CHAPTER ELEVEN
STABILITY OF COLUMNS

Learning objectives

1. Develop an appreciation of the phenomenon of buckling and the various types of structure instabilities.
2. Understand the use of buckling formulas in the analysis and design of structures.

Strange as it sounds, the column behind the steering wheel in Figure 11.1*a* is designed to fail: it is meant to *buckle* during a car crash, to prevent impaling the driver. In contrast, the columns of the building in Figure 11.1*b* are designed so that they do *not* buckle under the weight of a building.

Buckling is instability of columns under compression. Any axial members that support compressive axial loads, such as the weight of the building in Figure 11.1*b*, are called *columns*—and not all structural members behave the same. If a compressive axial force is applied to a long, thin wooden strip, then it bends significantly, as shown in Figure 11.1*c*. If the columns of a building were to bend the same way, the building itself would collapse. And when a column buckles, the collapse is usually sudden and catastrophic.

Under what conditions will a compressive axial force produce only axial contraction, and when does it produce bending? When is the bending caused by axial loads catastrophic? How do we design to prevent catastrophic failure from axial loads? As we shall see in this chapter, we can identify members that are likely to collapse by studying structure's equilibrium. Geometry, materials, boundary conditions, and imperfections all affect the stability of columns.

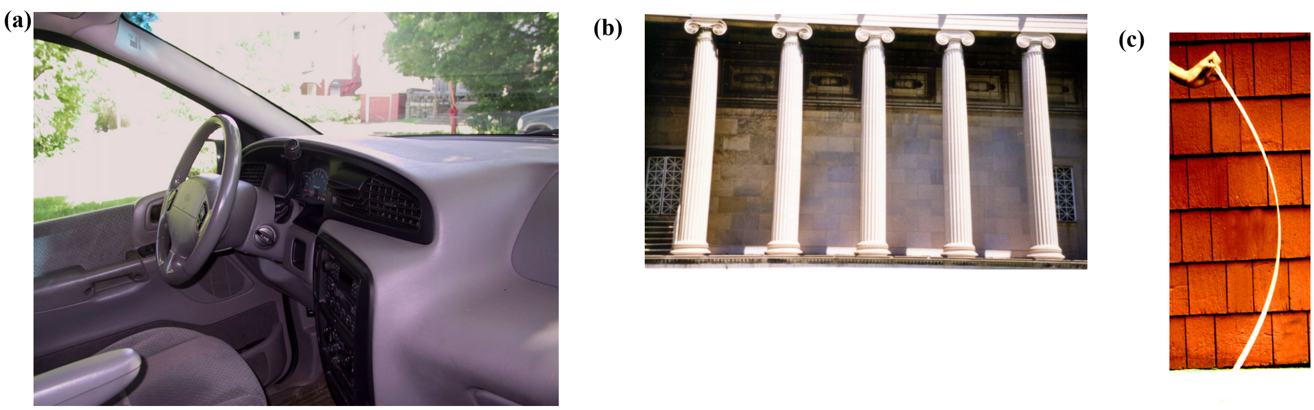


Figure 11.1 Examples of columns.

11.1 BUCKLING PHENOMENON

Buckling is an instability of equilibrium in structures that occurs from compressive loads or stresses. A structure or its components may fail due to buckling at loads that are far smaller than those that produce material strength failure. Very often buckling is a catastrophic failure. We discuss briefly some of the approaches and types of buckling in the following sections.

11.1.1 Energy Approach

We look at the energy approach using an analogy of a marble that is in equilibrium on different types of surfaces as shown in Figure 11.2. Left to itself, it will simply stay put. Suppose, however, that we disturb the marble to the shaded position in each case. When the surface is concave, as in Figure 11.2*a,* the marble will return to its equilibrium position — and it is said to be in *stable equilibrium*. When the surface is flat, as in Figure 11.2*b,* the marble will acquire a new equilibrium position. In this case the marble is said to be in a *neutral equilibrium*. Last when the surface is convex, as in Figure 11.2*c*, the marble will roll off. In this third case, a change in position also disturbs the equilibrium state and so the marble is said to be in *unstable equilibrium*.

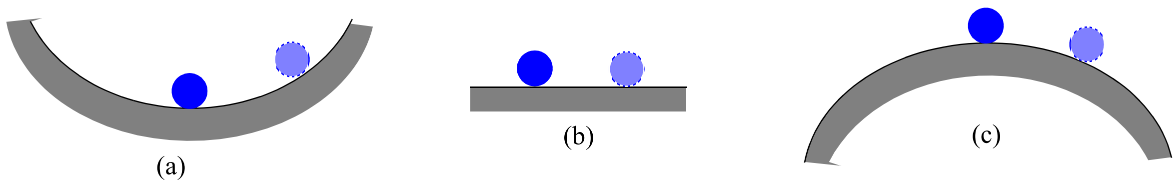


Figure 11.2 Equilibrium using marble (*a*) Stable. (*b*) Neutral. (*c*) Unstable.

The marble analogy in Figure 11.2 is useful in understanding one approach to the buckling problem, the *energy* method. Every deformed structure has a potential energy associated with it. This potential energy depends on the strain energy (the energy due to deformation) and on the work done by the external load. If the potential energy function is concave at the equilibrium position, then the structure is in stable equilibrium. If the potential energy function is convex, then the structure is in unstable equilibrium. The external load at which the potential energy function changes from concave to convex is called the *critical load* at which the buckling occurs. This energy method approach is beyond the scope of this book.

11.1.2 Eigenvalue Approach

To elaborate the eigenvalue approach in determining the load at which buckling occurs consider a rigid bar (Figure 11.3*a*) with a torsional spring at one end and a compressive axial load at the other end. Figure 11.3*b* shows the free-body diagram of the bar. Clearly, $\theta = 0$ is an equilibrium position. We call it a trivial solution to the problem. But at what value of P does there exist a nontrivial solution to the problem? This is the classical statement of an *eigenvalue problem,* and the *critical value* of P for which the nontrivial solution exists is called the *eigenvalue*. At this critical value of P the rod acquires a new equilibrium.

To determine the critical value of P, we consider the equilibrium of the moment at O in Figure 11.3*b.*

$$PL\sin\theta = K_\theta\theta \tag{11.1a}$$

For small angles we can approximate $\sin\theta \approx \theta$ and rewrite Equation (11.1a) as

$$(PL - K_\theta)\theta = 0 \tag{11.1b}$$

In Equation (11.1b) $\theta = 0$ is one solution, but if $PL = K_\theta$ then θ can have any non-zero value. Thus, the critical value of P is

$$P_{cr} = K_\theta / L \tag{11.1c}$$

You may be more familiar with eigenvalue problem in context of matrices. In problems 11.7 and 11.8 there are two unknown angles, and the problem can be cast in matrix form.

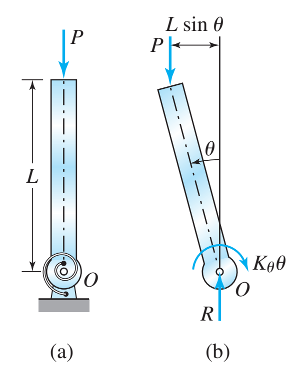


Figure 11.3 Eigenvalue problem.

11.1.3 Bifurcation Problem

To describe the bifurcation problem, we rewrite Equation (11.1a) as

$$PL/K_\theta = \theta/\sin\theta \tag{11.1d}$$

Figure 11.4 shows the plot of PL/K_θ versus θ. The equilibrium line separates the unstable region from the stable region. The bar remains in the vertical equilibrium position ($\theta = 0$) provided the load (P) increases are below point A and it will return to the vertical position if it is disturbed (rotated) slightly to the left or right. Any disturbance in equilibrium for load values above point A will send the bar to either to the left branch or to the right branch of the curve, where the it acquires a new equilibrium position. Point A is the *bifurcation point,* at which there are three possible solutions. The load P at the bifurcation point is called the *critical load*. Thus, we again see the same problem with a different perspective because of the methodology used in solving it.

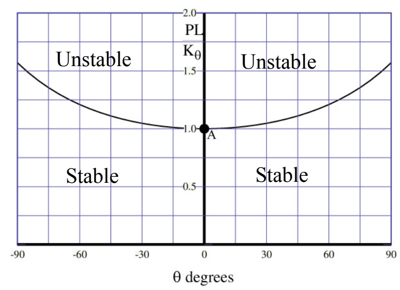


Figure 11.4 Bifurcation problem.

11.1.4 Snap Buckling

In *snap buckling* a structure jumps from one equilibrium configuration to a dramatically different equilibrium configuration. It is most often seen in shallow thin walled curved structures. To explain this phenomenon, consider a bar that can slide in a smooth slot. It has a spring attached to it at the right end and a force P applied to it at the left end, as shown in Figure 11.5. As we increase the force P, the inclination of the bar at the equilibrium position moves closer to the horizontal position. But there is an inclination at which the bar suddenly jumps across the horizontal line to a position below the horizontal line

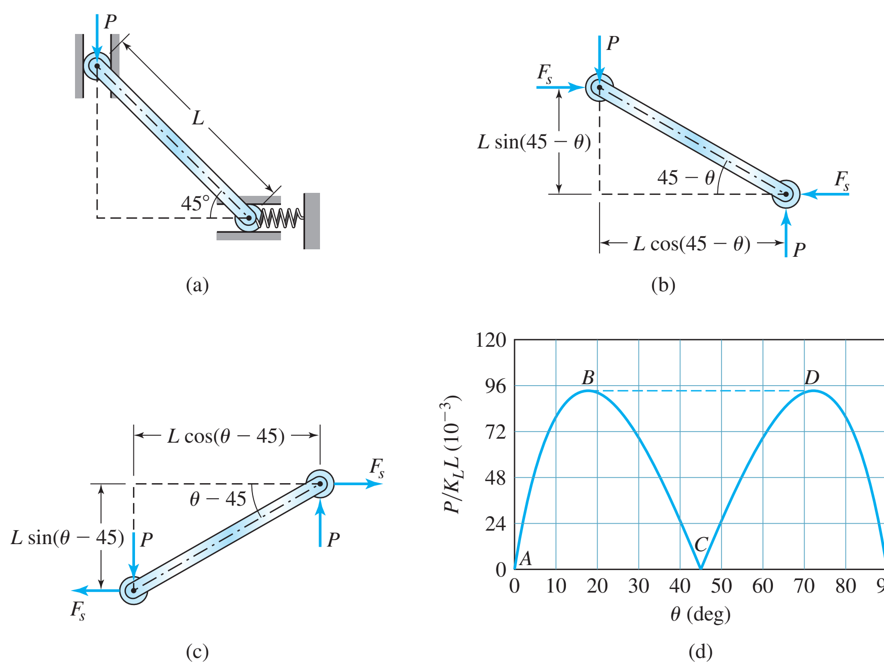


Figure 11.5 Snap buckling problem. (*a*) Undeformed position, $\theta = 0$. (*b*) $0 < \theta < 45°$. (*c*) $\theta > 45°$. (*d*) Load versus θ.

We consider the equilibrium of the bar before and after the horizontal line to understand the mathematics of snap buckling. Suppose the spring is in the instructed position, as shown in Figure 11.5*a*. We define the inclination of the bar by the angle θ measured from the undeformed position. Figure 11.5*b* and *c* shows the free-body diagrams of the bar before and after the horizontal position. The spring force must reverse direction as the bar crosses the horizontal position to ensure moment equilibrium. The deformation of the spring before the horizontal position is $L\cos(45^\circ - \theta) - L\cos 45^\circ$. Thus the spring force is $F_s = K_L[L\cos(45^\circ - \theta) - L\cos 45^\circ]$. By moment equilibrium we obtain

$$\frac{P}{K_L L} = [\cos(45^\circ - \theta) - \cos 45^\circ]\tan(45^\circ - \theta), \qquad 0 < \theta < 45^\circ \tag{11.2a}$$

In a similar manner, by considering the moment equilibrium in Figure 11.5*c*, we obtain

$$\frac{P}{K_L L} = [\cos(\theta - 45^\circ) - \cos 45^\circ]\tan(\theta - 45^\circ), \qquad \theta > 45^\circ \tag{11.2b}$$

Figure 11.5*d* shows a plot of $P/K_L L$ versus θ obtained from Equations (11.2a) and (11.2b). As we increase *P*, we move along the curve until we reach point *B*. At *B* rather than following paths *BC* and *CD*, the bar jumps (snaps) from point *B* to point *D*. It should be emphasized that each point on paths *BC* and *CD* represents an equilibrium position, but it is not a stable equilibrium position that can be maintained.

11.1.5 Local Buckling

The perspectives on the buckling problem in the previous sections were about structural stability. Besides the instability of a structure, however, we can have *local instabilities*. Figure 11.6*a* shows the crinkling of an aluminum can under compressive axial loads. This crinkling is the local buckling of the thin walls of the can. Figure 11.6*b* shows a thin cylindrical shaft under torsion. The stress cube at the top shows the torsional shear stresses. But if we consider a stress cube in principal coordinates, then we see that principal stress 2 is compressive. This compressive principal stress can also cause local buckling, though the orientation of the crinkles will be different than those from the crushing of the aluminum can.

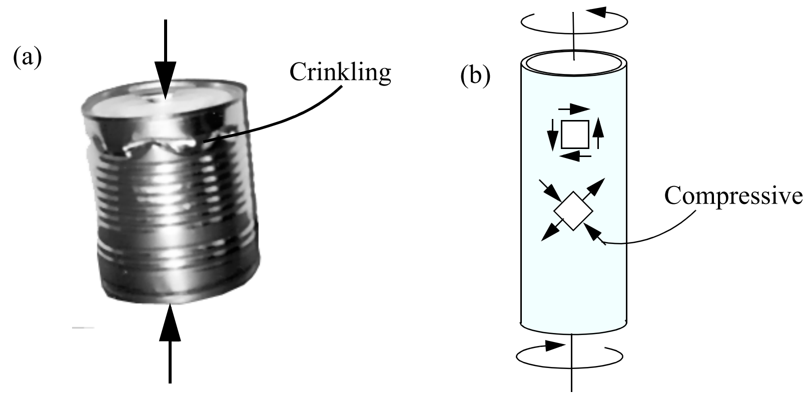


Figure 11.6 Local buckling. (*a*) Due to axial loads. (*b*) Due to torsional loads.

Consolidate your knowledge

1. Describe in your own words the various types of buckling.

PROBLEM SET 11.1

Stability of discrete systems

11.1 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.1. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}

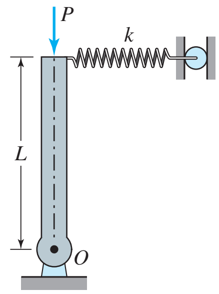


Figure P11.1

11.2 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.2. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}.

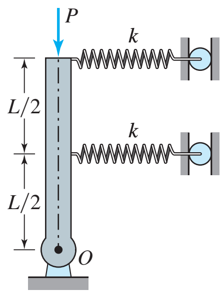


Figure P11.2

11.3 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.1. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}.

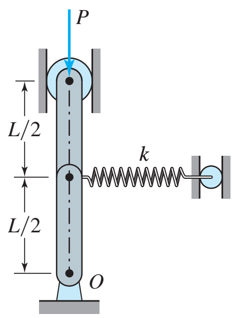


Figure P11.3

11.4 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.4. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

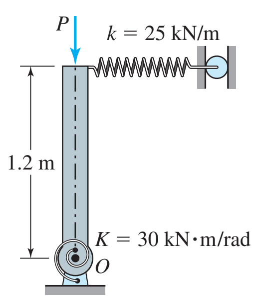


Figure P11.4

11.5 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.5. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

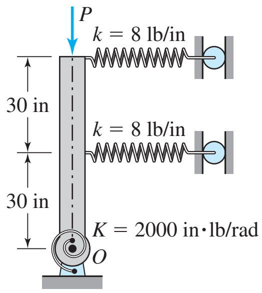


Figure P11.5

11.6 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.6. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

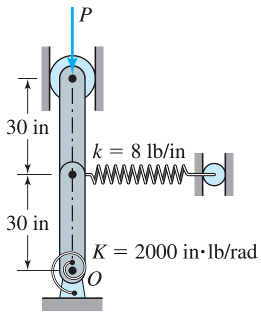


Figure P11.6

Stretch yourself

11.7 Two rigid bars are pin connected and supported as shown in Figure 11.7. The linear displacement spring constant is $k = 25$ kN/m and the linear rotational spring constant is K= 30 kN/rad. Using θ_1 and θ_2 as the angle of rotation of the bars AB and BC from the vertical, write the equilibrium equations in matrix form and determine the critical load P by finding the eigenvalues of the matrix. Assume small angles of rotation to simplify the calculations.

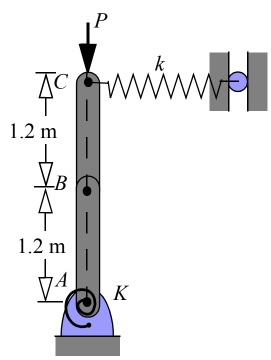


Figure P11.7

11.8 Two rigid bars are pin connected and supported as shown in Figure 11.8. The linear displacement spring constant is $k = 8$ lb/in. and the linear rotational spring constant is K= 2000 in.-lb/rad. Using θ_1 and θ_2 as the angle of rotation of the bars AB and BC from the vertical, write the equilibrium equations in matrix form and determine the critical load P by finding the eigenvalues of the matrix. Assume small angles of rotation to simplify the calculations.

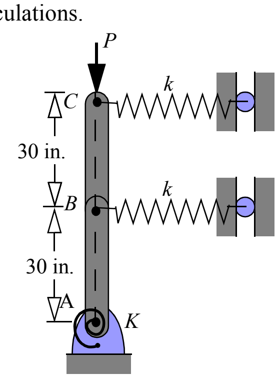


Figure P11.8

11.2 EULER BUCKLING

In this section we develop a theory for a straight column that is simply supported at either end. This theory was first developed by Leonard Euler (see Section 11.4) and is named after him.

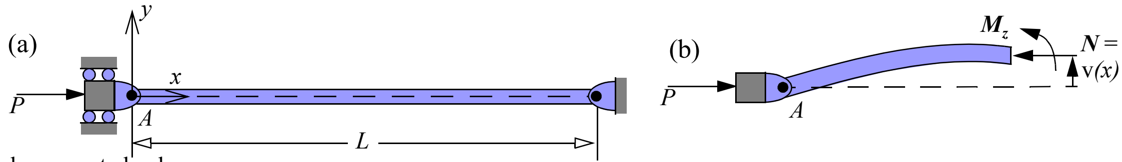


Figure 11.7 Simply supported column.

Figure 11.7*a* shows a simply supported column that is axially loaded with a force *P*. We shall initially assume that bending is about the *z* axis; as our equations in Chapter 7 on beam deflection were developed with just this assumption. We shall relax this assumption at the end to generate the formula for a critical buckling load.

Let the bending deflection at any location *x* be given by v(*x*), as shown in Figure 11.7*b*. An imaginary cut is made at some location *x*, and the internal bending moment is drawn according to our sign convention. The internal axial force N will be equal to *P*. By balancing the moment at point *A* we obtain $M_z + P\text{v} = 0$. Substituting the moment–curvature relationship of Equation (7.1), we obtain the differential equation:

$$EI_{zz}\frac{d^2\text{v}}{dx^2} + P\text{v} = 0 \tag{11.3a}$$

If buckling can occur about any axis and not just the *z* axis, as we initially assumed, then the subscripts *zz* in the area moment of inertia should be dropped. The boundary value problem can be written using Equation (11.3a) as

- **Differential Equation**

$$\frac{d^2\text{v}}{dx^2} + \lambda^2\text{v} = 0 \tag{11.3b}$$

where

$$\lambda = \sqrt{\frac{P}{EI}} \tag{11.3c}$$

- **Boundary Conditions**

$$\text{v}(0) = 0 \tag{11.4a}$$

$$\text{v}(L) = 0 \tag{11.4b}$$

Clearly $\text{v} = 0$ would satisfy the boundary-value problem represented by Equations (11.3a), (11.4a), and (11.4b). This trivial solution represents purely axial deformation due to compressive axial forces. Our interest is to find the value of *P* that would cause bending; in other words, a nontrivial ($\text{v} \neq 0$) solution to the boundary-value problem. Alternatively, at what value of *P* does a nontrivial solution exist to the boundary-value problem? As observed in Section 11.1, this is the classical statement of an eigenvalue problem.

The solution to the differential equation, Equation (11.3b), is

$$\text{v}(x) = A\cos\lambda x + B\sin\lambda x \tag{11.5}$$

From the boundary condition (11.4*a*) we obtain

$$\text{v}(0) = A\cos(0) + B\sin(0) = 0 \qquad \text{or} \qquad A = 0 \tag{11.6a}$$

From boundary condition (11.4*b*),[1] we obtain

$$\text{v}(L) = A\cos\lambda L + B\sin\lambda L = 0 \qquad \text{or} \qquad B\sin\lambda L = 0 \tag{11.6b}$$

If $B = 0$, then we obtain a trivial solution. For a nontrivial solution the sine function must equal zero:

$$\sin\lambda L = 0 \tag{11.7}$$

Equation (11.7) is called the *characteristic equation,* or the *buckling equation*.

Equation (11.7) is satisfied if $\lambda L = n\pi$. Substituting for λ and solving for *P*, we obtain

$$P_n = \frac{n^2\pi^2 EI}{L^2}, \qquad n = 1, 2, 3, \ldots \tag{11.8}$$

Equation (11.8) represents the values of load P (the eigenvalues) at which buckling would occur. What is the lowest value of P at which buckling will occur? Clearly, for the lowest value of P, n should equal 1 in Equation (11.8). Furthermore minimum value of I should be used. The critical buckling load is

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{11.9}$$

P_{cr}, the critical buckling load, is also called *Euler load*. *Buckling will occur about the axis that has minimum area moment of inertia*. The solution for v can be written as

$$v = B\sin\left(n\pi\frac{x}{L}\right) \tag{11.10}$$

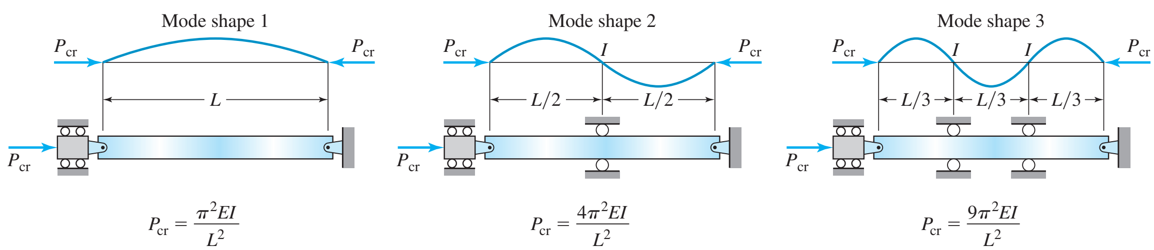


Figure 11.8 Importance of buckled modes.

Equation (11.10) represents the buckled mode (eigenvectors). Notice that the constant B in Equation (11.10) is undetermined. This is typical in eigenvalue problems. The importance of each buckled mode shape can be appreciated by examining Figure 11.8. If buckled mode 1 is prevented from occurring by installing a restraint (or support), then the column would buckle at the next higher mode at critical load values that are higher than those for the lower modes. Point I on the deflection curves describing the mode shapes has two attributes: it is an inflection point and the magnitude of deflection at this point is zero. Recall that the curvature d^2v/dx^2 at an inflection point is zero. Hence the internal moment $\boldsymbol{M_z}$ at this point is zero. If roller supports are put at any other points than the inflection points I, as predicted by Equation (11.10), then the boundary-value problem (see Problem 11.32) will have different eigenvalues (critical loads) and eigenvectors (mode shapes).

In many situations it may not be possible to put roller supports in order to change a mode to a higher critical buckling load. But buckling modes and buckling loads can also be changed by using elastic supports. Figure 11.9 shows a water tank on columns. The two rings are the elastic supports. Elastic supports can be modeled as springs and formulas for buckling loads developed as shown in Example 11.3.

Figure 11.9 Elastic supports on columns of a water tank.

[1]A matrix form may be more familiar for an eigenvalue problem. The boundary condition equations can be written in matrix form as

$$\begin{bmatrix} 1 & 0 \\ \cos\left(\sqrt{\frac{P}{EI_{zz}}}L\right) & \sin\left(\sqrt{\frac{P}{EI_{zz}}}L\right) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a nontrivial solution—that is, when A and B are not both zero—the condition is that the determinant of the matrix must be zero. This yields $\sin\left(\sqrt{(P/EI_{zz})}L\right) = 0$, in agreement with our solution.

Consolidate your knowledge

1. With the book closed derive the Euler buckling formula and comment on higher buckling modes.

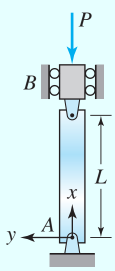
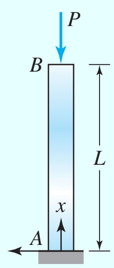
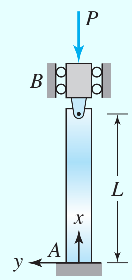
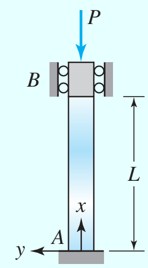
11.2.1 Effects of End Conditions

Equation (11.9) is applicable only to simply supported columns. However, the process used to obtain the formula can be used for other types of supports. Table 11.1 shows the critical elements in the derivation process and the results for three other supports. The formula for critical loads for all cases shown in Table 11.1 can be written as

$$\boxed{P_{\mathrm{cr}} = \frac{\pi^2 EI}{L_{\mathrm{eff}}^2}} \tag{11.11}$$

where L_{eff} is the effective length of the column. The effective length for each case is given in the last row of Table 11.1. This definition of effective length will permit us to extend results that will be derived in Section 11.3 for simply supported imperfect columns to imperfect columns with the supports shown in cases 2 through 4 in Table 11.1.

TABLE 11.1 **Buckling of columns with different supports**

	Case 1	Case 2	Case 3	Case 4[a]
	 Pinned at both ends	 One end fixed, other end free	 One end fixed, other end pinned	 Fixed at both ends
Differential equation	$EI\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = 0$	$EI\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = P\mathrm{v}(L)$	$EI\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = R_B(L-x)$	$EI\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = R_B(L-x) + M_B$
Boundary conditions	$\mathrm{v}(0) = 0$ $\mathrm{v}(L) = 0$	$\mathrm{v}(0) = 0$ $\frac{d\mathrm{v}}{dx}(0) = 0$	$\mathrm{v}(0) = 0$ $\frac{d\mathrm{v}}{dx}(0) = 0$ $\mathrm{v}(L) = 0$	$\mathrm{v}(0) = 0$ $\frac{d\mathrm{v}}{dx}(0) = 0$ $\mathrm{v}(L) = 0$ $\frac{d\mathrm{v}}{dx}(L) = 0$
Characteristic equation $\lambda = \sqrt{\frac{P}{EI}}$	$\sin\lambda L = 0$	$\cos\lambda L = 0$	$\tan\lambda L = \lambda L$[b]	$2(1-\cos\lambda L) - \lambda L\sin\lambda L = 0$
Critical load P_{cr}	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{4L^2} = \frac{\pi^2 EI}{(2L)^2}$	$\frac{20.13EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2}$	$\frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2}$
Effective length L_{eff}	L	$2L$	$0.7L$	$0.5L$

a. R_B and M_B are the force and moment reactions.

b. The roots of the equations have to be found iteratively. The two smallest roots of the equation are λL=4.4934 and λL=7.7253.

In Equation (11.9), I can be replaced by Ar^2, where A is the cross-sectional area and r is the minimum radius of gyration [see Equation (A.11)]. We obtain

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_{eff}/r)^2} \tag{11.12}$$

where L_{eff}/r is the *slenderness ratio* and σ_{cr} is the compressive axial stress just before the column would buckle.

Equation (11.12) is valid only in the elastic region—that is, if $\sigma_{cr} < \sigma_{yield}$. If $\sigma_{cr} > \sigma_{yield}$, then elastic failure will be due to stress exceeding the material strength. Thus $\sigma_{cr} = \sigma_{yield}$ defines the failure envelope for a column. Figure 11.10 shows the failure envelopes for steel, aluminum, and wood using the material properties given in Table D.1. As nondimensional variables are used in the plots in Figure 11.10, these plots can also be used for metric units. Note that the slenderness ratio is defined using effective lengths; hence these plots are applicable to columns with different supports.

The failure envelopes in Figure 11.10 show that as the slenderness ratio increases, the failure due to buckling will occur at stress values significantly lower than the yield stress. This underscores the importance of buckling in the design of members under compression.

The failure envelopes, as shown in Figure 11.10, depend only on the material property and are applicable to columns of different lengths, shapes, and types of support. These failure envelopes are used for classifying columns as short or long.[2] Short column design is based on using yield stress as the failure stress. Long column design is based on using critical buckling stress as the failure stress. The slenderness ratio at point A for each material is used for separating short columns from long columns for that material. Point A is the intersection point of the straight line representing elastic material failure and the hyperbola curve representing buckling failure.

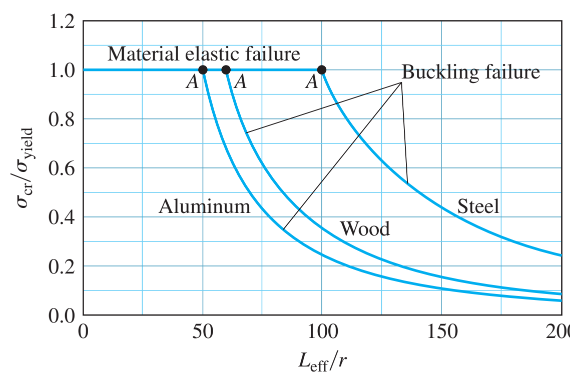


Figure 11.10 Failure envelopes for Euler columns.

EXAMPLE 11.1

A hollow circular steel column (E = 30,000 ksi) is simply supported over a length of 20 ft. The inner and outer diameters of the cross section are 3 in. and 4 in., respectively. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at the midpoint, what would be the new critical buckling load?

PLAN

(a) The area moment of inertia I for a hollow cylinder is same about all axes and can be found using the formula in Table C.2. From the value of I the radius of gyration can be found. The ratio of the given length to the radius of gyration gives the slenderness ratio. (b)In Equation (11.9) the given values of E and L, as well as the calculated value of I in part (a), can be substituted to obtain the critical buckling load P_{cr}. (c) Dividing P_{cr} by the cross-sectional area, the critical axial stress σ_{cr} can be found. (d)The column will buckle at the next higher buckling load, which can be found by substituting $n = 2$ and E, I, and L into Equation (11.8).

SOLUTION

(a) The outer diameter $d_o = 4$ in. and the inner diameter $d_i = 3$ in. From Table C.2 the area moment of inertia for the hollow cylinder, the cross-sectional area A, and the radius of gyration r can be calculated using Equation (A.11),

$$I = \frac{\pi(d_o^4 - d_i^4)}{64} = \frac{\pi[(4\text{ in.})^4 - (3\text{ in.})^4]}{64} = 8.590\text{ in.}^4 \qquad A = \frac{\pi(d_o^2 - d_i^2)}{4} = \frac{\pi[(4\text{ in.})^2 - (3\text{ in.})^2]}{4} = 5.498\text{ in.}^2 \tag{E1}$$

[2]Intermediate column is a third classification used if the critical stress is between yield stress and ultimate stress. See Equation (11.26) and Problems 11.64 and 11.65 for additional details.

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{8.590\ \text{in.}^4}{5.498\ \text{in.}^2}} = 1.250\ \text{in.} \tag{E2}$$

The length $L = 20$ ft = 240 in. Thus the slenderness ratio is $L/r = (240\ \text{in.})/(1.25\ \text{in.})$.

ANS. $L/r = 192$

(b) Substituting $E = 30{,}000$ ksi, $L = 240$ in., and $I = 8.59$ in.^{4} into Equation (11.9), we obtain the critical buckling load,

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(30{,}000\ \text{ksi})(8.590\ \text{in.})}{(240\ \text{in.})^2} \tag{E3}$$

ANS. $P_{cr} = 44.15$ kips

(c) The axial stress at the critical buckling load can be found as

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{44.15\ \text{kips}}{5.498\ \text{in.}^2} \tag{E4}$$

ANS. $\sigma_{cr} = 8.03$ ksi (C)

(d) With the support in the middle, the buckling would occur in mode 2. Substituting $n = 2$ and E, I, and L into Equation (11.8) we obtain the critical buckling load,

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2} = \frac{2^2\pi^2(30{,}000\ \text{ksi})(8.590\ \text{in.})}{(240\ \text{in.})^2} \tag{E5}$$

ANS. $P_{cr} = 176.6$ kips

COMMENTS

1. This example highlights the basic definitions of variables and equations used in buckling problems.
2. The middle support forces the column into the mode 2 buckling mode in part (d). Another perspective is to look at the column as two simply supported columns, each with an effective length of half the column or $L_{eff} = 120$ in. Substituting this into Equation (11.11), we obtain the same value as in part (d).

EXAMPLE 11.2

The hoist shown in Figure 11.11 is constructed using two wooden bars with modulus of elasticity $E = 1800$ ksi and ultimate stress of $\sigma_{ult} = 5$ ksi. For a factor of safety of $K = 2.5$, determine the maximum permissible weight W that can be lifted using the hoist for the two cases: (a) $L = 30$ in.; (b) $L = 60$ in.

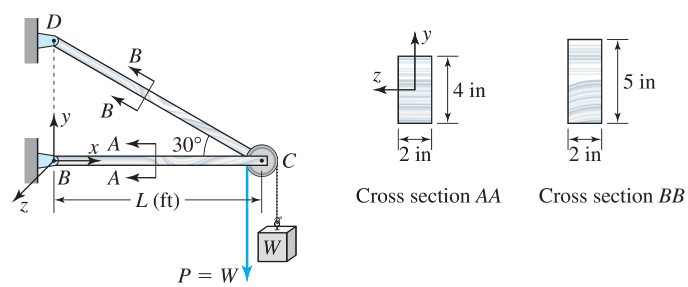


Figure 11.11 Hoist in Example 11.2.

PLAN

The axial stresses in the members can be found and compared with the calculated allowable values to determine a set of limits on W. By inspection we see that member BC will be in compression. Internal force in BC in terms of W can be found from free body diagram of the pulley and compared to critical buckling of BC to get another limit on W. The maximum value of W that satisfies the strength and buckling criteria can now be determined.

SOLUTION

The allowable stress in wood is

$$\sigma_{allow} = \sigma_{ult}/K = (5\ \text{ksi})/2.5 = 2\ \text{ksi} \tag{E1}$$

The free-body diagram of the pulley is shown in Figure 11.12 with the force in BC drawn as compressive and the force in CD as tensile. By equilibrium the internal axial forces

$$N_{CD}\ \sin 30° = 2W \qquad \text{or} \qquad N_{CD} = 4W \tag{E2}$$

$$N_{BC} = N_{CD} \cos 30° \quad \text{or} \quad N_{BC} = 3.464W \tag{E3}$$

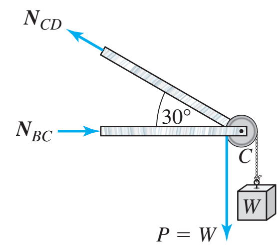


Figure 11.12 Free-body diagram in Example 11.2.

The cross-sectional areas for the two members are $A_{BC} = 8$ in.^{2} and $A_{CD} = 10$ in.^{2}. The axial stresses in terms of W can be found, and these should be less than the allowable stress of 2 ksi, from which we get two limits on W,

$$\sigma_{CD} = \frac{N_{CD}}{A_{CD}} = \frac{4W}{10\text{ in.}^2} \le 2\text{ ksi} \quad \text{or} \quad W \le 5.0\text{ kips} \tag{E4}$$

$$\sigma_{BC} = \frac{N_{BC}}{A_{BC}} = \frac{3.463\text{W}}{8\text{ in.}^2} \le 2\text{ ksi} \quad \text{or} \quad W \le 4.62\text{ kips} \tag{E5}$$

(a) We determine the minimum area moment of inertias for cross-section AA,

$$I_{yy} = \frac{1}{12}(4\text{ in.})(2\text{ in.})^3 = 2.667\text{ in.}^4 \qquad I_{zz} = \frac{1}{12}(2\text{ in.})(4\text{ in.})^3 = 10.67\text{ in.}^4 \tag{E6}$$

Substituting $E = 1800$ ksi, $L = 30$ in., and $I = 2.667$ in.^{4} into Equation (11.9), we obtain

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1800\text{ ksi})(2.667\text{ in.}^4)}{(30\text{ in.})^2} = 52.63\text{ kips} \tag{E7}$$

N_{BC} should be less than the critical load P_{cr} divided by factor of safety K,

$$N_{BC} \le (P_{cr}/K) \quad \text{or} \quad 3.464W \le [(52.63\text{ kips})/2.5] \quad \text{or} \quad W \le 6.08\text{ kips} \tag{E8}$$

The maximum value of W must satisfy Equations (E4), (E5), and (E8).

ANS. $W_{max} = 4.6$ kips

(b) Substituting $E = 1800$ ksi, $L = 60$ in., and $I = 2.667$ in.^{4} into Equation (11.9), we obtain

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1800\text{ ksi})(2.667\text{ in.}^4)}{(60\text{ in.})^2} = 13.159\text{ kips} \tag{E9}$$

N_{BC} should be less than the critical load P_{cr} divided by factor of safety K,

$$N_{BC} \le (P_{cr}/K) \quad \text{or} \quad 3.464W \le [(13.159\text{ kips})/2.5] \quad \textit{or} \quad W \le 1.52\text{ kips} \tag{E10}$$

The maximum value of W must satisfy Equations (E4), (E5), and (E10).

ANS. $W_{max} = 1.5$ kips

COMMENTS

1. This example highlights the importance of identifying compression members such as BC, so that buckling failure is properly accounted for in design.
2. The example also emphasizes that the minimum area moment of inertia that must be used is Euler buckling. Had we used I_{zz} instead of I_{yy}, we would have found $P_{cr} = 52.7$ kips and incorrectly concluded that the failure would be due to strength failure and not buckling in case (b)
3. In case (a) material strength governed the design, whereas in case (b) buckling governed the design. If we had several bars of different lengths and different cross-sectional dimensions (such as in Problems 11.18 and 11.19), then it would save a significant amount of work to calculate the slenderness ratio that would separate long columns from short columns. Substituting $\sigma_{cr} = \sigma_{allow} = 2$ ksi into Equation (11.12), we find that $L/r = 94.2$ is the ratio that separates long columns from short columns. It can be checked that the slenderness ratio in case (a) is 51.9, hence material strength governed W_{max}. In case (b) the slenderness ratio is 103.9, hence buckling governed W_{max}.

EXAMPLE 11.3

Linear springs are attached at the free end of a column, as shown in Figure 11.13. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. Show that the critical load P_{cr} for (b) $k = 0$ and (c) $k = \infty$ is as given in Table 11.1 for cases 2 and 3, respectively.

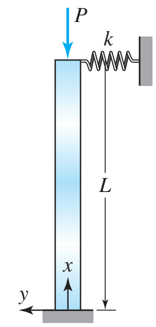


Figure 11.13 Column with elastic support in Example 11.3.

PLAN

The spring exerts a spring force $k\mathrm{v}_L$ at the upper end that must be incorporated into the moment equation, and hence into the differential equation. The boundary conditions are that the deflection and slope at $x = 0$ are zero. (a) The characteristic equation will be generated while solving the boundary-value problem. (b), (c) The roots of the characteristic equation for the two cases will give P_{cr}.

SOLUTION

By equilibrium of moment about point O in Figure 11.14, we obtain an expression for moment $\boldsymbol{M}_z$,

$$\boldsymbol{M}_z - P(\mathrm{v}_L - \mathrm{v}) + k\mathrm{v}_L(L - x) = 0 \qquad \text{or} \qquad \boldsymbol{M}_z + P\mathrm{v} = P\mathrm{v}_L - k\mathrm{v}_L(L - x) \tag{E1}$$

Substituting into Equation (7.1), we obtain the differential equation

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = P\mathrm{v}_L - k\mathrm{v}_L(L - x) \tag{E2}$$

(a) Using Equation (11.3c), Equation (E2) can be written as:

- **Differential equation:**

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = \lambda^2\mathrm{v}_L - \frac{k\mathrm{v}_L}{EI}(L - x) \tag{E3}$$

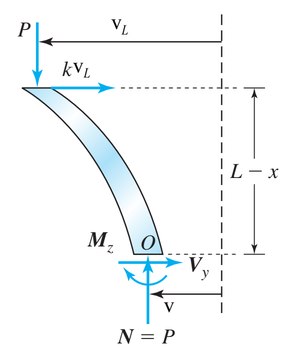


Figure 11.14 Free-body diagram in Example 11.3.

The zero deflection and slope boundary condition are also written to complete the statement of the boundary-value problem,

- **Boundary Conditions:**

$$\mathrm{v}(0) = 0 \tag{E4}$$

$$\frac{d\mathrm{v}}{dx}(0) = 0 \tag{E5}$$

The homogeneous solution v_H to Equation (E3) is given by Equation (11.5). The particular solution is

$$\mathrm{v}_P = \mathrm{v}_L - \frac{k\mathrm{v}_L}{\lambda^2 EI}(L - x) \tag{E6}$$

Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ can be written as

$$v(x) = A\cos\lambda x + B\sin\lambda x + v_L - \frac{kv_L}{\lambda^2 EI}(L-x) \qquad \text{(E7)}$$

Substituting $x = 0$ into Equation (E7) and using Equation (E4), we obtain

$$v(0) = A\cos(0) + B\sin(0) + v_L - \frac{kv_L}{\lambda^2 EI}(L-0) = 0 \quad \text{or} \quad A = \left(\frac{kL}{\lambda^2 EI} - 1\right)v_L \qquad \text{(E8)}$$

Differentiating Equation (E7), then substituting $x = 0$ and using Equation (E5), we obtain

$$\frac{dv}{dx}(0) = -\lambda A\sin(0) + B\lambda\cos(0) + \frac{kv_L}{\lambda^2 EI} = 0 \quad \text{or} \quad B = -\frac{k}{\lambda^3 EI}v_L \qquad \text{(E9)}$$

Substituting the values of A and B into Equation (E7), we obtain

$$v(x) = \left[\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda x - \frac{k}{\lambda^3 EI}\sin\lambda x + 1 - \frac{k}{\lambda^2 EI}(L-x)\right]v_L \qquad \text{(E10)}$$

Substituting $x = L$ into Equation (E7), we obtain

$$v(L) = \left[\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda L - \frac{k}{\lambda^3 EI}\sin\lambda L + 1 - 0\right]v_L = v_L \qquad \text{(E11)}$$

Since v_L is a common factor, Equation (E11) can be simplified to the following *characteristic equation:*

ANS. $\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda L - \frac{k}{\lambda^3 EI}\sin\lambda L = 0$

(b) Substituting $k = 0$ into Equation (E11), we obtain $\cos\lambda L = 0$, which is the characteristic equation for case 2 in Table 11.1. Thus the P_{cr} value corresponding to the smallest root will be as given in Table 11.1 for case 2.

(c) We rewrite Equation (E11) as

$$\tan\lambda L = \lambda L - \frac{\lambda^3 EI}{k} \qquad \text{(E12)}$$

As k tends to infinity, the second term tends to zero and we obtain $\tan\lambda L = \lambda L$, which is the characteristic equation for case 3 in Table 11.1. Thus the P_{cr} value corresponding to the smallest root will be as given in Table 11.1 for case 3.

COMMENTS

1. This example shows that a spring could simulate an imperfect support that provides some restraint to deflection. The restraining effect is more than zero (free end) but not as much as a roller support.
2. The spring could also represent other beams that are pin connected at the top end. These pin-connected beams provide elastic restraint to deflection but no restraint to the slope. If the beams were welded rather than pin connected, then we would have to include a torsional spring also at the end.
3. The example also demonstrates that the critical buckling loads can be changed by installing some elastic restraints, such as rings, to support the columns of the water tank in Figure 11.9.

EXAMPLE 11.4

Determine the maximum deflection of the column shown in Figure 11.15 in terms of the modulus of elasticity E, the length of the column L, the area moment of inertia I, the axial force P, and the intensity of the distributed force w.

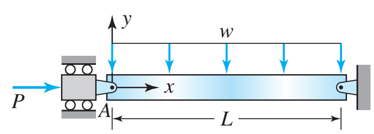


Figure 11.15 Buckling of beam with distributed load in Example 11.4.

PLAN

The moment from the distributed load can be added to the moment for case 1 in Table 11.1 and the differential equation written. The boundary conditions are that the deflection at $x = 0$ and $x = L$ is zero. The boundary-value problem can be solved, and the deflection at $x = L/2$ evaluated to obtain the maximum deflection.

SOLUTION

The reaction force in the y direction is half the total load wL acting on the beam. An imaginary cut at some location x can be made and the free-body diagram of the left part drawn as shown in Figure 11.16. By balancing the moment at point O, we obtain an expression for the moment $\boldsymbol{M}_z$,

$$M_z + P\mathrm{v}(x) - \frac{wL}{2}x + \frac{wx^2}{2} = 0 \tag{E1}$$

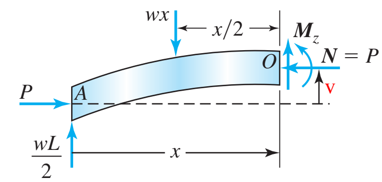


Figure 11.16 Free-body diagram in Example 11.4.

Substituting Equation (E1) into Equation (7.1), we obtain the differential equation

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = \frac{wL}{2}x - \frac{wx^2}{2} \tag{E2}$$

Using Equation (11.3c), Equation (E2) can be written as

- **Differential Equation**

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = \frac{wLx}{2EI} - \frac{wx^2}{2EI} \tag{E3}$$

The zero-deflection boundary conditions at either end are written to complete the statement of the boundary-value problem.

- **Boundary Conditions**

$$\mathrm{v}(0) = 0 \tag{E4}$$

$$\mathrm{v}(L) = 0 \tag{E5}$$

To find the particular solution, we substitute $\mathrm{v}_P = a + bx + cx^2$ into Equation (E3) and simplify,

$$2c + \lambda^2(a + bx + cx^2) = \frac{wLx}{2EI} - \frac{wx^2}{2EI} \quad \text{or} \quad (2c + \lambda^2 a) + \left(\lambda^2 b - \frac{wL}{2EI}\right)x + \left(\lambda^2 c + \frac{w}{2EI}\right)x^2 = 0 \tag{E6}$$

If Equation (E6) is to be valid for any value of x, then each of the terms in parentheses must be zero and we obtain the values of constants a, b, and c,

$$c = -\frac{w}{2\lambda^2 EI} \qquad b = \frac{wL}{2\lambda^2 EI} \qquad a = -\frac{2c}{\lambda^2} = \frac{w}{\lambda^4 EI} \tag{E7}$$

Hence the particular solution is

$$\mathrm{v}_P = \frac{w}{\lambda^4 EI} + \frac{wL}{2\lambda^2 EI}x - \frac{w}{2\lambda^2 EI}x^2 \tag{E8}$$

The homogeneous solution v_H to Equation (E3) is given by Equation (11.5). Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ can be written as

$$\mathrm{v}(x) = A\cos\lambda x + B\sin\lambda x + \frac{w}{\lambda^4 EI} + \frac{wL}{2\lambda^2 EI}x - \frac{w}{2\lambda^2 EI}x^2 \tag{E9}$$

Substituting $x = 0$ into Equation (E9) and using Equation (E4), we obtain

$$\mathrm{v}(0) = A\cos(0) + B\sin(0) + \frac{w}{\lambda^4 EI} + 0 - 0 = 0 \quad \text{or} \quad A = -\frac{w}{\lambda^4 EI} \tag{E10}$$

Substituting $x = L$ into Equation (E9) and using Equation (E5), we obtain

$$\mathrm{v}(L) = A\cos\lambda L + B\sin\lambda L + \frac{w}{\lambda^4 EI} + \frac{wL^2}{2\lambda^2 EI} - \frac{wL^2}{2\lambda^2 EI} = 0 \quad \text{or} \quad -\frac{w}{\lambda^4 EI}\cos\lambda L + B\sin\lambda L + \frac{w}{\lambda^4 EI} = 0 \tag{E11}$$

Since $\sin\lambda L = 2\sin(\lambda L/2)\cos(\lambda L/2)$ and $1 - \cos\lambda L = 2\sin^2(\lambda L/2)$ the above equation can be simplified as

$$B = -\frac{w}{\lambda^4 EI}\,\frac{1-\cos\lambda L}{\sin\lambda L} = -\frac{w}{\lambda^4 EI}\tan\left(\frac{\lambda L}{2}\right) \tag{E12}$$

By symmetry the maximum deflection will occur at midpoint. Substituting $x = L/2$, A and B into Equation (E9), we obtain

$$\mathrm{v}_{max} = \mathrm{v}\left(\frac{L}{2}\right) = A\cos\left(\frac{\lambda L}{2}\right) + B\sin\left(\frac{\lambda L}{2}\right) + \frac{w}{\lambda^4 EI} + \frac{wL^2}{4\lambda^2 EI} - \frac{wL^2}{8\lambda^2 EI} \quad \text{or}$$

$$\mathrm{v}_{max} = \frac{w}{\lambda^4 EI}\left[-\cos\left(\frac{\lambda L}{2}\right) - \tan\left(\frac{\lambda L}{2}\right)\sin\left(\frac{\lambda L}{2}\right)\right] + \frac{w}{\lambda^4 EI} + \frac{wL^2}{8\lambda^2 EI} \tag{E13}$$

Equation (E13) can be simplified by substituting the tangent function in terms of the sine and cosine functions to obtain

$$\mathrm{v}_{max} = -\frac{w}{\lambda^4 EI}\left[\sec\left(\frac{\lambda L}{2}\right) - 1\right] + \frac{wL^2}{8\lambda^2 EI} \tag{E14}$$

Substituting for λ, the maximum deflection can be written as

ANS. $$\mathrm{v}_{max} = -\frac{wEI}{P^2}\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] + \frac{wL^2}{8P}$$

COMMENTS

1. In Equation (E13), as $\lambda L \to \pi$, the secant function tends to infinity and the maximum displacement becomes unbounded, which means the column becomes unstable. $\lambda L = \pi$ corresponds to the Euler buckling load of Equation (11.9). Thus the *transverse distributed load does not change the critical buckling load* of a column.
2. However the failure mode can be significantly affected by the transverse distributed load. The maximum normal stress will be the sum of axial stress and maximum bending normal stress, $\sigma_{max} = P/A + M_{max}y_{max}/I$. The maximum bending moment will be at $x = L/2$ and can be found from Equation (E1) as $\boldsymbol{M}_{max} = wL^2/8 - P\mathrm{v}_{max}$. Substituting and simplifying gives the maximum normal stress:

$$\sigma_{max} = \frac{P}{A} + \frac{wEy_{max}}{P}\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] \tag{E15}$$

By equating the maximum normal stress to the yield stress, we obtain a failure envelope, which clearly depends on the value of w.

QUICK TEST 11.1

Time: 15 minutes/Total: 20 points

Answer true or false. If false, give the correct explanation. Each question is worth two points. Use the solutions given in Appendix E to grade yourself.

1. Column buckling can be caused by tensile axial forces.
2. Buckling occurs about an axis with minimum area moment of inertia of the cross section.
3. If buckling is avoided at the Euler buckling load by the addition of supports in the middle, then the column will not buckle.
4. By changing the supports at the column end, the critical buckling load can be changed.
5. The addition of uniform transversely distributed forces decreases the critical buckling load on a column.
6. The addition of springs in the middle of the column decreases the critical buckling load.
7. Eccentricity in loading decreases the critical buckling load.
8. Increasing the slenderness ratio increases the critical buckling load.
9. Increasing the eccentricity ratio increases the normal stress in a column.
10. Material strength governs the failure of short columns and Euler buckling governs the failure of long columns.

PROBLEM SET 11.2

Euler buckling

11.9 A hollow circular steel column ($E = 200$ GPa) is simply supported over a length of 5 m. The inner and outer diameters of the cross section are 75 mm and 100 mm. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at the midpoint, what would be new critical buckling load?

11.10 A 30-ft-long hollow square steel column ($E = 30{,}000$ ksi) is built into the wall at either end. The column is constructed from $\frac{1}{2}$-in.-thick sheet metal and has outer dimensions of 4 in. × 4 in. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.11 A 10 ft long lumber (E = 1,800 ksi) column with a rectangular cross section of 4 in. x 6 in. is pinned at both ends. (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.12 A 4 m long column is constructed from a steel (E = 210 GPa) sheet of thickness 10 mm. The sheet metal is bent to form a hollow rectangular cross section with outer dimension of 120 mm x 80 mm. One end of the column is fixed and the other is a free end as in case 2 of Table 11.1 (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.13 A 12 ft long lumber (E = 1,800 ksi) column with a rectangular cross section of 6 in. x 8 in. is pinned at one end and fixed at the other as in case 3 of Table 11.1. (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.14 A 5 m long column is constructed from a steel (E = 210 GPa) sheet metal of thickness 15 mm. The sheet metal is bent to form a hollow rectangular cross section with outer dimension of 120 mm x 90 mm. The ends of the column are fixed as in case 4 of Table 11.1. Determine the critical buckling load *P.*

11.15 A 20-ft-long wooden column ($E = 1800$ ksi) has cross-section dimensions of 8 in. × 8 in. The column is built in at one end and simply supported at the other end. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.16 A W12 × 35 steel section (see Appendix E) is used for a 21-ft column that is simply supported at each end. Use $E = 30{,}000$ ksi and determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at intervals of 7 ft, what would be the critical buckling load?

11.17 An S200 × 34 steel section (see Appendix E) is used as a 6-m column that is built in at each end. Use $E = 200$ GPa and determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.18 Columns made from alloy will be used in the construction of a frame. The cross section of the columns is a hollow square of 0.125-in. thickness and outer dimensions of a in. The modulus of elasticity $E = 9000$ ksi and the yield stress $\sigma_{yield} = 90$ ksi. Table 11.18 lists the lengths L and outer square dimensions a. Identify the long and short columns. Assume the ends will be simply supported.

TABLE P11.18 **Column geometric properties**

L **(ft)**	***a*** **(in.)**
1.0	1.125
1.5	1.500
2.0	1.750
2.5	2.750
3.0	3.000
3.5	3.000
4.0	3.000

11.19 Columns made from alloy will be used in the construction of a frame. The cross section of the columns is a hollow cylinder of 10-mm thickness and an outer diameter of d mm. The modulus of elasticity $E = 100$ GPa and the yield stress $\sigma_{yield} = 600$ MPa. Table P11.19 lists the lengths L and outer diameters d. Identify the long and short columns. Assume the ends of the column are built in.

TABLE P11.19 **Column geometric properties**

L **(m)**	***d*** **(mm)**
1	60
2	80
3	100
4	150
5	200
6	225
7	250

11.20 Three column cross sections are shown in Figure P11.20. The area of each of the three cross sections is equal to A. Determine the ratios of critical loads P_{cr1}: P_{cr2}: P_{cr3} assuming (a) the ends are simply supported; (b) the ends are built in. (c) How do you expect the ratios to change if the end conditions were as in cases 2 and 3 of Table 11.1?

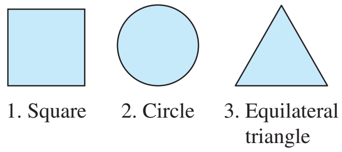


Figure P11.20

11.21 Figure P11.21 shows two steel (E = 30,000 ksi, σ_{yield} = 30 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 750 lb is applied. Bars AP and BP have lengths L_{AP} = 8 in. and L_{BP} = 10 in. Determine the factor of safety for the assembly.

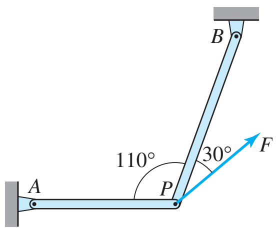


Figure P11.21

11.22 Figure P11.22 shows two steel (E = 30,000 ksi, σ_{yield} = 30 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 600 lb is applied. Bars AP and BP have lengths L_{AP} = 7 in. and L_{BP} = 10 in. Determine the factor of safety for the assembly.

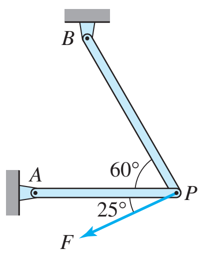


Figure P11.22

11.23 Figure P11.23 shows two copper (E = 15,000 ksi, σ_{yield} = 12 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 500 lb is applied. Bars AP and BP have lengths L_{AP} = 7 in. and L_{BP} = 9 in. Determine the factor of safety for the assembly.

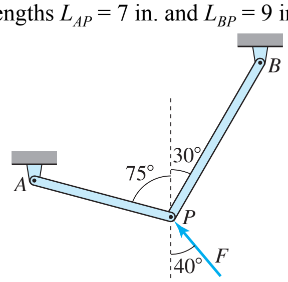


Figure P11.23

11.24 Figure P11.24 shows two (E = 200 GPa, σ_{yield} = 200 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = 350 mm. Determine the factor of safety for the assembly.

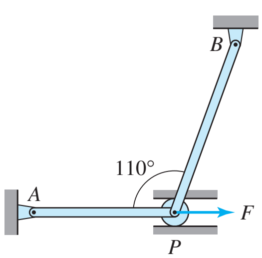


Figure P11.24

11.25 Figure P11.25 shows two (E = 200 GPa, σ_{yield} = 360 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = 300 mm. Determine the factor of safety for the assembly.

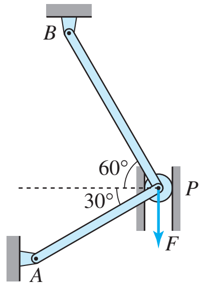


Figure P11.25

11.26 Figure P11.26 shows two (E = 200 GPa, σ_{yield} = 200 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = *300* mm. Determine the factor of safety for the assembly.

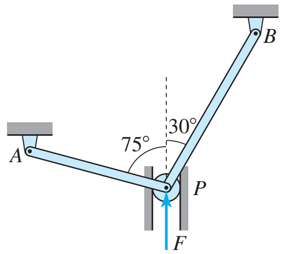


Figure P11.26

Formulation and solutions

11.27 (a) Solve the boundary-value problem for case 2 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.28 (a) Solve the boundary-value problem for case 3 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.29 (a) Solve the boundary-value problem for case 4 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.30 A torsional spring with a spring constant K is attached at one end of a column, as shown in Figure P11.30. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. (b) Show that for $K = 0$ and $K = \infty$ the critical load P_{cr} is as given in Table 11.1 for cases 1 and 3, respectively.

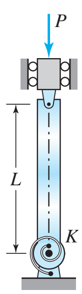


Figure P11.30

11.31 A torsional spring with a spring constant K is attached at one end of a column, as shown in Figure P11.31. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. (b) Show that for $K = 0$ the critical load P_{cr} is as given for case 2 in Table 11.1. (c) For $K = \infty$ obtain the critical load P_{cr}.

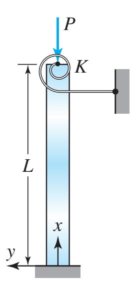


Figure P11.31

11.32 Consider the column shown in Figure P11.32. (a) Determine the critical buckling in terms of E, I, L, and α. (b) Show that when $\alpha = 0.5$, the critical load corresponds to mode 2, as shown in Figure 11.8.

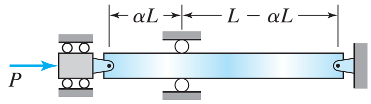


Figure P11.32

11.33 For the column shown in Figure P11.33 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

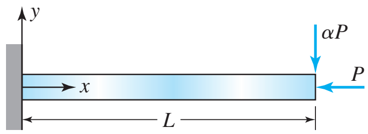


Figure P11.33

11.34 For the column shown in Figure P11.34 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

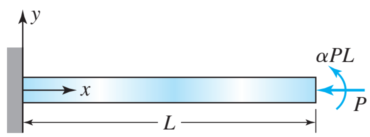


Figure P11.34

11.35 For the column shown in Figure P11.35 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

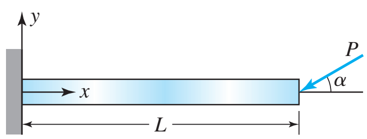


Figure P11.35

Design problems

11.36 Steel (E = 210 GPa) rectangular bars of 15 mm x 25 mm cross section form an assembly shown in Figure P11.36. Determine the maximum load P that can be applied without buckling of any bar. Use a = 1 m, b = 0.7 m, and c = 1 m.

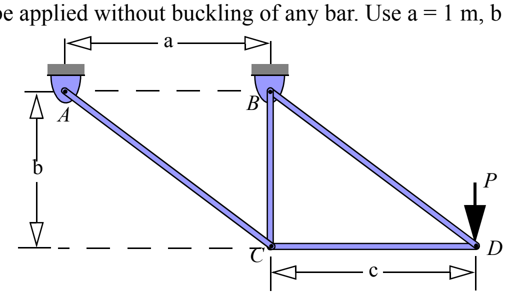


Figure P11.36

11.37 Steel (E = 210 GPa) rectangular bars of 15 mm x 25 mm cross section form an assembly shown in Figure P11.36. Determine the maximum load P that can be applied without buckling of any bar. Use a = 1 m, b = 0.7 m, and c = 1.4 m.

11.38 Steel (E= 30,000 ksi) rectangular bars of 1/2 in. x 1 in. cross section form an assembly shown in Figure P11.38. Determine the maximum load P that can be applied without buckling of any bar.

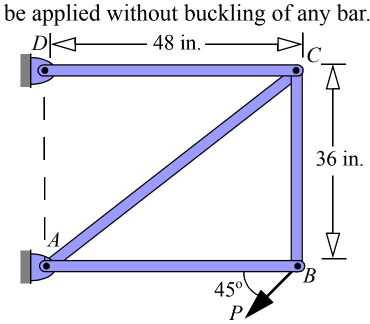


Figure P11.38

11.39 Steel (E= 30,000 ksi) rectangular bars of 1/2 in. x 1 in. cross section form an assembly shown in Figure P11.39. Determine the maximum load P that can be applied without buckling of any bar.

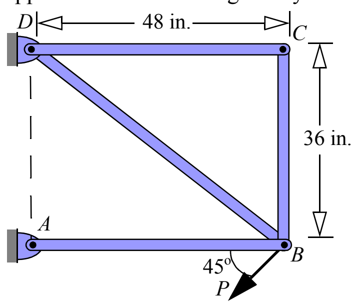


Figure P11.39

11.40 A hoist is constructed using two wooden bars to lift a weight of 5 kips, as shown in Figure P11.40. The modulus of elasticity for wood E = 1800 ksi and the allowable normal stress is 3.0 ksi. Determine the maximum value of L to the nearest inch that can be used in constructing the hoist.

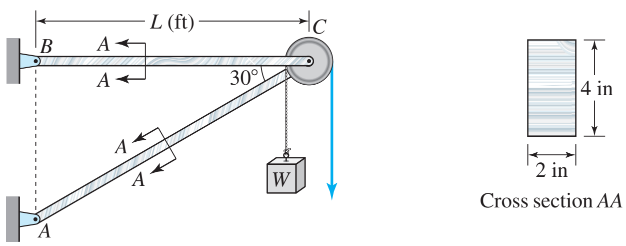


Figure P11.40

11.41 Two steel cylinders (E = 30,000 ksi and σ_{yield} = 30 ksi) AB and CD are loaded as shown in Figure P11.41. Determine the maximum load P to the nearest lb, if a factor of safety of 2 is desired. Model the ends of column AB as built in

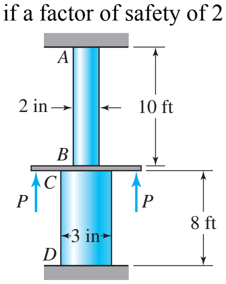


Figure P11.41

11.42 A spreader is to be made from an aluminum pipe (E = 10,000 ksi) of $\frac{1}{8}$-in. thickness and an outer diameter of 2 in., as shown in Figure P11.42. The pipe lengths available for design start from 4 ft in 6-in. steps up to 8 ft. The allowable normal stress is 40 ksi. Develop a table for the lengths of pipe and the maximum force F the spreader can support.

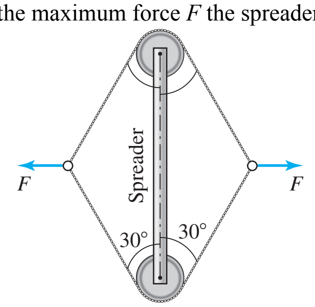


Figure P11.42

11.43 Two 200-mm × 50-mm pieces of lumber (E = 12.6 GPa) form a part of a deck that is modeled as shown in Figure P11.43. The allowable stress for the lumber is 18 MPa. (a) Determine the maximum intensity of the distributed load w. (b) What is the factor of safety for column BD corresponding to the answer in part (a)?

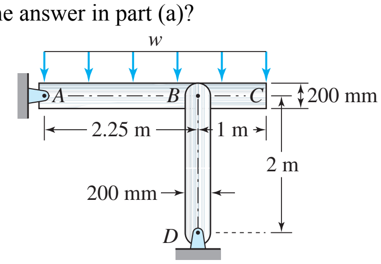


Figure P11.43

11.44 Two 200-mm × 50-mm pieces of lumber ($E = 12.6$ GPa) form a part of a deck that is modeled as shown in Figure P11.44. The allowable stress for the lumber is 18 MPa. (a) Determine the maximum intensity of the distributed load w. (b) What is the factor of safety for column BC corresponding to the answer in part (a)?

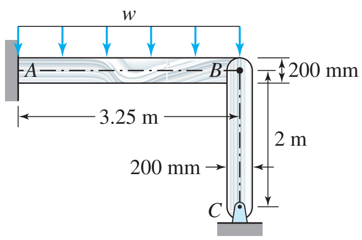


Figure P11.44

11.45 A rigid bar hinged at point O has a force P applied to it, as shown in Figure P11.45. Bars A and B are made of steel with a modulus of elasticity $E = 30{,}000$ ksi and an allowable stress of 25 ksi. Bars A and B have circular cross sections with areas $A_A = 1$ in.^{2} and $A_B = 2$ in.^{2}, respectively. Determine the maximum force P that can be applied.

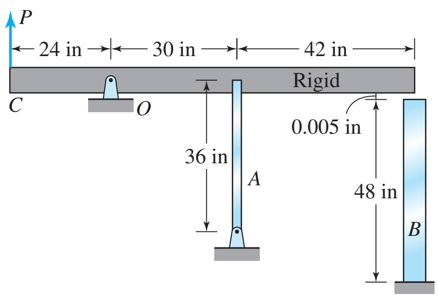


Figure P11.45

Stretch yourself

11.46 Show that for a beam with a constant bending rigidity EI, the fourth-order differential equation for solving buckling problems is given by

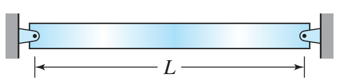
$$EI\frac{d^4 v}{dx^4} + P\frac{d^2 v}{dx^2} = p_y \tag{11.13}$$

where P is a compressive axial force and p_y is the distributed force in the y direction.

11.47 Using Equation (11.13), solve Example 11.4.

11.48 Show that the critical change of temperature at which the beam shown in Figure P11.48 will buckle is given by the equation below.

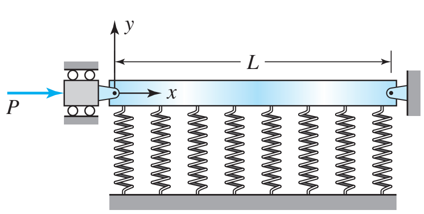
Figure P11.48



$$\Delta T_{crit} = \frac{\pi^2}{\alpha(L/r)^2}$$

where α is the thermal coefficient of expansion and r is the radius of gyration.

11.49 A column with a constant bending rigidity EI rests on an elastic foundation as shown in Figure P11.49. The foundation modulus is k, which exerts a spring force per unit length of kv. Show that the governing differential equation is given by Equation (11.15). (*Hint:* See Problems 7.48 and 11.46.)



$$EI\frac{d^4 v}{dx^4} + P\frac{d^2 v}{dx^2} + kv = 0 \tag{11.14}$$

Figure P11.49

11.50 Show that the buckling load for the column on an elastic foundation described in Problem 11.49 is given by the eigenvalues

$$P_n = \frac{\pi^2 EI}{L^2}\left[n^2 + \frac{1}{n^2}\left(\frac{kL^4}{\pi^4 EI}\right)\right], \qquad n = 1,2,3,\ldots \tag{11.15}$$

Note: For $n = 1$ and $k = 0$ Equation (11.15) gives the Euler buckling load.

11.51 For a simply supported column with a symmetric composite cross section, show that the critical load P_{cr} is given by

$$P_{cr} = \frac{\pi^2 \sum_{i=1}^{n} E_i I_i}{L_{eff}^2} \tag{11.16}$$

where L_{eff} = the effective length of the column, E_i is the modulus of elasticity for the i^{th} material, I_i is the area moment of inertia about the buckling axis, and n is the number of materials in the cross section. [See Equations (6.36) and (11.3a).]

11.52 A composite column has the cross section shown in Figure P11.52. The modulus of elasticity of the outside material is twice that of the inside material. In terms of E, d, and L, determine the critical buckling load.

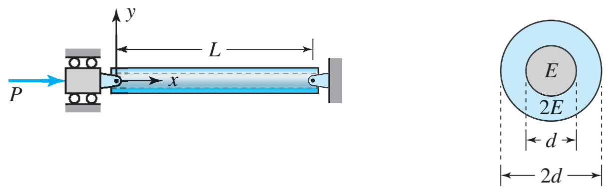


Figure P11.52

11.53 Two strips of material of a modulus of elasticity of $2E$ are attached to a material with a modulus of elasticity E to form a composite cross section of the column shown in Figure P11.53. In terms of E, a, and L, determine the critical buckling load. The column is free to buckle in any direction.

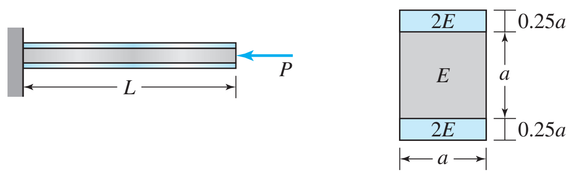


Figure P11.53

11.3* IMPERFECT COLUMNS

In the development of the theory for axial members and the symmetric bending of beams, we obtained that the condition for decoupling axial deformation from bending deformation for linear, elastic, and homogeneous material: the applied loads must pass through the centroid of the cross sections, and the centroids of all cross sections are on a straight line. However, the requirements for decoupling the axial from the bending problem may not be met for a number of reasons, some of which are given here:

- The column material may contain small holes, minute cracks, or other material inclusions. Hence the homogeneity requirement or the requirement that the centroids of all cross sections be on a straight line may not be met.
- The material processing may cause local strain hardening. Hence the condition of linear and elastic material behavior across the entire cross section may not be met.
- The theoretical design centroid and the actual centroid are offset due to manufacturing tolerances.
- Local conditions at the support cause the reaction force to be offset from the centroid.
- The transfer of loads from one member to another may not occur at the centroid.

This partial list can be considered as imperfections in the column, which cause the application of axial loads to be offset from the centroid of the cross section. This offset loading is termed **eccentric loading** on columns. In this section we study the impact of eccentricity in loading on buckling.

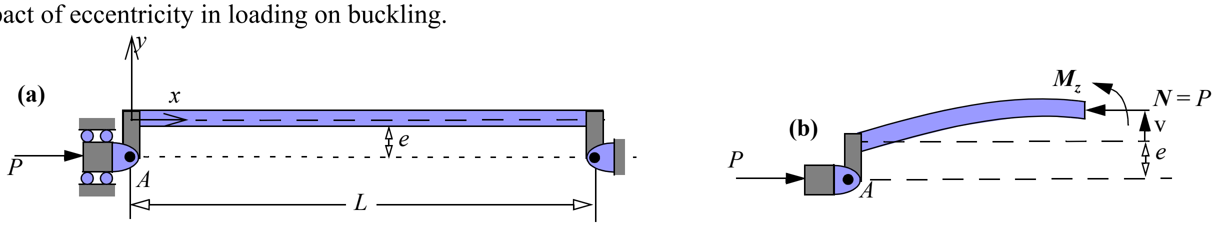


Figure 11.17 Eccentrically loaded column.

Figure 11.17a shows a simply supported column on which an eccentric compressive axial load is applied at a distance e from the centroid of the cross section. Figure 11.17b shows the free-body diagram of the column segment. By balancing the moment at point A we obtain $\boldsymbol{M}_z + P(\mathrm{v} + e) = 0$. Substituting the moment–curvature relationship of Equation (7.1), we obtain the differential equation

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = -\frac{Pe}{EI} \tag{11.17}$$

where λ is given by Equation (11.3c). The boundary conditions are that displacements at $x = 0$ and $x = L$ are zero, as given by Equation (11.4a) and (11.4b). The homogeneous solution to Equation (11.17) is given by Equation (11.5), that is, $\mathrm{v}_H(x) = A\cos\lambda x + B\sin\lambda x$. The particular solution to Equation (11.17) is $\mathrm{v}_P(x) = -e$. Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ is

$$\mathrm{v}(x) = A\cos\lambda x + B\sin\lambda x - e \tag{11.18}$$

From boundary condition (11.4*a*) we obtain

$$\mathrm{v}(0) = A\cos(0) + B\sin(0) - e = 0 \qquad \text{or} \qquad A = e \tag{11.19a}$$

From boundary condition (11.4*b*) we obtain

$$\mathrm{v}(L) = A\cos\lambda L + B\sin\lambda L - e = 0 \qquad \text{or} \tag{11.19b}$$

$$B = \frac{e(1-\cos\lambda L)}{\sin\lambda L} = \frac{e\left(2\sin^2\dfrac{\lambda L}{2}\right)}{2\sin\dfrac{\lambda L}{2}\cos\dfrac{\lambda L}{2}} = e\tan\frac{\lambda L}{2} \tag{11.19c}$$

Substituting for A and B in Equation (11.18), we obtain the deflection as

$$\mathrm{v}(x) = e\left[\cos\lambda x + \tan\left(\frac{\lambda L}{2}\right)\sin\lambda x - 1\right] \tag{11.20}$$

As $\lambda L/2 \to \pi/2$, the function $\tan(\lambda L/2) \to \infty$ and the displacement function $\mathrm{v}(x)$ becomes unbounded. Thus the critical load value can be found by substituting for λ in the equation $\lambda L/2 = \pi/2$ to obtain the same critical value as given by Equation (11.9). In other words, *the buckling load value does not change with the eccentricity of the loading*. We will make use of this observation to extend our formulas to other types of support conditions.

In the eigenvalue approach discussed in Section 11.2, we were unable to determine the displacement function because we had an undetermined constant B in Equation (11.10). But here the displacement function is completely determined by Equation (11.20). The maximum deflection (by symmetry) will be at the midpoint. Substituting $x = L/2$ into Equation (11.20), we obtain

$$\mathrm{v}_{\max} = e\left[\cos\left(\frac{\lambda L}{2}\right) + \tan\left(\frac{\lambda L}{2}\right)\sin\left(\frac{\lambda L}{2}\right) - 1\right] \tag{11.21a}$$

Using trigonometric identities, this equation can be simplified as $\mathrm{v}_{\max} = e[\sec(\lambda L/2) - 1]$. Substituting for λ from Equation (11.3c), we obtain

$$\mathrm{v}_{\max} = e\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] \tag{11.21b}$$

We can write

$$\sqrt{\frac{P}{EI}} = \sqrt{\frac{PP_{cr}}{P_{cr}EI}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}} \tag{11.21c}$$

We obtain the maximum deflection equation as

$$\boxed{\mathrm{v}_{\max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]} \tag{11.22}$$

The maximum normal stress is the sum of compressive axial stress and maximum compressive bending stress:

$$\sigma_{\max} = \frac{P}{A} + \frac{\boldsymbol{M}_{\max}y_{\max}}{I} \tag{11.23a}$$

The maximum bending moment will be at the midpoint of the column, and its value is $\boldsymbol{M}_{max} = P(e + v_{max})$. Substituting for v_{max} we obtain

$$\sigma_{max} = \frac{P}{A} + \frac{P y_{max}}{I} e\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)\right] \tag{11.23b}$$

Equation (11.23b) was derived for simply supported columns. We can extend the results to other supports by changing the length of the column to the effective length L_{eff}, as given in Table 11.1. We also substitute $y_{max} = c$, where c represents the maximum distance from the buckling (bending) axis to a point on the cross section. Substituting $I = Ar^2$, where A is the cross-sectional area and r is the radius of gyration, we obtain

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\frac{P}{EA}}\right)\right]} \tag{11.24}$$

Equation (11.24) is called the *secant formula.* The quantity ec/r^2 is called the *eccentricity ratio*.

By equating σ_{max} to failure stress σ_{fail} in Equation (11.24), we obtain the failure envelope for an imperfect column. The failure envelope equation can be written in nondimensional form as

$$\frac{P/A}{\sigma_{fail}}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\left(\frac{\sigma_{fail}}{E}\right)\frac{P/A}{\sigma_{fail}}}\right)\right] = 1 \tag{11.25}$$

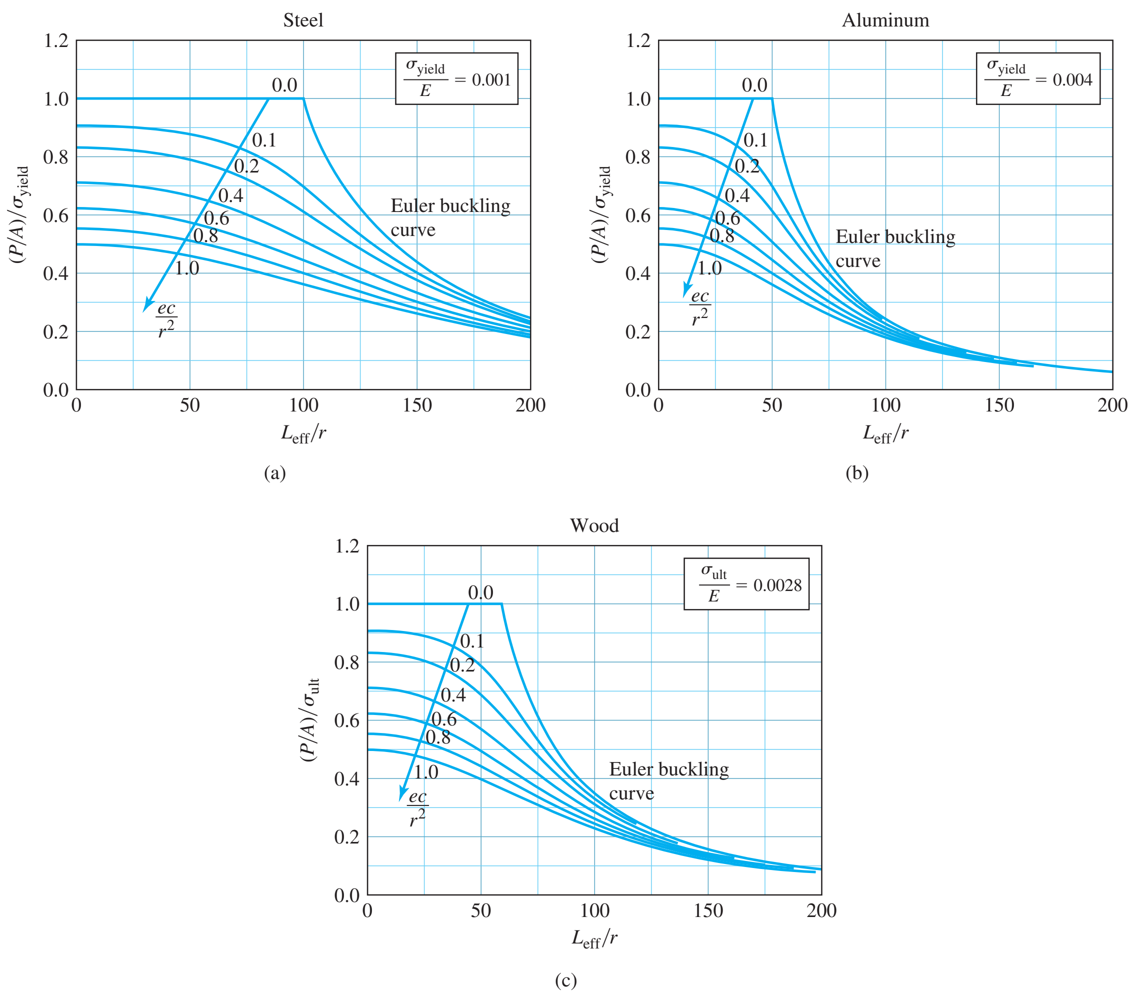


Figure 11.18 Failure envelopes for imperfect columns.

Equation (11.25) can be plotted for different materials, as shown in Figure 11.18. These curves can be used for metric as well for U.S. customary units, since the variables used in creating the plots are nondimensional. The curves can be used for any material that has the same value for σ_{yield}/E. The failure stress in the cases of steel and aluminum would be the yield stress σ_{yield}, whereas for wood it would be the ultimate stress σ_{ult}. The curves can also be used for different end conditions by using the appropriate L_{eff} as given in Table 11.1.

EXAMPLE 11.5

A wooden box column (E = 1800 ksi) is constructed by joining four pieces of lumber together, as shown in Figure 11.19. The load P = 80 kips is applied at a distance of e = 0.667 in. from the centroid of the cross section. (a) If the length is L = 10 ft, what are the maximum stress and the maximum deflection? (b) If the allowable stress is 3 ksi, what is the maximum permissible length L to the nearest inch?

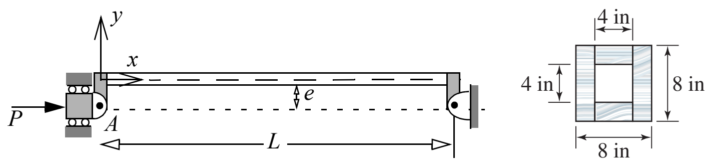


Figure 11.19 Eccentrically loaded box column.

PLAN

The cross-sectional area A, the area moment of inertia I, the radius of gyration r, and the maximum distance c from the bending (buckling) axis can be found from the cross-section dimensions. The effective length is the actual length L as the column is pin held at each end. (a) Substituting L_{eff} = 120 in. and the values of the other variables into Equations (11.22) and (11.24), we can find the maximum stress and the maximum deflection. (b) Equating σ_{max} in Equation (11.24) to 3 ksi and substituting the remaining variables, we find the length L.

SOLUTION

From the given cross section, the cross-sectional area A, the area moment of inertia I, and the radius of gyration r can be found:

$$A = (8\text{ in.})(8\text{ in.}) - (4\text{ in.})(4\text{ in.}) = 48\text{ in.}^2 \qquad I = \frac{1}{12}[(8\text{ in.})^4 - (4\text{ in.})^4] = 320\text{ in.}^4 \tag{E1}$$

$$r = \sqrt{\frac{I}{A}} = 2.582\text{ in.} \tag{E2}$$

(a) Since the column is pinned at both ends, $L_{eff} = L$ = 10 ft = 120 in. Substituting L_{eff}, I, and E = 1800 ksi into Equation (11.11) give the critical buckling load:

$$P_{cr} = \frac{\pi^2(1800\text{ ksi})(320\text{ in.}^4)}{(120\text{ in.})^2} = 394.8\text{ kips} \tag{E3}$$

Substituting e = 0.667in., P = 80 kips, and Equation (E3) into Equation (11.22), we obtain the maximum deflection,

$$v_{max} = (0.667\text{ in.})\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{80\text{ kips}}{394.8\text{ kips}}}\right) - 1\right] = 0.2103\text{ in.} \tag{E4}$$

ANS. v_{max} = 0.21 in.

Substituting c = 4 in., e = 0.667 in., r = 2.582 in., P = 80 kips, E = 1800 ksi, and A = 48 in.^{2} into Equation (11.24), we obtain the maximum normal stress,

$$\sigma_{max} = \frac{80\text{ kips}}{48\text{ in.}^2}\left[1 + \frac{(0.667\text{ in.})(4\text{ in.})}{(2.582\text{ in.})^2}\sec\left(\frac{120\text{ in.}}{2(2.582\text{ in.})}\sqrt{\frac{80\text{ kips}}{(1800\text{ ksi})(48\text{ in.}^2)}}\right)\right] = 2.544\text{ ksi} \tag{E5}$$

ANS. σ_{max} = 2.5 ksi (C)

(b) Substituting σ_{max} = 3 ksi, c = 4 in., e = 0.667 in., r = 2.582 in., P = 80 kips, E = 1800 ksi, and A = 48 in.^{2} into Equation (11.24), we can find $L_{eff} = L$ in. can be found,

$$3 = \frac{80\text{ kips}}{48\text{ in.}^2}\left[1 + \frac{(0.667\text{ in.})(4\text{ in.})}{(2.582\text{ in.})^2}\sec\left\{\frac{L\text{ in.}}{2(2.582\text{ in.})}\sqrt{\frac{80\text{ kips}}{(1800\text{ ksi})(48\text{ in.}^2)}}\right\}\right] \tag{E6}$$

$$\sec\{5.892(10^{-3})L\} = 2 \quad \text{or} \quad \cos(5.892\times10^{-3}L) = 0.5 \quad \text{or} \quad L = 177.7\text{ in.} \tag{E7}$$

Rounding downward, the maximum permissible length is: thus L = 177 in.

ANS. L = 177 in.

COMMENTS

1. The axial stress P/A = (80 kips)/(48 in.^{2}) = 1.667 ksi, but the normal stress due to bending from eccentricity causes the normal stress to be significantly higher, as seen by the value of σ_{max}.
2. If the right end of the column shown in Figure 11.19 were built in rather than held by a pin, then from case 3 in Table 11.1, $L_{eff} = 0.7L$ = 84 in. Using this value, we can find P_{cr} = 805.7 kips, v_{max} = 0.091 in., and σ_{max} = 2.42 ksi.
3. In Equation (E7) we rounded downward, as shorter columns will result in a stress that is less than allowable.

EXAMPLE 11.6

A wooden box column (E = 1800 ksi) is constructed by joining four pieces of lumber together, as shown in Figure 11.19. The ultimate stress is 5 ksi. Determine the maximum load P that can be applied.

PLAN

The eccentricity ratio and the slenderness ratio can be found using the values of the geometric quantities calculated in Example 11.5. Noting that $\sigma_{ult}/E = 0.0028$, the failure envelopes for wood that are shown in Figure 11.18 can be used and $(P/A)/\sigma_{ult}$ can be found, from which the maximum load P can be determined.

SOLUTION

From Equation (E2) in Example 11.5, r = 2.582 in. Thus the slenderness ratio L_{eff}/r = (120 in.)/(2.582 in.) = 46.48. From Figure 11.19, c = 4 in. and e = 0.667 in. Thus the eccentricity ratio $ec/r^2 = 0.400$.

For a slenderness ratio of 46.48 and an eccentricity ratio of 0.4, we estimate the value of $(P/A)/\sigma_{ult} = 0.6$ from the failure envelope for wood in Figure 11.18. Substituting σ_{ult} = 5 ksi and A = 48 in.^{2}, we obtain the maximum load P_{max} = (0.6) (5 ksi) (48in.^{2}).

ANS. P_{max} = 144 kips

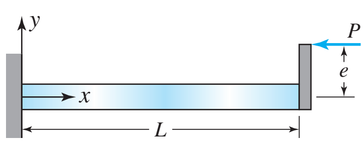
COMMENT

1. If we let x represent $(P/A)/\sigma_{ult}$ and substitute the remaining variables in Equation (11.25), we obtain the following nonlinear equation: $x[1 + 0.4\ \sec(1.2297\sqrt{x})] = 1$. The root of the equation can be found using a numerical method such as discussed in Section B.2.2. The value of the root to the third-place decimal is 0.593, which would yield a value of P_{max} = 142.3 kips, a difference of 1.18% from that reported in our example. The difference is small and an acceptable engineering approximation. Use of the plots in Figure 11.18 was a quick way of finding the load value with reasonable engineering approximation.

PROBLEM SET 11.3

Imperfect columns

11.54 A column built in on one end and free at the other end has a load that is eccentrically applied at a distance e from the centroid, as shown in Figure P11.54. Show that the deflection curve is given by the equation below.



$$v(x) = \frac{e(1 - \cos \lambda x)}{\cos \lambda L}$$

Figure P11.54

where λ is as given by Equation (11.3c).

11.55 On the cylinder shown in Figure P11.55 the applied load P = 3 kips, the length L = 5 ft, and the modulus of elasticity E = 30,000 ksi. What are the maximum stress and the maximum deflection?

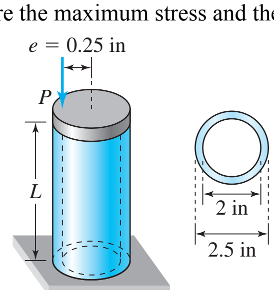


Figure P11.55

11.56 On the cylinder shown in Figure P11.55 the applied load $P = 3$ kips and the modulus of elasticity $E = 30{,}000$ ksi. If the allowable normal stress is 8 ksi, what is the maximum permissible length L of the cylinder?

11.57 The length of the cylinder shown in Figure P11.55 is $L = 5$ ft. The yield stress of steel used in the cylinder is 30 ksi, and the modulus of elasticity $E = 30{,}000$ ksi. Determine the maximum load P that can be applied. Use the plot for steel in Figure 11.18.

11.58 On the column shown in Figure P11.58 the applied load $P = 100$ kN, the length $L = 2.0$ m, and the modulus of elasticity $E = 70$ GPa. What are the maximum stress and the maximum deflection?

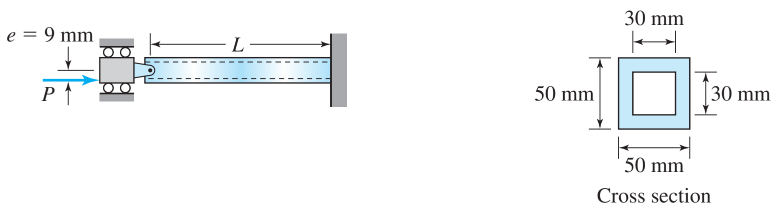


Figure P11.58

11.59 On the column shown in Figure P11.58 the applied load $P = 100$ kN and the modulus of elasticity $E = 70$ GPa. If the allowable normal stress is 250 MPa, what is the maximum permissible length L of the column?

11.60 The length of the column shown in Figure P11.58 is $L = 2.0$ m. The yield stress of aluminum used in the column is 280 MPa, and the modulus of elasticity $E = 70$ GPa. Determine the maximum load P that can be applied. Use the plot for aluminum in Figure 11.18.

11.61 A wide-flange W8 × 18 member is used as a column, as shown in Figure P11.61. The applied load $P = 20$ kips, the length $L = 9$ ft, and the modulus of elasticity $E = 30{,}000$ ksi. What are the maximum stress and the maximum deflection?

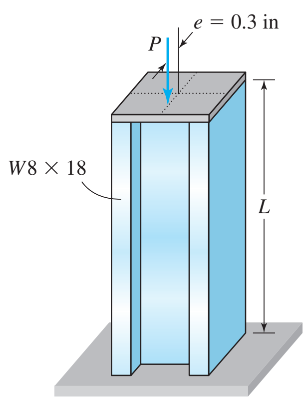


Figure P11.61

11.62 On the column shown in Figure P11.61 the applied load $P = 20$ kips and the modulus of elasticity $E = 30{,}000$ ksi. If the allowable normal stress is 24 ksi, what is the maximum permissible length L of the column?

11.63 The length of the column shown in Figure P11.61 is $L = 9$ ft. The yield stress of steel is 30 ksi, and the modulus of elasticity $E =$ 30,000 ksi. Determine the maximum load P that can be applied. Use the plot for steel in Figure 11.18.

Stretch yourself

In Problems 11.64 and 11.65, the critical stress in intermediate columns *is between yield stress and ultimate stress. The* tangent modulus theory *of buckling accounts for it by replacing the modulus of elasticity by the tangent modulus of elasticity (see Figure 3.7), that is,*

$$P_{cr} = \frac{\pi^2 E_t I}{L_{eff}^2} \tag{11.26}$$

where E_t is the tangent modulus, which depends on the stress level P_{cr}/A. Using an iterative trial and error procedure and Equation (11.26), the critical buckling load can be determined.

11.64 A simply supported 6-ft pipe has an outside diameter of 3 in. and a thickness of $\frac{1}{8}$ in. The pipe material has the stress–strain curve shown in Figure P11.64. Using Equation (11.26), determine the critical buckling load.

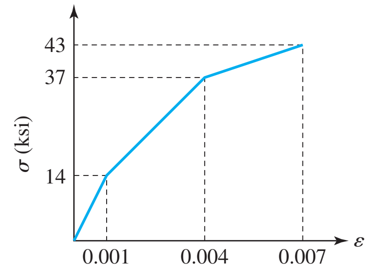


Figure P11.64

11.65 A square box column is constructed from a sheet of 10-mm thickness. The outside dimensions of the square are 75 mm × 75 mm and the column has a length of 0.75 m. The material stress–strain curve is approximated as shown in Figure P11.65. Using Equation (11.26), determine the critical buckling load.

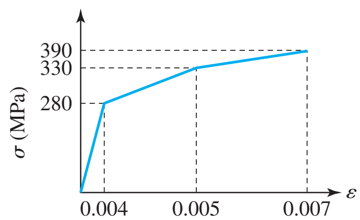


Figure P11.65

11.66 A column that is pin held at its ends has a small initial curvature, which is approximated by the sine function shown in Figure P11.66. Show that the elastic curve of the column is given by the equation below.

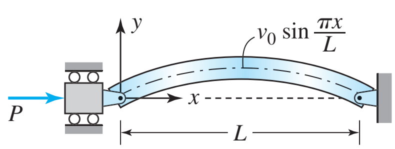


Figure P11.66

$$v(x) = \frac{v_0}{1 - P/P_{cr}} \sin\frac{\pi x}{L}$$

11.67 In *double modulus theory,* also known as *reduced modulus theory* for intermediate columns, it is recognized that the bending action during buckling increases the compressive axial stress on the concave side of the beam but decreases the compressive stress on the convex side of the beam. Thus the use of the tangent modulus of elasticity E_t is appropriate on the concave side, but on the convex side of the beam it may be better to use the original modulus of elasticity. Modeling the cross section material with the two moduli E_t and E and using Equation (11.26), show

$$P_{cr} = \frac{\pi^2 E_r I}{L_{eff}^2} \qquad E_r = E_t\frac{I_1}{I} + E\frac{I_2}{I} \tag{11.27}$$

where E_r is the *reduced modulus of elasticity,* I_1 and I_2 are the moments of inertia of the areas on the concave and convex sides of the axis passing through the centroid, and I is the moment of inertia of the entire cross section.

Computer problems

11.68 A circular marble column of 2-ft diameter and 20-ft length has a load P applied to it at a distance of 2 in. from the center. The modulus of elasticity is 8000 ksi and the allowable stress is 20 ksi. Determine the maximum load P the column can support, assuming that both ends are (a) pinned; (b) built in.

11.69 Determine the maximum load P to the nearest newton in Problem 11.60.

11.70 Determine the maximum load P to the nearest pound in Problem 11.63.

MoM in Action: Collapse of World Trade Center

On September 11, 2001, at 8:46 A.M, five terrorists flew a plane (Figure 11.20*a*) containing 10,000 gallons of fuel into tower 1 of the World Trade Center (WTC 1). Seventeen minutes later, five other terrorists flew a second plane containing 9,100 gallons of fuel into tower 2 (WTC 2). Within an hour, the floors of WTC 2 started collapsing vertically downward, and WTC 1 collapsed just 29 minutes later. A total of 2749 people apart from the terrorists died that day in New York. It is a tragic story of how social forces affect engineering design.

The construction of WTC complex began in 1968. The twin towers were to be the symbol of world commerce and for years the world's tallest buildings, at 110 stories each. Their innovative design maximized usable space by having all supporting columns only on the perimeter of each floor. There were 4 major structural subsystems: (i) the exterior wall (Figure 11.20*b*), with 59 columns on each side; (ii) a rectangular inner core of 47 columns; (iii) a system of bridging steel trusses (Figure 11.20*c*) on each floor, connecting the exterior wall to the inner core using angle clips. Viscoelastic dampers reduced the swaying motion on higher floors due to wind; and (iv) a truss system between 107th and 110th floor—further bridged the inner core to the exterior wall.

The exterior wall (like flanges in beam cross section increase area moment of inertia) was designed to resist the force of 140-mph hurricane winds. The inner core, like an axial column, supported most of the weight of building, equipment, and people. Insulation on the steel and a sprinkler system in the event of fire met building codes at that time. The design even planned for the impact of an airliner lost in fog, and it stood long enough so that most of the 14,500 people in the towers escaped that morning. But it was not designed for a Molotov cocktail of 10,000 gallons of jet fuel.

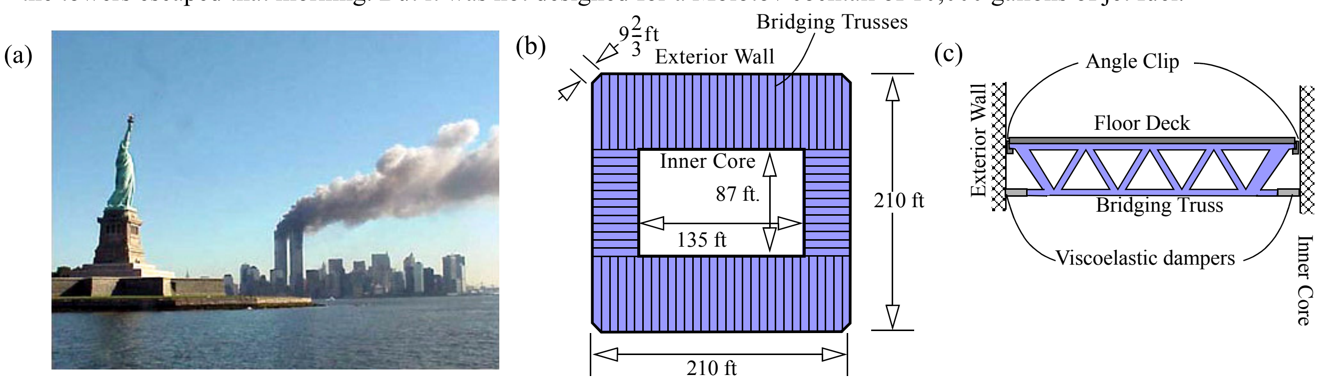


Figure 11.20 (a) World Trade Center Towers; (b) Plan form; (c) Floor.

Even so, it took several factors to initiate the collapse. First, the angle clips on the exterior wall of several floors—at the height of plane—broke, transferring the floors' weight as compressive loads to the inner core. Second, the breaking of the clips in turn removed elastic support from the core column, decreasing the critical buckling loads (see Figure 11.8). Third, the temperature increase from burning fuel introduced another mechanism of buckling failure (see Problem 11.48). Finally, the insulation of the inner core on the floors of impact broke, exposing the steel to high temperatures. This significantly decreased the modulus of elasticity, the critical buckling load, and the ultimate strength. The towers would have survived the first three failures. But the design did not take into account prolonged high temperature and its impact on the stiffness and strength of steel. No one could imagine a fuel-laden plane deliberatively flown into a building.

The floors suffering a direct impact buckled after nearly an hour of intense fire, and the floors above started falling on the weakened floors below. The moving mass of the floors gathered momentum, lending their downward motion to the floors below.

The WTC towers were well designed for the physical forces conceivable at the time. New skyscraper designs will incorporate greater insulation on the steel beams and columns to counter future threats. A collapse happened, but it will happen no more.

*11.4 CONCEPT CONNECTOR

As with the deflection of beams (Chapter 7), mathematicians played a key role in developing the theory of buckling. The history of buckling also shows that original ideas are not enough if the ideas cannot be communicated to others. The importance to engineering of oral and written skills in technical communications is a thus lesson over two hundred years old.

11.4.1 History: Buckling

Leonard Euler (1707–1782) is one of the most prolific mathematicians who ever lived (Figure 11.21). Born in Basel, he went to the University of Basel, then renowned for its research in mathematics. After studying under John Bernoulli (see Section 7.6), he started work in 1727 at the Russian Academy at St. Petersburg, where he developed analytical methods for solving mechanics problems. At the invitation of King Frederick II of Prussia, he moved to Berlin in 1741, where he wrote his book-length *Introduction to Calculus, Differential Calculus,* and *Integral Calculus*, in addition to his remarkable original contributions to mathematics. In 1766, Catherine II, the empress of Russia, wooed him back to St. Petersburg. Even as he was going blind from cataract, he continued his prolific publications with the help of assistants. In fact, with a bibliography that runs to 866 entries, one could easily miss his pioneering insight into buckling and the formula he derived [Equation (11.9)].


Leonard Euler.

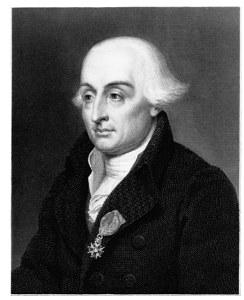

Joseph-Louis Lagrange.

Figure 11.21 Buckling theory pioneers.

Even after Euler, the early development of buckling was primarily mathematical. Joseph-Louis Lagrange (1736–1813), another pioneer in the establishment of analytical methods for mechanics (Figure 11.21), took the next step. He developed a complete set of buckling loads and the associated buckling modes given by Equations ((11.8) and (11.10). Columns with eccentric loads (Problem 11.54) and columns with initial curvatures (Problem 11.66) were first formulated and studied by Thomas Young (1773–1829). Young was also the first to consider columns of variable cross section. Unfortunately, he was neither a good teacher nor a writer, and much of his work went unappreciated. As his biographer, Lord Rayleigh, said,[3] "Young.... from various causes did not succeed in gaining due attention from his contemporaries. Positions that he had already occupied were in more than one instance reconquered by his successors at great expense of intellectual energy."

There was another reason why in the early 1800s developments in column buckling were unappreciated by the practicing engineer. Euler buckling did not accurately predict compression failure in the structural members then in use. The effects of end conditions and imperfections, as well as the formula's range of validity, were not yet understood It took the experiments of Eaton Hodgkinson in 1840 on cast-iron columns to give new life to the Euler buckling theory. In 1845, Anatole Henri Ernest Lamarle, a French engineer, proposed correctly that the Euler formula should be used below the proportional limit, while experimentally determined formulas should be used for shorter columns.

In 1889 F. Engesser, a German engineer, proposed the *tangent modulus theory* (see Problems 11.64 and 11.65), in which the elastic modulus is replaced by the tangent modulus of elasticity when proportional stress is exceeded. Also in 1889, the French engineer A. G. Considère, based on a series of tests, proposed that if buckling occurs above yield stress, then the elastic modulus in the Euler formula should be replaced by a reduced modulus of elasticity, between the elastic modulus and the tangent modulus. On learning of Considère's work, Engesser incorporated the suggestion into his *reduced modulus theory,* also known as *double modulus theory* (see Problem 11.66). Yet the two approaches competed for almost 50 years. In 1905 J. B. Johnson, C. W. Bryan, and F. E. Turneaure recommended a modification of the Euler formula for steel columns, using an

[3] Quotation is from S. P. Timoshenko, *History of Strength of Materials*.

experimentally determined constant for different supports. It was the beginning of the concept of *effective length* to account for different end conditions, and their text on *Theory and Practice of Modern Framed Structures* remained in print for ten editions. In 1946 F. R. Shanley, an American aeronautical engineering professor, refined these theories and finally resolved “the column paradox,” as he called it, that had separated proponents of the reduced modulus theory and the tangent modulus theory.

For all its refinements and limitations, the Euler buckling formula is still used three centuries later for column design and is still valid for long columns with pin-supported ends. Such is the power of logical thinking.

11.5 CHAPTER CONNECTOR

In past chapters, our analysis was based on the equilibrium of forces and moments. This chapter emphasized that not only equilibrium, but the *stability* of the equilibrium is an important consideration in design. There are many types of instabilities. We studied how coupled axial and bending deformation, for example, can produce buckling in columns. This case emphasizes the need for caution in decoupling phenomena for ease of understanding.

All our theories have relied on an equilibrium approach. An alternative approach can be used to replace the last link in the logic Figure 3.12. Though our theories will have the same assumptions and limitations, the energy method has a very different perspective from equilibrium methods, as discussed in the next and last chapter of this book.

POINTS AND FORMULAS TO REMEMBER

- Buckling is the instability in equilibrium of a structure due to compressive forces or stresses.
- Structural members that support compressive axial loads are called columns.
- Study of buckling as a *bifurcation* problem requires determining the critical buckling load at the point where two or more solutions exist for deformation.
- Study of buckling by the *energy method* requires determining the critical buckling load at the point the potential energy changes from a concave to a convex function.
- Study of buckling as an *eigenvalue* problem requires determining the critical buckling load at the point where a nontrivial solution exists for bending deformation due to axial loading.
- In *snap buckling* the structure snaps(or jumps) from one equilibrium configuration to a very different equilibrium configuration at the critical buckling load.
- *Local buckling* of thin structural members occurs due to compressive stresses.
- Buckling of columns occurs about an axis that has a minimum value of area moment of inertia.
- The Euler buckling load is $P_{cr} = \dfrac{\pi^2 EI}{L^2}$ (11.9)

  where P_{cr} is the critical buckling load, E is the modulus of elasticity, L is the length of the column, and I is the minimum area moment of inertia of the cross section.
- Equation (11.9) is valid only for elastic columns with pin-held ends.
- The effect of supports at the end can be incorporated by defining an effective length L_{eff} for a column and calculating the critical buckling load from

$$P_{cr} = \frac{\pi^2 EI}{L_{eff}^2} \qquad (11.11)$$

- The *Slenderness ratio* is defined as L_{eff}/r, where r is the radius of gyration about the buckling axis.
- The slenderness ratio at which the maximum normal stress is equal to the yield stress separates the short columns from the long columns in Euler buckling.
- The failure of short columns is governed by material strength.
- The failure of long columns is governed by Euler buckling loads.
- Eccentricity in loading does not affect the critical buckling load, but the maximum normal stress becomes significantly larger than the axial stress due to the addition of bending normal stress,

$$v_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \quad (11.22) \qquad \sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (11.24)$$

  where v_{max} is the maximum deflection, e is the eccentricity in loading, P is the applied axial load, P_{cr} is the Euler buckling load for the column, σ_{max} is the maximum normal stress in the column, r is the radius of gyration about the buckling (bending) axis, c is the maximum distance perpendicular to the buckling (bending) axis, A is the cross-sectional area, and L_{eff} is the effective length of the column.
- The *eccentricity ratio* is defined as ec/r^2.