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Section 2
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Summary of Concepts and Formulas

I. General Principles

1. Statics

   (a) External equilibrium (loads, reactions): $\sum \mathbf{F} = \mathbf{0}$, $\sum \mathbf{M} = \mathbf{0}$

   (b) Internal forces (free-body diagram)

   (c) Stress

      i. stress tensor $\begin{bmatrix} \sigma_x & \sigma_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$

      ii. Moment equilibrium: $\tau_{xy} = \tau_{yx}$, etc.

      iii. Mean stress $\sigma_0 = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$; pressure $p = -\sigma_0$

      iv. Equilibrium of thin shells of revolution: (cylindrical) $\sigma_z = pR/2t$, $\sigma_\theta = pR/t$, $\sigma_r \approx 0$; (spherical) $\sigma_\theta = \sigma_\phi = pR/2t$, $\sigma_r \approx 0$.

2. Geometry

   (a) Displacement field: $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$

   (b) Strain

      i. $\varepsilon_x = \dfrac{\partial u}{\partial x}$ (longitudinal), $\gamma_{xy} = \gamma_{yx} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$ (shear), etc.

      ii. strain tensor $\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$

3. Constitutive Properties

   (a) General: relation between stress, strain and temperature

   (b) Linear elastic isotropic

      i. general: $\varepsilon_x = \dfrac{1}{E}[\sigma_x - \nu\sigma_y - \nu\sigma_z] + \alpha\Delta T$ etc., $\gamma_{xy} = \dfrac{\tau_{xy}}{G}$ etc.; $G = \dfrac{E}{2(1+\nu)}$

      ii. plane stress (in xy-plane): $\sigma_z = \tau_{yz} = \tau_{zx} = 0$, $\sigma_x = \dfrac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$ etc.

      iii. plane strain: $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$, $\sigma_x = \dfrac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y]$ etc.

   (c) Elastic-perfectly plastic: stress-strain curve becomes horizontal after yield, unloading is elastic.

(d) Yield stress: uniaxial σ_{yp}, shear τ_{yp}

(e) Yield criteria

i. Tresca: $|\tau|_{\max} \le \tau_{\text{yp}} = \frac{1}{2}\sigma_{\text{yp}}$

ii. Mises, $\frac{1}{2}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \le \sigma_{\text{yp}}^2 = 3\tau_{\text{yp}}^2$

4. Work, energy

(a) Global work-energy relation: (elastic) $W = U$, $\bar{W} = \bar{U}$; (linear elastic) $U = \bar{U}$

i. $W = \int F\, d\Delta$ or $\int M\, d\theta$,

ii. $\bar{W} = \int \Delta\, dF$ or $\int \theta\, dM$

(b) Linear elastic

i. Local: $\frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) = U_o(\varepsilon_x, \ldots) = \bar{U}_o(\sigma_x, \ldots)$;

ii. Global: $U = \int_V U_o dV$ (strain energy), $\bar{U} = \int_V \bar{U}_o dV$ (complementary energy)

(c) Castigliano's theorems for elastic systems

i. 1st Theorem: $P_i = \dfrac{\partial U}{\partial \Delta_i}$, $M_i = \dfrac{\partial U}{\partial \theta_i}$

ii. 2nd Theorem: $\Delta_i = \dfrac{\partial \bar{U}}{\partial P_i}$, $\theta_i = \dfrac{\partial \bar{U}}{\partial M_i}$

(d) Reciprocal relations (linear elastic)

i. Finite number of degrees of freedom: $U = \frac{1}{2}\sum_i \sum_j k_{ij}\Delta_i\Delta_j$, $k_{ij} = k_{ji}$ ($[k_{ij}]$ = stiffness matrix, $\{\Delta_i\}$ include displacements and rotations)

ii. Finite number of loads: $\bar{U} = \frac{1}{2}\sum_i \sum_j f_{ij}F_iF_j$, $f_{ij} = f_{ji}$ ($[f_{ij}]$ = flexibility matrix, $\{F_i\}$ include both forces and moments)

(e) Potential energy: $\Pi = U + \Omega$

i. $\Omega = -\sum(F\Delta + M\theta)$ or $-\int_0^L qv\, dx$ etc.: potential energy of applied loads

ii. Minimum potential energy: $\delta\Pi = 0$ for equilibrium (finite number of degrees of freedom: $\partial\Pi/\partial\Delta_i = 0$, $\partial\Pi/\partial\theta_j = 0$)

5. Transformation of axes in two dimensions

(a) Vectors: $F_{x'} = \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{F}$, etc., $\begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$

(b) Tensors: $\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{x'y'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
where $\sigma_{xx} = \sigma_x$, $\sigma_{xy} = \tau_{xy}$ etc.

i. $\sigma_{x'} = \sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$, $\sigma_{y'} = \sigma_{\theta+\pi/2}$

ii. $\tau_\theta = \tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$

iii. For strains use $\varepsilon_{xx} = \varepsilon_x$, $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$ etc.

iv. Chain rule for derivatives: $\dfrac{\partial}{\partial x'} = \dfrac{\partial x}{\partial x'}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial x'}\dfrac{\partial}{\partial y}$, etc.

v. Strain rosettes: $\varepsilon_{\theta_i} = \frac{1}{2}[(\varepsilon_x + \varepsilon_y) + (\varepsilon_x - \varepsilon_y)\cos 2\theta_i + \gamma_{xy}\sin 2\theta_i]$, $i = 1, 2, 3$; solve for ε_x, ε_y, γ_{xy}

(c) Principal values

i. Definition: $\sigma_{x'} = \sigma_1$ and $\sigma_{y'} = \sigma_2$ when $\tau_{x'y'} = 0$; also $\dfrac{d\sigma_{x'}}{d\theta} = 0$; convention: $\sigma_1 > \sigma_2$ (note that $\sigma_1 = \sigma_2$ if and only if $\sigma_x = \sigma_y$ **and** $\tau_{xy} = 0$)

ii. Principal angles: $\tan 2\theta_{1,2} = 2\tau_{xy}/(\sigma_x - \sigma_y)$

iii. Principal values: $\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + \tau_{xy}^2}$

(d) Maximum shear

i. Definition: $\dfrac{d\sigma_{x'y'}}{d\theta} = 0$

ii. Maximum shear angle: $\tan 2\theta_s = -(\sigma_x - \sigma_y)/2\tau_{xy}$

iii. Maximum shear: $\tau_{\max} = \sqrt{[\frac{1}{2}(\sigma_x - \sigma_y)]^2 + \tau_{xy}^2} = \frac{1}{2}(\sigma_1 - \sigma_2)$

iv. Relation to principal angle: $\theta_s - \theta_{1,2} = \pm\pi/4$

(e) Mohr's circle

i. To draw: in the σ_θ-τ_θ plane, mark the points (σ_x, τ_{xy} and (σ_y, $-\tau_{xy}$), and draw the line between them. The intersection of this line with the σ_θ-axis, i.e. the point $(\frac{1}{2}[\sigma_x + \sigma_y], 0)$, is the center of the circle.

ii. The radius is $r = \tau_{\max}$.

iii. The circle intersects the σ_θ-axis at $(\sigma_1, 0)$ and $(\sigma_2, 0)$.

iv. Rotation on the circle is twice the physical rotation and in the opposite direction.

6. Transformation of axes in three dimensions

(a) Principal values and principal directions

i. In order to solve the system $\left\{\begin{array}{c} (\sigma_x - \sigma)n_x + \tau_{xy}n_y + \tau_{xz}n_z = 0, \\ \tau_{xy}n_x + (\sigma_y - \sigma)n_y + \tau_{yz}n_z = 0, \\ \tau_{xz}n_x + \tau_{yz}n_y + (\sigma_z - \sigma)n_z = 0 \end{array}\right\}$, it is necessary that $\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$. This is a cubic equation in σ whose roots σ_1, σ_2, σ_3 are the principal values (eigenvalues), and the vector $\mathbf{n} = \mathbf{i}n_x + \mathbf{j}n_y + \mathbf{k}n_z$ for each of these values (eigenvector) gives the direction of the corresponding principal axis.

ii. If one of the principal axes is known, two-dimensional analysis can be used to find the other two.

iii. When all three principal values are known, Mohr's circles can be drawn. The maximum shear is the radius of the largest Mohr's circle.

(b) Yield criteria in terms of principal stresses (no numbering convention in terms of value)

i. Tresca: $\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_1 - \sigma_3|) = 2\tau_{\text{yp}}$

ii. Mises: $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_{\text{yp}}^2$

II. Slender Bodies

1. Internal Forces
   (a) Axial force: $P = \int_A \sigma_x \, dA$
   (b) Torque (cylindrical shaft): $T = \int_A \tau_{z\theta} r \, dA$
   (c) Bending moment: $M_z = -\int_A \sigma_x y \, dA$, $M_y = \int_A \sigma_x z \, dA$
   (d) Shear force: $V_y = -\int_A \tau_{xy} \, dA$, $V_z = \int_A \tau_{xz} \, dA$
2. Elastic energy
   (a) Strain energy: $U' = \int_A U_o \, dA$, $U = \int_0^L U' \, dx$
   (b) Complementary energy:$\bar{U}' = \int_A \bar{U}_o \, dA$, $\bar{U} = \int_0^L \bar{U}' \, dx$
3. Axial loading
   (a) General
      i. Strain-displacement relation: $\varepsilon = \varepsilon_x = \dfrac{du}{dx}$
      ii. Elongation $\Delta L = u(L) - u(0) = \int_0^L \varepsilon \, dx$
      iii. Resultant force $P = \int_A \sigma \, dA$
      iv. Equilibrium (differential equation): $\dfrac{dP}{dx} + p = 0$ ($p =$ distributed axial load per unit length)
      v. Work of virtual load: $\bar{F}\Delta = \int_0^L \bar{P}\varepsilon \, dx$ ($= \bar{P}\Delta L$ if uniform), where $\bar{P} =$ bar force due to virtual load $\bar{F}$ conjugate to Δ
   (b) Linear elastic
      i. Hooke's Law $\varepsilon = \sigma / E$
      ii. Elongation $\Delta L = \int_0^L (P/EA) \, dx$ ($= PL/EA$ if uniform)
      iii. Strain energy: $U' = \frac{1}{2}(\int_A E\varepsilon^2 \, dA) dx$ ($= \frac{1}{2} EA\varepsilon^2$, $U = EA(\Delta L)^2/2L$ if uniform)
      iv. Complementary energy: $\bar{U}' = \int_A (\sigma^2/E) \, dA$ ($= P^2/2EA$ if homogeneous, $\bar{U} = P^2 L/2EA$ if uniform, $EA/L =$ axial spring constant)
      v. Work of virtual load: $\bar{F}\Delta = \int_0^L (\bar{P}P/EA) \, dx$ ($= \bar{P}PL/EA$ if uniform)
4. Torsion
   (a) General (cylindrical shaft)
      i. Strain-displacement(twist) relation: $\gamma = \gamma_{z\theta} = r\phi'$ ($\phi' = \dfrac{d\phi}{dz}$)
      ii. Total rotation $\Delta\phi = \phi(L) - \phi(0) = \int_0^L \phi' \, dz$
      iii. Resultant torque $T = 2\pi \int_b^c \tau r^2 \, dr$
      iv. Equilibrium (differential equation): $\dfrac{dT}{dz} + t = 0$ ($t =$ distributed torque per unit length)

v. Work of virtual load: $\bar{M}\theta$ or $\bar{F}\Delta = \int_0^L \bar{T}\phi'\,dz$

(b) Linear elastic (cylindrical shaft)

i. Hooke's Law $\gamma = \tau/G$

ii. Torque $T = GJ\phi'$, $J = \int_A r^2\,dA = \pi(c^4 - b^4)/2$ ($c =$ outer radius, $b =$ inner radius)

iii. Stress $\tau = Tr/J$

iv. Total rotation $\Delta\phi = \int_0^L (T/GJ)\,dz$

v. Strain energy: $U' = \frac{1}{2}\int_A G\gamma^2\,dA$ ($= \frac{1}{2}GJ\phi'^2$ if homogeneous, $U = GJ(\Delta\phi)^2/2L$ if uniform, $GJ/L =$ torsional spring constant)

vi. Complementary energy: $\bar{U}' = \int_A(\tau^2/2G)\,dAdz$ ($= T^2/2GJ$ if homogeneous, $\bar{U} = T^2L/2GJ$ if uniform)

vii. Work of virtual load: $\bar{M}\theta = \int_0^L(\bar{T}T/GJ)\,dz$ ($= \bar{T}TL/GJ$ if uniform)

(c) Elastic-perfectly plastic (cylindrical shaft)

i. Yield torque: $T_{\mathrm{yp}} = \tau_{\mathrm{yp}}J/c$

ii. Ultimate torque: $T_{\mathrm{u}} = \tau_{\mathrm{yp}}\int_A r\,dr = (2\pi/3)(c^3 - b^3)$

iii. Solid shaft: $T = T_{\mathrm{u}}[1 - (1/4)(r_{\mathrm{p}}/c)^3]$, $T_{\mathrm{u}} = (4/3)T_{\mathrm{yp}}$

iv. In unloading strain change is linear, leading to residual stress $\tau_{\mathrm{res}} = \tau_{\mathrm{init}} - T_{\mathrm{init}}r/J$, and residual twist $\phi'_{\mathrm{res}} = (d\phi/dz)_{\mathrm{init}} - T_{\mathrm{init}}/GJ$.

v. At elastic-plastic interface: $\gamma = \gamma_{\mathrm{yp}} = r_{\mathrm{p}}\phi'$

(d) Thin-walled closed tubes

i. Shear Flow $q = \tau t =$ constant

ii. Equilibrium $T = \oint rq\,ds = 2qA$ (where $A =$ area enclosed by mean curve of tube)

iii. Complementary elastic energy $\bar{U}' = \frac{1}{2}\oint(\tau^2/G)t\,ds\,dz = \frac{1}{2}q^2\oint(1/Gt)ds$

iv. Twist $\phi' = \frac{1}{2}[T/(2A)^2]\oint(1/Gt)ds$

v. Torsional stiffness: if $G =$ constant, $J = T/G\phi' = (2A)^2/\oint(1/t)ds$

5. Bending

(a) General (loading in xy-plane)

i. Geometry: assume section symmetric about y-axis, origin of yz axes at the centroid of the section

ii. Kinematics

A. Strain-curvature relation: $\varepsilon_x = -(y - y_0)\kappa$ (where y_0 is the y-coordinate of the neutral axis)

B. Curvature-displacement relation: $\kappa \approx v''$,

C. Rotation-displacement relation: $\theta \approx v'$

iii. Equilibrium (differential equations) $\dfrac{dV}{dx} = q$, $\dfrac{dM}{dx} = V$ (where $V = V_y$, $M = M_z$, $q =$ transverse load per unit length)

iv. Work of virtual force: $\bar{F}\Delta = \int_0^L \bar{M}\kappa\, dx$ ($\bar{M}(x)$ = bending moment due to virtual load $\bar{F}$, which may be either a force or a moment, conjugate to the displacement [rotation] Δ)

(b) Elastic pure bending about z-axis ($V = 0, P = 0, M_y = 0$)

i. Hooke's Law $\sigma = E\varepsilon$

ii. Moment $M = M_z = EI\kappa$, (transformed sections) $M = E_{\text{ref}}\hat{I}\kappa$

iii. Second moment of area ("moment of inertia"): $I = I_z = \int_A y^2\, dA$ (y measured from centroid); rectangle $I = bh^3/12$, circle $I = \pi c^4/4$; for sections composed of simple subsections use parallel-axis theorem $I = I_{\text{o}} + Ad^2$ (where I_{o} is calculated about the centroid of the subsection and d is the y-distance from there to the centroid of the whole section)

iv. Neutral axis ($\sum F_x = 0$): $y_0 = \int_A Ey\, dA / \int_A E\, dA$ ($= 0$ if homogeneous, i.e. $E =$ constant)

v. Stress $\sigma = -E\kappa(y - y_0)$ ($= -My/I$ if homogeneous)

vi. Strain energy: $U' = \frac{1}{2}\int_A E\varepsilon^2\, dA$ ($= \frac{1}{2}EI\kappa^2$ if homogeneous)

vii. Complementary energy: $\bar{U}' = \int_A (\sigma^2/2E)\, dA$ ($= M^2/2EI$ if homogeneous)

viii. Work of virtual load: $\bar{F}\Delta$ or $\bar{M}\theta = \int_0^L (\bar{M}M/EI)\, dx$

(c) Elastic bending with shear

i. Shear flow: $q = VQ/I$, $Q = \int_{A'} y\, dA = \bar{y}A'$

ii. Average shear stress: $\tau \approx q/t$; rectangular: $\tau(y) = (V/2I)[(h/2)^2 - y^2]$, $\tau_{\max} = 3V/2A$; I-beam: $\tau_{\max} \approx V/A_{\text{web}}$

(d) Elastic bending with axial loads

i. Tensile P, use superposition for stresses ($\sigma = P/A - My/I$)

ii. Compressive P, use superposition for stresses; check for buckling

(e) Elastic-perfectly plastic pure bending about z-axis

i. Initial yield: $M_{\text{yp}} = \sigma_{\text{yp}} I / y_{\max}$

ii. Rectangular section: $M_{\text{yp}} = \sigma_{\text{yp}} bh^2/6$, $M = M_{\text{u}}[1 - (1/3)(y_{\text{p}}/(h/2))^2]$, $M_{\text{u}} = (3/2)M_{\text{yp}}$

iii. Neutral axis: $T = C$, (ultimate state) $\int_{A_T} \sigma_{\text{yp}}^T\, dA = \int_{A_C} \sigma_{\text{yp}}^C\, dA$; $A_C = A_T = \frac{1}{2}A$ if $\sigma_{\text{yp}}^T = \sigma_{\text{yp}}^C = \sigma_{\text{yp}}$

iv. Ultimate moment: $Tl_T + Cl_C = M_{\text{u}}$; if $\sigma_{\text{yp}}^T = \sigma_{\text{yp}}^C = \sigma_{\text{yp}}$, $M_{\text{u}} = \sigma_{\text{yp}} Ad/2$, where d is the distance between the centroids of the half-areas

v. Unloading: strain change is linear, resulting in residual stress $\sigma_{\text{res}} = \sigma_{\text{init}} + M_{\text{init}} y/I$ and residual curvature $\kappa_{\text{res}} = \kappa_{\text{init}} - M_{\text{init}}/EI$.

vi. At elastic-plastic interface: $\varepsilon = \varepsilon_{\text{yp}} = -y_{\text{p}}\kappa$

(f) Singularity functions

i. Ramp function: $<x> = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}$

ii. General ($n > 0$): $<x - a>^n = \begin{cases} (x-a)^n, & x > a \\ 0, & x \le a \end{cases}$

   iii. Special: Heaviside $H(x-a) = (d/dx)<x-a>$, Dirac delta $\delta(x-a) = (d/dx)H(x-a)$, doublet $\delta'(x-a) = (d/dx)\delta(x-a)$.
   iv. Integration rules: $\int \delta'(x-a)\,dx = \delta(x-a) + C$, $\int \delta(x-a)\,dx = H(x-a) + C$, $\int H(x-a)\,dx = <x-a> + C$, $\int <x-a>^n\,dx = <x-a>^{n+1}/(n+1) + C$

(g) Deflections due to elastic bending
   i. Differential equation: $(EIv'')'' = q$ or $EIv'''' = q$ for constant EI
   ii. Boundary conditions
      A. Fixed (built-in): $\theta = 0,\ v = 0$,
      B. Pin or roller: $M = EIv'' = 0,\ v = 0$,
      C. Free: $M = EIv'' = 0,\ V = EIv''' = 0$;
      D. Intermediate roller support: $v = 0$ (v and v' are continuous),
      E. Intermediate hinge: $M = EIv'' = 0$ (v is continuous)
   iii. Particular results: $$|v|_{\max} = \begin{cases} FL^3/3EI & \text{end}-\text{loaded cantilever} \\ FL^3/48EI & \text{center}-\text{loaded simple beam} \\ 5wL^4/384EI & \text{uniformly loaded simple beam} \end{cases}$$

6. Elastic stability and buckling

(a) General
   i. Potential energy: $\Pi = U + \Omega$, where $\Omega = -P\Delta$ (Δ = displacement conjugate to P)
   ii. Equilibrium (single-degree-of-freedom system): $\dfrac{d\Pi}{d\theta} = 0$
   iii. Equilibrium is stable when $\dfrac{d^2\Pi}{d\theta^2} > 0$ and unstable when $\dfrac{d^2\Pi}{d\theta^2} < 0$, neutral (limiting condition) when $\dfrac{d^2\Pi}{d\theta^2} = 0$

(b) Linearized
   i. Critical load P_{cr} is (a) the value of P allowing non-zero solutions of the equilibrium equation (initially perfect system), (b) the asymptotic value of P as the deflection grows large (initially imperfect system)
   ii. Multi-degree-of-freedom systems: critical loads are eigenvalues of equilibrium equations, buckling modes are eigenvectors.

(c) Elastic column
   i. Linearized equilibrium: (general) $(EIv'')'' + Pv'' = 0$; (constant EI and P) $EIv'''' + Pv'' = 0$
   ii. General solution (constant EI and P): $v = A + Bx + C\cos\lambda x + D\sin\lambda x$, where $\lambda = \sqrt{P/EI}$; find $\lambda_n L$ ($n = 1, 2, \ldots$) such that the boundary conditions are satisfied with A, B, C, D not all zero
   iii. Critical loads for uniform columns: $P_{\text{cr}} = \lambda_1^2 EI = \pi^2 EI/L_e^2$, where $L_e = \pi/\lambda_1$ is the effective length (length of equivalent pinned-pinned column)
      A. Pinned-pinned (Euler): $P_{\text{cr}} = \pi^2 EI/L^2 = P_E$, $L_e = L$

B. Clamped-free (cantilever): $P_{\text{cr}} = P_E/4$, $L_e = 2L$
C. Pinned-clamped: $P_{\text{cr}} = 2.05 P_E$, $L_e = 0.699L$
D. Clamped-clamped (no sidesway): $P_{\text{cr}} = 4P_E$, $L_e = 0.5L$
E. Clamped-clamped (with sidesway): $P_{\text{cr}} = P_E$, $L_e = L$

iv. Critical stress $\sigma_{\text{cr}} = P_{\text{cr}}/A = \dfrac{\pi^2 E}{(L_e/r)^2}$, where $r = \sqrt{I/A}$

v. Elastic limit: $\sigma_{\text{cr}} = \sigma_{\text{yp}}$, so $L_e/r = (L_e/r)_{\text{cr}} = \pi\sqrt{E/\sigma_{\text{yp}}}$