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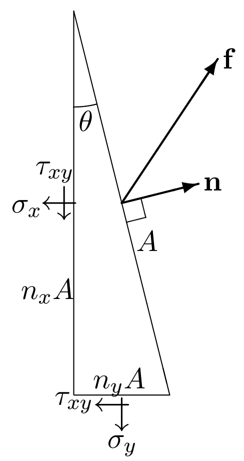
Principal stresses: the algebraic approach

We begin by using the same wedge as in the geometric approach, but we do not use it to define new axes x', y'. Instead we define the orientation of the inclined plane by the unit normal vector $\mathbf{n}$, where $n_x = \cos\theta$ and $n_y = \sin\theta$. We write the resultant force on the inclined plane as $\mathbf{f}A$, where A is the area of the inclined plane. Clearly $\mathbf{f}$ has dimensions of stress (force per unit area) but it is not technically a stress (since there is no reference to the axes defined by the inclined plane); it is sometimes called a stress vector, but the usual term for it is **traction vector**. Only if $\mathbf{f}$ is resolved into components along axes x', y' will these components coincide with the stress components $\sigma_{x'}$, $\tau_{x'y'}$.



The force equilibrium equations of the wedge with the respect to the axes x, y, after dividing by A, become

$$\begin{aligned} f_x &= \sigma_x n_x + \tau_{xy} n_y, \\ f_y &= \tau_{xy} n_x + \sigma_y n_y. \end{aligned}$$

The condition for the normal $\mathbf{n}$ to indicate a principal axis is that the shear stress on its plane vanishes, or equivalently that $\mathbf{f}$ is purely normal, that is, $\mathbf{f} = \sigma\mathbf{n}$, where σ is the normal stress on the plane. Consequently,

$$\begin{aligned} \sigma_x n_x + \tau_{xy} n_y &= \sigma n_x, \\ \tau_{xy} n_x + \sigma_y n_y &= \sigma n_y, \end{aligned}$$

or equivalently,

$$\begin{aligned} (\sigma_x - \sigma) n_x + \tau_{xy} n_y &= 0, \\ \tau_{xy} n_x + (\sigma_y - \sigma) n_y &= 0. \end{aligned} \tag{1}$$

For these equations to yield a solution for n_x and n_y, their respective coefficients must be proportional, that is, $(\sigma_x - \sigma)/\tau_{xy} = \tau_{xy}/(\sigma_y - \sigma)$. On cross-multiplying, this equation becomes

$$(\sigma_x - \sigma)(\sigma_y - \sigma) - \tau_{xy}^2 = 0. \tag{2}$$

This can be rewritten as the quadratic equation

$$\sigma^2 - (\sigma_x + \sigma_y)\sigma + \sigma_x\sigma_y - \tau_{xy}^2 = 0,$$

whose roots are the principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

Once each of the principal stresses are determined, its value can be introduced into **either one** of equations (1) in order to determine the ratio $n_y/n_x = \tan\theta$ which gives the direction of the corresponding principal axis.

Note that the left-hand side of equation (2) is just the determinant of the system (1). In fact, it is a fundamental principle of linear algebra that a system $[A]\{x\} = 0$ has a nontrivial solution $\{x\}$ only if the determinant of $[A]$ is zero. This makes it easy to extend the present approach to three-dimensional stress. Instead of a two-dimensional wedge we consider the three-dimensional tetrahedron with an inclined face of area A defined by the unit normal vector $\mathbf{n}$ and with the other three faces perpendicular to the x-, y- and z-axes respectively, and with the respective areas n_xA, n_yA and n_zA. The three-dimensional version of equations (1) is

$$\begin{aligned}
(\sigma_x - \sigma)n_x + \tau_{xy}n_y + \tau_{xz}n_z &= 0, \\
\tau_{xy}n_x + (\sigma_y - \sigma)n_y + \tau_{yz}n_z &= 0, \\
\tau_{xz}n_x + \tau_{yz}n_y + (\sigma_z - \sigma)n_z &= 0.
\end{aligned} \tag{3}$$

The condition that the determinant of this system is zero reduces to a cubic equation of the form

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0, \tag{4}$$

where I_1, I_2 and I_3 are known as the **principal invariants** of the stress tensor. They are called "invariants" because their values are independent of the axes in which the stress components are defined, in the same way that the magnitude of a vector is an invariant.

If, in particular, it is the principal axes that are used, then the matrix representing the stress tensor takes the diagonal form

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

and the invariants are

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad I_2 = \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2, \quad I_3 = \sigma_1\sigma_2\sigma_3.$$

Note that the invariants are functions of the principal stresses that are **symmetric** in the sense that they are independent of the numbering.

Note also that if one of the original axes is already known to be a principal axis, then the corresponding root of equation (4) is known, and only a quadratic equation needs to be solved.