MP202: Mechanics

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1. Newton's Laws of Motion

I. Any massive object maintains a state of uniform motion unless acted on by a force.

II. A force, $\mathbf{F}$, applied to any object changes its momentum, $\mathbf{p}$, at a rate

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{1}$$

III. To every action there is an equal and opposite reaction.

I. Newton's first law was in fact found by Galileo Galilei and it is often called *Galileo's Lay of Inertia.* At first sight it may look as though it is just a special case of the second law when $\mathbf{F} = 0$, but there are two important aspects to it. Firstly it is not at all obvious unless friction is eliminated, and this was Galileo's insight: once we realise that friction itself is a force we can include friction among the forces on the left hand side of (1) and in this sense Newton's first law is a special case of the second law.

A second, more important point, is that Newton's first Law is only true in a non-accelerating reference frame. If we do experiments in a train running on a smooth track with blinds on the windows then Newton's first law would not hold inside the train when it is accelerating, either changing its speed or going round corners at constant speed. There are 'funny' forces in accelerating reference frames *e.g.* centrifugal force in a rotating reference frame. Such forces are only due to the observer's motion and are not considered to be dynamical, they are often called 'fictitious forces' or 'pseudo-forces'. A reference frame which is not accelerating is called an *inertial reference frame* — because it is a reference frame in which Galileo's law of inertia holds true. From a modern perspective Newton's first Law should be viewed as stating that inertial reference frames occupy a very special place in the theory of mechanics.

II. Newton's second law is perhaps better known in the form when the momentum is written as $\mathbf{p} = m\mathbf{v}$, with m the mass and $\mathbf{v}$ the velocity of the object. Then, provided the mass is constant, we have

$$\mathbf{F} = m\mathbf{a},$$

where $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ is the acceleration. But this is only correct for constant mass, the form (1) is more general and is applicable even if the mass changes with time, *e.g.* a rocket using fuel at such a rate that its mass decreases significantly as it accelerates upward.

If there is more than one force acting on m then we should add them together using the rules of vector addition and the force $\mathbf{F}$ appearing in Newton's second law is the vector sum of all the separate forces acting on m.

The true power of Newton's second law only comes to the fore when we know something about $\mathbf{F}$. The forces on a mass m can depend on the position and velocity of the mass (*e.g.* friction depends on velocity), as well as explicitly on time. If the position of the object as a function of time is denoted by $\mathbf{r}(t)$ then, if we know the explicit form of $\mathbf{F}$ as a function of $\mathbf{r}$, $\mathbf{v} = \dot{\mathbf{r}}$,* and t, then Newton's second law, for constant m, is

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = m\ddot{\mathbf{r}}, \tag{2}$$

which is a *differential equation* for $\mathbf{r}$ as a function of time.

An example is the equation governing the motion of two masses, m_1 and m_2, attracting each other gravitationally due to Newton's other famous law — his universal law of gravitation. In this case $\mathbf{F}$ is independent of velocity and depends only on the separation $\mathbf{r}$ between the two masses, as an inverse square,

$$\mathbf{F}(\mathbf{r}) = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = -\frac{Gm_1m_2}{r^3}\mathbf{r},$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector in the direction $\mathbf{r}$ and $G = 6.67 \times 10^{-11}\ kg^{-1}m^3s^{-2}$ is Newton's universal constant of gravitation. If $m_1 >> m_2$ then we can consider m_1 to be fixed in space and choose our origin to coincide with the position of m_1, then $\mathbf{r}$ is just the position vector of m_2 and its acceleration is $\ddot{\mathbf{r}}$ so, applying Newton's second law to m_2, we get

$$m_2\ddot{\mathbf{r}} = -\frac{Gm_1m_2}{r^3}\mathbf{r}.$$

We shall see many other examples of (2) in the following.

Note that if we use a (non-accelerating) reference frame in which $\mathbf{r}$ is the position vector of a point mass then we can transfer to a different reference frame, moving with velocity $-\mathbf{u}$ relative to the first, and the position vector in the new reference frame will be

$$\mathbf{r}' = \mathbf{r} + \mathbf{u}t.$$

The velocity of the mass in the new reference frame is

$$\dot{\mathbf{r}}' = \dot{\mathbf{r}} + \dot{\mathbf{u}}t + \mathbf{u}$$

and its acceleration is

$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}} + \ddot{\mathbf{u}}t + 2\dot{\mathbf{u}}.$$

The accelerations are equal, $\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}$, for all t if and only if $\ddot{\mathbf{u}} = \dot{\mathbf{u}} = 0$, *i.e.* if and only if $\mathbf{u}$ is a constant. In other words the accelerations are equal if and only if the new reference frame is also non-accelerating. If we transfer to an accelerating reference frame then $\mathbf{u}$

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* We shall often denote time derivatives with dots, so $\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

must necessarily become a function of time, it is necessarily the case that $\ddot{\mathbf{r}}' \neq \ddot{\mathbf{r}}$, and Newton's second law will look different in the two references frames. Provided we stick to non-accelerating reference frames then $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}'$ and the right hand side of Newton's second law (2) is the same in all non-accelerating reference frames. Using an accelerating (*e.g.* a rotating reference frame) is not forbidden but it complicates Newton's second law and usually makes the analysis more tricky.

III. Newton's third law is often illustrated by considering someone stepping out of a small boat at a quay side. If the boat is not tied up and you put one leg on the quay side and push with the other to try and step onto the quay, watch out!

However in other situations Newton's third law is not always so obvious. Consider a planet orbiting the Sun. The Sun exerts a gravitational force on the planet and Newton's third law states that the planet must exert an equal and opposite force on the Sun. This force is in a line between the Sun and the planet but, as the planet is moving this line is rotating. Since it takes a finite time for any physical effect to pass between the planet and the Sun, due to the finite speed of light and Einstein's Special Theory of Relativity,[†] information about the planet's position always arrives at the Sun late. If the planet is a distance R from the Sun and c is the speed of light, no physical effect due to the planet can possibly influence the Sun in less than a time $t = R/c$. Is the force on the Sun due to the planet in the direction of where the planet is now or where it was at a time $t = R/c$ earlier? Answers to questions like this lie in the dynamical theory of gravity, developed by Einstein at the beginning of the 20th century and called the General Theory of Relativity. The answer is very surprising — the very notion of time is different for the planet and the Sun! However the speed of light is so large that, in everyday circumstances over distances that are not too large we can ignore such effects and treat the gravitational force as being instantaneous – an approximation known as "action at a distance".

Most of this course will be dedicated to solving Newton's second law for various kinds of common forces and we shall ignore the subtleties of General Relativity.

2. One Dimensional Motion

2.1 Forces which depend only on time.

We shall start with a class of problems where $\mathbf{F}(t)$ is independent of the position and velocity of the mass m. This class of problems is perhaps the simplest because we can just try to integrate Newton's second law directly.

For motion in one dimension we can label the position of the mass m by a single co-ordinate x and suppose the particle moves along a straight line. The position of m will in general be a function of time: $x(t)$, *e.g.* x could be increasing when m moves to the right and decreasing when m moves to the left along the line. The velocity is $v(t) = \dot{x}(t)$, which can be positive or negative depending on the direction of motion of m. Newton's second law is then

$$m\ddot{x} = F(t),$$

[†] The importance of inertial reference frames is central to Special Relativity — the theory is formulated in the class of inertial reference frames.

where the force only has one component and is a given function of time. For example if $F(t) = const$ then $m\ddot{x} = F$ and $\ddot{x} = \dot{v}$, so the acceleration

$$a = \dot{v} = \frac{F}{m} \tag{3}$$

is a constant and

$$\frac{dv}{dt} = a \qquad \Rightarrow \qquad v(t) = at + C$$

where C is a constant. Setting $t = 0$ we see that $C = v(0)$, the initial velocity of m. To save writing we shall denote the initial velocity by $v(0) = v_0$ (which could be either positive or negative, depending on the initial direction of m's motion), so

$$v(t) = at + v_0$$

changes linearly with time. What we have done here is to integrate equation (3),

$$\dot{v} = \frac{dv}{dt} = a \qquad \Rightarrow \qquad v(t) = \int_0^t a dt + v_0 = a \int_0^t dt + v_0 = at + v_0$$

and v_0 is a constant of integration. We can integrate again to get $x(t)$,

$$v(t) = \frac{dx}{dt} = at + v_0 \qquad \Rightarrow \qquad x(t) = \int_0^t (at + v_0)\, dt + x_0 = \frac{1}{2}at^2 + v_0 t + x_0,$$

where x_0 is a second constant of integration, $x_0 = x(0)$, which is the initial position of m at $t = 0$.

More generally, if $F(t)$ is some given non-trivial function of t (that is less trivial than the simple case of being just a constant, independent of t) we can try to integrate the equation

$$\frac{dv}{dt} = \frac{F(t)}{m} \qquad \Rightarrow \qquad v(t) = \int_0^t \frac{F(t)}{m} dt + v_0 = \frac{1}{m} \int_o^t F(t) dt + v_0$$

and, provided the integral $\int_0^t F(t)dt$ can be evaluated for the given function $F(t)$, we can evaluate $v(t)$. Once we have $v(t)$ explicitly then we can try the next step and evaluate $x(t) = \int_0^t v(t)dt + x_0$. For example suppose $F(t) = At + B$, with A and B constants, is a linear function. Then

$$v(t) = \frac{1}{2}At^2 + Bt + v_0$$

is quadratic and

$$x(t) = \frac{1}{6}At^3 + \frac{1}{2}Bt^2 + v_0 t + x_0$$

is cubic.

2.2 Forces which depend only on velocity

As a second illustration of the use of Newton's second law, consider the case of a mass m moving through a fluid, *eg.* air or water, subject to a frictional force which opposes its motion.

The velocity is $v(t) = \dot{x}(t)$, which can be positive or negative depending on the direction of motion of m. For small speeds it usually a good approximation to take the frictional force to be proportional to the speed, $F_{friction} = -c\dot{x} = -cv$, where c is a constant (called the *co-efficient of friction*). Then Newton's second law reads

$$m\ddot{x} = -c\dot{x} \qquad \Leftrightarrow \qquad m\dot{v} = -cv.$$

The sign is important here: if m is moving to the right, $\dot{x}$ is positive and the frictional force must oppose the motion, *i.e.* it must be negative; if m is moving to the left, $\dot{x}$ is negative and the frictional force must be positive. Hence c must be positive, otherwise m would accelerate in the same direction as it is moving, which is nonsense. Friction always makes objects decelerate, $F_{friction}$ must always have the opposite sign to that of v. Note that if v is ever zero then $\dot{v}$ is also zero and so v does not change with time and it stays zero, but if $v \neq 0$, then it necessarily must change and the speed decreases.

The equation $m\dot{v} = -cv$ is easy to solve for the unknown function $v(t)$. We write it suggestively as

$$m\frac{dv}{dt} = -cv \quad \Rightarrow \quad mdv = -cvdt \quad \Rightarrow \quad m\frac{dv}{v} = -cdt$$

as long as $v \neq 0$. Suppose that $v > 0$ and that the speed starts out as v_0 at time $t = 0$, then integrate both sides to get

$$m\int_{v_0}^{v} \frac{dv}{v} = -c\int_0^t dt \qquad \Rightarrow \qquad m(\ln v - \ln v_0) = -ct,$$

remembering that m and c are constants and can be taken outside the integrals. Here 'ln' signifies natural logarithms.* Since $\ln v - \ln v_0 = \ln(v/v_0)$ we have

$$\begin{aligned} \ln\left(\frac{v}{v_0}\right) &= -\frac{c}{m}t \\ \Rightarrow \qquad v(t) &= v_0 \exp\left(-\frac{c}{m}t\right). \end{aligned} \tag{4}$$

i.e. the velocity decreases exponentially toward zero as time increases.

Having obtained the velocity as a function of time it is now easy to integrate again to get $x(t)$. Suppose m starts off at $t = 0$ with speed v_0 from the point x_0, then

$$\frac{dx}{dt} = v_0 \exp\left(-\frac{ct}{m}\right) \qquad \Rightarrow \qquad dx = v_0 \exp\left(-\frac{ct}{m}\right) dt$$

* A subtle point to remember about problems of this type is that, if $v < 0$, then $\int \frac{dv}{v} = \ln|v| - \ln|v_0|$.

and, integrating again,

$$\int_{x_0}^{x} dx = v_0 \int_0^t \exp\left(-\frac{ct}{m}\right) dt \qquad \Rightarrow \qquad x - x_0 = -\frac{v_0 m}{c}\left(e^{-\frac{ct}{m}} - 1\right).$$

The x in this last equation is $x(t)$, so

$$x(t) = \frac{v_0 m}{c}\left(1 - e^{-\frac{ct}{m}}\right) + x_0. \tag{5}$$

While the velocity tends exponentially to zero but never reaches zero in any finite time, m only ever travels a finite distance even as $t \to \infty$,

$$x(t) - x_0 \underset{t\to\infty}{\longrightarrow} \frac{v_0 m}{c}.$$

Using (4) to eliminate the exponential in (5) we see that the velocity decreases linearly with x, until it vanishes,

$$v(x) = v_0 - \frac{c}{m}(x - x_0). \tag{6}$$

Notice that as the friction increases (c increases) the distance travelled decreases, while as the friction decreases (c decreases) the distance travelled increases. The constant c depends on a number of factors: it is greater in water than in air, for example, and even greater again in treacle. It also depends on the shape of the mass m, for example racing cars have spoilers on them to reduce air friction.

In this example the force depends only on v and is independent of t. For problems of this kind we can find $v(x)$ directly as follows. Since $v = \frac{dx}{dt}$ we can use the Chain Rule for differentiation to write

$$m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}.$$

So

$$m\dot{v} = F(v) \qquad \Leftrightarrow \qquad mv\frac{dv}{dx} = F(v) \qquad \Leftrightarrow \qquad m\int \frac{vdv}{F(v)} = \int dx.$$

In this case $F(v) = -cv$ is linear, so

$$-\frac{m}{c}\int dv = \int dx \qquad \Rightarrow \qquad -\frac{m}{c}(v - v_0) = x - x_0,$$

reproducing (6) above.

A slightly more complicated case is when the motion of m is vertical and the acceleration due to gravity must be taken into account. We shall use a co-ordinate y for vertical motion, reserving x for a horizontal motion when we discuss 2-dimensional problems later. Measuring y as increasing upwards the force due to gravity is then in the negative y-direction. In the absence of friction we have

$$m\ddot{y} = -mg \tag{7}$$

as the gravitational force is mg downwards. So

$$\ddot{y} = -g \qquad \Rightarrow \qquad \dot{y} = -gt + v_0 \qquad \Rightarrow \qquad y = -\frac{1}{2}gt^2 + v_0 t + y_0,$$

where $v_0 = \dot{y}(0)$ is the initial velocity (positive in the upwards direction) and $y_0 = y(0)$ the initial height, with $y = 0$ being ground level.

If m is dropped from a height h, then $y_0 = h$ and $v_0 = 0$, so

$$v(t) = -gt \qquad \text{and} \qquad y(t) = h - \frac{1}{2}gt^2$$

and m hits the ground when $y(t) = 0$, that is at time T with

$$T = \sqrt{\frac{2h}{g}}. \tag{8}$$

If m is thrown upwards from ground level, $y_0 = 0$, with velocity $v_0 > 0$, then

$$y(t) = v_0 t - \frac{1}{2}gt^2 \tag{9}$$

and the maximum height is reached when $\frac{dy}{dt} = 0$, that is when

$$t = t_{max=}v_0/g \qquad \Rightarrow \qquad y_{max} = \frac{v_0^2}{2g}. \tag{10}$$

The mass m hits the ground again when $y(t) = 0$ with $t > 0$ so, from (9), this happens at a time $T = \frac{2v_0}{g}$.

Including air friction equation (7) is modified to

$$m\ddot{y} = -mg - c\dot{y}, \tag{11}$$

with $c > 0$ so that the frictional force always has the opposite sign to that of $\dot{y}$, *i.e.* it opposes the motion.

The first thing to notice about equation (11) is that the acceleration vanishes when $\dot{y} = -mg/c$. When the motion is downward with constant velocity, $v_{term} = -mg/c$, the air friction exactly balances the gravitational force and m travels downwards with constant velocity: $|v_{term}|$ is known as the *terminal velocity* of the falling body.

Now we shall solve equation (11), which is a second order differential equation for the unknown function $y(t)$. It is simplest to use $v = \dot{y}$ and write

$$m\dot{v} = -mg - cv,$$

which is a first order differential equation for the unknown function $v(t)$. We can manipulate this equation as

$$m\frac{dv}{dt} = -mg - cv \quad \Rightarrow \quad mdv = -(mg + cv)dt \quad \Rightarrow \quad m\int_{v_0}^{v} \frac{dv}{mg + cv} = -\int_0^t dt$$

and integrate directly to get

$$\ln\left(\frac{mg+cv}{mg+cv_0}\right) = -\frac{c}{m}t \qquad \Rightarrow \qquad mg+cv = (mg+cv_0)\exp\left(-\frac{c}{m}t\right),$$

provided $v > -\frac{mg}{c}$ between time 0 and time t (this solution works for v_0 either positive or negative, provided only that $v_0 > -\frac{mg}{c}$). So

$$v(t) = \left(v_0 + \frac{mg}{c}\right)e^{-\frac{c}{m}t} - \frac{mg}{c} = v_0 e^{-\frac{c}{m}t} - \frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right) \qquad (12)$$
$$\xrightarrow[t\to\infty]{} \quad -\frac{mg}{c}.$$

Equation (12) can immediately be integrated to get $y(t)$:

$$\int_0^t v(t)dt = \left(v_0 + \frac{mg}{c}\right)\int_0^t e^{-\frac{c}{m}t}dt - \frac{mg}{c}\int_0^t dt = -\frac{m}{c}\left(v_0 + \frac{mg}{c}\right)\left(e^{-\frac{c}{m}t} - 1\right) - \frac{mg}{c}t$$

and

$$\int_0^t v(t)dt = \int_0^t \frac{dy}{dt}dt = y(t) - y(0)$$

so

$$y(t) = \frac{m}{c}\left(v_0 + \frac{mg}{c}\right)\left(1 - e^{-\frac{c}{m}t}\right) - \frac{mg}{c}t + y_0, \qquad (13)$$

where again $y_0 = y(0)$.

For example if m is dropped from an initial height h, then $y_0 = h$ and $v_0 = 0$, giving

$$v(t) = -\frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right)$$

and

$$y(t) = \frac{m^2 g}{c^2}\left(1 - e^{-\frac{c}{m}t}\right) - \frac{mg}{c}t + h.$$

Initially, when $t << m/c$, we can approximate $e^{-\frac{c}{m}t} \approx 1 - \frac{c}{m}t + \frac{1}{2}\frac{c^2}{m^2}t^2$ and

$$v(t) \approx -gt, \qquad y(t) \approx h - \frac{1}{2}gt^2,$$

so the motion starts out as if there was no air friction, because the velocity is initially zero and takes time to build up to a point where air friction becomes important.

At late times, when $t >> m/c$ the exponential is negligible so

$$y(t) \approx \frac{m^2 g}{c^2} - \frac{mg}{c}t + h,$$

and m falls with terminal velocity as if it had been dropped from a higher point $h + \frac{m^2 g}{c^2}$.

The mass m will hit the ground at a time T given by

$$0 = \frac{m^2 g}{c^2}\left(1 - e^{-\frac{c}{m}T}\right) - \frac{mg}{c}T + h$$

so

$$T = \frac{ch}{mg} + \frac{m}{c}\left(1 - e^{-\frac{c}{m}T}\right). \tag{14}$$

We cannot get $T(h)$ in closed form, but if $\frac{cT}{m} >> 1$ then we can ignore the exponential, because it is negligible, and

$$T = \frac{ch}{mg} + \frac{m}{c}.$$

At the other extreme, if $\frac{cT}{m} << 1$, we can expand the exponential as

$$e^{-\frac{c}{m}t} = 1 - \frac{ct}{m} + \frac{1}{2}\left(\frac{ct}{m}\right)^2 + \cdots$$

and in this case (14) gives

$$\begin{aligned} T &= \frac{ch}{mg} + \frac{m}{c}\left\{1 - \left(1 - \frac{cT}{m} + \frac{1}{2}\Big(\frac{cT}{m}\Big)^2 + \cdots\right)\right\} = \frac{ch}{mg} + T - \frac{1}{2}\frac{c}{m}T^2 + \ \cdots, \\ \Rightarrow \qquad 0 &= \frac{ch}{mg} - \frac{1}{2}\frac{c}{m}T^2 + \ \cdots. \end{aligned}$$

Ignoring terms of order c^2 or higher, which are indicated by the dots, this reduces to

$$0 = \frac{ch}{mg} - \frac{1}{2}\frac{c}{m}T^2 \qquad \Rightarrow \qquad T^2 = \frac{2h}{g}$$

which is the answer we got before (8), when we analysed the case with no friction ($c = 0$).

If m is thrown upwards from ground level at $t = 0$, with initial speed $v_0 > 0$, it reaches a maximum height when, using (12),

$$\frac{dy}{dt} = v(t) = 0 \quad \Rightarrow \quad \left(v_0 + \frac{mg}{c}\right)e^{-\frac{c}{m}t} - \frac{mg}{c} = 0 \quad \Rightarrow \quad t = -\frac{m}{c}\ln\left(\frac{mg}{cv_0 + mg}\right). \tag{15}$$

Putting this value of t into (13), with $y_0 = 0$, we get the maximum height

$$\begin{aligned} y_{Max} &= \frac{m}{c}\left(v_0 + \frac{mg}{c}\right)\left\{1 - \left(\frac{mg}{cv_0 + mg}\right)\right\} + \frac{m^2 g}{c^2}\ln\left(\frac{mg}{cv_0 + mg}\right) \\ &= \frac{mv_0}{c} - \frac{m^2 g}{c^2}\ln\left(\frac{cv_0 + mg}{mg}\right). \end{aligned} \tag{16}$$

Note that, when $\frac{ct}{m}$ is small, we can expand the exponential in (15) as

$$e^{-\frac{c}{m}t} = 1 - \frac{c}{m}t + \frac{1}{2}\Big(\frac{c}{m}t\Big)^2 + \ \cdots$$

to give

$$v_0\Big(1-\frac{c}{m}t+\cdots\Big)+\frac{mg}{c}\Big(1-\frac{c}{m}t+\frac{1}{2}\Big(\frac{c}{m}t\Big)^2+\ \cdots\Big)-\frac{mg}{c}=0$$
$$\Rightarrow \qquad v_0-gt+\frac{ct}{m}\left(\frac{1}{2}gt-v_0\right)+\ \cdots=0.$$

In the limit of zero viscosity, $c\to 0$, we are left with

$$v_0=gt \qquad \Rightarrow \qquad t=\frac{v_0}{g}$$

which is the answer we got before with $c=0$, when we ignored friction, (10). We can also find y_{Max} for small c, using

$$\ln\left(\frac{cv_0+mg}{mg}\right)=\ln\left(1+\frac{cv_0}{mg}\right)=\frac{cv_0}{mg}-\frac{1}{2}\Big(\frac{cv_0}{mg}\Big)^2+\frac{1}{3}\Big(\frac{cv_0}{mg}\Big)^3+\ \cdots$$

in (16) to give

$$y_{Max}=\frac{mv_0}{c}-\frac{m^2g}{c^2}\left\{\frac{cv_0}{mg}-\frac{1}{2}\Big(\frac{cv_0}{mg}\Big)^2+\frac{1}{3}\Big(\frac{cv_0}{mg}\Big)^3+\ \cdots\right\}=\frac{1}{2}\frac{v_0^2}{g}-\frac{1}{3}\left(\frac{cv_0^3}{mg^2}\right)+\ \cdots.$$

Again $c=0$ gives the previous result (10), $y_{Max}=\frac{v_0^2}{2g}$, and now we see that including a small amount of air friction reduces the maximum height by an amount proportional to v_0^3.

2.3 Forces which depend only on position.

The next example of the application of Newton's laws is when the force is independent of velocity and time and depends only on position, so $\mathbf{F}(\mathbf{r})$. Newton's second law is then

$$m\dot{v}=F(x).$$

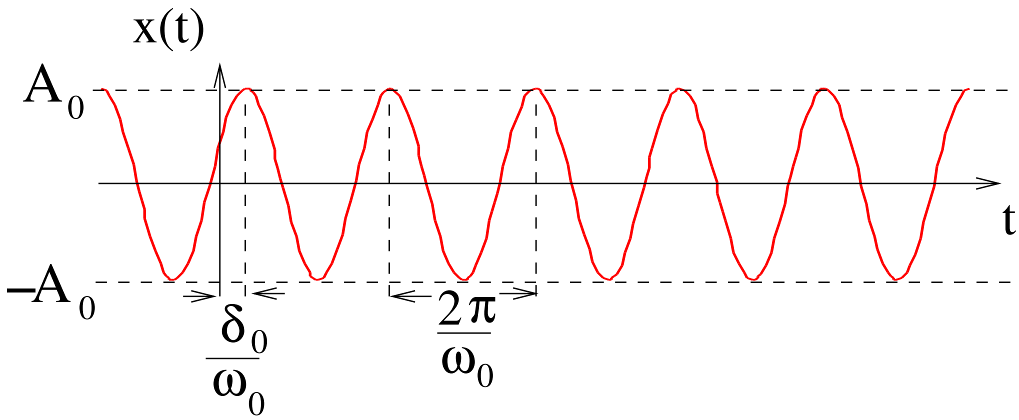
As an example consider Hooke's law, where $F=-kx$ is proportional to x, with $k>0$ a constant. If m is pushed away from $x=0$ then F will always push it back towards $x=0$ again. With $v=\dot{x}$ we have

$$m\ddot{x}=-kx \qquad \Rightarrow \qquad \ddot{x}=-\frac{k}{m}x$$

which has the general solution

$$x(t)=A_0\cos(\omega_0 t-\delta_0) \tag{17}$$

where $A_0\geq 0$ and $0\leq\delta_0<2\pi$ are constants, called the *amplitude* and the *phase* of the motion. The position oscillates as a function of time, between $x=A_0$ and $x=-A_0$, with angular frequency $\omega_0=\sqrt{\frac{k}{m}}$ (the frequency, in Hertz, is $\frac{\omega_0}{2\pi}$). Oscillatory motion of this kind is called *Simple Harmonic Motion* and is very common in Nature, provided the amplitude is not too large.



When the force depends only on position we can associate an energy with it, called the *potential energy*, and the total energy, kinetic plus potential, is conserved. The potential energy is derived as follows,

$$F(x) = m\frac{dv}{dt} \qquad \Rightarrow \qquad vF(x) = mv\frac{dv}{dt} = \frac{m}{2}\frac{d(v^2)}{dt} \qquad \Rightarrow \qquad \frac{dx}{dt}F(x) = \frac{m}{2}\frac{d(v^2)}{dt}. \quad (18)$$

Define

$$U(x) := -\int^x F(x)dx \qquad \Rightarrow \qquad F(x) = -\frac{dU}{dx}$$

so, using the Chain Rule,

$$\frac{dU}{dt} = \frac{dx}{dt}\frac{dU}{dx} = -\frac{dx}{dt}F(x).$$

Using this in (18) we get

$$-\frac{dU}{dt} = \frac{m}{2}\frac{d(v^2)}{dt} \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{1}{2}mv^2 + U(x)\right) = 0,$$

hence the total energy,

$$E = \frac{1}{2}mv^2 + U(x)$$

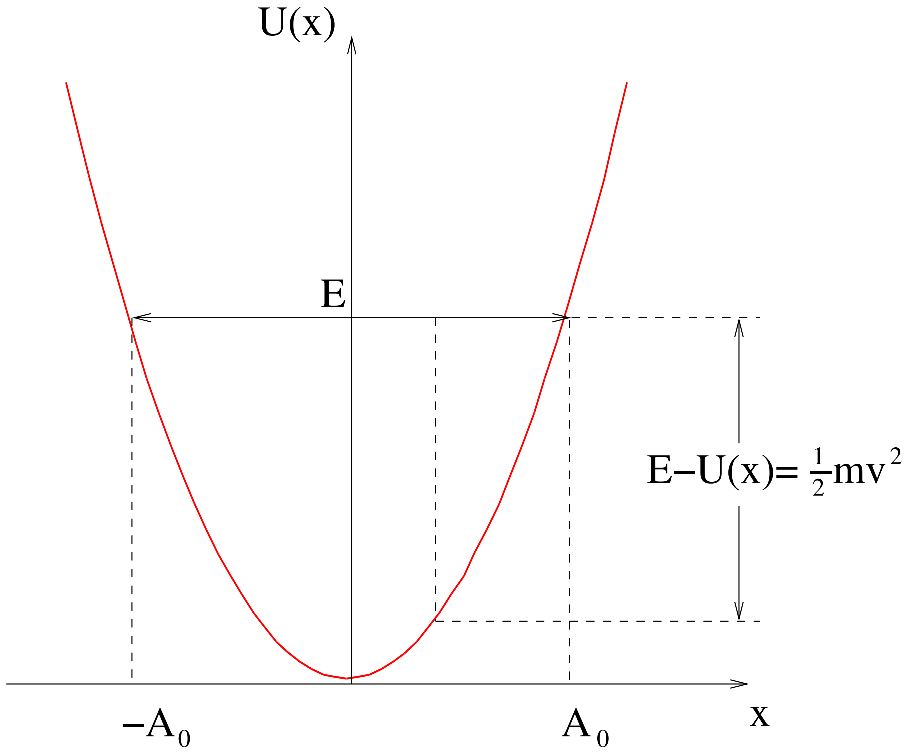
is constant during the motion. $U(x)$ is called the *potential energy* for the motion. For example Hooke's law has

$$F(x) = -kx = -\frac{d}{dx}\left(\frac{k}{2}x^2\right)$$

and the potential $U(x) = \frac{1}{2}kx^2$ is quadratic. With $x = A_0 \cos(\omega_0 t - \delta_0)$, $v = \dot{x} = -A_0\omega_0 \sin(\omega_0 - \delta_0)$ and

$$E = \frac{1}{2}\left(mA_0^2\omega_0^2 \sin^2(\omega_0 t - \delta_0) + kA_0^2 \cos^2(\omega_0 t - \delta_0)\right) = \frac{1}{2}mA_0^2\omega_0^2,$$

since $\omega_0^2 = k/m$. Hence the energy is a constant proportional to the amplitude squared.



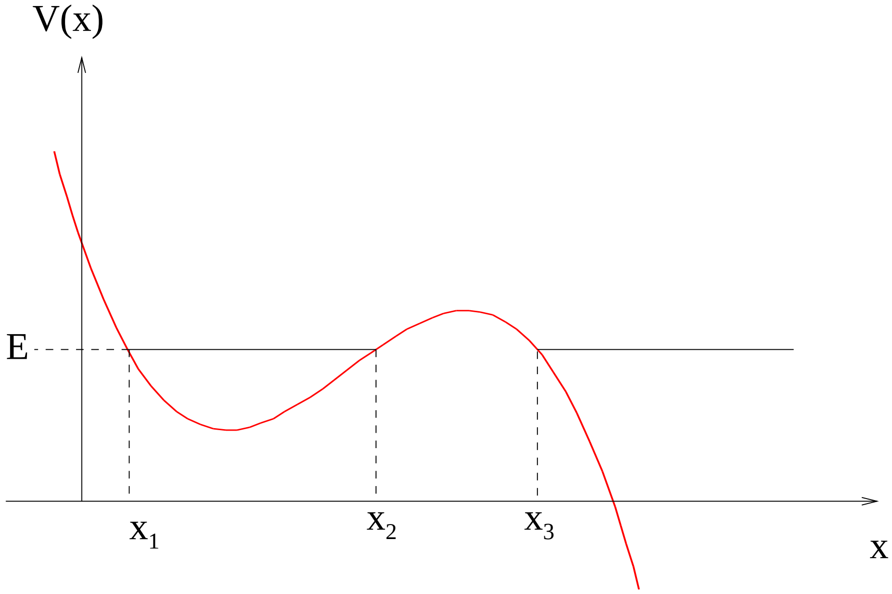
The mass oscillates between $x = A_0$ and $x = -A_0$. At any point $|x| \leq A_0$ the difference between E and $U(x)$ manifests itself as a non-zero velocity $v = \pm\sqrt{\frac{2\left(E-U(x)\right)}{m}}$, which vanishes at the end points $x = \pm A_0$. The speed is a maximum $|v| = \sqrt{\frac{2E}{m}}$ at $x = 0$.

Knowing the shape of the potential energy $U(x)$ as a function of x gives a qualitative understanding of the possible motion of a particle.

For example in the next figure if a particle has energy E then the region $x < x_1$ is forbidden, the particle will oscillate between $x = x_1$ and $x = x_2$ if placed between x_1 and x_2; the region $x_2 < x < x_3$ is forbidden; and the particle will accelerate to indefinitely large x if placed at any point $x > x_3$.

The particle has no forces on it at any point x_0 where the derivative of $U(x)$ vanishes, $F(x_0) = -U'(x_0) = 0$. The particle can be placed at rest at any such point and it will remain at rest in a state of equilibrium. If $U'(x_0) = 0$ and x_0 is a minimum of U, so $U''(x_0) > 0$, then $F(x) = -U'(x) < 0$ at any point $x > x_0$ close to x_0 on the right and $F(x) = -U'(x) > 0$ at any point $x < x_0$ close to x_0 on the left. Hence any displacement away from x_0 will result in a force that pushes the particle back to x_0. In a situation like this x_0 is a point of *stable equilibrium*. If x_0 is a maximum of U, so $U''(x_0) < 0$, then

$F(x) = -U'(x) > 0$ at any point $x > x_0$ close to x_0 on the right and $F(x) = -U'(x) < 0$ at any point $x < x_0$ close to x_0 on the left. Hence any displacement away from x_0 will result in a force that pushes the particle further away from x_0. In a situation like this x_0 is a point of *unstable equilibrium*. If $U'(x_0) = U''(x_0) = 0$ then x_0 is said to be a point of *neutral equilibrium*.



2.4 Forced Harmonic Oscillator

Frequently in physics we encounter situations in which a simple harmonic oscillator is driven by an external force. For example a child on a swing is a simple harmonic oscillator which will oscillate with some natural angular frequency ω_0 if left alone but if someone pushes the swing then they are applying a time-dependent external force $F(t)$. In such situations Newton's second law reads

$$m\ddot{x} = -kx + F(t) \qquad \Rightarrow \qquad m\ddot{x} + kx = F(t)$$

where the total force on the right hand side of the first equation is the sum of the force due to Hooke's law and the external force. For concreteness suppose the external force is of the form $F(t) = F_0 \cos(\Omega t)$ where F_0 and Ω are constants (Ω is called the forcing frequency) so the equation governing the motion is

$$m\ddot{x} + kx = F_0 \cos(\Omega t). \tag{19}$$

Our task is to solve this equation for the unknown function $x(t)$. Let us try looking for a solution of the form

$$x(t) = \widetilde{A} \cos(\Omega t)$$

with $\widetilde{A}$ a constant. Then $\dot{x} = -\Omega \widetilde{A} \sin(\Omega t)$ and $\ddot{x} = -\Omega^2 \widetilde{A} \cos(\Omega t)$, so equation (19) is

$$-m\Omega^2 \widetilde{A} \cos(\Omega t) + k\widetilde{A} \cos(\Omega t) = F_0 \cos(\Omega t) \qquad \Rightarrow \qquad (F_0 + m\widetilde{A}\Omega^2 - k\widetilde{A}) \cos(\Omega t) = 0.$$

We want this to be true for all t, which fixes $\widetilde{A}$ to be

$$\widetilde{A} = \frac{F_0}{k - m\Omega^2} = \frac{F_0}{m(\omega_0^2 - \Omega^2)} \tag{20}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the frequency of the unforced oscillator, that is the frequency found in section 2.3 when $F(t) = 0$. So we have found a solution of (19), namely

$$x(t) = \frac{F_0}{m(\omega_0^2 - \Omega^2)} \cos(\Omega t). \tag{21}$$

Note that $\widetilde{A}$ is positive for $\Omega < \omega_0$, negative for $\Omega > \omega_0$ and diverges for $\Omega = \omega_0$. * The magnitude of $\widetilde{A}$, $A(\Omega) = |\widetilde{A}|$, is called the *amplitude* of the oscillation. The mass m oscillates about $x = 0$ with maximum displacement equal to the amplitude $A(\Omega)$ and angular frequency equal to that of the forcing frequency Ω.

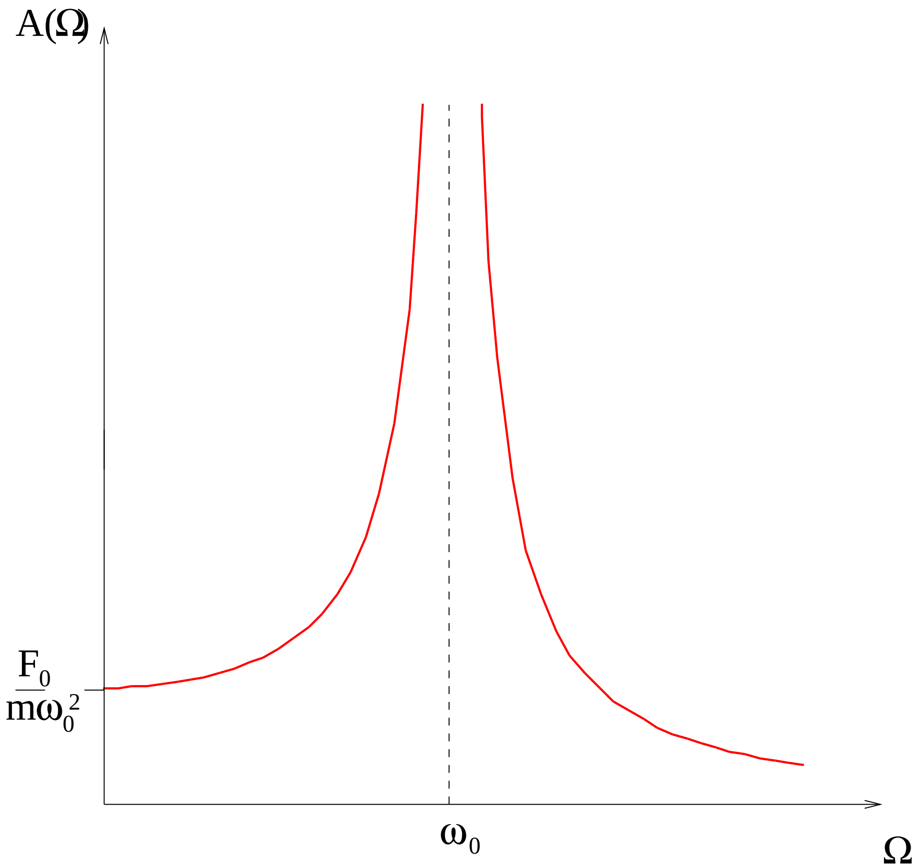
We can rewrite (21) in terms of the amplitude by introducing a phase into the argument of the cosine to take care of the sign,

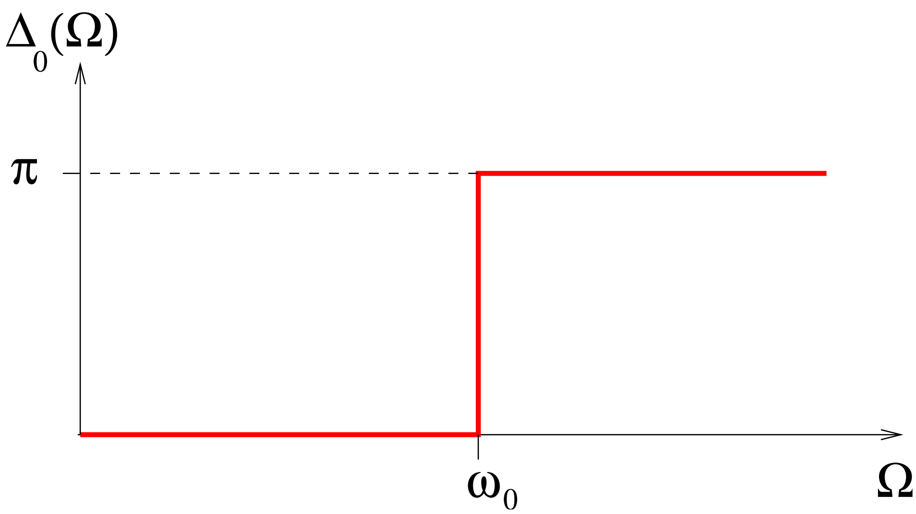
$$\Delta_0 = \begin{cases} 0, & \text{for } \Omega < \omega_0 \\ \pi, & \text{for } \Omega > \omega_0. \end{cases}$$

Then (21) can be written as

$$x(t) = \frac{F_0}{m\,|\omega_0^2 - \Omega^2|} \cos\big(\Omega t - \Delta_0\big) = A(\Omega)\cos\big(\Omega t - \Delta_0\big). \tag{22}$$

* Strictly speaking this solution is invalid when the forcing frequency Ω equals the natural frequency ω_0 but we can see that the amplitude of the oscillations gets arbitrarily large as Ω approaches ω_0. This is the phenomenon of *resonance* and it is very common in oscillatory systems. We shall tame the infinity at $\Omega = \omega_0$ in the next section where we shall consider a more realistic model by including friction.





The amplitude and phase are plotted above, as functions of the forcing frequency Ω. For $\Omega < \omega_0$ the motion of m is in phase with the driving force but for $\Omega > \omega_0$ it is 180° out of phase with the driving force.

We have found a solution to (19) but this is not the only one! We already know from section 2.3 that

$$x(t) = A_0 \cos(\omega_0 t - \delta_0) \tag{23}$$

satisfies

$$m\ddot{x} + kx = 0. \tag{24}$$

Hence adding (21) to (23) gives another solution of (19), since

$$\begin{aligned} x(t) &= \frac{F_0}{m(\omega_0^2 - \Omega^2)} \cos(\Omega t) + A_0 \cos(\omega_0 t - \delta_0) \qquad (25) \\ \Rightarrow \qquad m\ddot{x} + kx &= F_0 \cos(\Omega t) + 0 = F_0 \cos(\Omega t). \end{aligned}$$

For a given external driving force, with F_0 and Ω fixed, the solution (25) has two independent constants, A_0 and δ_0. By varying the constants A_0 and δ_0 in (25) we can change the initial position and velocity. For example suppose the mass starts off initially at $x(0) = 0$ and with initial speed zero, $\dot{x}(0) = 0$. Then

$$x(0) = \frac{F_0}{m(\omega_0^2 - \Omega^2)} + A_0 \cos(\delta_0) \tag{26}$$

and

$$\begin{aligned} \dot{x}(t) &= -\Omega \frac{F_0}{m(\omega_0^2 - \Omega^2)} \sin(\Omega t) - \omega_0 A_0 \sin(\omega_0 t - \delta_0) \\ \Rightarrow \qquad \dot{x}(0) &= \omega_0 A_0 \sin \delta_0 = 0. \qquad (27) \end{aligned}$$

Equation (26) tells us that $A_0 \neq 0$ and (27) then forces $\sin(\delta_0) = 0$. Choosing δ_0 in the range $0 \leq \delta_0 < 2\pi$ then leaves only two possibilities, $\delta_0 = 0$ and $\delta_0 = \pi$, giving $\cos(\delta_0) = 1$ and $\cos(\delta_0) = -1$ respectively. Suppose $\Omega < \omega_0$, then we can clearly satisfy (26) by setting $A_0 = \frac{F_0}{m(\omega_0^2 - \Omega^2)}$ and $\cos(\delta_0) = -1$. For $\Omega > \omega_0$, choose $A_0 = \frac{F_0}{m(\Omega^2 - \omega_0^2)}$ and $\cos(\delta_0) = +1$.

In either case (25) becomes

$$\begin{aligned} x(t) &= \frac{F_0}{m(\omega_0^2 - \Omega^2)} \big(\cos(\Omega t) - \cos(\omega_0 t)\big) \\ &= \frac{2F_0}{m(\omega_0^2 - \Omega^2)} \left\{ \sin\left(\frac{(\omega_0 - \Omega)t}{2}\right) \sin\left(\frac{(\omega_0 + \Omega)t}{2}\right) \right\}. \end{aligned}$$

Note that for $(\omega_0 - \Omega)t << 1$ small, so $\sin\left(\frac{(\omega_0 - \Omega)t}{2}\right) \approx \frac{(\omega_0 - \Omega)t}{2}$,

$$x(t) \approx \frac{F_0 t}{m(\omega_0 + \Omega)} \sin\left(\frac{(\omega_0 + \Omega)t}{2}\right).$$

In particular if $\Omega = \omega_0 + \delta\Omega$ with $\delta\Omega/\omega_0 << 1$ we are close to resonance and

$$x(t) \approx \frac{F_0 t}{2m\omega_0} \sin(\omega_0 t)$$

so the system oscillates with frequency ω_0 and the amplitude grows linearly with time. Eventually the amplitude becomes so large that Hooke's law breaks down and equation (19) is no longer valid.

2.6 Damped Harmonic Oscillator.

In the previous section we saw that the amplitude of the motion of a forced harmonic oscillator diverges when the forcing frequency Ω equals the natural frequency ω_0 of the oscillator. This is not physical as the amplitude can never be infinite. A more realistic model should include friction. If there is a frictional force proportional to velocity, $-c\dot{x}$ with $c > 0$, then Newton's second law (19) is modified to

$$m\ddot{x} = -kx - c\dot{x} + F_0 \cos(\Omega t) \qquad \Leftrightarrow \qquad m\ddot{x} + c\dot{x} + kx = F_0 \cos(\Omega t). \tag{28}$$

For simplicity we shall first study the case with no forcing, $F_0 = 0$,

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{29}$$

This is an example of a second order linear differential equation for the unknown function $x(t)$. It is second order because it involves the second derivative of $x(t)$ and it is linear because it only involves the function $x(t)$ and its derivatives linearly. This last feature is very important and makes the problem tractable. Non-linear differential equations, involving $x^2(t)$ or $\sin(x(t))$ are much harder and we shall not consider them in this course.

Before going on to find explicit solutions of (29) we first prove a useful little result that will help in the analysis. Suppose $x_1(t)$ are $x_2(t)$ are two solutions of (29) so we have that

$$\begin{aligned} m\ddot{x}_1 + c\dot{x}_1 + kx_1 &= 0 \\ m\ddot{x}_2 + c\dot{x}_2 + kx_2 &= 0. \end{aligned}$$

Now let $x(t) = a_1 x_1(t) + a_2 x_2(t)$ with a_1 and a_2 arbitrary constants. Then

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= m(a_1\ddot{x}_1 + a_2\ddot{x}_2) + c(a_1\dot{x}_1 + a_2\dot{x}_2) + k(a_1x_1 + a_2x_2) \\ &= a_1(m\ddot{x}_1 + c\dot{x}_1 + kx_1) + a_2(m\ddot{x}_2 + c\dot{x}_2 + kx_2) = 0. \end{aligned}$$

This means that, given two solutions of (29) any linear combination of them, with constant co-efficients, is another solution. In fact it can be shown, but will not be proven here, that it is sufficient to find two independent solutions (excluding the trivial case $x = 0$, which is clearly a solution) and *any* solution of (29) is a linear combination of them (x_1 and x_2 are *independent* if they are not constant multiples of each other, *i.e.* $x_2 = a_1 x_1$ is not independent of x_1 and so). This leads to the following result:

Theorem

The most general solution $x(t)$ of (29) is given by first finding any two independent solutions, $x_1(t)$ and $x_2(t)$, and then taking the linear combination

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

where a_1 and a_2 are arbitrary constants.

Notice the solution (25) that we found for (19) also had two independent constants, called A_0 and δ_0 there.

This is extremely useful. There is in fact an infinite number of solutions of (29), since there is an infinite number of pairs (a_1, a_2), but we only need to find two independent ones and we have them all! The two arbitrary constants a_1 and a_2 in the general solution of the differential equation (29) are associated with the fact that we lose information in differentiating a function — we throw away any constant part — so the solution of a differential equation is not unique. In a second derivative we lose two pieces of information, the constant part and the linear part. The arbitrary constants a_1 and a_2 represent two pieces of information about the function $x(t)$ that are lost in (29). The solution of a differential equation is not unique. We can however restrict to a unique solution by specifying more information, such as initial conditions for the motion. For example if we know the initial position and velocity of m, at $t = 0$, then this fixes the two constants a_1 and a_2 uniquely. Suppose we have found two solutions $x_1(t)$ and $x_2(t)$ and we are told that m starts out from $x_0 = x(0)$ at $t = 0$ with velocity $v_0 = \dot{x}(0)$. This tells us that

$$x(0) = a_1 x_1(0) + a_2 x_2(0) = x_0, \qquad \text{and} \qquad \dot{x}(0) = a_1 \dot{x}_1(0) + a_2 \dot{x}_2(0) = v_0$$

$$\Rightarrow \qquad \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

so, since we know x_0 and v_0, we can solve for a_1 and a_2 provided the matrix $\begin{pmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{pmatrix}$ is invertible:

$$a_1 = \frac{x_0 \dot{x}_2(0) - v_0 x_2(0)}{x_1(0)\dot{x}_2(0) - x_2(0)\dot{x}_1(0)}$$

$$a_2 = \frac{v_0 x_1(0) - x_0 \dot{x}_1(0)}{x_1(0)\dot{x}_2(0) - x_2(0)\dot{x}_1(0)},$$

which fixes the constants a_1 and a_2 uniquely in terms of the initial conditions.*

An example of this procedure is the solution presented in section 2.3, with $c = 0$ and no friction,

$$\ddot{x} + \omega_0^2 x = 0,$$

with $\omega_0^2 = \frac{k}{m}$. It is easy to check that $x_1(t) = \cos(\omega_0 t)$ and $x_2(t) = \sin(\omega_0 t)$ are two independent solutions of this equation. Hence the general solution is

$$x(t) = a_1 \cos(\omega_0 t) + a_2 \sin(\omega_0 t).$$

Indeed the solution (17)

$$A_0 \cos(\omega_0 t - \delta_0) = A_0 \big(\cos(\omega_0 t)\cos\delta_0 + \sin(\omega_0 t)\sin\delta_0\big) = A_0 \cos\delta_0 \cos(\omega_0 t) + A_0 \sin\delta_0 \sin(\omega_0 t)$$

is of precisely this form, with $a_1 = A_0 \cos\delta_0$ and $a_2 = A_0 \sin\delta_0$. Imposing, for example, the initial condition that the mass m starts off a distance d from the origin with zero velocity, $x(0) = d$ and $\dot{x}(0) = 0$, we see that

$$d = x(0) = a_1 \qquad \text{and} \qquad 0 = \dot{x}(0) = \omega_0 a_2$$

* This does not work if $x_1(0)\dot{x}_2(0) - x_2(0)\dot{x}_1(0) = 0$.

so $a_1 = d$ and $a_2 = 0$, giving the unique solution

$$x(t) = d\cos(\omega_0 t)$$

with these initial conditions.

Returning now to the case with friction (29) we need to find two independent solutions. To do this we shall use a mathematical trick which is often useful in problems involving oscillations, we shall use complex numbers. First we try to find solutions of the form

$$x(t) = e^{\lambda t},$$

with λ a constant. Using this form in (29) gives

$$(m\lambda^2 + c\lambda + k)e^{\lambda t} = 0,$$

since $\dot{x} = \lambda e^{\lambda t}$ and $\ddot{x} = \lambda^2 e^{\lambda t}$. For this to be true for all t it must be the case that

$$m\lambda^2 + c\lambda + k = 0.$$

Notice what this has achieved, we have succeeded in turning a differential equation for an unknown function $x(t)$ into an algebraic equation for one unknown constant λ which is easy to solve. Divide through by m and let $\gamma = c/m > 0$ so the equation becomes

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0$$

which has two solutions

$$\lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}.$$

We have thus found two solutions of the differential equation, namely

$$x_1(t) = e^{\lambda_+ t} \qquad \text{and} \qquad x_2(t) = e^{\lambda_- t},$$

and the general solution is a linear combination of these with constant co-efficients, which we shall call C_1 and C_2 here,

$$x(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}.$$

Notice however that these solutions are very different depending on whether $\gamma^2 - 4\omega_0^2$ is positive, negative or zero.

i) If the friction is small so c is not large, in particular if $\gamma < 2\omega_0$, then let $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ giving

$$\lambda_\pm = -\frac{\gamma}{2} \pm i\omega$$

so

$$x(t) = e^{-\frac{\gamma t}{2}}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}),$$

with

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

At first sight $x(t)$ now looks complex, but this need not be the case. We can get a real solution simply by choosing C_1 and C_2 to be complex numbers with C_2 the complex conjugate of C_1, so $C_2 = (C_1)^*$. This gives a solution with two real constants in it: denote the real part of C_1 by $\frac{a_1}{2}$ and the imaginary part by $-\frac{a_2}{2}$ (so $C_1 = \frac{a_1 - ia_2}{2}$, $C_2 = \frac{a_1+ia_2}{2}$) then

$$\begin{aligned} x(t) &= e^{-\frac{\gamma t}{2}} \left\{ \left(\frac{a_1 - ia_2}{2}\right)\big(\cos(\omega t) + i\sin(\omega t)\big) + \left(\frac{a_1 + ia_2}{2}\right)\big(\cos(\omega t) - i\sin(\omega t)\big) \right\} \\ &= e^{-\frac{\gamma t}{2}} (a_1\cos(\omega t) + a_2\sin(\omega t)) \end{aligned}$$

is a linear combination of two real solutions, namely $e^{-\frac{\gamma t}{2}}\cos(\omega t)$ and $e^{-\frac{\gamma t}{2}}\sin(\omega t)$.

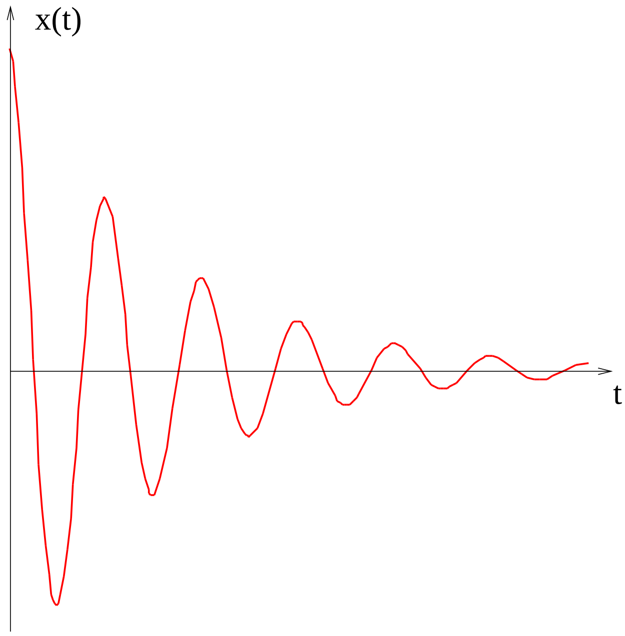
Alternatively we can express this in terms of two constants* $A_\gamma > 0$ and $0 \le \delta_\gamma < 2\pi$ defined as $A_\gamma^2 = a_1^2 + a_2^2$ and $\tan\delta_\gamma = a_2/a_1$, so $a_1 = A_\gamma\cos\delta_\gamma$ and $a_2 = A_\gamma\sin\delta_\gamma$, giving

$$x(t) = A_\gamma e^{-\frac{\gamma t}{2}}\cos(\omega t - \delta_\gamma). \tag{30}$$

This is oscillatory motion with an amplitude $A_\gamma e^{-\frac{\gamma t}{2}}$ that decreases to zero exponentially with time. The mass starts out a distance $d = A_\gamma\cos\delta_\gamma$ from the origin and oscillates with angular frequency $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, losing energy with time due to friction, the decay constant in the exponential $\frac{\gamma}{2}$ being proportional to the co-efficient of friction, $\frac{\gamma}{2} = \frac{c}{2m}$.

Behaviour like this is common in many oscillating systems, *e.g.* the oscillations of a clock pendulum will decrease with time and eventually cease if the clock is not wound up. The motion looks something like the following:

* The subscript γ on A_γ and δ_γ is just to distinguish them from the constants A_0 and δ_0 that were introduced in the solution of the undamped oscillator, $\gamma = 0$.



ii) If the friction is large enough and $\gamma > 2\omega_0$ then the roots $\lambda_\pm$ are both real and negative and the general solution is

$$x(t) = a_1 e^{\lambda_+ t} + a_2 e^{\lambda_- t}$$

with a_1 and a_2 real. The first term decays faster than the second and after a while, when $t >> 1/\lambda_+$,

$$x(t) \approx a_2 e^{\lambda_- t}$$

to a very good approximation. If the mass starts off at $t = 0$ a distance $d = a_1 + a_2 > 0$ from the origin it just relaxes exponentially to $x = 0$. It never overshoots and x never becomes negative because the friction is strong enough to dampen any oscillations.

iii) $\gamma = 2\omega_0$ is a critical value that separates oscillating motion, case i) above, with completely damped motion, case ii) above. When this happens $\lambda_+ = \lambda_- = -\gamma/2$ and we only have one solution,

$$x(t) = a_1 e^{-\frac{\gamma t}{2}}.$$

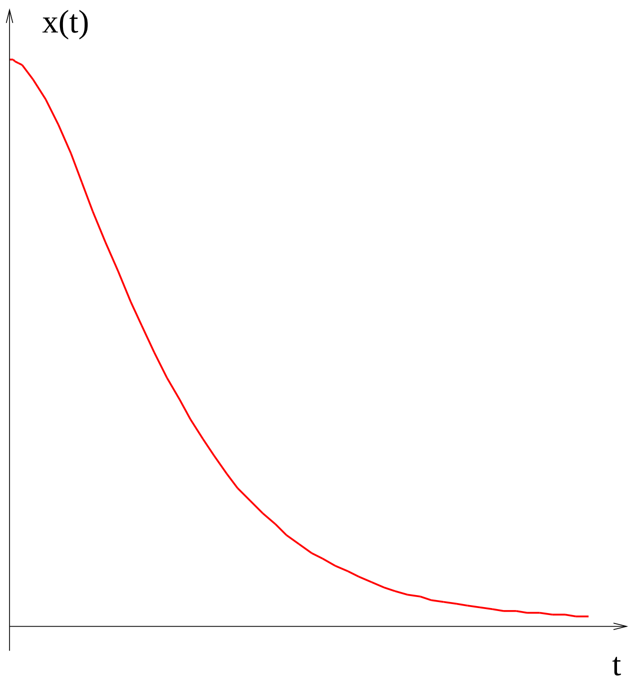
In this case, and only in this case, a second solution of (29) is $x = te^{-\frac{\gamma t}{2}}$ and the general solution is then

$$x(t) = a_1 e^{-\frac{\gamma t}{2}} + a_2 t e^{-\frac{\gamma t}{2}}.$$

If the mass starts at $t = 0$ from $x = d$ with zero velocity, then $a_1 = d$ and $-\frac{\gamma}{2} a_1 + a_2 = 0$, so

$$x(t) = d\left(1 + \frac{\gamma t}{2}\right) e^{-\frac{\gamma t}{2}}.$$

The mass relaxes back to $x = 0$ as shown below. This case is known as *critically damped* and can be important in many engineering situations: for example in a sensitive voltmeter we want the needle to be fast and responsive, so the friction should be low, but we also want oscillations to be damped so that the needle settles down to a stationary position as soon as possible after the voltage is varied, the optimal damping is critical.



2.7 Forced Damped Harmonic Oscillator

Now consider a damped harmonic oscillator that is driven by an external force of frequency Ω, equation (28). Dividing through by m this is

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\Omega t). \tag{31}$$

In section 2.4, for $\gamma = 0$, we found a solution by trying $x = \tilde{A} \cos(\Omega t)$, but this doesn't quite work here. Instead we try the slightly more general form

$$x = A \cos(\Omega t - \Delta),$$

with Δ a constant (the $\gamma = 0$ case was written this way in equation (22)). Substituting this guess into (31) gives

$$\begin{aligned}
& -\Omega^2 A \cos(\Omega t - \Delta) - \Omega\gamma A \sin(\Omega t - \Delta) + \omega_0^2 A \cos(\Omega t - \Delta) = \frac{F_0}{m} \cos(\Omega t) \\
\Rightarrow \quad & A(\omega_0^2 - \Omega^2) \cos(\Omega t - \Delta) - A\Omega\gamma \sin(\Omega t - \Delta) = \frac{F_0}{m} \cos(\Omega t) \\
\Rightarrow \quad & A\big\{(\omega_0^2 - \Omega^2) \cos\Delta + \Omega\gamma \sin\Delta\big\} \cos(\Omega t) + A\big\{(\omega_0^2 - \Omega^2) \sin\Delta - \Omega\gamma \cos\Delta\big\} \sin(\Omega t) \\
& \qquad = \frac{F_0}{m} \cos(\Omega t).
\end{aligned}$$

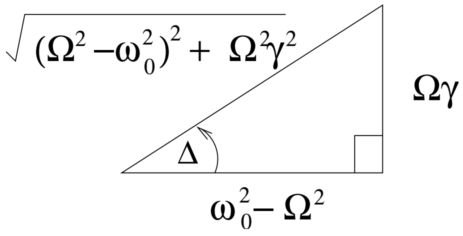
We can satisfy this for all t by setting

$$A = \frac{F_0}{m\left\{(\omega_0^2 - \Omega^2)\cos\Delta + \Omega\gamma\sin\Delta\right\}} \tag{32}$$

and

$$(\omega_0^2 - \Omega^2)\sin\Delta - \Omega\gamma\cos\Delta = 0 \qquad \Rightarrow \qquad \tan\Delta = \frac{\Omega\gamma}{\omega_0^2 - \Omega^2}. \tag{33}$$

Simple trigonometry allows us to calculate $\sin\Delta$ and $\cos\Delta$ from the figure below



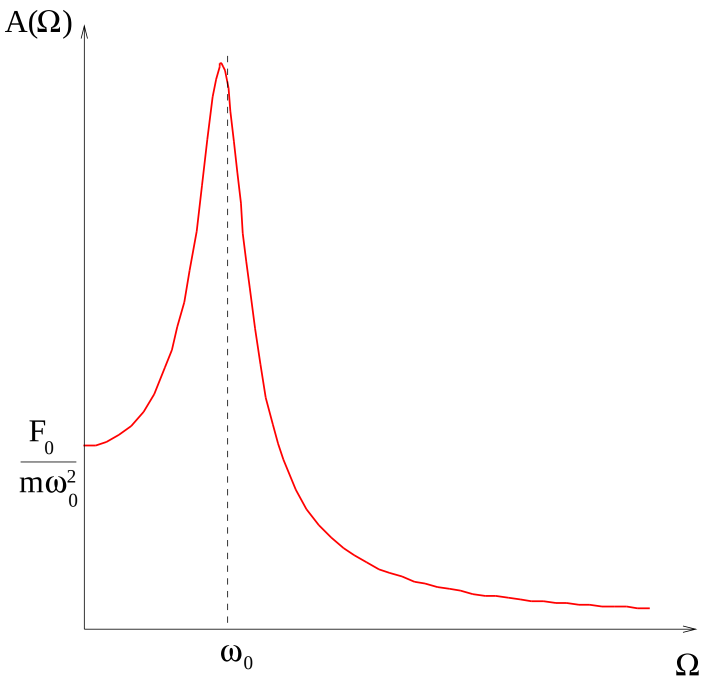
to get

$$\sin\Delta = \frac{\Omega\gamma}{\sqrt{(\Omega^2-\omega_0^2)^2+\Omega^2\gamma^2}}, \qquad \cos\Delta = \frac{\omega_0^2-\Omega^2}{\sqrt{(\Omega^2-\omega_0^2)^2+\Omega^2\gamma^2}}$$

and these can be used in (32) to give

$$A = \frac{F_0}{m\sqrt{(\Omega^2-\omega_0^2)^2+\Omega^2\gamma^2}}.$$

When the friction vanishes, $\gamma = 0$ and $\Delta = 0$, this reduces to (20), provided we choose $\sqrt{(\Omega^2-\omega_0^2)^2} = \Omega^2 - \omega_0^2$. When the friction is non-zero the amplitude is now finite when $\Omega = \omega_0$ and we see that friction removes the infinity that we found at resonance in the friction free case.



The amplitude peaks at a frequency

$$\Omega_{Max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} = \sqrt{\omega^2 - \frac{\gamma^2}{4}},$$

where $\frac{dA(\Omega)}{d\Omega} = 0$. For small damping, this very close to the natural undamped frequency ω_0.

So we have found a solution, namely

$$x(t) = \frac{F_0}{m\sqrt{(\Omega^2 - \omega_0^2)^2 + \Omega^2\gamma^2}} \cos(\Omega t - \Delta) \tag{34}$$

with Δ determined by (33). However this is not the only solution. In solving equation (19) for the forced harmonic oscillator without damping we found that the most general solution, (25), had two arbitrary constants, an amplitude A_0 and a phase δ_0, which we then fixed by choosing specific initial conditions ($A_0 \cos(\omega_0 t - \delta_0)$ remember was the general solution of the undamped equation with no forcing, (24)). A similar situation applies here. The general solution of the damped oscillator equation with no forcing is (30), $A_\gamma e^{-\frac{\gamma t}{2}} \cos(\omega t - \delta_\gamma)$. Adding this to the solution of (31) that we have just found, namely (34), gives

$$x(t) = A_\gamma e^{-\frac{\gamma t}{2}} \cos(\omega t - \delta_\gamma) + \frac{F_0}{m\sqrt{(\Omega^2 - \omega_0^2)^2 + \Omega^2\gamma^2}} \cos(\Omega t - \Delta). \tag{35}$$

This is a solution of (31), since

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 = 0 + \frac{F_0}{m}\cos(\Omega t) = \frac{F_0}{m}\cos(\Omega t).$$

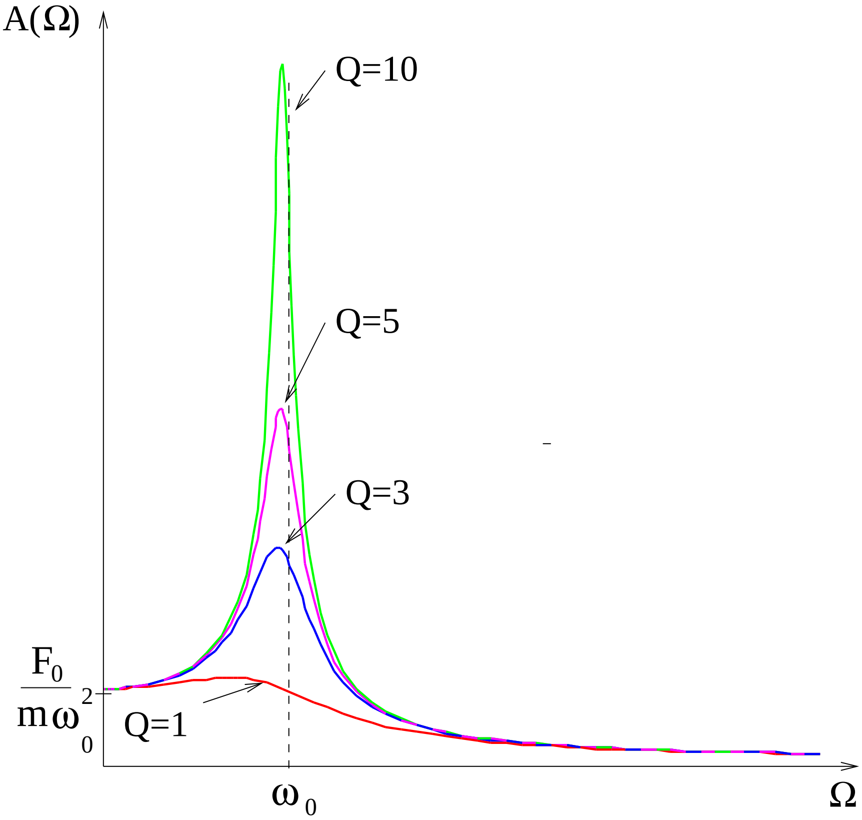
Equation (35) is in fact the most general solution of the forced damped harmonic oscillator equation (31). It has two arbitrary constants, an amplitude A_γ and a phase δ_γ which, as for the undamped oscillator, can be fixed by choosing specific initial conditions. Notice however that, because of the damping factor $e^{-\frac{\gamma t}{2}}$, the first term dies away with time and becomes negligible for $t >> \frac{2}{\gamma}$, at late times the solution becomes insensitive to the initial conditions and always settles down to (34). Perturbations of (34) induced by the first term on the right hand side of (35) are called *transients*.

Let us focus on the late time behaviour (34),

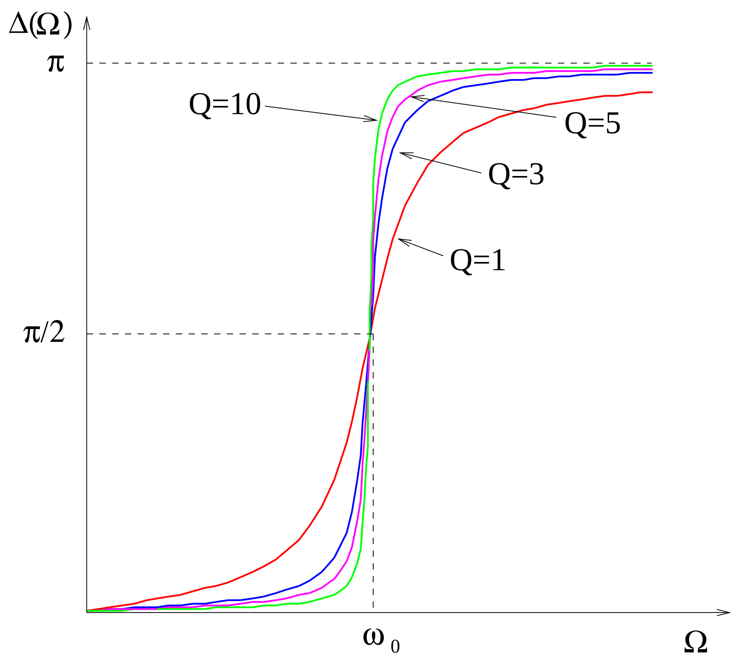
$$x(t) = \frac{F_0}{m\sqrt{(\Omega^2 - \omega_0^2)^2 + \Omega^2\gamma^2}} \cos(\Omega t - \Delta).$$

This has an amplitude

$$A(\Omega) = \frac{F_0}{m} \frac{1}{\sqrt{(\Omega^2 - \omega_0^2)^2 + \Omega^2\gamma^2}} = \frac{F_0}{m\Omega\omega_0} \frac{1}{\sqrt{\left(\frac{\Omega}{\omega_0} - \frac{\omega_0}{\Omega}\right)^2 + Q^{-2}}}$$



A(Ω)
Q=10
Q=5
Q=3
F0
mω0 2
Q=1
ω0
Ω



Δ(Ω)
π
Q=10
Q=5
Q=3
Q=1
π/2
ω0
Ω

and a phase

$$\Delta = \tan^{-1}\left(\frac{\Omega\gamma}{\omega_0^2-\Omega^2}\right) = \tan^{-1}\left(\frac{\Omega\omega_0}{Q\big(\omega_0^2-\Omega^2\big)}\right) = \tan^{-1}\left(\frac{(\Omega/\omega_0)}{Q\big(1-(\frac{\Omega^2}{\omega_0^2})\big)}\right),$$

where we have defined $Q := \frac{\omega_0}{\gamma}$ which is called the *quality factor* of the oscillator. A small Q means that the oscillator is heavily damped, a large Q means that there is very little damping, with $Q \to \infty$ the undamped case. The amplitude and phase, as functions of the forcing frequency Ω, are plotted above, for some different values of Q.

The amplitude of the late-time behaviour oscillations (34) is independent of time, but energy is being dissipated due to friction. This is because the external force is continually supplying energy to the oscillator to compensate for the frictional dissipation. We can determine the rate at which energy is supplied from the power absorbed by the oscillator, using the formula $Power = Force \times Velocity$. The external force is

$$F_0\cos(\Omega t)$$

while the velocity is

$$\dot{x} = -\frac{\Omega F_0}{m\sqrt{\big(\Omega^2-\omega_0^2\big)^2+\Omega^2\gamma^2}}\sin(\Omega t-\Delta)$$

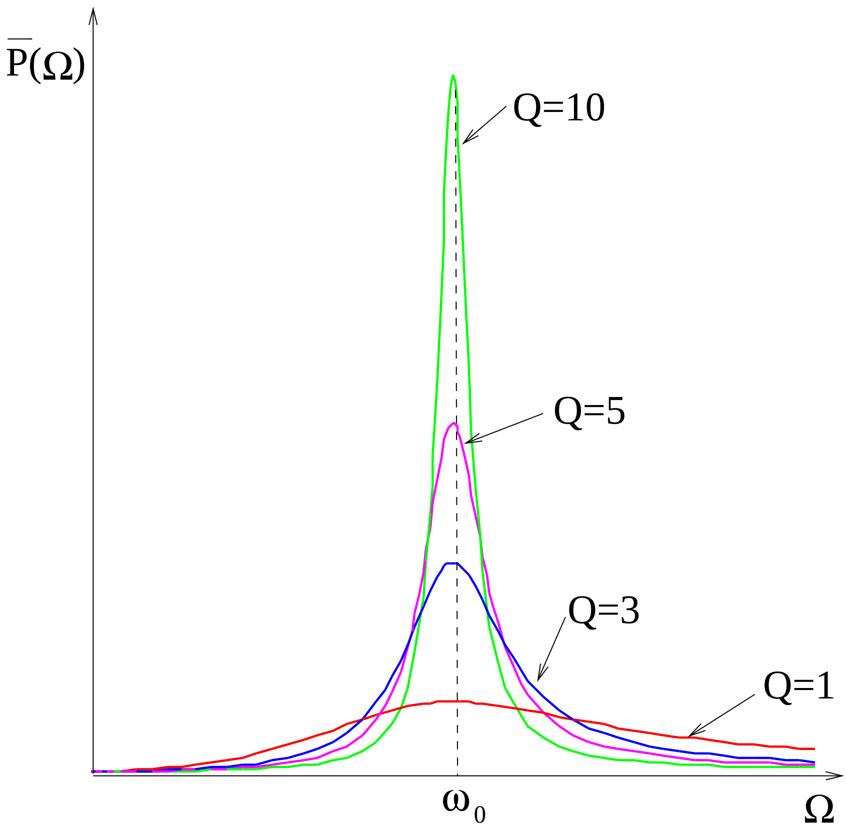
so the power is

$$\begin{aligned} P(t) &= -\frac{\Omega F_0^2}{m\sqrt{\big(\Omega^2-\omega_0^2\big)^2+\Omega^2\gamma^2}}\cos(\Omega t)\sin(\Omega t-\Delta) \\ &= -\frac{\Omega F_0^2}{m\sqrt{\big(\Omega^2-\omega_0^2\big)^2+\Omega^2\gamma^2}}\big(\cos(\Omega t)\sin(\Omega t)\cos(\Delta)-\cos^2(\Omega t)\sin(\Delta)\big). \end{aligned}$$

This is a complicated oscillating function of time and it is more useful to calculate the average power, $\overline{P}$, over a single cycle of period $T = \frac{2\pi}{\Omega}$, which is

$$\overline{P} = \frac{1}{T}\int_0^T P(t)dt = \frac{\Omega F_0^2}{2m\sqrt{\big(\Omega^2-\omega_0^2\big)^2+\Omega^2\gamma^2}}\sin(\Delta) = \frac{F_0^2}{2m\omega_0 Q}\frac{1}{\left\{\big(\frac{\Omega}{\omega_0}-\frac{\omega_0}{\Omega}\big)^2+Q^{-2}\right\}}.$$

The average power absorbed by the oscillator per cycle is shown below, as a function of the forcing frequency Ω, for different values of Q. The larger Q is the higher and narrower the peak, indicating that an oscillator with large Q will absorb a lot of energy only at driving frequencies close to the natural frequency of the undamped oscillator, ω_0.



We have been studying (31) which is a special case of a more general class of problems, which we write as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{36}$$

with $F(t)$ some specified function of time. Differential equations of this type are called *linear*, *second-order*, *inhomogeneous* equations: *linear* because the function x, and its derivatives, only appear linearly in the equation, there is nothing like x^2 or $\sin(\dot{x})$ in the equation, which would be non-linear; *second-order* because the highest derivative of x appearing in the equation is a second-order derivative; *inhomogeneous* because the presence of the function $F(t)$ on the right hand side means that, if we have a solution $x(t)$ then simply shifting t to get $x(t+c)$, with c a constant, is in general *not* another solution — this is in contrast to the *homogeneous* equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{37}$$

for which, having found a solution $x(t)$, $x(t+c)$ *is* another solution.

We shall describe some useful mathematical results for solving equations like (36). We already know from section **2.6** that the problem of finding the most general solution of a homogeneous equation is greatly simplified by first finding any two independent solutions and then taking linear combinations of them. This reduces the amount of work considerably because we only need to find two independent solutions and we are done. It is not quite so simple for an inhomogeneous equation, but it is not much harder either. We first note that, if $x_1(t)$ is a solution of (37) and $x_2(t)$ a solution of (36) then $x(t) = x_1(t) + x_2(t)$ is another solution of (36), since*

$$\begin{aligned}\ddot{x} + \gamma\dot{x} + \omega_0^2 x &= (\ddot{x}_1 + \ddot{x}_2) + \gamma(\dot{x}_1 + \dot{x}_2) + \omega_0^2(x_1 + x_2) \\ &= (\ddot{x}_1 + \gamma\dot{x}_1 + \omega_0^2 x_1) + (\ddot{x}_2 + \gamma\dot{x}_2 + \omega_0^2 x_2) = 0 + \frac{F(t)}{m} = \frac{F(t)}{m}.\end{aligned}$$

We shall state, without proof, the following theorem:

Theorem

> The most general solution $x(t)$ of (36) is given by first finding any solution of (36) and adding to this the most general solution of (37).

So, assuming we can find any two independent solutions, $x_1(t)$ and $x_2(t)$, of (37) and any solution, x_3, of (36) then the most general solution of (36) is

$$x(t) = a_1 x_1(t) + a_2 x_2(t) + x_3(t), \tag{38}$$

with a_1 and a_2 arbitrary constants. In this construction $x_3(t)$ is called a *particular* solution of (36).†

The problem now reduces to finding two independent solutions of (37) and one particular solution of (36). We have already done the first part in section **2.6** — to find two solutions of (37) try a solution of the form $x = e^{\lambda t}$ which reduces the differential equation to an algebraic one,

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0 \qquad \Rightarrow \qquad \lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}.$$

If $\gamma^2 > 4\omega_2$ both roots are real and the general solution of the homogeneous equation is

$$x = a_1 e^{\lambda_+ t} + a_2 e^{\lambda_- t};$$

if $\gamma^2 < 4\omega_2$ the roots are complex, one being the complex conjugate of the other, $\lambda_\pm = -\frac{\gamma}{2} \pm i\omega$ with $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, and the general solution is

$$x = e^{-\frac{\gamma}{2}t}\big(a_1\cos(\omega t) + a_2\sin(\omega t)\big);$$

---

* Note that this would *not* be true if $x_1(t)$ and $x_2(t)$ were both solutions of (36), since we would then have $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 2\frac{F(t)}{m}$, which is not equation (36).

† Note that a constant multiple, $a_3 x_3(t)$ with a_3 a constant, of $x_3(t)$ is *not* a solution of (36) unless $a_3 = 1$, since

$$a_3\ddot{x}_3 + a_3\gamma\dot{x}_3 + a_3\omega_0^2 x_3 = a_3\frac{F(t)}{m}.$$

if $\gamma^2 = 4\omega_2$ the general solution of the homogeneous equation is

$$x = \big(a_1 + a_2 t\big)e^{-\frac{\gamma}{2}t}.$$

Finally we must find particular solutions, $x_3(t)$, of the inhomogeneous equation (36). The form of these depends on the functional form of $F(t)$, we have already studied the case where $F(t) = F_0\cos(\Omega t)$ is an oscillating function and here we shall simply list a few other possibilities.

1) $F(t)$ a polynomial in t of order k with $F(0) \neq 0$: try $x_3(t)$ a polynomial of degree k and fix the co-efficients by requiring that $x(t)$ satisfy (36). An example with $k = 2$ is

$$\ddot{x} + 5\dot{x} + 6x = 6t^2 - \frac{1}{3}.$$

We obtain two independent solutions of the corresponding homogeneous equation

$$\ddot{x} + 5\dot{x} + 6x = 0$$

as we did before, try $x(t) = e^{\lambda t}$ for which

$$(\lambda^2 + 5\lambda + 6)e^{\lambda t} = 0 \qquad \Rightarrow \qquad (\lambda + 3)(\lambda + 2) = 0 \qquad \Rightarrow \qquad \lambda = -2 \quad \text{or} \quad -3,$$

which gives two independent solutions, $x_1(t) = e^{-3t}$ and $x_2(t) = e^{-2t}$.
For a particular solution try $x = At^2 + Bt + C$, with A, B and C constants. Then $\dot{x} = 2At + B$ and $\ddot{x} = 2A$, so

$$\ddot{x}+5\dot{x}+6x = 2A+5(2At+B)+6(At^2+Bt+C) = 6At^2+(10A+6B)t+(2A+5B+6C).$$

demanding that this equals $6t^2 - \frac{1}{3}$ requires, equating co-efficients,

$$\begin{aligned} &6At^2 + (10A + 6B)t + (2A + 5B + 6C) = 6t^2 - \frac{1}{3} \\ &\qquad \Rightarrow \qquad 6A = 1, \qquad 10A + 6B = 0, \qquad 2A + 5B + 6C = -\frac{1}{3}, \end{aligned}$$

so $A = 1$, $B = -\frac{5}{3}$ and $C = 1$, so a particular solution is

$$x = t^2 - \frac{5}{3}t + 1.$$

Of course this is only one solution, it is not the most general one, but we can use it as $x_3(t)$ in (38) to get the most general solution, namely

$$x(t) = a_1 e^{-3t} + a_2 e^{-2t} + t^2 - \frac{5}{3}t + 1.$$

Determining the two constants a_1 and a_2 requires more information, for example, suppose we are told that $x(0) = 1$ and $\dot{x}(0) = 0$. Then

$$x(0) = a_1 + a_2 + 1 = 1 \qquad \Rightarrow \qquad a_1 = -a_2$$

and

$$\dot{x}(t) = -3a_1 e^{-3t} - 2a_2 e^{-2t} + 2t - \frac{5}{3} \quad \Rightarrow \quad \dot{x}(0) = -3a_1 - 2a_2 - \frac{5}{3} = 0$$

$$\Rightarrow \qquad a_1 = -\frac{5}{3}, \qquad \text{since } a_2 = -a_1,$$

and the unique solution satisfying these initial conditions is

$$x(t) = \frac{5}{3}\left(e^{-2t} - e^{-3t}\right) + t^2 - \frac{5}{3}t + 1.$$

If $F(0) = 0$ then this will not quite work. If $F(t)$ is a polynomial of degree k with $F(0) = 0$, then $t = 0$ is a root of the equation $F(t) = 0$ and $F(t)$ must be of the form $F(t) = tG(t)$ where $G(t)$ is a polynomial of degree $k-1$. In this case the form $x(t) = ty(t)$, where $y(t)$ is a polynomial of degree k whose co-efficients are determined by substitution into (36), will give a solution, provided $G(0) \neq 0$.

2) If $F(t)$ is of the form $\sin(\Omega t)$ or $\cos(\Omega t)$ then try a solution of the form

$$x(t) = A\cos(\Omega t) + B\sin(\Omega t),$$

with A and B constants which can be determined by substituting this form in (36). We have already seen an example of this in section **2.7** and we will not go through it again. Note however that this will not work in the special case $\gamma = 0$ when $\Omega = \omega_0$, in this case we need a different substitution and

$$x(t) = t\big(A\cos(\omega_0 t) + B\sin(\omega_0 t)\big)$$

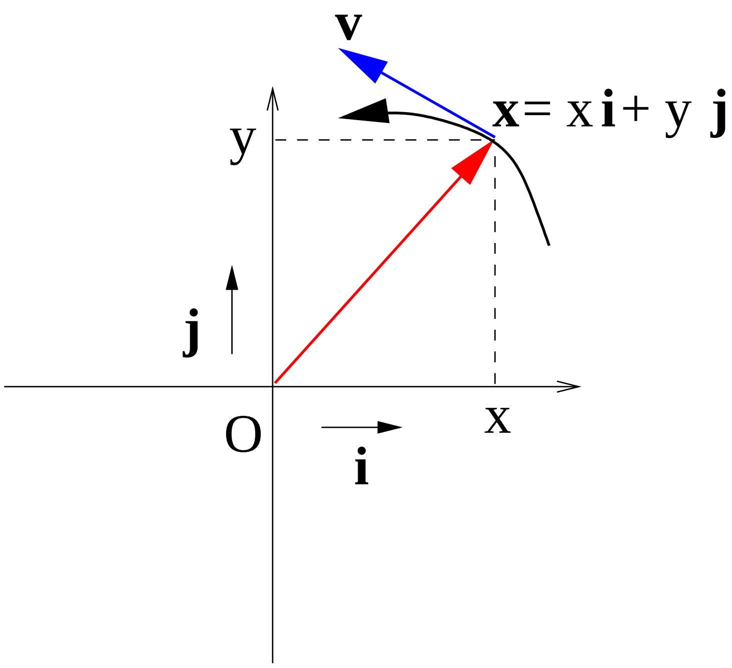
will work.

3. Two Dimensional Motion

We now consider motion in 2-dimensions. We shall use Cartesian co-ordinates (x, y) and denote positions in 2-dimensional space by vectors. Vector quantities will be indicated by bold-face letters and the position denoted by $\mathbf{x}$. Using an orthonormal basis $(\boldsymbol{i}, \boldsymbol{j})$, with $\boldsymbol{i}$ a unit vector in the x-direction and $\boldsymbol{j}$ a unit vector in the y-direction, so an arbitrary point in the 2-dimensional plane is denoted by $\mathbf{x} = x\boldsymbol{i} + y\boldsymbol{j}$. In general the position of a moving particle will be a function of time, and $\mathbf{x}(t)$ traces out a curve in the 2-dimensional plane whose points can be labelled by the values of t at the time when the particle is at that point. The co-ordinates are then functions of time $(x(t), y(t))$, but $\boldsymbol{i}$ and $\boldsymbol{j}$ are constant vectors, so the velocity is

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j},$$

which is a vector tangent to the curve $\mathbf{x}(t)$ at the time t.



The acceleration of the particle is

$$\ddot{\mathbf{x}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j},$$

so Newton's second law for a particle of mass m is

$$\mathbf{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} = m\ddot{\mathbf{x}} = m(\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}),$$

where F_x is the x-component and F_y the y-component of the force. Equating components gives two equations

$$F_x = m\ddot{x} \qquad F_y = m\ddot{y}.$$

If we know the explicit form of F_x and F_y in a given physical problem we can try to solve for $x(t)$ and $y(t)$ and hence predict the motion of m.

3.1 Projectiles

As a first example we consider projectiles in the absence of friction. Let x label a horizontal direction and y the vertical direction. The gravitational force is vertically downwards so we take $F_y = -mg$ and $F_x = 0$, giving

$$\mathbf{F} = -mg\,\boldsymbol{j} = m\ddot{\mathbf{x}}$$

or, in components,

$$m\ddot{y} = -mg \quad \Rightarrow \quad \ddot{y} = -g, \qquad\qquad m\ddot{x} = 0 \quad \Rightarrow \quad \ddot{x} = 0.$$

Taken individually we have already solved both of these equations, if the initial position is given $x(0) = x_0$, $y(0) = y_0$ and the initial velocity is $\dot{x}(0) = v_{x,0}$, $\dot{y}(0) = v_{y,0}$ with x_0, y_0, $v_{x,0}$ and $v_{y,0}$ given constants, then the subsequent motion is

$$x(t) = v_{x,0}t + x_0, \qquad y(t) = -\frac{1}{2}gt^2 + v_{y,0}t + y_0.$$

These are the equations for a parabola, written in parametric form with parameter t. As a concrete example, suppose m is thrown from the origin, $x_0 = y_0 = 0$ with speed v_0 at angle α to the horizontal at $t = 0$, then $v_{x,0} = v_0 \cos\alpha$ and $v_{y,0} = v_0 \sin\alpha$, then

$$\mathbf{x}(t) = \big(v_0 \cos(\alpha) t\big)\boldsymbol{i} + \Big(-\frac{1}{2}gt^2 + v_0 \sin(\alpha) t\Big)\boldsymbol{j}$$

and it hits the ground again at the time T when

$$y(T) = 0 \qquad \Rightarrow \qquad T = \frac{2v_0 \sin\alpha}{g} = \frac{2v_{y,0}}{g}$$

during which time it has travelled a distance

$$x(T) = v_0 \cos(\alpha) T = \frac{2v_0^2 \cos\alpha \sin\alpha}{g} = \frac{v_0^2 \sin(2\alpha)}{g} = \frac{2v_{x,0}v_{y,0}}{g}.$$

If a hurler can hit a sliotar with a maximum speed v_0, then he should hit it at an angle $\alpha = \frac{\pi}{2} = 45^\circ$ above the horizontal, so that $\sin(2\alpha) = 1$, if he wants to maximise the distance that it will travel before it hits the ground again (this conclusion will be modified when air friction is taken into account). The maximum height is achieved, for a fixed v_0 and α, when

$$\frac{dy}{dt} = 0 \qquad \Rightarrow \qquad -gt + v_0 \sin\alpha = 0 \qquad \Rightarrow \qquad t = \frac{v_0 \sin\alpha}{g}$$

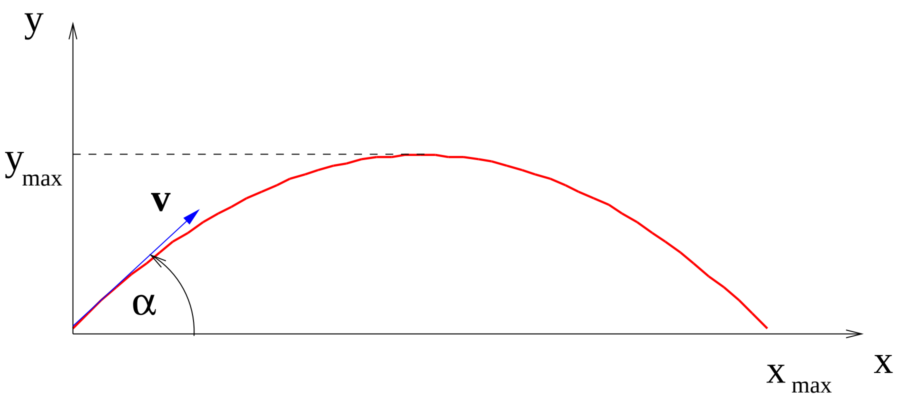
at which point

$$y_{Max} = -\frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2 + v_0\sin(\alpha)\left(\frac{v_0\sin\alpha}{g}\right) = \frac{1}{2}\frac{(v_0\sin\alpha)^2}{g} = \frac{v_{y,0}^2}{2g}.$$

Again, for a fixed v_0, the best one can do is

$$y_{Max} = \frac{v_0^2}{2g},$$

when $\alpha = \frac{\pi}{2}$.



Now we shall consider the same problem when friction is included. As before suppose the frictional force is such as to oppose the motion and is proportional to the velocity. This can be represented in vector notation as

$$\mathbf{F}_{Friction} = -c\mathbf{v} = -c\dot{\mathbf{x}},$$

with $c > 0$ a positive constant. Adding this to the gravitational force Newton's second law now reads

$$m\ddot{\mathbf{x}} = -mg\boldsymbol{j} - c\dot{\mathbf{x}}$$

or, in components

$$m\ddot{x} = -c\dot{x}, \qquad m\ddot{y} = -mg - c\dot{y}.$$

Again, taken individually we have already solved equations like these.. Using the same initial conditions as above, when air friction was ignored, set

$$x(0) = y(0) = 0 \qquad \text{and} \qquad \dot{x}(0) = v_0 \cos\alpha, \quad \dot{y}(0) = v_0 \sin\alpha$$

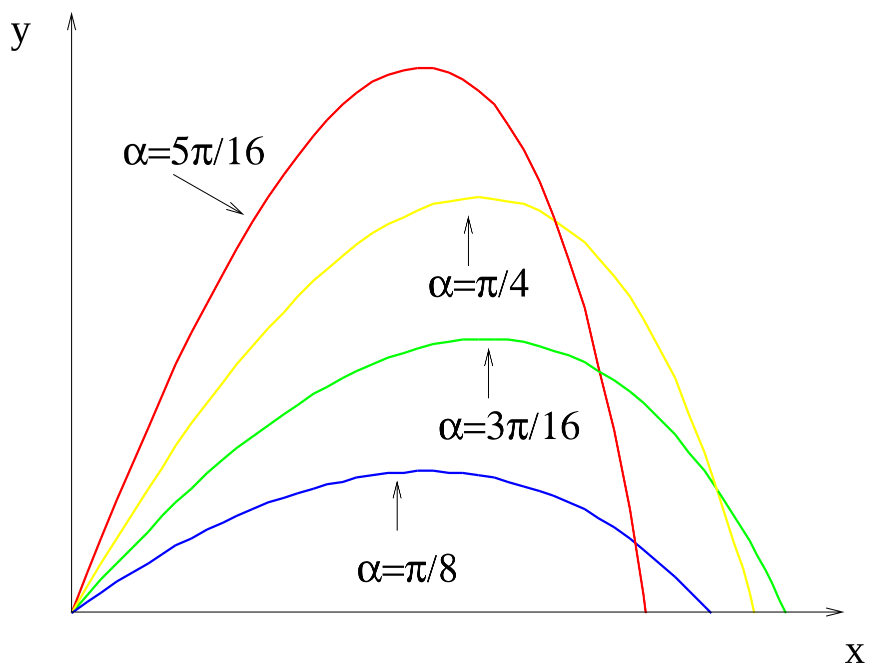
we can use the results of section 2.2, namely (5), with v_0 replaced by $v_0 \cos\alpha$ and $x_0 = 0$, to express the solution for $x(t)$ as

$$x(t) = \frac{mv_0 \cos\alpha}{c} \left(1 - e^{-\frac{ct}{m}}\right).$$

For $y(t)$ we use the solution (13), with v_0 replaced by $v_0 \sin\alpha$ and $y_0 = 0$, to give

$$y(t) = \frac{m}{c} \left(v_0 \sin\alpha + \frac{mg}{c}\right) \left(1 - e^{-\frac{ct}{m}}\right) - \frac{mg}{c} t.$$

A plot of the trajectory, for fixed v_0 and different α is shown below. Note that the curves are not symmetric about their maximum and the trajectory that extends farthest to the right is not the line with $\alpha = \frac{\pi}{4}$, but rather has $\alpha < \frac{\pi}{4}$. The angle, α, that maximises the horizontal distance traveled depends on c, but it is always less than $\frac{\pi}{4}$ for $c > 0$.



3.2 Central Forces

An important problem concerning motion in two dimensions is when there is a force acting on a mass m which is always directed toward a fixed point in the two dimensional plane. For example consider a particle carrying an electric charge Q_1 moving in the electric field of another charge, Q_2, which is fixed in space inside a dielectric material, such as air or water. Choosing the origin to coincide with the fixed position of Q_2 the Coulomb force on Q_1 is always directed toward or away from the origin, O, and is inversely proportional to the square of the distance from Q_1 to O. For example, if Q_1 and Q_2 are the same sign the force is repulsive and equals

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\epsilon}\frac{\hat{\mathbf{r}}}{r^2} = \frac{Q_1 Q_2}{4\pi\epsilon}\frac{\mathbf{r}}{r^3},$$

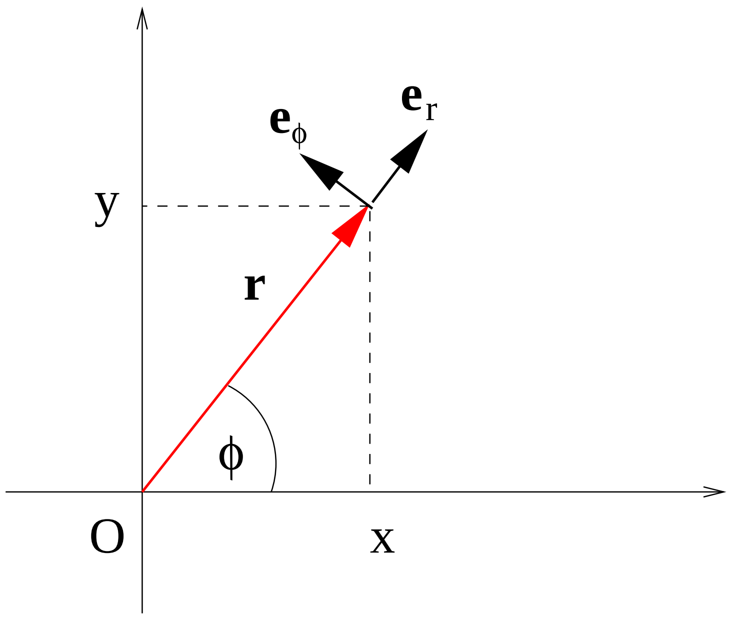
where $\mathbf{r}$ is the position of Q_1 relative to O and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ is a unit vector in the same direction as $\mathbf{r}$. The constant ϵ goes under the fancy name of *the electric permittivity of the medium* — it determines the strength of the Coulomb force in the dielectric: a small value of ϵ means that the Coulomb force is weak.

Another example is Newton's universal law of gravitation which states that the attractive gravitational force between two masses, m and M say, is again inversely proportional to their separation and in the same direction as a line joining the masses,

$$\mathbf{F} = -GmM\frac{\hat{\mathbf{r}}}{r^2} = GmM\frac{\mathbf{r}}{r^3},$$

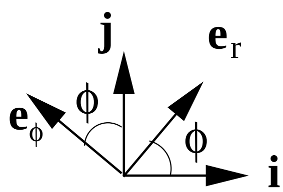
where $G = 6.67 \times 10^{-11}\ kg^{-1}m^3s^{-2}$ is Newton's universal constant of gravitation and $\mathbf{r}$ is the vector between the positions of m and M. If one of the masses, say M, is much larger than the other then the larger mass can be considered to be at rest and we can take M to be fixed at the origin with $\mathbf{r}$ the position vector of m.

Both of these situations are called central force problems, because in each case the force is directed to one central point, which can be chosen to be the origin. For central force problems it is usually convenient to use two dimensional polar co-ordinates, (r, ϕ), defined as $x = r\cos\phi$, $y = r\sin\phi$. Thus $r^2 = x^2 + y^2$ and the radial co-ordinate, $r = \sqrt{x^2 + y^2}$ is the distance of the point $\mathbf{r}$ from the origin while $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, with $0 \le \phi < 2\pi$, is the angle between the point $\mathbf{r}$ and the x-axis, ϕ is called the *azimuthal* angle of the point $\mathbf{r}$.



It is useful to define two mutually orthogonal unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\phi$ with $\mathbf{e}_r.\mathbf{e}_r = \mathbf{e}_\phi.\mathbf{e}_\phi = 1$ and $\mathbf{e}_r.\mathbf{e}_\phi = 0$. The first, $\mathbf{e}_r$, is a unit vector in the same direction as $\mathbf{r}$, so $\mathbf{e}_r = \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ and the second, $\mathbf{e}_\phi$, is a unit vector in the direction that $\mathbf{r}$ traces out when ϕ is increased, keeping r constant, *i.e.* $\mathbf{e}_\phi$ is a unit tangent vector at the point $\mathbf{r}$, in the anti-clockwise direction, to a circle of radius r centred on the origin. In terms of the Cartesian basis, $\boldsymbol{i}$ and $\boldsymbol{j}$, used before we see from the figure below that

$$\begin{aligned} \mathbf{e}_r &= \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}, & e_\phi &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j} \\ \boldsymbol{i} &= \cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi, & \boldsymbol{j} &= \sin\phi\,\mathbf{e}_r + \cos\phi\,e_\phi. \end{aligned} \tag{39}$$



For a moving point mass $\mathbf{r}(t)$ is a function of time, so $r(t)$ and $\phi(t)$ are also functions of time. An extra subtlety with polar-coordinates however is that $\mathbf{e}_r$ and $\mathbf{e}_\phi$ also depend on time. Since $\boldsymbol{i}$ and $\boldsymbol{j}$ are constant vectors we have

$$\begin{aligned} \dot{\mathbf{e}}_r &= \frac{d}{dt}(\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}) = \dot{\phi}(-\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}) = \dot{\phi}\,\mathbf{e}_\phi \\ \dot{\mathbf{e}}_\phi &= \frac{d}{dt}(-\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}) = \dot{\phi}(-\cos\phi\,\boldsymbol{i} - \sin\phi\,\boldsymbol{j}) = -\dot{\phi}\,\mathbf{e}_r. \end{aligned}$$

Hence the velocity of m is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r = \dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi. \tag{40}$$

$v_r = \dot{r}$ is the *radial* component of the velocity and $v_\phi = r\dot{\phi}$ the *angular* component.

The acceleration is

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \frac{d}{dt}(r\dot{\phi})\,\mathbf{e}_\phi + r\dot{\phi}\,\dot{\mathbf{e}}_\phi$$
$$= \ddot{r}\,\mathbf{e}_r + \dot{r}\dot{\phi}\,\mathbf{e}_\phi + (\dot{r}\dot{\phi} + r\ddot{\phi})\,\mathbf{e}_\phi - r^2\dot{\phi}^2\,\mathbf{e}_r$$
$$= (\ddot{r} - r\dot{\phi}^2)\,\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\mathbf{e}_\phi. \tag{41}$$

Hence the acceleration has radial component $a_r = \ddot{r} - r\dot{\phi}^2$ and angular component $a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$.

For a central force $\mathbf{F}$ is by definition in the radial direction, $\mathbf{F} = \pm F\,\mathbf{e}_r$ with F the magnitude of the force,* and Newton's second law immediately leads to the important conclusion that the angular acceleration is zero,

$$\mathbf{F} = m\mathbf{a} \qquad \Rightarrow \qquad \pm F\,\mathbf{e}_r = m\big\{(\ddot{r} - r\dot{\phi}^2)\,\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\mathbf{e}_\phi\big\}.$$

Equating the co-efficients of the two basis vectors we conclude that

$$\pm F = m(\ddot{r} - r\dot{\phi}^2), \qquad 0 = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}).$$

In particular

$$\frac{d}{dt}\big(mr^2\dot{\phi}\big) = rm(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0$$

so $mr^2\dot{\phi}$ is a constant, called the *angular momentum* of m. The momentum of m at any time is a vector $\mathbf{p} = m\mathbf{v}$ and the angular momentum is also a vector, defined to be

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}).$$

For motion confined to a two-dimensional plane $m(\mathbf{r} \times \mathbf{v})$ is a vector pointing perpendicularly out of the plane. Defining $\boldsymbol{k}$ to be a unit vector in the z-direction, so $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$, we have

$$m(\mathbf{r} \times \mathbf{v}) = m(r\,\mathbf{e}_r) \times (\dot{r}\,\mathbf{e}_r + r\dot{\phi}\,\mathbf{e}_\phi) = mr^2\dot{\phi}(\mathbf{e}_r \times \mathbf{e}_\phi) = mr^2\dot{\phi}\,\boldsymbol{k}.$$

The angular momentum always points in the same direction and has magnitude $L = mr^2\dot{\phi}$. Sometimes it will be convenient to use $l = L/m = r^2\dot{\phi}$, the angular momentum per unit mass.

So, for a particle moving under a central force, the angular momentum is constant. this is because the only thing that changes angular momentum is a torque and for a force directed toward the origin the torque about the origin is zero. In general only the combination $l = r^2\dot{\phi}$ is constant, not necessarily r or $\dot{\phi}$ independently. Thus if r decreases $\dot{\phi}$ must increase and *vice versa.*

Having established that angular momentum is conserved we now only have to think about the radial part of the acceleration,

$$\ddot{r} - r\dot{\phi}^2 = \ddot{r} - r\left(\frac{l^2}{r^4}\right) = \ddot{r} - \frac{l^2}{r^3}.$$

---

* The plus sign is for a repulsive force, the minus sign for an attractive force.

Newton's second law then gives

$$\mathbf{F} = m\mathbf{a} \qquad \Rightarrow \qquad F = m\left(\ddot{r} - \frac{l^2}{r^3}\right).$$

This is true for any central force, but we shall now specialise to the important case where $F(r)$ depends only on the distance of m from the origin, as in Coulomb's law or Newton's universal law of gravitation, where $F(r) \propto 1/r^2$.

For Newton's universal law of gravitation, for example, with M fixed at the origin (which is a good approximation only if $M >> m$, *eg.* a planet going round the Sun or a satellite orbiting the Earth),

$$F(r) = -\frac{GmM}{r^2} = m\left(\ddot{r} - \frac{l^2}{r^3}\right) \qquad \Rightarrow \qquad \ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3}. \tag{42}$$

It is remarkable that m drops out of the equation completely! In fact m plays two completely different roles here: in $\mathbf{F} = m\mathbf{a}$ it is the inertial mass of the moving particle while in $F = \frac{GmM}{r^2}$ it is the gravitational mass of the particle (analogous to electric charge in Coulomb's law). There is no a priori reason why the inertial mass and the gravitational mass of a body should be the same, but all the experimental observations imply that they are.* The experimental observation that inertial and gravitational are the same is one of the cornerstone assumptions of Einstein's general theory of relativity, that describes the dynamics of gravitational fields that change with time, but we shall not go into that here — we shall use the Newtonian description of gravity in which all gravitational fields are static.

Now let us focus on solving (42). We proceed by analogy with a problem we have already studied, particle motion in one dimension only. Consider a particle with unit mass moving in one dimension and label the position of the particle by $r(t)$, with r constrained so that $r > 0$. Then the equation

$$\ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3}$$

is exactly the same as that of a unit mass particle moving under a force

$$F_{Eff} = -\frac{GM}{r^2} + \frac{l^2}{r^3}.$$

This is called the *effective* force for the original problem, because it is not the real force in two dimensions, that only involved $-\frac{GM}{r^2}$. Nevertheless we can use the same technique as before to solve the one dimensional problem. Introduce an *effective* potential $U_{Eff}(r)$ by $F_{Eff} = -\frac{dU_{Eff}}{dr}$ so, for motion under gravity,

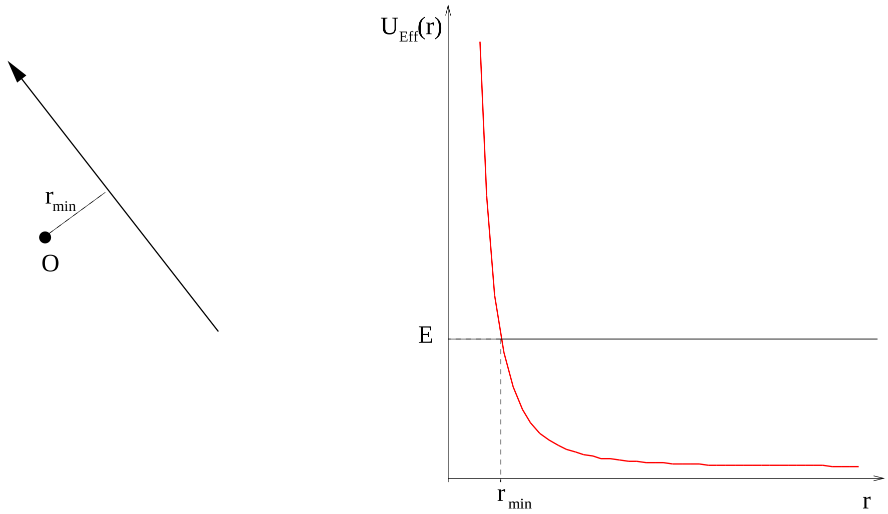
$$U_{Eff} = -\frac{GM}{r} + \frac{l^2}{2r^2}.$$

* Strictly speaking all the experiments tell is that the inertial mass and the gravitational mass are proportional to each other, we then chose a convention for G so that they are equal.

The first term is just the usual potential energy for a $1/r^2$ force, but the second term requires some explanation – it is due to the rotation of m about the origin in two dimensions and is related to centrifugal force. To understand the significance let's first turn off gravity and ignore the first term, we are now considering the motion of a particle in two dimensions moving under no force whatsoever, so the motion should be in a straight line. Nevertheless, unless the particle happens to be heading straight for the origin, it will have some non-zero angular momentum $l = r^2\dot{\phi} \neq 0$ and

$$U_{Eff} = \frac{l^2}{2r^2} \qquad \Rightarrow \qquad F_{Eff} = \frac{l^2}{r^3}$$

and, in the effective one dimensional motion, the particle experience a repulsive force from the origin, proportional to $1/r^3$, called the *centrifugal barrier*. This force diverges as $r \to 0$ and the particle can never get to the origin – but that is what we expect because it is not moving towards the two-dimensional origin if $l \neq 0$. Picture the motion as below,



in two dimensions the particle moves in a straight line, but when only r is considered the particle comes in from large r, reaches a point of closest approach to the origin and then recedes again. In the effective one dimensional problem the particle is repelled from the origin by an inverse cube force. In the absence of the gravitational force the total energy is just the knietic energy

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{1}{2}m\left(\dot{r}^2 + \frac{l^2}{r^2}\right),$$

and the point of closest approach to the origin, r_{min}, is easily seen from the effective potential as being the point at which $\dot{r} = 0$ where $E = \frac{m}{2}\frac{l^2}{r_{min}^2} = \frac{1}{2m}\frac{L^2}{r_{min}^2}$, hence the minimum distance from the origin depends on both the energy E and the angular momentum L,

$$r_{min} = \frac{L}{\sqrt{2mE}} = \frac{L}{mv}.$$

When the angular momentum is zero the particle is heading straight for the origin and will pass through O, so $r_{min} = 0$.

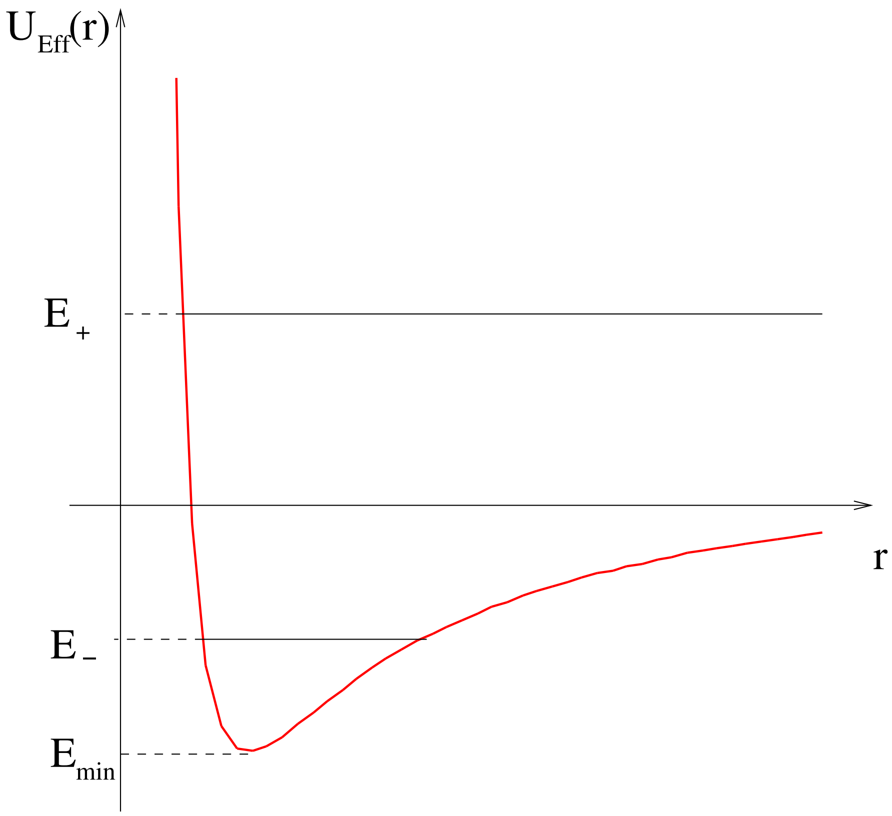
Now return to the full problem with $\ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3}$. We treat this as an effective one dimensional problem for a particle of unit mass, with

$$U_{Eff} = -\frac{GM}{r} + \frac{l^2}{2r^2}.$$

Note that the potential energy is negative when $r > \frac{l^2}{2GM}$ and has a minimum when $\frac{dU_{Eff}}{dr} = 0$, that is at

$$r = \frac{l^2}{GM},$$

where $U_{Eff} = -\frac{1}{2}\left(\frac{GM}{l}\right)^2$.



The total energy per unit mass is conserved,

$$E = \frac{\dot{r}^2}{2} - \frac{GM}{r} + \frac{l^2}{2r^2} \tag{43}$$

and the behaviour of m is very different for $E > 0$ and $E < 0$. For $E = E_+ > 0$, m is repelled from the origin and moves out to indefinitely large r. For $E = E_- < 0$, m is trapped in a potential well and moves between a minimum and a maximum value of r. In the limiting case of $E = E_{min} = -\frac{1}{2}\left(\frac{GM}{l}\right)^2$, r is constant at $r = \frac{l^2}{GM}$.

It should not be forgotten that, if $l \neq 0$, the full two dimensional motion involves rotation with $\dot{\phi} = l/r^2 \neq 0$. So the fact that r is constant for $E = E_{min} = -\frac{1}{2}\left(\frac{GM}{l}\right)^2$ does not mean that M is not moving in two dimensions, it only means that $\dot{\phi} = l/r^2 = \frac{(GM)^2}{l^3}$ is constant, so m moves in a circle of radius $\frac{l^2}{GM}$ with angular velocity $\frac{(GM)^2}{l^3}$. For $E_{min} < E < 0$ the particle orbits around the origin, moving in and out between two extreme values of r — this is called a *bound* orbit. For $E \geq 0$ the particle will eventually move off to arbitrarily large r in an *unbound* orbit.

We can go further and find the geometrical shape of the orbit by solving equation (43) for the unknown function $r(t)$,

$$\dot{r}^2 = 2E + \frac{2GM}{r} - \frac{l^2}{r^2}.$$

One way to proceed is to use $l = r^2\dot{\phi}$ to change the independent variable from t to ϕ and find $r(\phi)$ instead of $r(t)$. From the chain rule $\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{l}{r^2}\frac{dr}{d\phi}$ giving

$$\frac{l^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 = 2E + \frac{2GM}{r} - \frac{l^2}{r^2}. \tag{44}$$

A simplification arises from using $u(\phi) = 1/r(\phi)$, with

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \qquad \Rightarrow \qquad \left(\frac{du}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2,$$

and (44) can be re-written

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2E}{l^2} + \frac{2GMu}{l^2} - u^2 = -\left(u - \frac{GM}{l^2}\right)^2 + \frac{2E}{l^2} + \left(\frac{GM}{l^2}\right)^2. \tag{45}$$

Differentiating with respect to ϕ gives

$$\left(\frac{du}{d\phi}\right)\left(\frac{d^2u}{d\phi^2}\right) = -\left(\frac{du}{d\phi}\right)\left(u - \frac{GM}{l^2}\right) \qquad \Rightarrow \qquad \frac{d^2u}{d\phi^2} = -\left(u - \frac{GM}{l^2}\right).$$

This is nothing more than our old friend the harmonic oscillator equation! Shifting the origin of u to $\tilde{u} = u - \frac{GM}{l^2}$ we have simply

$$\frac{d^2\tilde{u}}{d\phi^2} = -\tilde{u}$$

with the most general solution

$$\tilde{u} = A_0\cos(\phi - \delta_0) \Rightarrow u = A_0\cos(\phi - \delta_0) + \frac{GM}{l^2}.$$

The constant A_0 can be related to the energy using (45),

$$\left(\frac{du}{d\phi}\right)^2 = A_0^2 \sin^2(\phi - \delta_0) = -A_0^2 \cos^2(\phi - \delta_0) + \frac{2E}{l^2} + \left(\frac{GM}{l^2}\right)^2 \Rightarrow A_0^2 = \frac{2E}{l^2} + \left(\frac{GM}{l^2}\right)^2.$$

Choosing $A_0 > 0$ we have

$$r(\phi) = \frac{1}{\sqrt{\left(\frac{GM}{l^2}\right)^2 + \frac{2E}{l^2}} \cos(\phi - \delta_0) + \frac{GM}{l^2}}.$$

If $E < 0$, the numerator never vanishes and $r(\phi)$ is finite for $0 \leq \phi < 2\pi$ giving a bound orbit, as expected. If $E = -\frac{1}{2}\frac{G^2M^2}{l^2}$, then $r = \frac{l^2}{GM}$ is a constant and the orbit is a circle. for $-\frac{1}{2}\frac{G^2M^2}{l^2} < E < 0$, r oscillates between a minimum value

$$r_{min} = \frac{1}{\frac{GM}{l^2} + \sqrt{\left(\frac{GM}{l^2}\right)^2 + \frac{2E}{l^2}}}$$

when $\cos(\phi - \delta_0) = 1$ and a maximum

$$r_{max} = \frac{1}{\frac{GM}{l^2} - \sqrt{\left(\frac{GM}{l^2}\right)^2 + \frac{2E}{l^2}}}$$

when $\cos(\phi - \delta_0) = -1$, returning to the same value for $\phi \to \phi + 2\pi$. The fact that $r(\phi) = r(\phi + 2\pi)$ means that m returns to exactly the same point in space after every revolution around M and the orbit is said to be *closed*. Not all types of central force give closed orbits, a $1/r^3$ force does not, for example. Indeed the only two kinds of central force that lead to closed orbits are inverse square, $1/r^2$, and quadratic r^2. Without losing any generality of the solution we can choose $\phi = 0$ to be the angle at which $r = r_{min}$, *ie.* $\delta_0 = 0$, then we have the solution of the orbit equation as

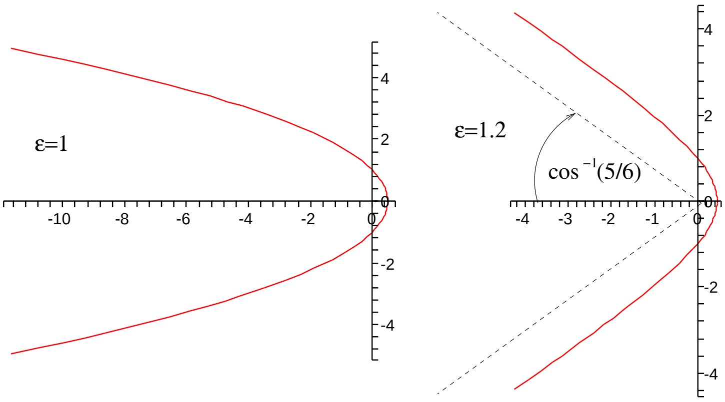
$$r(\phi) = \frac{l^2}{GM} \frac{1}{\big(1 + \epsilon \cos(\phi)\big)}, \tag{46}$$

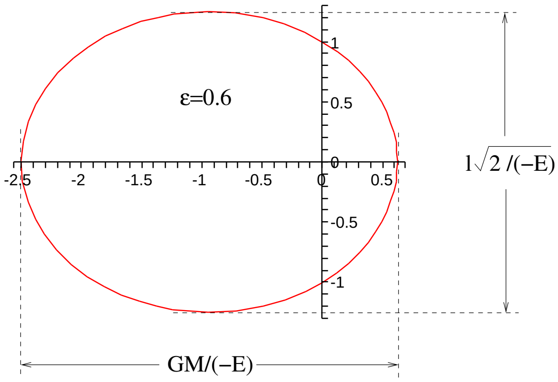
where we have defined

$$\epsilon := \sqrt{1 + \frac{2El^2}{G^2M^2}}.$$

For $-\frac{1}{2}\frac{G^2M^2}{l^2} < E < 0$ we have $0 < \epsilon < 1$ and equation (46) is actually the equation for an ellipse, with one focus centred on the origin and with eccentricity ϵ. We have thus established that Kepler's First Law of planetary motion, that the planets move in ellipses with the Sun at one focus, follows from Newton's inverse square law of gravitation.

If $\epsilon = 1$, then r is infinite for $\phi = \pm\pi$ and (46) is the equation of a parabola, such as the orbit of many irregular comets. If $\epsilon > 1$ $(E > 0)$ then r is infinite when $\phi = -\cos^{-1}(1/\epsilon)$, which occurs at two different values of ϕ in the range $-\pi < \phi < \pi$ and this corresponds to a trajectory that comes in from infinity and swings round M to recede back out to infinity in less than one complete revolution. Typical orbits are shown below:





The planets all have orbits with small eccentricities, the largest is that of Mercury with $\epsilon = 0.21$. The next largest is Mars with $\epsilon = 0.093$ which is difficult to distinguish from a circle just by looking at it — it requires careful measurement to see that it is not a perfect circle.

3.3 Conservative Forces

We saw in section **2.3** that, for frictionless motion in one dimension for a force $F(x)$ that depends only on position, we can define a potential energy $U(x)$ such that the total energy, kinetic plus potential, is conserved and the force is minus the derivative of the potential, $F(x) = -\frac{dU}{dx}$. The situation in two (or three) dimensions is a little more subtle, if the components of a force, $F_x(x,y)$ and $F_y(x,y)$, depend only on position we cannot conclude that the force is derivable from a potential. To see this suppose that $\mathbf{F} = F_x(x,y)\,\boldsymbol{i} + F_y(x,y)\,\boldsymbol{j}$ is derivable from a potential, so

$$F_x = -\frac{\partial U}{\partial x}, \qquad F_y = -\frac{\partial U}{\partial y}.$$

Then

$$\partial_y F_x = \frac{\partial F_x}{\partial y} = -\frac{\partial^2 U}{\partial y \partial x}, \qquad \partial_x F_y = \frac{\partial F_y}{\partial x} = -\frac{\partial^2 U}{\partial x \partial y}$$

and, assuming that U is at least twice differentiable,

$$\partial_x F_y = \partial_y F_x, \qquad \text{or} \qquad \partial_x F_y - \partial_y F_x = 0 \qquad \Rightarrow \qquad \nabla \times \mathbf{F} = 0,$$

so the curl of $\mathbf{F}$ must vanish. This is a necessary condition on $\mathbf{F}$ for a potential to exist. An example of a force depending only on position for which no potential exists is a central force that depends on the direction,

$$\mathbf{F} = F_r(r, \phi)\mathbf{e}_r,$$

with $\partial_\phi F_r \neq 0$. Because the force is central there is no torque, $F_\phi = 0$, but if the force were derivable from a potential we would have $F_r(r, \phi) = -\partial_r U(r, \phi)$ and $\partial_\phi F_r = -\partial_\phi \partial_r U \neq 0$ requires $\partial_\phi U \neq 0$, which then implies that $F_\phi \neq 0$, contrary to our assumption that the force is central. Hence no potential exists for such a force (angular momentum is still conserved though, since there is no torque).

If the force can be derived from a potential then energy is conserved. Suppose a potential exists then define the total energy to be

$$E = \frac{m}{2}\mathbf{v}.\mathbf{v} + U(x, y).$$

Its time derivative is

$$\frac{dE}{dt} = m\frac{d\mathbf{v}}{dt}.\mathbf{v} + \frac{dx}{dt}\frac{\partial U}{\partial x} + \frac{dy}{dt}\frac{\partial U}{\partial y} = \mathbf{F}.\mathbf{v} + \mathbf{v}.\nabla U = \mathbf{v}.(\mathbf{F} + \nabla U) = 0,$$

where $\nabla U = \partial_x U\, \boldsymbol{i} + \partial_y U\, \boldsymbol{j}$ is the gradient of $U(x, y)$.

The existence of a potential has an interesting consequence which is perhaps not immediately obvious. Consider the work done, W_{12}, by an external force $\mathbf{F}$ on a particle m moving between two position $\mathbf{r}_1$ and $\mathbf{r}_2$ along a specified curve, C_{12}. For an infinitesimal displacement $d\mathbf{r}$ at a point $\mathbf{r}$ of the curve the work done is $\mathbf{F}.d\mathbf{r}$ and the total work done on m in moving from $\mathbf{r}_1$ to $\mathbf{r}_2$ along the curve is

$$W_{12} = \int_{C_{12}} \mathbf{F}.d\,\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}.d\,\mathbf{r} = m\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{d\,\mathbf{v}}{dt}.\mathbf{v}\,dt = \frac{m}{2}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{d}{dt}(v^2)\,dt = \frac{m}{2}(v_2^2 - v_1^2),$$

where we have used $d\,\mathbf{r} = \frac{d\,\mathbf{r}}{dt}dt = \mathbf{v}dt$ and v_2, v_1 are the speeds at the end points $\mathbf{r}_2$ and $\mathbf{r}_1$ respectively. The last expression here shows that the work done is just the difference between the kinetic energies at $\mathbf{r}_2$ and that at $\mathbf{r}_1$ and, in general, this will depend on the path taken between $\mathbf{r}_1$ and $\mathbf{r}_2$. Now consider two different paths C_{12} and C'_{12} between $\mathbf{r}_1$ and $\mathbf{r}_2$ and suppose m first travels from $\mathbf{r}_1$ to $\mathbf{r}_2$ along C_{12} and then back from $\mathbf{r}_2$ to $\mathbf{r}_1$ along C'_{12} The difference

$$\int_{C_{12}} \mathbf{F}.d\,\mathbf{r} - \int_{C'_{12}} \mathbf{F}.d\,\mathbf{r} = \oint_C \mathbf{F}.d\,\mathbf{r}$$

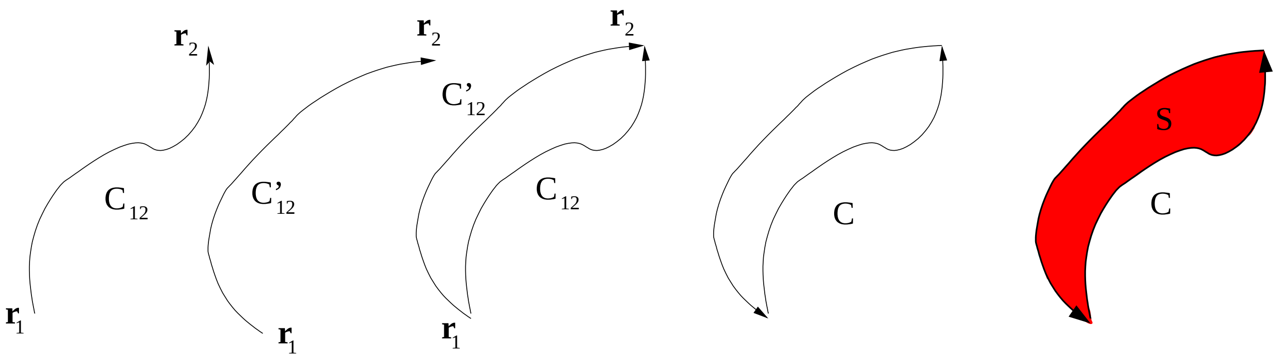
is an integral around the closed loop consisting of C_{12} followed by $-C'_{12}$, which we shall call C, and we remind ourselves that C is a closed curve by using the symbol $\oint_C(\cdots)d\,\mathbf{r}$ for the integral. Now let us assume that $\mathbf{F}$ is such that $\nabla\times\mathbf{F}=0$, then we can use Stokes' theorem to conclude

$$\oint_C \mathbf{F}.d\,\mathbf{r} = \int_S (\nabla\times\mathbf{F}).d\,\mathbf{S} = 0$$

where S is the area enclosed by C and $d\mathbf{S}$ is a vector normal to the surface whose magnitude is an infinitesimal area element at the point interior to S where the integrand is being evaluated. This means that

$$\nabla\times\mathbf{F}=0 \qquad \Rightarrow \qquad \int_{C_{12}} \mathbf{F}.d\,\mathbf{r} = \int_{C'_{12}} \mathbf{F}.d\,\mathbf{r},$$

i.e. the difference in kinetic energy between $\mathbf{r}_1$ and $\mathbf{r}_2$ is *independent* of the path taken between them.



Another way of seeing this is to use conservation of energy directly,

$$E = \frac{m}{2}v_1^2 + U(\mathbf{r}_1) = \frac{m}{2}v_2^2 + U(\mathbf{r}_2) \qquad \Rightarrow \qquad \frac{m}{2}v_2^2 - \frac{m}{2}v_1^2 = U(\mathbf{r}_2) - U(\mathbf{r}_1),$$

so the difference of the kinetic energies depends only on the points $\mathbf{r}_2$ and $\mathbf{r}_1$, not on any path between them. Indeed

$$\int_{C_{12}} \mathbf{F}.d\,\mathbf{r} = -\int_{C_{12}} \nabla U(\mathbf{r}).d\,\mathbf{r} = [U(\mathbf{r})]_{\mathbf{r}_1}^{\mathbf{r}_2} = U(\mathbf{r}_2) - U(\mathbf{r}_1).$$

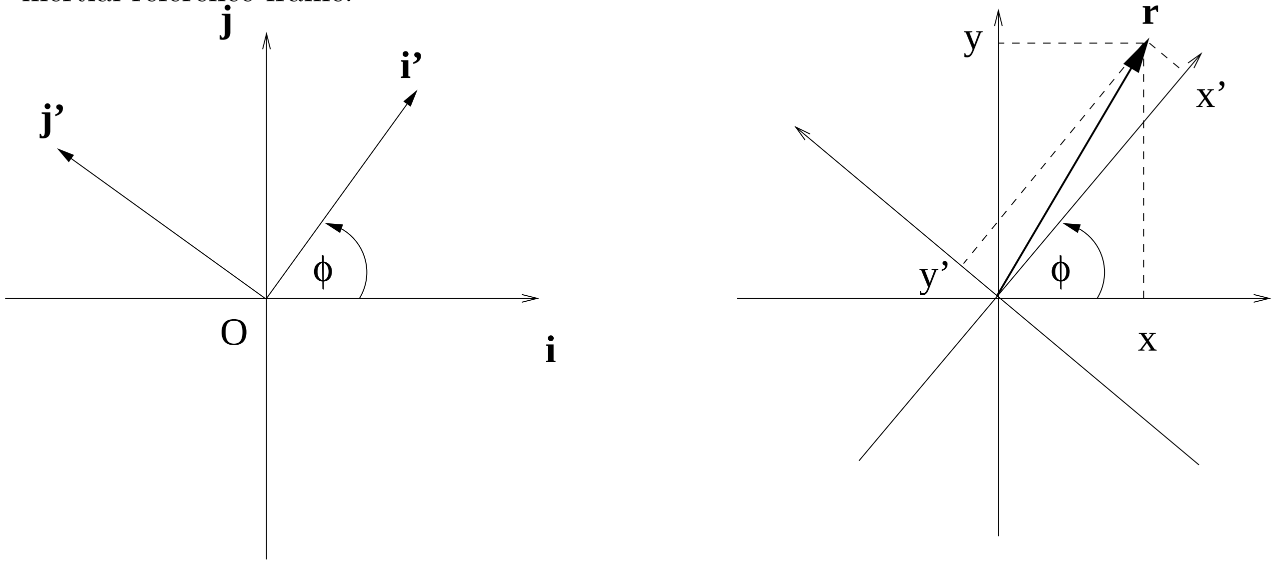
So an alternative way of characterising a conservative force is to say that the work done in transporting a particle under the influence of a conservative force around any closed loop is always zero.

3.4 Rotating Reference Frames

Sometimes it is useful to analyse natural phenomena from the point of view of a reference frame rotating with constant angular velocity, for example the Earth is rotating once on its axis every 24 hours so a laboratory fixed to the surface of the Earth is rotating. A rotating reference frame is not an inertial reference frame and this affects the way that the dynamics is described. We can still use Newton's laws of motion in a rotating reference frame, we just have to be careful with the analysis. Restricting to two dimensional motion for the moment we can describe vectors in a reference frame rotated about a fixed point O by an amount α by using the basis

$$\boldsymbol{i}' = \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j}, \qquad \boldsymbol{j}' = -\sin\alpha\,\boldsymbol{i} + \cos\alpha\,\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are fixed unit vectors, associated with Cartesian co-ordinates (x, y) in an inertial reference frame.



We can use either $(\boldsymbol{i}, \boldsymbol{j})$ or $(\boldsymbol{i}', \boldsymbol{j}')$ as a basis for two dimensional vectors, *eg.* for the position vector

$$\mathbf{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$$
$$\mathbf{r}' = x'\,\boldsymbol{i}' + y'\,\boldsymbol{j}'.$$

In fact

$$x' = r\cos(\phi - \alpha) = x\cos\alpha + y\sin\alpha, \qquad y' = r\sin(\phi - \alpha) = -x\sin\alpha + y\cos\alpha,$$

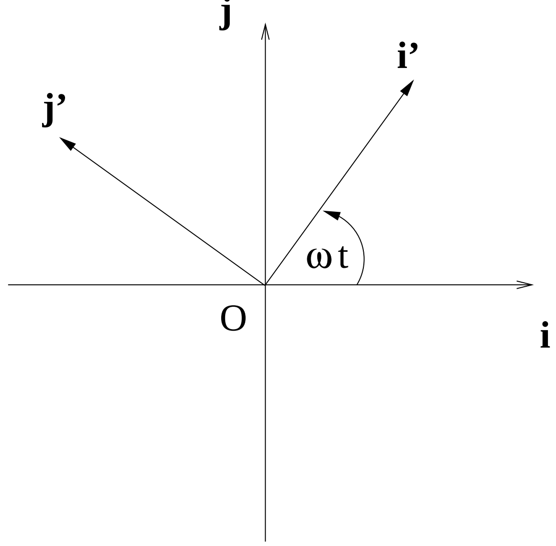
and $\mathbf{r} = \mathbf{r}'$, the position of a point is described by the same vector in both reference frames, though the co-ordinates are different because the bases are different.

If α is a constant then (x', y') are Cartesian co-ordinates in an inertial reference frame and $(\boldsymbol{i}', \boldsymbol{j}')$ is a basis in this inertial reference frame. If α is not a constant but a function of time, $\alpha(t)$, then (x', y') are not co-ordinates for an inertial reference frame, nevertheless they are still perfectly good co-ordinates, they are just not inertial. We can still use them, but Newton's second law will look a little unfamiliar using these co-ordinates. For example suppose (x', y') are co-ordinates in a reference frame that rotates with constant angular velocity $\omega = \dot{\alpha}$, so α is a linear function of t. We can always choose the x'-direction so that $\alpha(0) = 0$ and then $\alpha(t) = \omega t$. Now we have

$$\boldsymbol{i}' = \cos(\omega t)\,\boldsymbol{i} + \sin(\omega t)\,\boldsymbol{j}, \qquad \boldsymbol{j}' = -\sin(\omega t)\,\boldsymbol{i} + \cos(\omega t)\,\boldsymbol{j},$$

$$x' = x\cos(\omega t) + y\sin(\omega t), \qquad y' = -x\sin(\omega t) + y\cos(\omega t). \tag{47}$$

We shall refer to (x, y) as static co-ordinates and (x', y') as rotating co-ordinates.



The rotating basis vectors $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are not constant, we find

$$\frac{d\boldsymbol{i}'}{dt} = -\omega\sin(\omega t)\,\boldsymbol{i} + \omega\cos(\omega t)\,\boldsymbol{j} = \omega\,\boldsymbol{j}', \qquad \frac{d\boldsymbol{j}'}{dt} = -\omega\cos(\omega t)\,\boldsymbol{i} - \omega\sin(\omega t)\,\boldsymbol{j} = -\omega\,\boldsymbol{i}'. \tag{48}$$

Now consider a particle of mass m following a trajectory $\mathbf{r}(t)$, but watched by an observer in the rotating reference frame using co-ordinates (x', y'),

$$\mathbf{r}'(t) = x'(t)\,\boldsymbol{i}' + y'(t)\,\boldsymbol{j}' = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} = \mathbf{r}(t). \qquad (49).$$

The rotating observer, using co-ordinates (x', y'), would measure the velocity of m to be

$$\mathbf{v}' = \dot{x}'\boldsymbol{i}' + \dot{y}'\boldsymbol{j}' \tag{50}$$

and the acceleration to be

$$\mathbf{a}' = \ddot{x}'\boldsymbol{i}' + \ddot{y}'\boldsymbol{j}'.$$

In a rotating reference frame we expect centrifugal forces. For example if m is initially at rest in the rotating frame, at $x'(0) = x_0$ and $y'(0) = 0$ say with $\dot{x}'(0) = \dot{y}'(0) = 0$, then it must be moving in the static reference frame. Its initial position in the static reference frame is $x(0) = x_0$, $y(0) = 0$ and it is moving with speed ωx_0 in the positive y-direction. If there are no forces on m in the static frame then m continues to move with constant velocity in that frame, $\mathbf{v} = \omega x_0 \boldsymbol{j}$, and its subsequent position in the static frame is $x = x_0$, $y = \omega x_0 t$. Its motion in the rotating frame is then given by (47) to be

$$\begin{aligned}
&x' = x_0 \cos(\omega t) + \omega t x_0 \sin(\omega t), \qquad y' = -x_0 \sin(\omega t) + \omega t x_0 \cos(\omega t),\\
&\dot{x}' = \omega^2 t x_0 \cos(\omega t), \qquad \dot{y}' = \omega^2 t x_0 \sin(\omega t),\\
&\ddot{x}' = -\omega^3 t x_0 \sin(\omega t) + \omega^2 x_0 \cos(\omega t), \qquad \ddot{y}' = \omega^3 t x_0 \cos(\omega t) + \omega^2 x_0 \sin(\omega t).
\end{aligned}$$

Thus even though m experiences no forces in the static reference frame, there is an acceleration in the rotating reference frame. For example at $t = 0$, when $\mathbf{v}' = 0$, we see that

$$\ddot{x}' = \omega^2 x_0, \quad \ddot{y}' = 0 \qquad \Rightarrow \qquad \mathbf{a}'(0) = \omega^2 x_0 \boldsymbol{i}' = \omega^2 \mathbf{r}'(0).$$

There is an apparent force in the rotating reference frame, the centrifugal force, which at $t = 0$ is $\mathbf{F}'_{Cent} = m\omega^2 \mathbf{r}'$, though the story gets a little more complicated when $\mathbf{v}' \neq 0$.

Returning to the case of a general motion we have, using (48) and (49), since $\mathbf{r} = \mathbf{r}'$

$$\begin{aligned}
\dot{\mathbf{r}} = \dot{\mathbf{r}}' &= \dot{x}'\boldsymbol{i}' + x'\omega\boldsymbol{j}' + \dot{y}'\boldsymbol{j}' - y'\omega\boldsymbol{i}'\\
&= (\dot{x}' - \omega y')\boldsymbol{i}' + (\dot{y}' + \omega x')\boldsymbol{j}'\\
&= \mathbf{v}' - \omega(y'\boldsymbol{i}' - x'\boldsymbol{j}'),
\end{aligned}$$

$$\begin{aligned}
\ddot{\mathbf{r}} = \ddot{\mathbf{r}}' &= (\ddot{x}' - \omega\dot{y}')\boldsymbol{i}' + (\dot{x}' - \omega y')\omega\boldsymbol{j}' + (\ddot{y}' + \omega\dot{x}')\boldsymbol{j}' - (\dot{y}' + \omega x')\omega\boldsymbol{i}' \qquad (51)\\
&= (\ddot{x}' - 2\omega\dot{y}' - \omega^2 x')\boldsymbol{i}' + (\ddot{y}' + 2\omega\dot{x}' - \omega^2 y')\boldsymbol{j}'\\
&= \mathbf{a}' - \omega^2\mathbf{r}' - 2\omega(\dot{y}'\boldsymbol{i}' - \dot{x}'\boldsymbol{j}'),
\end{aligned}$$

To interpret this suppose that there are no forces on m in the static frame, then $\mathbf{a} = \ddot{\mathbf{r}} = 0$. Now $\mathbf{r} = \mathbf{r}'$, so $\ddot{\mathbf{r}}' = 0$ and

$$m\mathbf{a}' = m\omega^2\mathbf{r}' + 2m\omega(\dot{y}'\boldsymbol{i}' - \dot{x}'\boldsymbol{j}') \qquad (52).$$

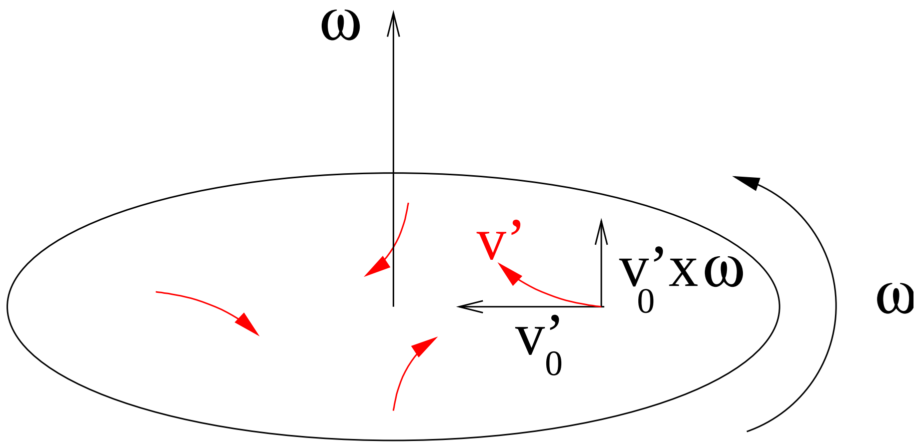
The first term on the right hand side is the centrifugal force due to the rotation of the reference frame, but what is the meaning of the second term? To interpret the last term we first write it in a more concise form. Note that the definition of angular velocity requires specifying not only the magnitude of the velocity, ω, but also the axis of rotation, which in the two dimensional examples considered here is the axis perpendicular to the (x, y)-plane, $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$, or equivalently the axis perpendicular to the (x', y')-plane since $\boldsymbol{k} = \boldsymbol{i}' \times \boldsymbol{j}'$. Angular velocity is actually a vector which here is $\boldsymbol{\omega} = \omega\boldsymbol{k}$. Now, using (50),

$$\mathbf{v}' \times \boldsymbol{\omega} = \omega(\dot{x}'\boldsymbol{i}' + \dot{y}'\boldsymbol{j}') \times \boldsymbol{k} = \omega(-\dot{x}'\boldsymbol{j}' + \dot{y}'\boldsymbol{i}')$$

and so (52) can be written as

$$m\mathbf{a}' = \omega^2\mathbf{r}' + 2m(\mathbf{v}' \times \boldsymbol{\omega}).$$

The last term here, $2m(\mathbf{v}' \times \boldsymbol{\omega})$ is called the *Coriolis force*. Like centrifugal force it is not a real physical force and exists only in rotating reference frames as a consequence of the rotation — it is an example of a pseudo-force. Nevertheless it has real physical consequences, it is responsible for the spiral shapes of tropical storms, which spiral anti-clockwise in the northern hemisphere and clockwise in the southern hemisphere. To understand this imagine a circular area of low pressure surrounded by a region of higher pressure. A wind will blow and air will start to flow radially in towards the centre of the low pressure, with velocity $\mathbf{v}'$ as measured relative to the surface of the Earth. But the Earth is rotating in an anti-clockwise direction about an axis from the south pole to the north pole, so $\boldsymbol{\omega}$ points out of the Earth's surface in the northern hemisphere and into the surface in the southern hemisphere. In the picture below, the wind starts to move radially with velocity $\mathbf{v}_0'$, but as it moves inwards its tangential velocity is too high at smaller radii for the motion to remain radial — it moves off to one side and misses the centre. This is due to the Coriolis acceleration, which is initially $2(\mathbf{v}_0' \times \boldsymbol{\omega})$. The paths followed are the curved trajectories.



4. Three Dimensional Motion

To describe motion in three dimensions we need a basis of three vectors. We can use a basis appropriate to Cartesian co-ordinates (x, y, z), with unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the x, y and z directions respectively, with $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, but for some problems it is more convenient to use an orthonormal basis adapted to spherical polar co-ordinates (r, θ, ϕ). These are defined as shown in the diagram below. In terms of Cartesians

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\theta$$

and

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}, \\
\mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}, \\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.
\end{aligned} \tag{53}$$

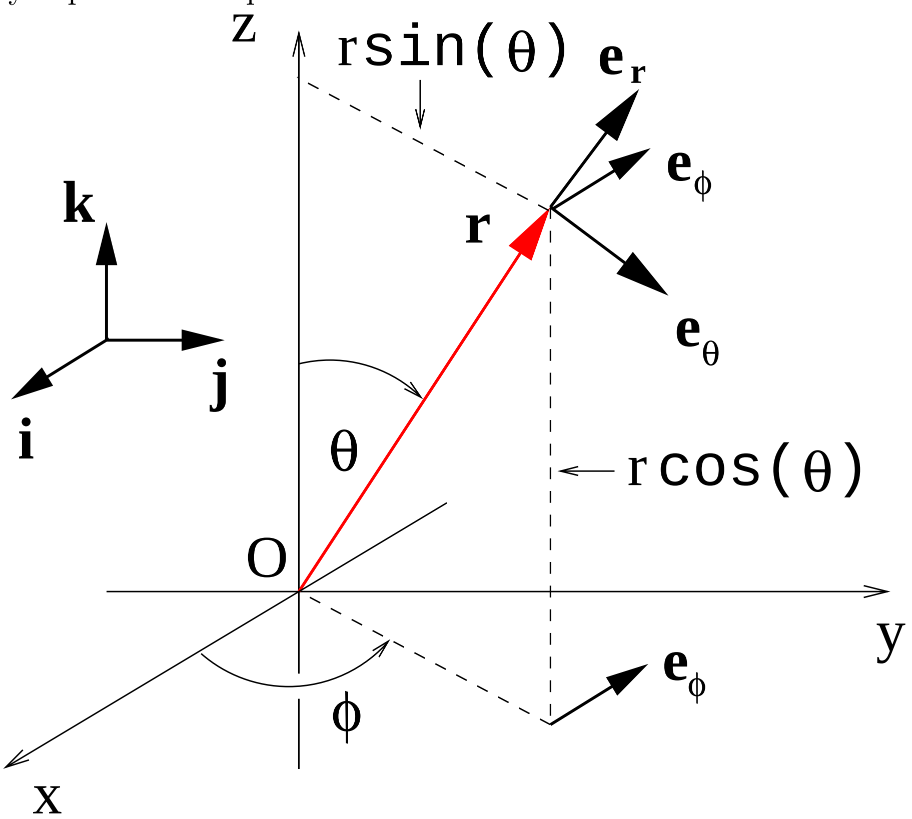
Some calculation involving vector products shows that

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi,$$
$$\mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r,$$
$$\mathbf{e}_\phi \times \mathbf{e}_r = \mathbf{e}_\theta.$$

The position of a particle in three dimensions, relative to the origin O, is then

$$\mathbf{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} = r\,\mathbf{e}_r.$$

To calculate the velocity and the acceleration in a basis adapted to polar co-ordinates we must bear in mind that $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are not constant vectors, they depend on time because they depend on the position $\mathbf{r}$ which is a function of time.



Orthogonal vectors associated with polar co-ordinates. The direction $\mathbf{e}_\phi$ associated with a point $\mathbf{r}$ is always perpendicular to $\boldsymbol{k}$ and is shown twice, at the point $\mathbf{r}$ itself and also in the $x-y$ plane.

Using (53) gives

$$\dot{\mathbf{e}}_r = \dot{\theta}\;\mathbf{e}_\theta + \dot{\phi}\sin\theta\;\mathbf{e}_\phi,$$
$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\;\mathbf{e}_r + \dot{\phi}\cos\theta\;\mathbf{e}_\phi,$$
$$\dot{\mathbf{e}}_\phi = -\dot{\phi}(\sin\theta\;\mathbf{e}_r + \cos\theta\;\mathbf{e}_\theta),$$

and

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\;\mathbf{e}_\theta + r\dot{\phi}\sin\theta\;\mathbf{e}_\phi, \tag{54}$$
$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta \tag{55}$$
$$+ (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\mathbf{e}_\phi.$$

As an example consider a central force problem for a particle of mass m moving under the influence of a force $\mathbf{F} = F(r, \theta, \phi)\, \mathbf{e}_r$ that is always directed toward the origin in three dimensions,

$$F(r, \theta, \phi)\, \mathbf{e}_r = m\, \ddot{\mathbf{r}}.$$

In particular $\mathbf{F}$ has no component in the $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ directions and so there is no acceleration in these directions either and (55) gives

$$2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta = 0 \qquad 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta = 0. \tag{56}$$

The second of these equations implies that

$$\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right) = 0 \qquad \Rightarrow \qquad l := r^2\dot{\phi}\sin^2\theta$$

is a constant, so we can eliminate $\dot{\phi}$ from the first using

$$\dot{\phi}^2 = \frac{l^2}{r^4\sin^4\theta}$$

to show that

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r^2\dot{\phi}^2\sin\theta\cos\theta = \frac{l^2\cos\theta}{r^2\sin^3\theta}.$$

The interpretation of these equations is the following: $r\sin\theta$ is the distance of the mass m from the z-axis, so l is the angular momentum per unit mass about the z-axis — this is a constant for a central force, since there is no torque. Referring to the diagram on the previous page, $r^2\dot{\theta}$ is the angular momentum per unit mass about the $\mathbf{e}_\phi$-axis, but in general this is not a constant if $\mathbf{e}_\phi$ is moving (that is if $l \neq 0$).

A solution of equations (56) is always given by setting $\theta = \pi/2$, then it is consistent to have θ stay at this value, with $\dot{\theta} = 0$, since $\cos\theta = 0$ and now $l = r^2\dot{\phi}$ is the total angular momentum per unit mass. The motion then occurs entirely in the (x, y)-plane and equations (54) and (55) reduce to

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\phi}\,, \qquad \mathbf{e}_\phi = (\ddot{r} - r\dot{\phi}^2)\,\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\mathbf{e}_\phi$$

which reproduces (40) and (41). Hence the central force problem for a single particle can always be reduced to the two-dimensional problem that we have already studied.

Angular momentum is actually a vector of course and another way to derive this result is to use vector notation. The linear momentum of m is defined to be the vector

$$\mathbf{p} = m\,\mathbf{v} = m\frac{d\mathbf{r}}{dt}$$

and the angular momentum about O is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) = m\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right).$$

Note that $\mathbf{L}$ is always at right angles to the plane defined by $\mathbf{r}$ and $\mathbf{v}$ at any moment of time. The rate of change of angular momentum is also a vector

$$\frac{d\mathbf{L}}{dt} = m\left(\frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt}\right) + m\left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) = 0 + \mathbf{r} \times \left(m\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r} \times \mathbf{F}.$$

The combination

$$\boldsymbol{\tau} := \mathbf{r} \times \mathbf{F}$$

is called the *torque*, it represents a force acting on m that tends to twist it about the origin. For a central force $\mathbf{F}$ is parallel to $\mathbf{r}$, the torque vanishes, and hence the angular momentum $\mathbf{L}$ is a constant vector. Since $\mathbf{L}$ is perpendicular to the plane in which $\mathbf{r}$ and $\mathbf{v}$ lie and $\mathbf{L}$ is constant, $\mathbf{r}$ and $\mathbf{v}$ must always lie in the same two dimensional plane for a central force. The initial position and velocity of m specify a two dimensional plane and we are free to choose our axes so that this plane is the (x, y)-plane, with $\mathbf{L}$ in the z-direction. The subsequent motion of m is always confined to the (x, y)-plane and we can treat the whole problem as two dimensional. In particular the analysis of the Kepler problem and planetary orbits that was presented in section **3.2** was completely general and applies to the full three dimensional situation.

5. Systems of Particles

Consider a system of N particles, labelled by an index $i = 1, \cdots, N$, each with a constant mass m_i and moving in three dimensions with positions $\mathbf{r}_i$. The velocity of each particle is $\mathbf{v}_i = \dot{\mathbf{r}}_i$ and the momentum $\mathbf{p}_i = m_i\mathbf{v}_i$. Newton's second law gives

$$\mathbf{F}_i = \dot{\mathbf{p}}_i = m_i\dot{v}_i = m_i\ddot{r}_i$$

where the force on the i-th particle is $\mathbf{F}_i$. In general $\mathbf{F}_i$ can be split into two parts — there could be a force external to the system of particles acting on each particle, which we shall denote by $\mathbf{F}_i^{(e)}$ (such as the force of gravity, for example), but there could also be forces between the particles. If particle j exerts a force F_{ji} on particle i, then the total force on particle i will be

$$\mathbf{F}_i = \mathbf{F}_i^{(e)} + \sum_{j \neq i} \mathbf{F}_{ji} = m_i\ddot{\mathbf{r}}_i,$$

where the sum extends over all the particles in the system *except* for particle i, because it does not exert a force on itself.

Note that, if $\mathbf{F}_{ji}$ is the force that particle j exerts on particle i, then Newton's third law implies that the force that particle i exerts on particle j, $\mathbf{F}_{ij}$, should be equal and opposite, *ie.*. $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$.

The total force on the system of particles is then

$$\mathbf{F}^{(e)} = \sum_{i=1}^{N} \mathbf{F}_i = \sum_{i=1}^{N}\Big(\mathbf{F}_i^{(e)} + \sum_{j \neq i} \mathbf{F}_{ji}\Big) = \sum_{i=1}^{N} \mathbf{F}_i^{(e)} + \sum_{i=1}^{N}\Big(\sum_{j \neq i} \mathbf{F}_{ji}\Big) = \sum_{i=1}^{N} \mathbf{F}_i^{(e)},$$

where in the last equality we have used the fact that the double sum always contains both $\mathbf{F}_{ij} + \mathbf{F}_{ji}$ for each pair of particles and these just cancel. So Newton's third law tells us that the forces between all the particles cancel out and the total force is just the sum of the external forces only.

5.1 Centre of mass motion

To analyse the motion of the particles in more detail it is useful to consider the position of each particle relative to the centre of mass of the system. The *centre of mass* of the system is the point

$$\mathbf{R} := \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{M},$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system of particles. Let $\mathbf{r}'_i$ be the position of particle i relative to the centre of mass,

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i.$$

Then

$$\ddot{\mathbf{R}} = \frac{\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i}{M} = \frac{\sum_{i=1}^{N} \mathbf{F}_i^{(e)}}{M} = \frac{\mathbf{F}^{(e)}}{M},$$

so

$$\mathbf{F}^{(e)} = \sum_{i=1}^{N} \mathbf{F}_i^{(e)} = M\ddot{\mathbf{R}} \tag{57}$$

and the centre of mass of the entire system moves in the same way as a point particle with mass $M = \sum_{i=1}^{N} m_i$ moving under the influence of a force $\mathbf{F}^{(e)} = \sum_{i=1}^{N} \mathbf{F}_i^{(e)}$. In terms of the total momentum of the system

$$\mathbf{P} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \frac{d}{dt}\Big(\sum_{i=1}^{N} m_i \mathbf{r}_i\Big) = \frac{d}{dt}\big(M\mathbf{R}\big) = M\dot{\mathbf{R}}$$

so (57) can be written

$$\mathbf{F}^{(e)} = \dot{\mathbf{P}},$$

since M is constant. Note the total momentum $\mathbf{P} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{p}_i$ is just the sum of the individual momenta of each particle, $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$.

In particular if the sum of the external forces vanishes, $\mathbf{F}^{(e)} = \sum_{i=1}^{N} \mathbf{F}_i^{(e)} = 0$ then the centre of mass momentum $\mathbf{P}$ is constant, as is the velocity of the centre of mass $\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M}\mathbf{P}$.

5.2 Angular momentum

Now consider the total angular momentum of the system of particles about the origin, O. The angular momentum of particle i about O is $\mathbf{r}_i \times \mathbf{p}_i$ and the total angular momentum, $\mathbf{L}$, of the system about O is the sum of the individual angular momenta of each particle,

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i.$$

This can be decomposed into the angular momentum of the system about is own centre of mass and the angular momentum of the centre of mass about O. Using $\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i$ we have $\sum_{i=1}^{N} m_i \mathbf{r}'_i = 0$ since, by definition $M\mathbf{R} = \sum_{i=1}^{N} m_i \mathbf{r}_i$,

$$M\mathbf{R} = \sum_{i=1}^{N} m_i \mathbf{r}_i = \sum_{i=1}^{N} m_i \mathbf{R} + \sum_{i=1}^{N} m_i \mathbf{r}'_i = M\mathbf{R} \quad \Rightarrow \quad \sum_{i=1}^{N} m_i \mathbf{r}'_i = 0.$$

Now

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i \qquad \Rightarrow \qquad \mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}'_i = \mathbf{V} + \mathbf{v}'_i,$$

where $\mathbf{v}'_i = \dot{\mathbf{r}}'_i$ is the velocity of particle i relative to the centre of mass of the system. Hence

$$\begin{aligned}
\mathbf{L} &= \sum_{i=1}^{N} m_i \big(\mathbf{r}_i \times \mathbf{v}_i\big) = \sum_{i=1}^{N} m_i \big(\mathbf{R} + \mathbf{r}'_i\big) \times \big(\mathbf{V} + \mathbf{v}'_i\big) \\
&= \sum_{i=1}^{N} m_i (\mathbf{R} \times \mathbf{V}) + \sum_{i=1}^{N} m_i \big(\mathbf{R} \times \mathbf{v}'_i\big) + \sum_{i=1}^{N} m_i \big(\mathbf{r}'_i \times \mathbf{V}\big) + \sum_{i=1}^{N} m_i \big(\mathbf{r}'_i \times \mathbf{v}'_i\big) \\
&= \sum_{i=1}^{N} m_i (\mathbf{R} \times \mathbf{V}) + \sum_{i=1}^{N} m_i \big(\mathbf{r}'_i \times \mathbf{v}'_i\big) = M(\mathbf{R} \times \mathbf{V}) + \sum_{i=1}^{N} m_i \big(\mathbf{r}'_i \times \mathbf{v}'_i\big),
\end{aligned}$$

where in the last equation we have used the fact that $\sum_{i=1}^{N} m_i \mathbf{r} p_i = 0$, which implies $\sum_{i=1}^{N} m_i \mathbf{v}'_i = 0$. Now the momentum of particle i relative to the centre of mass is $\mathbf{p}'_i = m_i \mathbf{v}'_i$ and the total momentum of the system is $\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i$ so

$$\mathbf{L} = (\mathbf{R} \times \mathbf{P}) + \sum_{i=1}^{N} \big(\mathbf{r}'_i \times \mathbf{p}'_i\big).$$

Thus the total angular momentum consists of two separate pieces

$$\mathbf{L} = (\mathbf{R} \times \mathbf{P}) + \sum_{i=1}^{N} \big(\mathbf{r}'_i \times \mathbf{p}'_i\big) := (\mathbf{R} \times \mathbf{P}) + \mathbf{L}', \tag{58}$$

where the first term is the angular momentum of the centre of mass of the system about the origin O and the second term, $\mathbf{L}' := \sum_{i=1}^{N} \big(\mathbf{r}'_i \times \mathbf{p}'_i\big)$, is the angular momentum of the system of particles about its own centre of mass.

The rate of change of the total angular momentum is

$$\dot{\mathbf{L}} = \frac{d}{dt}\left(\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i\right) = \sum_{i=1}^{N} \frac{d}{dt}\big(\mathbf{r}_i \times \mathbf{p}_i\big) = \sum_{i=1}^{N} \big\{(\dot{\mathbf{r}}_i \times \mathbf{p}_i) + (\mathbf{r}_i \times \dot{\mathbf{p}}_i)\big\} = \sum_{i=1}^{N} (\mathbf{r}_i \times \mathbf{F}_i),$$

since $\mathbf{p}_i$ is parallel to $\dot{\mathbf{r}}_i$ so $\dot{\mathbf{r}}_i \times \mathbf{p}_i = 0$. Breaking $\mathbf{F}_i$ up into its two parts,

$$\mathbf{F}_i = \mathbf{F}_i^{(e)} + \sum_{j\neq i} \mathbf{F}_{ji} \qquad \Rightarrow \qquad \sum_{i=1}^{N} (\mathbf{r}_i \times \mathbf{F}_i) = \sum_{i=1}^{N} \big(\mathbf{r}_i \times \mathbf{F}_i^{(e)}\big) + \sum_{i=1}^{N} \Big(\mathbf{r}_i \times \big(\sum_{j\neq i} \mathbf{F}_{ji}\big)\Big)$$

the sum $\sum_{i=1}^{N} \Big(\mathbf{r}_i \times \big(\sum_{j\neq i} \mathbf{F}_{ji}\big)\Big)$ consists of sums of pairs

$$\big(\mathbf{r}_i \times \mathbf{F}_{ji}\big) + \big(\mathbf{r}_j \times \mathbf{F}_{ij}\big) = \big(\mathbf{r}_i - \mathbf{r}_j\big) \times \mathbf{F}_{ji},$$

since Newton's third law implies that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. It is often (but not always)* the case that the force between any two particles is in the same direction as a straight line between the particles, so $\mathbf{F}_{ji} \propto \mathbf{r}_i - \mathbf{r}_j$, in which case $\big(\mathbf{r}_i - \mathbf{r}_j\big) \times \mathbf{F}_{ji} = 0$. For example this would be true for electrostatic or gravitational forces between the particles. When this is the case (and only then) we have

$$\dot{\mathbf{L}} = \sum_{i=1}^{N} \Big(\mathbf{r}_i \times \mathbf{F}_i^{(e)}\Big) = \sum_{i=1}^{N} \boldsymbol{\tau}_i^{(e)},$$

where $\boldsymbol{\tau}_i^{(e)} = \mathbf{r}_i \times \mathbf{F}_i^{(e)}$ is the external torque about the origin on particle i. The total external torque about O, $\boldsymbol{\tau}^{(e)}$, on the whole system of particles is the sum of the torques on the individual particles,

$$\boldsymbol{\tau}^{(e)} = \sum_{i=1}^{N} \boldsymbol{\tau}_i^{(e)},$$

so

$$\dot{\mathbf{L}} = \boldsymbol{\tau}^{(e)} \tag{59}$$

and the rate of change of the total angular momentum about the origin is equal to the total torque about O on the system of particles. In particular if the total torque vanishes then the total angular momentum is a constant.

Equations (57), (58) and (59) are very useful because they allow the motion of the system to be analysed in two parts, the motion of the centre of mass and the motion of the particles relative to the centre of mass. Using (57) and (59) we can deal with the centre of mass motion as though we had a single particle of mass M at the point $\mathbf{R}$ experiencing a force $\mathbf{F}^{(e)}$ and a torque $\boldsymbol{\tau}^{(e)}$ about O.

5.3 Energy

The total kinetic energy of the system, which we shall denote by T, is the sum of the individual kinetic energies of each particle,

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \big(\mathbf{V} + \mathbf{v}_i'\big).\big(\mathbf{V} + \mathbf{v}_i'\big)$$

---

* An exception is magnetic forces between moving charged particles.

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{V}.\mathbf{V} + \sum_{i=1}^{N} m_i \mathbf{V}.\mathbf{v}'_i + \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}'_i.\mathbf{v}'_i \\
&= \frac{1}{2}M(\mathbf{V}.\mathbf{V}) + \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}'_i.\mathbf{v}'_i, \qquad (60)
\end{aligned}
$$

where we have again used $\sum_{i=1}^{N} m_i \mathbf{v}'_i = 0$. The first term in the last equation here is the kinetic energy of the centre of mass and the second term is the total kinetic energy of the individual particles relative to the centre of mass, which can be thought of as the *internal* kinetic energy of the system viewed as a whole.

If the external forces are conservative then there is a potential energy function $U_i^{(e)}(\mathbf{r}_i)$ for each $\mathbf{F}_i^{(e)}$ — in Cartesian components, with $\mathbf{r}_i = x_i\,\boldsymbol{i} + y_i\,\boldsymbol{j} + z_i\,\boldsymbol{k}$,

$$
F_{i,x}^{(e)} = -\frac{\partial U_i^{(e)}(\mathbf{r}_i)}{\partial x_i}, \qquad F_{i,y}^{(e)} = -\frac{\partial U_i^{(e)}(\mathbf{r}_i)}{\partial y_i}, \qquad F_{i,z}^{(e)} = -\frac{\partial U_i^{(e)}(\mathbf{r}_i)}{\partial z_i}, \qquad i = 1, \ldots, N.,
$$

where x_i, y_i and z_i are the Cartesian co-ordinates of particle i.

If the internal forces are conservative then there is a potential energy $U_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ associated with every pair (i, j) of particles,

$$
F_{ji,x} = -\frac{\partial U_{ij}}{\partial x_i}, \qquad F_{ji,y} = -\frac{\partial U_{ij}}{\partial y_i}, \qquad F_{ji,z} = -\frac{\partial U_{ij}}{\partial z_i}.
$$

It is the same potential $U_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ that gives rise to both $\mathbf{F}_{ji}$ and $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, so it must be the case that

$$
\frac{\partial U_{ij}}{\partial x_i} = -\frac{\partial U_{ij}}{\partial x_j}, \qquad \frac{\partial U_{ij}}{\partial y_i} = -\frac{\partial U_{ij}}{\partial y_j}, \qquad \frac{\partial U_{ij}}{\partial z_i} = -\frac{\partial U_{ij}}{\partial z_j},
$$

which implies that $U_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ is really only a function of $\mathbf{r}_i - \mathbf{r}_j$ and not of $\mathbf{r}_i$ and $\mathbf{r}_j$ separately, and we express this by writing $U_{ij}(\mathbf{r}_i - \mathbf{r}_j)$. Note that, since $\mathbf{r}_i - \mathbf{r}_j = \mathbf{r}'_i - \mathbf{r}'_j$, $U_{ij}(\mathbf{r}_i - \mathbf{r}_j) = U_{ij}(\mathbf{r}'_i - \mathbf{r}'_j)$ and the internal potential energies are independent of the position of the centre of mass $\mathbf{R}$.

The total potential energy of the system is the sum of the external potential energies $U_i^{(e)}(\mathbf{r}_i)$ and the internal potential energies $U_{ij}(\mathbf{r}_i - \mathbf{r}_j)$,

$$
U = \sum_{i=1}^{N} U_i^{(e)} + \sum_{(i,j)} U_{ij} \qquad (61)
$$

where the second sum is over all possible pairs (i, j).

5.4 Rigid Body Motion

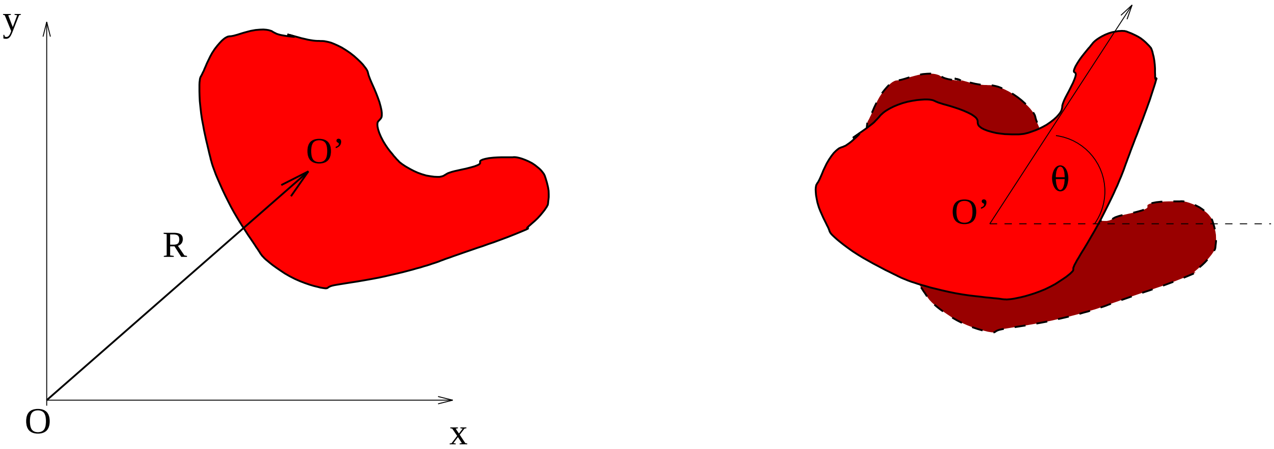
If the system of particles constitutes a *rigid body*, such as a rock or crystal, then the distances between all the particle pairs $|\mathbf{r}'_i - \mathbf{r}'_j|$ are fixed, as is the distance of particle i

from the centre of mass, $r'_i = |\mathbf{r}'_i|$. Our analysis will be based on equation (57), (58) and (59)

$$\mathbf{F}^{(e)} = M\ddot{R}, \qquad \boldsymbol{\tau}^{(e)} = \dot{\mathbf{L}}, \qquad \mathbf{L} = M(\mathbf{R}\times\mathbf{V}) + \mathbf{L}' \tag{62}$$

with $\mathbf{L}' = \sum_{i=1}^{N} \mathbf{r}'_i \times \mathbf{p}'_i$ the angular momentum about the centre of mass, $\mathbf{R}$.

For simplicity we shall consider a two dimensional rigid body, a solid flat object like a flat piece of metal. Choose the object to lie in the (x, y)-plane. The configuration of the object is completely specified by giving the position of the centre of mass and one angle to determine its orientation relative to the (x, y)-axes. For the latter we choose a fixed line in the body passing through the centre of mass, *eg.* a line parallel to the x-axis at a given time, then this line will rotate if the body is rotating and the rotation angle θ, which will be a function of time in general, determines the orientation of the body at any later time. Denoting the centre of mass by $\mathbf{R} = X\,\boldsymbol{i} + Y\,\boldsymbol{j}$ the configuration of the body at any time t is completely specified if we know $X(t)$, $Y(t)$ and $\theta(t)$.



Since the body is rigid every particle rotates about the centre of mass, which we shall denote by O', with the same angular velocity. In an infinitesimal time interval, δt, we have

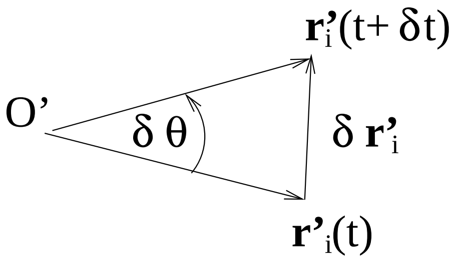
$$\delta\theta = \dot{\theta}\delta t, \qquad |\delta\mathbf{r}'_i| = r'_i\delta\theta = r'_i\dot{\theta}\delta t \qquad \Rightarrow \qquad v'_i = |\mathbf{v}'_i| = r'_i\dot{\theta},$$

and $\mathbf{v}'_i.\mathbf{r}'_i = 0$. The angular velocity of particle i about O' is defined to be the vector

$$\boldsymbol{\omega} = \frac{v'_i}{r'_i}\boldsymbol{k} = \dot{\theta}\boldsymbol{k},$$

in the z-direction if $\dot{\theta} > 0$ and in the $-z$-direction if $\dot{\theta} < 0$. The three vectors $\mathbf{r}'_i$, $\mathbf{v}'_i$ and $\boldsymbol{\omega}$ are mutually orthogonal for a rigid body and, using vector product notation,

$$\mathbf{v}'_i = \boldsymbol{\omega}\times\mathbf{r}'_i. \tag{63}$$



The angular momentum about the centre of mass is

$$\mathbf{L}' = \sum_{i=1}^{N} \mathbf{r}'_i \times \mathbf{p}'_i = \sum_{i=1}^{N} m_i \big(\mathbf{r}'_i \times \mathbf{v}'_i\big) = \sum_{i=1}^{N} m_i \Big\{ \boldsymbol{\omega} \big(r'_i\big)^2 - \big(\mathbf{r}'_i . \boldsymbol{\omega}\big)\mathbf{r}'_i \Big\} = \boldsymbol{\omega} \sum_{i=1}^{N} m_i \big(r'_i\big)^2.$$

The constant

$$I := \sum_{i=1}^{N} m_i \big(r'_i\big)^2$$

is a property of the body and is called the *moment of inertia* of M.* The equations (62) governing the motion now read

$$\mathbf{F}^{(e)} = M\ddot{R}, \qquad \boldsymbol{\tau}^{(e)} = \dot{\mathbf{L}}, \qquad \mathbf{L} = M(\mathbf{R} \times \mathbf{V}) + I\boldsymbol{\omega}. \tag{64}$$

It should be emphasised that the moment of inertia depends on the point about which it is calculated. If we calculate the moment of inertia about the origin O,

$$\begin{aligned} I_O &= \sum_{i=1}^{N} m_i (\mathbf{r}_i . \mathbf{r}_i) = \sum_{i=1}^{N} m_i (\mathbf{R} + \mathbf{r}'_i).(\mathbf{R} + \mathbf{r}'_i) \\ &= \sum_{i=1}^{N} m_i \big(R^2 + 2\mathbf{R}.\mathbf{r}'_i + (\mathbf{r}'_i)^2\big) = MR^2 + 2\mathbf{R}.\Big(\sum_{i=1}^{N} m_i \mathbf{r}'_i\Big) + I \\ &= MR^2 + I, \end{aligned}$$

hence

$$\boxed{I_O = MR^2 + I} \tag{65}$$

a result known as the *parallel axes theorem*.

The total energy of a rigid body follows from (60) and (63), with

$$\mathbf{v}'_i.\mathbf{v}'_i = (\boldsymbol{\omega} \times \mathbf{r}'_i).(\boldsymbol{\omega} \times \mathbf{r}'_i) = \omega^2 (r'_i)^2 \quad \Rightarrow \quad \sum_{i=1}^{N} m_i \mathbf{v}'_i.\mathbf{v}'_i = \omega^2 \sum_{i=1}^{N} m_i (r'_i)^2 = \omega^2 I.$$

* For a very large number of particles we can replace the discrete sum with a two dimensional integral, $\sum_{i=1}^{N} m_i \to \int \rho(\mathbf{r}')dS'$, where $\rho(\mathbf{r}')$ is the mass per unit area at the point $\mathbf{r}'$ and dS' is an infinitesimal area element. For example, using Cartesian co-ordinates

$$I = \int_{Area\ of\ body} \rho(x', y')dx'dy'.$$

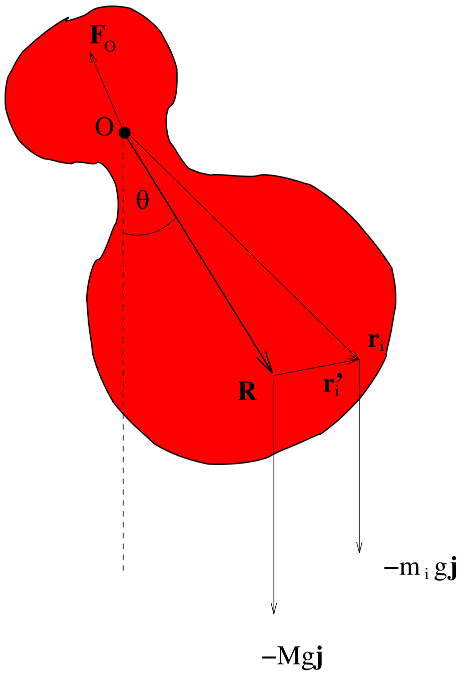
The total energy is therefore

$$E = \frac{M}{2}\mathbf{V}.\mathbf{V} + \frac{I}{2}\boldsymbol{\omega}.\boldsymbol{\omega}, \tag{66}$$

where the second term is the energy due to the rotation of the rigid body about its centre of mass, its rotational energy. The total energy is therefore sum of the kinetic energy of the centre of mass plus the rotational energy around the centre of mass.

5.5 The Compound Pendulum

As an example of the use of these equations we shall consider the motion of a compound pendulum, that is a pendulum that is not just a point mass on the end of a taut string or light rod but one that has a more general mass distribution. For simplicity we stay in two dimensions and consider a pendulum that consists of a flat shape pinned at a fixed point O about which it is free to rotate, without friction, in a vertical plane — the same plane as the object itself lies. The total force on the object is the force due to gravity plus the force due to the pin that keeps O fixed. The former acts on every point of the body while the latter only acts directly on the point O but its effect will be transmitted to other parts of the body because it is rigid. Denoting the angle between the vertical and the centre of mass by θ, then the value of θ at any time uniquely determines the configuration of the whole system at that time and our task is to determine $\theta(t)$ as a function of time.



Calculating $\ddot{R}$ directly using (57) is tricky because we would need to know the force acting on the hinge, $\mathbf{F}_O$, to determine $\mathbf{F}^{(e)}$ and $\mathbf{F}_O$ changes as the pendulum swings.

Since we don't yet know $\mathbf{F}_O$, it is easier to use (58) and (59). For almost all the particles the only external force on particle i is the force of gravity acting vertically downwards, $\mathbf{F}_i^{(e)} = -m_i g\boldsymbol{j}$ (the only exception to this is the point O, where the force $\mathbf{F}_O$ acts). Hence the total torque about O is

$$\boldsymbol{\tau}_i^{(e)} = \mathbf{r}_i \times \mathbf{F}_i^{(e)} = -m_i g(\mathbf{r}_i \times \boldsymbol{j})$$

($\mathbf{F}_O$ does not contribute to the torque about O because it acts precisely at the point O). So the total external torque is

$$
\begin{aligned}
\boldsymbol{\tau}^{(e)} &= -g\sum_{i=1}^{N} m_i\big(\mathbf{r}_i\times\boldsymbol{j}\big) = -g\sum_{i=1}^{N} m_i\Big(\big(\mathbf{R}+\mathbf{r}'_i\big)\times\boldsymbol{j}\Big)\\
&= -g\Big(\sum_{i=1}^{N} m_i\Big)(\mathbf{R}\times\boldsymbol{j}) - g\sum_{i=1}^{N}\big(m_i\mathbf{r}'_i\big)\times\boldsymbol{j}\\
&= -gM(\mathbf{R}\times\boldsymbol{j}),
\end{aligned}
$$

since $\sum_{i=1}^{N}\big(m_i\mathbf{r}'_i\big)=0$ from the definition of the centre of mass. This means that the total external torque acts as though it were acting on a point mass M at the centre of mass $\mathbf{R}$ of the body and (59) gives

$$\boldsymbol{\tau}^{(e)} = -gM(\mathbf{R}\times\boldsymbol{j}) = \dot{\mathbf{L}}.$$

Now

$$\mathbf{L} = M(\mathbf{R}\times\mathbf{V}) + I\boldsymbol{\omega}$$

and

$$\mathbf{V} = \boldsymbol{\omega}\times\mathbf{R} \quad\Rightarrow\quad \mathbf{R}\times\mathbf{V} = \mathbf{R}\times(\boldsymbol{\omega}\times\mathbf{R}) = R^2\boldsymbol{\omega} - (\mathbf{R}.\boldsymbol{\omega})\mathbf{R} = R^2\boldsymbol{\omega},$$

as $\mathbf{R}.\boldsymbol{\omega}=0$, so

$$\mathbf{L} = (MR^2+I)\boldsymbol{\omega} = I_O\boldsymbol{\omega},$$

where I_O is the moment of inertia about the pivot O and we have used the parallel axis theorem (66).

We now have

$$\boldsymbol{\tau}^{(e)} = -Mg(\mathbf{R}\times\boldsymbol{j}) = \frac{d\mathbf{L}}{dt} = I_O\dot{\boldsymbol{\omega}}.$$

So

$$Mg(\boldsymbol{j}\times\mathbf{R}) = I_O\dot{\boldsymbol{\omega}} = I_O\ddot{\theta}\boldsymbol{k}.$$

Now since

$$\mathbf{R} = X\,\boldsymbol{i} + Y\,\boldsymbol{j} = R(\sin\theta\,\boldsymbol{i} - \cos\theta\,\boldsymbol{j})$$

this is

$$
\begin{aligned}
I_O\ddot{\theta}\,\boldsymbol{k} &= MgR(\boldsymbol{j}\times\boldsymbol{i})\sin\theta = -(MgR\sin\theta)\,\boldsymbol{k}\\
\Rightarrow\qquad \ddot{\theta} &= -\left(\frac{MgR}{I_O}\right)\sin\theta.
\end{aligned}
$$

For small oscillations, $\theta << 1$ and $\sin\theta\approx\theta$ so

$$\boxed{\ddot{\theta} = -\left(\frac{MgR}{I_O}\right)\theta}$$

and $\ddot{\theta} = -\omega_0^2 \theta$ with $\omega_0^2 = \frac{MgR}{I_O}$. This is again the harmonic oscillator equation with general solution

$$\theta(t) = A_0 \cos(\omega_0 t - \delta_0).$$

The natural frequency of the pendulum is $\omega_0 = \sqrt{\frac{MgR}{I_O}}$. Note that for a simple pendulum, with all the mass M concentrated at the point $\mathbf{R}$, $I = 0$, $I_O = MR^2$ and ω_0 reduces to $\sqrt{\frac{g}{R}}$ which is a result with which you should be familiar — the frequency of a simple pendulum is independent of the mass and inversely proportional to the square root of the length. We see here that a compound pendulum yields a frequency that depends on how the mass is distributed.

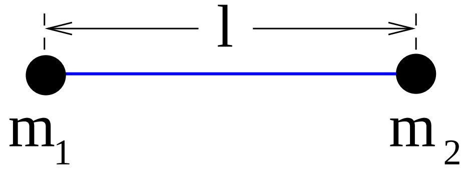
6. Lagrangian Formulation of Mechanics

6.1 Constrained Systems

A rigid body is an example of a *constrained system*. In general a system of N particles in three dimensions requires $3N$ co-ordinates to specify its configuration uniquely, for example (x_i, y_i, z_i), $i = 1, \ldots, N$ are $3N$ co-ordinates. But for a rigid body these are not all independent degrees of freedom as there are constraints between the co-ordinates, $|\mathbf{r}_i - \mathbf{r}_j|^2 = const.$ In general if there are k independent constraints among the $3N$ co-ordinates there are only $3N - k$ independent degrees of freedom in the problem.

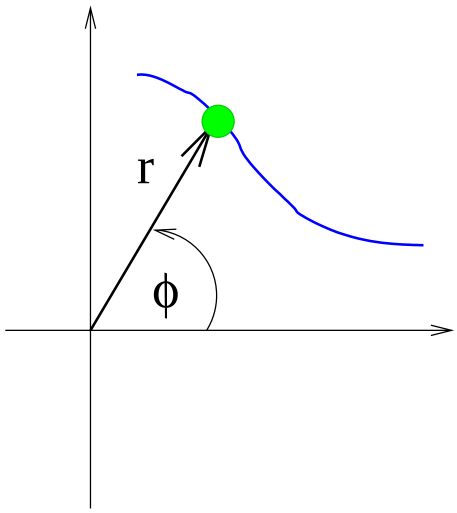
Here are some examples of constrained systems:

1) Two point masses, m_1 and m_2, fixed at the end of a light, rigid rod of length l, free to move in three dimensions . $\mathbf{r}_1$ and $\mathbf{r}_2$ involve 6 Cartesian co-ordinates, but there is one constraint, $|\mathbf{r}_1 - \mathbf{r}_2| = l$, so there are only 5 degrees of freedom in the problem. For example we could use the co-ordinates of the centre of mass $X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$, $Y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$ and $Z = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$, together with two angles to specify the direction in which the rod lies as our 5 degrees of freedom.



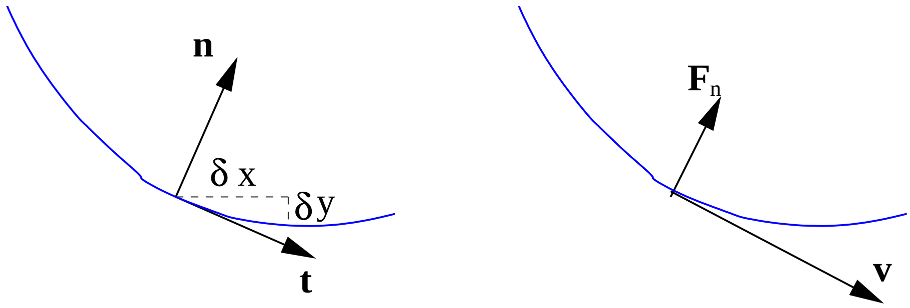
If the rod and masses moved in two dimensions there would only be 3 degrees of freedom, not 5. Two Cartesian co-ordinates for the centre of mass and one angle to specify the orientation of the rod.

2) A bead of mass m slides on a frictionless wire in a two dimensional plane. The Cartesian co-ordinates of the bead are (x, y), but they are not independent. If the wire is bent into a shape specified by some function $y = f(x)$, then there is only one degree of freedom in the problem. We could use x as the degree of freedom or we could use 2-dimensional polar co-ordinates with $x = r\cos\phi$, $y = r\sin\phi$ and choose ϕ as the independent degree of freedom, with r given by $r(\phi) = \sqrt{x^2 + y^2} = \sqrt{x^2 + f^2(x)}$.



In both these examples the $3N - k$ degrees of freedom that we use to describe the motion need not be simple Cartesian co-ordinates, they are more general quantities that are called *generalised co-ordinates.* We shall denote generalised co-ordinates by q^α, $\alpha = 1, \ldots, 3N - k$ in three dimensions. ($\alpha = 1, \ldots, 2N - k$ in two dimensions)*

Let us consider example 2 above in a little more detail. The wire is assumed to be frictionless and this means that there are no frictional forces opposing the motion of the bead, nevertheless there are still forces acting on the bead, the forces of constraint that force it to follow the wire. In the absence of friction the constraint forces on the bead are always normal to the wire, at the point where the bead is at any given time. Suppose the equation describing the shape of the wire is $y = f(x)$, then under an infinitesimal displacement $(\delta x, \delta y)$ of the bead, $\delta y = \frac{df(x)}{dx}\delta x = f'\delta x$.



The direction tangent to the wire at this point is $\delta \mathbf{x} = \delta x\, \boldsymbol{i} + \delta y\, \boldsymbol{j}$ and the unit vector tangent to the wire is

$$\mathbf{t} = \frac{\delta x\, \boldsymbol{i} + \delta y\, \boldsymbol{j}}{\sqrt{(\delta x)^2 + (\delta y)^2}} = \frac{\boldsymbol{i} + \frac{\delta y}{\delta x}\, \boldsymbol{j}}{\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}} \quad \underset{\delta x \to 0}{\longrightarrow} \quad \frac{\boldsymbol{i} + f'\, \boldsymbol{j}}{\sqrt{1 + (f')^2}}.$$

* Unconstrained motion in two dimensions can be thought of as motion in three dimensions with $k = N$ constraints, $2N = 3N - N$.

The unit normal to the wire, $\mathbf{n}$, is at right-angles to this

$$\mathbf{n} = \pm\left(\frac{-f'\,\boldsymbol{i} + \boldsymbol{j}}{\sqrt{1+(f')^2}}\right),$$

where we choose the sign so that $\mathbf{n}$ is the same direction as $\mathbf{F}_n$. When the constraining force is normal to the wire $\mathbf{F}_n.\mathbf{t} = 0$, or

$$\mathbf{F}_n.\delta\mathbf{x} = 0. \tag{67}$$

Of course $\mathbf{F}_n.\delta\mathbf{x}$ is just the work done on the bead by the constraining force when the bead moves through an infinitesimal displacement $\delta\mathbf{x}$ so this is telling us that the constraining forces do no work.

The velocity of the bead is

$$\mathbf{v} = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j} = \dot{x}(\boldsymbol{i} + f'\,\boldsymbol{j}) = \left(\dot{x}\sqrt{1+(f')^2}\right)\mathbf{t},$$

where we have used $\dot{y} = \dot{x}f'$. The constraint force is in the $\mathbf{n}$-direction

$$\mathbf{F}_n = F_n\mathbf{n} = \pm\frac{F_n}{\sqrt{1+(f')^2}}(-f'\,\boldsymbol{i} + \boldsymbol{j}), \tag{68}$$

where F_n is the magnitude of the constraining force.

Suppose the wire is in a vertical plane, with x along the horizontal axis and y up the vertical axis. Then the gravitational force on m is $-mg\,\boldsymbol{j}$ and Newton's second law reads

$$m\ddot{\mathbf{x}} = -mg\,\boldsymbol{j} + \mathbf{F}_n$$

or, in components,

$$\ddot{x} = \mp\frac{F_n}{m}\frac{f'}{\sqrt{1+(f')^2}}, \qquad \ddot{y} = -g \pm \frac{F_n}{m}\frac{1}{\sqrt{1+(f')^2}}. \tag{69}$$

Obviously we need to know F_n in order to solve these equations, but they are not independent because of the constraint. Since $y = f(x)$ the chain rule gives

$$\dot{y} = \frac{df}{dx}\dot{x} = f'\dot{x}, \qquad \text{and} \qquad \ddot{y} = f''\dot{x}^2 + f'\ddot{x},$$

and $\ddot{x} = \mp\frac{F_n}{m}\frac{f'}{\sqrt{1+(f')^2}}$ then gives

$$\ddot{y} = f''\dot{x}^2 + f'\ddot{x} = f''\dot{x}^2 \mp \frac{F_n}{m}\frac{(f')^2}{\sqrt{1+(f')^2}} = -g \pm \frac{F_n}{m}\frac{1}{\sqrt{1+(f')^2}}$$

$$\Rightarrow \qquad f''\dot{x}^2 = -g \pm \frac{F_n}{m}\sqrt{1+(f')^2} \quad \Rightarrow \quad \frac{F_n}{m} = \pm\frac{f''\dot{x}^2 + g}{\sqrt{1+(f')^2}}.$$

We can eliminate F_n from the $\ddot{x}$ equation in (69) to give

$$\ddot{x}\big(1+(f')^2\big) = -f'\big(f''\dot{x}^2+g\big), \tag{70}$$

but this is still going to be a very hard equation to solve for anything except the simplest functions $f(x)$.

Notice that the constraint force depends on $\dot{x}$, but it is still a conservative force, it does no work on the bead (67). Let's check that energy is conserved. The kinetic energy, T, is

$$T = \frac{m}{2}\big(\dot{x}^2+\dot{y}^2\big) = \frac{m}{2}\dot{x}^2\big(1+(f')^2\big). \tag{71}$$

Choose the zero of potential energy to be at $y=0$, then the potential energy due to the gravitational force is $U(x) = mgy = mgf(x)$ and the total energy is

$$\begin{aligned} & E = T+U = \frac{m}{2}\dot{x}^2\big(1+(f')^2\big) + mgf(x) \\ \Rightarrow \qquad & \frac{dE}{dt} = m\ddot{x}\dot{x}\big(1+(f')^2\big) + m\dot{x}^2\big(f'f''\dot{x}\big) + mgf'\dot{x} \\ & \quad = m\dot{x}\left\{\ddot{x}\big(1+(f')^2\big) + f'\big(\dot{x}^2\big(f''\big)+g\big)\right\} = 0, \end{aligned}$$

which vanishes due to the equation of motion (70). Hence the energy is indeed constant.

There is a quick way of getting at (70) without ever introducing the constraint force. For an unconstrained free particle moving under the influence of a force, $\mathbf{F} = -\nabla U$, the kinetic energy, $T(\dot{x},\dot{y}) = \frac{m}{2}\big(\dot{x}^2+\dot{y}^2\big)$ is a function of two independent quantities, $\dot{x}$ and $\dot{y}$. The x-component of the momentum, $p_x = m\dot{x}$ can be obtained from T by partial differentiation with respect to $\dot{x}$, keeping $\dot{y}$ fixed,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x},$$

and the x-component of Newton's second law can be written as

$$m\ddot{x} = \dot{p}_x = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\partial_x U.$$

For the bead on the wire $\dot{y}$ is not independent of $\dot{x}$, rather $\dot{y} = f'(x)\dot{x}$, and

$$T(x,\dot{x}) = \frac{m}{2}\dot{x}^2\big(1+(f')^2\big)$$

depends on $\dot{x}$ and x. Define the *generalised momentum* as being the partial derivative of $T(x,\dot{x})$ with respect to $\dot{x}$, keeping x fixed,

$$\pi_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}\big(1+(f')^2\big).$$

Then the rate of change of the generalised momentum is

$$\dot{\pi}_x = m\ddot{x}\big(1+(f')^2\big) + 2m\dot{x}^2 f''f'.$$

Now notice that the equation of motion (70) gives

$$\dot{\pi}_x = m\ddot{x}\big(1 + (f')^2\big) + 2m\dot{x}^2 f'' f' = m\dot{x}^2 f'' f' - mgf'.$$

The right hand side is not quite the negative of the derivative of the potential energy with respect to x, $-\partial_x U = -mgf'$, but the extra term

$$m\dot{x}^2 f'' f' = \frac{\partial T}{\partial x}$$

is the partial derivative of $T(x, \dot{x})$ with respect to x with $\dot{x}$ held fixed. Hence Newton's second law for the bead on the wire can be written as

$$\dot{\pi}_x = \frac{\partial}{\partial x}\big\{T(x, \dot{x}) - U(x)\big\}.$$

The function

$$L(x, \dot{x}) := T(x, \dot{x}) - U(x)$$

is called the *Lagrangian* for the system (after the French mathematician Joseph-Louis Lagrange, (1736-1813), who was the first person to formulate mechanics in this way). Since $U(x)$ is independent of $\dot{x}$ the generalised momentum can equally well be obtained from L as

$$\pi_x = \frac{\partial T}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}}.$$

In terms of the Lagrangian, Newton's second law for the bead reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{72}$$

Equation (72) is completely equivalent to Newton's second law for the constrained motion of the bead and has the advantage that it can be obtained purely from a knowledge of $T(x, \dot{x})$ and $U(x)$ — we never need to see the force of constraint, $\mathbf{F}_n$, explicitly. Three points to note here are:

1) The Lagrangian $L = T - U$ is *not* the same as the energy $E = T + U$ (unless of course $U = 0$). The Lagrangian is not a constant of the motion in general.
2) The derivative with respect to t here is the total derivative, it acts on both $\dot{x}(t)$ and $x(t)$.
3) A subtle point has been slipped in here. The Euler-Lagrange equation assumes that x and $\dot{x}$ are independent of each other, whereas for the actual motion of course $\dot{x} = \frac{dx}{dt}$ and $\dot{x}$ is not independent, it is derived from $x(t)$ once $x(t)$ is known. The philosophy of the Euler-Lagrange equation is to treat both x and $\dot{x}$ as independent until we have completely solved for the motion, they are varied separately in $L(x, \dot{x})$. Note that the initial position and velocity $x_0 = x(0)$ and $v_0 = \dot{x}(0)$ *are* independent, they must both be given to solve for the subsequent motion. There is still only one generalised co-ordinate in this problem, and that is x.

6.2 Lagrangian Formulation

More generally for a system with $n = 3N - k$ independent generalised co-ordinates q^α, with $\alpha = 1, \ldots, n$, we can express the equations of motion in Lagrangian form if we know the potential energy $U(q)$ and the kinetic energy $T(q, \dot{q})$. Then the Lagrangian is

$$\boxed{L(q, \dot{q}) = T(q, \dot{q}) - U(q)} \tag{73}$$

and Newton's second law can be written

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = \frac{\partial L}{\partial q^\alpha}\ ,} \tag{74}$$

which are known as the *Euler-Lagrange* equations of motion. There are n generalised momenta, $\pi_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$, one for each generalised co-ordinate, and there are n independent equations of motion in (74).

The Lagrangian formulation of Newton's equations is an extremely useful way of describing systems with constraints, though it is also useful for systems without constraints but whose dynamics is best described using generalised co-ordinates rather than Cartesian co-ordinates, polar co-ordinates for example. We only need to determine the number of independent degrees of freedom of the system and identify useful generalised co-ordinates and then calculate the Lagrangian as a function of our generalised co-ordinates. The dynamical equations follow from (74). All of the dynamics follows from a single function of the generalised co-ordinates and their time derivatives.

Here are some examples:

1) A very simple example is a single particle of mass m moving in three dimensions under the influence of a conservative force arising from a potential U. Using Cartesian co-ordinates (x, y, z)
$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$
and the force has components
$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}.$$
The Lagrangian is
$$L = T - U = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + +\dot{y}^2\right) - U(x, y, z).$$
The momenta are
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

and Newton's equations follow from (74),

$$\dot{p}_x = -\frac{\partial U}{\partial x}, \quad \dot{p}_y = -\frac{\partial U}{\partial y}, \quad \dot{p}_z = -\frac{\partial U}{\partial z}.$$

For Cartesian co-ordinates with no constraints Lagrange's equations are just another way of writing Newton's second law,

2) Sometimes Cartesians are not the best description of a dynamical system. Take a central force in three dimensions, where the force depends only on the distance from the origin. Then polar co-ordinates are more suited to the symmetry of the problem. In three dimensional polar co-ordinates the velocity is (54) so the kinetic energy of a particle of mass m is

$$T = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2(\sin\theta)^2\dot{\phi}^2\big).$$

The potential for a central force is a function of r only, $U(r)$, so $\mathbf{F} = F_r\mathbf{e}_r$ with $F_r = -\frac{dU}{dr}$. The Lagrangian is

$$L = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2(\sin\theta)^2\dot{\phi}^2\big) - U(r)$$

giving generalised momenta

$$\pi_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2(\sin\theta)^2\dot{\phi}$$

while

$$\frac{\partial L}{\partial r} = mr\big(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\big) - \frac{dU}{dr}, \quad \frac{\partial L}{\partial \theta} = mr^2(\sin\theta\cos\theta)\dot{\phi}^2, \quad \frac{\partial L}{\partial \phi} = 0.$$

The Euler-Lagrange equations are thus

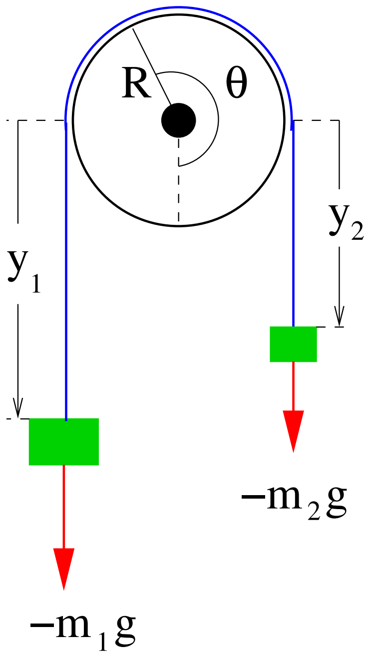
$$\begin{aligned}
\dot{\pi}_r &= m\ddot{r} = mr\big(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\big) - \frac{dU}{dr}, \\
\dot{\pi}_\theta &= m\frac{d}{dt}\big(r^2\dot{\theta}\big) = mr^2(\sin\theta\cos\theta)\dot{\phi}^2, \\
\dot{\pi}_\phi &= m\frac{d}{dt}\big(r^2(\sin\theta)^2\dot{\phi}\big) = 0.
\end{aligned}$$

The middle equation is satisfied by setting $\theta = \frac{\pi}{2} = const$, since then $\dot{\theta} = 0$ and $\cos\theta = 0$. The motion of the particle lies in a two dimensional plane, which we can choose to be the plane $\theta = \frac{\pi}{2}$ (*ie.* the (x, y)-plane). The last equation then implies that $ml := mr^2\dot{\phi} = const$, this is conservation of angular momentum (we do not use L for total angular momentum in this section, in order to avoid confusion with the Lagrangian). Finally, using l in the first equation above gives

$$\ddot{r} = \frac{ml^2}{r^3} - \frac{dU}{dr},$$

the first term on the right hand side representing centrifugal acceleration. All this obtained in a few lines from the Lagrangian!

3) As a last example we consider a system that has constraints. Let two masses, m_1 and m_2, be connected together by a light inextensible string of length l that hangs over a light circular pulley of radius R in a vertical plane, which is free to rotate about a horizontal axis through its centre. We assume the string does not slip on the pulley.



The vertical distances y_1 of mass m_1 and y_2 of mass m_2 downward from the axis of the pulley are not independent, but are constrained by $y_1 + \pi R + y_2 = l$, so there is only one degree of freedom, we shall choose to use $y = y_1$, and

$$y_2 = l - y_1 - \pi R \qquad \Rightarrow \qquad \dot{y}_2 = -\dot{y}_1.$$

Ignoring the masses of the string and the pulley, the kinetic energy is just the sum of the kinetic energies of m_1 and m_2,

$$T = \frac{m_1}{2}\dot{y}_2^2 + \frac{m_1}{2}\dot{y}_2^2 = \frac{m_1 + m_2}{2}\dot{y}^2.$$

The potential energy is the sum of the potential energies of m_1 and m_2,

$$U = -g(m_1 y_1 + m_2 y_2) = -g\big(m_1 y + m_2(l - \pi R - y)\big) = -g(m_1 - m_2)y + const.$$

Ignoring the constant in the potential the Lagrangian is

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{y}^2 - g(m_1 - m_2)y.$$

The generalised momentum is

$$\pi_y = \frac{\partial L}{\partial \dot{y}} = (m_1 + m_2)\dot{y}$$

and the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \quad \Rightarrow \quad \dot{\pi}_y = (m_1 + m_2)\ddot{y} = g(m_1 - m_2),$$

so the acceleration

$$\ddot{y} = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

is constant, y increases if $m_1 > m_2$ and decreases if $m_2 > m_1$. Suppose $m_1 > m_2$ and the system is released from rest at $t = 0$ with $y(0) = 0$ then the subsequent motion is

$$y(t) = \frac{g}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)t^2,$$

at least until y reaches $l - \pi R$.

It is even easy to include the mass and moment of inertia of the pulley. If the pulley has mass M and is of constant density $\rho = \frac{M}{\pi R^2}$, then its moment of inertia about the central axis is

$$I = 2\pi \int_0^R (\rho r^2) r dr = \frac{2\pi}{4}\rho R^4 = \frac{1}{2}MR^2.$$

Measure the angle of rotation from the vertical by θ, with $\theta = 0$ when $y = 0$, then θ is not an independent degree of freedom, $y = R\theta$. The kinetic energy of the rotating pulley is then given by (66), with $\mathbf{V} = 0$ since the centre of the pulley is fixed in space,

$$T_{Pulley} = \frac{I}{2}\dot{\theta}^2 = \frac{I}{2}\frac{\dot{y}^2}{R^2} = \frac{M}{4}\dot{y}^2$$

and we add this to the Lagrangian,

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\dot{y}^2 + g(m_1 - m_2)y.$$

The equation of motion is now

$$\ddot{y} = g\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{M}{2}}\right),$$

which is easily solved.

6.3 Time Dependent Constraints*

The Lagrangian formulation of mechanics is an extremely efficient way of describing dynamics in the presence of constraints and conservative forces, even when the constraints depend on time.

First consider a simple example with no constraints. Take N identical particles each of mass m moving in three dimensions, with Cartesian co-ordinates x^I, $i = 1, \cdots, 3N$, under the influence of a conservative force,

$$F_I = -\frac{\partial U}{\partial x^I},$$

with potential $U(x^I)$. We can use the Cartesian co-ordinates themselves as generalised co-ordinates, then the kinetic energy is

$$T = \frac{m}{2}\sum_{I=1}^{3N} \dot{x}^I \dot{x}^I \tag{75}$$

so the Lagrangian is

$$L(\dot{x}^I, x^I) = \frac{m}{2}\sum_{I=1}^{3N} \dot{x}^I \dot{x}^I - U(x^I).$$

The generalised momenta are the ordinary momenta in this case,

$$p_I = \frac{\partial L}{\partial \dot{x}^I} = \frac{\partial T}{\partial \dot{x}^I} = m\dot{x}^I$$

and the kinetic energy depends only on $\dot{x}^I$, not on x^I, so

$$\frac{\partial L}{\partial x^I} = -\frac{\partial U}{\partial x^I}.$$

The Euler-Lagrange equations then read

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^I}\right) &= \left(\frac{\partial L}{\partial x^I}\right) \\ \Rightarrow \qquad \frac{dp_I}{dt} &= -\left(\frac{\partial U}{\partial x^I}\right) = F_I, \end{aligned}$$

which simply reproduces Newton's second law.

The Lagrangian formulation even allows for time dependent constraints. Consider a system of N particles, labelled by an index $i = 1, \ldots, N$, moving in three dimensions. There are $3N$ degrees of freedom and we shall use $3N$ Cartesian co-ordinates which will be labelled by x^I, $I = 1, \ldots, 3N$. Let there be k constraints and choose $n = 3N - k$

* The material in this section was not covered in the lectures and is included for further interest.

generalised co-ordinates q^α, with $\alpha = 1, \ldots, n$. The Cartesian co-ordinates labelling the particle positions are then not all independent degrees of freedom, they can be expressed as functions of q^α and possibly t itself, if the constraints depend on time, $x^I(q^\alpha, t)$. Then

$$\dot{x}^I = \sum_{\alpha=1}^{n} \frac{\partial x^I}{\partial q^a} \dot{q}^\alpha + \frac{\partial x^I}{\partial t} \tag{76}$$

is in general a function of $\dot{q}^\alpha$, q^α and t, $\dot{x}^I(\dot{q}^\alpha, q^\alpha, t)$. However its dependence on $\dot{q}^\alpha$ is particularly simple, it is linear in $\dot{q}^\alpha$,

$$\frac{\partial \dot{x}^I}{\partial \dot{q}^a} = \frac{\partial x^I}{\partial q^a}. \tag{77}$$

Now decompose the force the force on each particle into an external force $\mathbf{F}^{(e)}$, *eg.* gravity, and a constraint force $\mathbf{F}^{(c)}$. Newton's second law is

$$\mathbf{F} = \mathbf{F}^{(e)} + \mathbf{F}^{(c)} = \dot{\mathbf{p}}$$

where $p^I = m\dot{x}^I$, assuming for simplicity that each particle has the same mass (this is not necessary). Then under a virtual displacement $\delta\mathbf{x}$ for which the constraints do no work, $\mathbf{F}^{(c)}.\delta\mathbf{x} = 0$, we have

$$0 = (\mathbf{F} - \dot{\mathbf{p}}).\delta\mathbf{x} = (\mathbf{F}^{(e)} + \mathbf{F}^{(c)} - \dot{\mathbf{p}}).\delta\mathbf{x} = (\mathbf{F}^{(e)} - \dot{\mathbf{p}}).\delta\mathbf{x}.$$

So, for virtual displacements,

$$(\mathbf{F}^{(e)} - \dot{\mathbf{p}}).\delta\mathbf{x} = 0.$$

Note that it is *not* true that $\mathbf{F}^{(e)} - \dot{\mathbf{p}} = 0$, unless of course there are no constraints and $\mathbf{F}^{(c)} = 0$.

If the external forces are conservative there is a potential energy function, $U(x^I)$, for which

$$F_I^{(e)} = -\frac{\partial U}{\partial x^I}$$

and we can write

$$\sum_{I=1}^{N} \left(\frac{\partial U}{\partial x^I} + \frac{dp^I}{dt} \right) \delta x^I = 0.$$

Virtual displacements cannot be in arbitrary directions, they must correspond to variations in the generalised co-ordinates $q^\alpha \to q^\alpha + \delta q^\alpha$, so the variation in x^I is constrained to be of the form

$$\delta x^I = \sum_{\alpha=1}^{n} \frac{\partial x^I}{\partial q^\alpha} \delta q^\alpha.$$

However the δq^α are arbitrary, so

$$\sum_{I=1}^{N} \left(\frac{\partial U}{\partial x^I} + \frac{dp^I}{dt} \right) \left(\frac{\partial x^I}{\partial q^\alpha} \right) = 0. \tag{78}$$

Now $L = T(\dot{x}^I, x^I) - U(x^I)$ with the kinetic energy given by (75), and using (77),

$$\sum_{I=1}^N p^I \frac{\partial x^I}{\partial q^\alpha} = \sum_{I=1}^N \frac{\partial L}{\partial \dot{x}^I} \frac{\partial x^I}{\partial q^\alpha} = \sum_{I=1}^N \frac{\partial L}{\partial \dot{x}^I} \frac{\partial \dot{x}^I}{\partial \dot{q}^\alpha} = \frac{\partial L}{\partial \dot{q}^\alpha},$$

so

$$\begin{aligned} \sum_{I=1}^N \frac{dp^I}{dt} \frac{\partial x^I}{\partial q^\alpha} &= \frac{d}{dt}\left(\sum_{I=1}^N p^I \frac{\partial x^I}{\partial q^\alpha}\right) - \sum_{I=1}^N p^I \frac{d}{dt}\left(\frac{\partial x^I}{\partial q^\alpha}\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \sum_{I=1}^N p^I \frac{d}{dt}\left(\frac{\partial x^I}{\partial q^\alpha}\right). \end{aligned} \tag{79}$$

While

$$\begin{aligned} \sum_{I=1}^N \frac{\partial U}{\partial x^I} \frac{\partial x^I}{\partial q^\alpha} = \frac{\partial U}{\partial q^\alpha} = -\frac{\partial L}{\partial q^\alpha} + \frac{\partial T}{\partial q^\alpha} \\ = -\frac{\partial L}{\partial q^\alpha} + \sum_{I=1}^N \frac{\partial T}{\partial \dot{x}^I} \frac{\partial \dot{x}^I}{\partial q^\alpha} = -\frac{\partial L}{\partial q^\alpha} + \sum_{I=1}^N p_I \frac{\partial \dot{x}^I}{\partial q^\alpha}. \end{aligned} \tag{80}$$

Putting (79) and (80) into (78) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} + \sum_{I=1}^N p_I \left(\frac{\partial \dot{x}^I}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial x^I}{\partial q^\alpha}\right)\right) = 0. \tag{81}$$

Now the magic is, using (76),

$$\frac{\partial \dot{x}^I}{\partial q^\alpha} = \sum_{\beta=1}^n \left(\frac{\partial^2 x^I}{\partial q^\alpha \partial q^\beta}\right) \dot{q}^\beta + \frac{\partial^2 x^I}{\partial q^\alpha \partial t}$$

and

$$\frac{d}{dt}\left(\frac{\partial x^I}{\partial q^\alpha}\right) = \sum_{\beta=1}^n \left(\frac{\partial^2 x^I}{\partial q^\beta \partial q^\alpha}\right) \dot{q}^\beta + \frac{\partial^2 x^I}{\partial t \partial q^\alpha},$$

so every term in the last summation in (81) vanishes leaving the Euler-Lagrange equations for the constrained system

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0.}$$

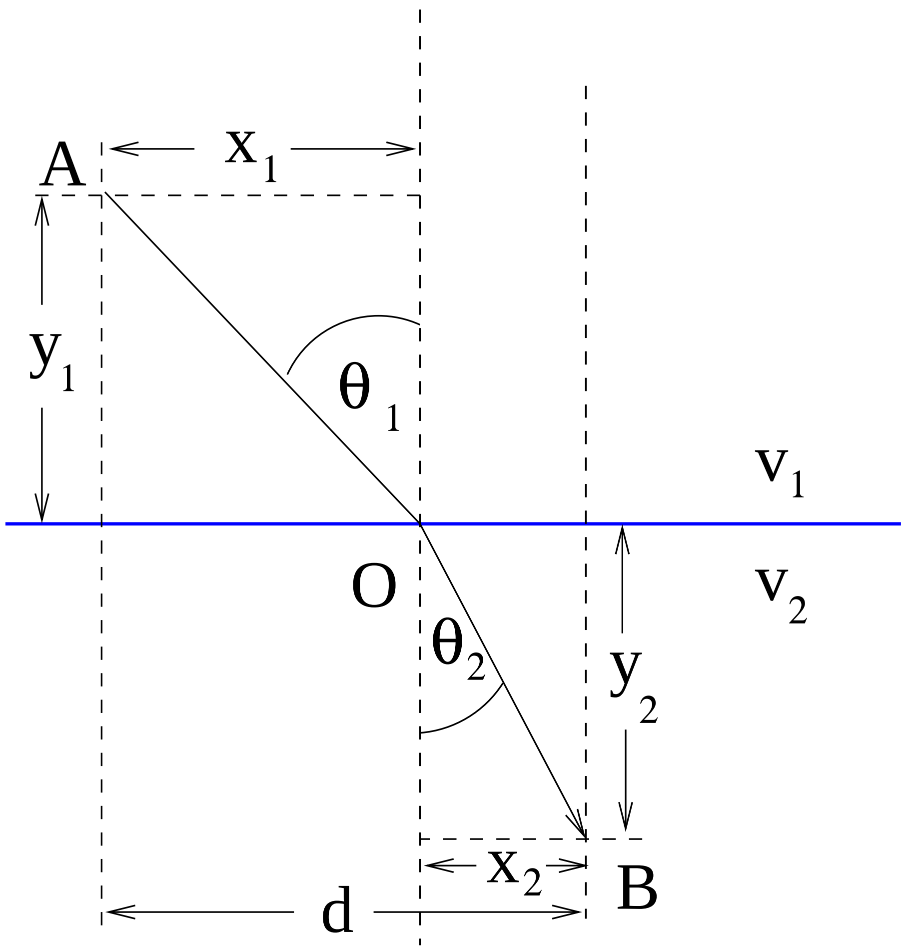
6.4 Variational Formulation and the Action

In static equilibrium a system always seeks out a configuration that minimises the energy, in particular statics requires that the kinetic energy vanishes and the potential energy be minimised with respect to vatriations of particle positions,

$$\frac{\partial U}{\partial q^\alpha} = 0.$$

Amazingly the Euler-Lagrange equations can be obtained from a similar principle, but it is not the energy that is minimised. Historically the origins of this concept are related to a minimisation principle in optics known the principle of least time — a ray of light travels in such a way as to minimise the time it takes to travel between any two given points.

Consider a beam of light traveling across a planar interface from a point A in one medium (*eg.* air) in which the speed of light is v_1, to a point B in a different medium (*eg.* water) in which the speed of light is v_2. What trajectory will minimise the time taken for the light to travel from A to B? The light will travel in a straight line in medium 1 and a straight line in medium 2, but we can vary the point O to try and minimise the time.



Since A and B are fixed y_1 and y_2 are fixed and $x_1 + x_2 = d$, but we can vary x_1 and x_2 by moving the point O though only one of them is independent as $x_2 = d - x_1$. The time taken for the light to travel from A to O, t_1, is the length of AO, which is $\sqrt{x_1^2+y_1^2}$, divided by the speed of light in the medium 1,

$$t_1 = \frac{\sqrt{x_1^2+y_1^2}}{v_1}.$$

Similarly the time taken for light to travel from O to B is

$$t_2 = \frac{\sqrt{x_2^2+y_2^2}}{v_2} = \frac{\sqrt{(d-x_1)^2+y_2^2}}{v_2}.$$

Hence the total time to travel from A to B is

$$T = t_1 + t_2 = \frac{\sqrt{x_1^2+y_1^2}}{v_1} + \frac{\sqrt{(d-x_1)^2+y_2^2}}{v_2}.$$

Now y_1 and y_2 are fixed and we can minimise $T(x_1)$ by varying x_1 and demanding that $\frac{dT}{dx_1} = 0$, so

$$\frac{dT}{dx_1} = \frac{x_1}{v_1\sqrt{x_1^2+y_1^2}} - \frac{(d-x_1)}{v_2\sqrt{(d-x_1)^2+y_2^2}} = \frac{x_1 v_2\sqrt{(d-x_1)^2+y_2^2} - (d-x_1)v_1\sqrt{x_1^2+y_1^2}}{v_1 v_2\sqrt{x_1^2+y_1^2}\sqrt{(d-x_1)^2+y_2^2}}.$$

This vanishes, for finite x_1, only when

$$\begin{aligned} & x_1 v_2\sqrt{(d-x_1)^2+y_2^2} = (d-x_1)v_1\sqrt{x_1^2+y_1^2} \qquad (82)\\ \Rightarrow \quad & x_1 v_2\sqrt{x_2^2+y_2^2} = x_2 v_1\sqrt{x_1^2+y_1^2} \\ \Rightarrow \quad & \frac{x_1}{\sqrt{x_1^2+y_1^2}}v_2 = \frac{x_2}{\sqrt{x_2^2+y_2^2}}v_2 \\ \Rightarrow \quad & \sin\theta_1 v_2 = \sin\theta_2 v_1, \end{aligned}$$

or

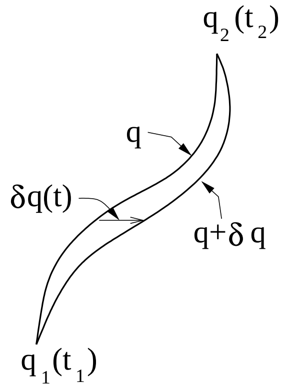
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$

which is Snell's law for refraction! Snell's law follows from the assumption that the light travels in a manner that *minimises* the time taken to go from A to B (we leave it as an exercise to show that this is a minimum and not a maximum, *ie.* check that $\frac{d^2T}{dx_1^2} > 0$ when x_1 is given by (82). This way of viewing refraction, as minimising the travel time of a light beam, is known in optics as *Fermat's principle* or the *principle of least time.* The law of reflection can be derived the same way.

We are about to show that Newton's 2nd law, in the Euler-Lagrange formulation (74), can also be derived by extremising a certain quantity known as the *action.* Suppose a particle travels from a point with generalised co-ordinates q_1^α at time t_1 to a point q_2^α at a time t_2 along a trajectory $q^\alpha(t)$, with $q_1^\alpha = q^\alpha(t_1)$ and $q_1^\alpha = q^\alpha(t_1)$. The action, S, is defined to be the integral of the Lagrangian with respect to time along the trajectory $q^\alpha(t)$,

$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}) dt.$$

The notation $S[q]$ is a conventional way of expressing the notion that the action depends on the path, $q^\alpha(t)$, taken between the initial point q_1^α and the final point q_2^α. The action is not a function of time, time has been integrated over in the definition of S, but it does depend on the path taken between the fixed points q_1 and q_2. It is a function of a function, called a *functional*, and is written $S[q]$ to emphasise the fact that it depends on the function $q(t)$. Now suppose we vary the path, keeping the end points fixed.



Then in general $S[q]$ will change. If the path is varied from $q(t)$ to $q(t) + \delta q(t)$, and also $\dot{q}(t) \to \dot{q}(t) + \delta\dot{q}(t)$, where $\delta q(t)$ is an infinitesimally small function with $\delta q(t_1) = 0$ and $\delta q(t_2) = 0$ so the end points do not move, then the change in the action will be

$$\begin{aligned} \delta S[q] = S[q + \delta q] - S[q] &= \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}) dt - \int_{t_1}^{t_2} L(q, \dot{q}) dt \\ &= \int_{t_1}^{t_2} \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha \right) dt + \cdots \\ &= \sum_{\alpha=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha \right) dt + \cdots , \end{aligned} \tag{83}$$

where the dots indicate quantities of order δq^2, which we shall ignore for infinitesimal δq.

The second term under the summation on the right hand side can be integrated by parts using,

$$\frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) \delta q^\alpha,$$

so

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha dt &= \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right) dt - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) \delta q^\alpha dt \\ &= \left[\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) \delta q^\alpha dt \end{aligned} .$$

But $\delta q(t_1) = \delta q(t_2) = 0$, since the end points are assumed to be held fixed, so the integrated term in square brackets above vanishes and

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) \delta q^\alpha dt.$$

Using this in (83) we find

$$\delta S[q] = \sum_{\alpha=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right)\right) \delta q^\alpha dt + \cdots .$$

The variation of $S[q]$ then vanishes, to first order in δq, for *any* $\delta q(t)$ with $\delta q_1 = 0$ fixed at t_1 and $\delta q_2 = 0$ fixed at t_2, if and only if

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = 0,$$

that is $\delta S[q] = 0$ if and only if the Euler-Lagrange equations are satisfied. The actual trajectory of the particle is such as to extremise the action. In fact the action is a maximum when $q(t)$ is the correct physical trajectory, rather than a minimum, but this just means that $-S$ is minimised. This is clear from a static situation where the kinetic energy is zero, so $L = -U$, and the second term above, involving the total time derivative, also vanishes then

$$\frac{\partial L}{\partial q^\alpha} = -\frac{\partial U}{\partial q^\alpha}$$

and a minimum of U is a maximum of L, and hence a maximum of S.