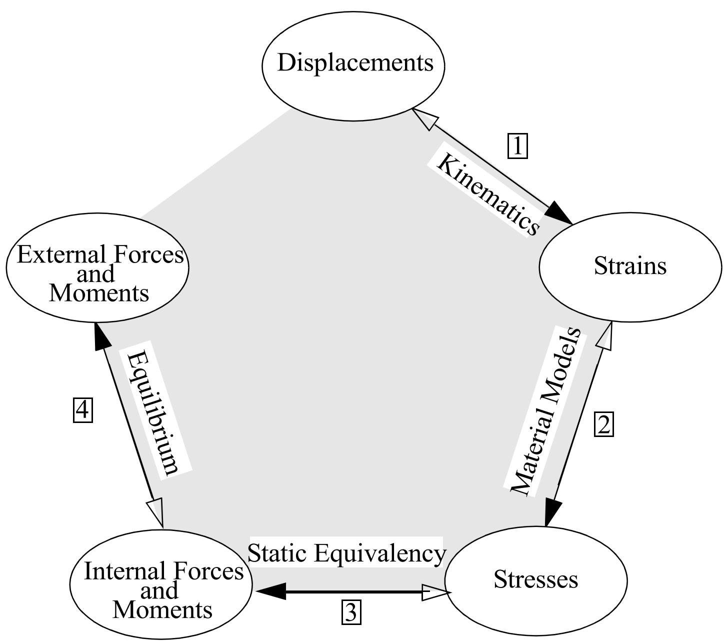
Inelastic Structural Behavior

The learning objectives of this chapter are:

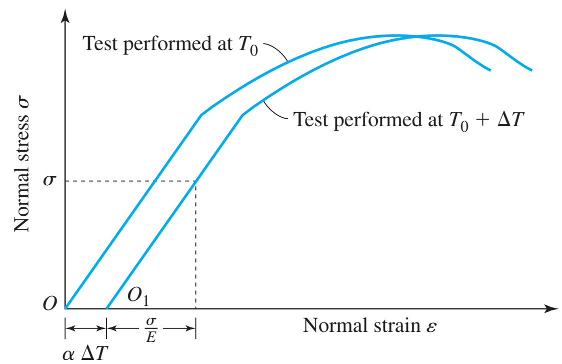
- Understand the incorporation of thermal and initial strains in the theory and analysis of axial members.
- Understand the analysis techniques for incorporating elastic-perfectly plastic material behavior in axial members, circular shafts, and symmetric beams.
- Understand the incorporation of non-linear material models into the basic simplified theories on structural members.



The following assumptions regarding material models will be dropped.

Assumption 5: There are no inelastic strains.
Assumption 6: Material is elastic.
Assumption 7: Stress and strains are linearly related.

Effects of Temperature



$$\varepsilon = \frac{\sigma}{E} + \alpha\Delta T$$

- No thermal stresses are produced in a homogenous, isotropic, unconstrained body due to uniform temperature changes.

$$\begin{aligned}
\varepsilon_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E + \alpha\ \ \Delta T \\
\varepsilon_{yy} &= [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E + \alpha\ \ \Delta T \\
\varepsilon_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E + \alpha\ \ \Delta T \\
\gamma_{xy} &= \tau_{xy}/G \\
\gamma_{yz} &= \tau_{yz}/G \\
\gamma_{zx} &= \tau_{zx}/G
\end{aligned}$$

Thermal Strain

Mechanical Strain

C5.1 The stress at a point, material properties, and change in temperature are as given below. Calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and σ_{zz} (a) assuming plane stress, and (b) assuming plane strain.

$$\sigma_{xx} = 300 \;\; MPa(C) \qquad \sigma_{yy} = 300 \;\; MPa(T) \qquad \tau_{xy} = 150 \;\; MPa$$

$$G = 15 \;\; GPa \qquad \nu = 0.2 \qquad \alpha = 26.0\mu/{}^{o}C \qquad \Delta T = 75^{o}C$$

Initial strain in axial members

Examples of initial strain & stresses.

Tightening a nut pretensions the bolt.

Tolerances during assembly may introduce strains and stresses in some members.

Steel bars may be prestressed in reinforced concrete to introduce residual stresses.

For thermal strains $\varepsilon_o = \alpha\Delta T$

Assume an initial strain of ε_o

Kinematics: $\varepsilon_{xx} = \frac{du}{dx}(x)$

Stresses: $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} + \varepsilon_o = \frac{du}{dx}$ or $\sigma_{xx} = E\left(\frac{du}{dx} - \varepsilon_o\right)$

Internal Axial Force: $\boldsymbol{N} = \int_A \sigma_{xx} dA$

Assume material homogeneity across the cross-section and ε_o is uniform.

$$\boldsymbol{N} = \int_A \left(E\frac{du}{dx} - E\varepsilon_o\right) dA = \frac{du}{dx}\int_A E dA - \int_A E\varepsilon_o dA = \frac{du}{dx}EA - EA\varepsilon_o$$

$$\frac{du}{dx} = \frac{\boldsymbol{N}}{EA} + \varepsilon_o \qquad \sigma_{xx} = \frac{\boldsymbol{N}}{A}$$

Assumptions 9 through 11 are assumed valid. ***N***, E and A are constant between x_1 and x_2

Assume ε_o also does not change with x

$$u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{EA} + \varepsilon_o(x_2 - x_1) \text{ or } \delta = \frac{\boldsymbol{N}L}{EA} + \varepsilon_o L$$

- *P*re-strains (stresses) can be analyzed by using ε_o as negative to the actual initial strain.
- δ is the deformation of the bar in the undeformed direction.
- If ***N*** is a tensile force then δ is elongation.
- If ***N*** is a compressive force then δ is contraction.
- The sign of $\varepsilon_o L$ must be consistent with the force ***N*** shown on the free body diagram.
- Deformation of a member shown in the drawing of approximate deformed geometry must be consistent with the internal force in the member that is shown on the free body diagram.
- No thermal stresses are produced in statically determinate structures.
- In displacement method displacements of points or deformation members are treated as unknowns.
- In force method reaction forces or internal forces are treated as unknowns.

General Procedure for Indeterminate Structure

The procedure outlined can be used for solving statically indeterminate structure problems by either the force method or by the displacement method.

1. If there is a gap, assume it will close at equilibrium.
2. Draw free-body diagrams, noting the tensile and compressive nature of internal forces.
3. Write equilibrium equations relating internal forces to each other.

   or

   Write internal forces in terms of reaction forces using equilibrium equations.
4. Draw an exaggerated approximate deformed shape, ensuring that the deformation is consistent with the free body diagrams of step 2. Write compatibility equations relating deformation of the bars to each other.

   or

   Write compatibility equations in terms of unknown displacements of points on the structure, if displacement method is to be used.
5. Write internal forces in terms of deformations using $\delta = \frac{NL}{EA} + \varepsilon_o L$ ensuring consistency in sign for the three terms.
6. Solve the equations of steps 3, 4, and 5 simultaneously for the unknown.
7. Check whether the assumption of gap closure in step 1 is correct.

C5.2 The rigid bar in Fig. C5.2 is horizontal when the unit is put together by finger-tightening the nut. The pitch of the threads is 0.125 inch. Develop a table in steps of quarter turn of the nut that can be used for prescribing the pre-tension in bar B. Maximum number of quarter turns is limited by the yield stress.

	Bar A	Bar B
Modulus of Elasticity	10,000 ksi	30,000 ksi
Yield Stress	24 ksi	30 ksi
Area of cross-section	0.5 in^2	0.75 in^2

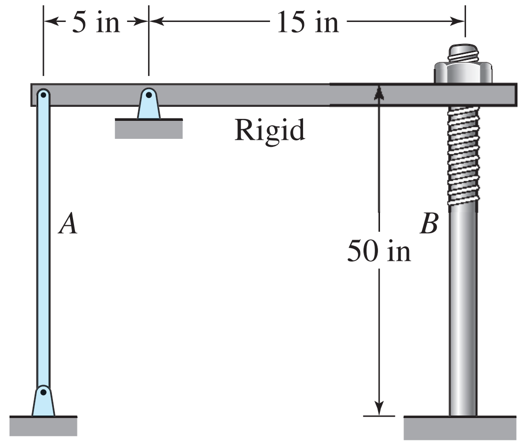


Fig. C5.2

C5.3 Determine the axial stress in bar A of problem C5.2 assuming that the nut is turned 1 full turn and the temperature of bar A is decreased by 50°F. The coefficients of thermal expansion for bar A is $\alpha_{st} = 22.5\ \mu / {}^oF$.

Class Problem 5.1

Write equilibrium equations, compatibility equations, and $\delta = \frac{NL}{EA} + \varepsilon_o L$ for each member using the given data. **No need to solve**.

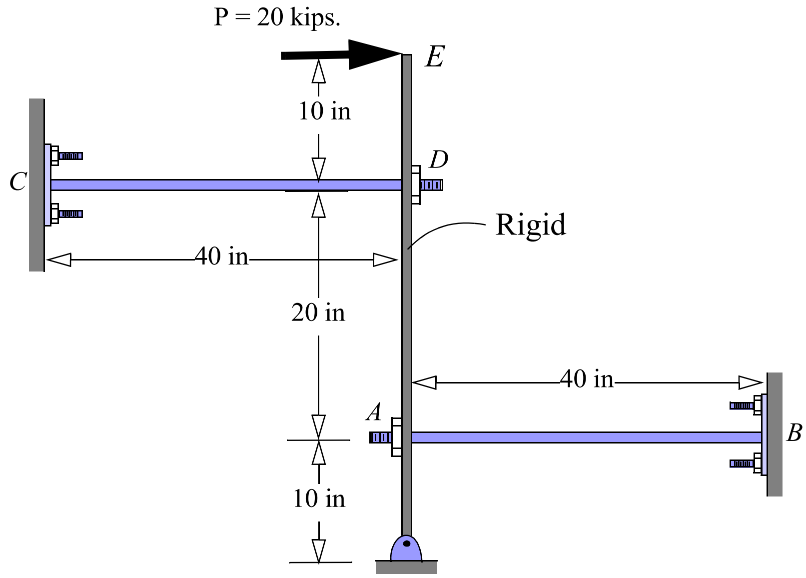
Data: $\Delta T = 100^oF$ $\alpha = 20\ \mu/^oF$ $E = 10{,}000$ ksi $A = 5\ in^2$

Use displacement of point E δ_E as unknown.

$\alpha \Delta T L = 0.08$

$EA = 50,000$

$L/EA = 0.8(10^{-3})$



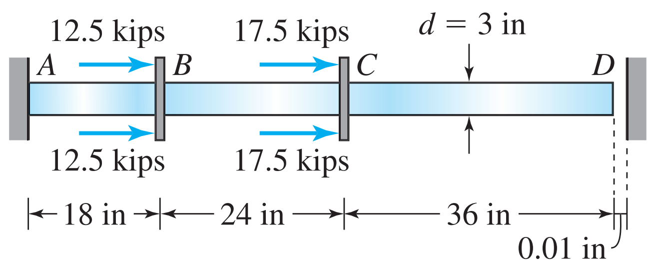
Class Problem 5.2

Write equilibrium equations, compatibility equations, and $\delta = \frac{NL}{EA} + \varepsilon_o L$ for each member using the given data. **No need to solve**. Use reaction force at A (R_A) as unknown.

$\Delta T = 100^oF$ $\alpha = 20\ \mu/^oF$ $E = 10{,}000$ ksi $A = 5$ in^2

$$\alpha\Delta T = 2(10^{-3})$$

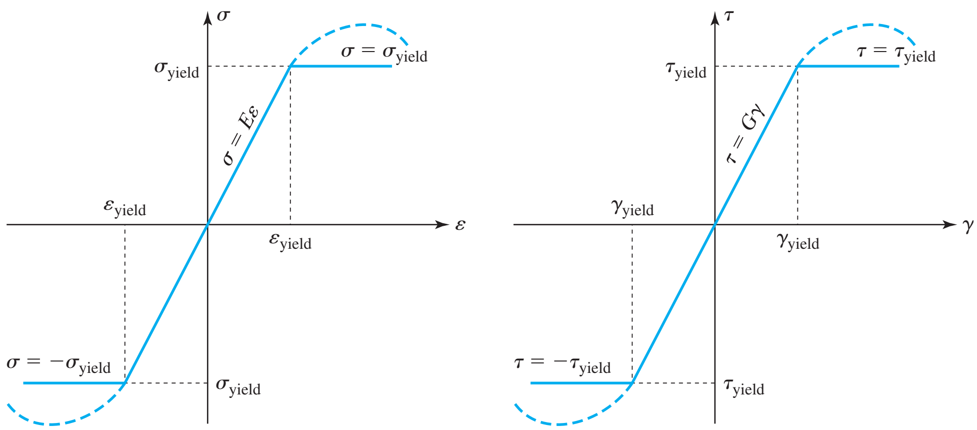
$$EA = 50,000$$



Non-linear material models

- Elastic-perfectly plastic in which the non-linearity is approximated by a constant.
- Linear strain hardening model in which the non-linearity is approximated by a linear function.
- Power law model in which the non-linearity is approximated by one term non-linear function.

Elastic-perfectly plastic

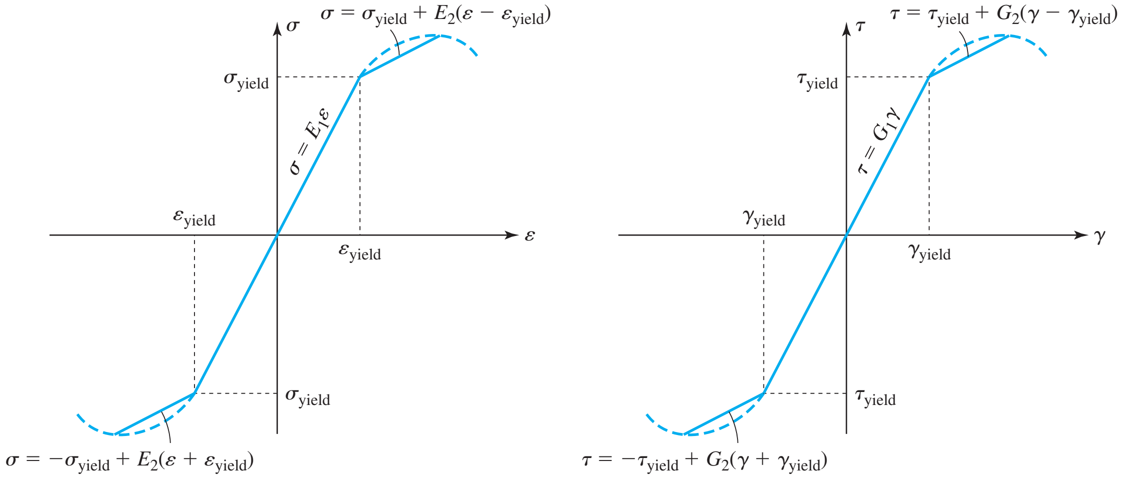


$$\sigma = \begin{cases} \sigma_{yield} & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} & \varepsilon \leq -\varepsilon_{yield} \end{cases} \qquad \tau = \begin{cases} \tau_{yield} & \gamma \geq \gamma_{yield} \\ G\gamma & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} & \gamma \leq -\gamma_{yield} \end{cases}$$

- The set of points forming the boundary between the elastic and plastic region on a body, is called the elastic-plastic boundary.

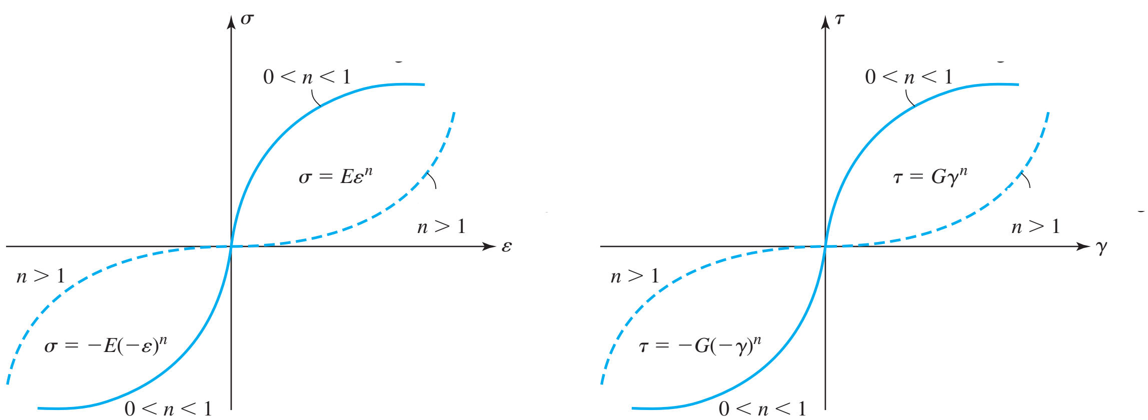
1. On the elastic-plastic boundary the strain must be equal to the yield strain, and stress equal to yield stress.
2. Deformations and strains are continuous at all points including points at the elastic plastic boundary.

Linear strain hardening material model



$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \end{cases}$$

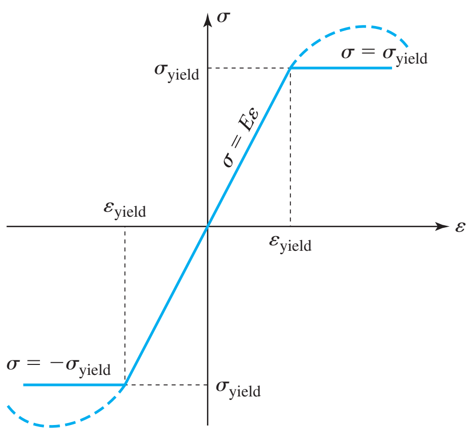
Power Law



$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \geq 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases}$$

Elastic-perfectly plastic axial members

- The plot of the applied force vs. the deflection at that point in the direction of the applied force is called the load deflection curve.
- The load at which the structure exhibits unbounded deformation is called the collapse load.



Elastic: $\delta = \frac{NL}{EA}$

C5.4 A force F is applied to the roller that slides inside a slot as shown in Fig. C5.4. Both bars have an area of cross-section of A = 100 mm^2, modulus of elasticity E = 200 GPa, and a yield stress of 250 MPa. Bar AP and BP have lengths of L_{AP}= 200 mm and L_{BP}= 250 mm respectively. Draw the load deflection curve and determine the collapse load.

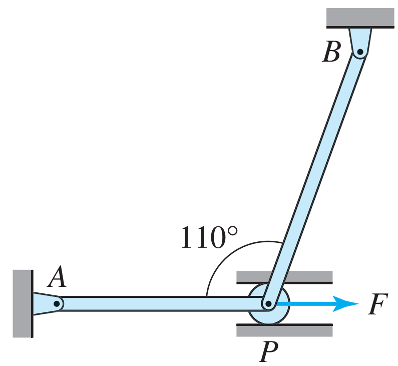


Fig. C5.4

C5.5 Three steel ($E = 200$ GPa, $\sigma_{yield} = 200$ MPa) bars shown in Fig. C5.5 have lengths of L_A=4 m, L_B=3 m and L_C= 2 m respectively. All bars have the same cross-sectional area of 500 mm^2. Draw the load deflection curve for the structure and determine the collapse load.

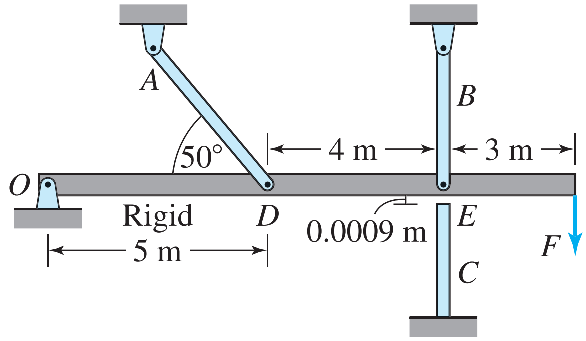
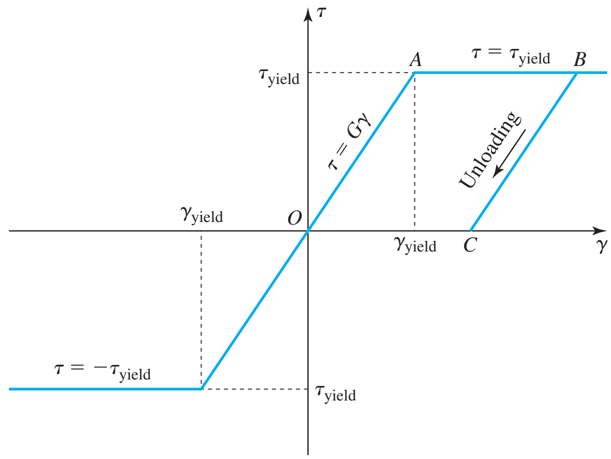


Fig. C5.5

Elastic-perfectly plastic circular shafts



- Before yield stress the material stress-strain relationship is represented by Hooke's Law and after yield stress the stress is assumed to be constant.
- To determine the strain (deformation) in the horizontal portion AB of the curve we have to use the requirement that deformation must be continuous.
- Unloading (elastic recovery) from a point in the plastic region is along line BC which is parallel to the linear portion of the stress strain curve OA.

Kinematic: $\gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$

Elastic: $\tau_{x\theta} = \frac{T\rho}{J}$ $\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{GJ}$

At the elastic-plastic boundary: $\gamma_{yield} = \rho_y\frac{d\phi}{dx}$

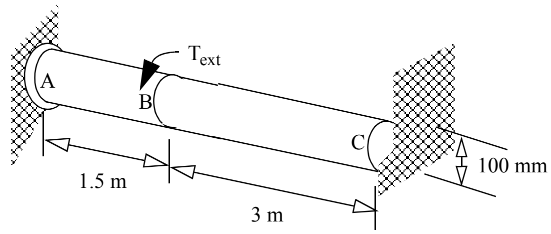
If ρ_y is not a function of x: $\phi_2 - \phi_1 = \frac{\gamma_{yield}}{\rho_y}(x_2 - x_1)$

Stress distribution: $\tau = \begin{cases} \frac{\tau_{yield}\rho}{\rho_y} & \rho \le \rho_y \\ \tau_{yield} & \rho \ge \rho_y \end{cases}$

Internal equivalent torque: $T = \int_A \rho\tau_{x\theta}dA$

C5.6 The shaft shown in Fig. C5.6 made from elastic - perfectly plastic material has a shear yield stress of 200 MPa and a shear modulus of G = 80 GPa. The plastic zone in section *AB* is 25 mm deep. Determine: (a) the torque T_{ext} (b) the rotation of section at *B*. (c) the residual stress in AB when the external torque T_{ext} is removed.

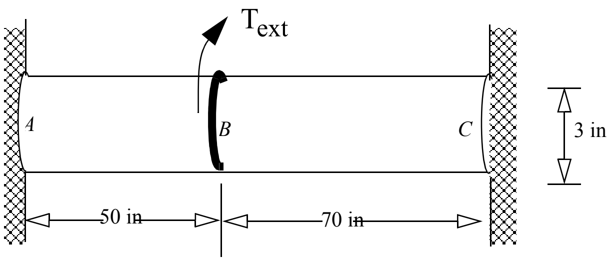
Fig. C5.6



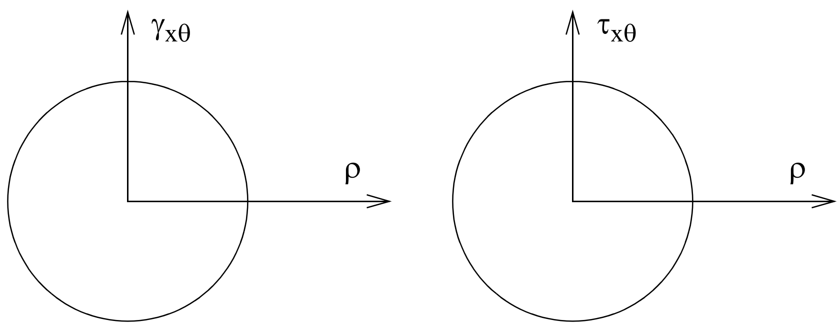
Class Problem 5.3

A shaft made from elastic-perfectly plastic. Due to the action of the torque the depth of plastic zone in *AB* was 1 in. **Draw the shear strain and shear stress distribution in section *AB* and *BC*.**

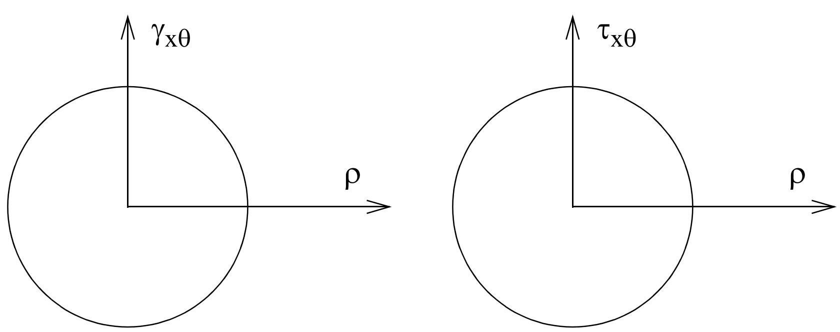
Data: $\gamma_y = 1.25(10^{-3})$ $\tau_y = 15$ ksi $G = 12,000$ ksi



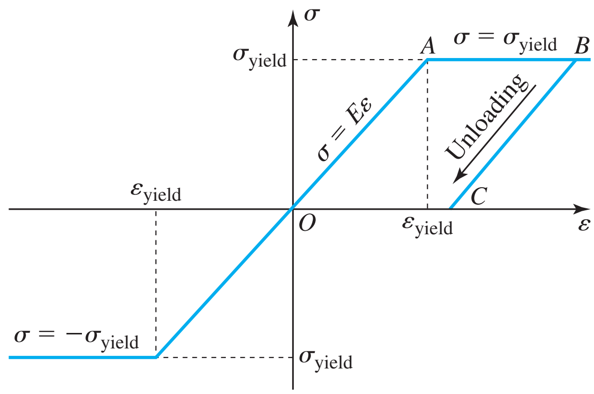
Section *AB*



Section *BC*

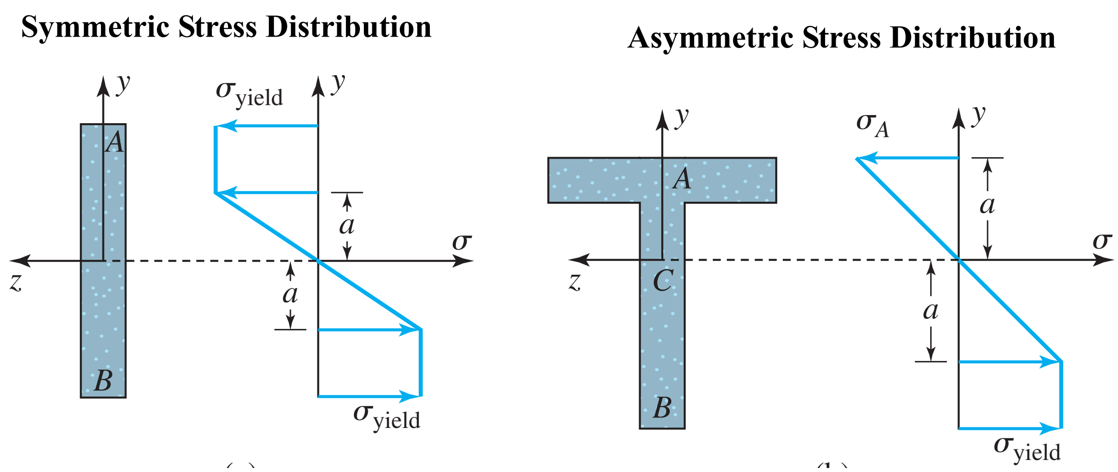


Elastic-perfectly plastic beams



- Before yield stress, the material stress-strain relationship is represented by Hooke's Law and after yield stress, the stress is assumed to be constant.
- To determine the strain (deformation) in the horizontal portion AB of the curve we have to use the requirement that deformation must be continuous.
- Unloading (elastic recovery) from a point in the plastic region is along line BC which is parallel to the linear portion of the stress-strain curve OA.

Stress distribution and changes in neutral axis location



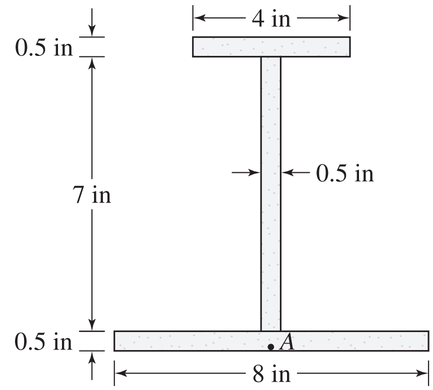
$$N = \int_A \sigma_{xx} dA = 0 \qquad M_z = -\int_A y\sigma_{xx} dA \qquad f = M_p / M_e$$

- The moment at which the maximum bending normal stress just reaches the yield stress is called the elastic moment and will be designated by M_e.
- The internal moment for which the entire cross-section becomes fully plastic is called the plastic moment and will be designated by M_p
- The ratio of the plastic moment to the elastic moment is called the shape factor for the cross-section and will be designated by f.

C5.7 An elastic - perfectly plastic material has a yield stress of σ_{yield} = 40 ksi. Point A in Fig. C5.8 is at yield stress due to bending of the beam. (a) Determine the location of the neutral axis assuming it is in the web.(b) The applied moment that produced the state of stress.

C5.8 For the cross-section shown in Fig. C5.8 determine the shape factor

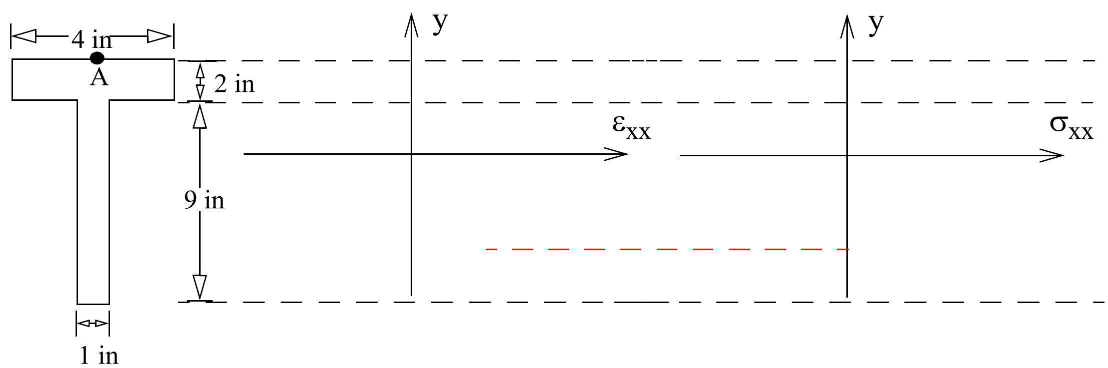
Fig. C5.8



Class Problem 5.4

Point *A* on the cross-section shown just reaches yield stress in *compression* at a given load. Parameter 'a' represent the location of the elastic plastic boundary from the neutral axis, which we assume is in the web. **Sketch the strain and stress distribution as a function of y**. Write coordinates at each line.

Data: $\sigma_{yield} = 30$ ksi $\quad \varepsilon_{yield} = 0.001 \quad E = 30,000$ ksi



Non-linear models in structural members

Kinematics

Axial	Bending	Torsion
$\varepsilon_{xx} = \frac{du}{dx}(x)$	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2}(x)$	$\gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$

$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \ge 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases}$$

$$\tau = \begin{cases} G\gamma^n & \gamma \ge 0 \\ -G(-\gamma)^n & \gamma < 0 \end{cases}$$

Static equivalency (Internal Forces and Moments)

Axial	Bending	Torsion
$\boldsymbol{N} = \int_A \sigma_{xx} dA$	$\boldsymbol{N} = \int_A \sigma_{xx} dA = 0$ $\boldsymbol{M}_z = -\int_A y\sigma_{xx} dA$ $\boldsymbol{V}_y = \int_A \tau_{xy} dA$	$\boldsymbol{T} = \int_A \rho\tau_{x\theta} dA$

C5.9 A circular solid shaft of radius R is made from a non-linear material that has a shear stress-shear strain relationship given by $\tau = K\,\gamma^{0.4}$. Show

$$\tau_{max} = 0.5411\frac{\boldsymbol{T}}{R^3} \qquad \phi_2 - \phi_1 = 0.2154(x_2 - x_1)\left(\frac{\boldsymbol{T}}{KR^{3.4}}\right)^{2.5}$$

where τ_{max} is the maximum shear stress at a section, ***T*** is the internal torque at the section, ϕ_2 , and ϕ_1 are the rotation of section at x_1 and x_2.

C5.10 The stress-strain curve in tension, for a material is given by $\sigma = K\varepsilon^{0.5}$. For a rectangular cross-section show that the bending normal stress is given by:

$$\sigma_{xx} = \begin{cases} \dfrac{-5\sqrt{2}}{bh^2}\left(\dfrac{y}{h}\right)^{0.5} \boldsymbol{M_z} & y > 0 \\ \dfrac{5\sqrt{2}}{bh^2}\left(\dfrac{-y}{h}\right)^{0.5} \boldsymbol{M_z} & y < 0 \end{cases}$$