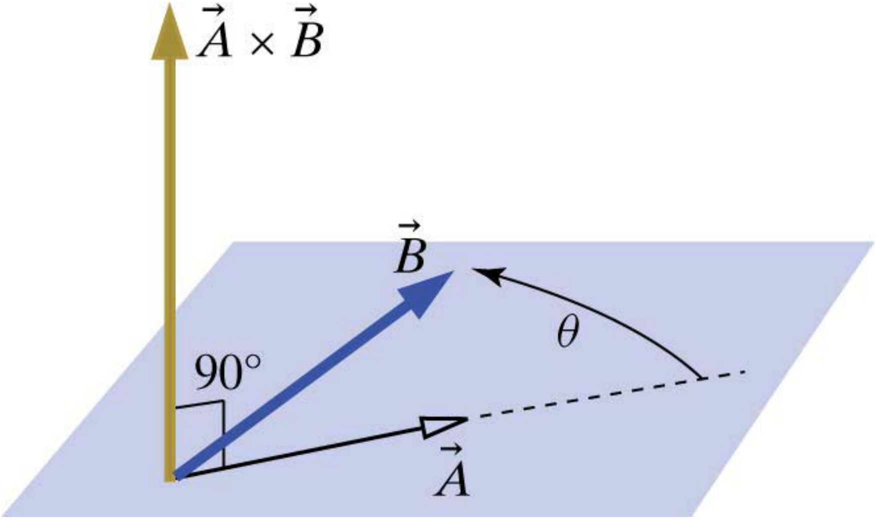


$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

In polar form in 2D:

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = AB\sin\theta$$

where θ is the angle between the tails of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

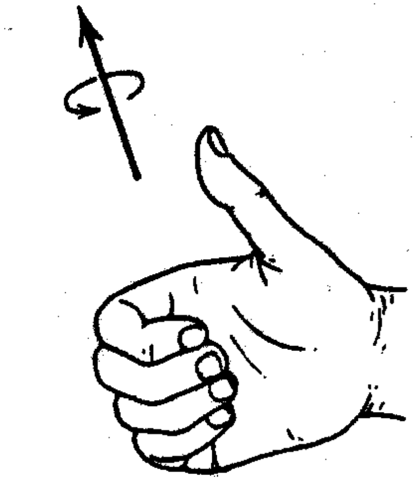


$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\left(\vec{\mathbf{B}}\times\vec{\mathbf{A}}\right)$$

$$\vec{\mathbf{A}}\times\vec{\mathbf{A}} = 0$$

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$$



Use **right hand** rule

1	2	3	4	5

What is the direction of

$< 0, 0, 3> \times < 0, 4, 0>$?

(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude

1	2	3	4	5

What is the direction of

$< 0, 4, 0> \times < 0, 0, 3>$?

(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude

1	2	3	4	5

What is the direction of

$< 0, 0, 6> \times < 0, 0, -3>$?

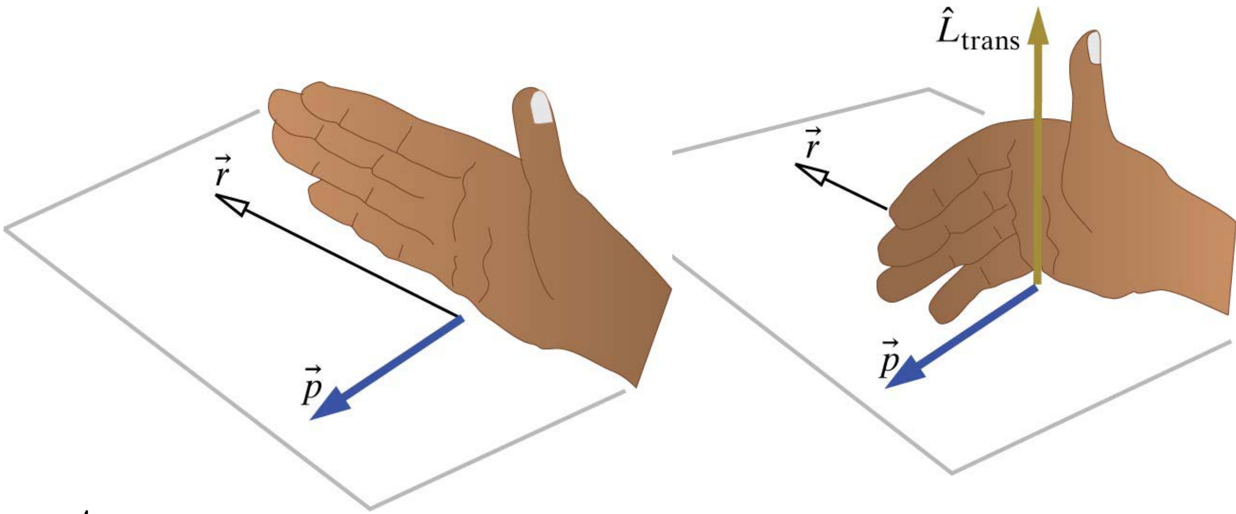
(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude

Translational angular momentum of an object relative to location A

$$\vec{\mathbf{L}}_{trans,A} = \vec{\mathbf{r}}_A \times \vec{\mathbf{p}}$$

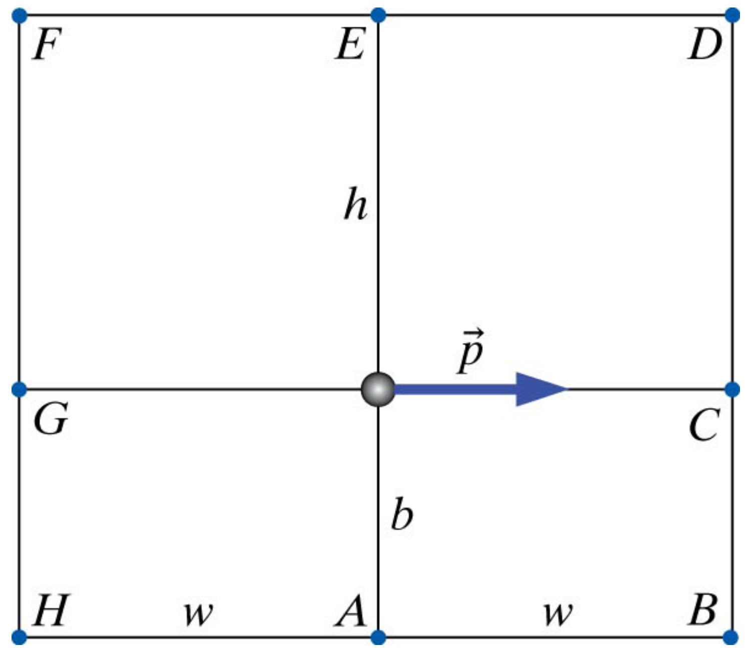
$$\left|\vec{\mathbf{L}}_{trans,A}\right| = \left|\vec{\mathbf{r}}_A\right|\left|\vec{\mathbf{p}}\right| \sin\theta$$

$$= r_A p \sin\theta$$

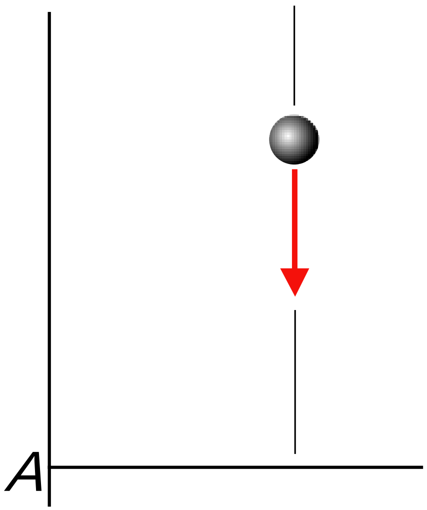


$p\sin\theta$ is the component of $\vec{\mathbf{p}}$ perpendicular to $\vec{\mathbf{r}}$.

See important worked example in *M&I* …
Calculate $\vec{\mathbf{L}}_{trans}$ for a particle moving relative to locations A, B and C.

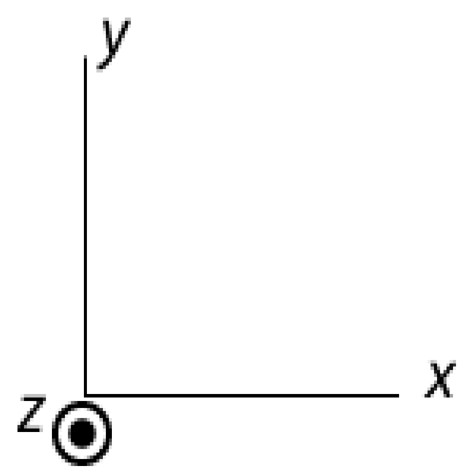


1	2	3	4	5



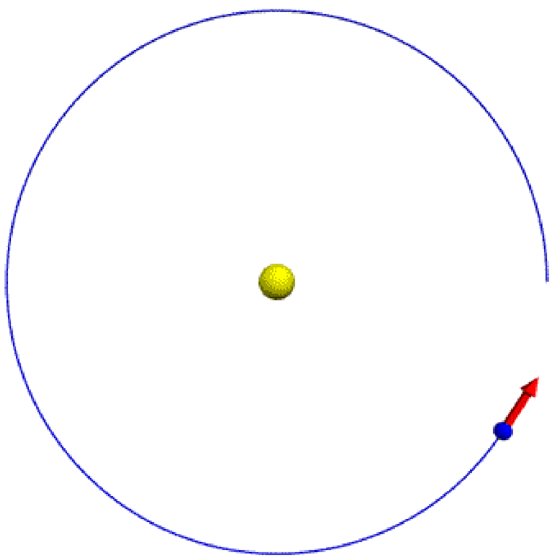
A ball falls straight down in the ***xy*** plane. Its momentum is shown by the red arrow. What is the direction of the ball's ***angular momentum*** about location *A*?

(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude



1 2 3 4 5

A planet orbits a star, in a circular orbit in the ***xy*** plane.
Its momentum is shown by the red arrow.

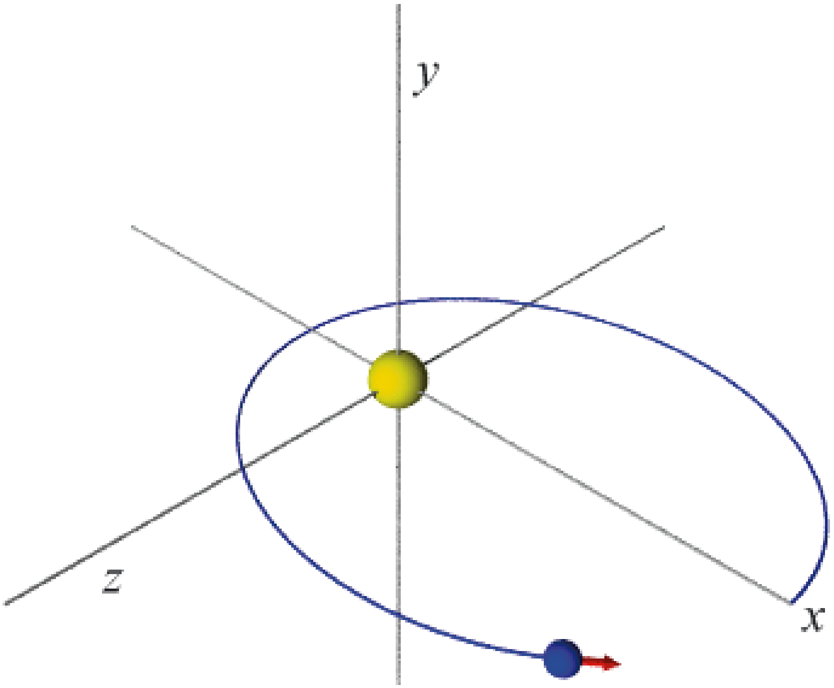


What is the direction of the angular momentum of the planet?

1. Same direction as $\vec{p}$
2. Opposite to $\vec{p}$
3. Into the page
4. Out of the page
5. Zero magnitude

1	2	3	4	5

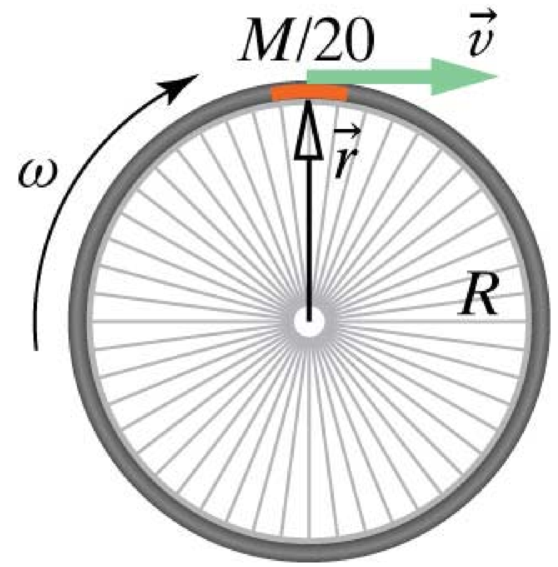
A comet orbits the Sun in the ***xz*** plane. Its momentum is shown by the red arrow. What is the direction of the comet's ***angular momentum*** about the Sun?



(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude

Rotational angular momentum

Consider a bicycle wheel spinning about its centre of mass with angular speed ω (in radians per second).
Nearly all mass M is in rim.
Divide rim into 20 pieces each of mass $M/20$.



$$\left|\vec{\mathbf{L}}_{CM}\right| = R(M/20)v\sin 90°$$

$$= R(M/20)\omega R \qquad \text{where } v = \omega R$$

For whole wheel $\left|\vec{\mathbf{L}}_{CM}\right| = 20(M/20)\omega R^2 = MR^2\omega = I\omega$

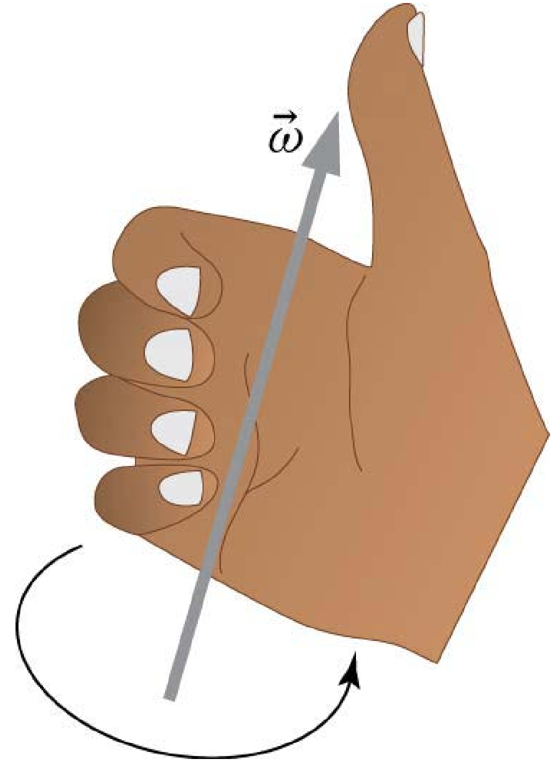
where I is the moment of inertia of the wheel (rim).

$$\left|\vec{\mathbf{L}}_{CM}\right| = I\omega$$

or

Rotational angular momentum

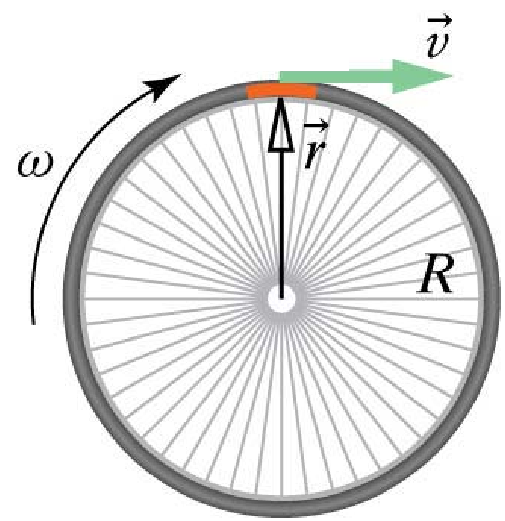
$$\vec{\mathbf{L}}_{rot} = I\vec{\boldsymbol{\omega}}$$



where $\vec{\boldsymbol{\omega}}$ is the angular velocity vector.

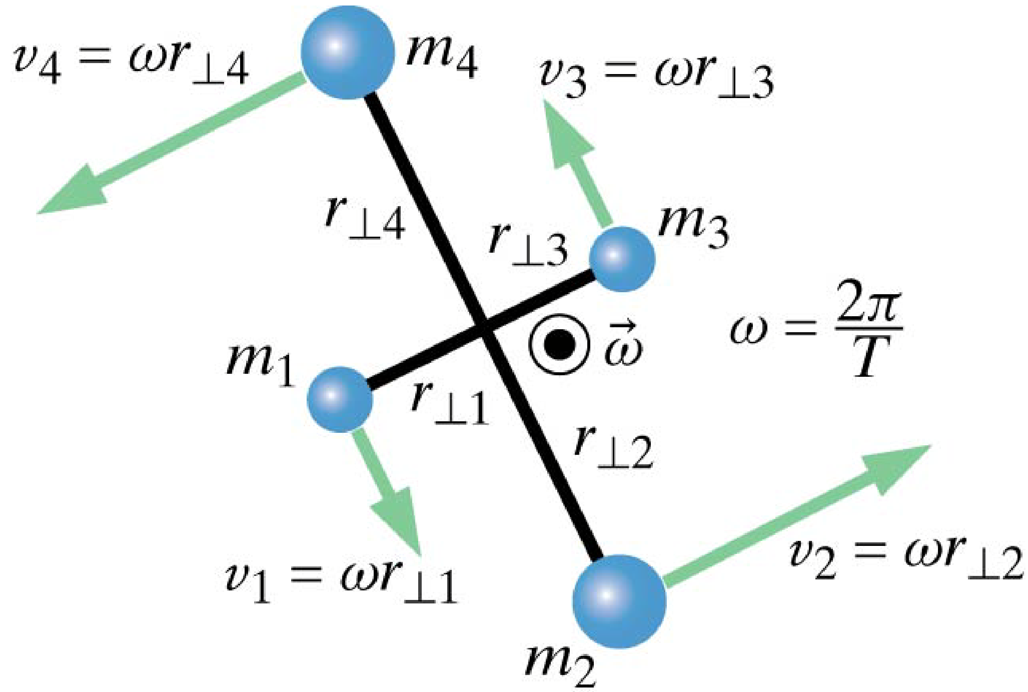
$$\omega = \frac{2\pi}{T}$$ and direction ….

$$v = \frac{2\pi r}{T} = \omega r$$



In general $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$

Rotational angular momentum: general case



Consider a collection of 4 masses rotating with the same ω about a common COM.

$$\left|\vec{\mathbf{L}}_{rot}\right| = r_{\perp 1}m_1 v + r_{\perp 2}m_2 v + r_{\perp 2}m_2 v + r_{\perp 2}m_2 v$$

$$= \left(m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + m_3 r_{\perp 3}^2 + m_4 r_{\perp 4}^2\right)\omega \quad = \quad I\omega$$

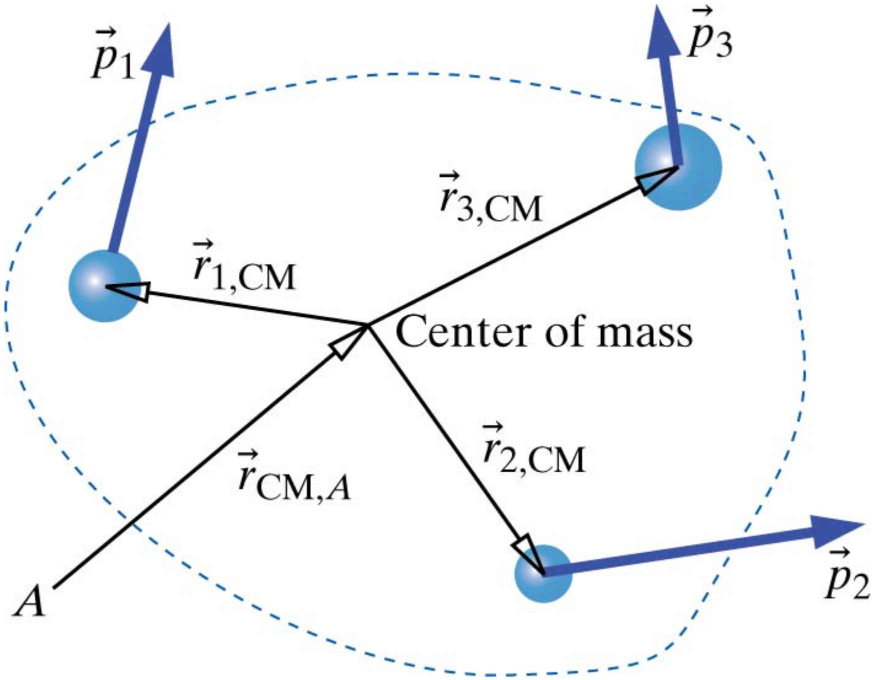
$$\therefore \vec{\mathbf{L}}_{rot} = I\vec{\boldsymbol{\omega}}$$

Also $\quad K_{rot} = \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{(I\omega)^2}{I} = \frac{L_{rot}^2}{2I}$

M&I
3E 11.3

Translational plus rotational angular momentum

For a system which is both translating (relative to A) and rotating about a centre of mass …



$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_{trans,A} + \vec{\mathbf{L}}_{rot}$$

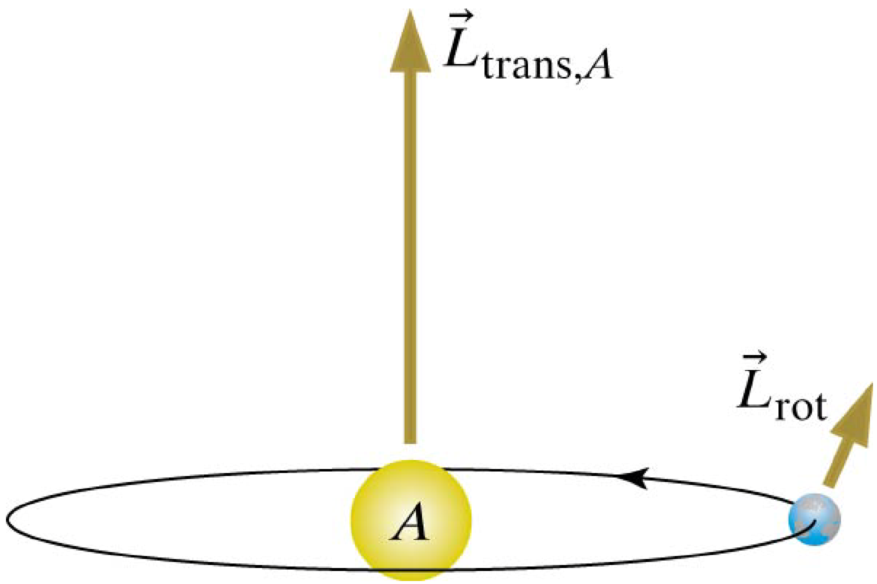
$$= \left[\vec{\mathbf{r}}_{CM,A} \times \vec{\mathbf{P}}_{tot}\right] + \left[\vec{\mathbf{r}}_{1,CM} \times \vec{\mathbf{p}}_1 + \vec{\mathbf{r}}_{2,CM} \times \vec{\mathbf{p}}_2 + \vec{\mathbf{r}}_{3,CM} \times \vec{\mathbf{p}}_3\right]$$

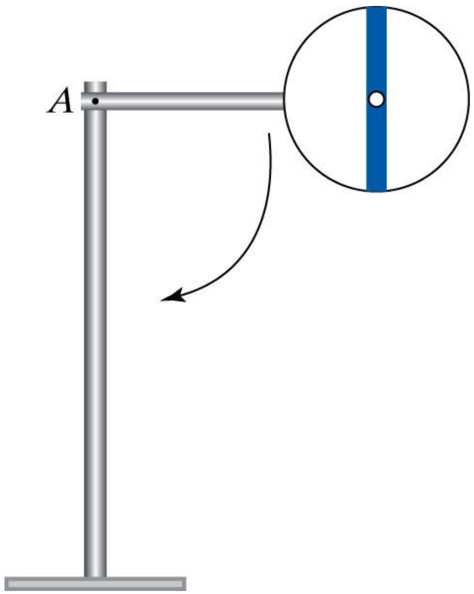
where $\vec{\mathbf{P}}_{tot} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3$

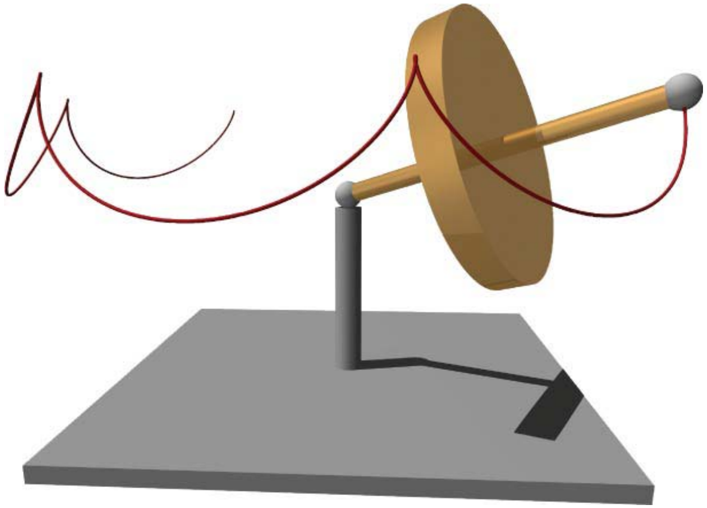
$\vec{\mathbf{L}}_{trans,A}$:translational (or "orbital") angular momentum

$\vec{\mathbf{L}}_{rot}$:rotational (or "spin") angular momentum

Translational plus rotational angular momentum: examples







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The angular momentum principle

$$\vec{\mathbf{L}}_{trans,A} = \vec{\mathbf{r}}_A \times \vec{\mathbf{p}}$$

$$\frac{d}{dt}\vec{\mathbf{L}}_{trans,A} = \frac{d}{dt}\left(\vec{\mathbf{r}}_A \times \vec{\mathbf{p}}\right)$$

$$= \frac{d\vec{\mathbf{r}}_A}{dt} \times \vec{\mathbf{p}} \quad + \quad \vec{\mathbf{r}}_A \times \frac{d\vec{\mathbf{p}}}{dt}$$

Now $\quad \dfrac{d\vec{\mathbf{r}}_A}{dt} \times \vec{\mathbf{p}} \;=\; \vec{\mathbf{v}}_A \times m\vec{\mathbf{v}}_A \;=\; 0$

$$\therefore \frac{d\vec{\mathbf{L}}_A}{dt} \;=\; \vec{\mathbf{r}}_A \times \frac{d\vec{\mathbf{p}}}{dt} \;=\; \vec{\mathbf{r}}_A \times \vec{\mathbf{F}} \;=\; \vec{\boldsymbol{\tau}}_A$$

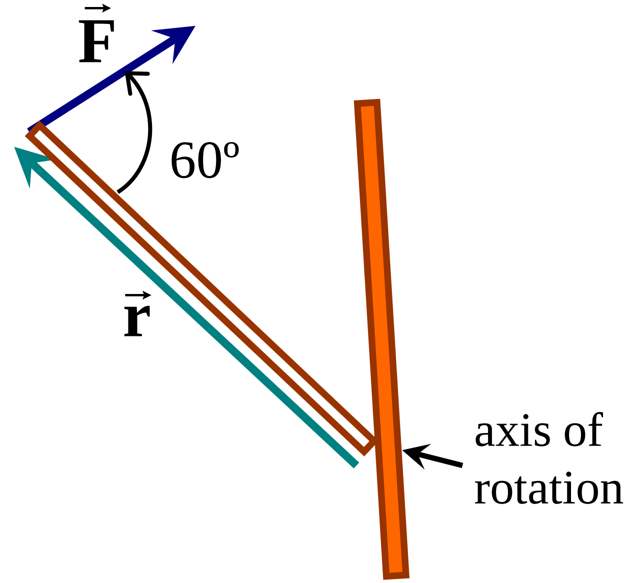
where the “**torque**” $\vec{\boldsymbol{\tau}}_A \equiv \vec{\mathbf{r}}_A \times \vec{\mathbf{F}}$

Torque

τ : "tau"

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Also called the moment of the force $\vec{\mathbf{F}}$ about the turning point (axis of rotation).

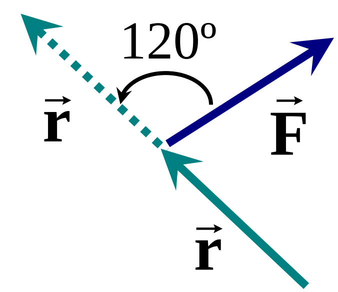


e.g. a metre stick is free to rotate about a fixed axis at one end as shown.

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

$$\left|\vec{\boldsymbol{\tau}}\right| = \tau = rF\sin\theta$$

θ is the angle between the tails of the $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$



$$\therefore \tau = (1)(4)\sin 120^\circ \quad = \quad 3.46\text{ N m}$$

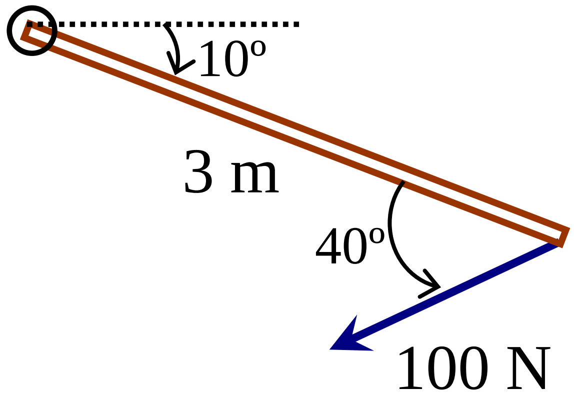
$$\vec{\boldsymbol{\tau}} = 3.46 \text{ N m into the page}$$

Examples

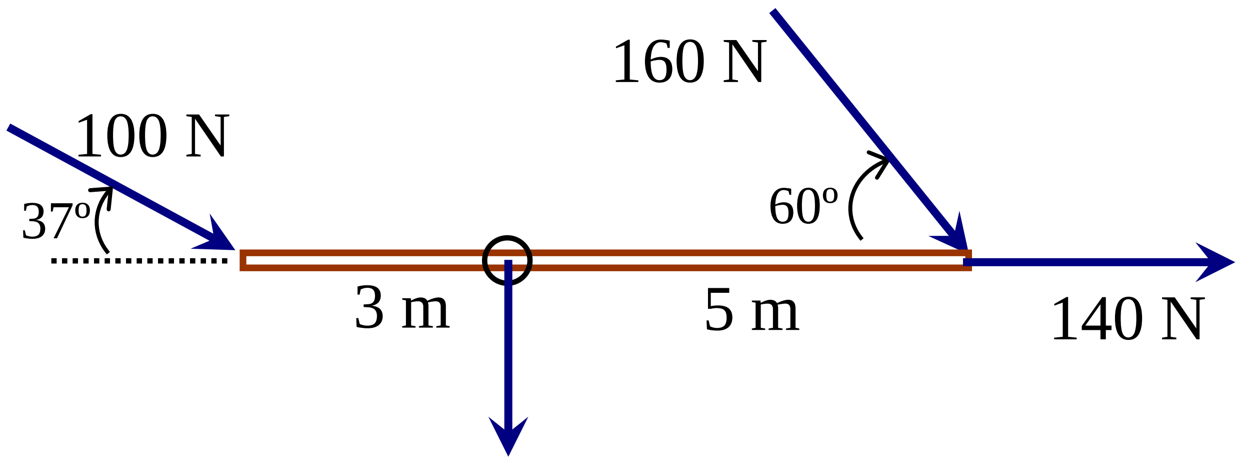
For each situation below, determine the resultant torque acting on the axis of rotation O.

Use a coordinate system with the $\hat{\mathbf{k}}$- axis out of the page.

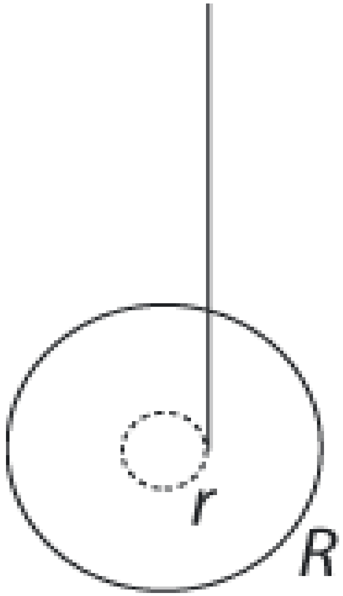
(a)



(b)

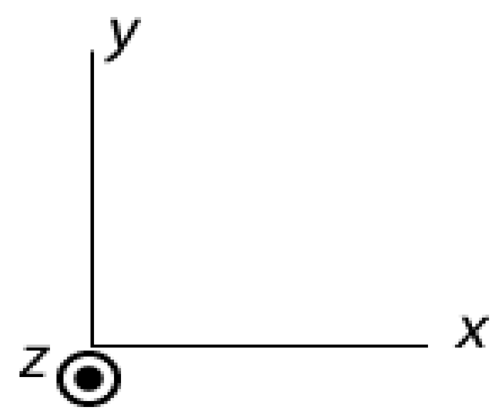


1	2	3	4	5



A yo-yo is in the ***xy*** plane. You pull up on the string with a force of magnitude 0.6 N.
What is the direction of the torque you exert on the yo-yo?

(1) $+x$
(2) $-x$
(3) $+y$
(4) $-y$
(5) $+z$
(6) $-z$
(7) zero magnitude



The angular momentum principle

$$\frac{d\vec{\mathbf{L}}_A}{dt} = \vec{\boldsymbol{\tau}}_A$$

finite time form: $\Delta\vec{\mathbf{L}}_A = \vec{\boldsymbol{\tau}}_{net,A}\Delta t$

Principle of conservation of angular momentum

$$\Delta\vec{\mathbf{L}}_{A,system} + \Delta\vec{\mathbf{L}}_{A,surroundings} = 0$$

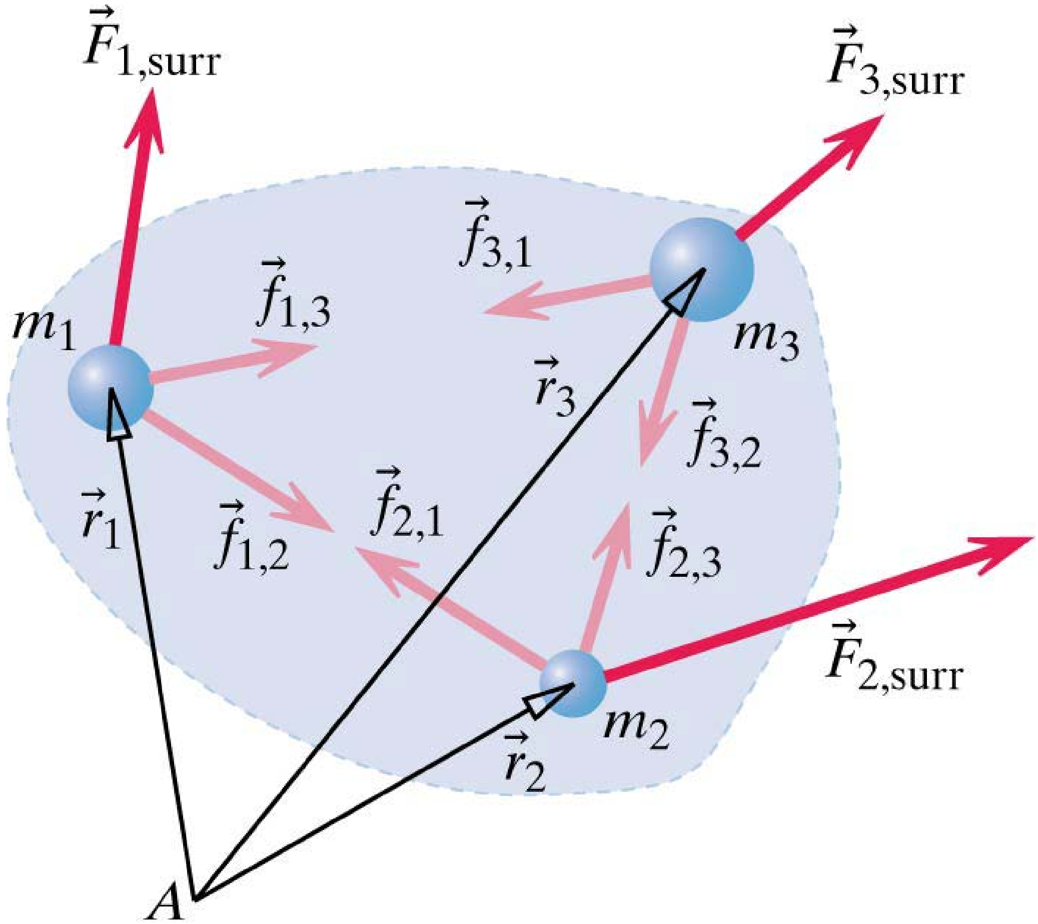
Angular momentum update formula for a closed system

$$\Delta\vec{\mathbf{L}}_A = \vec{\boldsymbol{\tau}}_{net,A}\Delta t$$

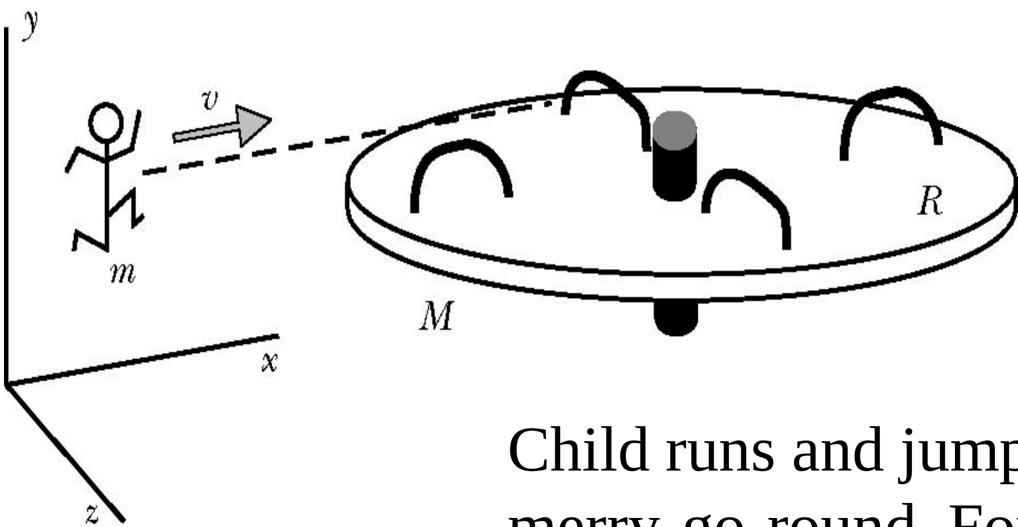
$$\vec{\mathbf{L}}_{A,f} = \vec{\mathbf{L}}_{A,i} + \vec{\boldsymbol{\tau}}_{net,A}\Delta t$$

M&I
3E 11.5

Angular momentum in multiparticle systems



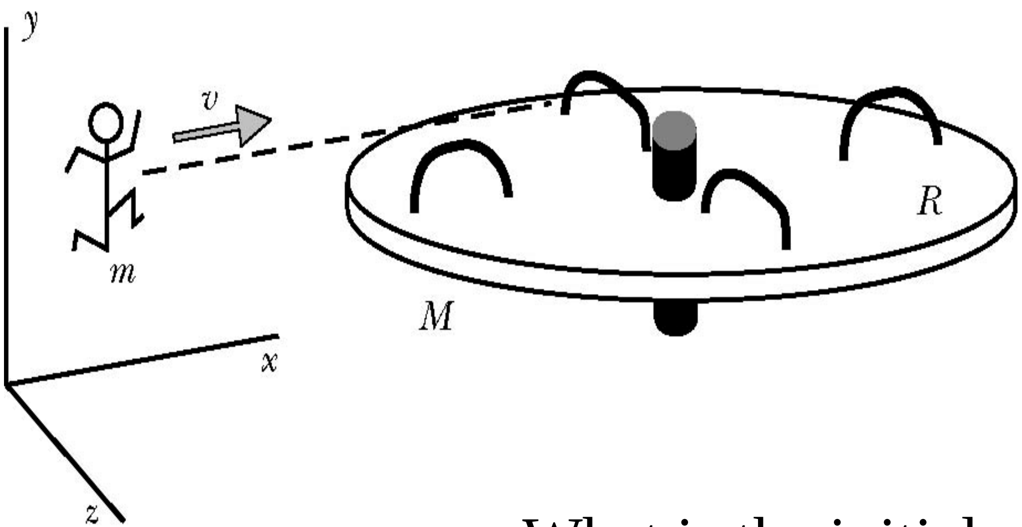
$$\Delta \vec{L}_{tot,A} = \vec{\tau}_{net,A}\Delta t$$



1	2	3	4	5

Child runs and jumps on playground merry-go-round. For the system of the child + disk (excluding the axle and the Earth), which statement is true from just before to just after impact?

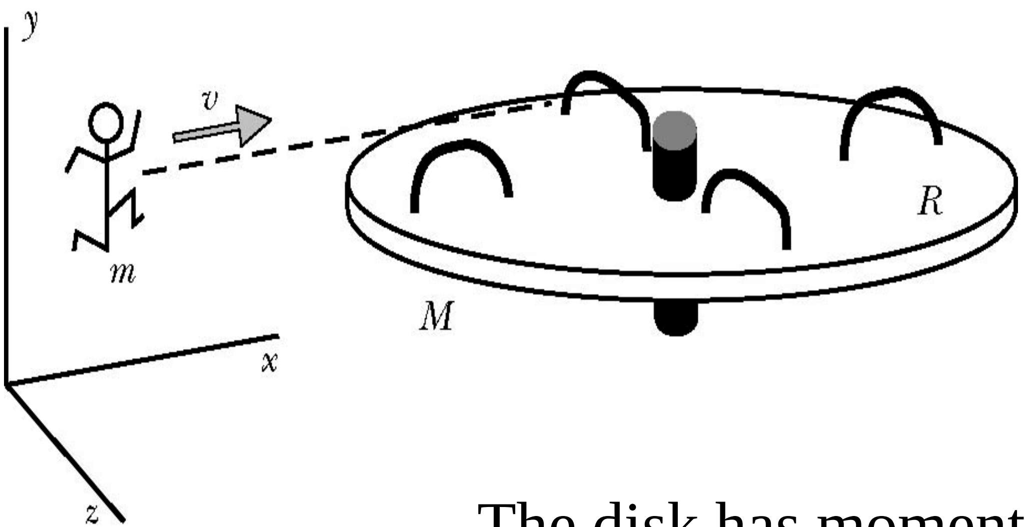
1. K, $\vec{P}$, and $\vec{L}$ do not change
2. $\vec{P}$ and $\vec{L}$ do not change
3. $\vec{L}$ does not change
4. K and $\vec{P}$ do not change
5. K and $\vec{L}$ do not change

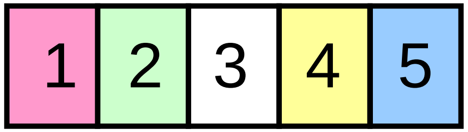


1	2	3	4	5

What is the initial angular momentum of the child + disk about the axle?

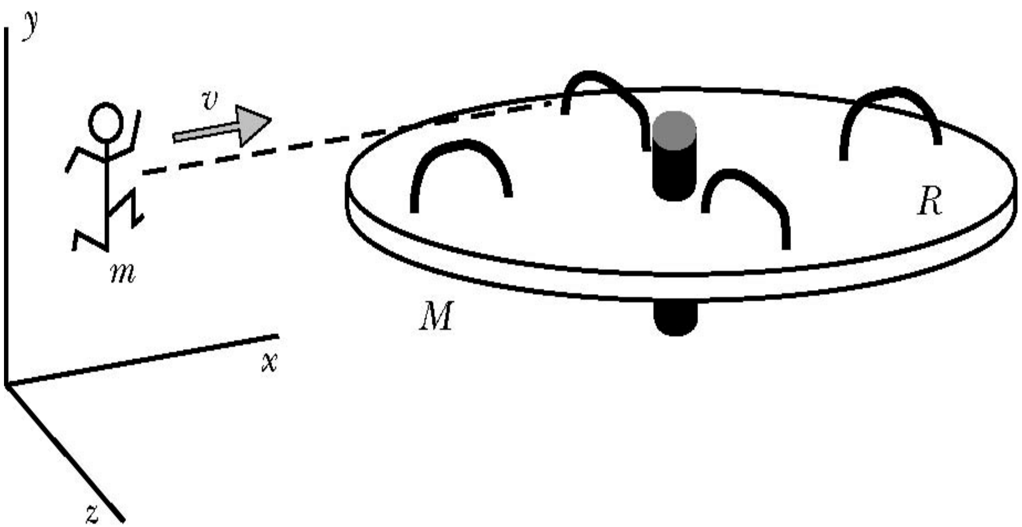
(1) < 0, 0, 0 >
(2) < 0, –Rmv, 0 >
(3) < 0, Rmv, 0 >
(4) < 0, 0, –Rmv >
(5) < 0, 0, Rmv >

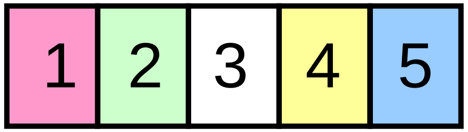




The disk has moment of inertia I, and after the collision it is rotating with angular speed $\boldsymbol{\omega}$. The rotational angular momentum of the disk alone (not counting the child) is

(1) $< 0, 0, 0 >$
(2) $< 0, -I\boldsymbol{\omega}, 0 >$
(3) $< 0, I\boldsymbol{\omega}, 0 >$
(4) $< 0, 0, -I\boldsymbol{\omega} >$
(5) $< 0, 0, I\boldsymbol{\omega} >$

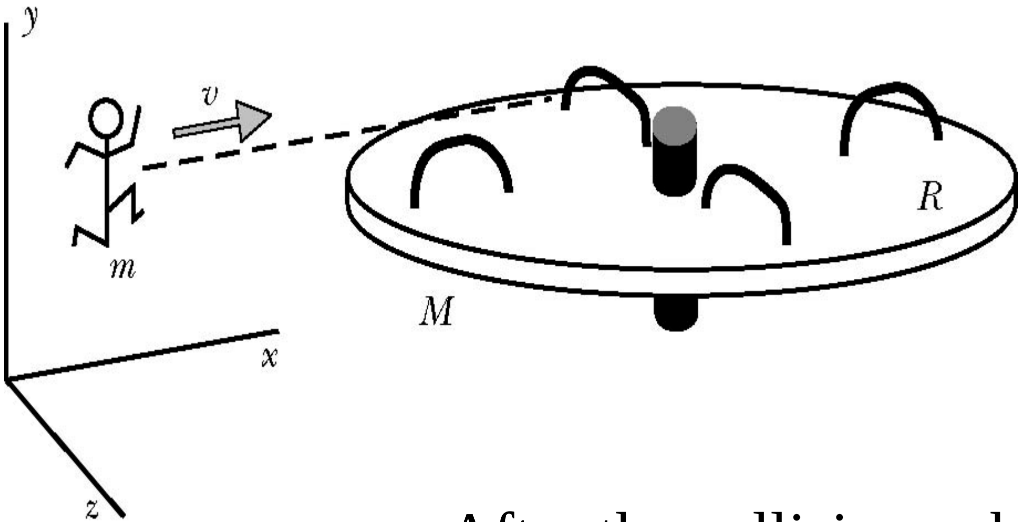




After the collision, what is the speed (in m/s) of the child?

(1) $\boldsymbol{\omega}\, R$
(2) $\boldsymbol{\omega}$
(3) $\boldsymbol{\omega}\, R^2$
(4) $\boldsymbol{\omega} / R$
(5) $\boldsymbol{\omega}^2\, R$

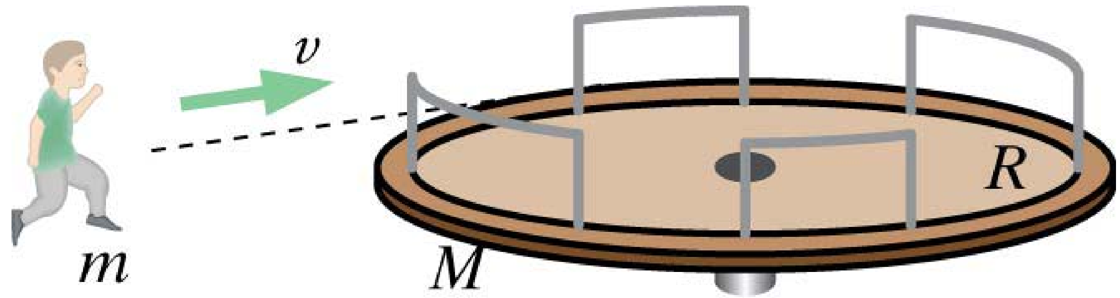
1	2	3	4	5



After the collision, what is the translational angular momentum of the child about the axle?

(1) < 0, 0, 0 >
(2) < 0, –$Rm\boldsymbol{\omega}$, 0 >
(3) < 0, $Rm\boldsymbol{\omega}$, 0 >
(4) < 0, –$Rm(\boldsymbol{\omega}R)$, 0 >
(5) < 0, $Rm(\boldsymbol{\omega}R)$, 0 >

Example



A playground ride consists of a uniform-density disk of mass 300 kg and radius 2 m mounted on a low friction axle. Starting from a distance of 5 m from the edge of the disk, a child of mass 40 kg runs at 3 m s^{-1} on a line tangential to the disk and jumps onto the outer edge of the disk. If the disk was initially at rest, how fast does it rotate just after the collision?

$$\vec{\mathbf{L}}_{A,f} = \vec{\mathbf{L}}_{A,i} + \vec{\boldsymbol{\tau}}_{net,A}\Delta t \qquad \vec{\boldsymbol{\tau}}_{net,A} = 0$$

$$L_{A,i} = rp\sin\theta = Rmv = (2)(40)(3) = 240 \text{ kg m}^2 \text{ s}^{-1}$$

$$L_{A,f} = I\omega = \left(mR^2 + \tfrac{1}{2}MR^2\right)\omega = L_{A,i} = 240 \text{ kg m}^2 \text{ s}^{-1}$$

$$\therefore \omega = 0.32 \text{ radians s}^{-1}$$

error in textbook

M&I
3E 11.6

Three fundamental principles of mechanics

Momentum	**Angular momentum**	**Energy**
$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}_{net}$	$\frac{d\vec{\mathbf{L}}_A}{dt} = \vec{\boldsymbol{\tau}}_{net,A}$	$\Delta E = W + Q$
If there are external forces, then momentum changes.	If there are external torques, angular momentum changes.	If there are energy inputs, then energy changes.
Location of object does not matter.	Location of object relative to point A is important.	Location of object does not matter.

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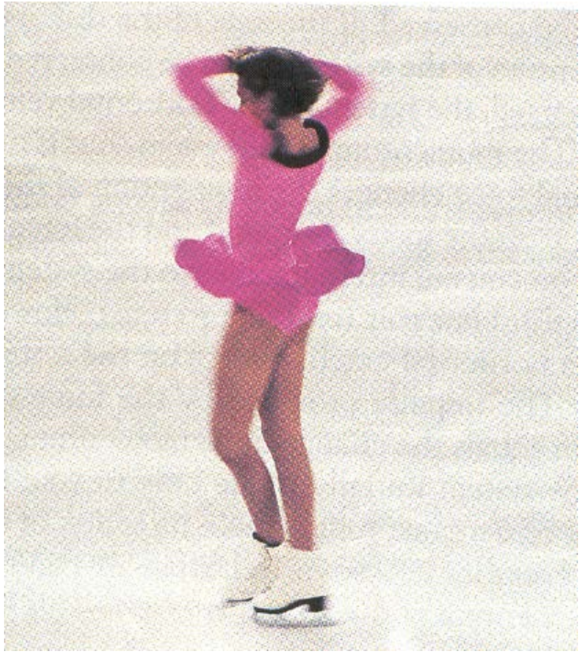
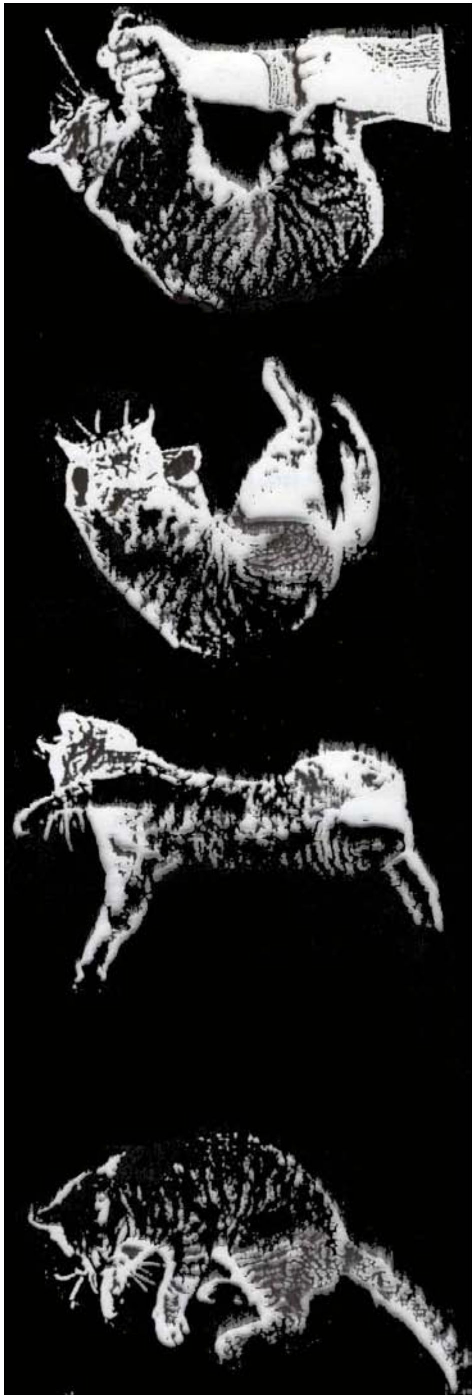
3E 11.7

Systems with zero torques

$$\Delta\vec{\mathbf{L}}_A = 0$$



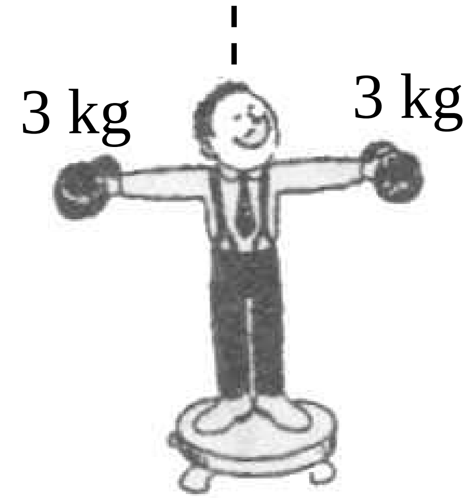


$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

Example

A man stands on a rotating stool which is free to rotate without friction. He holds 3 kg in each hand at a distance of 1 m from the centre of this body. Say that he is rotating at an angular speed of 10 radians per second.

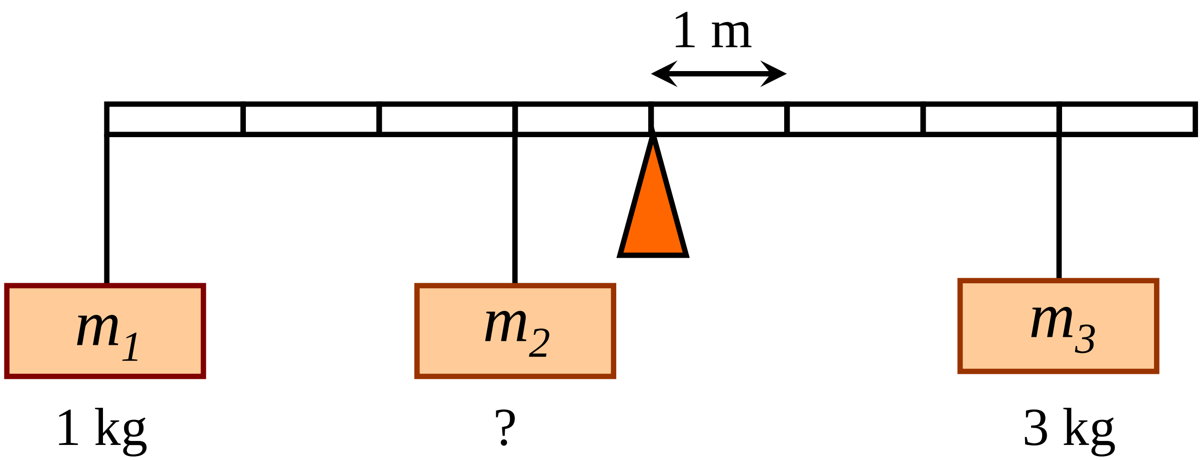


If the man brings the masses straight towards his chest until they are a distance of 0.2 m from the centre of his body, what will now be his angular speed?

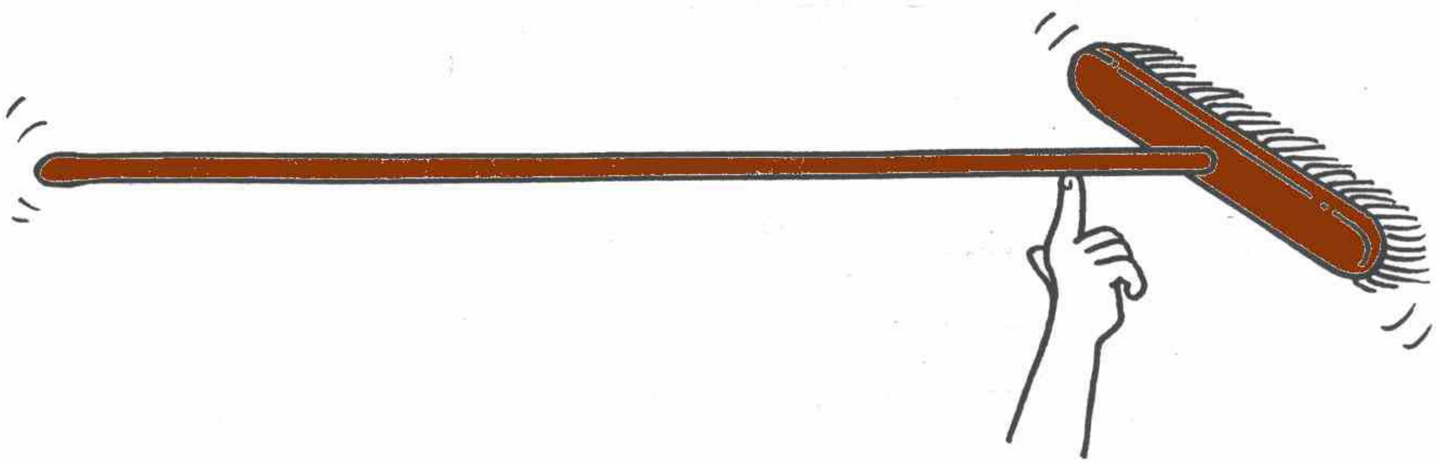
Systems with nonzero torques

Example 1

A beam is balanced on a fulcrum (triangle) with three masses hung from the positions shown.
What is the mass of m_2?
Assume that the beam is massless.



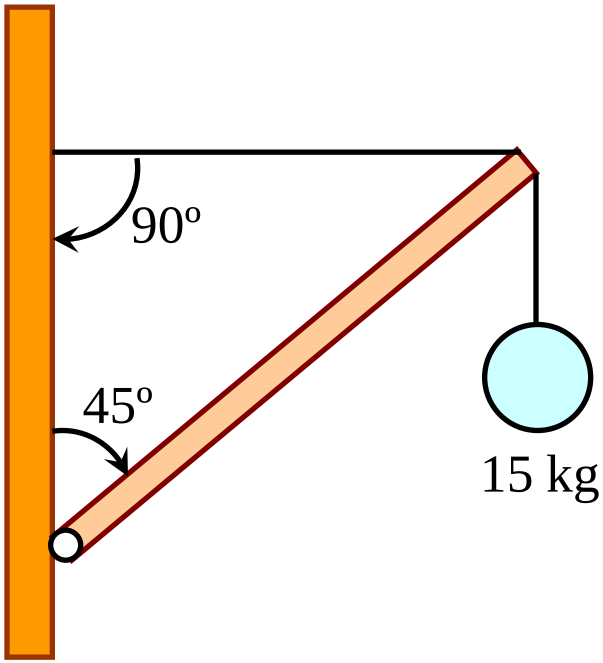
1	2	3	4	5



The broom balances at its centre of mass. If you cut the broom into two parts through the centre of mass and then weigh each part on a sale, which part will weigh more?

(A) the longer piece
(B) the shorter piece
(C) both the same

Example 2



A beam of mass 5 kg and length 10 m is held at an angle of 45º by a cable as shown. If a 15 kg mass hangs from the end of the beam, determine the force of the wall on the beam and the tension in the cable.

$$\frac{d\vec{\mathbf{p}}}{dt} = 0 \quad \rightarrow \quad \vec{\mathbf{F}}_{net} = 0$$

and

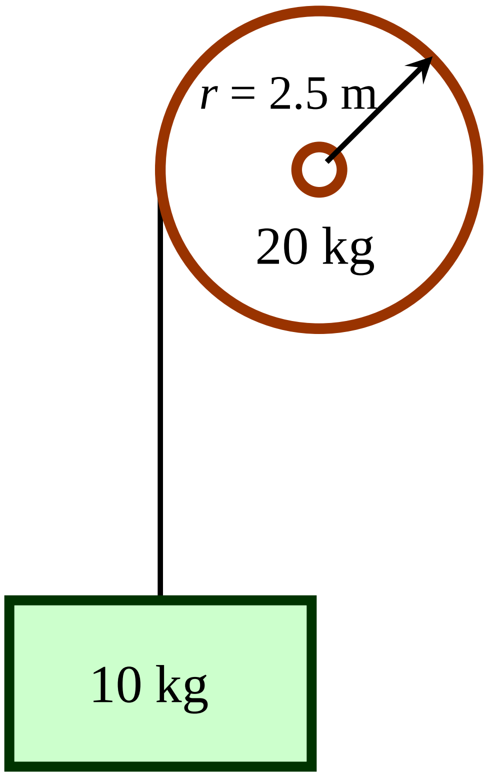
$$\frac{d\vec{\mathbf{L}}_A}{dt} = 0 \quad \rightarrow \quad \vec{\boldsymbol{\tau}}_{net,A} = 0$$

Example 3

A 10 kg block is attached to a light rope which is wound around a 20 kg cylindrical pulley of radius $r = 2.5$ m.

How long will it take for the speed of the 10 kg block to increase from zero to 4.0 m s^{-1} after the system is released?

Use $I_{cylinder} = 6$ kg m^2.



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Angular momentum quantization

The Bohr Model

Niels Bohr (Noble Prize in Physics 1922) proposed a revolutionary model which broke from classical theory.

For his model of the hydrogen atom, Bohr postulated that:

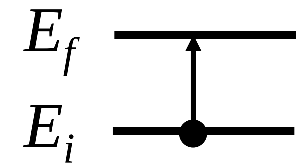
… the electron follows circular orbits around the positive nucleus. (Centripetal force = Coulomb attraction)

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r}$$

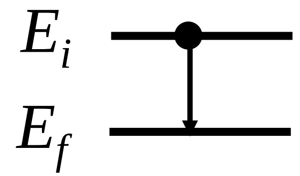
… the electron moves in certain allowed orbits without radiating energy. The electron is said to be in a "stationary state".

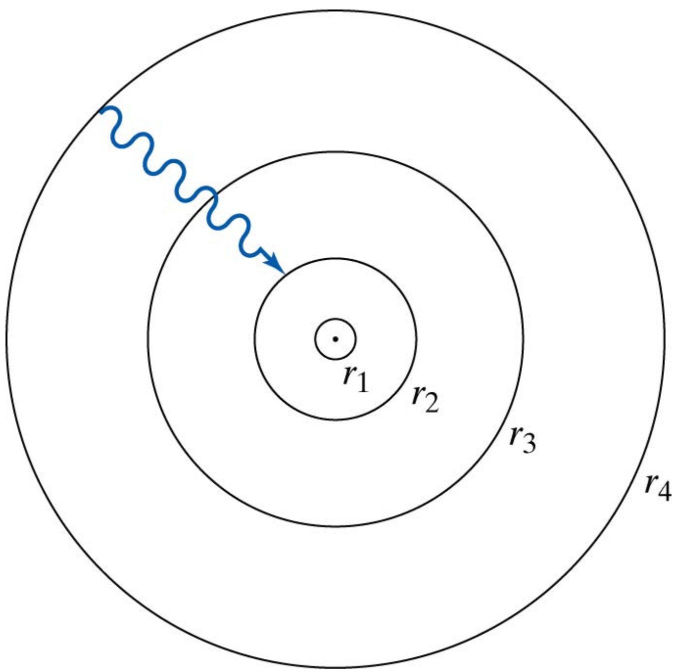
… the electron "jumps" (undergoes a transition) between stationary states by absorbing or emitting a quantum, *hf*, of energy.

Absorption: $hf = E_f - E_i$

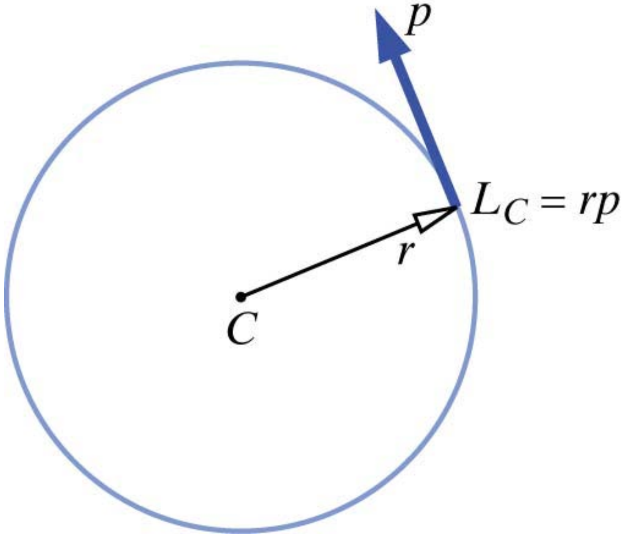


Emission: $hf = E_i - E_f$





Allowed radii of electron orbits



Bohr:

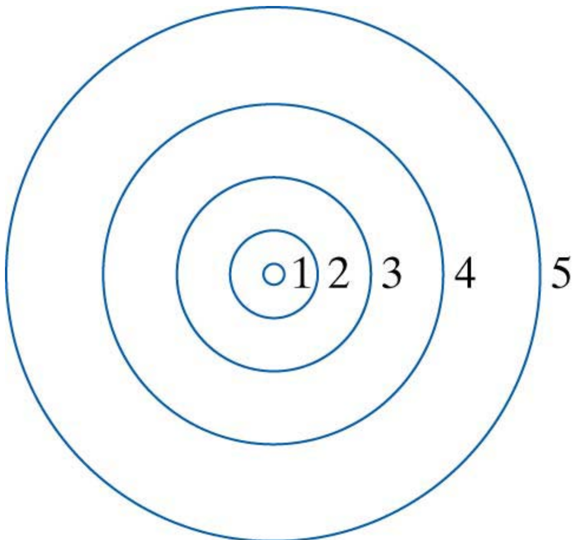
angular momentum is "quantized":

$$L_C = rp = N\hbar$$

$$N = 1, 2, 3, \ldots \text{ and } \quad \hbar = \frac{h}{2\pi}$$

Then combining with $\displaystyle \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r}$

gives $\displaystyle r_N = N^2 \frac{\hbar^2}{\left(\frac{1}{4\pi\varepsilon_0}\right)e^2 m}$



For $N = 1$, $r = 0.53 \times 10^{-10}$ m

For the hydrogen atom, Bohr was able to propose a formula for the allowed energy levels for the electron:

$$E_N = -\frac{13.6}{N^2}\ \mathrm{eV} \qquad ;\ N = 1, 2, 3, \ldots$$

... which allowed the calculation of the frequencies of the photon emitted when an electron undergoes a transition from an outer orbit to an inner one ...

… which in turn explained the experimental observations perfectly.

Bohr’s model of the hydrogen atom was a great triumph.

He extended his model to other (ionized) single-electron atoms, e.g. $\mathrm{He}^{+}, \mathrm{Li}^{++}, \mathrm{Be}^{+++}$

However ... the Bohr model was <u>not</u> satisfactory for other multi-electron atoms and did not explain the quantum postulates.