University of California at Berkeley — CE 130
Department of Civil and Environmental Engineering — Section 2
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Maxwell–Betti Reciprocal Relations

In a linearly elastic system subject to discrete loads $F_1, F_2, \ldots$, if the conjugate displacements are $\Delta_1, \Delta_2, \ldots$, the strain energy U and the complementary energy $\bar{U}$ are equal to

$$U = \bar{U} = \tfrac{1}{2}(F_1\Delta_1 + F_2\Delta_2 + \ldots)$$

The displacements can, in turn, be decomposed as

$$\Delta_1 = \Delta_{11} + \Delta_{12} + \ldots, \quad \Delta_2 = \Delta_{21} + \Delta_{22} + \ldots, \text{ etc.},$$

where Δ_{ij} is the part of Δ_i that is due to the load F_j, and can be expressed as

$$\Delta_{ij} = f_{ij}F_j,$$

f_{ij} being the corresponding *flexibility coefficient*.

According to the **Maxwell–Betti Reciprocal Theorem**,

$$F_i\Delta_{ij} = F_j\Delta_{ji}$$

(the work done by one load on the displacement due to a second load is equal to the work done by the second load on the displacement due to the first), or, equivalently,

$$f_{ij} = f_{ji}$$

(the flexibility matrix is symmetric).

To prove the theorem, it is sufficient to consider a system with only two loads. If only F_1 is applied first, the displacement Δ_1 has the value Δ_{11} (while Δ_2 has the value Δ_{21})and the strain energy at that stage is $\frac{1}{2}F_1\Delta_{11}$. Applying F_2 (with F_1 remaining in place) results in the additional displacements Δ_{12} and Δ_{22}. The work done by F_2 is $\frac{1}{2}F_2\Delta_{22}$, while the additional work done by F_1 is $F_1\Delta_{12}$ (note the absence of the factor of one-half, since F_1 remains constant in the process). The final value of the strain energy (or complementary energy) is therefore

$$U = \bar{U} = \tfrac{1}{2}F_1\Delta_{11} + \tfrac{1}{2}F_2\Delta_{22} + F_1\Delta_{12}.$$

If the order of application of the loads is reversed, the result is obviously

$$U = \bar{U} = \tfrac{1}{2}F_2\Delta_{22} + \tfrac{1}{2}F_1\Delta_{11} + F_2\Delta_{21}.$$

In a linear elastic system, however, the complementary energy is a function of the loads only and is independent of the order in which they are applied. Consequently,

$$F_1\Delta_{12} = F_2\Delta_{21},$$

and the theorem is proved.

It also follows that the stiffness matrix $[k_{ij}] = [f_{ij}]^{-1}$ is symmetric.